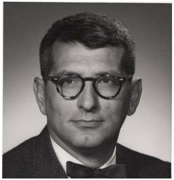
# CSI 445/660 – Part 6
# (Centrality Measures for Networks)

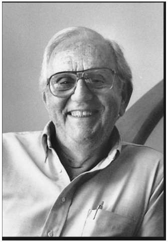
# References


1. L. Freeman, "Centrality in Social Networks: Conceptual Clarification", *Social Networks*, Vol. 1, 1978/1979, pp. 215–239.
2. S. Wasserman and K. Faust, *Social Network Analysis: Methods and Applications*, Cambridge University Press, New York, NY, 1994.
3. M. E. J. Newman, *Networks: An Introduction*, Oxford University Press, New York, NY, 2010.
4. Wikipedia entry on Centrality Measures:
   `https://en.wikipedia.org/wiki/Centrality`

# Some Pioneers on the Topic



- Alex Bavelas (1913–1993) **(??)**
- Received Ph.D. from MIT (1948) in Psychology.
- Dorwin Cartwright was a member of his Ph.D. thesis committee.
- Taught at MIT, Stanford and the University of Victoria (Canada).



- Harold Leavitt (1922–2007)
- Received Ph.D. from MIT.
- Authored an influential text ("Managerial Psychology") in 1958.
- Taught at Carnegie Mellon and Stanford.

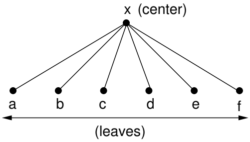
# Centrality Measures for Networks

**Centrality:**

- Represents a "measure of importance".
  - Usually for nodes.
  - Some measures can also be defined for edges (or subgraphs, in general).
- Idea proposed by Alex Bavelas during the late 1940's.
- Further work by Harold Leavitt (Stanford) and Sidney Smith (MIT) led to **qualitative** measures.
- **Quantitative** measures came years later. (Many such measures have been proposed.)

# Point Centrality – A Qualitative Measure

**Example:**




- The **center** node is "structurally more important" than the other nodes.

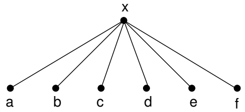
**Reasons for the importance of the center node:**

- The center node has the maximum possible degree.
- It lies on the shortest path ("geodesic") between any pair of other nodes (leaves).
- It is the closest node to each leaf.
- It is in the "thick of things" with respect to any communication in the network.

# Degree Centrality – A Quantitative Measure

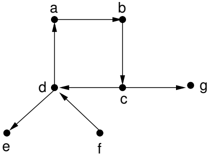
- For an **undirected** graph, the **degree** of a node is the number of edges incident on that node.
- For a **directed** graph, both **indegree** (i.e., the number of incoming edges) and **outdegree** (i.e., the number of outgoing edges) must be considered.

**Example 1:**



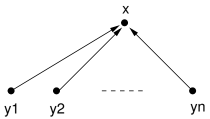

- Degree of $x = 6$.
- For all other nodes, degree $= 1$.

**Example 2:**




- Indegree of $b = 1$.
- Outdegree of $d = 2$.

**When does a large indegree imply higher importance?**

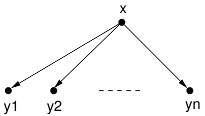



- Consider the Twitter network.
- Think of $x$ as a **celebrity** and the other nodes as followers of $x$.

- For a different context, think of each node in the directed graph as a web page.
- Each of the nodes $y_1$, $y_2$, ..., $y_n$ has a link to $x$.
- The larger the value of $n$, the higher is the "importance" of $x$ (a crude definition of **page rank**).

# Degree Centrality (continued)

**When does a large outdegree imply higher importance?**




- Consider the hierarchy in an organization.
- Think of $x$ as the manager of $y_1$, $y_2$, ..., $y_n$.
- Large outdegree may mean more "power".

