Problem Set 5

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PROBLEM 1

Show that for $(X, \mathscr{T})$ a compact Hausdorff space, any topology $\mathscr{T} \subsetneq \mathscr{T}'$ makes $(X, \mathscr{T}')$ no longer compact Hausdorff. Furthermore, for $\mathscr{T}' \subsetneq \mathscr{T}$, the space $(X, \mathscr{T}')$ is not compact Hausdorff.

Proof. For this proof, we will show that for $\mathscr{T}, \mathscr{T}'$ topologies on X with $\mathscr{T} \subset \mathscr{T}'$ and the property that X is compact Hausdorff under both these topologies, then they must be equal.

To do so, we consider the identity function $id : (X, \mathscr{T}') \to (X, \mathscr{T})$, which is continuous since $\mathscr{T} \subset \mathscr{T}'$. Now, for any C closed in $(X, \mathscr{T}')$, since $(X, \mathscr{T}')$ is compact and id is continuous, $id(C) = C$ is compact in $(X, \mathscr{T})$. However, since $(X, \mathscr{T})$ is Hausdorff, C must also be closed. Thus, id is a closed map, and thus a homeomorphism, and the two topologies must be equal. □

PROBLEM 2

Prove that for compact subsets A, B of X, their union $A \cup B$ is compact as well.

Proof. Let $\mathscr{O}$ be an open cover of $A \cup B$. In particular, $\mathscr{O}$ covers A, and has a finite subset $\mathscr{O}_A$ that covers A. Similarly, there is a finite subset $\mathscr{O}_B$ that covers B. Their union $\mathscr{O}_A \cup \mathscr{O}_B$ covers $A \cup B$, and is finite, as it is the union of finite sets. It is also a subcover, since both $\mathscr{O}_A$ and $\mathscr{O}_B$ are subsets of $\mathscr{O}$. Thus. $\mathscr{O}$ has a finite subcover, and $A \cup B$ is compact as desired. □

Problem 3

Suppose A is a subset of a metric space. Show that A is compact implies that A is closed and bounded. Give an example where the converse does not hold.

Proof. Suppose A is a compact subset of a metric space (X, d). Since all metric spaces are Hausdorff, it follows immediately that A is closed (since it is a compact subset of a Hausdorff space).

Now, fix $x \in A$, and consider the open cover $\mathscr{O} = \{V_n(x) \mid n \in \mathbb{N}\}$ of balls centered at x with radius n. This has a finite subcover, by compactness of A. So, let N be the largest radius in the finite subcover. For each $V_m(x)$ in the finite subcover, then, we have that $V_m(x) \subset V_N(x)$. Thus, since the subcover covers A, it follows that $V_N(x)$ covers A, and thus A is bounded.

For an example where the converse fails, consider the metric space $(\mathbb{R}, d_{LA})$ of the real numbers with the discrete (Los Angeles) metric. The set $\mathbb{R}$ itself is closed and bounded (since everything is at most distance 1 away from any particular point), but is clearly not compact, since it is an uncountable discrete set of points. ☐

Problem 4

Show that for X Hausdorff, A, B disjoint compact subsets of X, then there exist disjoint open sets U and V containing A and B respectively.

Proof. Let $a \in A$, and $b \in B$. Since A and B are disjoint, it must be that $a \neq b$. Thus, there exist disjoint open sets U_{ab}, V_{ab} such that $a \in U_{ab}$ and $b \in V_{ab}$. Now, since B is compact, the open cover $\mathscr{O}_B = \{V_{ab} \mid b \in B\}$ has a finite subcover $\{V_{ab_i}\}_{i=1}^n$. Furthermore, the open set $U_a = \bigcap_{i=1}^n U_{ab_i}$ does not intersect the open set $V_a = \bigcup_{i=1}^n V_{ab_i}$ which covers B,

So, consider the open cover $\mathscr{O}_A = \{U_a \mid a \in A\}$. By compactness of A, this has a finite subcover $\{U_{a_j}\}_{j=1}^m$. Now, we have that the open set $V = \bigcap_{j=1}^m V_{a_j}$ (which covers B, since each V_a covers B) does not intersect the open set $U = \bigcup_{j=1}^m U_{a_j}$ which covers A.

