Predicting GDP growth using AR, constant growth + hybrid models

Dan Lewis

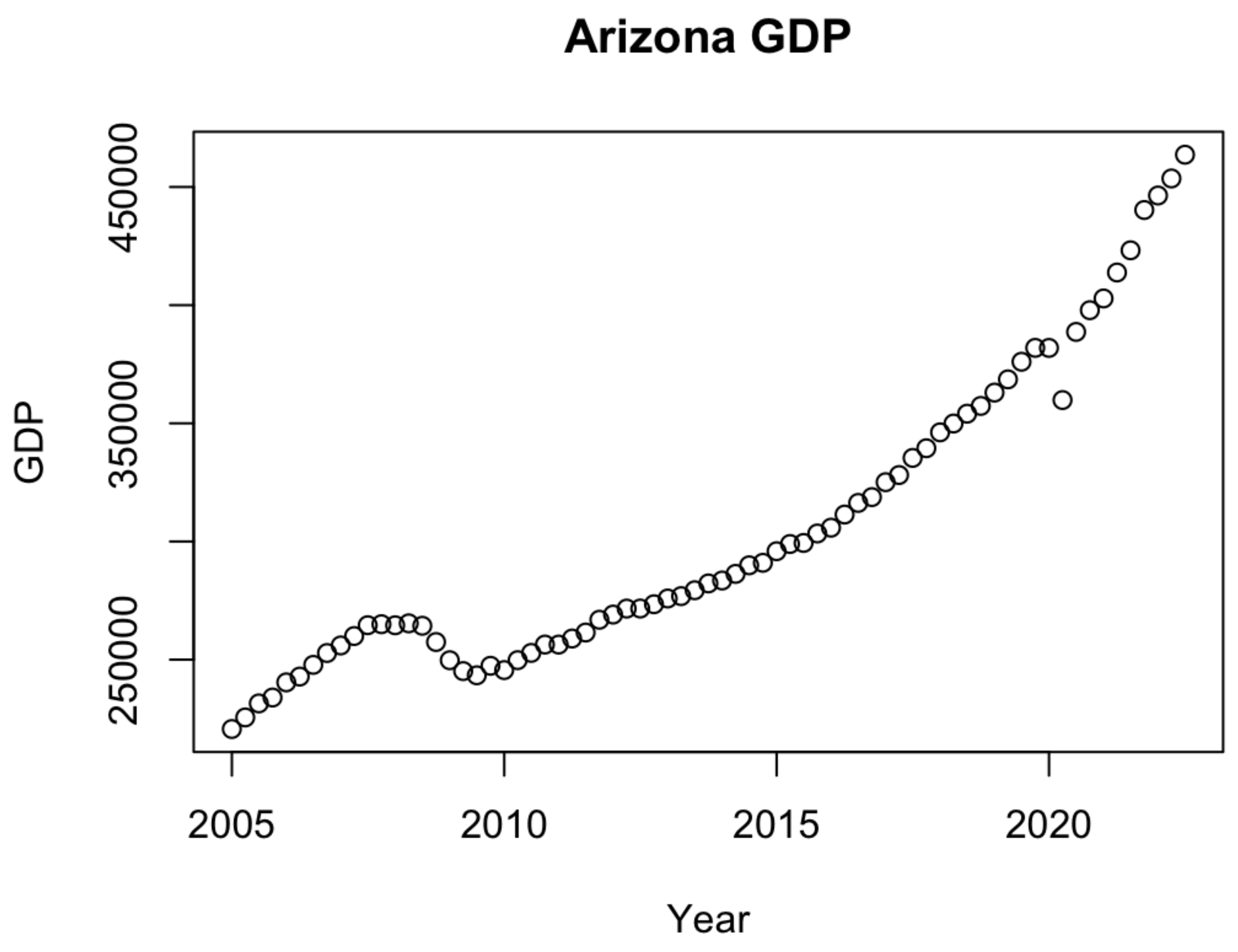
The dataset

FRED
ECONOMIC DATA | ST. LOUIS FED

- Obtained an updated dataset from the Federal Reserve Bank of St. Louis.

- Consists of quarterly data on the seasonally adjusted annual rate of the all industry GDP total, measured in millions of dollars, for ~~each of the 50 US states, together with the District of Columbia~~ the state of Arizona.

- The data stretches from 2005 Q1 to 2022 Q~~2~~ 3: ~~70~~ 71 quarters in all.



Recall the theory:

- An AR(4) model:

$$Y_t = \alpha + \rho_1 Y_{t-1} + \rho_2 Y_{t-2} + \rho_3 Y_{t-3} + \rho_4 Y_{t-4} + \epsilon_t \, .$$

- The **best** predictor of economic growth.
- Should outperform a model with fewer lags, since it approximates the data generating process (DGP) better.
- Should outperform a model with more lags since the coefficients will be estimated with more precision.

Yet, last semester…

- AR(2) performs better
  - BUT AR(16) performs best!
- Possible explanation:
  - Doesn't suffice to train the model on 50 data points to produce 20 predictions.
  - A more refined approach, e.g. k-fold cross-validation, may point to the AR(4) model as the best predictor.

-	MSFE	R^2	Runtime (s)
AR(1)	2979313641	0.4764882	0.47
AR(2)	1042557581	0.3654467	0.50
AR(3)	1334502169	0.3573387	1.00
AR(4)	1267756968	0.3614263	0.79
AR(5)	2090562081	0.3576215	0.91
AR(6)	1772792756	0.364209	1.00
AR(8)	2619002637	0.4082417	1.04
AR(12)	2952219077	0.485657	2.21
AR(16)	395146580	0.5845539	1.48

TABLE 8.2. Step two: we train each model on the first 50 data points in our sample and predict the remaining 20 data points.

Constant growth models

$$\tilde{Y}_{T+1} = \hat{\kappa} Y_T$$

where we estimate the growth rate κ as the mean

$$\hat{\kappa} = \frac{1}{T} \sum_{t=1}^{T-1} \frac{Y_{t+1}}{Y_t} .$$

<u>Hybrid models</u>

We consider a simple average of the AR(4) and constant growth predictions:

$$\check{Y}_{T+1} = \frac{\hat{Y}_{T+1} + \tilde{Y}_{T+1}}{2} .$$

Enhanced Plan of attack

- **<u>Step one:</u>**
  - Train each model on the first ~~69~~ 70 data points.
  - Compare the predicted GDP figures for 2022 Q~~2~~ 3 against the true figures.
- **<u>Step two:</u>**
  - Train the model on the first 50 datapoints in our sample, and use this to make predictions for the remaining ~~20~~ 21 datapoints.
  - Estimate the MSFE. Compare with constant growth and hybrid models.

Identification of AR(p) models

The Yule—Walker equations

$$\gamma_m = \sum_{k=1}^{p} \rho_k \gamma_{m-k} + \sigma_\epsilon^2 \delta_{m,0}$$

where $m = 0, \ldots, p$

Udny Yule (1871 — 1951)

Gilbert Walker (1868 — 1958)

Step 1 Solve

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_p \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_{-1} & \gamma_{-2} & \cdots \\ \gamma_1 & \gamma_0 & \gamma_{-1} & \cdots \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_p \end{pmatrix}$$

Step 2 Solve

$$\gamma_0 = \sum_{k=1}^{p} \rho_k \gamma_{-k} + \sigma_\epsilon^2$$

for σ_ϵ^2

Example: identifying AR(4)

$$\gamma_0 = \sum_{k=1}^{4} \rho_k \gamma_{-k} + \sigma_\epsilon^2$$

$$\gamma_1 = \sum_{k=1}^{4} \rho_k \gamma_{1-k}$$

$$\gamma_2 = \sum_{k=1}^{4} \rho_k \gamma_{2-k}$$

$$\gamma_3 = \sum_{k=1}^{4} \rho_k \gamma_{3-k}$$

$$\gamma_4 = \sum_{k=1}^{4} \rho_k \gamma_{4-k}$$

Or, in matrix form:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_{-1} & \gamma_{-2} & \gamma_{-3} \\ \gamma_1 & \gamma_0 & \gamma_{-1} & \gamma_{-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \gamma_{-1} \\ \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix}$$

(Together with the equation for γ_0 to the left)

Use that $\gamma_{-k} = \gamma_k$

So we obtain

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}$$

and

$$\sigma_\epsilon^2 = \gamma_0 - \sum_{k=1}^{p} \rho_k \gamma_{-k}$$

Bounds

A natural choice?

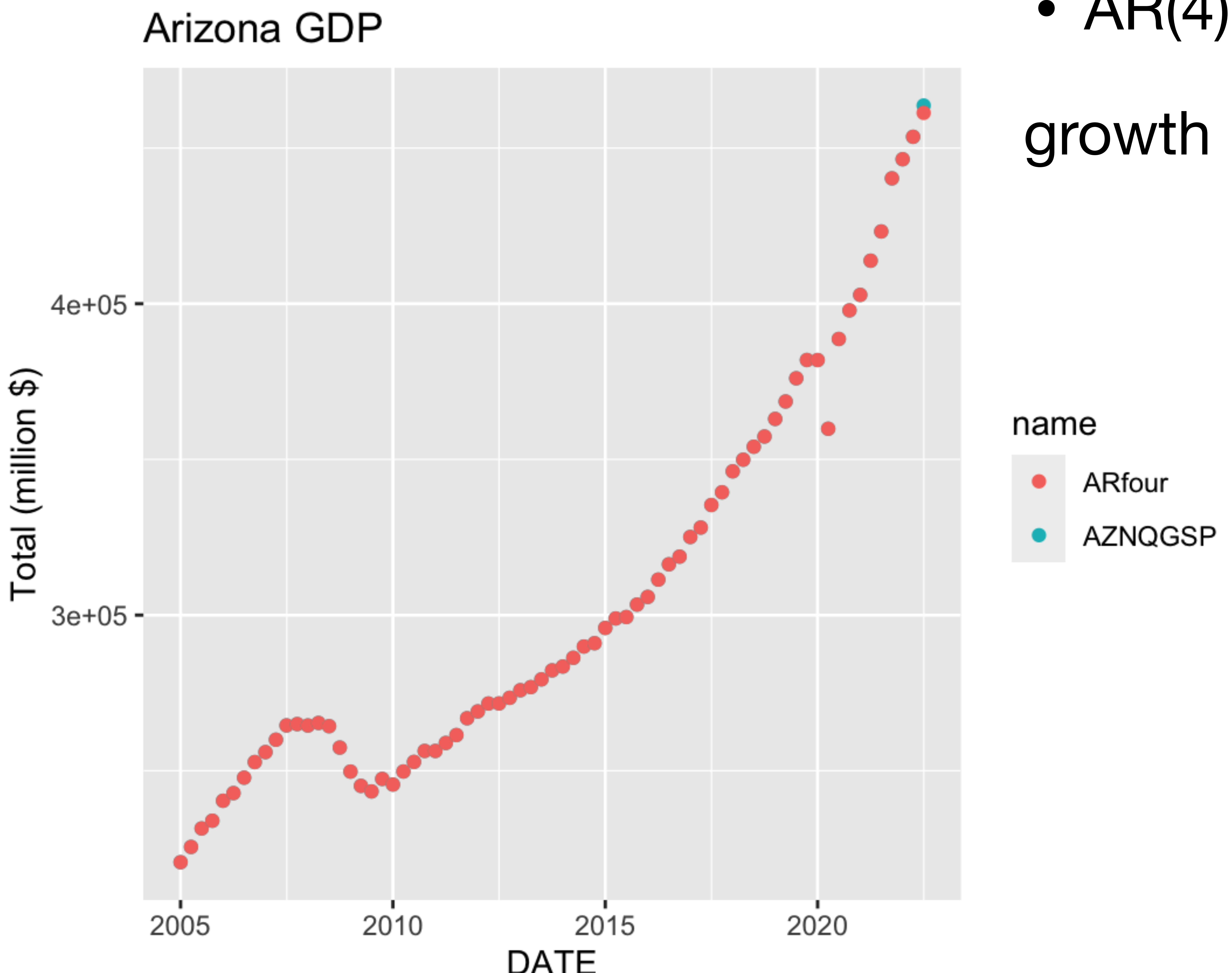
- The trend in GDP is increasing, so the estimate of GDP in any particular quarter should be at least that of the previous quarter.
- Provided the estimated growth rate $\hat{\kappa} \geq 0$, constant growth model is OK.

