Lecture 1 — Math 1360 — Section 5.5

Review of u-substitution:

First some <u>indefinite integrals</u> (antiderivatives)

<u>Ex:</u> $\int 3x^4 - 8x + \cos x + 5 \, dx =$

$\frac{3x^5}{5} - \frac{8x^2}{2} + (+\sin x) + 5x + C =$

$\frac{3}{5}x^5 - 4x^2 + \sin x + 5x + C$

<u>Ex:</u> $\int \frac{1}{x} + \sec^2 x - e^x \, dx =$

$\ln|x| + \tan x - e^x + C$

<u>Ex:</u> $\int \frac{1}{\sqrt{1-x^2}} - \sin x \, dx =$

$\arcsin(x) - (-\cos x) + C =$

$\underbrace{\arcsin(x)}_{\sin^{-1}(x)} + \cos x + C$

$\mathbb{P}$ Try $\int 3x^8 - \sqrt[5]{x} - \sec x \tan x \, dx$

Sol'n: $\frac{3x^9}{9} - \frac{x^{6/5}}{6/5} - \sec x + C =$

$$\boxed{\frac{1}{3}x^9 - \frac{5}{6}x^{6/5} - \sec x + C}$$

$\mathbb{P}$ Try $\int \frac{1}{x^2+1} \, dx$

Observe: $\int \frac{1}{x^2+1} \, dx \neq \ln|x^2+1| + C$

why? $\frac{d}{dx} \ln|x^2+1| = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1} \neq \frac{1}{x^2+1}$

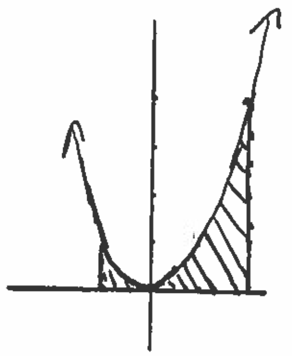
$\int \frac{1}{x^2+1} \, dx = \arctan(x) + C$ or $\tan^{-1}(x) + C$

since $\frac{d}{dx} \arctan(x) = \frac{1}{x^2+1}$.

Definite Integrals: (Area)

Ex:

$$\int_{-1}^{2} x^2\, dx = \frac{x^3}{3}\Big|_{-1}^{2} = \frac{2^3}{3} - \frac{(-1)^3}{3} = \frac{8}{3} + \frac{1}{3} = 3$$



More generally:

$$\int_a^b f(x)\, dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

Ex:

$$\int_{-3}^{4} 3x^2 - 6x\, dx = x^3 - 3x^2\Big|_{-3}^{4} =$$

$$(4^3 - 3\cdot 4^2) - \left[(-3)^3 - 3(-3)^2\right] =$$

$$(64 - 48) - (27 - 27) = 16 - 0 = 16.$$

P:

$$\int_0^2 x^3 - e^x\, dx$$

Soln:

$$\frac{x^4}{4} - e^x\Big|_0^2 = \left(\frac{2^4}{4} - e^2\right) - \left(\frac{0^4}{4} - e^0\right)$$

$$= (4 - e^2) - (0 - 1) = 5 - e^2 \approx -2.39$$

Ex: $\int_{-2}^{3} \frac{1}{x}\,dx = \ln|x| \Big|_{-2}^{3} =$

$\ln|3| - \ln|-2| = \ln 3 - \ln 2 = \ln(3/2) \approx .4$

is <u>completely wrong!</u> Why? Because

$f(x) = \frac{1}{x}$ is undefined on the

interval $[-2, 3]$ (specifically at zero).

When finding / calculating a definite integral we need the function to be defined on the entire interval of integration.

P: $\int_{-2}^{2} x^3 - 5x \, dx$

Soln: $\frac{x^4}{4} - \frac{5x^2}{2} \Big|_{-2}^{2} =$

$\left(\frac{2^4}{4} - \frac{5 \cdot 2^2}{2}\right) - \left[\frac{(-2)^4}{4} - \frac{5(-2)^2}{2}\right] = \left(\frac{16}{4} - \frac{20}{2}\right) - \left(\frac{16}{4} - \frac{20}{2}\right)$

$= (4-10) - (4-10) = 0$!

