Lecture 3 Math 1360 Section 6.1

Integration by Parts (IBP):

Recall that integration & differentiation are inverse operations (they undo each other)

$$\frac{\partial}{\partial x}\left[\int x^3 + \sin x \, \partial x\right] = \frac{\partial}{\partial x}\left(\frac{x^4}{4} - \cos x + C\right) = x^3 + \sin x$$

& $$\int\left[\frac{\partial}{\partial x} x^3 + \sin x\right]\partial x = \int 3x^2 + \cos x \, \partial x = x^3 + \sin x + \underline{\underline{C}}$$

slight difference.

Generally: $$\frac{\partial}{\partial x}\left[\int f(x)\,\partial x\right] = f(x)$$ &

$$\int\left[\frac{\partial}{\partial x} f(x)\right]\partial x = f(x) + C$$

Recall the product Rule:

$$\frac{\partial}{\partial x}\left[f(x)\cdot g(x)\right] = f(x)\,g'(x) + g(x)\,f'(x)$$

Now integrate both sides of the equation

$$\underbrace{\int \left[\frac{d}{dx} f(x)g(x)\right] dx}_{f(x)g(x)} = \underbrace{\int f(x)g'(x)\,dx}_{\text{solve for this}} + \int g(x)f'(x)\,dx$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

Using more standard notation letters we let

$$u = f(x) \longrightarrow \frac{du}{dx} = f'(x) \longrightarrow du = f'(x)\,dx$$

$$v = g(x) \longrightarrow \frac{dv}{dx} = g'(x) \longrightarrow dv = g'(x)\,dx$$

So $\displaystyle\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)dx}_{dv} = \underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)dx}_{du}$

becomes $$\boxed{\int u\,dv = uv - \int v\,du}$$

"ultra violet voodoo"

Ex: $\int x \sin x \, dx$

Let $u = x$ and $dv = \underbrace{\sin x \, dx}_{\text{the rest}}$ so that

$u \cdot dv = x \sin x \, dx$ is the entire expression after the integral symbol.

Then $\int x \sin x \, dx = \int u \, dv = uv - \int v \, du$

$= x \cdot (-\cos x) - \int (-\cos x) \, dx =$

$u = x \rightarrow du = dx$

$dv = \sin x \, dx \rightarrow v = -\cos x$

$-x \cos x + \int \cos x \, dx =$

$\boxed{-x \cos x + \sin x + C}$

Check: $\frac{d}{dx}\left[-x \cos x + \sin x + C\right] =$

$-1 \cdot \cos x + (-x) \cdot (-\sin x) + \cos x + 0 =$

$x \sin x$!

How do we choose u and dv?

① We want dv to be as "compicated" as possible while still being simple to antidifferentiate (since we need v immediately).

② We typically pick u in the following way while keeping in mind part ① for the leftovers:

L	I	A	T	E
og	nverse Trig	lgebra	rig	xponential

By Algebra we mean powers of x (in fact some people use a P instead of an A)

- We pick u (typically) to be the first (from left to right) type of function to occur.
- We use this method (IBP) when we have a <u>mixture</u> of two or more types of functions and when u-sub hasn't already worked.

Ex: $\int \frac{\sin(\ln x)}{x}\,dx$ is definitely a mixture of types of functions, but we would try u-sub first before attempting IBP.

$$\left.\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}\right\} \quad \int \frac{\sin(\ln x)}{x}\,dx = \int \sin(u)\,du$$

$$= -\cos(u) + c = \boxed{-\cos(\ln x) + c}$$

Assuming that a u-sub hasn't worked we try IBP. If the integral has a logarithm (usually $\ln x$) that is always our choice for u! If no logarithms we look to set u equal to an inverse trig function. If none of these are present then we try setting u equal to x to a power. If we have a mixture of exponential & trig it usually doesn't matter, but we'll discuss it.

