Problem Set 4: 1.9

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PROBLEM 1.8

Problem 1.8:
Obtain the classical implicit function theorem from the general implicit function theorem stated in theorem 1.38.

Theorem 1.38: Assume that $\psi : M^c \to N^d$ is C^∞, that $n \in N$, $P = \psi^{-1}(n)$ is nonempty, and that $d\psi$ is surjective for all $m \in P$. Then, P has a unique manifold structure such that (P, i) is a submanifold of M, and i is an imbedding, and $\dim(P) = c - d$.

Let $U \subset \mathbb{R}^{c-d} \times \mathbb{R}^d$ and $f : \mathbb{R}^c \to \mathbb{R}^d$. Furthermore, the jacobian is nonsingular in the first d dimensions at a point (r_0, s_0) such that $f(r_0, s_0) = 0$. Because of the continuity of the determinant of the first d dimensions of the Jacobian, there exists a rectangular neighborhood $V_0 \times W_0 \subset U$ of (r_0, s_0) such that that first d dimensions of the Jacobian is nonsingular on this neighborhood, implying that the Jacobian is surjective. Then by 1.38, $P = f^{-1}|_{V_0 \times W_0}(0)$ is an imbedded submanifold of $V_0 \times W_0$ of dimension $c - d$.

Now we construct $\pi_1 : V_0 \times W_0 \to V_0$ the projection function, and $\pi_1 \circ \iota$ is C^∞. Then, $d(\pi_1 \circ \iota) : T(P)_{(r_0, s_0)} \to T(V_0)_{r_0}$ is an isomorphism, and it follows from Theorem 1.30 that there exists an open set $(r_0, s_0) \in V \times W \subset V_0 \times W_0$, such that $(\pi_1 \circ \iota)^{-1}|_{P \cap V \times W} : V \to P \cap V \times W$ is bijective, C^∞.

Then, consider $g = (\pi_2 \circ \iota) \circ (\pi_1 \circ \iota)^{-1}|_{P \cap V \times W} : V \to W$, which satisfies the requirements of the implitict function theorem. Let $(p, q) \in V \times W$. Then, $f(p, q) = 0 \implies (p, q) \in P$, and then $g(p) = (\pi_2 \circ \iota) \circ (\pi_1 \circ \iota)^{-1}(p) = \pi_2 \circ \iota(p, q) = q$ for the same unique q because of the bijectivity of $(\pi_1 \circ \iota)^{-1}$.

In instead we have $q = g(p)$, then $f(p, g(p)) = 0$ since $(p, g(p)) \in P$ because of the restriction built into the g function.

Problem 1.9

Problem 1.9:
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1$$

For which points $p \in \{(0,0), (\frac{1}{3}, \frac{1}{3}), (\frac{-1}{3}, \frac{-1}{3})\}$ is $f^{-1}(f(p))$ an imbedded submanifold?

Solution:
The general Jacobian for this transformation is $\left[3x^2 + y, 3y^2 + x\right]$, which will be nonsurjective iff both components are zero. Solving for this condition yields two solutions: The Jacobian is zero if $x = y = 0$ and if $x = y = \frac{-1}{3}$

Consider first $p = (0,0)$. The resulting "level curve", $P(0,0) = f^{-1}(f(0,0))$ contains the point $(0,0)$. Since this point satisfies the condition for the Jacobian to be nonsurjective, the Jacobian is not everywhere nonsurjective for $P(0,0)$.

Consider as well the point $p = (\frac{1}{3}, \frac{1}{3})$. Does the resulting level curve $V(x^3 + xy + y^3 - \frac{5}{27}$ contain either $(0,0)$ or $(\frac{-1}{3}, \frac{-1}{3})$? It is simple to check that it does not contain either point, Therefore, this level curve is a imbedded submanifold.

Consider as well the point $p = (\frac{-1}{3}, \frac{-1}{3})$. Obviously, it's level curve contains $(\frac{-1}{3}, \frac{-1}{3})$. Therefore, the Jacobian is not everywhere nonsurjective, and the level curve is not an imbedded submanifold.

Problem 1.10

Problem 1.10:

Let M be a compact manifold of dimension n, and let $f : M \to \mathbb{R}^n$ be C^∞. Prove that f cannot be everywhere non-singular.

M compact implies that $f(M)$ is also compact, which also implies it is closed, and bounded in $\mathbb{R}^n$ by Heine Borel. Thus, f attains a maximum on M at some point $x \in M$. But then the first derivatives with respect to coordinate chart containing x must all be zero, implying the Jacobian at that point is singular, this contradicting the