Homework 5

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Problem 1

Let E, F be closed subspaces of a Hilbert space. Prove that $P_E P_F = P_E$ if and only if $E \subseteq F$.

Proof. Suppose first that $P_E P_F = P_E$. Then, in particular,

$$P_E^* = (P_E P_F)^* = P_F^* P_E^* = P_F P_E = P_E$$

by self-adjointness of projections. Thus, $P_E P_F = P_F P_E = P_E$, and the projections commute. Thus, the von Neumann algebra $W^*(P_E, P_F, I)$ is Abelian, and is isometrically $*$-isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ). In particular, the projections P_E, P_F get sent to self-adjoint idempotents $P_E \mapsto M_{\chi_S}$ and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$.

Now, the requirement $P_F P_E = P_E$ corresponds to the requirement

$$M_{\chi_{S'}} M_{\chi_S} = M_{\chi_S}$$

which means that $S \subset S'$. This, in turn, implies that $E \subset F$.

Indeed, E is the subspace of H on which P_E is the identity, which corresponds to the subspace

$$\tilde{E} = \int_S^\oplus H(x) d\mu(x)$$

on which M_{χ_S} is the identity. Similarly,

$$\tilde{F} = \int_{S'}^\oplus H(x) d\mu(x)$$

Clearly, $\tilde{E} \subset \tilde{F}$ (since $S \subset S'$) and so $E \subset F$ as well.

For the converse direction, assume that $E \subset F$. Then, on F, $P_F = I_F$, and $P_F P_E = I_F P_E = P_E$. Furthermore, on $F^\perp$, $P_F P_E = 0 = P_E$. Thus, on all of $H = F \oplus F^\perp$, $P_F P_E = P_E$ as desired. $\square$

Problem 2

Characterize the closed subspaces E, F of a Hilbert space H satisfy $P_F P_E = P_E P_F$.

Proof. I assert that P_E and P_F commute if and only if H can be decomposed into the four orthogonal components

$$H = E \cap F \oplus E \cap F^{\perp} \oplus E^{\perp} \cap F \oplus E^{\perp} \cap F^{\perp}$$

To see this, suppose first that P_E and P_F commute. Then, consider the Abelian von Neumann algebra $W^*(P_E, P_F, I)$. Like before, we use the Borel functional calculus to identify $W^*(P_E, P_F, I)$ with $L^\infty(X, \mu)$ acting on $\tilde{H} = \int_X^{\oplus} H(x) d\mu(x)$, which is unitarily equivalent to H (in the sense that there is a unitary transformation U such that $\tilde{H} = UH$ and $UW^*(P_E, P_F, I)U^* = L^\infty(X, \mu)$).

Under this identification, $P_E \mapsto M_{\chi_S}$, and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$. In particular, this decomposes $\tilde{H}$ into four orthogonal components

$$\begin{aligned}
\tilde{H} &= \int_X^{\oplus} H(x) d\mu(x) \\
&= \int_{S \cap S'}^{\oplus} H(x) d\mu(x) \oplus \int_{S \cap S'^c}^{\oplus} H(x) d\mu(x) \oplus \int_{S' \cap S^c}^{\oplus} H(x) d\mu(x) \oplus \int_{S^c \cap S'^c}^{\oplus} H(x) d\mu(x)
\end{aligned}$$

which translates into the decomposition on H as

$$H = E \cap F \oplus E \cap F^{\perp} \oplus E^{\perp} \cap F \oplus E^{\perp} \cap F^{\perp}$$

as desired.

Conversely, suppose H can be decomposed this way. Then, let $v \in H$ be decomposed as

$$v = v_1 + v_2 + v_3 + v_4$$

where $v_1 \in E \cap F$, $v_2 \in E \cap F^{\perp}$, $v_3 \in E^{\perp} \cap F$ and $v_4 \in E^{\perp} \cap F^{\perp}$.

