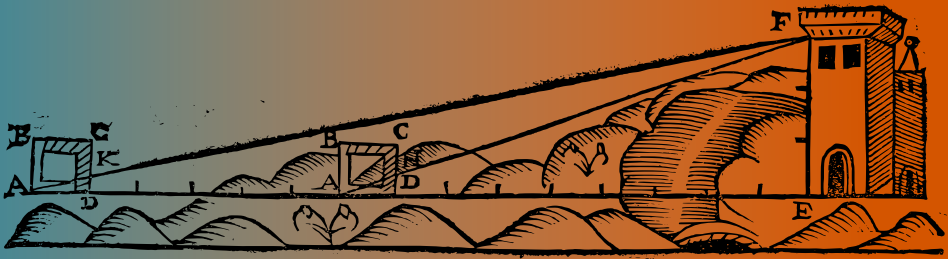
Yakov Perelman

GEOMETRY *for* ENTERTAINMENT



The Mir Titles Project

Geometry for Entertainment

Yakov Perelman

2024

Ya. I. Perelman

Geometry for Entertainment

The Mir Titles Project

Seventh Edition, Revised

Edited and supplemented by B. A. Kordemsky

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Translated from the Russian and typeset in LaTeX by *Damitr Mazanav*.

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Source files available at https://gitlab.com/mirtitles/perelman-geometry

Front cover: Woodcut from Cosimo Bartoli's *Del modo di misvrare* published in 1564. From Stillman Drake's collection https://archive.org/details/cosimobartolidel00bart.

Contents

Editor’s Preface

Geometry for Entertainment is written both for friends of mathematics and for those readers from whom many attractive aspects of mathematics have somehow been hidden.

More importantly, this book is intended for those readers who studied (or are currently studying) geometry only at the blackboard and therefore are not used to noticing familiar geometric relationships in the world of things and phenomena around us, have not learnt to use the acquired geometric knowledge in practise, in difficult cases of life, on a hike, in a bivouac or front-line situation.

To arouse the reader's interest in geometry or, in the words of the author, "to inspire a desire and cultivate a taste for its study is the objective of this book."

To this end, the author will take geometry "out of the walls of the school room into the free air, into the forest, field, to the river, on the road, in order to indulge in relaxed geometric studies without a textbook and tables in the open air …", and draws the reader's attention to the pages of L. N. Tolstoy and A. P. Chekhov, Jules Verne and Mark Twain. He finds a theme for geometric problems in the works of N. V. Gogol and A. S. Pushkin, and finally offers the reader "a motley selection of problems, curious in plot, unexpected in result."

The seventh edition of *Geometry for Entertainment* is published without the direct participation of the author. Ya. I. Perelman died in Leningrad in 1942.

The new edition of the book contains almost all the articles of the previous edition, newly illustrated, edited and supplemented with facts and information from our Soviet reality, as well as a considerable number (about 30) additional articles.

I was guided by the desire to increase the “utility coefficient” of Ya. Perelman’s book, to make it even more effective and interesting, involving new readers in the ranks of friends of mathematics.

To what extent this was possible, I hope to learn from readers at the address: Moscow, 64, Chernyshevsky Str., 81, Sq. 53, B. A. Kordemsky.

B. Kordemsky

Translator's Preface

Yakov Perelman's books have been a constant source of inspiration for me throughout my life. Though many of his works have been translated to English and other languages, several works remain untranslated. As a part of Mir Titles Project we endeavour to bring all such works to the people. This translation is a rather ambitious project and it brings me great pleasure to present this untranslated work of Perelman into an English version.

Illustrations

The beautiful and abundant illustrations in the form of woodcuts, are the heart of the book. Geometry being primarily reliant on illustrations, is brought to life in a variety of situations. Familiar geometrical shapes, lines, ratios are found amongst trees, rivers, homes, skies and other natural settings. This makes everyday objects the familiarly mathematical.

Each topic is complemented by relevant illustrations which make understanding them easier. Being woodcuts, it was easy for me to convert them to digital form. I have made no effort to change the images, except in some cases replacing the Russian letters with Roman ones.

Examples

In his discussions emphasises the geometrical relationships in the measurable and unknown quantities. This approach is historical in the sense that this is how geometry developed: to solve problems of measurement of unknown quantities. Thus we have problems related to measuring a variety of things, using direct measurement or very primitive instruments.

Translation

I have made use of machine translations for the bulk of text, and it has worked at a satisfactory level. At times I have made use of several translation services to make sure I am on right track and the meaning is not lost in translation.

Typesetting

During the course of typesetting this book, discussions posted on and help from kind people LaTeX forum at stackexchange has been of great help. I have typeset the book in a square profile with marginpar for smaller figures and notes.

If there are any mistakes in the mathematics or translation they are all mine. Any suggestions and criticisms to improve the translation are welcome. I hope that this English version finds enthusiastic readers and inspires many more brilliant minds in the generations to come.

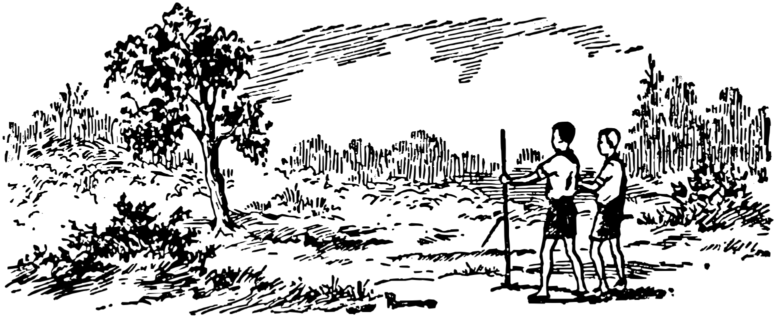
Damitr Mazanav

Part I.

Geometry In The Open Air

Nature speaks the language of mathematics: the letters of this language are circles, triangles and other mathematical shapes.

Galileo



1. Geometry In The Forest

1.1. By the length of the shadow

I remember now the amazement with which I looked

for the first time, he looked at a gray-haired forester, who, standing near a huge pine tree, measured its height with a small pocket device. When he aimed his square board at the top of the tree, I expected that the old man would now start climbing there with a measuring chain. Instead, he put the device back in his pocket and announced that the measurement was over. I thought it hadn't started yet …

I was very young then, and this way of measuring, when a person determines the height of a tree without cutting it down and climbing to the top, was in my eyes something like a small miracle. It was only later, when I was initiated into the rudiments of geometry, that I realised how simple such miracles are performed. There are many different ways to make such measurements using very simple instruments and even without any devices.

The easiest and most ancient way is, without a doubt, the one by which the Greek sage Thales determined the height of the pyramid in Egypt sixth century BC. He took advantage of the pyramid's 'shadow'. The priests and the pharaoh, gathered at the foot of the highest pyramid, looked puzzled at the northern newcomer, who guessed the height of the huge structure from the shadow. Thales, says the legend, chose a day and an hour when the length of his own shadow was equal to his height; at this moment, the height of the pyramid should also be equal to the length of the shadow cast by it[1]. This is perhaps the only case when a person benefits from his shadow …

[1] Of course, the length of the shadow had to be measured from the midpoint of the square base of the pyramid; Thales could directly measure the width of this base.

The task of the Greek sage now seems childishly simple to us, but let's not forget that we are looking at it from the height of a geometric building erected after Thales. He lived long before Euclid, the author of the wonderful book that taught geometry for two millennia after his death. The truths contained in it, which are now known to every schoolboy, were not yet discovered in the era of Thales. And in order to use the shadow to solve the problem of the height of the pyramid, it was necessary to already know some geometric properties of the triangle, namely the following two (of which Thales himself discovered the first):

1. that the angles at the base of an isosceles triangle are equal, and vice versa – that the sides lying opposite the equal angles of the triangle are equal to each other;
2. that the sum of the angles of any triangle (or at least a rectangular one) is equal to two right angles.

Only Thales, armed with this knowledge, had the right to conclude that when his own shadow is equal to his height, the sun's rays meet the flat ground at an angle of half a straight line, and therefore the top of the pyramid, the middle of its base and the end of its shadow should mark an isosceles triangle.

It would seem that this simple method is very convenient to use on a clear sunny day to measure lonely trees whose shadow does not merge with the shadow of neighbouring ones. But in our latitudes it is not as easy as in Egypt to waylay the right moment for this: The sun is low above the horizon, and the shadows are equal to the height of the objects casting them only in the afternoon hours of the summer months. Therefore, the Thales method in this form is not always applicable.

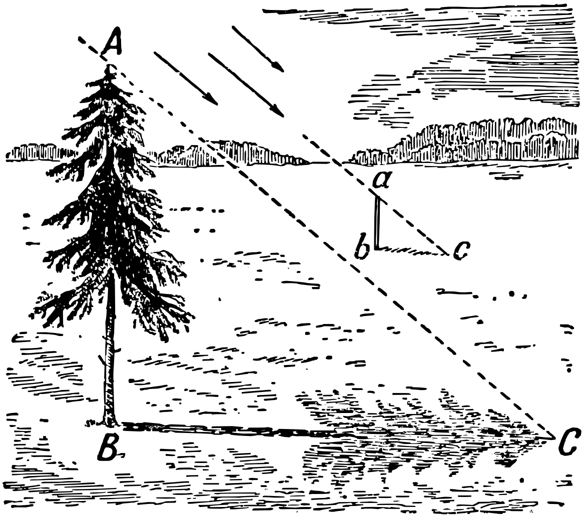


Figure 1.: Measuring the height of a tree by shadow.

It is not difficult, however, to modify this method so that on a sunny day, any shadow can be used, regardless of its length. Additionally, measuring both your own shadow and the shadow of a pole, the desired height is calculated from the proportion (Figure 1):

$$AB : ab = BC : bc,$$

meaning the height of the tree is as many times greater than your own height (or the height of the pole) as the shadow of the tree is longer than your shadow (or the shadow of the pole). This naturally follows from the geometric similarity of triangles ABC and abc (based on two angles).

Some readers may object that such an elementary technique does not need a geometric justification at all: is it really unclear even without geometry that how many times is a tree taller, how many times is its shadow longer? However, the matter is not as simple as it seems. Try to apply this rule to shadows cast by the light of a street lamp or lamp – it will not be justified. In Figure 2 you can see that the columns AB are about three times higher than the pedestal ab, and the shadow of the column is eight times larger than the shadow of the pedestal ($BC : bc$). It is impossible to explain why the method is applicable in this case, but not in the other, without geometry.

