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# Homework 5

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## PROBLEM 1

Let  $E, F$  be closed subspaces of a Hilbert space. Prove that  $P_E P_F = P_E$  if and only if  $E \subseteq F$ .

*Proof.* Suppose first that  $P_E P_F = P_E$ . Then, in particular,

$$P_E^* = (P_E P_F)^* = P_F^* P_E^* = P_F P_E = P_E$$

by self-adjointness of projections. Thus,  $P_E P_F = P_F P_E = P_E$ , and the projections commute. Thus, the von Neumann algebra  $W^*(P_E, P_F, I)$  is Abelian, and is isometrically  $*$ -isomorphic to  $L^\infty(X, \mu)$  for some measure space  $(X, \mu)$ . In particular, the projections  $P_E, P_F$  get sent to self-adjoint idempotents  $P_E \mapsto M_{\chi_S}$  and  $P_F \mapsto M_{\chi_{S'}}$  for some measurable subsets  $S, S' \subset X$ .

Now, the requirement  $P_F P_E = P_E$  corresponds to the requirement

$$M_{\chi_{S'}} M_{\chi_S} = M_{\chi_S}$$

which means that  $S \subset S'$ . This, in turn, implies that  $E \subset F$ .

Indeed,  $E$  is the subspace of  $H$  on which  $P_E$  is the identity, which corresponds to the subspace

$$\tilde{E} = \int_S^\oplus H(x) d\mu(x)$$

on which  $M_{\chi_S}$  is the identity. Similarly,

$$\tilde{F} = \int_{S'}^\oplus H(x) d\mu(x)$$

Clearly,  $\tilde{E} \subset \tilde{F}$  (since  $S \subset S'$ ) and so  $E \subset F$  as well.

For the converse direction, assume that  $E \subset F$ . Then, on  $F$ ,  $P_F = I_F$ , and  $P_F P_E = I_F P_E = P_E$ . Furthermore, on  $F^\perp$ ,  $P_F P_E = 0 = P_E$ . Thus, on all of  $H = F \oplus F^\perp$ ,  $P_F P_E = P_E$  as desired.  $\square$

## PROBLEM 2

Characterize the closed subspaces  $E, F$  of a Hilbert space  $H$  satisfy  $P_F P_E = P_E P_F$ .

*Proof.* I assert that  $P_E$  and  $P_F$  commute if and only if  $H$  can be decomposed into the four orthogonal components

$$H = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp$$

To see this, suppose first that  $P_E$  and  $P_F$  commute. Then, consider the Abelian von Neumann algebra  $W^*(P_E, P_F, I)$ . Like before, we use the Borel functional calculus to identify  $W^*(P_E, P_F, I)$  with  $L^\infty(X, \mu)$  acting on  $\tilde{H} = \int_X^\oplus H(x) d\mu(x)$ , which is unitarily equivalent to  $H$  (in the sense that there is a unitary transformation  $U$  such that  $\tilde{H} = UH$  and  $UW^*(P_E, P_F, I)U^* = L^\infty(X, \mu)$ ).

Under this identification,  $P_E \mapsto M_{\chi_S}$ , and  $P_F \mapsto M_{\chi_{S'}}$  for some measurable subsets  $S, S' \subset X$ . In particular, this decomposes  $\tilde{H}$  into four orthogonal components

$$\begin{aligned} \tilde{H} &= \int_X^\oplus H(x) d\mu(x) \\ &= \int_{S \cap S'}^\oplus H(x) d\mu(x) \oplus \int_{S \cap S'^c}^\oplus H(x) d\mu(x) \oplus \int_{S' \cap S^c}^\oplus H(x) d\mu(x) \oplus \int_{S^c \cap S'^c}^\oplus H(x) d\mu(x) \end{aligned}$$

which translates into the decomposition on  $H$  as

$$H = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp$$

as desired.

