Number of question only to be written in this margin

1a Let $u_j^n = \xi^n e^{ikx_j} = \xi^n e^{ikjh}$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$u_j^{n+1} = u_j^{n-1} - v\frac{\tau}{h}(u_{j+1}^n - u_{j-1}^n)$$

$$\Rightarrow \xi^{n+1}e^{ikjh} = \xi^{n-1}e^{ikjh} - v\frac{\tau}{h}\left(\xi^n e^{ik(j+1)h} - \xi^n e^{ik(j-1)h}\right)$$

$$\Rightarrow \xi^2 = 1 - v\frac{\tau}{h}\left(\xi e^{ikh} - \xi e^{-ikh}\right)$$

$$\Rightarrow \xi^2 + \xi v\frac{\tau}{h}\left(e^{ikh} - e^{-ikh}\right) - 1 = 0$$

$$-4v^2\frac{\tau^2}{h^2}\sin^2(kh) + 4$$

$$\Rightarrow \xi^2 + \xi v\frac{\tau}{h}\left(2i\sin(kh)\right) - 1 = 0$$

$$\xi = \frac{-v\frac{\tau}{h}(2i\sin(kh)) \pm \sqrt{\left(v\frac{\tau}{h}(2i\sin(kh))\right)^2 + 4}}{2}$$

$$= -vi\frac{\tau}{h}\sin(kh) \pm \sqrt{1 - v^2\frac{\tau^2}{h^2}\sin^2(kh)}$$

$$|\xi| = \sqrt{v^2\frac{\tau^2}{h^2}\sin^2(kh) \pm 1 - v^2\frac{\tau^2}{h^2}\sin^2(kh)}$$ Let $r = v^2\frac{\tau^2}{h^2}$

complex?

$\cancel{= v\frac{\tau^2}{h^2}\sin^2(kh)}$

So

$|\xi| = 1$ if $v^2\frac{\tau^2}{h^2} < 1$

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$$\Rightarrow v\frac{\tau}{h} < 1 \quad \Rightarrow \frac{\tau}{h} < \frac{1}{V}$$

so $C = \frac{1}{|V|}$, C is the caurant number.

1b $T_j^n = \frac{1}{\tau}\left(u(x_j, t_{n+1}) - u(x_j, t_{n-1})\right) + v\frac{1}{h}\left(u(x_{j+1}, t_n) - u(x_{j-1}, t_n)\right)$

$$u(x_j, t_{n+1}) = u + \tau u_t + \tfrac{1}{2}\tau^2 u_{tt} + O(\tau^3)$$
$$u(x_j, t_{n-1}) = u - \tau u_t + \tfrac{1}{2}\tau^2 u_{tt} + O(\tau^3)$$
$$u(x_{j+1}, t_n) = u + h u_x + \tfrac{1}{2}h^2 u_{xx} + O(h^3)$$
$$u(x_{j+1}, t_n) = u - h u_x + \tfrac{1}{2}h^2 u_{xx} + O(h^3)$$

$\frac{1}{6}\tau^3 u_{ttt}$

$$T_j^n = \frac{1}{\tau}\left[u + \tau u_t + \tfrac{1}{2}\tau^2 u_{tt} + \dots - u + \tau u_t - \tfrac{1}{2}\tau^2 u_{tt} + \dots\right]$$
$$+ v\frac{1}{h}\left[u + h u_x + \tfrac{1}{2}h^2 u_{xx} + \dots - u + h u_x - \tfrac{1}{2}h^2 u_{xx} + \dots\right]$$

$$= 2u_t + \frac{1}{3}\tau^2 u_{ttt} + \dots + 2v u_x + \frac{1}{3}h^2 u_{xxx} + \dots$$

pde: $u_t + v u_x = 0$

$$\Rightarrow T_j^n = \frac{1}{3}\tau^2 u_{ttt} + \frac{1}{3}h^2 u_{xxx} + \dots$$

8 Hence the truncation error has order $O(h^2, \tau^2)$ second-order in space and time.

1c since we only have initial condition u_0, we cannot perform the leapfrog scheme on the first timestep. as we don't have $n-1$. We ~~could~~ use ~~FTCS~~ centred differencing, for example, on the first and ~~use~~ ~~last~~ and use leapfrog everywhere else.

