Lecture 2 | Math 1360 | Section 5.5

- Hard u-substitution
- Back substitution
- Definite u-substitution.

Ex: $\int \frac{e^x}{1+e^{2x}}\,dx$

Let $u = e^x$,

$\underbrace{du = e^x\,dx}_{\text{good! we have that!}}$

$$\int \frac{\overbrace{e^x\,dx}^{du}}{1+e^{2x}} = \int \frac{du}{1+e^{2x}} = \int \frac{1}{1+(\underset{u}{e^x})^2}\,du$$

$$= \int \frac{1}{1+u^2}\,du = \int \frac{1}{u^2+1}\,du = \arctan(u)+C$$

$$= \boxed{\arctan(e^x)+C}$$

PP: Try $\int \frac{4x^3}{\sqrt{1-x^8}}\,dx$ w/ $u = x^4$

Answer: $\arcsin(x^4)+C$ or $\sin^{-1}(x^4)+C$

Ex: $\int \underbrace{\cos^6 x}_{u^6} \underbrace{\sin x \, dx}_{-du}$

$u = \cos x$

($\cos x$ is raised to a power)

$$= -\int u^6 \, du$$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= -\frac{u^7}{7} + C = \boxed{-\frac{\cos^7 x}{7} + C}$$

P: Try $\int \tan^3 x \underbrace{\sec^2 x \, dx}_{du}$

$\left.\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array}\right\}$ $\int \tan^3 x \, du = \int u^3 \, du$

$$= \frac{u^4}{4} + C = \boxed{\frac{\tan^4 x}{4} + C}$$

* it also works (here) to let $u = \sec x$ as we'll see in section 6.2. Although the answer requires trig identities and is not nearly as nice.

Back Substitution:

Ex: $\int x^9 \sqrt[3]{x^5+4}\, dx$

The only plausible choice is $u = x^5 + 4$

$du = 5x^4\, dx$

$\frac{du}{5} = x^4\, dx$

$$\int \underbrace{x^9}_{x^5 \cdot x^4} \sqrt[3]{x^5+4}\, dx = \int x^5 \sqrt[3]{x^5+4}\, \underbrace{x^4\, dx}_{du/5}$$

$$= \frac{1}{5}\int \underbrace{x^5}_{u-4} \sqrt[3]{\underbrace{x^5+4}_{u}}\, du = \frac{1}{5}\int (u-4)\sqrt[3]{u}\, d$$

$$= \frac{1}{5}\int (u-4)\, u^{1/3}\, du = \frac{1}{5}\int u^{4/3} - 4u^{1/3}\, du$$

$$= \frac{1}{5}\left[\frac{3}{7}u^{7/3} - \underbrace{3}_{\frac{4}{4/3}} u^{4/3}\right] + C$$

$$= \boxed{\frac{3}{35}(x^5+4)^{7/3} - \frac{3}{5}(x^5+4)^{4/3} + C}$$

P: $\int x^3 (x^2-9)^{98}\, dx$

Soln: $u = x^2 - 9 \longrightarrow du = 2x\,dx$

$\downarrow$ $\qquad \frac{du}{2} = x\,dx$

$x^2 = u + 9$

$$\int x^3 (x^2-9)^{98}\, dx = \int \underbrace{x^2}_{u+9} (\underbrace{x^2 \quad}_{u})^{98} \underbrace{x\,dx}_{\frac{du}{2}}$$

$$= \frac{1}{2}\int (u+9)\, u^{98}\, du = \frac{1}{2}\int u^{99} + 9u^{98}\, du$$

$$= \frac{1}{2}\left[\frac{u^{100}}{100} + \frac{9u^{99}}{99}\right] + c =$$

$$\frac{1}{2}\left[\frac{(x^2-9)^{100}}{100} + \frac{u^{99}}{11}\right] + c = \frac{(x^2-9)^{100}}{200} + \frac{(x^2-9)^{99}}{22} + c$$

