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A solar azimuth formula that renders circumstantial treatment unnecessary without compromising mathematical rigor: Mathematical setup, application and extension of a formula based on the *subsolar point* and `atan2` function

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ABSTRACT

A conceptually and mathematically concise formula for computing the solar azimuth angle has been used by a subgroup of scientists, but for lack of documentation and publication, it has not been well circulated. This paper introduces this formula which is based on the idea of a unit vector, ***S***, originating from the observer's location and pointing toward the center of the Sun. The vector is completely determined by the coordinates of the *subsolar point* and of the observer. The x- and y-components of the vector determine the solar azimuth angle, and their use along with the function **atan2**, which is available in a number of programming/scripting languages, including **Fortran** and **Python**, renders any circumstantial treatment absolutely unnecessary. The z-component of the vector, at the same time, determines the solar zenith angle. The use of the unit vector also facilitates a figure that can be used as a full 3D graphic depiction of the daily and annual cycles of the Sun's position for any given location, and this figure can be called "wreath of analemmas".
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1. Introduction

To calculate the solar azimuth angle, three formulas are commonly used, namely, the ones that express the sine, cosine and tangent of the solar azimuth angle as functions of the declination and the zenith angle of the Sun, and the latitude and hour angle of the observer's point, and they can be found in nearly all the listed references. Each of these formulas needs circumstantial treatment, which can be ponderous and not easy to understand, to put the result from the inversion of a trigonometric function into the right quadrant. Refs. [2–5] provide warnings about using these formulas and discuss at length how the treatment is performed.

Among the algorithms we reviewed, three that used the sine function to calculate the azimuth angle of the Sun give erroneous results in different ranges of latitude: 1.) Ref. [6] which used the equations in *The American Ephemeris and Nautical Almanac* [7], one of two predecessors of *The Astronomical Almanac*, to calculate the declination and right ascension of the Sun; 2.) Ref. [8] which used a conventional approximate formula for the declination of the Sun that does not consider year-to-year changes; and 3.) Ref. [9] which used the equations from the *Astronomical Almanac for the Year 1986*. The three algorithms treat the result from the inverse of the sine function differently though. We tested the azimuth angles from all three algorithms on an hourly basis for the average day of each month [10] at points ranging from North Pole to South Pole and found that 1.) Ref. [6] is correct only when $90^\circ > \varphi \geq 0^\circ$ where φ is the latitude; in other words Ref. [6], is incorrect at the North Pole and the entire Southern Hemisphere; 2.) Ref. [8] is correct only when $90^\circ \geq \varphi \geq 23.45^\circ$; in other words, Ref. [8] is incorrect to the south of the Tropic of Cancer; and 3.) Ref. [9] is correct when $90^\circ > \varphi > -90^\circ$; in other words, Ref. [9] is correct for the entire globe except North Pole and South Pole. This indicates that the sine-based formula for azimuth is at least prone to errors. In passing, we also point out that when Refs. [6,8] are incorrect, their answers and the correct answer normally sum to 180° or -180°.

Ref. [11] also provides the sine formula for the azimuth angle of the Sun and explains how the result from the inverse sine function needs to be treated differently depending on whether the Sun is to the North or South of the East-West line and whether it is before or

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after the noon. In addition, the algorithm requires the use of South-Clockwise convention in Northern Hemisphere and North-Clockwise convention in Southern Hemisphere for the definition of the azimuth angle. Thus, caution is advised especially when multiple sites located in both hemispheres are being investigated simultaneously.

Refs. [12,13] modified the algorithm of Ref. [6] by adding the refraction correction of the azimuth angle and by correcting the right ascension calculation beyond summer solstice, respectively. Ref. [14] developed a new algorithm for calculating the declination and right ascension of the Sun to the accuracy of existing algorithms while making the calculation simpler; the algorithm uses the tangent function to calculate the azimuth angle of the Sun.

Ref. [15] presented the most sophisticated algorithm for declination and right ascension of the Sun based on Ref. [16] that reduces the errors of zenith and azimuth down to $\pm 0.0003°$ over the period from the year -2000 to 6000. The function **atan2** is used to calculate the azimuth angle, but its arguments are not the x- and y-components of the unit vector that will be defined later in this paper. Algorithms proposed and discussed in Refs. [17,18] were based on Ref. [15] and were aimed to speed up the computation and used the same azimuth angle formula as Ref. [15]. In Ref. [19], approximations were used to replace the original equations in Ref. [15] for faster calculation and higher accuracy but over much shorter time span around the present; the formula used for the azimuth angle was the same as that of Ref. [15].

