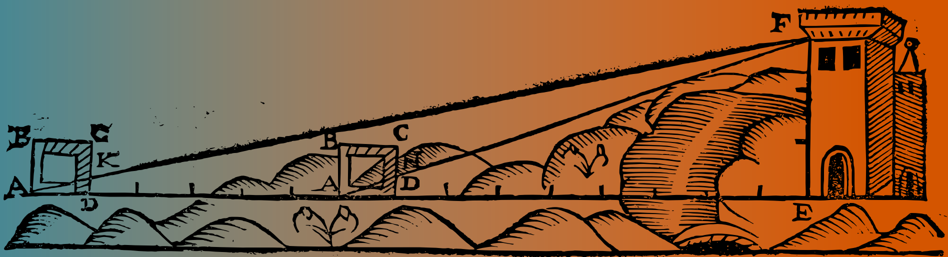
Yakov Perelman

GEOMETRY *for* ENTERTAINMENT



The Mir Titles Project

Geometry for Entertainment

Yakov Perelman

2024

Ya. I. Perelman

Geometry for Entertainment

The Mir Titles Project

Seventh Edition, Revised

Edited and supplemented by *B.A. Kordemsky*

First Published by State Publishing House Of Technical And Theoretical Literature Moscow – 1950 – Leningrad.

Original scan in Russian by the *Russian Lutherean* on The Internet Archive
https://archive.org/details/20220910_perelman_geometry/.

Translated from the Russian and typeset in Linux Libertine using LaTeX by *Damitr Mazanav.*

This fully electronic English translation released in 2024 by THE MIR TITLES PROJECT https://mirtitles.org .

Source files available at
https://gitlab.com/mirtitles/perelman-geometry

Front cover: Woodcut from Cosimo Bartoli's *Del modo di misvrare* published in 1564.
https://archive.org/details/cosimobartolidel00bart.

Contents

Editor’s Preface

Geometry for Entertainment is written both for friends of mathematics and for those readers from whom many attractive aspects of mathematics have somehow been hidden.

More importantly, this book is intended for those readers who studied (or are currently studying) geometry only at the blackboard and therefore are not used to noticing familiar geometric relationships in the world of things and phenomena around us, have not learnt to use the acquired geometric knowledge in practise, in difficult cases of life, on a hike, in a bivouac or front-line situation.

To arouse the reader's interest in geometry or, in the words of the author, "to inspire a desire and cultivate a taste for its study is the objective of this book."

To this end, the author will take geometry "out of the walls of the school room into the free air, into the forest, field, to the river, on the road, in order to indulge in relaxed geometric studies without a textbook and tables in the open air ...", and draws the reader's attention to the pages of L. N. Tolstoy and A. P. Chekhov, Jules Verne and Mark Twain. He finds a theme for geometric problems in the works of N. V. Gogol and A. S. Pushkin, and finally offers the reader "a motley selection of problems, curious in plot, unexpected in result."

The seventh edition of *Geometry for Entertainment* is published without the direct participation of the author. Ya. I. Perelman died in Leningrad in 1942.

The new edition of the book contains almost all the articles of the previous edition, newly illustrated, edited and supplemented with facts and information from our Soviet reality, as well as a considerable number (about 30) additional articles.

I was guided by the desire to increase the “utility coefficient” of Ya. Perelman’s book, to make it even more effective and interesting, involving new readers in the ranks of friends of mathematics.

To what extent this was possible, I hope to learn from readers at the address: Moscow, 64, Chernyshevsky Str., 81, Sq. 53, B. A. Kordemsky.

B. Kordemsky

Translator’s Preface

Yakov Perelman’s books have been a constant source of inspiration for me throughout my life. Though many of his works have been translated to English and other languages, several works remain untranslated. As a part of Mir Titles Project we endeavour to bring all such works to the people. This translation is a rather ambitious project and it brings me great pleasure to present this untranslated work of Perelman into an English version.

Illustrations

The beautiful and abundant illustrations in the form of woodcuts, are the heart of the book. Geometry being primarily reliant on illustrations, is brought to life in a variety of situations. Familiar geometrical shapes, lines, ratios are found amongst trees, rivers, homes, skies and other natural settings. This makes everyday objects the familiarly mathematical.

Each topic is complemented by relevant illustrations which make understanding them easier. Being woodcuts, it was easy for me to convert them to digital form. I have made no effort to change the images, except in some cases replacing the Russian letters with Roman ones.

Examples

In his discussions emphasises the geometrical relationships in the measurable and unknown quantities. This approach is historical in the sense that this is how geometry developed: to solve problems of measurement of unknown quantities. Thus we have problems related to measuring a variety of things, using direct measurement or very primitive instruments.

Translation

I have made use of machine translations for the bulk of text, and it has worked at a satisfactory level. At times I have made use of several translation services to make sure I am on right track and the meaning is not lost in translation.

Typesetting

During the course of typesetting this book, discussions posted on and help from kind people LaTeX forum at stackexchange has been of great help. I have typeset the book in a square profile with marginpar for smaller figures and notes.

If there are any mistakes in the mathematics or translation they are all mine. Any suggestions and criticisms to improve the translation are welcome. I hope that this English version finds enthusiastic readers and inspires many more brilliant minds in the generations to come.

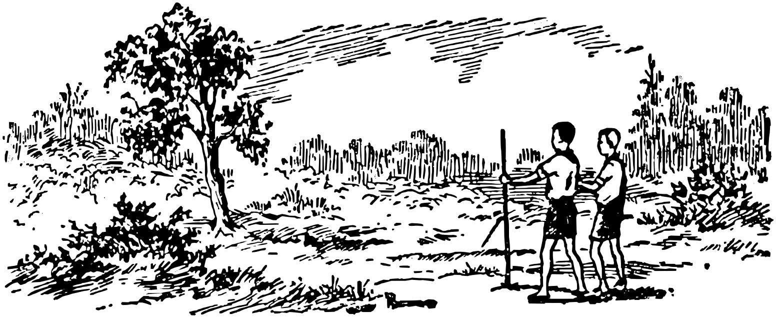
Damitr Mazanav

Part I.

Geometry In The Open Air

Nature speaks the language of mathematics: the letters of this language are circles, triangles and other mathematical shapes.

Galileo



1. Geometry In The Forest

1.1. By the length of the shadow

I remember now the amazement with which I looked

for the first time, he looked at a gray-haired forester, who, standing near a huge pine tree, measured its height with a small pocket device. When he aimed his square board at the top of the tree, I expected that the old man would now start climbing there with a measuring chain. Instead, he put the device back in his pocket and announced that the measurement was over. I thought it hadn't started yet …

I was very young then, and this way of measuring, when a person determines the height of a tree without cutting it down and climbing to the top, was in my eyes something like a small miracle. It was only later, when I was initiated into the rudiments of geometry, that I realised how simple such miracles are performed. There are many different ways to make such measurements using very simple instruments and even without any devices.

The easiest and most ancient way is, without a doubt, the one by which the Greek sage Thales determined the height of the pyramid in Egypt sixth century BC. He took advantage of the pyramid's 'shadow'. The priests and the pharaoh, gathered at the foot of the highest pyramid, looked puzzled at the northern newcomer, who guessed the height of the huge structure from the shadow. Thales, says the legend, chose a day and an hour when the length of his own shadow was equal to his height; at this moment, the height of the pyramid should also be equal to the length of the shadow cast by it[1]. This is perhaps the only case when a person benefits from his shadow …

[1] Of course, the length of the shadow had to be measured from the midpoint of the square base of the pyramid; Thales could directly measure the width of this base.

The task of the Greek sage now seems childishly simple to us, but let's not forget that we are looking at it from the height of a geometric building erected after Thales. He lived long before Euclid, the author of the wonderful book that taught geometry for two millennia after his death. The truths contained in it, which are now known to every schoolboy, were not yet discovered in the era of Thales. And in order to use the shadow to solve the problem of the height of the pyramid, it was necessary to already know some geometric properties of the triangle, namely the following two (of which Thales himself discovered the first):

1. that the angles at the base of an isosceles triangle are equal, and vice versa – that the sides lying opposite the equal angles of the triangle are equal to each other;

2. that the sum of the angles of any triangle (or at least a rectangular one) is equal to two right angles.

Only Thales, armed with this knowledge, had the right to conclude that when his own shadow is equal to his height, the sun's rays meet the flat ground at an angle of half a straight line, and therefore the top of the pyramid, the middle of its base and the end of its shadow should mark an isosceles triangle.

It would seem that this simple method is very convenient to use on a clear sunny day to measure lonely trees whose shadow does not merge with the shadow of neighbouring ones. But in our latitudes it is not as easy as in Egypt to waylay the right moment for this: The sun is low above the horizon, and the shadows are equal to the height of the objects casting them only in the afternoon hours of the summer months. Therefore, the Thales method in this form is not always applicable.

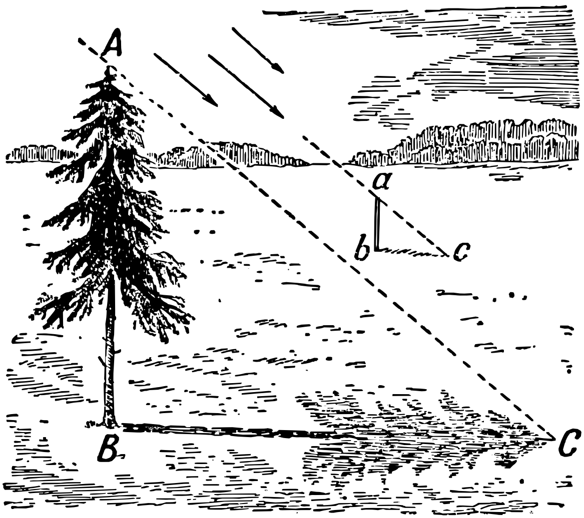


Figure 1.: Measuring the height of a tree by shadow.

It is not difficult, however, to modify this method so that on a sunny day, any shadow can be used, regardless of its length. Additionally, measuring both your own shadow and the shadow of a pole, the desired height is calculated from the proportion (Figure 1):

$$AB : ab = BC : bc,$$

meaning the height of the tree is as many times greater than your own height (or the height of the pole) as the shadow of the tree is longer than your shadow (or the shadow of the pole). This naturally follows from the geometric similarity of triangles ABC and abc (based on two angles).

Some readers may object that such an elementary technique does not need a geometric justification at all: is it really unclear even without geometry that how many times is a tree taller, how many times is its shadow longer? However, the matter is not as simple as it seems. Try to apply this rule to shadows cast by the light of a street lamp or lamp – it will not be justified. In Figure 2 you can see that the columns AB are about three times higher than the pedestal ab, and the shadow of the column is eight times larger than the shadow of the pedestal ($BC : bc$). It is impossible to explain why the method is applicable in this case, but not in the other, without geometry.

Question Let's take a closer look at what the difference is. The essence of the matter boils down. to the fact that the

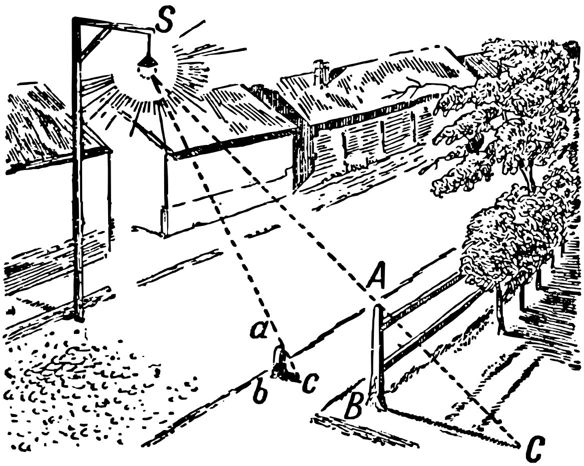


Figure 2.: When such a measurement is impossible. (Is the method applicable for a shadow cast by a streetlamp?)

sun's rays are parallel to each other, the rays of the lantern are not parallel. After that, we have the right to consider the rays of the Sun parallel, although they certainly intersect in the place from which they originate.

Answer The rays of the Sun falling on the Earth can be considered parallel because the angle between them is extremely small, almost imperceptible. A simple geometric calculation will convince you of this. Imagine two rays coming from some point of the Sun and falling on the Earth at a distance of, say, one kilo-meter from each other. So, if we put one leg of a compass at this point of the Sun, and with the other we described a circle with a radius equal to the distance from the Sun to the Earth (i.e., with a radius of 150,000,000 km), then an arc of one kilometer in length would appear between our two radii rays. The total length of this gigantic circle would be equal to $2\pi \times 150{,}000{,}000$ km = 940,000,000 km. One degree of it, of course, is 360 times less, i.e. about 2,600,000 km; one arc minute is 60 times less than a degree, i.e. equal to 43,000 km, and one arc second is another 60 times less, i.e. 720 km. But our arc is only 1 km in length, so it corresponds to an angle of $1/720 \approx 0.001{,}38''$ seconds. This angle is elusive even for the most accurate astronomical instruments; therefore, in practise we can consider the rays of the Sun falling on the Earth as parallel lines.[2]

[2] Another thing is the rays directed from some point of the Sun to the ends of the earth's diameter; the angle between them is large enough to measure (about $17''$); the definition of this angle gave astronomers one of the means to establish how great the distance from the Earth to the Sun is.

Trying to apply the method of shadows in practise, you will immediately be convinced, however, of its unreliability. Shadows are not delimited so clearly that measuring their length can be done quite accurately. Each shadow cast by the light of the Sun has an indistinctly outlined grey border of penumbra, which gives the border of the shadow uncertainty. This is because the Sun is not a point, but a large luminous body emitting rays from many points. Figure 3 illustrates why, as a result of this, the shadow of tree AB also has an additional component in the form of half-shadow CD, gradually fading away.

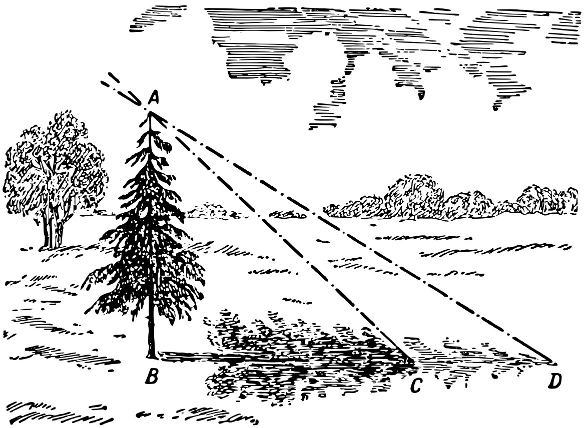


Figure 3.: How penumbra is formed.

The angle of the *CAD* between the extreme boundaries of the penumbra is equal to the angle at which we always see the solar disk, i.e. half a degree. The error resulting from the fact that both shadows are not measured quite accurately can reach 5% or more when the Sun is not too low. This error is added to other unavoidable errors – from uneven soil, etc. – and makes the final result little reliable. In mountainous terrain, for example, this method is completely inapplicable.

1.2. Two More Methods

It is entirely possible to measure height without relying on shadows. There are many methods; let's start with two simple ones.

Firstly, we can utilise the properties of an isosceles right triangle. For this purpose, we can make use of a very simple tool, which can be easily crafted from a piece of board and three pins. On a board of any shape, even a piece of bark with a flat side, mark three points to form the vertices of a right triangle – and insert a pin at each point (see Figure 4). Suppose you don't have a drafting triangle to construct a right angle, nor a compass to mark equal sides. In that case, fold any piece of paper once, and then fold it again across the first fold so that both parts of the first fold coincide – and you'll obtain a right angle. The same piece of paper can be used instead of a compass to measure equal distances.

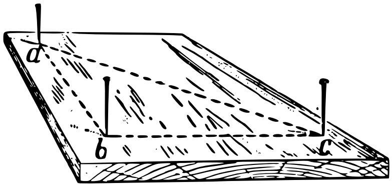


Figure 4.: Pin height measuring device.

As you can see, the tool can be entirely crafted in a makeshift environment.

If you don't have a drafting triangle on hand to construct a right angle, nor a compass to mark equal sides, then simply fold any scrap of paper once, and then fold it again across the first fold so that both parts of the first fold coincide—and you'll obtain a right angle. The same piece of paper can be used instead of a compass to measure equal distances.

As you can see, the tool can be entirely crafted in a makeshift environment.

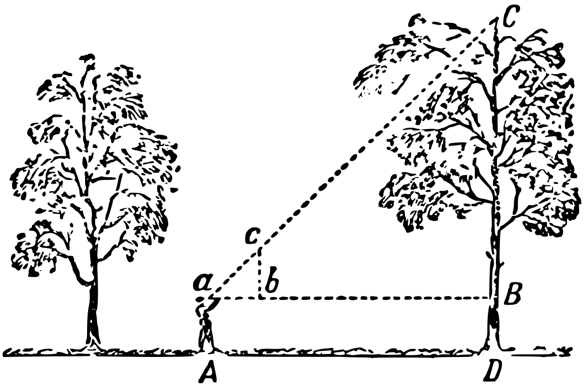


Figure 5.: The scheme of application of the pin device.