As a robustness check?

- Can assess visually — expect inaccurate long-run projections
- Constant growth model as a lower bound on AR predictions?

Results

Step one:



- AR(4) prediction underestimates GDP growth in 2022 Q3 by 2383.544 ($m)

Results

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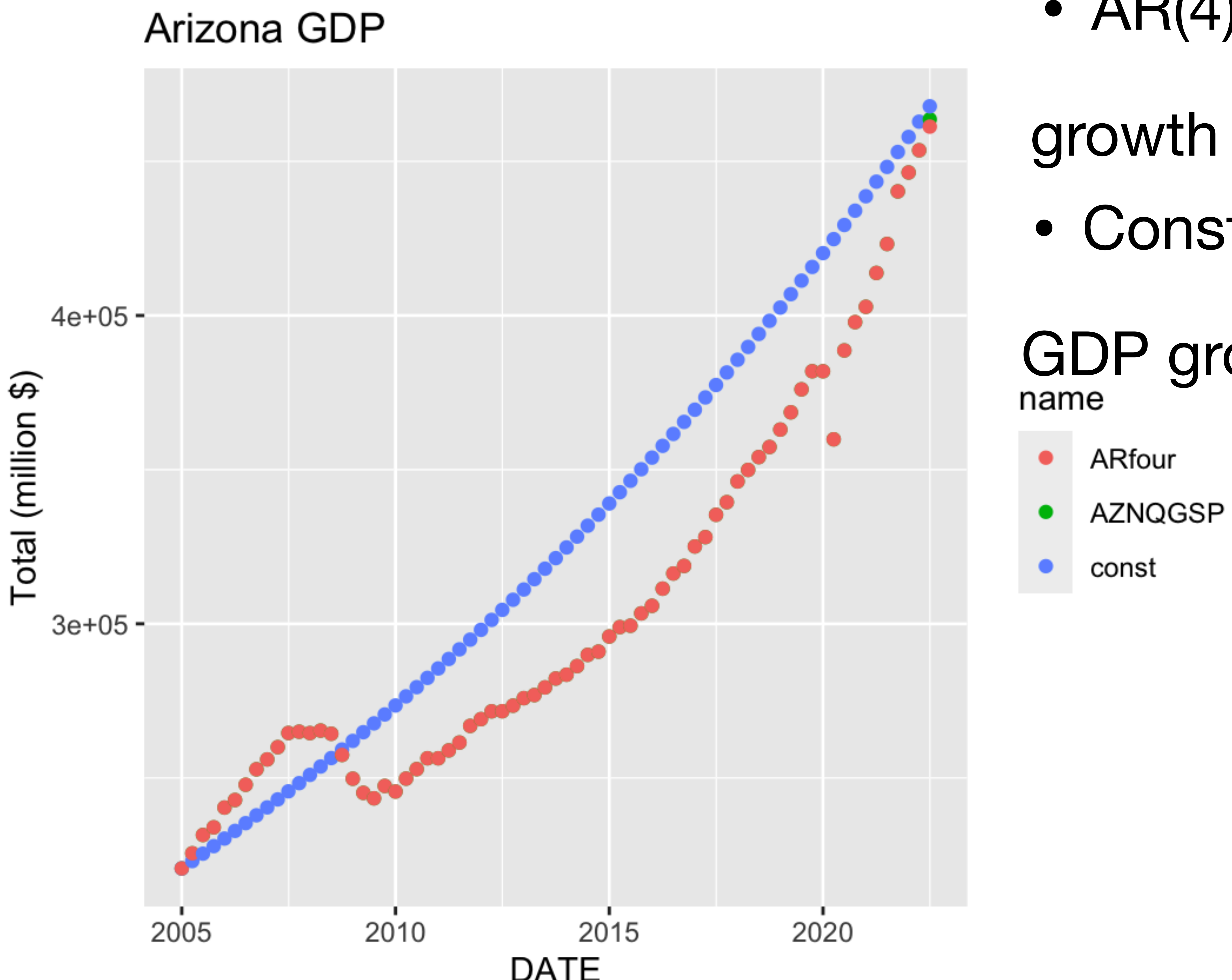
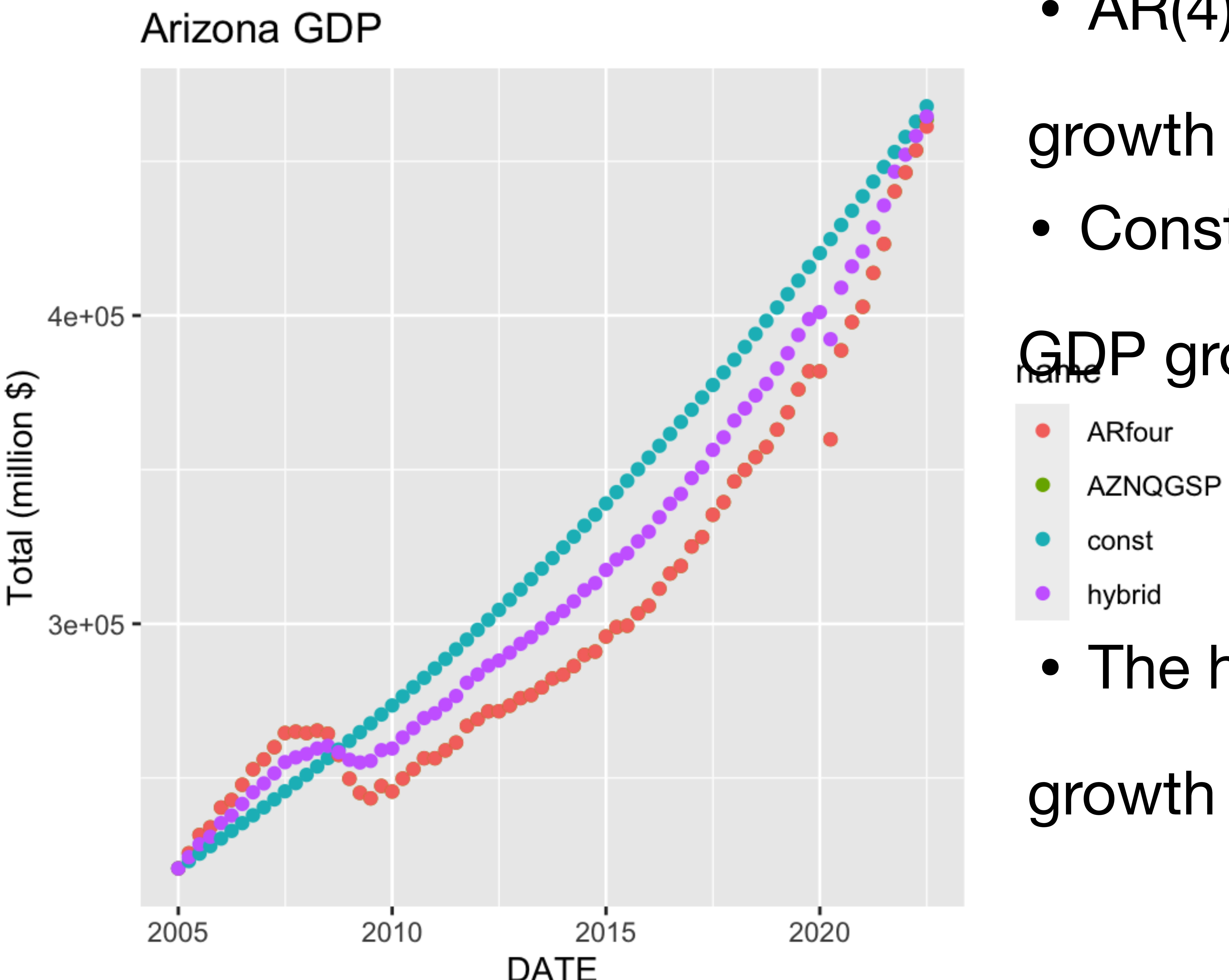


- AR(4) prediction <u>underestimates</u> GDP growth in 2022 Q3 by 2383.544 ($m)
- Constant growth prediction <u>overestimates</u> GDP growth in 2022 Q3 by 4228.378 ($m)

