

TOPIC 4:

REASONING ABOUT IMPOSSIBILIA

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If 2 weren't even, what would the smallest even number be?

If $7 + 5$ were 11, I would have gotten a perfect score on the test

(Williamson '07)

If P were equal to NP (and someone proved it), modern cryptography would be compromised

- true

If P were equal to NP (and someone proved it), modern cryptography would not be compromised

- false

Now if 6 turned out to be 9
I don't mind, I don't mind
If all the hippies cut off all their hair
I don't care, I don't care



(Jimi Hendrix, 'If 6 was 9')

From Lewis Carroll, 'Through the looking-glass'

'I can't believe that!' said Alice.

'Can't you?' the Queen said in a pitying tone. 'Try again: draw a long breath, and shut your eyes.'

Alice laughed. 'There's no use trying,' she said: 'one can't believe impossible things.'

'I daresay you haven't had much practice,' said the Queen. 'When I was your age, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast.'



Suppose $ab = 0$. Prove by reductio that either $a = 0$ or $b = 0$.

Student: ‘... $ab = 0$ with a different from 0 and b different from 0, that is against my normal beliefs and I must pretend it to be true ...’

(Antonini & Marotti '08, Dutilh Novaes '16)

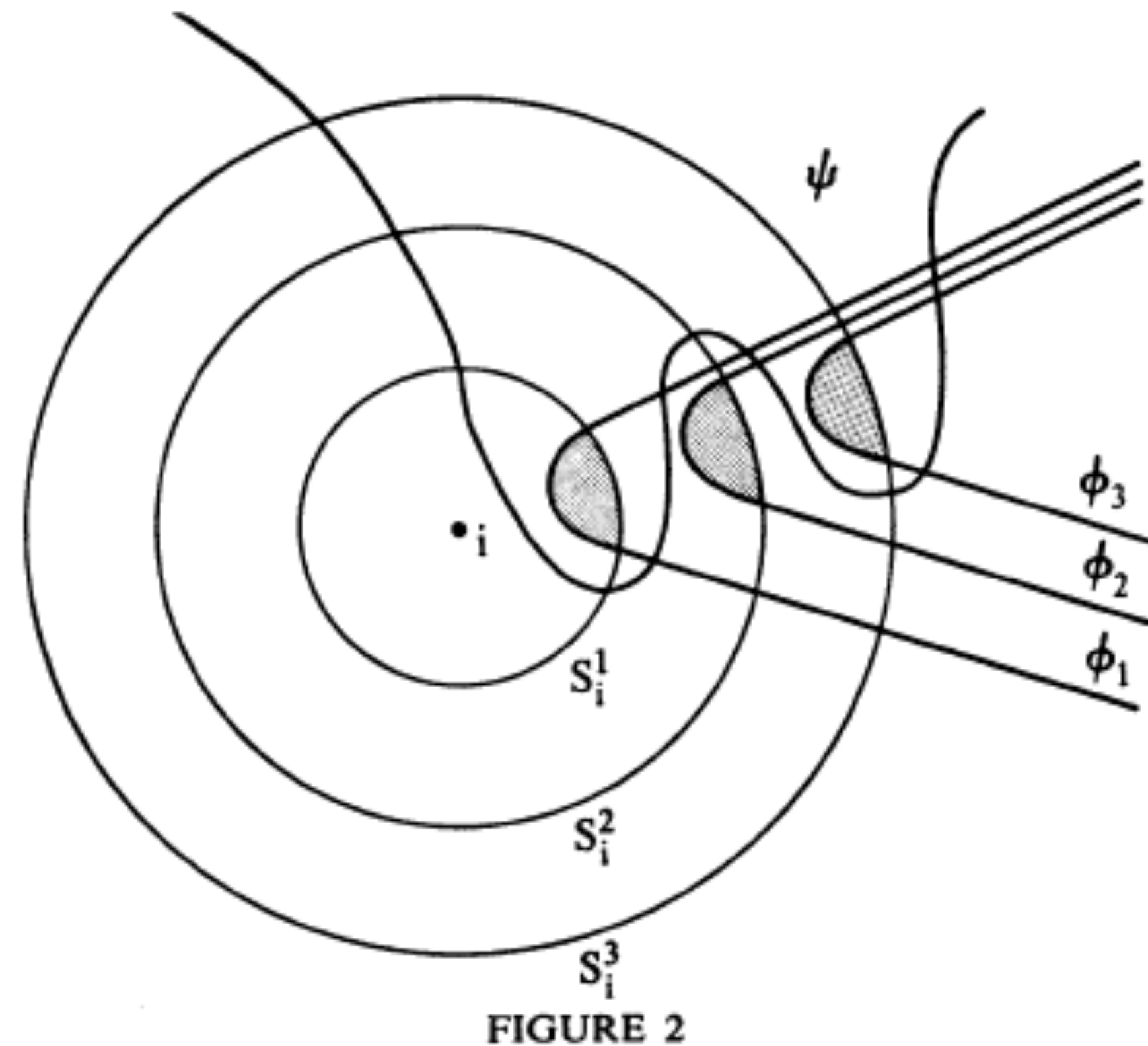
Locally coherent, globally incoherent

M.C. Escher, 'Waterfall', 1961



The Lewis/Stalnaker semantics

Stalnaker '78, Lewis '73



‘If were A, would C’

=> in all (the) closest A-world(s), C

If A is true in no possible worlds:
Trivially true or presupposition
failure

Counterpossible *wh*-questions
are presupposition failures

‘If 4 weren’t even, **what** would ...’

