Ph22.2 Three Body Problem

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1. We define the variables x_1, x_2, x_3, x_4 that represent the following variables:

$$
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= \dot{x} \\
x_4 &= \dot{y}
\end{aligned}
$$

so that they satisfy the following equations of motion:

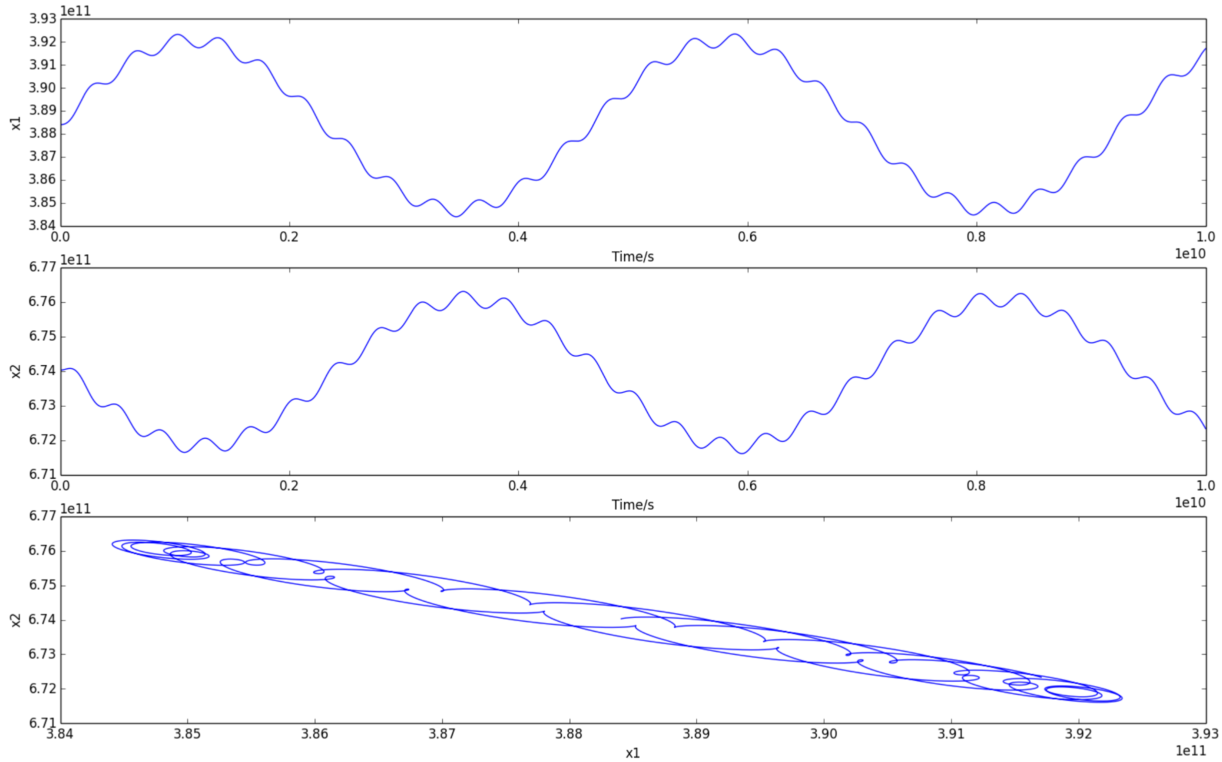
$$
\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{GM_1}{\left[\left(x_1 - \frac{M_2 R}{M_1+M_2}\right)^2 + x_2^2\right]^{3/2}}\left(x_1 - \frac{M_2 R}{M_1 + M_2}\right) - \frac{GM_2}{\left[\left(x_1 + \frac{M_1 R}{M_1+M_2}\right)^2 + x_2^2\right]^{3/2}}\left(x_1 + \frac{M_1 R}{M_1 + M_2}\right) + 2\Omega x_4 + \Omega^2 x_1 \\
\dot{x}_4 &= -\frac{GM_1}{\left[\left(x_1 - \frac{M_2 R}{M_1+M_2}\right)^2 + x_2^2\right]^{3/2}} x_2 - \frac{GM_2}{\left[\left(x_1 + \frac{M_1 R}{M_1+M_2}\right)^2 + x_2^2\right]^{3/2}} x_2 - 2\Omega x_3 + \Omega^2 x_2
\end{aligned}
$$