Ref. [20], a popular reference for solar energy research, uses the classic cosine formula for the solar azimuth angle and offers verbal explanation for determining the sign of the angle. The author also warns about the use of the sine formula because it gives improper values when the absolute value of the azimuth angle is larger than 90°. Ref. [21], another widely used reference, uses the same formula except adding the sign function of the hour angle as a multiplicative factor to the cosine function. Confusion may arise, though, when hour angle is defined differently, *e.g.*, from 0° to 360° westward [4].

Neither the sine nor the cosine function alone can uniquely determine the solar azimuth angle, since neither is an injective, or one-to-one, function in the range [-π, π], or [-180°, 180°]. The sine and cosine functions joined together as a pair, on the other hand, can uniquely determine the solar azimuth angle. However, dividing the sine function by the cosine function, which creates the tangent function, causes the information to degenerate, as the tangent function is not a one-to-one function in the full possible range of the azimuth angle, which is why the value from the arctangent function also needs further treatment to get the final angle in the right quadrant. What is provided in *The Nautical Almanac* [22] is one that uses the tangent function.

Meanwhile, a subgroup of scientists has been using a formula based on the x-, y- and z-components of a unit vector, $\boldsymbol{S}$, which is entirely determined by the coordinates of the observer and the *subsolar point* which is the point where the Sun is overhead; the unit vector originates from the observer's coordinates, or the latitude and longitude, and points toward the Sun. Now the solar zenith angle is just the **arccosine** of the z-component, and the solar azimuth angle is simply the **atan2** of the x- and y-components, and the **atan2** here is a function that takes two separate arguments and is available in languages such as **Fortran** and **Python**. Besides the simplicity of this method, the equation of time is conveniently incorporated into the expression for the longitude of the subsolar point. It is not clear, however, who first derived the formula and, in addition, for lack of documentation and publication, it has not been widely circulated. Coincidentally, Ref. [3] derived the said unit vector using vector analysis and obtained identically the same x-, y- and z-components mentioned above; from there, however, the author did not proceed to use the method discussed here. Although the algorithms we reviewed also used the **atan2** function, their arguments are not the x- and y-components of the unit vector as defined here.

The variables involved in deriving solar geometry can be put into two categories: 1) those that fall within the area of astronomy and can be termed, appropriately, *ephemerides*, and the relevant ones in this category are the declination of the Sun, the Earth-Sun distance, and the equation of time; 2) those that purely belong to 3D geometry and/or spherical trigonometry, and the ones in this category are solar zenith and azimuth angles and can be derived from those in Category 1).

For first approximations, there is a simple formula for the declination of the Sun which neglects the year-to-year change; the Earth-Sun distance is considered a constant, namely, 1 au; and the *equation of time* is ignored. Since it is easy and straightforward to use those formulas in *The Astronomical Almanac*, as has been done by Ref. [9]; there is not a good reason to opt for simpler ones with more uncertainties. It should be noted here that although different formulas for the astronomical ephemerides used in solar geometry give different results which cause differences in subsequent solar zenith and azimuth angles, the focus of this paper is not about these minor differences or about which formula is more accurate; instead it is about the azimuth angle formula that does not need the circumstantial treatments that are error-prone yet inevitable as in the cases of the traditional formulas. With the same input, these formulas do give the same output as the formula to be presented in this paper, but only if the circumstantial treatments are executed correctly.

In this paper, we will first provide formulas from Ref. [1] for variables in Category 1), and then give formulas for those in Category 2). To illustrate how these formulas work, we put some results in two figures. Finally, our subroutine in **Fortran 90** is deployed in Appendix A.

