Problem Set 3: 1.7

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September 30, 2016

PROBLEM 1.7

Problem 1.7: Prove the following:
(a) Let M be a differentiable manifold, and $A \subset M$. Fix a topology on A. Then, there is at most one differentiable structure on A such that (A, i) is a submanifold.
Suppose that there are two families of coordinate charts $\mathscr{F}_1$ and $\mathscr{F}_2$ on A such that (A, i) is a submanifold of M. We will show that these two families are compatible, and by the maximality of the differentiable structure, they share the same differentiable structure.

Let A_1 be the manifold A under $\mathscr{F}_1$, and A_2 be the manifold A under $\mathscr{F}_2$. Then, by theorem 1.32, there exists a unique mapping $\psi_0 : A_1 \to A_2$ such that $i \circ \psi_0 = i$, where i is the inclusion map that makes A a submanifold. By this relation, however, it must be that $\psi_0 = id$. Since A_1 and A_2 have the same topology, id is a homeomorphism, and by 1.32a, ψ_0 is C^∞.
So, let (U_1, ϕ_1) be a coordinate chart on A_1, and (U_2, ϕ_2) a coordinate chart on A_2. By the existence of ψ_0, the map $\phi_2 \circ \psi_0 \circ \phi_1^{-1}$ exists and is C^∞. Furthermore, since $\psi_0 = id$, $\phi_2 \circ \psi_0 \circ \phi_1^{-1} = \phi_2 \circ \phi_1^{-1}$ and is C^∞. Thus, since this holds for arbitrary coordinate charts on A_1 and A_2, $\mathscr{F}_1$ is compatible with $\mathscr{F}_2$.

Thus, if A has a differentiable structure that makes (A, i) a submanifold, it is unique.

(b) Let $A \subset M$. If in the relative topology, A has a differentiable structure such that (A, i) is a submanifold of M, then A has a unique submanifold structure such that (A, i) is a submanifold of M.
Let A_r be A under the relative topology with a differentiable structure such that it is a submanifold under the inclusion map i_r. Then, let A_t be A with any manifold structure that

makes it a submanifold under the inclusion map i_t. To show that A_r is the unique manifold structure on A that makes it a submanifold, we will show that A_r and A_t are diffeomorphic to each other. Furthermore, the diffeomorphism will be the identity map, and the two spaces are actually equal.

By theorem 1.32, there exists a unique $\psi_0 : A_t \to A_r$ such that $i_r \circ \psi_0 = i_t$. We will show that such a ψ_0 is a diffeomorphism. ψ_0 is automatically C^∞ since i_r is an imbedding (theorem 1.32b). By the result of problem 6, if ψ_0 is bijective, and $d\psi_0$ is everywhere nonsingular, then ψ_0 is a diffeomorphism.
ψ_0 is bijective, since $\psi_0 = i_r^{-1} \circ i_t$, which as a set operation is bijective. (i_t is bijective onto its range, and $\text{dom}(i_r^{-1} = \text{ran}(i_t))$.

To see that $d\psi_0$ is everywhere nonsingular, note that $d\psi_0 = d(i_r^{-1} \circ i_t)$. Since i_r^{-1} is a homeomorphism, and i_r is an immersion, di_r^{-1} is nonsingular. Similarly, since i_t is an immersion, di_t is nonsingular.

Therefore, ψ_0 is a diffeomorphism. Since $\psi_0 = i_r^{-1} \circ i_t$, which is the identity, the spaces are actually equal.
Thus, the manifold structure on A is unique.