Homework 1

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1 Warmup

Consider a degree 3 and a degree 4 polynomial:

$$A_1 + B_1x + C_1x^2 + D_1x^3 \tag{1}$$
$$A_2 + B_2x + C_2x^2 + D_2x^3 + E_2x^4 \tag{2}$$

In order to intersect in five points, there must exist five solutions to the equation:

$$A_1 + B_1x + C_1x^2 + D_1x^3 = A_2 + B_2x + C_2x^2 + D_2x^3 + E_2x^4$$

Rearranging makes it clear that there are only four possible soltions to the equation:

$$(A_1 - A_2) + (B_1 - B_2)x + (C_1 - C_2)x^2 + (D_1 - D_2)x^3 - E_2x^4 = 0$$

Thus there can not be more than four intersections between two third and fourth polynomials.

2 Interpolation Accuracy

An interpolating polynomial for $ln(x)$ is obtained from a two degree Lagrange polynomial:

$$\begin{aligned} L_3(x) &= y_1\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \\ &= -ln2\frac{(x-1)(x-4)}{2} + ln4\frac{(x-1)(x-2)}{6} \end{aligned} \tag{3}$$

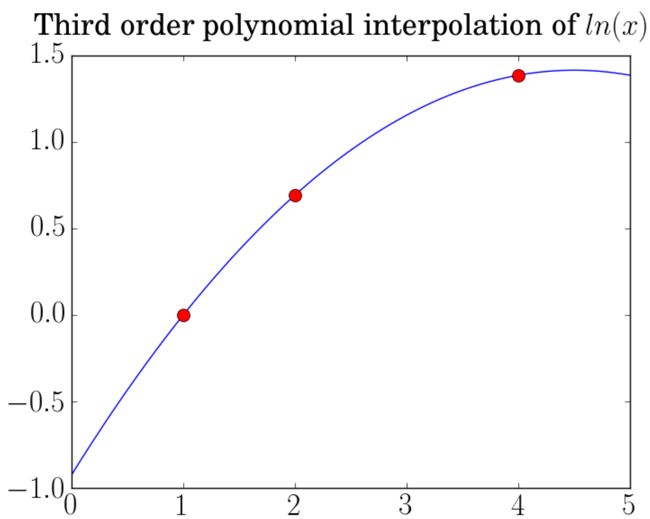


Figure 1: the interpolated function (eq 3) for the given data points of $ln(x)$

3 Bezier Curves

$$P_0 = (1, 1)$$
$$P_1 = (1, \frac{2}{3})$$
$$P_2 = (3, \frac{1}{3})$$
$$P_3 = (9, 1)$$