Conversely, suppose  $H$  can be decomposed this way. Then, let  $v \in H$  be decomposed as

$$v = v_1 + v_2 + v_3 + v_4$$

where  $v_1 \in E \cap F$ ,  $v_2 \in E \cap F^\perp$ ,  $v_3 \in E^\perp \cap F$  and  $v_4 \in E^\perp \cap F^\perp$ .

We compute the effect of  $P_E P_F$  and  $P_F P_E$  directly.

$$\begin{aligned} P_E P_F(v) &= P_E P_F(v_1 + v_2 + v_3 + v_4) \\ &= P_E(v_1 + v_3) \\ &= v_1 \\ P_F P_E(v) &= P_F P_E(v_1 + v_2 + v_3 + v_4) \\ &= P_F(v_1 + v_2) \\ &= v_1 \end{aligned}$$

and thus  $P_E P_F = P_F P_E$  as desired. □

### PROBLEM 3

Let  $E, F$  be closed subspaces of a Hilbert space  $H$ . An operator  $U$  is said to be a partial isometry from  $E$  to  $F$  if  $U|_E$  is an isometry onto  $F$ , and  $U|_{E^\perp} = 0$ . Prove that  $U$  is a partial isometry  $\iff U^*U$  is a projection  $\iff UU^*$  is a projection.

*Proof.* Suppose first that  $U$  is a partial isometry from  $E$  to  $F$ . I assert that  $U^*U = P_E$ . To see this, suppose  $e \in E$ ,  $v \in H$  and let  $v = e' + v'$  where  $e' \in E$  and  $v' \in E^\perp$ . Then,

$$\begin{aligned}\langle U^*Ue|v \rangle &= \langle U^*Ue|e' + v' \rangle \\ &= \langle Ue|Ue' \rangle + \langle Ue|Uv' \rangle \\ &= \langle e|e' \rangle + 0 \\ \implies \langle U^*Ue - e|v \rangle &= 0\end{aligned}$$

and since this holds for all  $v \in H$ ,  $U^*Ue - e = 0$  and thus  $U^*Ue = e$  and  $U^*U$  is the identity on  $E$ .

Furthermore, for  $v' \in E^\perp$ ,

$$U^*Uv' = U^*(0) = 0$$

and so  $U^*U$  is the zero map on  $E^\perp$ . Thus,  $U^*U$  agrees with  $P_E$  at all points, so  $U^*U = P_E$  as desired.

Conversely, suppose  $U^*U$  is a projection  $P_E$  for some closed subspace  $E$ . Define  $F = U(E)$ . We will first show that  $U$  is an isometry of  $E$  onto  $F$ . To see this, suppose  $e, e' \in E$ . We calculate directly

$$\begin{aligned}\langle Ue|Ue' \rangle &= \langle U^*Ue|e' \rangle \\ &= \langle P_Ee|e' \rangle \\ &= \langle e|e' \rangle\end{aligned}$$

and thus  $U$  is an isometry from  $E$  to  $F$ . Note that this immediately implies that  $F$  is a closed subspace. Finally, we show that  $U|_{E^\perp} = 0$ . Let  $v' \in E^\perp$ , Then,

$$\begin{aligned}\|Uv'\|^2 &= \langle Uv'|Uv' \rangle \\ &= \langle U^*Uv'|v' \rangle \\ &= \langle 0|v' \rangle = 0\end{aligned}$$

and so  $U^*U|_{E^\perp} = 0$  as desired. Thus,  $U$  is a partial isometry.