Question

Let's take a closer look at what the difference is. The essence of the matter boils down. to the fact that the sun's rays are

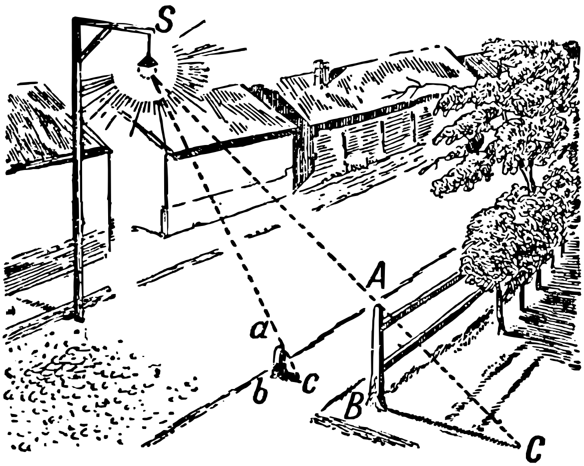


Figure 2.: When such a measurement is impossible. (Is the method applicable for a shadow cast by a streetlamp?)

parallel to each other, the rays of the lantern are not parallel. After that, we have the right to consider the rays of the Sun parallel, although they certainly intersect in the place from which they originate.

Answer

The rays of the Sun falling on the Earth can be considered parallel because the angle between them is extremely small, almost imperceptible. A simple geometric calculation will convince you of this. Imagine two rays coming from some point of the Sun and falling on the Earth at a distance of, say, one kilo-meter from each other. So, if we put one leg of a compass at this point of the Sun, and with the other we described a circle with a radius equal to the distance from the Sun to the Earth (i.e., with a radius of 150,000,000 km), then an arc of one kilometer in length would appear between our two radii rays. The total length of this gigantic circle would be equal to $2\pi \times 150{,}000{,}000$ km $= 940{,}000{,}000$ km. One degree of it, of course, is 360 times less, i.e. about 2,600,000 km; one arc minute is 60 times less than a degree, i.e. equal to 43,000 km, and one arc second is another 60 times less, i.e. 720 km. But our arc is only 1 km in length, so it corresponds to an angle of $1/720 \approx 0.001{,}38''$ seconds. This angle is elusive even for the most accurate astronomical instruments; therefore, in practise we can consider the rays of the Sun falling on the Earth as parallel lines.[2]

[2] Another thing is the rays directed from some point of the Sun to the ends of the earth's diameter; the angle between them is large enough to measure (about $17''$); the definition of this angle gave astronomers one of the means to establish how great the distance from the Earth to the Sun is.

Trying to apply the method of shadows in practise, you will immediately be convinced, however, of its unreliability. Shadows are not delimited so clearly that measuring their length can be done quite accurately. Each shadow cast by the light of the Sun has an indistinctly outlined grey border of penumbra, which gives the border of the shadow uncertainty. This is because the Sun is not a point, but a large luminous body emitting rays from many points. Figure 3 illustrates why, as a result of this, the shadow of tree AB also has an additional component in the form of half-shadow CD, gradually fading away.

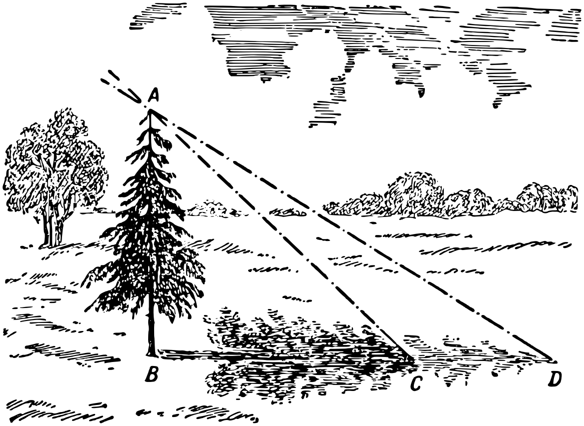


Figure 3.: How penumbra is formed.

The angle of the *CAD* between the extreme boundaries of the penumbra is equal to the angle at which we always see the solar disk, i.e. half a degree. The error resulting from the fact that both shadows are not measured quite accurately can reach 5% or more when the Sun is not too low. This error is added to other unavoidable errors – from uneven soil, etc. – and makes the final result little reliable. In mountainous terrain, for example, this method is completely inapplicable.

1.2. Two More Methods

It is entirely possible to measure height without relying on shadows. There are many methods; let's start with two simple ones.

Firstly, we can utilise the properties of an isosceles right triangle. For this purpose, we can make use of a very simple tool, which can be easily crafted from a piece of board and three pins. On a board of any shape, even a piece of bark with a flat side, mark three points to form the vertices of a right triangle – and insert a pin at each point (see Figure 4). Suppose you don't have a drafting triangle to construct a right angle, nor a compass to mark equal sides. In that case, fold any piece of paper once, and then fold it again across the first fold so that both parts of the first fold coincide – and you'll obtain a right angle. The same piece of paper can be used instead of a compass to measure equal distances.

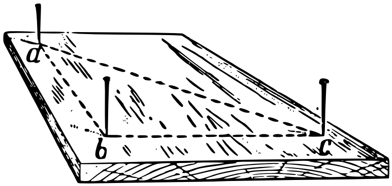


Figure 4.: Pin height measuring device.

As you can see, the tool can be entirely crafted in a makeshift environment.

If you don't have a drafting triangle on hand to construct a right angle, nor a compass to mark equal sides, then simply fold any scrap of paper once, and then fold it again across the first fold so that both parts of the first fold coincide—and you'll obtain a right angle. The same piece of paper can be used instead of a compass to measure equal distances.

As you can see, the tool can be entirely crafted in a makeshift environment.

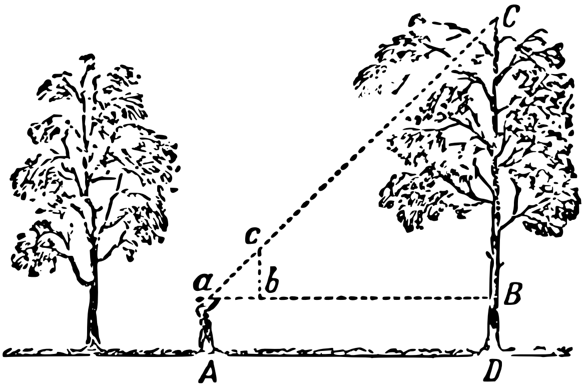


Figure 5.: The scheme of application of the pin device.

Handling it is no more difficult than crafting it. Stepping away from the tree being measured, hold the tool so that one of the legs of the triangle is perpendicular. You can use a string or a weight tied to the top pin. Approaching or moving away from the tree, you will always find a spot A (see Figure 5), from which, looking at pins a and c, you will see that they cover the top C of the tree: this means that the extension of the hypotenuse ac passes through point C. Then, obviously, the distance aB is equal to CB, since angle $a = 45°$.

Consequently, by measuring distance aB (or, at another location, distance AD) and adding BD to it, i.e., the elevation of point a above the ground, you will obtain the desired height of the tree.

Another method does not even require a pin device. Here you need a pole, which you will have to insert vertically into the ground so that the protruding part is exactly at your height. The location for the pole must be chosen so that, lying down as shown in Figure 6, you see the top of the tree in a straight line with the upper point of the pole. Since triangle Aba is isosceles and right-angled, angle $A = 45°$, and therefore AB equals BC, i.e., the desired height of the tree.

1.3. The Method of Jules Verne

The next, also quite simple, method for measuring tall objects is vividly described by Jules Verne in his famous novel *The*

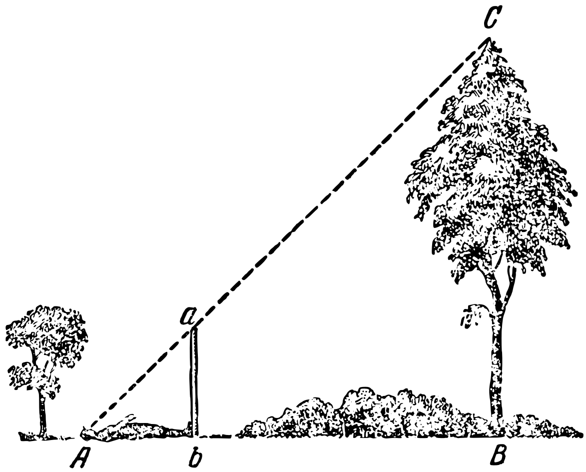


Figure 6.: Another way to determine the height.

Mysterious Island.

“Today we need to measure the height of the Far View platform,” said the engineer.

“Will you need a tool for that?” asked Herbert.

“No, we won’t. We’ll proceed somewhat differently, resorting to a somewhat simpler and more accurate method.”

The young man, eager to learn as much as possible, followed the engineer, who descended from the granite wall to the rocky shore.

Taking a straight pole, twelve feet long, the engineer measured it as precisely as possible, comparing it to his own height, which he knew well. Meanwhile, Herbert held a plumb bob given to him by the engineer: just a stone attached to the end of a rope.

Not reaching five hundred feet from the granite wall, which rose vertically, the engineer drove the pole two feet into the sand and firmly secured it, placing it vertically with the help of the plumb bob.

Then he moved away from the pole to a distance where, lying on the sand, one could see both the end of the pole and the edge of the ridge in a straight line (see Figure 7). He carefully marked this point with a stake.

"Are you familiar with the basics of geometry?" he asked Herbert as he rose from the ground.

"Yes."

"Do you remember the properties of similar triangles?"

"Their corresponding sides are proportional."

"Exactly. So now I'll construct two similar right triangles. In the smaller one, one leg will be the plumb-line pole, and the other will be the distance from the stake to the base of

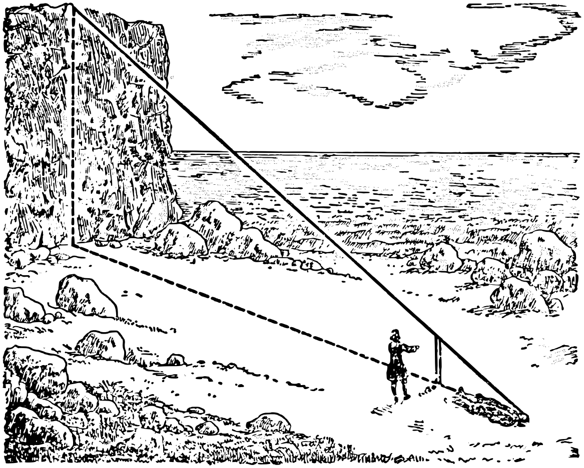


Figure 7.: How the heroes of Jules Verne measured the height of the cliff.

the pole; the hypotenuse will be my line of sight. In the other triangle, the legs will be: the granite wall, the height of which we want to determine, and the distance from the stake to the base of this wall; the hypotenuse will be my line of sight, coinciding with the direction of the hypotenuse of the first triangle."