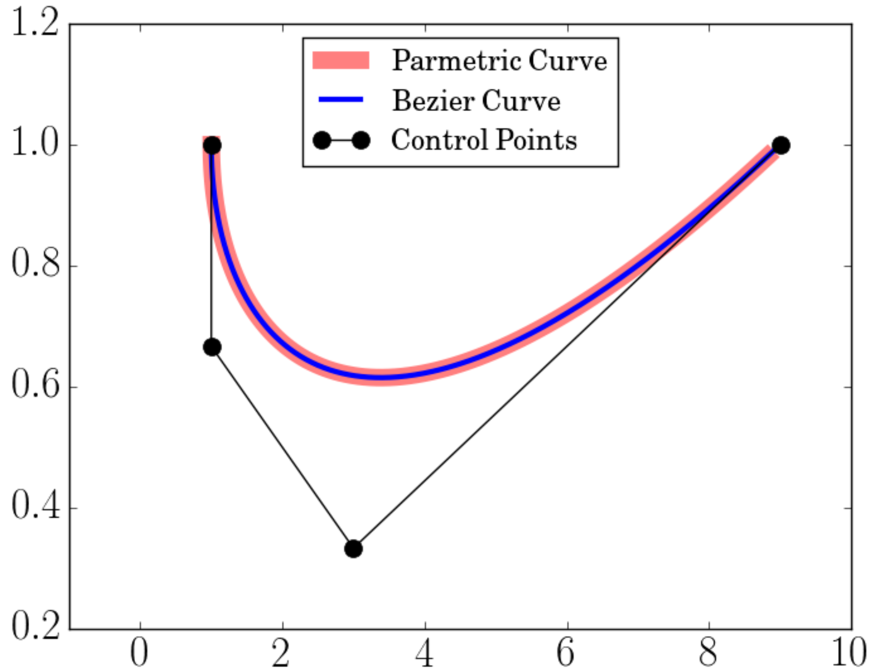


Figure 2: Piecewise function with its Bezier representation

4 Spline Theory

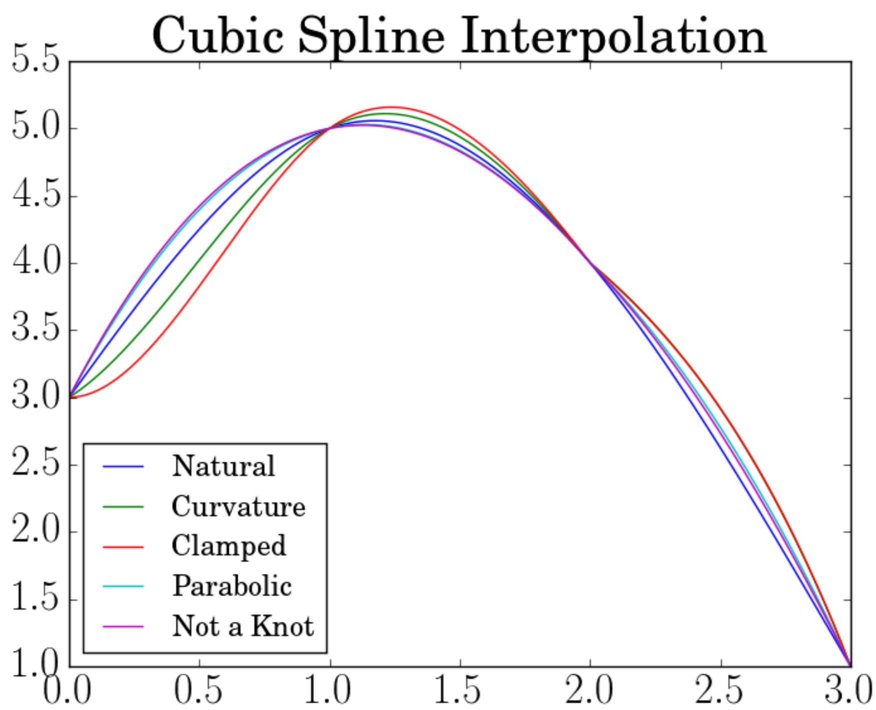


Figure 3: Cubic Spline Interpolation using different boundary conditions

Natural

$$\boldsymbol{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0.25 & 3.75 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ -4 \\ -2 \\ 0 \end{bmatrix} \boldsymbol{b} = \begin{bmatrix} 0 \\ -18 \\ -12 \\ 0 \end{bmatrix}$$

Curvature

Using $y_1'' = 5$, $y_4'' = -5$

$$\boldsymbol{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0.25 & 3.75 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 5 \\ -5.66 \\ -0.33 \\ -5 \end{bmatrix} \boldsymbol{b} = \begin{bmatrix} 5 \\ -18 \\ -12 \\ -5 \end{bmatrix}$$

Clamped

Using $y_1' = 5$, $y_4' = -5$

$$\boldsymbol{a} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0.5 & 3.5 & 1 & 0 \\ 0 & 0.285 & 3.714 & 1 \\ 0 & 0 & 0.269 & 1.73 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -7.86 \\ -2.26 \\ --1.06 \\ -5.46 \end{bmatrix} \boldsymbol{b} = \begin{bmatrix} -18 \\ -18 \\ -12 \\ -12 \end{bmatrix}$$

Parabolic

$$\boldsymbol{a} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 0.2 & 3.8 & 1 \\ 0 & 0 & -0.263 & 1.263 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -3.25 \\ -3.25 \\ -1.75 \\ -1.75 \end{bmatrix} \boldsymbol{b} = \begin{bmatrix} 0 \\ -18 \\ -12 \\ 0 \end{bmatrix}$$

Not-a-Knot

$$\boldsymbol{a} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0.166 & 4 & 1 \\ 0 & 0.166 & -0.5 & 1.5 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -4 \\ -3 \\ -2 \\ -1 \end{bmatrix} \boldsymbol{b} = \begin{bmatrix} 0 \\ -18 \\ -12 \\ 0 \end{bmatrix}$$

5 Runge's Phenomenon

Lagrange interpolating polynomials, natural cubic splines, and chebyshev polynomials were used to interpolate $1/(1+t^2)$. Error was evaluated using RMS error and tended to dercease for increasing numbers of points, as expected. The spline interpolation was best for all numbers of points in terms of the RMSE.