Ex: $$\int x^2 \ln x \, dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{x^2 dx}_{dv}$$

$u = \ln x \qquad du = \frac{1}{x} dx$

$dv = x^2 dx \qquad v = \frac{x^3}{3}$

$$\int x^2 \ln x \, dx = \underbrace{\ln x}_{u} \cdot \underbrace{\frac{x^3}{3}}_{v} - \int \underbrace{\frac{x^3}{3}}_{v} \cdot \underbrace{\frac{1}{x} dx}_{du}$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^2 dx = \frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + C$$

$$= \boxed{\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C}$$

Usually $\ln x$ only gets paired w/ algebraic terms.
multiplied

$\int \ln x \cdot \sin x \, dx$ is not integrable! $\qquad u = \ln x \quad du = \frac{1}{x} dx$

$dv = \sin x \, dx \quad v = -\cos x$

$$= \ln x (-\cos x) - \int -\cos x \cdot \frac{1}{x} dx =$$

$$-\ln x \cos x + \underbrace{\int \frac{\cos x}{x} dx}_{\text{no elementary integral}}$$

Ex: Find $\int x^2 \cos x \, dx$

$u = x^2 \qquad du = 2x\,dx$

$dv = \cos x \, dx \qquad v = \sin x$

$$\int x^2 \cos x \, dx = x^2 \sin x - \int \sin x \cdot 2x \, dx$$

$$= x^2 \sin x - 2 \underbrace{\int x \cdot \sin x \, dx}_{\text{IBP again!}}$$

$u = x \qquad du = dx$

$dv = \sin x\,dx \qquad v = -\cos$

$$= x^2 \sin x - 2\left[-x\cos x + \sin x + C\right]$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

not $-2C$.

Since $-2C$ is still just a constant, we rename it C.

IP: Check

$$\frac{d}{dx}\left(x^2 \sin x + 2x\cos x - 2\sin x\right) =$$

$$\cancel{2x\sin x} + x^2\cos x + \cancel{2\cos x} + \cancel{2x(-\sin x)} - \cancel{2\cos x}$$

$$= x^2 \cos x \,!$$

Integrals of the following forms:

$$\int x^n e^x \, dx, \quad \int x^n \sin x \, dx, \quad \int x^n \cos x \, dx$$

all require n iterations of the IBP process

<u>Ex:</u> $\int \underbrace{x^3}_{u} \underbrace{e^x \, dx}_{dv}$

$u = x^3 \quad du = 3x^2 dx$

$dv = e^x dx \quad v = e^x$

$$\int x^3 e^x \, dx = x^3 e^x - \int e^x \cdot 3x^2 \, dx$$

$u = x^2 \quad du = 2x dx$

$$= x^3 e^x - 3 \int \underbrace{x^2}_{u} \underbrace{e^x \, dx}_{dv}$$

$dv = e^x dx \quad v = e^x$

$$= x^3 e^x - 3 \left[x^2 e^x - \int e^x \cdot 2x \, dx \right]$$

$$= x^3 e^x - 3x^2 e^x + 3 \int 2x e^x \, dx$$

$u = x \quad du = dx$

$dv = e^x dx \quad v = e^x$

$$= x^3 e^x - 3x^2 e^x + 6 \int \underbrace{x}_{u} \underbrace{e^x \, dx}_{dv}$$

$$= x^3 e^x - 3x^2 e^x + 6 \left[x e^x - \int e^x \, dx \right]$$

$$= \boxed{x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C}$$

For these types of integrals we can use the tabular method:

<u>Ex:</u> $\int x^3 e^x \, dx$

differentiate to 0 ↓   ↓ anti-differentiate to same level

u		dv
x^3	$+$	e^x
$3x^2$	$-$	e^x
$6x$	$+$	e^x
6	$-$	e^x
0		e^x

so $\int x^3 e^x \, dx =$

$$+x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

Just as before!

<u>Ex:</u> $\int x^2 \sin(3x) \, dx$

u		dv
x^2	$+$	$\sin(3x)$
$2x$	$-$	$-\cos(3x)/3$
2	$+$	$-\sin(3x)/9$
0		$\cos(3x)/27$

$$\int x^2 \sin(3x) \, dx =$$

$$+x^2\left(-\frac{\cos(3x)}{3}\right) - 2x\left(\frac{-\sin(3x)}{9}\right) + 2\left(\frac{\cos(3x)}{27}\right) + C$$

$$= \boxed{-\frac{x^2\cos(3x)}{3} + \frac{2x\sin(3x)}{9} + \frac{2\cos(3x)}{27} + C}$$

or $-\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + C.$

Check: (Unnecessary, but nice to see)

$$\frac{d}{dx}\left[-\frac{1}{3}x^2\cos(3x)+\frac{2}{9}x\sin(3x)+\frac{2}{27}\cos(3x)\right]=$$

$$-\frac{1}{3}\left(2x\cos(3x)-x^2\sin(3x)\cdot 3\right)+\frac{2}{9}\left(\sin(3x)+x\cos(3x)3\right)+\frac{2}{27}(-\sin(3x)3)$$