**Undirected graphs:**

- High degree nodes are called **hubs** (e.g. airlines).
- High degree may also also represent higher **risk**.

  **Example:** In disease propagation, a high degree node is more likely to get infected compared to a low degree node.

# Normalized Degree

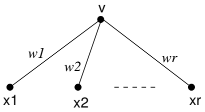
**Definition:** The **normalized degree** of a node $x$ is given by

$$\text{Normalized Degree of } \; x \; = \; \frac{\text{Degree of } \; x}{\text{Maximum possible degree}}$$

- Useful in comparing degree centralities of nodes between two networks.

**Example:** A node with a degree of 5 in a network with 10 nodes may be relatively more important than a node with a degree of 5 in a network with a million nodes.

**Weighted Degree Centrality (Strength):**




- Weighted degree (or strength) of $v = w_1 + w_2 + \ldots + w_r$.

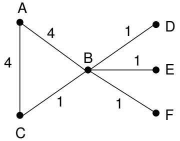
# Degree Centrality (continued)

Assuming an **adjacency list** representation

- for an undirected graph $G(V, E)$, the degree (or weighted degree) of all nodes can be computed in **linear** time (i.e., in time $O(|V| + |E|)$) and
- for a directed graph $G(V, E)$, the indegree or outdegree (or their weighted versions) of all nodes can be computed in **linear** time.

**Combining degree and strength:** (**[Opsahl et al. 2009]**)

**Motivating Example:**




- A and B have the same strength.
- However, B seems more central than A.

# Combining Degree and Strength (continued)

**Proposed Measure by Opsahl et al.:**

- Let $d$ and $s$ be the degree and strength of a node $v$ respectively.
- Let $\alpha$ be a parameter satisfying the condition $0 \leq \alpha \leq 1$.
- The combined measure for node $v = d^{\alpha} \times s^{1-\alpha}$.
- When $\alpha = 1$, the combined measure is the **degree**.
- When $\alpha = 0$, the combined measure is the **strength**.
- A suitable value of $\alpha$ must be chosen for each context.

# Farness and Closeness Centralities

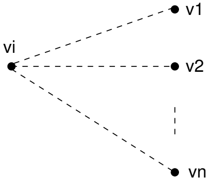
**Assumptions:**

- Undirected graphs. (Extension to directed graphs is straightforward.)
- Connected graphs.
- No edge weights. (Extension to weighted graphs is also straightforward.)

**Notation:**

- Nodes of the graph are denoted by $v_1$, $v_2$, ..., $v_n$.
- For any pair of nodes $v_i$ and $v_j$, $d_{ij}$ denotes the number of edges in a shortest path between $v_i$ and $v_j$.

# Farness and Closeness Centralities (continued)




- A schematic showing shortest paths between node $v_i$ and the other nodes of an undirected graph.

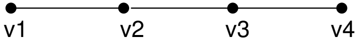
**Definition:** The **farness centrality** $f_i$ of node $v_i$ is given by

$$
\begin{aligned}
f_i &= \text{Sum of the distances between } v_i \text{ and the other nodes} \\
&= \sum_{v_j \in V - \{v_i\}} d_{ij}
\end{aligned}
$$

**Definition:** The **closeness centrality** (or **nearness centrality**) $\eta_i$ of node $v_i$ is given by $\eta_i = 1/f_i$.

**Note:** If a node $x$ has a larger closeness centrality value compared to a node $y$, then $x$ is more central than $y$.

**Example 1:**

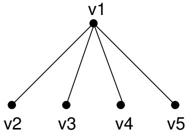



- $f_1 = 1 + 2 + 3 = 6$. So, $\eta_1 = 1/6$.
- $f_2 = 1 + 1 + 2 = 4$. So, $\eta_2 = 1/4$.
- $f_3 = 2 + 1 + 2 = 4$. So, $\eta_3 = 1/4$.
- $f_4 = 3 + 2 + 1 = 6$. So, $\eta_4 = 1/6$.

