

## Problem Set 5

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### PROBLEM 2.1

(a)

Let  $U, V, W$  be vector spaces, with  $\phi : V \times W \rightarrow V \otimes W$  the natural mapping,  $l : V \times W \rightarrow U$  bilinear.

NTS: exists unique  $\tilde{l} : V \otimes W \rightarrow U$  such that  $\tilde{l} \circ \phi = l$ .

Define  $\tilde{l}$  on decomposable tensors of the form  $v \otimes w$  as  $\tilde{l}(v \otimes w) = l(v, w)$  and extend to all of  $V \otimes W$  by linearity.

It is clear that  $\tilde{l} \circ \phi(v, w) = \tilde{l}(v \otimes w) = l(v, w)$  and the diagram commutes.

Uniqueness: Suppose  $\tilde{l}'$  is another linear lifting of  $l$ . Then, for  $(v_0, w_0)$ ,  $\tilde{l} \circ \phi(v_0, w_0) = \tilde{l}(v_0 \otimes w_0) = l(v_0, w_0) = \tilde{l}' \circ \phi(v_0, w_0) = \tilde{l}'(v_0 \otimes w_0)$ , and thus  $\tilde{l}' = \tilde{l}$ .

Now we also prove universality within a general category. In a general category  $C$ , let  $X$  and  $Y$  satisfy the mapping property for  $A$ .

(b)

$V \otimes W \cong W \otimes V$ . Define the isomorphism as, for  $\psi : V \times W \rightarrow W \times V$  the canonical isomorphism,  $\psi_0 : V \otimes W \rightarrow W \otimes V$ .

Let  $\phi$  be the bilinear map from part (a) of  $V \times W$  into  $V \otimes W$  and  $\phi'$  the bilinear map of  $W \times V$  into  $W \otimes V$ . Then,  $\psi_0 = \phi' \circ \psi$ , with natural inverse  $\psi_0^{-1} = \phi \circ \psi^{-1}$  where  $\psi_0$  is extended to all of  $V \otimes W$  via linearity.

(c)

$U \otimes (V \otimes W) = (U \otimes V) \otimes W$ . Apply the same lifting as (b) on  $\psi : U \times (V \times W) \rightarrow (U \times V) \times W$ .

(d)

$\alpha$  is injective by linearity  $\alpha(v_1) - \alpha(v_2) = 0 \rightarrow \alpha(v_1 - v_2) = 0$  and triviality of the kernel.

Let  $T : V \rightarrow W$  be an element of  $\text{Hom}(V, W)$ .  $T(x_i) = \sum c_j y_j = w_i$ . Then,  $T(V) = T(\sum c_i y_i) = \sum c_i T(x_i) = \sum c_i (\sum c_j y_j) = \sum_i w_i$ . Let  $f_i = \pi_i$  be the  $i$ -th coordinate projection. Then  $T(V) = \sum f_i(v) w_i = \sum \alpha(f_i \otimes w_i)(v) = \alpha(\sum (f_i \otimes w_i)(v))$ . Then  $\alpha$  is surjective as well.

(e)

Suppose  $(v \otimes w) \in V \otimes W$ . Then  $(v \otimes w) = (\sum c_i e_i) \otimes (\sum d_j f_j) = \sum_i ((c_i e_i) \otimes (\sum d_j f_j)) = \sum_i c_i (e_i \otimes (\sum d_j f_j)) = \sum_i \sum_j c_i (e_i \otimes (d_j f_j)) = \sum_i \sum_j c_i d_j (e_i \otimes f_j)$ . Thus the desired set is a basis.

## PROBLEM 2.2

### 0.1 PART A

Provide an example of a homogeneous tensor that is not decomposable

*Proof.* Let  $V$  be a vector space, and  $V \otimes V$  the corresponding tensor product space. Furthermore, let  $v, w$  be vectors in  $V$ . Then, the tensor  $v \otimes w + w \otimes v$  is homogeneous of degree two, but is not decomposable.  $\square$

### 0.2 PART B

Show that for  $\dim(V) \leq 3$ , every homogeneous element of  $\Lambda(V)$  is decomposable.