Handling it is no more difficult than crafting it. Stepping away from the tree being measured, hold the tool so that one of the legs of the triangle is perpendicular. You can use a string or a weight tied to the top pin. Approaching or moving away from the tree, you will always find a spot A (see Figure 5), from which, looking at pins a and c, you will see that they cover the top C of the tree: this means that the extension of the hypotenuse ac passes through point C. Then, obviously, the distance aB is equal to CB, since angle $a = 45^\circ$.

Consequently, by measuring distance aB (or, at another location, distance AD) and adding BD to it, i.e., the elevation of point a above the ground, you will obtain the desired height of the tree.

Another method does not even require a pin device. Here you need a pole, which you will have to insert vertically into the ground so that the protruding part is exactly at your height. The location for the pole must be chosen so that, lying down as shown in Figure 6, you see the top of the tree in a straight line with the upper point of the pole. Since triangle Aba is isosceles and right-angled, angle $A = 45^\circ$, and therefore AB equals BC, i.e., the desired height of the tree.

1.3. The Method of Jules Verne

The next, also quite simple, method for measuring tall objects is vividly described by Jules Verne in his famous novel *The*

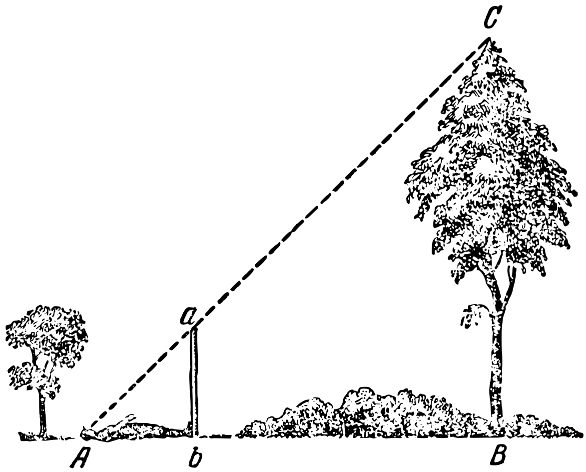


Figure 6.: Another way to determine the height.

Mysterious Island.

“Today we need to measure the height of the Far View platform,” said the engineer.

“Will you need a tool for that?” asked Herbert.

“No, we won’t. We’ll proceed somewhat differently, resorting to a somewhat simpler and more accurate method.”

The young man, eager to learn as much as possible, followed the engineer, who descended from the granite wall to the rocky shore.

Taking a straight pole, twelve feet long, the engineer measured it as precisely as possible, comparing it to his own height, which he knew well. Meanwhile, Herbert held a plumb bob given to him by the engineer: just a stone attached to the end of a rope.

Not reaching five hundred feet from the granite wall, which rose vertically, the engineer drove the pole two feet into the sand and firmly secured it, placing it vertically with the help of the plumb bob.

Then he moved away from the pole to a distance where, lying on the sand, one could see both the end of the pole and the edge of the ridge in a straight line (see Figure 7). He carefully marked this point with a stake.

"Are you familiar with the basics of geometry?" he asked Herbert as he rose from the ground.

"Yes."

"Do you remember the properties of similar triangles?"

"Their corresponding sides are proportional."

"Exactly. So now I'll construct two similar right triangles. In the smaller one, one leg will be the plumb-line pole, and the other will be the distance from the stake to the base of

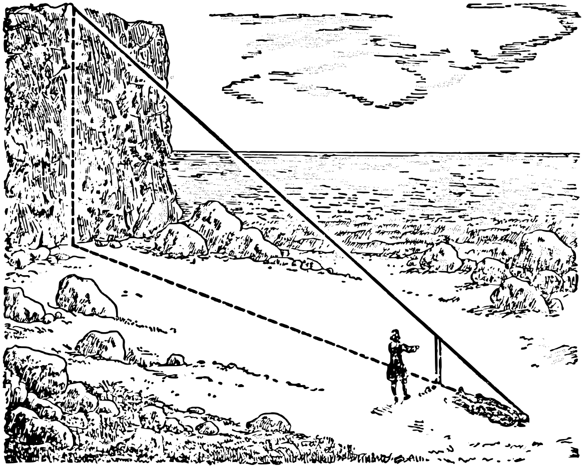


Figure 7.: How the heroes of Jules Verne measured the height of the cliff.

the pole; the hypotenuse will be my line of sight. In the other triangle, the legs will be: the granite wall, the height of which we want to determine, and the distance from the stake to the base of this wall; the hypotenuse will be my line of sight, coinciding with the direction of the hypotenuse of the first triangle."

"Understood!" exclaimed the youth. "The distance from the stake to the pole is related to the distance from the stake to the base of the wall, as the height of the pole is to the height of the wall."

"Yes. And consequently, if we measure the first two distances, then, knowing the height of the pole, we can calculate the fourth, unknown term of the proportion, i.e., the height of the wall. In this way, we can manage without directly measuring the height."

Both horizontal distances were measured: the smaller one was 15 feet, the larger one was 500 feet.

At the end of the measurements, the engineer made the following record:

$$15 : 500 = 10 : x,$$
$$500 \times 10 = 5000,$$
$$5000 : 15 = 333.3.$$

Thus, the height of the granite wall was 333 feet.

1.4. How Sergeant Popov Acted

Some of the methods described for measuring height are inconvenient as they require lying on the ground. This inconvenience can, of course, be avoided.

Here's a story from one of the fronts of the Great Patriotic War. Lieutenant Ivanyuk's unit was ordered to build a bridge across a mountain river. On the opposite bank were entrenched fascists. To scout the location for the bridge, the lieutenant assigned a reconnaissance group led by Senior Sergeant Popov. In the nearest forest, they measured the diameter and height of the most typical trees and counted the number of trees that could be used for construction.

They measured the height of the trees using a pole (stick) as shown in Figure 8.

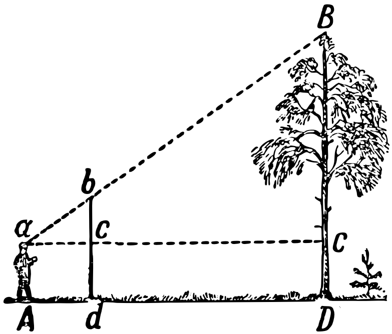


Figure 8.: Measuring the height of the trees with a pole.

This method works as follows: Armed with a pole taller than your own height, drive it into the ground vertically at some distance from the tree being measured (see Figure 8). Step back from the pole along the line dD until you reach point A, from where, looking at the top of the tree, you'll see the upper point B of the pole aligned with it. Then, without changing the position of your head, look along the horizontal line aC, noting the point C where your line of sight intersects the pole and the tree trunk. Ask your assistant to mark these points, and the observation is complete. Then, based on the similarity of triangles abc and aBC, calculate BC from the proportion

$$\frac{BC}{bc} = \frac{aC}{ac},$$

and thus

$$BC = bc \cdot \frac{aC}{ac}$$

The distances bc, aC, and ac can be easily measured directly. To obtain the actual height of the tree, add the distance BC to the distance CD, which is also measured directly.

To determine the number of trees, the senior sergeant ordered the soldiers to measure the area of the forest. Then he counted the number of trees in a small area measuring 50 by 50 meters and multiplied accordingly.

Based on all the data collected by the scouts, the unit commander determined where and what kind of bridge needed to be built. The bridge was completed on time, and the combat mission was successfully accomplished![3]

[3] The episodes of the Great Patriotic War described here and further are narrated by A. Demidov in the journal *Military Knowledge* No. 8, 1949, in the article *River Reconnaissance.*

1.5. Using a Notebook

As a device for an approximate estimate of the inaccessible height, you can also use your pocket back book, if it is equipped with a pencil stuck in a cover or a loop with a book. It will help you to build in space those two similar triangles, from which the desired height is obtained. The book should be held near the eyes as shown in the simplified Figure 9. It should be in the vertical book so that, looking from the point a, you can see the top of the tree B covered with the tip of the pencil b. Then, due to the similarity of the triangles abc and aBC, the height of the BC will be determined from the proportion

$$\frac{BC}{bc} = \frac{aC}{ac}.$$

The distances of bc, ac and aC are measured directly. To the resulting value of the BC, add the length of CD, which is, on level ground, the height of the eyes above the ground

Since the width of the ac book is unchanged, if you always stand at the same distance from the measured tree (for example, 10 m), the height of the tree will depend only on the extended part of the pencil. Therefore, you can calculate in advance what height corresponds to a particular extension, and put these numbers on the pencil. Your notebook will then turn into a simplified altimeter, since you can use it to determine heights immediately, without calculations.

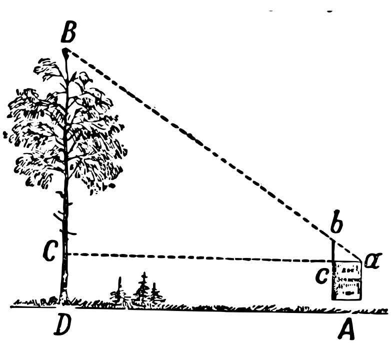


Figure 9.: Height measurement using a notebook.

1.6. Without Approaching The Tree

Sometimes it may be inconvenient to get close to the base of the tree being measured. Can its height still be determined in such a case?

Absolutely. For this purpose, a clever device has been devised, which, like the previous ones, is easy to make by yourself. Two planks, *ab* and *cd* (top of Figure 10), are fastened together at right angles so that *ab* equals *bc*, and *bd* equals half of *ab*. That's the whole device.

To measure height with it, hold it in your hands, directing plank CD vertically (for which it has a plumb line with a weight), and stand precisely in two places: first (Figure 10) at point *A*, where the device is positioned with end *c* up, and then at point A', a bit farther away, where the device is held with end d up. Point A is chosen so that, looking from a to the end of a, it is seen on the same line as the top of the tree. Point A' is found so that, looking from a' to point d', it is seen coinciding with B.

The discovery of these two points A and A'[4] constitutes all the measurement because the desired part of the tree's height, BC, is equal to the distance DA'. The equality follows easily from the fact that $aC = BC$ and $a'C = 2BC$; thus,

$$a'C - aC = BC.$$

[4] These points must necessarily lie in a straight line with the base of the tree.

You can see that using this simple device, we measure the tree's height without approaching closer than its height. It goes without saying that if it's possible to approach the trunk, it's sufficient to find just one of the points – A or A' – to determine its height.

Instead of two planks, you can use four pins, arranging them on a board properly; in this form, the "device" is even simpler.

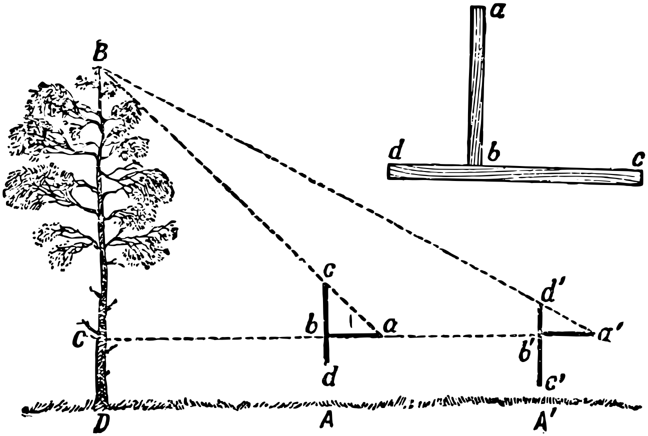


Figure 10.: The use of a simple altimeter consisting of two planks.

1.7. Forest Rangers' Altimeter

It's time to explain how the "real" altimeters, used in practise by forest workers, are constructed. I'll describe one of these altimeters, slightly modifying it so that the device can be easily crafted at home.

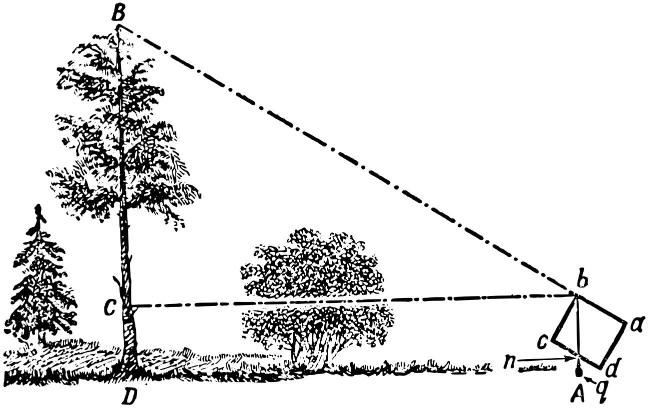


Figure 11.: The scheme of using the altimeter of foresters.

The essence of the device is visible in Figure 11. A cardboard or wooden rectangle, *abcd*, is held in the hand so that, looking along edge *ab*, the tip *B* of the tree is in line with it. A weight, *q*, is suspended from point *b* on a thread. We note the point *n* where the thread intersects line *dc*. Triangles *bBC* and *bnc* are similar because they are both rectangular and have equal acute angles bBC and bnc (with corresponding parallel sides). Therefore, we can write the proportion:

$$\frac{BC}{nc} = \frac{bC}{bc}; \text{ hence}$$
$$BC = bC \cdot \frac{nc}{bc}.$$

Since bC, nc, and bc can be measured directly, it is easy to obtain the desired height of the tree by adding the length of the lower part CD to the trunk (the height of the device above the ground).

A few details remain to be added. If the edge of the board bc is made, for example, exactly 10 cm, and centimeter divisions are marked on edge dc, then the ratio nc/bc will always be expressed as a decimal fraction, directly indicating what fraction of the distance bC represents the height of the tree BC. For example, let's say the thread stops against the 7th division mark (i.e., $nc = 7$ cm); this means that the height of the tree above eye level is 0.7 times the observer's distance from the trunk.

The second improvement relates to the method of observation: to make it convenient to look along line ab, you can fold down two squares with holes drilled in them at the upper corners of the cardboard rectangle: one smaller one for the eye

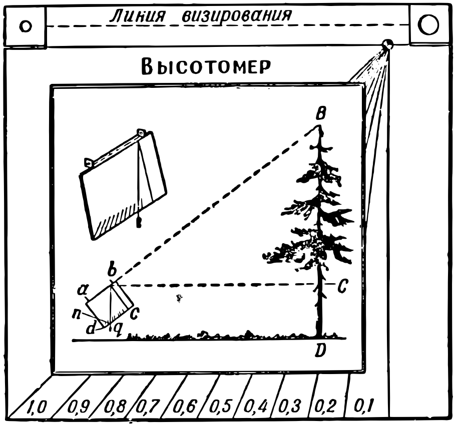


Figure 12.: The forest rangers' altimeter.

and one larger one for sighting the tree top (see Figure 11). Further enhancement is represented by the device shown almost to scale in Figure 12. It is easy and quick to make it in this form; no special skill is required. Occupying little space in the pocket, it will provide you with the ability to quickly determine the heights of encountered objects during excursions—trees, poles, buildings, and so on. (This tool is part of the *Geometry in the Open Air* kit developed by the author of this book.)

Question Is it possible to use the altimeter described now to measure trees that cannot be approached closely? If possible, what should be done In such cases?

Answer The device should be aimed at the top of the tree B, as shown in Figure 13, from two points, A and A'.

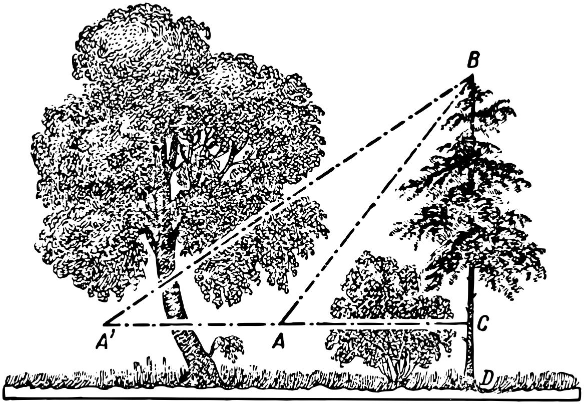


Figure 13.: How to measure the height of a tree without approaching it.

Let's say at point A we determined that $BC = 0.9\,AC$, and at point A' we determined that $BC = 0.4\,A'C$. Then we know that:

$$AC = \frac{BC}{0.9}, \quad A'C = \frac{BC}{0.4}$$

So that we can write

$$AA' = A'C - AC = \frac{BC}{0.4} - \frac{BC}{0.9} = \frac{25}{18}\,BC.$$

Hence,

$$\begin{aligned} AA' &= \frac{25}{18}\,BC, \\ \therefore\ BC &= \frac{18}{25}\,AA' \\ &= 0.72\,AA'. \end{aligned}$$

You can see that by measuring the distance AA' between both observation points and taking a certain fraction of this value, we can determine the desired and inaccessible height.

1.8. Using a Mirror

Question Here's another unconventional method for determining the height of a tree using a mirror. At some distance (see Figure 14) from the tree being measured, on level ground at point C, place a small mirror horizontally and step back to point D, from where the observer can see the top of tree's point A in the mirror. Then, the tree (AB) is as many times taller than the observer's height (ED) as the distance BC from the mirror to the tree is greater than the distance CD from the mirror to the observer. Why?

