CSI 445/660 – Part 4
(Positive and Negative Relationships)

Ref: Chapter 5 of [Easley & Kleinberg].

Positive and Negative Relationships

- So far: Edges in a network represent **friendship** information (positive relationships).
- We also need to consider **conflicts** (negative relationships).
- The combination leads to the notion of **structural balance**.
- Provides another illustration of how **local** structure (i.e., a property involving a few nodes at a time) may have a **global** effect.

Model:

- The underlying graph is a **clique**; that is, each person has a positive or negative relationship with every other person. (General graphs will be considered later.)
- Each edge has a **label**: '$+$' (indicating a positive relationship) or '-' (indicating a negative relationship).
- A common model for studying international conflicts.

Model (continued):

- Ideas developed (in the sociological context) by Fritz Heider.

- Fritz Heider (1896–1988)
- Austrian Sociologist.
- Taught at the University of Kansas for many years.

- The mathematical development is due to Dorwin Cartwright and Frank Harary.

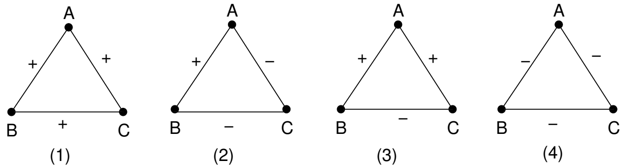
Positive and Negative ... (continued)

- Dorwin Cartwright (1915–2008)
- Areas: Psychology and Mathematics.
- One of the founders of Group Dynamics.
- University of Michigan, Ann Arbor, MI.

- Frank Harary (1921–2005)
- Mathematician who specialized in Graph Theory and its Applications.
- University of Michigan, Ann Arbor, MI and later New Mexico State University, Las Cruces, NM.

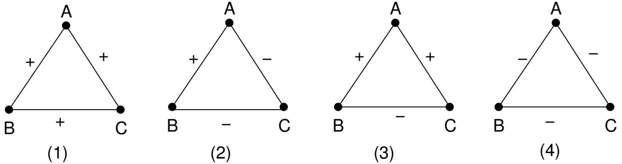
Structural Balance

Possible Edge Labelings for Three People:



- Labelings (1) and (2) have an **odd** number of '+' labels.
- Labelings (3) and (4) have an **even** number of '+' labels.
- Labeling (1): Three mutual friends; causes no problem.
- Labeling (2): Two friends and they both dislike the third; causes no problem.
- So, Labelings (1) and (2) have **structural balance**.

Structural Balance (continued)



- Labeling (3): A has two friends who don't like each other. This may be a source of "stress" for A. (It may cause A to lose the friendship with B or C.)

  Note: Recall (from the slides for Part 2) the study by Bearman & Moody [2004] involving the health records of teenage girls.

- Labeling (4): Here, two of the people may "team up" against the third person (i.e., there may be forces to change the label of one of the edges to '$+$').

- So, Labelings (3) and (4) have **structural imbalance**.

Structural Balance (continued)

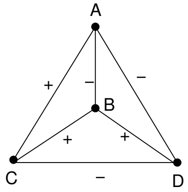
Balance condition for Three People:

- A labeled triangle is **balanced** if and only if the number of '$+$' labels is **odd**.

Extension – Structural Balance for Cliques:

- A labeled clique is **balanced** if and only if each of its triangles is balanced (i.e., in each triangle, the number of '$+$' labels is **odd**).

Example:



- 4-clique.
- Not balanced.
- Triangle BCD has two edges labeled '$+$' (and so does triangle ABC).

Structural Balance (continued)

Testing the Structural Balance – An Easy Algorithm:

Input: A clique G with n nodes where each edge has a '$+$' or '-' label.

Output: "Yes" if G is balanced and "No" otherwise.

Outline of the Algorithm:

1. **for** each triple of nodes x, y and z **do**
   if (triangle $\{x, y, z\}$ is **not** balanced)
   Output "No" and **stop**.
2. Output "Yes".

Running time: $O(n^3)$ (since there are $\binom{n}{3} = O(n^3)$ triangles in a clique with n nodes).