Frequent in logical and mathematical reasoning

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THE PRINCIPLES OF LOGIC

BOOK I

§ 57. It is true that, if you make a series longer, you decrease the chance of irregularity. It is true that, *if per impossibile* the series were so long that, in comparison with its length, every possible abnormal run was a period which other periods might easily balance in the completed cycle—if, I say, *per impossibile* this phantom could be real, it is true that the above chance of irregularity would vanish. If we as-

- Reductio proofs can be framed using counterfactuals (Lewis '73)
- Relative computability theory (Jenny '16)

Well, this is awkward

Yagisawa '88, Vacek '13

7 instances in Lewis' book 'On the Plurality of Worlds':

- '...if, per impossibile, the method of dominance had succeeded in ranking some false theories above others, it could still have been challenged by those who care little about truth'
- 'If, per impossible, you knew which row contained the mystery number, you should then conclude that it is almost certainly prime'
- 'The same would have been true if all different alterations had appeared in different parts of one big world'
- '...even if, per impossibile, the job could be done, I would still find it very peculiar if it turned out that before we can finish analyzing modality, we have to analyze talking-donkeyhood as well!'
- 'Suppose, per impossibile, that you knew which equivalence class contains the actual world'
- 'Suppose, per impossibile, that the ersatzer did produce the requisite axioms; and what is still more marvellous, that he persuaded us that he had them right'
- 'Suppose, per impossibile, that spherical shape is not the intrinsic property it seems to be, but rather is a relation that things sometimes bear to worlds of which they are parts'

and more

Lewis, 'What experience teaches':

- 'If two possible locations in our region agree in their x coordinate, then no amount of x-information can eliminate one but not both. If, per impossibile, two possible locations agreed in all their coordinates, then no information whatsoever could eliminate one but not both ...'

Lewis, 'Rearrangement of particles':

- 'But then you have to draw me bent and also straight, which you can't do; and if per impossibile you could, you still wouldn't have done anything to connect the bentness to t_1 and the straightness to t_2 ...'
- 'You have to draw them at two different distances apart, which you can't do; and if per impossibile you could, you still wouldn't have done anything to connect one distance to t_1 , and the other to t_2 ...'

Getting metaphysics out of the way

Williamson '07: trivial truth of counterpossibles is logically, metaphysically desirable

- If $7+4$ were 12, $7+5$ would be 13
- If $7+4$ were 12, $7+5$ would be 200

Why do they feel different? ‘... only the former counterfactual is assertable in a context in which for dialectical purposes the possibility of the antecedent is not excluded, and this is what the antecedent requires.’

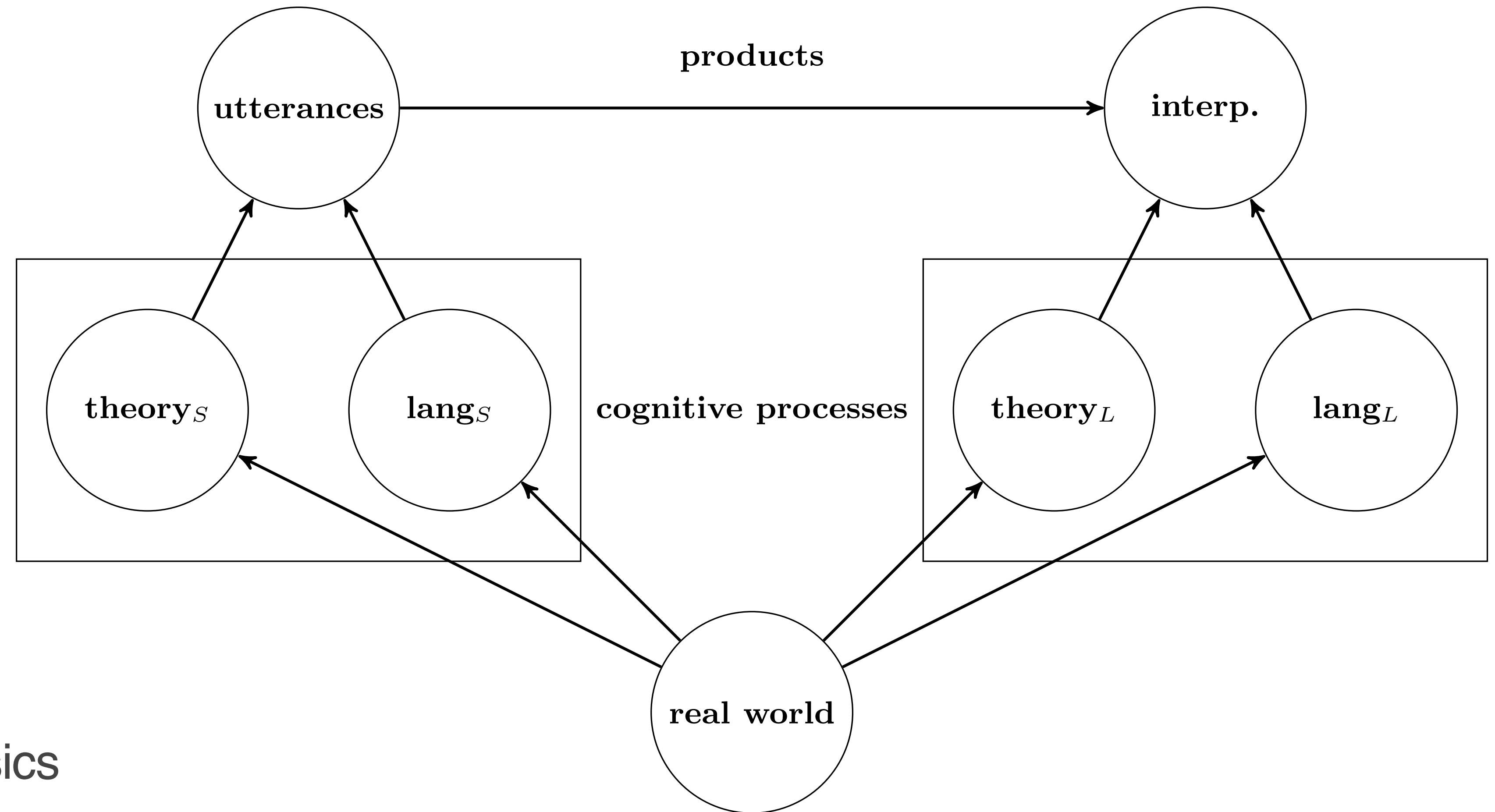
Where does language understanding fit into this picture of meaning?

