Problem Set 2: 1.5-1.6

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PROBLEM 1.5

Problem 1.5: Prove the following:

(a) If (U,ϕ) and $(V,\psi) \in \mathscr{F}$, then $\widetilde{\psi} \circ \tilde{\phi}^{-1}$ is C^∞. It is easily seen that $\tilde{\phi} = \phi \times d\phi$. Then the composition is $\widetilde{\psi} \circ \tilde{\phi}^{-1} = (\psi \circ \phi^{-1}, d(\phi) \circ d(\psi)^{-1}) = (\psi \circ \phi^{-1}, d(\phi \circ \psi^{-1}))$ using the chain rule on differentials. $\psi \circ \phi^{-1}$ is C^∞ as the composition of C^∞ functions, and the differential preserves the C^∞ property of $\psi \circ \phi^{-1}$, so $\widetilde{\psi} \circ \tilde{\phi}^{-1}$ is C^∞.

(b) The collection $\{\tilde{\phi^{-1}}(W) : W \text{ open in } \mathbb{R}^{2d}, (U,\phi) \in \mathscr{F}\}$ forms a basis for a topology on $T(M)$ which makes $T(M)$ into a 2d-dimensional, second countable, locally Euclidean space. Locally Euclidean and 2d-dimensional properties are obvious from the definition of $\tilde{\phi}$ since it defines a homeomorphism between $T(M)$ and $\mathbb{R}^{2d}$. To show it is a basis, notice that $T(M)$ has the initial topology generated by $< \tilde{\phi} >$ for all $(U,\phi) \in \mathscr{F}$ which has the canonical subbasis $\{\tilde{\phi^{-1}}(W) : W \text{ open in } \mathbb{R}^{2d}, (U,\phi) \in \mathscr{F}\}$. To show this a basis we have to show it contains its finite intersections. Let $\tilde{\phi_1^{-1}}(W_1)$ and $\tilde{\phi_2^{-1}}(W_2)$ be elements of the subbasis. Then

$$\tilde{\phi_1^{-1}}(W_1) \cap \tilde{\phi_2^{-1}}(W_2) = \tilde{\phi_1^{-1}}(\tilde{\phi}_1(\tilde{\phi_1^{-1}}(W_1) \cap \tilde{\phi_2^{-1}}(W_2))$$

and then define $\tilde{\phi_3^{-1}}(W_3)$ to be $\tilde{\phi_1^{-1}}$ restricted to $\tilde{\phi}_1(\tilde{\phi_1^{-1}}(W_1) \cap \tilde{\phi_2^{-1}}(W_2)$. $\tilde{\phi_3^{-1}}$ is defined since it is just a restriction of $\tilde{\phi_1^{-1}}$ and it is in the canonical subbasis since $\tilde{\phi_3^{-1}}(W_3) = \tilde{\phi_1^{-1}}(W_3)$.

To show it is second countable, consider a countable basis B_C in $\mathbb{R}^{2d}$. Let $\{(U_\alpha, \phi_\alpha)\}_\alpha$ be a differentiable structure on M. Using a partition of unity we can construct a countable subset of $\{(U_\alpha, \phi_\alpha)\}_\alpha$ that charts the whole manifold. Then, for each ϕ_i in this countable subset there exists a $\tilde{\phi}_i$ such that $\{\tilde{\phi_i^{-1}}(b_j)\}$ for each $b_j \in B_C$ is a countable basis.

(c) It is clear that the collection covers the whole space $T(M)$ since the collection $\{\tilde{\phi}^{-1}(\mathbb{R}^{2d})\}$ for each $\phi \in \mathscr{F}$ covers the whole space.

PROBLEM 1.6

Problem 1.6: Prove that if ψ is one-to-one, onto, and everywhere non-singular, then ψ is a diffeomorphism.

Suppose for a contradiction that ψ is not a diffeomorphism. Then, by contrapositive of Cor. A of Inverse Function Theorem, $d\psi$ is not an isomorphism on a neighborhood of a point m. But then, since $d\psi$ is everywhere nonsingular, in particular at m, then $d\psi$ must not be surjective at m. But if $d\psi$ is not surjective at m and $d\psi$ is a linear transformation between vector spaces, $\dim(M) < \dim(N)$. Call $p = \dim(M)$ and $d = \dim(N)$. Then, let (U, ϕ) be a coordinate system on N such that $\phi(U) = \mathbb{R}^d$. Then, range$(\phi \circ \psi) = \mathbb{R}^d$ as well since ψ is onto.

To show this is a contradiction we show that the range of $(\phi \circ \psi)$ has measure zero in $\mathbb{R}^d$. First, there exists a countable collection of coordinate charts (V_i, τ_i) that cover M by the Lindelhof property of second countable spaces. Then, we consider $f : \mathbb{R}^p \to M \to \mathbb{R}^d$ defined as $f_i = \psi \circ \psi \circ \tau_i^{-1}$. Note that each f_i is C^1 which implies that range of each f_i has measure zero. Because each f_i factors through $V_i \subset M$, then, $\phi \circ \psi(V_i)$ has measure zero for each V_i.

Now consider $\phi \circ \psi(M)$. Breaking M up into its countable cover of V_i's,

$$\phi \circ \psi(M) = \phi \circ \psi(\bigcup_{i=1}^{\infty} V_i) = \bigcup_{i=1}^{\infty} \phi \circ \psi(V_i)$$

Then,

$$\mu(\phi \circ \psi(M)) = \mu(\bigcup_{i=1}^{\infty} \phi \circ \psi(V_i)) \leq \bigcup_{i=1}^{\infty} \mu(\phi \circ \psi(V_i)) = \sum 0 = 0$$

Therefore, the measure of the range is zero, so the range cannot be all of $\mathbb{R}^d$.