We also define the following initial conditions:

$$
\begin{aligned}
x_1(0) &= R\frac{M_2 - M_1}{M_2 + M_1}\cos\alpha \\
x_2(0) &= R\sin\alpha \\
x_3(0) &= 0 \\
x_4(0) &= 0
\end{aligned}
$$

We perform the numerical simulation for a timestep of 10^5 seconds, which is a small fraction of the orbital period of 3.7×10^8 seconds. We use 100 000 time steps to examine the behaviour over around 27 orbits of Jupiter around the Sun. The plots of x and y against time and y against x are shown below for various α values.

For $\alpha = \pi/3$,



We note that there seems to be a long-term periodic oscillation superimposed with another short-term oscillation that occurs with the Jupiter-Sun orbit.

When we change $\alpha = -\frac{\pi}{3}$ we see a similar structure reflected about the origin.

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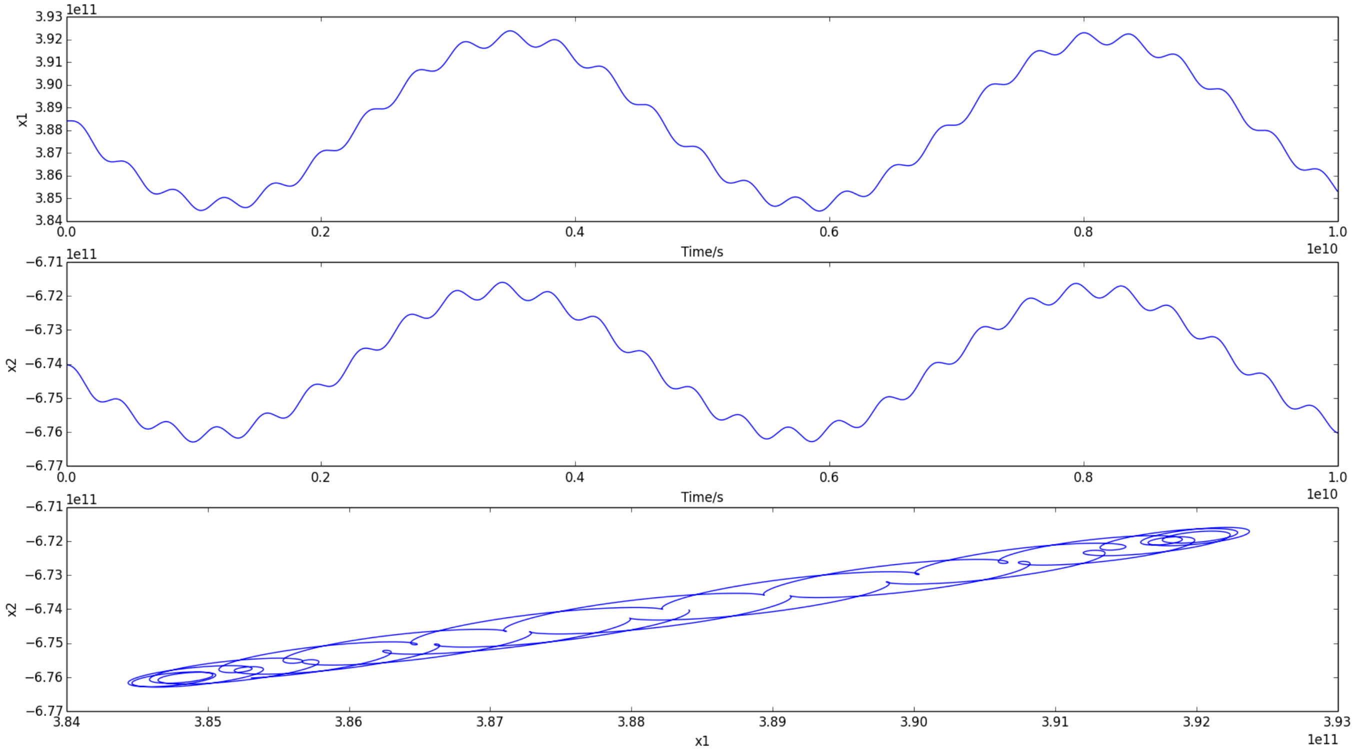
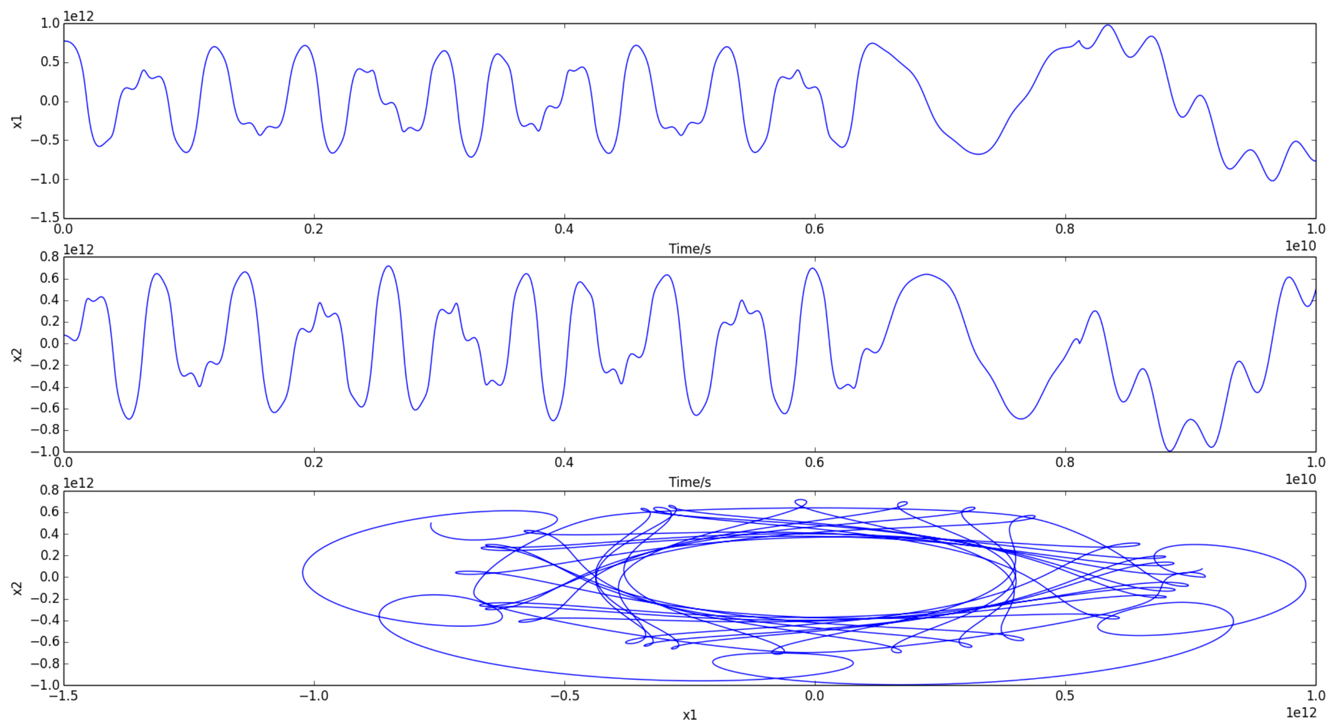


For an angle that is not $\pm\frac{\pi}{3}$, we see less regular variations ($\alpha = 0.1$)



2. We define the dynamical variables accordingly:

$$
\begin{aligned}
x_1 &= r_{1,x}\\
x_2 &= r_{1,y}\\
x_3 &= r_{2,x}\\
x_4 &= r_{2,y}\\
x_5 &= r_{3,x}\\
x_6 &= r_{3,y}\\
x_7 &= v_{1,x}\\
x_8 &= v_{1,y}\\
x_9 &= v_{2,x}\\
x_{10} &= v_{2,y}\\
x_{11} &= v_{3,x}\\
x_{12} &= v_{3,y}
\end{aligned}
$$

