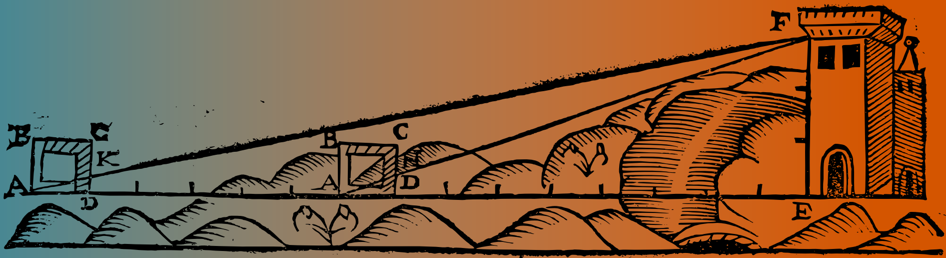
Yakov Perelman

GEOMETRY for ENTERTAINMENT



THE MIR TITLES PROJECT

Ya. I. Perelman

Geometry for Entertainment

The Mir Titles Project

Translated from the Russian by *Damitr Mazanav*

Translated from the Russian and typeset in LaTeX using Linux Libertine by *Damitr Mazanav* from the Seventh revised edition published in 1950.

This fully electronic English translation released in 2024 by THE MIR TITLES PROJECT https://mirtitles.org .

Link to the source files.

Front cover: Woodcut from Cosimo Bartoli's *Del modo di misvrare* published in 1564.

Перельман Я.И. Занимательная геометрия. Москва-Ленинград: Государственное издательство технико-теоретической литературы, 1950

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Edited and supplemented by *B.A. Kordemsky*

ABOUT MIR TITLES PROJECT

The Mir Titles project aims to preserve the wealth of knowledge encapsulated in various books published during the Soviet era for future generations. This extensive collection features works on science, mathematics, philosophy, popular science, and history, as well as large number of books on Soviet, Russian, and children's literature in several languages. The project has been made possible through the generous support and contributions from friends and supporters worldwide.

https://mirtitles.org
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Editor's Preface

Geometry for Entertainment is written both for friends of mathematics and for those readers from whom many attractive aspects of mathematics have somehow been hidden.

More importantly, this book is intended for those readers who studied (or are currently studying) geometry only at the blackboard and therefore are not used to noticing familiar geometric relationships in the world of things and phenomena around us, have not learnt to use the acquired geometric knowledge in practise, in difficult cases of life, on a hike, in a bivouac or front-line situation.

To arouse the reader's interest in geometry or, in the words of the author, "to inspire a desire and cultivate a taste for its study is the objective of this book."

To this end, the author will take geometry "out of the walls of the school room into the free air, into the forest, field, to the river, on the road, in order to indulge in relaxed geometric studies without a textbook and tables in the open air ...", and draws the reader's attention to the pages of L. N. Tolstoy and A. P. Chekhov, Jules Verne and Mark Twain. He finds a theme for geometric problems in the works of N. V. Gogol and A. S. Pushkin, and finally offers the reader "a motley selection of problems, curious in plot, unexpected in result."

The seventh edition of *Geometry for Entertainment* is published without the direct participation of the author. Ya. I. Perelman died in Leningrad in 1942.

The new edition of the book contains almost all the articles of the previous edition, newly illustrated, edited and supplemented with facts and information from our Soviet reality, as well as a considerable number (about 30) additional articles.

I was guided by the desire to increase the “utility coefficient” of Ya. Perelman’s book, to make it even more effective and interesting, involving new readers in the ranks of friends of mathematics.

To what extent this was possible, I hope to learn from readers at the address: Moscow, 64, Chernyshevsky Str., 81, Sq. 53, B. A. Kordemsky.

B. Kordemsky

Translator’s Preface

Yakov Perelman’s books have been a constant source of inspiration for me throughout my life. Though many of his works have been translated into English and other languages, several remain untranslated. As the overarching aim of the Mir Titles Project, we endeavour to bring all such works to the public. This translation was a rather ambitious project, and I was not sure of the time it would require. But the project got completed in little over a months time. It brings me great pleasure to present this English translation of untranslated work of Perelman.

Examples

Perelman, in his discussions, emphasises the geometrical relationships in measurable and unknown quantities. This approach is historical in the sense that this is how geometry developed: to solve problems of measurement of unknown quantities.

The book has two sections, the first one, *Geometry In The Open Air* is oriented towards the "field" aspect, involving measurements of variety of things, using direct and indirect measurements using very basic instruments, including our body parts.

The second part of the book, *Between Seriousness And Joke In Geometry*, delves into more whimsical and application aspects of geometry. Many discussions are based on legends and literature in which Perelman investigates geometrical aspects of the situations depicted. A lot of the content of the book focuses on practical aspects of applying geometry to solve problems that we might face in open fields, by the river, in the sky, in the ocean, during constructions, in scaling objects for measuring angles, lengths, areas, and volumes, among other things. I am sure that after reading this book, you will see mathematics embedded in many familiar objects.

Illustrations

The beautiful and abundant illustrations are the heart of the book. Geometry, being primarily reliant on graphics, is brought to life in a variety of situations. Familiar geometrical shapes, lines, and ratios are found among trees, rivers, homes, skies, and other natural settings.

Each topic is complemented by relevant illustrations, which makes visualising and understanding them easier. I have made no effort to change the images, except in some cases where I replaced the Russian words with English ones. Only one image had to be completely redrawn, as the original was not of good quality.

Translation

I have made use of machine translations for the bulk of the text, and the results are quite satisfactory. At times, I have used several translation services to ensure I am on the right track and that the meaning is not lost in translation. Though, of course, there might be places where I have not translated correctly. I learnt to read the Russian in a very rudimentary way in the process. :)

At places I have added my own notes for clarification which are indicated by my initials at the end – DM.

Typesetting

I have typeset the book in a square profile with marginpar for figures and notes. This design helps in making the book more readable by trying to keep the figures and notes on the same page. During the course of typesetting this book, several challenges in LaTeX were resolved by the discussions and questions posted on tex.stackexchange. I am really grateful to the experts in the forum whose answers helped me learn a lot and resolve the typographical and technical challenges.

If there are any mistakes in the mathematics or translation, they are all mine. Any suggestions and criticisms to improve the translation and the design of the book are welcome. Translating this book was a great learning experience for me in several ways. I hope that this English version finds enthusiastic readers and inspires many more brilliant minds in the generations to come.

Please drop your suggestions on the Mir Titles blog or write to mirtitles@gmail.com

Damitr Mazanav
MUMBAI, MAY 2024

Contents

Editor's Preface **vii**

Translator's Preface **xi**

I. Geometry In The Open Air 1

1. Geometry In The Forest **5**

By the Length of the Shadow 5

Two More Methods 13
The Method of Jules Verne 15
How Sergeant Popov Acted 19
Using a Notebook 22
Without Approaching the Tree 23
Forest Rangers' Altimeter 25
Using a Mirror 29
Two Pines 32
The Shape of the Tree Trunk 33
Universal Formula 35
Volume and Weight of a Tree at the Root 38
Leaf Geometry 43
Six-legged Heroes 45

2. Geometry By The River **49**
Measuring the Width of the River 49
Using a Visor 56
The Length Of An Island 59
A Pedestrian On the Opposite Bank 60
Simple Rangefinders 63
The Energy of the River 67
The Flow Rate 69
How Much Water Flows in The River? 72
Water Wheel 78
Rainbow Film 79
Circles on the Water 81

Fantastic Shrapnel 83
The Keel Wave 85
Speed of Projectiles 89
Finding Pond Depth 91
Starry Sky in the River 93
Path Across the River 95
To Construct Two Bridges 97

3. Geometry In The Open Field **99**
Visible Sizes of the Moon 99
Angle of View 103
Plate and Moon 105
Moon and Copper Coins 106
Sensational Photographs 107
Reservoir Set Decoration 110
Living Protractor 112
Jacob’s Staff 116
Rake Angle Gauge 118
Artilleryman’s Angle 120
Your Visual Acuity 123
The Limiting Minute 125
The Moon and Stars at the Horizon 128
The length of the Moon’s Shadow 132
How High is the Cloud Above the Ground? 134
Tower Height from Photograph 142
Exercises 144

4. Geometry on the Road **147**
The Art of Measuring by Steps 147
Eye-meter 149
Slopes 155
Heap of Gravel 158
The Proud Hill 161
At a Road Curve 163
Radius of Curvature 165
The Bottom of the Ocean 168
Do Water Mountains Exist? 171

5. Field Trigonometry Without Formulas and Tables **175**
Calculation of the Sine 175
Extraction of the Square Root 181
To Find an Angle From its Sine 183
Height of the Sun Problem 185
Distance to the Island Problem 186
Lake Width Problem 188
Triangle Area Problem 190
Determining the Magnitude of an Angle Without Any Instruments 192

6. Where Heaven and Earth Converge **195**
Horizon 195
Ship on the Horizon 199

The Distance to the Horizon 200
Gogol's Tower Problem 208
Pushkin's Hill 209
Where the Rails Meet 210
Lighthouse Problems 211
Lightning Problem 213
Sailboat Problem 214
Horizon on the Moon 215
In the Lunar Crater 216
On Jupiter 217

7. The Geometry Of The Robinsons **219**
The Geometry of the Starry Sky 219
The Latitude of the 'Mysterious Island' 225
Determining Geographic Longitude 229

II. Between Seriousness And Joke In Geometry **233**

8. Geometry In The Dark **237**
Into the Depths of the Hold 237
Measuring the Barrel 238
Measuring Ruler (The Mayne-Reid Problem) . . . 240
What Needed to be Done 243
Verification of the Calculation 246

The Night Journey of Mark Twain 252
Mysterious Circumnavigation 255
Measuring with Bare Hands 267
Straight Angle in the Dark 270

9. Old And New About The Circle 273
Practical Geometry of the Egyptians and Romans . 273
"I know this and I remember it perfectly." 275
Jack London's Error 282
Dropping a Needle 283
Straightening a Circle Problem 286
Squaring the Circle 288
Triangle of Bing 294
Head or Feet 295
Wire Along the Equator Problem 297
Facts and Calculations 298
The Girl on the Tightrope 303
The Path Through the Pole 307
For the Drive Belt 315
The Raven's Cleverness 319

10. Geometry Without Measurements And Calculations 323
Building Without a Compass 323
Centre of Gravity of a Plate 325
Napoleon's Task 326

Simple Trisector 328
Clock Trisector 330
Dividing a Circle 331
Billiard Ball Problem 334
The "Smart" Ball 337
In One Stroke 346
The Seven Bridges of Königsberg 351
Geometrical Joke 352
Form Verification Problem 354
Game 355

11. Big And Small In Geometry 359
In a Thimble 359
Volume and Pressure 362
Thinner than a Spider Web but Stronger than Steel 366
Two Jars 370
Gigantic Cigarette 370
Ostrich Egg Task 371
Elephant Bird Egg 372
Eggs of Russian Birds 373
Weight of the Eggshell 375
The Sizes of Our Coins 376
Coin Worth a Million Roubles 377
Visual Representations 377
Our Normal Weight 382
Gulliver's Geometry 384

Why Does Dust and Clouds Float in the Air? . . . 388

12. Geometric Economy **393**

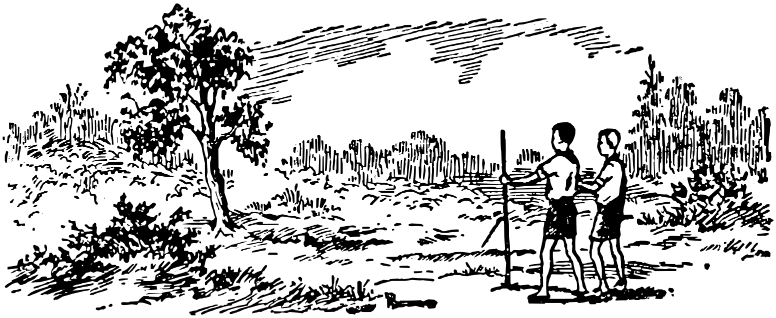
How Did Pahom Buy Land 393
The Problem of Leo Tolstoy 398
Trapezoid or Rectangle 401
The Remarkable Property of a Square 403
Plots of Different Shapes 405
Figures with the Greatest Area 407
Nails 411
Body of Greatest Volume 411
Product of Equal Factors 412
Triangle with the Greatest Area 415
The Heaviest Beam 416
From a Cardboard Triangle 417
The Tinsmith’s Dilemma 419
The Turner’s Dilemma 422
How to Lengthen a Board? 425
The Shortest Path 428

Part I.

Geometry In The Open Air

Nature speaks the language of mathematics: the letters of this language are circles, triangles and other mathematical shapes.

Galileo



1. Geometry In The Forest

By the Length of the Shadow

I remember now the amazement with which I looked for the first time, he looked at a gray-haired forester, who, standing near a huge pine tree, measured its height with a small pocket device. When he aimed his square board at the top of the tree, I expected that the old man would now start climbing there with a measuring chain. Instead, he put the device back in his pocket and announced that the measurement was over. I thought it hadn't started yet …

I was very young then, and this way of measuring, when a person determines the height of a tree without cutting it down and climbing to the top, was in my eyes something like a small miracle. It was only later, when I was initiated into the rudiments of geometry, that I realised how simple such miracles are performed. There are many different ways to make such measurements using very simple instruments and even without any devices.

The easiest and most ancient way is, without a doubt, the one by which the Greek sage Thales determined the height of the pyramid in Egypt sixth century BC. He took advantage of the pyramid's 'shadow'. The priests and the pharaoh, gathered at the foot of the highest pyramid, looked puzzled at the northern newcomer, who guessed the height of the huge structure from the shadow. Thales, says the legend, chose a day and an hour when the length of his own shadow was equal to his height; at this moment, the height of the pyramid should also be equal to the length of the shadow cast by it[1]. This is perhaps the only case when a person benefits from his shadow …

The task of the Greek sage now seems childishly simple to

[1] Of course, the length of the shadow had to be measured from the midpoint of the square base of the pyramid; Thales could directly measure the width of this base.

us, but let's not forget that we are looking at it from the height of a geometric building erected after Thales. He lived long before Euclid, the author of the wonderful book that taught geometry for two millennia after his death. The truths contained in it, which are now known to every schoolboy, were not yet discovered in the era of Thales. And in order to use the shadow to solve the problem of the height of the pyramid, it was necessary to already know some geometric properties of the triangle, namely the following two (of which Thales himself discovered the first):

1. that the angles at the base of an isosceles triangle are equal, and vice versa – that the sides lying opposite the equal angles of the triangle are equal to each other;
2. that the sum of the angles of any triangle (or at least a rectangular one) is equal to two right angles.

Only Thales, armed with this knowledge, had the right to conclude that when his own shadow is equal to his height, the sun's rays meet the flat ground at an angle of half a straight line, and therefore the top of the pyramid, the middle of its base and the end of its shadow should mark an isosceles triangle.

It would seem that this simple method is very convenient to use on a clear sunny day to measure lonely trees whose shadow does not merge with the shadow of neighbouring ones. But in our latitudes it is not as easy as in Egypt to waylay the right moment for this: The sun is low above the horizon, and the shadows are equal to the height of the objects casting them only in the afternoon hours of the summer months. Therefore, the Thales method in this form is not always applicable.

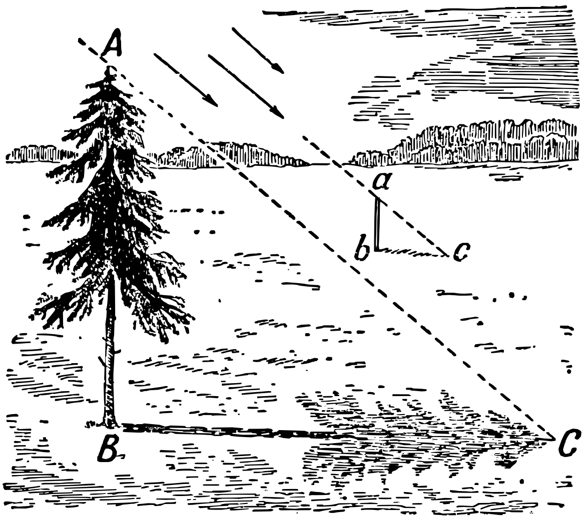


Figure 1: Measuring the height of a tree by shadow.

It is not difficult, however, to modify this method so that on a sunny day, any shadow can be used, regardless of its length. Additionally, measuring both your own shadow and the shadow of a pole, the desired height is calculated from the proportion (Figure 1):

$$AB : ab = BC : bc,$$

meaning the height of the tree is as many times greater than your own height (or the height of the pole) as the shadow of the tree is longer than your shadow (or the shadow of the pole). This naturally follows from the geometric similarity of triangles ABC and abc (based on two angles).

Some readers may object that such an elementary technique does not need a geometric justification at all: is it really unclear even without geometry that how many times is a tree taller, how many times is its shadow longer? However, the matter is not as simple as it seems. Try to apply this rule to shadows cast by the light of a street lamp or lamp – it will not be justified. In Figure 2 you can see that the columns AB are about three times higher than the pedestal ab, and the shadow of the column is eight times larger than the shadow of the pedestal ($BC : bc$). It is impossible to explain why the method is applicable in this case, but not in the other, without geometry.

Question Let's take a closer look at what the difference is. The essence of the matter boils down. to the fact that the sun's rays are parallel to each other, the rays of the lantern

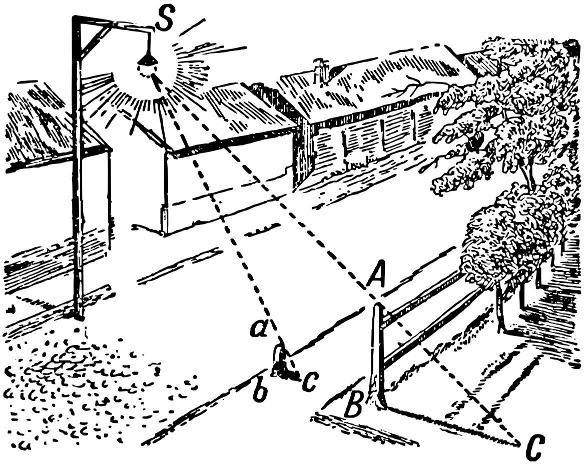


Figure 2: When such a measurement is impossible. (Is the method applicable for a shadow cast by a streetlamp?)

are not parallel. After that, we have the right to consider the rays of the Sun parallel, although they certainly intersect in the place from which they originate.

Answer The rays of the Sun falling on the Earth can be considered parallel because the angle between them is extremely small, almost imperceptible. A simple geometric calculation will convince you of this. Imagine two rays coming from some point of the Sun and falling on the Earth at a distance of, say, one kilo-meter from each other. So, if we put one leg of a compass at this point of the Sun, and with the other we described a circle with a radius equal to the distance from the Sun to the Earth (i.e., with a radius of 150,000,000 km), then an arc of one kilometer in length would appear between our two radii rays. The total length of this gigantic circle would be equal to $2\pi \times 150{,}000{,}000\,\text{km} = 940{,}000{,}000\,\text{km}$. One degree of it, of course, is 360 times less, i.e. about 2,600,000 km; one arc minute is 60 times less than a degree, i.e. equal to 43,000 km, and one arc second is another 60 times less, i.e. 720 km. But our arc is only 1 km in length, so it corresponds to an angle of $1/720 \approx 0.001{,}38''$ seconds. This angle is elusive even for the most accurate astronomical instruments; therefore, in practise we can consider the rays of the Sun falling on the Earth as parallel lines.[2]

[2] Another thing is the rays directed from some point of the Sun to the ends of the earth's diameter; the angle between them is large enough to measure (about $17''$); the definition of this angle gave astronomers one of the means to establish how great the distance from the Earth to the Sun is.

Trying to apply the method of shadows in practise, you will immediately be convinced, however, of its unreliability. Shadows are not delimited so clearly that measuring their length can be done quite accurately. Each shadow cast by the light of the Sun has an indistinctly outlined grey border of penumbra, which gives the border of the shadow uncertainty. This is because the Sun is not a point, but a large luminous body emitting rays from many points. Figure 3 illustrates why, as a result of this, the shadow of tree AB also has an additional component in the form of half-shadow CD, gradually fading away.

The angle of the CAD between the extreme boundaries of

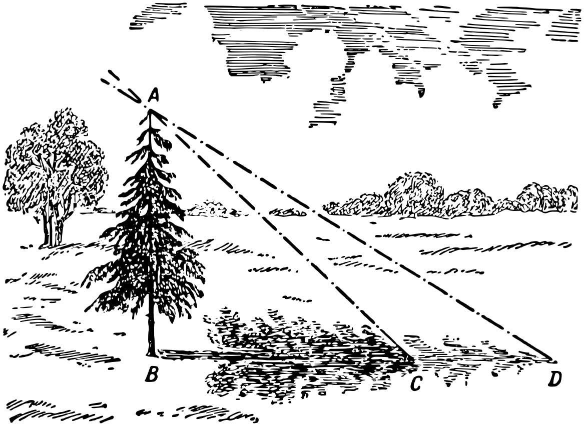


Figure 3: How penumbra is formed.

the penumbra is equal to the angle at which we always see the solar disk, i.e. half a degree. The error resulting from the fact that both shadows are not measured quite accurately can reach 5% or more when the Sun is not too low. This error is added to other unavoidable errors – from uneven soil, etc. – and makes the final result little reliable. In mountainous terrain, for example, this method is completely inapplicable.

Two More Methods

It is entirely possible to measure height without relying on shadows. There are many methods; let's start with two simple ones.

Firstly, we can utilise the properties of an isosceles right triangle. For this purpose, we can make use of a very simple tool, which can be easily crafted from a piece of board and three pins. On a board of any shape, even a piece of bark with a flat side, mark three points to form the vertices of a right triangle – and insert a pin at each point (see Figure 4). Suppose you don't have a drafting triangle to construct a right angle, nor a compass to mark equal sides. In that case, fold any piece of paper once, and then fold it again across the first fold so that both parts of the first fold coincide – and you'll obtain a right angle. The same piece of paper can be used instead of a compass to measure equal distances.

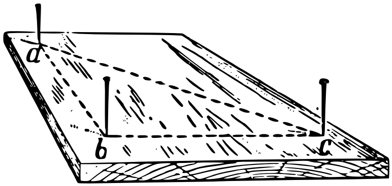


Figure 4: Pin height measuring device.

As you can see, the tool can be entirely crafted in a makeshift environment.

If you don't have a drafting triangle on hand to construct a right angle, nor a compass to mark equal sides, then simply fold any scrap of paper once, and then fold it again across the first fold so that both parts of the first fold coincide—and you'll obtain a right angle. The same piece of paper can be used instead of a compass to measure equal distances.

As you can see, the tool can be entirely crafted in a makeshift environment.

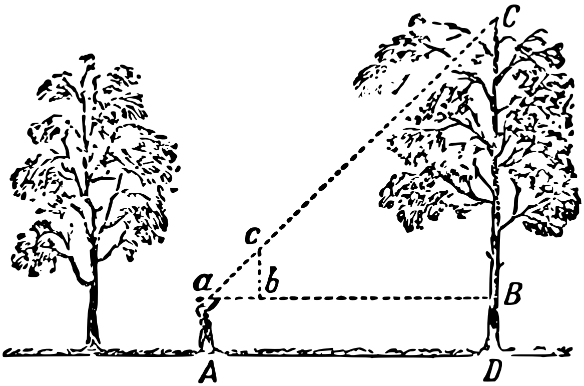


Figure 5: The scheme of application of the pin device.

Handling it is no more difficult than crafting it. Stepping away from the tree being measured, hold the tool so that one of the legs of the triangle is perpendicular. You can use a string or a weight tied to the top pin. Approaching or moving away from the tree, you will always find a spot A (see Figure 5), from which, looking at pins a and c, you will see that they cover the top C of the tree: this means that the extension of the hypotenuse ac passes through point C. Then, obviously, the distance aB is equal to CB, since angle $a = 45°$.

Consequently, by measuring distance aB (or, at another location, distance AD) and adding BD to it, i.e., the elevation of point a above the ground, you will obtain the desired height of the tree.

Another method does not even require a pin device. Here you need a pole, which you will have to insert vertically into the ground so that the protruding part is exactly at your height. The location for the pole must be chosen so that, lying down as shown in Figure 6, you see the top of the tree in a straight line with the upper point of the pole. Since triangle Aba is isosceles and right-angled, angle $A = 45°$, and therefore AB equals BC, i.e., the desired height of the tree.

The Method of Jules Verne

The next, also quite simple, method for measuring tall objects is vividly described by Jules Verne in his famous novel *The*

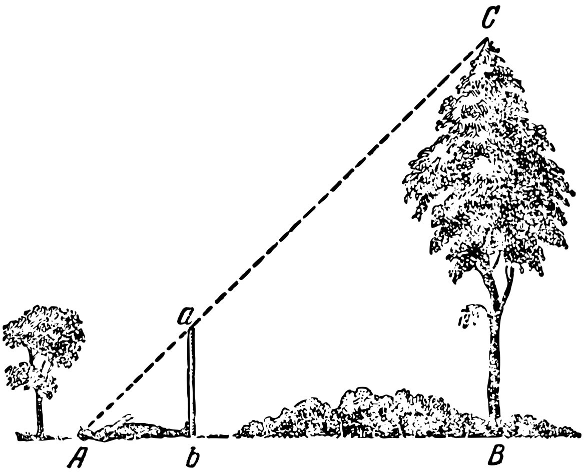


Figure 6: Another way to determine the height.

Mysterious Island.

"Today we need to measure the height of the Far View platform," said the engineer.

"Will you need a tool for that?" asked Herbert.

"No, we won't. We'll proceed somewhat differently, resorting to a somewhat simpler and more accurate method."

The young man, eager to learn as much as possible, followed the engineer, who descended from the granite wall to the rocky shore.

Taking a straight pole, twelve feet long, the engineer measured it as precisely as possible, comparing it to his own height, which he knew well. Meanwhile, Herbert held a plumb bob given to him by the engineer: just a stone attached to the end of a rope.

Not reaching five hundred feet from the granite wall, which rose vertically, the engineer drove the pole two feet into the sand and firmly secured it, placing it vertically with the help of the plumb bob.

Then he moved away from the pole to a distance where, lying on the sand, one could see both the end of the pole and the edge of the ridge in a straight line (see Figure 7). He carefully marked this point with a stake.

"Are you familiar with the basics of geometry?" he asked Herbert as he rose from the ground.

"Yes."

"Do you remember the properties of similar triangles?"

"Their corresponding sides are proportional."

"Exactly. So now I'll construct two similar right triangles. In the smaller one, one leg will be the plumb-line pole, and the other will be the distance from the stake to the base of

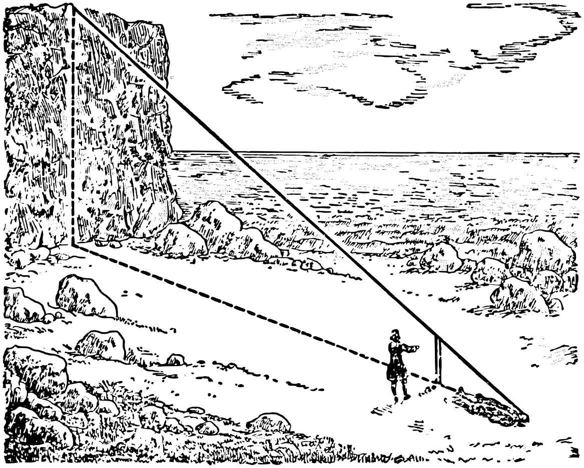


Figure 7: How the heroes of Jules Verne measured the height of the cliff.

the pole; the hypotenuse will be my line of sight. In the other triangle, the legs will be: the granite wall, the height of which we want to determine, and the distance from the stake to the base of this wall; the hypotenuse will be my line of sight, coinciding with the direction of the hypotenuse of the first triangle."

"Understood!" exclaimed the youth. "The distance from the stake to the pole is related to the distance from the stake to the base of the wall, as the height of the pole is to the height of the wall."

"Yes. And consequently, if we measure the first two distances, then, knowing the height of the pole, we can calculate the fourth, unknown term of the proportion, i.e., the height of the wall. In this way, we can manage without directly measuring the height."

Both horizontal distances were measured: the smaller one was 15 feet, the larger one was 500 feet.

At the end of the measurements, the engineer made the following record:

$$15 : 500 = 10 : x,$$
$$500 \times 10 = 5000,$$
$$5000 : 15 = 333.3.$$

Thus, the height of the granite wall was 333 feet.

How Sergeant Popov Acted

Some of the methods described for measuring height are inconvenient as they require lying on the ground. This inconvenience can, of course, be avoided.

Here's a story from one of the fronts of the Great Patriotic War. Lieutenant Ivanyuk's unit was ordered to build a bridge across a mountain river. On the opposite bank were entrenched fascists. To scout the location for the bridge, the lieutenant assigned a reconnaissance group led by Senior Sergeant Popov. In the nearest forest, they measured the diameter and height of the most typical trees and counted the number of trees that could be used for construction.

They measured the height of the trees using a pole (stick) as shown in Figure 8.

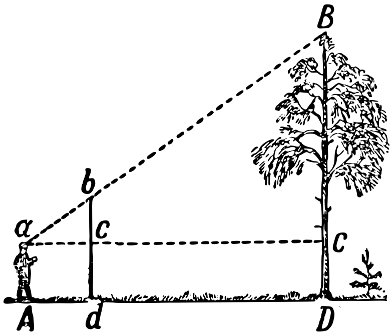


Figure 8: Measuring the height of the trees with a pole.

This method works as follows: Armed with a pole taller than your own height, drive it into the ground vertically at some distance from the tree being measured (see Figure 8). Step back from the pole along the line *dD* until you reach point *A*, from where, looking at the top of the tree, you'll see the upper point *B* of the pole aligned with it. Then, without changing the position of your head, look along the horizontal line *aC*, noting the point C where your line of sight intersects the pole and the tree trunk. Ask your assistant to mark these points, and the observation is complete. Then, based on the similarity of triangles abc and aBC, calculate BC from the proportion

$$\frac{BC}{bc} = \frac{aC}{ac},$$

and thus

$$BC = bc \cdot \frac{aC}{ac}$$

The distances bc, aC, and ac can be easily measured directly. To obtain the actual height of the tree, add the distance BC to the distance CD, which is also measured directly.

To determine the number of trees, the senior sergeant ordered the soldiers to measure the area of the forest. Then he counted the number of trees in a small area measuring 50 by 50 meters and multiplied accordingly.

Based on all the data collected by the scouts, the unit commander determined where and what kind of bridge needed to be built. The bridge was completed on time, and the combat mission was successfully accomplished![3]

[3] The episodes of the Great Patriotic War described here and further are narrated by A. Demidov in the journal *Military Knowledge* No. 8, 1949, in the article *River Reconnaissance.*

Using a Notebook

As a device for an approximate estimate of the inaccessible height, you can also use your pocket back book, if it is equipped with a pencil stuck in a cover or a loop with a book. It will help you to build in space those two similar triangles, from which the desired height is obtained. The book should be held near the eyes as shown in the simplified Figure 9. It should be in the vertical book so that, looking from the point a, you can see the top of the tree B covered with the tip of the pencil b. Then, due to the similarity of the triangles abc and aBC, the height of the BC will be determined from the proportion

$$\frac{BC}{bc} = \frac{aC}{ac}.$$

The distances of bc, ac and aC are measured directly. To the resulting value of the BC, add the length of CD, which is, on level ground, the height of the eyes above the ground

Since the width of the ac book is unchanged, if you always stand at the same distance from the measured tree (for example, 10 m), the height of the tree will depend only on the extended part of the pencil. Therefore, you can calculate in advance what height corresponds to a particular extension, and put these numbers on the pencil. Your notebook will then turn into a simplified altimeter, since you can use it to determine heights immediately, without calculations.

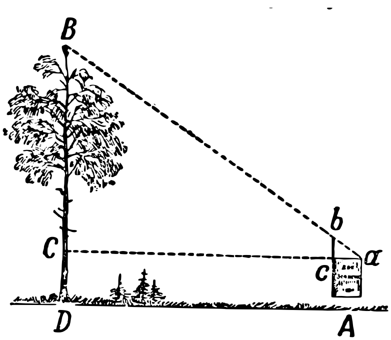


Figure 9: Height measurement using a notebook.

Without Approaching the Tree

Sometimes it may be inconvenient to get close to the base of the tree being measured. Can its height still be determined in such a case?

Absolutely. For this purpose, a clever device has been devised, which, like the previous ones, is easy to make by yourself. Two planks, *ab* and *cd* (top of Figure 10), are fastened together at right angles so that *ab* equals *bc*, and *bd* equals half of *ab*. That's the whole device.

To measure height with it, hold it in your hands, directing plank CD vertically (for which it has a plumb line with a weight), and stand precisely in two places: first (Figure 10) at point *A*, where the device is positioned with end *c* up, and then at point A', a bit farther away, where the device is held with end d up. Point A is chosen so that, looking from a to the end of a, it is seen on the same line as the top of the tree. Point A' is found so that, looking from a' to point d', it is seen coinciding with B.

The discovery of these two points A and A'[4] constitutes all the measurement because the desired part of the tree's height, BC, is equal to the distance DA'. The equality follows easily from the fact that $aC = BC$ and $a'C = 2BC$; thus,

[4] These points must necessarily lie in a straight line with the base of the tree.

$$a'C - aC = BC.$$

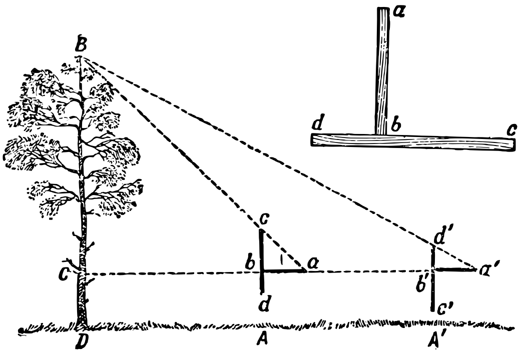


Figure 10: The use of a simple altimeter consisting of two planks.

You can see that using this simple device, we measure the tree's height without approaching closer than its height. It goes without saying that if it's possible to approach the trunk, it's sufficient to find just one of the points – A or A' – to determine its height.

Instead of two planks, you can use four pins, arranging them on a board properly; in this form, the "device" is even simpler.

Forest Rangers' Altimeter

It's time to explain how the "real" altimeters, used in practise by forest workers, are constructed. I'll describe one of these altimeters, slightly modifying it so that the device can be easily crafted at home.

The essence of the device is visible in Figure 11. A cardboard or wooden rectangle, $abcd$, is held in the hand so that, looking along edge ab, the tip B of the tree is in line with it. A weight, q, is suspended from point b on a thread. We note the point n where the thread intersects line dc. Triangles bBC and bnc are similar because they are both rectangular and have equal acute angles bBC and bnc (with corresponding parallel sides). Therefore, we can write the proportion:

$$\frac{BC}{nc} = \frac{bC}{bc}; \text{ hence,}$$

$$BC = bC \cdot \frac{nc}{bc}.$$

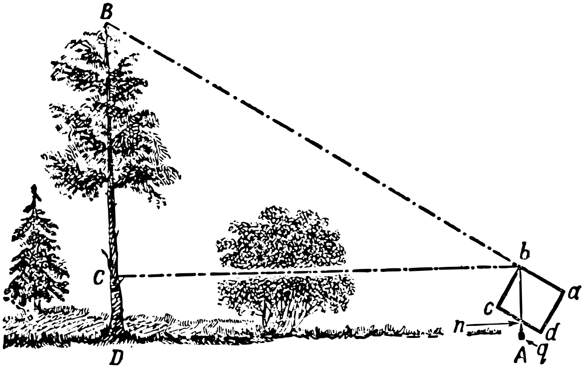


Figure 11: The scheme of using the altimeter of foresters.

Since bC, nc, and bc can be measured directly, it is easy to obtain the desired height of the tree by adding the length of the lower part CD to the trunk (the height of the device above the ground).

A few details remain to be added. If the edge of the board bc is made, for example, exactly 10 cm, and centimeter divisions are marked on edge dc, then the ratio nc/bc will always be expressed as a decimal fraction, directly indicating what fraction of the distance bC represents the height of the tree BC. For example, let's say the thread stops against the 7th division mark (i.e., $nc = 7$ cm); this means that the height of the tree above eye level is 0.7 times the observer's distance from the trunk.

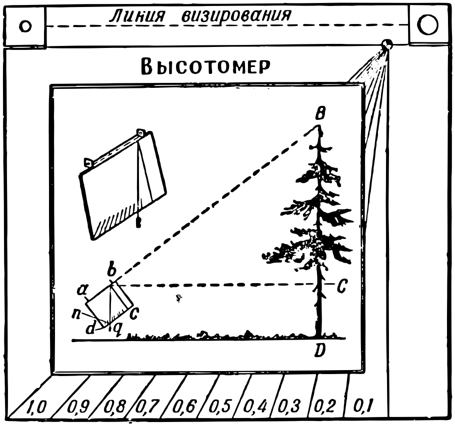


Figure 12: The forest rangers' altimeter.

The second improvement relates to the method of observation: to make it convenient to look along line *ab*, you can fold down two squares with holes drilled in them at the upper corners of the cardboard rectangle: one smaller one for the eye and one larger one for sighting the tree top (see Figure 11). Further enhancement is represented by the device shown almost to scale in Figure 12. It is easy and quick to make it in this form; no special skill is required. Occupying little space in the pocket, it will provide you with the ability to quickly determine the heights of encountered objects during excursions—trees, poles, buildings, and so on. (This tool is part of the *Geometry in the Open Air* kit developed by the author of this book.)

Question Is it possible to use the altimeter described now to measure trees that cannot be approached closely? If possible, what should be done In such cases?

Answer The device should be aimed at the top of the tree B, as shown in Figure 13, from two points, A and A'.

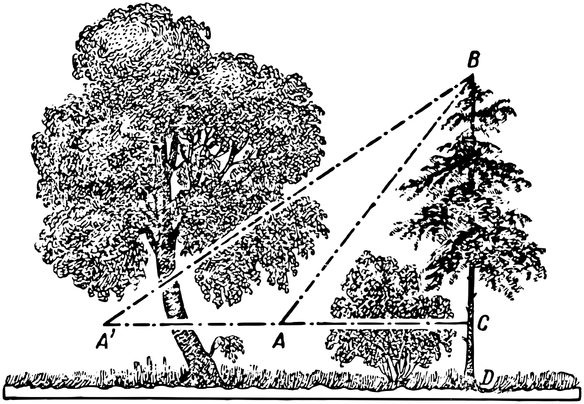


Figure 13: How to measure the height of a tree without approaching it.

Let's say at point A we determined that $BC = 0.9\,AC$, and at point A' we determined that $BC = 0.4\,A'C$. Then we know that:

$$AC = \frac{BC}{0.9}, \quad A'C = \frac{BC}{0.4},$$

So that we can write

$$AA' = A'C - AC = \frac{BC}{0.4} - \frac{BC}{0.9} = \frac{25}{18}\,BC.$$

Hence,

$$\begin{aligned} AA' &= \frac{25}{18}\,BC, \\ \therefore\ BC &= \frac{18}{25}\,AA' \\ &= 0.72\,AA'. \end{aligned}$$

You can see that by measuring the distance AA' between both observation points and taking a certain fraction of this value, we can determine the desired and inaccessible height.

Using a Mirror

Question Here's another unconventional method for determining the height of a tree using a mirror. At some distance (see Figure 14) from the tree being measured, on level ground at point C, place a small mirror horizontally and step back to point D, from where the observer can see the top of tree's point A in the mirror. Then, the tree (AB) is as many times taller than the observer's height (ED) as the distance BC from the mirror to the tree is greater than the distance CD from the mirror to the observer. Why?

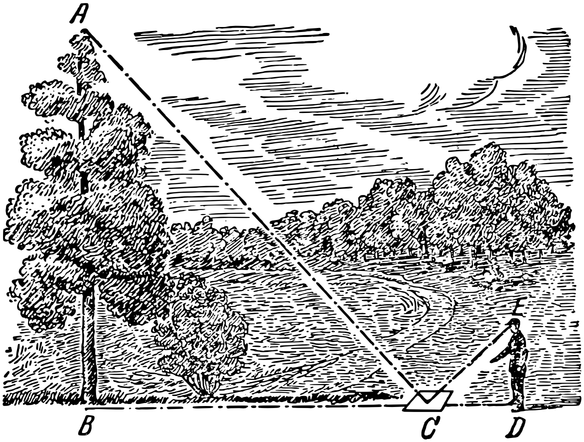


Figure 14: Height measurement using a mirror.

Answer The method is based on the law of reflection of light. The top A (Figure 15) is reflected at point A' in such a way that $AB = A'B$. From the similarity of triangles BCA' and CED, it follows that

$$\frac{A'B}{ED} = \frac{BC}{CD}.$$

In this, simply replace $A'B$ with AB to justify the relationship stated in the problem. This convenient and effortless method can be applied in any weather, but not in dense vegetation, only to a solitary tree.

Question However, what should be done when it is impossible to approach the tree being measured closely for some reason?

Answer This is an ancient problem dating back over 500 years. It is discussed by the medieval mathematician Antonius de Cremona in his work *On Practical Land Measurement* (1400).

The problem is solved by the dual application of the method described earlier – placing the mirror in two locations. By making the appropriate construction, it is easy to deduce from the similarity of triangles that the sought-after height of the tree is equal to the observer's eye level multiplied by the ratio of the distance between the mirror positions to the difference in distances from the mirror to the observer.

Before concluding the discussion on measuring the height of trees, I propose to the reader another "forest" problem.

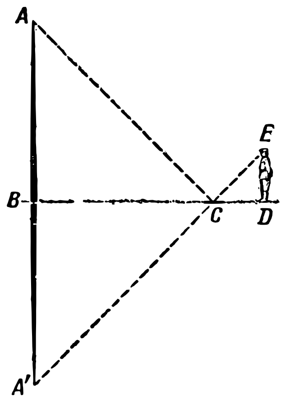


Figure 15: Geometric construction for the method of measuring the height using a mirror.

Two Pines

Question Two pine trees grow 40 meters apart. You measured their heights: one turned out to be 31 meters tall, while the other, younger one, is only 6 meters tall. Can you calculate the distance between their tops?

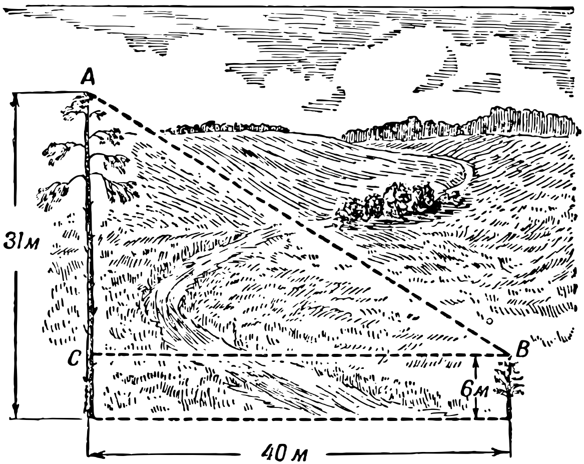


Figure 16: What is the distance between the tops of the pines?

Answer The desired distance between the tops of the pine trees (see Figure 16) according to the Pythagorean theorem is

$$\sqrt{40^2 + 25^2} = 47 \text{ m}.$$

The Shape of the Tree Trunk

Now you can already, walking through the forest, determine – in almost half a dozen different ways – the height of any tree. It will probably be interesting for you to determine its volume as well, calculate how many cubic meters of wood it contains, and at the same time weigh it. To find out if, for example, it would be possible to take away such a trunk on one cart. Both of these tasks are no longer as simple as determining height; experts have not found ways to accurately resolve it and are content with only a more or less approximate estimate. Even for a felled trunk, which lies in front of you cleared of branches, the task is far from easy.

The thing is, a tree trunk, even the smoothest one without bulges, does not represent either a cylinder, a complete cone, a truncated cone, or any other geometric solid whose volume we can calculate using formulas. The trunk is certainly not a cylinder — it tapers towards the top (it has "runoff", as foresters say) — but it is also not a cone because its "generating line" is not a straight line, but a curve, and moreover, not a circular arc, but some other curve, convex towards the axis of the tree.[5]

[5] The curve that fits closest to this is called the "semicubical parabola" ($y^3 = ax^2$); the solid obtained by rotating this parabola is called a "neiloid" (named after the ancient mathematician Neil, who found a way to determine the length of the arc of such a curve). The shape of a tree trunk grown in the forest approximates that of a neiloid. Calculating the volume of a neiloid is done using advanced mathematical techniques.

Therefore, a more or less accurate calculation of the volume of a tree trunk can only be done using the tools of integral calculus. To some readers, it may seem strange that the measurement of a simple log requires resorting to the services of higher mathematics. Many think that higher mathematics is only relevant to some special subjects, whereas in everyday life, only elementary mathematics is applicable. This is completely incorrect: one can fairly accurately calculate the volume of a star or a planet using elements of geometry, whereas an exact calculation of the volume of a long log or a beer barrel is impossible without analytical geometry and integral calculus.

However, our book does not assume that the reader is familiar with higher mathematics; therefore, here we will have to be content with only an approximate calculation of the volume of the trunk. We will assume that the volume of the trunk is more or less close either to the volume of a truncated cone, or – for a trunk with a pointed end — to the volume of a complete cone, or, finally, – for short logs — to the volume of a cylinder. The volume of each of these three solids can be easily calculated. Could we find a formula for the volume that would be suitable for all three of these named solids for the sake of consistency in calculation? Then we would approximately calculate the volume of the trunk without caring about what it resembles more — a cylinder or a cone, complete or truncated.

Universal Formula

Such a formula exists; moreover, it is not only suitable for cylinders, complete and truncated cones, but also for all kinds of prisms, pyramids complete and truncated, and even for spheres. Here is this remarkable formula, known in mathematics as Simpson's formula:

$$v = \frac{h}{6}\,(b_1 + 4b_2 + b_3),$$

where h is the height of the solid, b_1 is the area of the lower base, b_2 is the area of the middle section[6], b_3 is the area of the upper base.

[6] That is, the cross-sectional area of the body in the middle of its height.

Question Prove that with this formula, one can calculate the volume of the following seven geometric solids: prism, pyramid complete, pyramid truncated, cylinder, cone complete, cone truncated, sphere.

Answer It is very easy to verify the correctness of this formula by simply applying it to the listed solids. Then, we obtain for the prism and cylinder (see Figure 17 a):

$$v = \frac{h}{6}\,(b_1 + 4b_2 + b_3) = b_1 h;$$

for the pyramid and cone (see Figure 17 b):

$$v = \frac{h}{6}\,(b_1 + 4\frac{b_2}{4} + 0) = \frac{b_1 h}{3};$$

for the truncated cone (see Figure 17 c):

$$\begin{aligned} v &= \frac{h}{6}\left[\pi R^2 + 4\pi\frac{(R+r)^2}{2} + \pi r^2\right], \\ &= \frac{h}{6}\left[\pi R^2 + \pi R^2 + 2\pi Rr + \pi r^2 + \pi r^2\right], \\ &= \frac{\pi h}{3}\left[R^2 + Rr + r^2\right], \end{aligned}$$

for the truncated pyramid, the proof proceeds similarly; finally, for the sphere (see Figure 17 d):

$$v = \frac{2R}{6}\left(0 + 4\pi R^2 4 + 0\right) = \frac{4}{3}\pi R^3.$$

Question Let's note another interesting feature of our universal formula: it is also suitable for calculating the area of plane figures: parallelograms, trapezoids, and triangles, if by

§ h we mean, as before, the height of the figure,
§ by b_1 the length of the lower base,
§ by b_2 the length of the middle base and
§ by b_3 the length of the lower base.

How can we confirm this?

Answer Applying the formula, we have: for a parallelogram

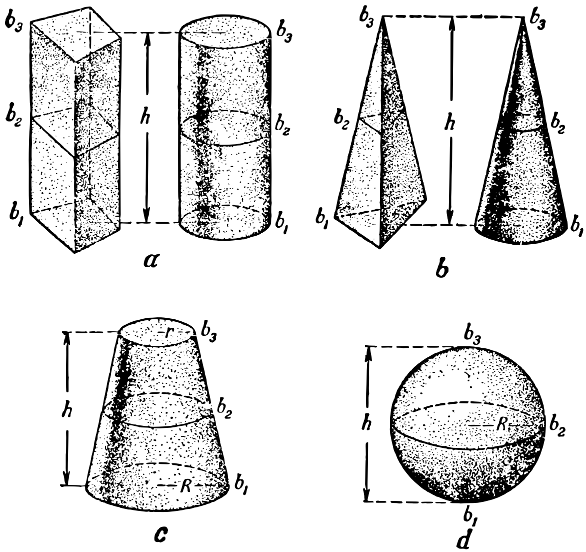


Figure 17: Geometric bodies whose volumes can be calculated using a single formula.

(square, rectangle) (see Figure 18 a)

$$S = \frac{h}{6}\left(b_1 + 4b_1 + b_1\right) = b_1 h;$$

for a trapezoid (see Figure 18 b)

$$S = \frac{h}{6}\left(b_1 + 4\frac{b_1 + b_2}{2} + b_3\right) = \frac{h}{2}\left(b_1 + b_3\right);$$

for a triangle (see Figure 18 c)

$$S = \frac{h}{6}\left(b_1 + 4\frac{b_1}{2} + 0\right) = \frac{b_1 h}{2}.$$

You can see that our formula has enough right to be called universal.

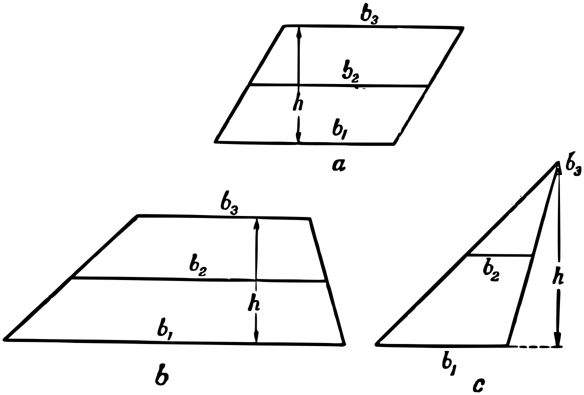


Figure 18: The universal formula is also suitable for calculating the areas of these figures.

Volume and Weight of a Tree at the Root

So, you have a formula with which you can approximately calculate the volume of a felled tree trunk without worrying about what geometric shape it resembles: a cylinder, a complete cone, or a truncated cone. For this, four measurements are needed – the length of the trunk and three diameters: the lower cut, the upper, and in the middle of the length. Measuring the lower and upper diameters is very simple; however, determining the average diameter without a special device ("measuring fork" used by foresters, see Figure 19 and Figure 20[7] is quite difficult. But the difficulty can be overcome by encircling the trunk with a rope and dividing its length by 3 1/7 to get the diameter.

[7] A similar principle is applied in the well-known device for measuring the diameter of round objects – the caliper Figure 20, to the right).

Figure 19: Measuring the diameter of a tree with a measuring fork.

The volume of a felled tree trunk obtained in this way is accurate enough for many practical purposes. In short, but less accurately, this problem can be solved by calculating the volume of the trunk as the volume of a cylinder, the diameter of the base of which is equal to the diameter of the trunk in the middle of its length; however, the result obtained is underestimated, sometimes by 12%. But if you mentally divide the trunk into two-meter segments and determine the volume of each of these almost cylindrical parts to then add them up, the result will be much better: it errs on the side of underestimation by no more than 2–3%.

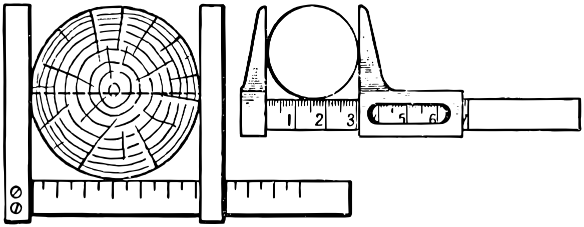


Figure 20: Measuring fork (left) and caliper (right).

However, all this is completely inapplicable to a tree at the root: if you are not going to climb it, then you can only measure the diameter of its lower part. In this case, to determine the volume, you will have to be satisfied with only a very approximate estimate, comforting yourself with the fact that professional foresters usually proceed in a similar way. They also use a table of so-called "species numbers," i.e., numbers that show what proportion of the volume of the measured tree is compared to the volume of a cylinder of the same height and diameter, measured at the height of a grown man's chest, i.e., 130 cm (this height is the most convenient for measuring).

Figure 21 illustrates this clearly. Of course, "species numbers" vary for trees of different species and heights, as the shape of the trunk is variable. However, the fluctuations are not particularly great: for pine and fir trunks (grown in dense plantations), "species numbers" range from 0.45 to 0.51, i.e., are approximately half.

Thus, without much error, it can be assumed that the volume of a coniferous tree at the root is half the volume of a cylinder of the same height with a diameter equal to the diameter of the tree at chest height.

This is, of course, only an approximate estimate, but it is not too far from the true result: up to 2% in the overestimation direction and up to 10% in the underestimation direction.[8]

[8] It must be remembered that "species numbers" refer only to trees that have grown in the forest, i.e. to tall and thin (smooth, without nodes); for free-standing branched trees, such general rules for calculating volume cannot be specified.

From here, it is only one step towards estimating the weight of the tree at the root. For this, it is enough to know that 1 cubic meter of fresh pine or fir wood weighs about 600–700 kg. For example, suppose you are standing next to a fir tree, the height of which you have determined to be 28 m, and the circumference of the trunk at chest height is 120 cm. Then the area of the corresponding circle is 1,100 cm^2, or 0.11 m^2, and the volume of the trunk is $1/2 \times 0.11 \times 28 = 1.5\,\mathrm{m}^3$.

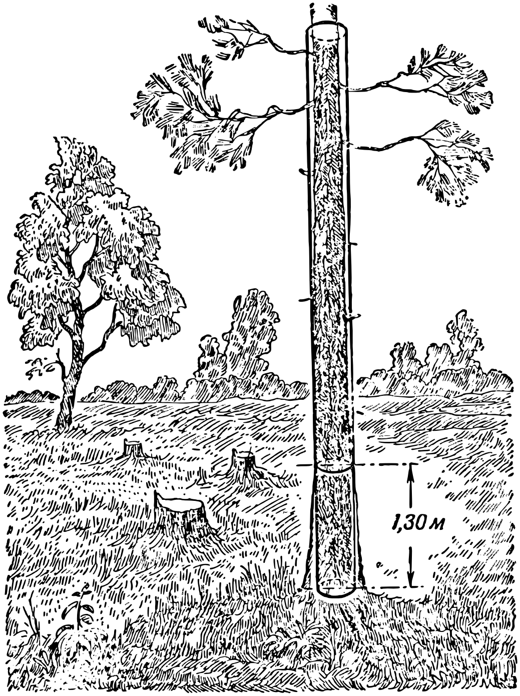


Figure 21: What is a “species number”?

Assuming that 1 cubic meter of fresh fir wood weighs on average 650 kg, we find that 1.0 cubic meter should weigh about a ton (1,000 kg).

Leaf Geometry

Question In the shadow of a silver poplar from its roots, a thicket has grown. Pick a leaf and notice how large it is compared to the leaves of the parent tree, especially those that grew in bright sunlight. The shaded leaves compensate for the lack of light with the size of their area, capturing sunlight rays. Understanding this is the task of botany. But the geometer can also have a say here: he can determine exactly how many times the area of the thicket leaf is larger than the area of the parent tree leaf.

How would you solve this problem?

Answer You can go two ways. First, determine the area of each leaf separately and find their ratio. The area of the leaf can be measured by covering it with transparent grid paper, each square of which corresponds, for example, to 4 square millimeters (a sheet of transparent grid paper used for this purpose is called a pallet). This is a perfectly correct but overly laborious method.[9]

[9] However, this method has an advantage: using it, you can compare the areas of leaves with different shapes, which cannot be done according to the method described below.

A shorter method is based on the fact that both leaves, different in size, still have the same or almost the same shape: in other words, they are geometrically similar figures. We know that the areas of such figures are related as the squares of their linear dimensions. Therefore, by determining how many times one leaf is longer or wider than the other, we can find the ratio of their areas simply by squaring this number. Let the thicket leaf be 15 cm long, and the leaf from the tree branch only 4 cm long; the ratio of their linear dimensions is 15/4, and therefore, in terms of area, one is larger than the other by 225/16 times, or about 14. Rounding off (since full accuracy cannot be achieved here), we can say that the thicket leaf is approximately 15 times larger than the tree leaf in terms of area.

Let us consider another example.

Figure 22: Determine the ratio of the areas of these leaves.

Question At a dandelion grown in shade, a leaf is 31 cm long. At another specimen grown in sunlight, the leaf blade is only 3.3 cm long. Approximately how many times is the area of the first leaf larger than the area of the second?

Answer We proceed as before. The ratio of the areas is

$$\frac{31^2}{3.3^2} = \frac{960}{10.9} = 87;$$

so one leaf is approximately 90 times larger than the other in terms of area.

It is easy to find in the forest many pairs of leaves of the same shape but different sizes, thus providing interesting material for geometric problems on the ratio of areas of similar figures. It always seems strange to an unaccustomed eye that a relatively small difference in the length and width of leaves results in a noticeable difference in their areas. For example, if two leaves, geometrically similar in shape, differ in length by 20%, then the ratio of their areas is

$$1.2^2 \approx 1.4,$$

meaning the difference is 40%. And with a difference in width of 40%, One leaf exceeds the other in area by

$$1.4^2 \approx 2,$$

or nearly twice.

Question We invite the reader to determine the ratio of the areas of the leaves depicted in Figure 22 and Figure 23.