Results

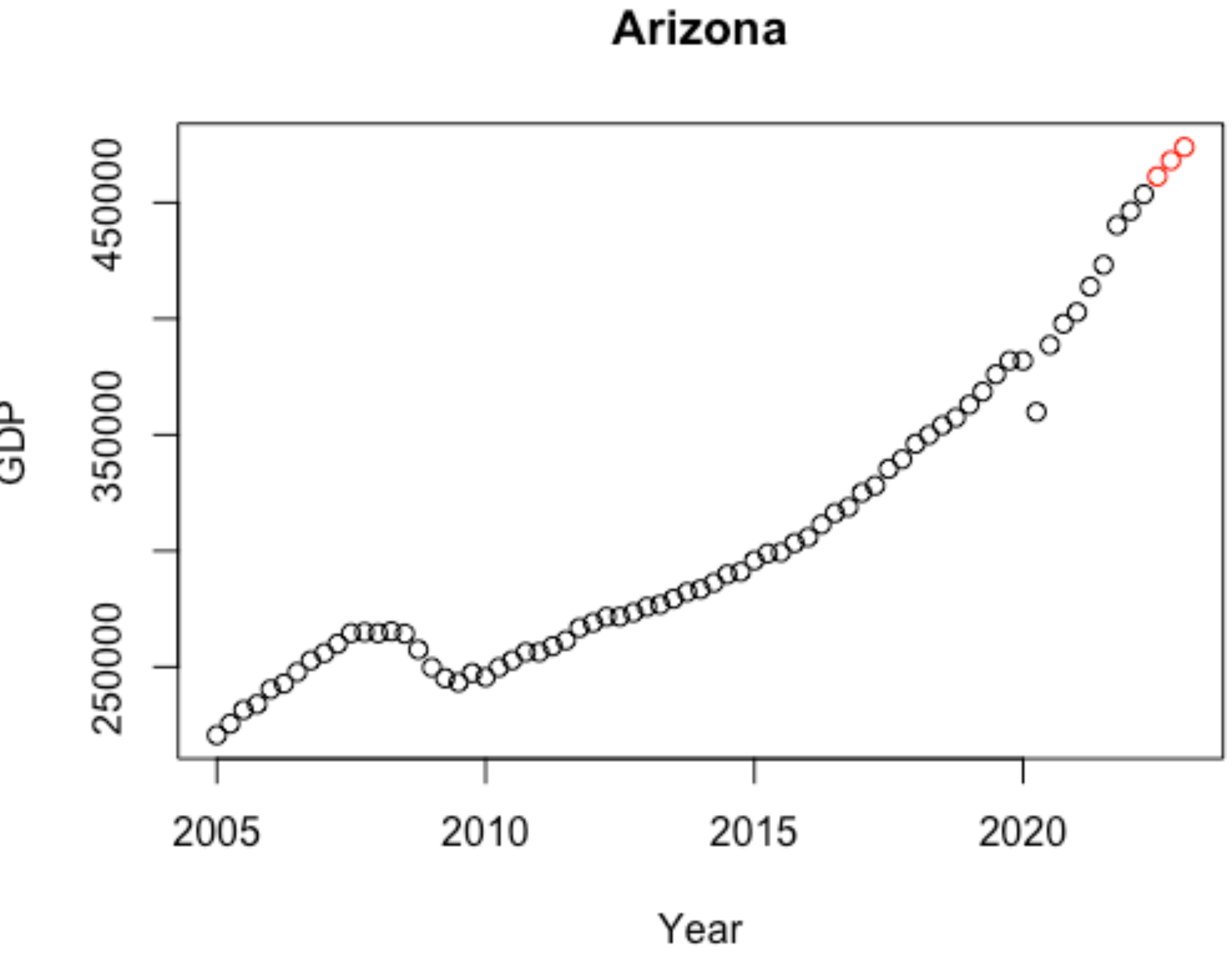
Step one:



- AR(4) prediction underestimates GDP growth in 2022 Q3 by 2383.544 ($m)
- Constant growth prediction overestimates GDP growth in 2022 Q3 by 4228.378 ($m)
- The hybrid prediction overestimates GDP growth in 2022 Q3 by 922.417 ($m)

Strange, wasn't AR(4) an overestimate last time?

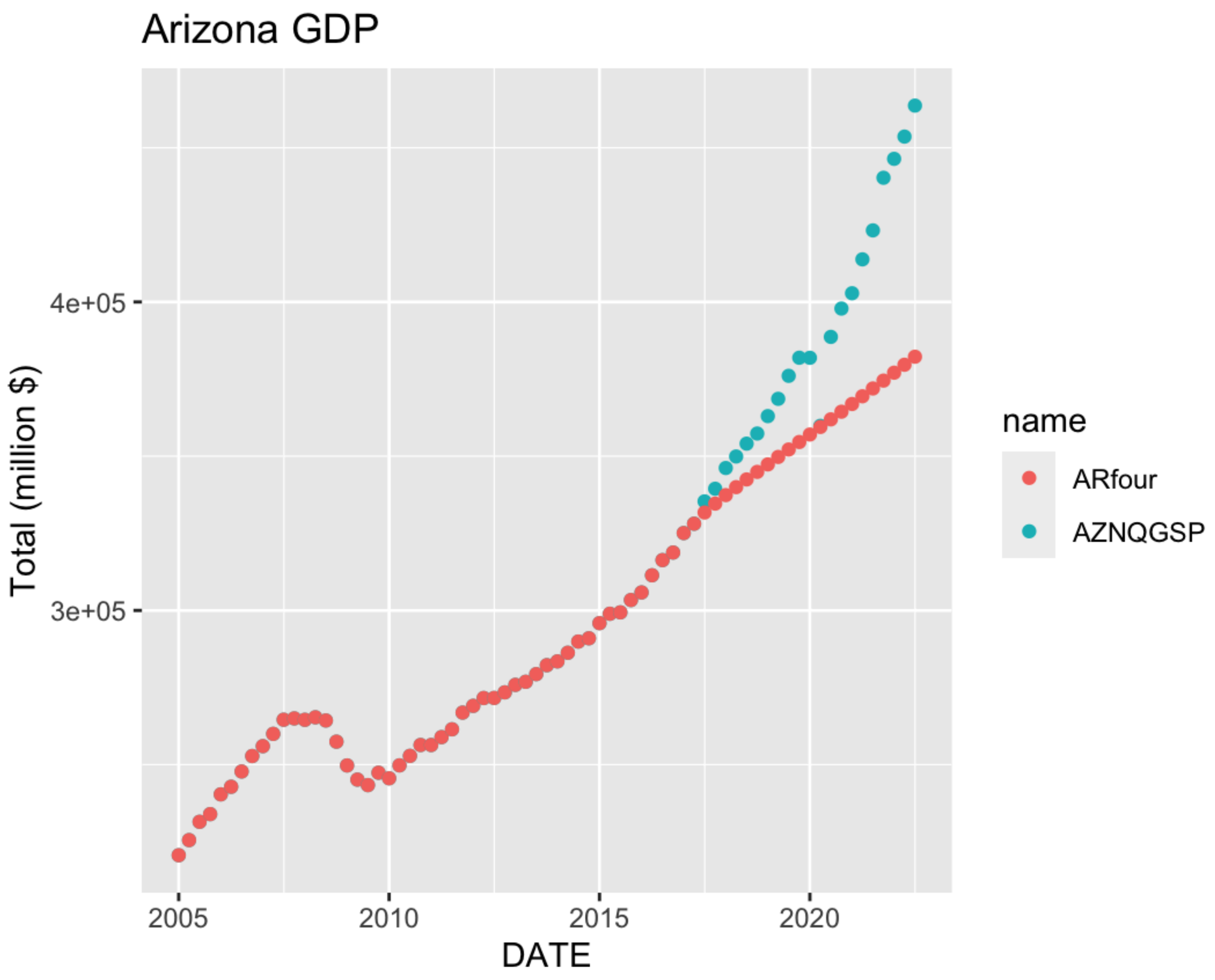
- **Step one:**
  - AR3 performs better!
  - BUT: very weak evidence (N = 1)



-	Forecast	FE	Runtime (s)
True GDP	453602.3	-	-
AR(1)	451282.5	-2319.802	0.93
AR(2)	451375.5	-2226.806	1.05
AR(3)	453340	-262.2576	1.07
AR(4)	453943.2	340.8569	0.99
AR(5)	454395.6	793.3188	1.05
AR(6)	454404.8	802.4994	0.92
AR(8)	452720.7	-881.6492	1.11
AR(12)	458074.9	4472.624	1.24
AR(16)	460015.5	6413.18	0.96

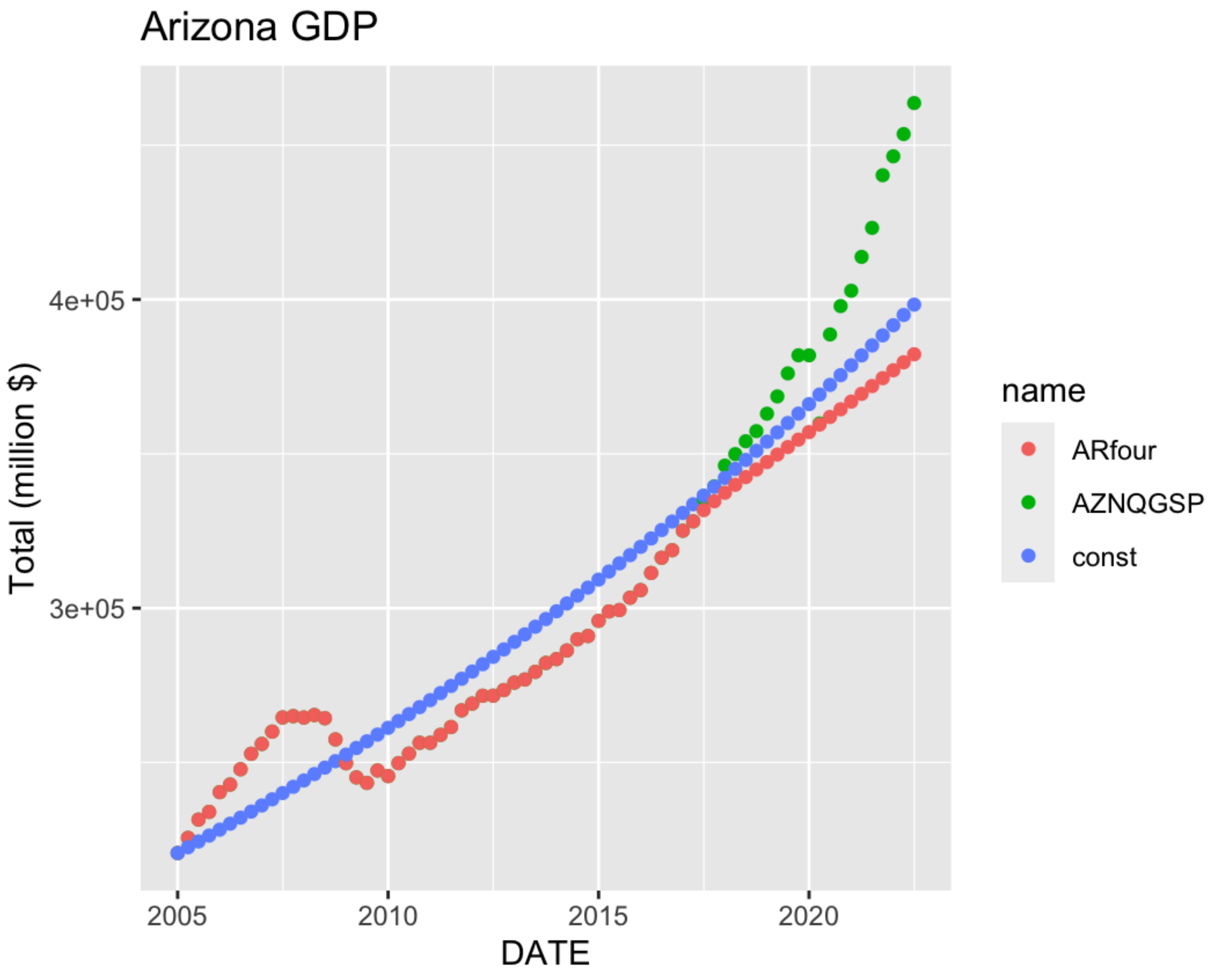
TABLE 8.1. Step one: we compare the predicted Arizona GDP figures for 2022 Q2 against the true figures provided in our dataset.