2. The formulas

2.1. The declination of the Sun ($\boldsymbol{\delta}$), the Earth-Sun distance ($\boldsymbol{R_{ES}}$), and the equation of time ($\boldsymbol{E}$)

The following formulas are straight from Ref. [1] except that the time argument *n* has been modified for an arbitrary year, preferably from 1950 to 2050; they are in the order such that results from earlier expressions can be used by later expressions. Only those equations of immediate interest are numbered.

$$n = -1.5 + (Y_{in} - 2000) \cdot 365 + N_{leap} + \text{ Day of Year} + \text{ Fraction of Day from 0:00 UT (day)},$$

$$L = 280.466 + 0.9856474n \ (°),$$

$$g = 357.528 + 0.9856003n \ (°),$$

$$\lambda = L + 1.915 \sin g + 0.020 \sin(2g) \ (°),$$

$$\varepsilon = 23.440 - 0.0000004n \ (°),$$

$$\alpha = \tan^{-1}(\cos\varepsilon\tan\lambda)\cdot 180\Big/\pi\ (^\circ),$$

$$\delta = \sin^{-1}(\sin\varepsilon\sin\lambda)\cdot 180\Big/\pi\ (^\circ), \tag{1}$$

$$R_{ES} = 1.00014 - 0.01671\cos g - 0.00014\cos(2g)\ (\text{au}), \tag{2}$$

$$E_{min} = (L-\alpha)\cdot 4\ \ (\text{min}). \tag{3}$$

where.

n is the number of days of Terrestrial Time (TT) from J2000.0 UT;
Y_{in} is the input year;
N_{leap} is the number of leap years;
L is the mean longitude of the Sun corrected for aberration;
g is the mean anomaly;
λ is the ecliptic longitude;
ε is the obliquity of ecliptic;
α is the right ascension;
δ is the declination of the Sun;
R_{ES} is the Earth-Sun distance;
E_{min} is the equation of time.

Note that L and g as well as λ given above can be either positive or negative, but computationally they need to be put in the range 0°–360°; this can be accomplished by using the **modulo** function; α needs to be in the same quadrant as λ, and this can be done by using the **atan2** function, which takes two arguments, instead of the **atan** function, which takes only one. All these treatments have been properly taken care of in the code in Appendix A.

According to the *Almanac*, the errors of the right ascension and declination of the Sun given by these formulas are less than $(1/60)^\circ$, and the error of the equation of time is less than 3.5 s, if the input year is between 1950 and 2050.

2.2. The solar zenith and azimuth angles

Suppose the observer's coordinates, or latitude and longitude, are (φ_o, λ_o), and the *subsolar point*'s coordinates are (φ_s, λ_s), then the x-, y- and z-components of the unit vector, $\boldsymbol{S}$, pointing from the observer to the center of the Sun are as follows:

$$\varphi_s = \delta, \tag{4}$$

$$\lambda_s = -15(T_{GMT} - 12 + E_{min}/60), \tag{5}$$

$$S_x = \cos\varphi_s\sin(\lambda_s - \lambda_o), \tag{6}$$

$$S_y = \cos\varphi_o\sin\varphi_s - \sin\varphi_o\cos\varphi_s\cos(\lambda_s - \lambda_o), \tag{7}$$

$$S_z = \sin\varphi_o\sin\varphi_s + \cos\varphi_o\cos\varphi_s\cos(\lambda_s - \lambda_o) \tag{8}$$

Here the effect of parallax is ignored, which is equivalent to assuming that the Earth-Sun distance is infinity. It can be shown that there exists $S_x^2 + S_y^2 + S_z^2 = 1$.

The derivation of Eqs. (6)–(8) is straightforward if working in the Earth-Centered Earth-Fixed (ECEF) coordinate system, which is a geocentric right-handed Cartesian system, as follows: 1.) At the subsolar point in the ECEF system, construct a unit vector pointing upward; 2.) At the observer's coordinates in the ECEF system, construct three unit vectors pointing in the east, north and upward, respectively, and compute the dot product of each with the vector in Step 1.) to get the right sides of Eqs. (6)–(8). Refer to Fig. 1a for additional information about the geometric configuration.

Ref. [3] did not use the idea of the subsolar point, but used essentially the same method to derive the x-, y- and z-components of $\boldsymbol{S}$, and they are exactly the same as the S_x, S_y and S_z here, noticing that $\lambda_s - \lambda_o$ differs from the hour angle, ω, by only a negative sign.