0 E.g.

$n=1$: ~~$u_j^1 = \frac{u_j^2 - u^1}{2}$~~ $u_j^2 = u_{j+1}^1 - 2u_j^1 + u_{j-1}^1$?

$n=2$: u_2

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2a
$$\frac{u_j^{n+1} - u_j^n}{\tau} = \frac{1}{2h^2}\left[D_{j+\frac12}(u_{j+1}^{n+1} - u_j^{n+1}) - D_{j-\frac12}(u_j^{n+1} - u_{j-1}^{n+1})\right] \quad (*)$$

$$+ \frac{1}{2h^2}\left[D_{j+\frac12}(u_{j+1}^n - u_j^n) - D_{j-\frac12}(u_j^n - u_{j-1}^n)\right]$$

4 $h = \frac{1}{N+1}$, with N unknowns.
The main advantage is that the Crank-nicohlson scheme is unconditionally stable. ✓

2b. Let $\alpha = \frac{\tau}{2h^2}$. Then $(*) \Rightarrow$

$$-\alpha D_{j-\frac12} u_{j-1}^{n+1} + (1 + \alpha(D_{j+\frac12} + D_{j-\frac12}))u_j^{n+1} - \alpha D_{j+\frac12} u_{j+1}^{n+1}$$

$$= \alpha D_{j-\frac12} U_{j-1}^n + (1 - \alpha(D_{j+\frac12} + D_{j-\frac12}))u_j^n + \alpha D_{j+\frac12} u_{j+1}^n$$

~~At $x=0, j=0$ $\partial u = \beta(t)$.~~

~~introduce u_{j-1} term~~

~~$\Rightarrow -\alpha D_{-\frac12}$~~

~~$u_{-1} = \frac{u_1 - u_0}{2}$~~ ~~$u_0 = \frac{u_1 - u_{-1}}{2h}$~~

~~$\Rightarrow u_1 = u_1 - 2u_0$~~

So $a_j = \alpha D_{j-\frac12}$ ✓ $b_j = 1 + \alpha(D_{j+\frac12} + D_{j-\frac12})$ ✓

~~$-\alpha D_{-\frac12} u_{-1}^{n+1} + (1$~~

$c_j = \alpha D_{j+\frac12}$ ✓

4 $d_j = \alpha D_{j-\frac12} U_{j-1}^n + (1 - \alpha(D_{j+\frac12} + D_{j-\frac12}))U_j^n + \alpha D_{j+\frac12} U_{j+1}^n$ ✓

$1 - 2\beta\alpha$

$x=0, j=0$

introduce U_{-1} term? $u^1 = u^{-1} - \alpha(u^0_{j+1} - u^0_{j-1})$

$U_0 = \frac{1}{h}(U_1 - U_{-1})$ $u^{-1} = u^1$

$hU_0 - U_1 = U_{-1}$

~~$d_0 = \alpha(D_{j-\frac12} - \alpha)U^0_{j-1}$~~

~~$a_0 = 0$, $b_0 = (1 - \alpha(D_{\frac12} + D_{-\frac12}))U_0^0 + u_1^0 \alpha D_{j-\frac12}$~~

$c_0 = \alpha D_{\frac12}$ ~~$d_0 = \alpha D_{\frac12} U^0_{j+1}$~~

Number of question only to be written in this margin

2c $u_{k-1}^{n+1} - e_{k-1} u_k^{n+1} = f_{k-1} \Rightarrow u_{k-1}^{n+1} = e_{k-1} u_k^{n+1} + f_{k-1}.$

$$-a_k u_{k-1}^{n+1} + b_k u_k^{n+1} - c_k u_{k+1}^{n+1} = d_k.$$

$$\Rightarrow -a_k(e_{k-1} u_k^{n+1} + f_{k-1}) + b_k u_k^{n+1} - c_k u_{k+1}^{n+1} = d_k.$$

$$\Rightarrow (b_k - a_k e_{k-1}) u_k^{n+1} - c_k u_{k+1}^{n+1} = d_k + a_k f_{k-1}$$

$$\Rightarrow u_k^{n+1} - \frac{c_k}{b_k - a_k e_{k-1}} u_{k+1}^{n+1} = \frac{d_k + a_k f_{k-1}}{b_k - a_k e_{k-1}}$$

