Midterm

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Problem 1

Construct explicitly a linear isometric bijection between ℓ^1 and c_0^*.

Proof. We define the linear bijection as

$$\begin{aligned}\phi :&\ell^1 \to c_0^* \\ \phi(y) = (x \mapsto &\sum_{n=1}^{\infty} y_n x_n)\end{aligned}$$

First, we observe that $\phi(y) \in c_0^*$. To see this, note that $\phi(y)$ is clearly linear, since

$$\begin{aligned}\phi(y)(\alpha x + \beta z) &= \sum_{n=1}^{\infty} y_n(\alpha x_n + \beta z_n) \\ &= \alpha \sum_{n=1}^{\infty} x_n y_n + \beta \sum_{n=1}^{\infty} z_n y_n \\ &= \alpha\phi(y)(x) + \beta\phi(y)(z)\end{aligned}$$

as desired.

Next, we show that $\phi(y)$ is bounded. To see this, we compute directly

$$\begin{aligned}\|\phi(y)(x)\| &= \|\sum_{n=1}^{\infty} y_n x_n\| \\ &\leq \sum_{n=1}^{\infty} \|y_n x_n\| \\ &\leq \|x\|_\infty \sum_{n=1}^{\infty} \|y_n\| \\ &= \|x\|_\infty \|y\|_1\end{aligned}$$

and thus $\phi(y)$ is bounded. Therefore, $\phi(y) \in c_0^*$.

Now, we show that ϕ is a linear isometric bijection. First, observe that ϕ is linear, since

$$\begin{aligned}
\phi(\alpha y + \beta z) &= \left(x \mapsto \sum_{n=1}^{\infty} (\alpha y_n + \beta z_n) x_n \right) &&= \left(x \mapsto \sum_{n=1}^{\infty} (\alpha y_n x_n + \beta z_n x_n) \right) \\
&= \left(x \mapsto \alpha \sum_{n=1}^{\infty} y_n x_n + \beta \sum_{n=1}^{\infty} z_n x_n \right) \\
&= \alpha \phi(y) + \beta \phi(z)
\end{aligned}$$

Next, we observe that ϕ is an isometry. From the proof that $\phi(y)$ is bounded, we ascertained that

$$\|\phi(y)\| \leq \|y\|_1$$

Now, we just need to show that there is a sequence of elements $x_i \in c_0$ of norm 1 for which $\|\phi(y)(x_i)\| \to \|y\|_1$.

We define $(x_i)_n$ as

$$(x_i)_n = \begin{cases} \text{sign}(y_n), & n \leq i \\ 0, & n > i \end{cases}$$

Thus,

$$\begin{aligned}
|\phi(y)(x_i)| &= |\sum_{n=1}^{\infty} y_n (x_i)_n| \\
&= |\sum_{n=1}^{i} y_n \text{sign}(y_n)| \\
&= |\sum_{n=1}^{i} |y_n|| \\
&= \|y\|_1 - \sum_{n=i+1}^{\infty} |y_n|
\end{aligned}$$

and since $y \in \ell^1$, we know that the tail goes to zero as $i \to \infty$. Thus, $\lim_i |\phi(y)(x_i)| = \|y\|_1$ as desired.

Clearly, ϕ is a linear isometry. Now, we just need to show its bijective.

To see ϕ is injective, we just need to show its kernel is trivial. So, suppose $y \in \ell^1$ is such that $\phi(y)$ is the zero map. This means that for all $x \in c_0$, $\phi(y)(x) = 0$. So, consider the standard basis sequences e_i with a one in the ith spot, and zeroes elsewhere. Since

$$\phi(y)(e_i) = \sum_{n=1}^{\infty} y_n (e_i)_n = y_i$$

it follows that $\phi(y) = 0$ implies that $y_i = 0$ for all i, and thus $y = 0$. So, the kernel is trivial, and ϕ is injective.

Now, let $f \in c_0^*$. Define a sequence y as $y_i = f(e_i)$ where e_i is the standard basis sequence defined above. Then, since every sequence in c_0 can be uniquely written as a linear combination of e_i, we have

$$\begin{aligned}
f(x) &= f(\sum x^i e_i) \\
&= \sum x^i f(e_i) \quad \text{by linearity of } f \\
&= \sum x^i y_i \\
&= \phi(y)(x)
\end{aligned}$$

and so ϕ is surjective.

Thus, ϕ is a linear isometric bijection, as desired. $\square$

Problem 2

Prove that c_0 is not reflexive.

