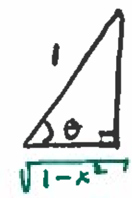
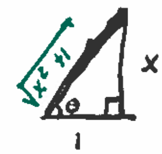
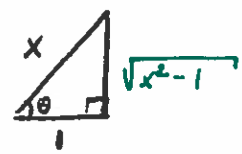
Lecture 5 Math 1360 Section 6.2

Trig substitution:

If we see	Let	B/c	
$\sqrt{1-x^2}$	$x = \sin\theta$	$1-\sin^2\theta = \cos^2\theta$	
$\sqrt{x^2+1}$	$x = \tan\theta$	$\tan^2\theta + 1 = \sec^2\theta$	
$\sqrt{x^2-1}$	$x = \sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	

* It is VITAL to change all occurences of the variable x to θ!

Ex: $$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta \Rightarrow \sin^{-1}x = \theta$

$dx = \cos\theta\, d\theta$

$$= \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta = \int \frac{\cos\theta}{\cos\theta}\,d\theta = \int 1\, d\theta$$

$$= \theta + C = \boxed{\sin^{-1}x + C}$$

Ex: $\int \sqrt{1-x^2}\, dx$ $\qquad x = \sin\theta$, $\; dx = \cos\theta\, d\theta$

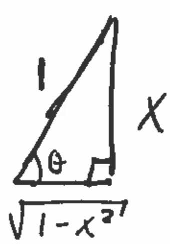
$$= \int \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta = \int \cos\theta \cdot \cos\theta \, d\theta$$

$$= \int \cos^2\theta \, d\theta = \int \frac{1+\cos(2\theta)}{2} \, d\theta =$$

$$\frac{1}{2}\int 1 + \cos(2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C$$

$$= \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4} + C = \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C$$

Now What? $\qquad x = \sin\theta \Rightarrow \theta = \arcsin(x)$
$\qquad\qquad\qquad\qquad\qquad\quad = \sin^{-1}(x)$



$\sin\theta = x = \frac{x}{1}$

$= \frac{opp}{hyp}$; $\cos\theta = \sqrt{1-x^2}$

So $\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C$

$$= \boxed{\frac{\arcsin(x)}{2} + \frac{x\sqrt{1-x^2}}{2} + C}$$

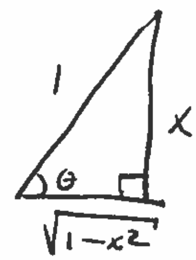
$\underline{P}$: Try $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$$\int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta\, d\theta = \int \csc^2\theta - 1 \;\; d\theta$$

$$= -\cot\theta - \theta + C =$$



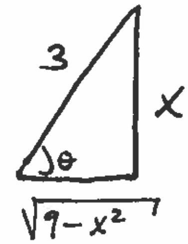
$$= \boxed{-\frac{\sqrt{1-x^2}}{x} - \arcsin(x) + C}$$

$\underline{Ex}$: $\displaystyle\int \frac{1}{x\sqrt{9-x^2}}\,dx$

$x = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$

$$= \int \frac{1}{3\sin\theta\sqrt{9(1-\sin^2\theta)}} \cdot 3\cos\theta\, d\theta = \frac{1}{3}\int \frac{1}{\sin\theta}\, d\theta =$$

$$\frac{1}{3}\int \csc\theta\, d\theta = -\frac{1}{3}\ln\left|\csc\theta + \cot\theta\right| + C$$



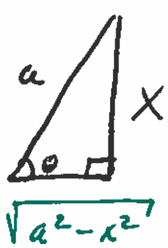
$$= \boxed{-\frac{1}{3}\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + C}$$

Generalizing:

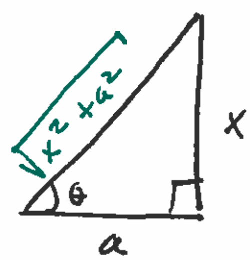
If we see	Let	B/c
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{x^2+a^2}$	$x = a\tan\theta$	$a^2\tan^2\theta + a^2 = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

w/ triangles:

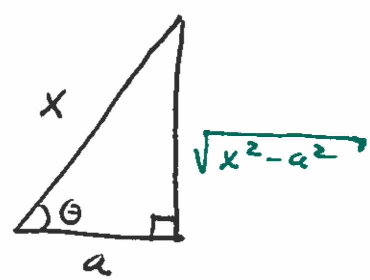
$x = a\sin\theta$ or $\sin\theta = \frac{x}{a}$



$x = a\tan\theta$ or $\tan\theta = \frac{x}{a}$

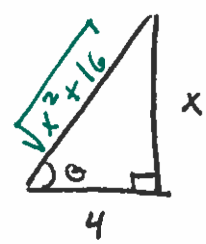


$x = a\sec\theta$ or $\sec\theta = \frac{x}{a}$



Ex: $\displaystyle\int \frac{1}{x^2\sqrt{x^2+16}}\,dx$ $\quad x = 4\tan\theta, \quad dx = 4\sec^2\theta\, d\theta$

$$= \int \frac{1}{16\tan^2\theta\sqrt{16\sec^2\theta}} \cdot 4\sec^2\theta\, d\theta =$$



$$\frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{16}\int \csc\theta\cot\theta\, d\theta$$

$$= \frac{1}{16}(-\csc\theta) + C = \boxed{-\frac{1}{16}\cdot\frac{\sqrt{x^2+16}}{x} + C}$$

$\underline{\text{Ex:}}$ $\displaystyle\int \frac{x}{\sqrt{x^2+25}}\,dx$

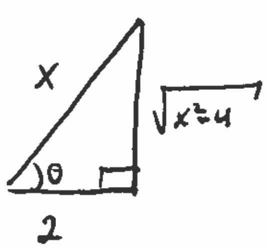
* Always look for an easy u-sub before moving on to trig-sub!