Hence $e_k = \dfrac{c_k}{b_k - a_k e_{k-1}}$ and $f_k = \dfrac{d_k + a_k f_{k-1}}{b_k - a_k e_{k-1}}.$

6 $e_0 = \dfrac{c_0}{b_0 - a_0 e_{-1}}$ $f_0 = \dfrac{d_0 + a_0 f_{-1}}{b_0 - a_0 e_{-1}}$

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Number of question only to be written in this margin

3a Start with centred differencing on points

~~$x_{i,j}$~~

~~$u_{i,j}$ at $x_{i,j}$ at~~

$$x_i = ih_x \;,\; y_j = jh_y \;,\; h_x = \frac{1}{N_x} \;,\; h_y = \frac{1}{N_y}$$

$$u_{i,j} = u(x_i, y_j)$$

where N_x, N_y are the number of points in x and y respectively.

Then $u(x_i, y_j)$ can be approximated by

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} = -k^2 u_{i,j}$$

This forms a block linear system

$$A\underline{x} = \underline{b}$$

with

$$A = \begin{bmatrix} [A_{11}] & [A_{12}] & & \\ [A_{21}] & [A_{22}] & [A_{23}] & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \end{bmatrix}$$

with diagonal blocks

$$A_{i,i} = \begin{bmatrix} -\frac{2}{h_x^2} - \frac{2}{h_y^2} + \tilde{k} & \frac{1}{h_x^2} & 0 \\ \frac{1}{h_x^2} & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix} \quad \text{with } N_x \times N_x$$

and off-diagonal blocks

$$A_{i-1,i} = A_{i+1,i} = \begin{bmatrix} \frac{1}{h_y^2} & 0 \\ 0 & \ddots \end{bmatrix} \quad \text{with } N_y \text{ such rows of this block.}$$

Number of question only to be written in this margin

3a $\underline{b} = -k^2 \cancel{u(x,y)}$ + boundary conditions

Here we have boundary conditions $u = 0$, so

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$\underline{b} = -k^2 u(x,y)$ with $u(x,y)$ having entries $u(x_i, y_i) = U_{i+jN_x}$.

3b First we decompose A into D - diagonal, $-L$ - low triangular and $-U$ - upper triangular.

$A = D - U - L$ $\quad (D-L)^{-1}$ $\quad (D-L)\underline{x}_{n+1} - U\underline{x}_n$

1 for jacobi ~~matrix~~ method we form an iterative sequence

$$\underline{x}_{n+1} = [D]^{-1} \cancel{\ldots} ((L+U)\underline{x}_n) + \underline{b}$$

hence $G = (-U x_n [D]^{-1}$ $\quad c = L[D]^{-1}$.

when $\underline{x}_{n+1} = x_n$ then $A\underline{x}_n = \underline{b}$

D^{-1} is triviall

3c $\rho(G) = \underline{m} \; \max_i |\lambda_i(G)|$

1 Assume G has a full set of eigenvectors then

$$G = S \Lambda S^{-1}$$

by ~~divergence theorem~~

~~$\underline{x}_{n+1} = [D]^{-1}(U x_n - L)$~~

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Number of question only to be written in this margin

5a ~~Let~~ for local basis functions we have

$$u(x) = \sum_{i=0}^{N} u_i \phi_i(x)$$

with ϕ_i being local basis functions

we have

$$\phi_i = \begin{cases} \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \\[2ex] \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\[2ex] 0, & \text{otherwise} \end{cases}$$

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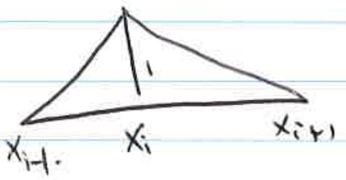
with $x \in [a,b]$ we have $[a = x_0, \dots, x_{N+1} = b]$.



5b For weak form multiply by ϕ_i and itegrate w.r.t Ω

$$\int_\Omega \phi \, \nabla^2 u \, dA = \int f(x,y) \, dA$$

2

integration by parts.

$$\phi \int_\Omega \nabla^2 u \, dA - \int \phi' \left(\int \nabla^2 u \, dA \right) dA$$

=> by

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(4)