"Understood!" exclaimed the youth. "The distance from the stake to the pole is related to the distance from the stake to the base of the wall, as the height of the pole is to the height of the wall."

"Yes. And consequently, if we measure the first two distances, then, knowing the height of the pole, we can calculate the fourth, unknown term of the proportion, i.e., the height of the wall. In this way, we can manage without directly measuring the height."

Both horizontal distances were measured: the smaller one was 15 feet, the larger one was 500 feet.

At the end of the measurements, the engineer made the following record:

$$15 : 500 = 10 : x,$$
$$500 \times 10 = 5000,$$
$$5000 : 15 = 333.3.$$

Thus, the height of the granite wall was 333 feet.

1.4. How Sergeant Popov Acted

Some of the methods described for measuring height are inconvenient as they require lying on the ground. This inconvenience can, of course, be avoided.

Here's a story from one of the fronts of the Great Patriotic War. Lieutenant Ivanyuk's unit was ordered to build a bridge across a mountain river. On the opposite bank were entrenched fascists. To scout the location for the bridge, the lieutenant assigned a reconnaissance group led by Senior Sergeant Popov. In the nearest forest, they measured the diameter and height of the most typical trees and counted the number of trees that could be used for construction.

They measured the height of the trees using a pole (stick) as shown in Figure 8.

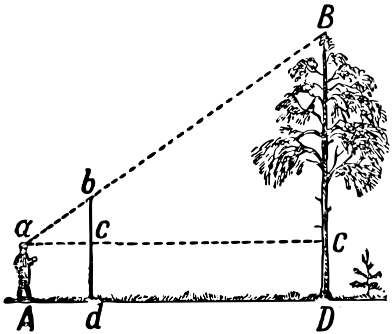


Figure 8.: Measuring the height of the trees with a pole.

This method works as follows: Armed with a pole taller than your own height, drive it into the ground vertically at some distance from the tree being measured (see Figure 8). Step back from the pole along the line dD until you reach point A, from where, looking at the top of the tree, you'll see the upper point B of the pole aligned with it. Then, without changing the position of your head, look along the horizontal line aC, noting the point C where your line of sight intersects the pole and the tree trunk. Ask your assistant to mark these points, and the observation is complete. Then, based on the similarity of triangles abc and aBC, calculate BC from the proportion

$$\frac{BC}{bc} = \frac{aC}{ac},$$

and thus

$$BC = bc \cdot \frac{aC}{ac}$$

The distances bc, aC, and ac can be easily measured directly. To obtain the actual height of the tree, add the distance BC to the distance CD, which is also measured directly.

To determine the number of trees, the senior sergeant ordered the soldiers to measure the area of the forest. Then he counted the number of trees in a small area measuring 50 by 50 meters and multiplied accordingly.

Based on all the data collected by the scouts, the unit commander determined where and what kind of bridge needed to be built. The bridge was completed on time, and the combat mission was successfully accomplished![3]

[3] The episodes of the Great Patriotic War described here and further are narrated by A. Demidov in the journal *Military Knowledge* No. 8, 1949, in the article *River Reconnaissance*.

1.5. Using a Notebook

As a device for an approximate estimate of the inaccessible height, you can also use your pocket back book, if it is equipped with a pencil stuck in a cover or a loop with a book. It will help you to build in space those two similar triangles, from which the desired height is obtained. The book should be held near the eyes as shown in the simplified Figure 9. It should be in the vertical book so that, looking from the point *a*, you can see the top of the tree *B* covered with the tip of the pencil *b*. Then, due to the similarity of the triangles *abc* and *aBC*, the height of the *BC* will be determined from the proportion

$$\frac{BC}{bc} = \frac{aC}{ac}.$$

The distances of *bc*, *ac* and *aC* are measured directly. To the resulting value of the *BC*, add the length of *CD*, which is, on level ground, the height of the eyes above the ground

Since the width of the *ac* book is unchanged, if you always stand at the same distance from the measured tree (for example, 10 m), the height of the tree will depend only on the extended part of the pencil. Therefore, you can calculate in advance what height corresponds to a particular extension, and put these numbers on the pencil. Your notebook will then turn into a simplified altimeter, since you can use it to determine heights immediately, without calculations.

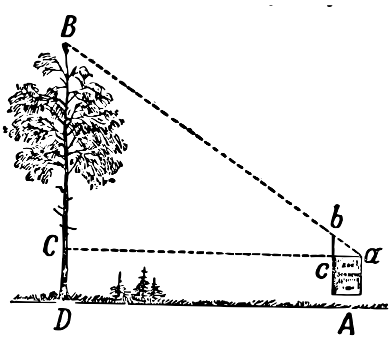


Figure 9.: Height measurement using a notebook.

1.6. Without Approaching The Tree

Sometimes it may be inconvenient to get close to the base of the tree being measured. Can its height still be determined in such a case?

Absolutely. For this purpose, a clever device has been devised, which, like the previous ones, is easy to make by yourself. Two planks, *ab* and *cd* (top of Figure 10), are fastened together at right angles so that *ab* equals *bc*, and *bd* equals half of *ab*. That's the whole device.

To measure height with it, hold it in your hands, directing plank CD vertically (for which it has a plumb line with a weight), and stand precisely in two places: first (Figure 10) at point *A*, where the device is positioned with end *c* up, and then at point A', a bit farther away, where the device is held with end d up. Point A is chosen so that, looking from a to the end of a, it is seen on the same line as the top of the tree. Point A' is found so that, looking from a' to point d', it is seen coinciding with B.

The discovery of these two points A and A'[4] constitutes all the measurement because the desired part of the tree's height, BC, is equal to the distance DA'. The equality follows easily from the fact that $aC = BC$ and $a'C = 2BC$; thus,

[4] These points must necessarily lie in a straight line with the base of the tree.

$$a'C - aC = BC.$$

You can see that using this simple device, we measure the tree's height without approaching closer than its height. It goes without saying that if it's possible to approach the trunk, it's sufficient to find just one of the points – A or A' – to determine its height.

Instead of two planks, you can use four pins, arranging them on a board properly; in this form, the "device" is even simpler.

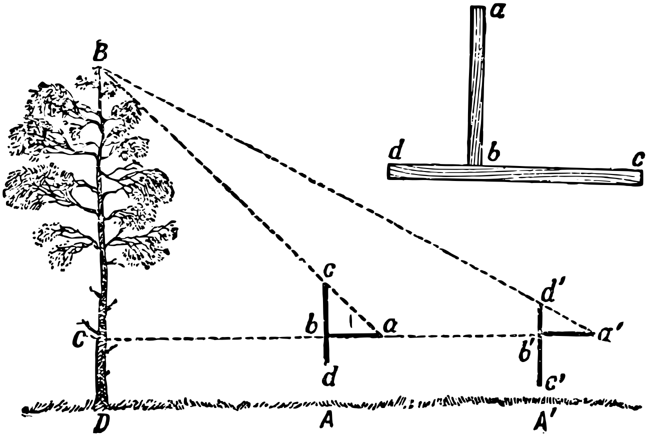


Figure 10.: The use of a simple altimeter consisting of two planks.

1.7. Forest Rangers’ Altimeter

It’s time to explain how the “real” altimeters, used in practise by forest workers, are constructed. I’ll describe one of these altimeters, slightly modifying it so that the device can be easily crafted at home.

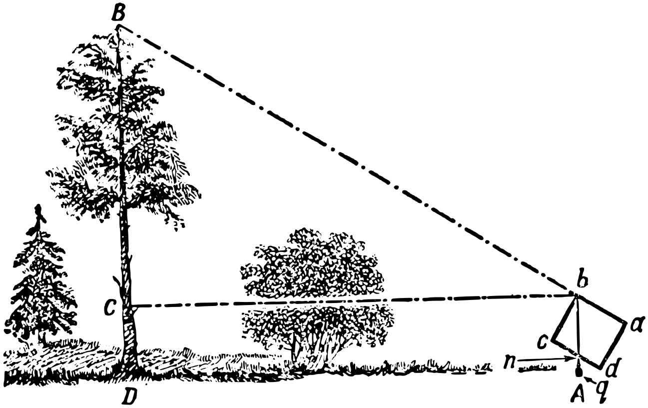


Figure 11.: The scheme of using the altimeter of foresters.

The essence of the device is visible in Figure 11. A cardboard or wooden rectangle, *abcd*, is held in the hand so that, looking along edge *ab*, the tip *B* of the tree is in line with it. A weight, *q*, is suspended from point *b* on a thread. We note the point *n* where the thread intersects line *dc*. Triangles *bBC* and *bnc* are similar because they are both rectangular and have equal acute angles bBC and bnc (with corresponding parallel sides). Therefore, we can write the proportion:

$$\frac{BC}{nc} = \frac{bC}{bc}; \text{ hence}$$
$$BC = bC \cdot \frac{nc}{bc}.$$

Since bC, nc, and bc can be measured directly, it is easy to obtain the desired height of the tree by adding the length of the lower part CD to the trunk (the height of the device above the ground).

A few details remain to be added. If the edge of the board bc is made, for example, exactly 10 cm, and centimeter divisions are marked on edge dc, then the ratio nc/bc will always be expressed as a decimal fraction, directly indicating what fraction of the distance bC represents the height of the tree BC. For example, let's say the thread stops against the 7th division mark (i.e., $nc = 7$ cm); this means that the height of the tree above eye level is 0.7 times the observer's distance from the trunk.

The second improvement relates to the method of observation: to make it convenient to look along line ab, you can fold down two squares with holes drilled in them at the upper corners of the cardboard rectangle: one smaller one for the eye

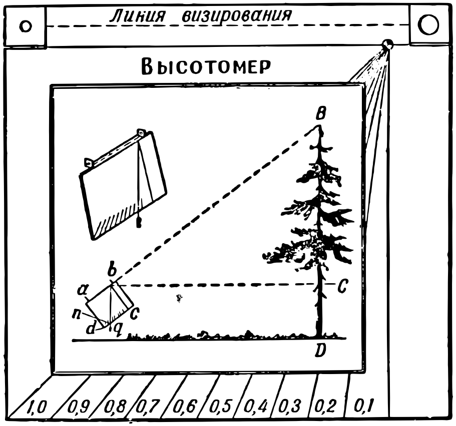


Figure 12.: The forest rangers' altimeter.

and one larger one for sighting the tree top (see Figure 11). Further enhancement is represented by the device shown almost to scale in Figure 12. It is easy and quick to make it in this form; no special skill is required. Occupying little space in the pocket, it will provide you with the ability to quickly determine the heights of encountered objects during excursions—trees, poles, buildings, and so on. (This tool is part of the *Geometry in the Open Air* kit developed by the author of this book.)