$$=\cancel{-\frac{2}{3}x\cos(3x)}+x^2\sin(3x)+\cancel{\frac{2}{9}\sin(3x)}+\cancel{\frac{2}{3}x\cos(3x)}\cancel{-\frac{2}{9}\sin(3x)}$$

$$=x^2\sin(3x).\quad\checkmark$$

P Use tabular to find $\int x^4 e^{-2x}\,dx$

u	sign	dv
x^4	$+$	e^{-2x}
$4x^3$	$-$	$-\frac{1}{2}e^{-2x}$
$12x^2$	$+$	$\frac{1}{4}e^{-2x}$
$24x$	$-$	$-\frac{1}{8}e^{-2x}$
24	$+$	$\frac{1}{16}e^{-2x}$
0		$-\frac{1}{32}e^{-2x}$

$$x^4\left(-\frac{1}{2}e^{-2x}\right)-4x^3\left(\frac{1}{4}e^{-2x}\right)+12x^2\left(-\frac{1}{8}e^{-2x}\right)-24x\left(\frac{1}{16}e^{-2x}\right)+24\left(-\frac{1}{32}e^{-2x}\right)+C$$

$$=\boxed{-\frac{1}{2}x^4e^{-2x}-x^3e^{-2x}-\frac{3}{2}x^2e^{-2x}-\frac{3}{2}xe^{-2x}-\frac{3}{4}e^{-2x}+C}$$

or

$$-e^{-2x}\left(\frac{1}{2}x^4+x^3+\frac{3}{2}x^2+\frac{3}{2}x+\frac{3}{4}\right)+C$$

* or maybe $\int x^3e^{-2x}\,dx$

Ex: $\int x^4 \ln x \, dx$ does not require tabular!

Only uses IBP once!

$$\left.\begin{aligned} u &= \ln x & du &= \tfrac{1}{x}\,dx \\ dv &= x^4\,dx & v &= \tfrac{x^5}{5} \end{aligned}\right\} \underbrace{\ln x}_{u}\underbrace{\left(\frac{x^5}{5}\right)}_{v} - \int \underbrace{\frac{x^5}{5}}_{v} \cdot \underbrace{\frac{1}{x}\,dx}_{du}$$

$$= \frac{x^5 \ln x}{5} - \frac{1}{5}\int x^4\,dx = \boxed{\frac{x^5 \ln x}{5} - \frac{x^5}{25} + C}$$

Ex: $\int \ln x \, dx$

$$\begin{aligned} u &= \ln x & du &= \tfrac{1}{x}\,dx \\ dv &= 1 \cdot dx & v &= x \end{aligned}$$

$$\int \ln x \, dx = \ln x \cdot x - \int x \cdot \frac{1}{x}\,dx$$

$$= x \ln x - \int 1 \cdot dx$$

$$= \boxed{x \ln x - x + C}$$

* Probably worth committing to memory

Ex: $\int \arcsin(x)\,dx$

$u = \arcsin x \qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dv = dx \qquad v = x$

$$\int \arcsin(x)\,dx = x \arcsin x - \underbrace{\int \frac{x}{\sqrt{1-x^2}}\,dx}_{u-\text{sub}}$$

$u = 1-x^2$

$du = -2x\,dx \rightarrow \frac{du}{-2} = x\,dx$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-1/2}\,du$$

$$= -\frac{1}{2}\,\frac{u^{1/2}}{1/2} = -u^{1/2} = -\sqrt{1-x^2}$$

So, $\int \arcsin(x)\,dx = x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$$= x \arcsin(x) - \left(-\sqrt{1-x^2}\right) + C$$

$$= \boxed{x \arcsin(x) + \sqrt{1-x^2} + C}$$

So what is $\int_{-1/2}^{1} \arcsin(x)\,dx$?

$$\int_{-1/2}^{1} \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2}\,\Big|_{-1/2}^{1} =$$

$$\left[1\cdot\underbrace{\arcsin(1)}_{\pi/2} + \sqrt{1-1^2}\right] - \left[-\tfrac{1}{2}\underbrace{\arcsin(-\tfrac{1}{2})}_{-\pi/6} + \sqrt{1-(-\tfrac{1}{2})^2}\right]$$

$$= \frac{\pi}{2} + 0 - \left[\frac{\pi}{12} + \sqrt{3/4}\right] = \boxed{\frac{5\pi}{12} - \frac{\sqrt{3}}{2}}$$

P: $\int \arctan(x)\,dx$ $\quad u = \arctan(x) \quad du = \frac{1}{1+x^2}dx$

$dv = dx \quad v = x$

$$\int \arctan(x)\,dx = x\arctan x - \underbrace{\int \frac{x}{1+x^2}\,dx}_{u\text{-sub}}$$

$$= \boxed{x\arctan x - \tfrac{1}{2}\ln(1+x^2) + C}$$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

Why not $\ln|1+x^2|$?

