Problem Set 3

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Problem 1

Part 1

Prove 1.18.ii from the notes.

Proof. (Copied from homework 1)

Note that if the composite function (f, g) is measurable, then this statement reduces to part i, and the proof is complete.

So, let's prove that (f, g) is measurable, given f, g are each individually measurable. (Note that this construction works for general products of measurable spaces, where the product σ-algebrais given by $\sigma(\mathscr{A}_1 \times \mathscr{A}_2)$. Generally, this says that the product measurable space has the universal property of product spaces).

Let $E, F \in B(\mathbb{R})$ be measurable sets, and consider the product $E \times F$. The inverse image $(f, g)^{-1}(E \times F) = f^{-1}(E) \cap g^{-1}(F)$ is the intersection of measurable sets (since f and g are both individually measurable), and is measurable.

Now, consider the "good set"

$$\mathscr{E} = \{G | G \in B(\mathbb{R}^2) \text{and } (f, g)^{-1}(G) \in \mathscr{A}\}$$

It is clear from above that we have the inclusion relations

$$B(\mathbb{R}) \times B(\mathbb{R}) \subset \mathscr{E} \subset B(\mathbb{R}^2)$$

Now, $\mathscr{E}$ is clearly a σ-algebra, since both conditions on $\mathscr{E}$ preserve complements and unions. Therefore, taking σ of the inclusion relations yields:

$$\begin{aligned} \sigma(B(\mathbb{R}) \times B(\mathbb{R})) \subset \mathscr{E} &\quad \subset B(\mathbb{R}^2) \\ \implies B(\mathbb{R}^2) \subset \mathscr{E} &\quad \subset B(\mathbb{R}^2) \end{aligned}$$

Thus, $\mathscr{E}$ is actually the whole Borel set $B(\mathbb{R}^2)$, and thus (f, g) is a measurable function, as desired. □

Part 2

Complete 2.6 from the notes.

Describe all measurable functions $f : \mathbb{N} \to [0, \infty]$ that are finite μ_c-almost everywhere in the counting measure, and find

$$\int_{\mathbb{N}} f d\mu_c$$

Proof. Note that the only subset of $\mathbb{N}$ with zero measure is $\emptyset$, so if f is finite μ_c-almost everywhere, then f is finite on all subsets of $\mathbb{N}$ i.e. f is bounded.

Then, the integral becomes

$$\int_{\mathbb{N}} f d\mu_c = \sum_{i=1}^{\infty} f_i$$

This is clear to see by approximating f with simple functions that converge monotonically to f. Let

$$\phi_i = \sum_{j=1}^{i} f_j \chi_{\{j\}}$$

It is clear that $\phi_{i+1} \geq \phi_i$, since $\phi_i(n) = \phi_{i+1}(n)$ for all $n < i$, and for $n > i$, we have

$$\begin{aligned} \phi_i(n) &= 0 \\ &\leq \phi_{i+1}(n) \end{aligned}$$

since $\phi_i > 0$ for all i (definition of simple function).

It's also clear that $\phi_i \to f$ pointwise, since for fixed $x \in \mathbb{N}$, $\phi_{x+j}(x) = f(x)$ for all $j > 0$.

Thus, the monotone convergence theorem tells us that

$$\begin{aligned} \lim_{i\to\infty} \int_{\mathbb{N}} \phi_i(x) d\mu_c(x) &= \int_{\mathbb{N}} \lim_{i\to\infty} \phi_i(x) d\mu_c(x) \\ \lim_{i\to\infty} \sum_{j=1}^{i} f_j \mu_c(\{j\}) &= \int_{\mathbb{N}} f(x) d\mu_c(x) \\ \sum_{j=1}^{\infty} f_j &= \int_{\mathbb{N}} f(x) d\mu_c(x) \end{aligned}$$

as desired. □

Part 3

Do the same thing for the δ-measure μ_{δ_p} on Ω.

Proof. Note that any subset that does not contain p has measure zero, Thus, f is finite μ_{δ_c}-almost everywhere if and only if $f(p)$ is finite. This is clear, since $\Omega \setminus \{p\}$ has measure zero, so f can do whatever it wants on $\Omega \setminus \{p\}$. However, the measure of $\{p\}$ is not zero, so f must be finite on $\{p\}$.

To compute the integral, we first observe the general fact that changing a function on a set of measure zero does not change the integral.

Lemma. *For measurable functions f and g from a measurable space Ω such that $f = g$ μ-almost everywhere, $\int f = \int g$.*

Proof.

$$\begin{aligned}
\int_\Omega f d\delta - \int_\Omega g d\delta &= \int_\Omega (f-g) d\delta \\
&= \int_{\{x|f(x)=g(x)\}} (f-g) d\delta + \int_{\{x|f(x)\neq f(g)\}} (f-g) d\delta \\
&= \int_{\{x|f(x)=g(x)\}} (0) d\delta + 0 \\
&= 0
\end{aligned}$$

□

Thus, we have that

$$\int_\Omega f d\delta = \int_\Omega f(p)\chi_{\{p\}} d\delta$$

and the second integral is the integral of a simple function, and is just $f(p)\delta(\{p\}) = f(p)$. Thus,

$$\int_\Omega f d\delta = f(p)$$

□

Part 4

Show that the Dirichlet function is measurable, and calculate its integral.

Proof. The Dirichlet function is defined as the characteristic function on $\mathbb{Q}$. To see this is measurable, observe that $\mathbb{Q}$ is measurable, since it is a countable disjoint union of points, which are all measurable.