Thus, U and V are disjoint open sets containing A and B respectively, as desired. ☐

Problem 5

Show that for X and Y topological spaces with Y compact, the projection $\pi_x : X \times Y \to X$ is a closed map.

Proof. Let $C \subset X \times Y$ be closed, and let $C_X = \pi_X(C)$. Now, for $x \in X \setminus C_X$, we know that the slice $\{x\} \times Y$ does not intersect C, and thus $\{x\} \times Y \subset X \times Y \setminus C$, which is an open subset of $X \times Y$. Since the slice $\{x\} \times Y$ is contained in this open set, and Y is compact, we have by the tube lemma that there is some neighborhood U_x of x such that $U_x \times Y$ is completely contained in $X \times Y \setminus C$. Therefore, since $U_x \times Y \cap C = \emptyset$, it must be that $U_x \cap C_x = \emptyset$ as well. Thus, each point x in the complement of C_x has a neighborhood that is also in the complement of C_x, and so the complement of C_x is open, and C_x is closed, as desired. ☐

Problem 6

Suppose Y is compact Hausdorff, and $f : X \to Y$ a function of sets. Prove that f is continuous if and only if its graph is a closed subset of $X \times Y$.

Proof. ($\Longrightarrow$) Suppose f is continuous. Specifically, we know that for every net x_α in X that converges to some point $x \in X$, the net $f(x_\alpha)$ in Y converges to $f(x) \in Y$.

So, consider a net $(x_\alpha, f(x_\alpha))$ in the graph $\{(x, f(x)) \mid x \in X\}$ that converges to some $(x, y) \in X \times Y$. Now, since the projection maps π_x, π_y are continuous, they preserve nets, so

$$\pi_x(x_\alpha, f(x_\alpha)) = x_\alpha \to \pi_x(x, y) = x$$

and

$$\pi_y(x < \alpha, f(x_\alpha)) = f(x_\alpha) \to \pi_y(x, y) = y$$

but we know that $f(x_\alpha) \to f(x)$ by continuity of f, and since Y is Hausdorff, nets have unique limits, so $y = f(x)$, and the net $(x_\alpha, f(x_\alpha))$ converges to $(x, f(x)) \in \{(x, f(x) \mid x \in X\}$, and so the graph of f is closed, as desired.

($\Longleftarrow$) Suppose f is not continuous. That is, suppose there exists some net x_α in X converging to some $L \subset X$ such that $f(x_\alpha)$ does not converge to $f(L)$.

Now, consider the net $(x_\alpha, f(x_\alpha))$ in $X \times Y$. In particular, consider the projection $\pi_y((x_\alpha, f(x_\alpha))) = f(x_\alpha)$. Since Y is compact, it follows that this net has a convergent subnet $f(x_{\alpha_\beta}) \to y$ for some $y \in Y$.

Now, in the product space, we have the net

$$(x_{\alpha_\beta}, f(x_{\alpha_\beta}))$$

which converges in the first coordinate to some subset $L' \subset L$, and in the second coordinate to y. Now, suppose $y \notin f(L)$. Then, we have that

$$(x_{\alpha_\beta}, f(x_{\alpha_\beta})) \to L' \times \{y\}$$

but since $y \notin f(L)$, it follows that

$$L' \times \{y\} \not\subset \{(x, f(x)) \mid x \in X\}$$

and thus the graph is not closed.

Now, suppose that $y \in f(L)$. Then, it follows that, since $f(x_\alpha) \not\to f(L)$, there is some subnet $f(x_{\alpha_\gamma})$ for which y is not an accumulation point. In particular, this subnet has a convergent sub-subnet which does not converge to y. Applying the argument above to this sub-subnet yields a net in the graph that does not converge in the graph, as desired. $\square$