Problem Set 1: 1.1-1.4

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PROBLEM 1.1

Problem 1.1: Prove that in Example 1.5(d) one does indeed obtain a differentiable structure on S^d.

Example 1.5(d): The $d-sphere$ is the set

$$S^d = \{x \in \mathbb{R}^{d+1} | \sum_{i=1}^{d+1} x_i^2 = 1\} \tag{0.1}$$

Let $n = (0,\dots,0,1)$ and $s = (0,\dots,0,-1)$. Then the standard differentiable structure on S^d is obtained by taking $\mathscr{F}$ to be the maximal collection containing $(S^d - n, p_n)$ and $(S^d - s, p_s)$, where p_n and p_s are stereographic projections from n and s respectively.

$\mathscr{F}$ is a differentiable structure of class C^∞, a collection of coordinate systems $\{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ satisfying:
(a) NTS: $\bigcup_{\alpha \in A} U_\alpha = S^d$

$(S^d - n) \subset S^d$ and $(S^d - s) \subset S^d \Rightarrow (S^d - n) \bigcup (S^d - s) \subset S^d$. Since $n \neq s, S^d \subset (S^d - n) \bigcup (S^d - s) \subset \bigcup_{\alpha \in A} U_\alpha$. Then, $\bigcup_{\alpha \in A} U_\alpha = S^d$ since $\bigcup_{\alpha \in A} U_\alpha \subset S^d$ because by definition, $U_\alpha \subset S^d\ \forall \alpha \in A$.

(b) NTS: $\phi_\alpha \circ \phi_\beta^{-1}$ is C^∞ for all $\alpha, \beta \in A$.

Since the example defines $\mathscr{F}$ to be the maximal collection containing $(S^d - n, p_n)$ and $(S^d - s, p_s)$, it is sufficient to show that $p_s \circ p_n^{-1}$ and $p_n \circ p_s^{-1}$ are C^∞.
Using the standard stereographic projections we have:

$p_n : S^d \to \mathbb{R}^d; (x_1, x_2, \ldots, x_d, x_{d+1}) \mapsto \frac{1}{1-x_{d+1}}(x_1, x_2, \ldots, x_d)$
$p_s : S^d \to \mathbb{R}^d; (x_1, x_2, \ldots, x_d, x_{d+1}) \mapsto \frac{1}{1+x_{d+1}}(x_1, x_2, \ldots, x_d)$
Then the inverses are:
$p_n^{-1} : \mathbb{R}^d \to S^d; (x_1, x_2, \ldots, x_d) \mapsto (1 - x_{d+1})(x_1, x_2, \ldots, x_d, \frac{x_{d+1}}{1-x_{d+1}})$
$p_s^{-1} : \mathbb{R}^d \to S^d; (x_1, x_2, \ldots, x_d) \mapsto (1 + x_{d+1})(x_1, x_2, \ldots, x_d, \frac{x_{d+1}}{1+x_{d+1}})$
where $x_{d+1} = \frac{(\sum_{i=1}^d x_i^2)-1}{(\sum_{i=1}^d x_i^2)+1}$ for p_n^{-1} since $(1-x_{d+1})(x_1, x_2, \ldots, x_d, \frac{x_{d+1}}{1+x_{d+1}}) \in S^d \Rightarrow (1-x_{d+1})^2(\sum_{i=1}^d x_i^2 + \frac{x_{d+1}^2}{(1-x_{d+1})^2}) = 1$. Then, $(1+(\sum_{i=1}^d x_i^2))x_{d+1}^2 - (2(\sum_{i=1}^d x_i^2))x_{d+1} + (\sum_{i=1}^d x_i^2 - 1) = 0$, and applying the quadratic equation we obtain the previous result, $x_{d+1} = \frac{(\sum_{i=1}^d x_i^2)-1}{(\sum_{i=1}^d x_i^2)+1}$. The other solution to the quadratic simply gives the position of the pole of the sphere where the stereographic line intersects the sphere. Similarly, for p_s^{-1}, $x_{d+1} = -\frac{(\sum_{i=1}^d x_i^2)-1}{(\sum_{i=1}^d x_i^2)+1}$.