Then the equations of motion are:

$$
\begin{aligned}
\dot{x}_1 &= x_7\\
\dot{x}_2 &= x_8\\
\dot{x}_3 &= x_9\\
\dot{x}_4 &= x_{10}\\
\dot{x}_5 &= x_{11}\\
\dot{x}_6 &= x_{12}\\
\dot{x}_7 &= \frac{-GM_1M_2}{\left[(x_1-x_3)^2+(x_2-x_4)^2\right]^{3/2}}(x_1-x_3) - \frac{GM_1M_3}{\left[(x_1-x_5)^2+(x_2-x_6)^2\right]^{3/2}}(x_1-x_5)\\
\dot{x}_8 &= \frac{-GM_1M_2}{\left[(x_1-x_3)^2+(x_2-x_4)^2\right]^{3/2}}(x_2-x_4) - \frac{GM_1M_3}{\left[(x_1-x_5)^2+(x_2-x_6)^2\right]^{3/2}}(x_2-x_6)\\
\dot{x}_9 &= \frac{-GM_2M_1}{\left[(x_3-x_1)^2+(x_4-x_2)^2\right]^{3/2}}(x_3-x_1) - \frac{GM_2M_3}{\left[(x_3-x_5)^2+(x_4-x_6)^2\right]^{3/2}}(x_3-x_5)\\
\dot{x}_{10} &= \frac{-GM_2M_1}{\left[(x_3-x_1)^2+(x_4-x_2)^2\right]^{3/2}}(x_4-x_2) - \frac{GM_2M_3}{\left[(x_3-x_5)^2+(x_4-x_6)^2\right]^{3/2}}(x_4-x_6)\\
\dot{x}_{11} &= \frac{-GM_3M_1}{\left[(x_5-x_1)^2+(x_6-x_2)^2\right]^{3/2}}(x_5-x_1) - \frac{GM_3M_2}{\left[(x_5-x_3)^2+(x_6-x_4)^2\right]^{3/2}}(x_5-x_3)\\
\dot{x}_{12} &= \frac{-GM_3M_1}{\left[(x_5-x_1)^2+(x_6-x_2)^2\right]^{3/2}}(x_6-x_2) - \frac{GM_3M_2}{\left[(x_5-x_3)^2+(x_6-x_4)^2\right]^{3/2}}(x_6-x_4)
\end{aligned}
$$

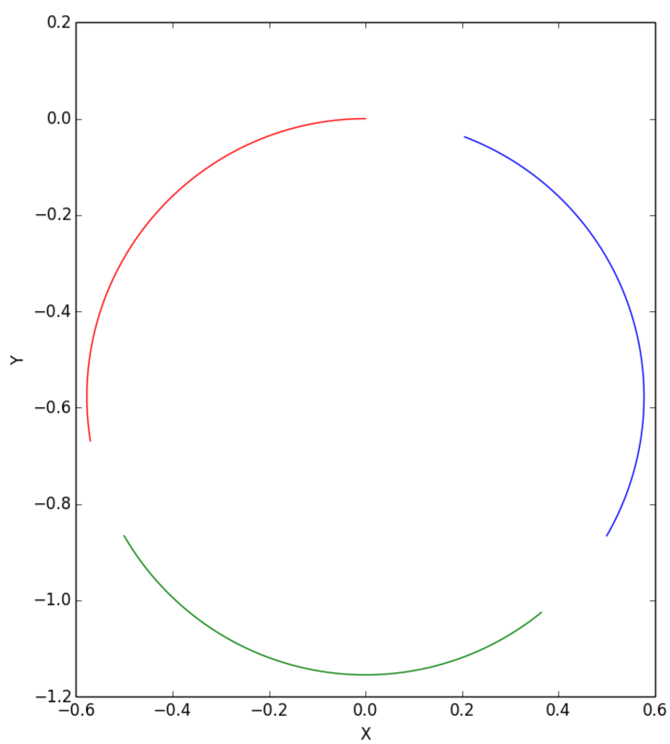
We first verify that the Lagrange solution is true. Consider an equilateral triangle of length d and with equal masses M. Let the masses have velocities v that are perpendicular to the line connecting each mass to the centre of the equilateral triangle. Then each mass is $R = \frac{d}{\sqrt{3}}$ away from the equilateral triangle. Consider one mass. By symmetry, the resultant force vector due to the other two masses has magnitude $\frac{GM^2}{d^2} 2\cos 30°$ and points towards the centre of the equilateral triangle. Hence the masses exhibit circular motion. Equating the centripetal acceleration to that of the resultant acceleration, we obtain: $M\frac{v^2}{R} = M\frac{v^2\sqrt{3}}{d} = \frac{GM^2 \times 2\frac{\sqrt{3}}{2}}{d^2} \implies v = \sqrt{\frac{GM}{d}}$.