Figure 23: Determine the ratio of the areas of these leaves.

Six-legged Heroes

Amazing creatures, ants! Swiftly climbing up stems with a burden much heavier than their tiny size (Figure 24), ants present an intriguing puzzle to observant individuals: where does the insect derive the strength to effortlessly carry a load ten times its own weight? Indeed, a human might struggle to climb stairs while carrying, for instance, a piano (Figure 24), with the weight ratio of the load to the body being roughly similar to that of an ant. Thus, it seems that the ant is relatively stronger than a human!

But is it really so?

Without geometry, this cannot be understood. Let's listen to what the expert (Professor A.F. Brandt) has to say, primarily about the strength of muscles, and then about the current question regarding the comparison of forces between the insect and the human: "A muscle resembles a resilient cord; however, its contraction is based not on elasticity, but on other reasons, and is normally manifested under the influence of nervous excitation, as demonstrated in physiological experiments involving the application of electric current to the corresponding nerve or directly to the muscle."

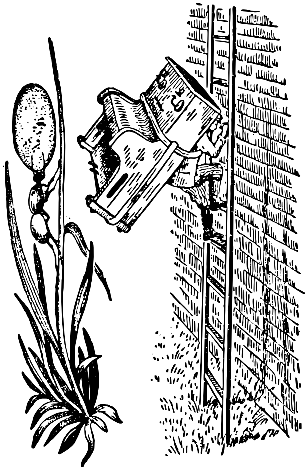


Figure 24: The six-legged hero.

"These experiments are easily conducted on muscles excised from a freshly killed frog, as the muscles of cold-blooded animals retain their vital properties for a long time even outside the organism, even at ordinary temperatures. The experiment is very simple. The main calf muscle, which extends the hind leg, is excised together with a piece of the femur bone from which it originates, and together with the terminal tendon. This muscle is found to be the most convenient due to its size, shape, and ease of preparation. A hook is passed through the tendon, and a weight is attached to it."

"If wires from a galvanic element are touched to such a muscle, it instantly contracts, shortens, and lifts the load. By gradually adding additional weights, the maximum lifting capacity of the muscle can be easily determined. Now, if we bind together in length two, three, or four identical muscles and stimulate them simultaneously, we will not achieve greater lifting force; the load will only be lifted to a greater height, corresponding to the sum of the contractions of individual muscles. However, if we bundle two, three, or four muscles together, the entire system will lift a weight many times greater when stimulated. The same result, obviously, would be obtained if the muscles were fused together. Thus, we conclude that the lifting force of muscles depends not on their length or total mass, but only on their thickness, i.e., *cross-sectional area.*"

"After this digression, let's turn to the comparison of similarly structured, geometrically similar, but differently sized animals. Let's imagine two animals: the original and one that has been doubled in size in all linear dimensions. In the second animal, the volume and weight of the entire body, as well as each of its organs, will be eight times greater; however, all corresponding planar dimensions, including the cross-sectional area of muscles, will be only four times greater. It turns out that as the animal grows to twice the length and eight times the weight, its muscular strength increases only fourfold, i.e., the animal becomes relatively weaker. Based on this reasoning, an animal that is three times longer (with cross-sectional areas three times larger and a weight 27 times greater) would be relatively three times weaker, and one that is four times longer would be four times weaker, and so on."

"The law of unequal growth in volume and weight of the animal, and thus of muscular strength, explains why insects – as observed in ants, predatory wasps, and others – can carry loads 30 to 40 times their own weight, whereas a human can typically carry excluding gymnasts and porters – only about 9/10 times their own weight, – and a horse, which we view as a magnificent living work machine, even less, namely, only about 7/10 of its own weight."[10]

[10] For more details, see *Fun with Physics* by Ya. I. Perelman, Chapter X *Mechanics in the Living World.*

After these explanations, we will look at the feats of that ant-giant with different eyes, about whom I.A. Krylov mockingly wrote:

> *Some ant had extraordinary strength,*
> *Such as was unheard of even in ancient times;*
> *He even (says his faithful historian)*
> *Could lift two barley grains.*

2. Geometry By The River

Measuring the Width of the River

When crossing a river, measuring its width is just as easy for those who know geometry, how to determine the height of a tree, without climbing to the top. The inaccessible distance is measured the same techniques that we used to measure the inaccessible height. In both cases, the definition of the desired distance is replaced an example of another distance that is easily measurable directly.

Of the many ways to solve this problem, let's look at some of the simplest ones.

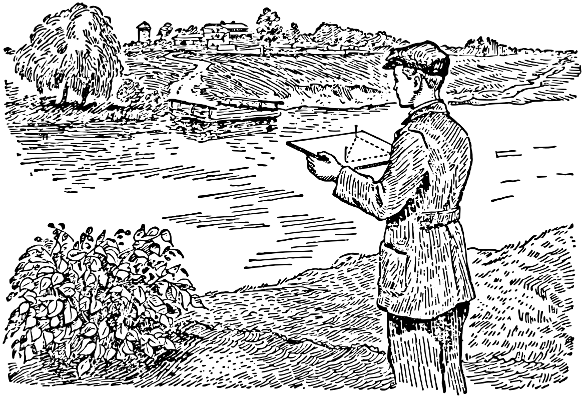


Figure 25: Measuring the width of the river with a pin device.

1. The first method requires the familiar "device" with three pins at the vertices of an isosceles right triangle (Figure 25). Let's say we need to determine the width of river AB (Figure 26), standing on the bank where point B is, without crossing to the opposite bank. Standing somewhere at point C, hold the pin device close to your eye so that, looking with one eye along the two

pins, you see both covering points B and A. It's clear that when you manage this, you will be exactly on the extension of line AB.

Now, without moving the plank of the device, look along the other two pins (perpendicular to the previous direction) and notice any point D covered by these pins, i.e., lying on the line perpendicular to AC. After this, insert a pin at point C, leave this place, and go with your instrument along line CD until you find a point E (Figure 27), where you can simultaneously cover point C for one eye with pin b and point A with pin a. This means you have found the third vertex of triangle ACE on the shore, where angle C is a right angle, and angle E is opposite to the acute angle of the pin device, i.e., half the right angle (45°). Obviously, angle A is also half right angle, i.e., $AC = CE$. If you measure the distance CE even by steps, you will know the distance AC, and by subtracting BC, which is easy to measure, you will determine the desired width of the river.

It is quite inconvenient and difficult to hold the pin device still in hand; therefore, it is better to attach this plank to a stick with a pointed end and insert it vertically into the ground.

2. The second method is similar to the first. Here also, find point C on the extension of AB and mark line CD perpendicular to CA using the pin device. But then

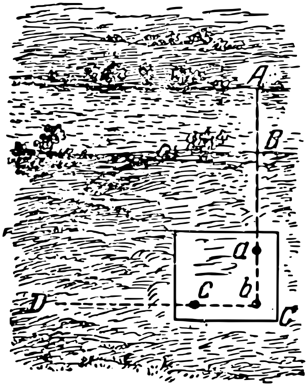


Figure 26: First position of the pin device.

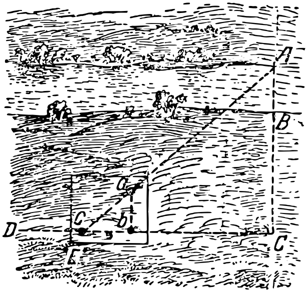


Figure 27: Second position of the pin device.

proceed differently (Figure 28). Equal distances CE and EF of arbitrary length are measured on the straight line CD, and pegs are inserted at points E and F. Then, standing at point F with a pin device, the direction FG is marked out perpendicular to FC. Now, walking along FG, find a point H on this line from which point A seems to be covered by point E. This will mean that points H, E, and A lie on the same straight line.

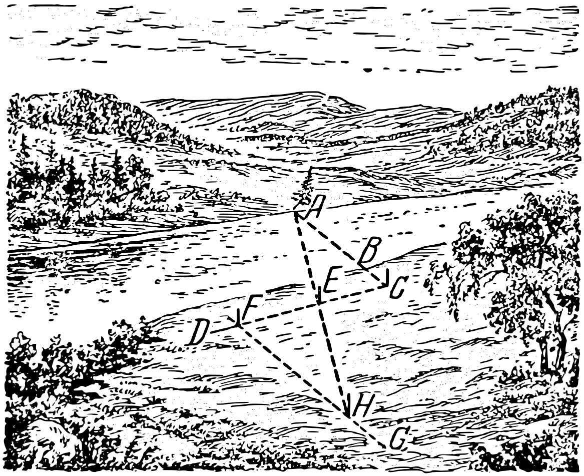


Figure 28: Using the congruence criteria of triangles to find the width of the river.

The problem is solved: the distance FH is equal to the distance AC, from which it is only necessary to subtract BC to find the desired width of the river (the reader, of course, will guess for himself why FH is equal to AC).

This method requires more space than the first one; if the terrain allows executing both methods, it is useful to verify one result by another.

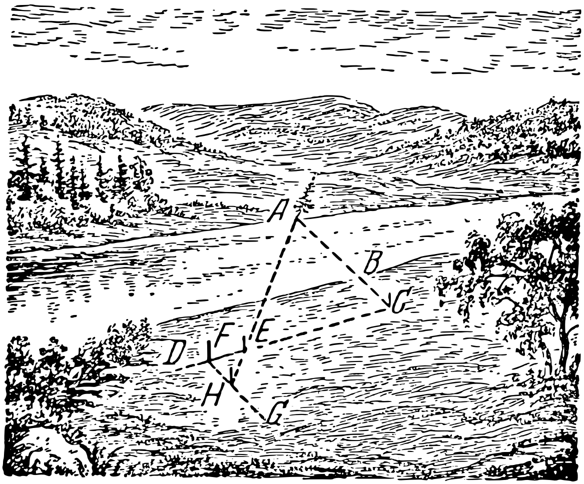


Figure 29: Using the similarity criteria of triangles to find the width of the river.

3. The method described above can be modified: instead of measuring equal distances on the straight line CF,

measure one distance several times smaller than the other. For example (Figure 29), FE is measured four times less than EC, and then we proceed as before: in the direction FG, perpendicular to FC, we find a point H from which the peg E appears to cover point A. But now FH is no longer equal to AC, but four times smaller than this distance: triangles ACE and EFH are not congruent here, but similar (they have equal angles with unequal sides). From the similarity of triangles follows the proportion:

$$\frac{AC}{FH} = \frac{CE}{EF} = \frac{4}{1}.$$

Therefore, by measuring FH and multiplying the result by 4, we get the distance AC, and by subtracting BC, we find the desired width of the river.
This method, as we can see, requires less space and is therefore more convenient to perform than the previous one.

4. The fourth method is based on the property of a right triangle that if one of its acute angles is 30°, then the length of the cathetus is half the hypotenuse. It is very easy to verify the correctness of this.
   Let angle B of right triangle ABC (Figure 30, left) be 30°; we will prove that in this case, $AC = 1/2\,AB$. Rotate triangle ABC around BC so that it is symmetric with

its initial position (Figure 30, right), forming figure ABD; line AC is straight because both angles at point C are right angles. In triangle ABD, angle $\angle A = 60°$, angle ABD, composed of two $30°$ angles, is also equal to $60°$. Therefore, $AD = BD$ as sides opposite equal angles. But $AC = 1/2\,AD$, therefore,

$$AC = \frac{1}{2}AB.$$

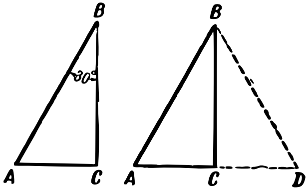


Figure 30: When the cathetus is half the hypotenuse.

Wishing to take advantage of this property of the triangle, we must arrange the pins on the board so that their bases represent a right triangle in which the cathetus is half the hypotenuse. With this device, we place ourselves at point C (Figure 31) so that the direction AC coincides with the hypotenuse of the pin triangle. Looking along the short cathetus of this triangle, mark the direction CD and find a point E on it so that the direction EA is perpendicular to CD (this is done using the same pin device). It is easy to see that the distance CE – the cathetus lying opposite the angle of $30°$ – is equal to half of AC. Therefore, by measuring CE, doubling this distance and subtracting BC, we obtain the desired width of the AB river.

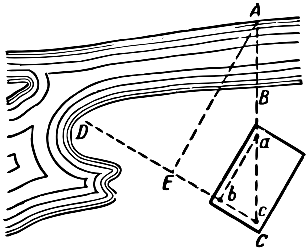


Figure 31: The scheme of application of a right-angled triangle with a $30°$ angle.

Here are four easily executable methods, with which it is always possible, without crossing to the other bank, to measure the width of the river with quite satisfactory accuracy.

We will not consider methods that require the use of more complex instruments (even homemade ones) here.

Using a Visor

Here's how this method came in handy for Senior Sergeant Kupriyanov in frosty conditions.[11] His detachment was ordered to measure the width of the river, across which they were to organise a crossing...

[11] See the footnote on page 21.

Approaching a bush near the river, Kupriyanov's detachment took cover, and Kupriyanov himself, along with soldier Karpov, moved closer to the riverbank, from where the fascist-occupied shore was clearly visible. In such conditions, measuring the width of the river had to be done by eye.

"Come on, Karpov, how much?" Kupriyanov asked.

"I think no more than 100-110 meters," Karpov replied. Kupriyanov agreed with his scout, but for control, he decided to measure the width of the river using a "visor."

This method is simple. You have to face the river and pull the visor over your eyes so that the lower edge of the visor precisely aligns with the line of the opposite bank (see Figure 32). The visor can be replaced with the palm of your hand or a notepad, tightly pressed edge to your forehead.

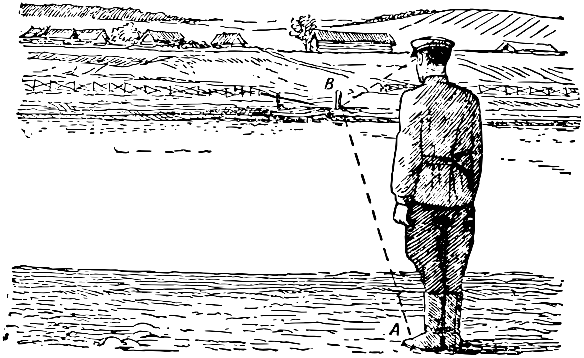


Figure 32: Observing a point on the opposite bank from under the visor.

Then, without changing the position of your head, you need to turn to the right or left, or even backward (towards the side where the area available for measuring the distance is more level) and notice the farthest point visible from under the visor (palm, notepad).

The distance to this point will be approximately equal to the width of the river.

Kupriyanov utilized this method. He quickly stood up in the bushes, pressed a notepad to his forehead, then quickly turned and aimed at the distant point. Then, together with Karpov, he crawled to that point, measuring the distance with a rope. It turned out to be 105 meters.

Kupriyanov reported the data he obtained to the command.

Question Provide a geometric explanation for the “visor” method.

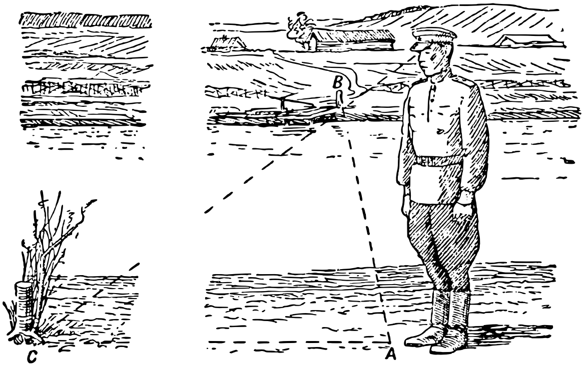


Figure 33: In the same way, you can aim at a point on your own bank.

Answer The line of sight, touching the edge of the visor (palm, notepad), is initially directed towards the line of the opposite bank (see Figure 32). When a person turns, the line of sight, like the leg of a compass, describes a circle, and then $AC = AB$ as the radii of the same circle (see Figure 33).

The Length Of An Island

Question Now we are faced with a more challenging task. Standing by the river or lake, you see an island (see Figure 34) whose length you wish to measure without leaving the shore. Is it possible to carry out such a measurement?

Although in this case, both ends of the measured line are inaccessible to us, the problem is still entirely solvable, and without complex instruments.

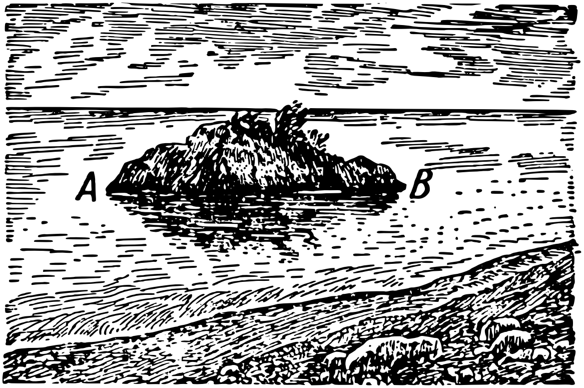


Figure 34: How to determine the length of the island.

Answer To measure the length of an island without leaving the shore, you can use the following method. Choose arbitrary points P and Q on the shore and place stakes in them. Then find points M and N on the line PQ such that the directions AM and BM form right angles with the direction of PQ (this can be done using a compass). In the middle of the distance MN, place a stake O and find on the extension of the line AM a point C from which the stake O appears to cover point B. Similarly, on the extension of BN, find point D from which stake O appears to cover the end A of the island. The distance CD will be the desired length of the island.

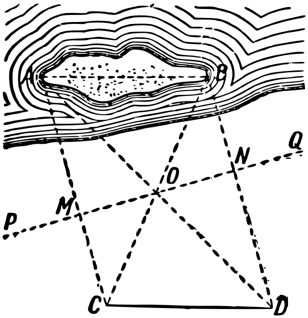


Figure 35: We use the properties of congruent right triangles to find the length of an isalnd.

This can be easily proved. Consider the right triangles AMO and OND; in them, the legs MO and NO are equal, and the angles AOM and NOD are also equal, therefore, the triangles are equal, and $AO = OD$. Similarly, it can be proved that $BO = OC$. By comparing the triangles ABO and COD, it can be seen that their distances AB and CD are equal.

A Pedestrian On the Opposite Bank

Question As you walk along the riverbank, you see a person on the other side, and you can clearly distinguish their steps. Can you, without moving from your spot, determine at least approximately the distance between them and you? You have no instruments at hand.

Answer You don't have any instruments, but you have eyes and hands – that's enough. Extend your arm forward towards

Figure 36: How to determine the distance to a pedestrian walking on the other side of the river.

the pedestrian and look at the tip of your finger with one eye if the pedestrian is moving towards your right hand, and with the other eye if they're moving towards your left hand. At the moment when the distant pedestrian is covered by your finger (see Figure 36), close the eye that was looking and open the other: the pedestrian will appear to you as if they've moved backward. Count how many steps they take before they align again with your finger. You'll get all the data needed for an approximate determination of the distance. Let's explain how to use them.

Suppose in Figure 36 (inset), your eyes are marked as a and b, point M is the tip of your finger extended, point A is the initial position of the pedestrian, and B is the final position. The triangles abM and ABM are similar (you should turn towards the pedestrian so that ab is approximately parallel to their direction of movement). Therefore, $BM : bM = AB : ab$ – is a proportion in which only one term, BM, is unknown, but all others can be directly determined. Indeed, bM is the length of your extended arm, ab is the distance between the pupils of your eyes, and AB is measured in steps taken by the pedestrian (assuming an average step to be around 3/4 metres). Therefore, the unknown distance from you to the pedestrian on the opposite bank, AB, equals

$$MB = AB\,\frac{bM}{ab}.$$

For example, if the distance between your eye pupils ab is 6 cm, the length of bM from the end of your extended arm to the eye is 60 cm, and the pedestrian takes, say, 14 steps from A to B, then their distance from you would be $MB = 14 \cdot 60/6 = 140$ steps, or 105 meters.

It's enough for you to measure in advance the distance between your eye pupils and bM – the distance from the eye to the end of your extended arm – so that you can quickly determine the distance of inaccessible objects by remembering their ratio. On average, for most people, bM/ab is around 10 with slight fluctuations. The difficulty will only be in somehow determining the distance AB. In our case, we used the steps of a distant person. But you can also use other references. For instance, if you're measuring the distance to a distant freight train, you can estimate *AB* in comparison to the length of a freight car, which is usually known (7.6 meters between buffers). If you're determining the distance to a house, you can estimate *AB* by comparing it to the width of a window, the length of a brick, etc.

The same method can be applied to determine the size of a distant object if its distance from the observer is known. For this purpose, you can also use other "rangefinders", which we will describe next.

Simple Rangefinders

In the first chapter, we described the simplest instrument for determining inaccessible heights – the altimeter. Now, let's describe the simplest device for measuring inaccessible distances – the 'rangefinder.' The simplest rangefinder can be made from an ordinary matchstick. To do this, you just need to mark millimeter divisions on one of its sides, alternating between light and dark (see Figure 37).

Figure 37: The match is a rangefinder.

You can use this primitive "rangefinder" to estimate the distance to a distant object only in those cases when the dimensions of that object are known to you (see Figure 38). However, more sophisticated rangefinders can also be used under the same condition. Suppose you see a person in the distance and set yourself the task of determining the distance to them. Here, the matchstick rangefinder can come in handy. Holding it in your outstretched arm and looking with one eye, you bring its free end into coincidence with the top of the distant figure. Then, slowly moving your thumbnail along the matchstick, you stop it at the point that projects onto the base of the human figure. All you have to do now is to find out, by bringing the matchstick closer to the eye, at which mark your thumbnail stopped – and then you have all the data to solve the problem.

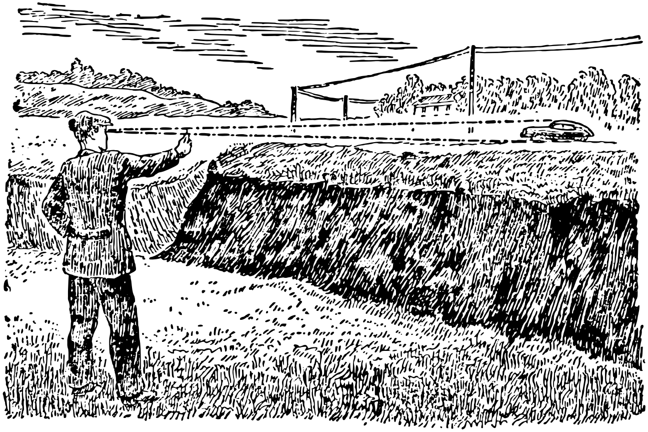


Figure 38: The use of a rangefinder match to determine inaccessible distances.

You can easily verify the correctness of the proportion:

$$\frac{\text{desired distance}}{\begin{array}{c}\text{distance from the eye}\\ \text{to the matchstick}\end{array}} = \frac{\text{average height of a person}}{\begin{array}{c}\text{measured part}\\ \text{of the matchstick}\end{array}}$$

From here, it's easy to calculate the desired distance. For example, if the distance to the matchstick is 60 cm, the height of the person is 1.7 m, and the measured part of the matchstick is 12 mm, then the determined distance would be:

$$60 \cdot \frac{1700}{12} = 8{,}500\ \text{cm} = 85\ \text{m}.$$

To gain some skill in using this rangefinder, measure the height of someone from your group and, asking them to move away a certain distance, try to determine how many steps they took away from you.

With the same method, you can determine the distance to a rider (average height 2.2 m), a cyclist (wheel diameter 75 cm), a telegraph pole along the railway track (height 8 m), vertical distance between adjacent insulators (90 cm), to a train, a brick house, and similar objects whose dimensions can be estimated with sufficient accuracy. There can be quite a few such cases during excursions.

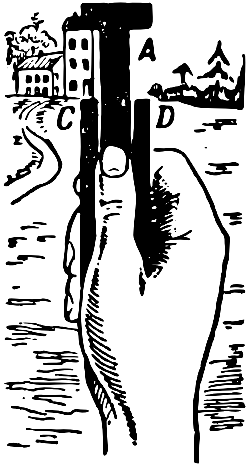


Figure 39: The retractable rangefinder in action.

For those skilled in crafting, making a more convenient device of the same type, intended for estimating distances based

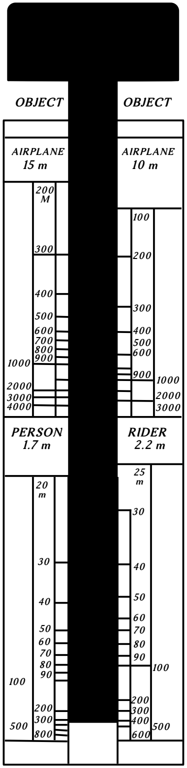


Figure 40: The design of the retractable rangefinder.

on the size of a distant human figure, won't be much trouble.

The device is clear in Figure 39 and Figure 40. The observed object is placed precisely in the gap *A*, formed when the extension part of the device is raised. The size of the gap can be conveniently determined by the divisions on the part *C* and *D* of the board. To avoid the need for any calculations, you can directly mark on strip *C* the distances corresponding to the divisions if the observed object is a human figure (the device for measuring the distance of the outstretched arm). On the right strip *D*, you can mark distances, pre-calculated for cases where a rider is observed (2.2 m). For telegraph poles (height 8 m), planes with a wingspan of 15 m, and other larger objects, you can use the upper, free parts of strips *C* and *D*. Then the device will look like the one presented in Figure 40.

Of course, the accuracy of such distance estimation is low. It's just an estimate, not a measurement. In the example discussed earlier, where the distance to the human figure was estimated at 85 m, an error of 1 mm in measuring the matchstick portion would result in a deviation of 7 m (1/12 out of 85). But if the person stood four times farther away, and we measured only 3 mm on the matchstick, then an error of even 1/2 mm would cause a change in the result by 57 m. Therefore, our example is reliable only for relatively short distances – in the range of 100–200 m. When estimating larger distances, it's necessary to choose larger objects.

The Energy of the River

You know the edge where everything breathes abundance,
Where rivers flow purer than silver,
Where the steppe breeze sways the feather grass,
Where villages are nestled in cherry orchards.

A.K. Tolstoy

A river, the length of which is no more than 100 km, is considered small. Do you know how many such small rivers there are in the USSR? A lot – 48 thousand!

If these rivers were stretched into a single line, it would result in a ribbon 13,800,000 km long. With such a ribbon, you could encircle the Earth at the equator thirty times (the length of the equator is approximately 40,009 km).

The flow of these rivers is leisurely, but it conceals an inexhaustible supply of energy within it. Specialists believe that if the hidden potential of all the small rivers flowing through our homeland were combined, an impressive number would be obtained – 34 million kilowatts! This gifted energy needs to be widely utilised for electrifying the economy of settlements located near rivers.

Let the river flow freely,
If the plan says so,
A dam with a stone ridge across all depths
Will block the way forever.

S. Shchipachev

You know that this is achieved through hydroelectric power stations (HPS), and you can show a lot of initiative and provide real assistance in preparing for the construction of small HPS. Indeed, the builders of HPS will be interested in everything related to the river regime: its width and flow rate ("water flow"), the area of the cross-section of the riverbed ("active section"), and what water head the banks allow. And all this can be measured with available means and represents a relatively simple geometric problem.

We will now proceed to solving this problem.

But first, let's present here a practical advice from specialists, engineers V. Yarosh and I. Fedorov, regarding the selection of a suitable location on the river for the construction of a future dam.

They recommend building a small hydroelectric power station with a capacity of 15-20 kilowatts "no further than 5 km from the village."

"The dam of an HPS should be built no closer than 10-15 km and not farther than 20-40 km from the source of the river because moving away from the source entails the costly reinforcement of the dam, which is caused by a large influx of water. If the dam is located closer than 10-15 km from the source, due to the small water flow and insufficient head, the hydroelectric power station will not be able to provide the necessary power. The chosen stretch of the river should not be abundant in great depths, which also increases the cost of construction, requiring a heavy foundation."

The Flow Rate

Between village and mountain grove,
Winds a river like a bright ribbon.

A. Fet

How much water flows in such a river in a day? It's easy to calculate if you first measure the speed of the water flow in the river. The measurement is performed by two people. One person holds a watch, the other holds some noticeable float, for example, a half-empty bottle with a flag. They choose a straight section of the river and place two stakes A and B along the bank at a distance, for example, 10 m from each other (see Figure 41).

Two more stakes C and D are placed on lines perpendicular to AB. One of the participants in the measurement with the

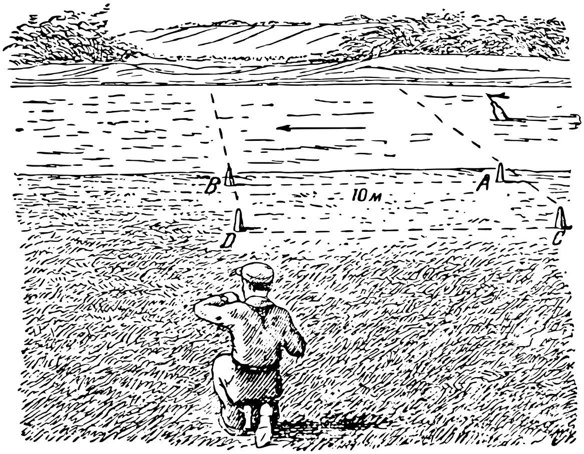


Figure 41: Measurement of the river flow velocity.

watch stands behind stake *D*. The other, with the float, goes a bit upstream of stake *A*, throws the float into the water, and then stands behind stake *C*. Both observers look along the directions *CA* and *DB* towards the water surface. At the moment when the float crosses the extension of the line *CA*, the first observer waves his hand. Upon this signal, the second observer starts the timer for the first time and then again when the float crosses the direction of *DB*.

Let's assume that the time difference is 20 seconds.

Then the speed of the water flow in the river is:

$$\frac{10}{20} = 0.5\,\mathrm{m/s}.$$

Usually, the measurement is repeated about ten times[12], throwing the float into different points on the river surface. Then the obtained numbers are summed up and divided by the number of measurements. This gives the average speed of the surface layer of the river.

[12] Instead of throwing one float ten times, you can immediately throw 10 floats at some distance from each other.

Deeper layers flow slower, and the average speed of the entire flow is approximately 4/5 times the surface speed. In our case, therefore, it's 0.4 m/s.

You can determine the surface speed by another – albeit less reliable – method.

Sit in a boat and paddle 1 km (measured along the shore) against the current, and then back – with the current, trying to paddle with the same force all the time.

Let's say you paddled these 1,000 m against the current in 18 minutes, and with the current in 6 minutes. Denoting the desired speed of the river current as x, and the speed of your movement in still water as y, you form the equations:

$$\frac{1000}{y - x} = 18, \quad \text{and} \quad \frac{1000}{y + x} = 6.$$

Rearranging we get:

$$y + x = \frac{1000}{6}, \quad \text{and} \quad y - x = \frac{1000}{18}.$$

Solving for x, we get $2x = 110$, and $x = 55$. The speed of the water flow on the surface is 55 m per minute, and therefore, the average speed is about $5/6$ m/s.

How Much Water Flows in The River?

To measure the amount of water flowing in a river, you can always determine the speed at which the water flows. The more challenging part of the preparatory work needed to calculate the quantity of flowing water is to determine the cross-sectional area of the water. To find the magnitude of this area, known as the "live cross-section" of the river, you need to make a drawing of this section. Such work is done as follows.

First Method: At the point where you measured the width of the river, you drive a stake into the ground on both banks, right at the water's edge. Then, with a companion, you get into a boat and row from one stake to the other, trying to keep a straight line connecting the stakes. An inexperienced rower will not be able to handle such a task, especially in a river with a fast current. Your companion must be a skilled rower; besides, a third participant in the work should stand on the bank, ensuring that the boat stays on the correct course and giving the rower signals indicating which way to turn when necessary. During the first crossing of the river, you only need to count how many strokes of the oars it took and from there figure out how many strokes move the boat 5 or 10 meters. Then, for the second crossing, armed with a sufficiently long rake with markings on it, you plunge the rake vertically to the bottom every 5-10 meters (measured by the number of oar strokes) and record the depth of the river at that point.

This method can only measure the live cross-section of a small river; for a wide, multi-water river, more complex methods are needed, which are performed by specialists. An amateur must choose a task that suits their modest measuring means.

Second Method: On a narrow and shallow river, you don't need a boat.

Between the stakes, you stretch a cord perpendicular to the current with marks or knots made on it every 1 or 2 meters, and by lowering a ruler to the bottom at each knot, you measure the depth of the riverbed. When all measurements are done, you first draw a millimeter paper or a grid paper sketch of the cross-section profile of the river. You will get a figure similar to the one shown in Figure 42. It is quite easy to determine the area of this figure since it can be divided into a series of trapezoids (where both bases and the height are known) and two side triangles, also with known base and height. If the scale of the drawing is 1:100, then the result will be obtained directly in square meters.

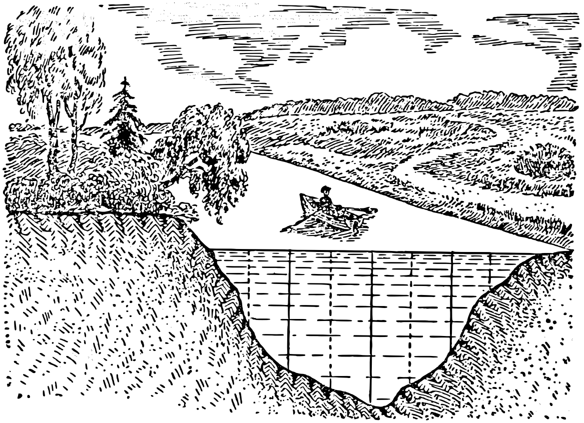


Figure 42: “Live Cross-Section” of a River.

Now you have all the data needed to calculate the amount of flowing water. Obviously, through the live cross-section of the river, a volume of water equal to the volume of a prism is passing every second, where the cross-section serves as the base and the average second-by-second flow rate serves as the height. For example, if the average flow rate of water in the river is 0.4 meters per second, and the area of the live cross-section is, let's say, 3.5 square meters, then every second, through this section, there will be a transfer of

$$3.5 \times 0.4 = 1.4 \text{ cubic meters of water,}$$

or the same amount in tons.[13] This amounts to

$$1.4 \times 3600 = 5040 \text{ cubic meters of water per hour, and}$$
$$5040 \times 24 = 120{,}960 \text{ cubic meters of water per day,}$$

which is over a hundred thousand cubic meters. And yet, a river with a live cross-section of 3.5 square meters is a small river; it could be, for example, 3.5 meters wide and 1 meter deep at a fordable point. But even it contains energy capable of transforming into mighty electricity. So how much water flows per day in a river like the Neva, through which 3300 cubic meters of water pass every second through its live cross-section! This is the "average flow rate" of water in the Neva at Leningrad. The "average flow rate" of water in the Dnieper at Kiev is 700 cubic meters.

Young explorers and future dam builders also need to determine the maximum head the banks can allow, i.e., the difference in water levels that the dam can create (Figure 43). To do this, stakes are driven into the banks 5-10 meters away from the water, as usual, along a line perpendicular to the

[13] 1 cubic meter of fresh water weighs 1000 kg.

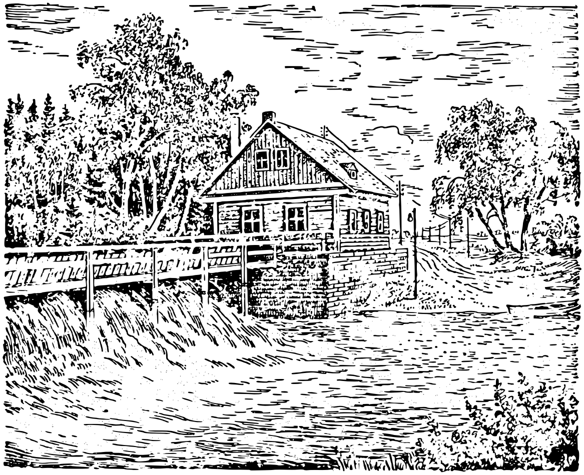


Figure 43: The hydroelectric power station with a capacity of 80 kilowatts belongs to the Burmakin Agricultural Collective Farm and provides energy to seven collective farms.

river's current. Then, moving along this line, small stakes are placed at points of characteristic bends in the banks (Figure 44). Using rulers with markings, the elevation of one stake above the other and the distances between them are measured. Based on the measurement results, a profile of the banks is drawn similar to the profile of the riverbed. The bank profile can indicate the magnitude of the allowable head.

Suppose the water level can be raised by the dam by 2.5

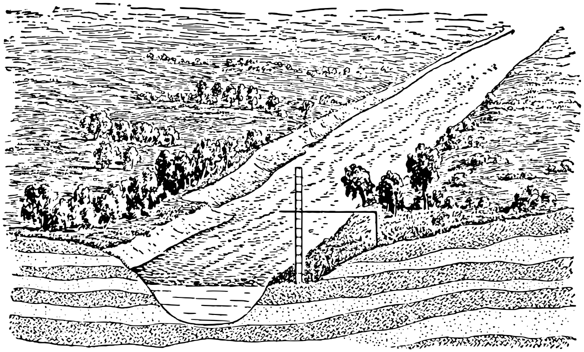


Figure 44: Coast profile measurement.

meters. In this case, you can estimate the potential power of your future hydroelectric power plant.

For this, energy experts recommend multiplying 1.4 (the second-by-second flow rate of the river) by 2.5 (the water level height) and by 6 (the coefficient, which varies depending on energy losses in machines). The result will be in kilowatts. Thus,

$$1.4 \times 2.5 \times 6 = 21 \text{ kilowatts.}$$

Since the river levels, and consequently the flow rates, vary throughout the year, it is necessary to calculate the value of the flow rate that is characteristic of the river for most of the year.

Water Wheel

Question A wheel with blades is installed near the bottom of the river so that it can rotate easily. In which direction will it rotate if the current is flowing from right to left (see Figure 45)?

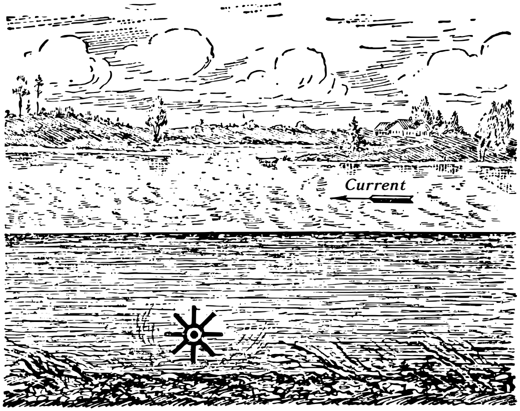


Figure 45: In which direction will the wheel rotate?

Answer The wheel will rotate counterclockwise. The velocity of the deeper layers of water is lower than the velocity of the upper layers, therefore, the pressure on the upper blades will be greater than on the lower ones.

Rainbow Film

On the river, into which water flows from the factory, you can often notice beautiful colourful iridescence near the outlet. Oil (for example, motor oil) flowing into the river along with the water from the plant remains on the surface as it is lighter and spreads out in an extremely thin layer. Can the thickness of such a film be measured or at least estimated?

The task seems intricate, but solving it is not particularly difficult. You already guess that we will not bother with the hopeless task of directly measuring the thickness of the film. We will measure it indirectly, in other words, calculate it.

Take a certain amount of motor oil, for example, 20 litres, and pour it onto the water, far from the shore (from a boat). When the oil spreads over the water in the form of a more or less clearly defined circular spot, measure the diameter of this circle at least approximately. Knowing the diameter, calculate the area. And since you also know the volume of the oil taken (it is easy to calculate by weight), then the thickness of the film will naturally be determined from here. Let's consider an example.

Question One gram of kerosene, spreading over water, covers a circle with a diameter of 30 cm.[14] What is the thickness of the kerosene film on the water? One cubic centimetre of

[14] The standard oil consumption for covering water bodies to destroy malaria mosquito larvae is 400 kilogrammes per hectare.

kerosene weighs 0.8 grams.

Answer Let's find the volume of the film, which is certainly equal to the volume of the kerosene taken. If one cubic centimetre of kerosene weighs 0.8 grams, then for 1 gram we have

$$\frac{1}{0.8} = 1.25\,\text{cm}^3 \quad \text{or} \quad 1{,}250\,\text{mm}^3.$$

The area of the circle with a diameter of 30 cm, or 300 mm, is 70,000 mm^2. The desired thickness of the film is equal to the volume divided by the area of the base:

$$\frac{1250}{70000} = 0.018\,\text{mm}.$$

In other words, less than 0.02 mm. Direct measurement of such thickness using conventional means is, of course, impossible. Oil and soap films spread even thinner, reaching 0.009,1 mm or less. "Once," recounts the English physicist Boys in the book *Soap Bubbles*, "I conducted such an experiment on a pond. A spoonful of olive oil was poured onto the water surface. Immediately a large spot was formed, about 20-30 meters in diameter. Since the spot was a thousand times longer and a thousand times wider than the spoon, the thickness of the oil layer on the water surface should have been approximately one millionth of the thickness of the oil layer in the spoon, or about 0.000002 millimetres."

Circles on the Water

Question You've surely observed with curiosity the circles created by a stone thrown into calm water (see Figure 46). Explaining this instructive natural phenomenon has probably never been difficult for you: disturbance spreads from the point of origin in all directions at the same speed, so at any moment, all the points of disturbance must be equidistant from the source, forming circles.

Figure 46: Circles on the water.

But what about in flowing water? Should the waves from a stone thrown into a fast-flowing river also form circles, or will their shape be elongated?

At first glance, it might seem that in flowing water, circular waves should elongate in the direction of the current: the disturbance is transmitted downstream faster than upstream and sideways. Therefore, the disturbed parts of the water surface should, it seems, arrange themselves along some elongated closed curve, at least not in a circle.

However, in reality, this is not the case. By throwing stones into the swiftest river currents, you can see that the waves formed are strictly circular – exactly the same as in still water. Why is this?

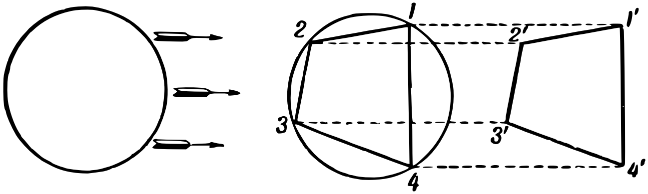


Figure 47: The flow of water does not change the shapes of the waves.

Answer Let's reason as follows. If the water were not flowing, the waves would be circular. What change does the flow introduce? It carries each point of this circular wave in the direction indicated by the arrows (see Figure 47, left), and all points are moved along parallel lines at the same speed, i.e., the same distance. And "parallel displacement" (translation) does not change the shape of the figure. Indeed, as a result of such displacement, point *1* (Figure 47, right) will be at point *1′*, point *2* will be at point *2′*, and so on; quadrilateral *1234* will be replaced by quadrilateral *1′2′3′4′*, which is equal to it, as can be easily seen from the formed parallelograms *122′1′*, *233′2′*, *344′3′*, and so on. By taking not just four but more points on the circumference, we would also get equal polygons; finally, by taking an infinite number of points, i.e., a circle, we would get an equal circle after parallel displacement.

That's why the downstream movement of water does not change the shapes of the waves – they remain circular even in flowing water. The difference is only that on the surface of a lake, the circles do not move (if we don't consider that they diverge from their stationary centre), while on the surface of a river, the circles move together with their centre at the speed of the water flow.

Fantastic Shrapnel

Question Let's tackle a problem that seems unrelated at first but, as we will see, is closely related to the topic at hand.

Imagine a shrapnel projectile flying high in the air. It begins to descend and suddenly explodes; the fragments scatter in different directions. Let's assume that all of them are thrown by the explosion with the same force and travel without encountering any obstacles from the air. The question is: how will the fragments be arranged one second after the explosion if during this time they have not yet reached the ground?

Answer The problem resembles the problem of circles on water. And here it seems that the fragments should be arranged in a shape elongated downwards, in the direction of descent; after all, the fragments thrown upwards fly slower than those thrown downwards. However, it is easy to prove that the fragments of our imaginary shrapnel should be arranged on the surface of a sphere. Let's momentarily imagine that there is no gravity; then, of course, all fragments will fly from the explosion site to the same distance within a second, i.e., they will be arranged on the surface of a sphere. Now let's introduce the force of gravity. Under its influence, the fragments should descend; but since all bodies, as we know, fall at the same speed, the fragments should descend to the same distance within a second, along parallel lines.[15] But such parallel displacement does not change the shape of the figure—the sphere remains a sphere.

[15] The differences are due to the air resistance, which we have excluded in our task.

So, the fragments of the fantastic shrapnel should form a sphere, which, as if inflating, descends downward at the speed of a freely falling body.

The Keel Wave

Let's return to the river. (Standing on a bridge, pay attention to the wake left by a fast-moving boat. You will see how two water crests diverge at an angle from the bow (see Figure 48). Where do they come from? And why is the angle between them sharper the faster the boat moves?

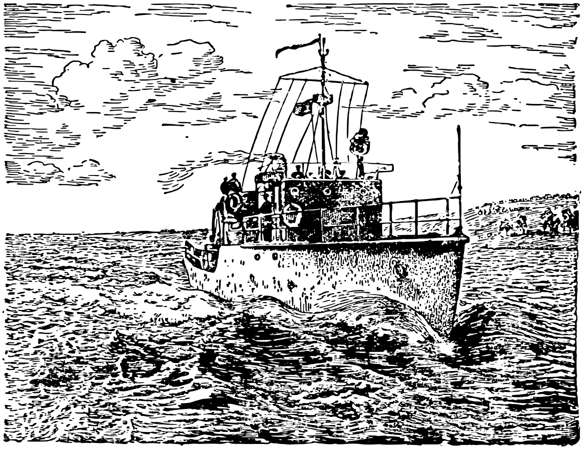


Figure 48: The keel wave.

To understand the reason for the emergence of these crests, let's once again turn to the diverging circles formed on the water surface by stones thrown into it. By throwing stones into the water at certain intervals, you can see circles of different sizes on the water surface; the later a stone is thrown, the smaller the circle it creates.

If you throw stones along a straight line, the circles formed collectively generate a wave similar to the one at the bow of a ship. The smaller and more frequent the stones are thrown, the more noticeable the similarity becomes. By immersing a stick in the water and moving it along the water surface, you effectively replace the intermittent falling of stones with continuous motion, and then you see exactly the wave that forms at the bow of a ship. To make this vivid picture even clearer, let's add a bit more.

Each moment, the bow of the ship, plunging into the water, generates the same circular wave as a thrown stone. The circle expands in all directions, but meanwhile, the ship moves forward and creates a second circular wave, followed immediately by a third, and so on. The intermittent formation of circles caused by stones is perceived as continuous due to their continuous occurrence, resulting in the pattern shown in Figure 49.

Meeting each other, the crests of neighbouring waves break each other: only two small segments of the complete circle remain intact, which are located on their outer parts. These outer segments merge to form two continuous crests, positioned as external tangents to all circular waves (Figure 49, right).

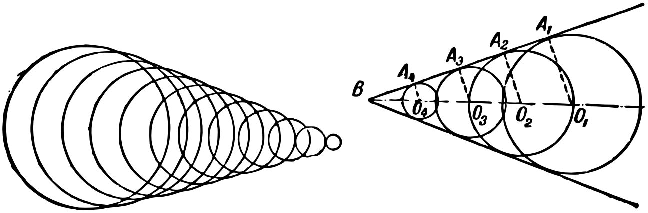


Figure 49: How the keel wave is formed.

This is how the origin of the water crests visible behind the boat, behind any body moving rapidly along the water surface, occurs.

It follows directly from this that this phenomenon is possible only when the body moves faster than the water waves. If you slowly move a stick through the water, you will not see any crests: the circular waves will be arranged one inside the other, and it will be impossible to draw a common tangent to them.

Diverging crests can also be observed when the body stands still and the water flows past it. If the current of the river is fast enough, such wakes are formed in the water flowing around bridge piers. The shape of the waves is even more

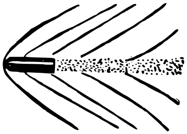


Figure 50: A head wave (shock wave) in the air formed by a flying projectile.

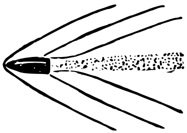


Figure 51: Another example of head wave (shock wave) in the air formed by a flying projectile.

distinct here than, for example, from a steamship, as their regularity is not disrupted by the action of a propeller.

Having clarified the geometric aspect of the matter, let's try to solve such a problem.

Question What determines the magnitude of the angle between both branches of the keel wave of the steamer?

Answer The angle between the branches of the keel wave of a steamship depends on several factors, primarily the speed of the ship relative to the speed of the wave propagation in the water.

Let's draw radii from the centre of the circular waves (Figure 49, right) to the corresponding points of the straight wave crest, i.e., to the points of common tangency. It's easy to understand that O_1B represents the distance travelled by the ship's bow in some time, and O_1A_1 – represents the distance over which the wave propagates in the same time period. The ratio O_1B/O_1A_1 is the sine of angle O_1BA_1, which, in turn, is the ratio of the wave propagation velocity to the ship's velocity. Therefore, the angle B between the crests of the keel wave is nothing else but twice the angle whose sine is the ratio of the wave propagation velocity to the ship's velocity.

The speed of wave propagation in water is approximately the same for all vessels. Therefore, the angle between the branches of the keel wave primarily depends on the speed of the ship. In general, the sine of half the angle is proportional to this speed. Conversely, the angle's magnitude indicates how many times the ship's speed exceeds the speed of the waves. For instance, if the angle between the branches of the keel wave is 30°, as is common for most cargo and passenger ships, then the sine of half this angle (approximately 0.26) suggests that the ship's speed exceeds the wave speed by roughly four times.

Speed of Projectiles

Question Waves, similar to those considered now, are generated in the air by a flying bullet or artillery shell.

There are methods to photograph a projectile in flight; Figure 50 and Figure 51 reproduce two such images of projectiles moving at different speeds. In both drawings, the "head wave"[16] that interests us is clearly visible (as it is called in this case). Its origin is the same as that of the bow wave of a ship. And here the same geometric relationships apply, namely: the sine of half the angle of divergence of the head waves is equal to the ratio of the speed of wave propagation in the air to the speed of the projectile itself. But wave propagation in the air occurs at a speed close to the speed of sound,

[16] The term *shock wave* is also commonly used. – DM

i.e., 330 m/s. Therefore, it is easy, having a photograph of a flying projectile, to approximately determine its speed. How to do this for the two images provided here?

Answer Let's measure the angle of divergence of the head wave branches in Figure 50 and Figure 51. In the first case, it is about 80°, and in the second case – approximately 55°. Half of them are 40° and 27.5°. For 40°, $\sin(40°) = 0.64$, and for 27.5°, $\sin(27.5°) = 0.46$. Therefore, the speed of wave propagation in the air, i.e., 330 m/s, is 0.64 of the projectile's flight speed in the first case and 0.46 in the second. Hence, $330/0.64 = 520$ m/s for the speed of the first projectile, and $330/0.46 = 720$ m/s for the speed of the second projectile.

You see that quite simple geometric considerations, with some support from physics, helped us solve a problem that, at first glance, seemed very intricate; to determine the speed of a projectile at the moment of photography. (However, this calculation is only approximately correct, since some secondary circumstances are not taken into account here.)

For those who want to independently perform such a speed calculation for projectiles, here are three reproductions of photographs of shells flying at different speeds (Figure 52).

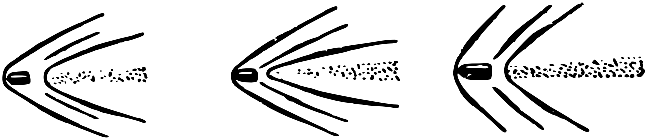


Figure 52: How to determine the speed of flying projectiles?

Finding Pond Depth

The ripples on the water distracted us for a while in the field of artillery. Let's go back to the river again and consider the Hindu lotus problem,

The ancient Hindus had a custom of offering tasks and rules in verse, Here is one of such tasks:

Question

Above the tranquil lake,
With a half-foot size, the lotus flower rose.
It grew alone. And the wind, in a gust,
Bent it sideways.
No more flower above the wave,
But the fisherman found it, in early spring,
Two feet from where it grew.
So, I propose a question:
How deep is the water of the lake?

Translated by *V. I. Lebedev*

Answer Let's denote the depth of the pond CD as x Figure 53. Then, according to the Pythagorean theorem, we

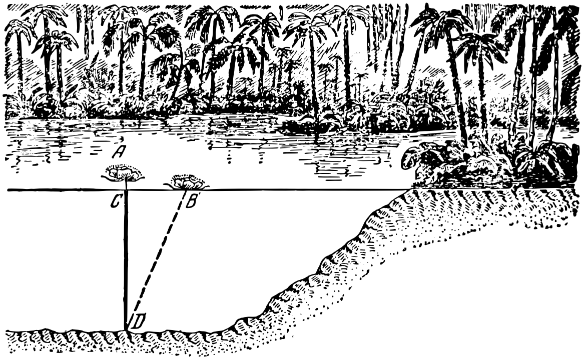


Figure 53: The Hindu problem of the lotus flower.

have:

$$BD^2 - x^2 = BC^2.$$

Thus,

$$x^2 = \left(x + \frac{1}{2}\right)^2 - 2^2.$$

From this, we get:

$$x^2 = x^2 + x + \frac{1}{4} - 4, \quad x = 3\,\frac{3}{4}.$$

The depth is approximately 3 3/4 feet.

Near the riverbank or a shallow pond, you can find a water plant that will provide you with real material for a similar problem: without any tools, without even getting your hands wet, you can determine the depth of the water at that spot.

Starry Sky in the River

The river also offers the geometer problems at night. Remember Gogol's description of the Dnieper: "The stars burn and shine over the world and all are reflected in the Dnieper at once. The Dnieper holds them all in its dark womb: not one can escape from it, unless it goes out in the sky." Indeed, when standing on the bank of a wide river, it seems that the entire starry dome is reflected in the water mirror. But is it really so? Do all the stars "surrender" to the river?

Let's make a drawing (Figure 54): A is the eye of the observer standing on the riverbank, at the edge of the cliff, MN is the water surface. What stars can the observer see in the water from point AD? To answer this question, let's drop a perpendicular AD from point A to the line MN and extend it to an equal distance to point A'.

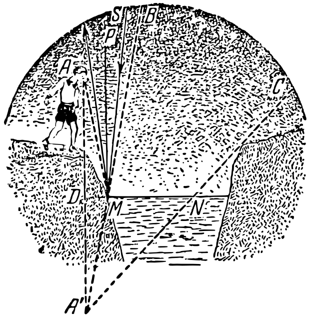


Figure 54: Which part of the starry sky can be seen in the water channel of the river.

If the observer's eye were at A', he could only see the part of the starry sky that fits inside angle $BA'C$. The actual observer's field of view from point A is the same. Stars outside this angle are not visible to the observer; their reflected rays pass by his eyes.

How to make sure of this? How to prove that, for example, star S lying outside angle $BA'C$ is not visible to our observer in the water mirror of the river? Let's follow its ray, falling close to the shore, to point M; it will reflect according to the laws of physics at an angle to perpendicular MP, which is equal to the angle of incidence SMP and therefore less than angle PMA (this is easy to prove based on the equality of triangles ADM and $A'DM$); thus, the reflected ray must pass by A. Therefore, the rays of star S reflected in points beyond point M will also pass by the observer's eye.

Thus, Gogol's description contains exaggeration: not all stars are reflected in the Dnieper, or, at least, less than half of the starry sky. Even more interesting is that the extent of the reflected part of the sky does not prove that you are facing a wide river. In a narrow river with low banks, you can see almost half of the sky (which seems like a wide river) if you lean close to the water. You can easily verify this by constructing a field of view for such a case (Figure 55).

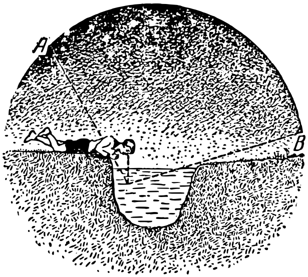


Figure 55: In a narrow river with low banks, you can see more stars.

In a narrow river with low banks, you can see more stars.

Path Across the River

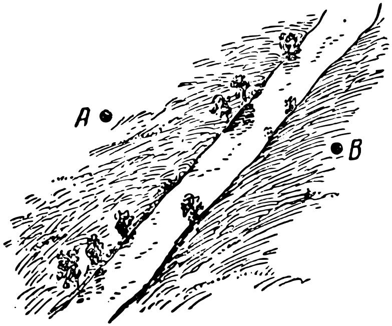


Figure 56: Where can we build a bridge at right angles to the river banks so that the road from A to B is the shortest?

Question Between points *A* and *B*, a river (or canal) flows with approximately parallel banks (Figure 56). It is necessary to construct a bridge across the river at right angles to its banks. Where should the bridge be placed to make the path from *A* to *B* the shortest?

Answer Drawing a straight line through point *A* (Figure 57), perpendicular to the direction of the river, and marking off segment *AC* equal to the width of the river from *A*, we connect *C* to *B*. The bridge should be constructed at point *D* to make the path from *A* to *B* the shortest.

Indeed, by constructing bridge *DE* (Figure 58) and connecting *E* to *A*, we obtain the path *AEDB*, in which segment *AE* is

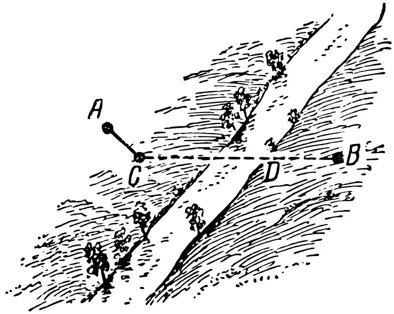


Figure 57: Choosing he location for building the bridge.

parallel to *CD* (*AEDC* is a parallelogram, as its opposite sides *AC* and *ED* are equal and parallel). Therefore, the length of path *AEDB* equals the length of path *ACB*. It is easy to show that any other path is longer than this one.

