Problem Set 4: 1.9

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PROBLEM 2.1

Let U, V, W be vector spaces, with $\phi : V \times W \to V \otimes W$ the natural mapping, $l : V \times W \to U$ bilinear.

NTS: exists unique $\widetilde{l} : V \otimes W \to U$ such that $\widetilde{l} \circ \phi = l$.

Define $\widetilde{l}$ on decomposable tensors of the form $v \otimes w$ as $\widetilde{l}(v \otimes w) = l(v, w)$

PROBLEM 2.2

0.1 PART A

Provide an example of a homogeneous tensor that is not decomposable

Proof. Let V be a vector space, and $V \otimes V$ the corrseponding tensor product space. Furthermore, let v, w be vectors in V. Then, the tensor $v \otimes w + w \otimes v$ is homogeneous of degree two, but is not decomposable. □

0.2 PART B

Show that for $dim(V) \leq 3$, every homogeneous element of $\Lambda(V)$ is decomposable.

Proof. Let V be a three dimensional vector space with basis $\{v_1, v_2, v_3\}$. Then, the corresponding exterior algebra has basis elements

$$\begin{array}{ccc} & v_1 \wedge v_2 \wedge v_3 & \\ v_1 \wedge v_2 & v_1 \wedge v_3 & v_2 \wedge v_3 \\ v_1 & v_2 & v_3 \\ & 1 & \end{array}$$

It suffices to check for degree two elements of $\Lambda(V)$ that they are decomposable.
To this end, let $c_1 v_1 \wedge v_2 + c_2 v_1 \wedge v_3 + c_3 v_2 \wedge v_3$ be an arbitrary degree two element of the exterior algebra. Then, it is easy to see that

$$\begin{aligned} c_1 v_1 \wedge v_2 + c_2 v_1 \wedge v_3 + c_3 v_2 \wedge v_3 &= v_1 \wedge (c_1 v_2 + c_2 v_3) + c_3 v_2 \wedge v_3 \\ &= (v_1 - \frac{c_1}{c_3} v_3) \wedge (c_1 v_2 + c_2 v_3) \end{aligned}$$

□

0.3 Part C

Give an example of a homogeneous indecomposable element of $\Lambda(V)$.

Proof. The element $v_1 \wedge v_2 + v_3 \wedge v_4$ for linearly independent $v_1 ... v_4$ is indecomposable. □

0.4 Part D

Is $\alpha \wedge \alpha = 0$?

Proof. Since $\alpha \wedge \alpha = -\alpha \wedge \alpha$, this implies $\alpha \wedge \alpha = 0$. □