Question

Is it possible to use the altimeter described now to measure trees that cannot be approached closely? If possible, what should be done In such cases?

Answer

The device should be aimed at the top of the tree B, as shown in Figure 13, from two points, A and A'.

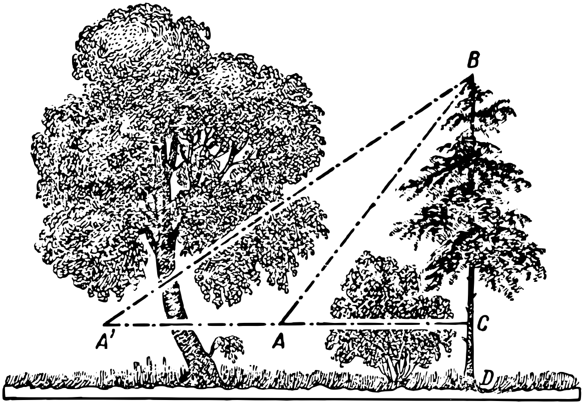


Figure 13.: How to measure the height of a tree without approaching it.

Let's say at point A we determined that $BC = 0.9\,AC$, and at point A' we determined that $BC = 0.4\,A'C$. Then we know that:

$$AC = \frac{BC}{0.9}, \quad A'C = \frac{BC}{0.4}$$

So that we can write

$$AA' = A'C - AC = \frac{BC}{0.4} - \frac{BC}{0.9} = \frac{25}{18}\,BC.$$

Hence,

$$
\begin{aligned}
AA' &= \frac{25}{18}\,BC,\\
\therefore\ BC &= \frac{18}{25}\,AA'\\
&= 0.72\,AA'.
\end{aligned}
$$

You can see that by measuring the distance AA' between both observation points and taking a certain fraction of this value, we can determine the desired and inaccessible height.

1.8. Using a Mirror

Question

Here's another unconventional method for determining the height of a tree using a mirror. At some distance (see Figure 14) from the tree being measured, on level ground at point C, place a small mirror horizontally and step back to point D, from where the observer can see the top of tree's point A in the mirror. Then, the tree (AB) is as many times taller than the observer's height (ED) as the distance BC from the mirror to the tree is greater than the distance CD from the mirror to the observer. Why?

Answer

The method is based on the law of reflection of light. The top A (Figure 15) is reflected at point A' in such a way that $AB = A'B$. From the similarity of triangles BCA' and CED,

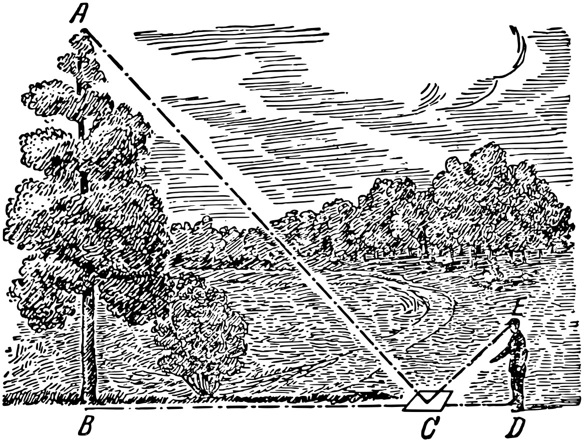


Figure 14.: Height measurement using a mirror.

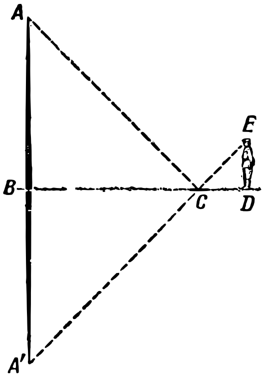


Figure 15.: Geometric construction for the method of measuring the height using a mirror.

it follows that

$$\frac{A'B}{ED} = \frac{BC}{CD}.$$

In this, simply replace $A'B$ with AB to justify the relationship stated in the problem. This convenient and effortless method can be applied in any weather, but not in dense vegetation, only to a solitary tree.

Question

However, what should be done when it is impossible to approach the tree being measured closely for some reason?

Answer

This is an ancient problem dating back over 500 years. It is discussed by the medieval mathematician Antonius de Cremona in his work *On Practical Land Measurement* (1400).

The problem is solved by the dual application of the method described earlier – placing the mirror in two locations. By making the appropriate construction, it is easy to deduce from the similarity of triangles that the sought-after height of the tree is equal to the observer's eye level multiplied by the ratio of the distance between the mirror positions to the difference in distances from the mirror to the observer.

Before concluding the discussion on measuring the height of trees, I propose to the reader another "forest" problem.

1.9. Two Pines

Question

Two pine trees grow 40 meters apart. You measured their heights: one turned out to be 31 meters tall, while the other, younger one, is only 6 meters tall. Can you calculate the distance between their tops?

Answer

The desired distance between the tops of the pine trees (see

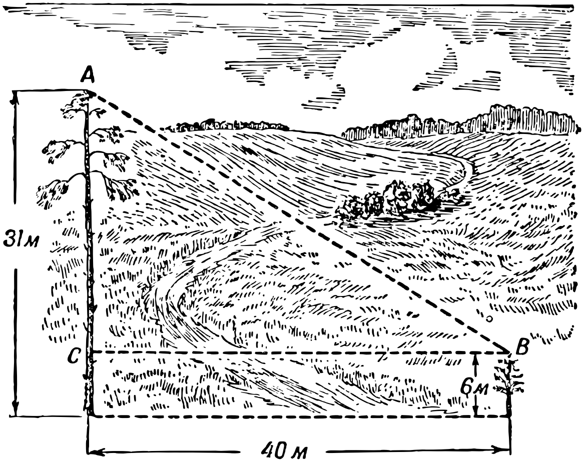


Figure 16.: What is the distance between the tops of the pines?

Figure 16) according to the Pythagorean theorem is

$$\sqrt{40^2 + 25^2} = 47 \text{ m}.$$

1.10. The shape of the tree trunk

Now you can already, walking through the forest, determine – in almost half a dozen different ways – the height of any tree. It will probably be interesting for you to determine its volume as well, calculate how many cubic meters of wood it contains, and at the same time weigh it. To find out if, for example, it would be possible to take away such a trunk on one cart. Both of these tasks are no longer as simple as determining height; experts have not found ways to accurately resolve it and are content with only a more or less approximate estimate. Even for a felled trunk, which lies in front of you cleared of branches, the task is far from easy.

The thing is, a tree trunk, even the smoothest one without bulges, does not represent either a cylinder, a complete cone, a truncated cone, or any other geometric solid whose volume we can calculate using formulas. The trunk is certainly not a cylinder — it tapers towards the top (it has ”runoff,” as foresters say) — but it is also not a cone because its ”generating line” is not a straight line, but a curve, and moreover, not a circular arc, but some other curve, convex towards the axis of the tree.[5]

[5] The curve that fits closest to this is called the “semicubical parabola” ($y^3 = ax^2$); the solid obtained by rotating this parabola is called a “neiloid” (named after the ancient mathematician Neil, who found a way to determine the length of the arc of such a curve). The shape of a tree trunk grown in the forest approximates that of a neiloid. Calculating the volume of a neiloid is done using advanced mathematical techniques.

Therefore, a more or less accurate calculation of the volume of a tree trunk can only be done using the tools of integral calculus. To some readers, it may seem strange that the measurement of a simple log requires resorting to the services of higher mathematics. Many think that higher mathematics is only relevant to some special subjects, whereas in everyday life, only elementary mathematics is applicable. This is completely incorrect: one can fairly accurately calculate the volume of a star or a planet using elements of geometry, whereas an exact calculation of the volume of a long log or a beer barrel is impossible without analytical geometry and integral calculus.

However, our book does not assume that the reader is familiar with higher mathematics; therefore, here we will have to be content with only an approximate calculation of the volume of the trunk. We will assume that the volume of the trunk is more or less close either to the volume of a truncated cone, or – for a trunk with a pointed end — to the volume of a complete cone, or, finally, – for short logs — to the volume of a cylinder. The volume of each of these three solids can be easily calculated. Could we find a formula for the volume that would be suitable for all three of these named solids for the sake of consistency in calculation? Then we would approximately calculate the volume of the trunk without caring about what it resembles more — a cylinder or a cone, complete or truncated.

1.11. Universal Formula

Such a formula exists; moreover, it is not only suitable for cylinders, complete and truncated cones, but also for all kinds of prisms, pyramids complete and truncated, and even for spheres. Here is this remarkable formula, known in mathematics as Simpson's formula:

$$v = \frac{h}{6}\left(b_1 + 4b_2 + b_3\right)$$

where h is the height of the solid, b_1 is the area of the lower base, b_2 is the area of the middle section[6], b_3 is the area of the upper base.

[6] That is, the cross-sectional area of the body in the middle of its height.

Question

Prove that with this formula, one can calculate the volume of the following seven geometric solids: prism, pyramid complete, pyramid truncated, cylinder, cone complete, cone truncated, sphere.

Answer

It is very easy to verify the correctness of this formula by simply applying it to the listed solids. Then, we obtain for the prism and cylinder (see Figure 17 a):

$$v = \frac{h}{6}\left(b_1 + 4b_2 + b_3\right) = b_1 h;$$

for the pyramid and cone (see Figure 17 b):

$$v = \frac{h}{6}\left(b_1 + 4\frac{b_2}{4} + 0\right) = \frac{b_1 h}{3};$$

for the truncated cone (see Figure 17 c):

$$\begin{aligned} v &= \frac{h}{6}\left[\pi R^2 + 4\pi\frac{(R+r)^2}{2} + \pi r^2\right] \\ &= \frac{h}{6}\left[\pi R^2 + \pi R^2 + 2\pi Rr + \pi r^2 + \pi r^2\right] \\ &= \frac{\pi h}{3}\left[R^2 + Rr + r^2\right] \end{aligned}$$

for the truncated pyramid, the proof proceeds similarly; finally, for the sphere (see Figure 17 d):

$$v = \frac{2R}{6}\left(0 + 4\pi R^2 4 + 0\right) = \frac{4}{3}\pi R^3.$$

Question

Let's note another interesting feature of our universal formula: it is also suitable for calculating the area of plane figures: parallelograms, trapezoids, and triangles, if by

§ h we mean, as before, the height of the figure,

§ by b_1 the length of the lower base,

§ by b_2 the length of the middle base and

§ by b_3 the length of the lower base.

How can we confirm this?

