Problem Set 5: 2.1, 2.2, 2.3

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PROBLEM 2.1

(a)
Let U, V, W be vector spaces, with $\phi : V \times W \to V \otimes W$ the natural mapping, $l : V \times W \to U$ bilinear.

NTS: exists unique $\widetilde{l} : V \otimes W \to U$ such that $\widetilde{l} \circ \phi = l$.

Define $\widetilde{l}$ on decomposable tensors of the form $v \otimes w$ as $\widetilde{l}(v \otimes w) = l(v, w)$ and extend to all of $V \otimes W$ by linearity.

It is clear that $\widetilde{l} \circ \phi(v, w) = \widetilde{l}(v \otimes w) = l(v, w)$ and the diagram commutes.
Uniqueness: Suppose $\widetilde{l}'$ is another linear lifting of l. Then, for (v_0, w_0), $\widetilde{l} \circ \phi(v_0, w_0) = \widetilde{l}(v_0 \otimes w_0) = l(v_0, w_0) = \widetilde{l}' \circ \phi(v_0, w_0) = \widetilde{l}'(v_0 \otimes w_0)$, and thus $\widetilde{l}' = \widetilde{l}$.

Now to prove isomorphism. The universal mapping property can be summarized in a commutative diagram: For some A, B is said to satisfy the universal mapping property if $\forall C \forall l, \exists! \widetilde{l}$ such that the following diagram commutes.

$$\begin{tikzcd}
B \arrow[dr, dashed, "\widetilde{l}"] & \\
A \arrow[u, "\phi_B"] \arrow[r, "l"] & C
\end{tikzcd}$$

To prove uniqueness, let (X, ϕ_X) be another object that satisfies the mapping property for A. Then, by applying the mapping property of B to X, we get the following diagram.

$$\begin{tikzcd}
B \arrow[dr, dashed, "\widetilde{\phi_X}"] & \\
A \arrow[u, "\phi_B"] \arrow[r, "\phi_X"] \arrow[d, "\phi_B"'] & X \arrow[dl, dashed, "\widetilde{\phi_B}"] \\
B &
\end{tikzcd}$$

Then, from the diagram, since $\widetilde{\phi_X} \circ \phi_B = \phi_X$ and $\widetilde{\phi_B} \circ \phi_X = \phi_B$, it follows that $\phi_X = \widetilde{\phi_X} \circ \widetilde{\phi_B} \circ \phi_X$ and $\phi_B = \widetilde{\phi_B} \circ \widetilde{\phi_X} \circ \phi_B$. Thus, $\widetilde{\phi_B}$ and $\widetilde{\phi_X}$ are inverses of each other that compose to the identity, and form an isomorphism of X and B.
(b) Now we also prove universality within a general category. In a general category C, let X and Y satisfy the mapping property for A.

$V \otimes W \cong W \otimes V$. Define the isomorphism as, for $\psi : V \times W \to W \times V$ the canonical isomorphism, $\psi_0 : V \otimes W \to W \otimes V$.

Let ϕ be the bilinear map from part (a) of $V \times W$ into $V \otimes W$ and ϕ' the bilinear map of $W \times V$ into $W \otimes V$. Then, $\psi_0 = \phi' \circ \psi$, with natural inverse $\psi_0^{-1} = \phi \circ \psi^{-1}$ where ψ_0 is extended to all of $V \otimes W$ via linearity.

(c)
$U \otimes (V \otimes W) = (U \otimes V) \otimes W$. Apply the same lifting as (b) on $\psi : U \times (V \times W) \to (U \times V) \times W$.

(d)
α is injective by linearity $\alpha(v_1) - \alpha(v_2) = 0 \to \alpha(v_1 - v_2) = 0$ and triviality of the kernel.

Let $T : V \to W$ be an element of $\mathrm{Hom}(V, W)$. $T(x_i) = \sum c_j y_j = w_i$. Then, $T(V) = T(\sum c_i y_i) = \sum c_i T(x_i) = \sum c_i (\sum (c_j y_j)) = \sum_i w_i$. Let $f_i = \pi_i$ be the i-th coordinate projection. Then $T(V) = \sum f_i(v) w_i = \sum \alpha(f_i \otimes w_i)(v) = \alpha(\sum (f_i \otimes w_i)(v)$. Then α is surjective as well.

(e)
Suppose $(v \otimes w) \in V \otimes W)$. Then $(v \otimes w) = (\sum c_i e_i) \otimes (\sum d_j f_j) = \sum_i ((c_i e_i) \otimes (\sum d_j f_j)) = \sum_i c_i (e_i \otimes (\sum d_j f_j)) = \sum_i \sum_j c_i (e_i \otimes (d_j f_j)) = \sum_i \sum_j c_i d_j (e_i \otimes f_j)$. Thus the desired set is a basis.