Ex: $\displaystyle\int \frac{x^3}{\sqrt{1-x^8}}\,dx$

$u = 1-x^8$ seems like a good choice,

but $du = -8x^7\,dx$ is problematic...

we don't have $x^7\,dx$ and getting

it, although possible, is unwieldy.

we have $x^3\,dx$, so let $u = x^4$

to get $du = 4x^3\,dx \rightarrow \frac{du}{4} = x^3 dx$

$$\int \frac{\overbrace{x^3\,dx}^{du/4}}{\sqrt{1-\underbrace{x^8}_{u^2}}} = \frac{1}{4}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{4}\arcsin(u) + C = \boxed{\frac{1}{4}\arcsin(x^4) + C}$$

Definite Integrals w/ u-sub:

$$\int_{-3}^{3} \frac{x}{x^2+1}\,dx$$

① $u = x^2+1 \longrightarrow du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$\int_{x=-3}^{x=3} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_{x=-3}^{x=3} \frac{du}{u} =$$

$$\frac{1}{2}\ln|u|\Big|_{x=-3}^{x=3} = \frac{1}{2}\ln|x^2+1|\Big|_{-3}^{3}$$

$$= \frac{1}{2}\ln|3^2+1| - \frac{1}{2}\ln|(-3)^2+1| = 0$$

② $$\int_{x=-3}^{x=3} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_{u=10}^{u=10} \frac{1}{u}\,du = 0$$

③ $f(x) = \frac{x}{x^2+1}$ is odd! $\int_{-3}^{3} \frac{x}{x^2+1}\,dx = 0.$

Ex:
$$\int_0^{\pi/2} \sin^4 x \cos x \, dx$$

$u = \sin x \qquad x = 0 \longrightarrow u = \sin(0) = 0$

$du = \cos x \, dx \qquad x = \pi/2 \longrightarrow u = \sin(\pi/2) = 1$

$u=1$

$$\int_0^{\pi/2} \sin^4 x \, \underbrace{\cos x \, dx}_{du} = \int_0^1 u^4 \, du$$

$u=0 \qquad u^4$

$$= \frac{u^5}{5}\Big|_0^1 = \frac{1^5}{5} - \frac{0^5}{5} = \boxed{\frac{1}{5}}$$

P
$$\int_0^1 x^2 (x^3-1)^{10} \, dx$$

$\frac{du}{3}$

$u = x^3 - 1$

$du = 3x^2 dx$

$\frac{du}{3} = x^2 dx$

$x = 0 \longrightarrow u = 0^3 - 1 = -1$

$x = 1 \longrightarrow u = 1^3 - 1 = 0$

$$\frac{1}{3}\int_{-1}^0 u^{10} \, du = \frac{1}{3} \cdot \frac{u^{11}}{11}\Big|_{-1}^0$$

$$= \frac{u^{11}}{33}\Big|_{-1}^0 = \frac{0^{11}}{33} - \frac{(-1)^{11}}{33} = \boxed{\frac{1}{33}}$$

$$\int_{-1}^{1} 3x^5 \left(x^3+1\right)^4 \, dx$$

$u = x^3+1 \rightarrow x^3 = u-1$

$du = 3x^2 \, dx$

$x=-1 \rightarrow u = (-1)^3+1 = 0$

$x = 1 \rightarrow u = 1^3+1 = 2$

$$= \int_{-1}^{1} \underset{u-1}{x^3} \Big(\underbrace{x^3+1}_{u}\Big)^4 \underbrace{3x^2 dx}_{du}$$

$$= \int_0^2 (u-1)\cdot u^4 \, du = \int_0^2 u^5 - u^4 \, du =$$

$$\frac{u^6}{6} - \frac{u^5}{5} \Bigg|_0^2 = \left(\frac{2^6}{6} - \frac{2^5}{5}\right) - \Bigg(\underbrace{\frac{0^6}{6} - \frac{0^5}{5}}_{0}\Bigg)$$

$$= \frac{64}{6} - \frac{32}{5} = \frac{32}{3} - \frac{32}{5} = 32\Bigg(\underbrace{\frac{1}{3} - \frac{1}{5}}_{2/15}\Bigg) = \boxed{\frac{64}{15}}$$