Cognition screens off language use from metaphysics

‘Dialectical purposes’
are the object of interest!

Metaphysics is only
indirectly relevant

- intuitive physics vs. physics
- moral reasoning for meta-ethical nihilists



Today's main ideas in brief

Mathematical reasoning is (often) procedural

- and relies on models & metaphors drawn from everyday life

Procedures support counterfactuals

- if you define counterfactuals in terms of interventions
- partiality is key

This gives us a non-trivial interventionist semantics for mathematical counterfactuals

Models and language understanding

Goal: a psychologically realistic theory of lg. understanding that incorporates key insights of model-theoretic semantics

What kind of 'possible worlds' serve as our points of evaluation?

- Metaphysically possible worlds: no
- Impossible worlds? maybe, but ...
- Partial worlds simulated using generative models
 - representing knowledge of the world
 - formulated in procedural terms, as programs
 - formalizes intuitive physics, metaphysics, psychology, etc.

Thinking as model-building

Craik 1943, *The Nature of Explanation*



Kenneth Craik

By a model we thus mean any physical or chemical system which has a similar relation-structure to that of the process it imitates. By 'relation-structure' I do not mean some obscure non-physical entity which attends the model, but the fact that it is a physical working model which works in the same way as the process it parallels, in the aspects under consideration at any moment. Thus, the model need not resemble the real object pictorially; Kelvin's tide-predictor, which consists of a number of pulleys on levers, does not resemble tide in appearance, but it works in the same way in certain essential respects—it combines oscillations of various frequencies so as to produce an oscillation which closely resembles in amplitude at each moment the variation in tide

level at any place. Again, since the physical object is 'translated' into a working model which gives a prediction, is retranslated into terms of the original object, we can say that the model invariably either precedes or succeeds the external object it models. The only logical distinction is the ground of cheapness, speed, and convenience. The *Mary* is designed with the aid of a model in a tank because of the greater cheapness and convenience of the latter; we do not design toy boats by trying out the different plans on boats the size of Atlantic liners. In the same way, in the particular case of our own nervous systems, the reason I regard them as modelling the real process is that they permit trial of alternatives, in, e.g. bridge design, to proceed on a cheaper and smaller scale than if each bridge turn were built and tried by sending a train over it, to see whether it was sufficiently strong.