Method	RMSE 5pt	RMSE 10pt	RMSE 15pt
Lagrange	0.27918	0.10959	1.77154
Splines	0.14108	0.04083	0.00075
Chebyshev	0.22641	0.07909	0.02549

$$f(t) = \frac{1}{1+t^2} \tag{4}$$

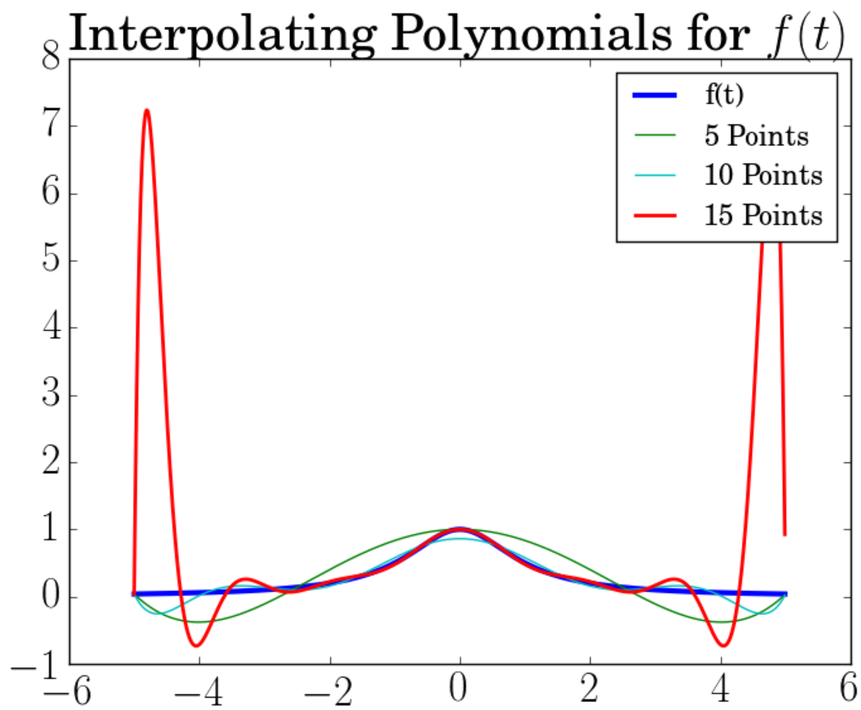


Figure 4: Lagrange Interpolating Polynomials

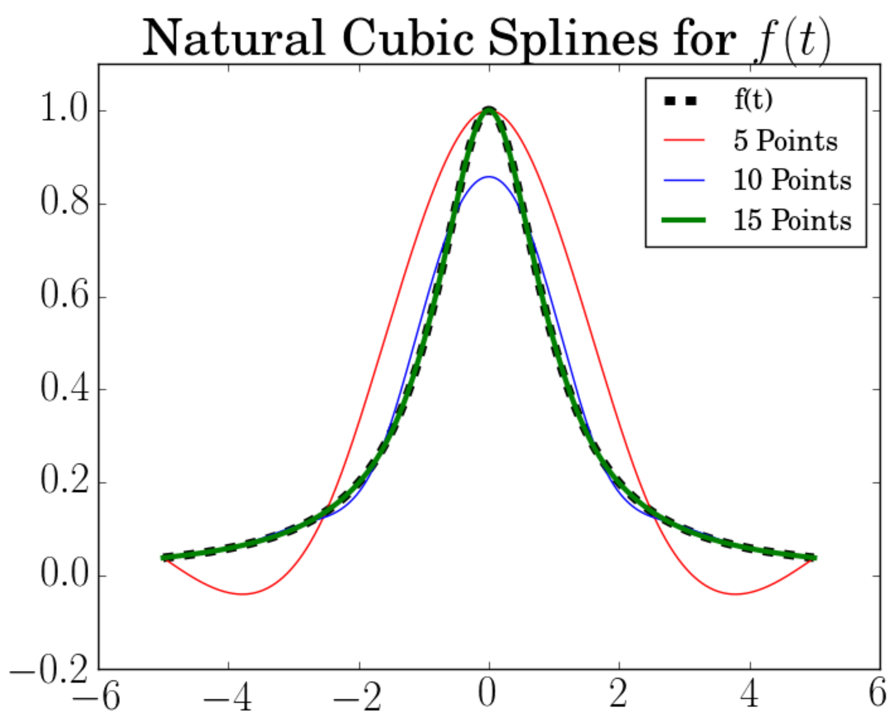


Figure 5: Natural cubic spline interpolation for varying numbers of sample points

