Hillsdale College

Final Exam: Differentiable Manifolds

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1 The Torus Skew Line

Problem 1: The torus is the manifold $S^1 \times S^1$. Consider S^1 as the unit circle in the complex plane. We define $\phi : \mathbb{R} \to S^1 \times S^1$ by

$$\phi(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$$

where α is an irrational number. Prove that $(\mathbb{R}, \phi)$ is a dense submanifold of $S^1 \times S^1$.
Solution: This proof will be done in two parts. First I will show that $(\mathbb{R}, \phi)$ is in fact a submanifold of the torus, then I will show that it is dense in the torus.

$(\mathbb{R}, \phi)$ is a submanifold of $S^1 \times S^1$

A submanifold is a pair (X, f) of a manifold X and an immersion f that is one-to-one. $\mathbb{R}$ is already known to be a manifold, so it suffices to show that ϕ is a one-to-one immersion.
In order for ϕ to be an immersion, $d\phi_x$ must be nonsingular $\forall x \in \mathbb{R}$. Since there is a global coordinate system on both the torus and $\mathbb{R}$, it suffices to check non-singularity in these coordinate systems. To do so, $d\phi$ must be known. So,

$$
\begin{aligned}
d\phi_x :& \mathbb{R}_x \to \mathbb{T}^2_{\phi(x)} \\
d\phi_x\left(\frac{\partial}{\partial t}_x\right) &= \frac{\partial(e^{2\pi i t})}{\partial t}_x \frac{\partial}{\partial y_1}_{\phi(x)} + \frac{\partial(e^{2\pi i \alpha t})}{\partial t}_x \frac{\partial}{\partial y_2}_{\phi(x)} \\
&= 2\pi i t e^{2\pi i t} \frac{\partial}{\partial y_1}_{\phi(x)} + 2\pi i \alpha t e^{2\pi i \alpha t} \frac{\partial}{\partial y_2}_{\phi(x)} \\
&= 2\pi i t \left(e^{2\pi i t} \frac{\partial}{\partial y_1}_{\phi(x)} + \alpha e^{2\pi i \alpha t} \frac{\partial}{\partial y_2}_{\phi(x)} \right)
\end{aligned}
$$

In order for this to be nonsingular, both the $\frac{\partial}{\partial y_1}_{\phi(x)}$ coordinate and the $\frac{\partial}{\partial y_2}_{\phi(x)}$ coordinate must not both be zero. Such a condition occurs when $e^{2\pi i t} = \alpha e^{2\pi i \alpha t} = 0$. But, this implies that

$$
\begin{aligned}
&\exists n_1, n_2 \in \mathbb{Z}, \\
&\qquad t = n_1 \\
&\qquad \alpha t = n_2 \\
&\qquad\quad \implies \alpha t - t \in \mathbb{Z} \\
&\qquad\quad \implies \alpha - 1 \in \mathbb{Z}
\end{aligned}
$$

But this violates the assumption that α is irrational. So, $d\phi$ must be nonsingular everywhere.
Next, ϕ must be an injective map. But, ϕ is easily seen to be nonsingular, by a similar argument to the one for $d\phi$. Thus, ϕ must be injective.

$\phi(\mathbb{R})$ Is Dense In $\mathbb{T}^2$

For the skew line to be dense in the torus, it must be that every open set in $\mathbb{T}^2$ intersects $\phi(\mathbb{R})$. This will be shown by showing that any point in $\mathbb{T}^2$ is arbitrarily close to the skew line.
To do this, I will view $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$, and my maps as $\phi(t) \to [t, \alpha t]$. Here, $(x_1, y_1) \sim (x_2, y_2) \iff (x_2 - x_1, y_2 - y_1) \in \mathbb{Z} \times \mathbb{Z}$.

Consider a point $[x, y]$. To show the skew line is dense in $\mathbb{T}^2$, I will show that it intersects the region $\{[x, y] | y \in (a, b)\}$, where a and b are chosen as the endpoints of the $\frac{1}{2^n}$ partition of the unit interval that contains y.

Suppose ther exists an n such that the skew line does not intersect the box. Then, by the integer translational symmetry of this quotient space, the skew line does not intersect any of the other partitions of the interval. Thus, the skew line must be passing through the boundaries of the partition.

However, there are only finitely many boundaries, so by the pidgeonhole principle, there must be at least one for which the skew line passes through twice. This violates the assumption that the skew line imbedding was one-to-one, and so it must not be the case that the skew line does not intersect the neighborhood of $[x, y]$. Therefore, the skew line is dense in $\mathbb{T}^2$.

2 The Exactness of Left Wedging

Problem 2: Let $\xi \in V$. Prove that the composition

$$\Lambda_p(V) \to^{\xi\wedge} \Lambda_{p+1}(V) \to^{\xi\wedge} \Lambda_{p+2}(V)$$

is exact.

Solution: First, I will prove that the sequence is a cochain complex. Then, I will prove that the sequence is exact at $p+1$.
It is easy to see that the composition of any two sequential arrows is null. To see this, observe that the composition looks like

$$\xi \wedge \xi \wedge v$$

For some $v \in \Lambda_p(V)$. But, since $\xi \wedge \xi = 0$, the whole product equals zero.