Results



Arizona GDP
Total (million $)
4e+05
3e+05
2005
2010
2015
2020
DATE
name
ARfour
AZNQGSP

Results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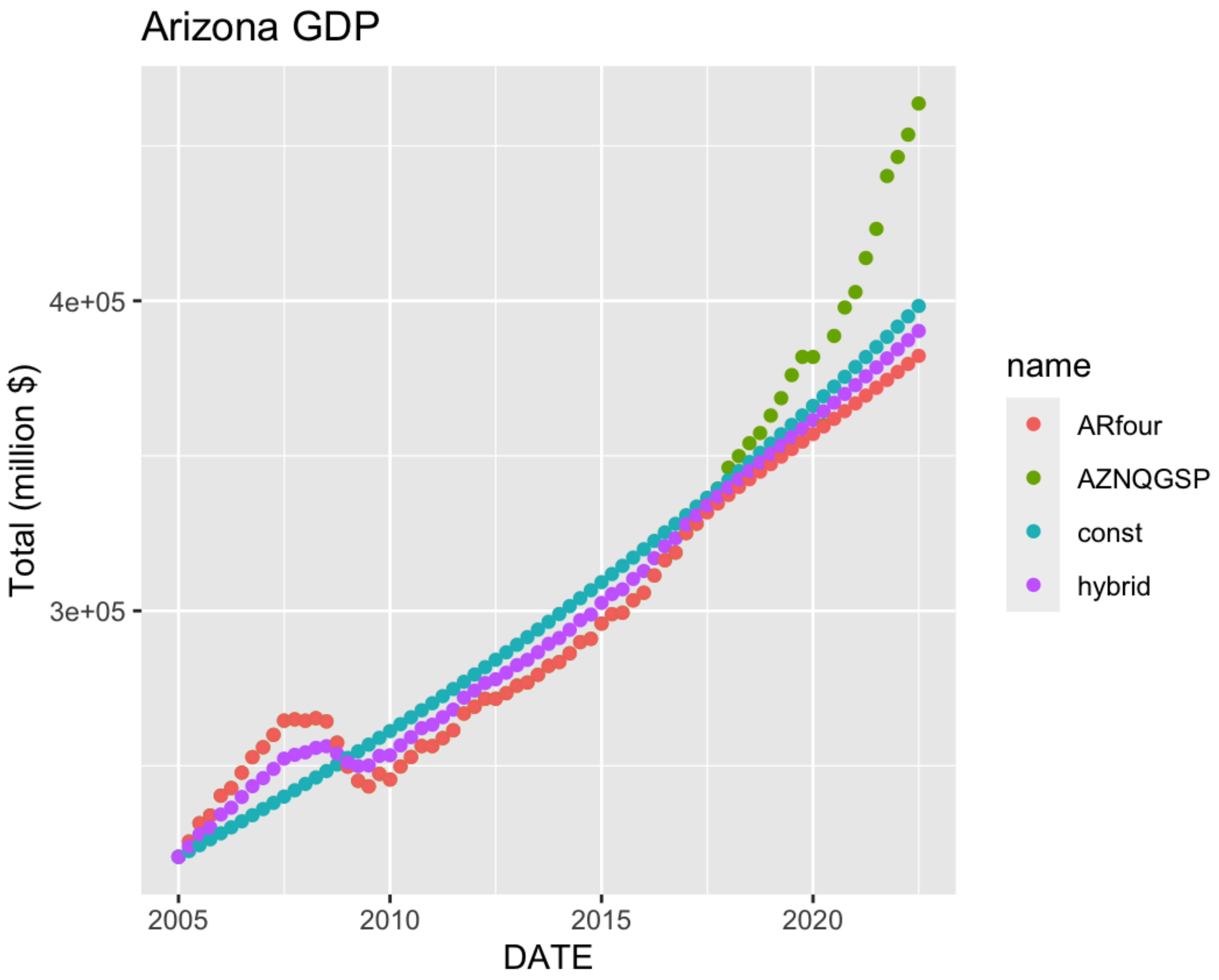
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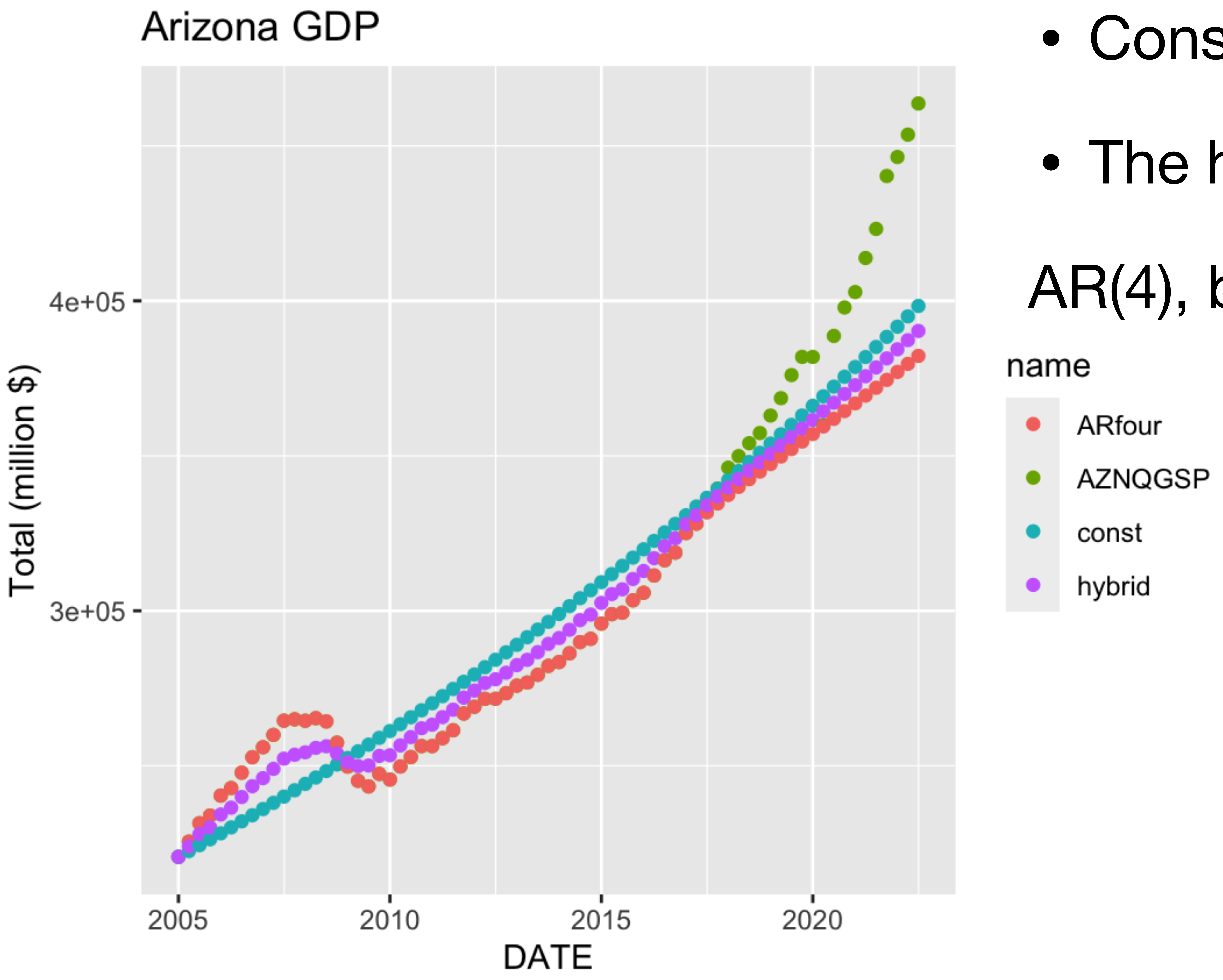
const
hybrid

Results



- Constant growth performs better
- The hybrid model fits the data better than AR(4), but less well than constant growth.

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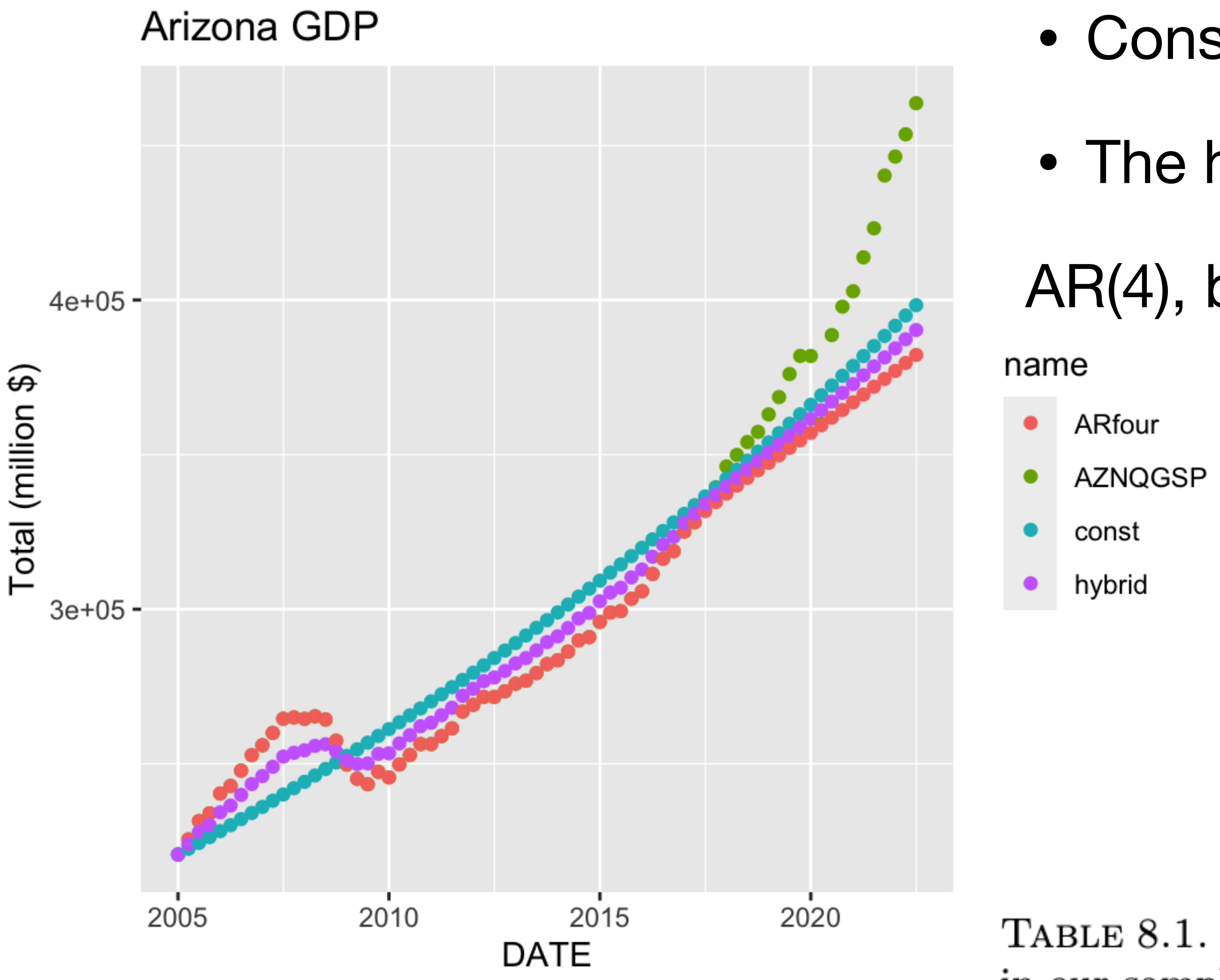

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Model	MSFE	Runtime (s)
AR(4)	1522965460	0.69
Const.	880893619	0.94
Hybrid	1176194559	1.90

TABLE 8.1. Step 2: we train each model on the first 50 data points in our sample and predict the remaining 21 data points.