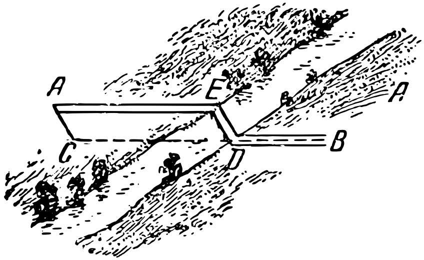


Figure 58: The bridge is built.

Suppose we suspect that some path *AMNB* (Figure 59) is shorter than *AEDB*, i.e., shorter than *ACB*. Connecting *C* to *N*, we see that *CN* equals *AM*. Thus, path *AMNB* = *ACNB*. But *CNB* is obviously greater than *CB*; therefore, *ACNB* is greater than *ACB*, and consequently, greater than *AEDB*.

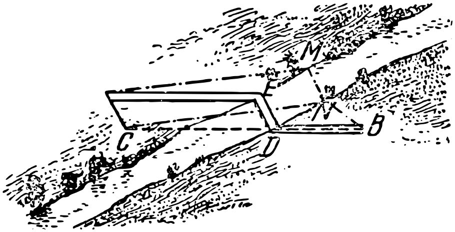


Figure 59: The *AEDB* path is indeed the shortest.

Thus, the path *AMNB* turns out to be not shorter but longer than the path *AEDB*. This reasoning applies to any position of the bridge that does not coincide with *ED*; in other words, the path *AEDB* is indeed the shortest.

To Construct Two Bridges

Question A more complex case may arise when it is necessary to find the shortest path from *A* to *B* across the river, which needs to be crossed twice at right angles to the banks (see Figure 60). In what places should bridges be built across the rivers?

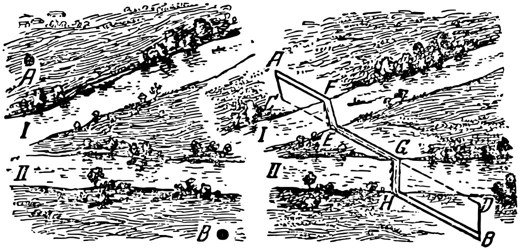
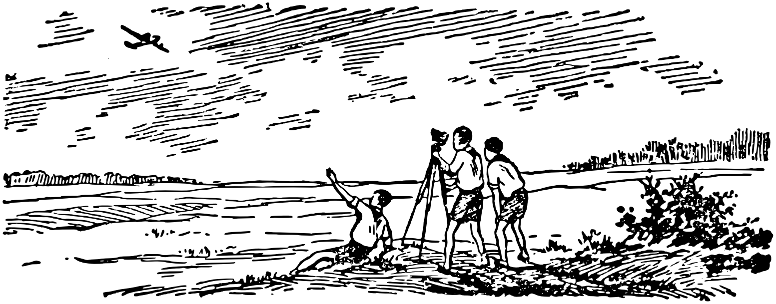


Figure 60: Two bridges are built.

Answer It is necessary to draw a line segment *AC* from point *A* (to the right in the Figure 60), equal to the width of the river at point *C*, and perpendicular to its banks. From point *B*, draw a line segment *BD* equal to the width of the river at point *D*, also perpendicular to the banks. Connect points *C* and *D* with a straight line. Build bridge *EF* at point *E* and bridge *GH* at point *G*. The path *AFEGHB* is the sought shortest path from *A* to *B*.

The reader will, of course, understand this if reasoning in this case is conducted in the same way as we reasoned in the previous problem.



3. Geometry In The Open Field

Visible Sizes of the Moon

What size does the full moon in the sky seem to you? Different people give quite different answers to this question.

Estimates like "the size of a plate," "the size of an apple," "the size of a human face," and so on, are extremely vague, indefinite, indicating only that those answering do not understand the essence of the question.

The correct answer to such an apparently ordinary question can only be given by someone who clearly understands what exactly needs to be understood by the "apparent" or "visible" size of an object. Few suspect that here we are referring to the magnitude of a certain angle – precisely the angle formed by two straight lines drawn to our eye from the extreme points of the object under consideration; this angle is called the "angle of view" or "angular size of the object" (Figure 61).

Figure 61: What is the angle of view or angular size of the object.

And when the apparent size of the Moon in the sky is assessed by comparing it with the sizes of a plate, an apple, and so forth, such answers are either completely meaningless or should imply that the Moon is visible in the sky at the same angle as the plate or apple. But such an indication alone is still insufficient: after all, we see a plate or an apple under different angles depending on their distance: up close – under larger angles, far away – under smaller ones. To introduce clarity, it is necessary to specify from what distance the plate or apple is being observed. Comparing the sizes of distant objects with the sizes of others, the distance of which is not indicated, is a very common literary technique employed by even first-rate writers. It creates a certain impression due to its closeness to the familiar psychology of most people, but it does not produce a clear image… Here's an example from Shakespeare's *King Lear*; it describes (by Edgar) the view from a high cliff by the sea:

> *How terrifying! How my head spins!*
> *How low to cast my gaze…*
> *The rooks and crows that flutter there in the air at mid-distance,*
> *Seem barely as large as flies. Halfway down Hangs a man gathering seaweed…*
> *What a dreadful trade! He seems to me no bigger than his head.*
> *The fishermen walking along the shore – like mice;*
> *and that tall ship at anchor has shrunk to the size of its boat;*
> *its boat – a floating dot, As if too small for sight…*

These comparisons would provide a clear idea of distance if accompanied by indications of the degree of remoteness of the objects being compared (mice, the human head, crows, boat ...). Similarly, when comparing the size of the moon with that of a plate or apples, indications of how far these everyday objects should be from the eye are necessary.

This distance turns out to be much greater than commonly thought. Holding an apple at arm's length, you not only obscure the Moon but also a significant portion of the sky. Suspend the apple on a string and gradually move away from it until it just covers the full lunar disk: in this position, the apple and the Moon will have the same apparent size for you. By measuring the distance from your eye to the apple, you will find that it is approximately 10 meters. This is how far you would need to move the apple away from you for it to truly seem to be the same size as the Moon in the sky! A plate, on the other hand, would need to be moved about 80 meters away from you, or about fifty steps.

What is said may seem unbelievable to anyone hearing it for the first time; however, it is indisputable and follows from the fact that we perceive the Moon at an angle of only *half a degree*. We rarely have to estimate angles in our everyday lives, and therefore most people have a very vague idea of the magnitude of an angle with a small number of degrees, such as an angle of 1°, 29°, or 59° (not to mention surveyors, draftsmen, and other specialists accustomed to practically measuring angles). We only estimate large angles more or less reasonably, especially if we manage to compare them with angles familiar to us between the hands of a clock; everyone, of course, is familiar with angles of 90°, 60°, 30°, 120°, 150°, which we are so accustomed to seeing on a dial (at 8 o'clock, 2 o'clock, 1 o'clock, 4 o'clock, 5 o'clock) that even without distinguishing the numbers, we guess the time based on the size of the angle between the hands. But we usually see small and individual objects at much smaller angles and therefore completely lack the ability to even approximately estimate angles of view.

Angle of View

To provide a concrete example of a one-degree angle, let's calculate how far an average-height person (1.7 meters) should move away from us to appear at such an angle. Translating the problem into the language of geometry, let's say we need to calculate the radius of a circle, the arc of which at 1° has a length of 1.7 meters (strictly speaking, not an arc, but a chord, but for small angles, the difference between the lengths of the arc and the chord is negligible). We reason as follows: if the arc at 1° equals 1.7 meters, then the full circumference containing 360° will have a length of $1.7 \times 360 = 610$ m, and the radius will be $1/2\pi$ the length of the circumference; if we take the value of π as approximately $^{22}/_{7}$, then the radius will be equal to

$$\frac{610}{44/7} \approx 98\,\mathrm{m}.$$

So, a person appears at an angle of 1° if they are approx-

Figure 62: The human figure is visible from a distance of about hundred meters at an angle of 1°.

imately at a distance of 100 meters from us (Figure 62). If they move twice as far away – to 200 meters – they will be seen at an angle of half a degree; if they approach to a distance of 50 meters, the angle of view will increase to 2°, and so on. It is also easy to calculate that a stick of 1 meter in length should appear to us at an angle of 1° at a distance of $360/(44/7) = 57\,\mathrm{m}$.

At the same angle, we perceive an object of 1 cm from a distance of 57 cm, 1 km from a distance of 57 km, and so on – in general, any object from a distance 57 times greater than its diameter. If we remember this number – 57, we can quickly and easily perform all calculations related to the angular size of an object. For example, if we want to determine how far we need to move an apple with a diameter of 9 cm to see it at an angle of 1°, it is sufficient to multiply 9 by 57 – we get 513 cm, or about 5 meters; from twice the distance, it is perceived at half the angle – half a degree, i.e., it appears the size of the Moon.

In the same way, for any object, we can calculate the distance at which it appears to be the same size as the lunar disk.

Plate and Moon

Question At what distance should a plate with a diameter of 25 cm be moved away to appear the same size as the Moon in the sky?

Answer $25\ \text{cm} \times 57 \times 2 = 2{,}850\ \text{cm} = 28\ \text{m}$.

Moon and Copper Coins

Question Perform the same calculation for a five-kopeck coin (diameter 25 mm) and a three-kopeck coin (22 mm).

Answer For the five-kopeck coin:

$$0.025 \text{ m} \times 57 \times 2 = 2.85 \text{ meters.}$$

For the three-kopeck coin:

$$0.022 \text{ m} \times 57 \times 2 = 2.514 \text{ meters.}$$

If it seems unbelievable to you that the Moon appears no larger than a two-kopeck coin from a distance of four steps or an ordinary pencil from a distance of 80 cm, – hold the pencil at arm's length against the full Moon disk: it will easily cover it. Strangely enough, the most suitable comparison object for the Moon in terms of perceived size is not a plate, not an apple, not even a cherry, but a pea or, even better, a match head! Comparing it to a plate or an apple implies moving them to an unusually large distance; we see an apple in our hands or a plate on the dining table ten to twenty times larger than the lunar disk. And only a match head, which we examine at a distance of 25 cm from the eye ('distance of distinct vision'), is seen at an angle of half a degree, i.e., the same size as the Moon.

The fact that the lunar disk deceptively appears to grow in the eyes of most people by 10 to 20 times is one of the most curious optical illusions. It depends, one might think, mostly on the *brightness* of the Moon: the full moon stands out against the sky much more sharply than plates, apples, coins, and other comparison objects do amidst the surrounding environment.[17]

[17] For the same reason, the incandescent filament of an electric light bulb seems to us much thicker than in a cold, non-luminous state.

This illusion is imposed on us with such irresistible force that even artists, distinguished by a keen eye, succumb to it alongside others and depict the full moon in their paintings much larger than it should be. It is enough to compare the landscape painted by an artist with a photograph to be convinced of this.

The same applies to the Sun, which we see from Earth at the same angle of half a degree; although the true diameter of the solar sphere is 400 times larger than that of the lunar one, its distance from us is also 400 times greater.

Sensational Photographs

To explain the important concept of the angle of view, let's deviate a bit from our main topic – geometry in open fields – and provide a few examples from the realm of photography.

On the movie screen, you've surely seen such catastrophes as train collisions or such incredible scenes as a car driving on water.

Recall the movie *Captain Grant's Children.* What a strong impression – isn't it? – the scenes of the shipwreck during the storm or the sight of crocodiles surrounding the boy stuck in the swamp left on you. Of course, no one thinks that such photographs were taken directly from real life. But how were they obtained?

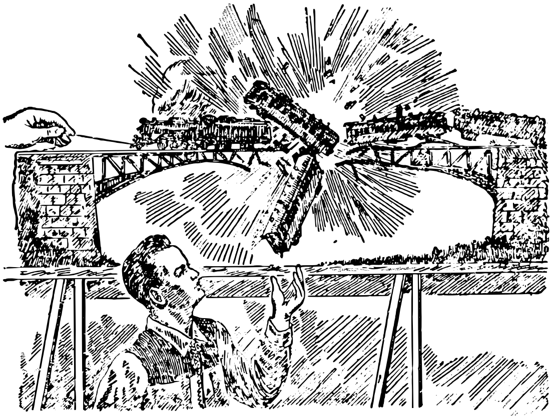


Figure 63: Preparing a train accident for filming.

The secret is revealed by the illustrations attached here. In

Figure 63, you see the 'catastrophe' of a toy train in a toy setting; in Figure 64 – a toy car being pulled on a string behind an aquarium. This is the 'nature' from which the film was shot. But why do we succumb to the illusion when we see these images on the screen, as if we were looking at real trains and cars? After all, here, in the illustrations, we would immediately notice their miniature size, even if we couldn't compare them with the size of other objects. The reason is simple: toy trains and cars are filmed for the screen from a very close distance; therefore, they appear to the viewer at approximately the same angle of view as we usually see real trains and cars. That's the whole secret of the illusion.

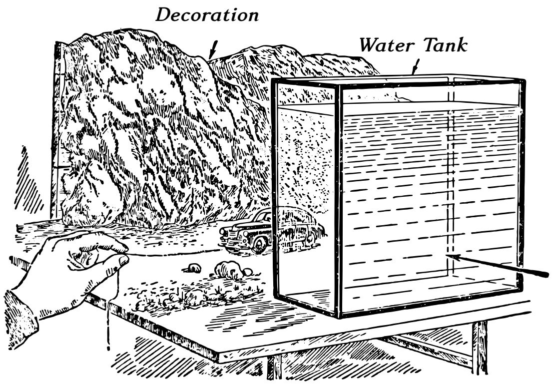


Figure 64: The underwater road trip.

Or here's another frame from the movie *Ruslan and Ludmila* (Figure 65). A huge head and a small Ruslan on a horse. The head is placed on a model field close to the camera. And Ruslan on the horse – at a considerable distance. That's the entire secret of the illusion.

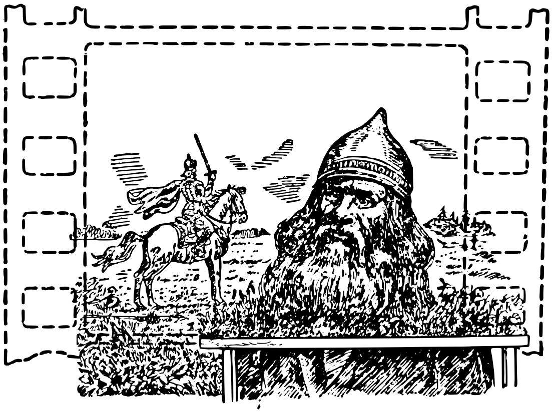


Figure 65: A shot from the movie *Ruslan and Lyudmila.*

Reservoir Set Decoration

Figure 66 is another example of an illusion based on the same principle. You see a strange landscape reminiscent of the nature of ancient geological epochs: bizarre trees resembling giant mosses, on them – huge water drops, and in the foreground – a gigantic monster resembling harmless frogs. Despite such an unusual view, the drawing is made with subtlety: it's nothing but a small patch of soil in the forest, only drawn from an unusual angle of view. We never see moss stems, water drops, frogs, etc., at such a large angle of view, and therefore the drawing seems so alien, unfamiliar to us. Before us is a landscape as we would see it if we were shrunk to the size of an ant.

Figure 66: Mysterious landscape depicted from nature.

Swindlers from bourgeois newspapers act in the same way to create fake reportage photographs. One foreign newspaper once published a note criticising the city administration for allowing huge snow mountains to form on the city streets. To support this, an impressive photo of one such mountain was provided (Figure 67, left). Upon examination, it turned out that the nature for the photograph was a small snowdrift, taken by the 'joker' photographer from a very close distance, i.e., at an unusually large angle of view (Figure 67, right).

Another time, the same newspaper reproduced a photo of a wide crevice in the rock near the city; it served, according to the newspaper, as the entrance to an extensive underground cave, where a group of careless tourists who dared to enter the cave for exploration disappeared without a trace. A volunteer search party equipped to search for the lost discovered that the crevice was photographed… from a barely

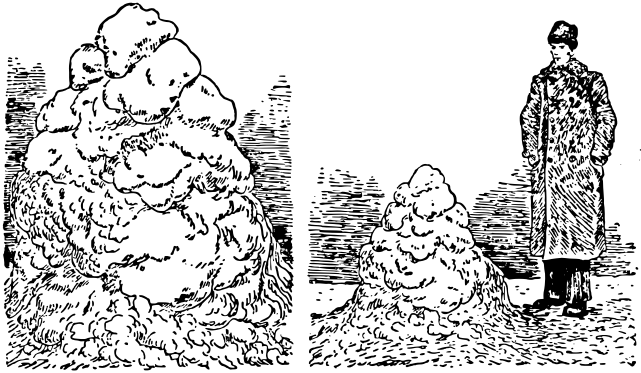


Figure 67: Snow mountain in a photograph (left) and in nature (right).

noticeable crack in the icy wall, a centimetre wide!

Living Protractor

Making a simple protractor device yourself is not very difficult, especially if you use a protractor. But sometimes even a homemade protractor may not be at hand during a countryside walk. In such cases, you can rely on the services of a "living protractor" that is always with us. These are our own fingers. To use them for a rough estimate of viewing angles, you just need to make a few preliminary measurements and calculations.

First of all, you need to determine at what angle we see the fingernail of our outstretched index finger. The usual width of a nail is 1 cm, and its distance from the eye in such a position is about 60 cm; therefore, we see it at an angle of about 1° (slightly less because an angle of 1° would be at a distance of 57 cm). For teenagers, the nail is smaller, but the arm is shorter, so the viewing angle for them is approximately the same – 15°. The reader would do well to perform this measurement and calculation for themselves, relying on book data, to make sure the result is not too far from 15°; if the deviation is significant, you should try another finger.

Knowing this, you have a way to estimate small viewing angles literally with your bare hands. Each distant object, which is just covered by the fingernail of your outstretched index finger, is seen by you at an angle of 1° and, therefore, is 57 times farther away than its width. If the nail covers half of the object, it means its angular size is 2°, and the distance is equal to 28 widths.

The Full Moon covers only half of the nail, i.e., it is seen at an angle of half a degree, meaning it is 114 times its width away from us; here is a valuable astronomical measurement made literally with bare hands!

For larger angles, use the knuckle of your thumb, holding it bent on your outstretched hand. For an adult, the length (note: length, not width) of this joint is about 3.5 cm, and the distance from the eye with an outstretched arm is about 55 cm. It is easy to calculate that the angular size in this position should be about 4°. This provides a means to estimate angles of 4° (and therefore 8°).

Here, you should also add two more angles that can be measured with your fingers – namely, those under which the intervals between fingers are seen when your middle and index fingers are spread as wide as possible; and between your thumb and index finger, also spread to the maximum. It is easy to calculate that the first angle is approximately 7°-8°, and the second is 15°-16°.

There can be many cases to apply your living protractor during walks in open spaces. Suppose a freight car is visible in the distance, which is covered by approximately half of the knuckle of your outstretched thumb, i.e., it is visible at an angle of about 2°. Since the length of a freight car is known (about 6 m), you can easily find out how far you are from it: $6 \times 28 \approx 170$ m or so. A method that does not seem to promise good results, but after a short exercise you will learn to appreciate the services of this “living ecker”[18], the measurement is, of course, rough, but still more reliable than an ungrounded estimate just by sight.

[18] An “ekker” is a surveying instrument for drawing lines on the ground at right angles.

Additionally, using your living protractor, you can, in the ab-

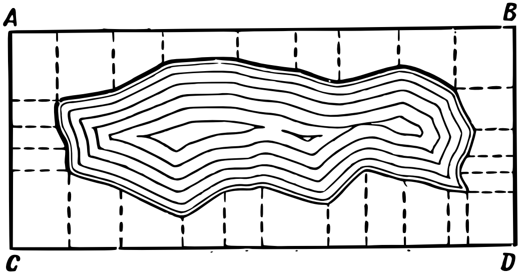


Figure 68: Mapping of the lake on the plan.

sence of any tools, measure the angular height of luminaries above the horizon, the mutual separation of stars in degrees, the apparent sizes of a meteor's trail, etc. Finally, knowing how to make right angles on the ground without instruments, you can draw up a plan of a small area using the method whose essence is clear from the illustration, for example, when surveying a lake (Figure 68), measure rectangle *ABCD*, as well as the lengths of the perpendiculars dropped from prominent points on the shore, and the distances from their bases to the vertices of rectangle *ABCD*. In short, being in Robinson Crusoe's situation, knowing how to use your own hands to measure angles (and your feet to measure distances) could be useful for a variety of needs.

Jacob's Staff

If you wish to have more accurate angle measures than the simple "living protractor" described earlier, you can make yourself a simple and convenient device that was once used by our ancestors. This is called "Jacob's staff" after its inventor – a device that was widely used by sailors until the 18th century (Figure 69), before it was gradually replaced by even more convenient and precise instruments (sextants).

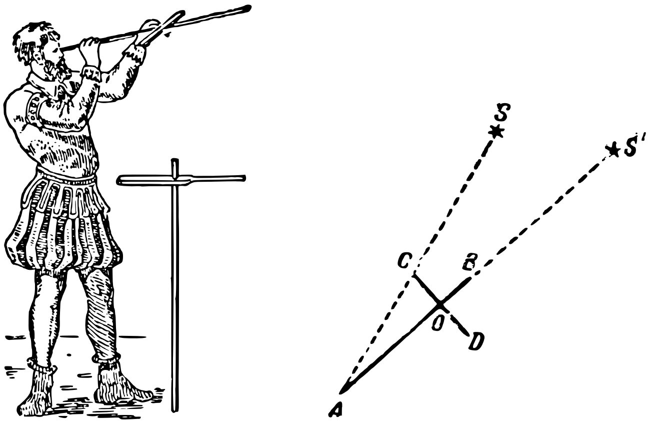


Figure 69: Jacob's staff and a diagram of its use.

It consists of a long ruler AB, about 170–100 cm, along which a perpendicular block CD can slide; both parts CO and OD of the sliding block are equal to each other. If you want to determine the angular distance between the stars S and S' using this block (Figure 69), you attach the end A of the ruler to your eye (where a perforated plate is attached for convenience of observation) and direct the ruler so that the star S' is visible at the end B of the ruler; then you move the crosspiece CO along the ruler until the star S is just visible at the end C (Figure 69). Now all that remains is to measure the distance AO in order to calculate the value of the angle SAS' using the length of the CO. Those familiar with trigonometry will understand that the tangent of the desired angle is equal to the ratio of CO/AO. Our "field trigonometry", presented in the fifth chapter, is also sufficient for performing this calculation: you calculate the length AC using the Pythagorean theorem, then find angle C, whose sine is equal to CO/AC.

Finally, you can find the desired angle graphically; by drawing triangle ACO on paper to scale, you measure angle A with a protractor, or if you don't have one, by the method described in our "field trigonometry" (see Chapter 5).

What is the other half of the crosspiece for? In case the angle to be measured is too large to be measured by the method described above. In that case, instead of directing the ruler AB toward the star S', you aim segment AD toward the point

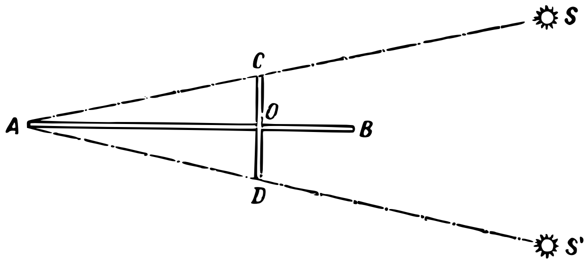


Figure 70: Determination of the angular distance between stars using Jacob's staff.

S', moving the crosspiece so that its end C coincides with the star S at the same time (Figure 70). Finding the angle SAS' by calculation or construction is, of course, not difficult.

To avoid having to make calculations or constructions each time you measure, you can perform them in advance, even when making the device, and mark the results on the ruler AB; then, when aiming the device at the stars, you only need to read the reading recorded at point O — this is the value of the measured angle.

Rake Angle Gauge

It's even easier to make another device for measuring angular magnitude – the so-called 'rake angle gauge,' which indeed resembles a rake in appearance (Figure 71). Its main part is a board of any shape, with a drilled plate attached at one end;

the observer aligns their eye with its hole. At the opposite end of the board, a row of thin pins[19] (commonly used for insect collections) is inserted, the gaps between which make up the 57th part of their distance from the hole of the drilled plate. It is already known that each interval is observed at an angle of one degree. One can also place the pins using another method, yielding a more precise result: two parallel lines are drawn on the wall, one meter apart, and stepping back from the wall perpendicular to them by 57 m, these lines are viewed through the hole in the drilled plate; the pins are inserted into the board so that each pair of adjacent pins covers the lines drawn on the wall.

[19] Instead of pins, you can use a frame with strands stretched on it.

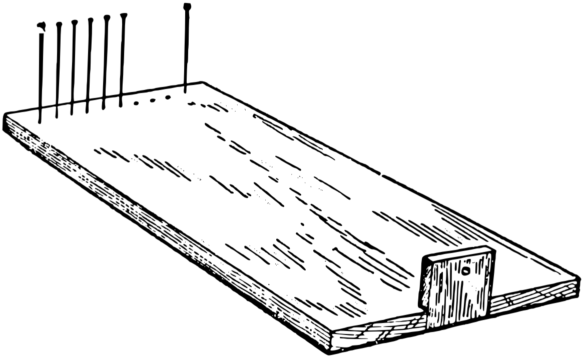


Figure 71: A rake protractor.

Once the pins are placed, some of them can be removed to obtain angles of 2°, 3°, 5°. The method of using this angle gauge is, of course, understandable to the reader even without explanations. By using this angle gauge, angles of view can be measured with quite high accuracy, no less than 0.25°.

Artilleryman's Angle

An artilleryman does not shoot 'blindly'. Knowing the position of the target, he determines its angular magnitude and calculates the distance to the target; in another case, he determines at what angle he needs to turn the gun to shift fire from one target to another. He solves such tasks quickly and mentally. How? Look at Figure 72, AB is the arc of circle from $OA = D$; ab - the arc of circle with radius $Oa = r$. From the similarity of the two sectors AOB and aOB it follows

$$\frac{AB}{D} = \frac{ab}{r}, \quad \text{or} \quad ,$$

$$AB = \frac{ab}{r} D.$$

The ratio ab/r characterises the angle of view AOB; knowing this ratio, it is easy to calculate AB from a known D or D from a known AB.

Artillerymen facilitate their calculations by dividing the circle

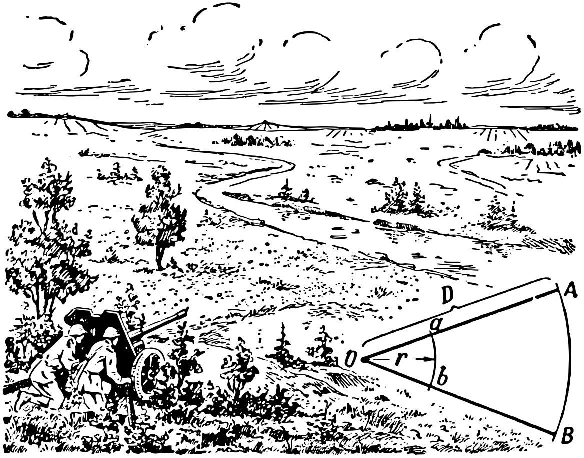


Figure 72: A rake protractor.

not into 360 parts, as usual, but into 6000 equal arcs. Then the length of each division is approximately 1/1000 times the radius of the circle. Indeed, let, for example, the arc ab of the angle circle O (Figure 72) represent one unit of division; then the length of the entire circumference $2\pi r \approx 6r$, and the length of the arc $ab = 6r/6000 = 1/1000\, r$.

In artillery, it is called a 'thousandth'. So,

$$AB = \frac{0.001\, r}{r}\, D = 0.001\, D,$$

i.e., to find out what distance AB on the ground corresponds to one division of the angle-measurer (angle in one 'thousandth'), it is sufficient in the distance D to move the decimal point three digits to the right.

When transmitting commands or observation results by field telephone or radio, the number of 'thousandths' is pronounced like a telephone number, for example: an angle of 105 'thousandths' is pronounced: 'one zero five', and written as:

$$1-05;$$

an angle of 8 'thousandths' is pronounced: 'zero zero eight', and written as:

$$1-08.$$

Now you can easily solve such an artillery problem.

Question The tank (in terms of height) is visible from the anti-tank gun at an angle of $0-05$. Determine the distance to the tank, assuming its height is 2 meters.

Answer

$$\text{5 divisions of the angle-measurer} = 2\,\text{m},$$

$$\text{so 1 division of the angle-measurer} = \frac{2\,\text{m}}{5} = 0.4\,\text{m}.$$

Since one division of the angle-measurer corresponds to one thousandth of the distance, the entire distance is therefore a thousand times greater, i.e.,

$$D = 0.4 \times 1000 = 400\,\mathrm{m}.$$

If the commander or scout does not have an angle-measuring device at hand, they use their palm, fingers, or any other improvised means, as described in our book (Living Protractor). However, their "value" must be given to the artilleryman not in degrees, but in "thousandths". Here is the approximate "value" in "thousandths" of some items:

Object	Angle in 'thousandths'
Palm of the hand (average)	1 – 20
Middle, index, or ring finger	0 – 30
Round pencil (thickness)	0 – 12
Three-kopeck or twenty-kopeck coin (diameter)	0 – 40
Matchstick (length)	0 – 75
Matchstick (thickness)	0 – 03

Your Visual Acuity

Once you grasp the concept of angular magnitude of an object, you will understand how visual acuity is measured, and even perform such measurements yourself.

Draw on a piece of paper 20 equally spaced black lines each as long as a matchstick (5 cm) and one millimetre thick, so that they fill a square (see Figure 73). Attach this drawing to a well-lit wall and step away from it until you notice that the lines are no longer individually distinguishable, but merge into a continuous grey background. Measure this distance and calculate – as you already know how to do – the angle of vision under which you cease to distinguish the stripes at 1 mm thickness. If this angle is $1'$ (one minute), then your vision acuity is normal; if it is three minutes – your acuity is "below normal" and so on.

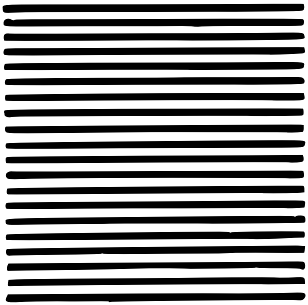


Figure 73: To measure visual acuity.

Question Lines in Figure 73 merge for your eye at a distance of 2 m. Is vision acuity normal?

Answer We know that from a distance of 57 mm, a stripe with a width of 1 mm is visible at an angle of 1°, i.e., $60'$. Therefore, from a distance of 2,000 mm, it is visible at an angle x which is determined from the proportion

$$\frac{x}{60} = \frac{57}{2000}, \qquad x = 1.7'.$$

Vision acuity is below normal and amounts to

$$\frac{1}{1.7} \approx 0.6.$$

The Limiting Minute

We've just mentioned that stripes viewed at an angle of less than one minute cease to be individually distinguishable by a normal eye. This holds true for any object: regardless of the contours of the observed object, they cease to be distinguishable by a normal eye if they are visible at an angle less than $1'$. Each object becomes a barely distinguishable point, "too small for sight" (Shakespeare), a speck without size or shape. This is a property of the normal human eye: one angular minute is the average limit of its acuity. What causes this is a special question related to the physics and physiology of vision. We only discuss the geometric aspect of the phenomenon here.

The same applies equally to large but distant objects and to close but too small ones. We cannot distinguish with the naked eye the shapes of dust particles floating in the air: illuminated by sunlight, they appear to us as identical tiny dots, although in reality, they have a variety of shapes. We cannot discern small details of an insect's body for the same reason, because we see them at an angle less than $1'$. For the same reason, without a telescope, we do not see the details on the surface of the Moon, planets, and other celestial bodies.

The world would seem completely different to us if the boundary of natural vision were pushed further. A person whose limit of visual acuity is not $1'$, but, for example, $0.5'$, would see the surrounding world deeper and farther than we do. Chekhov vividly described this advantage of sharp eyesight in his story *The Steppe.*

"His (Vasya's) vision was remarkably sharp. He saw so well that the brown barren steppe was always full of life and content for him. He only had to look into the distance to see a fox, a hare, a bustard, or some other animal, keeping itself away from people. It was not difficult to see a fleeing hare or a flying bustard – anyone passing through the steppe could see them – but not everyone could see wild animals in their domestic life, when they are not running, hiding, or looking around anxiously. And Vasya saw playing foxes, hares washing their paws, bustards spreading their wings, and snipes drumming. Thanks to such keen vision, besides the world that everyone saw, Vasya had another world, his own, inaccessible to anyone else, and probably very good, because when he looked and admired, it was difficult to envy him."

It's strange to think that such a remarkable change is sufficient by simply lowering the limit of discernibility from $1'$ to $0.5'$ or so …

The magical effect of microscopes and telescopes is also due to the same reason. The purpose of these instruments is to alter the course of rays from the observed object so that they enter the eye with a more divergent beam; thanks to this, the object appears at a larger angle of vision. When it is said that a microscope or telescope magnifies by 100 times, it means that with their help, we see objects at an angle 100 times larger than with the naked eye. And then the details hidden from the naked eye beyond the limit of acuity become accessible to our sight.

We see the full moon at an angle of 30′; and since the diameter of the Moon is 3,500 km, then each area of the Moon with a diameter of 3500/30, i.e., about 120 km, merges into a barely distinguishable point for the naked eye. In a telescope, which magnifies by 100 times, much smaller areas with a diameter of $120/100 = 1.2$ km will already be indistinguishable, and in a telescope with a 1000-fold magnification – an area of 120 m wide. Hence, among other things, if there were such constructions on the Moon as our large plants or ocean liners, we could see them through modern telescopes.[20]

The rule of the limiting minute has a significant effect on our ordinary everyday observations as well. Due to this feature of our vision, every object, distant by 3400 (i.e., 57×60) of its diameter, ceases to be distinguished by us in its outlines and merges into a point. Therefore, if someone claims that they recognised a person's face with the naked eye from a

[20] Under the condition of full transparency and uniformity of our atmosphere. In reality, the air is not homogeneous and not entirely transparent; therefore, at high magnifications, the visible picture becomes hazy and distorted. This imposes a limit on the use of very strong magnifications and prompts astronomers to build observatories in the clear air of high mountain peaks.

distance of a quarter of a kilometre, do not believe them – unless they have phenomenal vision. After all, the distance between a person's eyes is only 8 cm; this means that both eyes merge into a point already at a distance of $3 \times 3{,}400$ cm, i.e., 100 m. Artillerymen use this for visual estimation of distance. According to their rules, if a person's eyes appear as two separate points from a distance, then the distance to them does not exceed 100 steps (i.e., 60-70 m). We obtained a greater distance – 100 m: this shows that the military sign indicates somewhat reduced (to 30%) visual acuity.

Question Can a person with normal vision distinguish a rider at a distance of 10 km, using binoculars magnifying three times?

Answer The height of the rider is 2.2 m. His figure turns into a point for the naked eye at a distance of $2.2 \times 8400 = 7$ km; in binoculars magnifying three times – at a distance of 21 km. Therefore, it is possible to distinguish him at distance of 10 km with such binoculars (if the air is sufficiently transparent).

The Moon and Stars at the Horizon

Even the most inattentive observer knows that the full moon, hanging low at the horizon, appears noticeably larger than when it is high in the sky. The difference is so significant that it's hard not to notice. The same holds true for the Sun; it's well-known how large the solar disk appears at sunset or sunrise compared to its size high in the sky, for example, when it shines through the clouds (directly looking at the unobscured sun is harmful to the eyes).

For stars, this peculiarity manifests in the way the distances between them increase as they approach the horizon. Anyone who has seen the beautiful constellation Orion in winter (or Cygnus in summer) high in the sky and low near the horizon cannot help but be struck by the enormous difference in the constellation's size in both positions.

All this is made more mysterious by the fact that when we look at celestial bodies at sunrise or sunset, they are not only not closer but, on the contrary, farther away (by the size of the Earth's radius), as easily understood from Figure 74: at the zenith, we view the celestial body from point A, and at the horizon – from points B or C. So why do the Moon, the Sun, and constellations appear larger at the horizon?

One might answer, "Because it's not true." It's a visual deception. With the help of a protractor or another angle-measuring instrument, it's easy to verify that the lunar disk is visible in both cases at the same angular size (half a degree)[21]. Using the same device or "Jacob's staff," one can ascertain that the angular distances between stars do not change, re-

[21] Measurements made with more precise instruments show that the visible diameter of the Moon is even smaller when the Moon is near the horizon, due to the fact that refraction slightly flattens the disk.

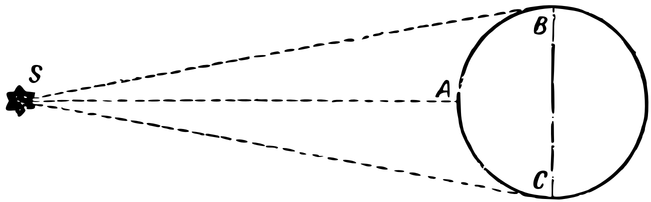


Figure 74: Why is the Sun, being on the horizon, farther from the observer than being in the middle of the sky.

gardless of whether the constellation is at the zenith or the horizon. Therefore, the enlargement is an optical illusion to which all people are subject without exception.

What explains such a strong and universal visual deception? Science has not yet provided a definitive answer to this question, as far as we know, although it has been striving to resolve it for 2000 years, since the time of Ptolemy. The illusion is related to the fact that we perceive the entire celestial sphere not as a hemisphere in the geometric sense but as a spherical segment, the height of which is 2–3 times smaller than the radius of the base. This is because with our head and eyes in their usual position, distances in the horizontal direction and close to it are perceived by us as more significant compared to vertical ones: in the horizontal direction, we view an object “directly”, while in any other direction – with eyes raised upwards or lowered downwards. If one observes the Moon while lying on one’s back, it will appear larger when it is at the zenith than when it stands low above the horizon.[22]

[22] In previous editions of *Entertaining Geometry*, Ya. I. Perelman explained the apparent enlargement of the Moon at the horizon by the fact that at the horizon, we see it next to distant objects, while in the empty celestial vault, we see it alone. However, the same illusion is observed on the unfilled horizon of the sea, so the previously proposed explanation of the described effect must be considered unsatisfactory.

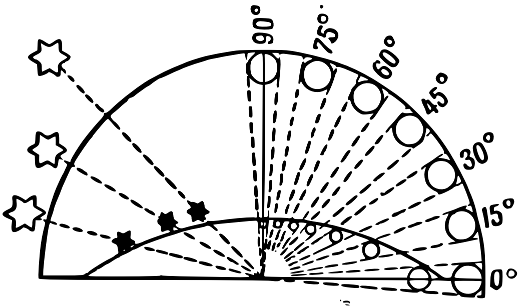


Figure 75: The influence of the flatness of the firmament on the apparent size of the celestial objects.

Psychologists and physiologists are faced with the task of explaining why the visible size of an object depends on the orientation of our eyes. As for the influence of the apparent flattening of the celestial sphere on the size of celestial bodies in its different parts, it becomes quite understandable from the scheme depicted in Figure 75. The lunar disk is always visible at an angle of half a degree in the celestial vault, whether the Moon is at the horizon (at an altitude of 0°) or at the zenith (at an altitude of 90°). However, our eye refers this disk not always to the same distance: the Moon at the zenith is perceived by us as being closer than at the horizon, and therefore its size seems different – within the same angle, a smaller circle is placed closer to the vertex than farther from it. On the left side of the same figure, it is shown how, due to this reason, the distances between stars seem to stretch as they approach the horizon: when the angular distances between them are the same, they seem different.

There is another instructive aspect here. Admiring the huge lunar disk near the horizon, have you noticed any new features on it that you couldn't discern on the disk of the moon standing high? No. But before you is an enlarged disk, so why aren't new details visible? Because there is no enlargement here as provided, for example, by binoculars: the angle at which the object is perceived by us is not enlarged here. Only the enlargement of this angle helps us distinguish new details; any other "enlargement" is simply a visual deception, utterly useless for us.[23]

[23] For more information, see the book by the same author *Physics for Entertainment*, Book 2, Ch. IX.

What is the Length of the Moon's Shadow and the Shadow of the Stratosat?

A rather unexpected application for the angle of view has been found by me in tasks to calculate the length of the shadow cast by various bodies in space. For example, the Moon casts a cone-shaped shadow in the world space, which accompanies it everywhere.

How far does this shadow extend?

To perform this calculation, there is no need, based on the similarity of triangles, to compose a proportion in which the diameters of the Sun and the Moon, as well as the distance between the Moon and the Sun, are included. The calculation can be made much simpler. Imagine that your eye is placed at the point where the cone of the lunar shadow ends, at the vertex of this cone, and you are looking from there at the Moon. What will you see? A black circle of the Moon, covering the Sun. The viewing angle under which we see the disk of the Moon (or the Sun) is known: it is equal to half a degree. But we already know that an object seen at an angle of half a degree is removed from the observer by $2 \times 57 = 114$ of its diameters. Therefore, the apex of the cone of the lunar shadow is located from the Moon at 114 lunar diameters. Hence the length of the lunar shadow is

$$3500 \times 114 = 400{,}000\,\text{km}.$$

The shadow is longer than the distance from the Earth to the Moon; hence total solar eclipses can occur (for locations on the Earth's surface that fall into this shadow).

It is easy to calculate the length of the Earth's shadow in space: it is as many times larger than the lunar shadow as the diameter of the Earth exceeds the diameter of the Moon, i.e., approximately four times.

The same principle applies to calculating the length of the spatial shadows of smaller objects. For example, let's find out how far the cone-shaped shadow cast by the *SOAK-1* stratosat extended in the air at the moment when its envelope was inflated into a sphere. Since the diameter of the stratosat's sphere is 36 meters, the length of its shadow (the angle at the vertex of the shadow cone is the same, half a degree) is

$$36 \times 114 \approx 4{,}100 \text{ m},$$

or about 4 km.

In all cases considered, we were, of course, talking about the length of the total shadow, not the penumbra.

How High is the Cloud Above the Ground?

Remember how you were amazed by the long, winding white trail when you first saw it high in the clear blue sky. Now, of course, you know that this cloud strip is a kind of "autograph" of the airplane, left in the airspace as a "memory" of its whereabouts.

In the cooled, moist, and dusty air, fog easily forms.

A flying airplane continuously throws out small particles – the products of engine operation, and these particles are the points around which water vapor condenses; a cloud forms.

If the height of this cloud can be determined before it melts, then one can also find approximately how high our brave pilot has climbed in his airplane.

Question How to determine the height of a cloud above the ground if it is not even above our heads yet?

Answer To determine large heights, it is necessary to enlist the help of an ordinary camera – a device that is quite complex nowadays and quite popular among young people.

In this case, two cameras with the same focal lengths are needed. (Focal lengths are usually engraved on the rim of the camera lens.)

Both cameras are installed on elevations of more or less equal height.

In the field, these can be tripods, in the city – towers on the roofs of houses. The distance between the elevations should be such that one observer can see the other directly or through binoculars. This distance (base) is measured or determined from a map or plan of the area.

The cameras are installed so that their optical axes are parallel. They can be directed, for example, at the zenith.

When the photographed object is in the field of view of the camera lens, one observer gives a signal to the other, for example, by waving a handkerchief, and at this signal both observers simultaneously take pictures.

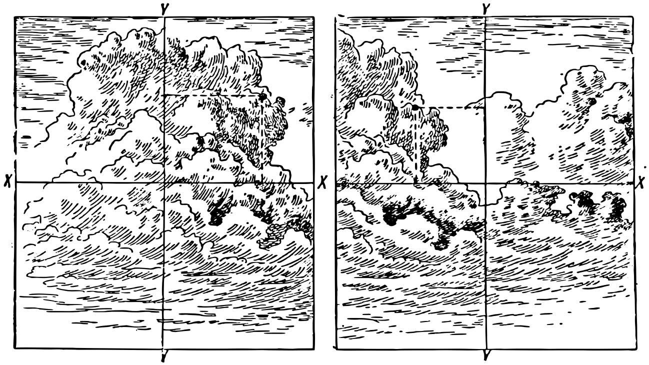


Figure 76: An image of two photo prints of the cloud.

On the prints, which in size should be exactly equal to the photographic plates, they draw straight lines YY and XX, connecting the midpoints of the opposite edges of the pictures (Figure 76). Then they mark on each picture the same point of the cloud and calculate its distances (in pixels) from the lines YY and XX. These distances are denoted respectively by the letters x_1, y_1 for one picture and x_2, y_2 for the other.

If the marked points on the pictures turn out to be on different sides of the line YY (as in Figure 76), then the height of the cloud H is calculated by the formula

$$H = b \cdot \frac{F}{x_1 + x_2},$$

where b is the length of the base (in m), F is the focal length (in mm).

If the marked points turn out to be on the same side of the lineYY, then the height of the cloud is determined by the formula

$$H = b \cdot \frac{F}{x_1 - x_2}$$

As for the distances y_1 and y_2, they are not needed to calculate H, but by comparing them with each other, one can determine the correctness of the shooting.

If the plates lay in the cassettes tightly and symmetrically, then y_1 will be equal to y_2. In practise, however, they will, of course, be slightly different.

Let us say, for example, that the distances from the lines YY and XX to the marked point of the cloud on the *original* photographs are as follows:

$$x_1 = 32\,\text{mm}, \quad y_1 = 29\,\text{mm},$$
$$x_2 = 23\,\text{mm}, \quad y_y = 25\,\text{mm}.$$

The focal lengths of the lenses $F = 135$ mm and the distance between the cameras (base) $H = 937$ m.[24] Photographs show that to determine the height of the cloud, it is necessary to use the formula

$$\begin{aligned} H &= b \cdot \frac{F}{x_1 + x_2}, \\ &= 937\,\text{m} \frac{135}{32 + 23}, \\ &\approx 2{,}300\,\text{m} \end{aligned}$$

So, the cloud was at a height of about 2.3 km from the ground.

Those who want to understand the derivation of the formula for determining the height of the cloud can use the diagram shown in Figure 77.

The drawing shown in Figure 77 must be imagined in space (spatial imagination is developed in the study of that part of geometry called *stereometry*).

Figures *I* and *II* – images of photographic plates; F_1 and F_2 - optical centres of camera lenses; N - observed point of the

[24] According to the experience described in the book by N. F. Platonov, *The application of mathematical analysis to solving practical problems.* In the article *Cloud Height*, N. F. Platonov provides a derivation of the formula for calculating M, describes the possible installations of devices for photographing fbblack and gives a number of practical tips.

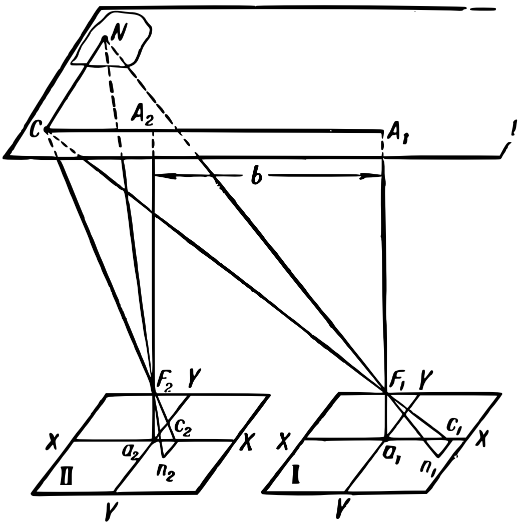


Figure 77: A diagram of the image of the cloud point on the plates of two cameras aimed at the zenith.

cloud; n_1 and n_2 – images of point N on the photographic plates; a_1A_1, and a_2A_2 – perpendiculars raised from the midpoint of each photographic plate to the level of the cloud; $A_1A_2 = a_1a_2 = b$ the size of the baseline.

If we move from the optical centre F_1 upwards to point A_1, then from point A_1 along the baseline to such a point C that will be the vertex of a right angle A_1CN, and finally, from point C to point N, then segments F_1A_1, A_1C_1, and CN in the camera will correspond to segments $F_1a_1 = F$ (focal length), $a_1c_1 = x_1$ and $c_1n_1 = y_1$.

Similar constructions are used for the second camera. The proportions follow from the similarity of triangles

$$\frac{A_1C}{x_1} = \frac{A_1F_1}{F} = \frac{C_1F_1}{F_1c} = \frac{CN}{y_1} \text{ and,}$$
$$\frac{A_2C}{x_2} = \frac{A_2F_2}{F} = \frac{C_2F_2}{F_2c} = \frac{CN}{y_2}.$$

Comparing these proportions and bearing in mind the obvious equality of $A_2F_2 = A_1F_1$, we find, firstly, that $y_1 = y_2$ (a sign of correct shooting), secondly, that

$$\frac{A_1C}{x_1} = \frac{A_2C}{x_2};$$

according to the drawing $A_2C = A_1C - b$, therefore,

$$\frac{A_1C}{x_1} = \frac{A_1C - b_1}{x_2}; \text{ or,}$$
$$A_1C = b \cdot \frac{x_1}{x_1 - x_2}, \text{ finally,}$$
$$A_1F_1 \approx H = b \cdot \frac{F}{x_1 - x_2}.$$

If n_1 and n_2-— the images on the plates of point N — turned out to be on different sides of the straight line, this would indicate that point C is between points A_1 and A_2, and then $A_2C = b - A_1C$, and the desired height

$$H = b \cdot \frac{F}{x_1 + x_2}.$$

These formulas apply only to the case when the optical axes of the cameras are aimed at the zenith. If the cloud is far from the zenith and near the horizon? If the vision of the devices does not fall, then you can give the devices a different position (while maintaining the parallelism of the optical axes), for example, direct them horizontally and, moreover, perpendicular to the basis or along the basis.

For each position of the devices, it is necessary to first build an appropriate drawing and derive formulas for determining the height of the cloud.

Question Here, “in broad daylight”, noticeable feathery, highly layered clouds appeared in the sky of a whitish colour. Determine their height two or three times at certain intervals. If it turns out that the clouds have descended, this is a sign of worsening weather: expect rain in a few hours.

Take a picture of a floating balloon or stratostat and determine its height.

Tower Height from Photograph

Question Using a camera, one can determine not only the height of a cloud or a flying airplane but also the height of a ground structure: towers, masts, antennas, etc.

In Figure 78 – a photograph of the wind turbine of the *Central Wind Turbine Institute*, installed in Crimea near Balaklava. At the base of the tower is a square, the side length of which, let's assume, you know as a result of direct measurement: 6 meters.

Make the necessary measurements on the photograph and determine the height of the entire wind turbine installation.

Answer The photograph of the tower and its true outline are geometrically similar to each other. Therefore, the height of the tower in real life is as many times greater than the height of its image as the image of the height is greater than the image of the base.

Measurements of the image: the length of the least distorted diagonal of the base is 23 mm, the height of the entire installation is 71 mm.

Since the length of the side of the square base of the tower is 6 m, the diagonal of the base is $6^2 + 6^2 = 6\sqrt{2} \approx 8.48$ m.

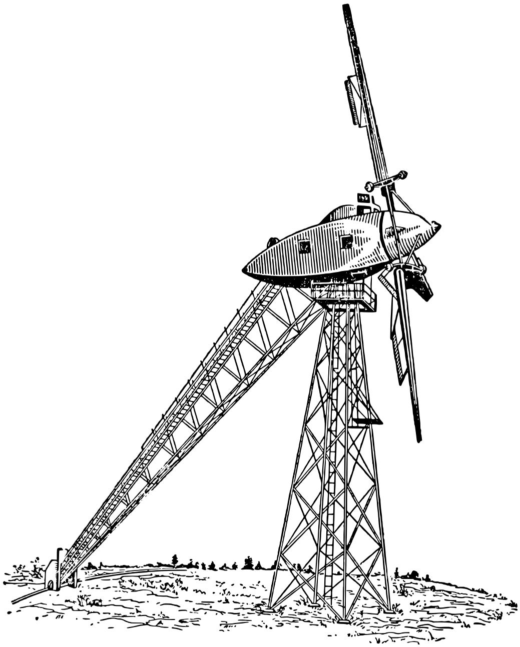


Figure 78: The wind turbine of the Central Wind Turbine Institute in Crimea.

Therefore,

$$\frac{71}{23} = \frac{h}{8.48}, \text{ then,}$$
$$h = \frac{71 \times 8.48}{23} \approx 26\,\text{m}.$$

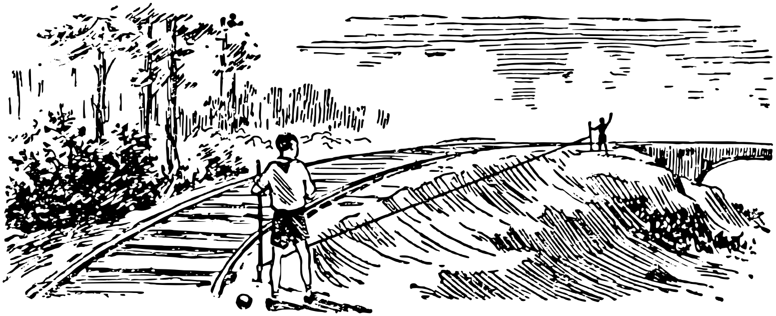
Of course, not every photograph is suitable; only one in which the proportions are not distorted, as can happen with inexperienced photographers.

Exercises

Let the reader now apply the information gleaned from this chapter to solve a series of diverse problems:

1. A person of average height (1.7 m) is seen from a distance at an angle of $12'$. Find the distance to him.
2. A cavalryman on horseback (2.2 m) is seen from a distance at an angle of $9'$. Find the distance to him.
3. A telegraph pole (8 m) is seen at an angle of $22'$. Find the distance to it.
4. A lighthouse with a height of 42 m is visible from a ship at an angle of $1^\circ 10'$. At what distance from the lighthouse is the ship?
5. The Earth is observed from the Moon at an angle of $1^\circ 54'$. Determine the distance from the Moon to the Earth.
6. A building is visible from a distance of 2 km at an angle of $12'$. Find the height of the building.
7. The Moon is visible from the Earth at an angle of $30'$. Knowing that the distance to the Moon is 380,000 km, determine its diameter.

8. How large should the letters on the classroom board be so that students sitting at their desks see them as clearly as the letters in their books (at a distance of 25 cm from the eye)? Take the distance from the desks to the board as 5 m.
9. A microscope magnifies 50 times. Can it be used to observe human blood cells, the diameter of which is 0.007 mm?
10. If there were people on the Moon of our height, what magnification of a telescope would be required to distinguish them from Earth?
11. How many 'thousandths' are there in one degree?
12. How many degrees are there in one 'thousandth'?
    13. An airplane, moving perpendicular to the line of our observation, travels a distance visible at an angle of 300 'thousandths' in 10 seconds. Determine the speed of the airplane if the distance to it is 2000 m?



4. Geometry on the Road

The Art of Measuring by Steps

While out for a countryside walk along a railway track or on a highway, you can perform a series of interesting geometric exercises.

First, use the highway to measure the length of your step and walking speed. This will allow you to – measure distances by steps – an art that is acquired quite easily after a short practise. The main thing here is to get used to making steps of the same length every time, i.e., to adopt a certain 'measured' gait.

On the highway, every 100 meters, there is a white stone; by walking such a 100-meter interval with your usual 'measured' step and counting the number of steps, you can easily find the average length of your step. Such measurement should be repeated annually, for example, every spring, because the length of a step, especially in young people, does not remain constant.

It is worth noting an interesting relationship discovered by repeated measurements: the average length of an adult's step is approximately half of their height, measured to eye level. For example, if a person's height to their eyes is 1 meter 40 centimetres, then the length of their step is about 70 centimetres. It's interesting to verify this rule whenever possible.

In addition to knowing the length of your stride, it's also useful to know your walking speed – the number of kilometres you cover per hour. Sometimes the following rule is used for this: we walk as many kilometeres per hour as we take steps in three seconds; for example, if we take four steps in three seconds, then we walk 4 km per hour. However, this rule is only applicable when the length of the stride is known. It's not difficult to determine it: denoting the length of the stride in meters as x, and the number of steps in three seconds as p, we have the equation

$$\frac{3600}{3} \cdot nx = n \cdot 1000,$$

from which $1200x = 1000$ and $x = 1000/1200$m, i.e., about 80—85 cm. This is a relatively large stride; people of tall stature make such strides. If your stride differs from 80—85 cm, then you will need to measure your walking speed in a different way, by determining the time it takes you to cover the distance between two roadside posts using a clock.

Eye-meter

It's pleasant and useful not only to measure distances without a measuring tape, but also to estimate them directly by eye without measurement. This skill is achieved only through practice. In my school years, when I participated in summer excursions out of town with a group of friends, such exercises were very common for us. They were carried out in the form of a special sport, invented by ourselves – in the form of a competition for the accuracy of the eye-meter. When we went out on the road, we would visually mark some roadside tree or other distant object, and the competition would begin.

“How many steps to the tree?” someone from the participants would ask.

The rest would give their estimated number of steps, and then together count the steps to determine whose estimate was closer to the truth – that person would be the winner. Then it was their turn to mark an object for the eye-meter distance estimation.

Whoever determined the distance more accurately than the others would receive one point. After 10 times, the points were counted: the one with the most points was considered the winner of the competition.

I remember that at first our distance estimates were given with rough errors. But very soon, much faster than expected, we sharpened our skills in the art of estimating distances by eye, making very few mistakes. Only with a sharp change in surroundings, for example, when transitioning from an open field to sparse forest or to a bush-covered clearing, when returning to dusty, narrow city streets, or at night, with the deceptive illumination of the moon, would we catch each other making significant errors. However, we soon learned to adapt to all circumstances, mentally taking them into account during eye-meter estimates. Finally, our group reached such perfection in eye-meter distance estimation that we had to completely abandon this sport: everyone guessed equally well, and the competitions lost their interest. But we acquired a decent eye-meter, which served us well during our wanderings in the countryside.