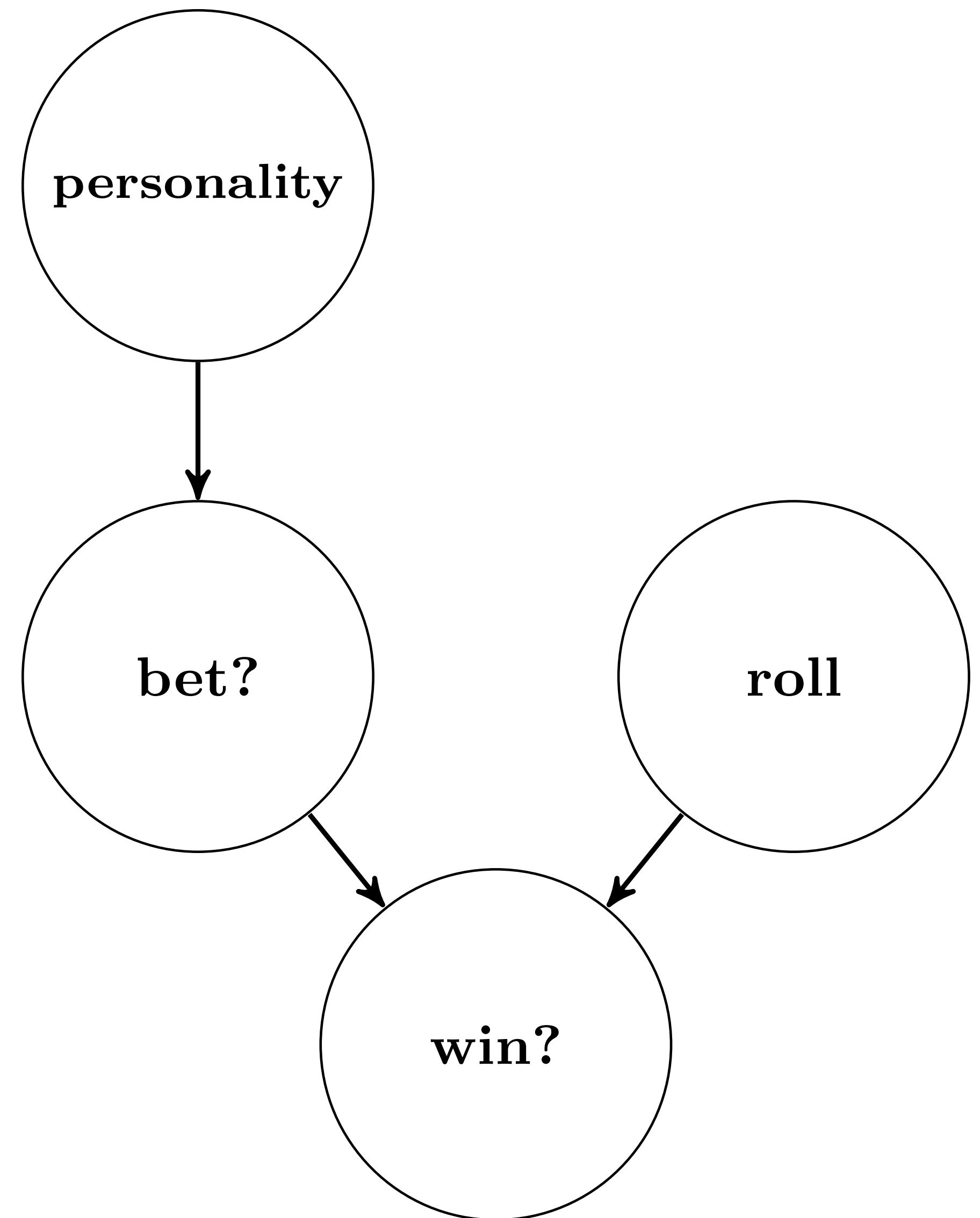
First pass: Causal models

Spirtes et al. '93, Pearl '00

‘variables’ = questions (partitions on W)