So, in the above example, nodes $v_2$ and $v_3$ are more central than nodes $v_1$ and $v_4$.

# Farness and Closeness Centralities (continued)

**Example 2:**




- $f_1 = 4$. So, $\eta_1 = 1/4$.
- For every other node, the farness centrality value $= 7$; so the closeness centrality value $= 1/7$.
- Thus, $v_1$ is more central than the other nodes.

**Remarks:**

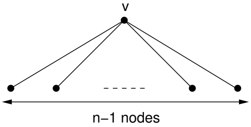
- For any graph with $n$ nodes, the **farness centrality** of each node is **at least** $n - 1$.

  **Reason:** Each of the other $n - 1$ nodes must be at a distance of at least 1.

# Farness and Closeness Centralities (continued)

**Remarks (continued):**

- Since the farness centrality of each node is at least $n-1$, the **closeness centrality** of any node must be **at most** $1/(n-1)$.




- For the star graph on the left, the closeness centrality of the center node $v$ is exactly $1/(n-1)$.

- If $G$ is an $n$-clique, then the closeness centrality of each node of $G$ is $1/(n-1)$.

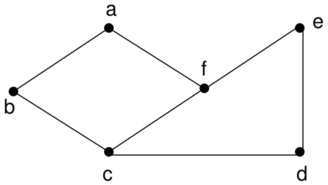
# An Algorithm for Computing Farness and Closeness

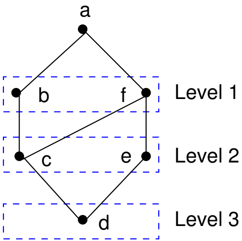
**Assumptions:** The given undirected graph is **connected** and does **not** have edge weights.

**Computing Farness (or closeness) Centrality (Idea):**

- A Breadth-First-Search (BFS) starting at a node $v_i$ will find shortest paths to all the other nodes.

**Example:**



# An Algorithm for Farness ... (continued)

Let $G(V, E)$ denote the given graph.

- Recall that the time for doing a BFS on $G = O(|V| + |E|)$.
- So, farness (or closeness) centrality for any node of $G$ can be computed in $O(|V| + |E|)$ time.
- By carrying out a BFS from each node, the time to compute farness (or closeness) centrality for **all** nodes of $G$ $= O(|V|(|V| + |E|))$.
- The time is $O(|V|^3)$ for **dense** graphs (where $|E| = \Omega(|V|^2)$) and $O(|V|^2)$ for **sparse** graphs (where $|E| = O(|V|)$).

# Eccentricity Measure

- Recall that **farness centrality** of a node $v_i$ is given by

$$f_i \;=\; \sum_{v_j \in V - \{v_i\}} d_{ij}$$

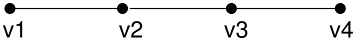
- The **eccentricity** $\mu_i$ of node $v_i$ is defined by replacing the **summation** operator $\left(\sum\right)$ by the **maximization** operator; that is,

$$\mu_i \;=\; \max_{v_j \in V - \{v_i\}} \{d_{ij}\}$$

- This measure was studied by two graph theorists (Gert Sabidussi and Seifollah L. Hakimi).
- **Interpretation:** If $\mu_i$ denotes the eccentricity of node $v_i$, then every other node is within a distance of **at most** $\mu_i$ from $v_i$.
- If the eccentricity of node $x$ is less than that of $y$, then $x$ is more central than $y$.

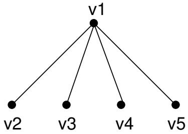
# Examples: Eccentricity Computation

**Example 1:**




- $\mu_1 = \max\{1, 2, 3\} = 3.$
- $\mu_2 = \max\{1, 1, 2\} = 2.$
- $\mu_3 = \max\{2, 1, 1\} = 2.$
- $\mu_4 = \max\{3, 2, 1\} = 3.$

**Example 2:**

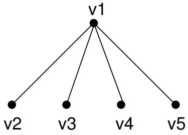



- $\mu_1 = 1.$
- For every other node, eccentricity $= 2$.

