## Final Exam

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March 17, 2018

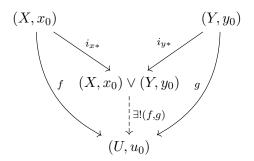
## Problem 1

Carefully prove a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is contractible.

*Proof.* For this proof, we will begin by proving a couple of useful lemmas.

**Lemma 1.** The wedge sum is the coproduct in the category  $Top_*$  of pointed topological spaces.

*Proof.* To show this is the coproduct, we need to show that it satisfies the universal property for coproducts. That is for  $(X, x_0), (Y, y_0)$  pointed topological spaces, and  $(U, u_0)$  any other pointed topological space with arrows  $f: (X, x_0) \to (U, u_0)$  and  $g: (Y, y_0) \to (U, u_0)$ ,

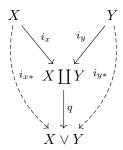


where (f,g) is the unique arrow that makes the diagram commute. Now, recall that the wedge sum is defined as the quotient

$$(X,x_0)\vee (Y,y_0)=X\prod Y/x_0\sim y_0$$

so we can bootstrap by using the fact that the disjoint union is the coproduct in Top. That is,

we define  $i_{x*}$  and  $i_{y*}$  to be the unique maps given by the universal property



Note that these maps preserve basepoints, since

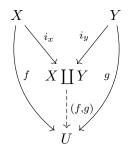
$$i_{x*}(x_0) = q \circ i_x(x_0) = q(x_0)$$

and similarly for  $y_0$ .

Now, suppose we have maps  $f:(X,x_0)\to (U,u_0)$  and  $g:(Y,y_0)\to (U,u_0)$ . Define (f,g) to be the unique map

$$(f,g): X \coprod Y \to U$$

that makes the diagram



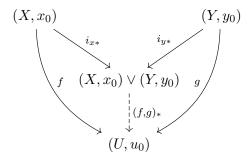
commute. Now, since  $f(x_0) = g(y_0) = u_0$ , it follows that (f,g) is constant on the fibers of  $q: X \coprod Y \to X \vee Y$  and thus by the universal property of quotient maps factors through  $X \vee Y$ . That is,

$$X \coprod Y$$

$$(f,g) \qquad q$$

$$U \leftarrow \cdots \qquad X \lor Y$$

commutes for a unique  $(f,g)_*$ . Thus, we have for f,g arrows from  $(X,x_0)$  and  $(Y,y_0)$  to  $(U,u_0)$  a unique arrow  $(f,g)_*$  from  $X\vee Y$  which makes



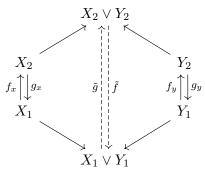
commute as desired.

**Lemma 2.** The wedge product is stable with respect to (basepoint-preserving) homotopy. That is, if  $X_1 \simeq X_2$  and  $Y_1 \simeq Y_2$  relative to basepoints, then

$$X_1 \vee Y_1 \simeq X_2 \vee Y_2$$

*Proof.* Since the wedge sum is the coproduct of pointed topological spaces, we construct the homotopy equivalence as follows:

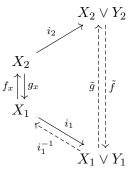
Let  $f_x: X_1 \to X_2$  and  $g_x: X_2 \to X_1$  be homotopy equivalences of  $X_2$  and  $X_1$ . That is,  $f_x$  and  $g_x$  are homotopy inverses of each other. Furthermore, let  $f_y: Y_1 \to Y_2$  and  $g_y: Y_2 \to Y_1$  be homotopy inverses as well. Then, by the universal property of coproducts, we have the commutative diagram



where  $\tilde{f}$  is defined in terms of the universal property of coproducts with respect to the compositions  $X_1 \xrightarrow{f_x} X_2 \longrightarrow X_2 \vee Y_2$  and  $Y_1 \xrightarrow{f_y} Y_2 \longrightarrow {}_2 \vee Y_2$  (and similarly for  $\tilde{g}$ ).

I assert that  $\tilde{f}$  and  $\tilde{g}$  are homotopy inverses. It should be clear that by symmetry of the problem, I need only check that  $\tilde{f} \circ \tilde{g} \simeq \mathbb{1}$ .

Suppose  $x \in X_1 \vee Y_1$ , and without loss of generality let x be in the inclusion of  $X_1$  to  $X_1 \vee Y_1$ . Then, we have the diagram



(where  $i_1^{-1}$  is only defined on the image of  $i_1$ , but we are assuming that  $x \in i_1(X_1)$  for this diagram chase, so this arrow exists). From here, it is clear that  $\tilde{g}(x) = i_2 \circ f_x(i_1^{-1}(x))$ , and if we identify  $X_1$  and  $X_2$  as subspaces of their wedge product, we have  $\tilde{g}(x) = f_x(x)$ . Similarly, we have  $\tilde{f}(x) = g_x(x)$ . Thus,

$$\tilde{f} \circ \tilde{g}(x) = g \circ f(x) \simeq \mathbb{1}(x)$$

as desired.

Thus,  $\tilde{f}$  and  $\tilde{g}$  are homotopy inverses, and  $X_1 \vee Y_1 \simeq X_2 \vee Y_2$  as desired.

Now, we prove the main result.

Let Z be a CW complex which satisfies the hypotheses. In particular, let X and Y be such that  $Z = X \cup Y$ , X and Y are both contractible, and their intersection  $A = X \cap Y$  is contractible as well.

Since A is a contractible subcomplex of Z, we know (via Hatcher prop 0.16 and 0.17) that Z is homotopy equivalent to Z/A. Thus, we only need to show that Z/A is contractible.

Let  $q: Z \to Z/A$  be the canonical quotient map. Since  $Z = X \cup Y$ , we know that  $q(Z) = Z/A = q(X) \cup q(Y)$ . In particular, q(X) = X/A and q(Y) = Y/A (this follows from the definition of the quotient). Thus,  $Z/A = X/A \cup Y/A$ . Since the quotient map is the identity on  $Z \setminus A$  and

collapses A to a point, it follows that  $X/A \cap Y/A = A/A = \{a_0\}$  where  $a_0$  is the point q(A). Thus, Z/A is actually the wedge sum

$$Z/A = X/A \vee Y/A$$

Now, since A is contractible, we know that  $X/A \simeq X$  and  $Y/A \simeq Y$ . In particular, since X and Y are contractible, so is X/A and Y/A. Thus,  $X/A \simeq Y/A \simeq \{\cdot\}$ . By lemma 2, this implies that

$$X/A \vee Y/A \simeq \{\cdot\} \vee \{\cdot\} = \{\cdot\}$$

and thus,  $X/A \vee Y/A$  is contractible. Thus,  $Z/A = X/A \vee Y/A$  is contractible as well, and Z itself is contractible as desired.