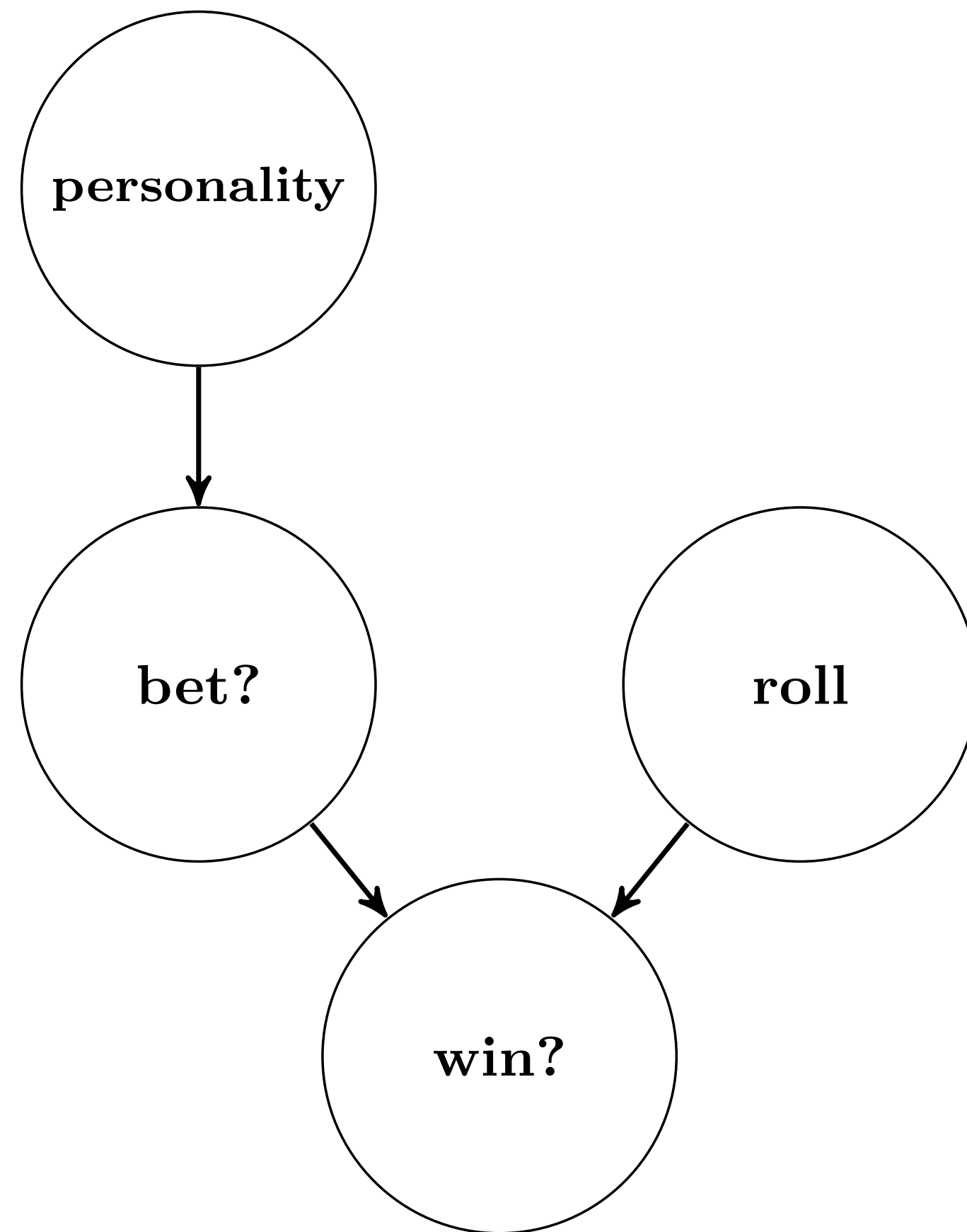
arrows = direct causal links

inference by conditioning on observations



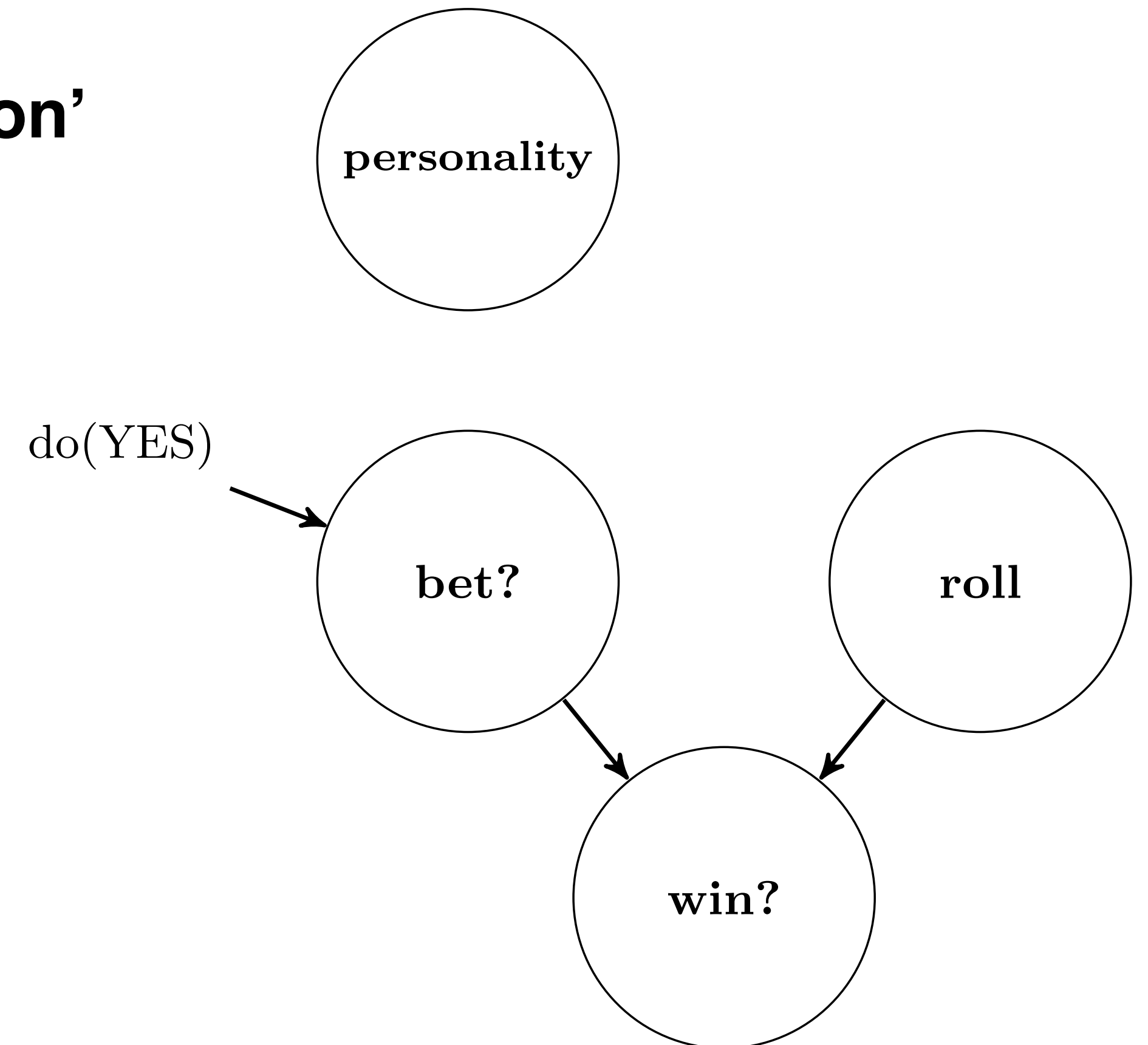
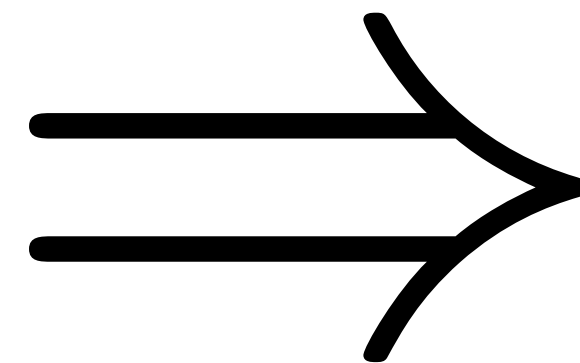
Counterfactual reasoning as intervention