Exactness of the sequence involves showing that $\ker(\xi\wedge) = \mathrm{im}(\xi\wedge)$. However, the kernel of this map is precisely those exterior products in $\Lambda_{p+1}(V)$ that contain ξ as one of their terms. (That is, vectors that have a component along ξ) It is clear that the image of this map is the same thing. Since the wedge product is alternating, it is possible to move the ξ along any point of the wedge product. Thus, the image of $\xi\wedge$ is the elements of $\Lambda_{p+1}(V)$ that have ξ as an element of their product.
Thus, the sequence is exact.

3 Lie Groups and Second Countability

Problem 3: a Lie group is a differentiable manifold which also has a group structure such that the map $G \times G \to G$ defined by $(\sigma, \tau) \to \sigma\tau^{-1}$ is C^∞. Prove the assumption of second countability in a connected Lie group is redundant.

Solution: For this proof, we will use the result that the Lie group is the union of powers of any neighborhood of the identity. One can construct, then, a countable neighborhood basis for the Lie group by taking countable powers of the neighborhood.

4 Wedge Product Preservatives

Problem 4: Show that if α and β are closed forms, then $\alpha \wedge \beta$ is also closed. Furthermore, if they are exact, then their wedge is also exact.

Solution: I will begin by showing the preservation of closedness. if α and β are closed, then $d\alpha = 0$, and $d\beta = 0$. Then,

$$\begin{aligned} d(\alpha \wedge \beta) &= (d\alpha) \wedge \beta + (-1)^{\text{some power}} \alpha \wedge (d\beta) \\ &= 0 \wedge \beta + (-1)^{\text{some power}} \alpha \wedge 0 \\ &= 0 \end{aligned}$$

So, $\alpha \wedge \beta$ is closed.
Next, let's show exactness is preserved. Suppose $\alpha = d\lambda$ and$\beta = d\mu$. Then,

$$\begin{aligned} \alpha \wedge \beta &= (d\lambda) \wedge (d\mu) \\ &= d(\lambda \wedge (d\mu)) \end{aligned}$$

So, the wedge is exact.

5 Exact Sequence Tensors

Problem 5: PRove that if you tensor a long exact sequence of torsionless sheaves with a sheaf S, then the resulting long sequence is still exact.

Solution: Let

$$\ldots \to^{1} T_{q-1} \to^{2} T_q \to^{3} T_{q+1} \to^{4} \ldots$$

be a long exact sequence of torsionless sheaves, and let S be a sheaf. Then, let's examine the tensor product

$$\ldots \to^{1\otimes id} T_{q-1} \otimes S \to^{2\otimes id} T_q \otimes S \to^{3\otimes id} T_{q+1} \otimes S \to^{4\otimes id} \ldots$$

Furthermore, let's examine exactness of the sequence at q.

We know from the exactness of the first sequence that $\ker(3) = \mathrm{im}(2)$. Then, $\ker(3 \otimes id)$ is

$$\left\{t \otimes s | t \in T_q, s \in S, 3(t) = 0\right\} \cup \left\{t \otimes s | t \in T_q, s \in S, id(s) = s = 0\right\}$$

Since T_q is torsionless, nothing else from S can cause the tensor product to collapse.

But, $\left\{t \otimes s | t \in T_q, s \in S, id(s) = s = 0\right\} = \{t \otimes 0\} = \{0\}$. So, $\ker(3 \otimes id) = \ker(3) \otimes S$.

Examining $\mathrm{im}(2 \otimes id)$, we see that

$$\begin{aligned} \mathrm{im}(2 \otimes id) &= \left\{t \otimes s | t = 2(t'), t' \in T_{q-1}, s \in S\right\} \\ &= \mathrm{im}(2) \otimes S \\ &= \ker(3) \otimes S \end{aligned}$$

Therefore, since $\ker(3 \otimes id) = \mathrm{im}(2 \otimes id)$, the tensor sequence is exact.

6 Stars and Triangles Forever

Problem 6: Let $\star : E^p(M) \to E^{n-p}(M)$ be the Hodge dual operator, which satisfies $\star\star = (-1)^{p(n-p)}$, and let $\delta : E^p(M) \to E^{p-1}(M)$ be defined as

$$\delta = (-1)^{n(p+1)+1} \star d \star$$

Then, the Laplace operator is $\Delta = \delta d + d\delta$, and is a linear operator on $E^p(M)$ for each p with $0 \le p \le n$.