Problem 2.2

(a)
Provide an example of a homogeneous tensor that is not decomposable

Proof. Let V be a vector space, and $V \otimes V$ the corrseponding tensor product space. Furthermore, let v, w be vectors in V. Then, the tensor $v \otimes w + w \otimes v$ is homogeneous of degree two, but is not decomposable. □

(b)
Show that for $dim(V) \leq 3$, every homogeneous element of $\Lambda(V)$ is decomposable.

Proof. Let V be a three dimensional vector space with basis $\{v_1, v_2, v_3\}$. Then, the corresponding exterior algebra has basis elements

$$\begin{array}{ccc} & v_1 \wedge v_2 \wedge v_3 & \\ v_1 \wedge v_2 & v_1 \wedge v_3 & v_2 \wedge v_3 \\ v_1 & v_2 & v_3 \\ & 1 & \end{array}$$

It suffices to check for degree two elements of $\Lambda(V)$ that they are decomposable.
To this end, let $c_1 v_1 \wedge v_2 + c_2 v_1 \wedge v_3 + c_3 v_2 \wedge v_3$ be an arbitrary degree two element of the exterior algebra. Then, it is easy to see that

$$\begin{aligned} c_1 v_1 \wedge v_2 + c_2 v_1 \wedge v_3 + c_3 v_2 \wedge v_3 &= v_1 \wedge (c_1 v_2 + c_2 v_3) + c_3 v_2 \wedge v_3 \\ &= (v_1 - \frac{c_1}{c_3} v_3) \wedge (c_1 v_2 + c_2 v_3) \end{aligned}$$

□

(c)
Give an example of a homogeneous indecomposable element of $\Lambda(V)$.

Proof. The element $v_1 \wedge v_2 + v_3 \wedge v_4$ for linearly independent $v_1 ... v_4$ is indecomposable. □

(d)
Is $\alpha \wedge \alpha = 0$?

Proof. Decomposable elements of α wedge together to zero, meaning $\alpha \wedge \alpha = 0$. □

PROBLEM 2.3

(a)
Let $u \in \Lambda_k(V)$ and $v \in \Lambda_l(V)$. Then, u and v are homogeneous and the wedge product $u \wedge v$ is an element of $\Lambda_{k+l}(V)$ by the definition of the exterior algebra as $C(V)/I(V)$, following from the definition of the tensor product of homogeneous tensors. Since u, v are homogenous, $u = u_1 \wedge ... \wedge u_k$ and $v = v_1 \wedge ... \wedge v_l$, and thus $u \wedge v = u_1 \wedge ... \wedge u_k \wedge v_1 \wedge ... \wedge v_l = (-1)^k v_1 \wedge u_1 \wedge ... \wedge u_k \wedge v_2 \wedge ... \wedge v_l = (-1)^k (-1)^l v_1 \wedge ... \wedge v_l \wedge u_1 \wedge ... \wedge u_k \wedge = (-1)^{kl} v \wedge u$

(b)
First we show that $\{e_\Phi\}$ is a basis of $\Lambda(V)$. Observe that the elements of $\{e_\Phi\}$ span $\Lambda(V)$. To show linear independence of the set, consider $\sum a_\Phi e_\Phi = 0$. Then, $(\sum a_\Phi e_\Phi) \wedge (e_1 \wedge ... \wedge e_n) = a_\emptyset (e_1 \wedge ... \wedge e_n) = 0$. Then we induct; assume that $a_s = 0 \; \forall s$ such that $|s| \leq k$.
Then, for $|s| \geq k+1$, $\sum a_s e_s = 0$, $(\sum a_\Phi e_\Phi) \wedge e_{\Phi^c} = a_\Phi (e_1 \wedge ... \wedge e_n) = 0$ which implies that $a_\Phi = 0$ for $|\Phi| = k+1$, where Φ^c is the set theoretic complement of Φ.

Furthermore, it is obvious that $\Lambda_d(V)$ is a 1 dimensional vector space and is isomorphic to $\mathbb{R}$, and that $\Lambda_{d+j}(V)$ is isomorphic to $\{0\}$ and that $\dim(\Lambda_k(V)) = 2^d$ and $\dim(\Lambda_k(V)) = \binom{n}{k}$.

(c)
Define $\widetilde{l}$ on decomposable states as $\widetilde{l}(v_1 \wedge ... \wedge v_k) = l(v_1, ... v_k)$ so that $\widetilde{l} \circ \phi = l$, and extend $\widetilde{l}$ to all of $\Lambda_k(V)$ by linearity. Universality follows from the result in problem 2.1 (b).