Pearl '00



$O = \{\text{no bet, roll} = 6, \text{no win}\}$

“If I’d bet, I would have won”



$O = \{\text{roll} = 6\}$

Causal models as generative models as programs

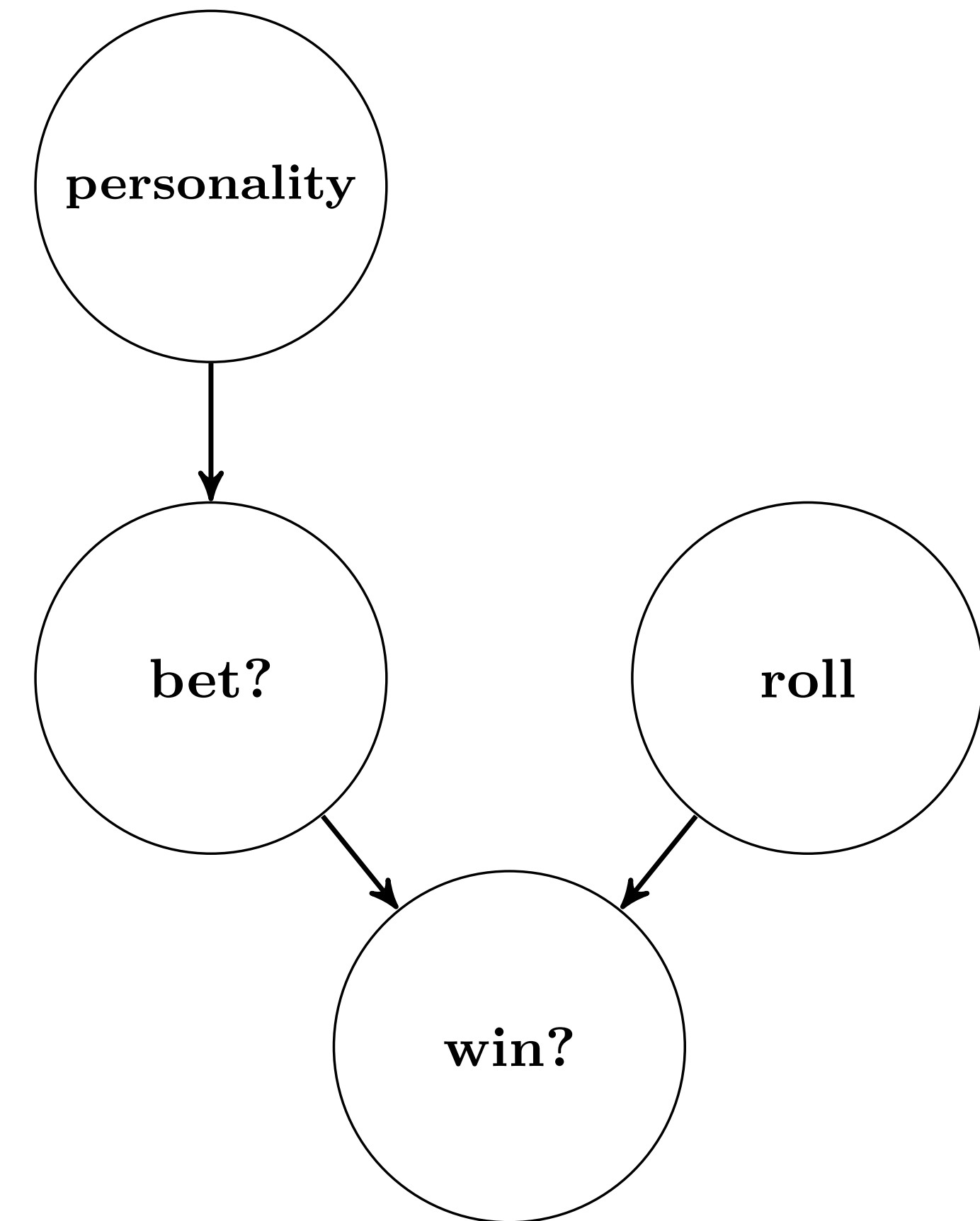
Tenenbaum et al. '11, Oaksford & Chater '13, Goodman et al '16

```
personality = ['risk-seeking', 'risk-averse'].random()
```

```
bet = ifelse(personality == 'risk-seeking', True, False)
```

```
roll = [1, 2, 3, 4, 5, 6].random()
```

```
win = if (bet && even(roll)) True  
      else False
```