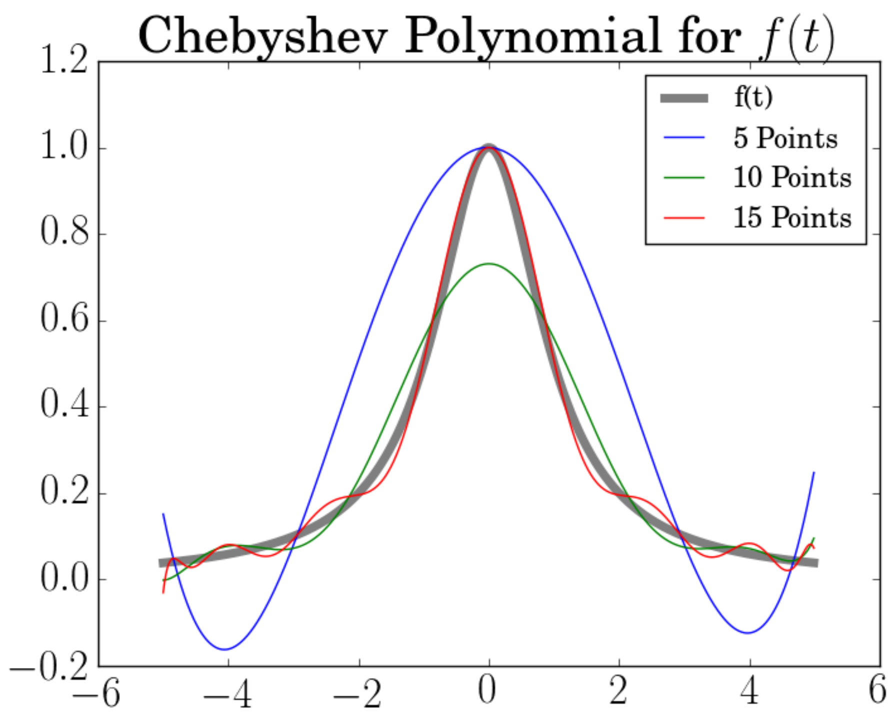


Figure 6: Chebyshev Polynomial for varying numbers of sample points.