# Eccentricity – Additional Definitions

**Definition:** A node $v$ of a graph which has the smallest eccentricity among all the nodes is called a **center** of the graph.

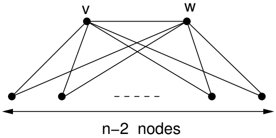
**Example:**




- The center of this graph is $v_1$.
  (The eccentricity of $v_1 = 1$.)

**Note:** A graph may have two or more centers.

**Example:**



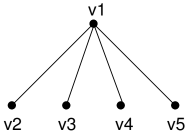

- Both $v$ and $w$ are centers of this graph.
  (Their eccentricities are $= 1$.)
- If $G$ is clique on $n$ nodes, then every node of $G$ is a center.

# Eccentricity – Additional Definitions (continued)

**Definition:** The smallest eccentricity value among all the nodes is called the **radius** of the graph.

**Note:** The value of the radius is the eccentricity of a center.

**Example:**




- The radius of this graph is 1 (since $v_1$ is the center of this graph and the eccentricity of $v_1 = 1$.)

**Facts:**

- The **largest eccentricity value** is the **diameter** of the graph.
- For any graph, the diameter is at most twice the radius. **(Students should try to prove this result.)**

# An Algorithm for Computing Eccentricity

Let $G(V, E)$ denote the given graph.

- **Recall:** By carrying out a BFS from node $v_i$, the shortest path distances between $v_i$ and all the other nodes can be found in $O(|V| + |E|)$ time.
- So, the eccentricity of any node of $G$ can be computed in $O(|V| + |E|)$ time.
- By repeating the BFS for each node, the time to compute eccentricity for **all** nodes of $G = O(|V|(|V| + |E|))$.
- So, the radius, diameter and all centers of $G$ can be found in $O(|V|(|V| + |E|))$ time.

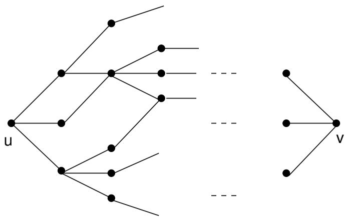
# Random Walk Based Centrality (Brief Discussion)

**Ref:** **[Noh & Rieger 2004]**

**Motivation:**

- Definitions of centrality measures (such as **closeness** centrality) assume that "information" propagates along shortest paths.
- This may not be appropriate for certain other types of propagation. For example, propagation of diseases is a **probabilistic** phenomenon.

**Idea of Random Walk Distance in a Graph:**



# Random Walk ... (Brief Discussion)

**Random Walk Algorithm – Outline:**

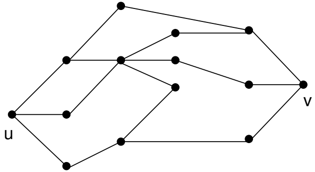
- Suppose we want to find the random walk distance from $u$ to $v$.
- **Initialize:** Current Node $= u$ and No. of steps $= 0$.
- **Repeat**
  1. Randomly choose a neighbor $x$ of the Current Node.
  2. No. of steps $=$ No. of steps $+ 1$.
  3. Set Current Node $= x$.

  **Until** Current Node $= v$.

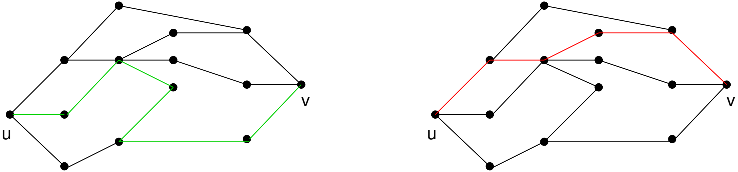
**Note:** In Step 1 of the loop, if the Current Node has degree $d$, probability of choosing any neighbor is $1/d$.