Proof. This proof relies on the following useful lemma

Lemma 1. *For X, Y normed linear spaces, if $X \cong Y$ by a linear isometry, then $X^* \cong Y^*$.*

Proof. Let $\phi : X \to Y$ be a linear isometric bijection between X and Y. Then, for each bounded linear functional $f \in Y^*$, the pullback $\phi^*(f) = f \circ \phi$ is a bounded linear functional on X. The fact that $\phi^*(f)$ is linear is clear, since it is the composition of linear functions. Furthermore, $\phi^*(f)$ is clearly bounded, since

$$\begin{aligned} \|\phi^*(f)(x)\| &= \|f(\phi(x))\| \\ &\leq \|\phi(x)\|\|f\| \\ &= \|x\|\|f\| \end{aligned}$$

Thus, ϕ^* defines a map from Y^* to X^*. It should be clear that this is an isometry, since

$$\begin{aligned} \|\phi^*(f)\| &= \sup_{x\in X, \|x\|=1} \|\phi^*(f)(x)\| \\ &= \sup_{x\in X, \|x\|=1} \|f(\phi(x))\| \\ &= \sup_{y\in Y, \|y\|=1} \|f(y)\| \qquad \text{since } \phi \text{ an isometric bijection} \\ &= \|f\| \end{aligned}$$

as desired.

By the symmetry of this problem ϕ^{-1} also induces an isometry $\phi^{-1*} : X^* \to Y^*$. This map clearly inverts ϕ^*, since

$$\phi^{-1*} \circ \phi^*(f) = (y \mapsto f(\phi(\phi^{-1}(y)))) = y \mapsto f(y) = f$$

and thus ϕ^* is a linear isometric bijection between X^* and Y^* as desired. □

Thus, since $\ell^1 \cong c_0^*$, we have that

$$\ell^{1*} \cong c_0^{**}$$

but $\ell^{1*} \cong \ell^\infty \ncong c_0$ and so $c_0 \ncong c_0^{**}$ and thus is not reflexive.

To see that $\ell^\infty \ncong c_0$, we will show that their duals are not isometric. It then follows immediately from the lemma that c_0 and ℓ^∞ are not isometric.

To see this, we note first that $c_0^* \cong \ell^1$, and ℓ^1 is seperable, since $\mathbb{Q}^\infty$ the set of all rational sequences with compact support is dense in ℓ^1. Indeed, if x_n is a sequence in ℓ^1, and $\varepsilon > 0$ is arbitrary, we can set N large enough so that $\sum_{n=N}^\infty |x_n| < \frac{\varepsilon}{2}$. Then, we can choose rationals r_n so that $|r_n - x_n| < \frac{\varepsilon}{2N}$. Then,

$$\begin{aligned} \|r_n - x_n\|_1 &= \sum_{n=1}^{N} |r_n - x_n| + \sum_{n=N}^{\infty} |x_n| \\ &\leq N\frac{\varepsilon}{2N} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and so $\mathbb{Q}^\infty$ is dense.

Now, we just need to show that $\ell^{\infty*}$ is not seperable. To show, this, we will find an uncountable discrete subset.

Consider the Stone-Cech compactification of $\mathbb{N}$. First, we show that this space is uncountable. To see this, recall that the space $\{0,1\}^{\mathbb{R}}$ is compact Hausdorff, and is seperable. So, let S be a countable dense subset of $\{0,1\}^{\mathbb{R}}$. Define a surjection $f : \mathbb{N} \to S$, which is trivially continuous, since $\mathbb{N}$ has the discrete topology. Thus, by the universal property of the Stone-Cech compactification, f extends to a function $\beta f : \beta\mathbb{N} \to S$, which extends to a surjection $g : \beta\mathbb{N} \to \{0,1\}^{\mathbb{R}}$. Thus, $|\beta N| \geq |\{0,1\}^{\mathbb{R}}|$ and $\beta\mathbb{N}$ is uncountable.

Next, we show that $\ell^\infty \cong C(\beta\mathbb{N})$. Let $x \in \ell^\infty$. Then, x is a function from $\mathbb{N}$ to $\mathbb{R}$, but since x is bounded, it can be thought of as a function from $\mathbb{N}$ to B for some closed ball $B \subset \mathbb{R}$. Thus, since B is compact Hausdorff, there is a unique extension $\beta x : \beta\mathbb{N} \to B \subset \mathbb{R}$. Thus, this defines an injection from ℓ^∞ to $C(\beta\mathbb{N})$. Furthermore, if we take B to be the smallest such ball, we see that the extension $\beta x : \beta\mathbb{N} \to B$ remains bounded by B, and thus $\|\beta x\|_{\text{sup}} = \|x\|_\infty$ and the injection is actually an isometry.