$$\left.\begin{aligned} u &= x^2+25 \\ du &= 2x\,dx \\ \tfrac{1}{2}\,du &= x\,dx \end{aligned}\right\} \quad \int \frac{x}{\sqrt{x^2+25}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C$$

$$= \sqrt{u} + C = \boxed{\sqrt{x^2+25} + C}$$

$\underline{\text{P}}$: Try $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$ even though we know the answer is $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C$!

$$\left.\begin{aligned} x &= 2\sec\theta \\ dx &= 2\sec\theta\tan\theta\,d\theta \end{aligned}\right\} \quad \int \frac{1}{x\sqrt{x^2-4}}\,dx = \int \frac{1}{2\sec\theta\sqrt{4\tan^2\theta}}\cdot 2\sec\theta\tan\theta\,d\theta$$



$$= \int \frac{1}{2}\,d\theta = \frac{1}{2}\theta + C$$

$$= \boxed{\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C}$$

since $\sec\theta = \frac{x}{2}$

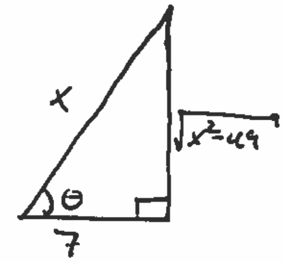
$\underline{P:}$

$$\int \frac{1}{x^2\sqrt{x^2-49}}\,dx$$

$x = 7\sec\theta$

$dx = 7\sec\theta\tan\theta$

$\sec\theta = \frac{x}{7}$



$$= \int \frac{1}{49\sec^2\theta \cdot 7\tan\theta} \cdot 7\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{49}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{49}\int \cos\theta\,d\theta = \frac{1}{49}\sin\theta + C$$

$$= \boxed{\frac{1}{49}\cdot\frac{\sqrt{x^2-49}}{x} + C}$$

$\underline{\text{Check:}}$ $$\frac{d}{dx}\,\frac{1}{49}\,\frac{(x^2-49)^{1/2}}{x} = \frac{1}{49}\left[\frac{x\cdot\frac{1}{2}(x^2-49)^{-1/2}\cdot 2x - (x^2-49)^{1/2}}{x^2}\right]$$

$$= \frac{1}{49}\left[\frac{x^2(x^2-49)^{-1/2} - (x^2-49)^{1/2}}{x^2}\right] =$$

$$\frac{1}{49}\left[\frac{(x^2-49)^{-1/2}\left(x^2-(x^2-49)\right)}{x^2}\right] = \frac{1}{49}\left[\frac{49}{x^2\sqrt{x^2-49}}\right]$$

$$= \frac{1}{x^2\sqrt{x^2-49}} \checkmark$$

Ex: $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{\tan^3\theta \cdot \sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta \tan^3\theta\, d\theta$$

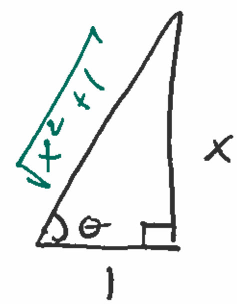
$u = \sec\theta$

$du = \sec\theta\tan\theta\, d\theta$

$$= \int \underbrace{\tan^2\theta}_{u^2-1} \cdot \underbrace{\sec\theta\tan\theta\, d\theta}_{du}$$

$$= \int u^2 - 1\, du = \frac{u^3}{3} - u + C =$$

$$\frac{\sec^3\theta}{3} - \sec\theta + C =$$



$$\boxed{\frac{(x^2+1)^{3/2}}{3} - \sqrt{x^2+1} + C}$$

or $\quad u = x^2+1 \quad du = 2x\,dx, \quad \dfrac{du}{2x} = dx$

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{x^3}{\sqrt{x^2+1}} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int u^{1/2} - u^{-1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right] + C$$

$$= \frac{1}{3}(x^2+1)^{3/2} - \sqrt{x^2+1} + C.$$ Same!

$$\int_0^{3/2} \sqrt{4x^2+9}\, dx$$

We see an "x^2+a^2" form, we need

$4x^2+9 = 9\tan^2\theta + 9 = 9\sec^2\theta$

So $4x^2 = 9\tan^2\theta \longrightarrow 2x = 3\tan\theta$

$\longrightarrow x = \frac{3}{2}\tan\theta$

$\longrightarrow dx = \frac{3}{2}\sec^2\theta\, d\theta$

Limits of int:

$x = 3/2 \longrightarrow \frac{3}{2}\tan\theta = \frac{3}{2} \longrightarrow \theta = \pi/4$

$x = 0 \longrightarrow \frac{3}{2}\tan\theta = 0 \longrightarrow \theta = 0$

$$\int_0^{\pi/4} \sqrt{9\sec^2\theta} \cdot \frac{3}{2}\sec^2\theta\, d\theta = \frac{9}{2}\int_0^{\pi/4} \sec^3\theta\, d\theta$$

$$= \frac{9}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right]$$

$$= \frac{9}{4}\left[\left(\sec\pi/4 \tan\pi/4 - \ln|\sec\pi/4 + \tan\pi/4|\right) - \left(\sec 0 \tan 0 - \ln|\sec 0 + \tan 0|\right)\right]$$

$$= 9/4\left[\left(\sqrt{2} - \ln(\sqrt{2})\right) - \underbrace{\left(1\cdot 0 - \ln(1+0)\right)}_{0}\right] =$$

$$\boxed{9/4\left(\sqrt{2} - \ln\sqrt{2}\right)}$$