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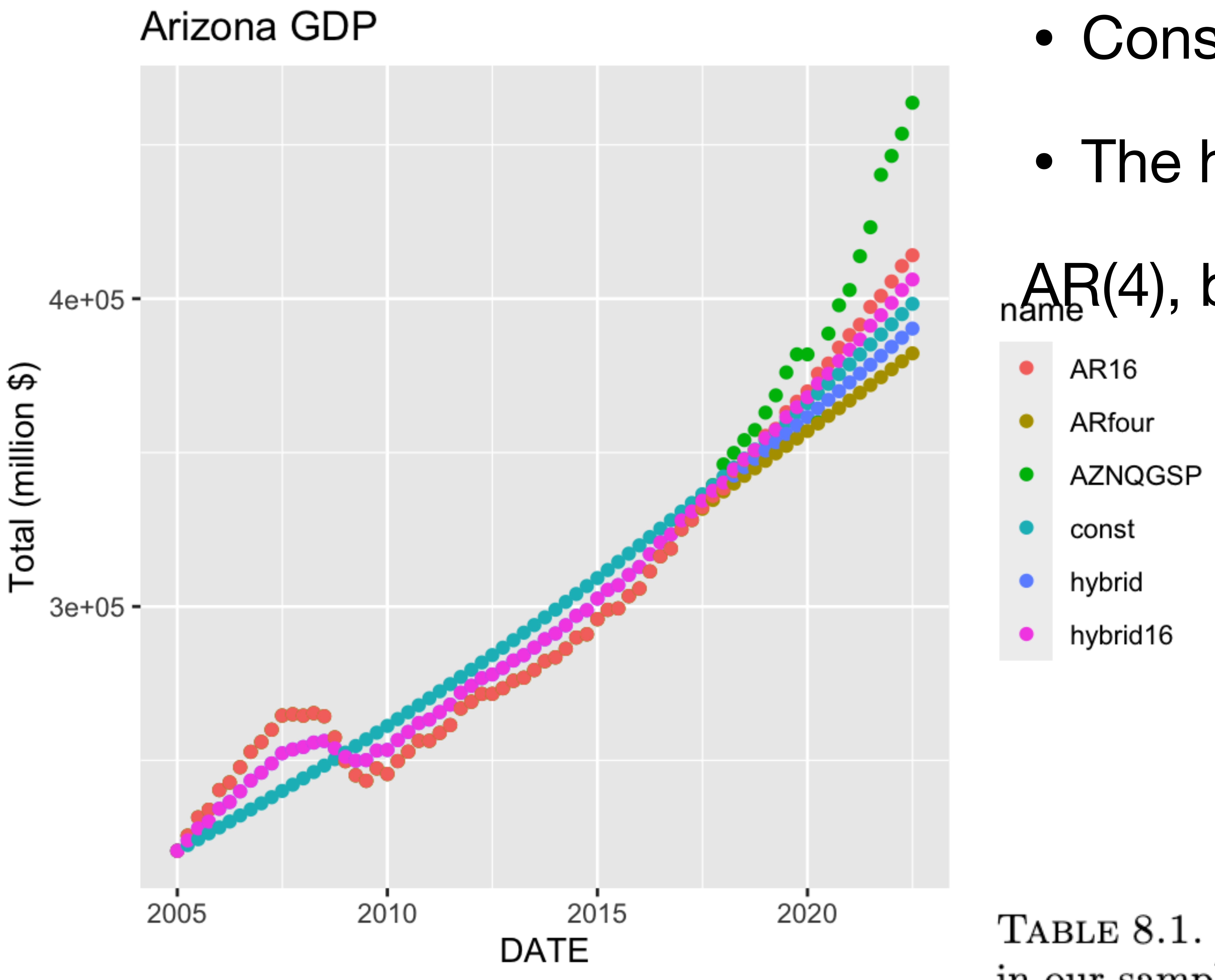

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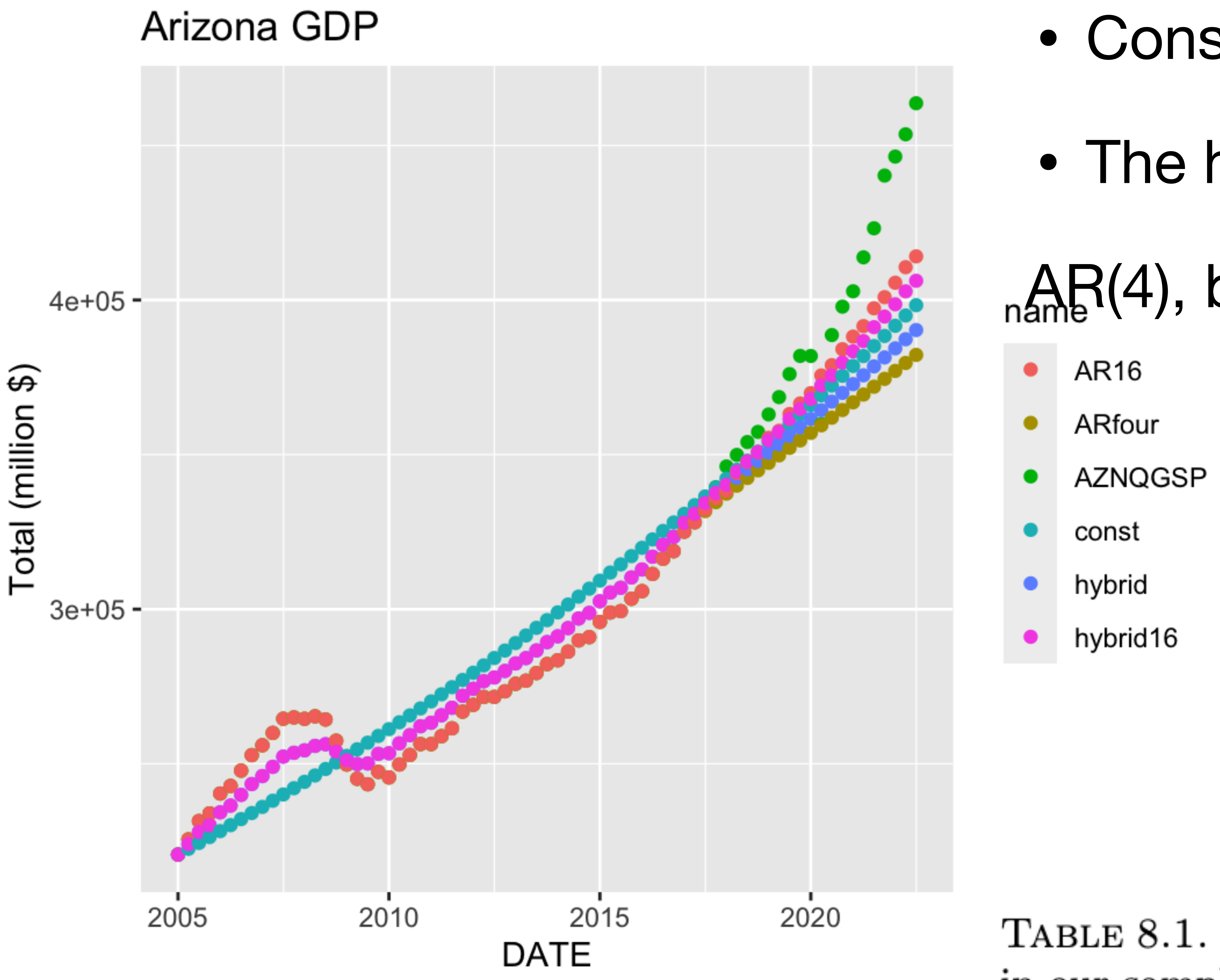

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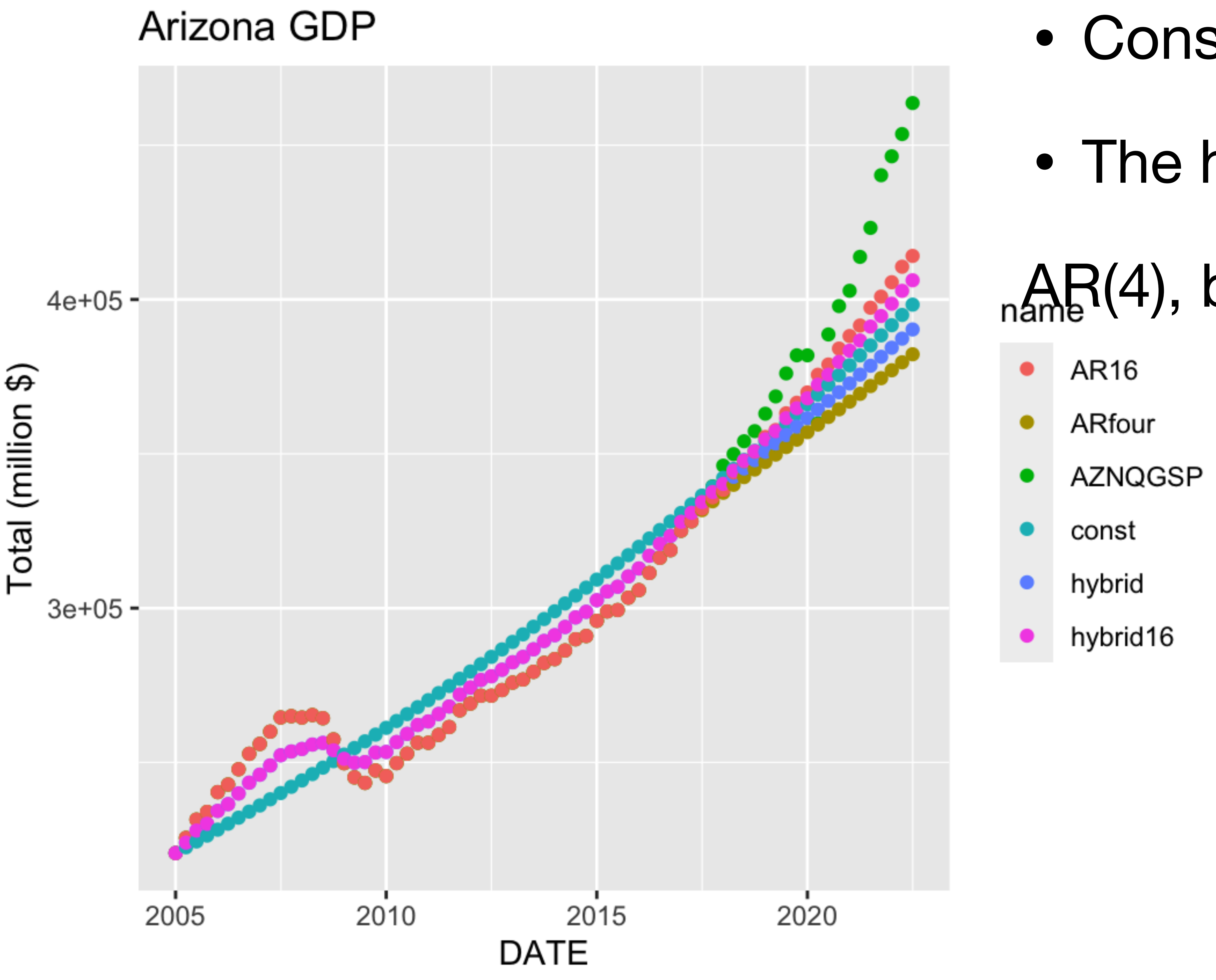