We can compose these formulas to obtain:
$p_n \circ p_s^{-1} : \mathbb{R}^d \to \mathbb{R}^d; (x_1, x_2, \ldots, x_d) \mapsto \frac{(1+x_{d+1})}{(1-x_{d+1})}(x_1, x_2, \ldots, x_d)$
$p_s \circ p_n^{-1} : \mathbb{R}^d \to \mathbb{R}^d; (x_1, x_2, \ldots, x_d) \mapsto \frac{(1-x_{d+1})}{(1+x_{d+1})}(x_1, x_2, \ldots, x_d)$
From the formulas above, $p_n \circ p_s^{-1}$ and $p_s \circ p_n^{-1}$ simply multiplies $(x_1, x_2, \ldots, x_d)$ by a factor that is a composition of rational functions with domain $\mathbb{R}$ since the restrictions on x_{d+1} gaurantee that the denominators do not go to zero $\Rightarrow p_n \circ p_s^{-1}$ and $p_s \circ p_n^{-1}$ are C^∞ since each component function is C^∞.
(c) NTS $\mathscr{F}$ is maximal w.r.t. (b). This is satisfied by definition of the example.

Problem 1.2

Problem 1.2: The usual differentiable structure on the real line $\mathbb{R}$ was obtained by taking $\mathscr{F}$ to be the maximal collection containing the identity map. Let $\mathscr{F}_1$ be the maximal collection (w.r.t 1.4(b)) containing the map $t \mapsto t^3$. Prove that $\mathscr{F} \neq \mathscr{F}_1$, but that $(\mathbb{R}, \mathscr{F})$ and $(\mathbb{R}, \mathscr{F}_1)$ are diffeomorphic.

First we show $\mathscr{F} \neq \mathscr{F}_1$ by showing that $(\mathbb{R}, t) \in \mathscr{F}$ but $(\mathbb{R}, t) \notin \mathscr{F}_1$. $t \circ (t^3)^{-1} = t^{1/3} \notin C^\infty$ on the intersection of their domains $(\mathbb{R})$, so it fails condition b) for $\mathscr{F}$. Therefore this coordinate map is not in $\mathscr{F}_1$ so $\mathscr{F} \neq \mathscr{F}_1$.

However, they are diffeomorphic. The diffeomorphism is given by $\psi : (\mathbb{R}, \mathscr{F}) \to (\mathbb{R}, \mathscr{F}_1); t \mapsto t^{1/3}$. ψ is bijective by definition, so we need to show that ψ and ψ^{-1} are both C^∞.

To show ψ is C^∞ we let $\phi \in \mathscr{F}, \tau \in \mathscr{F}_1$. Then, $\tau \circ \psi \circ \phi^{-1}$ is $\tau \circ t^{1/3} \circ t \circ \phi^{-1} = (\tau \circ t^{1/3}) \circ (t \circ \phi^{-1})$, which is C^∞ since it is the composition of $(\tau \circ t^{1/3})$ and $(t \circ \phi^{-1})$ which are C^∞ by definition of $\mathscr{F}$ and $\mathscr{F}_1$.

Similarly, to show ψ^{-1} is C^∞, we check $\phi \circ \psi^{-1} \circ \tau^{-1} = \phi \circ t^3 \circ \tau^{-1} = \phi \circ t \circ \psi^{-1} \circ \tau^{-1} = (\phi \circ t) \circ (t^3 \circ \tau^{-1})$, which is C^∞ for the same reasons.

PROBLEM 1.3

Problem 1.3: Let U_α be an open cover of a manifold M. Prove that there exists a refinement V_α such that $\overline{V_\alpha} \subset U_\alpha$ for each α.
We first prove the following lemma:

Lemma 1.1: For continuous function $\phi : M \to \mathbb{R}$ on a manifold M, $\phi(\partial \text{supp}(\phi)) = \{0\}$. To show this, observe that if $x \in \partial \text{supp}(\phi))$, then $\exists$ net $\{x_\nu\}$ converging to x such that $\{x_\nu\} \not\in \text{supp}(\phi))$ which implies that $\phi(x_\nu) = 0$. Since $x_\nu \to x$, $\phi(x_\nu) \to \phi(x)$. But $\phi(x_\nu) \to 0$. Since $\mathbb{R}$ is Hausdorff, convergence is unique and $\phi(x) = 0$.