Counterfactuals in programs

Oaksford & Chater '13, Goodman et al. '16, Icard '17

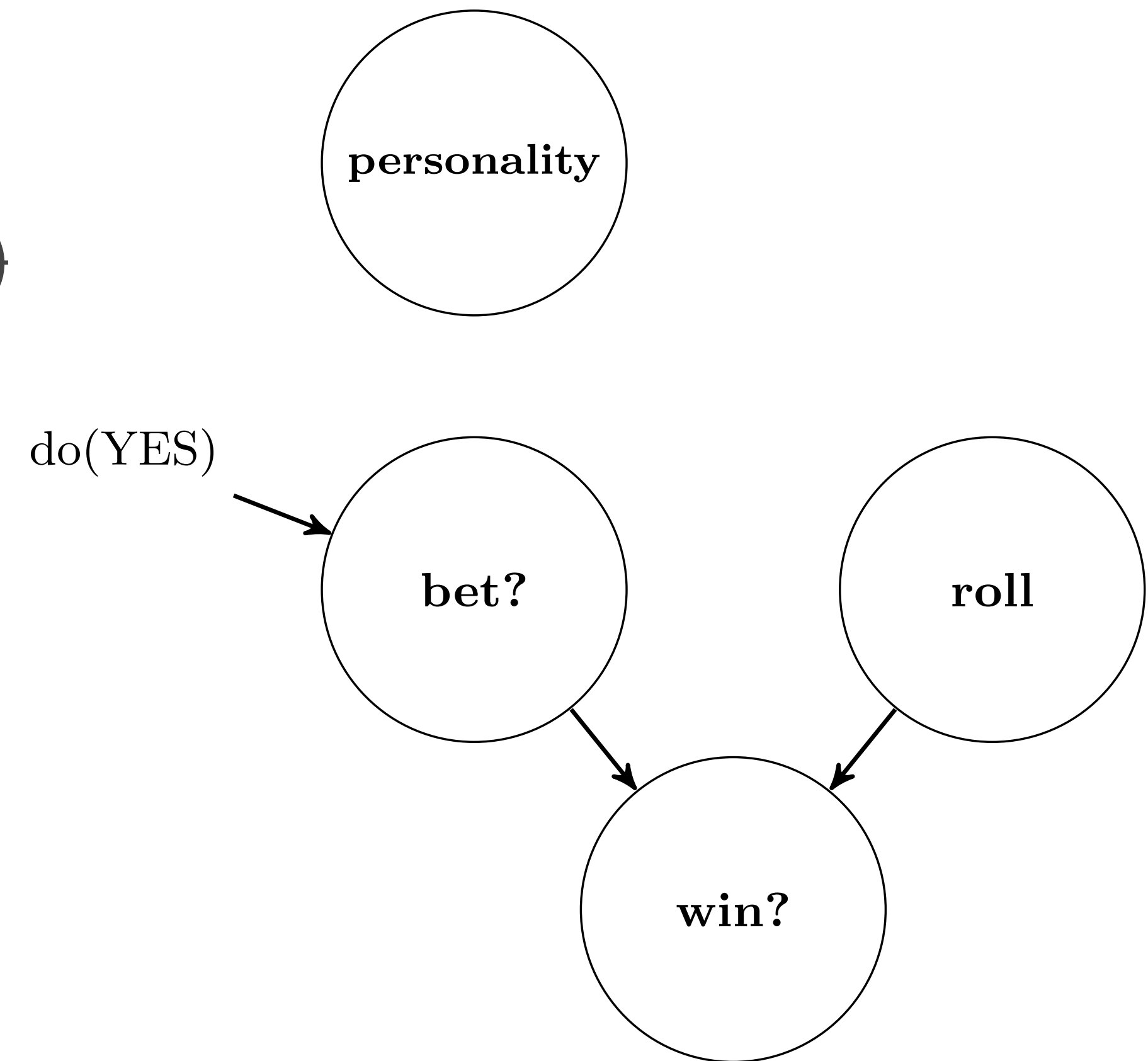
```
personality = ['risk-seeking', 'risk-averse'].random()
```

```
bet = ifelse(personality == 'risk-seeking', True, False)
```

```
bet = True
```

```
roll = [1, 2, 3, 4, 5, 6].random()
```

```
win = if (bet && even(roll)) True  
      else False
```



Mathematical reasoning as procedural

e.g., Nesher et al. '82; Vergnaud '82, Greer '92, Siegler & Alibaba '05

Children learning math acquire both

- analogies to ordinary causal knowledge
- content-blind procedures for manipulating numbers