Answer The method is based on the law of reflection of light. The top A (Figure 15) is reflected at point A' in such a way that $AB = A'B$. From the similarity of triangles BCA'

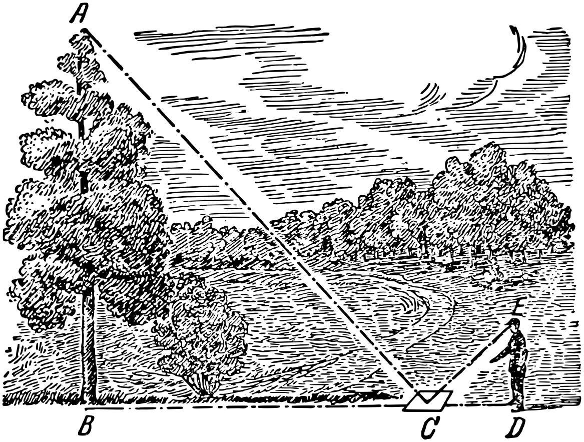


Figure 14.: Height measurement using a mirror.

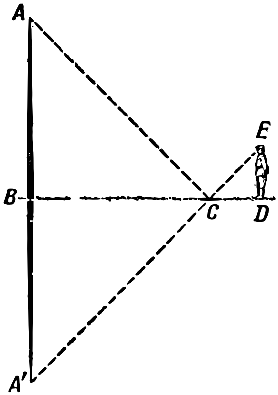


Figure 15.: Geometric construction for the method of measuring the height using a mirror.

and CED, it follows that

$$\frac{A'B}{ED} = \frac{BC}{CD}.$$

In this, simply replace $A'B$ with AB to justify the relationship stated in the problem. This convenient and effortless method can be applied in any weather, but not in dense vegetation, only to a solitary tree.

Question However, what should be done when it is impossible to approach the tree being measured closely for some reason?

Answer This is an ancient problem dating back over 500 years. It is discussed by the medieval mathematician Antonius de Cremona in his work *On Practical Land Measurement* (1400).

The problem is solved by the dual application of the method described earlier – placing the mirror in two locations. By making the appropriate construction, it is easy to deduce from the similarity of triangles that the sought-after height of the tree is equal to the observer's eye level multiplied by the ratio of the distance between the mirror positions to the difference in distances from the mirror to the observer.

Before concluding the discussion on measuring the height of trees, I propose to the reader another "forest" problem.

1.9. Two Pines

Question Two pine trees grow 40 meters apart. You measured their heights: one turned out to be 31 meters tall, while the other, younger one, is only 6 meters tall. Can you calculate the distance between their tops?

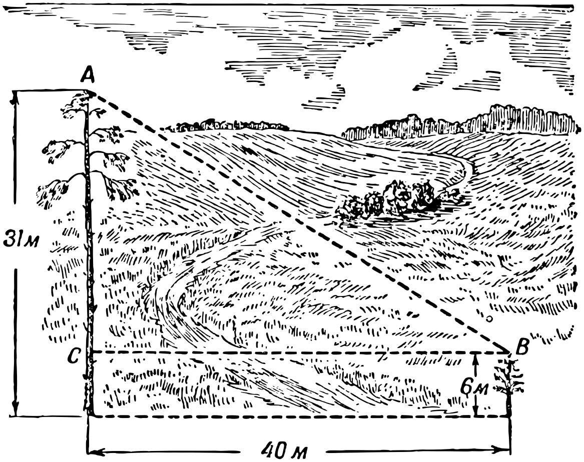


Figure 16.: What is the distance between the tops of the pines?

Answer The desired distance between the tops of the pine trees (see Figure 16) according to the Pythagorean theorem is

$$\sqrt{40^2 + 25^2} = 47 \text{ m}.$$

1.10. The shape of the tree trunk

Now you can already, walking through the forest, determine – in almost half a dozen different ways – the height of any tree. It will probably be interesting for you to determine its volume as well, calculate how many cubic meters of wood it contains, and at the same time weigh it. To find out if, for example, it would be possible to take away such a trunk on one cart. Both of these tasks are no longer as simple as determining height; experts have not found ways to accurately resolve it and are content with only a more or less approximate estimate. Even for a felled trunk, which lies in front of you cleared of branches, the task is far from easy.

The thing is, a tree trunk, even the smoothest one without bulges, does not represent either a cylinder, a complete cone, a truncated cone, or any other geometric solid whose volume we can calculate using formulas. The trunk is certainly not a cylinder — it tapers towards the top (it has ”runoff,” as foresters say) — but it is also not a cone because its ”generating line” is not a straight line, but a curve, and moreover, not a circular arc, but some other curve, convex towards the axis of the tree.[5]

[5] The curve that fits closest to this is called the “semicubical parabola” ($y^3 = ax^2$); the solid obtained by rotating this parabola is called a “neiloid” (named after the ancient mathematician Neil, who found a way to determine the length of the arc of such a curve). The shape of a tree trunk grown in the forest approximates that of a neiloid. Calculating the volume of a neiloid is done using advanced mathematical techniques.

Therefore, a more or less accurate calculation of the volume of a tree trunk can only be done using the tools of integral calculus. To some readers, it may seem strange that the measurement of a simple log requires resorting to the services of higher mathematics. Many think that higher mathematics is only relevant to some special subjects, whereas in everyday life, only elementary mathematics is applicable. This is completely incorrect: one can fairly accurately calculate the volume of a star or a planet using elements of geometry, whereas an exact calculation of the volume of a long log or a beer barrel is impossible without analytical geometry and integral calculus.

However, our book does not assume that the reader is familiar with higher mathematics; therefore, here we will have to be content with only an approximate calculation of the volume of the trunk. We will assume that the volume of the trunk is more or less close either to the volume of a truncated cone, or – for a trunk with a pointed end — to the volume of a complete cone, or, finally, – for short logs — to the volume of a cylinder. The volume of each of these three solids can be easily calculated. Could we find a formula for the volume that would be suitable for all three of these named solids for the sake of consistency in calculation? Then we would approximately calculate the volume of the trunk without caring about what it resembles more — a cylinder or a cone, complete or truncated.

1.11. Universal Formula

Such a formula exists; moreover, it is not only suitable for cylinders, complete and truncated cones, but also for all kinds of prisms, pyramids complete and truncated, and even for spheres. Here is this remarkable formula, known in mathematics as Simpson's formula:

[6] That is, the cross-sectional area of the body in the middle of its height.

$$v = \frac{h}{6}\left(b_1 + 4b_2 + b_3\right)$$

where h is the height of the solid, b_1 is the area of the lower base, b_2 is the area of the middle section[6], b_3 is the area of the upper base.

Question Prove that with this formula, one can calculate the volume of the following seven geometric solids: prism, pyramid complete, pyramid truncated, cylinder, cone complete, cone truncated, sphere.

Answer It is very easy to verify the correctness of this formula by simply applying it to the listed solids. Then, we obtain for the prism and cylinder (see Figure 17 a):

$$v = \frac{h}{6}\left(b_1 + 4b_2 + b_3\right) = b_1 h;$$

for the pyramid and cone (see Figure 17 b):

$$v = \frac{h}{6}\left(b_1 + 4\frac{b_2}{4} + 0\right) = \frac{b_1 h}{3};$$

for the truncated cone (see Figure 17 c):

$$
\begin{aligned}
v &= \frac{h}{6}\left[\pi R^2 + 4\pi\frac{(R+r)^2}{2} + \pi r^2\right] \\
&= \frac{h}{6}\left[\pi R^2 + \pi R^2 + 2\pi Rr + \pi r^2 + \pi r^2\right] \\
&= \frac{\pi h}{3}\left[R^2 + Rr + r^2\right]
\end{aligned}
$$

for the truncated pyramid, the proof proceeds similarly; finally, for the sphere (see Figure 17 d):

$$v = \frac{2R}{6}\left(0 + 4\pi R^2 4 + 0\right) = \frac{4}{3}\pi R^3.$$

Question Let's note another interesting feature of our universal formula: it is also suitable for calculating the area of plane figures: parallelograms, trapezoids, and triangles, if by

§ h we mean, as before, the height of the figure,

§ by b_1 the length of the lower base,

§ by b_2 the length of the middle base and

§ by b_3 the length of the lower base.

How can we confirm this?

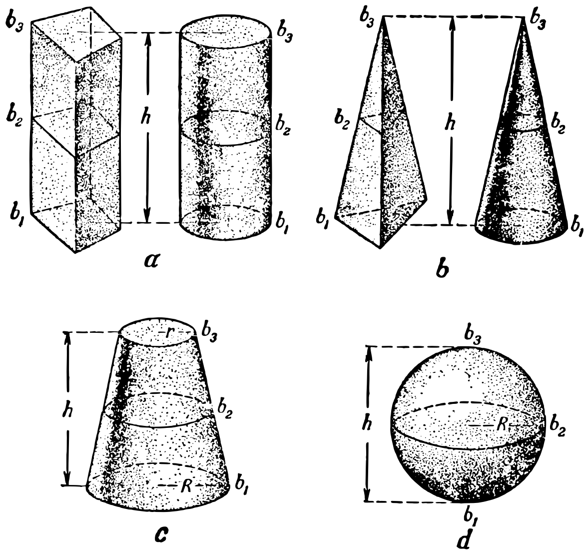


Figure 17.: Geometric bodies whose volumes can be calculated using a single formula.

Answer Applying the formula, we have: for a parallelogram (square, rectangle) (see Figure 18 a)

$$S = \frac{h}{6}\left(b_1 + 4b_1 + b_1\right) = b_1 h;$$

for a trapezoid (see Figure 18 b)

$$S = \frac{h}{6}\left(b_1 + 4\frac{b_1 + b_2}{2} + b_3\right) = \frac{h}{2}\left(b_1 + b_3\right);$$

for a triangle (see Figure 18 c)

$$S = \frac{h}{6}\left(b_1 + 4\frac{b_1}{2} + 0\right) = \frac{b_1 h}{2}.$$

You can see that our formula has enough right to be called universal.

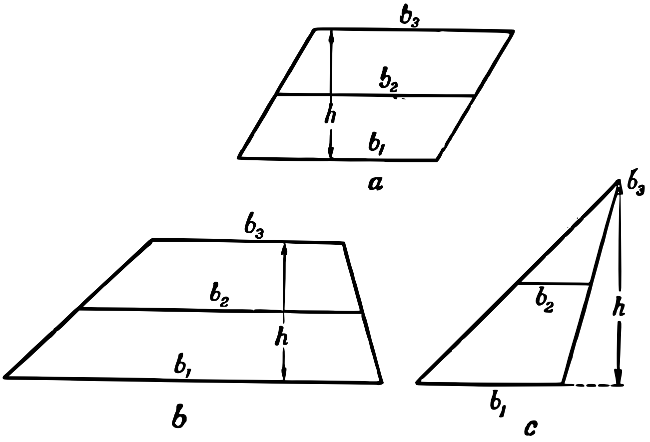


Figure 18.: The universal formula is also suitable for calculating the areas of these figures.

1.12. Volume and Weight of a Tree at the Root

So, you have a formula with which you can approximately calculate the volume of a felled tree trunk without worrying about what geometric shape it resembles: a cylinder, a complete cone, or a truncated cone. For this, four measurements are needed – the length of the trunk and three diameters: the lower cut, the upper, and in the middle of the length. Measuring the lower and upper diameters is very simple; however, determining the average diameter without a special device ("measuring fork" used by foresters, see Figure 19 and Figure 20[7] is quite difficult. But the difficulty can be overcome by encircling the trunk with a rope and dividing its length by 3 1/7 to get the diameter.

[7] A similar principle is applied in the well-known device for measuring the diameter of round objects – the caliper Figure 20, to the right).

The volume of a felled tree trunk obtained in this way is accurate enough for many practical purposes. In short, but less accurately, this problem can be solved by calculating the volume of the trunk as the volume of a cylinder, the diameter of the base of which is equal to the diameter of the trunk in the middle of its length; however, the result obtained is underestimated, sometimes by 12%. But if you mentally divide the trunk into two-meter segments and determine the volume of each of these almost cylindrical parts to then add them up, the result will be much better: it errs on the side of

Figure 19.: Measuring the diameter of a tree with a measuring fork.

underestimation by no more than 2–3%.

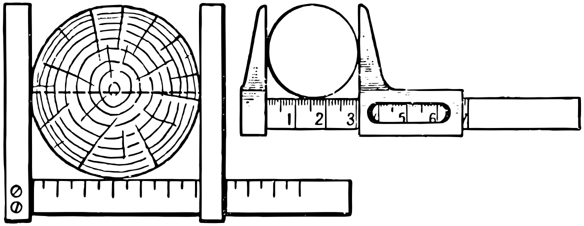


Figure 20.: Measuring fork (left) and caliper (right).

However, all this is completely inapplicable to a tree at the root: if you are not going to climb it, then you can only measure the diameter of its lower part. In this case, to determine the volume, you will have to be satisfied with only a very approximate estimate, comforting yourself with the fact that professional foresters usually proceed in a similar way. They also use a table of so-called “species numbers,” i.e., numbers that show what proportion of the volume of the measured tree is compared to the volume of a cylinder of the same height and diameter, measured at the height of a grown man’s chest, i.e., 130 cm (this height is the most convenient for measuring).

Figure 21 illustrates this clearly. Of course, “species numbers” vary for trees of different species and heights, as the shape of the trunk is variable. However, the fluctuations are not particularly great: for pine and fir trunks (grown in dense plantations), “species numbers” range from 0.45 to 0.51, i.e., are approximately half.

Thus, without much error, it can be assumed that the volume of a coniferous tree at the root is half the volume of a cylinder of the same height with a diameter equal to the diameter of the tree at chest height.

This is, of course, only an approximate estimate, but it is not too far from the true result: up to 2% in the overestimation direction and up to 10% in the underestimation

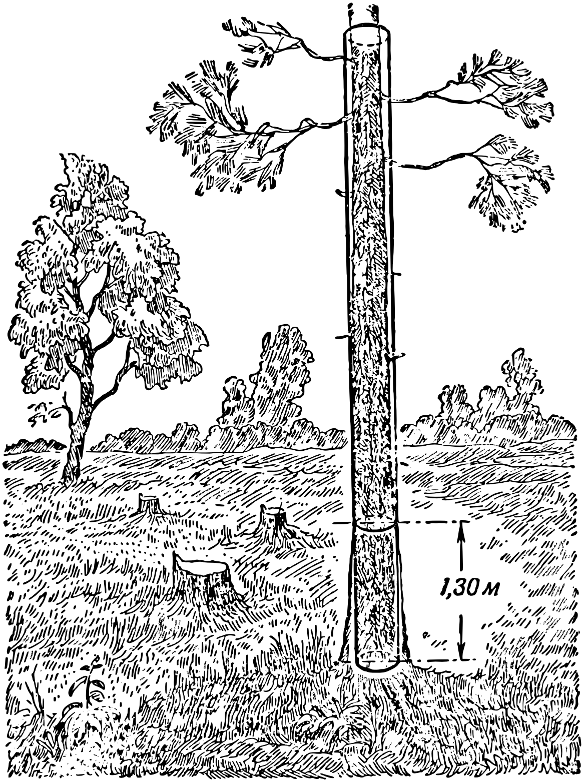


Figure 21.: What is a “species number”?

direction.[8]

From here, it is only one step towards estimating the weight of the tree at the root. For this, it is enough to know that 1 cubic meter of fresh pine or fir wood weighs about 600–700 kg. For example, suppose you are standing next to a fir tree, the height of which you have determined to be 28 m, and the circumference of the trunk at chest height is 120 cm. Then the area of the corresponding circle is 1,100 cm^2, or 0.11 m^2, and the volume of the trunk is $1/2 \times 0.11 \times 28 = 1.5\,\mathrm{m}^3$. Assuming that 1 cubic meter of fresh fir wood weighs on average 650 kg, we find that 1.0 cubic meter should weigh about a ton (1,000 kg).

[8] It must be remembered that "species numbers" refer only to trees that have grown in the forest, i.e. to tall and thin (smooth, without nodes); for free-standing branched trees, such general rules for calculating volume cannot be specified.

1.13. Leaf Geometry

Question In the shadow of a silver poplar from its roots, a thicket has grown. Pick a leaf and notice how large it is compared to the leaves of the parent tree, especially those that grew in bright sunlight. The shaded leaves compensate for the lack of light with the size of their area, capturing sunlight rays. Understanding this is the task of botany. But the geometer can also have a say here: he can determine exactly how many times the area of the thicket leaf is larger than the area of the parent tree leaf.

How would you solve this problem?

Answer You can go two ways. First, determine the area of each leaf separately and find their ratio. The area of the leaf can be measured by covering it with transparent grid paper, each square of which corresponds, for example, to 4 square millimeters (a sheet of transparent grid paper used for this purpose is called a pallet). This is a perfectly correct but overly laborious method.[9]

[9] However, this method has an advantage: using it, you can compare the areas of leaves with different shapes, which cannot be done according to the method described below.