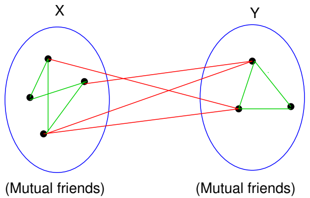
Characterizing Structural Balance

Note: The following trivial cases are ignored in the discussion.

- All edges of G are labeled '$+$': G is balanced.
- All edges of G are labeled '-': G is unbalanced.

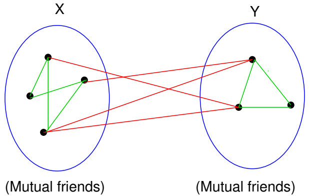
Idea of Battling Factions:

- Suppose we can partition the nodes of G into two sets X and Y such that the following conditions hold:
  - All edges inside X or inside Y are labeled '$+$' and
  - all edges that join a node in X to a node in Y are labeled '-'.



- Not all edges are shown.
- Each **green** edge has the label '$+$' and each **red** edge has the label '-'.

Characterizing Structural Balance (continued)



- Not all edges are shown.

- X and Y are called **battling factions**.
- In this structure, every triangle is balanced.
- **Key idea:** In any balanced clique, such a structure exists.

Terminology:

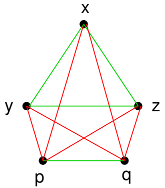
- **Internal edge:** An edge that joins two nodes in X or two nodes in Y.
- **External edge:** An edge that joins a node in X to a node in Y.

Characterizing Structural Balance (continued)

Theorem: **[Cartwright & Harary]**
If a labeled complete graph G is balanced, then

- either all the edge labels in G are '$+$' or
- the nodes of G can be partitioned into two sets X and Y such that
  1. each internal edge is labeled '$+$' and
  2. each external edge is labeled '-'.

Example:



- This 5-clique is balanced.
- Partition: $X = \{x, y, z\}$ and $Y = \{p, q\}$.

Proof Sketch for the Cartwright-Harary Theorem

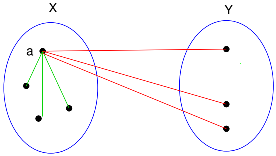
Notes:

- We are ignoring the (trivial) case where all edge labels are '+'.
- So, assume that at least one edge has the label '-'.
- The proof actually constructs the **battling factions** partition.

Construction:

- Choose any node a of G.
- Let the set X consist of a and all the nodes which are **friends** of a.
- Let Y be the remaining set of nodes (i.e., the **enemies** of a).

An Illustration:

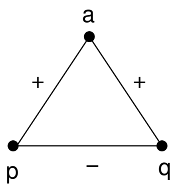


- Not all nodes/edges are shown.

Proof Sketch ... (continued)

Part 1: We must show that each internal edge in X has the label '$+$'.

- Consider any two nodes p and q in X.
- If one of p and q is the node a, the conclusion follows since all nodes in X are friends of a.
- So, assume that p and q are different from a.
- If p and q are enemies, we get the following **unbalanced** triangle in G:

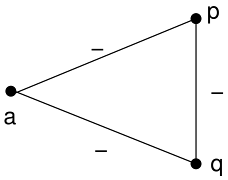


- This contradicts the assumption that G is balanced.

Proof Sketch ... (continued)

Part 2: We must show that each internal edge in Y has the label '$+$'.

- Consider any two nodes p and q in Y.
- If p and q are enemies, we get the following **unbalanced** triangle in G:

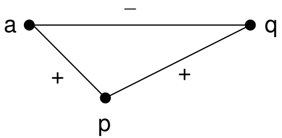


- A contradiction.

Proof Sketch ... (continued)

Part 3: We must show that each external edge has the label '-'.

- Consider any two nodes $p \in X$ and $q \in Y$.
- If p and q are friends, we get the following **unbalanced** triangle in G:



- A contradiction.

This completes the proof of the theorem.

Note: The theorem leads to an $O(n^2)$ algorithm for the problem of testing whether an edge labeled clique is balanced.