The solar zenith angle is now simply

$$Z = \cos^{-1}S_z, \tag{9}$$

and the solar azimuth angle following the *South-Clockwise convention* is

$$\gamma_s = \text{atan2}\ \left(-S_x,\ -S_y\right). \tag{10}$$

Eq. (10) gives an unambiguous solar azimuth angle and it is final, and it works everywhere from pole to pole. In other words, the solar azimuth angle is in the right quadrant. The standard **atan2**(y, x) function, which follows the *East-Clockwise convention*, is available in programming/scripting languages **Fortran**, **Python**, etc. and it gives the angle in the range [-π, π], which can be converted to [-180°, 180°] if so desired.

The *North-Clockwise convention* can be achieved using γ_s = **atan2**$(S_x,\ S_y)$, and the East-Counterclockwise convention can be realized using γ_s = **atan2**$(S_y,\ S_x)$, though the latter is rarely used in solar energy analysis.

The advantage of Eq. (10) is that it is not only concise, but never fails. When both S_x and S_y are 0, or, when the subsolar point and the observer's point are the same and, therefore, the azimuth angle cannot be defined, Eq. (10) gives 0 and keeps the job running. This treatment is physically inconsequential, because the term involving azimuth angles in solar energy analysis also becomes 0.

The *subsolar point* is not an unfamiliar concept, but the way it is used as in the method discussed here dispels some of the mysterious aura around solar geometry, and the Earth-Sun configuration is thus mentally more accessible.

It can be easily shown that $S_y/\sqrt{S_x^2+S_y^2}$, $S_x/\sqrt{S_x^2+S_y^2}$ and S_y/S_x are equivalent to the three staple formulas traditionally used, namely, the sine, cosine and tangent of the azimuth angle, respectively. Both signs of S_x and S_y are required to determine the azimuth angle, but each of the three formulas causes two possible distinct cases to degenerate into one, which is why none alone can give an unambiguous result. Ref. [3] derived the expressions of S_x, S_y and S_z using vector analysis, but did not use Eq. (10) to compute the solar azimuth angle; rather, the author proceeded to discuss the three traditional formulas at length.

This unit vector defined as such determines the zenith angle and the azimuth angle of the Sun simultaneously, and, in addition, its terminal point traces out the analemma for any given location and given hour, as we will see later.

Note these formulas are based on geocentric latitudes and no correction for the atmospheric refraction has been applied to the solar zenith angle, Z. The atmospheric refraction depends on the temperature profile of the atmosphere and surface elevation and can be evaluated subsequently.

Note that the equation of time, E_{min}, appears in the expression for λ_s, the longitude of the *subsolar point*.

A word of warning is that in computer languages, the input angles of trigonometric functions and output angles of inverse trigonometric functions are all in radians.

Finally, Eq. (10) is not limited to the computer. The idea behind it is applicable even when one uses a calculator, and it can be done as follows: 1) Calculate the values of S_x and S_y; 2) Determine which quadrant the angle falls in by inspecting the signs of S_x and S_y; 3) If the angle is in the 1st and 4th quadrants, $\gamma_s = \tan^{-1}(S_y/S_x)$; otherwise, $\gamma_s = \tan^{-1}(S_y/S_x) + 180°$, supposing the calculator is in the `degree` mode.

2.3. When the observed body is a satellite

It has been assumed that the Sun is a body with an altitude of infinity. In the case of a satellite with a finite altitude above the surface of the Earth, H, the effect of parallax cannot be ignored. It can be shown, however, that Eqs. (6)–(10) can still be used, assuming now that (φ_s, λ_s) are the coordinates of the *sub-satellite point* which is the point where the satellite is overhead and are known, except a ΔZ_H needs to be added to the Z in Eq. (9) and, ΔZ_H to be calculated as follows

$$\Delta Z_H = \tan^{-1}\left[\frac{R\sin Z}{R(1-\cos Z)+H}\right], \tag{11}$$

where R is the radius of the Earth, assuming a spherical Earth, and H is the altitude of the satellite above the surface of the Earth. The zenith angle of the satellite is thus $Z + \Delta Z_H$. Note that when H is infinity, $\Delta Z_H = 0$. Also note that if H is set to the actual Earth-Sun distance, Eq. (11) becomes an additive correction to the zenith angle of the Sun due to parallax. See Fig. 1a for the geometric configuration of this case.