Answer

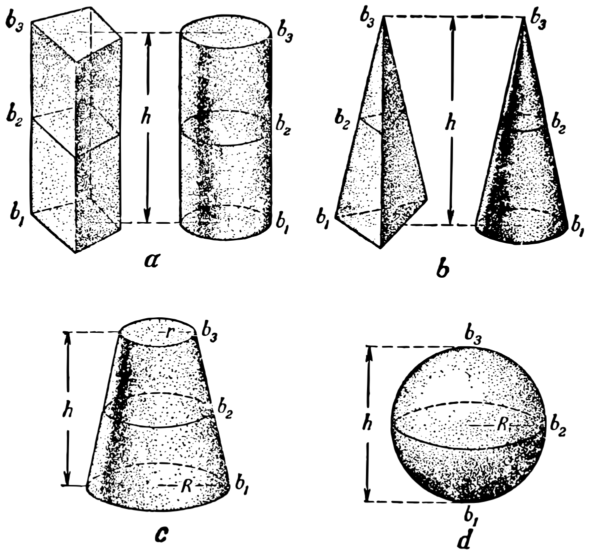


Figure 17.: Geometric bodies whose volumes can be calculated using a single formula.

Applying the formula, we have: for a parallelogram (square, rectangle) (see Figure 18 a)

$$S = \frac{h}{6}\,(b_1 + 4b_1 + b_1) = b_1 h;$$

for a trapezoid (see Figure 18 b)

$$S = \frac{h}{6}\left(b_1 + 4\frac{b_1 + b_2}{2} + b_3\right) = \frac{h}{2}\,(b_1 + b_3);$$

for a triangle (see Figure 18 c)

$$S = \frac{h}{6}\left(b_1 + 4\frac{b_1}{2} + 0\right) = \frac{b_1 h}{2}.$$

You can see that our formula has enough right to be called universal.

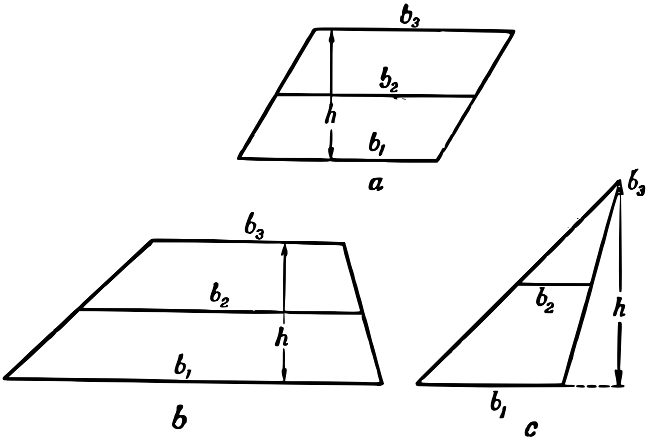


Figure 18.: The universal formula is also suitable for calculating the areas of these figures.

1.12. Volume and Weight of a Tree at the Root

So, you have a formula with which you can approximately calculate the volume of a felled tree trunk without worrying about what geometric shape it resembles: a cylinder, a complete cone, or a truncated cone. For this, four measurements are needed – the length of the trunk and three diameters: the lower cut, the upper, and in the middle of the length. Measuring the lower and upper diameters is very simple; however, determining the average diameter without a special device ("measuring fork" used by foresters, see Figure 19 and Figure 20[7] is quite difficult. But the difficulty can be overcome by encircling the trunk with a rope and dividing its length by 3 1/7 to get the diameter.

[7] A similar principle is applied in the well-known device for measuring the diameter of round objects – the caliper Figure 20, to the right).

The volume of a felled tree trunk obtained in this way is accurate enough for many practical purposes. In short, but less accurately, this problem can be solved by calculating the volume of the trunk as the volume of a cylinder, the diameter of the base of which is equal to the diameter of the trunk in the middle of its length; however, the result obtained is underestimated, sometimes by 12%. But if you mentally divide the trunk into two-meter segments and determine the volume of each of these almost cylindrical parts to then add them up, the result will be much better: it errs on the side of

Figure 19.: Measuring the diameter of a tree with a measuring fork.

underestimation by no more than 2–3%.

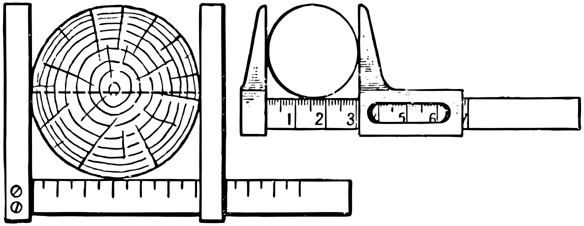


Figure 20.: Measuring fork (left) and caliper (right).

However, all this is completely inapplicable to a tree at the root: if you are not going to climb it, then you can only measure the diameter of its lower part. In this case, to determine the volume, you will have to be satisfied with only a very approximate estimate, comforting yourself with the fact that professional foresters usually proceed in a similar way. They also use a table of so-called “species numbers,” i.e., numbers that show what proportion of the volume of the measured tree is compared to the volume of a cylinder of the same height and diameter, measured at the height of a grown man’s chest, i.e., 130 cm (this height is the most convenient for measuring).

Figure 21 illustrates this clearly. Of course, “species numbers” vary for trees of different species and heights, as the shape of the trunk is variable. However, the fluctuations are not particularly great: for pine and fir trunks (grown in dense plantations), “species numbers” range from 0.45 to 0.51, i.e., are approximately half.

Thus, without much error, it can be assumed that the volume of a coniferous tree at the root is half the volume of a cylinder of the same height with a diameter equal to the diameter of the tree at chest height.

This is, of course, only an approximate estimate, but it is not too far from the true result: up to 2% in the overestimation direction and up to 10% in the underestimation

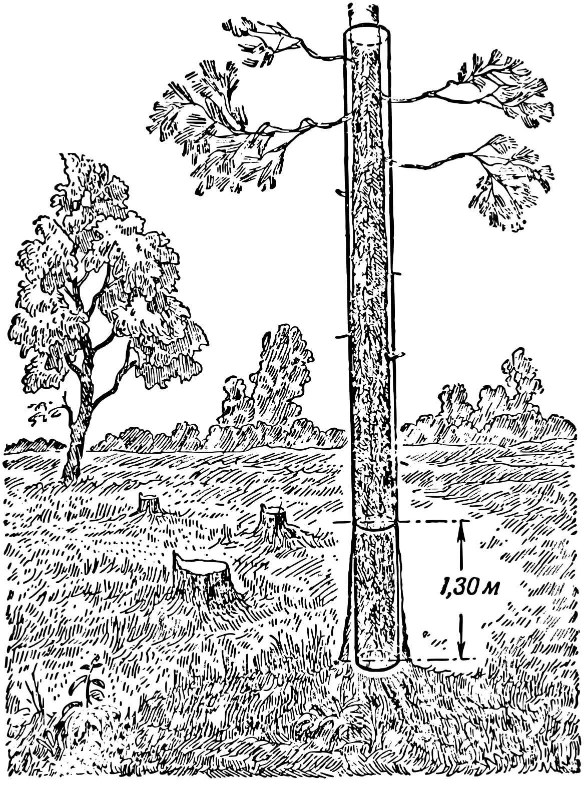


Figure 21.: What is a "species number"?

direction.[8]

From here, it is only one step towards estimating the weight of the tree at the root. For this, it is enough to know that 1 cubic meter of fresh pine or fir wood weighs about 600–700 kg. For example, suppose you are standing next to a fir tree, the height of which you have determined to be 28 m, and the circumference of the trunk at chest height is 120 cm. Then the area of the corresponding circle is 1,100 cm^2, or 0.11 m^2, and the volume of the trunk is $1/2 \times 0.11 \times 28 = 1.5\,\mathrm{m}^3$. Assuming that 1 cubic meter of fresh fir wood weighs on average 650 kg, we find that 1.0 cubic meter should weigh about a ton (1,000 kg).

[8] It must be remembered that "species numbers" refer only to trees that have grown in the forest, i.e. to tall and thin (smooth, without nodes); for free-standing branched trees, such general rules for calculating volume cannot be specified.

1.13. Leaf Geometry

Question

In the shadow of a silver poplar from its roots, a thicket has grown. Pick a leaf and notice how large it is compared to the leaves of the parent tree, especially those that grew in bright sunlight. The shaded leaves compensate for the lack of light with the size of their area, capturing sunlight rays. Understanding this is the task of botany. But the geometer can also have a say here: he can determine exactly how many times the area of the thicket leaf is larger than the area of the parent tree leaf.

How would you solve this problem?

Answer

You can go two ways. First, determine the area of each leaf separately and find their ratio. The area of the leaf can be measured by covering it with transparent grid paper, each square of which corresponds, for example, to 4 square millimeters (a sheet of transparent grid paper used for this purpose is called a pallet). This is a perfectly correct but overly laborious method.[9]

[9] However, this method has an advantage: using it, you can compare the areas of leaves with different shapes, which cannot be done according to the method described below.

A shorter method is based on the fact that both leaves, different in size, still have the same or almost the same shape: in other words, they are geometrically similar figures. We know that the areas of such figures are related as the squares of their linear dimensions. Therefore, by determining how many times one leaf is longer or wider than the other, we can find the ratio of their areas simply by squaring this number. Let the thicket leaf be 15 cm long, and the leaf from the tree branch only 4 cm long; the ratio of their linear dimensions is 15/4, and therefore, in terms of area, one is larger than the other by 225/16 times, or about 14. Rounding off (since full accuracy cannot be achieved here), we can say that the thicket leaf is approximately 15 times larger than the tree leaf in terms of area.

Let us consider another example.

Question

At a dandelion grown in shade, a leaf is 31 cm long. At another specimen grown in sunlight, the leaf blade is only 3.3 cm long. Approximately how many times is the area of the first leaf larger than the area of the second?

Answer

We proceed as before. The ratio of the areas is

$$\frac{31^2}{3.3^2} = \frac{960}{10.9} = 87;$$

so one leaf is approximately 90 times larger than the other in terms of area.

Figure 22.: Determine the ratio of the areas of these leaves.

It is easy to find in the forest many pairs of leaves of the same shape but different sizes, thus providing interesting material for geometric problems on the ratio of areas of similar figures. It always seems strange to an unaccustomed eye that a relatively small difference in the length and width of leaves results in a noticeable difference in their areas. For example, if two leaves, geometrically similar in shape, differ in length by 20%, then the ratio of their areas is

$$1.2^2 \approx 1.4,$$

meaning the difference is 40%. And with a difference in width of 40%, One leaf exceeds the other in area by

$$1.4^2 \approx 2,$$

or nearly twice.

We invite the reader to determine the ratio of the areas of the leaves depicted in Figure 22 and Figure 23.

Question

Figure 23.: Determine the ratio of the areas of these leaves.