6 Gibb's Phenomenon

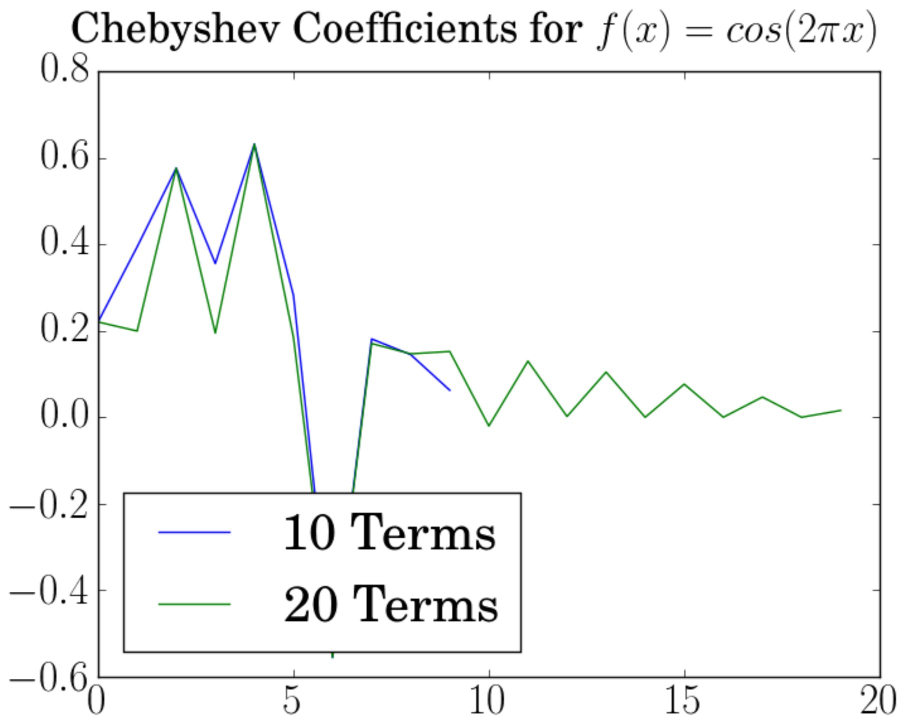


Figure 7: Ringing phenomenon in Chebyshev approximations

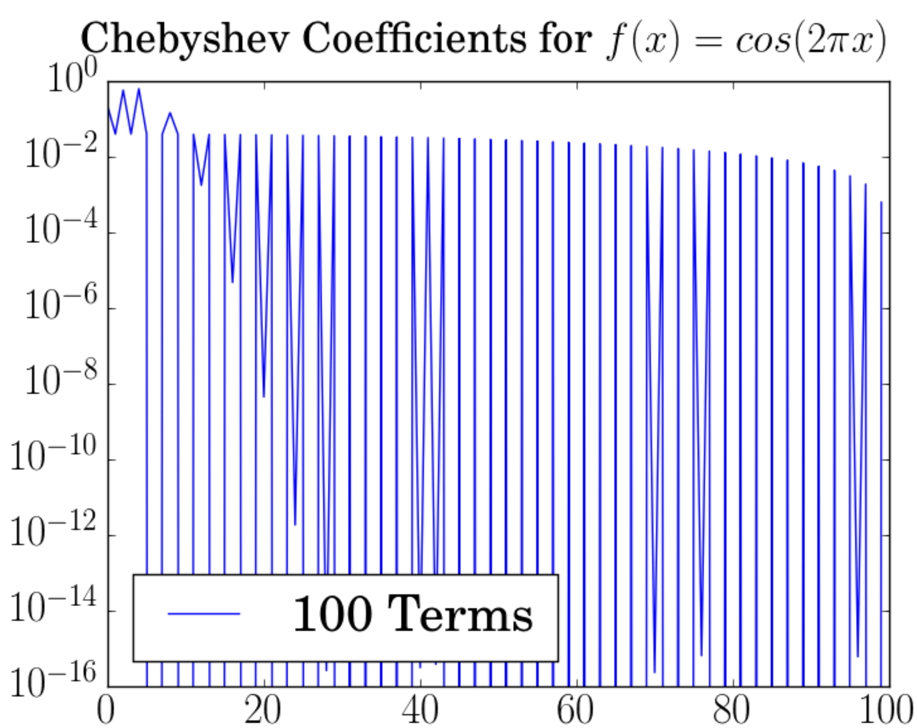


Figure 8: 100 Coefficients displaying a clear power law relationship

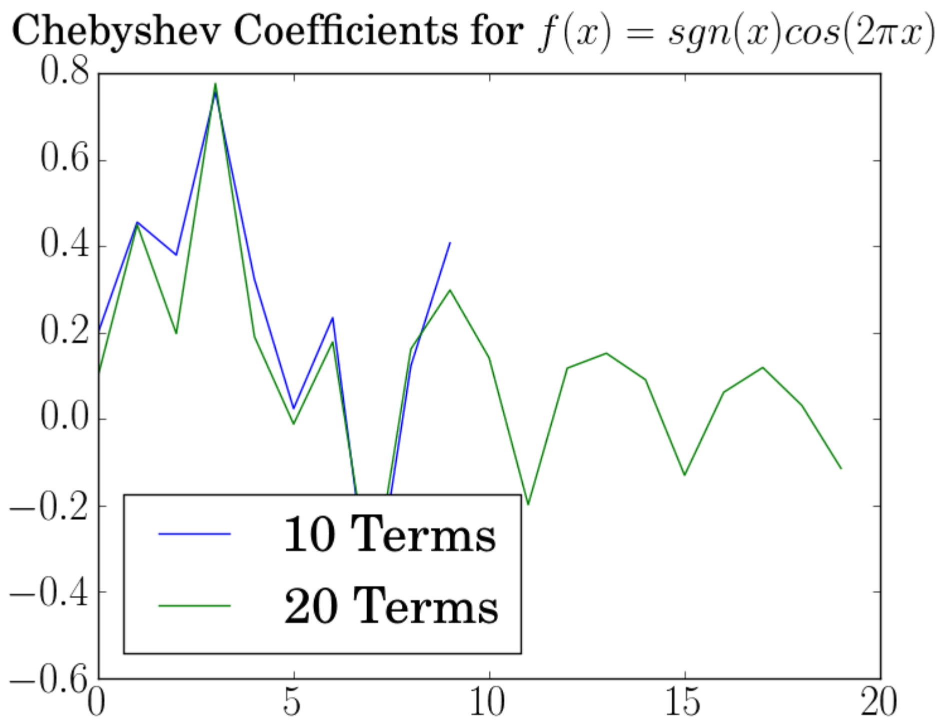


Figure 9: Ringing phenomenon in Chebyshev approximations

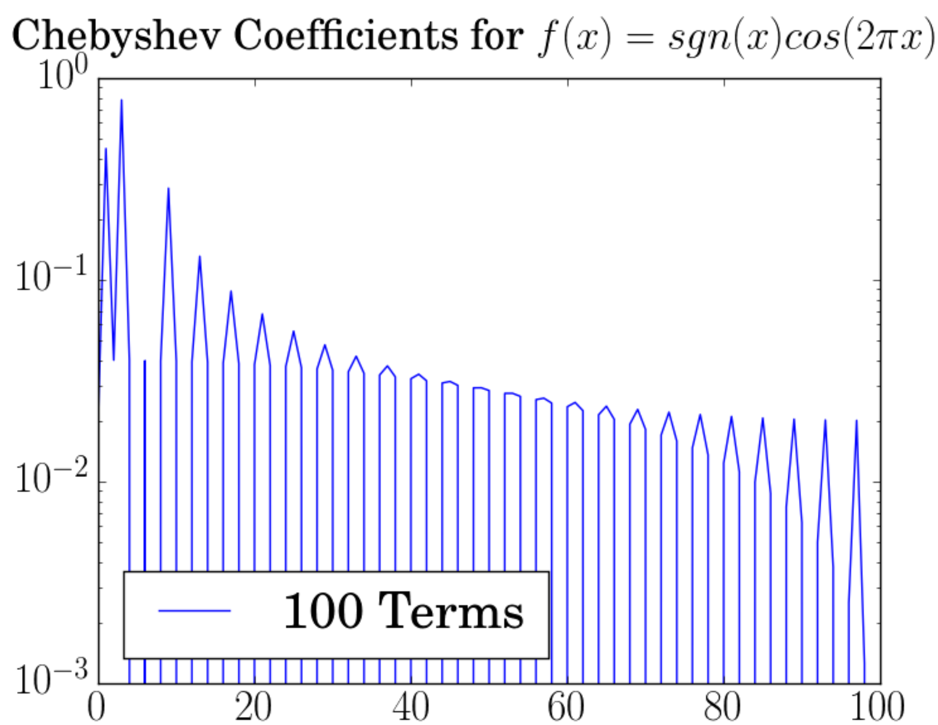