Early stages involve causal metaphors. Examples:

- ‘put’, ‘take’, ‘get’, ‘give’, ‘increase’; ‘sequences of events ordered in time’
- ‘multiplication makes bigger, division makes smaller’
- ‘3rd and 4th graders ... believe that the equal sign is simply a signal to execute an arithmetic operation’ in word problems
- $3 + \underline{\quad} = 7$ harder than mathematically equivalent $7 - 3 = \underline{\quad}$

Partiality via laziness

e.g., Haskell: Bird & Wadler '88

lazy evaluation ('non-strict', 'call-by-need')

- only build objects that you're going to use
- leave everything else implicit in procedures
- useful: e.g., can represent infinite lists

effect: counterfactual assumptions don't need to be globally consistent

Lazy evaluation

Bird & Wadler '88

`even = [2, ...]` : a procedure with potential to generate partial lists as needed

example: get fourth element of `even`

- `fourth(even)`
- \Rightarrow `fourth(2:tail(even))`
- \Rightarrow `fourth(2:4:tail(tail(even)))`
- \Rightarrow `fourth(2:4:6:tail(tail(tail(even))))`
- \Rightarrow `fourth(2:4:6:8:tail(tail(tail(tail(even)))))`
- \Rightarrow 8

efficiency depends on how `even` is computed

Intervening on a program with lazy evaluation

‘If 4 were not even, what would the second smallest even number be?’

- intervene: modify even procedure so that 4 is excluded
- return the result of applying second to mutated even

- `second(even[exclude 4])`
- `=> second(2:tail(even[exclude 4]))`
- ~~`=> second(2:4:tail(tail(even[exclude 4])))`~~
- `=> second(2:6:tail(tail(even[exclude 4])))`
- `=> 6`

The importance of partiality

cf. Baron, Colyvan & Ripley 2017

If $7 + 5$ were 11, I would have gotten a perfect score on the test

- Williamson '07: no coherent world where antecedent is true
- lazy approach: force ' $7 + 5 = 11$ ', ignoring variables not mentioned
- no need to
 - create a full, coherent 'world' with this property
 - consider other number-theoretic consequences

[After the proof of $P \neq NP$:] If P were equal to NP , ...

- there's a whole field devoted to examining downstream consequences of such suppositions

Potential for (quasi-)subjectivity

Every mathematical object can be build using a variety of procedures

- none is privileged

`odd = [1, 3, 5, 7, ...]`

`odd = concat(reverse([5, 3, 1]), [7, 9, ...])`

`odd = concat(concat([1], append([3, 5], 7)), [9, 11, ...]))`

Examples where procedure matters to intuitive interpretation of counterfactual?

A variant of (part of) Euclid's proof

cf. Lewis '73, Williamson '07

Let $f(x)$ = the product of all prime numbers less than or equal to x .

If there were a largest prime p , then $f(p) + 1$ would be not be prime.

Why?

- $f(p)$ is larger than any prime
- Every number is either prime or composite
- So, $f(p) + 1$ would be composite, not prime.

On the other hand

A variant of (part of) Euclid's proof

cf. Lewis '73, Williamson '07

Let $f(x)$ = the product of all prime numbers less than or equal to x .

If there were a largest prime p , then $f(p) + 1$ would be prime.

Why?

- $f(p)$ is divisible by all primes up to p .
- so, $f(p)$ is divisible by every prime number.
- so, $f(p) + 1$ is divisible by no prime number.
- so, $f(p) + 1$ is prime (by the unique prime factorization theorem)

The reductio step

‘If there were a largest prime p , then $f(p) + 1$ would be prime’ is true

‘If there were a largest prime p , then $f(p) + 1$ would not be prime’ is true

- the reductio requires us to construct partial models satisfying each
- no absurdity until we try to create a larger model satisfying both

Locally coherent, globally incoherent

M.C. Escher, 'Waterfall', 1961

Williamson '07, p.172:

Examples of false counterpossibles
'are quite unpersuasive. First, they tend
to fall apart when thought through ...'



Levels of analysis: Counterfactuals are special

Computational theory	Representation and algorithm	Hardware implementation
What is the goal of the computation, why is it appropriate, and what is the logic of the strategy by which it can be carried out?	How can this computational theory be implemented? In particular, what is the representation for the input and output, and what is the algorithm for the transformation?	How can the representation and algorithm be realized physically?

Figure 1–4. The three levels at which any machine carrying out an information-processing task must be understood.

Generally, formal semantics is about the computational level

Counterfactuals give us a glimpse into the process by which the product is constructed (algorithms)

Summary: How to reason about six impossible things before breakfast

Mathematical counterfactuals are interesting – and totally non-trivial – for a theory of language understanding

General framework for analysis:

- interventionist theory of counterfactuals
- cognitive theory build around generative models

Counterfactuals may allow us to see inside the procedures people use to reason about math & logic

Laziness & partiality may help make sense of how people reason about impossibilities

Overall trajectory

1. Probability & generative models
2. Indicative conditionals: probability & trivalence
3. Counterfactuals & causal models
4. Reasoning about impossibilia

Thank you for a wonderful time at Paris VII and for all the great discussion!

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