Lecture 4 Math 1360 Section 6.2

Trigonometric Integrals:

Powers of sine & cosine:

- If only one of them is raised to an even power let u be that trig function:

$\int \sin^3 x \cos^2 x \, dx$ $\quad$ $\int \cos^5 x \, dx$ $\quad$ $\int \sin^3 x \, dx$

$u = \cos x$ $\quad$ $u = \sin x$ $\quad$ $u = \cos x$

- If both are raised to odd powers let u be the trig function raised to a higher power.

$\int \sin^5 x \cos^3 x \, dx$, $u = \sin x$

- If both are raised to even powers use:

$\sin^2 x = \dfrac{1 - \cos(2x)}{2}$, $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$

possibly multiple times.

$$\int \sin^3 x \cos^2 x \, dx$$

$u = \cos x \qquad du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= \int \underbrace{\sin^2 x}_{1-u^2} \ \underbrace{\cos^2 x}_{u^2} \ \underbrace{\sin x \, dx}_{-du}$$

$\cos^2 x + \sin^2 x = 1 \longrightarrow$

$\sin^2 x = 1 - \cos^2 x = 1 - u^2$

$$= -\int (1-u^2)u^2 \, du = \int (u^2-1)u^2 \, du = \int u^4 - u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$$

P: Try $\int \sin^3 x \, dx \qquad u = \cos x \qquad du = -\sin x \, dx$

$$= \int \sin^2 x \ \sin x \, dx = \int (1-u^2) \cdot (-du) =$$

$$\int u^2 - 1 \, du = \frac{u^3}{3} - u + C = \boxed{\frac{\cos^3 x}{3} - \cos x + C}$$

$$\int \cos^5 x \, dx \qquad u = \sin x \qquad du = \cos x \, dx$$

$$\cos^2 x = 1 - \sin^2 x = 1 - u^2$$

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx$$

$$= \int \underbrace{\cos^2 x}_{1-u^2} \cdot \underbrace{\cos^2 x}_{1-u^2} \underbrace{\cos x \, dx}_{du} = \int (1-u^2)^2 du$$

$$= \int 1 - 2u^2 + u^4 \, du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}.$$

Ex: $\int \sin^5 x \cos^3 x \, dx \qquad u = \sin x \qquad du = \cos x \, dx$

$$= \int \underbrace{\sin^5 x}_{u^5} \underbrace{\cos^2 x}_{1-u^2} \underbrace{\cos x \, dx}_{du} = \int u^5 (1-u^2) \, du$$

$$= \int u^5 - u^7 \, du = \frac{u^6}{6} - \frac{u^8}{8} + C =$$

$$\boxed{\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C}$$

<u>Note:</u> We could do this w/ $u = \cos x$, but it requires more effort

$$\int \sin^5 x \cos^3 x \, dx \qquad \begin{aligned} u &= \cos x \\ -du &= +\sin x \, dx \end{aligned}$$

$$= \int \underbrace{\sin^4 x}_{(1-u^2)^2} \underbrace{\cos^3 x}_{u^3} \underbrace{\sin x \, dx}_{-du} = -\int (1-2u^2+u^4) u^3 \, du$$

$$= \int -u^3 + 2u^5 - u^7 \, du = -\frac{u^4}{4} + \frac{2u^6}{6} - \frac{u^8}{8} + C$$

$$= \boxed{-\frac{1}{4}\cos^4 x + \frac{1}{3}\cos^6 x - \frac{1}{8}\cos^8 x + C}$$

<u>Ex:</u> $\displaystyle\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} \, dx$

$$= \frac{1}{2}\int 1 + \cos(2x) \, dx = \frac{1}{2}\left(x + \frac{\sin(2x)}{2}\right) + C$$

$$= \boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

$\underline{P}$: $\int_0^{\pi} \sin^2 x \, dx = \int_0^{\pi} \frac{1-\cos(2x)}{2} \, dx$

$= \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^{\pi} = \frac{1}{2}\left[\left(\pi - \frac{\sin(2\pi)}{2}\right) - \left(0 - \frac{\sin(0)}{2}\right)\right]$

$= \frac{1}{2}\left[(\pi - 0) - (0-0)\right] = \boxed{\frac{\pi}{2}}$

$\underline{Ex}$: $\int \sin^2 x \cos^2 x \, dx =$

$\int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} \, dx = \frac{1}{4}\int 1 - \cos^2(2x) \, dx$

$= \frac{1}{4}\int 1 - \left(\frac{1+\cos(4x)}{2}\right) dx = \frac{1}{4}\int 1 - \frac{1}{2} - \frac{\cos(4x)}{2} \, dx$

$= \frac{1}{4}\int \frac{1}{2} - \frac{\cos(4x)}{2} \, dx = \frac{1}{8}\int 1 - \cos(4x) \, dx =$