We compute the effect of $P_E P_F$ and $P_F P_E$ directly.

$$\begin{aligned}
P_E P_F(v) &= P_E P_F(v_1 + v_2 + v_3 + v_4) \\
&= P_E(v_1 + v_3) \\
&= v_1 \\
P_F P_E(v) &= P_F P_E(v_1 + v_2 + v_3 + v_4) \\
&= P_F(v_1 + v_2) \\
&= v_1
\end{aligned}$$

and thus $P_E P_F = P_F P_E$ as desired. □

Problem 3

Let E, F be closed subspaces of a Hilbert space H. An operator U is said to be a partial isometry from E to F if $U|_E$ is an isometry onto F, and $U|_{E^\perp} = 0$. Prove that U is a partial isometry $\iff$ U^*U is a projection $\iff$ UU^* is a projection.

Proof. Suppose first that U is a partial isometry from E to F. I assert that $U^*U = P_E$. To see this, suppose $e \in E$, $v \in H$ and let $v = e' + v'$ where $e' \in E$ and $v' \in E^\perp$. Then,

$$\begin{aligned} \langle U^*Ue|v\rangle &= \langle U^*Ue|e' + v'\rangle \\ &= \langle Ue|Ue'\rangle + \langle Ue|Uv'\rangle \\ &= \langle e|e'\rangle + 0 \\ \implies \langle U^*Ue - e|v\rangle &= 0 \end{aligned}$$

and since this holds for all $v \in H$, $U^*Ue - e = 0$ and thus $U^*Ue = e$ and U^*U is the identity on E.

Furthermore, for $v' \in E^\perp$,

$$U^*Uv' = U^*(0) = 0$$

and so U^*U is the zero map on $E^\perp$. Thus, U^*U agrees with P_E at all points, so $U^*U = P_E$ as desired.

Conversely, suppose U^*U is a projection P_E for some closed subspace E. Define $F = U(E)$. We will first show that U is an isometry of E onto F. To see this, suppose $e, e' \in E$. We calculate directly

$$\begin{aligned} \langle Ue|Ue'\rangle &= \langle U^*Ue|e'\rangle \\ &= \langle P_E e|e'\rangle \\ &= \langle e|e'\rangle \end{aligned}$$

and thus U is an isometry from E to F. Note that this immediately implies that F is a closed subspace. Finally, we show that $U|_{E^\perp} = 0$. Let $v' \in E^\perp$, Then,

$$\begin{aligned} \|Uv'\| &= \langle Uv'|Uv'\rangle \\ &= \langle U^*Uv'|v'\rangle \\ &= \langle 0|v'\rangle = 0 \end{aligned}$$

and so $U^*U|_{E^\perp} = 0$ as desired. Thus, U is a partial isometry.

Finally, we show that if U is a partial isometry, then U^* is a partial isometry, which will complete the proof. So, suppose U is a partial isometry. We will show that U^* is a partial isometry from F to E. First, we check that $U^*|_F$ is an isometry. Now, for each $f \in F$, there is some $e \in E$ with $Ue = f$. Thus, for $e, e' \in E$, $f, f' \in F$ with $Ue = f, Ue' = f'$,

$$\begin{aligned} \langle U^*f|U^*f'\rangle &= \langle U^*Ue|U^*Ue\rangle \\ &= \langle P_E e|P_E e'\rangle \\ &= \langle e|e'\rangle \\ &= \langle Ue|Ue'\rangle \\ &= \langle f|f'\rangle \end{aligned}$$

and so U^* is an isometry from F to E. Finally, we show that $U^*|_{F^\perp} = 0$. This is immediate, since

$$\ker U^* = U(H)^\perp = F^\perp$$

where we used the identity $\ker A^* = A(H)^\perp$ for all bounded operators A.

Note that this completes the proof. If U is a partial isometry, then U^* is a partial isometry, which implies that $U^{**}U^* = UU^*$ is a projection. Similarly, if $UU^* = U^{**}U^*$ is a projection, then U^* is a partial isometry, and thus $U^{**} = U$ is a partial isometry as well. □

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