Problem Set 4: 1.9

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PROBLEM 2.1

(a)
Let U, V, W be vector spaces, with $\phi : V \times W \to V \otimes W$ the natural mapping, $l : V \times W \to U$ bilinear.

NTS: exists unique $\tilde{l} : V \otimes W \to U$ such that $\tilde{l} \circ \phi = l$.

Define $\tilde{l}$ on decomposable tensors of the form $v \otimes w$ as $\tilde{l}(v \otimes w) = l(v, w)$ and extend to all of $V \otimes W$ by linearity.

It is clear that $\tilde{l} \circ \phi(v, w) = \tilde{l}(v \otimes w) = l(v, w)$ and the diagram commutes.
Uniqueness: Suppose $\tilde{l}'$ is another linear lifting of l. Then, for (v_0, w_0), $\tilde{l} \circ \phi(v_0, w_0) = \tilde{l}(v_0 \otimes w_0) = l(v_0, w_0) = \tilde{l}' \circ \phi(v_0, w_0) = \tilde{l}'(v_0 \otimes w_0)$, and thus $\tilde{l}' = \tilde{l}$.

(b)

$V \otimes W \cong W \otimes V$. Define the isomorphism as, for $\psi : V \times W \to W \times V$ the canonical isomorphism, $\psi_0 : V \otimes W \to W \otimes V$.

Let ϕ be the bilinear map from part (a) of $V \times W$ into $V \otimes W$ and ϕ' the bilinear map of $W \times V$ into $W \otimes V$. Then, $\psi_0 = \phi' \circ \psi$, with natural inverse $\psi_0^{-1} = \phi \circ \psi^{-1}$ where ψ_0 is extended to all of $V \otimes W$ via linearity.

(c)

$U \otimes (V \otimes W) = (U \otimes V) \otimes W$. Apply the same lifting as (b) on $\psi : U \times (V \times W) \to (U \times V) \times W$.

(d)

α is injective by linearity $\alpha(v_1) - \alpha(v_2) = 0 \to \alpha(v_1 - v_2) = 0$ and triviality of the kernel.

Let $T : V \to W$ be an element of $\mathrm{Hom}(V, W)$. $T(x_i) = \sum c_j y_j = w_i$. Then, $T(V) = T(\sum c_i y_i) = \sum c_i T(x_i) = \sum c_i (\sum (c_j y_j)) = \sum_i w_i$. Let $f_i = \pi_i$ be the i-th coordinate projection. Then $T(V) = \sum f_i(v) w_i = \sum \alpha(f_i \otimes w_i)(v) = \alpha(\sum (f_i \otimes w_i)(v)$. Then α is surjective as well.

(e)

Suppose $(v \otimes w) \in V \otimes W)$. Then $(v \otimes w) = (\sum c_i e_i) \otimes (\sum d_j f_j) = \sum_i ((c_i e_i) \otimes (\sum d_j f_j)) = \sum_i c_i (e_i \otimes (\sum d_j f_j)) = \sum_i \sum_j c_i (e_i \otimes (d_j f_j)) = \sum_i \sum_j c_i d_j (e_i \otimes f_j)$. Thus the desired set is a basis.