$\frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C = \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$

Similar, but worse: $\int \sin^4 x \, dx =$

$$\int \sin^2 x \cdot \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1-\cos(2x)}{2} \, dx$$

$$= \frac{1}{4} \int 1 - 2\cos(2x) + \cos^2(2x) \, dx =$$

$$\frac{1}{4} \int 1 - 2\cos(2x) + \frac{1+\cos(4x)}{2} \, dx =$$

$$\frac{1}{4} \int \frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x) \, dx =$$

$$\frac{1}{4} \left(\frac{3}{2}x - \frac{\cancel{2}\sin(2x)}{\cancel{2}} + \frac{1}{2} \frac{\sin(4x)}{4} \right) + C =$$

$$\boxed{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$$

<u>P:</u> Try $\int \cos^4 x \, dx$ or even

$\int \sin^6 x \, dx$. (Gross)

Powers of secant & tangent:

- If $\sec\theta$ is to an even power let $u = \tan\theta$

$$\int \sec^2\theta \tan\theta \, d\theta \qquad\qquad \int \sec^4\theta \tan^2\theta \, d\theta$$

$$u = \tan\theta \qquad\qquad u = \tan\theta$$

- If $\tan\theta$ is to an odd power let $u = \sec\theta$

$$\int \sec^2\theta \tan\theta \, d\theta \qquad\qquad \int \sec^3\theta \tan^3\theta \, d\theta$$

$$u = \sec\theta \qquad\qquad u = \sec\theta$$

- If Both try letting u be the one raised to a higher power (but either will work)

- If neither then we have to do IBP & [redacted] trig identity

* The idea is if we let $u = \tan\theta$, then $du = \sec^2\theta \, d\theta$. We want the remaining secant to be raised to an <u>even</u> power so we can change it to tangent. Same if $u = \sec\theta$, $du = \sec\theta \tan\theta$. Remaining tangent should be to an even power.

Ex: $\int \sec^4\theta \tan^2\theta \, d\theta \qquad u = \tan\theta, \; du = \sec^2\theta \, d\theta$

$$= \int \underbrace{\sec^2\theta}_{1+u^2} \underbrace{\tan^2\theta}_{u^2} \underbrace{\sec^2\theta \, d\theta}_{du}$$

The leftover $\sec^2\theta$ can easily be converted to something w/ u

$$= \int (1+u^2) u^2 \, du =$$

$$\int u^2 + u^4 \, du = \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$\sec^2\theta = 1 + \tan^2\theta$$
$$= 1 + u^2$$

$$= \boxed{\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C}$$

Note: $u = \sec\theta$, $du = \sec\theta \tan\theta$ does not work here!

$$\int \sec^4\theta \tan^2\theta \, d\theta = \int \underbrace{\sec^3\theta}_{u^3} \tan\theta \underbrace{\sec\theta \tan\theta \, d\theta}_{du}$$

we really want the "leftovers" to be raised to an <u>even</u> power!

$\tan^2\theta = \sec^2\theta - 1$
$= u^2 - 1$

$\tan\theta = \sqrt{u^2-1}$

$\int u^3 \cdot \sqrt{u^2-1} \, du$ is <u>not</u> nice.

$\underline{\mathbb{P}}$: Try $\int \sec^2\theta \tan\theta \, d\theta$ w/ $u = \tan\theta$

We get $\int u \, du = \frac{u^2}{2} + C = \frac{\tan^2\theta}{2} + C$

$\underline{Ex:}$ $\int \sec^3\theta \tan^3\theta \, d\theta$

$u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$$\int \underbrace{\sec^2\theta}_{u^2} \underbrace{\tan^2\theta}_{u^2-1} \underbrace{\sec\theta \tan\theta \, d\theta}_{du}$$

$\tan^2\theta = \sec^2\theta - 1$

$= u^2 - 1$

$$= \int u^2(u^2-1) \, du = \int u^4 - u^2 \, du =$$

$$\frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + C}$$

$\underline{\mathbb{P}}$: $\int \sec^2\theta \tan\theta \, d\theta$ w/ $u = \sec\theta$

We get $\int u \, du = \frac{u^2}{2} + C = \frac{\sec^2\theta}{2} + C$

$$= \frac{\tan^2\theta + 1}{2} + C = \frac{\tan^2\theta}{2} + \underbrace{\frac{1}{2} + C}_{\text{new } C}$$

$$\int \sec^4\theta \tan^5\theta \, d\theta \qquad u = \tan\theta$$
$$du = \sec^2\theta \, d\theta$$

$$\int \underbrace{\sec^2\theta}_{u^2+1} \underbrace{\tan^5\theta}_{u^5} \underbrace{\sec^2\theta \, d\theta}_{du} = \int (u^2+1) u^5 \, du$$

$$= \frac{u^8}{8} + \frac{u^6}{6} + C = \frac{\tan^8\theta}{8} + \frac{\tan^6\theta}{6} + C$$

* $u = \sec\theta$ also works, but $\tan^4\theta = (\tan^2\theta)^2 = (u^2-1)^2$ is worse.