Let $\{U_\alpha\}$ be any open cover of manifold M. Then by Theorem 1.11, there exists a partition of unity ϕ_α subordinate to open cover $\{U_\alpha\}$. Let

$$V_\alpha = \text{Int}(\text{supp}(\phi_\alpha))$$

so that

$$\overline{V_\alpha} = \text{supp}(\phi_\alpha) \subset U_\alpha.$$

We now show that $\{V_\alpha\}$ is a refinement of $\{U_\alpha\}$. First, observe $V_\alpha \subset U_\alpha \forall \alpha$. This follows directly from the definition of V_α. Next we show that $\{V_\alpha\}$ covers M. For a contradiction, suppose there exists $x \in M$ such that x is not covered by $\{V_\alpha\}$, that is $x \not\in V_\alpha \forall \alpha$. Then, $x \not\in \text{Int}(\text{supp}(\phi_\alpha)) \forall \alpha$. But, this means $x \in \partial \text{supp}(\phi)$ for some (possibly more than one) β with the property $\sum \phi_\beta = 1$. However by lemma 1.1, $\phi_\beta(x) = 0\ \forall \beta$, so $\sum \phi_\beta = 0$, a contradiction.

PROBLEM 1.4

Problem 1.4: Use the fact that manifolds are regular and paracompact to prove that manifolds are normal topological spaces.

Let C and D be closed subsets of M. Then, by the regularity of $M, \forall d \in D\ \exists V_d \ni d, U_d \supset C$, where V_d, U_d open and $U_d \cap V_d = \emptyset$. We call V_d and U_d "regular pairs."

Then observe that the set $\{V_d\}$ forms an open cover of D, and then $\{V_d\} \cup (M - D)$ covers M. By paracompactness of M, there is a locally finite refinement $\{Q_\alpha\}$ of this cover. Let $\mathscr{O} = \{Q_\alpha : Q_\alpha \cap D \neq \emptyset\}$. $\mathscr{O}$ is then a locally finite open cover of D and $Q \in \mathscr{O} \implies Q \subset V_d$ for some d.

Then, let

$$U_D = \bigcup_{Q \in \mathscr{O}} Q$$

By the local finiteness of $\mathscr{O}$, $\forall c \in C$, there exists a neighborhood W_c of c such that W_c intersects only finitely many elements Q in $\mathscr{O}$. If W_c intersects no elements Q in $\mathscr{O}$, then let $U_c = W_c$. Else, define $T = \{Q_t\}_{t=1}^{n}$ for Q_t the elements of $\mathscr{O}$ that intersect W_t. Observe that for each t, $Q_t \subset V_t$ for some V_t. Let U_t be the corresponding regular pair of V_t. Recall that for regular pairs, $U_t \cap V_t = \emptyset$ and $U_t \supset C$. Then, since $Q_t \subset V_t$, $U_t \cap Q_t = \emptyset$. Furthermore, since $C \subset U_t, c \in C \implies c \in U_t \forall t$.

We can now construct our normal open set containing C.
Let

$$U_c = W_c \cap \bigcap_T U_t$$

It is easy to verify that for all $c \in C$, U_c is open and contains c. Furthermore, we can show that $U_c \cap U_D = \emptyset$. To see this, observe that $\forall Q \in \mathscr{O}, U_c \cap Q = (W_c \cap \bigcap_T U_t) \cap Q$ and either $W_c \cap Q = \emptyset$ or $U_t \cap Q = \emptyset$.

Then, let

$$U_C = \bigcup_{c \in C} U_c$$

It is easy to verify that U_C and U_D form a normal pair.