It's curious that eye-metering seems to be independent of visual acuity. Among our group was a nearsighted boy who not only didn't lag behind the others in the accuracy of eye-meter distance estimation but sometimes even emerged as the winner of the competitions. Conversely, a boy with perfectly normal vision couldn't master the art of determining distances by eye at all. Later, I had to observe the same phenomenon when estimating the height of trees using the eye-meter method: practising this with students – not for play anymore, but for the needs of future professions – I noticed that nearsighted individuals mastered this skill just as well as others. This can be a consolation for the nearsighted: without having perfect vision, they are still capable of developing quite satisfactory eye-metering skills.

Practising eye-metering distance estimation can be done at any time of the year, in any setting. Walking along the city streets, you can set yourself eye-metering tasks, trying to guess how many steps to the nearest lamp-post or to various objects along the way. In bad weather, you can effectively fill the time with such activities while traversing empty streets.

The military pay a lot of attention to eye-metering distance estimation: a good eye-meter is necessary for a scout, a marksman, an artilleryman. It's interesting to learn about the signs they use in the practise of eye-metering estimations. Here are a few notes from the artillery textbook:

"On-eye distances are determined either by the skill of distinguishing visible objects at different distances from the observer to a certain degree of clarity, or by estimating distances using some familiar visual stretch of 100–200 steps, which seems smaller the further away it is from the observer."

"When determining distances based on the clarity of visible objects, it should be noted that objects illuminated or brighter in colour appear closer in terrain or on water; objects positioned higher than others; groups compared to individual objects and generally larger objects."

"You can follow the following signs: up to 50 steps, you can clearly distinguish people's eyes and mouth; at 100 steps, eyes appear as dots; at 200 steps, buttons and uniform details can still be distinguished; at 300 steps, the face is visible; at 400 steps, leg movements can be discerned; at 500 steps, the colour of the uniform is visible."

At the same time, the most experienced eye may make an error of up to 10% in either direction of the determined dis-

Figure 79: The tree behind the hillock seems close.

tance.

There are cases, however, when eye-metering errors are much more significant. Firstly, when determining distance on a uniform, completely one-coloured surface – on the smooth surface of a river or lake, on a clean sandy plain, on

Figure 80: Climb up the hill, and the tree is still the same distance away.

a densely overgrown field. Here, the distance always seems smaller than the true one; estimating it by eye, we make an error of two-fold, if not more. Secondly, errors are easily possible when determining the distance to an object whose base is obscured by a railway embankment, a small hill, a building, or any other elevation. In such cases, we involuntarily consider the object to be not behind the elevation, but on it itself, and therefore make an error again in the direction of reducing the determined distance (Figure 79 and Figure 80).

In such cases, relying on the eye-meter is dangerous, and resorting to other methods of distance estimation, which we have already discussed and will continue to discuss, becomes necessary.

Slopes

Alongside the railway track, besides the mile (more precisely – kilometre) posts, you also see other low poles with inscriptions on small boards, like those shown in Figure 81.

Figure 81: "Slope Signs."

These are 'slope signs.' In the first one, for example, the upper number 0.002 means that the gradient of the track here (in which direction – indicated by the position of the board) is equal to 0.002: the track rises or descends by 2 mm for every thousand millimetres. And the lower number 140 indicates that such a gradient extends for 140 m, where another sign is placed with a notation of a new gradient. (When roads were not yet reorganised according to the metric system, such a board indicated that over a distance of 140 fathoms, the track rises or descends every fathom by 0.002 fathoms.) The second board with the inscription 0.006/55 indicates that over the next 55 m, the track rises or descends by 6 mm for every meter.

Knowing the meaning of the slope signs, you can easily calculate the difference in height between two adjacent points on the track marked by these signs. In the first case, for example, the height difference is $0.002 \times 140 = 0.28$ m; in the second case – $0.006 \times 55 = 0.33$ m.

In railway practise, as you can see, the magnitude of the track gradient is not determined in degrees. However, it is easy to convert these track gradient markings into degrees. If AB (Figure 81) is the path line, BC is the height difference between points A and B, then the inclination of the path line AB to the horizontal line AC will be marked on the post by the ratio BC/AC. Since angle A is very small, we can take AB and AC as radii of a circle, the arc of which is BC.[25] Then the calculation of angle A, if the ratio $BC : AB$ is known, will not be difficult. With a slope, for example, marked as 0.002, we

[25] To another reader, it may seem perhaps unacceptable to consider the inclined AB equal to the perpendicular AC; Therefore, it is instructive to ensure how small the difference in length between AC and AB is when BC is, for example, 0.01 of AB. By the Pythagorean theorem, we have:

$$\begin{aligned} AC^2 &= \sqrt{AB^2 - \frac{AB^2}{100}} \\ &= \sqrt{0.9999\,AB^2} \approx 0.99995\,AB^2. \end{aligned}$$

The difference in length is only 0.00005. For approximate calculations, such a small error can certainly be neglected.

reason as follows: with an arc length equal to 1/57 radius, the angle is 1° (see page 104); what angle corresponds to an arc of 0.002 radii? We find its value x from the proportion

$$\frac{x}{1^\circ} = \frac{0.002}{1/57}, \text{ from where,}$$
$$x = 0.002 \times 57 = 0.11^\circ,$$

i.e., about 7′.

Only very small gradients are allowed on railway tracks. We have set the maximum gradient at 0.008, i.e., in degrees, 0.008×57 – less than 0.5°: this is the maximum gradient. Only for the Transcaucasian Railway, gradients up to 0.025 are allowed as an exception, which corresponds to almost 1.5° in degrees.

Such insignificant gradients are completely unnoticed by us. A pedestrian begins to feel the slope under their feet only when it exceeds 57/24: this corresponds to approximately 2.5° in degrees.

Having walked along the railway track for several kilometres and noted the observed gradient signs, you can calculate how much you have ascended or descended in total, i.e., what is the difference in height between the starting and ending points.

Question You started a walk along the railway track at a post marked with an ascent of 0.004/153 miles and encountered the following signs further along[26]:

Platform	Ascent	Ascent	Platform	Descent
$\frac{0.000}{60}$	$\frac{0.0017}{84}$	$\frac{0.0032}{121}$	$\frac{0.000}{45}$	$\frac{0.004}{210}$

[26] Sign 0.000 means a horizontal section of the path ("platform").

You finished the walk at another slope sign. What distance did you travel, and what is the difference in height between the first and last signs?

Answer Total distance travelled:

$$153 + 60 + 84 + 121 + 45 + 210 = 673\,\text{m}.$$

You went up by

$$0.004 \times 153 + 0.0017 \times 84 + 0.0032 \times 121 = 1.15\,\text{m},$$

and descended:

$$0.004 \times 210 = 0.84\,\text{m}.$$

Therefore, in total, you ended up higher than the starting point by $1.15 - 0.84 = 0.31$ m.

Heap of Gravel

Heap of gravel at the edges of the road also presents an object worthy of attention for the "geometer in the open air." By asking what volume the heap in front of you contains, you set yourself a geometric problem, quite intricate for someone accustomed to overcoming mathematical difficulties only on paper or on the blackboard.

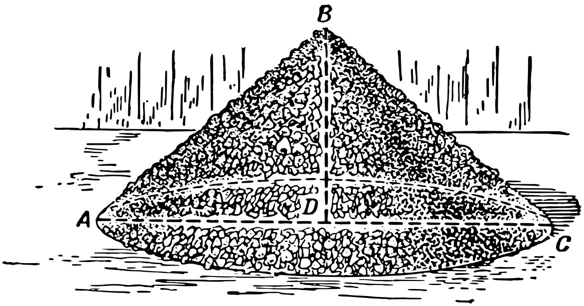


Figure 82: For the problem of the gravel heap.

You have to calculate the volume of a cone, the height and radius of which are not directly measurable. However, nothing prevents you from determining their value indirectly. You will find the radius by measuring the circumference of the base with a ruler or string and dividing it by 2π.[27]

[27] In practice, this action is replaced by multiplication by the reciprocal, if looking for the diameter, and by 0.159 if calculating the radius.

The height poses a greater challenge (Figure 82): you have to measure the length of the generatrix *AB* or, as the road surveyors do, both generatrices *ABC* at once (by laying the measuring tape over the apex of the heap), and then, knowing the radius of the base, calculate the height *BD* using the

Pythagorean theorem. Let's consider an example.

Question The circumference of the base of a conical heap of gravel is 12.1 m; the length of the two generatrices is 4.6 m. What is the volume of the heap?

Answer The radius of the base of the heap is

$$12.1 \times 0.159 \text{ (instead of } 12.1/6.28) = 1.9\,\text{m}.$$

The height is

$$\sqrt{(2.3^2 - 1.9^2)} = 1.2\,\text{m},$$

hence the volume of the heap is

$$\frac{1}{3} \times 3.14 \times 1.9^2 \times 1.2 = 4.6\,\text{m}^3$$

(approximately 1/2 cubic fathoms in previous measurements).

Typically, the standard volume measurements for gravel heaps on our roads were 1/2, 1/4, and 1/8 cubic fathoms, i.e., in metric units, 4.8, 2.4, and 1.2 cubic meters.

The Proud Hill

Looking at conical heaps of gravel or sand brings to mind an ancient legend of Eastern peoples, retold by Pushkin in *The Miserly Knight*:

> *I read somewhere,*
> *That once a king ordered his warriors*
> *To carry earth by handfuls into a heap, –*
> *And a proud hill rose up,*
> *And the king could from the heights with joy survey*
> *Both the valley, covered with white tents,*
> *And the sea, where ships sailed.*

This is one of those few legends in which, despite its seeming plausibility, there is not a grain of truth. It can be proven by geometric calculation that if some ancient despot had decided to undertake such an endeavour, he would have been discouraged by the meagerness of the result: before him would have risen such a pitiful pile of earth that no imagination could inflate it into the legendary "proud hill".

Let's make an approximate calculation. (How many warriors could an ancient king have? Ancient armies were not as numerous as they are today. An army of 100,000 people was already very impressive in size. Let's stick with this number, i.e., assume that the hill was composed of 100,000 handfuls. Grab the largest handful of earth and fill a glass with it: you won't fill it with just one handful. Let's assume that the handful of an ancient warrior equalled in volume 1/5 (dam^3). Hence, the volume of the hill is:

$$\frac{1}{5} \times 100000 = 20,000\ \mathrm{dam}^3 = 20\mathrm{m}^3.$$

Thus, the hill represented a cone with a volume of no more than 20 cubic meters. Such a modest volume is already disappointing. But let's continue the calculations to determine the height of the hill. To do this, we need to know the angle formed by the generatrices of the cone with its base. In our case, we can assume it to be equal to the angle of natural slope, i.e., 45°: steeper slopes cannot be allowed as the earth would collapse (it would be more plausible to take an even shallower slope, for example, one and a half). Stopping at an angle of 45°, we conclude that the height of such a cone is equal to the radius of its base; therefore,

$$20 = \frac{\pi x^3}{3}, \text{ hence,}$$
$$x = \sqrt[3]{\frac{60}{\pi}} = 2.4\mathrm{m}.$$

One must possess a rich imagination to call a pile of earth 2.4 meters (1.5 human height) a "proud hill." By making the calculation for the case of a one and a half slope, we would have obtained an even more modest result.

Attila had the most numerous army known in the ancient world. Historians estimate it to be 700,000 people. If all these warriors participated in building the hill, a heap taller than the one we calculated would have been formed, but not significantly: since the volume would be seven times larger than ours, the height would exceed the height of our heap by only 1.7 times; it would be equal to $2.4 \times 1.9 = 4.6$ meters. It is doubtful that a mound of such dimensions could satisfy the ambition of Attila.

From such small elevations, it was easy, of course, to see "the valley covered with white tents"," but surveying the sea was possible only if the affair took place not far from the shore.

We will discuss how far one can see from a certain height in the sixth chapter.

At a Road Curve

Neither highways nor railways ever make sharp turns; they always transition smoothly from one direction to another, without breaks, in an arc. This arc is usually part of a circle arranged so that the straight sections of the road serve as tangents to it.

For example, in Figure 83, the straight sections *AB* and *CD*

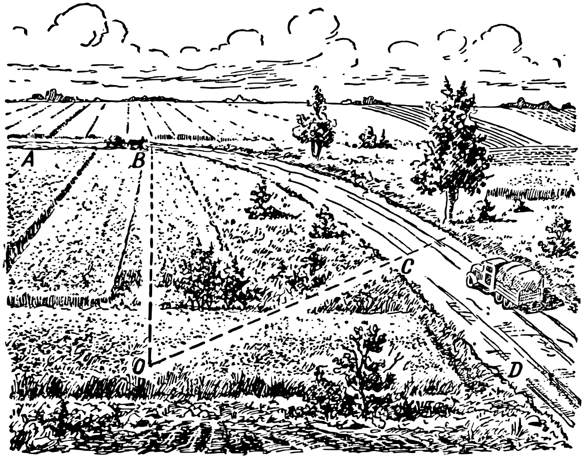


Figure 83: Road rounding.

of the road are connected by the arc BC so that AB and CD touch (geometrically) this arc at points B and C, i.e., AB forms a right angle with radius OB, and CD forms the same angle with radius OC. This is done, of course, to smoothly transition the path from a straight direction to a curved part and back again.

The radius of a road curve is usually taken to be quite large – on railways, not less than 600 meters; the most common radius of curvature on the main railway track is 1000 or even 2000 m.

Radius of Curvature

Standing near one of such curves, could you determine the magnitude of its radius? This is not as easy as finding the radius of an arc drawn on paper. On a drawing, it's simple: you draw two arbitrary chords and draw perpendiculars from their midpoints: the point of their intersection is known to be the centre of the arc; the distance from it to any point on the curve is the desired length of the radius.

But to make a similar construction on the ground would, of course, be very inconvenient: after all, the centre of the curve is 1-2 meters from the road, often in an inaccessible place. It would be possible to make the construction on a plan, but taking the curves off onto a plan is also not an easy task.

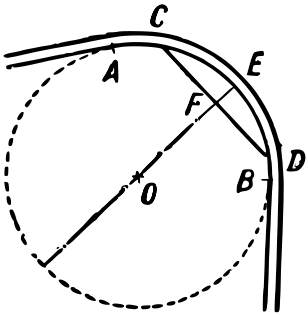


Figure 84: To calculate the radius of curvature.

All these difficulties are eliminated if one resorts not to constructing, but to calculating the radius. For this purpose, the following method can be used. We mentally supplement the arc *AB* of the curve (Figure 84) to a circle. Connecting arbitrary points *C* and *D* of the arc of the curve, we measure the chord *CD*, as well as the "arrow" *EF* (i.e., the height of the segment CED). From these two data, it is already not difficult to calculate the desired length of the radius.

Considering the lines *CD* and the diameter of the circle as intersecting chords, we denote the length of the chord by a, the length of the arrow by h, the radius by R; we have:

$$\frac{a^2}{4} = h(2R - h), \text{ from where,}$$
$$\frac{a^2}{4} = 2Rh - h^2, \text{and the desired radius,}^{28}$$
$$R = \frac{a^2 + 4h^2}{8h}.$$

For example, with an arrow of 0.5 m and a chord of 48 m, the desired radius

$$R = \frac{48^2 + 4 \times 0.5^2}{8 \times 0.5} = 580\,\text{m}.$$

This calculation can be simplified if we consider $2R - h$ equal to $2R$ - an allowable liberty, since h is very small compared to R (after all, R is hundreds of meters, and h is units of them). Then we get a very convenient approximation formula for calculations

$$R = \frac{a^2}{8h}.$$

Applying it to the case we have just considered, we would have obtained the same value $R = 580$ m.

Having calculated the length of the radius of curvature and knowing, in addition, that the centre of the curvature lies on the perpendicular to the middle of the chord, you can

[28] This could have been obtained in another way — from a right triangles COF, where $OS = R$, $CF = a/2$, $OF = R - h$. According to the Pythagorean theorem

$$R^2 = (R-h)^2 + \left(\frac{a}{2}\right)^2, \text{ from where,}$$

$$R^2 = R^2 - 2Rh + h^2 + \frac{a^2}{4},$$

$$R = \frac{a^2 + 4h^2}{8h}.$$

approximately mark the place where the centre of the curved part of the road should lie.

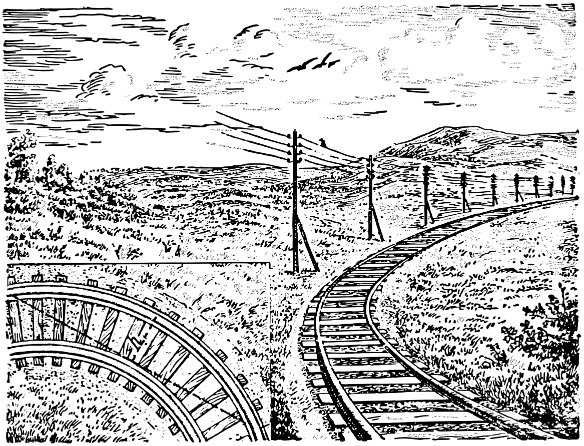


Figure 85: Calculating the radius of a railway curve.

If rails are laid on the road, then finding the radius of curvature is simplified. In fact, by stretching a rope along the tangent to the inner rail, we get the chord of the arc of the outer rail, the arrow of which h (Figure 85) is equal to the gauge of the track – 1.52 m. The radius of curvature in this case (if a is the length of the chord) is approximately

$$R = \frac{a^2}{8 \times 1.52} = \frac{a^2}{12.2}.$$

For $a = 120$ m, the radius of curvature is 1,200 m.[29]

[29] In practise, this method presents the disadvantage that, due to the large radius of rounding, the rope for the chord requires a very long one.

The Bottom of the Ocean

The analogy of a road curve to the bottom of the ocean seems like a somewhat unexpected leap, at least not immediately clear. However, geometry ties both topics together quite naturally.

We are talking about the curvature of the ocean floor, what shape it takes: concave, flat, or convex. To many, it will undoubtedly seem incredible that despite their immense depth, the oceans do not form basins on the Earth's surface; as we will see, their bottom is not only not concave but even convex. Considering the ocean as "bottomless and boundless", we forget that its "boundlessness" is hundreds of times greater than its "bottomlessness", i.e., the water depth of the ocean represents a far-reaching layer that, of course, mirrors the curvature of our planet.

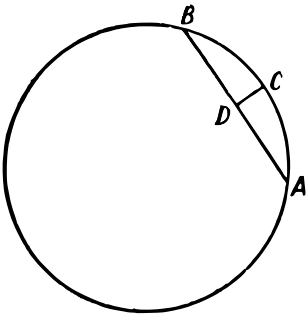


Figure 86: Is the ocean floor flat.

Let's take the example of the Atlantic Ocean. Its width near the equator is approximately one-sixth of the full circumference. If the circle in Figure 86 is the equator, then arc *ACB* represents the water expanse of the Atlantic Ocean. If its bottom were flat, the depth would equal CD, the arrow of arc ACB. Knowing that arc $AB = 1/6$ of the circumference and, consequently, chord AB is a side of a regular inscribed hexagon (which, as is known, equals the radius of the circle R), we can calculate CD from the previously derived formula for road curves:

$$R = \frac{a^2}{8h}, \quad \text{or} \quad h = \frac{a^2}{8R}.$$

Knowing that $a = R$, we get for this case we get $h = R/8$. With $R = 6{,}400$ km, we have: $h = 800$ km.

Thus, for the bottom of the Atlantic Ocean to be flat, its greatest depth should be 800 km. In reality, however, it does not even reach 10 km. Hence the straightforward conclusion: the bottom of this ocean, in its general form, exhibits convexity, slightly less curved than its water surface.

This holds true for other oceans as well: their bottoms represent areas of reduced curvature on the Earth's surface, hardly deviating from its overall spherical shape.

Our formula for calculating the radius of road curvature shows that the larger the water basin, the more convex its bottom. Looking at formula $h = a^2/8R$, we directly see that with increasing width a of the ocean or sea, its depth

h must increase very rapidly, proportional to the square of width a. However, when transitioning from smaller water basins to larger ones, depth does not increase in such a swift progression.

An ocean may be a hundred times wider than a sea, but its depth is not a 100×100, i.e., ten thousand times deeper. Therefore, relatively smaller basins have bottoms more depressed than oceans. The bottom of the Black Sea between Crimea and Asia Minor is not convex like oceans, not even flat but slightly concave. The water surface of this sea represents an arc approximately 2° (more precisely, 1/170 of the Earth's circumference). The depth of the Black Sea is quite uniform and equals 2.2 km. Equating the arc to the chord in this case, we find that for it to have a flat bottom, the sea should have a maximum depth of approximately

$$h = \frac{40000^2}{170^2 \times 8R} = 1.1\,\text{km}.$$

Thus, the actual bottom of the Black Sea lies more than one kilometre $(2.2 - 1.1)$ below the imagined plane drawn through the extreme points of its opposite shores, i.e., it represents a basin, not convexity.

Do Water Mountains Exist?

The previously derived formula for calculating the radius of road curvature will help us answer this question.

The previous problem has already prepared us for the answer. Water mountains exist, but not in the physical sense of these words, but rather in their geometric meaning. Not only each sea but even each lake represents, in a way, a water mountain.

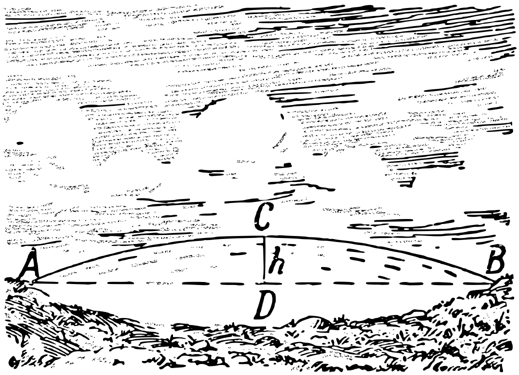


Figure 87: “The Water Mountain.”

When you stand at the shore of a lake, you are separated from the opposite shore by a convexity of water, the height of which is greater the wider the lake. We can calculate this height: using the formula $R = a^2/8h$, we have the value of the depth $h = a^2/8R$. Here, a is the distance between the shores in a straight line, which we can equate to the width of the lake (chord-arc). If this width, let's say, is 100 km, then the height of the water "mountain" is about 200 meters.

$$h = \frac{10000}{8 \times 6400} = 200\,\mathrm{m}.$$

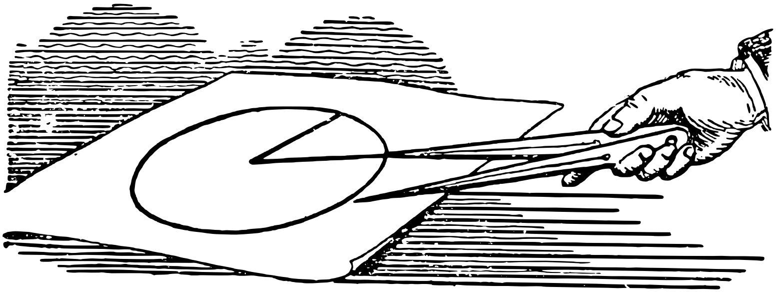
A water mountain of impressive height!

Even a small lake with a width of 10 km elevates the peak of its convexity above the straight line connecting its shores by 2 meters, i.e., higher than human height.

But are we right to call these water convexities "mountains"? In the physical sense, no: they do not rise above the horizontal surface, so they are plains. It is erroneous to think that the line AB (Figure 87) is a horizontal line over which arc ACB rises. The horizontal line here is not AB, but ACB, coinciding with the free surface of calm water. The line ADB is inclined to the horizon: AD slopes downward underground to point D, its deepest point, and then rises up again, emerging from the ground (or water) at point B. If pipes were laid along the line AB, a ball placed at point A would not stay here, but would roll (when the pipe walls are smooth) to point D and from there, picking up speed, would ascend to point B; then, unable to stay there, it would roll back to D, run to A

again, roll down again, and so on. An ideally smooth ball in an ideally smooth pipe (with no air hindering movement) would roll back and forth endlessly…

So, although it may seem to the eye (Figure 87) that *ACB* is a mountain, in the physical sense of the word, it is a flat area. The mountain – if you will – exists here only in a geometric sense.



5. Field Trigonometry Without Formulas and Tables

Calculation of the Sine

In this chapter, we will demonstrate how to calculate the sides of a triangle with an accuracy of 2% and angles with an accuracy of 1°, using only the concept of sine and without resorting to tables or formulas. Such simplified trigonometry can be useful during outdoor walks when tables are not at hand, and formulas are half-forgotten. Robinson Crusoe on his island could successfully employ such trigonometry.

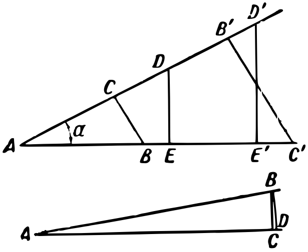


Figure 88: What is the sine of an acute angle?

So, imagine that you have not yet studied trigonometry or have forgotten it entirely – a state that some readers may easily imagine. Let's begin to acquaint ourselves with it again. What is the sine of an acute angle? It is the ratio of the opposite side to the hypotenuse in the triangle formed by dropping a perpendicular from the angle to one of its sides. For example, the sine of angle a (Figure 88) is BC/AB, or ED/AD, or $D'E'/AD'$, or $B'C'/AC'$. It is easy to see that due to the similarity of the triangles formed here, all these ratios are equal to each other.

Let's find the sine values of various angles from 1° to 90°. How can we do this without having tables at hand? Quite simply: we need to create our own table of sines. That's exactly what we'll do now.

Let's start with angles whose sine values we know from geometry. Firstly, the angle of 90°, whose sine is obviously 1. Then the angle of 45°, the sine of which can be easily calculated using the Pythagorean theorem; it equals $\sqrt{2}/2$, which is approximately 0.707. Next, we know the sine of 30°; since the side opposite such an angle is half the hypotenuse, the sine of $30^\circ = 1/2$.

So, we know the sines (or, as commonly denoted, sin) of three angles:

$$\begin{aligned}\sin 30^\circ &= 0.5,\\ \sin 45^\circ &= 0.707, \quad \text{and}\\ \sin 90^\circ &= 1.\end{aligned}$$

Of course, this is not enough; we need to know the sines of all intermediate angles at least through every degree. For very small angles, we can approximate the sine by taking the ratio of the arc to the radius without much error: from Figure 88 (lower), shows that the ratio $\overline{BC}/AB$ differs little from the $\widehat{BD}/AB$ ratio. For example, for an angle of 1°, the arc $BD = 2\pi R/360$ and, therefore, sin 1° can be taken equal to

$$\frac{2\pi R}{360, R} = \frac{\pi}{180} = 0.0175.$$

Using the same method, we find:

$$\begin{aligned}\sin 2^\circ &= 0.0349,\\ \sin 3^\circ &= 0.0524,\\ \sin 4^\circ &= 0.0698,\\ \sin 5^\circ &= 0.0873.\end{aligned}$$

However, we need to determine how far we can extend this table without introducing significant errors. If we were to compute sin 30° using this method, we would obtain 0.524 instead of 0.500; the difference would already be in the second significant digit, and the margin of error would be 24/500, i.e. about 5%.

This is too crude even for basic field trigonometry. To find the limit to which we can calculate sine values using the approximate method described, let's try to find sin 15° using a more precise method. To do this, we'll employ the following relatively straightforward construction (Figure 89).

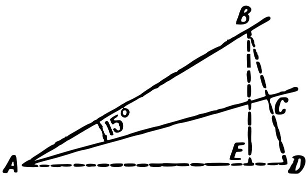


Figure 89: How to calculate sin 15°?

Let $\sin 15^\circ = BC/AB$. Let us extend BC by an equal distance to the point D; connect A with D, then we get two equal triangles: ADC and ABC, with the angle BAE equal to 30° degrees. We'll drop a perpendicular BE on AD from point B, creating a right triangle BAE with a 30° angle ($< BAE$), then $BE = AB/2$. Next, we'll calculate AE using the Pythagorean theorem in triangle ABE:

$$AE^2 = AB^2 - \left(\frac{AB}{2}\right)^2 = \frac{3}{4}AB^2,$$

$$AE = \frac{AB}{2}\sqrt{3} = 0.866\,AB.$$

Thus, $ED = AD - AE = AB - 0.866\,AB = 0.134\,AB$. Now, from triangle BED, we'll calculate BD:

$$BD^2 = BE^2 + ED^2 = \left(\frac{AB}{2}\right)^2 + (0.134\,AB)^2 = 0.268\,AB^2$$

$$BD = \sqrt{0.268\,AB^2} = 0.518\,AB.$$

Half of BD, i.e., BC, equals $0.259\,AB$, therefore the sine of angle BAD is the same as the sine of 15°.

$$\sin 15^\circ = \frac{BC}{AB} = \frac{0.259\,AB}{AB} = 0.259.$$

This is the tabular value of $\sin 15^\circ$ if we limit ourselves to three significant figures. The approximate value we would have found using the previous method is 0.262. Comparing the values 0.259 and 0.262, we see that by rounding to two significant figures, we obtain 0.26 and 0.26, which are identical results. The error in replacing the more accurate value (0.259) with the approximate one (0.26) is 1/1000 times the difference, i.e., about 0.4%. This margin of error is acceptable for field calculations, so we are justified in calculating the sines of angles from 1 to 15 degrees using our approximate method.

For the range from 15° to 30° degrees, we can compute the sines using proportions. We reason as follows: the difference between $\sin 30^\circ$ and $\sin 15^\circ$ is $0.50 - 0.26 = 0.24$. Therefore, we can assume that for each degree increase, the sine increases by approximately 1/15 of this difference, i.e., 0.24/15 = 0.016. Strictly speaking, this is not precisely accurate, but the deviation from the rule only becomes apparent in the third significant digit, which we discard anyway. Thus, by adding 0.016 successively to sin 15°, we get the sines of 16°, 17°, 18°, etc.:

$$\sin 16^\circ = 0.26 + 0.016 = 0.28,$$
$$\sin 17^\circ = 0.26 + 0.032 = 0.29,$$
$$\sin 18^\circ = 0.26 + 0.048 = 0.31,$$
$$\sin 25^\circ = 0.26 + 0.16 = 0.42, \text{ and so on.}$$

All these sines are accurate to the first two decimal places, which is sufficient for our purposes; they differ from the true sines by less than half the value of the last digit.

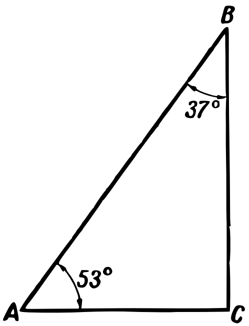


Figure 90: To calculate the sine of an angle greater than 45°.

Using the same method, we calculate angles within the range between 30° and 45° degrees. The difference between sin 45° − sin 30° = 0.707 − 0.5 = 0.207. Dividing this difference by 15, we get 0.014. We'll add this value successively to sin 30°, resulting in:

$$\sin 31^\circ = 0.5 + 0.014 = 0.51,$$
$$\sin 32^\circ = 0.5 + 0.028 = 0.52,$$
$$\ldots\ldots$$
$$\sin 40^\circ = 0.5 + 0.014 = 0.64, \text{ and so on.}$$

We also need to find the sines of acute angles larger than 45°. The Pythagorean theorem can help us with this. Let's say we want to find $\sin 53^\circ$ (Figure 90), which corresponds to the ratio of side BC/AB. Since angle $B = 37^\circ$, we can compute its sine using the method described earlier: it equals $0.5 + 7 \times 0.014 = 0.6$. On the other hand, since we know that $\sin B = AC/AB$, so $AC/AB = 0.6$ or $AC = 0.6\,AB$. Knowing AC, we can easily calculate AC. This segment equals $\sqrt{(AB^2 - BC^2)} = \sqrt{(AB^2 - (0.6\,AB)^2)} = 0.8\,AB$. Overall, the calculation is straightforward; we just need to be able to compute square roots.

Extraction of the Square Root

The method for extracting square roots mentioned in algebra courses is easily forgotten. But one can do without it. In many geometry textbooks, an ancient simplified method for computing square roots using the division method is presented. Here, I will share another old method, which is also more straightforward than the one considered in algebra courses.

Let's say we need to compute the square root of 13. It lies between the square roots of 9 and 16, therefore, it is equal to 3 plus a fraction denoted by x.

So,

$$\sqrt{13} = 3 + x,$$

hence,

$$13 = 9 + 6x + x^2.$$

Since the square of the fraction x is a small fraction that can be neglected in the first approximation, we have:

$$\begin{aligned} 13 &= 9 - 6x, \text{ this leads to,} \\ 6x &= 4, \text{ hence,} \\ x &= \frac{4}{6} = 0.67. \end{aligned}$$

Therefore, approximately, $\sqrt{13} = 3.67$. If we want to determine the value of the root more precisely, we can write the equation $\sqrt{13} = 3\,2/3 + +y$, where y is a small positive or negative fraction. From this equation, we have

$$13 = \frac{121}{9} + \frac{22}{3}y + y^2.$$

Neglecting y^2, we find that y is approximately equal to -0.06. Thus, in the second approximation, $\sqrt{13} = 3.67 - 0.06 = 3.61$.

The third approximation can be found using the same method, and so on. Using the conventional method taught in algebra courses, we would find the square root of 13 accurate to 0.01, also as 3.61.

To Find an Angle From its Sine

So, we have the ability to compute the sine of any angle from 0° to 90° with two decimal places. The need for a ready-made table is eliminated; for approximate calculations, we can always create it ourselves if desired.

But to solve trigonometric problems, one also needs to be able to compute angles from a given sine. This is also not difficult. Let's say we need to find the angle whose sine is 0.38. Since this sine is less than 0.5, the angle we seek is less than 30°. But it is greater than 15°, as the sine of 15°, we know, is 0.26. To find this angle, which lies between 15° and 30°, we proceed as explained on page 180:

$$\begin{aligned} 0.38 - 0.26 &= 0.12, \\ \frac{0.12}{0.016} &= 7.5^\circ, \\ 15^\circ + 7.5^\circ &= 22.5^\circ. \end{aligned}$$

So, the angle we seek is approximately 22.5°. Another example: find the angle whose sine is 0.62.

$$\begin{aligned} 0.62 - 0.50 &= 0.12, \\ \frac{0.12}{0.014} &= 8.6^\circ, \\ 30^\circ + 8.6^\circ &= 38.6^\circ. \end{aligned}$$

The angle we seek is approximately 38.65°.

Finally, the third example: find the angle whose sine is 0.91.

Since this sine lies between 0.71 and 1, the angle we seek lies between 45° and 90°. In Figure 91, BC is the sine of angle A, if $BA = 1$.

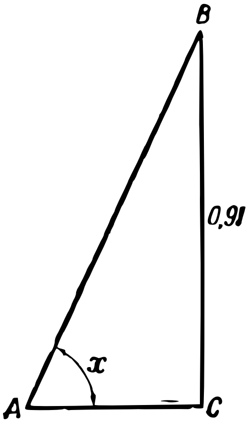


Figure 91: To calculate an acute angle by its sine.

Knowing BC, it is easy to find the sine of angle B: BC

$$AC^2 = 1 - BC^2 = 1 - 0.91^2,$$
$$AC^2 = 1 - 0.83 = 0.17,$$
$$AC = \sqrt{0.17} = 0.42.$$

Now let's find the value of angle B, whose sine is 0.42; after that, it will be easy to find angle A, which is equal to $90° - B$. Since 0.42 lies between 0.26 and 0.5, angle B lies between 15° and 30°. It is determined as follows:

$$0.42 - 0.26 = 0.16,$$
$$\frac{0.16}{0.016} = 10°,$$
$$B = 15° + 10° = 25°.$$

And that means the angle $A = 90° - B = 90° - 25° = 65°$.

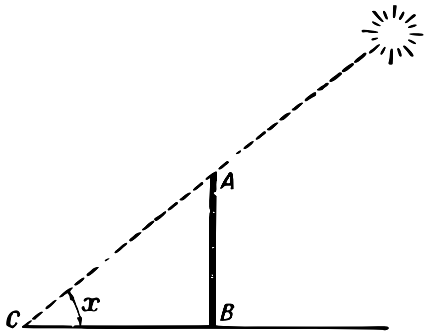


Figure 92: To determine the height of the Sun above the horizon.

We are now well-equipped to approximately solve trigonometric problems, as we can find sines from angles and angles from sines with an accuracy sufficient for outdoor purposes.

But is the sine alone sufficient for this? Will we not need the other trigonometric functions – cosine, tangent, etc.? We will now demonstrate with several examples that for our simplified trigonometry, one can easily manage with just the sine.

Height of the Sun Problem

Question The shadow *BC* (Figure 92) of the vertical pole *AB*, with a height of 4.2 m, measures 6.5 m. What is the height of the Sun above the horizon, i.e., what is the angle C?

Answer It is easy to see that the sine of angle C is equal to AB/AC. Since

$$\begin{aligned} AC &= \sqrt{AB^2 + BC^2}, \\ &= \sqrt{4.2^2 + 6.5^2}, \\ &= 7.74. \end{aligned}$$

Therefore, the sine we are seeking is approximately $4.2/7.74 = 0.55$. Using the method described earlier, we find the corresponding angle: 33°. The height of the Sun is approximately 33° with an accuracy of 0.5°.

Distance to the Island Problem

While wandering with a compass near the river, you noticed an island A on it (Figure 93) and wish to determine its distance from point B on the shore. To do this, you determine the angle ABN, formed with the north-south direction (NS), by the straight line BA using the compass. Then, you measure the straight line BC and determine the angle NBC between it and NS. Finally, you do the same at point C for the straight line AC. Let's assume that you obtained the following data:

The direction AB deviates from NS to the east by 52°. The direction BC deviates from NS to the east by 110°. The direction CA deviates from NS to the east by 27°.

The length of BC is 137 m. How can you calculate the distance BA using this data?

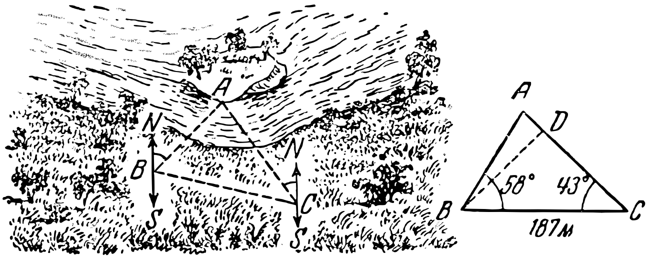


Figure 93: How to calculate the distance to the island?

Answer In triangle ABC, we know the side BC. Angle $ABC = 110° - 52° = 58°$; angle $ACB = 180° - 110° - 27° = -43°$. Let's consider the height BD in this triangle (Figure 98, to the right): $\sin C = \sin 43° = BD/187$. By calculating the $\sin 43°$ using the method mentioned earlier, we obtain approximately 0.68. Hence,

$$BD = 187 \times 0.68 = 127.$$

Now, in triangle ABD, we know the side BD; angle $A = 180° - (58° + 43°) = 79°$, and angle $ABD = 90°79° = 11°$. We can calculate the $\sin 11^\circ$: it is 0.19. Therefore, $AD/AB = 0.19$. Using the Pythagorean theorem,

$$AB^2 = BD^2 + AD^2.$$

Substituting 0.19 AB for AD and 127 for BD, we have:

$$AB^2 = 127^2 + (0.19AB)^2.$$

from which $AB \approx 128$. Thus, the distance to the island is approximately 128 m. The reader, I believe, would not have difficulty in calculating the direction AC if needed.

Lake Width Problem

To determine the width of lake AB (see Figure 94), you found from the compass that line AC veers westward by 21°, and BC veers eastward by 22°. With $BC = 68$ m and $AB = 35$ m, calculate the width of the lake.

Answer In triangle ABC, we know angle $ABC = 43^\circ$ and the lengths of its sides are 68 m and 35 m. Dropping a perpendicular from A to D (to the right), we get a right triangle ADC where the angle C is 43°. Hence $\sin 43^\circ = AD/AC$. Calculating, $\sin 43^\circ = 0.68$, hence $AD = \sin 43^\circ \times AC = 0.68 \times 35 = 24$. Next, we compute CD and BD:

$$CD^2 = AC^2 - AD^2 = 35^2 - 24^2 = 649; \therefore CD = 25.5;$$

$$BD = BC - CD = 68 - 25.5 = 42.5.$$

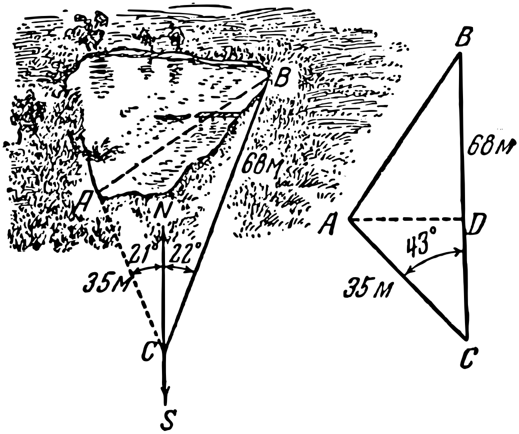


Figure 94: How to calculate the width of the lake?

Now, from triangle ABD,

$$AB^2 = AD^2 + BD^2 = 24^2 + 42.5^2,$$

$$AB^2 = 2380;$$

$$AB \approx 49.$$

Thus, the sought width of the lake is about 49 m.

If we had to calculate the other two angles in triangle ABC, having found $AB = 49$, we would proceed as follows:

$$\sin B = \frac{AD}{AB} = \frac{24}{49} = 0.49, \text{ hence}$$

$$B = 29^\circ.$$

The third angle C is found by subtracting the sum of angles 29° and 43° from 180°; it equals 108°.

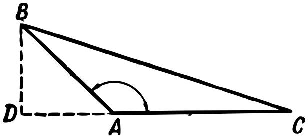


Figure 95: To the solution of the obtuse angle.

It may happen that in the case of solving a triangle (on two sides and the angle between them), this angle is not sharp, but obtuse. If, for example, an obtuse angle A and two sides, AB and AC, are known in the ABC triangle (Figure 95), then the calculation of the remaining elements is as follows. Lowering the height BD, determine BD and AD from the triangle BDA; then, knowing $DA + AC$, find BC and $\sin C$ by calculating the ratio BD/BC.

Triangle Area Problem

Question During an excursion, we measured the sides of a triangular area in steps and found them to be 43, 60, and 54 steps, respectively. What are the angles of this triangle?

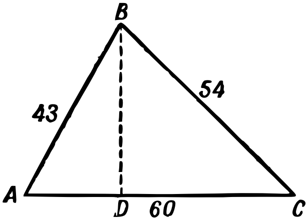


Figure 96: Find the angles of this triangle 1) by calculation, 2) using a protractor.

Answer This is the most complex case of solving a triangle: by three sides. However, it can be managed without resorting to other functions besides sine. Dropping BD (see Figure 96) the perpendicular from B to the longest side AC, we have:

$$BD^2 = 43^2 - AD^2, \quad BD^2 = 54^2 - DC^2,$$

hence,

$$
\begin{aligned}
43^2 - AD^2 &= 54^2 - DC^2, \\
DC^2 - AD^2 &= 54^2 - 43^2 = 1070. \\
\text{But } DC^2 - AD^2 &= (DC + AD)(DC - AD) = 60(DC - AD). \\
\therefore\ \ 60(DC - AD) &= 1070 \quad \text{and} \quad DC - AD = 17.8.
\end{aligned}
$$

From the equations $DC - AD = 17.8$ and $DC + AD = 60$, we get: $2DC = 77.8$, i.e. $DC = 38.9$. Now, it's easy to calculate the height:

$$BD = \sqrt{54^2 - 38.9^2} = 37.4,$$

from which we find:

$$
\begin{aligned}
\sin A &= \frac{BD}{AB} = \frac{37.4}{43} = 0.87; \quad A \approx 60^\circ, \\
\sin C &= \frac{BD}{BC} = \frac{37.4}{54} = 0.69; \quad A \approx 44^\circ.
\end{aligned}
$$

Then we can find angle $B = 180^\circ - (A + C) = 76^\circ$.

If we were to calculate this using tables, according to the rules of "real" trigonometry, we would get angles expressed in degrees and minutes. However, these minutes would be inherently erroneous, as the sides measured in steps involve an error of no less than 2–3%. Hence, to avoid self-deception, the obtained "accurate" angle values should be rounded to at least whole degrees. And then we would arrive at the same result as we did by resorting to simplified methods. The benefit of our "field" trigonometry is quite evident here.

Determining the Magnitude of an Angle Without Any Instruments

To measure angles on the ground, we need at least a compass, and sometimes just our own fingers or a matchbox. But there may be a need to measure an angle drawn on paper, on a plan, or on a map.

Of course, if a protractor is at hand, the problem is easily solved. But what if there is no protractor, for example, in field conditions? A geometer should not be at a loss in this case either. How would you solve the following problem:

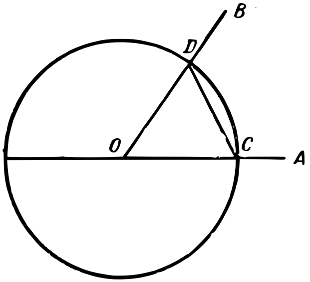


Figure 97: 97. How to determine the value of the depicted angle of the *AOB* using only a compass?

Question Angle *AOB* (see Figure 97) is depicted, measuring 180°. Determine its magnitude without measurements.

Answer One could drop a perpendicular from any point on side *BO* to side *AO*, measure the catheti and the hypotenuse in the resulting right-angled triangle, find the sine of the angle, and then the magnitude of the angle itself (see page 183). But such a solution would not meet the strict condition of not measuring anything!

Let's use the solution proposed in 1946 by J. Rupeika from Kaunas.

From vertex O, as from the centre, using the compass, let's draw a complete circle with any radius. Points C and D, the intersections of the circle with the sides of the angle, are then connected by a straight line.

Now, using only the compass, starting from the initial point C on the circle, we will sequentially lay off the chord CD in the same direction until the compass leg coincides again with the starting point C.

While laying off the chords, we must count how many times the circle is circumvented during this time and how many times the chord is laid off.

Let's assume that we circled the circle n times and during this time laid off the chord CD S times. Then the desired angle will be equal to

$$\angle AOB = \frac{360^\circ \cdot n}{S}.$$

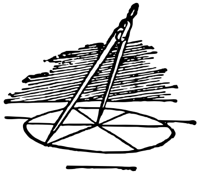
Indeed, let the given angle contain x°; by laying off the chord CD on the circle S times, we, so to speak, increased the angle x° by S times, but since the circle was circled n times in the process, this angle will be $360^\circ \times n$, i.e., $x^\circ \cdot S = 360^\circ \cdot n$;

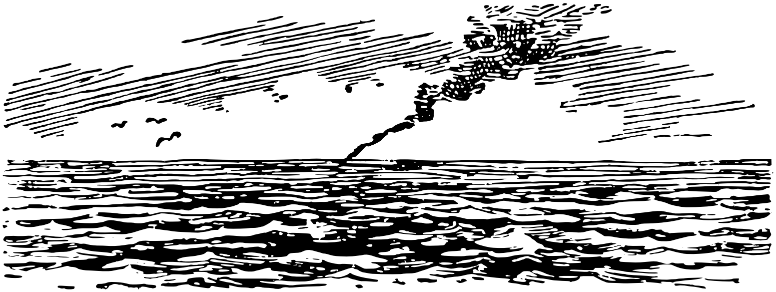
hence,

$$x^\circ = \frac{360 \cdot n}{S}.$$

For the angle depicted in the drawing, $n = 3$, $S = 20$ (check!); therefore, $\angle AOB = 54^\circ$. In the absence of a compass, a circle can be drawn using a pin and a strip of paper; the chord can also be laid off using the same paper strip.

Question Determine the angles of the triangle in Figure 96 using this method.





6. Where Heaven and Earth Converge

Horizon

In the countryside or on a level field, you see yourself at the centre of a circle that bounds the earth's surface visible to your eye. This is the *horizon.* The horizon line is elusive: as you approach it, it recedes from you. Yet, although inaccessible, it still exists in reality; it is not an optical illusion or a mirage. For every point of observation, there is a definite boundary of the earth's surface visible from it, and the distance to this boundary is easy to calculate.

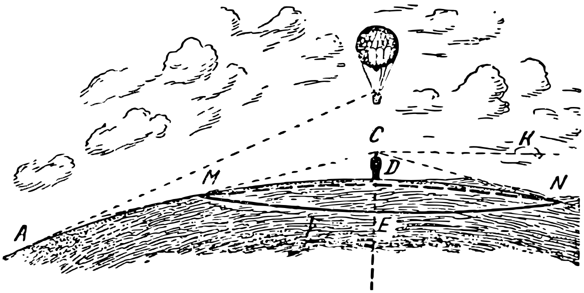


Figure 98: The horizon and its dependence on the height of the observation.

To understand the geometric relationships associated with the horizon, let's refer to Figure 98, depicting a portion of the earth. At point C, the observer's eye is placed at a height CD above the earth's surface. How far does this observer see around them on a level surface? Obviously, only to points M and N, where the line of sight touches the earth's surface: beyond this, the earth lies below the line of sight. These points M, N (and others lying on the circle MEN) represent the boundary of the visible part of the earth's surface, i.e. they form the horizon line. The observer must feel as though the sky meets the earth here because at these points, they simultaneously see both the sky and earthly objects.

Perhaps you might think that Figure 98 does not give an accurate depiction of reality: after all, in reality, the horizon always remains at eye level, whereas in the illustration, the circle clearly lies below the observer. Indeed, we always perceive the horizon line to be at the same level as our eyes and even rising with us as we ascend. But this is an optical illusion: in reality, the horizon line is always below the eye, as shown in Figure 98. However, the angle formed by the straight lines *CN* and *CM* with the line *CK*, perpendicular to the radius at point *C* (this angle is called the 'dip of the horizon'), is very small, and it is impossible to perceive it without instruments.

Incidentally, let's note another interesting circumstance. We just mentioned that when the observer is raised above the earth's surface, for example, in an aeroplane, the horizon line seems to remain at eye level, i.e., it appears to rise with the observer. If one ascends high enough, it will seem as if the ground beneath the aeroplane lies below the horizon line – in other words, the earth appears concave, with the horizon line serving as its edges. This is well described and explained by Edgar Allan Poe in the fantastical 'The Balloon-Hoax'.

"More than anything else," his aeronaut hero recounts, "I was astonished by the circumstance that the surface of the earth seemed concave. I expected to see it convex during

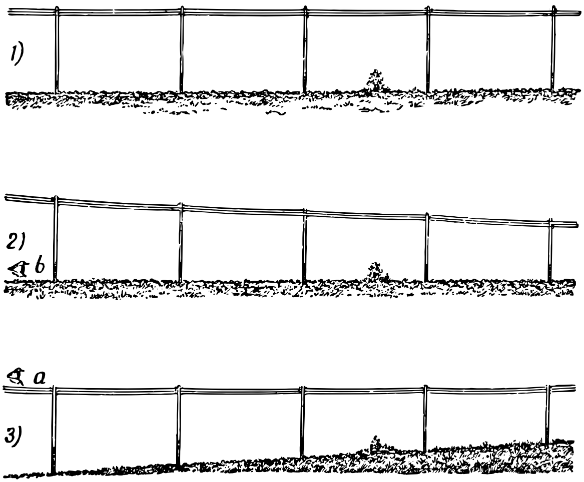


Figure 99: What does an eye observing the row of telegraph poles see?

ascent; only through reflection did I find an explanation for this phenomenon. A plumb-line drawn from my balloon to the earth would form the cathetus of a right triangle, the base of which would be the line from the base of the plumb to the horizon, and the hypotenuse would be the line from the horizon to my balloon. But my height was insignificant compared to the field of view; in other words, the base and hypotenuse of the imaginary right triangle were so great compared to the plumb-line cathetus that they could be considered almost parallel. Therefore, every point directly beneath the aeronaut always seems to lie below the horizon level. Hence the impression of concavity. And this should continue until the height of ascent becomes so significant that the base of the triangle and the hypotenuse cease to appear parallel."

In addition to this explanation, let's add the following example. Imagine a straight row of telegraph poles (Figure 99). For an eye placed at point *b*, at the level of the bases of the poles, the row appears as indicated by the number *2*. But for an eye at point *a*, at the level of the tops of the poles, the row appears as *3*, i.e., the ground seems to him as if rising at the horizon.

Ship on the Horizon

When we observe a ship from the shore of the sea or a large lake, emerging from below the horizon, it seems to us that we see the vessel not at the point (Figure 100) where it actually is, but much closer, at point *B*, where our line of sight glides over the convexity of the sea. When observed with the naked eye, it is difficult to shake the impression that the ship is at point *B*, rather than farther beyond the horizon (compare with what was said in the fourth chapter about the influence of a hill on judging distance).

However, through a telescope, this different distance of the

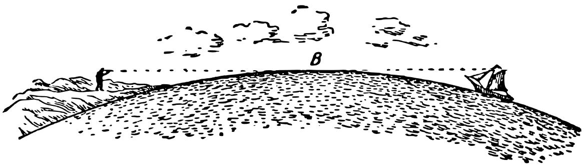


Figure 100: Ship beyond the horizon.

ship is perceived much more distinctly. The telescope does not show objects nearby and far away equally clearly: through a telescope aimed into the distance, nearby objects appear blurred, and conversely, a telescope aimed at nearby objects shows distant ones in a haze. Therefore, if one directs a telescope (with sufficient magnification) at the water's horizon and adjusts it so that the water surface is clearly visible, the ship will appear in blurred outlines, revealing its greater distance from observation (Figure 101). Conversely, setting the telescope to sharply show the outlines of the ship, partially hidden below the horizon, we will notice that the water surface at the horizon loses its previous clarity and appears as if in a haze (Figure 102).

The Distance to the Horizon

How far does the horizon line lie from the observer? In other words, what is the radius of the circle centred around ourselves on level ground? How to calculate the distance to the horizon, knowing the elevation of the observer above the

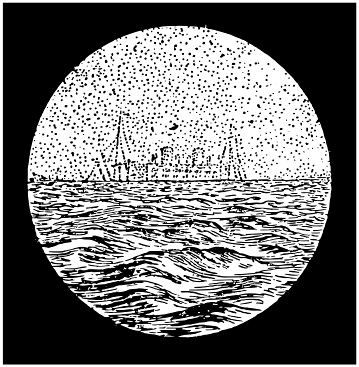


Figure 101: A ship over the horizon, viewed through a telescope with focus on water surface.

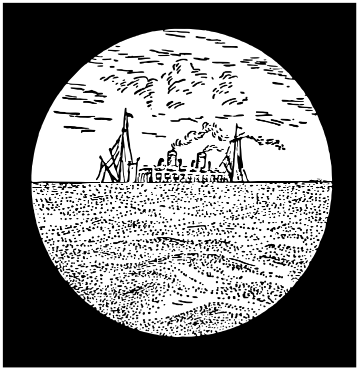


Figure 102: A ship over the horizon, viewed through a telescope with focus on the ship.

earth's surface?

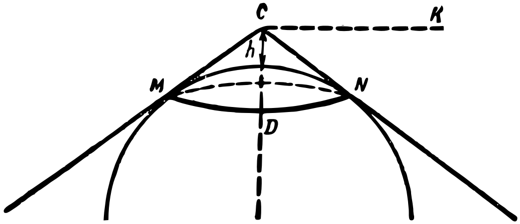


Figure 103: The problem of the distance to the horizon.

The problem boils down to calculating the length of segment CN (Figure 103), the tangent drawn from the observer's eye to the earth's surface. The square of the tangent – as we know from geometry – equals the product of the external segment h by the secant to the entire length of this secant, i.e., by $h + 2R$, where R is the radius of the earth. Since the elevation of the observer's eye above the ground is usually extremely small compared to the diameter ($2R$) of the earth, for example, for the highest altitude of an aeroplane, it is about 0.001 of its fraction, then $h + 2R$ can be taken as approximately 2R, and then the formula simplifies to:

$$CN^2 = h2R.$$

Therefore, the distance to the horizon can be calculated by a very simple formula:

$$\text{distance to the horizon} = \sqrt{2Rh},$$

where R is the radius of the earth (about 6,400 km[30]), and h is the elevation of the observer's eye above the earth's surface.

Since $\sqrt{6400} = 80$, the formula can be expressed as follows:

$$\text{distance to the horizon} = 80\sqrt{2h} = 113\sqrt{h},$$

where h should be expressed in kilometres.

This calculation is purely geometrical and simplified. If we wish to refine it considering physical factors influencing the distance to the horizon, we must take into account the so-called 'atmospheric refraction'. Refraction, i.e., the bending of light rays in the atmosphere, increases the distance to the horizon by approximately 1/15 of the calculated distance (by about 6%). The number 6% is just an average. The distance to the horizon is slightly increased or decreased depending on many conditions, namely:

Increases	**Decreases**
at high pressure,	at low pressure,
near the Earth's surface,	at heights,
in cold weather,	in warm weather,
in the morning and evening,	during the day,
in humid weather,	in dry weather,
over the sea,	over land.

[30] More precisely, 6371 km.

Question How far can a person standing on a plain see the earth?

Answer Assuming that the eyes of an adult are elevated by 1.6 meters, or 0.0016 km, we have:

$$\text{distance to the horizon} = 113 \times \sqrt{0.0016} = 4.52\,\text{km}.$$

The Earth's atmosphere, as mentioned earlier, bends the path of rays, causing the horizon to recede on average by 6%. Further than that distance obtained by the formula. To account for this correction, we need to multiply 4.52 km by 1.06; we get:

$$4.52 \times 1.06 = 4.8\,\text{km}.$$

Thus, a person of average height can see no further than 4.8 km on a level surface. The diameter of the circle he observes is only 9.6 km, and the area is 72 km^2. This is much less than what people usually think when they describe the distant expanse of the steppes, scanned by the eye.