# Examples of Random Walks

**A graph for carrying out a random walk:**




**Examples of random walks on the above graph:**



# Random Walk ... (Brief Discussion)

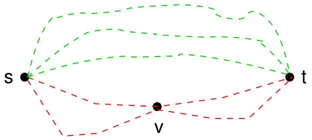
**Definition:** The **random walk distance** (or **hitting time**) from $u$ to $v$ is the expected number of steps used in a random walk that starts at $u$ and ends at $v$.

- One can define farness/closeness centrality measures based on random walk distances.
- **Weakness:** Even for undirected graphs, the random walk distances are **not symmetric**; that is, the random walk distance from $u$ to $v$ may **not** be the same as the random walk distance from $v$ to $u$.

# Betweenness Centrality (for Nodes)

- Measures the importance of a node using the **number of shortest paths** in which the node appears.
- Suggested by Bavelas; however, he didn't formalize it.
- The measure was developed by Linton Freeman and J. M. Anthonisse.

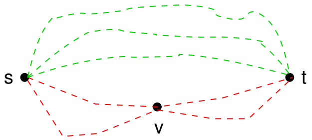
Consider a node $v$ and two other nodes $s$ and $t$.




- Each shortest path between $s$ and $t$ shown in **green doesn't** pass through node $v$.
- Each shortest path between $s$ and $t$ shown in **red** passes through node $v$.

# Betweenness Centrality ... (continued)

**Notation:** Any shortest path between nodes $s$ and $t$ will be called an $s$-$t$ **shortest path**.




- Let $\sigma_{st}$ denote the number of all $s$-$t$ shortest paths.
- Let $\sigma_{st}(v)$ denote the number of all $s$-$t$ shortest paths that pass through node $v$.

Consider the ratio $\dfrac{\sigma_{st}(v)}{\sigma_{st}}$ :

- This gives the fraction of $s$-$t$ shortest paths passing through $v$.
- The larger the ratio, the more important $v$ is with respect to the pair of nodes $s$ and $t$.
- To properly measure the importance of a node $v$, we need to consider all pairs of nodes (not involving $v$).

## Betweenness Centrality ... (continued)

**Definition:** The **betweenness centrality** of a node $v$, denoted by $\beta(v)$, is defined by

$$\beta(v) \;=\; \sum_{\substack{s\,,\,t \\ s \neq v\,,\,t \neq v}} \left[\frac{\sigma_{st}(v)}{\sigma_{st}}\right]$$

**Interpreting the above formula:** Suppose we want to compute $\beta(v)$ for some node $v$. The formula suggests the following steps.

- Set $\beta(v) = 0$.
- For each pair of nodes $s$ and $t$ such that $s \neq v$ and $t \neq v$,
  1. Compute $\sigma_{st}$ and $\sigma_{st}(v)$.
  2. Set $\beta(v) \;=\; \beta(v) + \sigma_{st}(v)/\sigma_{st}$.
- Output $\beta(v)$.

**Note:** For two nodes $x$ and $y$, if $\beta(x) > \beta(y)$, then $x$ is more central than $y$.

# Examples: Betweenness Computation

**Example 1:**

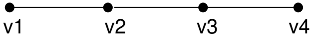

**Note:** Here, there is **only one** path between any pair of nodes. (So, that path is also the shortest path.)

Consider the computation of $\beta(v_2)$ first.

- The $s$-$t$ pairs to be considered are: $(v_1, v_3)$, $(v_1, v_4)$ and $(v_3, v_4)$.
- For the pair $(v_1, v_3)$:
  - The number of shortest paths between $v_1$ and $v_3$ is 1; thus, $\sigma_{v_1,v_3} = 1$.
  - The (only) path between $v_1$ and $v_3$ passes through $v_2$; thus, $\sigma_{v_1,v_3}(v_2) = 1$.
  - So, the ratio $\sigma_{v_1,v_3}(v_2)/\sigma_{v_1,v_3} \ = \ 1$.
- In a similar manner, for the pair $(v_1, v_4)$, the ratio $\sigma_{v_1,v_4}(v_2)/\sigma_{v_1,v_4} \ = \ 1$.