P: $\displaystyle\int_0^{\pi/4} \sec^2 x \, e^{5\overbrace{\tan x}^{u}} \, dx$ (du)

$u = \tan x$

$du = \sec^2 x \, dx$

$x = 0 \rightarrow u = \tan 0 = 0$

$x = \pi/4 \rightarrow u = \tan \pi/4 = 1$

$$= \int_0^1 e^{5u} \, du$$

$$= \frac{e^{5u}}{5} \Bigg|_0^1 = \frac{e^5}{5} - \frac{e^0}{5} = \boxed{\frac{e^5-1}{5}}$$

An interesting (and necessary) u-sub.

$\int \frac{1}{x^2+9}\,dx$ looks similar to $\int \frac{1}{x^2+1}\,dx = \arctan(x) + C.$

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{9\left(\frac{x^2}{9}+1\right)}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx$$

$u = \frac{x}{3}$

$du = \frac{1}{3}\,dx$

$3\,du = dx$

$$= \frac{1}{9}\int \frac{1}{u^2+1}\cdot 3\,du = \frac{1}{3}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{3}\arctan(u) + C = \boxed{\frac{1}{3}\arctan\left(\frac{x}{3}\right) + C}.$$

In a similar way we can see:

$$\int \frac{1}{x^2+25}\,dx = \frac{1}{5}\arctan\left(\frac{x}{5}\right) + C$$

$$\int \frac{1}{x^2+7}\,dx = \frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C$$

<u>Generally:</u> $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$

We will see another way to do this in section 6.2

What about $\int \frac{1}{\sqrt{49-x^2}}\,dx$?

We know $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$

Similar process, slightly different answer than previous.

$$\int \frac{1}{\sqrt{49-x^2}}\,dx = \int \frac{1}{\sqrt{49\left(1-\frac{x^2}{49}\right)}}\,dx$$

$$= \int \frac{1}{\underbrace{\sqrt{49}}_{7}\cdot\sqrt{1-\left(\frac{x}{7}\right)^2}}\,\underbrace{dx}_{7\,du}$$

u

$u = \frac{x}{7}$

$7u = x$

$7\,du = dx$

$$= \int \frac{1}{7\cdot\sqrt{1-u^2}}\cdot 7\,du = \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \arcsin(u) + C = \boxed{\arcsin\left(\frac{x}{7}\right) + C}$$

<u>Not</u> $\frac{1}{7}\arcsin\left(\frac{x}{7}\right) + C$

P Find $\int \frac{1}{\sqrt{36-x^2}}\,dx = \arcsin\left(\frac{x}{6}\right)+C$

$$\int \frac{1}{\sqrt{5-x^2}}\,dx = \arcsin\left(\frac{x}{\sqrt{5}}\right)+C$$

Generally: $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C.$

We could also show that

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C,$$

but we'll wait and use an alternate method in 6.2.

You could try $\int \frac{1}{x\sqrt{x^2-4}}\,dx$ if you want.

Soln: $\int \frac{1}{x\sqrt{x^2-4}}\,dx = \int \frac{1}{x\sqrt{4\left(\frac{x^2}{4}-1\right)}}\,dx =$

$$\int \frac{1}{2x\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx = \int \frac{1}{2\cdot 2u\sqrt{u^2-1}}\cdot 2\,du = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du$$

$u = \frac{x}{2} \rightarrow x = 2u$

$dx = 2du$

$$= \frac{1}{2}\sec^{-1}(u)+C = \boxed{\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)+C}$$