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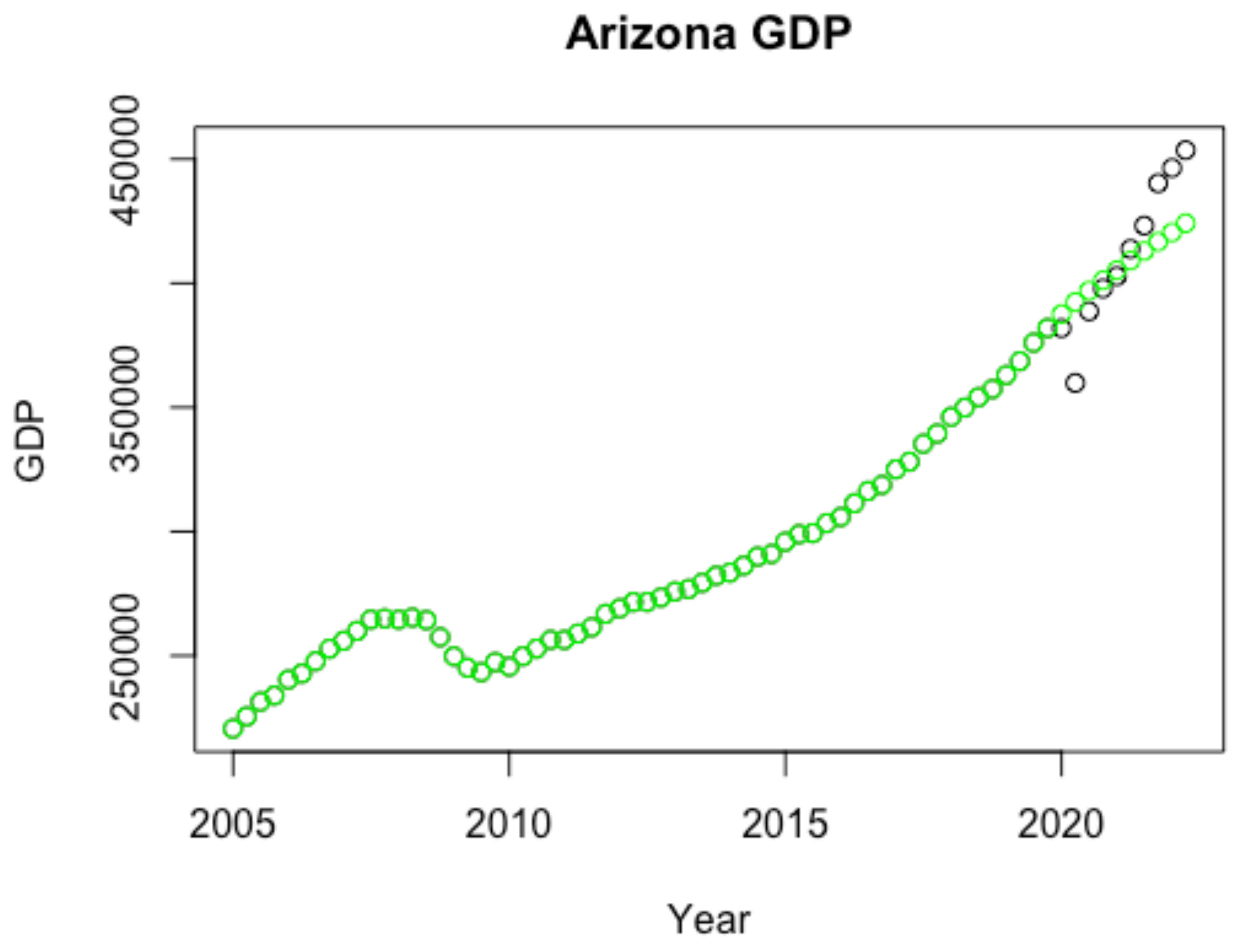
Hybrid16	670115229	2.32

Side quest

- **What if the Covid-19 Pandemic hadn't occurred?**
- Our AR(4) model (green) predicts continued economic growth, without the 2020 slump.
- However, by 2021 actual growth catches these predictions, and in 2022 is significantly higher than our model predicts.



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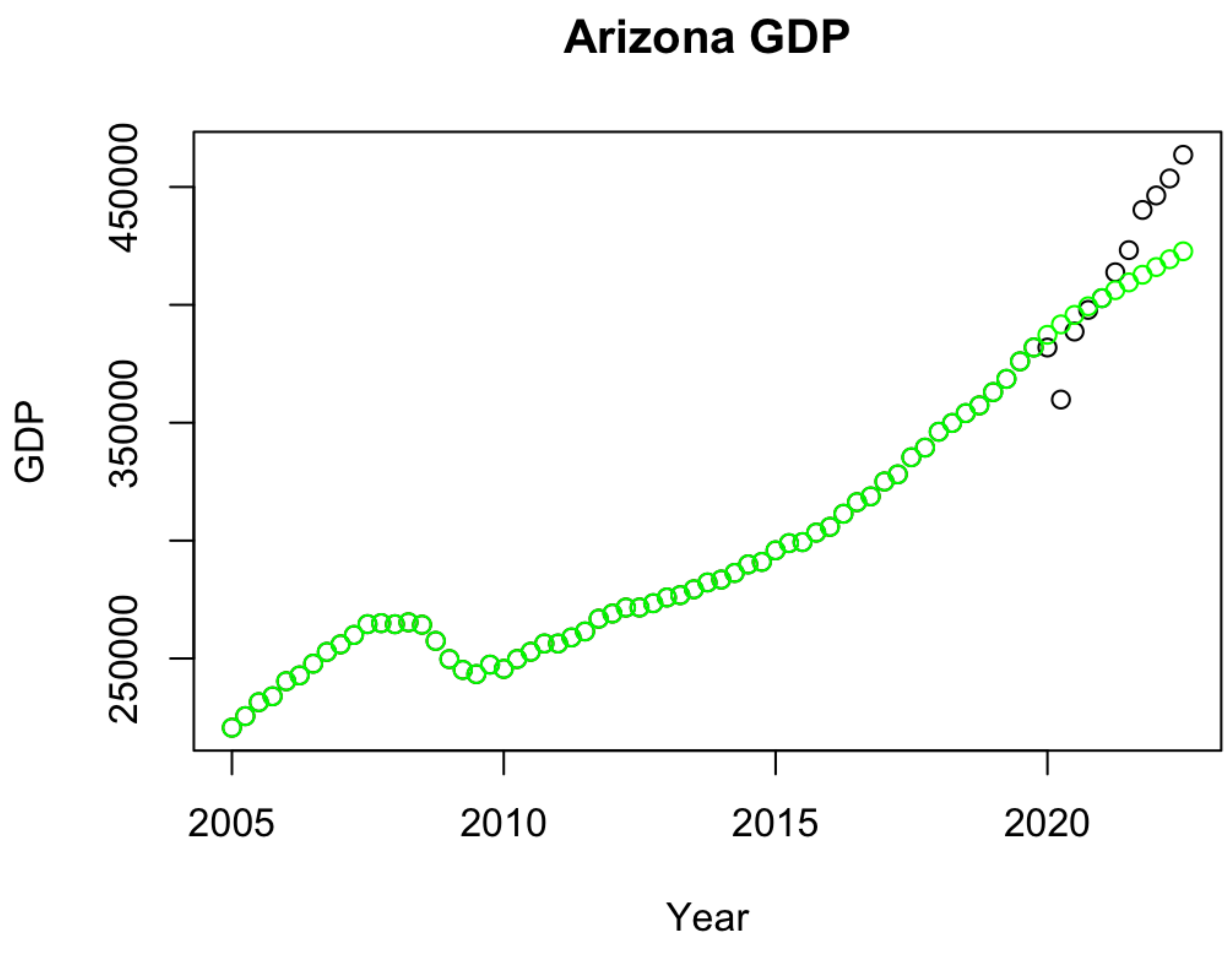