For completeness, the x-, y- and z-components of the unit vector originating from the observer and pointing toward the satellite are

$$S_x^{sat} = S_x \frac{\sin(Z+\Delta Z_H)}{\sin Z}, \tag{12}$$

$$S_y^{sat} = S_y \frac{\sin(Z+\Delta Z_H)}{\sin Z}, \tag{13}$$

$$S_z^{sat} = S_z \frac{\cos(Z+\Delta Z_H)}{\cos Z}, \tag{14}$$

where S_x, S_y, S_z and Z are from Eqs. (6)–(9), respectively, assuming that (φ_s, λ_s) are the coordinates of the *sub-satellite point* and are known. The azimuth angle of the satellite is

$$\gamma_{sat} = \mathrm{atan2}\left(-S_x^{sat}, \ -S_y^{sat}\right). \tag{15}$$

Since S_x^{sat} and S_y^{sat} in Eq. (15) differ from S_x and S_y in Eq. (10) by the same factor, they give the same result.

2.4. When the observer is a satellite and the observed body is the Sun

Now let's assume that the coordinates of the *sub-satellite point* and the *subsolar point* are (φ_o, λ_o) and (φ_s, λ_s), respectively, Eqs. (6)–(10) can be used for the solar zenith and solar azimuth angles from the perspective of the satellite, assuming the satellite uses the terrestrial convention for directions. The exception is that the solar zenith angle at sunset is now $\frac{\pi}{2} + \cos^{-1}\left(\frac{R}{R+H}\right)$. In the case of a

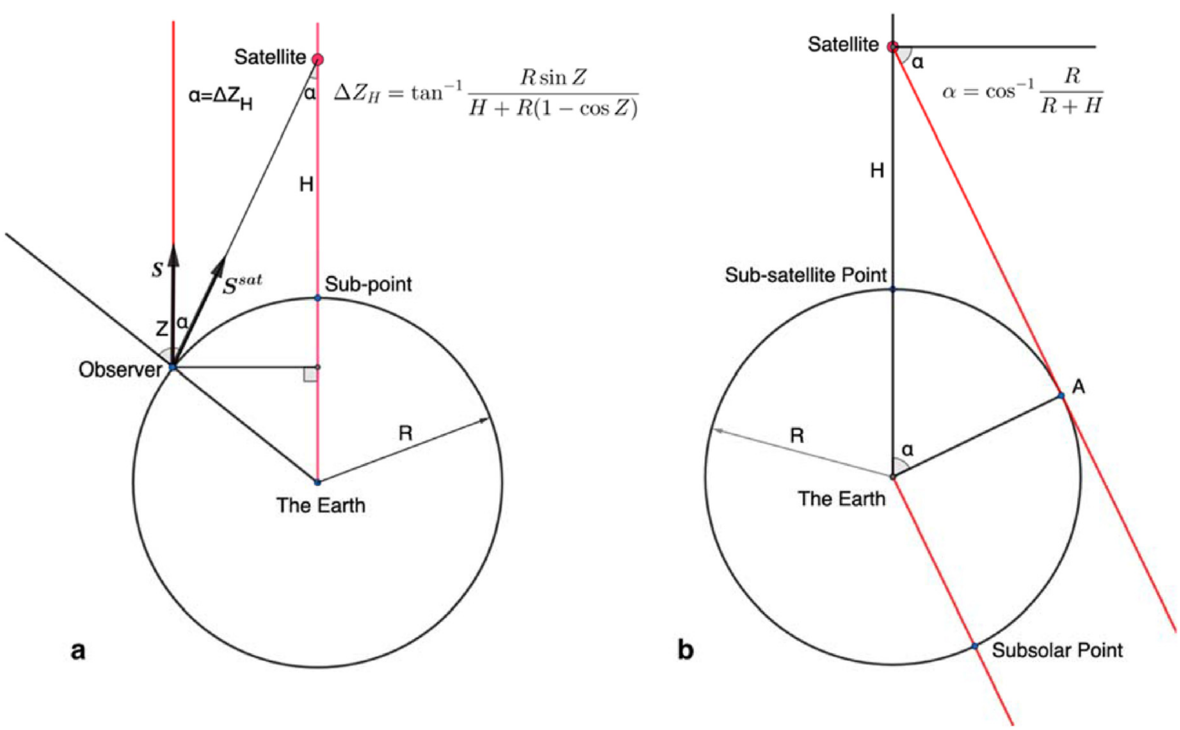

Fig. 1. a. When the observed body is a satellite. The diagram shows the plane containing the center of the Earth, the observer and the satellite. If the observed body is the Sun, the satellite in the diagram is substituted for the Sun, and H becomes infinity and, consequently, ΔZ_H, or α, becomes 0; **b**. When the observer is a satellite and the observed body is the Sun. The diagram shows the plane containing the center of the Earth, the satellite and the Sun.