Figure 10: 100 Coefficients displaying a clear power law relationship

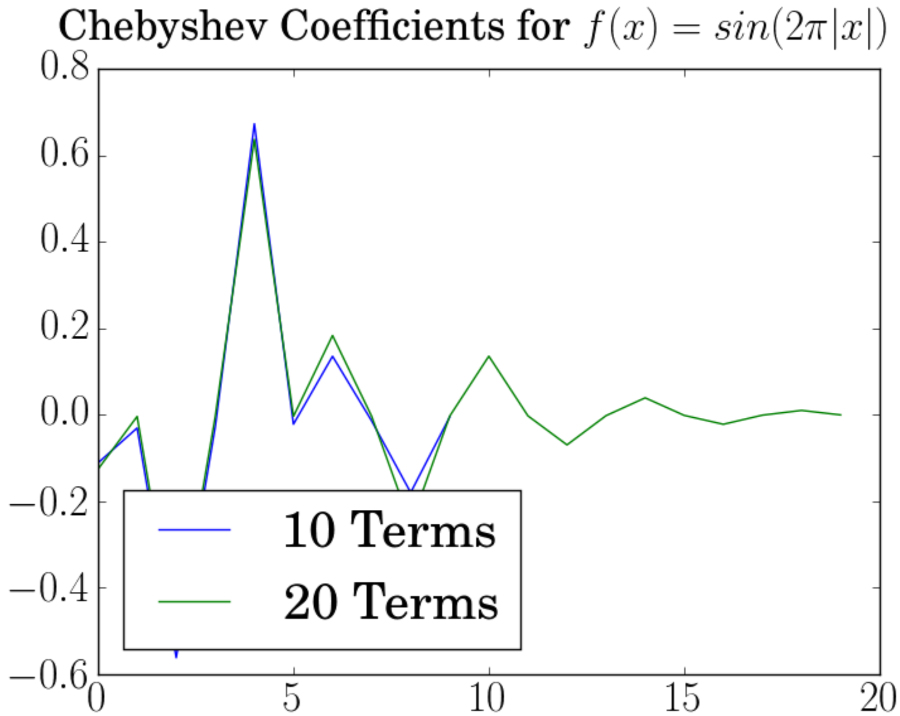


Figure 11: Ringing phenomenon in Chebyshev approximations

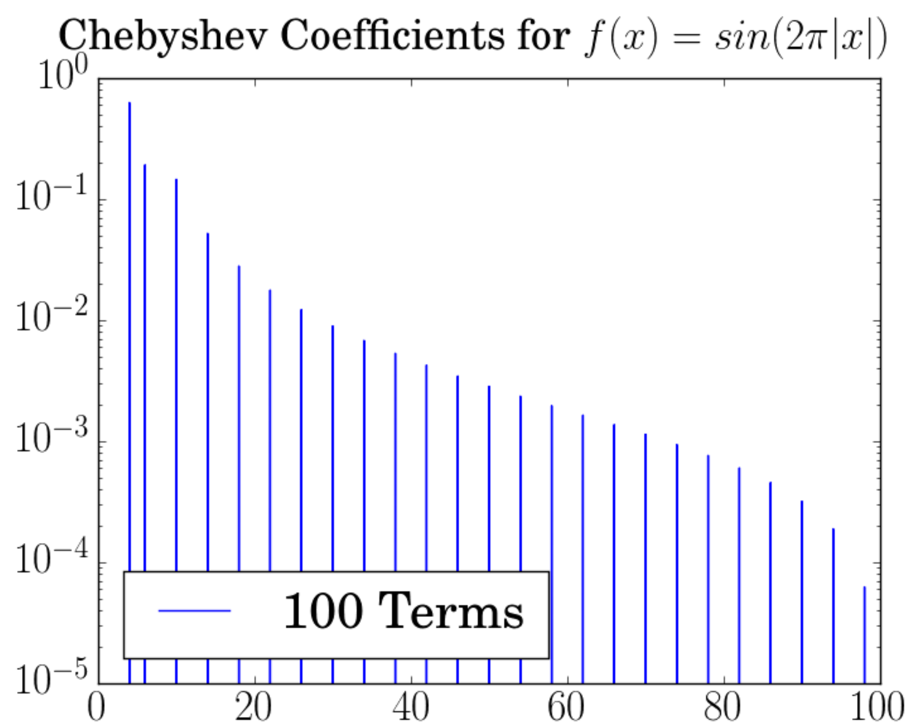


Figure 12: 100 Coefficients displaying a clear power law relationship

7 Chebyshev Calculus

7.1 First Derivatives

$$\begin{aligned}
T'_{10}(x) &= 100x - 1600x^3 + 6720 * x^5 - 10240 * x^7 + 5120 * x^9 \\
T'_{15}(x) &= -15 + 1680 * x^2 - 30240 * x^4 - 633600x^8 + 1013760x^{10} - 798720x^{12} + 245760x^{14} \\
T'_{20}(x) &= -400x + 26400x^3 - 506880x^5 + 4392960x^7 - 20500480x^9 + 55910400x^{11} \\
&\quad - 91750400x^{13} + 89128960x^{15} - 47185920x^{17} + 10485760x^{19}
\end{aligned} \tag{5}$$