We now simulate the orbits and pick $G = 1, M = 1, d = 1$ for convenience. Hence we have $v = 1$ as well. We place mass 1 at the origin and use the following initial conditions:

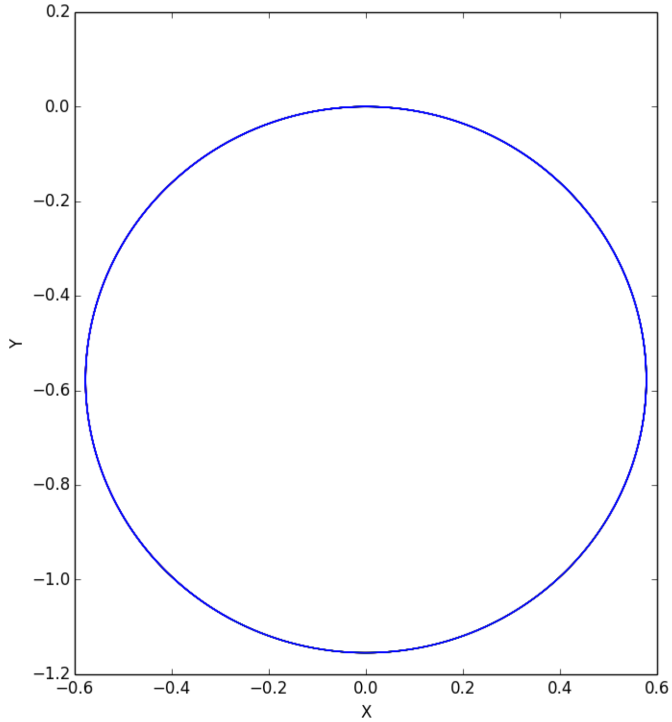
$$x_1(0) = 0$$
$$x_2(0) = 0$$
$$x_3(0) = -\frac{1}{2}$$
$$x_4(0) = -\frac{\sqrt{3}}{2}$$
$$x_5(0) = \frac{1}{2}$$
$$x_6(0) = -\frac{\sqrt{3}}{2}$$
$$x_7(0) = -1$$
$$x_8(0) = 0$$
$$x_9(0) = \frac{1}{2}$$
$$x_{10}(0) = -\frac{\sqrt{3}}{2}$$
$$x_{11}(0) = \frac{1}{2}$$
$$x_{12}(0) = \frac{\sqrt{3}}{2}$$

This corresponds to placing each of the masses on an equilateral triangle of side length 1, with an initial velocity of 1 perpendicular to the line connecting it to the centre of the triangle.