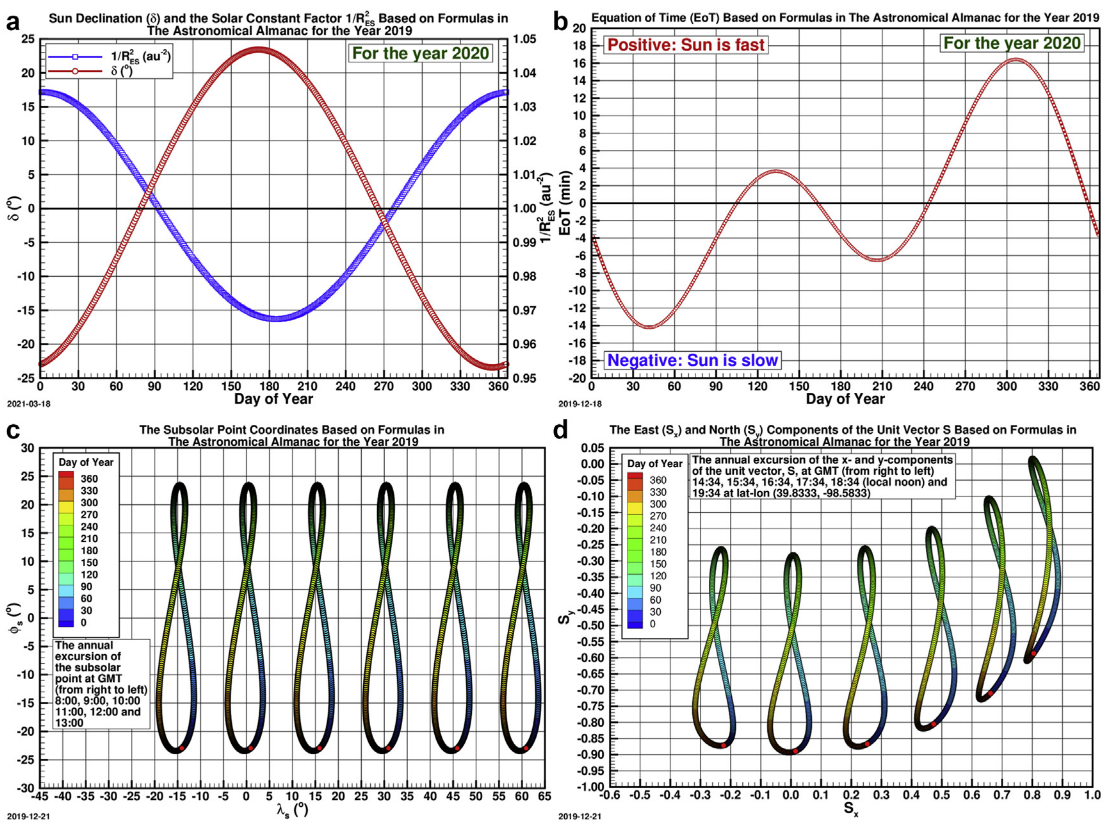

Fig. 2. a. The declination of the Sun, δ, and the solar constant factor, $1/R_{ES}^2$ for the year 2020 at 1-day step; **b**. The equation of time, E_{min}, for the year 2020 at 1-day step; **c**. The track of the subsolar point at GMT 8:00, 9:00, 10:00, 11:00, 12:00 and 13:00 throughout the year 2020; **d**. The x- and y-components of the unit vector, S, at a site in the US at GMT 14:34, 15:34, 16:34, 17:34, 18:34 (around local noon) and 19:34.