Question How far can a person sitting in a boat see the sea?

Answer If we assume that the elevation of the eyes of a person sitting in a boat above the water level is 1 meter, or 0.001 km, then the distance to the horizon is approximately

$$113\sqrt{0.001} = 3.58\,\text{km},$$

or considering the average atmospheric refraction, about 3.8 km. Objects located farther away are visible only in their upper parts; their bases are hidden below the horizon.

At a lower position of the eye, the horizon narrows: for half a meter, for example, up to 2 km. On the contrary, when observed from elevated points (from the mast), the horizon range increases: for 4 m, for example, up to 7 km.

Question How far in all directions did the land stretch for the aviators observing from the gondola of the stratosphere balloon *SOAK-1* when it was at the highest point of its ascent?

Answer Since the balloon was at a height of 22 km, the horizon distance for such an elevation is:

$$113\sqrt{22} = 530\,\text{km},$$

and with refraction considered – 580 km.

Question How high should a pilot ascend to see a 50 km radius around themselves?

Answer From the horizon distance formula, in this case, we have the equation:

$$50 = 113\sqrt{2Rh}, \text{Hence, solving for } h,$$

$$h = \frac{50^2}{2R} = \frac{2500}{12800} = 0.2\,\text{km}.$$

So, it's enough to ascend only 200 m.

To account for the correction, subtracting 6% of 50 km, we get 47 km:

$$h = \frac{47^2}{2R} = \frac{2200}{12800} = 0.17\,\text{km}.$$

So, approximately 170 m (instead of 200).

The twenty-six-story building of the University (Figure 104) – the world's largest educational and scientific centre – is being constructed at the highest point of the Lenin Hills in Moscow.

It will rise 200 m above the level of the Moscow River.

Therefore, from the windows of the upper floors of the University, a panorama up to 50 km in radius will be visible.

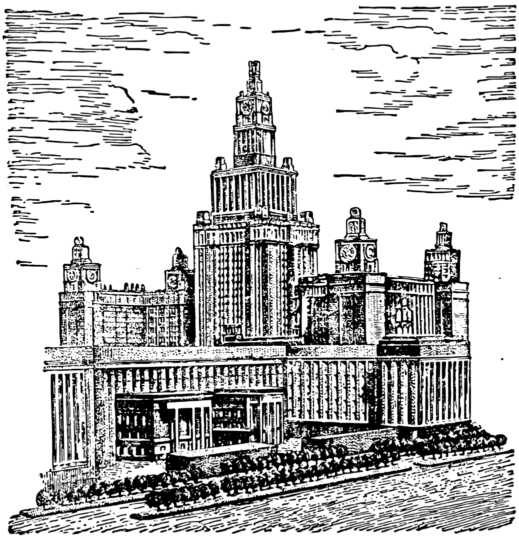


Figure 104: Moscow University (drawing from the project of the building under construction).

Gogol's Tower Problem

It's interesting to know what increases faster: the height of elevation or the horizon distance? Many people think that with the elevation of the observer, the horizon increases extraordinarily fast. This was also thought by Gogol, who wrote in his article *On the Architecture of Our Time*:

> Huge, colossal towers are necessary in the city… Here we usually limit them to a height that allows for viewing only the city, whereas for a capital, it is necessary to see, at least, a hundred and fifty versts[31] in all directions, and for this, perhaps just one or two extra floors – and everything changes. The volume of the horizon expands extraordinarily with elevation.

[31] 1 verst is 1.0668 km; 150 versts ~ 160 km.

Is this really the case?

Answer It's enough to look at the formula

$$\text{horizon distance} = \sqrt{2Rh},$$

to immediately understand the fallacy of the claim that the "volume of the horizon" increases very quickly with the elevation of the observer. On the contrary, the horizon distance grows slower than the height of elevation: it is proportional to the square root of the height. When the observer's elevation increases by 100 times, the horizon only moves 10 times further away; when the height becomes 1000 times larger, the horizon only moves 31 times further away. Therefore, it is erroneous to assert that "just one or two extra floors – and everything changes". If two more floors are added to an eight-story building, the horizon distance will increase by $\sqrt{10/8}$, i.e., by 1.1 times – only by 10%. Such an addition is hardly noticeable.

As for the idea of constructing a tower from which one could see, "at least, a hundred and fifty versts," i.e., 160 km, it is completely unattainable. Gogol, of course, did not realize that such a tower would have to have an enormous height.

Indeed, from the equation:

$$160 = \sqrt{2Rh}, \text{ we get,}$$

$$h = \frac{160^2}{2R} = \frac{25600}{12800} = 3{,}268\,\text{m}.$$

This is the height of a large mountain. The highest of the currently designed buildings in the capital of our Homeland is an 32-story administrative building, the gilded spire of which is planned to rise 280 meters from the base of the building – seven times lower than the heights envisioned by Gogol.[32]

[32] As of 2024, *Burj Khalifa* in Dubai is the tallest building in the world at 828 m. The horizon visible from top of this building is equal to $\sqrt{12800 \times 0.828} \approx 93.6$ km. – DM

Pushkin's Hill

Pushkin makes a similar mistake when speaking in *The Covetous Knigh* about the distant horizon opening up from the top of the "proud hill":

> And the king could joyfully behold from the height
> Both the valley covered with white tents, And the sea,
> where the ships were running...

We have already seen how modest the height of this "proud" hill was: even the hordes of Attila could not erect a hill higher than 4.5 meters by this method. Now we can complete the calculations by determining how much this hill expanded the horizon of the observer placed on its peak. The eye of such a spectator would be elevated above the ground by $4.5+1.5 = 6$ meters, and therefore, the horizon distance would be

$$\sqrt{2 \times 6400 \times 0.006} = 8.8\,\text{km}.$$

This is only 4 km more than what can be seen from standing on flat ground.

Where the Rails Meet

Question Certainly, you have noticed how the railway tracks narrowing into the distance. But have you ever seen the point where both rails finally meet each other? And is it possible to see such a point? You now have enough knowledge to solve this problem.

Answer Let's remember that each object turns into a point for the normal eye when it is seen at an angle of $1'$, i.e., when it is at a distance of 1/3400 of its diameter. The width of the railway track is 1.52 m. Therefore, the gap between the rails should merge into a point at a distance of $1.52 \times 3400 = 5.2$ km. So, if we could follow the rails for 5.2 km, we would see how they both converge at one point. But on flat ground, the horizon lies closer than 5.2 km, precisely at a distance of only 4.4 km. Therefore, a person with normal vision standing on flat ground cannot see the point where the rails meet. They could only observe it under one of the following conditions:

1. if their visual acuity is reduced so that objects merge into a point for them at an angle of more than $1'$;
2. if the railway track is not horizontal;
3. if the observer's eye is elevated above the ground by more than $5.2^2/2R = 27/12800 = 0.0021$km, i.e., 210 cm.

Lighthouse Problems

Question There is a lighthouse on the shore, the top of which rises 40 meters above the water's surface.

From what distance will this lighthouse be visible to a sailor-observer ("mastman") on the "mast" of the ship at a height of 10 meters above the water's surface?

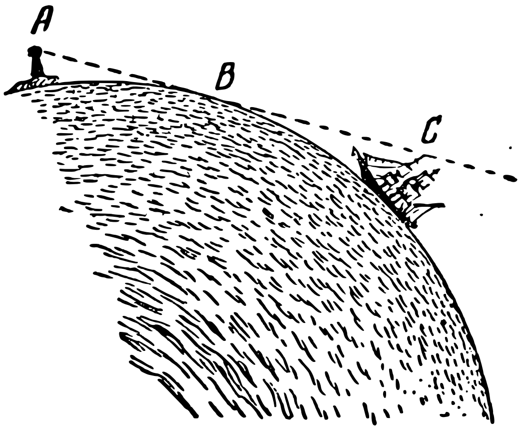


Figure 105: Lighthouse problems.

Answer From Figure 105, it can be seen that the problem reduces to calculating the length of the straight line AC, composed of two parts, AB and BC.

Part AB is the horizon distance of the lighthouse at a height above the ground of 40 meters, and BC is the horizon distance of the "mastman" at a height of 10 meters. Therefore, the required distance is

$$113\sqrt{0.04} + 113\sqrt{0.01} = 113\sqrt{0.2 + 0.01} = 34\,\text{km}.$$

Question What part of this lighthouse will be visible to the same “mastman” from a distance of 30 km?

Answer From Figure 105, the solution of the problem is clear: first, it is necessary to calculate the length of BC, then subtract the obtained result from the total length of AC, i.e., from 30 km, to find the distance AB. Knowing AB, we will calculate the height from which the horizon distance is equal to AB. Let’s perform all these calculations:

$$BC = 113\sqrt{0.01} - 11.3\,\text{km};$$

$$30 - 11.3 = 18.7\,\text{km}$$

$$\text{height} = \frac{18.7^2}{2R} = \frac{350}{12800} = 0.027\,\text{km}.$$

So, from a distance of 30 km, 27 m of the lighthouse’s height remains invisible; only 13 m is visible.

Lightning Problem

Question A lightning struck above your head at a height of 1.5 km. At what distance from your location could the lightning still be visible?

Solution: We need to calculate (see Figure 106) the horizon distance for a height of 1.5 km. It equals

$$113\sqrt{1.5} = 138\,\text{km}.$$

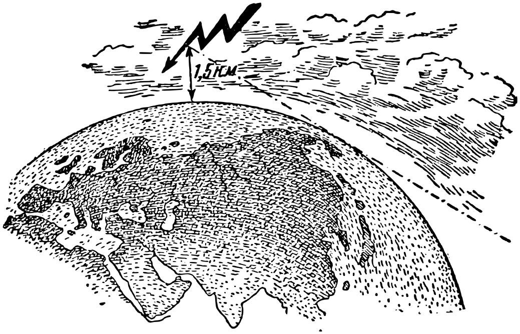


Figure 106: Lightning problem.

Thus, if the terrain is flat, the lightning would have been visible to a person whose eyes are at ground level at a distance of 138 km (with a 6% correction, it would be 146 km). At points 146 km away, it would have been visible right at the horizon, and since sound does not carry over such distances, it would have been observed here as sheet lightning – lightning without thunder.

Sailboat Problem

Question You are standing on the shore of a lake or sea, right by the water, observing a sailboat sailing away from you. You know that the top of the mast rises 6 m above sea level. At what distance from you will the sailboat start to appear to descend into the water (i.e., beyond the horizon), and at what distance will it disappear completely?

Answer The sailboat will start to disappear below the horizon (see Figure 100) at point B – at the horizon distance for a person of average height, i.e., 4.4 km. It will completely disappear below the horizon at a point where the distance from B equals

$$113\sqrt{0.006} = 8.7\,\text{km}.$$

Thus, the sailboat will disappear below the horizon at a distance from the shore of

$$4.4 + 8.7 = 13.1\,\text{km}.$$

Horizon on the Moon

Question So far, all our calculations have related to the Earth's sphere. But how would the horizon distance change if the observer were on another planet, for example, on one of the plains of the Moon?

Answer The problem is solved using the same formula; the horizon distance equals $\sqrt{2Rh}$, but in this case, instead of $2R$, we need to substitute the diameter of the Moon's sphere. Since the diameter of the Moon is 3,500 km, then at an eye elevation of 1.5 m above the ground, we have:

$$\text{horizon distance} = \sqrt{3500 \times 0.0015} = 2.3\,\text{km}.$$

On the lunar plain, we would see into the distance only up to 2.3 km.

In the Lunar Crater

Question Observing the Moon through even modest-sized telescopes, we see numerous so-called ring mountains – formations that do not exist on Earth. One of the greatest ring mountains – " Copernicus Crater" – has an outer diameter of 124 km and an inner diameter of 90 km. The highest points of the ring wall rise above the inner crater floor by 1,500 m. But if you were in the middle part of the inner crater, would you see this ring wall from there?

Answer To answer the question, we need to calculate the horizon distance for the top of the wall, i.e., for a height of 1.5 km. It equals on the moon to $\sqrt{3500 \times 1.5} = 23$ km. Adding the horizon distance for an average-height person, we get the distance at which the ring wall is hidden below the observer's horizon:

$$23 + 2.3 \approx 25\,\text{km}.$$

And since the centre of the wall is 45 km away from its edges, it is impossible to see this wall from the centre −unless you climb the slopes of the central mountains, rising on the floor of this crater to a height of 600 m.[33]

[33] See the book by the same author *Astronomy for Entertainment*, Chapter 3, article *Lunar Landscapes*.

On Jupiter

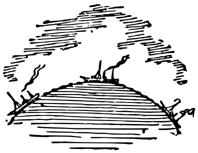
Question How far is the horizon on Jupiter, whose diameter is 11 times greater than Earth's?

Answer If Jupiter is covered by a solid crust and has a flat surface, then a person transported to its plain could see into the distance for:

$$\sqrt{11 \times 12800 \times 0.0016} = 14.4\,\text{km}.$$

For Independent Exercises

1. Calculate the horizon distance for the periscope of a submarine, raised above the calm surface of the sea by 30 cm.
2. How high should a pilot fly over Lake Ladoga to see both shores, separated by a distance of 210 km?
3. How high should a pilot fly between Leningrad and Moscow to see both cities at once? The distance between Leningrad and Moscow is 640 km.



7. The Geometry Of The Robinsons

(A few pages from Jules Verne)

The Geometry of the Starry Sky

The abyss has opened, full of stars;
To the stars, there is no count, to the abyss, no bottom.

-Lomonosov

There was a time when the author of this book was preparing himself for a somewhat unusual future: a career as a shipwrecked man. In short, I was thinking of becoming a Robinson Crusoe. If that had come to pass, the present book might have been more interestingly compiled than it is now, but perhaps it would not have been written at all. I didn't have to become Robinson, which I don't regret now. However, in my youth, I fervently believed in my calling as Robinson and prepared for it quite seriously. After all, even the most mediocre Robinson must possess many different types of knowledge and skills not required by people of other professions.

What, above all, does a person cast away by shipwreck on a deserted island have to do? Of course, determine the geographical position of their involuntary abode—latitude and longitude. Unfortunately, this is mentioned only briefly in most stories of old and new Robinsons. In the complete edition of the original 'Robinson Crusoe,' you'll find just one line about it, and that's in parentheses:

"In the latitude of $9^\circ 22'$ minutes north of the equator..."

This frustrating brevity drove me to despair when I was stocking up on the information necessary for my imagined future. I was ready to give up on the career of being the sole inhabitant of a deserted island when the secret was revealed to me in Jules Verne's *The Mysterious Island*.

I don't want to disappoint my readers in Robinsons, but I still think it's worth pausing here to discuss the simplest methods of determining geographical latitude. This skill may come in handy not only for the inhabitants of unknown islands. We still have so many inhabited places not marked on maps (and is there always a detailed map at hand?), that the task of determining geographical latitude may arise for many of my readers. True, we cannot assert, like Lermontov once did, that even: "Tambov is not always marked on the general map by the Circle"; but many villages and collective farms are still not marked on the general maps even today. There's no need to embark on maritime adventures to find yourself in the role of Robinson, for the first time determining the geographical position of your whereabouts.

The matter at hand is fundamentally relatively straightforward. Observing the clear starry sky at night, you will notice that the stars slowly describe inclined circles on the celestial sphere, as if the entire dome of the sky were smoothly rotating around an obliquely positioned invisible axis. In reality, however, you yourself, rotating together with the Earth, describe circles around its axis in the opposite direction. The only point of the starry dome in our northern hemisphere that remains stationary is the one where the imaginary extension of the Earth's axis meets, which is the northern "pole of the world" not far from the bright star at the end of the tail of the Little Dipper – the Pole Star. By locating it in our northern sky, we thereby find the position of the northern pole of the world. Finding it is not difficult if you first locate the position of the well-known constellation Ursa Major: draw a straight line through its extreme stars, as shown in Figure 107, and, continuing it for a distance approximately equal to the length of the entire constellation, you will arrive at the Pole Star.

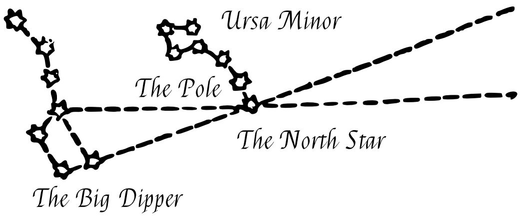


Figure 107: The search for the North Star.

This is one of those points on the celestial sphere that we will need to determine geographical latitude. The second is the so-called "zenith", which is the point in the sky directly above your head. In other words, the zenith is the point in the sky where the imaginary extension of the radius of the Earth, which passes through your current location, meets. The degree distance along the celestial arc between your zenith and the Pole Star is at the same time the degree distance of your location from the Earth's pole. If your zenith is 30° away from the Pole Star, then you are 30° away from the Earth's pole, and therefore 60° away from the equator; in other words, you are located on the 60th parallel.

Therefore, to find the latitude of any location, you simply need to measure in degrees (and its fractions) the "zenith distance" to the Pole Star; after this, subtract this value from 90° – and the latitude is determined. Alternatively, you can practically approach it differently. Since the arc between the zenith and the horizon contains 90°, subtracting the zenith distance to the Pole Star from 90° gives us nothing else but the length of the celestial arc from the Pole Star to the horizon; in other words, we obtain the "altitude" of the Pole Star above the horizon. Therefore, the geographical latitude of any location is equal to the height of the Pole Star above the horizon of that location.

Now you understand what needs to be done to determine the latitude. Waiting for a clear night, you locate the Pole Star in the sky and measure its angular height above the horizon; the result immediately gives you the sought latitude of your location. If you want to be precise, you should take into account that the Pole Star does not strictly coincide with the pole of the world but is offset from it by 1.25°. Therefore, the Pole Star does not remain completely stationary: it describes a small circle around the stationary celestial pole, moving higher and lower, to the right and left, by 1.25°. By determining the height of the Pole Star in its highest and lowest positions (an astronomer would say: at its upper and lower "culminations"), you take the average of both measurements. This is the true height of the pole, and consequently, the latitude of the location sought.

But if that's the case, there's no need to choose the Pole Star specifically; you can stop at any non-setting star and, by measuring its height in both extreme positions above the horizon, take the average of these measurements. As a result, you will obtain the height of the pole above the horizon, i.e., the latitude of the location. However, you must be able to catch the moments of the highest and lowest positions of the chosen star, which complicates matters; and it's not always possible to observe this within one night. That's why for the initial approximate measurements, it's better to work with the Pole Star, disregarding its slight deviation from the pole.

Up to now, we've imagined ourselves in the northern hemisphere. How would you proceed if you found yourself in the southern hemisphere? Exactly the same way, with just one difference: here, you need to determine the height of the southern pole instead of the northern pole. Unfortunately, near this pole, there is no bright star like the Pole Star in our hemisphere. The famous Southern Cross shines quite far from the southern pole, and if we want to use the stars of this constellation to determine the latitude, we will have to take the average of two measurements – at the highest and lowest positions of the star.

The heroes of Jules Verne's novel, *The Mysterious Island*, used precisely this beautiful constellation of the southern sky to determine the latitude of their "mysterious island".

It's instructive to reread the passage in the novel where the whole procedure is described. At the same time, we'll learn how the new Robinsons coped with their task without having a sextant.

The Latitude of the 'Mysterious Island'

It was 8 o'clock in the evening. The moon had not yet risen, but the horizon was already shimmering with delicate pale shades, which could be called lunar dawn. In the zenith, the constellations of the southern hemisphere sparkled, and among them was the Southern Cross. Engineer Smith observed this constellation for some time.

"Herbert," he said after some thought, "is today April 15th?"

"Yes," the youth replied.

"If I'm not mistaken, tomorrow is one of those four days in the year when true time equals mean time: tomorrow, the Sun will cross the meridian exactly at noon according to our clocks.[34] If the weather is clear, I'll be able to approximately determine the longitude of the island."

[34] Our clocks do not strictly correspond to solar time: there is a discrepancy between "true solar time" and the "mean time" shown by accurate clocks, which equals zero only four days a year: around April 16th, June 14th, September 1st, and December 24th. (See *Astronomy for Entertainment* by Ya. I. Perelman.)

"Without instruments?"

"Yes. The evening is clear, so today I will try to determine the latitude of our island by measuring the height of the stars of the Southern Cross, i.e., the height of the southern pole above the horizon. And tomorrow at noon, I will determine the longitude of the island."

"If the engineer had had a sextant – an instrument allowing precise measurement of angular distances between objects using the reflection of light rays – the task would have been no trouble at all. Having determined the height of the pole that evening, and the next day at noon – the moment when the Sun passed through the meridian, he would have obtained the geographic coordinates of the island: latitude and longitude. But there was no sextant, so it had to be replaced."

"The engineer entered the cave. By the light of the campfire, he cut out two rectangular planks, which he connected at one end in the shape of a compass, so that the legs could be shifted and spread apart. For the hinge, he used a strong acacia thorn found among the debris by the campfire."

"When the instrument was ready, the engineer returned to the shore. He needed to measure the height of the pole above the horizon, clearly defined, i.e., above sea level. For his observations, he went to the platform of the Distant View – taking into account also the height of the platform above sea level. This last measurement could be done on another day using elementary geometry techniques."

"The horizon, illuminated from below by the first rays of the moon, was sharply outlined, providing all the conveniences for observation."

"The constellation of the Southern Cross shone in the sky upside down: the star alpha, indicating its base, lies closer to the South Pole (of the world)."

"This constellation is not located as close to the South Pole as the Pole Star is to the North. The star alpha is 279° away from the pole; the engineer knew this and planned to incorporate this distance into his calculations. He awaited the moment when the star passed through the meridian – which facilitates the operation."

"Smith directed one leg of his wooden compass horizontally, the other towards the star alpha of the Cross, and the hole formed by the angle gave the angular height of the star above the horizon. To fix this angle reliably, he nailed a third plank across both planks using acacia thorns, so that the figure maintained its unchanged form."

"All that remained was to determine the magnitude of the angle obtained, relating the observation to sea level, i.e., taking into account the lowering of the horizon, for which it was necessary to measure the height of the cliff.[35] The magnitude of the angle would give the height of the star alpha Crucis, and therefore the height of the pole above the horizon, i.e., the geographical latitude of the island, since the latitude of any place on Earth is equal to the height of the pole above the horizon of that place. These calculations were planned to be carried out the next day."

[35] Since the measurement was taken by the engineer not at sea level but from a high cliff, the line of sight from the observer's eye to the edge of the horizon did not strictly coincide with the perpendicular to the Earth's radius but formed some angle with it. However, this angle was so small that it could be safely neglected for this case (at a height of 100 meters, it barely constitutes a third of a degree; therefore, there was no need for Smith, or rather Jules Verne, to complicate the calculation by introducing this correction.)

How the measurement of the height of the cliff was done, my readers already know from the excerpt provided in the first chapter of this book. Skipping this part of the novel here, let's follow the further work of the engineer:

"The engineer took the compass, which he had constructed the day before and with which he had determined the angular distance between the star alpha of the Southern Cross and the horizon. He carefully measured the magnitude of this angle using a circle divided into 360 parts and found that it was equal to 10°. Hence, the height of the pole above the horizon – after adding 10° to the 27° that separate the named star from the pole, and bringing the height of the cliff, from the top of which the measurement was made, to sea level – was found to be 37°. Smith concluded that the island of Lincoln was located at 37° South latitude, or – considering the imperfection of the measurement — between the 35th and 40th parallels."

All that remained was to find out its longitude. The engineer planned to determine it on the same day, at noon, when the Sun would pass through the meridian of the island.

Determining Geographic Longitude

"But how will the engineer determine the moment when the Sun crosses the meridian of the island, without having any instruments for it? This question greatly puzzled Herbert."

"The engineer arranged everything necessary for his astronomical observation. He chose a perfectly clear place on the sandy shore, levelled by the sea tide. A six-foot pole driven into this place was perpendicular to this platform."

"Herbert then understood how the engineer intended to act to determine the moment of the Sun's passage through the meridian of the island, or, in other words, to determine the local noon. He wanted to determine it by observing the shadow cast by the pole on the sand. This method, of course, is not very accurate, but in the absence of instruments, it still gave quite satisfactory results.

“The moment when the shadow of the object becomes the shortest will be noon. It is enough to carefully monitor the movement of the shadow’s tip to notice the moment when the shadow, having stopped decreasing, begins to lengthen again. In this case, the shadow played the role of a clock hand on a dial.”

“When, according to the engineer’s calculation, it was time for observation, he knelt down and, driving small stakes into the sand, began to mark the gradual shortening of the shadow cast by the pole.”

“The journalist (one of the engineer’s companions) held his chronometer in his hand, preparing to notice the moment when the shadow became the shortest. Since the engineer conducted the observation on April 16, one of those days when true noon coincides with mean noon, the moment observed by the journalist on his chronometer would be established by the time of the Washington meridian (the departure point for the travelers).”

The Sun moved slowly. The shadow gradually shortened. Finally noticing that it began to lengthen, the engineer asked:

‘What time is it?’

‘Five hours and one minute,’ replied the journalist.

“The observation was completed. Only a simple calculation remained to be done.”

“The observation established that the time difference between the Washington meridian and the meridian of Lincoln Island was almost exactly 5 hours. This means that when it is noon on the island, it is already 5 p.m. in Washington. The Sun in its apparent daily motion around the globe covers 1° in 4 minutes, and in an hour, 15°. Thus in one hour it equals $4 \times 15^\circ = 75^\circ$ minutes”

“Washington lies on the meridian $77^\circ 53' 11''$ west of the Greenwich meridian, accepted by both Americans and the English as the initial one. This means that the island lay approximately on the 152nd west longitude.”

Taking into account the insufficient accuracy of the observations, it could be asserted that the island lies between the 35th and 40th parallels of south latitude and between the 150th and 155th meridians west of Greenwich.

In conclusion, it should be noted that there are several quite diverse methods for determining geographic longitude. The method used by the characters of Jules Verne is just one of them (known as the ‘method of carrying chronometers’).

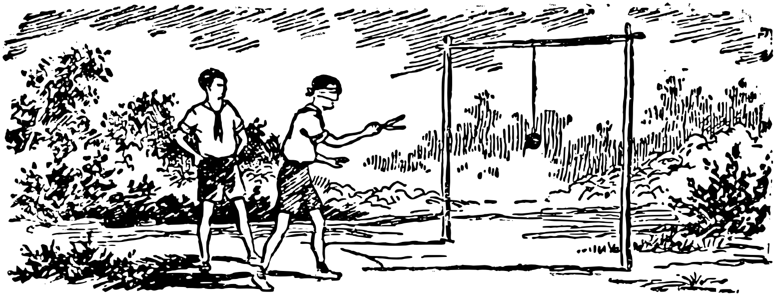
Similarly, there are other, more accurate methods for determining latitude than the one described here (not suitable for navigation, for example).

Part II.

Between Seriousness And Joke In Geometry

The subject of mathematics is so serious that it is useful to seize the opportunity to make it a little entertaining.

Blaise Pascal



8. Geometry In The Dark

Into the Depths of the Hold

Let's transport ourselves from the open air of fields and sea to the cramped and dark hold of an old ship, where a young hero in one of Mayne Reid's novels successfully solved a geometric problem in an environment where, perhaps, none of our readers have ever had to do mathematics. In the novel *The Boy Sailor* (or *In the Depths of the Hold*), Mayne Reid tells the story of a young lover of sea adventures (Fig. 108), who, having no means to pay for his passage, sneaked into the hold of an unfamiliar ship and there unexpectedly found himself sealed in for the duration of the sea voyage. Rummaging through the luggage that filled his dungeon, he came across a crate of crackers and a barrel of water. The sensible boy understood that this limited supply of food and drink had to be used as sparingly as possible, and therefore decided to divide it into daily portions.

Counting the crackers was not difficult, but how to establish the portions of water without knowing its total supply? This was the problem facing Mayne Reid's young hero. Let's see how he coped with it.

Measuring the Barrel

"I needed to establish a daily water ration for myself. To do this, I needed to find out how much water was in the barrel and then divide it into portions."

"Fortunately, in the village school, the teacher taught us some basic information from geometry in arithmetic lessons: I had the concept of a cube, a pyramid, a cylinder, a sphere; I also knew that a barrel can be considered as two truncated cones stacked with their large bases."

"To determine the capacity of my barrel, it was necessary to know its height (or, essentially, half of this height), then the circumference of one of the ends and the circumference of the midsection, i.e., the widest part of the barrel. Knowing these three quantities, I could accurately determine how many cubic units are contained in the barrel."

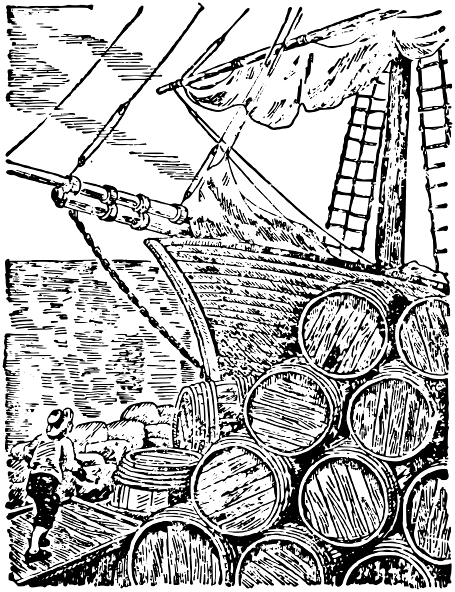


Figure 108: The young adventurer from the novel by Mein-Reid.

"I needed to establish my daily water ration. To do this, I needed to find out how much water was in the barrel and then divide it into portions."

"All I had to do was measure these quantities — but that's where all the difficulty lay."

"How to perform these measurements?"

"Finding out the height of the barrel was not difficult: it was right in front of me; but as for the circumferences, I couldn't reach them. I was too short to reach the top; besides, there were boxes standing on the sides."

"There was another difficulty: I had neither a scale nor a string that could be used for measurement; how could I determine the quantities if I had no measure? However, I decided not to give up my plan until I thought it through from all sides."

Measuring Ruler (The Mayne-Reid Problem)

"While contemplating the barrel, determined to measure it, I suddenly realised what I was lacking. A rod of sufficient length would assist me, one that could pass through the barrel at its widest point. If I inserted the rod into the barrel and pressed it against the opposite wall, I would know the diameter. Then, all I needed to do was triple the length of the rod to obtain the circumference. It's not perfectly precise, but it's sufficient for practical measurements. And since the hole I had previously made in the barrel was in its widest part, by inserting the rod into it, I would have the diameter I needed."

"But where to find such a rod? It was not difficult. I decided to use a plank from a box of biscuits and immediately got to work. Indeed, the plank was only 60 cm long, while the barrel was more than twice as wide. But this posed no problem; I just needed to prepare three sticks and tie them together to obtain a rod of sufficient length."

"Cutting the plank along the grain, I prepared three well-rounded and smoothed sticks. How to bind them together? I used shoelaces from my shoes, each nearly a meter long. Tying the sticks together, I obtained a plank of sufficient length – about one and a half meters."

"I began to measure, but encountered a new obstacle. It was impossible to insert my rod into the 60-cm hole: the space was too tight. I couldn't bend the rod either – it would likely break."

"Soon I devised a way to insert my measuring rod into the barrel: I disassembled it into parts, inserted the first part, and then tied the second part to its protruding end; then, after pushing through the second part, I tied the third."

"I positioned the rod so that it pressed against the opposite wall directly across from the hole and made a mark on it level with the surface of the barrel. Subtracting the thickness of the walls, I obtained the measurement necessary for my calculations."

"I pulled out the rod in the same order as I inserted it, being careful to mark the places where the individual parts were tied together, so that later I could restore the rod to the same length it had in the barrel. A small error could result in a significant deviation in the final result."

"So I had the diameter of the bottom base of the truncated cone. Now I needed to find the diameter of the barrel's top, which served as the upper base of the cone. I placed the rod on the barrel and pressed it against the opposite point of the edge, marking the diameter. It took no more than a minute."

"All that remained was to measure the height of the barrel. You might say I could simply place a stick vertically next to the barrel and mark the height on it. But my space was quite dark, and when I placed the stick vertically, I couldn't see where the upper bottom of the barrel reached. I could only rely on touch. I would have to feel the barrel with my hands and the corresponding spot on the stick. Additionally, the stick, while rotating around the barrel, could tilt, giving me an incorrect measurement for the height."

"After giving it some thought, I found a way to overcome this difficulty. I tied together only two planks, and I placed the third one on the top of the barrel so that it protruded beyond its edge by 30-40 cm; then I leaned a long rod against it to form a right angle with it and, consequently, be parallel to the height of the barrel. Making a mark at the point of the barrel that protruded the most, i.e., in the middle, and subtracting the thickness of the bottom, I thus obtained half of the barrel's height, or – the same thing – the height of one truncated cone."

"Now I had all the necessary data to solve the problem."

What Needed to be Done

"Expressing the volume of the barrel in cubic units and then converting it into gallons[36] was a simple arithmetic calculation, which I could easily handle. True, I didn't have any writing materials for calculations, but they would have been useless anyway since I was in complete darkness. I often had to perform all four arithmetic operations mentally without pen and paper. Now I had to deal with not very large numbers, and the task didn't bother me at all."

[36] A gallon is a measure of capacity. An English gallon contains 277 cubic inches (about 4.5 litres). There are 4 "quarts" in a gallon; 2 "pints" in a quart.

“But I encountered a new difficulty. I had three pieces of data: the height and both bases of the truncated cone; but what were the numerical values of these data? Before computing, I needed to express these values in numbers.”

“At first, this obstacle seemed insurmountable to me. Since I didn’t have any feet, meters, or any measuring ruler, I thought I had to give up on solving the problem.”

“However, I remembered that in the port, I measured my height, which turned out to be four feet. How could this information be useful to me now? Quite simply: I could lay out four feet on my rod and use that as a basis for my calculations.”

“To mark my height, I stretched out on the floor, then placed the rod on myself so that one end touched my feet, and the other lay on my forehead. I held the rod with one hand and, with the other hand free, marked the spot on it opposite to where my temple was.”

“Further —— new difficulties. A rod equal to 4 feet is useless for measurement if it’s not marked with small divisions – inches. It’s easy in theory to divide 4 feet into 48 parts (inches) and mark these divisions on the ruler. In practise, however, especially in the darkness I was in, it wasn’t so easy and straightforward.”

"How to find the midpoint of these 4 feet on the rod? How to divide each half of the rod in half again, and then each of the feet into 12 inches, all precisely equal to each other?"

"I started by preparing a stick slightly longer than 2 feet. Comparing it with the rod, where 4 feet were marked, I found that the inch-length of the stick was slightly longer than 4 feet. Shortening the stick and repeating the operation several times, on the fifth attempt, I reached the point where the double length of the stick was exactly 4 feet."

"This took a lot of time. But time was something I had plenty of: I was even glad to have something to fill it with."

"However, I figured out how to streamline the process by replacing the stick with a string that could easily be folded in half. The shoelaces from my shoes came in handy for this. Tying them with a secure knot, I got to work — and soon I could cut a piece exactly 1 foot long. Folding it in half was easy. Folding it in thirds was harder. But I managed, and soon I had three pieces, each 4 inches long, in my hands. All that remained was to fold them in half again, and once more, to get a piece 1 inch long."

"Now I had what I was earlier lacking to mark inch divisions on the rod; carefully aligning the pieces of my makeshift ruler, I made 48 notches representing inches. Then I had in my hands a ruler divided into inches, with which I could measure the lengths obtained. Only now could I complete the task."

"I immediately set about this calculation. Measuring both diameters, I took the average of their lengths, then found the area corresponding to this average diameter. This gave me the size of the base of a cylinder equivalent to two cones of equal height. Multiplying these results by the height, I determined the cubic content of the sought-after volume."

"Dividing the number of cubic inches obtained by 69 (the number of cubic inches in one quart)[37], I found out how many quarts were in my barrel."

[37] See the note on page 243.

"It held over a hundred gallons – 108, to be exact."

Verification of the Calculation

A reader knowledgeable in geometry will undoubtedly notice that the method used by the young hero Mayne-Reid to calculate the volume of two truncated cones is not entirely accurate. If (Figure 109) we denote the radius of the smaller bases as r, the radius of the larger base as R, and the height of the barrel, i.e., twice the height of each truncated cone, as h, then the volume obtained by the boy can be expressed by the formula:

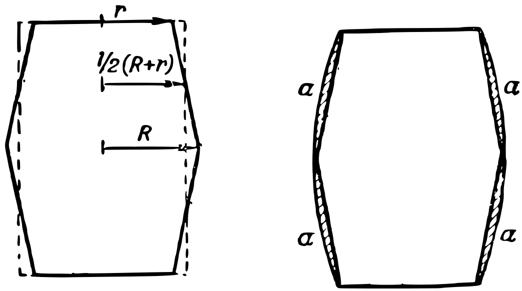


Figure 109: Checking the calculation of the young man.

$$\pi \left(\frac{R+r}{h} \right) h = \frac{\pi h}{4} (R^2 + r^2 + 2Rr).$$

However, following the rules of geometry, i.e., applying the formula for the volume of a truncated cone, we would obtain the expression for the sought-after volume:

$$\frac{\pi h}{3} (R^2 + r^2 + Rr).$$

Both expressions are not identical, and it is easy to see that the second one is greater than the first by:

$$\frac{\pi h}{12} (R - r)^2.$$

Those familiar with algebra will understand that the difference $(h/12)\,(R - r)^2$ is a positive quantity. That is, the method used by the Mayne-Reid boy resulted in an underestimated result.

It is interesting to determine approximately how much this underestimation is. Barrels are usually designed in such a way that their maximum width exceeds the diameter of the base by 1/5, i.e., $R - r = R/5$.

Assuming that the barrel in the Mayne-Reid novel was of this shape, we can find the difference between the obtained and the true volume of the truncated cones:

$$\frac{\pi h}{12}(R-r)^2 = \frac{\pi h}{12}\left(\frac{R}{5}\right)^2 = \frac{\pi h R^2}{300}.$$

That is, approximately $hR^2/300$ (if we consider $\pi \approx 3$). The error is equal to, as we see, the volume of the cylinder whose base radius is the radius of the largest section of the barrel, and its height is one three-hundredth of its height.

However, this does not reflect the true result, as the volume of the barrel is inherently greater than the volume of the two truncated cones inscribed within it. This is evident from Figure 109 (right), where it can be seen that with the specified method of measuring the barrel, a part of its volume, denoted by the letters a, a, a, a, is disregarded.

The young mathematician Mayne-Reid did not invent this formula for calculating the volume of a barrel; it is presented in some elementary geometry manuals as a convenient method for approximate determination of barrel contents. It's worth noting that measuring the volume of a barrel with absolute precision is a very difficult task. This was contemplated by the great Kepler himself, who left in his mathematical works a special study on the art of measuring barrels. A simple and precise geometric solution to this problem has not been found to this day: there are only established practical methods that provide results with greater or lesser approximation. For instance, in southern France, a simple formula, well justified by experience, is used:

$$\text{Volume of barrel} = 3.2\, hRr,$$

where: h is the height of the barrel, R is the radius of the larger base, r is the radius of the smaller base.

It is also interesting to consider the question: why, in fact, barrels are given such an inconvenient shape for measuring – a cylinder. with bulging sides? Wouldn't it be easier to make strictly cylindrical barrels? Such cylindrical barrels, however, are made, but not wooden, but metal (for kerosene, for example). So, in front of us:

Question Why are wooden barrels made with convex sides? What is the advantage of this shape?

Answer The benefit lies in the fact that by fitting hoops onto the barrels, they can be tightened tightly and securely by a very simple method: by pushing them closer to the wider part. Then the hoop can be easily tightened around the staves, providing the barrel with the necessary strength.

For the same reason, wooden buckets, tubs, vats, etc., are usually given a shape not of a cylinder, but of a truncated cone: here too, tight fitting of hoops around the product is achieved simply by pushing them onto the wider part (Figure 110).

Here it would be appropriate to acquaint the reader with the opinions on this subject expressed by Johannes Kepler. In the period between the discovery of the 2nd and 3rd laws of planetary motion, the great mathematician devoted attention to the question of the shape of barrels and even compiled a whole mathematical treatise on this topic. Here is how this is called: *Stereometry of Barrels*:

> Wine barrels have been assigned a round shape according to the requirements of the material, construction, and use. The liquid, when kept in metal vessels for a long time, spoils due to rust; glass and clay are inadequate in size and unreliable; stone vessels are unsuitable for use due to their weight—thus, wines remain to be poured and stored in wooden barrels. From a single whole log, it is also not easy to prepare vessels large enough and in sufficient quantities, and even if it is possible, they crack. Therefore, barrels should be constructed from many pieces of wood

Figure 110: Tight wrapping of the barrel with hoops is achieved by pushing them over a wide part.

connected to each other, and it is impossible to prevent the leakage of liquid through the gaps between individual pieces neither by using any material nor by any other method except compressing them with bindings...

If it were possible to construct vessels resembling spheres from wooden planks, then spherical vessels would be the most desirable. But since it is impossible to compress the planks into a sphere with bindings, a cylinder takes its place. But this cylinder cannot be perfectly smooth

> because weakened bindings would immediately become useless and could not be tightened any stronger, if the barrel did not have a conical shape, tapering on both sides from its belly. This shape is convenient for rolling and transportation in carts and, consisting of two similar halves on a common base, it is the most advantageous for rocking and aesthetically pleasing.[38]

[38] It should not be assumed that Kepler's treatise on measuring barrels is a mathematical triviality, a pastime of genius during leisure hours. No, it is a serious work in which infinitesimal quantities and the principles of integral calculus are introduced into geometry for the first time. The wine barrel and the practical problem of measuring its capacity served as an occasion for him to engage in deep and fruitful mathematical reflections. The translation of *Stereometry of Wine Barrels* was published in 1935.

The Night Journey of Mark Twain

The resourcefulness shown by the boy from the Mayne Reid story in his dire situation is truly remarkable. In the pitch darkness he found himself in, most people wouldn't even be able to orient themselves properly, let alone perform any measurements or calculations. It is instructive to compare Reid's story with a humorous tale of aimless wandering in a dark hotel room, an adventure seemingly experienced by the famous compatriot of Mayne Reid, the humorist Mark Twain. This story effectively highlights how difficult it is to form a correct understanding of the layout of objects in the dark, even in an ordinary room, if the environment is unfamiliar. Below is a condensed rendition of this amusing episode from *The Innocents Abroad* by Mark Twain.

> I woke up and felt thirsty. A wonderful idea occurred to me – to get dressed, go out to the garden, and refresh myself by washing up at the fountain.

I got up slowly and started searching for my things. Found one sock. Where the other one was, I couldn't imagine. Carefully lowering myself to the floor, I began to feel around, but to no avail. I searched further, groping and scooping. I moved further and further, but couldn't find the sock and only bumped into furniture. When I went to bed, there was much less furniture around; now the room was full of it, especially chairs, which seemed to be everywhere. Had two more families moved in during this time? I hadn't seen any of these chairs in the darkness, but kept bumping my head into them.

Finally, I decided I could live without one sock. Standing up, I headed towards the door, or so I thought – but unexpectedly saw my dim reflection in the mirror.

It was clear that I was lost and had no idea where I was. If there had been just one mirror in the room, it would have helped me orient myself, but there were two, and that was just as bad as a thousand.

I wanted to make my way to the door along the wall. I began my attempts again – and knocked over the picture. The noise was not great, but it sounded as loud as a whole panorama. Harris (my roommate, sleeping in the other bed) didn't move, but I felt that if I continued in the same manner, I would definitely wake him up. I'll try another way. I'll find the round table again – I had been near it several times – and from there, I'll try to make my way to my bed; if I find the bed, I'll find the pitcher of water, and then, at least, I'll quench my unbearable thirst. It's best to crawl on hands and knees; I had already tried this method and trusted it more.

Finally, I managed to stumble upon the table – felt it with my head – with a relatively small noise. Then I stood up again and walked, balancing with my arms outstretched and fingers spread. Found a chair. Then a lamp. Another chair. Then a sofa. My stick. Another sofa. This surprised me; I knew perfectly well that there was only one sofa in the room. Stumbled upon the table again and received a new blow. Then bumped into a new row of chairs.

It was only then that it occurred to me, as it should have long ago: the table was round and therefore could not serve as a starting point for my wanderings. By chance, I went into the space between the chairs and the sofa – but found myself in an entirely unfamiliar area, dropping the candlestick by the fireplace along the way. After the candlestick, I dropped the lamp, and after the lamp, the decanter crashed to the floor with a clang.

"Ah," I thought, "finally found you, my dear!"

"Thieves! Robbers!" yelled Harris.

The noise and cries woke up the entire house. The owner, guests, and servants appeared with candles and lanterns.

I looked around. It turned out I was standing next to Harris's bed. Only one sofa was against the wall; only one chair was positioned to bump into – I circled around it like a planet, colliding with it like a comet for a whole half of the night.

> After checking my pedometer, I confirmed that I had covered 47 miles during the night.

The last statement is exaggerated beyond measure: it is impossible to walk 47 miles on foot in just a few hours. However, the other details of the story are quite plausible and accurately depict the comedic difficulties one typically encounters when wandering aimlessly and haphazardly in the darkness of an unfamiliar room. Moreover, we should especially appreciate the remarkable methodical approach and the spirit of the young hero, Mayne-Reid, who not only managed to orient himself in complete darkness but also solved a challenging mathematical problem under such conditions.

Mysterious Circumnavigation

Regarding Twain's circling in the dark room, it's interesting to note a mysterious phenomenon observed in people wandering with their eyes closed: they cannot walk in a straight line but invariably veer off to the side, describing an arc, while imagining that they are moving straight ahead (see Figure 111).

It has long been noticed that travellers wandering without a compass in the desert, across the steppe in a blizzard, or in foggy weather—generally in all cases where there is no opportunity to orient themselves—deviate from the straight

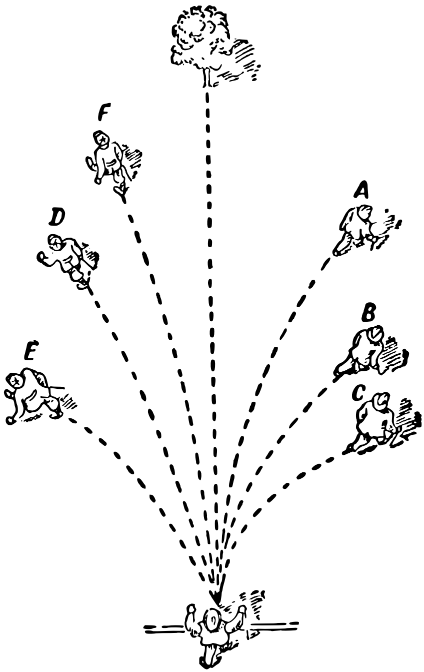


Figure 111: Walking with your eyes closed.

path and wander in circles, repeatedly returning to the same place. The radius of the circle described by the pedestrian is about 60–100 meters; the faster the walking, the smaller the radius of the circle, i.e., the tighter the circles closed.

Special experiments have even been conducted to study people's tendency to deviate from a straight path to a circular one. Here's what Hero of the Soviet Union I. Spirin reports on such experiments:

> On a smooth green aerodrome, a hundred future pilots were lined up. Blindfolds were placed on all of them, and they were instructed to walk straight ahead. The people started walking… At first, they walked straight; then some started veering to the right, others to the left, gradually began to make circles, returning to their old tracks.

A similar experiment is known in Venice in St. Mark's Square. People were blindfolded, placed at one end of the square, directly opposite the cathedral, and asked to reach it. Although they only had to walk a mere 175 meters, none of the subjects reached the facade of the building (which is 82 meters wide), and all veered to the side, describing an arc and bumping into one of the 60 stone colonnades (see Figure 112).

Anyone who has read Jules Verne's novel *The Adventures of Captain Hatteras* probably remembers the episode where the travellers stumbled upon someone's tracks in a snowy uninhabited desert:

> 'Those are our tracks, my friends!' exclaimed the doctor. 'We got lost in the fog and stumbled upon our own tracks …'

A classic description of such circular wandering was given to us by Leo Tolstoy in *Master and Man*:

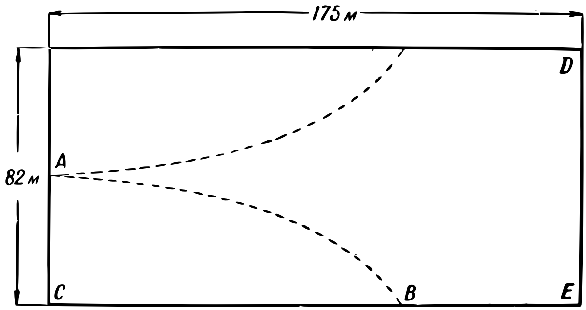


Figure 112: The schematic depiction of the experiment at St. Mark's Square in Venice.

Vasiliy Andreyich urged the horse towards where he, for some reason, assumed there was a forest and a watchtower. The snow blinded him, and the wind seemed to want to stop him, but he, leaning forward, continued to drive the horse forward without stopping.

For about five minutes he rode, as it seemed to him, straight ahead, seeing nothing except the horse's head and the white desert.

Suddenly something darkened before him. His heart joyfully pounded, and he rode towards this blackness, already seeing in it the walls of village houses. But black? It turned out to be a tall thistle grown on the boundary... And for some reason, the sight of this thistle, tormented by the merciless wind, made Vasily Andreyich shudder, and he hastily began to drive the horse, not noticing that, as he approached the thistle, he completely changed his previous direction.

> Again something darkened ahead of him. It was another boundary, overgrown with thistle. The dry grass was also desperately thrashed by the wind. Beside it was a horse's hoof print, blown by the wind. Vasily Andreyich stopped, leaned over, looked closely: it was a slightly blown horse's print and could be no one else's but his own. He was evidently circling in a small space.

Norwegian physiologist Guldborg, who dedicated a special study to wandering (1896), collected a number of carefully verified testimonies about genuine cases of this kind. Let's cite two examples.

Three travellers intended to leave the guard post on a snowy night and exit the valley, which was 4 km wide, to reach their home, located in the direction indicated by the dashed line on the attached diagram (Figure 113). Along the way, they imperceptibly veered to the right, following the curve marked by the arrow. After covering some distance, they, according to their calculations, believed they had reached their destination – when in fact they found themselves back at the same guard post they had left. Setting out on the path again, they veered even further off course and again arrived at the starting point. The same thing happened for the third and fourth times. In desperation, they made a fifth attempt – but with the same result. After the fifth round, they gave up further attempts to leave the valley and waited for morning.

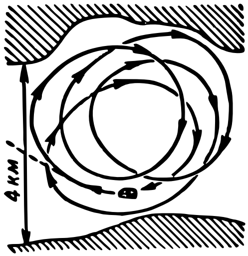


Figure 113: The schematic depiction of the wanderings of three travelers.

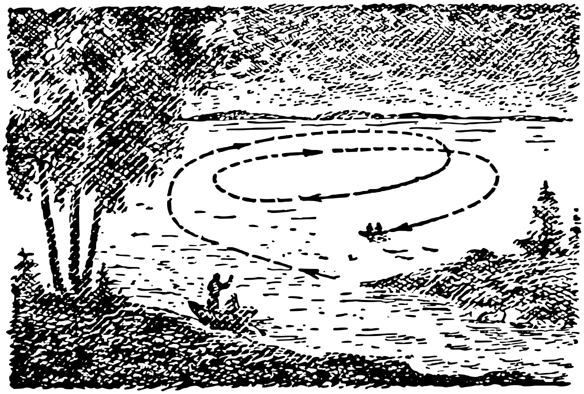


Figure 114: How the rowers attempted to cross the strait in foggy weather.

Even more difficult is rowing straight on the sea in a dark, starless night or in thick fog. There is a case, one of many similar ones, when rowers, intending to cross a strait 4 miles wide in foggy weather, twice ended up on the opposite shore, but did not reach it, and unconsciously circled twice before finally... landing back at their point of departure (Figure 114).

The same happens with animals. Polar travellers tell of circles drawn in the snow deserts by animals harnessed to sleds. Dogs, allowed to swim with their eyes blindfolded, also describe circles in the water. Blinded birds also fly in circles. A frightened animal, deprived of the ability to orient itself from fear, does not save itself in a straight line but in a spiral.

Zoologists have found that tadpoles, crabs, jellyfish, even microscopic amoebas in a drop of water all move in a circle.

What explains the mysterious preference of humans and animals for circles, the inability to maintain a straight direction in the dark? The question will immediately lose its seeming mystery in our eyes if we put it correctly.

Let's not ask why animals move in circles, but what they need to move in a straight line? Think about how a wind-up toy cart moves. Sometimes the cart doesn't roll straight but veers to the side. In this curved movement, no one sees anything mysterious; everyone guesses why this happens: obviously, the right wheels are not equal to the left ones.

It's clear that a living creature can only move in a straight line without the help of eyes if the muscles on its right and left sides work exactly the same. But the point is that the symmetry of the human and animal body is incomplete. In the vast majority of people and animals, the muscles on the right side of the body are not equally developed as the muscles on the left. Naturally, a pedestrian who always steps a little further with his right leg than his left will not be able to keep a straight line; if his eyes don't help him correct his path, he will inevitably veer to the left. Similarly, a rower, when deprived of the ability to orient himself due to fog, will inevitably veer to the left if his right arm works stronger than his left. This is a geometric necessity.

Imagine, for example, that when lifting the left leg, a person takes a step one millimetre longer than with the right leg. Then, by alternating each leg for a thousand steps, the person will describe a path with the left leg that is 1,000 mm, or a whole meter, longer than with the right leg. This is impossible on straight parallel paths, but quite feasible on concentric circles.

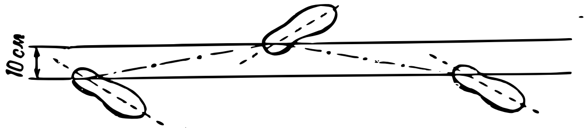


Figure 115: The lines of the prints of the right and left feet when walking.

We can even, using the plan of the circular movement described above in the snowy valley, calculate how much longer the step of those travellers was with the left leg compared to the right (since the path curved to the right, it is clear that it was the left leg that took longer steps). The distance between the lines of imprints of the right and left legs during walking (see Figure 115) is approximately 10 cm, or 0.1 m. When a person completes one full circle, their right leg covers a distance of $2\pi R$, and the left leg covers $2\pi(R + 0.1)$, where R is the radius of this circle in meters.

The difference is $2\pi(R + 0.1) - 2\pi R = 2\pi(0.1)$, which equals 0.62 m, or 620 mm, due to the difference in length between the left and right steps, repeated as many times as steps were taken. From Figure 113, we can deduce that our travellers described circles with a diameter of approximately 3.5 km, or a length of about 10,000 m. With an average step length of 0.7 m over a distance of 10,000 m, this equals $10000/0.7 = 14000$ steps; of these, 7000 were with the right leg and the same with the left. Thus, we find that 7000 “left” steps exceeded 7000 “right” steps by 620 mm.

Hence, one left step is longer than one right step by 620/7000 mm, or less than 0.1 mm. Here’s how such a tiny difference in steps can result in such a striking outcome!

The radius of the circle that the wanderer describes depends on the difference in lengths between the “right” and “left” steps. This dependency is easy to establish. The total number of steps taken over one circle, with a step length of 0.7 m , is $2\pi R/0.7$, where R is the radius of the circle in meters; of these, there are $2\pi R/2 \cdot 0.7$ “left” steps and the same number of “right” steps. Multiplying this number by the magnitude of the difference x in the length of the steps, we obtain the difference in the lengths of those concentric circles that are described by the left and right legs, i.e.:

$$\frac{2\pi Rx}{2 \cdot 0.7} = 2\pi \cdot 0.1 \quad \text{or} \quad Rx = 0.14,$$

where R and x are in meters.

By this simple formula, it is easy to calculate the radius of the circle when the difference in steps is known, and vice versa. For example, for the participants of the experiment in Piazza San Marco in Venice, we can establish the maximum radius of the circles described by them while walking. Indeed, since none of them reached the facade DE of the building (see Figure 112), then according to the "arrow" $AC = 41$ m and the chord BC, which does not exceed 175 m, we can calculate the maximum radius of the arc AB. It is determined by the equality

$$2R = \frac{BC^2}{AC} = \frac{175^2}{41} = 750\,\text{m},$$

from which R, the maximum radius, will be about 370 m.

Knowing this, we determine the minimum value of the difference in step lengths from the formula obtained earlier:

$$370x = 0.14, \ \text{hence} \ x = 0.4\,\text{mm}.$$

Sometimes you have to read and hear that the fact of circling while walking blindly depends on the difference in length between the right and left legs; since the left leg is longer for most people, they must inevitably deviate to the right from the straight direction while walking. Such an explanation is based on a geometric error. The important factor is the difference in step lengths, not leg lengths. From Figure 116, it is clear that even with a difference in leg length, one can still make strictly identical steps if each leg is extended at the same angle while walking, i.e., stepping so that $\angle B_1 = \angle B$. Since in this case always $A_1B_1 = AB$ and $B_1C_1 = BC$, then $\triangle A_1B_1C_1 \cong \triangle ABC$, and therefore $AC = C_1A_1$. Conversely, with strictly identical leg lengths, steps can be of different lengths if one leg is extended farther while walking than the other.

For a similar reason, a boatman rowing stronger with his right hand than with his left must inevitably lead the boat in a circle, bending to the left. Animals making uneven steps with their right or left legs, or birds making unequal strokes with their right and left wings, must also move in circles whenever they are unable to control straight-line direction with sight. Here too, a very slight difference in the strength of the arms, legs, or wings is sufficient.

With this perspective, the mentioned phenomena lose their mystery and become entirely natural. It would be astonishing if people and animals, on the contrary, could maintain a straight direction without controlling it with their eyes.