A shorter method is based on the fact that both leaves, different in size, still have the same or almost the same shape: in other words, they are geometrically similar figures. We know that the areas of such figures are related as the squares of their linear dimensions. Therefore, by determining how many times one leaf is longer or wider than the other, we can find the ratio of their areas simply by squaring this number. Let the thicket leaf be 15 cm long, and the leaf from the tree branch only 4 cm long; the ratio of their linear dimensions is 15/4, and therefore, in terms of area, one is larger than the other by 225/16 times, or about 14. Rounding off (since full accuracy cannot be achieved here), we can say that the thicket leaf is approximately 15 times larger than the tree leaf in terms of area.

Let us consider another example.

Question At a dandelion grown in shade, a leaf is 31 cm long. At another specimen grown in sunlight, the leaf blade is only 3.3 cm long. Approximately how many times is the area of the first leaf larger than the area of the second?

Answer We proceed as before. The ratio of the areas is

$$\frac{31^2}{3.3^2} = \frac{960}{10.9} = 87;$$

so one leaf is approximately 90 times larger than the other in terms of area.

Figure 22.: Determine the ratio of the areas of these leaves.

It is easy to find in the forest many pairs of leaves of the same shape but different sizes, thus providing interesting material for geometric problems on the ratio of areas of similar figures. It always seems strange to an unaccustomed eye that a relatively small difference in the length and width of leaves results in a noticeable difference in their areas. For example, if two leaves, geometrically similar in shape, differ in length by 20%, then the ratio of their areas is

$$1.2^2 \approx 1.4,$$

meaning the difference is 40%. And with a difference in width of 40%, One leaf exceeds the other in area by

$$1.4^2 \approx 2,$$

or nearly twice.

Question We invite the reader to determine the ratio of the areas of the leaves depicted in Figure 22 and Figure 23.

Figure 23.: Determine the ratio of the areas of these leaves.

1.14. Six-legged heroes

Amazing creatures, ants! Swiftly climbing up stems with a burden much heavier than their tiny size (Figure 24), ants present an intriguing puzzle to observant individuals: where does the insect derive the strength to effortlessly carry a load ten times its own weight? Indeed, a human might struggle to climb stairs while carrying, for instance, a piano (Figure 24), with the weight ratio of the load to the body being roughly similar to that of an ant. Thus, it seems that the ant is relatively stronger than a human!

But is it really so?

Without geometry, this cannot be understood. Let's listen to what the expert (Professor A.F. Brandt) has to say, primarily about the strength of muscles, and then about the current question regarding the comparison of forces between the insect and the human: "A muscle resembles a resilient cord; however, its contraction is based not on elasticity, but on

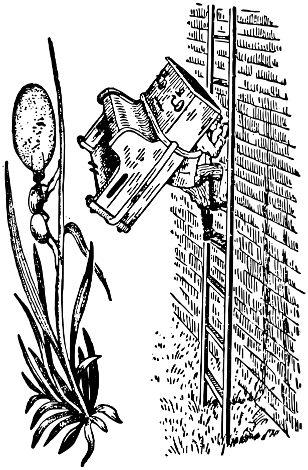


Figure 24.: The six-legged hero.

other reasons, and is normally manifested under the influence of nervous excitation, as demonstrated in physiological experiments involving the application of electric current to the corresponding nerve or directly to the muscle."

"These experiments are easily conducted on muscles excised from a freshly killed frog, as the muscles of cold-blooded animals retain their vital properties for a long time even outside the organism, even at ordinary temperatures. The experiment is very simple. The main calf muscle, which extends the hind leg, is excised together with a piece of the femur bone from which it originates, and together with the terminal tendon. This muscle is found to be the most convenient due to its size, shape, and ease of preparation. A hook is passed through the tendon, and a weight is attached to it."

"If wires from a galvanic element are touched to such a muscle, it instantly contracts, shortens, and lifts the load. By gradually adding additional weights, the maximum lifting capacity of the muscle can be easily determined. Now, if we bind together in length two, three, or four identical muscles and stimulate them simultaneously, we will not achieve greater lifting force; the load will only be lifted to a greater height, corresponding to the sum of the contractions of individual muscles. However, if we bundle two, three, or four muscles together, the entire system will lift a weight many times greater when stimulated. The same result, obviously, would be obtained if the muscles were fused together. Thus, we conclude that the lifting force of muscles depends not on their length or total mass, but only on their thickness, i.e., *cross-sectional area*."

"After this digression, let's turn to the comparison of similarly structured, geometrically similar, but differently sized animals. Let's imagine two animals: the original and one that has been doubled in size in all linear dimensions. In the second animal, the volume and weight of the entire body, as well as each of its organs, will be eight times greater; however, all corresponding planar dimensions, including the cross-sectional area of muscles, will be only four times greater. It turns out that as the animal grows to twice the length and eight times the weight, its muscular strength increases only fourfold, i.e., the animal becomes relatively weaker. Based on this reasoning, an animal that is three times longer (with cross-sectional areas three times larger and a weight 27 times greater) would be relatively three times weaker, and one that is four times longer would be four times weaker, and so on."

"The law of unequal growth in volume and weight of the animal, and thus of muscular strength, explains why insects – as observed in ants, predatory wasps, and others – can carry loads 30 to 40 times their own weight, whereas a human can typically carry excluding gymnasts and porters – only about 9/10 times their own weight, – and a horse, which we view as a magnificent living work machine, even less, namely, only about 7/10 of its own weight."[10]

[10] For more details, see *Fun with Physics* by Ya. I. Perelman, Chapter X *Mechanics in the Living World.*

After these explanations, we will look at the feats of that ant-giant with different eyes, about whom I.A. Krylov mockingly wrote:

Some ant had extraordinary strength,
Such as was unheard of even in ancient times;
He even (says his faithful historian)
Could lift two barley grains.

2. Geometry By The River

2.1. Measuring the width of the river

When crossing a river, measuring its width is just as easy for those who know geometry, how to determine the height of a tree, without climbing to the top. The inaccessible distance is measured the same techniques that we used to measure the inaccessible height. In both cases, the definition of the desired distance is replaced an example of another distance that is easily measurable directly.

Of the many ways to solve this problem, let's look at some of the simplest ones.

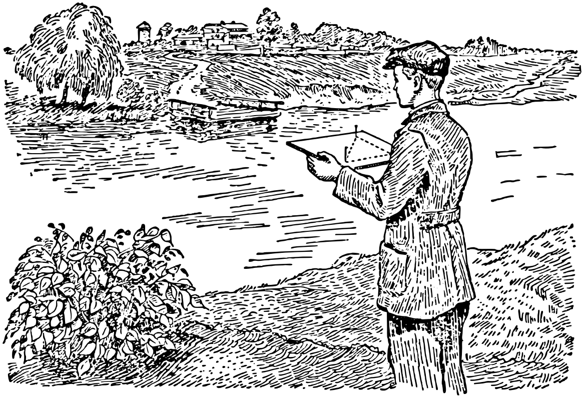


Figure 25.: Measuring the width of the river with a pin device.

1. The first method requires the familiar "device" with three pins at the vertices of an isosceles right triangle (Figure 25). Let's say we need to determine the width of river *AB* (Figure 26), standing on the bank where point *B* is, without crossing to the opposite bank. Standing somewhere at point *C*, hold the pin device close to your eye so that, looking with one eye along the two

pins, you see both covering points B and A. It's clear that when you manage this, you will be exactly on the extension of line AB.

Now, without moving the plank of the device, look along the other two pins (perpendicular to the previous direction) and notice any point D covered by these pins, i.e., lying on the line perpendicular to AC. After this, insert a pin at point C, leave this place, and go with your instrument along line CD until you find a point E (Figure 27), where you can simultaneously cover point C for one eye with pin b and point A with pin a. This means you have found the third vertex of triangle ACE on the shore, where angle C is a right angle, and angle E is opposite to the acute angle of the pin device, i.e., half the right angle (45°). Obviously, angle A is also half right angle, i.e., $AC = CE$. If you measure the distance CE even by steps, you will know the distance AC, and by subtracting BC, which is easy to measure, you will determine the desired width of the river.

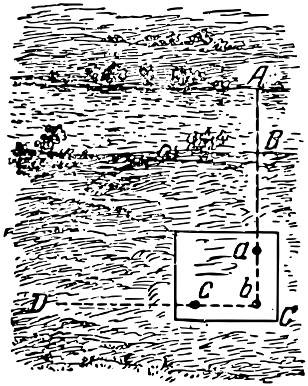


Figure 26.: First position of the pin device.

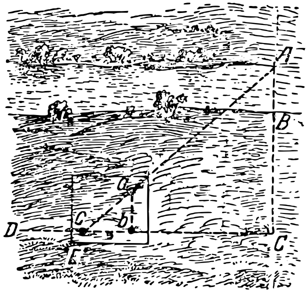


Figure 27.: Second position of the pin device.

It is quite inconvenient and difficult to hold the pin device still in hand; therefore, it is better to attach this plank to a stick with a pointed end and insert it vertically into the ground.

2. The second method is similar to the first. Here also,

find point C on the extension of AB and mark line CD perpendicular to CA using the pin device. But then proceed differently (Figure 28). Equal distances CE and EF of arbitrary length are measured on the straight line CD, and pegs are inserted at points E and F. Then, standing at point F with a pin device, the direction FG is marked out perpendicular to FC. Now, walking along FG, find a point H on this line from which point A seems to be covered by point E. This will mean that points H, E, and A lie on the same straight line.

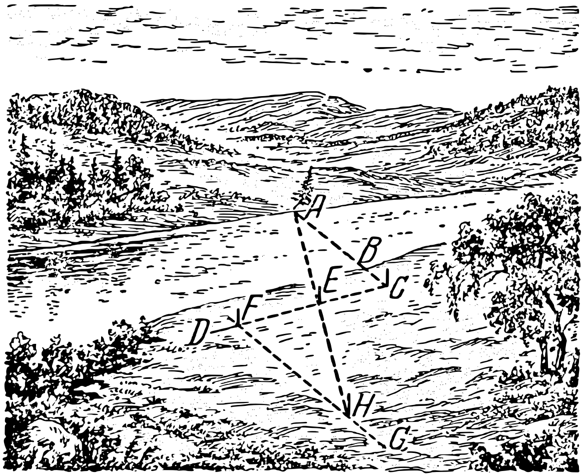


Figure 28.: Using the congruence criteria of triangles to find the width of the river.

The problem is solved: the distance FH is equal to the distance AC, from which it is only necessary to subtract BC to find the desired width of the river (the reader, of course, will guess for himself why FH is equal to AC).

This method requires more space than the first one; if the terrain allows executing both methods, it is useful to verify one result by another.

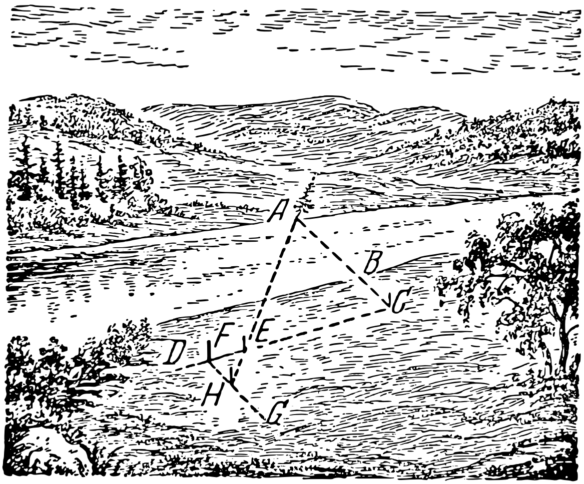


Figure 29.: Using the similarity criteria of triangles to find the width of the river.

3. The method described above can be modified: instead of measuring equal distances on the straight line CF, measure one distance several times smaller than the other. For example (Figure 29), FE is measured four times less than EC, and then we proceed as before: in the direction FG, perpendicular to FC, we find a point H from which the peg E appears to cover point A. But now FH is no longer equal to AC, but four times smaller than this distance: triangles ACE and EFH are not congruent here, but similar (they have equal angles with unequal sides). From the similarity of triangles follows the proportion:

$$\frac{AC}{FH} = \frac{CE}{EF} = \frac{4}{1}$$

Therefore, by measuring FH and multiplying the result by 4, we get the distance AC, and by subtracting BC, we find the desired width of the river.

This method, as we can see, requires less space and is therefore more convenient to perform than the previous one.

4. The fourth method is based on the property of a right triangle that if one of its acute angles is 30°, then the length of the cathetus is half the hypotenuse. It is very easy to verify the correctness of this.

Let angle B of right triangle ABC (Figure 30, left) be 30°; we will prove that in this case, $AC = 1/2\, AB$. Rotate triangle ABC around BC so that it is symmetric with its initial position (Figure 30, right), forming figure ABD; line AC is straight because both angles at point C are right angles. In triangle ABD, angle $\angle A = 60°$, angle ABD, composed of two 30° angles, is also equal to 60°. Therefore, $AD = BD$ as sides opposite equal angles. But $AC = 1/2\, AD$, therefore,

$$AC = \frac{1}{2} AB.$$

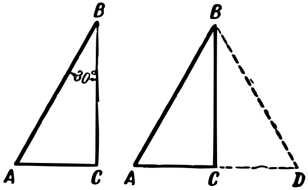


Figure 30.: When the cathetus is half the hypotenuse.

Wishing to take advantage of this property of the triangle, we must arrange the pins on the board so that their bases represent a right triangle in which the cathetus is half the hypotenuse. With this device, we place ourselves at point C (Figure 31) so that the direction AC coincides with the hypotenuse of the pin triangle. Looking along the short cathetus of this triangle, mark the direction CD and find a point E on it so that the direction EA is perpendicular to CD (this is done using the same pin device). It is easy to see that the distance CE – the cathetus lying opposite the angle of 30° – is equal to half of AC. Therefore, by measuring CE, doubling this distance and subtracting BC, we obtain the desired width of the AB river.

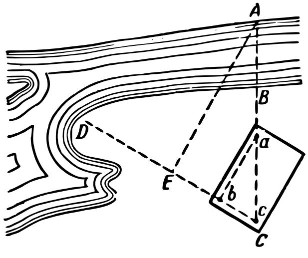


Figure 31.: The scheme of application of a right-angled triangle with a 30° angle.

Here are four easily executable methods, with which it is always possible, without crossing to the other bank, to measure the width of the river with quite satisfactory accuracy. We will not consider methods that require the use of more complex instruments (even homemade ones) here.

2.2. Using a visor

Here's how this method came in handy for Senior Sergeant Kupriyanov in frosty conditions.[11] His detachment was ordered to measure the width of the river, across which they were to organise a crossing...

[11] See the footnote on page 21.

Approaching a bush near the river, Kupriyanov's detachment took cover, and Kupriyanov himself, along with soldier Karpov, moved closer to the riverbank, from where the fascist-occupied shore was clearly visible. In such conditions, measuring the width of the river had to be done by eye.

"Come on, Karpov, how much?" Kupriyanov asked.

"I think no more than 100-110 meters," Karpov replied. Kupriyanov agreed with his scout, but for control, he decided to measure the width of the river using a "visor."

This method is simple. You have to face the river and pull the visor over your eyes so that the lower edge of the visor

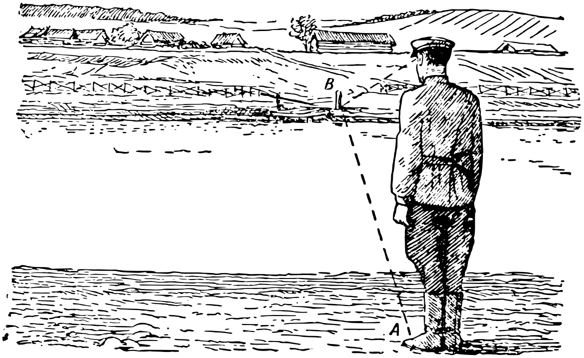


Figure 32.: Observing a point on the opposite bank from under the visor.

precisely aligns with the line of the opposite bank (see Figure 32). The visor can be replaced with the palm of your hand or a notepad, tightly pressed edge to your forehead. Then, without changing the position of your head, you need to turn to the right or left, or even backward (towards the side where the area available for measuring the distance is more level) and notice the farthest point visible from under the visor (palm, notepad).

The distance to this point will be approximately equal to the width of the river.

Kupriyanov utilized this method. He quickly stood up in the bushes, pressed a notepad to his forehead, then quickly turned and aimed at the distant point. Then, together with Karpov, he crawled to that point, measuring the distance with a rope. It turned out to be 105 meters.

Kupriyanov reported the data he obtained to the command.

Question Provide a geometric explanation for the "visor" method.