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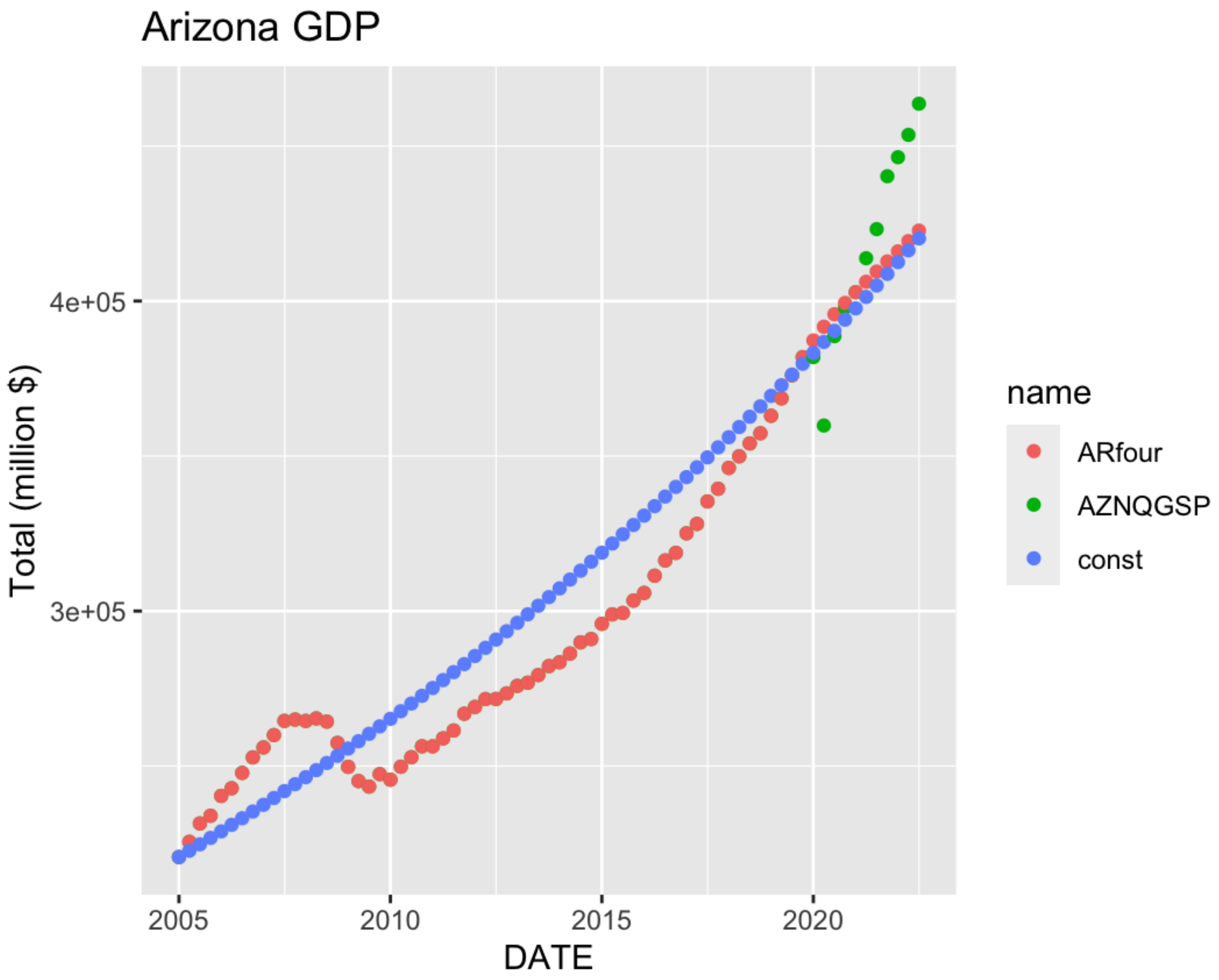
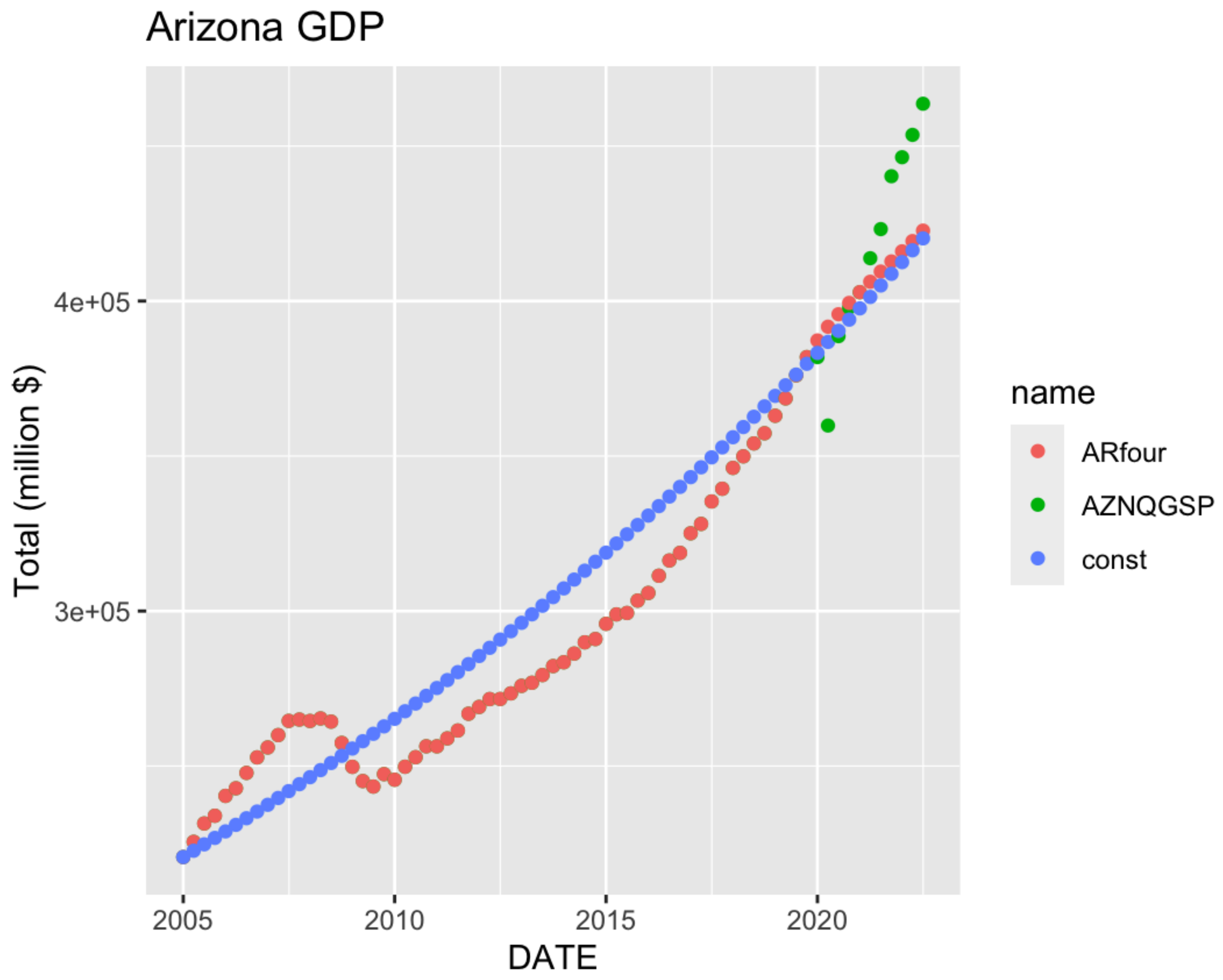


Side quest

- **What if the Covid-19 Pandemic hadn't occurred?**
- The constant growth model also predicts continued economic growth, without the 2020 slump.
- Again, by 2021 actual growth catches these predictions, and in 2022 is somewhat higher than our model predicts.
- However, contrary to previous observations, the constant growth model here predicts true growth <u>less well</u> than AR(4).



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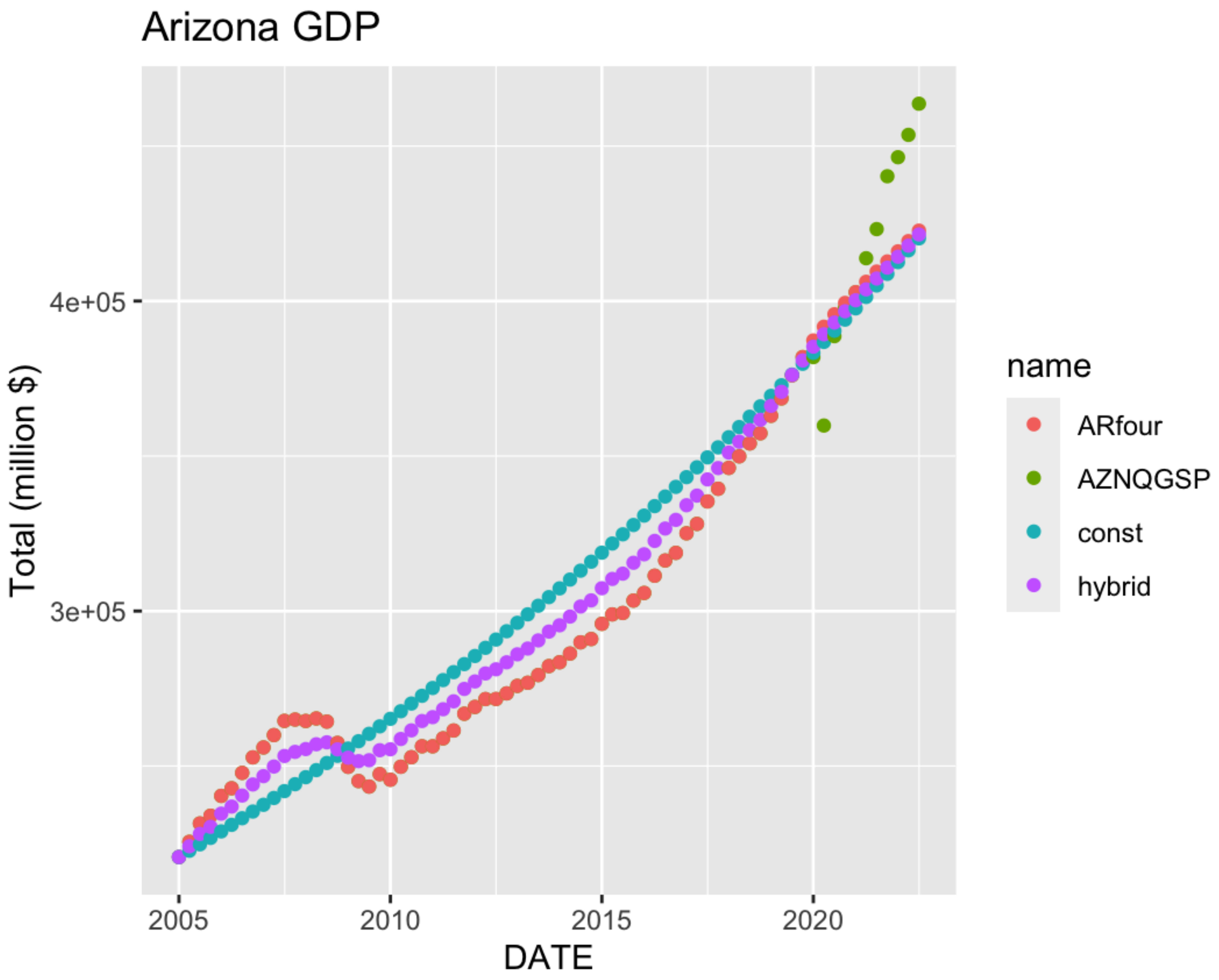