Generally: If $f(x)$ is odd & defined on the interval $[-a, a]$ then we have

$$\int_{-a}^{a} f(x)\,dx = 0$$

Ex: $\int_{-7}^{7} x^2 \sin(x)\,dx = 0$ since

$f(x) = x^2 \sin(x)$ is an odd function.

$$f(-x) = (-x)^2 \sin(-x) = x^2 \cdot (-\sin(x)) = -x^2 \sin x$$
$$= -f(x).$$

P: Show that $\int_{-1}^{1} \frac{x^4 \tan x}{1+x^6}\,dx = 0$

by showing that $f(x)$ is odd.

$$f(-x) = \frac{(-x)^4 \tan(-x)}{1+(-x)^6} = \frac{x^4 \cdot (-\tan x)}{1+x^6} =$$

$$-\frac{x^4 \tan x}{1+x^6} = -f(x).$$ So $\int_{-1}^{1} \frac{x^4 \tan x}{1+x^6}\,dx = 0.$

Note: $\sin(x)$, $\tan(x)$, $\csc(x)$, and $\cot(x)$ are all odd functions. So, for example, we can just state that $\csc(-x) = -\csc(x)$. Similarly $\cos(x)$ and $\sec(x)$ are even functions, so $\cos(-x) = \cos(x)$ & $\sec(-x) = \sec(x)$.

We can use these facts without justification to show, for example, that

$$\int_{-5}^{5} x^3 \cos(x)\, dx = 0$$ since

$f(x) = x^3 \cos(x)$ is odd:

$$f(-x) = (-x)^3 \cos(-x) = -x^3 \cdot \cos(x) = -f(x).$$

Note: $\int_{-4}^{4} \frac{1}{x}\, dx \neq 0$ even though $f(x) = \frac{1}{x}$ is an odd function. Since $f(x) = \frac{1}{x}$ is undefined on $[-4, 4]$ we don't yet have the tools to attack this. (Section 6.6)

$\underline{Ex:}$ $$\int_{\pi/4}^{\pi/3} \sec x \tan x \, dx = \sec x \Big|_{\pi/4}^{\pi/3} =$$

$$\sec(\pi/3) - \sec(\pi/4) = \frac{1}{\cos(\pi/3)} - \frac{1}{\cos(\pi/4)}$$

$$= \frac{1}{1/2} - \frac{1}{\sqrt{2}/2} = 2 - \frac{1}{1/\sqrt{2}} = \boxed{2-\sqrt{2}}$$

* remember $\cos(\pi/4) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$,

to calculate $\sec(\pi/4)$ it's easier as $\frac{1}{1/\sqrt{2}} = \sqrt{2}$.

$\underline{P}$ Solve: $$\int_{1/2}^{\sqrt{3}/2} \frac{-1}{\sqrt{1-x^2}} \, dx$$

(1.) $$-\arcsin(x) \Big|_{1/2}^{\sqrt{3}/2} = -\arcsin(\sqrt{3}/2) - (-\arcsin(1/2))$$

$$= -\pi/3 + \pi/6 = \boxed{-\pi/6}$$

(2.) $$\arccos(x) \Big|_{1/2}^{\sqrt{3}/2} = \arccos(\sqrt{3}/2) - \arccos(1/2)$$

$$= \pi/6 - \pi/3 = \boxed{-\pi/6}$$

u- Substitution:

Ex: $\int x^2 (1-x^3)^{10} dx$

Two things to consider when choosing u:

1. The derivative of u (or a constant multiple of the derivative of u) is also in the integrand.

Here, if we pick $u = 1-x^3$ then

$\frac{du}{dx} = -3x^2 \implies du = -3x^2 dx$

We see $x^2 dx$ is in the integrand & $du = -3x^2 dx$ is a constant (-3) multiple of that.

2. If the first criteria is hard to meet choose u to be something raised to a power or inside another function (trig, exponential, root, log, etc.)

* Here these choices are the same, yay!

$$\int x^2(1-x^3)^{10}\,dx$$

$u = 1-x^3$

$du = -3x^2\,dx$

$\frac{du}{-3} = x^2\,dx$

$$\int (1-x^3)^{10}\underbrace{x^2\,dx}_{\frac{du}{-3}} = \int (\underbrace{1-x^3}_{u})^{10}\cdot\frac{du}{-3} = -\frac{1}{3}\int u^{10}\,du$$

convert dx to du __before__ anything else!

$$-\frac{1}{3}\int u^{10}\,du = -\frac{1}{3}\cdot\frac{u^{11}}{11} + C =$$

$$-\frac{u^{11}}{33} + C = \boxed{\frac{-(1-x^3)^{11}}{33} + C}$$

EP: $\int x^3\sqrt{1+x^4}\,dx$ ($x^3\,dx$: $du/4$)

$u = 1+x^4$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$$= \int \sqrt{1+x^4}\,du = \int \sqrt{u}\,du =$$

$$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{3}(1+x^4)^{3/2} + C}$$

We are essentially undoing the chain rule!