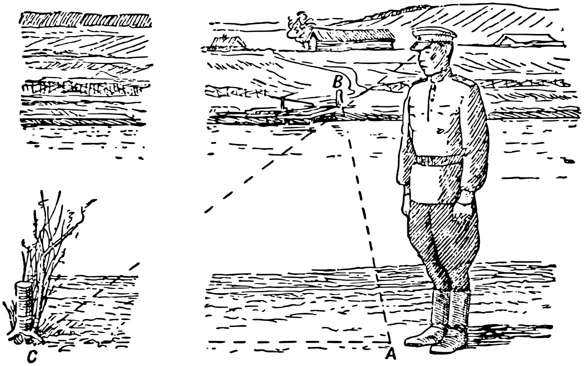


Figure 33.: In the same way, you can aim at a point on your own bank.

Answer The line of sight, touching the edge of the visor (palm, notepad), is initially directed towards the line of the opposite bank (see Figure 32). When a person turns, the line of sight, like the leg of a compass, describes a circle, and then $AC = AB$ as the radii of the same circle (see Figure 33).

2.3. The Length Of An Island

Question Now we are faced with a more challenging task. Standing by the river or lake, you see an island (see Figure 34) whose length you wish to measure without leaving the shore. Is it possible to carry out such a measurement?

Although in this case, both ends of the measured line are inaccessible to us, the problem is still entirely solvable, and without complex instruments.

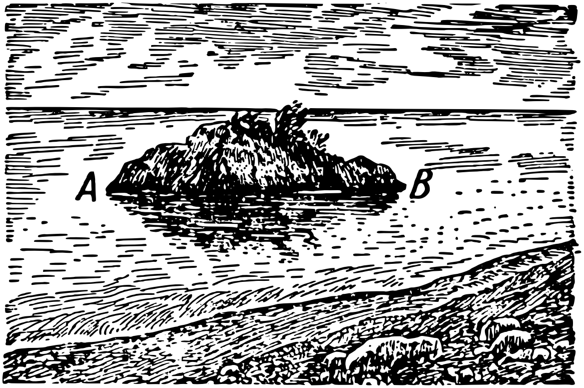


Figure 34.: How to determine the length of the island.

Answer To measure the length of an island without leaving the shore, you can use the following method. Choose arbitrary points P and Q on the shore and place stakes in them. Then find points M and N on the line PQ such that the directions AM and BM form right angles with the direction of PQ (this can be done using a compass). In the middle of the distance MN, place a stake O and find on the extension of the line AM a point C from which the stake O appears to cover point B. Similarly, on the extension of BN, find point D from which stake O appears to cover the end A of the island. The distance CD will be the desired length of the island.

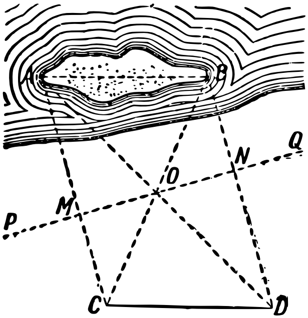


Figure 35.: We use the properties of congruent right triangles to find the length of an isalnd.

This can be easily proved. Consider the right triangles AMO and OND; in them, the legs MO and NO are equal, and the angles AOM and NOD are also equal, therefore, the triangles are equal, and $AO = OD$. Similarly, it can be proved that $BO = OC$. By comparing the triangles ABO and COD, it can be seen that their distances AB and CD are equal.

2.4. A pedestrian on the opposite bank

Question As you walk along the riverbank, you see a person on the other side, and you can clearly distinguish their steps. Can you, without moving from your spot, determine at least approximately the distance between them and you? You have no instruments at hand.

Figure 36.: How to determine the distance to a pedestrian walking on the other side of the river.

Answer You don't have any instruments, but you have eyes and hands – that's enough. Extend your arm forward towards the pedestrian and look at the tip of your finger with one eye if the pedestrian is moving towards your right hand, and with the other eye if they're moving towards your left hand. At the moment when the distant pedestrian is covered by your finger (see Figure 36), close the eye that was looking and open the other: the pedestrian will appear to you as if they've moved backward. Count how many steps they take before they align again with your finger. You'll get all the data needed for an approximate determination of the distance. Let's explain how to use them.

Suppose in Figure 36 (inset), your eyes are marked as *a* and *b*, point M is the tip of your finger extended, point A is the initial position of the pedestrian, and B is the final position. The triangles abM and ABM are similar (you should turn towards the pedestrian so that ab is approximately parallel to their direction of movement). Therefore, $BM : bM = AB : ab$ – is a proportion in which only one term, BM, is unknown, but all others can be directly determined. Indeed, bM is the length of your extended arm, ab is the distance between the pupils of your eyes, and AB is measured in steps taken by the pedestrian (assuming an average step to be around 3/4 metres). Therefore, the unknown distance from you to the pedestrian on the opposite bank, AB, equals

$$MB = AB\,\frac{bM}{ab}$$

For example, if the distance between your eye pupils ab is 6 cm, the length of bM from the end of your extended arm to the eye is 60 cm, and the pedestrian takes, say, 14 steps from A to B, then their distance from you would be $MB = 14 \cdot 60/6 = 140$ steps, or 105 meters.

It's enough for you to measure in advance the distance between your eye pupils and bM – the distance from the eye to the end of your extended arm – so that you can quickly determine the distance of inaccessible objects by remembering their ratio. On average, for most people, bM/ab is around 10 with slight fluctuations. The difficulty will only be in somehow determining the distance *AB*. In our case, we used the steps of a distant person. But you can also use other references. For instance, if you're measuring the distance to a distant freight train, you can estimate *AB* in comparison to the length of a freight car, which is usually known (7.6 meters between buffers). If you're determining the distance to a house, you can estimate *AB* by comparing it to the width of a window, the length of a brick, etc.

The same method can be applied to determine the size of a distant object if its distance from the observer is known. For this purpose, you can also use other "rangefinders", which we will describe next.

2.5. Simple Rangefinders

In the first chapter, we described the simplest instrument for determining inaccessible heights – the altimeter. Now, let's describe the simplest device for measuring inaccessible distances – the 'rangefinder.' The simplest rangefinder can be made from an ordinary matchstick. To do this, you just need to mark millimeter divisions on one of its sides, alternating between light and dark (see Figure 37).

Figure 37.: The match is a rangefinder.

You can use this primitive "rangefinder" to estimate the distance to a distant object only in those cases when the dimensions of that object are known to you (see Figure 38).

However, more sophisticated rangefinders can also be used under the same condition. Suppose you see a person in the distance and set yourself the task of determining the distance to them. Here, the matchstick rangefinder can come in handy. Holding it in your outstretched arm and looking with one eye, you bring its free end into coincidence with the top of the distant figure. Then, slowly moving your thumbnail along the matchstick, you stop it at the point that projects onto the base of the human figure. All you have to do now is to find out, by bringing the matchstick closer to the eye, at which mark your thumbnail stopped – and then you have all the data to solve the problem.

You can easily verify the correctness of the proportion:

$$\frac{\text{desired distance}}{\begin{array}{c}\text{distance from the eye}\\ \text{to the matchstick}\end{array}} = \frac{\text{average height of a person}}{\begin{array}{c}\text{measured part}\\ \text{of the matchstick}\end{array}}$$

From here, it's easy to calculate the desired distance. For example, if the distance to the matchstick is 60 cm, the height of the person is 1.7 m, and the measured part of the matchstick is 12 mm, then the determined distance would be:

$$60 \cdot \frac{1700}{12} = 8{,}500\ \text{cm} = 85\ \text{m}.$$

To gain some skill in using this rangefinder, measure the height of someone from your group and, asking them to

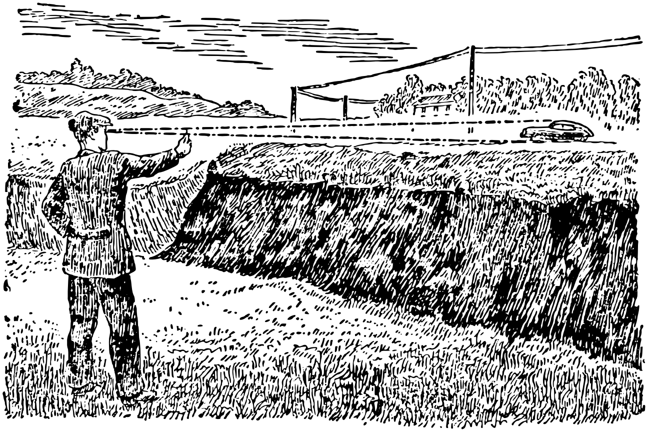


Figure 38.: The use of a rangefinder match to determine inaccessible distances.

move away a certain distance, try to determine how many steps they took away from you.

With the same method, you can determine the distance to a rider (average height 2.2 m), a cyclist (wheel diameter 75 cm), a telegraph pole along the railway track (height 8 m), vertical distance between adjacent insulators (90 cm), to a train, a brick house, and similar objects whose dimensions can be estimated with sufficient accuracy. There can be quite a few such cases during excursions.

For those skilled in crafting, making a more convenient device of the same type, intended for estimating distances based on the size of a distant human figure, won't be much trouble.

The device is clear in Figure 39 and Figure 40. The observed object is placed precisely in the gap *A*, formed when the extension part of the device is raised. The size of the gap can be conveniently determined by the divisions on the part *C* and *D* of the board. To avoid the need for any calculations, you can directly mark on strip *C* the distances corresponding to the divisions if the observed object is a human figure (the device for measuring the distance of the outstretched arm). On the right strip *D*, you can mark distances, pre-calculated for cases where a rider is observed (2.2 m). For telegraph poles (height 8 m), planes with a wingspan of 15 m, and other larger objects, you can use the upper, free parts of strips *C* and *D*. Then the device will look like the one presented in Figure 40.

Of course, the accuracy of such distance estimation is low. It's just an estimate, not a measurement. In the example discussed earlier, where the distance to the human figure was estimated at 85 m, an error of 1 mm in measuring the matchstick portion would result in a deviation of 7 m (1/12 out of 85). But if the person stood four times farther away, and we measured only 3 mm on the matchstick, then an error

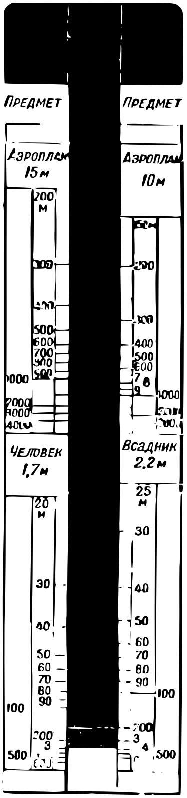


Figure 40.: The design of the re-

of even 1/2 mm would cause a change in the result by 57 m. Therefore, our example is reliable only for relatively short distances – in the range of 100–200 m. When estimating larger distances, it's necessary to choose larger objects.

2.6. The energy of the river

You know the edge where everything breathes abundance,
Where rivers flow purer than silver,
Where the steppe breeze sways the feather grass,
Where villages are nestled in cherry orchards.

A.K. Tolstoy

A river, the length of which is no more than 100 km, is considered small. Do you know how many such small rivers there are in the USSR? A lot - 48 thousand!

If these rivers were stretched into a single line, it would result in a ribbon 13,800,000 km long. With such a ribbon, you could encircle the Earth at the equator thirty times (the length of the equator is approximately 40,009 km).

The flow of these rivers is leisurely, but it conceals an inexhaustible supply of energy within it. Specialists believe that if the hidden potential of all the small rivers flowing through our homeland were combined, an impressive number would be obtained – 34 million kilowatts! This gifted energy needs to be widely utilised for electrifying the economy of settlements located near rivers.

Let the river flow freely,
If the plan says so,
A dam with a stone ridge across all depths
Will block the way forever.

S. Shchipachev

You know that this is achieved through hydroelectric power stations (HPS), and you can show a lot of initiative and provide real assistance in preparing for the construction of small HPS. Indeed, the builders of HPS will be interested in everything related to the river regime: its width and flow rate ("water flow"), the area of the cross-section of the riverbed ("active section"), and what water head the banks allow. And all this can be measured with available means and represents a relatively simple geometric problem.

We will now proceed to solving this problem.

But first, let's present here a practical advice from specialists, engineers V. Yarosh and I. Fedorov, regarding the selection of a suitable location on the river for the construction of a future dam.

They recommend building a small hydroelectric power station with a capacity of 15-20 kilowatts "no further than 5 km from the village."

"The dam of an HPS should be built no closer than 10-15 km and not farther than 20-40 km from the source of the river because moving away from the source entails the costly reinforcement of the dam, which is caused by a large influx of water. If the dam is located closer than 10-15 km from the source, due to the small water flow and insufficient head, the hydroelectric power station will not be able to provide the necessary power. The chosen stretch of the river should not be abundant in great depths, which also increases the cost of construction, requiring a heavy foundation."

2.7. The Flow Rate

Between village and mountain grove,
Winds a river like a bright ribbon.

A. Fet

How much water flows in such a river in a day? It's easy to calculate if you first measure the speed of the water flow in the river. The measurement is performed by two people. One person holds a watch, the other holds some noticeable float, for example, a half-empty bottle with a flag. They choose a straight section of the river and place two stakes A and B

along the bank at a distance, for example, 10 m from each other (see Figure 41).

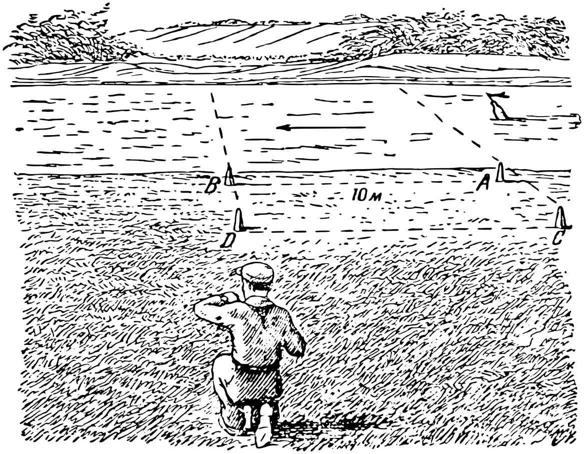


Figure 41.: Measurement of the river flow velocity.

Two more stakes C and D are placed on lines perpendicular to AB. One of the participants in the measurement with the watch stands behind stake D. The other, with the float, goes a bit upstream of stake A, throws the float into the water, and then stands behind stake C. Both observers look along the directions CA and DB towards the water surface. At the moment when the float crosses the extension of the line CA, the first observer waves his hand. Upon this signal, the second observer starts the timer for the first time and then again when the float crosses the direction of DB.

Let's assume that the time difference is 20 seconds.

Then the speed of the water flow in the river is:

$$\frac{10}{20} = 0.5\,\mathrm{m/s}.$$

Usually, the measurement is repeated about ten times[12], throwing the float into different points on the river surface. Then the obtained numbers are summed up and divided by the number of measurements. This gives the average speed of the surface layer of the river.

[12] Instead of throwing one float ten times, you can immediately throw 10 floats at some distance from each other.

Deeper layers flow slower, and the average speed of the entire flow is approximately 4/5 times the surface speed. In our case, therefore, it's $0.4\,\mathrm{m/s}$.

You can determine the surface speed by another – albeit less reliable – method.

Sit in a boat and paddle 1 km (measured along the shore) against the current, and then back – with the current, trying to paddle with the same force all the time.

Let's say you paddled these 1,000 m against the current in 18 minutes, and with the current in 6 minutes. Denoting the desired speed of the river current as x, and the speed of your movement in still water as y, you form the equations:

$$\frac{1000}{y-x} = 18, \quad \text{and} \quad \frac{1000}{y+x} = 6.$$

Rearranging we get:

$$y + x = \frac{1000}{6}, \quad \text{and} \quad y - x = \frac{1000}{18}.$$

Solving for x, we get $2x = 110$, and $x = 55$. The speed of the water flow on the surface is 55 m per minute, and therefore, the average speed is about 5/6 m/s.

2.8. How Much Water Flows In The River?

To measure the amount of water flowing in a river, you can always determine the speed at which the water flows. The more challenging part of the preparatory work needed to calculate the quantity of flowing water is to determine the cross-sectional area of the water. To find the magnitude of this area, known as the “live cross-section” of the river, you need to make a drawing of this section. Such work is done as follows.

First Method: At the point where you measured the width of the river, you drive a stake into the ground on both banks, right at the water's edge. Then, with a companion, you get into a boat and row from one stake to the other, trying to keep a straight line connecting the stakes. An inexperienced rower will not be able to handle such a task, especially in a river with a fast current. Your companion must be a skilled rower; besides, a third participant in the work should stand on the bank, ensuring that the boat stays on the correct course and giving the rower signals indicating which way to turn when necessary. During the first crossing of the river, you only need to count how many strokes of the oars it took and from there figure out how many strokes move the boat 5 or 10 meters. Then, for the second crossing, armed with a sufficiently long rake with markings on it, you plunge the rake vertically to the bottom every 5-10 meters (measured by the number of oar strokes) and record the depth of the river at that point.

This method can only measure the live cross-section of a small river; for a wide, multi-water river, more complex methods are needed, which are performed by specialists. An amateur must choose a task that suits their modest measuring means.

Second Method: On a narrow and shallow river, you don't need a boat.