1.14. Six-legged heroes

Amazing creatures, ants! Swiftly climbing up stems with a burden much heavier than their tiny size (Figure 24), ants present an intriguing puzzle to observant individuals: where does the insect derive the strength to effortlessly carry a load ten times its own weight? Indeed, a human might struggle to climb stairs while carrying, for instance, a piano (Figure 24), with the weight ratio of the load to the body being roughly similar to that of an ant. Thus, it seems that the ant is relatively stronger than a human!

But is it really so?

Without geometry, this cannot be understood. Let's listen to what the expert (Professor A.F. Brandt) has to say, primarily about the strength of muscles, and then about the current question regarding the comparison of forces between the insect and the human: "A muscle resembles a resilient cord; however, its contraction is based not on elasticity, but on other reasons, and is normally manifested under the influence of nervous excitation, as demonstrated in physiological experiments involving the application of electric current to the corresponding nerve or directly to the muscle."

"These experiments are easily conducted on muscles excised from a freshly killed frog, as the muscles of cold-blooded animals retain their vital properties for a long time even outside the organism, even at ordinary temperatures. The experiment is very simple. The main calf muscle, which extends the hind leg, is excised together with a piece of the femur bone from which it originates, and together with the terminal tendon. This muscle is found to be the most convenient due to its size, shape, and ease of preparation. A hook is passed through the tendon, and a weight is attached to it."

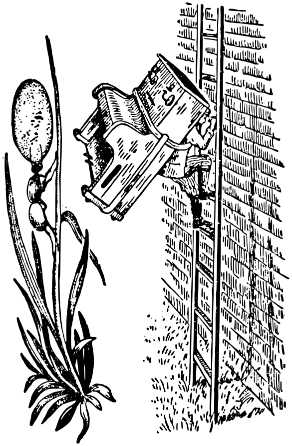


Figure 24.: The six-legged hero.

"If wires from a galvanic element are touched to such a muscle, it instantly contracts, shortens, and lifts the load. By gradually adding additional weights, the maximum lifting capacity of the muscle can be easily determined. Now, if we bind together in length two, three, or four identical muscles and stimulate them simultaneously, we will not achieve greater lifting force; the load will only be lifted to a greater height, corresponding to the sum of the contractions of individual muscles. However, if we bundle two, three, or four muscles together, the entire system will lift a weight many times greater when stimulated. The same result, obviously, would be obtained if the muscles were fused together. Thus, we conclude that the lifting force of muscles depends not on their length or total mass, but only on their thickness, i.e., *cross-sectional area*."

"After this digression, let's turn to the comparison of similarly structured, geometrically similar, but differently sized animals. Let's imagine two animals: the original and one that has been doubled in size in all linear dimensions. In the second animal, the volume and weight of the entire body, as well as each of its organs, will be eight times greater; however, all corresponding planar dimensions, including the cross-sectional area of muscles, will be only four times greater. It turns out that as the animal grows to twice the length and eight times the weight, its muscular strength increases only fourfold, i.e., the animal becomes relatively weaker. Based on this reasoning, an animal that is three times longer (with cross-sectional areas three times larger and a weight 27 times greater) would be relatively three times weaker, and one that is four times longer would be four times weaker, and so on."

"The law of unequal growth in volume and weight of the animal, and thus of muscular strength, explains why insects – as observed in ants, predatory wasps, and others – can carry loads 30 to 40 times their own weight, whereas a human can typically carry excluding gymnasts and porters – only about

9/10 times their own weight, – and a horse, which we view as a magnificent living work machine, even less, namely, only about 7/10 of its own weight."[10]

[10] For more details, see *Fun with Physics* by Ya. I. Perelman, Chapter X *Mechanics in the Living World.*

After these explanations, we will look at the feats of that ant-giant with different eyes, about whom I.A. Krylov mockingly wrote:

Some ant had extraordinary strength,
Such as was unheard of even in ancient times;
He even (says his faithful historian)
Could lift two barley grains.

2. Geometry By The River

2.1. Measuring the width of the river

When crossing a river, measuring its width is just as easy for those who know geometry, how to determine the height of a tree, without climbing to the top. The inaccessible distance is measured the same techniques that we used to measure the inaccessible height. In both cases, the definition of the desired distance is replaced an example of another distance that is easily measurable directly.

Of the many ways to solve this problem, let's look at some of the simplest ones.

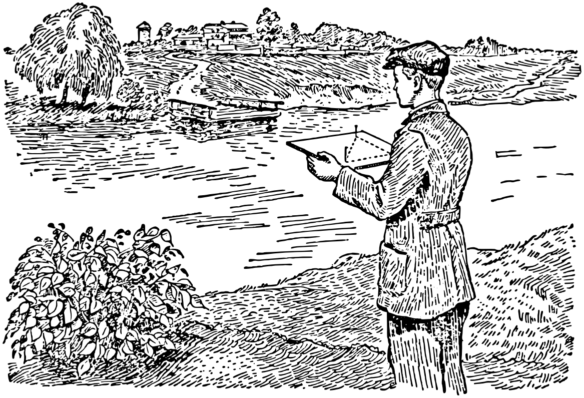


Figure 25.: Measuring the width of the river with a pin device.

1. The first method requires the familiar "device" with three pins at the vertices of an isosceles right triangle (Figure 25). Let's say we need to determine the width of river *AB* (Figure 26), standing on the bank where point *B* is, without crossing to the opposite bank. Standing somewhere at point *C*, hold the pin device close to your eye so that, looking with one eye along the two

pins, you see both covering points B and A. It's clear that when you manage this, you will be exactly on the extension of line AB.

Now, without moving the plank of the device, look along the other two pins (perpendicular to the previous direction) and notice any point D covered by these pins, i.e., lying on the line perpendicular to AC. After this, insert a pin at point C, leave this place, and go with your instrument along line CD until you find a point E (Figure 27), where you can simultaneously cover point C for one eye with pin b and point A with pin a. This means you have found the third vertex of triangle ACE on the shore, where angle C is a right angle, and angle E is opposite to the acute angle of the pin device, i.e., half the right angle (45°). Obviously, angle A is also half right angle, i.e., $AC = CE$. If you measure the distance CE even by steps, you will know the distance AC, and by subtracting BC, which is easy to measure, you will determine the desired width of the river.

It is quite inconvenient and difficult to hold the pin device still in hand; therefore, it is better to attach this plank to a stick with a pointed end and insert it vertically into the ground.

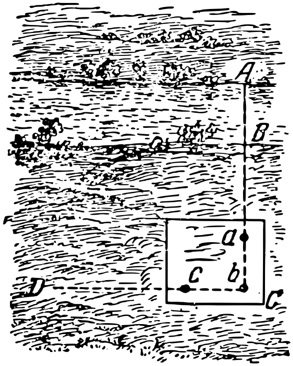


Figure 26.: First position of the pin device.

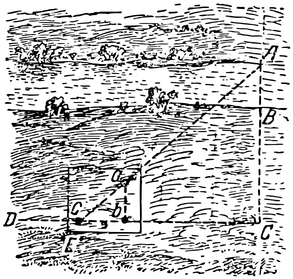


Figure 27.: Second position of the pin device.

2. The second method is similar to the first. Here also,

find point C on the extension of AB and mark line CD perpendicular to CA using the pin device. But then proceed differently (Figure 28). Equal distances CE and EF of arbitrary length are measured on the straight line CD, and pegs are inserted at points E and F. Then, standing at point F with a pin device, the direction FG is marked out perpendicular to FC. Now, walking along FG, find a point H on this line from which point A seems to be covered by point E. This will mean that points H, E, and A lie on the same straight line.

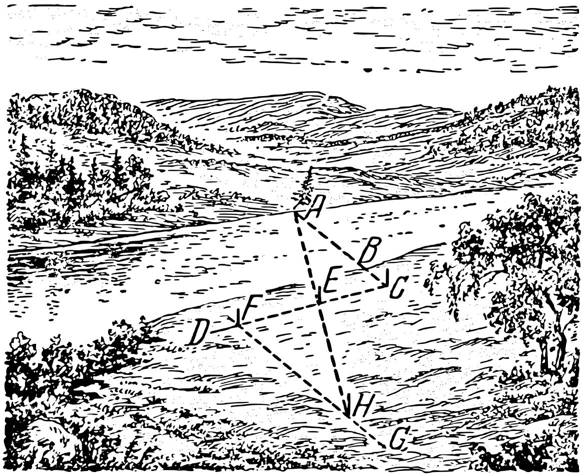


Figure 28.: Using the congruence criteria of triangles to find the width of the river.

The problem is solved: the distance FH is equal to the distance AC, from which it is only necessary to subtract BC to find the desired width of the river (the reader, of course, will guess for himself why FH is equal to AC).

This method requires more space than the first one; if the terrain allows executing both methods, it is useful to verify one result by another.

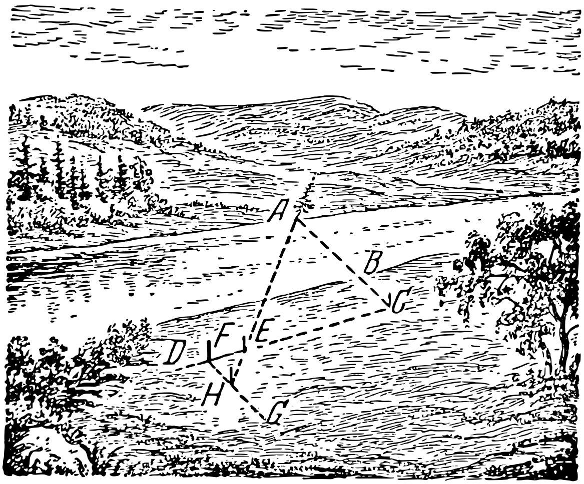


Figure 29.: Using the similarity criteria of triangles to find the width of the river.

3. The method described above can be modified: instead of measuring equal distances on the straight line *CF*, measure one distance several times smaller than the other. For example (Figure 29), *FE* is measured four times less than *EC*, and then we proceed as before: in the direction *FG*, perpendicular to *FC*, we find a point *H* from which the peg *E* appears to cover point *A*. But now *FH* is no longer equal to *AC*, but four times smaller than this distance: triangles *ACE* and *EFH* are not congruent here, but similar (they have equal angles with unequal sides). From the similarity of triangles follows the proportion:

$$\frac{AC}{FH} = \frac{CE}{EF} = \frac{4}{1}$$

Therefore, by measuring *FH* and multiplying the result by 4, we get the distance *AC*, and by subtracting *BC*, we find the desired width of the river.

This method, as we can see, requires less space and is therefore more convenient to perform than the previous one.

4. The fourth method is based on the property of a right triangle that if one of its acute angles is 30°, then the length of the cathetus is half the hypotenuse. It is very easy to verify the correctness of this.