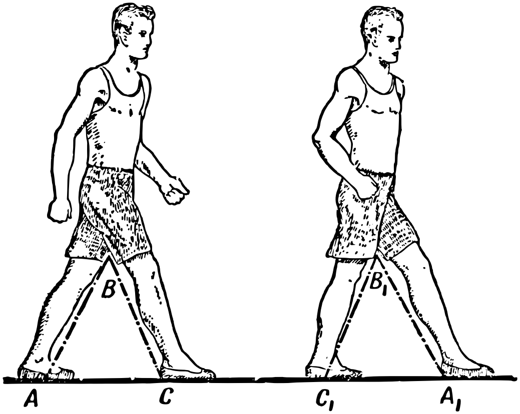


Figure 116: If the angle of each step is the same, then the steps will be exactly the same.

After all, a necessary condition for this is strict geometric symmetry of the body, absolutely impossible for living organisms. The slightest deviation from mathematically perfect symmetry must inevitably result in circling. The miracle is not what we marvel at here but rather that we were willing to consider it natural.

The inability to maintain a straight path is not a significant obstacle for humans: compasses, roads, and maps save them in most cases from the consequences of this deficiency.

For animals, especially those inhabiting deserts, steppes, and vast expanses of the sea, the asymmetry of their bodies, which compels them to describe circles instead of straight lines, is an important factor in their lives. It's as if an invisible chain binds them to their birthplace, depriving them of the ability to move away from it significantly. The gazelle, daring to venture far into the desert, inevitably returns back. Seagulls, leaving their native cliffs to fly into the open sea, cannot help but return to their nests (which makes the long flights of birds that cross continents and oceans in a straight direction all the more mysterious).

Measuring with Bare Hands

The boy from Mayne-Reid's story was able to solve his geometric problem successfully only because he had measured his height shortly before the journey and remembered the measurement results. It would be beneficial for each of us to have such a 'living meter' so that we could use it for measurements when needed. It is also useful to remember that for most people, the distance between the ends of their outstretched arms is equal to their height (see Figure 117) – a fact marked by the ingenious artist and scientist Leonardo da Vinci; this makes it more convenient to use our 'living meters' than what the boy did in Mayne-Reid's story.

On average, the height of an adult (of Slavic descent) is about 1.7 meters, or 170 centimeters: this is easy to remember.

However, it is not advisable to rely on the average value: everyone should measure their height and the span of their arms.

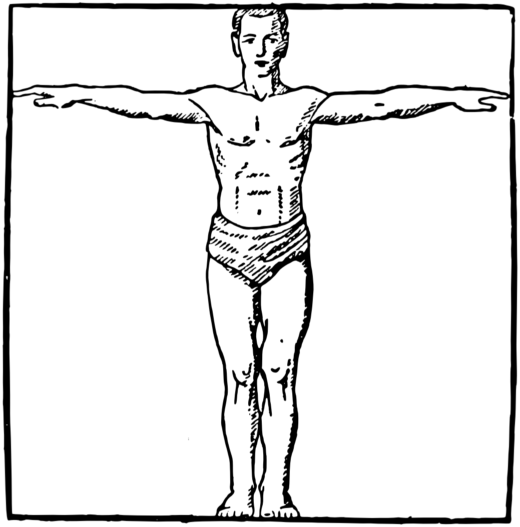


Figure 117: The Leonardo da Vinci rule – the vitruvian man.

For measuring small distances without a scale, it is useful to remember the length of one's "quarter", i.e., the distance between the tips of the thumb and little finger when fully stretched (see Figure 118). For an adult man, it is about 18 centimetres – roughly one “arshin”(hence the name “quarter”); however, it is smaller in young people and gradually increases with age (up to 25 years).

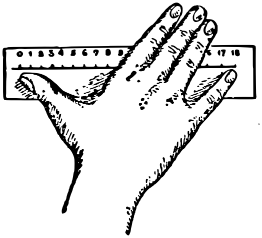


Figure 118: Measuring the distance between the ends of the fingers.

Furthermore, for the same purpose, it is useful to measure and remember the length of your index finger, considering it in two ways: from the tip to the middle finger (see Figure 119) and from the base of the thumb.

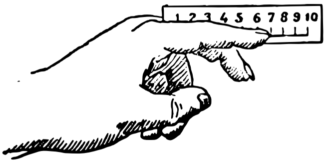


Figure 119: Measuring the length of the index finger.

Similarly, you should know the maximum distance between the tips of the index and middle fingers, which is about 19 centimeters for an adult (see Figure 120). Finally, you need to know the width of your fingers. The width of three middle fingers, tightly clenched, is approximately 5 centimeters.

Figure 120: Measuring the distance between the ends of two fingers.

Armed with all this information, you will be able to perform various measurements quite satisfactorily literally with your bare hands, even in the dark. An example is presented in Figure 121: here, the circumference of a glass is measured using fingers. Based on average values, it can be said that the circumference of the glass is equal to 18 + 5, i.e., 23 centimeters.

Straight Angle in the Dark

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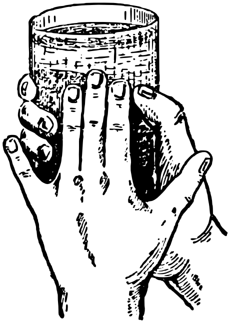


Figure 121: Measuring the circumference of the glass with "bare hands".

Question Returning once again to the Mayne-Reid mathematician, let's set ourselves a task: how should he have proceeded to reliably obtain a right angle? "I placed a long rod against it (the projecting plank) so that it formed a right angle with it," we read in the novel. Doing this in the dark, relying only on muscular sensations, we can make quite a few mistakes. However, the boy had a way to construct a right angle much more reliably. How?

Answer One should resort to using the Pythagorean theorem and construct a triangle from the planks, giving its sides such lengths that the triangle becomes rectangular. The simplest way to do this is to take planks with lengths of 3, 4, and 5 arbitrary equal segments (see Figure 122).

This is an ancient Egyptian method used in the land of pyra-

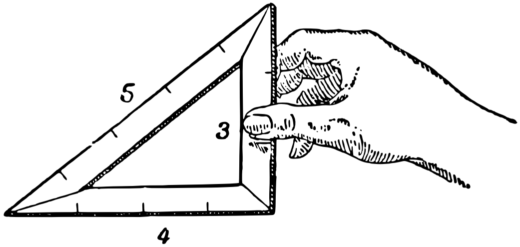
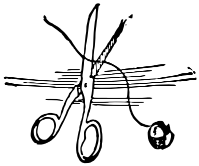


Figure 122: The simplest right-angled triangle, the lengths of whose sides correspond to these numbers.

mids thousands of years ago. However, even in our days, this method is often used in construction work.



9. Old And New About The Circle

Practical Geometry of the Egyptians and Romans

Any schoolchild can now calculate the circumference based on its diameter much more accurately than the wisest priest of the ancient land of pyramids or the most skilled architect of great Rome. The ancient Egyptians believed that the circumference was 3.16 times longer than the diameter, while the Romans believed it was 3.12 times longer. However, the correct ratio is 3.14159... Egyptian and Roman mathematicians established the ratio of the circumference to the diameter not through strict geometric calculation, as later mathematicians did, but simply through experience. But why did they make such errors? Couldn't they just stretch a piece of string around a circular object and then straighten the string to measure it?

Undoubtedly, they did just that, but it should not be assumed that such a method would necessarily yield good results. Imagine, for example, a vase with a round bottom with a diameter of 100 mm. The circumference of the bottom should be 314 mm. However, in practise, measuring with a string, you are unlikely to get this length: it is easy to make a mistake of one millimetre, and then π will turn out to be equal to 3.13 or 3.15. And if you consider that the diameter of the vase cannot be measured perfectly accurately either, and that an error of 1 mm is quite probable here too, then you will see that for π, quite wide limits are obtained between

$$\frac{313}{101} \quad \text{and} \quad \frac{315}{99},$$

i.e., in decimal fractions, between

$$3.09 \quad \text{and} \quad 3.18.$$

You see, by determining π in this way, we can get a result that does not coincide with 3.14: once it's 3.1, another time 3.12, the third time 3.17, and so on. Among them, by chance, there may be 3.14, but in the eyes of the calculator, this number will not carry more weight than the others.

Such an empirical approach cannot provide a somewhat acceptable value for π. In this regard, it becomes more understandable why the ancient world did not know the correct ratio of the circumference to the diameter and why it took the genius of Archimedes to find the value of $3\ 1/7$ for π without measuring, relying solely on reasoning.

"I know this and I remember it perfectly."

In the *Algebra* of the ancient Arab mathematician Mohammed ibn-Musa, we read the following lines about calculating the circumference:

> The best way is to multiply the diameter by $3\ 1/7$. This is the fastest and easiest way. God knows the best.

Now we also know that Archimedes' number $3\ 1/7$ does not perfectly express the ratio of the circumference to the diameter. It has been theoretically proven that this ratio cannot be expressed as any exact fraction. We can only write it with some approximation, which, however, far exceeds the accuracy required for the strictest demands of practical life. The mathematician Ludolph in the 16th century, in Leiden, had the patience to calculate π with 35 decimal places and bequeathed to have this value for π carved on his tombstone[39]! (see Figure 123).

[39] At that time, the notation π was not yet in use: it was introduced only from the middle of the 18th century by the famous Russian academician and mathematician Leonard Pavlovich Euler.

Here it is:

$$3.141{,}592{,}653{,}589{,}793{,}238{,}462{,}643{,}383{,}279{,}502{,}88\ldots$$

A certain Shanks in 1873 published a value of π in which there were 707 decimal places after the comma! Such long numbers approximately expressing the value of π have neither practical nor theoretical value. Only out of idleness and in pursuit of inflated "records" could there arise in our time a desire to "outdo" Shanks: in 1946-1947, Ferguson (Manchester City) and, independently of him, Weaver (from Washington) calculated 808 decimal places for the number π and were pleased to find errors in Shanks's calculations starting from the 528th digit.[40]

[40] As of 2024, the value of π has been calculated to 62,831,853,071,796 digits, that is **62 trillion** digits! For several different and modern methods to find value of π, please see the Wikipedia page https://en.wikipedia.org/wiki/Pi. – DM

If, for example, we wished to calculate the length of the Earth's equator with an accuracy of 1 cm, assuming that we know the length of its diameter precisely, it would be sufficient for us to take only 9 digits after the decimal point in the number π. And by taking twice as many digits (18),

Figure 123: Mathematical epitaph.

we could calculate the circumference with a precision of not more than 0.000,1 mm (100 times smaller than the thickness of a hair!) for a circle with a radius equal to the distance from the Earth to the Sun.

Our compatriot, the mathematician Grave, vividly demonstrated the absolute uselessness of even the first hundred decimal places of π. He calculated that if we imagine a sphere with a radius equal to the distance from the Earth to Sirius, i.e., a number of kilometres equal to 132 followed by ten zeros: 132×10^{10}, and fill this sphere with microbes, placing one billion (10^{10}) microbes in each cubic millimetre of the sphere, and then arrange all these microbes in a straight line so that the distance between each pair of adjacent microbes again equals the distance from Sirius to Earth, then, taking this fantastic segment as the diameter of a circle, it would be possible to calculate the length of the resulting gigantic circumference with microscopic precision — up to 0.000,001 mm, by using 100 digits after the decimal point in the number π. French astronomer Arago correctly observes in this regard that "in terms of accuracy, we would gain nothing if there were a relationship between the circumference and the diameter expressed by a number with complete accuracy."

For ordinary calculations involving π, it is sufficient to remember two digits after the decimal point (3.14), and for more precise calculations, four digits (3.1416: we take the last digit as 6 instead of 5 because the following digit is greater than 5).

Small poems or vivid phrases are better retained in memory than numbers, so special poems or individual phrases are invented to memorise some numerical value of π. In works of this kind of mathematical poetry, words are chosen so that the number of letters in each word sequentially coincides with the corresponding digit of the number π.

There is a famous poem in English – in 13 words, hence providing 12 digits after the decimal point in the number π; in German – in 24 words, and in French, in 30 words! (and there are even ones in 126 words).

They are curious but too large, cumbersome. Among the students, E. Y. Terskov, a mathematics teacher at one of the secondary schools in the Moscow region, enjoys popularity for inventing the following stanza:

«Это я знаю и помню прекрасно».
3 1 4 1 5 9

('I know this and remember it perfectly.')[41]

And one of his students, Elya Cherikover, with the resourcefulness typical of our schoolchildren, composed a witty, slightly ironic continuation:

«Пи многие знаки мне лишни, напрасны»,
2 6 5 3 5 8
('Pi, many digits are unnecessary for me, in vain,')

In total, a twelve-word couplet is obtained:

[41] This sentence and the next ones in Russian are not really translatable with the number of characters in the words intact and corresponding to the numbers in π. I am giving the translations in English which do not match. I am also adding some alternative mnemonics in English. – DM

«Это я знаю и помню прекрасно, Пи многие знаки мне
3 1 4 1 5 9 2 6 5 3

лишни, напрасны».
5 8

(‘I know this and remember it perfectly, Pi, many digits are unnecessary for me, in vain.’)

The author of this book, not daring to invent a poem, in turn suggests a simple and also quite sufficient prose phrase:

«Что я знаю о кругах?»
3 1 4 1 6

(‘What do I know about circles?’) – a question, secretly containing the answer.

Some English mnemonics:

See I have a rhyme assisting, My feeble brain, its tasks ofttimes
3 1 4 1 5 9 2 6 5 3 5 8

resisting.
9

Wow! I made a great discovery!
3 1 4 1 5 9

Can I have a small container of coffee?
3 1 4 1 5 9 2 6

How I want a drink, alcoholic of course, after the heavy lectures
3 1 4 1 5 9 2 6 5 3 5 8

involving quantum mechanics.
9 7 9

A German menmonic (24 digits) 314,159,265,358,979,323,746,264:

Wie o dies π, Macht ernstlich, so vielen viele Müh'! Lernt
3 1 4 1 5 9 2 6 5 3 5

immerhin, Jünglinge, leichte Verselein, Wie so zum Beispiel dies
8 9 7 9 3 2 3 7 4

dürfte zu merken sein.
6 2 6 4

A French mnemonic (30 digits) 3,141,592,653,589,793,237,462,643,383,279:

Que j'aime à faire apprendre un nombre utile aux sages!
3 1 4 1 5 9 2 6 5 3

Jmmortel Archimède, sublime ingénieur, Qui de ton jugement
5 8 9 7 9 3 2 3

peut sonder la valeur? Pour moi ton problème eut de pareils
7 4 6 2 6 4 3 3 8 3 2

avantages.
7

Jack London's Error

The next passage from Jack London's novel *The Little Lady of the Big House* provides material for geometric calculation:

> In the middle of the field stood a steel pole, driven deep into the ground. From the top of the pole to the edge of the field stretched a cable, attached to a tractor. The mechanics pressed a lever – and the engine started.
>
> The machine moved forward on its own, describing a circle around the pole, which served as its center.
>
> "To truly perfect the machine," Graham said, "you need to turn the circle it describes into a square."
>
> "Yes, on a square field, this system wastes a lot of land."
>
> Graham made some calculations, then remarked:
>
> "We lose approximately three acres out of every ten."
>
> "No less."

Readers are invited to verify this calculation.

Answer The calculation is incorrect: less than 0.3 of the total land is lost. Let's assume that the side of the square is a. The area of such a square is a^2. The diameter of the inscribed circle is also a, and its area is $\pi a^2/4$. The lost part of the square plot is

$$a^2 - \frac{\pi a^2}{4} = \left(1 - \frac{\pi}{4}\right) a^2 = 0.22\, a^2.$$

We can see that the unused part of the square field amounts not to 30% as the characters in the American novelist's story believed, but only about 22%.

Dropping a Needle

The most original and unexpected method for approximating the number π is as follows. Take a short (about two centimeters) sewing needle – preferably with a blunted tip to ensure uniform thickness – and draw a series of thin parallel lines on a sheet of paper, with each line spaced apart by twice the length of the needle.

Then, drop the needle onto the paper from a certain (arbitrary) height and observe whether the needle crosses one of the lines or not (see Figure 124, left). To prevent the needle from bouncing, place a piece of thin paper or cloth under the paper sheet. Repeat the dropping of the needle many times, for example a hundred or, even better, a thousand times, each time noting whether there was a crossing or not.[42] If you then divide the total number of needle drops by the number of instances where a crossing was observed, the result should approximate the value of π, more or less accurately.

[42] The intersection should also be considered the case when the needle only rests against the end of the drawn line.

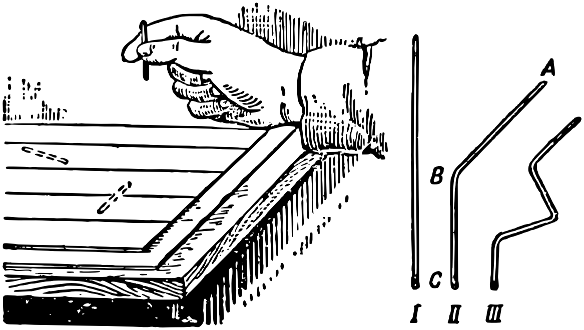


Figure 124: Buffon's needle-throwing experiment.

Let's explain why this happens. Let the most likely number of needle crossings be denoted as K, and the length of our needle be 20 mm. In the case of a crossing, the point of intersection must, of course, lie on one of these millimeters, and no one millimeter, nor any part of the needle, has any advantage in this regard over the others. Therefore, the most likely number of crossings for each individual millimeter is $K/20$. For a segment of the needle of 3 mm, it is $3K/20$, for a segment of 11 mm – $11K/20$, and so on. In other words, the most likely number of crossings is directly proportional to the length of the needle.

This proportionality is maintained even if the needle is bent. Let the needle be bent in the form shown in figure Figure 124, (*II* right), with segment $AB = 11$ mm and $BC = 9$ mm. For segment AB, the most likely number of crossings is $11K/20$, and for segment BC it is $9K/20$. For the entire needle it is $11K/20+9K/20$, which still equals K. We can bend the needle in a more intricate way (Figure 124, (*III* right)) – this would not change the number of crossings. (Note that with a bent needle, crossings are possible with two or more parts of the needle at once; such crossings should be counted as 2, 8, etc., because the first is counted for one part of the needle, the second for another, and so on.)

Now imagine that we are dropping a needle, bent into the shape of a circle with a diameter equal to the distance between the lines (it is twice the length of our needle). Such a ring must intersect any line twice each time (or touch two lines once each – in any case, two encounters occur). If the total number of drops is N, then the number of encounters is $2N$. Our straight needle is shorter than this ring by as many times as the radius is less than the circumference, which is 2π times. But we have already established that the most likely number of crossings is proportional to the length of the needle. Therefore, the most likely number (K) of crossings of our needle should be less than $2M$ by 2π times, i.e., equal to N/π. Hence,

$$\pi = \frac{\text{number of drops}}{\text{number of crossings}}.$$

The more drops observed, the more accurate the expression for π becomes. The Swiss astronomer R. Wolff in the middle of the last century observed 5000 needle drops on grid paper and obtained π as 3.159... an expression, however, less precise than the Archimedean number.

As you can see, the ratio of the circumference to the diameter is determined here experimentally, and what is more curious – neither a circle nor a diameter is drawn, meaning no compass is used. A person with no knowledge of geometry or even circles can determine π using this method if they patiently perform a very large number of needle drops.

Straightening a Circle Problem

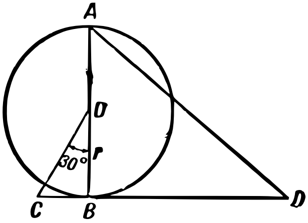


Figure 125: An approximate geometric method of rectifying a circle.

For many practical purposes, it is sufficient to take π as 3 and 1/7 straighten the circle by laying out its diameter on any straight line 3 1/7 times (dividing the segment into seven equal parts can be done quite accurately, as is well known). There are other approximate methods for straightening a circle used in practise by craftsmen such as carpenters, tinsmiths, and so on. We won't discuss them here, but we'll mention one fairly simple method of straightening that yields results with extremely high accuracy.

If it is necessary to straighten a circle O with radius r (see Figure 125), then draw the diameter AB, and at point B, draw line CD perpendicular to it. From the centre O, draw line OC at an $30°$ angle to AB. Then, on line CD from point C, mark off three radii of the given circle and connect the resulting point D to A: the length of segment AD equals the length of half the circumference. If segment AD is doubled in length, an approximately straightened circle O is obtained. The error is less than $0.0002\,r$.

On what basis is this construction founded?

Answer According to the Pythagorean theorem,

$$CB^2 + OB^2 = OC^2.$$

Denoting the radius OB as r and considering that $CB = OC/2$ (as a leg, lying opposite an $30°$ angle), we get:

$$CB^2 + r^2 = 4CB^2,$$

from which,

$$CB = \frac{r\sqrt{3}}{3}.$$

Next, in triangle ABD:

$$BD = CD - CB = 3r - \frac{r\sqrt{3}}{3}.$$

$$AD = \sqrt{BD^2 + 4r^2} = \sqrt{\left(3r - \frac{r\sqrt{3}}{3}\right)^2 + 4r^2},$$

$$= \sqrt{9r^2 - 2r^2\sqrt{3} + r^2/3 + 4r^2},$$
$$= 3.14153\, r.$$

Comparing this result with one obtained with high precision for π (approximately ($\pi = 3.14153$), we see that the difference is only $0.00006\, r$. If we were to straighten a circle with a radius of 1 unit using this method, the error would be only 0.00006 meters for the semicircle, and for the full circle, it would be 0.00012 meters, or 0.12 millimetres (roughly three times the thickness of a hair).

Squaring the Circle

It is unlikely that any reader has never heard of the "squaring of the circle" – that most famous problem in geometry that mathematicians have been working on for twenty centuries. I am even confident that among the readers there are those who have tried to solve this problem themselves. However, even more readers may wonder what exactly the difficulty lies in resolving this classic unsolvable problem. Many, accustomed to repeating from others that the problem of squaring the circle is unsolvable, do not have a clear understanding of the essence of the problem or the difficulty of its resolution.

In mathematics, there are many problems much more interesting both theoretically and practically than the problem of squaring the circle. However, none has gained such popularity as this problem, which has long become a byword. For two millennia, outstanding professional mathematicians and countless crowds of amateurs have worked on it.

To square the circle" means to draw a square whose area is exactly equal to the area of a given circle. Practically, this problem arises very often, but precisely in practise, it can be solved with any degree of accuracy. The famous ancient problem, however, requires that the drawing be done perfectly using only two types of drawing operations:

1. drawing a circle of a given radius around a given point;
2. drawing a straight line through two given points.

In short, it is necessary to make a drawing using only two drawing instruments: a compass and a straightedge.

In other words, it is necessary to make a drawing using only two drawing instruments: a compass and a straightedge.

Among non-mathematicians, there is a widespread belief that all the difficulty lies in the fact that the ratio of the circumference to its diameter (the famous number π) cannot be expressed by a finite number of digits. This is true only to the extent that the solvability of the problem depends on the peculiar nature of the number π. Indeed: transforming a rectangle into a square with equal area is an easily and precisely solvable problem. But the problem of squaring the circle is reduced to constructing – with a compass and a straightedge – a rectangle equal in area to the given circle. From the formula for the area of a circle, $S = \pi r^2$, or (which is the same thing) $S = \pi \times r \times r$, it is clear that the area of the circle is equal to the area of such a rectangle, where one side is r and the other is π times larger. As you know, π is not exactly equal to 3 1/7, or 3.14, or even 3.14159. The series of digits expressing this number goes to infinity.

The particularity of the number π, its irrationality[43], was established by mathematicians Lambert and Legendre back in the 18th century. Yet, the knowledge of the irrationality of π did not stop the efforts of those versed in mathematics, the "quadraturists". They understood that the irrationality of π itself does not make the problem hopeless. There are irrational numbers that geometry can "construct" perfectly accurately. For example, suppose we need to draw a segment that is longer than a given one by $\sqrt{2}$ times. The number $\sqrt{2}$, like π, is irrational. Nevertheless, nothing could be easier than drawing the desired segment: let's remember that $\sqrt{2}$ is the side of a square inscribed in a circle with a radius of 1, because it amounts to constructing a regular 64-gon.

[43] The peculiarity of an irrational number is that it cannot be expressed as any exact fraction.

Every schoolchild also easily copes with constructing the segment $a\sqrt{3}$ (the side of an equilateral inscribed triangle). Even constructing such a seemingly complex irrational expression presents no particular difficulty:

$$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}.$$

As we can see, an irrational multiplier, entering into the expression, does not always render this expression impossible to construct with a compass and straightedge. The unsolvability of squaring the circle is not entirely due to the fact that the number π is irrational, but to another peculiarity of this same number. Namely, the number π is not algebraic, i.e., it cannot be obtained as the solution to any equation with rational coefficients. Such numbers are called "transcendental".

A mathematician of the 19th century proved that the number π is transcendental. This expression for π would solve the problem of squaring the circle if the number of operations involved were finite (then the expression could be geometrically constructed). But since the number of square root extractions in this expression is infinite, Vieta's expression does not help the cause.

So, the unsolvability of the problem of squaring the circle is due to the transcendence of the number π, i.e., it cannot be the result of solving an equation with rational coefficients. This peculiarity of the number π was rigorously prooven in 1889 by the German mathematician Lindemann. In essence, this scientist should be considered the only person who solved the squaring of the circle problem, despite the fact that his solution is negative – it asserts that the desired construction is geometrically impossible. Thus, in 1889 the centuries-long efforts of mathematicians in this direction were concluded; however, unfortunately, the fruitless attempts of numerous amateurs, insufficiently familiar with the problem, have not ceased.

This is the state of affairs in theory. As for practise, it does not require an exact resolution of this famous problem at all. The belief of many that solving the problem of squaring the circle would have a huge significance for practical life is a profound misconception. For everyday needs, it is perfectly sufficient to have good approximate solutions to this problem.

Practically, the quest for squaring the circle became futile once the first 7-8 accurate digits of the number π were found. For the needs of practical life, it is entirely sufficient to know that π is approximately equal to 3.1415926. No measurement of length can provide a result expressed in more than seven significant digits. Therefore, having more than eight digits for π is useless: it does not improve the precision of calculations[44]. Even if the radius is expressed with seven significant

[44] See *Entertaining Mathematics* by Yakov I. Perelman.

digits, the circumference will not contain more than seven reliable digits, even if π is taken to a hundred digits. The fact that ancient mathematicians expended enormous effort to obtain possibly longer values of π has no practical significance. Moreover, the scientific value of these works is negligible. It's simply a matter of patience. If you have the inclination and leisure, you can find up to 1000 digits for π using, for example, the following infinite series discovered by Leibniz[45]:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

But this would be an unnecessary arithmetic exercise, far from contributing to the resolution of the famous geometric problem.

The French astronomer Arago, mentioned earlier, wrote the following about this:

> The seekers of the squaring of the circle continue to engage in solving a problem whose impossibility is now positively proven and which, even if it could be achieved, would be of no practical interest. It is not worth dwelling on this subject: the mentally ill, striving for the discovery of the squaring of the circle, are not swayed by any arguments. This mental illness has existed since ancient times.

And he ironically concludes:

[45] A lot of patience would be required for such a calculation because to obtain, for example, a six-digit π , it would be necessary to take in the specified series not a few, not a little – 2,000,000 terms.

The academies of all countries, fighting against the seekers of quadrature, have noticed that this disease usually intensifies in the spring.

Triangle of Bing

Let's consider one of the approximate solutions to the problem of squaring the circle, very convenient for practical purposes.

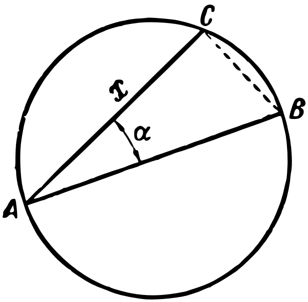


Figure 126: The method of the Russian engineer Bing (1836).

The method consists of computing (see Figure 126) the angle α, under which it is necessary to draw, to the diameter AB, the chord $AC = x$, which is one side of the desired square. To find the value of this angle, one will have to resort to trigonometry:

$$\cos\alpha = \frac{AC}{AB} = \frac{x}{2r},$$

where r is the radius of the circle.

This means that the side of the desired square $x = 2r\cos\alpha$, and its area is equal to $4r^2 cos^2\alpha$. On the other hand, the area of the square is equal to the area of the given circle πr^2.

Thus,

$$4r^2\cos^2\alpha = \pi r^2, \text{ from which,}$$

$$cos^2\alpha = \frac{\pi}{4}, \text{ and,}$$

$$\cos\alpha = \frac{1}{2}\sqrt{\pi} = 0.886.$$

By looking up in the tables, we find:

$$\alpha = 27^\circ 36'.$$

So, by drawing in this circle a chord at an angle of $27^\circ 36'$ to the diameter, we immediately obtain the side of the square whose area is equal to the area of the given circle. Practically, one prepares a drawing triangle[46], one of the acute angles of which is $27^\circ 36'$ (and the other is $62^\circ 24'$). Having such a triangle, one can immediately find the side of a square equal to it for each given circle.

[46] This convenient method was proposed in 1886 by the Russian engineer Bing; the mentioned drawing triangle is named after its inventor as the 'Triangle of Bing.'

For those who want to make such a drawing triangle for themselves, the following instruction may be useful.

Since the tangent of the angle $27^\circ 36'$ is 0.5923, or $\approx 23/44$, then the legs of such a triangle are related as 23:44. Therefore, by making a triangle with one leg, for example, 22 cm, and the other 11.5 cm, we will have what is required. It goes without saying that such a triangle can be used as an ordinary drawing instrument.

Head or Feet

It seems that one of Jules Verne's characters calculated which part of his body travelled the longest distance during his circumnavigation – his head or the tips of his feet. This is a very instructive geometric problem if framed in a certain way. We'll present it as follows:

Question Imagine that you've travelled around the Earth along the equator. By how much has the top of your head travelled a greater distance than the tip of your feet?

Answer The feet have travelled a distance of $2\pi R$, where R is the radius of the Earth. Meanwhile, the top of the head has travelled a distance of $2\pi(R + 1.7)$, where 1.7 meters is the height of a person. The difference in distances is $2\pi(R + 1.7) - 2\pi R = 2\pi(1.7) = 10.7$ m. Thus, the head has travelled 10.7 meters more than the feet.

Interestingly, the final answer does not involve the radius of the Earth. Therefore, the result would be the same on Earth, Jupiter, or even the smallest planetoid. In general, the difference in lengths between two concentric circles depends only on the distance between them, not their radii. Adding one centimetre to the radius of the Earth's orbit would increase its length by the same amount as an equal increase in the radius of any other circle.

This geometric paradox[47] underlies the following interesting problem, found in many collections of geometric puzzles.

[47] A paradox is a truth that seems implausible, in contrast to sophism – a false position that has the appearance of being true.

If a wire is stretched around the Earth's equator and its length is increased by 1 meter, would a mouse be able to pass between the wire and the Earth?

The usual answer is that the gap would be thinner than a hair: what is one meter compared to 40 million meters of the Earth's equator! However, the actual size of the gap is:

$$\frac{100}{2\pi}\,\text{cm} = 16\,\text{cm}.$$

Not only a mouse, but even a large cat could pass through such a gap.

Wire Along the Equator Problem

Question Now, imagine that the Earth is tightly encased along the equator with a steel wire. What would happen if this wire cooled by 1 °C? As the wire cools, it should shorten. If it doesn't break or stretch, how deeply will it embed into the soil?

Answer At first glance, such a slight decrease in temperature, just by 1 °C, might not seem to cause a noticeable embedding of the wire into the ground. However, calculations show otherwise.

Cooling by 1 °C, the steel wire shortens by one hundred thousandth of its length. With a length of 40 million meters (the length of the Earth's equator), the wire should shorten, as easily calculated, by 400 meters. However, the radius of this circle made by the wire will decrease not by 400 meters, but much less. To determine how much the radius will decrease, we need to divide 400 meters by 6.28, i.e., by 2π.

The result is about 64 m. Therefore, the wire, cooling by just 1 °C under the given conditions, would embed into the ground not just by a few millimeters, as one might think, but by more than 60 m!

Facts and Calculations

Question You have eight equal circles in front of you (see Figure 127). Seven of them are shaded – immovable, while the eighth (light one) rolls over them without slipping. How many revolutions will it make after going around the immovable circles once?

You can certainly figure this out practically: place eight coins of equal value on the table, for example, eight coins, arrange them as shown in the picture, and, pressing down on seven coins, roll the eighth one over them. To determine the number of revolutions, for example, watch the position of the number written on the coin. Every time the number returns to its initial position, the coin will have completed one full revolution around its centre.

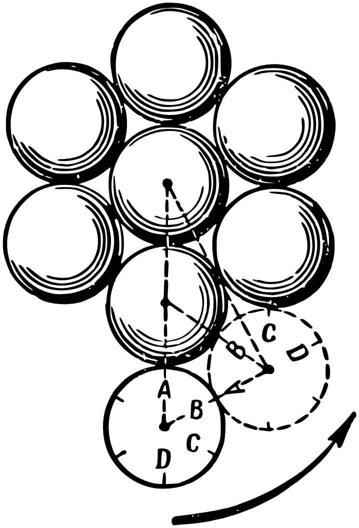


Figure 127: How many turns will the light circle make, bypassing the fixed shaded circles?

Perform this experiment not just in your imagination but in reality, and you will find that the coin makes a total of four revolutions.

Now let's try to arrive at the same answer through reasoning and calculations. Let's find out, for example, how much arc each immovable circle covers as the rolling circle moves. To do this, imagine the movement of the rolling circle from the "hill" *A* to the nearest "valley" between two immovable circles (shown by the dashed line in Figure 127).

From the diagram, it is easy to determine that the arc AB, along which the circle rolled, measures 60°. On the circumference of each immovable circle, there are two such arcs; together, they make up an arc of 120° or $1/3$ of the circumference.

Therefore, the rolling circle completes $1/3$ of a revolution, covering $1/3$ of each immovable circle. There are a total of six immovable circles; thus, the rolling circle will complete $1/3 \times 6 = 2$ revolutions.

This results in a discrepancy with the observations! But 'facts are stubborn things.' If observation does not confirm the calculation, then there must be a defect in the calculation.

Find the defect in the reasoning provided.

Answer The point is that when the circle rolls without slipping along a straight segment equal to $1/3$ of its circumference, it indeed completes $1/3$ of a revolution around its centre. This statement becomes untrue, not reflecting reality, when the circle rolls along an arc of a curved line. In the problem under consideration, as the rolling circle traverses an arc, for example, one-third of the length of its circumference, it completes not $1/3$ but $2/3$ of a revolution. Therefore, passing through six such arcs, it completes

$$6 \times \frac{2}{3} = 4 \text{ revolutions!}$$

You can visually confirm this. The dashed line in Figure 127 depicts the position of the rolling circle after it has rolled along the arc AB $(= 60^\circ)$ of an immovable circle, i.e., along an arc constituting 1/6 of the circumference. In the new position, the highest point on its circumference is occupied not by point A but by point C, which, as you can see, corresponds to a rotation of the points on the circumference by 120°, i.e., by 1/3 of a full revolution. A "path" of 120° corresponds to 2/3 of a full revolution of the rolling circle.

So, if the circle rolls along a curved (or piecewise linear) path, it completes a different number of revolutions than when it rolls along a straight path of the same length.

Let's dwell a little longer on the geometric aspect of this remarkable fact, especially since the usual explanation given for it is not always convincing.

Let a circle of radius r roll along a straight line. It completes one revolution along the segment AB, the length of which is equal to the circumference of the rolling circle, $2\pi r$. Let's fold the segment AB at its midpoint C (see Figure 128) and rotate the link CB by an angle α relative to its initial position.

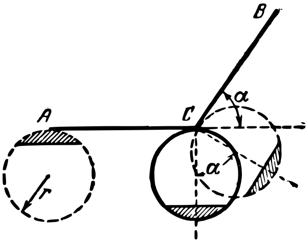


Figure 128: How an additional turn appears when the circle is rolling along a polyline.

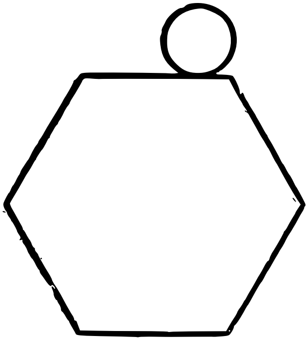


Figure 129: How an additional turn appears when the circle is rolling along a poly line.

Now, after making half a turn, the circle reaches the vertex C, and to assume such a position where it will touch the line CB at point C, it will rotate together with its centre by an angle equal to α (these angles are equal, as they have mutually perpendicular sides).

During this rotation, the circle rolls without moving along the segment. This creates the additional part of the full revolution compared to rolling along the straight line.

The additional rotation constitutes such a fraction of the full revolution as the angle α constitutes of the angle 2π, i.e., $\alpha/2\pi$. Along the segment CB, the circle also makes half a turn, so in total, when moving along the broken line ACB, it will complete $1 + \alpha/2\pi$ revolutions.

Now it's not difficult to imagine how many revolutions a circle rolling on the outside along the sides of a convex regular hexagon (see Figure 129) should make. It is obvious that it will make as many revolutions as it would wrap around on a straight path equal to the perimeter (i.e., the sum of the sides) of the hexagon, plus the number of revolutions equal to the sum of the external angles of the hexagon divided by 2π. Since the sum of the external angles of any convex polygon is constant and equal to $4d$ or 2π, then $2\pi/2\pi = 1$.

Thus, when circling the hexagon, as well as any convex polygon, the circle always makes one more revolution than when moving along a straight path equal to the perimeter of the polygon. With an infinite increase in the number of sides, a regular convex polygon approaches a circle. Therefore, all the considerations expressed remain valid for a circle as well. For example, according to the initially posed problem, if one circle rolls along an arc of 120° of another circle equal to it, then the assertion that the moving circle makes not 1/3 but 2/3 of a revolution becomes completely geometrically clear.

The Girl on the Tightrope

When a circle rolls along any line lying in the same plane as it, each point of the circle moves in the plane and has its own trajectory, as they say.

Trace the trajectory of any point on a circle rolling along a straight line or a circle, and you will see various curves.[48] Some of them are depicted in Figure 130 and 131.

[48] The reader will find a lot of useful and interesting information, as well as examples related to this issue, in the interesting book by G.N. Berman *Cycloid* (1948).

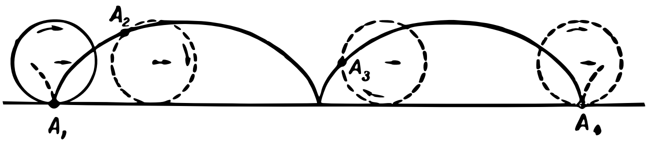


Figure 130: A cycloid is the trajectory of point A of the circumference of a disk rolling in a straight line without sliding.

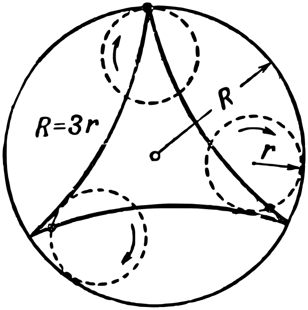


Figure 131: Three-point hypocycloid – the trajectory of points on the circumference of a disk rolling from the inside along the large circumference, where $R = 3r$.

The question arises: can a point on a circle rolling along the "inner side" of the circumference of another circle (131) describe not a curved line, but a straight one? At first glance, it seems impossible. However, I have seen such a construction with my own eyes. It was a toy – "the girl on the tightrope" (132). You can easily make it yourself. On a sheet of thick cardboard or plywood, draw a circle with a diameter of 30 cm so that there are margins on the sheet, and extend one of the diameters on both sides.

Insert a needle with a threaded thread into the ends of the diameter, stretch the thread horizontally, and attach both ends to the cardboard (front and back). Cut out the drawn circle, and place another cardboard (or plywood) circle with a diameter of 15 cm in the resulting window. At the very edge of the small circle, insert a needle as shown in Figure 132, cut out a figure of a girl acrobat from heavy paper, and glue her leg to the head of the needle with sealing wax.

Now try rolling the small circle while pressing it against the edges of the window; the head of the needle, and along with it the figure of the girl, will slide back and forth along the stretched thread. This can only be explained by the fact that the point on the rolling circle to which the needle is attached moves strictly along the diameter of the window.

But why, in a similar case depicted in Figure 131, does the

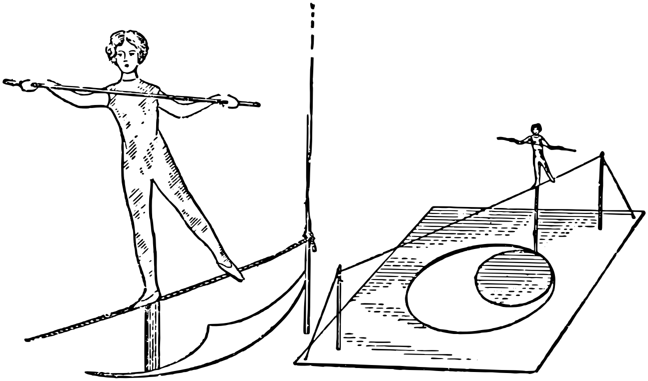


Figure 132: "A girl on a tightrope." There are points on a rolling circle that move in a straight line.

point on the rolling circle describe not a straight line, but a curved one (it is called a hypocycloid)? It all depends on the ratio of the diameters of the large and small circles.

Question Prove that if a circle with a diameter half that of a larger circle rolls along the circumference of the larger circle, then during this motion, any point on the circumference of the smaller circle will move along a straight line that is a diameter of the larger circle.

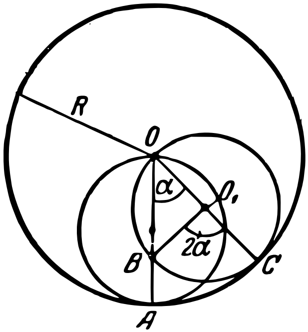


Figure 133: Geometric explanation of "Girl on a tightrope".

Answer If the diameter of circle O is half the diameter of circle O_1 (see Figure 133), then at any moment during the motion of circle O, one of its points is at the centre of circle

O_1.

Let's track the movement of point A. Suppose the small circle has rolled along arc AC. Where will point A be in the new position of circle O_1?

Obviously, it should be at a point B on its circumference such that arcs AC and BC are equal in length (the circle rolls without slipping). Let $OA = R$ and $\angle AOC = \alpha$. Then $AC = R\alpha$; therefore, $BC = R\alpha$. But since $O_1C = R/2$, then $\angle BO_1C = R \cdot \alpha/(R/2) = 2\alpha$, so $\angle BOC$ is half of $2\alpha/2 = \alpha$, i.e., point B remains on ray OA.

The toy described here represents a primitive mechanism for converting rotary motion into linear motion.

The construction of such mechanisms (also called inverters) has interested mechanical engineers since the time of the Ural mechanic Ivan Ivanovich Polzunov, the first inventor of the steam engine. Typically, these mechanisms, which impart linear motion to a point, have a hinge device.

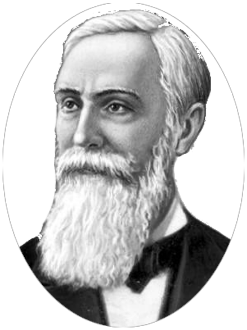


Figure 134: Pafnuty Lvovich Chebyshev (1821–1894).

A significant contribution to the mathematical theory of mechanisms was made by the brilliant Russian mathematician Pafnuty Lvovich Chebyshev (1821–1894) (Figure 134). He was not only a mathematician but also an outstanding mechanic. He built a model of a “walking” machine himself (it is still kept in the USSR Academy of Sciences), a mechanism for a self-propelled chair, and the best counting mechanism of the time – an arithmetic

The Path Through the Pole

You surely remember the famous flight of the Hero of the Soviet Union M.M. Gromov and his friends from Moscow to San Jacinto via the North Pole, when M.M. Gromov conquered two world records for non-stop flight – one for a straight-line flight (10,200 km) and another for a circuitous route (11,500 km) in 62 hours and 17 minutes.

Do you think the plane of the heroes who flew over the pole rotated along with the Earth's axis? This question is often asked, but the correct answer is not always given. Any aircraft, including one flying over the pole, must undoubtedly participate in the rotation of the Earth. This occurs because the flying plane is only separated from the solid part of the Earth but remains connected to the atmosphere and is carried along in the rotational motion around the Earth's axis.

Thus, during the flight over the pole from Moscow to America, the aircraft simultaneously rotated with the Earth around its axis. But what was the path of this flight?

To answer this question correctly, it is necessary to keep in mind that when we say "a body is in motion," it means that the position of the body changes relative to other bodies. The question of the path and movement would not make sense if the frame of reference, or simply put, the body with respect to which the movement occurs, is not specified (or at least implied), as mathematicians say.

Relative to the Earth, M.M. Gromov's aircraft moved almost along Moscow's meridian. Moscow's meridian, like any other, rotates with the Earth around its axis. Therefore, the aircraft adhered to the meridian during the flight and rotated with the Earth. However, this rotation is not reflected in the shape of the flight path for a ground observer because it occurs relative to another body, not the Earth.

Therefore, for those firmly connected to the Earth, the path of this heroic flight through the pole is an arc of a great circle if we assume that the plane moved precisely along the meridian and remained at the same distance from the centre of the Earth.

Now let's pose the question this way: we have the movement of the aircraft relative to the Earth, and we know that the aircraft and the Earth together rotate around the Earth's axis. That is, we have the movement of the aircraft and the Earth relative to some third body. What will be the flight path for an observer connected to this third body?

Let's simplify this unusual problem a bit. Let's imagine the polar region of our planet as a flat disk lying on a plane perpendicular to the Earth's axis. Let this imaginary plane be the "body" around which the disk rotates around the Earth's axis, and let a clockwork trolley roll uniformly along one of the diameters of the disk. It represents the plane flying along the meridian through the pole.

What line will be depicted on our plane to represent the path of the trolley (more precisely speaking, any single point of the trolley, for example, its centre of gravity)?

The time it takes for the trolley to travel from one end of the diameter to the other depends on its speed.

We will consider three cases:

1. The trolley completes its path in 12 hours;
2. It completes the path in 24 hours; and
3. It completes the same path in 48 hours.

The disk completes a full revolution in all cases within 24 hours.

The first case (see Figure 135). The trolley travels along the diameter of the disk in 12 hours. During this time, the disk completes half a revolution, i.e., it turns by 180°, and points A and A' switch places. In Figure 135, the diameter is divided into eight equal segments, each of which the trolley covers in 12/8 = 1.5 hours. Let's track where the trolley will be located 1.5 hours after the start of the movement. If the disk were not rotating, the trolley, starting from point A, would reach point b after 1.5 hours. But the disk rotates, and in 1.5 hours, it turns by $180°/8 = 45°$. During this time, point b of the disk moves to point b'. An observer standing on the disk and rotating with it would not notice its rotation and would only see that the trolley moved from point A to point b. But an observer who is outside the disk and does not participate in its rotation would see something else: for them, the trolley would move along a curved path from point A to point b'. Another 1.5 hours later, the observer standing outside the disk would see the trolley at point c'. Over the next 1.5 hours, the trolley would move along the arc $c'd'$, and after another 1.5 hours, it would reach the centre e.

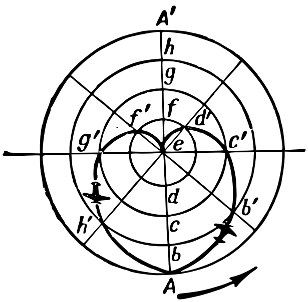


Figure 135: Curves that will be described on a fixed plane by a point involved in two movements.

Continuing to track the movement of the trolley, an observer standing outside the disk would see something completely unexpected: the trolley would describe a curve $ef'g'h'A$ for them, and the movement, strangely enough, would not end at the opposite point of the diameter, but at the starting point.

The solution to this unexpectedness is very simple: during the six hours of the trolley's journey along the second half of the diameter, this radius manages to turn together with the disk by 180° and occupy the position of the first half of the diameter. The trolley rotates together with the disk even at the moment when it passes over its centre. The trolley cannot fit entirely into the centre of the disk; it can only coincide with the centre at one point, and at that moment, it rotates entirely with the disk around this point. The same should happen with the plane when it flies over the pole. So, the journey of the trolley along the diameter of the disk from one end to the other appears differently to different observers. To someone standing on the disk and spinning with it, this path appears as a straight line. But to a stationary observer who is not participating in the rotation of the disk, the movement of the trolley appears as a curve, depicted in Figure 135 and resembling the outline of a heart.

Each of you would see the same curve if, for example, you were observing the flight of the plane from the centre of the Earth relative to an imaginary plane perpendicular to the Earth's axis, under the fantastic condition that the Earth is transparent, and you and the plane are not participating in the Earth's rotation, and if the flight across the pole of the observed plane lasted 12 hours.

Here we have an interesting example of the addition of two motions.

In fact, the flight across the pole from Moscow to the diametrically opposite point – the same parallel – did not last 12 hours, so we will now focus on analysing another prepara-

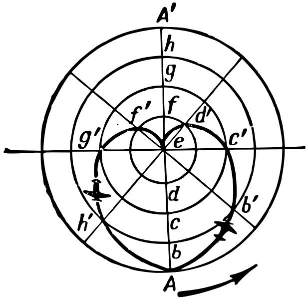


Figure 136: Curves that will be described on a fixed plane by a point involved in two movements.

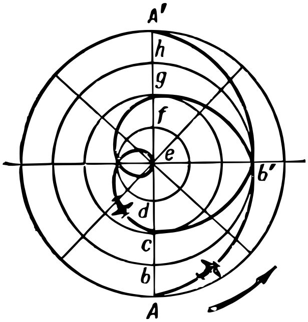


Figure 137: Two more curves resulting from the addition of two movements.

tory task of the same kind.

Second case (Figure 136): The trolley traverses the diameter in 24 hours. During this time, the disk completes a full revolution, and then, for an observer stationary relative to the disk, the path of the trolley's movement will take the form of the curve depicted in Figure 136.

Third case (Figure 137): The disk still completes a full revolution in 24 hours, but the trolley travels from one end to the other of the diameter in 48 hours.

This time, the trolley covers 1/8 diameters in 48 hours, which equals $48 : 8 = 6$ hours per diameter. During the same six hours, the disk manages to turn a quarter of a full revolution – 90°. Therefore, after six hours from the start of the movement, the trolley will move along the diameter (Figure 137) to point b, but the rotation of the disk will shift this point to b'. After another six hours, the trolley will reach point g, and so on. In 48 hours, the trolley covers the entire diameter, while the disk completes two full revolutions. The result of adding these two movements appears to a stationary observer as an intricate curve, depicted in Figure 137 as a solid line.

The case considered now brings us closer to the true conditions of the flight over the pole. The flight from Moscow to the pole by M. M. Gromov took approximately 24 hours; therefore, an observer at the center of the Earth would see this part of the trajectory as a line almost identical to the first half of the curve shown in Figure 137.

As for the second part of M. M. Gromov's flight, it lasted approximately one and a half times longer. Additionally, the distance from the pole to San Jacinto is also one and a half times longer than the distance from Moscow to the North Pole. Therefore, the trajectory of the second part of the journey would appear to a stationary observer as a line of the same shape as the first part of the journey but one and a half times longer.

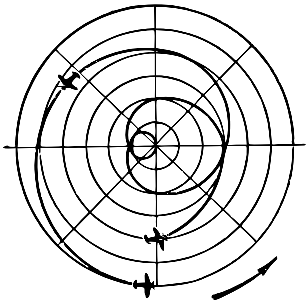


Figure 138: The flight path from Moscow to San Jacinto, which would appear to an observer who is not involved in either flight or the rotation of the Earth.

The resulting curve in the end is shown in Figure 138.

Many might be puzzled by the fact that the starting and ending points of the flight are shown in such close proximity on this diagram.

However, it should not be overlooked that the drawing does not show the simultaneous positions of Moscow and San Jacinto but rather their separation by a time interval of 2.5 hours.

So, this is roughly what the trajectory of M. M. Gromov's flight through the pole would have looked like if it were possible to observe the flight, for example, from the center of the Earth. Can we call this complex twist the true path of the flight over the pole, as opposed to the relative one depicted on maps? No, this motion is also relative: it is referred to some body not participating in the rotation of the Earth around its axis, just as the usual depiction of the flight path is related to the surface of the rotating Earth.

If we could observe the same flight from the Moon or the Sun[49], the flight path would appear to us in yet other ways.

[49] That is, relative to the coordinate system associated with the Moon or the Sun.

The Moon does not share the Earth's daily rotation, but it orbits our planet once a month. During the 62 hour flight from Moscow to San Jacinto, the Moon would have travelled about 30°, and this could not have gone unnoticed by the trajectory of the flight for a lunar observer. The shape of the air plane's trajectory, considered relative to the Sun, would also be influenced by a third movement – the rotation of the Earth around the Sun.

"There is no movement of a single body; there is only relative movement," says Friedrich Engels in *Dialectics of Nature*.

The task we have just considered convinces us of this in the most vivid way.

For the Drive Belt

When the students of the trade school finished their work, the master "in farewell" suggested to those willing to solve such

Question "For one of the new installations in our workshop", said the master, "we need to sew a drive belt—not just for two pulleys, as is often the case, but immediately for three", – and the master showed the students a drive diagram (Figure 139).

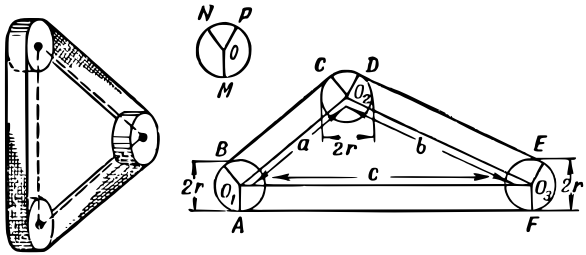


Figure 139: The drive circuit. How to determine the length of the drive belt using only the specified dimensions.

"All three pulleys," he continued, "have the same dimensions. Their diameter and the distances between their axes are indicated on the diagram."

"How, knowing these dimensions and without making any additional measurements, can we quickly determine the length of the drive belt?"

The students pondered. Soon, one of them said, "It seems to me that the only difficulty here is that the sizes of the arcs *AB*, *CD*, and *EF*, around which the belt wraps each pulley, are not indicated on the drawing. To determine the length of each of these arcs, we need to know the magnitude of the corresponding central angle, and it seems to me, we can't do without a protractor."

"The angles you're talking about", replied the master, "can even be calculated using the dimensions indicated on the drawing with the help of trigonometric formulas and tables, but this is a long and complicated path. A protractor is not needed here either, as there is no need to know the length of each arc of interest separately, it is sufficient to know ..."

"Their sum", some of the boys chimed in, realizing what was at stake.

"Now, go home and bring me your solutions tomorrow."

Don't rush, reader, to find out what solution the master's students brought him.

After all that has been said by the master, this problem is not difficult to solve independently.

Answer Indeed, the length of the drive belt is determined very simply: to the sum of the distances between the axes of the pulleys, you need to add the length of the circumference of one pulley. If the length of the belt is l, then:

$$l = a + b + c + 2\pi r.$$

Almost all those who attempted the problem guessed that the sum of the lengths of the arcs with which the belt comes into contact equals the entire circumference of one pulley, but not everyone managed to prove it.

From the solutions presented to the master, the following was recognised as the most concise.

Let BC, DE, and FA be tangents to the circles (see Figure 139). Draw radii to the points of tangency. Since the circles of the pulleys have equal radii, the figures O_1BCO_2, O_2DEO_3, and O_1O_3FA are rectangles, therefore, $BC + DE + FA = a + b + c$. It remains to show that the sum of the lengths of the arcs $AB + CD + EF$ equals the total circumference.

To do this, let's construct a circle O with radius r (see diagram, top). Draw $OM \parallel O_1A$, $ON \parallel O_1B$, and $OP \parallel O_2D$. Then $\angle MON = \angle AO_1B$, $\angle NOP = \angle CO_2D$, and $\angle ROM = \angle EO_3F$ as angles with parallel sides.

Hence, $AB + CD + EF = MN + NP + PR = 2\pi r$.

Thus, the length of the belt $l = a + b + c + 2\pi r$.

In the same way, it can be shown that not only for three, but also for any number of equal pulleys, the length of the drive belt will be equal to the sum of the distances between their axes plus the circumference of one pulley.

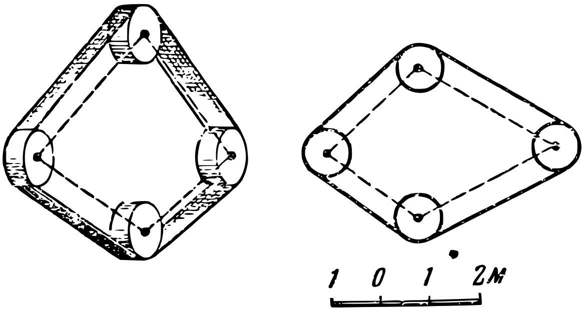


Figure 140: Extract the necessary dimensions from the drawing and calculate the length of the conveyor belt.

Answer On the diagram in Figure 140, a scheme of a conveyor with four equal rollers is shown (there are intermediate rollers, but they are omitted on the diagram as they do not affect the solution of the problem). Using the scale indicated on the diagram, take the necessary dimensions from the drawing and calculate the length of the conveyor belt.

The Raven's Cleverness

In our school readers in the native language, there is a funny story about a "clever raven". This ancient tale tells of a raven suffering from thirst who found a pitcher of water. There was little water in the pitcher, and the raven couldn't reach it with its beak. However, the raven seemed to figure out how to help itself: it started dropping pebbles into the pitcher. As a result of this trickery, the water level rose to the brim of the pitcher, and the raven could quench its thirst.

We won't delve into discussing whether the raven could display such resourcefulness. The case interests us from a geometric perspective. Let's consider the following problem:

Question Would the raven be able to drink if the water in the pitcher was filled only halfway?