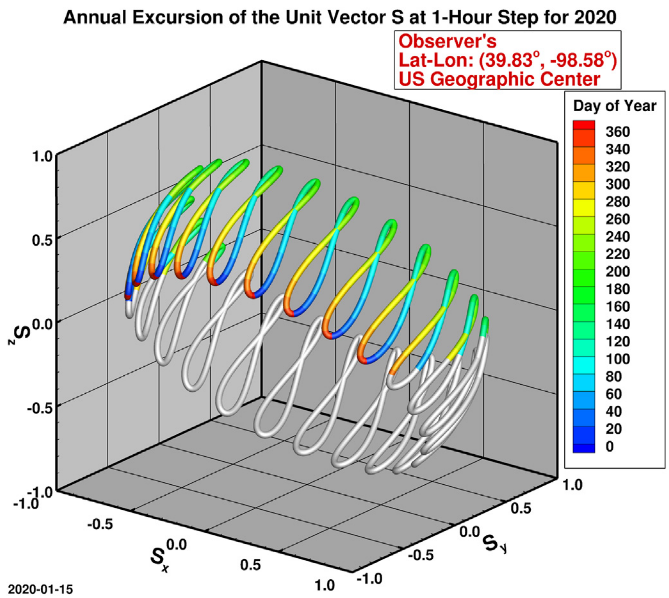

Fig. 3. The "wreath of analemmas": Annual excursion of the unit vector **S** at 1-h step throughout the year 2020 at a US site. Note that nighttime is in gray color and occurs when $S_z < 0$. Each figure "8" corresponds to a unique hour and is literally an analemma in the celestial sphere. The "8" figure at the lowest position represents the hour around local midnight, and the hour progresses counterclockwise. Note that the color bar also applies to nighttime analemmas, which means that corresponding points in all the congruent "8" figures, including those partly or completely in nighttime, have the same color, or the same day of year.

surface-based observer, $H = 0$, and the solar zenith angle at sunset is simply $\frac{\pi}{2}$ in the absence of atmospheric refraction. Fig. 1b shows the geometric configuration of this case.

3. Sample results

Shown in Fig. 2a and b are the declination of the Sun, δ, the solar constant factor, $1/R_{ES}^2$, and the equation of time, E_{min} for the year 2020 at 1-day step. These are taken directly from Ref. [1].

Fig. 2c shows the annual path of the subsolar point at GMT hours 8:00 to 13:00 at 1-h interval in the year 2020. The "8" figures are due to the equation of time. Fig. 2d shows the x- and y-components of the unit vector, $\boldsymbol{S}$, at a site in the US. The rightmost "8" figure gets distorted the most, because the corresponding hour is the farthest from the local noon, and because these "8" figures are actually the horizontal projections of the analemmas shown in Fig. 3.

Fig. 3 shows the x-, y- and z-components of the unit vector, $\boldsymbol{S}$, throughout the year 2020 at 1-h step at the same site as in Fig. 1c and d. This is a complete depiction of the position of the Sun for an entire year, because the z-component determines the solar zenith angle and the x- and y-components determine the solar azimuth angle. As the vector, $\boldsymbol{S}$, moves in time, its tip traces out the familiar "8" figures, and these figures are literally the analemmas that can be observed in the celestial sphere. This type of figure can be generated for any given location, so that the Sun's position gets visualized in practically the whole domain of time. This figure can be called, aptly, a "wreath of analemmas".

4. Summary

This paper presented a solar azimuth angle formula derived from a somewhat different perspective. The result is an algorithm that is conceptually more concise and computationally much simpler than the three traditional formulas, yet the algorithm has the same mathematical rigor and accuracy as any of the three traditional formulas. This algorithm also leads to better visualization of the annual excursion of the Sun's position for any given terrestrial location. The 5 instances of the function `modulo` and 2 instances of the function `atan2` make the code in Appendix A shorter by many lines than it otherwise would be.

The resulting one-step solar azimuth angle formula renders unnecessary the error-prone circumstantial treatments that are inextricably associated with the traditional sine, cosine and tangent formulas, and this will facilitate not only practical application but teaching and education in future. In the not very likely event that one does not have access to the `atan2` function, the x- and y-components of the unit vector S tell unequivocally which quadrant the solar azimuth angle falls in, and one can then proceed with basic trigonometry.