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\int f'(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\, dx}_{du}$$

$u = g(x)$

$\frac{du}{dx} = g'(x)$

$du = g'(x)\, dx$

$$= \int f'(u)\, du = f(u) + C$$

$$= f(g(x)) + C.$$

$\underline{\mathbb{P}}$: Try $\displaystyle\int \frac{(\ln x)^4}{x}\, dx$.

$u = \ln x$

$du = \frac{1}{x} dx$

$$\int \frac{(\ln x)^4}{x} dx = \int (\ln x)^4 \cdot \underbrace{\frac{1}{x}\, dx}_{du}$$

u

$$= \int u^4 du = \frac{u^5}{5} + C = \frac{(\ln x)^5}{5} + C$$

$$* \ \frac{d}{dx} \frac{(\ln x)^5}{5} = \frac{5(\ln x)^4}{5} \cdot (\ln x)' = (\ln x)^4 \cdot \frac{1}{x} = \frac{(\ln x)^4}{x}$$

$\underline{\text{Ex:}}$ $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

Both $\sin x$ & $\cos x$ <u>seem</u> like good choices.

$u = \sin x \longrightarrow du = \cos x \, dx$

$\int \frac{\sin x}{\cos x} dx = ???$ we don't have $\cos x \, dx$ to replace with du we have $\int \sin x \cdot \frac{1}{\cos x} dx$

$$\left.\begin{aligned} &\underline{u = \cos x} \\ &du = -\sin x \, dx \\ -&du = \sin x \, dx \end{aligned}\right\} \int \frac{\sin x}{\cos x} dx = \int \frac{\overbrace{\sin x \, dx}^{-du}}{\underbrace{\cos x}_{u}}$$

$$= \int \frac{-du}{u} = -\int \frac{du}{u} = -\int \frac{1}{u} du$$

$$= -\ln|u| + C = \boxed{-\ln|\cos x| + C}$$

$\underline{\text{or}}$ $-\ln|\cos x| + C = -1 \cdot \ln|\cos x| + C$

$$= \ln\left(|\cos x|\right)^{-1} + C = \ln\left(\frac{1}{|\cos x|}\right) + C$$

$$= \boxed{\ln|\sec x| + C}$$

"Mini u-substitution"

Ex: $\int e^{4x}\,dx$

$u = 4x$

$du = 4\,dx \longrightarrow \frac{1}{4}du = dx$

$\downarrow$

$\frac{1}{4}\int e^{u}\,du = \frac{1}{4}e^{u} + C = \frac{1}{4}e^{x} + C$

Ex: $\int \sin(5x)\,dx$

$u = 5x$

$du = 5\,dx \longrightarrow \frac{du}{5} = dx$

$\downarrow$

$\frac{1}{5}\int \sin(u)\,du = \frac{1}{5}(-\cos(u)) + C = \frac{-\cos(5x)}{5} + C$

Ex: $\int \cos(6x)\,dx = \frac{\sin(6x)}{6} + C$

b/c the derivative of sin(6x) will multiply by 6.

Ex: $\int e^{-4x+3}\,dx = \boxed{\frac{e^{-4x+3}}{-4} + C}$

or $\boxed{-\frac{1}{4}\cdot e^{(-4x+3)} + C}$

Ex:

$$\int \sec^2(5x)\,dx = \frac{\tan(5x)}{5} + C$$

we know $\int \sec^2(x)\,dx = \tan(x) + C$

b/c $\frac{d}{dx}\tan(x) = \sec^2 x$.

$$\frac{d}{dx}\tan(5x) = \sec^2(5x)\cdot 5,$$

so $\frac{d}{dx}\frac{\tan(5x)}{5} = \frac{d}{dx}\frac{1}{5}\cdot\tan(5x)$

$$= \frac{1}{5}\sec^2(5x)\cdot 5 = \sec^2(5x)!$$

IP:

$$\int \cos(3x+4)\,dx = \frac{\sin(3x+4)}{3} + C$$

$$\int \sec(3x)\tan(3x)\,dx = \frac{\sec(3x)}{3} + C$$

$$\int \frac{1}{2x+3}\,dx = \frac{\ln|2x+3|}{2} + C$$

$$\int \frac{1}{5-4x}\,dx = \frac{\ln|5-4x|}{-4} + C$$