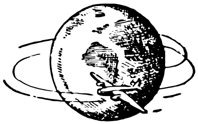
Answer The analysis of the problem will convince us that the method employed by the raven achieves its goal only under certain initial water levels in the pitcher.

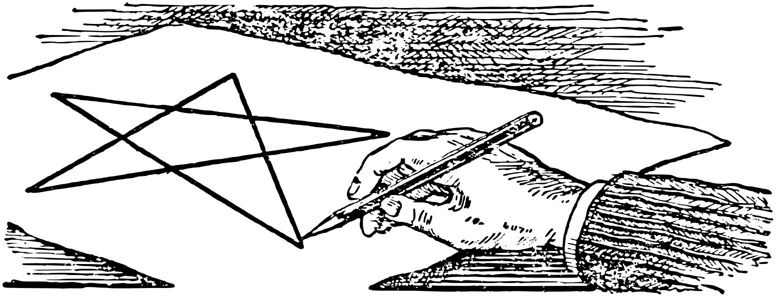
For simplicity, let's assume that the pitcher has the shape of a rectangular prism, and the pebbles are spherical and of equal size. It is easy to understand that the water will rise above the level of the pebbles only if the initial water supply occupies a larger volume than all the gaps between the pebbles: then the water will fill the gaps and protrude above the pebbles. Let's try to calculate the volume of these gaps. The easiest calculation is done when the pebbles are arranged in such a way that the centre of each lies on the same vertical line as the centres of the upper and lower pebbles. Let the diameter of a pebble be d, and therefore its volume be $1/6\pi d^3$, and the volume of the cube circumscribed around it be d^3. The difference in their volumes, $d^3 - 1/6\pi d^3$, is the volume of the unfilled part of the cube, and the ratio

$$\frac{d^3 - \frac{1}{6}\pi d^3}{d^3} = 0.48$$

means that the unfilled part of each cube constitutes 0.48 of its volume. The sum of the volumes of all the voids relative to the volume of the pitcher is much less than half. The situation hardly changes if the pitcher has a non-prismatic shape and the pebbles are not spherical. In all cases, it can be asserted that if initially the water in the pitcher was poured to below half, the raven would not have been able to raise the water to the brim by dropping pebbles.

If the raven were stronger – strong enough to pack the pebbles tightly in the pitcher and achieve their dense arrangement – it would have been able to raise the water more than twice the initial level. But this is beyond its capabilities, and by assuming a loose arrangement of the pebbles, we have not deviated from the real conditions. Moreover, pitchers are usually bulged in the middle; this should also reduce the height of the water rise and corroborate the correctness of our conclusion: if the water stood below half the height of the pitcher, the raven would not have been able to drink.





10. Geometry Without Measurements And Without Calculations

Building Without a Compass

When solving geometric construction problems, rulers and compasses are usually used. However, we will now see that sometimes it is possible to do without a compass in such cases where at first glance it seems absolutely necessary.

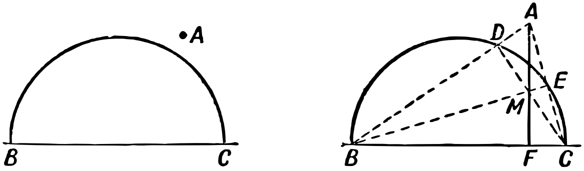


Figure 141: The task of building and solving it. The first case.

Question From point A (Figure 141, left), lying outside the given semicircle, drop a perpendicular to its diameter without using a compass. The position of the centre of the semicircle is not indicated.

Answer We will use the property of a triangle that all its altitudes intersect at a single point. Connect A with B and C; we get points D and E (Figure 141, right). The lines BE and CD are obviously the altitudes of triangle ABC. The third altitude, which is the required perpendicular to BC, must pass through the intersection point of the other two, i.e., point M. By drawing a line through points A and M with a ruler, we meet the requirements of the task without using a compass.

If the point is positioned such that the required perpendicular falls on the extension of the diameter (Figure 142), then the problem is solvable only if a full circle, rather than a

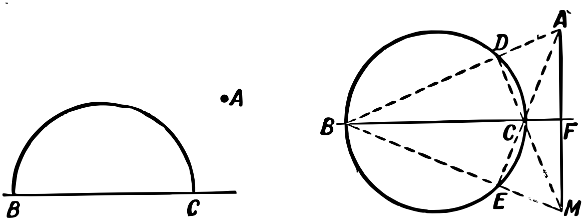


Figure 142: The same task. Second case.

semicircle, is given. Figure 142 shows that the solution is no different from the one we are already familiar with; only the altitudes of triangle *ABC* intersect not inside, but outside of it.

Centre of Gravity of a Plate

Question You probably know that the centre of gravity of a thin, uniform plate, which has the shape of a rectangle or a rhombus, is located at the intersection of the diagonals, and if the plate is triangular, it is at the intersection of the medians. If the plate is circular, it is at the centre of the circle.

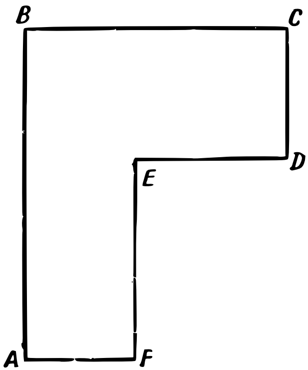


Figure 143: Using only a ruler, find the centre of gravity of the depicted plate.

Now try to figure out how to find the centre of gravity of a plate made up of two arbitrary rectangles joined into one shape, as shown in Figure 143. Let's agree to use only a ruler and not to measure or calculate anything.

Answer Extend side *DE* to intersect with *AB* at point *N* and side FE to intersect with BC at point M (Figure 144). We will first consider the given figure as composed of the rectangles $ANEF$ and $NBCD$. The centre of gravity of each of them is at the intersection points of their diagonals, O_1 and O_2.

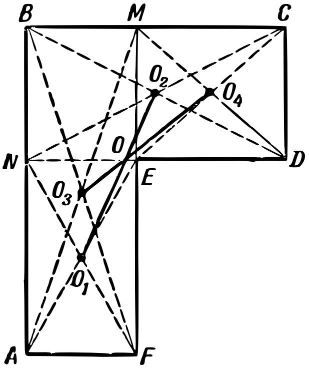


Figure 144: The centre of gravity of the plate found.

Therefore, the centre of gravity of the entire figure lies on the line O_1O_2. Now, consider the same figure as composed of the rectangles $ABMF$ and $EMCD$, whose centres of gravity are at the intersection points of their diagonals, O_3 and O_4. The centre of gravity of the entire figure lies on the line O_3O_4. Hence, it lies at the point O where the lines O_1O_2 and O_3O_4 intersect. All these constructions are indeed carried out only with the help of a ruler.

Napoleon's Task

We have just been dealing with a construction performed using only a ruler, without resorting to a compass (on the condition that one circle is given in the drawing in advance). Now let us consider several problems in which the opposite restriction is introduced: the use of a ruler is prohibited, and all constructions must be performed using only a compass. One such problem interested Napoleon (who, as is well known, was interested in mathematics). After reading a book on such constructions by the Italian scientist Mascheroni, he posed the following problem to French mathematicians:

Question Divide a given circle into four equal parts without using a ruler. The position of the centre of the circle is given.

Answer Suppose we need to divide circle O (Figure 145) into four parts. From an arbitrary point A, we lay off three times the radius of the circle along the circumference: we obtain points B, C, and D. It is easy to see that the distance AC, the chord of the arc constituting 1/3 of the circle, is the side of the inscribed equilateral triangle and therefore equals $r\sqrt{3}$, where r is the radius of the circle. AD is obviously the diameter of the circle. From points A and D, with a radius equal to AC, we draw arcs intersecting at point M. We will show that the distance MO is equal to the side of the square inscribed in our circle. In triangle AMO, the leg MO equals $\sqrt{AM^2 - AO^2} = \sqrt{3r^2 - r^2} = r\sqrt{2}$, i.e., the side of the inscribed square. Now, using the compass set to MO, we lay off four points on the circle in succession to obtain the vertices of the inscribed square, which will obviously divide the circle into four equal parts.

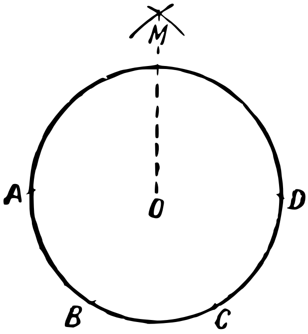


Figure 145: Divide the circle into four equal parts using the ruler.

Question Here is another, easier problem of the same kind. Without a ruler, increase the distance between given points A and B (see Figure 146) by five times, or more generally, by any given factor.

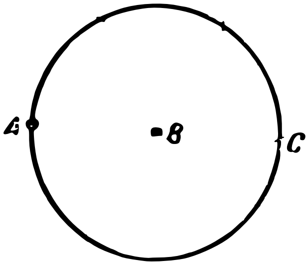


Figure 146: How can we increase the distance between points A and B by a factor n (an integer) using only a compass?

Answer From point B, draw a circle with radius AB (see Figure 146). Along this circle, measure the distance AB three times from point A: you get point C, which is obviously diametrically opposite to A. The distance AC is twice the distance AB. By drawing a circle from point C with radius BC, we can similarly find a point diametrically opposite to B and, therefore, three times the distance AB from A, and so on.

Simple Trisector

Using only a compass and an unmarked ruler, it is impossible to divide an arbitrarily given angle into three equal parts. However, mathematics does not completely rule out the possibility of performing this division using some other devices. Many mechanical devices have been invented to achieve this goal. These devices are called trisectors. You can easily make a simple trisector from stiff paper, cardboard, or thin metal. It will serve as an auxiliary drawing tool.

In Figure 146, the trisector is shown at full scale (shaded figure). The strip AB adjacent to the semicircle is equal in length to the radius of the semicircle. The edge of the strip BD forms a right angle with the line AC; it touches the semicircle at point B; the length of this strip is arbitrary. The same figure shows how to use the trisector. For example,

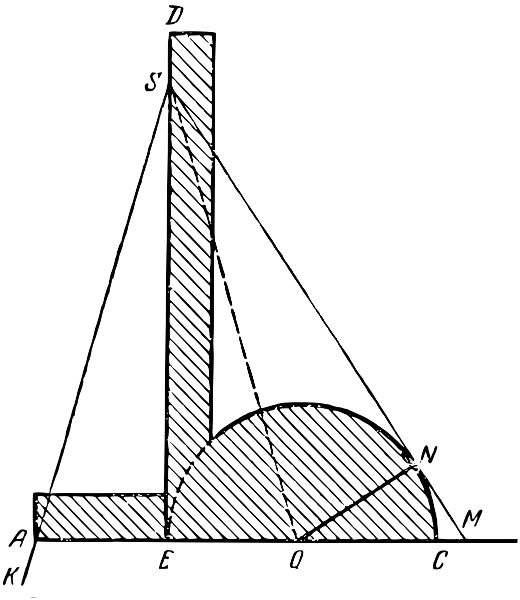


Figure 147: Trisector and its usage scheme.

let's say you need to divide the angle KSM (Figure 146) into three equal parts.

Place the trisector so that the vertex of the angle S is on the line BD, one side of the angle passes through point A, and the other side touches the semicircle at point[50]. Then draw

[50] The possibility of fitting our trisector into a given angle follows from a simple property of the points on the rays dividing the angle into three equal parts: if from any point O on the ray SO you draw segments $ON \perp SM$ and $OA \perp SB$ (Figure 147), then we have: $AB = OB = ON$. The reader can easily prove this for themselves.

straight lines SB and SO, and the division of the given angle into three equal parts is complete. To prove this, connect the centre of the semicircle O to the point of tangency N with a straight line segment. It is easy to see that triangle ASB is equal to triangle SBO, and triangle SBO is equal to triangle OSN. From the equality of these three triangles, it follows that angles ASB, BSO, and OSN are equal to each other, which is what was required to be proved.

This method of angle trisection is not purely geometric; it can be called mechanical.

Clock Trisector

Question Is it possible to divide a given angle into three equal parts using a compass, a ruler, and a clock?

Answer Yes, it is possible. Transfer the figure of the given angle onto transparent paper and, at the moment when both clock hands overlap, place the drawing onto the clock face so that the vertex of the angle coincides with the centre of the clock hands' rotation and one side of the angle aligns with the hands (Figure 148).

At the moment when the minute hand moves to coincide with the direction of the second side of the given angle (or move it manually), draw a ray from the vertex of the angle in

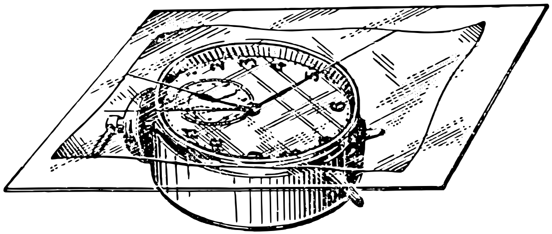


Figure 148: Clock Trisector.

the direction of the hour hand. This creates an angle equal to the angle of the hour hand's movement. Now, using a compass and a ruler, double this angle and then double the doubled angle again (the method for doubling an angle is well known in geometry). The angle obtained in this way will be one-third of the given angle.

Indeed, every time the minute hand describes a certain angle α, the hour hand moves by an angle that is 12 times smaller: $\alpha/12$, and after quadrupling this angle, you get $\alpha/12 \times 4 = \alpha/3$.

Dividing a Circle

Radio enthusiasts, designers, builders of various models, and enthusiasts sometimes face the practical problem of cutting a given sheet into a regular polygon with a specified number of sides.

This task can be reduced to:

Dividing a circle into n equal parts, where n is an integer.

Let's put aside the obvious solution using a protractor—since that's essentially an "eyeball" method—and consider a geometric solution using a compass and a ruler.

First, let's address the question: into how many equal parts can a circle theoretically be divided using a compass and a ruler? Mathematicians have fully resolved this: not into any number of parts.[51]

[51] For details, see the geometry textbook.

Possible divisions:

$$2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 15,\ 16,\ 17,\ 257,\ \text{etc.}$$

Impossible divisions:

$$7,\ 9,\ 11,\ 13,\ 14,\ \text{etc.}$$

Furthermore, there is no single method for constructing these divisions; the technique for dividing into 15 parts is different from that for 12 parts, and so on, making it difficult to remember all methods.

A practical geometric method is needed – one that is approximate but simple and general enough for dividing a circle into any number of equal arcs.

Unfortunately, geometry textbooks do not yet address this issue, so here we will present an interesting method for an approximate geometric solution to this problem.

For example, let's say we need to divide a given circle (Figure 149) into nine equal parts. Construct a scalene triangle ACB on some diameter AB of the circle and divide the diameter AB at point D in the ratio $AD : AB = 2 : 9$ (generally $AD : AB = 2 : n$).

Connect points C and D with a line segment and extend it to intersect the circle at point E. Then, the arc $\overset{\frown}{AE}$ will be approximately 1/9 of the circle (generally $\overset{\frown}{AE} = 360°/n$), or chord AE will be the side of a regular inscribed nonagon (n-gon).

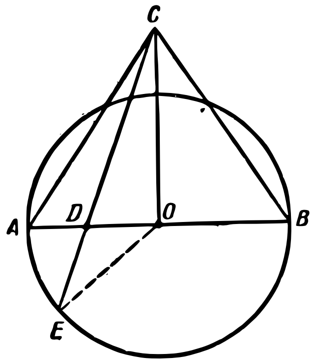


Figure 149: An approximate geometric method of dividing a circle into n equal parts.

The relative error in this method is about 0.8%.

If we express the relationship between the central angle AOE, formed in the described construction, and the number of divisions n, we get the following exact formula:

$$\tan \widehat{AOE} = \frac{\sqrt{3}}{2} \cdot \frac{n^2 + 16n - 32 - n}{n - 4}.$$

For larger values of n, this can be approximated by the formula:

$$\tan \widehat{AOE} = 4\sqrt{3} \cdot \left(\frac{1}{n} - \frac{2}{n^2}\right).$$

On the other hand, for exact division of the circle into n equal parts, the central angle should be equal to $360°/n$. By comparing the angle $360°/n$ with the angle AOE, we obtain the error made when considering the arc AE as $1/n$ part of the circle. The following table shows the error for some values of n.

n	3	4	5	6	7	8	10	20	60
$360°/n$	120°	90°	72°	60°	51°26′	45°	36°	18°	6°
$\widehat{AOE}$	120°	90°	71°57′	60°	51°31′	45°11′	36°21′	18°38′	6°26′
Error %	0	0	0.07	0	0.17	0.41	0.97	3.5	7.2

As can be seen from the table, this method can approximately divide the circle into 5, 7, 8, or 10 parts with a small relative error ranging from 0.07% to 1%, which is quite acceptable for most practical purposes. However, as the number of divisions n increases, the accuracy of the method decreases noticeably, i.e., the relative error increases, but studies show that for any n, the error does not exceed 10%.

Direction of the Shot (Billiard Ball Problem)

Hitting a billiard ball into a pocket by making it bounce off one, two, or even three rails of the table means, first of all, solving a geometric "construction" problem in your mind.

It's crucial to accurately "eyeball" the first point of impact on the rail; the subsequent path of the elastic ball on a good table will be determined by the law of reflection (*the angle of incidence equals the angle of reflection*).

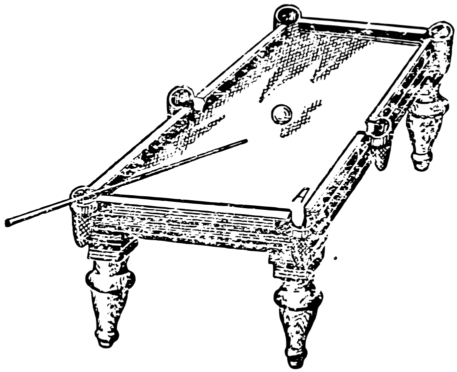


Figure 150: A geometric problem on a billiard table.

Question What geometric concepts can help you find the direction of the shot so that a ball located, for example, in the middle of the billiard table, after three bounces, lands in pocket A? (Figure 150).

Answer You need to imagine that three more tables are placed along the short side of the billiard table, and aim in the direction of the farthest pocket on the third imaginary table.

Figure 151 helps clarify this statement. Let $OabcA$ be the path of the ball. If you flip the "table" $ABCD$ around CD by 180° degrees, it will occupy position I, then flip it again around AD and once more around BC, it will occupy position III. As a result, pocket A will be at the point marked A_1.

Based on the obvious equality of the triangles, you can easily prove that $ab_1 = ab$, $b_1c_1 = bc$, and $c_1A_1 = cA$, i.e., the length of the straight line OA_1 is equal to the length of the broken line $OabcA$.

Therefore, aiming at the imaginary point A_1 will cause the ball to travel along the path $OabcA$, and it will land in pocket A.

Let's consider another question: under what condition will the sides OE and A_1E of the right triangle A_1EO be equal?

It is easy to establish that $OE = 5/2\,AB$ and $A_1E = 3/2\,BC$. If $OE = A_1E$, then $5/2AB = 3/2\,BC$, or $AB = 3/5\,BC$.

Thus, if the short side of the billiard table is 3/5 of the long side, then $OE = EA_1$; in this case, you can direct the shot of a ball located in the middle of the table at a 45° angle to the rail.

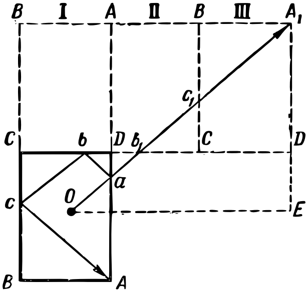


Figure 151: A geometric problem on a billiard table.

The "Smart" Ball

Simple geometric constructions just helped us solve a problem involving a billiard ball. Now let's have the same billiard ball solve an interesting, old problem on its own.

Is that even possible? – A ball can't think. True, but when a calculation needs to be performed, and we know the operations and their order, such a calculation can be entrusted to a machine, which will perform it quickly and without error.

Many mechanisms have been invented for this purpose, from simple mechanical calculators to complex electronic machines.

In leisure time, people often entertain themselves with the problem of how to pour a specific amount of water from a filled container of known capacity using two other empty containers, also of known capacities.

Here is one of many problems of this kind:

Divide the contents of a 12-bucket barrel in half using two empty barrels of nine buckets and five buckets.

To solve this problem, you don't need to experiment with actual barrels. All necessary "pourings" can be done on paper with a simple scheme:

9 Buckets	0	7	7	2	2	0	9	6	6
5 Buckets	5	5	0	5	0	2	2	5	0
12 Buckets	7	0	5	5	10	10	1	1	6

In each column, record the result of the current pouring.

Fill the five-bucket barrel. The nine-bucket barrel is empty (0), and seven buckets remain in the 12-bucket barrel.

Pour the seven buckets from the 12-bucket barrel into the nine-bucket barrel, and so on.

The scheme has nine columns, indicating that nine pourings were needed to solve the problem.

Try to find your own solution to the proposed problem, establishing a different order of pourings.

After several attempts, you will undoubtedly succeed, as the proposed pouring scheme is not the only possible one; however, with a different order of pourings, you might need more than nine.

In this context, it is interesting to explore the following:

1. Can a specific order of pourings be established that can be followed in all cases, regardless of the capacities of the containers?

2. Can any possible amount of water be poured from a third container using two empty containers, e.g., for example, from a 12-bucket barrel using barrels in 9 and 5 buckets, one bucket of water, or two buckets, or three, four, etc. to 11?

The "smart" ball will answer all these questions if we now construct a special "billiard table" for it.

Draw a sheet of paper in a slant grid so that the cells are equal rhombuses with acute angles of 60°, and construct the figure *OABCD* as shown in Figure 152.

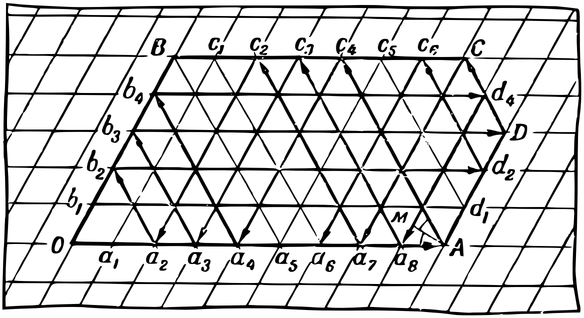


Figure 152: The "mechanism" of the "smart" ball.

This will be the "billiard table". If you push the billiard ball along *OA*, it will bounce off the rail *AD* exactly according to the law *the angle of incidence equals the angle of reflection*

($\angle OAM = \angle MAc_4$), and the ball will roll along the straight line Ac_4 connecting the vertices of the small rhombuses; it will bounce at point c_4 from the edge BC and roll along the line c_4a_4, then along the lines a_4b_4, b_4d_4, d_4a_8, and so on.

According to the conditions of the task, we have three barrels: nine, five And 12 buckets. In accordance with this, we will construct the figure so that the side OA contains nine cells, OB contains five cells, AD contains three cells ($12 - 9 = 3$), BC contains seven cells[52] ($12 - 5 = 7$).

Note that each point on the sides of the figure is separated by a certain number of cells from the sides OB and OA. For example, from point c_4 – it is four cells to OB and five cells to OA, from point a_4 – it is four cells to OB and zero cells to OA (because it lies on OA), from point d_4 –, it is eight cells to OB and four cells to OA, and so on.

Thus, each point on the sides of the figure where the billiard ball hits determines two numbers.

Let us agree that the first number, i.e., the number of cells separating the point from OB, indicates the number of buckets of water in the nine-bucket barrel, and the second number, i.e., the number of cells separating the same point from OA, indicates the number of buckets of water in the five-bucket barrel. The remaining amount of water will obviously be in

[52] A filled barrel is always the largest of the three. Let the capacity of empty barrels a and b be filled with c. If $c \geqslant a+b$, then the "billiard table" should be built in the form of a parallelogram with the sides a and b of the cells.

the 12-bucket barrel.

Now everything is prepared to solve the problem using the billiard ball.

Send it again along OA and, decoding each point where it hits the edge as indicated, follow its movement at least to point a_6 (Figure 152).

The first point of impact is $A\,(9; 0)$; this means the first transfer should result in the following distribution of water:

9 Buckets	9
5 Buckets	0
12 Buckets	3

This is feasible. The second point of impact is $c_4\,(4; 5)$; therefore, the ball recommends the following result for the second transfer:

9 Buckets	9	4
5 Buckets	0	5
12 Buckets	3	3

This is also feasible. The third point of impact is $a_4\,(4; 0)$; for the third transfer, the ball suggests returning five buckets to the 12-bucket barrel:

9 Buckets	9	4	4
5 Buckets	0	5	0
12 Buckets	3	3	8

The fourth point of impact is b_4 $(0; 4)$; the result of the fourth transfer:

9 Buckets	9	4	4	0
5 Buckets	0	5	0	4
12 Buckets	3	3	8	8

The fifth point of impact is d_4 $(8; 4)$, and the ball insists on transferring eight buckets to the empty nine-bucket barrel:

9 Buckets	9	4	4	0	8
5 Buckets	0	5	0	4	4
12 Buckets	3	3	8	8	0

Continue to follow the ball, and you will get the following table:

9 Buckets	9	4	4	0	8	8	3	3	0	9	7	7	2	2	0	9	6	6
5 Buckets	0	5	0	4	4	0	5	0	3	3	5	0	5	0	2	2	5	0
12 Buckets	3	3	8	8	0	4	4	9	9	0	0	5	5	10	10	1	1	6

So, after a series of transfers, the goal is achieved: six buckets of water in each of the two barrels. The ball has solved the problem!

However, the ball turned out not to be very smart.

It solved the problem in 18 moves, while we managed to solve it in nine moves (see the first table).

However, the ball can also shorten the series of transfers. Push it first along *OB*, stop it at point *B*, then push it again along *BC*, and then let it move as agreed – according to the law of *the angle of incidence equals the angle of reflection.* This will result in a shorter series of transfers.

If you allow the ball to continue moving after point a_6, you can easily verify that in the case under consideration, it will cover all marked points on the sides of the figure (and generally all vertices of the rhombuses) and only after that will return to the starting point *O*. This means that from the 12-bucket barrel, you can pour any integer number of buckets from one to nine into the nine-bucket barrel, and from one to five into the five-bucket barrel.

But a task of this kind may not have the required solution.

How does the ball detect this?

Very simply: in this case, it will return to the starting point *O* without hitting the desired point.

Figure 153 shows the mechanism for solving the problem for barrels of nine, seven, and twelve buckets:

9 Buckets	9	2	2	0	9	4	4	0	8	8	1	1
7 Buckets	0	7	0	2	2	7	0	4	4	0	7	0
12 Buckets	3	3	10	10	1	1	8	8	0	4	4	11
contd.												
9 Buckets	0	9	3	3	0	9	5	5	0	7	7	0
7 Buckets	1	1	7	0	3	3	7	0	5	5	0	7
12 Buckets	11	2	2	9	9	0	0	7	7	0	5	5

The "mechanism" shows that from a filled twelve-bucket barrel, using empty nine- and seven-bucket barrels, you can pour any number of buckets except half of its contents, i.e., except six buckets.

6 Buckets	6	3	3	0
3 Buckets	0	3	0	3
8 Buckets	2	2	5	5

Figure 154 shows the mechanism for solving the problem for barrels of three, six, and eight buckets. Here, the ball makes four bounces and returns to the starting point O.

The corresponding table shows that in this case, it is impossible to pour four buckets or one bucket from the eight-bucket barrel.

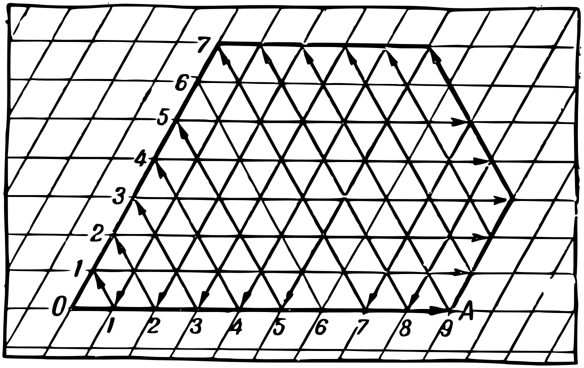


Figure 153: The "mechanism" shows that a full barrel of 12 buckets cannot be poured in half using empty barrels of nine and seven buckets.

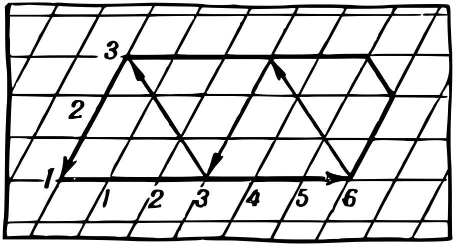


Figure 154: The "mechanism" for solving another transfer problem.

Thus, our "billiards" with the "smart" ball indeed serves as an interesting and peculiar calculating machine, quite adept at solving pouring problems.

In One Stroke

Question Redraw on a sheet of paper the five figures shown in Figure 155 and try to trace each of them in one stroke, i.e., without lifting your pencil from the paper and without tracing the same line more than once.

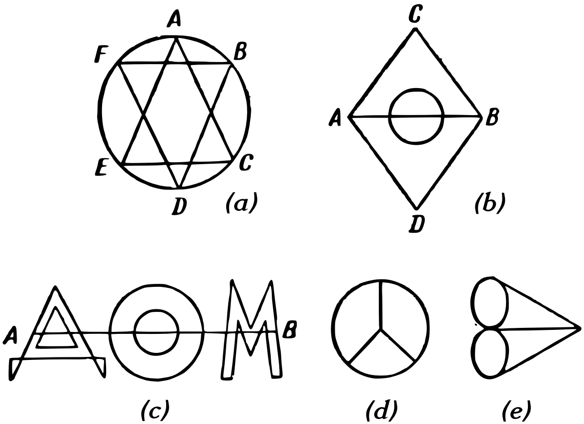


Figure 155: Try to draw each shape with a single stroke, without drawing the same line more than once.

Many of those who were given this problem started with figure *(d)*, as it appeared to be the simplest. However, all their attempts to draw this figure in one stroke failed. Disheartened, they approached the remaining figures with less confidence and, to their surprise and satisfaction, managed to solve the first two figures and even the intricate third one, which represents the crossed-out word *AOM* *(c)*, without much difficulty. But no one managed to trace the fifth figure *(e)*, just like the fourth figure *(d)*, in one stroke.

Why is it possible to solve the problem for some figures but not for others? Is it perhaps only because our ingenuity is lacking in some cases, or maybe the problem itself is generally unsolvable for certain figures? Is there any way to indicate a criterion that would allow us to determine in advance whether we can trace a given figure in one stroke or not?

Answer Let's call each intersection where the lines of a given figure meet a node. We will call a node even if an even number of lines meet at that point, and odd if an odd number of lines meet there. In figure *(a)*, all nodes are even; in figure *(b)*, there are two odd nodes (points A and B); in figure *(c)*, the ends of the segment crossing out the word *AOM* are odd nodes; in figures *(d)* and *(e)*, there are four odd nodes each.

Let's first consider a figure in which all nodes are even, such as figure *(a)*. We start our path from any point S. Passing through node A, for example, we draw two lines: one leading to A and one leading out from A. Since each even node has as many exits as it has entrances, as we move from node to node, the number of unmarked lines decreases by two each time. Therefore, in principle, it is quite possible to return to the starting point S after traversing all the lines.

But suppose we return to the starting point and there are no more exits from it, yet there is still an unmarked line on the figure, originating from some node B that we have already visited. This means we need to adjust our path: upon reaching node B, we first need to mark the missed lines and, returning to B, continue along the original path.

For example, let's say we decide to traverse figure *(a)* as follows: first along the sides of the triangle ACE, then, returning to point A, along the perimeter $ABCDEFA$ (see Figure 155). Since this leaves the triangle BDF unmarked, before we leave node B and proceed along arc BC, we should first traverse triangle BDF.

Therefore, if all nodes of a given figure are even, then starting from any point in the figure, it is always possible to mark the entire figure with a single stroke. In this case, the traversal of the figure should end at the same point from which it started.

Now, let's consider a figure that has two odd nodes.

Figure *(b)*, for example, has two odd nodes A and B.

It can also be traced with a single stroke.

In fact, let's start the detour from odd node № 1 and follow some line to odd node № 2, for example, from *A* to *B* along the *ACB* in figure *(b)* (Figure 155).

By drawing this line, we thereby exclude one line from each odd node, as if this line did not exist in the figure. Both odd nodes then become even. Since there were no other so-called even nodes in the figure, now we have a figure with only even nodes; in figure *(b)*, for example, after drawing the *ACB* line, a triangle with a circle remains.

So, if the figure contains two odd nodes, then a successful stroke should start in one of them and end in the other.

One additional note: starting from the odd node № 1, it is necessary to choose the path leading to the odd node № 2 so that no figures isolated from this figure are formed. [53]. For example, when drawing figure *(b)* in Figure 155, it would be unsuccessful to hurry from the odd node *A* to the odd node *B* in a straight line *AB*, since in this case the circle would remain isolated from the rest of the figure and not drawn.

[53] The inquisitive reader will find the details related to the issue under discussion in topology textbooks.

So, if the figure contains two odd nodes, then a successful stroke should start at one of them and end at the other.

Thus, the ends of the stroke are separated.

From here, it follows in turn that if a figure has four odd nodes, then it can be traced not with one stroke, but with two, but this no longer corresponds to the condition of our problem. Such are, for example, figures *(d)* and *(e)* in Figure 155.

As you can see, if one learns to reason correctly, much can be anticipated, thereby saving oneself from unnecessary expenditure of effort and time, and correct reasoning teaches, among other things, geometry.

Perhaps, dear reader, you may have found the reasoning presented here somewhat tiresome, but your efforts are rewarded by the advantage that knowledge brings over ignorance.

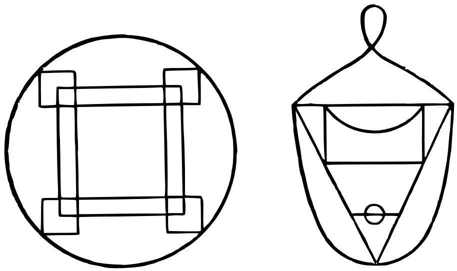


Figure 156: Draw each shape with a single stroke.

You can always determine in advance whether the task of traversing a given figure is solvable and know from which node to start its traversal.

Moreover, you can now easily come up with as many as you want intricate figures of this kind for your friends. As a conclusion, draw a couple more figures depicted in Figure 156.

The Seven Bridges of Königsberg

Two hundred years ago, in the city of Königsberg[54], there were seven bridges connecting the banks of the Pregel River (see Figure 157).

[54] At that time it was called Königsberg.

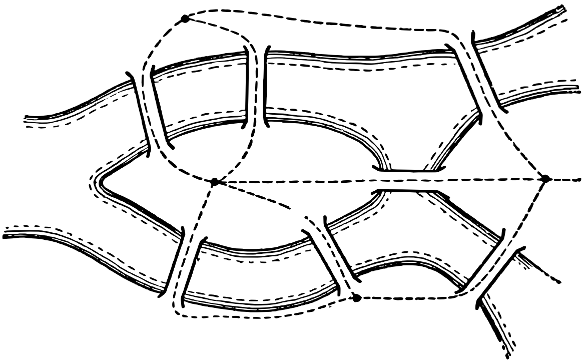


Figure 157: Is it possible to cross all these seven bridges, having visited each of them only once?

In 1736, the eminent mathematician of the time, Leonhard Euler (then around 30 years old), became interested in the following question: is it possible, while strolling through the city, to cross all seven bridges, but each only once?

It is easy to see that this problem is equivalent to the recently discussed problem of tracing a figure.

Let's depict the possible paths (dotted lines in Figure 157). It turns out to be one of the figures from the previous problem with four odd nodes (see Figure 155, figure *(e)*). As you now know, it cannot be traced with a single stroke, and therefore, it is impossible to cross all seven bridges, passing over each of them only once. Euler then proved this.

Geometrical Joke

After you and your companions have learned the secret of successfully tracing a figure with a single stroke, announce to your friends that you are still willing to draw a figure with four odd nodes, for example, a circle with two diameters (see Figure 158), without lifting the pencil from the paper and without tracing any line twice.

You know perfectly well that this is impossible, but you can insist on your sensational statement. Let me now teach you a little trick.

Start drawing the circle from point *A* (Figure 158). As soon as you draw a quarter of the circle – arc *AB*, place another sheet of paper at point *B* (or fold the bottom part of the sheet

Figure 158: Geometric joke.

on which you are making the construction) and continue tracing the lower part of the semicircle to point *D*, opposite point *B*.

Now remove the placed piece of paper (or unfold your sheet). On the front side of your sheet of paper, only the arc *AB* will be drawn, but the pencil will be at point *D* (even though you did not lift it from the paper!).

It's easy to complete the figure: first draw arc DA, then diameter AC, arc CD, diameter DB, and finally arc BC. You can also choose another route from point D; find it.

Form Verification Problem

Question Wanting to check whether a cut piece of fabric has the shape of a square, a seamstress ensures that when folding along the diagonals, the edges of the fabric coincide. Is this check sufficient?

Answer By this method, the seamstress only ensures that all sides of the quadrilateral piece of fabric are equal. Not only does a square possess this property among convex quadrilaterals, but also any rhombus, and a rhombus represents a square only when its angles are right angles. Therefore, the check applied by the seamstress is insufficient. It is necessary to visually confirm at least that the angles at the vertexes of the piece of fabric are right angles. For this purpose, for example, the piece can be additionally folded along its median line and the angles adjacent to one side can be observed to coincide.

Game

For the game, a rectangular sheet of paper and figures of identical and symmetrical shapes are needed, such as dominoes or coins of equal value, or matchboxes, etc. The number of figures should be sufficient to cover the entire sheet of paper. Two players participate. Players take turns placing figures in any position on any free space on the sheet of paper until they have nowhere left to place them.

It is not allowed to move the placed figures on the paper. The player who places the item last is considered the winner.

Question Find a way to conduct the game in which the player who starts always wins.

Answer The player starting the game should, on their first move, occupy the central area of the sheet by placing their figure in such a way that its centre of symmetry, if possible, coincides with the centre of the sheet of paper. They should then continue to place their figure symmetrically to the opponent's figure (see Figure 159).

Adhering to this rule, the player who starts the game will always find a place for their figure on the sheet of paper and will inevitably win.

The geometric essence of the mentioned way of conducting

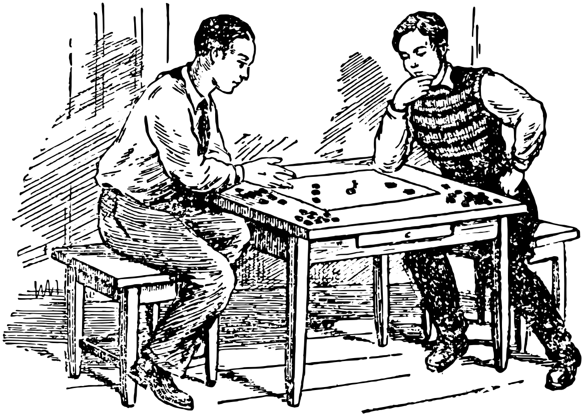
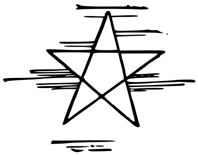


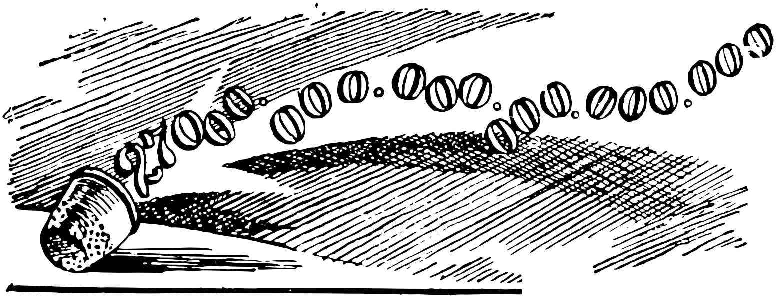
Figure 159: Geometric game. The winner is the one who places the item last.

the game is as follows: a rectangle has a centre of symmetry, i.e., a point at which all straight line segments passing through it are divided in half and divide the figure into two equal parts. Therefore, every point or area of the rectangle corresponds to a symmetric point or area belonging to the same figure, and only the centre of the rectangle does not have a symmetric point.

From this, it follows that if the first player occupies the central area, then, no matter where the opponent chooses to place their figure, there will always be a free area on the rectangular sheet of paper that is symmetric to the area occupied by the opponent’s figure.

Since the second player has to choose a place for their figure each time, eventually there will be no space left on the paper specifically for their figures, and the first player will win the game.





11. Big And Small In Geometry

In a Thimble

The number twenty-seven with eighteen zeros, written in the margin, can be read in different ways. Some will say: this is 27 trillion; others, for example financial workers, will read it as 27 quintillion, and still others will write it shorter:

21,000,000,000,000,000,000

27×10^{18} and read it as 27 multiplied by ten to the eighteenth power.

But what can fit in such an incredible quantity in one thimble?

We are talking about particles of the air surrounding us. Like all substances in the world, air consists of molecules. Physicists have established that in every cubic centimetre (i.e., approximately in a thimble) of the air surrounding us at a temperature of $0\,^{\circ}\mathrm{C}$, there are 27 trillion molecules. This is a numerical giant. To imagine it in any meaningful way is beyond the ability of even the liveliest imagination. Indeed, what can be compared to such a multitude? To the number of people in the world? But there are "only" two billion people on the globe (2×10^9), which is thirteen thousand million times smaller than the number of molecules in a thimble.

Even if all the stars in the universe, visible to the most powerful telescope, were surrounded by planets like our Sun, and if each of these planets were inhabited as our Earth is, then even the number of inhabitants would not equal the molecular population of one thimble! If you attempted to count this invisible population, counting continuously, for example, at a rate of a hundred molecules per minute, it would take you at least five hundred billion years to count.

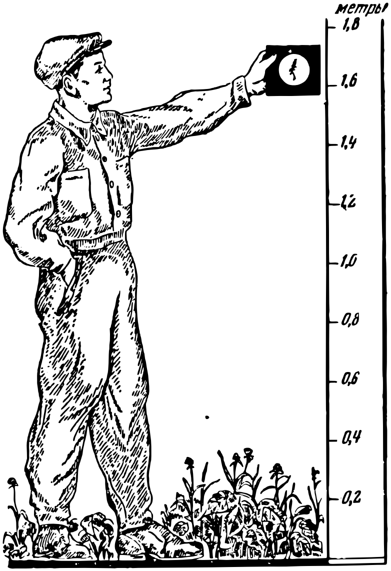


Figure 160: The young man looks at the typhus bacillus, magnified 1000 times.

Not everyone can clearly imagine even more modest numbers.

What do you envision when you are told, for example, about a microscope that magnifies by 1000 times? Not such a large number, just a thousand, but nevertheless, a thousandfold magnification is not perceived by everyone as it should be.

We often fail to appreciate the true smallness of the objects we see under a microscope at such magnification. A typhoid bacterium, magnified by 1000 times, seems to us the size of a fly (see Figure 160) viewed at a distance of clear vision, i.e., 25 cm.

But how small is this bacterium really? Imagine that along with the magnification of the bacterium, you also magnified yourself by 1000 times. This means that your height would reach 1700 m! Your head would be above the clouds, and any of the new skyscrapers being built in Moscow would seem much lower than your knees (see Figure 161). The bacterium is as much smaller than the tiny fly as you are smaller than this imaginary giant.

Volume and Pressure

One might think – isn't it too cramped for 27 trillion air molecules in a thimble? Not at all! An oxygen or nitrogen molecule has a diameter of 3/10,000,000 mm (or 3×10^{-7} mm). If we assume the volume of a molecule to be the cube of its diameter, we get:

$$\left(\frac{3}{10^7}\,\text{mm}\right)^3 = \frac{27}{10^{21}}\,\text{mm}^3.$$

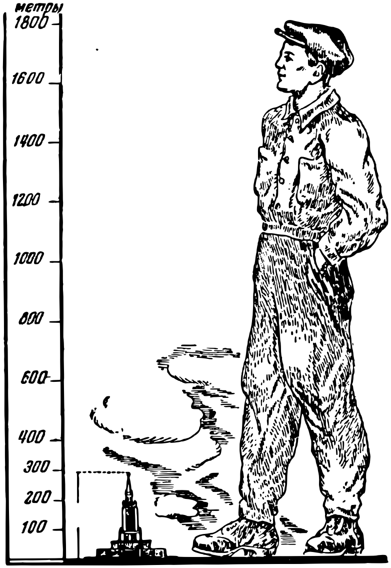


Figure 161: A young man magnified 1000 times.

There are 27×10^{18} molecules in a thimble. Thus, the volume occupied by all the inhabitants is approximately:

$$\frac{27}{10^{21}} \times 27 \times 10^{18} = \frac{729}{10^3}\ \mathrm{mm}^3.$$

That is, about 1 mm^3, which constitutes only one-thousandth of a cubic centimetre. The gaps between the molecules are many times larger than their diameters, so there is plenty of space for the molecules to move around. Indeed, as you know, air particles do not lie still, clustered together, but continuously and chaotically move from place to place, traversing the space they occupy. Oxygen, carbon dioxide, hydrogen, nitrogen, and other gases have industrial significance, but storing them in large quantities would require enormous reservoirs. For example, one ton (1000 kg) of nitrogen at normal pressure occupies a volume of 800 cubic meters, which means that to store just one ton of pure nitrogen, you would need a box measuring 20 m × 20 m × 20 m. And to store one ton of pure hydrogen, a tank with a capacity of 10,000 cubic meters would be needed.

Can we make gas molecules squeeze together? Engineers do just that – by compressing them, they force the molecules to become denser. But this is not an easy task. Don't forget that the pressure applied to the gas is met with equal force exerted by the gas on the vessel walls. Very strong walls are needed, which are also chemically resistant to the gas.

Modern chemical apparatus, manufactured domestically from alloy steels, can withstand enormous pressures, high temperatures, and the harmful chemical effects of gases.

Today, our engineers compress hydrogen to 1163 times its original density, so that one ton of hydrogen, which at atmospheric pressure occupies a volume of 10,000 cubic meters, fits into a relatively small cylinder with a capacity of about 9 cubic meters (see Figure 162).

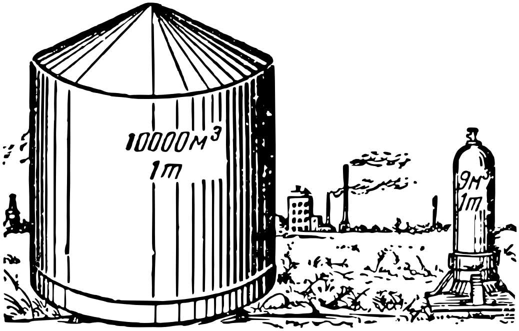


Figure 162: A ton of hydrogen at atmospheric pressure (to the left) and at a pressure of 5,000 atm (to the right). (The drawing is not to scale.)

How do you think hydrogen was subjected to reduce its volume by 1163 times? Recalling from physics that the volume of gas decreases proportionally to the increase in pressure, you might suggest this answer: the pressure on the hydrogen was increased by 1163 times. Is that actually the case? No. In reality, hydrogen had to be subjected to a pressure of 5000 atmospheres, i.e., the pressure was increased by 5000 times, not 1163 times. The fact is, this proportionality only holds for relatively low pressures.

At very high pressures, this relationship does not apply. For example, when one ton of nitrogen at our chemical plants is subjected to a pressure of 1000 atmospheres, the entire ton fits into a volume of 1.7 cubic meters instead of the 800 cubic meters it occupies at normal atmospheric pressure. Moreover, when the pressure is further increased to 5000 atmospheres, or five times, the volume of nitrogen decreases to just 1.1 cubic meters.

Thinner than a Spider Web but Stronger than Steel

The cross-section of a thread, wire, or even a spider web, no matter how small, still has a definite geometric shape, most often a circle. The diameter of the cross-section, or let's say, the thickness of a single spider web is about 5 microns (5/1000 mm). Is there anything thinner than a spider web? Who is the most skilled "thin-spinner"? The spider or perhaps the silkworm? No. The diameter of a natural silk thread is 18 microns, meaning the thread is 3.5 times thicker than a spider web.

People have long dreamt of surpassing the craftsmanship of the spider and the silkworm. There is an ancient legend about a marvellous weaver, the Greek Arachne[55].

She mastered the weaving craft so perfectly that her fabrics

[55] Arachne is the protagonist of a tale in Greek mythology known primarily from the version told by the Roman poet Ovid. – DM

were as fine as a spider web, transparent as glass, and light as air. Even the goddess Athena – the goddess of wisdom and patron of crafts – could not compete with her.

This legend, like many other ancient myths and fantasies, has become reality in our time. The modern Arachne, the most skilled "thin-spinner", turned out to be chemical engineers who created an extraordinarily thin and remarkably strong artificial fibre from ordinary wood. Silk threads produced by the copper-ammonia industrial method, for example, are 2.5 times thinner than a spider web and almost as strong as natural silk threads. Natural silk can withstand a load of up to 30 kg per square mm of cross-section, while copper-ammonia silk can withstand up to 25 kg per square mm.

A curious method of producing copper-ammonia silk involves transforming wood into cellulose, then dissolving the cellulose in an ammonia solution of copper. Streams of this solution are extruded through fine openings into water, which removes the solvent. The resulting fibres are then wound onto appropriate devices. The thickness of copper-ammonia silk thread is 2 microns. One micron thicker is the so-called acetate silk, which is also artificial. Remarkably, some types of acetate silk are stronger than steel wire! If steel wire can withstand a load of 110 kg per square millimetre of cross-section, acetate silk thread can withstand 126 kg per square millimetre.

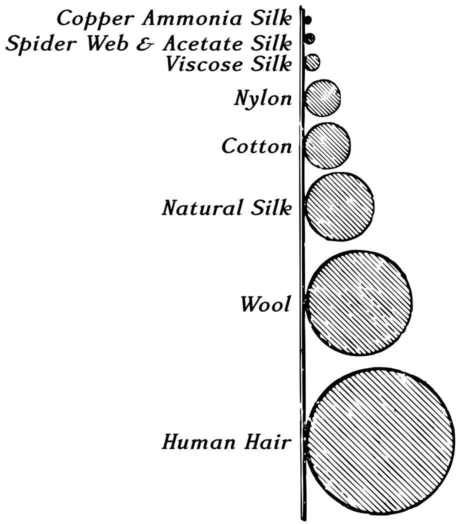


Figure 163: The comparative thickness of the fibres.

The well-known viscose silk has a thread thickness of about 4 microns and a tensile strength ranging from 20 to 62 kg per square millimetre of cross-section. Figure 163 shows the comparative thicknesses of spider silk, human hair, various artificial fibres, as well as wool and cotton fibres, while Figure 164 illustrates their strength in kilogrammes per square millimetre. Artificial, or synthetic, fibre is one of the major modern technological discoveries and has significant economic importance. Engineer Buyanov shares the following:

> Cotton grows slowly, and its quantity depends on climate and harvest. The producer of natural silk, the silkworm, is extremely limited in its capabilities. Throughout its life, it spins a cocoon containing only 0.5 grams of silk thread …
>
> The amount of artificial silk obtained through chemical processing from 1 cubic meter of wood can replace 320,000 silk cocoons or the annual wool yield from 30 sheep, or the average cotton harvest from half a hectare. This quantity of fibre is sufficient to produce four thousand pairs of women's stockings or 1500 meters of silk fabric.

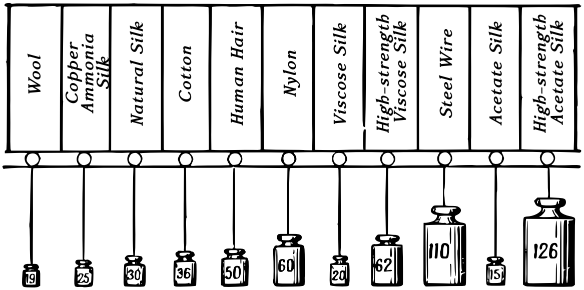


Figure 164: The ultimate strength of fibres (in kg per 1 sq. m of cross section)

Two Jars

We have an even poorer understanding of large and small quantities in geometry, where we must compare not just numbers but also surfaces and volumes. Everyone would unhesitatingly say that 5 square meters of jam is more than 3 square meters, but they might not immediately tell which of the two jars on the table holds more.

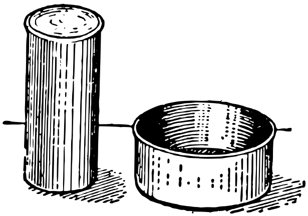


Figure 165: Which jar holds more?

Question Which of the two jars (Figure 165) holds more—the right, wider one, or the left, three times taller but half as wide one?

Answer For many, it will probably be surprising that in this case, the tall jar holds less than the wide one. However, it's easy to verify this with a calculation. The base area of the wide jar is 2×2, that is, four times greater than that of the narrow jar; its height is only three times less. Therefore, the volume of the wide jar is 4/3 times greater than the narrow one. If you pour the contents of the tall jar into the wide one, it will fill only 3/4 of it (Figure 166).

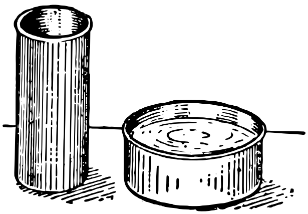


Figure 166: Result of pouring the contents of the tall jar into the wide one.

Gigantic Cigarette

Question In the window of a tobacco store, there is an enormous cigarette displayed, 15 times longer and 15 times thicker than a regular one. If stuffing a normal-sized cigarette requires half a gram of tobacco, how much tobacco was needed to stuff the gigantic cigarette in the window?

Answer

$$\frac{1}{2} \times 15 \times 15 \times 15 = 1700 \text{ gram},$$

i.e., about 1.7 kilogrammes.

Ostrich Egg Task

Question In Figure 166, a chicken egg is shown on the right and an ostrich egg on the left (the middle image is an egg of the extinct elephant bird, which will be discussed in the next problem). Look closely at the drawing and estimate how many times the volume of the ostrich egg is larger than that of the chicken egg. At first glance, it may not seem like a significant difference. The result obtained through proper geometric calculation is quite astonishing.

Answer By direct measurement on the drawing, we see that the ostrich egg is 2.5 times longer than the chicken egg. Consequently, the volume of the ostrich egg is greater than the volume of the chicken egg by

$$2.5 \times 2.5 \times 2.5 = 15,$$

i.e., approximately 15 times.

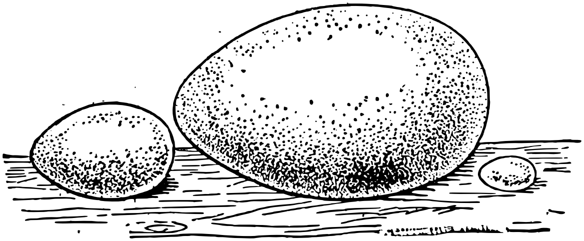


Figure 167: Comparative sizes of ostrich, elephant bird, and chicken eggs.

One such egg could provide breakfast for a family of five, assuming each person is satisfied with an omelet made from three eggs.

Elephant Bird Egg

Question In Madagascar, there once lived giant ostriches known as elephant birds, which laid eggs 28 cm long (the middle figure in Figure 166). Meanwhile, a chicken egg is 5 cm long. How many chicken eggs correspond in volume to one egg of the Madagascan ostrich?

Answer By multiplying $28/5 \times 28/5 \times$, we get approximately 170. One elephant bird egg is equivalent to almost 200 chicken eggs! More than fifty people could be satisfied with one such egg, which, as easy to calculate, weighed 8-9 kg. (Let us remind the reader of the clever fantasy story by Herbert Wells about the elephant bird egg.)

Eggs of Russian Birds

The most striking contrast in sizes, however, will be when we turn to our native nature and compare the eggs of the mute swan and the goldcrest, the smallest of all Russian birds. In Figure 168, the outlines of these eggs are shown in life size. What is the ratio of their volumes?

Answer By measuring the length of both eggs, we get 125 mm and 13 mm. By also measuring their width, we get 80 mm and 9 mm. It is easy to see that these numbers are almost proportional; verifying the proportion

$$\frac{125}{80} \approx \frac{13}{9},$$

and comparing the products of the extremes and means, we have: 1125 and 1040 – numbers that differ little. Hence, we conclude that assuming these eggs are geometrically similar bodies, we will not make a significant error. Therefore, the ratio of their volumes is approximately equal to

$$\frac{80^3}{9^3} = \frac{512000}{750} \approx 700.$$

Thus, the volume of a swan egg is about 700 times that of a goldcrest egg!

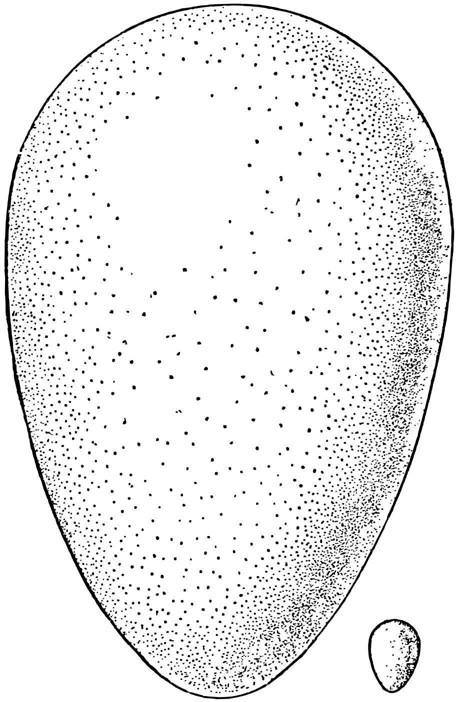


Figure 168: Swan's egg and goldcrest (full size). How many times is one larger than the other in volume?