<u>Ex:</u> $$\int \tan^3\theta \, d\theta = \int (\underbrace{\sec^2\theta - 1}_{\tan^2\theta}) \tan\theta \, d\theta$$

$$= \int \sec^2\theta \tan\theta \, d\theta - \int \tan\theta \, d\theta$$

$$= \frac{\sec^2\theta}{2} + \ln|\cos\theta| + C$$

could be $\frac{\tan^2\theta}{2}$

could be $-\ln|\sec\theta|$

Ex: $\int \csc^2\theta \cot^2\theta \, d\theta$ Same rules apply!

$u = \cot\theta$

$-du = +\csc^2\theta \, d\theta$

$$\int u^2 \cdot (-du) = -\frac{u^3}{3} + C = \boxed{\frac{-\cot^3\theta}{3} + C}$$

$$* \int \sec\theta \, d\theta = \int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\tan\theta + \sec\theta} \, d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\tan\theta + \sec\theta} \, d\theta$$

Let $u = \tan\theta + \sec\theta$, $du = (\sec^2\theta + \sec\theta\tan\theta)\,d\theta$

So we get $\int \frac{du}{u} = \ln|u| + C$,

So $\boxed{\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C}$

Observe:

$$\int \tan\theta \, d\theta = -\ln|\cos\theta| + C = \ln|\sec\theta| + C$$

$$\int \cot\theta \, d\theta = \ln|\sin\theta| + C = -\ln|\csc\theta| + C$$

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| + C$$
$$= \ln|\csc\theta - \cot\theta| + C.$$

Check: $\dfrac{d}{dx}\left(-\ln|\csc\theta + \cot\theta|\right) =$

$$-\frac{1}{\csc\theta + \cot\theta} \cdot \left(-\csc\theta\cot\theta - \csc^2\theta\right) =$$

$$\frac{\csc\theta\left(\cot\theta + \csc\theta\right)}{\left(\csc\theta + \cot\theta\right)} = \csc\theta \quad \checkmark$$

• $\dfrac{d}{dx}\ln|\csc\theta - \cot\theta| = \dfrac{1}{\csc\theta - \cot\theta} \cdot \left(-\csc\theta\cot\theta + \csc^2\theta\right)$

$$= \frac{\csc\theta\left(-\cot\theta + \csc\theta\right)}{\left(\csc\theta - \cot\theta\right)} = \csc\theta \quad \checkmark$$

Terrible integrals w/ $\sec\theta$ & $\tan\theta$:

$\int \sec^3\theta \, d\theta$

<u>IBP:</u>

$u = \sec\theta$ $\quad dv = \sec^2\theta \, d\theta$

$du = \sec\theta\tan\theta \, d\theta$ $\quad v = \tan\theta$

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \tan\theta \cdot \sec\theta\tan\theta \, d\theta$$

$$= \sec\theta\tan\theta - \int \tan^2\theta \cdot \sec\theta \, d\theta$$

$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta \, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

$$+\int \sec^3\theta \, d\theta \qquad\qquad + \int \sec^3\theta \, d\theta$$

$$2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \int \sec\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$

We could try $\int \sec x \tan^2 x \, dx$,

but really it's the same problem:

$$\int \sec x \tan^2 x \, dx = \int \sec x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \sec^3 x - \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx$$

$$= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x| \right) - \ln|\sec x + \tan x| + C$$

$$= \frac{1}{2}\left(\sec x \tan x - \ln|\sec x + \tan x| \right).$$

Ex: $\int \sec^3 x \tan^2 x \, dx$ is truly terrible

& should be completely avoided.

$$\int \sec^3 x \tan^2 x \, dx =$$

$$\int \sec^3 x \, (\sec^2 x - 1) \, dx = \underbrace{\int \sec^5 x \, dx} - \int \sec^3 x \, dx$$

$u = \sec^3 x$

$du = 3\sec^2 x \sec x \tan x \, dx$

$dv = \sec^2 x \, dx \qquad v = \tan x$

$$= \sec^3 x \tan x \underbrace{- 3\int \sec^3 x \tan^2 x \, dx}_{\int v\,du} - \underbrace{\int \sec^3 x \, dx}_{\text{we know this}}$$

So $\quad 4\int \sec^3 x \tan^2 x \, dx = \sec^3 x \tan x - \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right]$

So $\quad \int \sec^3 x \tan^2 x \, dx =$

$$\boxed{\frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x| + C}$$

So Gross