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln|1+x^2|$$

P Show $\int_{-\sqrt{3}}^{\sqrt{3}} \arctan x\,dx = 0$

* $\int_{-2}^{2} \arctan(x)\,dx = 0$

$$\int e^{\theta} \sin\theta \, d\theta \qquad u = \sin\theta \qquad dv = e^{\theta} d\theta$$

$$du = \cos\theta \, d\theta \qquad v = e^{\theta}$$

$$\int e^{\theta} \sin\theta \, d\theta = e^{\theta} \sin\theta - \underbrace{\int e^{\theta} \cos\theta \, d\theta}$$

$$u = \cos\theta \qquad dv = e^{\theta} d\theta$$

$$du = -\sin\theta \, d\theta \qquad v = e^{\theta}$$

$$= e^{\theta} \sin\theta - \left[e^{\theta} \cos\theta - \int e^{\theta} (-\sin\theta) \, d\theta \right]$$

So $\underline{\int e^{\theta} \sin\theta \, d\theta} = e^{\theta} \sin\theta - e^{\theta} \cos\theta - \underline{\int e^{\theta} \sin\theta \, d\theta}$

$$2 \int e^{\theta} \sin\theta \, d\theta = e^{\theta} \sin\theta - e^{\theta} \cos\theta$$

$$\int e^{\theta} \sin\theta \, d\theta = \frac{1}{2} \left(e^{\theta} \sin\theta - e^{\theta} \cos\theta \right) + C$$

* you can let $u = e^{\theta}$ each time instead of the trig function.

Ex: $\int e^{3x}\cos(x)\,dx$

$u = e^{3x}$ $\quad du = 3e^{3x}dx$

$dv = \cos x\,dx$ $\quad V = \sin x$

$$\int e^{3x}\cos(x)\,dx = e^{3x}\sin x - 3\underbrace{\int e^{3x}\sin x\,dx}$$

$u = e^{3x}$ $\quad du = 3e^{3x}dx$

$dv = \sin x\,dx$ $\quad V = -\cos x$

$$= e^{3x}\sin x - 3\left[-e^{3x}\cos x - 3\int e^{3x}(-\cos x)\,dx\right];$$ so

$$\int e^{3x}\cos(x)\,dx = e^{3x}\sin x + 3e^{3x}\cos x - 9\int e^{3x}\cos x\,dx$$

$+9\int e^{3x}\cos(x)\,dx$ $\qquad\qquad$ $+9\int e^{3x}\cos x\,dx$

$$10\int e^{3x}\cos x\,dx = e^{3x}\sin x + 3e^{3x}\cos x$$

$$\int e^{3x}\cos x\,dx = \boxed{\frac{e^{3x}\sin x + 3e^{3x}\cos x}{10} + C}$$

Versus: $\int e^{3x}\cos(x)\,dx$

$u = \cos(x) \qquad du = -\sin(x)\,dx$

$dv = e^{3x}dx \qquad v = e^{3x}/3$

$$\int e^{3x}\cos x\,dx = \frac{e^{3x}\cos x}{3} - \frac{1}{3}\int e^{3x}(-\sin x)\,dx$$

$$= \frac{e^{3x}\cos x}{3} + \frac{1}{3}\underbrace{\int e^{3x}\sin x\,dx}$$

$u = \sin x \qquad du = \cos x\,dx$

$dv = e^{3x}dx \qquad v = e^{3x}/3$

$$= \frac{e^{3x}\cos x}{3} + \frac{1}{3}\left[\frac{e^{3x}\sin x}{3} - \frac{1}{3}\int e^{3x}\cos x\,dx\right]$$

$$\frac{9}{9}\int e^{3x}\cos x\,dx = \frac{e^{3x}\cos x}{3} + \frac{e^{3x}\sin x}{9} - \frac{1}{9}\int e^{3x}\cos x\,d$$

$$\frac{10}{9}\int e^{3x}\cos(x)\,dx = \frac{e^{3x}\cos x}{3} + \frac{e^{3x}\sin x}{9}$$

$$\int e^{3x}\cos(x)\,dx = \boxed{\frac{3e^{3x}\cos x + e^{3x}\sin x}{10} + C}$$