Let angle B of right triangle ABC (Figure 30, left) be 30°; we will prove that in this case, $AC = 1/2\, AB$. Rotate triangle ABC around BC so that it is symmetric with its initial position (Figure 30, right), forming figure ABD; line AC is straight because both angles at point C are right angles. In triangle ABD, angle $\angle A = 60°$, angle ABD, composed of two 30° angles, is also equal to 60°. Therefore, $AD = BD$ as sides opposite equal angles. But $AC = 1/2\, AD$, therefore,

$$AC = \frac{1}{2}AB.$$

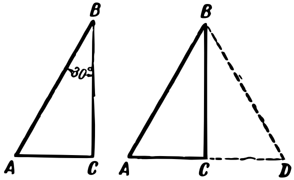


Figure 30.: When the cathetus is half the hypotenuse.

Wishing to take advantage of this property of the triangle, we must arrange the pins on the board so that their bases represent a right triangle in which the cathetus is half the hypotenuse. With this device, we place ourselves at point C (Figure 31) so that the direction AC coincides with the hypotenuse of the pin triangle. Looking along the short cathetus of this triangle, mark the direction CD and find a point E on it so that the direction EA is perpendicular to CD (this is done using the same pin device). It is easy to see that the distance CE – the cathetus lying opposite the angle of 30° – is equal to half of AC. Therefore, by measuring CE, doubling this distance and subtracting BC, we obtain the desired width of the AB river.

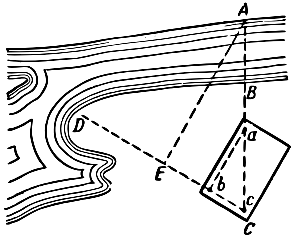


Figure 31.: The scheme of application of a right-angled triangle with a 30° angle.

Here are four easily executable methods, with which it is always possible, without crossing to the other bank, to measure the width of the river with quite satisfactory accuracy. We will not consider methods that require the use of more complex instruments (even homemade ones) here.

2.2. Using a visor

Here's how this method came in handy for Senior Sergeant Kupriyanov in frosty conditions.[11] His detachment was ordered to measure the width of the river, across which they were to organise a crossing...

[11] See the footnote on page 21.

Approaching a bush near the river, Kupriyanov's detachment took cover, and Kupriyanov himself, along with soldier Karpov, moved closer to the riverbank, from where the fascist-occupied shore was clearly visible. In such conditions, measuring the width of the river had to be done by eye.

"Come on, Karpov, how much?" Kupriyanov asked.

"I think no more than 100-110 meters," Karpov replied. Kupriyanov agreed with his scout, but for control, he decided to measure the width of the river using a "visor."

This method is simple. You have to face the river and pull the visor over your eyes so that the lower edge of the visor

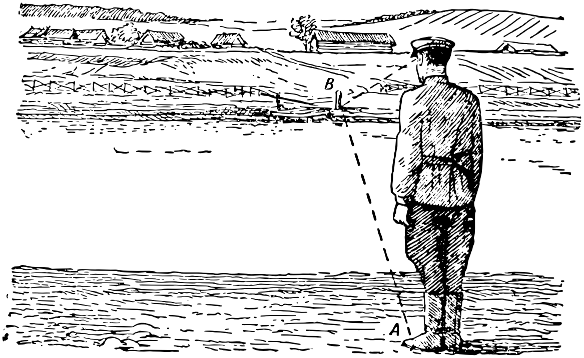


Figure 32.: Observing a point on the opposite bank from under the visor.

precisely aligns with the line of the opposite bank (see Figure 32). The visor can be replaced with the palm of your hand or a notepad, tightly pressed edge to your forehead. Then, without changing the position of your head, you need to turn to the right or left, or even backward (towards the side where the area available for measuring the distance is more level) and notice the farthest point visible from under the visor (palm, notepad).

The distance to this point will be approximately equal to the width of the river.

Kupriyanov utilized this method. He quickly stood up in the bushes, pressed a notepad to his forehead, then quickly turned and aimed at the distant point. Then, together with Karpov, he crawled to that point, measuring the distance with a rope. It turned out to be 105 meters.

Kupriyanov reported the data he obtained to the command.

Question

Provide a geometric explanation for the “visor” method.

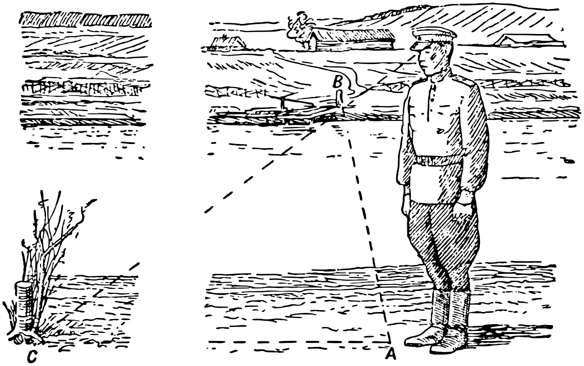


Figure 33: In the same way, you can aim at a point on your own bank.

Answer

The line of sight, touching the edge of the visor (palm, notepad), is initially directed towards the line of the opposite bank (see Figure 32). When a person turns, the line of sight, like the leg of a compass, describes a circle, and then $AC = AB$ as the radii of the same circle (see Figure 33).

2.3. The Length Of An Island

Question

Now we are faced with a more challenging task. Standing by the river or lake, you see an island (see Figure 34) whose length you wish to measure without leaving the shore. Is it possible to carry out such a measurement?

Although in this case, both ends of the measured line are inaccessible to us, the problem is still entirely solvable, and without complex instruments.

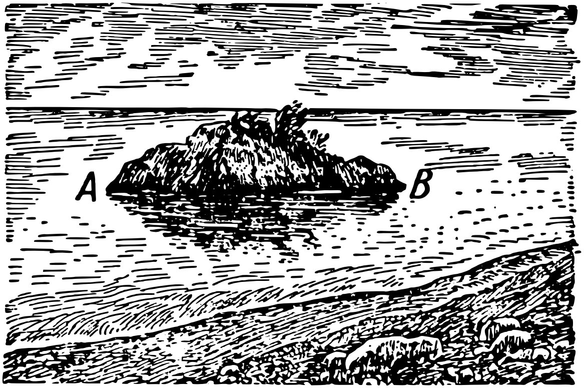


Figure 34.: How to determine the length of the island.

Answer

To measure the length of an island without leaving the shore, you can use the following method. Choose arbitrary points P and Q on the shore and place stakes in them. Then find points M and N on the line PQ such that the directions AM and BM form right angles with the direction of PQ (this can be done using a compass). In the middle of the distance MN, place a stake O and find on the extension of the line AM a point C from which the stake O appears to cover point B. Similarly, on the extension of BN, find point D from which stake O appears to cover the end A of the island. The distance CD will be the desired length of the island.

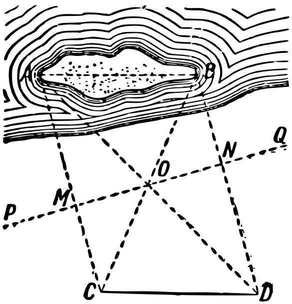


Figure 35.: We use the properties of congruent right triangles to find the length of an isalnd.

This can be easily proved. Consider the right triangles AMO and OND; in them, the legs MO and NO are equal, and the angles AOM and NOD are also equal, therefore, the triangles are equal, and $AO = OD$. Similarly, it can be proved that $BO = OC$. By comparing the triangles ABO and COD, it can be seen that their distances AB and CD are equal.

2.4. A pedestrian on the opposite bank

Question

As you walk along the riverbank, you see a person on the other side, and you can clearly distinguish their steps. Can you, without moving from your spot, determine at least approximately the distance between them and you? You have no instruments at hand.

Answer

You don't have any instruments, but you have eyes and hands

Figure 36.: How to determine the distance to a pedestrian walking on the other side of the river.

– that's enough. Extend your arm forward towards the pedestrian and look at the tip of your finger with one eye if the pedestrian is moving towards your right hand, and with the other eye if they're moving towards your left hand. At the moment when the distant pedestrian is covered by your finger (see Figure 36), close the eye that was looking and open the other: the pedestrian will appear to you as if they've moved backward. Count how many steps they take before they align again with your finger. You'll get all the data needed for an approximate determination of the distance. Let's explain how to use them.

Suppose in Figure 36 (inset), your eyes are marked as *a* and *b*, point *M* is the tip of your finger extended, point *A* is the initial position of the pedestrian, and B is the final position. The triangles abM and ABM are similar (you should turn towards the pedestrian so that ab is approximately parallel to their direction of movement). Therefore, $BM : bM = AB : ab$ – is a proportion in which only one term, BM, is unknown, but all others can be directly determined. Indeed, bM is the length of your extended arm, ab is the distance between the pupils of your eyes, and AB is measured in steps taken by the pedestrian (assuming an average step to be around 3/4 metres). Therefore, the unknown distance from you to the pedestrian on the opposite bank, AB, equals

$$MB = AB\,\frac{bM}{ab}$$

For example, if the distance between your eye pupils ab is 6 cm, the length of bM from the end of your extended arm to the eye is 60 cm, and the pedestrian takes, say, 14 steps from A to B, then their distance from you would be $MB = 14 \cdot 60/6 = 140$ steps, or 105 meters.

It's enough for you to measure in advance the distance between your eye pupils and bM – the distance from the eye to the end of your extended arm – so that you can quickly determine the distance of inaccessible objects by remembering their ratio. On average, for most people, bM/ab is around 10 with slight fluctuations. The difficulty will only be in somehow determining the distance AB. In our case, we used the steps of a distant person. But you can also use other references. For instance, if you're measuring the distance to a distant freight train, you can estimate *AB* in comparison to the length of a freight car, which is usually known (7.6 meters between buffers). If you're determining the distance to a house, you can estimate *AB* by comparing it to the width of a window, the length of a brick, etc.

The same method can be applied to determine the size of a distant object if its distance from the observer is known. For this purpose, you can also use other "rangefinders", which we will describe next.

2.5. Simple Rangefinders

In the first chapter, we described the simplest instrument for determining inaccessible heights – the altimeter. Now, let's describe the simplest device for measuring inaccessible distances – the 'rangefinder.' The simplest rangefinder can be made from an ordinary matchstick. To do this, you just need to mark millimeter divisions on one of its sides, alternating between light and dark (see Figure 37).

Figure 37.: The match is a rangefinder.

You can use this primitive "rangefinder" to estimate the distance to a distant object only in those cases when the dimensions of that object are known to you (see Figure 38). However, more sophisticated rangefinders can also be used under the same condition. Suppose you see a person in the distance and set yourself the task of determining the distance to them. Here, the matchstick rangefinder can come in handy. Holding it in your outstretched arm and looking with one eye, you bring its free end into coincidence with the top of the distant figure. Then, slowly moving your thumbnail along the matchstick, you stop it at the point that projects onto the base of the human figure. All you have to do now is to find out, by bringing the matchstick closer to the eye, at which mark your thumbnail stopped – and then you have all the data to solve the problem.

