Problem Set 1: 0.2, 0.4, 0.10, 0.14, 0.16, 0.17

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PROBLEM 0.2

Problem: Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ i onto S^{n-1}.

Solution: The retraction will be a linear (path-straight) retraction from x to $\frac{x}{|x|}$. That is,

$$F(x,t) = (1-t)x + t\left(\frac{x}{|x|}\right)$$

PROBLEM 0.4

Problem: A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in the weak sense, then the inclusion $A \to X$ is a homotopy equivalence.

Solution: Let f_t be the deformation retraction from X to A as specified in the problem. Then, it is clear that the inclusion map i_A is a homotopy equivalence. Since, $i_A \circ f_1 = f_1 \simeq f_0 = \mathbb{1}$, and $f_1 \circ i_A = f_1|_A \simeq f_0|_A = \mathbb{1}$, i_A is an equivalence relation.

PROBLEM 0.10

Problem: Show that a space X is contractable iff every map $f : X \to Y$ for arbitrary Y is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

Solution:

$$X \to Y$$

(=>) Let X be a contractable space. Then, there exists a contraction c_t from X to x_0. Let $f : X \to Y$ be any map from X to an arbitrary space Y. Then, consider the homotopy $f \circ c_t$, which is f at $t = 0$, and $f(x_0)$ at $t = 1$. Thus, f is nullhomotopic.
(<=) Let X be such a space that every map $f : X \to Y$ is nullhomotopic. Then, the identity map $id : X \to X$ is nullhomotopic, and X is contractable.

$$Y \to X$$

(=>) Let X be a contractable space. Then, there exists a contraction c_t from X to x_0. Let $f : Y \to X$ be any map from an arbitrary space Y to X. Then, consider the homotopy $c_t \circ f$, which is f at $t = 0$, and x_0 at $t = 1$. Thus, f is nullhomotopic.
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PROBLEM 0.14

Problem: given positive integers v, e, f such that $v - e + f = 2$, construct a cell structure on S^2 with V 0-cells, e 1-cells, and f 2-cells.

Solution: Let $n = v - e$, which must be at most 1. If $v - e = 1$, then arrange the 0-cells in a line, and connect adjacent ones with 1-cells to make a line. Then, since $f = 2 - n$, attach the single 2-cell to the 1-cell by "wrapping" the 2-cell around the line, and gluing the boundary to it.
Suppose $n < 1$. In step one, arrange the v 0-cells in a line, and connect adjacent ones with 1-cells. This leaves $1 - n$ 1-cells to attach. In step two, attach all of them in the same way: one end attaches to the start of the line formed in the previous step, and the other attaches to the end of the line. Then, with the $2 - n$ 2-cells to attach, use two of them to attach to the line formed in step one along with the first and last (respectively) 1-cells attached in step two. Then, with the remaining $-n$ 2-cells, fill in the spaces between the adjacent 1-cells attached in step two.

PROBLEM 0.16

Problem: Show that S^∞ is contractable.

Solution: Let S^∞ be constructed as $\bigcup_n S^n$. Then, let $x_0 \in S^0$. This proof will show that the n-sphere is contractable to x_0 when it is considered as the equator of the $n+1$-sphere.
Consider S^n as the equator of S^{n+1}, where S^{n+1} is constructed by attaching two $n+1$-cells to S^n. Then, S^n is identified with ∂D^{n+1}. Since D^{n+1} is contractable, S^n is contractable in this space as well.
Since each $x \in S^\infty$ is a member of some S^n, each point is contractable to x_0, and thus S^∞ is contractable.

PROBLEM 0.17

Problem: Show that the mapping cylinder of every map $f : S^1 \to S^1$ is a CW complex, and construct a 2-dimensional CW complex that contains both the annulus $S^1 \times I$ and a mobius band as deformation retracts.

Solution: Let f be a homotopy equivalence from the annulus to the mobius band. Then, the mapping cylinder of f is a space that is deformation retractable to both the annulus and the mobius band. TODO: find such a homotopy equivalence.