Between the stakes, you stretch a cord perpendicular to the current with marks or knots made on it every 1 or 2 meters, and by lowering a ruler to the bottom at each knot, you measure the depth of the riverbed. When all measurements are done, you first draw a millimeter paper or a grid paper sketch of the cross-section profile of the river. You will get a figure similar to the one shown in Figure 42. It is quite easy to determine the area of this figure since it can be divided into a series of trapezoids (where both bases and the height are known) and two side triangles, also with known base and height. If the scale of the drawing is 1:100, then the result will be obtained directly in square meters.

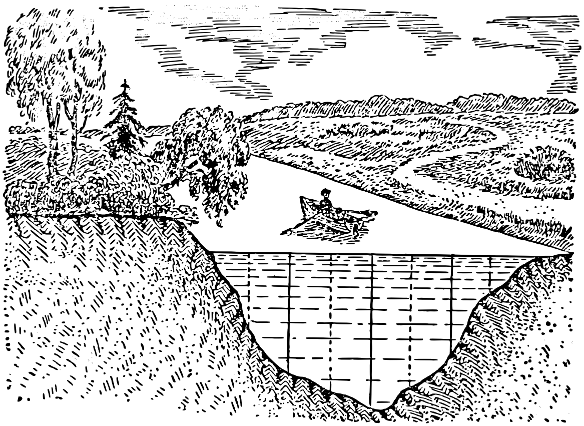


Figure 42.: “Live Cross-Section” of a River.

Now you have all the data needed to calculate the amount of flowing water. Obviously, through the live cross-section of the river, a volume of water equal to the volume of a prism is passing every second, where the cross-section serves as the base and the average second-by-second flow rate serves as the height. For example, if the average flow rate of water in the river is 0.4 meters per second, and the area of the live cross-section is, let's say, 3.5 square meters, then every second, through this section, there will be a transfer of

$$3.5 \times 0.4 = 1.4 \text{ cubic meters of water,}$$

or the same amount in tons[13] This amounts to

$$1.4 \times 3600 = 5040 \text{ cubic meters of water per hour, and}$$
$$5040 \times 24 = 120{,}960 \text{ cubic meters of water per day,}$$

which is over a hundred thousand cubic meters. And yet, a river with a live cross-section of 3.5 square meters is a small river; it could be, for example, 3.5 meters wide and 1 meter deep at a fordable point. But even it contains energy capable of transforming into mighty electricity. So how much water flows per day in a river like the Neva, through which 3300 cubic meters of water pass every second through its live cross-section! This is the "average flow rate" of water in the Neva at Leningrad. The "average flow rate" of water in the Dnieper at Kiev is 700 cubic meters.

[13] 1 cubic meter of fresh water weighs 1000 kg.

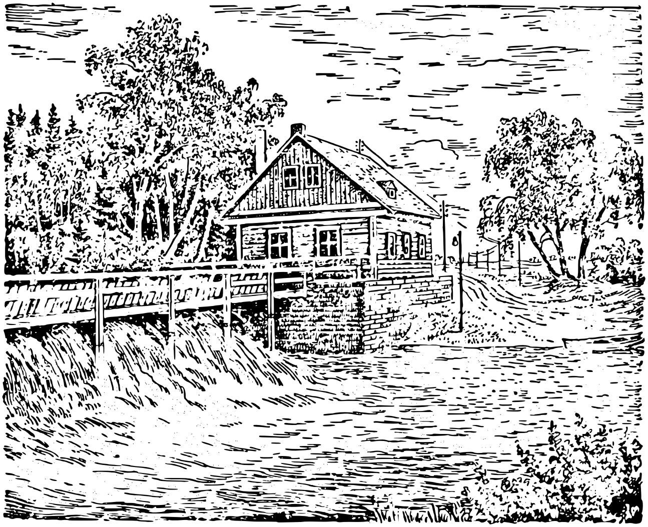


Figure 43.: The hydroelectric power station with a capacity of 80 kilowatts belongs to the Burmakin Agricultural Collective Farm and provides energy to seven collective farms.

Young explorers and future dam builders also need to determine the maximum head the banks can allow, i.e., the difference in water levels that the dam can create (Figure 43). To do this, stakes are driven into the banks 5-10 meters away from the water, as usual, along a line perpendicular to the river's current. Then, moving along this line, small stakes are placed at points of characteristic bends in the banks (Figure 44). Using rulers with markings, the elevation of one stake above the other and the distances between them are measured. Based on the measurement results, a profile of the banks is drawn similar to the profile of the riverbed. The bank profile can indicate the magnitude of the allowable head.

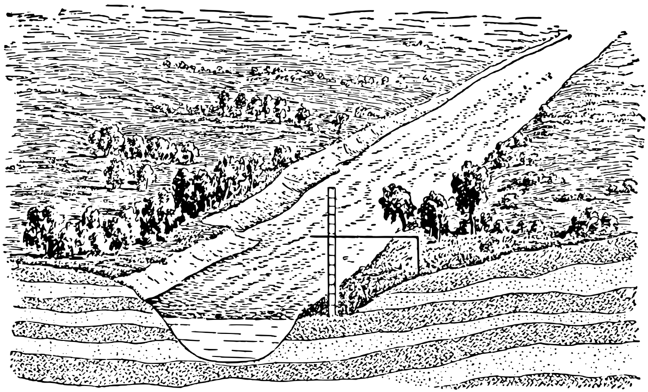


Figure 44.: Coast profile measurement.

Suppose the water level can be raised by the dam by 2.5 meters. In this case, you can estimate the potential power of your future hydroelectric power plant.

For this, energy experts recommend multiplying 1.4 (the second-by-second flow rate of the river) by 2.5 (the water level height) and by 6 (the coefficient, which varies depending on energy losses in machines). The result will be in kilowatts. Thus,

$$1.4 \times 2.5 \times 6 = 21 \text{ kilowatts.}$$

Since the river levels, and consequently the flow rates, vary throughout the year, it is necessary to calculate the value of the flow rate that is characteristic of the river for most of the year.

2.9. Water Wheel

Question A wheel with blades is installed near the bottom of the river so that it can rotate easily. In which direction will it rotate if the current is flowing from right to left (see Figure 45)?

Answer The wheel will rotate counterclockwise. The velocity of the deeper layers of water is lower than the velocity of the upper layers, therefore, the pressure on the upper blades will be greater than on the lower ones.

2.10. Rainbow Film

On the river, into which water flows from the factory, you can often notice beautiful colourful iridescence near the outlet.

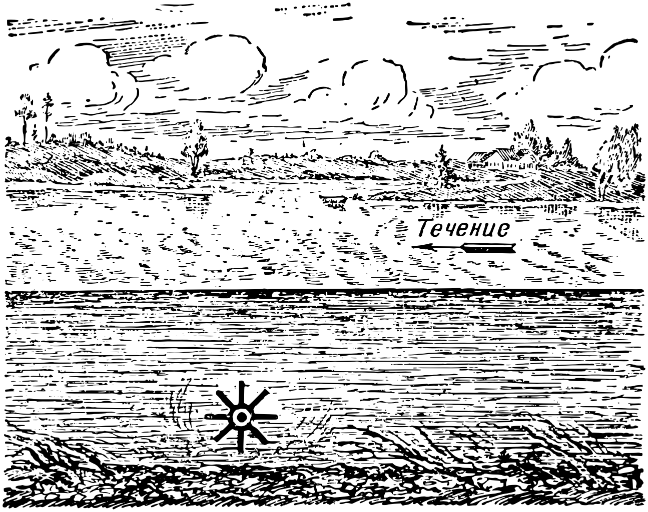


Figure 45.: In which direction will the wheel rotate?

Oil (for example, motor oil) flowing into the river along with the water from the plant remains on the surface as it is lighter and spreads out in an extremely thin layer. Can the thickness of such a film be measured or at least estimated?

The task seems intricate, but solving it is not particularly difficult. You already guess that we will not bother with the hopeless task of directly measuring the thickness of the film. We will measure it indirectly, in other words, calculate it.

Take a certain amount of motor oil, for example, 20 litres, and pour it onto the water, far from the shore (from a boat). When the oil spreads over the water in the form of a more or less clearly defined circular spot, measure the diameter of this circle at least approximately. Knowing the diameter, calculate the area. And since you also know the volume of the oil taken (it is easy to calculate by weight), then the thickness of the film will naturally be determined from here. Let's consider an example.

Question One gram of kerosene, spreading over water, covers a circle with a diameter of 30 cm.[14] What is the thickness of the kerosene film on the water? One cubic centimetre of kerosene weighs 0.8 grams.

Answer Let's find the volume of the film, which is certainly equal to the volume of the kerosene taken. If one cubic centimetre of kerosene weighs 0.8 grams, then for 1 gram we have

$$\frac{1}{0.8} = 1.25\,\text{cm}^3 \quad \text{or} \quad 1{,}250\,\text{mm}^3.$$

The area of the circle with a diameter of 30 cm, or 300 mm, is $70{,}000\,\text{mm}^2$. The desired thickness of the film is equal to

[14] The standard oil consumption for covering water bodies to destroy malaria mosquito larvae is 400 kilograms per hectare.

the volume divided by the area of the base:

$$\frac{1250}{70000} = 0.018\,\text{mm.mm}^3$$

In other words, less than 0.02 mm. Direct measurement of such thickness using conventional means is, of course, impossible. Oil and soap films spread even thinner, reaching 0.009,1 mm or less. "Once," recounts the English physicist Boys in the book *Soap Bubbles*, "I conducted such an experiment on a pond. A spoonful of olive oil was poured onto the water surface. Immediately a large spot was formed, about 20-30 meters in diameter. Since the spot was a thousand times longer and a thousand times wider than the spoon, the thickness of the oil layer on the water surface should have been approximately one millionth of the thickness of the oil layer in the spoon, or about 0.000002 millimetres."

2.11. Circles on the Water

Question You've surely observed with curiosity the circles created by a stone thrown into calm water (see Figure 46). Explaining this instructive natural phenomenon has probably never been difficult for you: disturbance spreads from the point of origin in all directions at the same speed, so at any moment, all the points of disturbance must be equidistant from the source, forming circles.

Figure 46.: Circles on the water.

But what about in flowing water? Should the waves from a stone thrown into a fast-flowing river also form circles, or will their shape be elongated?

At first glance, it might seem that in flowing water, circular waves should elongate in the direction of the current: the disturbance is transmitted downstream faster than upstream and sideways. Therefore, the disturbed parts of the water surface should, it seems, arrange themselves along some elongated closed curve, at least not in a circle.

However, in reality, this is not the case. By throwing stones into the swiftest river currents, you can see that the waves formed are strictly circular – exactly the same as in still water. Why is this?

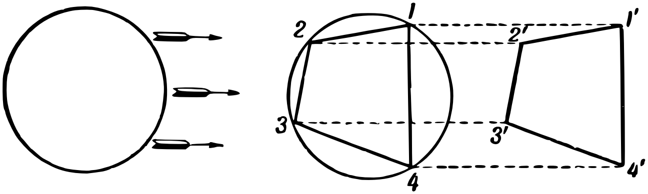


Figure 47.: The flow of water does not change the shapes of the waves.

Answer Let's reason as follows. If the water were not flowing, the waves would be circular. What change does the flow introduce? It carries each point of this circular wave in the direction indicated by the arrows (see Figure 47, left), and all points are moved along parallel lines at the same speed, i.e., the same distance. And "parallel displacement" (translation) does not change the shape of the figure. Indeed, as a result of such displacement, point *1* (Figure 47, right) will be at point *1′*, point *2* will be at point *2′*, and so on; quadrilateral *1234* will be replaced by quadrilateral *1′2′3′4′*, which is equal to it, as can be easily seen from the formed parallelograms *122′1′*, *233′2′*, *344′3′*, and so on. By taking not just four but more points on the circumference, we would also get equal polygons; finally, by taking an infinite number of points, i.e., a circle, we would get an equal circle after parallel displacement.

That's why the downstream movement of water does not change the shapes of the waves – they remain circular even in flowing water. The difference is only that on the surface of a lake, the circles do not move (if we don't consider that they diverge from their stationary centre), while on the surface of a river, the circles move together with their centre at the speed of the water flow.

2.12. Fantastic Shrapnel

Question Let's tackle a problem that seems unrelated at first but, as we will see, is closely related to the topic at hand.

Imagine a shrapnel projectile flying high in the air. It begins to descend and suddenly explodes; the fragments scatter in different directions. Let's assume that all of them are thrown by the explosion with the same force and travel without encountering any obstacles from the air. The question is: how will the fragments be arranged one second after the explosion if during this time they have not yet reached the ground?

Answer The problem resembles the problem of circles on water. And here it seems that the fragments should be arranged in a shape elongated downwards, in the direction of descent; after all, the fragments thrown upwards fly slower than those thrown downwards. However, it is easy to prove that the fragments of our imaginary shrapnel should be arranged on the surface of a sphere. Let's momentarily imagine that there is no gravity; then, of course, all fragments will fly from the explosion site to the same distance within a second, i.e., they will be arranged on the surface of a sphere. Now let's introduce the force of gravity. Under its influence, the fragments should descend; but since all bodies, as we know, fall at the same speed, the fragments should descend to the same distance within a second, along parallel lines.[15] But such parallel displacement does not change the shape of the figure—the sphere remains a sphere.

[15] The differences are due to the air resistance, which we have excluded in our task.

So, the fragments of the fantastic shrapnel should form a sphere, which, as if inflating, descends downward at the speed of a freely falling body.

2.13. The Keel Wave

Let's return to the river. (Standing on a bridge, pay attention to the wake left by a fast-moving boat. You will see how two water crests diverge at an angle from the bow (see Figure 48).

Where do they come from? And why is the angle between them sharper the faster the boat moves?

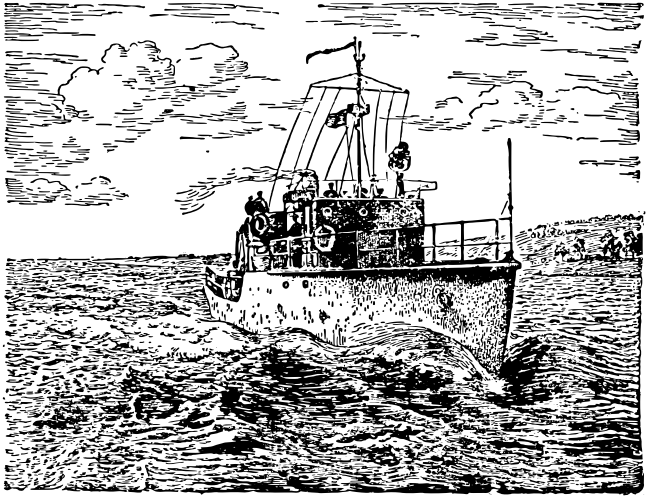


Figure 48.: The keel wave.

To understand the reason for the emergence of these crests, let's once again turn to the diverging circles formed on the water surface by stones thrown into it. By throwing stones into the water at certain intervals, you can see circles of different sizes on the water surface; the later a stone is thrown, the smaller the circle it creates.

If you throw stones along a straight line, the circles formed collectively generate a wave similar to the one at the bow of a ship. The smaller and more frequent the stones are thrown, the more noticeable the similarity becomes. By immersing a stick in the water and moving it along the water surface, you effectively replace the intermittent falling of stones with continuous motion, and then you see exactly the wave that forms at the bow of a ship. To make this vivid picture even clearer, let's add a bit more.

Each moment, the bow of the ship, plunging into the water, generates the same circular wave as a thrown stone. The circle expands in all directions, but meanwhile, the ship moves forward and creates a second circular wave, followed immediately by a third, and so on. The intermittent formation of circles caused by stones is perceived as continuous due to their continuous occurrence, resulting in the pattern shown in Figure 49.

Meeting each other, the crests of neighbouring waves break each other: only two small segments of the complete circle remain intact, which are located on their outer parts. These outer segments merge to form two continuous crests, positioned as external tangents to all circular waves (Figure 49, right).

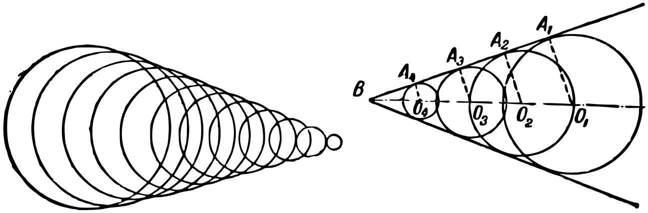


Figure 49.: How the keel wave is formed.

This is how the origin of the water crests visible behind the boat, behind any body moving rapidly along the water surface, occurs.

It follows directly from this that this phenomenon is possible only when the body moves faster than the water waves. If you slowly move a stick through the water, you will not see any crests: the circular waves will be arranged one inside the other, and it will be impossible to draw a common tangent to them.

Diverging crests can also be observed when the body stands still and the water flows past it. If the current of the river is fast enough, such wakes are formed in the water flowing around bridge piers. The shape of the waves is even more distinct here than, for example, from a steamship, as their regularity is not disrupted by the action of a propeller.

Having clarified the geometric aspect of the matter, let's try to solve such a problem.