Determine the Weight of the Eggshell Without Breaking the Eggs

Question You have two eggs of the same shape but different sizes. You need to approximately determine the weight of their shells without breaking them. What measurements, weightings, and calculations are necessary to perform this task? The thickness of the shells can be considered identical for both eggs.

Answer Measure the length of the major axis of each egg: let these be D and d. Let the weight of the shell of the first egg be x, and the second egg be y. The weight of the shell is proportional to its surface area, which is proportional to the square of its linear dimensions. Therefore, assuming the shell thickness of both eggs is the same, we set up the proportion:

$$\frac{x}{y} = \frac{D^2}{d^2}.$$

Weigh the eggs: let these weights be P and p. The weight of the egg's content can be considered proportional to its volume, i.e., the cube of its linear dimensions:

$$\frac{(P - x)}{(p - y)} = \frac{D^3}{d^3}.$$

We have a system of two equations with two unknowns;

solving it, we find:

$$x = \frac{p \cdot D^3 - P \cdot d^3}{d^2 \cdot (D - d)}; \quad y = \frac{p \cdot D^3 - P \cdot d^3}{D^2(D - d)}.$$

The Sizes of Our Coins

The weight of our coins is proportional to their value, i.e. a two—kopeck coin weighs twice as much as a kopeck coin, a three-kopeck coin weighs three times more, etc. The same is true for silver in exchange; a two-kopeck piece, for example, is twice as heavy as a dime. And since homogeneous coins usually have a geometrically similar shape, knowing the diameter of one change coin, you can calculate the dimensions of others that are homogeneous with it. Here are examples of such calculations.

Question The diameter of a five-kopeck coin is 25 mm. What is the diameter of a three-kopeck coin?

Answer The weight, and therefore the volume, of a three-kopeck coin is $3/5$, i.e., 0.6 of the volume of a five-kopeck coin. Therefore, its linear dimensions should be $\sqrt[3]{0.6}$ times smaller, i.e., 0.84 the size of a five-kopeck coin. Hence, the required diameter of the three-kopeck coin should be 0.84×25, i.e., 21 mm (actually 22 mm).

Coin Worth a Million Roubles

Question Imagine a fantastic silver coin worth a million roubles, which has the same shape as a 2-kopek coin, but correspondingly heavier. What would be the approximate diameter of such a coin? If it were placed on its edge next to a car, how many times taller would it be than the car?

Answer The dimensions of the coin would not be as enormous as one might think. Its diameter would be only about 3.8 m – just above one floor. Indeed, since its volume is larger than that of the two-kopek coin by 5,000,000 times, its diameter (as well as thickness) would be larger by $\sqrt[3]{5000000}$, i.e., only 172 times.

Multiplying 22 mm by 172, we get approximately 3.8 m – unexpectedly modest dimensions for a coin of such value.

Question Calculate which coin would be equivalent to a two-hryvnia coin enlarged to the size of a four-story building (in height) (Figure 169.

Visual Representations

The reader who has acquired the skill of comparing volumes of geometrically similar bodies based on their linear dimensions from the previous examples will no longer be caught off guard by questions of this kind. Therefore, they can easily

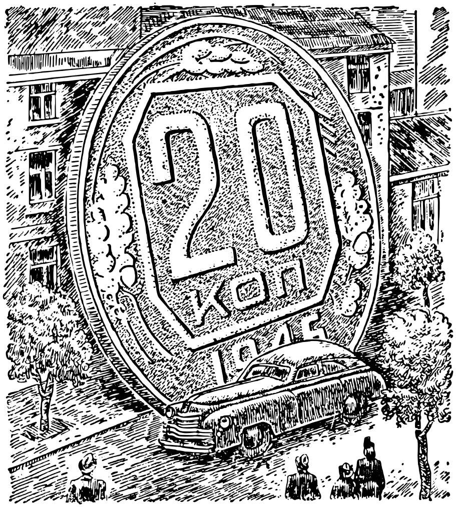


Figure 169: What coin is equivalent to this giant two-kopeck piece?

avoid the mistakes of some seemingly vivid images that often appear in illustrated magazines.

Question Here is an example of such images. If a person consumes an average of 400 grams of meat per day, then over 60 years of life, this would amount to about 9 tons. And since the weight of a bull is about 0.5 ton, a person could claim to have consumed 18 bulls by the end of their life.

In the accompanying Figure 170, reproduced from an English magazine, this gigantic bull is depicted next to a person consuming it over a lifetime. Is the picture accurate? What would be the correct scale?

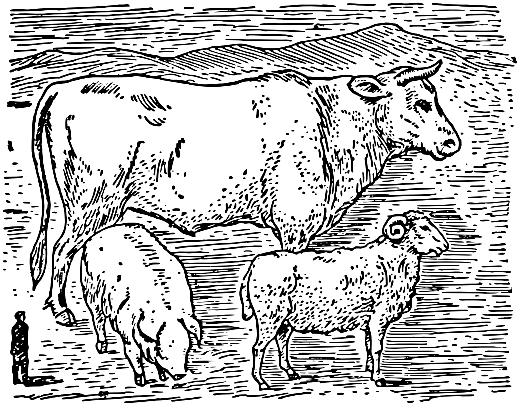


Figure 170: How much meat a person eats during their lifetime (to detect an error in the image).

Answer The picture is inaccurate. The bull depicted here is 18 times taller than normal and, of course, 18 times longer and thicker as well. Therefore, in terms of volume, it is $18 \times 18 \times 18 = 5832$ times larger than a normal bull. For a person to consume such a bull, they would have to live for at least two thousand years!

The correctly depicted bull should be taller, longer, and thicker than the normal one by only $\sqrt[3]{18}$, or 2.6 times; this would not be as impressive in the picture, so it wouldn't serve as a striking illustration of the amount of meat consumed by humans.

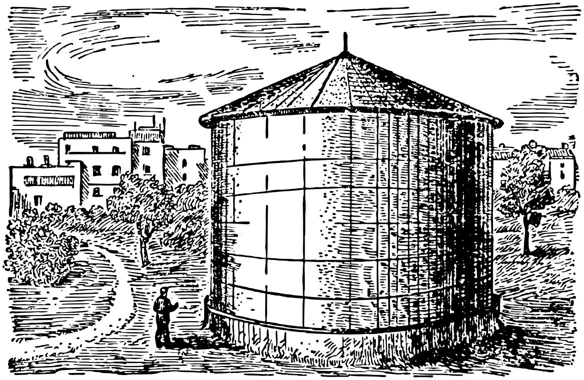


Figure 171: How much water does a person drink over a lifetime (what mistake did the artist make?).

Question In Figure 171, another illustration from the same field is reproduced. A person consumes 1.5 litres of various liquids per day (7-8 glasses). Over 70 years of life, this amounts to about 40,000 litres. Since a bucket holds 12 litres, the artist needed to depict some vessel that is 3300 times larger than a bucket. He thought he had done this in Figure 171. Is he correct?

Answer In the picture, the dimensions of the tank are greatly exaggerated. The vessel should be taller and wider than a normal bucket by only $\sqrt[3]{3300} = 14.9$, rounded to 15 times. If the height and width of a normal bucket are 30 cm, then to accommodate all the water we drink in our lifetime, a bucket with a height of 4.5 meters and the same width would suffice. Figure 172 depicts this vessel in the correct scale.

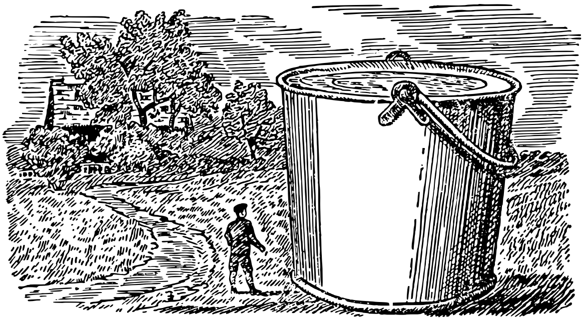


Figure 172: The same (see Figure 171) - correct depiction.

The examples considered show, among other things, that representing statistical numbers in the form of volumetric bodies is not sufficiently illustrative and does not produce the impression usually expected. Bar charts have an undeniable advantage in this regard.

Our Normal Weight

If we assume that all human bodies are geometrically similar (which is only true on average), then we can calculate people's weight based on their height (the average height of a person is 1.75 meters, and the average weight is 65 kg). The results obtained from such "calculations" may be surprising to many.

Let's assume you are shorter than average by 10 cm. What weight is considered normal for you?

Often, in everyday life, this problem is solved by subtracting from the normal weight the percentage that corresponds to 10 cm from the average height. In this case, for example, subtracting 65 kg by 10/175, the resulting weight – 62 kg – is considered normal.

This is an incorrect calculation.

The correct result can be obtained by calculating it from the proportion:

$$\frac{65}{x} = \frac{1.75^3}{1.65^3}, \text{ from which,}$$
$$x \approx 54\,\text{kg}.$$

The difference from the usually obtained result is quite significant – 8 kg.

Conversely, for a person whose height is 10 cm above average, the normal weight is calculated from the proportion:

$$\frac{65}{x} = \frac{1.75^3}{1.85^3}.$$

From this, $x = 78$ kg, which is 13 kg more than average. This increase is much more significant than commonly thought.

Undoubtedly, such correctly performed calculations should have significant importance in medical practise for determining normal weight, calculating medication dosage, and so on.

What then should be the relationship between the weight of a giant and a dwarf? To many, I'm sure, it would seem improbable that a giant could be 50 times heavier than a dwarf. However, correct geometric calculations lead to this.

One of the tallest giants, whose existence is well documented, was the Austrian Winkelmeier at a height of 278 cm; another, from Alsace, Krau, was 275 cm tall; the third, an Englishman named O'Brick, was reported to have reached 268 cm. All of them were a whole meter taller than a person of normal height. In contrast, dwarfs reach about 75 cm in adulthood – a meter shorter than normal height. What then is the ratio of the volume and weight of a giant to the volume and weight of a dwarf? It is equal to

$$\frac{275^3}{75^3}, \quad \text{or} \quad \frac{11^3}{3^3} = 49.$$

So, a giant is nearly fifty times heavier than a dwarf!

And if we believe the report about the Arab dwarf Agiba, who was 38 cm tall, then this ratio becomes even more striking: the tallest giant is seven times taller than this dwarf and, therefore, 343 times heavier. More reliably, Buffon's report measures a dwarf at 43 cm tall: this dwarf is 260 times lighter than the giant.

Gulliver's Geometry

The author of *Gulliver's Travels* carefully avoided the danger of getting lost in geometric relationships. The reader undoubtedly remembers that in the land of Lilliputians, our foot corresponded to an inch, while in the land of giants, conversely, an inch corresponded to a foot. In other words, in Lilliput, everything—people, things, and natural phenomena—were twelve times smaller than normal, while in the land of giants, they were twelve times larger. These seemingly simple relationships, however, became significantly more complicated when it came to solving questions like the following:

1. By how many times did Gulliver eat more for lunch than a Lilliputian?
2. By how many times did Gulliver need more fabric for a suit than the Lilliputians?
3. How much did an apple from the land of giants weigh?

The author of *Gulliver's Travels* managed these tasks quite successfully in most cases. He correctly calculated that since a Lilliputian was twelve times smaller in height than Gulliver, the volume of his body was smaller by $12 \times 12 \times 12 = 1728$ times; therefore, to satisfy Gulliver's body, 1728 times more food was needed than for a Lilliputian. And we read in *Gulliver's Travels* the following description of Gulliver's lunch:

> Three hundred cooks prepared meals for me. Around my house, sheds were set up where the cooking took place, and the cooks lived there with their families. When lunchtime came, I would pick up twenty servants and place them on the table, while a hundred served from the floor: some served the food, and others brought barrels of wine and other drinks on poles, carried from shoulder to shoulder. Those standing on top, as needed, lifted all this onto the table using ropes and pulleys …

Swift correctly calculated the amount of material for Gulliver's suit. The surface area of his body is greater than that of the Lilliputians, by $12 \times 12 = 144$ times; he therefore needs that much more material, tailors, and so on. All of this is accounted for by Swift, who narrates on behalf of

Figure 173: Midget tailors take measurements from Gulliver.

Gulliver, mentioning that "three hundred Lilliputian tailors (see Figure 173) were assigned to me, with instructions to make a complete suit of clothes according to local standards." (The urgency of the task required double the number of tailors.)

The need to make such calculations arose for Swift on almost every page. And generally speaking, he executed them correctly. If, as one critic claims, Pushkin's *Eugene Onegin* has "time calculated by the calendar," then in Swift's *Travels*, all dimensions are consistent with the rules of geometry. Only occasionally was the proper scale not maintained, especially in describing the land of the giants. Errors are sometimes encountered here.

Gulliver recounts,

> Once, a court dwarf accompanied us into the garden. Seizing a convenient moment when I found myself walking under one of the trees, he grabbed a branch and shook it over my head. A hail of apples, each the size of a good-sized barrel, rained down noisily to the ground; one struck me in the back and knocked me off my feet …

Gulliver managed to get up after this blow. However, it is easy to calculate that the impact from the fall of such an apple should have been truly crushing: after all, the apple is 1728 times heavier than ours, i.e., weighs 80 kg, and fell from a height twelve times greater. The energy of the impact should have exceeded the energy of the fall of an ordinary apple by 20,000 times and could only be compared to the energy of an artillery projectile …

Swift made his most significant mistake in calculating the muscular strength of the giants. As we have already seen in the first chapter, the power of large animals is not proportional to their size. If we apply the considerations outlined there to Swift's giants, it turns out that although their muscular strength was 144 times greater than Gulliver's, their body weight was 1728 times greater. And if Gulliver was able to lift not only the weight of his own body but also approximately the same load, then the giants would not have been able to even overcome the weight of their enormous bodies. They would have had to lie motionless in one place, powerless to make any significant movement. Their might, so vividly described by Swift, could only have resulted from an incorrect calculation.[56]

[56] See more detailed discussion on this in *Entertaining Physics* by Yakov Perelman.

Why Does Dust and Clouds Float in the Air?

"Their weight is lighter than air", is the usual answer, which seems so unquestionable to many that it leaves no room for doubt. However, such an explanation, despite its tempting simplicity, is completely erroneous. Dust particles are not only not lighter than air, but they are heavier than it by hundreds, even thousands of times.

What is a "dust particle"? These are the smallest particles of various heavy substances: fragments of stone or glass, specks of coal, wood, metals, fibres of fabrics, and so on. Are all these materials lighter than air? A simple reference to the table of specific gravities will convince you that each of them is either several times heavier than water or only 2-3 times lighter. And water is heavier than air by a factor of 800; therefore, dust particles are heavier than air by several hundred, if not thousand, times. Now the inconsistency of the common notion about the reason for the floating of dust particles in the air becomes obvious.

What is the true reason then? First of all, it should be noted that we usually misunderstand the phenomenon by considering it as floating. Only bodies that weigh less than an equal volume of air (or liquid) can float in the air (or liquid). Dust particles, however, exceed this weight by many times; therefore, they cannot float in the air. They don't float but rather hover, meaning they descend slowly, hindered in their fall by air resistance. A falling dust particle must carve a path for itself between air particles, either pushing them aside or carrying them along. Both actions consume energy during the fall. The energy expenditure is greater the larger the surface area of the body (more precisely, the cross-sectional area) compared to its volume. When large, massive bodies fall, we do not notice the slowing effect of air resistance because their weight significantly outweighs the resisting force.

Now let's see what happens when the body size decreases.

Geometry will help us understand this. It is easy to understand that as the volume of the body decreases, the weight decreases much more than the cross-sectional area: the decrease in weight is proportional to the third power of the linear reduction, while the weakening of resistance is proportional to the surface area, i.e., the second power of the linear reduction.

What significance does this have in our case? This is clear from the following example. Let's take a croquet ball with a diameter of 10 cm and a tiny ball made of the same material with a diameter of 1 mm. The ratio of their linear dimensions is 100, because 10 cm is 100 times larger than one millimetre. The small ball is lighter than the large one by 100^3 times, i.e., a million times; however, the resistance it encounters when moving through the air is weaker by only 100^2 times, i.e., ten thousand times. Clearly, the small ball should fall slower than the large one. In short, the reason dust particles stay in the air is their "floatability" due to their small size, not that they are supposedly lighter than air. A water droplet with a radius of 0.001 mm falls uniformly through the air at a speed of 0.1 mm per second; it takes only a negligible, imperceptible air current to disrupt such a slow descent.

This is why in a room where there is a lot of activity, dust settles less than in unused rooms, and during the day it settles less than at night, although it would seem that the opposite should happen: settling is hindered by the turbulent currents in the air, which are usually almost absent in calm, less frequented rooms.

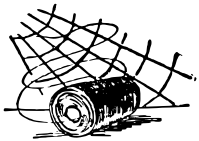
If a stone cube 1 cm high is crushed into cubic dust particles 0.0001 mm high, then the total surface area of the same mass of stone will increase by 10,000 times, and the air resistance to its movement will increase by the same factor. Dust particles often reach such sizes, and it is understandable that greatly increased air resistance completely changes the picture of their descent.

For the same reason, clouds “float” in the air. The outdated view that clouds consist of water bubbles filled with water vapour has long been rejected. Clouds are clusters of a huge number of extremely small but solid water droplets. Although these particles, although heavier than air by a factor of 800, almost do not fall; they descend at a barely noticeable speed. The significantly slowed descent is explained, as with dust particles, by their enormous surface area compared to their weight.

The weakest up draft of air is therefore capable not only of halting the extremely slow descent of clouds, maintaining them at a certain level, but also of lifting them upward.

The main reason underlying all these phenomena is the presence of air: in a vacuum, both dust particles and clouds (if they could exist) would fall as swiftly as heavy stones.

It is needless to add that the slow descent of a person with a parachute (about 5 m/s) belongs to phenomena of a similar order.



12. Geometric Economy

How Did Pahom Buy Land (Leo Tolstoy's Task)

We will begin this chapter, whose unusual title will become clear to the reader later on, with an excerpt from Leo Tolstoy's well-known story *How Much Land Does a Man Need?*

"And what will the price be?" Pakhom asks.

"The price is the same for everyone: 1000 roubles for a day."

Pakhom didn't understand.

"What kind of measure is a day? How many desyatinas[57] is that?"

[57] A unit of land area. – DM

"We don't know how to count that way. We sell by the day; however much you can walk around in a day, that's yours, and the price is 1000 roubles."

Pakhom was astonished.

"But you can walk around a lot of land in a day," he says.

The chief laughed.

"It's all yours," he says. "But there's one condition: if you don't return by sunset to the place where you started, you lose your money."

"How will I mark the places I walk?" Pakhom asks.

"We'll stand at the place you choose to start from; we'll stay there, and you go, make a circle, and take a scraper with you to mark where necessary, dig small holes, lay down turf at the corners; later, we'll plow from hole to hole. Choose any circle you want, but you must return to the starting place by sunset. Whatever you circle will be yours."

The Bashkirs dispersed. They promised to gather at dawn the next day and head to the place before sunrise.

They arrived in the steppe as dawn was breaking. The chief approached Pakhom and pointed with his hand.

“Here”, he said, “everything you can see is ours. Choose any part.”

The chief took off his fox fur hat and placed it on the ground.

“This will be the marker”, he said. “Start from here and come back to here. Whatever you circle will be yours”.

As soon as the sun’s rose, Pakhom slung the scraper over his shoulder and set off into the steppe.

He walked about a mile, stopped, and dug a small hole. He continued walking. After covering another mile, he dug another hole.

He had walked about five miles. He looked at the sun; it was already breakfast time. “One stretch is done,” thought Pakhom. “And there are four stretches in a day, it’s too early to turn yet. I’ll walk another five versts and then start turning left.” He continued straight ahead.

“Well”, he thought, “I’ve covered enough ground in this direction; it’s time to turn.’ He stopped, dug a larger hole, and turned sharply to the left.”

He walked a long way along this side as well, then turned the second corner. Pakhom glanced back at the shikhan (hill): it was hazy from the heat, and through the haze, he could barely see the people on the shikhan.

"Well", he thought, "I've taken long sides, I need to make this one shorter". He started on the third side. He looked at the sun—it was already nearing midday, and he had only covered about two versts on the third side. And the distance back to the starting point was still about 15 versts.

"No", he thought, "even if it's a crooked plot, I need to go straight to make it in time."

Pakhom quickly dug a hole and turned straight towards the shikhan.

Pakhom walked straight towards the shikhan, and it was becoming hard for him. He wanted to rest but couldn't – he wouldn't make it back before sunset. And the sun was already close to the horizon.

Pakhom kept walking, finding it increasingly difficult, but still quickening his pace. He walked and walked—still far to go; then he began to trot… Pakhom ran, his shirt and trousers sticking to his sweaty body, his mouth dry. His chest heaved like a blacksmith's bellows, and his heart pounded like a hammer.

Figure 174: Pakhom ran with his last strength, and the sun was already near the horizon.

Pakhom ran with his last strength, and the sun was just about to set. Any moment now it would start disappearing (see

Figure 174).

The sun was close, and the place was also very near. He saw the fox fur hat on the ground and the chief sitting on the ground.

Pakhom looked at the sun; it had already touched the ground and was just starting to set. Summoning his last bit of strength, Pakhom pushed himself, gathered his breath, and ran up the shikhan. He saw the hat. His legs gave way, and he fell forward, reaching the hat with his hands.

“Well done!” shouted the chief. “You have acquired a lot of land”.

An assistant ran up, trying to lift him, but blood was flowing from Pakhom’s mouth, and he lay there dead...

The Problem of Leo Tolstoy

Let’s set aside the grim conclusion of this story and focus on its geometric aspect. Is it possible to determine, based on the scattered information in this story, approximately how many desyatinas of land Pakhom walked around? The task, which at first glance seems impossible, is actually solved quite simply.

Answer By carefully rereading the story and extracting all geometric indications, it is not difficult to convince oneself that the data provided is sufficient for a comprehensive answer to the question. One can even draw a plan of the land parcel that Pakhom walked around.

First of all, it is clear from the story that Pakhom ran along the sides of a quadrilateral. Regarding the first side, we read:

"He walked about five versts... "I'll go another five versts, then I'll turn left..."

Thus, the first side of the quadrilateral was about 10 versts long.

For the second side, which forms a right angle with the first, no numerical indications are given in the story.

The length of the third side, evidently perpendicular to the second, is directly stated in the story: "Along the third side, he had walked only about two versts."

"The length of the third side—clearly perpendicular to the second—is directly stated in the story: 'Along the third side, he had walked only about two versts.'

The length of the fourth side is also directly provided: "To the place it is still the same 15 versts."[58]

Using this data, we can draw the plan of the land parcel that

[58] It is unclear here how Pakhom could see people on the hill from such a distance.

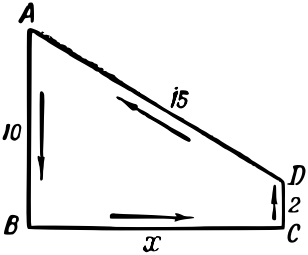


Figure 175: Tracing Pakhom's route.

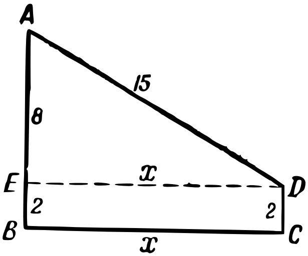


Figure 176: Pakhom's route clarification.

Pakhom walked around (see Figure 175). In the resulting quadrilateral $ABCD$, the side $AB = 10$ versts; $CD = 2$ versts; $AD = 15$ versts; and angles B and C are right angles. The length x of the unknown side BC can be easily calculated by drawing the perpendicular DE from D to AB (see Figure 176). Then in the right triangle AED, we know the leg $AE = 8$ versts and the hypotenuse $AD = 15$ versts. The unknown leg $ED = \sqrt{15^2 - 8^2} = 13$ versts.

So, the second side was about 13 versts long. Obviously, Pakhom was mistaken, thinking the second side was shorter than the first.

As you can see, it is quite possible to accurately draw the plan of the area Pakhom walked around. Undoubtedly, L.N. Tolstoy had a drawing similar to Figure 175 in front of him when writing his story.

Now it is easy to calculate the area of the trapezoid $ABCD$, consisting of the rectangle $EBCD$ and the right triangle AED. It equals:

$$2 \times 13 + \frac{1}{2} \times 8 \times 13 = 78 \text{ square versts.}$$

Calculating using the trapezoid formula would, of course, give the same result:

$$\frac{(AB + CD)}{2} \times BC = \frac{(10 + 2)}{2} \times 13 = 78 \text{ square versts.}$$

We have determined that Pakhom walked around a vast area of 78 square versts, or about 8000 desyatinas. Each desyatin cost him 12.5 kopecks.

Trapezoid or Rectangle

Question On the fateful day of his life, Pakhom walked $10+13+2+15 = 40$ versts along the sides of a trapezoid. His initial intention was to walk along the sides of a rectangle; the trapezoid resulted accidentally due to poor calculation. It is interesting to determine whether he gained or lost from the fact that his plot turned out to be a trapezoid rather than a rectangle. In which case would he have obtained a larger area of land?

Question Rectangles with a perimeter of 40 versts can vary widely, each having a different area. Here are several examples:

$$
\begin{aligned}
14 \times 6 &= 84 \text{ square versts,} \\
13 \times 7 &= 91 \text{ square versts,} \\
12 \times 8 &= 96 \text{ square versts,} \\
11 \times 9 &= 99 \text{ square versts.}
\end{aligned}
$$

We see that all these figures, with the same perimeter of 40 versts, have a larger area than our trapezoid. However, there can also be rectangles with a perimeter of 40 versts whose area is smaller than that of the trapezoid:

$$18 \times 2 = 36 \text{ square versts,}$$
$$19 \times 1 = 19 \text{ square versts,}$$
$$19.5 \times 0.5 = 9.75 \text{ square versts.}$$

Therefore, we cannot give a definitive answer to the problem. There are rectangles with a larger area than the trapezoid, but there are also rectangles with a smaller area, given the same perimeter. However, we can definitively answer the question: which rectangle with a given perimeter encloses the largest area? Comparing our rectangles, we notice that the smaller the difference in the length of the sides, the larger the area of the rectangle. It is natural to conclude that when there is no difference at all, i.e., when the rectangle becomes a square, the area of the figure reaches its maximum. It will then be $10 \times 10 = 100$ square versts. It is easy to see that this square indeed exceeds in area any rectangle with the same perimeter. Pakhom should have walked along the sides of a square to obtain the largest plot of land – 22 square versts more than what he managed to encompass.

The Remarkable Property of a Square

The remarkable property of a square – enclosing the largest area compared to all other rectangles with the same perimeter – is not widely known. Therefore, let's provide a rigorous proof of this statement.

Let's denote the perimeter of a rectangular figure by P. If we take a square with this perimeter, then each side must be $P/4$. We will prove that by shortening one side of the square by some amount b while lengthening the adjacent side by the same amount, we get a rectangle with the same perimeter but a smaller area. In other words, we will prove that the area of the square $(P/4)^2$ is greater than the area of the rectangle $(P/4 - b)(P/4 + b)$:

$$(P/4)^2 > (P/4 - b)(P/4 + b).$$

Since the right side of this inequality equals $(P/4)^2 - b^2$, the entire expression becomes:

$$0 > -b^2 \quad \text{or} \quad b^2 > 0.$$

But the latter inequality is obvious: the square of any number, whether positive or negative, is greater than 0. Therefore, the initial inequality, which led us to this conclusion, is also true.

Thus, a square has the largest area of all rectangles with the same perimeter.

This implies, among other things, that of all rectangular figures with equal areas, the square has the smallest perimeter. We can confirm this through the following reasoning. Suppose this is not true and there exists a rectangle *A*, which has an equal area to square *B* but a smaller perimeter. Then, by drawing a square *C* with the same perimeter as rectangle *A*, we would get a square with a larger area than *A*, and therefore larger than square *B*. What do we have as a result? That square *C* has a smaller perimeter than square *B* but a larger area. This is clearly impossible: if the side of square *C* is smaller than the side of square *B*, its area must also be smaller. Hence, it was not possible to assume the existence of rectangle *A*, which, with equal area, has a smaller perimeter than a square. In other words, of all rectangles with the same area, the square has the smallest perimeter.

Familiarity with these properties of the square would have helped Pakhom correctly calculate his strength and obtain a rectangular plot of the largest area. Knowing that he could walk 36 versts in a day without strain, he would have followed the boundary of a square with a side of 9 versts and by evening would have owned a plot of 81 square versts, 3 square versts more than he acquired with fatal exertion. Conversely, if he had initially limited himself to a specific area of a rectangular plot, say 36 square versts, he could have achieved the result with the least effort by following the boundary of a square with a side of 6 versts.

Plots of Different Shapes

But perhaps it would have been even more advantageous for Pakhom to carve out a plot of land not in a rectangular shape, but in some other form – quadrilateral, triangular, pentagonal, etc.?

This question can be considered strictly mathematically; however, out of concern for tiring our voluntary reader, we will not delve into this consideration here and will only acquaint him with the results.

Firstly, it can be proven that of *all quadrilaterals* with equal perimeter, the square has the largest area. Therefore, in seeking to have a quadrilateral plot, Pakhom could not have acquired more than 100 square versts by any means (assuming his maximum daily walk was 40 versts).

Secondly, it can be proven that a square has a larger area than any triangle of equal perimeter. An equilateral triangle with the same perimeter would have a side of $40/3 = 13\quad 1/3$ versts, and its area (using the formula $S = a^2\sqrt{3}/4$, where S is the area and a is the side) would be:

$$S = \frac{1}{4}\left(\frac{40}{3}\right)^2\sqrt{3} = 77 \text{ square versts.}$$

i.e., even less than that of the trapezoid Pakhom walked around. Further (on page 325), it will be proven that of all triangles with equal perimeters, the scalene triangle has the largest area. This means that if even the largest triangle has an area smaller than that of the square, then all other triangles with the same perimeter will have areas smaller than the square.

But if we compare the area of the square with that of a pentagon, hexagon, etc., with the same perimeter, the square's supremacy ends here: a regular pentagon has a larger area, a regular hexagon even larger, and so on. This can be easily verified with the example of a regular hexagon. With a perimeter of 40 versts, the side of the hexagon is 40/6 versts, and its area (using the formula $S = (3a^2\sqrt{3})/2$ is:

$$\frac{3}{2}\left(\frac{40}{6}\right)^2\sqrt{3} = 115 \quad \text{square versts.}$$

If Pakhom had chosen a regular hexagon for his plot, he would have acquired an area 115 - 78, i.e., 37 square versts more than he actually did, and 15 square versts more than he would have gotten from a square plot. (However, to achieve this, he would have needed to embark on his journey with surveying instruments).

Question Arrange six matches to form a figure with the largest possible area.

Answer With six matches, one can create a variety of shapes: an equilateral triangle, a rectangle, numerous parallelograms, several irregular pentagons, various irregular hexagons, and finally, a regular hexagon. A geometer, without comparing the areas of these shapes, already knows which one has the largest area: a regular hexagon.

Figures with the Greatest Area

It can be proven geometrically that the greater the number of sides in a regular polygonal area, the larger the area it encloses for the same boundary length. The shape with the largest area for a given perimeter is a circle. If Pakhom had run in a circle, covering the same 40 versts, he would have obtained an area of

$$\pi\left(\frac{40}{2\pi}\right)^2 = 127 \quad \text{square versts.}$$

No other shape, whether straight-edged or curved, can have a larger area with the same perimeter.

We will take some time to focus on this remarkable property of the circle to enclose a greater area than any other figure of any shape with the same perimeter. Some readers might be curious to know how such properties are proven. Below is a proof—though not entirely rigorous—proposed by mathematician Jakob Steiner. It is quite lengthy, but those who

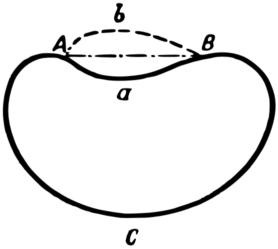


Figure 177: Establishing that the figure with the maximum area must be convex.

find it tedious can skip it without losing the understanding of what follows.

It is necessary to prove that the figure with the maximum area for a given perimeter is a circle. First, let's establish that the desired figure must be convex. This means that every chord of the figure must lie entirely inside it. Suppose we have a figure $AaBC$ (Figure 177) with an external chord AB. Let's replace the arc a with an arc b, which is symmetrical to it. This replacement will not change the perimeter of the figure ABC, but it will clearly increase the area. Therefore, figures like $AaBC$ cannot be the ones that enclose the maximum area for the same perimeter.

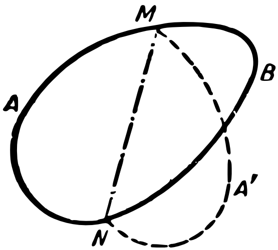


Figure 178: If a chord bisects the perimeter of a convex figure with the largest area, then it bisects the area.

Thus, the desired figure is convex. Furthermore, we can also establish another property of this figure: any chord that bisects its perimeter must also bisect its area. Let the figure $AMBN$ (Figure 178) be the desired one, and let the chord MN bisect its perimeter. We need to prove that the area AMN is equal to the area MBN.

Indeed, if one of these parts were larger than the other, for example, if $AMN > MNB$, then by folding the figure AMN along MN, we would obtain the figure $AMA'N$, whose area is larger than the original figure $AMB'M$, while the perimeter remains the same. Therefore, the figure $AMBN$, in which a chord bisecting the perimeter does not divide the area equally, cannot be the desired one (i.e., it cannot have the maximum area for a given perimeter).

Before proceeding further, let's prove the following auxiliary theorem: among all triangles with two given sides, the one with these sides enclosing a right angle has the greatest area. To prove this, let's recall the trigonometric expression for the area S of a triangle with sides a and b and the angle C between them:

$$S = \frac{1}{2}ab \ \sin C.$$

This expression is obviously maximised (for given sides) when $\sin C$ takes its maximum value, i.e., when it equals one. But the angle whose sine is 1 is a right angle, which is what needed to be proven.

Now we can proceed to the main task – proving that among all figures with perimeter p, the circle encloses the greatest area. To confirm this, let's try to assume the existence of a non-circular convex figure $MANB$ (Figure 179) that possesses this property. Let's draw a chord MN in it, bisecting its perimeter; as we know, it will also bisect the area of the figure.

We will bend half of the polygon MKN along the line MN so that it is positioned symmetrically ($MK'N$). Notice that the figure $MNK'M$ has the same perimeter and area as the

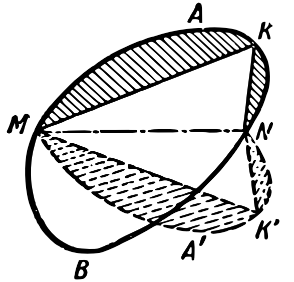


Figure 179: We assume the existence of a non-circular convex shape with the largest area.

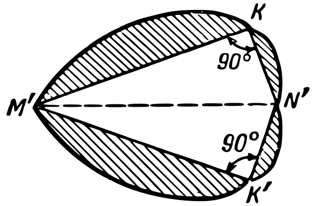


Figure 180: We establish that of all the figures with a given perimeter, the largest area is limited by a circle.

original figure $MNKM$. Since the arc MNK is not a semicircle (otherwise there would be nothing to prove), there must be points on it from which the segment MN is not visible at a right angle.

Let K be such a point, and K' be its symmetric point, i.e., angles K and K' are not right angles. By spreading (or shifting) the sides MK, KN, MK', NK', we can make the angle enclosed between them right, thus obtaining equal right-angled triangles. Adding these triangles along the hypotenuses, as shown in Figure 180, and attaching the shaded segments in corresponding positions, we obtain the figure $M'KMN'K'$, which has the same perimeter as the original but, obviously, a larger area (because the right-angled triangles $M'KN'$ and $M'K'N'$ have a greater area than the non-right-angled MKN and $MK'N$). Therefore, no non-circular figure can have the greatest area with a given perimeter. And only in the case of a circle, using the method described, could we not construct a figure with the same perimeter and a greater area.

This is how we can prove that a circle is the figure that has the greatest area with a given perimeter.

It is easy to prove the validity of such a position: among all figures of equal area, the circle has the smallest perimeter. To do this, we need to apply to the circle the reasoning that we previously applied to the square (see page 403).

Nails

Question Which nail is harder to pull out – round, square, or triangular – if they are driven in equally deep and have the same cross-sectional area?

Answer Let's start by assuming that the nail that has more surface area in contact with the surrounding material holds tighter. Which of our nails has the largest lateral surface area? We already know that with equal areas, the perimeter of a square is less than the perimeter of a triangle, and the circumference is less than the perimeter of a square. If we take the side of the square as one unit, then the calculation gives us the following values for these three shapes: 4.53, 4 (for triangle), and 3.55. Therefore, the triangular nail should hold tighter than the others.

However, such nails are not manufactured, at least they are not found for sale. The reason is probably that such nails are easier to bend and break.

Body of Greatest Volume

Similar to the property of a circle, a spherical surface possesses the largest volume for a given surface area. Conversely, of all bodies of equal volume, the sphere has the least surface area. These properties are not devoid of significance in practical life. A spherical samovar[59] has less surface area than a cylindrical one or one of any other shape that can hold the same number of cups. Since a body loses heat only from its surface, a spherical samovar cools down more slowly than any other of the same volume. Conversely, the reservoir of a thermometer heats up and cools down quickly (i.e., reaches the temperature of surrounding objects) when given a shape other than a sphere, such as a cylinder.

[59] A samovar is a metal container traditionally used to heat and boil water. – DM

For the same reason, the Earth, consisting of a solid shell and a core, must decrease in volume, i.e., compress and become denser, due to all the factors that alter its surface shape: its internal contents must become tighter whenever its external shape undergoes any change, deviating from that of a sphere. Perhaps this geometric fact is related to earthquakes and tectonic phenomena in general, but geologists should have judgement on this.

Product of Equal Factors

Problems like the ones we've been discussing approach the question from an economic perspective: given a certain expenditure of effort (for example, travelling a 40-verst distance), how can one achieve the most advantageous outcome (covering the largest area)? Hence the title of this section of the book: *Geometric Economy*. However, this is a popularisation, in mathematics, questions of this kind are known by a different name: problems *on maxima and minima.* They can vary greatly in plot and difficulty level. Many can only be solved using advanced mathematical techniques, but there are also many that require only elementary knowledge for their solution. Later on, a series of such problems from the field of geometry will be considered, which we will solve using one curious property of the product of equal factors.

For the case of two factors, this property is already familiar to us. We know that the area of a square is greater than the area of any rectangle with the same perimeter. If we translate this geometric statement into arithmetic language, it means the following: when it is necessary to divide a number into two parts such that their product is the greatest, one should divide it in half. For example, among all products:

$$3 \times 17,\ \ 16 \times 14,\ \ 12 \times 18,\ \ 11 \times 19,\ \ 10 \times 20,\ \ 15 \times 15,$$

and so on, with the sum of the factors being equal to 30, the greatest product will be 15×15, even when comparing products of fractional numbers (like 14.5×15.5), and so forth.

The same applies to products of three factors having a constant sum: their product reaches its maximum value when the factors are equal to each other. This directly follows from the previous reasoning. Let three factors x, y, z, with a sum equal to a:

$$x + y + z = a.$$

Suppose x and y are not equal to each other. If we replace each of them with half the sum $(x + y)/2$, the sum of the factors will not change:

$$\frac{x+y}{2} + \frac{x+y}{2} + z = x + y + z = a.$$

But since according to the previous reasoning

$$\frac{x+y}{2} \times \frac{x+y}{2} > xy,$$

then the product of the three factors

$$\frac{x+y}{2} \times \frac{x+y}{2} \times z,$$

is greater than the product xyz:

$$\frac{x+y}{2} \times \frac{x+y}{2} \times z > xyz.$$

In general, if among the factors x, y, z there are at least two unequal ones, we can always find numbers that, without changing the total sum, will give a greater product than xyz. And only when all three factors are equal, such replacement cannot be made. Therefore, for $x + y + z = a$, the product xyz will be maximum when

$$x = y = z.$$

Let's use this property of equal factors to solve several interesting problems.

Triangle with the Greatest Area

Question What shape should a triangle have to have the greatest area given the sum of its sides?

We have already noticed earlier (page 398) that this property belongs to an equilateral triangle. But how can we prove it?

Answer The area S of a triangle with sides a, b, c and perimeter $a + b + c = 2p$ is expressed, as known from the course of geometry, as:

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

from which

$$\frac{S^2}{p} = (p-a)(p-b)(p-c).$$

The area S of the triangle will be maximum when its square S^2 becomes maximum, or the expression S^2/p: where p, the semi-perimeter, is according to the condition a constant quantity. But since both parts of the equation simultaneously achieve their maximum value, the question boils down to when the product

$$(p - a)\,(p - b)\,(p - c),$$

becomes maximum. Noticing that the sum of these three factors is a constant,

$$(p - a) + (p - b) + (p - c) = 3p - (a + b + c) = 3p - 2p = p,$$

we conclude that the product reaches its maximum value when the factors are equal, i.e., when the equality is achieved:

$$p - a = p - b = p - c$$

from which

$$a = b = c$$

Thus, a triangle has the greatest area for a given perimeter when its sides are equal to each other.

The Heaviest Beam

Question From a cylindrical log, a beam of the greatest weight needs to be cut. How can this be done?

Answer The problem, obviously, boils down to inscribing a rectangle with the greatest area in a circle. Although after all that has been said, the reader may already suspect that such a rectangle will be a square, it is still interesting to rigorously prove this proposition.

Let's denote one side of the sought rectangle (Figure 181) as x; then the other side can be expressed as $\sqrt{4R^2 - x^2}$, where R is the radius of the circular cross-section of the log. The area of the rectangle is $S = x\sqrt{4R^2 - x^2}$ from which we get

$$S^2 = x^2(4R^2 - x^2)$$

Since the sum of the factors x^2 and $4R^2 - x^2$ is a constant value ($x^2 + 4R^2 - x^2 = 4R^2$), then the product of S^2 will be maximum when $x^2 = 4R^2 - x^2$, i.e., when $x = R\sqrt{2}$. At the same time, S, i.e., the area of the sought rectangle, also reaches its maximum value.

Thus, one side of the rectangle with the greatest area is equal to $R\sqrt{2}$, i.e., to the side of the inscribed square. The beam has the greatest volume when its cross-section is a square inscribed in the cross-section of the cylindrical log.

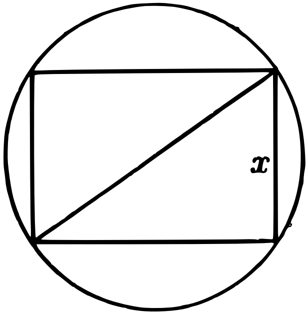


Figure 181: To the problem of the heaviest beam.

From a Cardboard Triangle

Question There is a triangular piece of cardboard. It is necessary to cut out from it a rectangle parallel to the given base and height, with the greatest possible area.

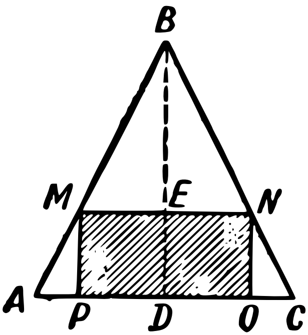


Figure 182: To insert a rectangle of maximum area into a triangle.

Answer Let ABC be the given triangle (Figure 182), and $MNOP$ be the rectangle that should remain after cutting. From the similarity of triangles ABC and NBM, we have:

$$\frac{BD}{BE} = \frac{AC}{NM},$$

hence

$$NM = \frac{BE \cdot AC}{BD}.$$

Denoting one side NM of the sought rectangle by y, its distance BE from the vertex of the triangle by x, the base AC of the given triangle by a, and its height BD by h, we rewrite the previously obtained expression in the following form:

$$y = \frac{ax}{h}.$$

The area S of the sought rectangle $MNOP$ is equal to:

$$\begin{aligned} S &= MN \cdot NO = MN \cdot (BD - BE) \\ &= (h - x)y = (h - x)\frac{ax}{h}, \end{aligned}$$

therefore,

$$\frac{Sh}{a} = (h - x)x.$$

The area S will be maximum when the product Sh/a is maximum, and consequently, when the product of the factors $(h - x)$ and x reaches its maximum value. But the sum $h - x + x = h$ is a constant value. Hence, the product is maximum when

$$h - x = x,$$

hence

$$x = h/2.$$

We find that the side MN of the sought rectangle passes through the midpoint of the triangle's height, and consequently, connects the midpoints of its sides. Therefore, this side of the rectangle is equal to $a/2$, and its other side is equal to $h/2$.

The Tinsmith's Dilemma

Question A tinsmith was asked to make a box without a lid from a square sheet of tin 60 cm wide, with the condition that the box should have the maximum possible capacity. The tinsmith pondered for a long time about the width to which the edges should be folded, but could not come to a definite solution (Figure 183). Perhaps the reader can help solve his dilemma?

Answer Let the width of the folded strips be x (Figure 184). Then the width of the square base of the box will be $60 - 2x$,

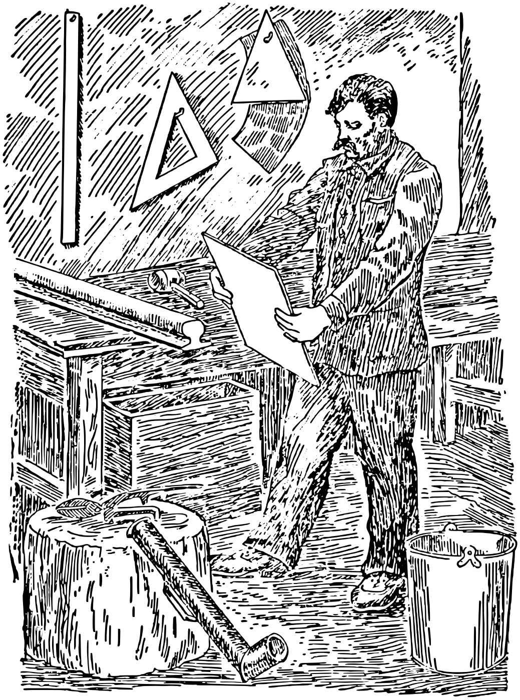


Figure 183: The tinsmith's dilemma.

and the volume V of the box can be expressed as:

$$v = (60 - 2x)(60 - 2x)\,x.$$

At what x does this product have the greatest value? If the sum of the three factors were constant, the product would be greatest when they are equal. However, in this case, the sum of the factors

$$60 - 2x + 60 - 2x + x = 120 - 3x,$$

is not a constant value, as it changes with x. However, it is not difficult to make the sum of the three factors constant: it is enough to multiply both parts of the equation by 4. We get:

$$4v = (60 - 2x)(60 - 2x)4x.$$

The sum of these factors is

$$60 - 2x + 60 - 2x + 4x = 120$$

a constant value. Hence, the product of these factors reaches its maximum value when they are equal, i.e., when

$$60 - 2x = 4x,$$

from which

$$x = 10.$$

Then $4v$, and therefore v, reach their maximum.

Thus, the box will have the greatest volume if the edges of the tin sheet are folded to 10 cm. This maximum volume

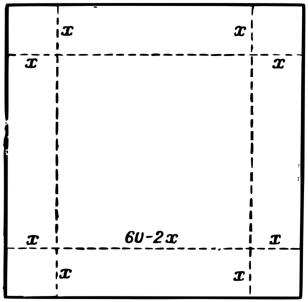


Figure 184: Solution to the tinsmith's problem.

is equal to $40 \times 40 \times 10 = 16{,}000\,\text{cm}^3$. Folding 1 cm more or less will decrease the volume of the box in both cases. Indeed,

$$9 \times 42 \times 42 = 15{,}900\,\text{cm}^3$$
$$11 \times 38 \times 38 = 15{,}900\,\text{cm}^3$$

both of which are less than $16{,}000\,\text{cm}^3$.[60]

[60] Solving the problem in general, we find that for a square sheet of width a, to obtain a box with the maximum volume, the strips of width $x = a/6$ should be folded, because the product $(a-2x)(a-2x)x$, or $(a-2x)(a-2x)\,4x$, is greatest when $a - 2x = 4x$.

The Turner's Dilemma

Question A turner is given a cone and tasked with turning it into a cylinder so that as little material as possible is removed (Figure 185).

The turner pondered the shape of the desired cylinder: should it be tall but narrow (Figure 186) or wide but short (Figure 187). He could not decide for a long time what shape would result in the cylinder with the largest volume, i.e., with the least amount of material removed. What should he do?

Answer The problem requires careful geometric consideration. Let ABC (Figure 188) be the cross-section of the cone, BD its height, denoted as h; the radius of the base $AD = DC$ denoted as R. The cylinder that can be turned from the cone has a cross-section $MNOP$. We need to find at what distance

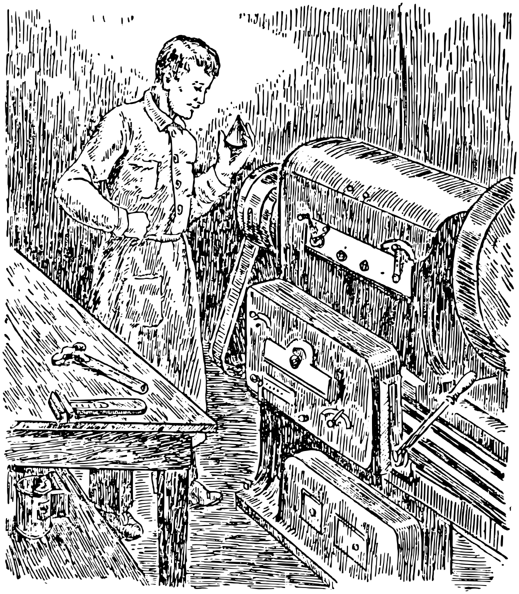


Figure 185: The turner's dilemma.

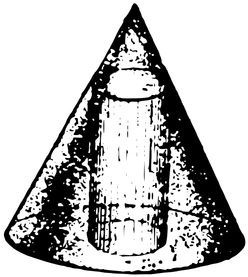


Figure 186: A cylinder can be turned from the cone to be tall but narrow or wide but short. In which case will less material be removed?

$BE = x$ from the vertex B the top base of the cylinder should be located so that its volume is maximised.

The radius r of the base of the cylinder (PD or ME) is easily found from the proportion:

$$\frac{ME}{AD} = \frac{BE}{BD},$$

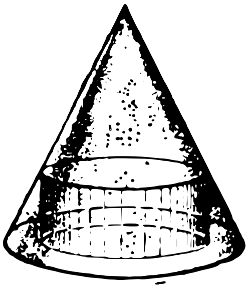


Figure 187: A cylinder can be turned from the cone to be tall but narrow or wide but short. In which case will less material be removed?

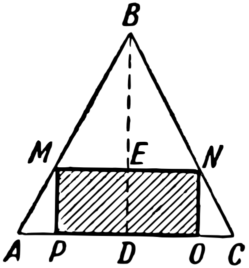


Figure 188: Axial section of the cone and the cylinder.

i.e.,

$$\frac{r}{R} = \frac{x}{h} \text{ from which,}$$
$$r = \frac{Rx}{h}.$$

The height ED of the cylinder is equal to $h-x$. Consequently, its volume is

$$v = \pi \left(\frac{Rx}{h}\right)^2 (h-x) = \pi \frac{R^2x^2}{h^2}(h-x).$$

from which

$$v\frac{h^2}{\pi R^2} = x^2(h-x).$$

In the expression $vh^2/\pi R^2$, the values h, π, and R are constants, and only v is a variable. We want to find such an x for which v becomes the greatest. But obviously, v will be the greatest at the same time as $vh^2/\pi R^2$, that is, $x^2\,(h-x)$. When does this latter expression become the greatest? Here, we have three variable factors: x, x, and $(h-x)$. If their sum were constant, the product would be greatest when the factors were equal. This constancy of the sum is easy to achieve if both parts of the last equation are multiplied by 2. Then we get:

$$2\frac{vh^2}{\pi R^2} = x^2\,(2h-2x).$$

Now the three factors on the right side have a constant sum:

$$x + x + 2h - 2x = 2h.$$

Therefore, their product will be the greatest when all the factors are equal, that is,

$$x = 2h - 2x, \quad \text{and} \quad x = \frac{2h}{3}.$$

Then the expression $2vh^2/\pi R^2$ will also be maximised, and with it the volume of the cylinder.

Now we know how the desired cylinder should be turned: its upper base should be located at 2/3 of the cone's height from the vertex.

How to Lengthen a Board?

When making something in a workshop or at home, it sometimes happens that the material at hand does not have the required dimensions.

In such cases, it is worth attempting to modify the material's dimensions through appropriate processing, and much can be achieved with some geometric and construction ingenuity and calculation.

Figure 189: How to lengthen the board by means of three sawing and one gluing?

Imagine this situation: you need a board of precise dimensions for making a bookshelf, specifically, 1 meter in length and 20 centimetres in width, but you have a shorter yet wider board, for example, 75 centimetres in length and 30 centimetres in width (see Figure 189 on the left).

What should you do?

Of course, you could saw off a strip 10 centimetres wide along the length of the board (dashed line), cut it into three equal pieces each 25 centimetres long, and use two of them to extend the board (see Figure 189 at the bottom).

This solution would be inefficient in terms of the number of operations (three cuts and three gluing points) and would not satisfy the strength requirements (strength would be reduced at the points where the strips are glued to the board).

You need to (see Figure 190) cut the board $ABCD$ along the diagonal AC and shift one half (for example, ABC) along the diagonal, parallel to itself, by the distance C_1E, which equals the missing length, i.e., 25 cm. The total length of the two halves will then be 1 meter. Now these halves need to be glued along the line AC_1, and the excess (shaded triangles) should be cut off. This will result in a board of the required dimensions.

Indeed, from the similarity of triangles ADC and C_1EC, we have:

$$\frac{AD}{DC} = \frac{C_1E}{EC}, \text{ from which}$$

$$EC = \frac{DC}{AD} \times C_1E, \quad \text{or}$$

$$EC = \frac{30}{75} \times 25 = 10\,\text{cm};$$

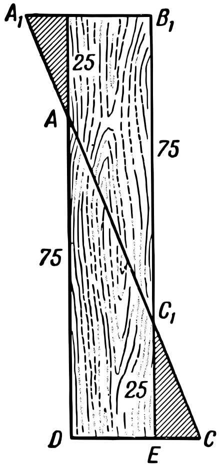


Figure 190: Solving the problem of lengthening the board.

$$DE = DC - EC = 30\text{ cm} - 10\text{ cm} = 20\text{ cm}.$$

The Shortest Path

In conclusion, let's consider a problem of "maxima and minima", which can be solved with a very simple geometric construction.

Question Along the bank of a river, a water tower needs to be built to supply water via pipes to villages A and B (see Figure 191).

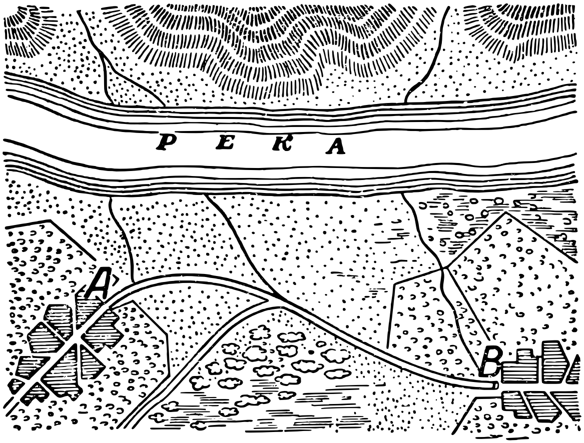


Figure 191: The problem of the water tower.

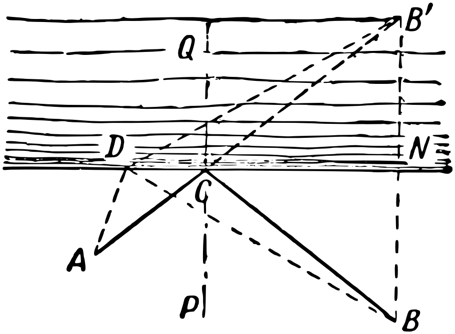


Figure 192: Geometric solution to the problem of finding the shortest path.

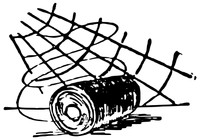
Where should it be built so that the total length of the pipes from the tower to both villages is the shortest?

Answer The problem reduces to finding the shortest path from A to the bank and then to B.

Suppose the shortest path is ACB (see Figure 191). Fold the diagram along line CN. We get point B'. If ACB is the shortest path, then, since $CB' = CB$, the path ACB' must be shorter than any other (e.g., ADB'). Thus, to find the shortest path, we only need to locate the point C where the straight line AB' intersects the riverbank. Then, connecting C and B, we find both parts of the shortest path from A to B.

By drawing a perpendicular at point C to CN, it's easy to see that the angles ACP and BCP, formed by both parts of the shortest path with this perpendicular, are equal ($\angle ACP = \angle B'CQ = \angle BCP$).

This is known as the law of reflection of a light ray off a mirror: *the angle of incidence equals the angle of reflection.* Hence, a light ray chooses the shortest path when it reflects, a conclusion known to the ancient physicist and geometer Heron of Alexandria two thousand years ago.



FINIS

Yakov Perelman

GEOMETRY *for* ENTERTAINMENT

This book is written both for friends of mathematics and for those readers from whom many attractive aspects of mathematics have somehow been hidden.

More importantly, this book is intended for those readers who studied (or are currently studying) geometry only at the blackboard and therefore are not used to noticing familiar geometric relationships in the world of things and phenomena around us, have not learnt to use the acquired geometric knowledge in practise, in difficult cases of life, on a hike, in a bivouac or front-line situation.

To arouse the reader's interest in geometry or, in the words of the author, ``to inspire a desire and cultivate a taste for its study is the objective of this book.''

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