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## Problem Set 5

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November 29, 2017

### PROBLEM 1

Show that for  $(X, \mathcal{T})$  a compact Hausdorff space, any topology  $\mathcal{T} \subsetneq \mathcal{T}'$  makes  $(X, \mathcal{T}')$  no longer compact Hausdorff. Furthermore, for  $\mathcal{T}' \subsetneq \mathcal{T}$ , the space  $(X, \mathcal{T}')$  is not compact Hausdorff.

*Proof.* For this proof, we will show that for  $\mathcal{T}, \mathcal{T}'$  topologies on  $X$  with  $\mathcal{T} \subset \mathcal{T}'$  and the property that  $X$  is compact Hausdorff under both these topologies, then they must be equal.

To do so, we consider the identity function  $id : (X, \mathcal{T}') \rightarrow (X, \mathcal{T})$ , which is continuous since  $\mathcal{T} \subset \mathcal{T}'$ . Now, for any  $C$  closed in  $(X, \mathcal{T}')$ , since  $(X, \mathcal{T}')$  is compact and  $id$  is continuous,  $id(C) = C$  is compact in  $(X, \mathcal{T})$ . However, since  $(X, \mathcal{T})$  is Hausdorff,  $C$  must also be closed. Thus,  $id$  is a closed map, and thus a homeomorphism, and the two topologies must be equal.  $\square$

### PROBLEM 2

Prove that for compact subsets  $A, B$  of  $X$ , their union  $A \cup B$  is compact as well.

*Proof.* Let  $\mathcal{O}$  be an open cover of  $A \cup B$ . In particular,  $\mathcal{O}$  covers  $A$ , and has a finite subset  $\mathcal{O}_A$  that covers  $A$ . Similarly, there is a finite subset  $\mathcal{O}_B$  that covers  $B$ . Their union  $\mathcal{O}_A \cup \mathcal{O}_B$  covers  $A \cup B$ , and is finite, as it is the union of finite sets. It is also a subcover, since both  $\mathcal{O}_A$  and  $\mathcal{O}_B$  are subsets of  $\mathcal{O}$ . Thus,  $\mathcal{O}$  has a finite subcover, and  $A \cup B$  is compact as desired.  $\square$

### PROBLEM 3

Suppose  $A$  is a subset of a metric space. Show that  $A$  is compact implies that  $A$  is closed and bounded. Give an example where the converse does not hold.

*Proof.* Suppose  $A$  is a compact subset of a metric space  $(X, d)$ . Since all metric spaces are Hausdorff, it follows immediately that  $A$  is closed (since it is a compact subset of a Hausdorff space).

Now, fix  $x \in A$ , and consider the open cover  $\mathcal{O} = \{V_n(x) \mid n \in \mathbb{N}\}$  of balls centered at  $x$  with radius  $n$ . This has a finite subcover, by compactness of  $A$ . So, let  $N$  be the largest radius in the finite subcover. For each  $V_m(x)$  in the finite subcover, then, we have that  $V_m(x) \subset V_N(x)$ . Thus, since the subcover covers  $A$ , it follows that  $V_N(x)$  covers  $A$ , and thus  $A$  is bounded.

For an example where the converse fails, consider the metric space  $(\mathbb{R}, d_{LA})$  of the real numbers with the discrete (Los Angeles) metric. The set  $\mathbb{R}$  itself is closed and bounded (since everything is at most distance 1 away from any particular point), but is clearly not compact, since it is an uncountable discrete set of points.  $\square$

## PROBLEM 4

Show that for  $X$  Hausdorff,  $A, B$  disjoint compact subsets of  $X$ , then there exist disjoint open sets  $U$  and  $V$  containing  $A$  and  $B$  respectively.