# Examples: Betweenness Computation (continued)

**Computation of $\beta(v_2)$ continued:**




- For the pair $(v_3, v_4)$:
  - The number of shortest paths between $v_3$ and $v_4$ is 1; thus, $\sigma_{v_3,v_4} = 1$.
  - The (only) path between $v_3$ and $v_4$ **does not** pass through $v_2$; thus, $\sigma_{v_3,v_4}(v_2) = 0$.
  - So, the ratio $\sigma_{v_3,v_4}(v_2)/\sigma_{v_3,v_4} = 0$.

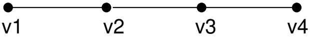
Therefore,

$$\begin{aligned} \beta(v_2) &= \quad 1 \quad \text{(for the pair } (v_1, v_3)) \\ &\quad + 1 \quad \text{(for the pair } (v_1, v_4)) \\ &\quad + 0 \quad \text{(for the pair } (v_3, v_4)) \\ &= 2. \end{aligned}$$

**Note:** In a similar manner, $\beta(v_3) = 2$.

# Examples: Betweenness Computation

**Example 1: (continued)**



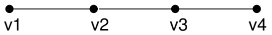

Now, consider the computation of $\beta(v_1)$.

- The $s$-$t$ pairs to be considered are: $(v_2, v_3)$, $(v_2, v_4)$ and $(v_3, v_4)$.
- For each of these pairs, the number of shortest paths is 1.
- $v_1$ **doesn't** lie on any of these shortest paths.
- Thus, for each pair, the fraction of shortest paths that pass through $v_1 = 0$.
- Therefore, $\beta(v_1) = 0$.

**Note:** In a similar manner, $\beta(v_4) = 0$.

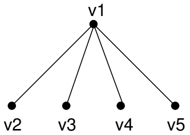
# Examples: Betweenness Computation (continued)

**Summary for Example 1:**




- $\beta(v_1) = \beta(v_4) = 0.$
- $\beta(v_2) = \beta(v_3) = 2.$

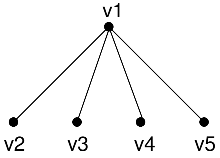
**Example 2:**




- Here also, there is **only one** path between any pair of nodes.
- Consider the computation of $\beta(v_1)$ first.

# Examples: Betweenness Computation (continued)

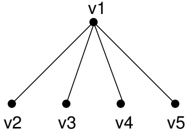
**Computation of $\beta(v_1)$ (continued):**




- We must consider all pairs of nodes from $\{v_2, v_3, v_4, v_5\}$.
- The number of such pairs $= 6$. (They are: $(v_2, v_3)$, $(v_2, v_4)$, $(v_2, v_5)$, $(v_3, v_4)$, $(v_3, v_5)$, $(v_4, v_5)$.)
- For each pair, there is only one path between them and the path passes through $v_1$.
- Therefore, the ratio contributed by each pair is 1.
- Since there are 6 pairs, $\beta(v_1) = 6$.

# Examples: Betweenness Computation (continued)

**Computation of** $\beta(v_2)$:




- We must consider all pairs of nodes from $\{v_1, v_3, v_4, v_5\}$.
- The number of such pairs $= 6$.

- For each pair, there is only one path between them and the path **doesn't** pass through $v_2$.
- Therefore, $\beta(v_2) = 0$.

**Notes:**

- In a similar manner, $\beta(v_3) = \beta(v_4) = \beta(v_5) = 0$.
- Summary for Example 2:
  - $\beta(v_1) = 6$ and
  - $\beta(v_i) = 0$, for $i = 2, 3, 4, 5$.