7.2 Second Derivatives

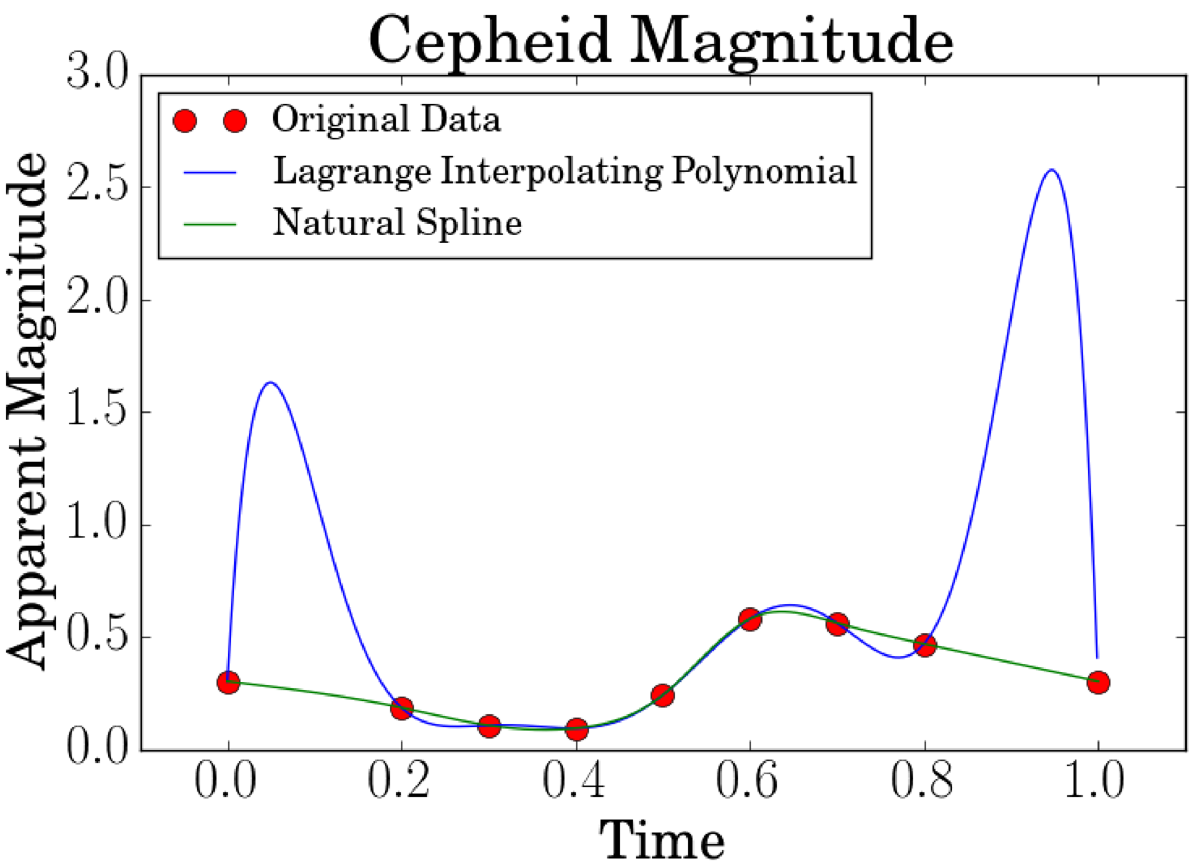
$$\begin{aligned}
T''_{10}(x) &= 100 - 4800x^2 + 33600x^4 - 71680x^6 + 46080x^8 \\
T''_{15}(x) &= 3360x - 120960x^3 + 1209600x^5 - 5068800x^7 + 10137600x^9 - 9584649x^{11} + 3440640x^{13} \\
T''_{20}(x) &= -400 + 79200x^2 + 30750720x^6 - 184504320x^8 + 615014400x^{10} - \\
&\quad 1192755200x^{12} + 1336934400x^{14} - 802160640x^{16} + 199229440x^{18}
\end{aligned} \tag{6}$$

7.3 Integral

$$\begin{aligned}
\int T_{10}(x)dx &= -x + \frac{50}{3} - 80x^5 + 160x^7 - \frac{1280}{9}x^9 + \frac{512}{11}x^{11} \\
\int T_{15}(x)dx &= -\frac{15}{2}x^2 + 140x^4 - 1008x^6 + 3600x^8 - 7040x^{10} + 7680x^{12} - \frac{30720}{7}x^{14} + 1024x^{16} \\
\int T_{20}(x)dx &= \frac{200}{3}x^3 + 1320x^5 - \frac{84480}{7}x^7 + \frac{183040}{3}x^9 - \\
&\quad 186368x^{11} + 358400x^{13} - \frac{1310720}{3}x^{15} + 327680x^{17} - \frac{2621440}{19}x^{19} + \frac{524288}{21}x^{21}
\end{aligned} \tag{7}$$

8 Cephied Lightcurve

The spline fit is much more reasonable for the edge cases compared to the Lagrange polynomial.


Cepheid Magnitude
Original Data
Lagrange Interpolating Polynomial
Natural Spline
Apparent Magnitude
Time
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.2
0.4
0.6
0.8
1.0