Question What determines the magnitude of the angle between both branches of the keel wave of the steamer?

Answer The angle between the branches of the keel wave of a steamship depends on several factors, primarily the speed of the ship relative to the speed of the wave propagation in the water.

Let's draw radii from the centre of the circular waves (Figure **??**, right) to the corresponding points of the straight wave crest, i.e., to the points of common tangency. It's easy to understand that O_1B represents the distance travelled by the ship's bow in some time, and O_1A_1 – represents the distance over which the wave propagates in the same time period. The ratio O_1B/O_1A_1 is the sine of angle O_1BA_1, which, in turn, is the ratio of the wave propagation velocity to the ship's velocity. Therefore, the angle B between the crests of the keel wave is nothing else but twice the angle whose sine is the ratio of the wave propagation velocity to the ship's velocity.

The speed of wave propagation in water is approximately the same for all vessels. Therefore, the angle between the branches of the keel wave primarily depends on the speed of the ship. In general, the sine of half the angle is proportional to this speed. Conversely, the angle's magnitude indicates how many times the ship's speed exceeds the speed of the waves. For instance, if the angle between the branches of the keel wave is 30°, as is common for most cargo and passenger ships, then the sine of half this angle (approximately 0.26) suggests that the ship's speed exceeds the wave speed by roughly four times.

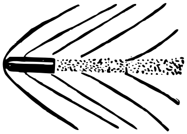


Figure 50.: A head wave (shock wave) in the air formed by a flying projectile.

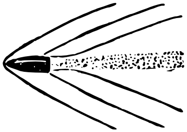


Figure 51.: Another example of head wave (shock wave) in the air formed by a flying projectile.

2.14. Speed of Projectiles

Question Waves, similar to those considered now, are generated in the air by a flying bullet or artillery shell.

There are methods to photograph a projectile in flight; Figure 50 and Figure 51 reproduce two such images of projectiles moving at different speeds. In both drawings, the "head wave"[16] that interests us is clearly visible (as it is called in this case). Its origin is the same as that of the bow wave of a ship. And here the same geometric relationships apply, namely: the sine of half the angle of divergence of the head waves is equal to the ratio of the speed of wave propagation in the air to the speed of the projectile itself. But wave propagation in the air occurs at a speed close to the speed of sound, i.e., 330 m/s. Therefore, it is easy, having a photograph of a flying projectile, to approximately determine its speed. How to do this for the two images provided here?

[16] The term shock wave is commonly used now. DM

Answer Let's measure the angle of divergence of the head wave branches in Figure 50 and Figure 51. In the first case, it is about 80°, and in the second case – approximately 55°. Half of them are 40° and 27.5°. For 40°, $\sin(40°) = 0.64$, and for 27.5°, $\sin(27.5°) = 0.46$. Therefore, the speed of wave propagation in the air, i.e., 330 m/s, is 0.64 of the projectile's flight speed in the first case and 0.46 in the second. Hence, $330/0.64 = 520$ m/s for the speed of the first projectile, and $330/0.46 = 720$ m/s for the speed of the second projectile.

You see that quite simple geometric considerations, with some support from physics, helped us solve a problem that, at first glance, seemed very intricate; to determine the speed of a projectile at the moment of photography. (However, this calculation is only approximately correct, since some secondary circumstances are not taken into account here.)

For those who want to independently perform such a speed calculation for projectiles, here are three reproductions of photographs of shells flying at different speeds (Figure 52).

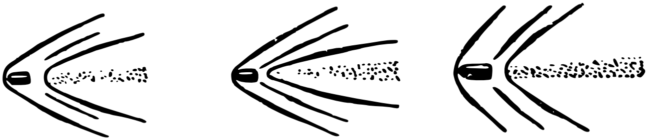


Figure 52.: How to determine the speed of flying projectiles?

2.15. Finding Pond Depth

The ripples on the water distracted us for a while in the field of artillery. Let's go back to the river again and consider the Hindu lotus problem,

The ancient Hindus had a custom of offering tasks and rules in verse, Here is one of such tasks:

Question

Above the tranquil lake,
With a half-foot size, the lotus flower rose.
It grew alone. And the wind, in a gust,
Bent it sideways.
No more flower above the wave,
But the fisherman found it, in early spring,
Two feet from where it grew.
So, I propose a question:
How deep is the water of the lake?

Translated by *V. I. Lebedev*

Answer Let's denote the depth of the pond CD as x Figure 53. Then, according to the Pythagorean theorem, we have:

$$BD^2 - x^2 = BC^2.$$

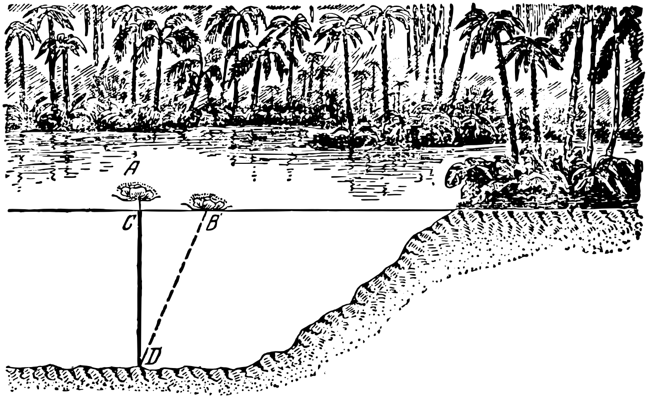


Figure 53.: The Hindu problem of the lotus flower.

Thus,

$$x^2 = \left(x + \frac{1}{2}\right)^2 - 2^2.$$

From this, we get:

$$x^2 = x^2 + x + \frac{1}{4} - 4, \quad x = 3\,\frac{3}{4}$$

The depth is approximately $3\,\frac{3}{4}$ feet.

Near the riverbank or a shallow pond, you can find a water plant that will provide you with real material for a similar problem: without any tools, without even getting your hands wet, you can determine the depth of the water at that spot.

2.16. Starry Sky in the River

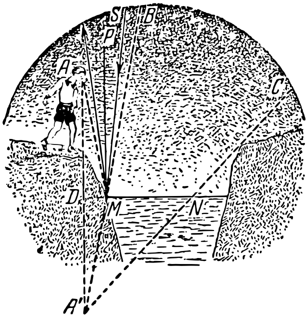


Figure 54.: Which part of the starry sky can be seen in the water channel of the river.

The river also offers the geometer problems at night. Remember Gogol's description of the Dnieper: "The stars burn and shine over the world and all are reflected in the Dnieper at once. The Dnieper holds them all in its dark womb: not one can escape from it, unless it goes out in the sky." Indeed, when standing on the bank of a wide river, it seems that the entire starry dome is reflected in the water mirror. But is it really so? Do all the stars "surrender" to the river?

Let's make a drawing (Figure 54): A is the eye of the observer standing on the riverbank, at the edge of the cliff, MN is the water surface. What stars can the observer see in the water from point AD? To answer this question, let's drop a perpendicular AD from point A to the line MN and extend it to an equal distance to point A'. If the observer's eye were at A', he could only see the part of the starry sky that fits inside angle $BA'C$. The actual observer's field of view from point A is the same. Stars outside this angle are not visible to the observer; their reflected rays pass by his eyes.

How to make sure of this? How to prove that, for example, star S lying outside angle $BA'C$ is not visible to our observer in the water mirror of the river? Let's follow its ray, falling close to the shore, to point M; it will reflect according to the laws of physics at an angle to perpendicular MP, which is equal to the angle of incidence SMP and therefore less than angle PMA (this is easy to prove based on the equality of triangles ADM and $A'DM$); thus, the reflected ray must pass by A. Therefore, the rays of star S reflected in points beyond point M will also pass by the observer's eye.

Thus, Gogol's description contains exaggeration: not all stars are reflected in the Dnieper, or, at least, less than half of the starry sky. Even more interesting is that the extent of the reflected part of the sky does not prove that you are facing a wide river. In a narrow river with low banks, you can see almost half of the sky (which seems like a wide river) if you lean close to the water. You can easily verify this by constructing a field of view for such a case (Figure 55).

In a narrow river with low banks, you can see more stars.

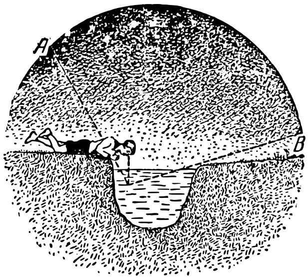


Figure 55.: In a narrow river with low banks, you can see more stars.

2.17. Path Across the River

Question Between points A and B, a river (or canal) flows with approximately parallel banks (Figure 56). It is necessary to construct a bridge across the river at right angles to its banks. Where should the bridge be placed to make the path from A to B the shortest?

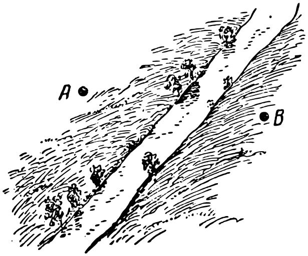


Figure 56.: Where can we build a bridge at right angles to the river banks so that the road from A to B is the shortest?

Answer Drawing a straight line through point A (Figure 57), perpendicular to the direction of the river, and marking off segment AC equal to the width of the river from A, we connect C to B. The bridge should be constructed at point D to make the path from A to B the shortest.

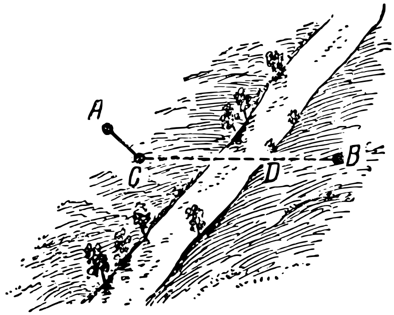


Figure 57.: Choosing he location for building the bridge.

Indeed, by constructing bridge DE (Figure 58) and connecting E to A, we obtain the path $AEDB$, in which segment AE is parallel to CD ($AEDC$ is a parallelogram, as its opposite sides AC and ED are equal and parallel). Therefore, the length of path $AEDB$ equals the length of path ACB. It is easy to show that any other path is longer than this one.

Suppose we suspect that some path $AMNB$ (Figure 59) is

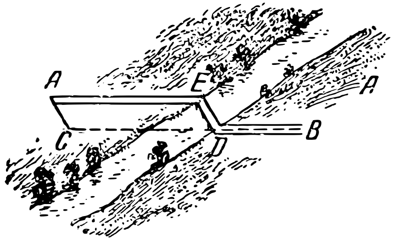


Figure 58.: The bridge is built.

shorter than $AEDB$, i.e., shorter than ACB. Connecting C to N, we see that CN equals AM. Thus, path $AMNB = ACNB$. But CNB is obviously greater than CB; therefore, $ACNB$ is greater than ACB, and consequently, greater than $AEDB$. Thus, the path $AMNB$ turns out to be not shorter but longer than the path $AEDB$. This reasoning applies to any position of the bridge that does not coincide with ED; in other words, the path $AEDB$ is indeed the shortest.

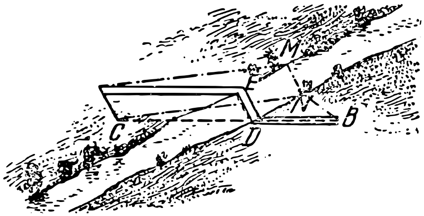


Figure 59.: The *AEDB* path is indeed the shortest.

2.18. To Construct Two Bridges

Question A more complex case may arise when it is necessary to find the shortest path from A to B across the river, which needs to be crossed twice at right angles to the banks (see Figure 60). In what places should bridges be built across the rivers?

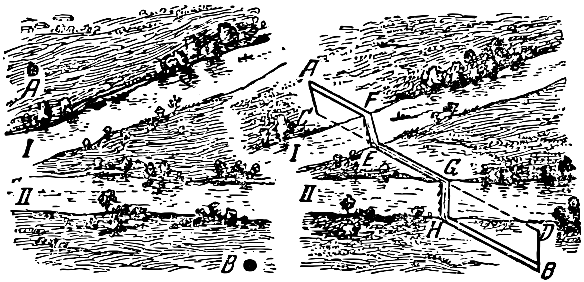
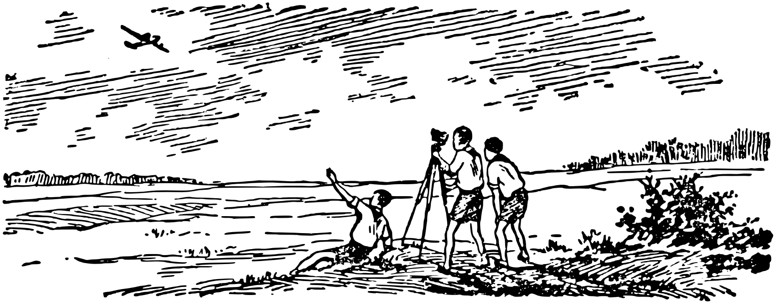


Figure 60.: Two bridges are built.

Answer It is necessary to draw a line segment AC from point A (to the right in the Figure 60), equal to the width of the river at point C, and perpendicular to its banks. From point B, draw a line segment BD equal to the width of the river at point D, also perpendicular to the banks. Connect points C and D with a straight line. Build bridge EF at point E and bridge GH at point G. The path $AFEGHB$ is the sought shortest path from A to B.

The reader will, of course, understand this if reasoning in this case is conducted in the same way as we reasoned in the previous problem.



3. Geometry In The Open Field

3.1. Visible sizes of the Moon

What size does the full moon in the sky seem to you? Different people give quite different answers to this question.

Estimates like “the size of a plate,” “the size of an apple,” “the size of a human face,” and so on, are extremely vague, indefinite, indicating only that those answering do not understand the essence of the question.

The correct answer to such an apparently ordinary question can only be given by someone who clearly understands what exactly needs to be understood by the “apparent” or “visible” size of an object. Few suspect that here we are referring to the magnitude of a certain angle – precisely the angle formed by two straight lines drawn to our eye from the extreme points of the object under consideration; this angle is called the “angle of view” or “angular size of the object” (Figure 61).

Figure 61.: What is the angle of view or angular size of the object.

And when the apparent size of the Moon in the sky is assessed by comparing it with the sizes of a plate, an apple, and so forth, such answers are either completely meaningless or should imply that the Moon is visible in the sky at the same angle as the plate or apple. But such an indication alone is still insufficient: after all, we see a plate or an apple under different angles depending on their distance: up close – under larger angles, far away – under smaller ones. To introduce clarity, it is necessary to specify from what distance the plate or apple is being observed. Comparing the sizes of distant objects with the sizes of others, the distance of which is not indicated, is a very common literary technique employed by even first-rate writers. It creates a certain impression due to its closeness to the familiar psychology of most people, but it does not produce a clear image... Here's an example from Shakespeare's *King Lear*; it describes (by Edgar) the view from a high cliff by the sea:

How terrifying! How my head spins!
How low to cast my gaze...
The rooks and crows that flutter there in the air at mid-distance,
Seem barely as large as flies. Halfway down Hangs a man gathering seaweed...
What a dreadful trade! He seems to me no bigger than his head.

> *The fishermen walking along the shore – like mice;*
> *and that tall ship at anchor has shrunk to the size*
> *of its boat;*
> *its boat – a floating dot, As if too small for sight…*

These comparisons would provide a clear idea of distance if accompanied by indications of the degree of remoteness of the objects being compared (mice, the human head, crows, boat …). Similarly, when comparing the size of the moon with that of a plate or apples, indications of how far these everyday objects should be from the eye are necessary.

This distance turns out to be much greater than commonly thought. Holding an apple at arm's length, you not only obscure the Moon but also a significant portion of the sky. Suspend the apple on a string and gradually move away from it until it just covers the full lunar disk: in this position, the apple and the Moon will have the same apparent size for you. By measuring the distance from your eye to the apple, you will find that it is approximately 10 meters. This is how far you would need to move the apple away from you for it to truly seem to be the same size as the Moon in the sky! A plate, on the other hand, would need to be moved about 80 meters away from you, or about fifty steps.

What is said may seem unbelievable to anyone hearing it for the first time; however, it is indisputable and follows from the fact that we perceive the Moon at an angle of only *half a degree*. We rarely have to estimate angles in our everyday lives, and therefore most people have a very vague idea of the magnitude of an angle with a small number of degrees, such as an angle of 1°, 29°, or 59° (not to mention surveyors, draftsmen, and other specialists accustomed to practically measuring angles). We only estimate large angles more or less reasonably, especially if we manage to compare them with angles familiar to us between the hands of a clock; everyone, of course, is familiar with angles of 90°, 60°, 30°, 120°, 150°, which we are so accustomed to seeing on a dial (at 8 o'clock, 2 o'clock, 1 o'clock, 4 o'clock, 5 o'clock) that even without distinguishing the numbers, we guess the time based on the size of the angle between the hands. But we usually see small and individual objects at much smaller angles and therefore completely lack the ability to even approximately estimate angles of view.