*Proof.* Let  $V$  be a three dimensional vector space with basis  $\{v_1, v_2, v_3\}$ . Then, the corresponding exterior algebra has basis elements

$$\begin{array}{ccccc} & & v_1 \wedge v_2 \wedge v_3 & & \\ & v_1 \wedge v_2 & & v_1 \wedge v_3 & v_2 \wedge v_3 \\ & v_1 & & v_2 & v_3 \\ & & & 1 & \end{array}$$

It suffices to check for degree two elements of  $\Lambda(V)$  that they are decomposable.

To this end, let  $c_1 v_1 \wedge v_2 + c_2 v_1 \wedge v_3 + c_3 v_2 \wedge v_3$  be an arbitrary degree two element of the exterior algebra. Then, it is easy to see that

$$\begin{aligned} c_1 v_1 \wedge v_2 + c_2 v_1 \wedge v_3 + c_3 v_2 \wedge v_3 &= v_1 \wedge (c_1 v_2 + c_2 v_3) + c_3 v_2 \wedge v_3 \\ &= (v_1 - \frac{c_1}{c_3} v_3) \wedge (c_1 v_2 + c_2 v_3) \end{aligned}$$

□

### 0.3 PART C

Give an example of a homogeneous indecomposable element of  $\Lambda(V)$ .

*Proof.* The element  $v_1 \wedge v_2 + v_3 \wedge v_4$  for linearly independent  $v_1 \dots v_4$  is indecomposable. □

### 0.4 PART D

Is  $\alpha \wedge \alpha = 0$ ?

*Proof.* Decomposable elements of  $\alpha$  wedge together to zero, meaning  $\alpha \wedge \alpha = 0$ . □

## PROBLEM 2.3

(a)

Let  $u \in \Lambda_k(V)$  and  $v \in \Lambda_l(V)$ . Then,  $u$  and  $v$  are homogeneous and the wedge product  $u \wedge v$  is an element of  $\Lambda_{k+l}(V)$  by the definition of the exterior algebra as  $C(V)/I(V)$ , following from the definition of the tensor product of homogeneous tensors. Since  $u, v$  are homogeneous,  $u = u_1 \wedge \dots \wedge u_k$  and  $v = v_1 \wedge \dots \wedge v_l$ , and thus  $u \wedge v = u_1 \wedge \dots \wedge u_k \wedge v_1 \wedge \dots \wedge v_l = (-1)^k v_1 \wedge u_1 \wedge \dots \wedge u_k \wedge v_2 \wedge \dots \wedge v_l = (-1)^k (-1)^l v_1 \wedge \dots \wedge v_l \wedge u_1 \wedge \dots \wedge u_k = (-1)^{kl} v \wedge u$

(b)

First we show that  $\{e_\Phi\}$  is a basis of  $\Lambda(V)$ . Observe that the elements of  $\{e_\Phi\}$  span  $\Lambda(V)$ . To show linear independence of the set, consider  $\sum a_\Phi e_\Phi = 0$ . Then,  $(\sum a_\Phi e_\Phi) \wedge (e_1 \wedge \dots \wedge e_n) = a_\emptyset (e_1 \wedge \dots \wedge e_n) = 0$ . Then we induct; assume that  $a_s = 0 \forall s$  such that  $|s| \leq k$ . Then, for  $|s| \geq k+1$ ,  $\sum a_s e_s = 0$ ,  $(\sum a_\Phi e_\Phi) \wedge e_{\Phi^c} = a_\Phi (e_1 \wedge \dots \wedge e_n) = 0$  which implies that  $a_\Phi = 0$  for  $|\Phi| = k+1$ , where  $\Phi^c$  is the set theoretic complement of  $\Phi$ .

Furthermore, it is obvious that  $\Lambda_d(V)$  is a 1 dimensional vector space and is isomorphic to  $\mathbb{R}$ , and that  $\Lambda_{d+j}(V)$  is isomorphic to  $\{0\}$ . Furthermore, that  $\dim(\Lambda_k(V)) = 2^d$