*Proof.* Let  $a \in A$ , and  $b \in B$ . Since  $A$  and  $B$  are disjoint, it must be that  $a \neq b$ . Thus, there exist disjoint open sets  $U_{ab}, V_{ab}$  such that  $a \in U_{ab}$  and  $b \in V_{ab}$ . Now, since  $B$  is compact, the open cover  $\mathcal{O}_B = \{V_{ab} \mid b \in B\}$  has a finite subcover  $\{V_{ab_i}\}_{i=1}^n$ . Furthermore, the open set  $U_a = \bigcap_{i=1}^n U_{ab_i}$  does not intersect the open set  $V_a = \bigcup_{i=1}^n V_{ab_i}$  which covers  $B$ ,

So, consider the open cover  $\mathcal{O}_A = \{U_a \mid a \in A\}$ . By compactness of  $A$ , this has a finite subcover  $\{U_{a_j}\}_{j=1}^m$ . Now, we have that the open set  $V = \bigcap_{j=1}^m V_{a_j}$  (which covers  $B$ , since each  $V_a$  covers  $B$ ) does not intersect the open set  $U = \bigcup_{j=1}^m U_{a_j}$  which covers  $A$ .

Thus,  $U$  and  $V$  are disjoint open sets containing  $A$  and  $B$  respectively, as desired.  $\square$

## PROBLEM 5

Show that for  $X$  and  $Y$  topological spaces with  $Y$  compact, the projection  $\pi_x : X \times Y \rightarrow X$  is a closed map.

*Proof.* Let  $C \subset X \times Y$  be closed, and let  $C_X = \pi_X(C)$ . Now, for  $x \in X \setminus C_X$ , we know that the slice  $\{x\} \times Y$  does not intersect  $C$ , and thus  $\{x\} \times Y \subset X \times Y \setminus C$ , which is an open subset of  $X \times Y$ . Since the slice  $\{x\} \times Y$  is contained in this open set, and  $Y$  is compact, we have by the tube lemma that there is some neighborhood  $U_x$  of  $x$  such that  $U_x \times Y$  is completely contained in  $X \times Y \setminus C$ . Therefore, since  $U_x \times Y \cap C = \emptyset$ , it must be that  $U_x \cap C_X = \emptyset$  as well. Thus, each point  $x$  in the complement of  $C_X$  has a neighborhood that is also in the complement of  $C_X$ , and so the complement of  $C_X$  is open, and  $C_X$  is closed, as desired.  $\square$

## PROBLEM 6

Suppose  $Y$  is compact Hausdorff, and  $f : X \rightarrow Y$  a function of sets. Prove that  $f$  is continuous if and only if its graph is a closed subset of  $X \times Y$ .

*Proof.* ( $\implies$ ) Suppose  $f$  is continuous. Specifically, we know that for every net  $x_\alpha$  in  $X$  that converges to some point  $x \in X$ , the net  $f(x_\alpha)$  in  $Y$  converges to  $f(x) \in Y$ .

So, consider a net  $(x_\alpha, f(x_\alpha))$  in the graph  $\{(x, f(x)) \mid x \in X\}$  that converges to some  $(x, y) \in X \times Y$ . Now, since the projection maps  $\pi_x, \pi_y$  are continuous, they preserve nets, so

$$\pi_x(x_\alpha, f(x_\alpha)) = x_\alpha \rightarrow \pi_x(x, y) = x$$

and

$$\pi_y(x < \alpha, f(x_\alpha)) = f(x_\alpha) \rightarrow \pi_y(x, y) = y$$

but we know that  $f(x_\alpha) \rightarrow f(x)$  by continuity of  $f$ , and since  $Y$  is Hausdorff, nets have unique limits, so  $y = f(x)$ , and the net  $(x_\alpha, f(x_\alpha))$  converges to  $(x, f(x)) \in \{(x, f(x)) \mid x \in X\}$ , and so the graph of  $f$  is closed, as desired.

( $\Leftarrow$ ) Suppose  $f$  is such that its graph  $\{(x, f(x)) \mid x \in X\} \subset X \times Y$  is closed. □