CRediT authorship contribution statement

Taiping Zhang: Conceptualization, Methodology, Writing – original draft, preparation. **Paul W. Stackhouse:** Funding acquisition, Investigation, Supervision, Project administration. **Bradley Macpherson:** Data curation, Software. **J. Colleen Mikovitz:** Data curation, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Subroutine in `Fortran 90`

```
! 2019-12-26  !Solar Geometry using subsolar point and atan2.
!              by Taiping Zhang.
! Input variables:
!     inyear: 4-digit year, e.g., 1998, 2020;
!     inmon: month, in the range of 1 - 12;
!     inday: day, in the range 1 - 28/29/30/31;
!     gmtime: GMT in decimal hour, e.g., 15.2167;
!     xlat: latitude in decimal degree, positive in Northern Hemisphere;
!     xlon: longitude in decimal degree, positive for East longitude.
!
! Output variables:
!     solarz: solar zenith angle in deg;
!     azi: solar azimuth in deg the range -180 to 180, South-Clockwise
!          Convention.
!
! Note: The user may modify the code to output other variables.

Subroutine sunpos_ultimate_azi_atan2(inyear, inmon, inday, gmtime, &
     xlat, xlon, solarz, azi)

  implicit none

  integer:: inyear, inmon, inday, nday(12), julday(0:12), xleap, i, &
       dyear, dayofyr
  real:: gmtime, xlat, xlon
  real:: n, L, g, lambda, epsilon, alpha, delta, R, EoT
  real:: solarz, azi
  real, parameter:: rpd=acos(-1.0)/180
  real:: sunlat, sunlon, PHIo, PHIs, LAMo, LAMs, Sx, Sy, Sz
  data nday / 31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31 /

  if((mod(inyear, 100)/=0 .and. mod(inyear, 4)==0) .or. &
       (mod(inyear, 100)==0.and.mod(inyear, 400)==0)) then
     nday(2)=29
  else
     nday(2)=28
  endif

  julday(0)=0
  do i=1, 12
     julday(i)=julday(i-1)+nday(i)
  enddo
! Note: julday(12) is equal to either 365 or 366.

  dyear=inyear-2000
  dayofyr=julday(inmon-1)+inday
  if(dyear<=0) then
     xleap=int(real(dyear)/4) !Note: xleap has the SAME SIGN as dyear. !!!
  elseif(dyear>0) then
     if(mod(dyear, 4)==0) then
        xleap=int(real(dyear)/4)   ! For leap-years.
     else
          xleap=int(real(dyear)/4)+1 !"+1" is for year 2000.
       endif
    endif

   ! --- Astronomical Almanac for the Year 2019, Page C5 ---
    n=-1.5+dyear*365.0+xleap*1.0+dayofyr+gmtime/24
    L=modulo(280.460+0.9856474*n, 360.0)
    g=modulo(357.528+0.9856003*n, 360.0)
    lambda=modulo(L+1.915*sin(g*rpd)+0.020*sin(2*g*rpd), 360.0)
    epsilon=23.439-0.0000004*n
    alpha=modulo(atan2(cos(epsilon*rpd)*sin(lambda*rpd), &
         cos(lambda*rpd))/rpd, 360.0) !alpha in the same quadrant as lambda.
    delta=asin(sin(epsilon*rpd)*sin(lambda*rpd))/rpd
    R=1.00014-0.01671*cos(g*rpd)-0.00014*cos(2*g*rpd)
    EoT=modulo((L-alpha)+180.0, 360.0)-180.0 !In deg.

   ! --- Solar geometry ---
    sunlat=delta  !In deg.
    sunlon=-15.0*(gmtime-12.0+EoT*4/60)
    PHIo=xlat*rpd
    PHIs=sunlat*rpd
    LAMo=xlon*rpd
    LAMs=sunlon*rpd
    Sx=cos(PHIs)*sin(LAMs-LAMo)
    Sy=cos(PHIo)*sin(PHIs)-sin(PHIo)*cos(PHIs)*cos(LAMs-LAMo)
    Sz=sin(PHIo)*sin(PHIs)+cos(PHIo)*cos(PHIs)*cos(LAMs-LAMo)

    solarz=acos(Sz)/rpd      !In deg.
    azi=atan2(-Sx, -Sy)/rpd  !In deg. South-Clockwise Convention.

   Endsubroutine sunpos_ultimate_azi_atan2
```

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