You can easily verify the correctness of the proportion:

$$\frac{\text{desired distance}}{\begin{array}{c}\text{distance from the eye}\\ \text{to the matchstick}\end{array}} = \frac{\text{average height of a person}}{\begin{array}{c}\text{measured part}\\ \text{of the matchstick}\end{array}}$$

From here, it's easy to calculate the desired distance. For example, if the distance to the matchstick is 60 cm, the height of the person is 1.7 m, and the measured part of the matchstick is 12 mm, then the determined distance would be:

$$60 \cdot \frac{1700}{12} = 8{,}500\,\text{cm} = 85\,\text{m}.$$

To gain some skill in using this rangefinder, measure the height of someone from your group and, asking them to

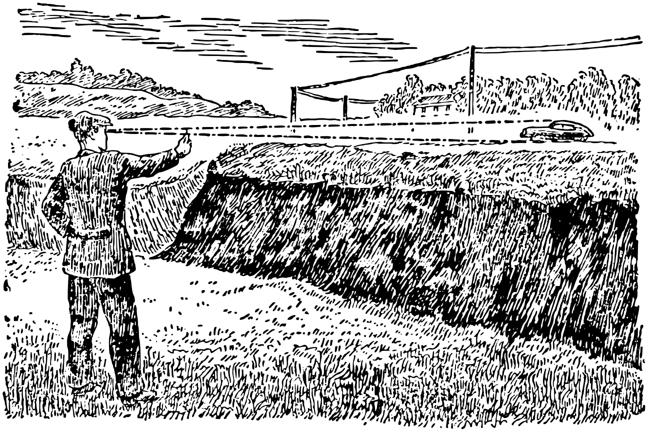


Figure 38.: The use of a rangefinder match to determine inaccessible distances.

move away a certain distance, try to determine how many steps they took away from you.

With the same method, you can determine the distance to a rider (average height 2.2 m), a cyclist (wheel diameter 75 cm), a telegraph pole along the railway track (height 8 m), vertical distance between adjacent insulators (90 cm), to a train, a brick house, and similar objects whose dimensions can be estimated with sufficient accuracy. There can be quite a few such cases during excursions.

For those skilled in crafting, making a more convenient device of the same type, intended for estimating distances based on the size of a distant human figure, won't be much trouble.

The device is clear in Figure 39 and Figure 40. The observed object is placed precisely in the gap *A*, formed when the extension part of the device is raised. The size of the gap can be conveniently determined by the divisions on the part *C* and *D* of the board. To avoid the need for any calculations, you can directly mark on strip *C* the distances corresponding to the divisions if the observed object is a human figure (the device for measuring the distance of the outstretched arm). On the right strip *D*, you can mark distances, pre-calculated for cases where a rider is observed (2.2 m). For telegraph poles (height 8 m), planes with a wingspan of 15 m, and other larger objects, you can use the upper, free parts of strips *C* and *D*. Then the device will look like the one presented in Figure 40.

Of course, the accuracy of such distance estimation is low. It's just an estimate, not a measurement. In the example discussed earlier, where the distance to the human figure was estimated at 85 m, an error of 1 mm in measuring the matchstick portion would result in a deviation of 7 m (1/12 out of 85). But if the person stood four times farther away, and we measured only 3 mm on the matchstick, then an error

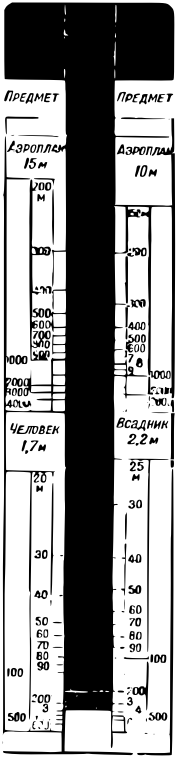


Figure 40.: The design of the retractable rangefinder.

of even 1/2 mm would cause a change in the result by 57 m. Therefore, our example is reliable only for relatively short distances – in the range of 100–200 m. When estimating larger distances, it's necessary to choose larger objects.

2.6. The energy of the river

You know the edge where everything breathes abundance,
Where rivers flow purer than silver,
Where the steppe breeze sways the feather grass,
Where villages are nestled in cherry orchards.

A.K. Tolstoy

A river, the length of which is no more than 100 km, is considered small. Do you know how many such small rivers there are in the USSR? A lot - 48 thousand!

If these rivers were stretched into a single line, it would result in a ribbon 13,800,000 km long. With such a ribbon, you could encircle the Earth at the equator thirty times (the length of the equator is approximately 40,009 km).

The flow of these rivers is leisurely, but it conceals an inexhaustible supply of energy within it. Specialists believe that if the hidden potential of all the small rivers flowing through our homeland were combined, an impressive number would be obtained – 34 million kilowatts! This gifted energy needs to be widely utilised for electrifying the economy of settlements located near rivers.

Let the river flow freely,
If the plan says so,
A dam with a stone ridge across all depths
Will block the way forever.

S. Shchipachev

You know that this is achieved through hydroelectric power stations (HPS), and you can show a lot of initiative and provide real assistance in preparing for the construction of small HPS. Indeed, the builders of HPS will be interested in everything related to the river regime: its width and flow rate ("water flow"), the area of the cross-section of the riverbed ("active section"), and what water head the banks allow. And all this can be measured with available means and represents a relatively simple geometric problem.

We will now proceed to solving this problem.

But first, let's present here a practical advice from specialists, engineers V. Yarosh and I. Fedorov, regarding the selection of a suitable location on the river for the construction of a future dam.

They recommend building a small hydroelectric power station with a capacity of 15-20 kilowatts "no further than 5 km from the village."

"The dam of an HPS should be built no closer than 10-15 km and not farther than 20-40 km from the source of the river because moving away from the source entails the costly reinforcement of the dam, which is caused by a large influx of water. If the dam is located closer than 10-15 km from the source, due to the small water flow and insufficient head, the hydroelectric power station will not be able to provide the necessary power. The chosen stretch of the river should not be abundant in great depths, which also increases the cost of construction, requiring a heavy foundation."

2.7. The Flow Rate

Between village and mountain grove,
Winds a river like a bright ribbon.

A. Fet

How much water flows in such a river in a day? It's easy to calculate if you first measure the speed of the water flow in the river. The measurement is performed by two people. One person holds a watch, the other holds some noticeable float, for example, a half-empty bottle with a flag. They choose a straight section of the river and place two stakes *A* and *B* along the bank at a distance, for example, 10 m from each other (see Figure 41).

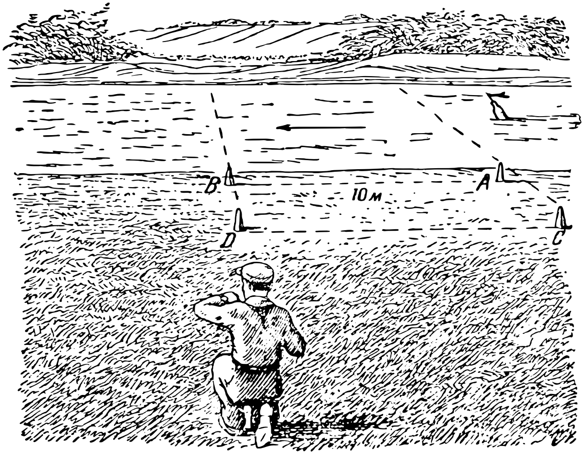


Figure 41.: Measurement of the river flow velocity.

Two more stakes C and D are placed on lines perpendicular to AB. One of the participants in the measurement with the watch stands behind stake D. The other, with the float, goes a bit upstream of stake A, throws the float into the water, and then stands behind stake C. Both observers look along the directions CA and DB towards the water surface. At the moment when the float crosses the extension of the line CA, the first observer waves his hand. Upon this signal, the second observer starts the timer for the first time and then again when the float crosses the direction of DB.

Let's assume that the time difference is 20 seconds.

Then the speed of the water flow in the river is:

$$\frac{10}{20} = 0.5\,\mathrm{m/s}.$$

Usually, the measurement is repeated about ten times[12], throwing the float into different points on the river surface. Then the obtained numbers are summed up and divided by the number of measurements. This gives the average speed of the surface layer of the river.

[12] Instead of throwing one float ten times, you can immediately throw 10 floats at some distance from each other.

Deeper layers flow slower, and the average speed of the entire flow is approximately 4/5 times the surface speed. In our case, therefore, it's 0.4 m/s.

You can determine the surface speed by another – albeit less reliable – method.

Sit in a boat and paddle 1 km (measured along the shore) against the current, and then back – with the current, trying to paddle with the same force all the time.

Let's say you paddled these 1,000 m against the current in 18 minutes, and with the current in 6 minutes. Denoting the desired speed of the river current as x, and the speed of your movement in still water as y, you form the equations:

$$\frac{1000}{y-x} = 18, \quad \text{and} \quad \frac{1000}{y+x} = 6.$$

Rearranging we get:

$$y + x = \frac{1000}{6}, \quad \text{and} \quad y - x = \frac{1000}{18}.$$

Solving for x, we get $2x = 110$, and $x = 55$. The speed of the water flow on the surface is 55 m per minute, and therefore, the average speed is about 5/6 m/s.

2.8. How Much Water Flows In The River?

To measure the amount of water flowing in a river, you can always determine the speed at which the water flows. The more challenging part of the preparatory work needed to calculate the quantity of flowing water is to determine the cross-sectional area of the water. To find the magnitude of this area, known as the "wetted cross-section" of the river, you need to make a drawing of this section. Such work is done as follows.

First Method: At the point where you measured the width of the river, you drive a stake into the ground on both banks, right at the water's edge. Then, with a companion, you get into a boat and row from one stake to the other, trying to keep a straight line connecting the stakes. An inexperienced rower will not be able to handle such a task, especially in a river with a fast current. Your companion must be a skilled rower; besides, a third participant in the work should stand on the bank, ensuring that the boat stays on the correct course and giving the rower signals indicating which way to turn when necessary. During the first crossing of the river, you only need to count how many strokes of the oars it took and from there figure out how many strokes move the boat 5 or 10 meters. Then, for the second crossing, armed with a sufficiently long rake with markings on it, you plunge the rake vertically to the bottom every 5-10 meters (measured by the number of oar strokes) and record the depth of the river at that point.

This method can only measure the wetted cross-section of a small river; for a wide, multi-water river, more complex methods are needed, which are performed by specialists. An amateur must choose a task that suits their modest measuring means.