3.2. Angle of View

To provide a concrete example of a one-degree angle, let's calculate how far an average-height person (1.7 meters) should move away from us to appear at such an angle. Translating the problem into the language of geometry, let's say we need to calculate the radius of a circle, the arc of which at 1° has a length of 1.7 meters (strictly speaking, not an arc, but a chord, but for small angles, the difference between the lengths of the arc and the chord is negligible). We reason as follows: if the arc at 1° equals 1.7 meters, then the full circumference containing 360° will have a length of $1.7 \times 360 = 610\,\text{m}$, and the radius will be $1/2\pi$ the length of the circumference; if we take the value of π as approximately 22/7, then the radius will be equal to

$$\frac{610}{44/7} \approx 98\,\text{m}.$$

So, a person appears at an angle of 1° if they are approx-

Figure 62.: The human figure is visible from a distance of about hundred meters at an angle of 1°.

imately at a distance of 100 meters from us (Figure 62). If they move twice as far away – to 200 meters – they will be seen at an angle of half a degree; if they approach to a distance of 50 meters, the angle of view will increase to 2°, and so on. It is also easy to calculate that a stick of 1 meter in length should appear to us at an angle of 1° at a distance of $360/(44/7) = 57$ m.

At the same angle, we perceive an object of 1 cm from a distance of 57 cm, 1 km from a distance of 57 km, and so on – in general, any object from a distance 57 times greater than its diameter. If we remember this number – 57, we can quickly and easily perform all calculations related to the angular size of an object. For example, if we want to determine how far we need to move an apple with a diameter of 9 cm to see it at an angle of 1°, it is sufficient to multiply 9 by 57 – we get 513 cm, or about 5 meters; from twice the distance, it is perceived at half the angle – half a degree, i.e., it appears the size of the Moon.

In the same way, for any object, we can calculate the distance at which it appears to be the same size as the lunar disk.

3.3. Plate and Moon

Question At what distance should a plate with a diameter of 25 cm be moved away to appear the same size as the Moon in the sky?

Answer $25\ \text{cm} \times 57 \times 2 = 2{,}850\ \text{cm} = 28\ \text{m}.$

3.4. Moon and Copper Coins

Question Perform the same calculation for a five-kopeck coin (diameter 25 mm) and a three-kopeck coin (22 mm).

Answer For the five-kopeck coin: $0.025\ \text{m} \times 57 \times 2 = 2.85$ meters, For the three-kopeck coin: $0.022\ \text{m} \times 57 \times 2 = 2.514$ meters.

If it seems unbelievable to you that the Moon appears no larger than a two-kopeck coin from a distance of four steps or an ordinary pencil from a distance of 80 cm, – hold the pencil at arm's length against the full Moon disk: it will easily cover it. Strangely enough, the most suitable comparison object for the Moon in terms of perceived size is not a plate, not an apple, not even a cherry, but a pea or, even better, a match head! Comparing it to a plate or an apple implies moving them to an unusually large distance; we see an apple in our hands or a plate on the dining table ten to twenty times larger than the lunar disk. And only a match head, which we examine at a distance of 25 cm from the eye ('distance of distinct vision'), is seen at an angle of half a degree, i.e., the same size as the Moon.

The fact that the lunar disk deceptively appears to grow in the eyes of most people by 10 to 20 times is one of the most curious optical illusions. It depends, one might think, mostly on the *brightness* of the Moon: the full moon stands out against the sky much more sharply than plates, apples, coins, and other comparison objects do amidst the surrounding environment.[17]

[17] For the same reason, the incandescent filament of an electric light bulb seems to us much thicker than in a cold, non-luminous state.

This illusion is imposed on us with such irresistible force that even artists, distinguished by a keen eye, succumb to it alongside others and depict the full moon in their paintings much larger than it should be. It is enough to compare the landscape painted by an artist with a photograph to be convinced of this.

The same applies to the Sun, which we see from Earth at the same angle of half a degree; although the true diameter of the solar sphere is 400 times larger than that of the lunar one, its distance from us is also 400 times greater.

3.5. Sensational Photographs

To explain the important concept of the angle of view, let's deviate a bit from our main topic — geometry in open fields — and provide a few examples from the realm of photography.

On the movie screen, you've surely seen such catastrophes as train collisions or such incredible scenes as a car driving on water.

Recall the movie 'Captain Grant's Children.' What a strong impression – isn't it? – the scenes of the shipwreck during the storm or the sight of crocodiles surrounding the boy stuck in the swamp left on you. Of course, no one thinks that such photographs were taken directly from real life. But how were they obtained?

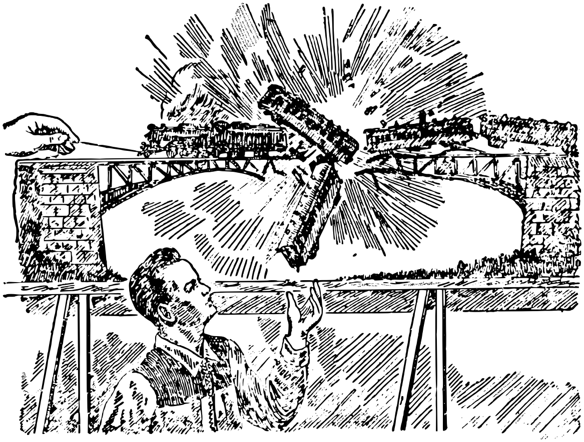


Figure 63.: Preparing a train accident for filming.

The secret is revealed by the illustrations attached here. In

Figure 63, you see the ‘catastrophe’ of a toy train in a toy setting; in Figure 64 – a toy car being pulled on a string behind an aquarium. This is the ‘nature’ from which the film was shot. But why do we succumb to the illusion when we see these images on the screen, as if we were looking at real trains and cars? After all, here, in the illustrations, we would immediately notice their miniature size, even if we couldn’t compare them with the size of other objects. The reason is simple: toy trains and cars are filmed for the screen from a very close distance; therefore, they appear to the viewer at approximately the same angle of view as we usually see real trains and cars. That’s the whole secret of the illusion.

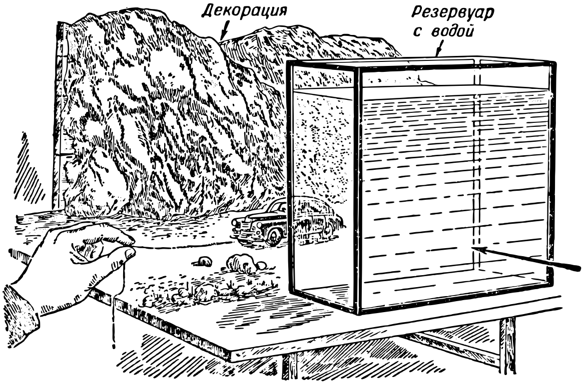


Figure 64.: The underwater road trip.

Or here's another frame from the movie 'Ruslan and Ludmila' (Figure 65). A huge head and a small Ruslan on a horse. The head is placed on a model field close to the camera. And Ruslan on the horse – at a considerable distance. That's the entire secret of the illusion.

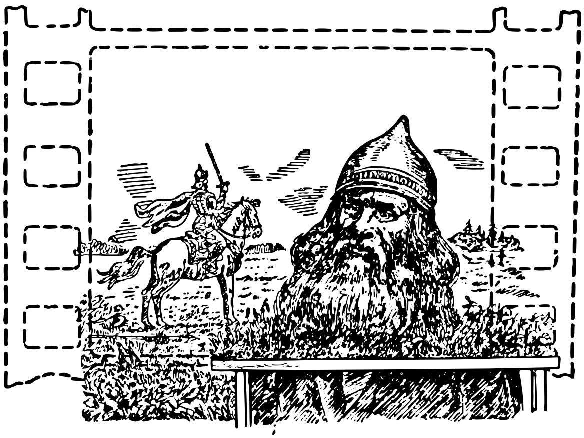


Figure 65.: A shot from the movie *Ruslan and Lyudmila*.

3.6. Reservoir Set Decoration

Figure 66 is another example of an illusion based on the same principle. You see a strange landscape reminiscent of the nature of ancient geological epochs: bizarre trees resembling giant mosses, on them – huge water drops, and in the foreground – a gigantic monster resembling harmless frogs. Despite such an unusual view, the drawing is made with subtlety: it's nothing but a small patch of soil in the forest, only drawn from an unusual angle of view. We never see moss stems, water drops, frogs, etc., at such a large angle of view, and therefore the drawing seems so alien, unfamiliar to us. Before us is a landscape as we would see it if we were shrunk to the size of an ant.

Figure 66.: Mysterious landscape depicted from nature.

Swindlers from bourgeois newspapers act in the same way to create fake reportage photographs. One foreign newspaper once published a note criticising the city administration for allowing huge snow mountains to form on the city streets. To support this, an impressive photo of one such mountain was provided (Figure 67, left). Upon examination, it turned out that the nature for the photograph was a small snowdrift, taken by the 'joker' photographer from a very close distance, i.e., at an unusually large angle of view (Figure 67, right).

Another time, the same newspaper reproduced a photo of a wide crevice in the rock near the city; it served, according to the newspaper, as the entrance to an extensive underground cave, where a group of careless tourists who dared to enter the cave for exploration disappeared without a trace. A volunteer search party equipped to search for the lost dis-

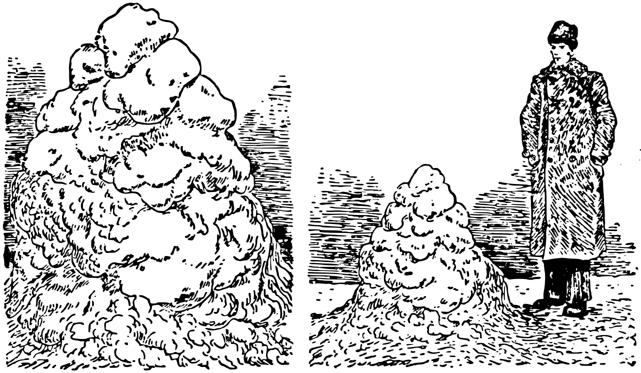


Figure 67.: Snow mountain in a photograph (left) and in nature (right).

covered that the crevice was photographed… from a barely noticeable crack in the icy wall, a centimetre wide!

3.7. Living Protractor

Making a simple protractor device yourself is not very difficult, especially if you use a protractor. But sometimes even a homemade protractor may not be at hand during a countryside walk. In such cases, you can rely on the services of a "living protractor" that is always with us. These are our own fingers. To use them for a rough estimate of viewing angles, you just need to make a few preliminary measurements and calculations.

First of all, you need to determine at what angle we see the fingernail of our outstretched index finger. The usual width of a nail is 1 cm, and its distance from the eye in such a position is about 60 cm; therefore, we see it at an angle of about 1° (slightly less because an angle of 1° would be at a distance of 57 cm). For teenagers, the nail is smaller, but the arm is shorter, so the viewing angle for them is approximately the same – 15°. The reader would do well to perform this measurement and calculation for themselves, relying on book data, to make sure the result is not too far from 15°; if the deviation is significant, you should try another finger.

Knowing this, you have a way to estimate small viewing angles literally with your bare hands. Each distant object, which is just covered by the fingernail of your outstretched index finger, is seen by you at an angle of 1° and, therefore, is 57 times farther away than its width. If the nail covers half of the object, it means its angular size is 2°, and the distance is equal to 28 widths.

The Full Moon covers only half of the nail, i.e., it is seen at an angle of half a degree, meaning it is 114 times its width away from us; here is a valuable astronomical measurement made literally with bare hands!

For larger angles, use the knuckle of your thumb, holding it bent on your outstretched hand. For an adult, the length (note: length, not width) of this joint is about 3.5 cm, and the distance from the eye with an outstretched arm is about 55 cm. It is easy to calculate that the angular size in this position should be about 4°. This provides a means to estimate angles of 4° (and therefore 8°).

Here, you should also add two more angles that can be measured with your fingers – namely, those under which the intervals between fingers are seen when your middle and index fingers are spread as wide as possible; and between your thumb and index finger, also spread to the maximum. It is easy to calculate that the first angle is approximately 7°-8°, and the second is 15°-16°.

There can be many cases to apply your living protractor during walks in open spaces. Suppose a freight car is visible in the distance, which is covered by approximately half of the knuckle of your outstretched thumb, i.e., it is visible at an angle of about 2°. Since the length of a freight car is known (about 6 m), you can easily find out how far you are from it: $6 \times 28 \approx 170$ m or so. A method that does not seem to promise good results, but after a short exercise you will learn to appreciate the services of this "living ecker"[18], the measurement is, of course, rough, but still more reliable than an ungrounded estimate just by sight.

[18] An "ekker" is a surveying instrument for drawing lines on the ground at right angles.

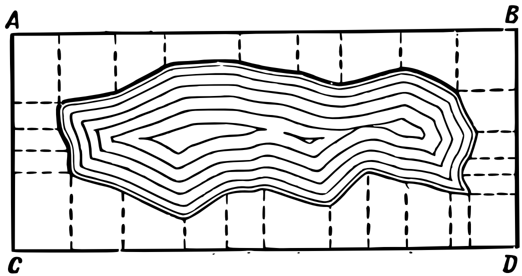


Figure 68.: Mapping of the lake on the plan.

Additionally, using your living protractor, you can, in the absence of any tools, measure the angular height of luminaries above the horizon, the mutual separation of stars in degrees, the apparent sizes of a meteor's trail, etc. Finally, knowing how to make right angles on the ground without instruments, you can draw up a plan of a small area using the method whose essence is clear from the illustration, for example, when surveying a lake (Figure 68), measure rectangle *ABCD*, as well as the lengths of the perpendiculars dropped from prominent points on the shore, and the distances from their bases to the vertices of rectangle *ABCD*. In short, being in Robinson Crusoe's situation, knowing how to use your own hands to measure angles (and your feet to measure distances) could be useful for a variety of needs.

3.8. Jacob’s Staff

If you wish to have more accurate angle measures than the simple “living protractor” described earlier, you can make yourself a simple and convenient device that was once used by our ancestors. This is called “Jacob’s staff” after its inventor – a device that was widely used by sailors until the 18th century (Figure 69), before it was gradually replaced by even more convenient and precise instruments (sextants).

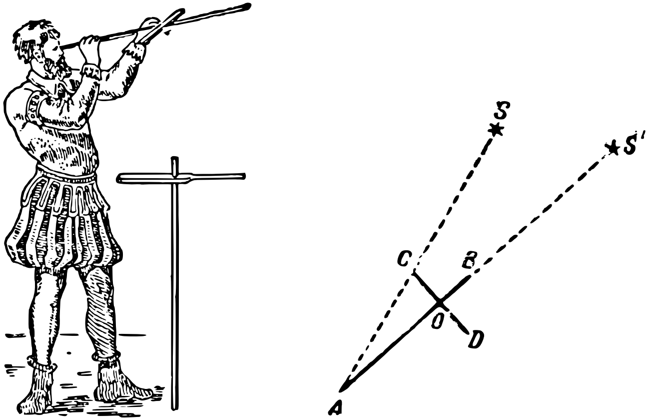


Figure 69.: Jacob’s staff and a diagram of its use.

It consists of a long ruler AB, about 170–100 cm, along which a perpendicular block CD can slide; both parts CO and OD of the sliding block are equal to each other. If you want to determine the angular distance between the stars S and S' using this block (Figure 69), you attach the end A of the ruler to your eye (where a perforated plate is attached for convenience of observation) and direct the ruler so that the star S' is visible at the end B of the ruler; then you move the crosspiece CO along the ruler until the star S is just visible at the end C (Figure 69). Now all that remains is to measure the distance AO in order to calculate the value of the angle SAS' using the length of the CO. Those familiar with trigonometry will understand that the tangent of the desired angle is equal to the ratio of CO/AO. Our "field trigonometry", presented in the fifth chapter, is also sufficient for performing this calculation: you calculate the length AC using the Pythagorean theorem, then find angle C, whose sine is equal to CO/AC.

Finally, you can find the desired angle graphically; by drawing triangle ACO on paper to scale, you measure angle A with a protractor, or if you don't have one, by the method described in our "field trigonometry" (see Chapter **??**).

What is the other half of the crosspiece for? In case the angle to be measured is too large to be measured by the method described above. In that case, instead of directing the ruler AB toward the star S', you aim segment AD toward the point

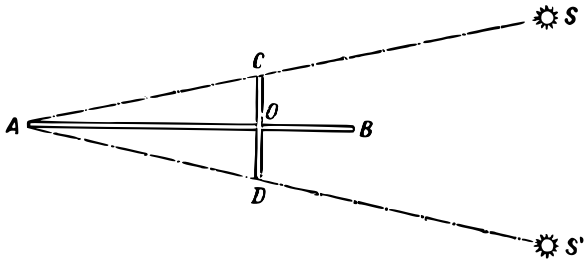


Figure 70.: Determination of the angular distance between stars using Jacob's staff.

S', moving the crosspiece so that its end C coincides with the star S at the same time (Figure 70). Finding the angle SAS' by calculation or construction is, of course, not difficult.

To avoid having to make calculations or constructions each time you measure, you can perform them in advance, even when making the device, and mark the results on the ruler AB; then, when aiming the device at the stars, you only need to read the reading recorded at point O — this is the value of the measured angle.