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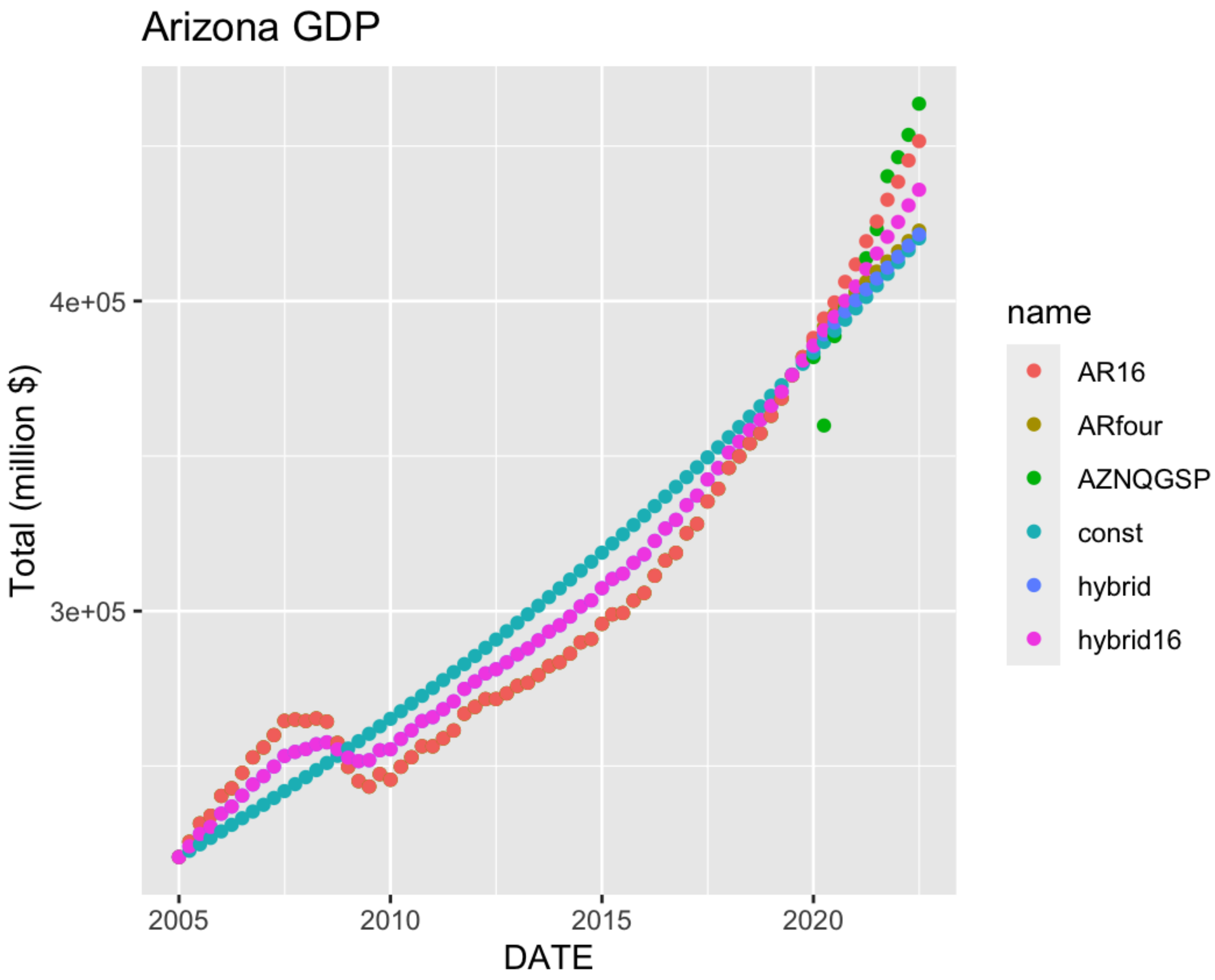
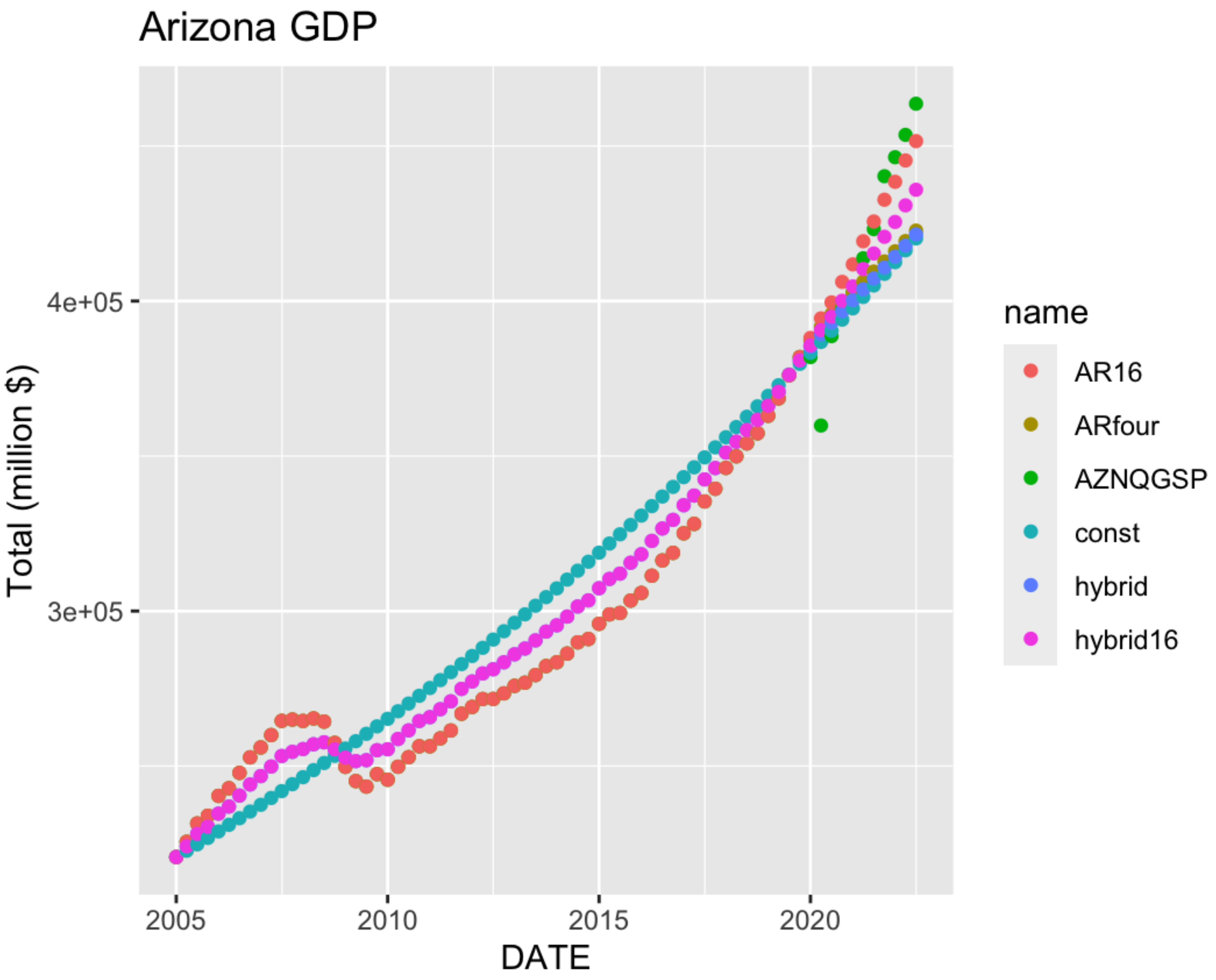


Side quest

- **What if the Covid-19 Pandemic hadn't occurred?**
- The AR(16) model provides a **remarkably accurate** projection of economic growth under this counterfactual.
- MSFE = 289876618

$$= \frac{1}{2.3} \times \text{previous best}$$



Conclusions

- We apply autoregressive, constant growth and hybrid models to real-world data to predict future economic growth.
- Our experimental evidence continues to point to the AR(16) model as a better predictor (lower MSFE) contrary to theory, which says AR(4) is best.
- Why? Likely due to too small a sample size. Possible remedy: k-fold cross validation.
- **Again, we have insufficient evidence to reject the hypothesis that AR(4) models provide the best prediction of future economic growth.**
- The task still warrants further investigation!

Reproducibility

- Description of computing infrastructure — All computation was run on the author's Macbook Air with a 1.1 GHz Dual-Core Intel Core i3 processor and 8GB memory.
- Average runtime for each approach — Included in tables.
- Details of train/validation/test splits — Described in ‘Plan of attack’
- Corresponding validation performance for each reported test result — There are not separate validation and test sets in this paper.
- A link to implemented code — See https://github.com/danlewis92/GDPproject: constantGrowth.R, hybrid.R, sideQuests.R, stepOne.R

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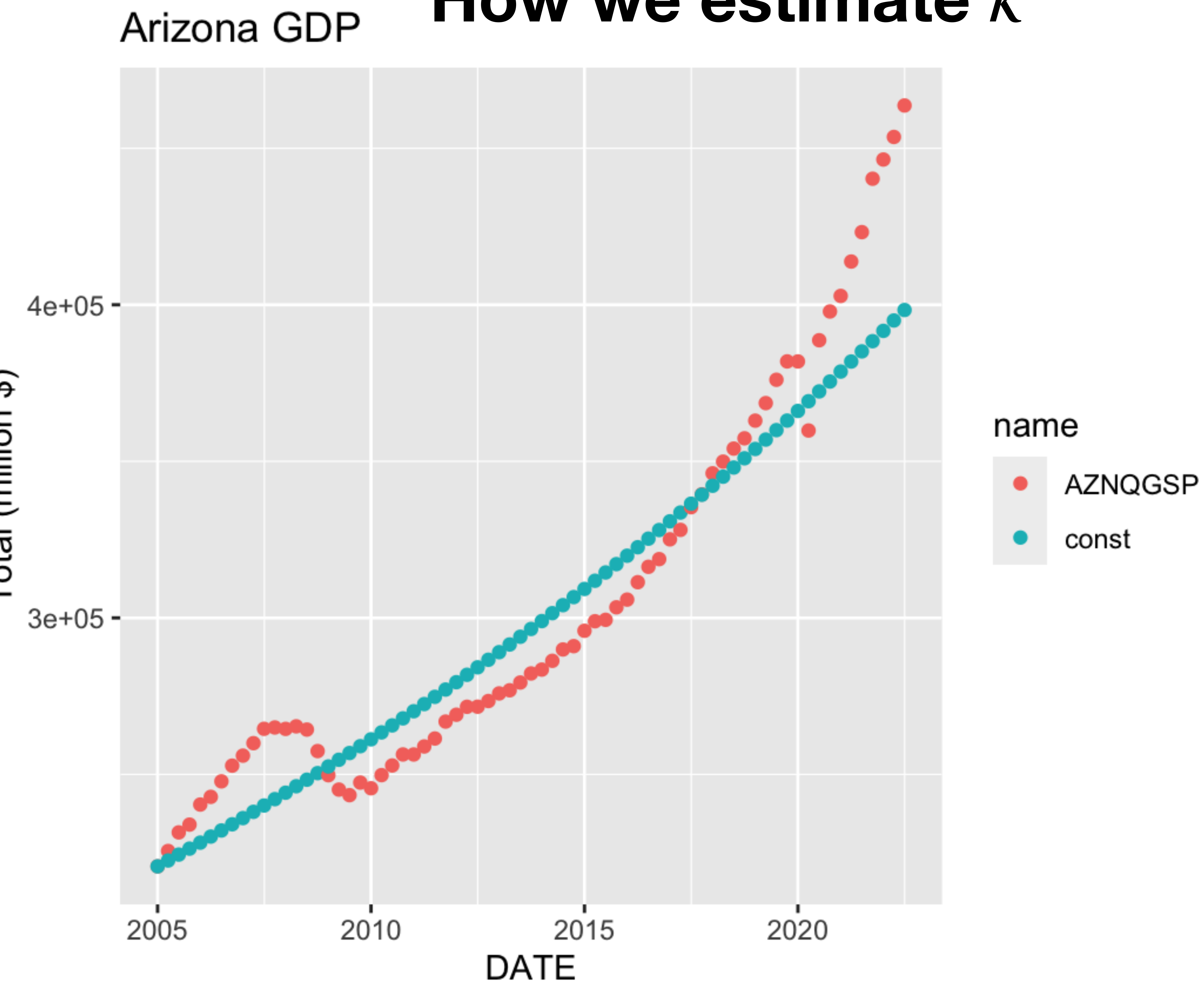
Why we don't use OLS to estimate κ

How we estimate κ



Estimating κ by OLS