The Laplacian on 0-forms

Part a: Show that on $E^0(\mathbb{R}^n)$, the Laplacian is

$$\Delta = (-1) \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Solution: It is ~~easy~~ doable to show this identity computationally. Let $f \in E^0(\mathbb{R}^n)$. Then,

$$\begin{aligned}
\Delta f &= \delta d f + d \delta f \\
&= \delta \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \right) + d(0) \\
&= (-1)^{n(0+1)+1} \star d \star \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \right) \\
&= (-1)^{n+1} \star d \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i^\star \right) \text{where } x_i^\star \text{ is the dual of } x_i \\
&= (-1)^{n+1} \star \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_i} dx_i \wedge dx_i^\star \right) \\
&= (-1)^{n+1} \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \star (dx_i \wedge dx_i^\star) \right) \\
&= (-1)^{n+1} \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} (-1)^{i-1} (-1)^{i-1} \right)
\end{aligned}$$

Finding $[\star, \Delta]$

Part b: Prove that $\star$ commutes with Δ.

Solution: Again, this is provable by direct computation.

$$
\begin{aligned}
\star\Delta &= \star\,(\delta d + d\delta) \\
&= \star\delta d + \star d\delta \\
&= \star(-1)^{n(p+1)+1} \star d \star d + \star d(-1)^{n(p+1)+1} \star d\star \\
&= (-1)^{n(p+1)+1} \star\star d \star d + (-1)^{n(p+1)+1} \star d \star d\star \\
&= d(-1)^{n(p+1)+1} \star d \star\star + (-1)^{n(p+1)+1} \star d \star d\star \\
&= d\delta \star + \delta d\star \\
&= \Delta\star
\end{aligned}
$$

Where $\star\star$ was able to commute with other operators due to the fact that it is a scalar.

7 Integration on Manifolds

Problem 7: Give your favorite example of an integral on manifolds computation.

Solution: For this problem, we will compute the integral

$$\int_{S^1} \frac{x\,dy - y\,dx}{x^2 + y^2}$$

For this integral, we will use the co-ordinate maps

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

which, by setting r to 1 and taking the exterior derivative, yields the identities

$$\begin{aligned} dx &= -\sin(\theta)\,d\theta \\ dy &= \cos(\theta)\,d\theta \end{aligned}$$

Then, the integral becomes

$$\begin{aligned} \int_{S^1} (\sin^2(\theta) + \cos^2(\theta))\,d\theta &= \int_{S^1} d\theta \\ &= 2\pi \end{aligned}$$

What makes this example interesting is the fact that the form integrated is actually a closed form on the space $\mathbb{R}^2 \setminus \{0\}$. Furthermore, since the integral is nonzero, the form is not exact. This demonstrates that there exist closed 1-forms on S^1 that are not exact, which means that the first cohomology group of S^1 is nontrivial. Since S^1 is a typical example of a simple space that differs from Euclidean space, this form gives an example of something that breaks the triviality of the cohomology of Euclidean space. In particular, it suggests that singularities (like the one this form has at zero) have something to do with cohomology (and homology) breaking.

8 THE STORY OF MANIFOLDS

Problem 8: Tell the story of differentiable manifolds.
Solution: The study of topology allows mathematics to talk about more general spaces than just $\mathbb{R}^n$. In particular, there is a certain class of spaces that seem locally similar to Euclidean space, but globally differ in some meaningful way. (Think of S^2, which is locally homeomorphic to some subset of $\mathbb{R}^2$, but is a compact space, unlike $\mathbb{R}^2$) Euclidean space has a rich structure of differentiability that allows calculus to be done on it, and it seems natural to ask if such a structure can also be given to more general topological spaces.

To that end, we define a differentiable manifold as a topological space that is locally Euclidean, along with a series of "co-ordinate systems" on open subsets of the space that satisfy the properties of a differentiable structure. In particular, the co-ordinate systems need to cover the whole space, they need to match up on overlaps in a C^∞ way, and the set of co-ordinate maps must be maximal in the sense that it contains all possible compatible maps.

With such a structure, it becomes possible to talk about generalized derivatives. By looking at linear derivations of germs of functions from the manifold to $\mathbb{R}$, we develop the idea of tangent vectors, which locally look like directional derivatives. Furthermore, by looking at wedge products of (duals of) these tangent vectors, we can develop a more general theory of differential forms on the manifold, which allow for integration over subsets of the manifold.

In the category of differentiable manifolds, the objects are the manifolds, and the arrows are C^∞ maps (in the sense that they preserve the C^∞ness of the coordinates). Then, one can define a functor, known as exterior differentiation, from this category to the category of forms on differentiable manifolds.

With these tools, we define integration of a differential form (which, in this case can be thought of as a generalized multidimensional function, vector field, etc.) on a manifold by locally "pulling back" the form to normal Euclidean space, and integrating the form in the usual way in $\mathbb{R}^N$.

Not that the theory of manifolds is all analysis, though. Homology and cohomology information can be pulled from the manifold. For cohomology, the algebraic spaces are the spaces of p-dimensional forms on the manifold, and the arrows are exterior differentiation. For homology, the spaces are free combinations of maps of Euclidean shapes into the manifold, with the arrows being the boundary operation. From this, we get the powerful result that the qth differential cohomology of a manifold is isomorphic to the dual of the qth simplicial homology, a result known as deRham's theorem.

All in all, the theory of differentiable manifolds is a way to use local Euclideanness of certain topological spaces to develop calculus and algebra on those spaces.