The integral can easily be seen to be zero, since

$$\begin{aligned}
\mu(\mathbb{Q}) &= \mu\left(\bigcup_{q\in\mathbb{Q}} \{q\}\right) \\
&= \sum_{q\in\mathbb{Q}} \mu(\{q\}) \\
&= 0
\end{aligned}$$

So, by the lemma of the previous problem,

$$\int_\mathbb{R} \chi_\mathbb{Q} d\mu = \int_\mathbb{R} 0 d\mu = 0$$

□

Part 5

Construct a sequence of functions $\{f_n\}$ satisfying the assumptions of Fatou's lemma such that

$$\lim \int f_n \neq \int \lim f_n$$

Proof. Let $f_n = \chi_{[n,n+1]}$. Then, $\int_{\mathbb{R}} f_n d\mu = 1$ for all n, but $f_n(x) \to 0$ for all x, so f_n converges pointwise to zero, and thus

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}} f_n d\mu &= 1\\ &\neq 0\\ &= \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu\end{aligned}$$

□

Problem 2

Part A

Prove that there exists constants $C_1(n), C_2(n)$ such that

$$C_1(n)r^n \leq \lambda^n B_r \leq C_2(n)r^n$$

Proof. To begin with, we note that the (∞-norm) cube of radius r, defined as the set

$$\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid |x_i| < r\ \forall i\}$$

contains the ball B_r. Thus, the volume of the cube $V = (2r)^n$ is an upper bound for the Lebesgue measure of B_n (this follows from the monotonicity of the measure, and the fact that the Lebesgue measure preserves the standard volume of boxes).

Note also that the (1-norm) cube of radius r, defined as the set

$$\{(x_1\dots,x_n) \mid \sum_{i=1}^{n}|x_i| < r\}$$

is contained in the ball B_r. This is clear, since the furthest away from the origin a point in the 1-cube of radius r can get is when exactly one coordinate is r with the rest zero. Since this is contained in B_r, every other point is as well.

The volume of the 1-cube of radius r is a bit trickier to compute. We know it looks like a cube with diagonal of length $2r$, which leads to a volume of $V = (\frac{2r}{\sqrt{n}})^n$ which works as a lower bound for the Lebesgue measure for B_n by a dual argument to the one above.

Thus, we have

$$\left(\frac{2}{\sqrt{n}}\right)^n r^n \leq \lambda^n B_n \leq (2)^n r^n$$

so, $C_1(n) = \left(\frac{2}{\sqrt{n}}\right)^n$ and $C_2(n) = 2^n$ function as the desired lower and upper bounds. □

Part B

For $k = 1, 2, \dots$ and a fixed $A \in (0,1)$, define $I_k = (k, k + \frac{A}{2^k})$. Find

$$\lambda^1\left(\bigcup_{k=1}^{\infty} I_k\right)$$

Proof. To begin with, we note that each I_k is disjoint from any other. This is clear, since

$$\frac{A}{2^k} < 1\ \forall k$$

so

$$k + \frac{A}{2^k} < k + 1 \ \forall k$$

Therefore, by σ-additivity of λ^1, we have

$$\lambda^1\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \lambda^1(I_k)$$

Now, by the definition of the Lebesgue measure, $\lambda^1((k, k + \frac{A}{2^k})) = k + \frac{A}{2^k} - k = \frac{A}{2^k}$. So, the sum becomes

$$\begin{aligned}\sum_{k=1}^{\infty} \lambda^1(I_k) &= \sum_{k=1}^{\infty} \frac{A}{2^k} \\ &= A\sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= A\end{aligned}$$

Thus,

$$\lambda^1\left(\bigcup_{k=1}^{\infty} I_k\right) = A$$

as desired. □

Part c

Problem 3

For a measurable function $f : (\Omega, \mathscr{A}, \mu) \to \mathbb{R}$, with $\mu(\Omega) < \infty$, which is finite μ-almost everywhere, prove that

$$\lim_{n\to\infty} \mu(\{|f| > n\}) = 0$$

Proof. For this problem, let $I(n) : \mathbb{R}^+ \to [0, \mu(\Omega)]$ defined as $I(n) = \mu(\{|f| > n\})$.

Now, suppose for a contradiction that there existed a sequence $k_i \to \infty$ such that $I(k_i) \to L$ for some $L \neq 0$. In particular, this would mean that any subsequence of k_i would converge to L as well. So, let $\{k_{i_j}\}$ be a monotonically increasing subsequence of $\{k_i\}$. In particular, note that $\{|f| > k_{i_j}\} \supset \{|f| > k_{i_{j+1}}\}$, so the sequence of subsets being measured by μ is monotonic as well. In this case, we can apply limit theorems to the measure to get that

$$\begin{aligned}\lim_{j\to\infty} \mu\{|f| > k_{i_j}\}) &= \mu\left(\lim_{j\to\infty}\{|f| > k_{i_j}\}\right) \\ &= \mu(\{|f| = \infty\}) \\ &= 0\end{aligned}$$

Where the last line is from the fact that f is finite almost everywhere. However, $0 \neq L$, which is a contradiction.

Thus, if $\{k_i\}$ is a sequence toward infinity such that $\{I(k_i)\}$ converges, it converges to zero.

So, suppose for a contradiction that there exists a sequence $\{n_i\}$ such that $\{I(n_i)\}$ does not converge. In particular, this would imply that there is some neighborhood of zero N_0 such that there is a subsequence $\{n_{i_j}\}$ that is never in N_0. However, since the target space $[0, \mu(\Omega)]$ is compact, such a subsequence must have a convergent sub-subsequence. By the earlier observation, this sub-subsequence must converge to zero. But all of the subsequence $\{n_{i_j}\}$ avoids N_0 and thus any sub-subsequence cannot converge to zero, a contradiction.

Thus, every sequence $k_i \to \infty$ defines a convergent sequence $I(k_i)$ that converges to zero, so the limit

$$\lim_{n\to\infty} \mu(\{|f| > n\}) = 0$$

as well. $\square$