Finally, we note that since $\beta\mathbb{N}$ is compact, any $f : \beta\mathbb{N} \to \mathbb{R}$ must map into a compact set, and is thus bounded. So, f restricts on $\mathbb{N}$ to an element of ℓ^∞, and the map is surjective as well.

Finally, we invoke the Riesz representation theorem to identify $\ell^{\infty*} \cong C(\beta\mathbb{N})^*$ with the set of finite Borel measures on $\beta\mathbb{N}$. In particular, the set $\{\delta_x : x \in \beta\mathbb{N}\}$ is uncountable, and is clearly discrete. Thus, $\ell^{\infty*}$ is not seperable, and in particular is not homeomorphic to $\ell^1 = c_0^*$ as desired.

Thus, $c_0 \ncong \ell^\infty$, as desired. $\square$

Problem 3

For X a Banach space, suppose $x_n \to x$ strongly for $x_n, x \in X$, and $\phi_n \to \phi$ in weak-$*$ for $\phi_n, \phi \in X^*$. Prove that $\phi_n(x_n) \to \phi(x)$. Prove by counterexample that this does not hold if $x_n \to x$ weakly.

Proof. We wish to evaluate $\lim_n \|\phi_n(x_n) - \phi(x)\|$, which we will directly show is zero. So, let $\varepsilon > 0$ be arbitrary, and let N be such that $\|x_n - x\| < \varepsilon$ and $\|\phi_n(x) - \phi(x)\| < \varepsilon$ for all $n > N$ (we can do this, since $x_n \to x$ strongly, and $\phi_n(x) \to \phi(x)$ since $\phi_n \to \phi$ in weak-$*$). Now, since $\phi_n \to \phi$ in weak-$*$, we know the set $\{\phi_n\}$ is strongly bounded (we proved this in homework 5). Let C be such a bound. That is, for all n, $\|\phi_n\| \leq C$. Then, we have for $n > N$

$$\begin{aligned}
\|\phi_n(x_n) - \phi(x)\| &= \|\phi_n(x_n) - \phi_n(x) + \phi_n(x) - \phi(x)\| \quad \leq \|\phi_n(x_n) - \phi_n(x)\| + \|\phi_n(x) - \phi(x)\| \\
&= \|\phi_n(x_n - x)\| + \|\phi_n(x) - \phi(x)\| \\
&\leq \|\phi_n\|\|x_n - x\| + \|\phi_n(x) - \phi(x)\| \\
&\leq C\varepsilon + \varepsilon = 2C\varepsilon
\end{aligned}$$

and thus, $\lim_n \|\phi_n(x_n) - \phi(x)\| = 0$ as desired.

Now for a counterexample to show this does not hold if $x_n \to x$ weakly. Let $X = \ell^p$, and let $x_n = e_n$ the standard basis sequences. Furthermore, let $\phi_n((x_i)) = x_n = \sum (e_n)_i x_i$. Now, ϕ_n can be identified with the sequence e_n, which is in ℓ^q and thus represents an element of ℓ^{p*}. However, $x_n \to 0$, $\phi_n \to 0$ in weak and weak-$*$ (respectively), but $\phi_n(x_n) = 1$ for all n, which converges to $1 \neq \phi(x) = 0$. $\square$

Problem 4

Show that X^* separates points.

Proof. Let $x, y \in X$ be distinct points. We wish to find a functional $\phi \in X^*$ for which $\phi(x) \neq \phi(y)$. We will consider two cases.

First, suppose $y = \lambda_0 x$ for some scalar λ_0. Then, construct ϕ on $\operatorname{span}(x)$ as

$$\phi(\lambda x) = \lambda$$

Clearly, this is a bounded linear functional, since

$$\|\phi(\lambda x)\| = |\lambda| = \frac{|\lambda|\|x\|}{\|x\|} = \frac{1}{\|x\|}\|\lambda x\|$$

Thus, we can extend it to the whole space X using Hahn-Banach. This defines a functional $\phi \in X^*$ with $\phi(x) = 1$ and $\phi(y) = \lambda_0$, and so ϕ separates x and y as desired.

Now, suppose x and y are linearly independent. Again, we define a functional on $\operatorname{span}(x) \oplus \operatorname{span}(y)$ as

$$\phi(\alpha x + \beta y) = \alpha$$

clearly this is a bounded linear functional, since it is just projection onto $\operatorname{span}(x)$ composed with the bounded linear functional defined before. Since the projection operator is a bounded linear operator, it follows that the composition is a bounded linear operator into $\mathbb{C}$ i.e. a bounded linear functional.

Thus, ϕ can be extended to the whole space, and $\phi(x) = 1$, and $\phi(y) = 0$. Thus, ϕ separates x and y as desired. $\square$