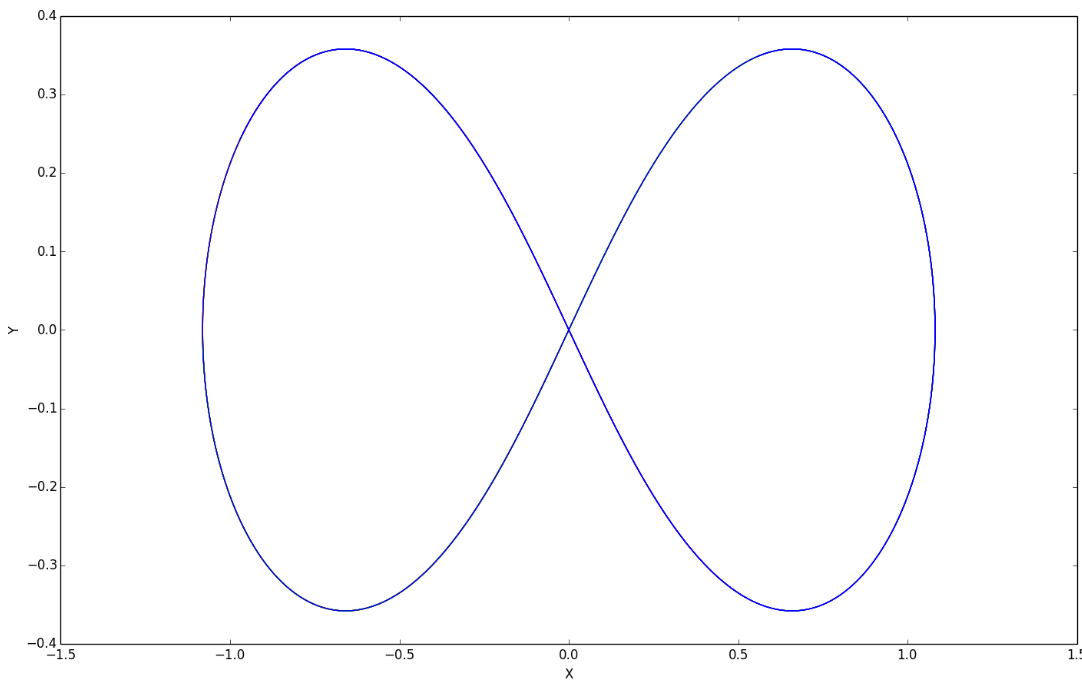
The orbits are indeed a circle (step size = 10^{-3}, 1000 steps), Mass 1 = Red, Mass 2 = Green, Mass 3 = Blue:



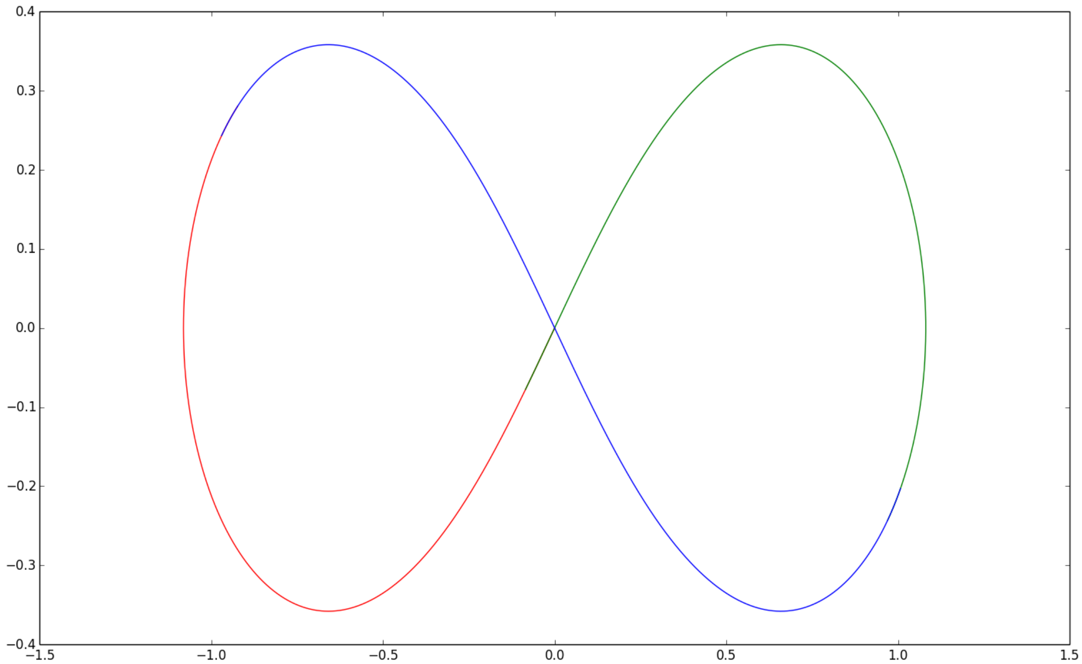
Running the simulation for a shorter time step reveals that the orbits indeed converge to a circle (step size = 10^{-2}, 1000 steps):



3. We simulate the orbit in accordance with the given initial conditions (step size $= 10^{-3}$, 10000 steps)



To see the individual orbits, we decrease the number of steps (step size 10^{-3}, 2200 steps):



I will attach the raw data of the simulation to the assignment submission (q22-2-3-choreographicrawdata.csv).