Finally, we show that if  $U$  is a partial isometry, then  $U^*$  is a partial isometry, which will complete the proof. So, suppose  $U$  is a partial isometry. We will show that  $U^*$  is a partial isometry from  $F$  to  $E$ . First, we check that  $U^*|_F$  is an isometry. Now, for each  $f \in F$ , there is some  $e \in E$  with  $Ue = f$ . Thus, for  $e, e' \in E$ ,  $f, f' \in F$  with  $Ue = f, Ue' = f'$ ,

$$\begin{aligned}\langle U^*f|U^*f' \rangle &= \langle U^*Ue|U^*Ue' \rangle \\ &= \langle P_Ee|P_Ee' \rangle \\ &= \langle e|e' \rangle \\ &= \langle Ue|Ue' \rangle \\ &= \langle f|f' \rangle\end{aligned}$$

and so  $U^*$  is an isometry from  $F$  to  $E$ . Finally, we show that  $U^*|_{F^\perp} = 0$ . This is immediate, since

$$\ker U^* = U(H)^\perp = F^\perp$$

where we used the identity  $\ker A^* = A(H)^\perp$  for all bounded operators  $A$ .

Note that this completes the proof. If  $U$  is a partial isometry, then  $U^*$  is a partial isometry, which implies that  $U^{**}U^* = UU^*$  is a projection. Similarly, if  $UU^* = U^{**}U^*$  is a projection, then  $U^*$  is a partial isometry, and thus  $U^{**} = U$  is a partial isometry as well.  $\square$

## PROBLEM 4

Prove that the subspace of self-adjoint operators on  $B(\mathbb{C}^2)$  is a subset of the linear subspace spanned (over  $\mathbb{R}$ ) by  $I$  and the three Pauli matrices

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and prove that the span of  $\{\sigma_i\}_{i=1}^3$  is isometrically isomorphic to a three-dimensional real Hilbert space under the operator norm.

*Proof.* We observe first that for any  $T \in B(\mathbb{C}^2)$  self-adjoint, we have the condition

$$T_j^i = \overline{T_i^j}$$

In particular,

$$\begin{aligned} T_1^1 &= \overline{T_1^1} \\ T_2^2 &= \overline{T_2^2} \end{aligned}$$

and so  $T_i^i$  is real. Furthermore, we have the relation

$$T_2^1 = \overline{T_1^2}$$

Finally, we observe that we can decompose  $T$  into a scalar part and a trace-free part by

$$\begin{aligned} C &= T - \text{tr}(T)I \\ T &= \text{tr}(T)I + C \end{aligned}$$

Now we are ready to show that  $T$  is a linear combination of  $I$  and the Pauli matrices. In particular, we will show that  $C$  is a linear combination of the Pauli matrices.

For  $C$ , we have the relations

$$\begin{aligned} C_1^1 &= -C_2^2 \\ C_2^1 &= \overline{C_1^2} \end{aligned}$$

as well as the requirement that  $C_1^1$  is real. Thus, we have three degrees of freedom. That is,

$$C = \begin{bmatrix} \alpha & \beta + \gamma i \\ \beta - \gamma i & -\alpha \end{bmatrix}$$

which decomposes as

$$C = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3$$

and so

$$T = \text{tr}(T)I + C$$

is a linear combination of  $I$  and  $\{\sigma_i\}_{i=1}^3$  as desired.

Next, we show that the span of  $\{\sigma_i\}_{i=1}^3$  under the operator norm is isometrically isomorphic to  $\mathbb{R}^3$ . To do so, we will show that the operator norm coincides with the pullback metric from  $\mathbb{R}^3$ . Specifically, we define a linear isomorphism

$$\begin{aligned} \Phi : \langle \sigma_i \rangle_{i=1}^3 &\rightarrow \mathbb{R}^3 \\ \Phi(\sigma_i) &= e_i \end{aligned}$$

where  $\{e_i\}$  is an orthonormal basis for  $\mathbb{R}^3$  with its usual inner product. In particular, if we denote the inner product of  $\mathbb{R}^3$  as  $\eta$ , we can define  $\Phi^*(\eta)$  by

$$\Phi^*(\eta)(u, v) = \eta(\Phi(u), \Phi(v))$$

In order to show that this metric coincides with the operator norm, we have to show that

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \Phi^*(\eta)(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3, \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3) = \alpha^2 + \beta^2 + \gamma^2$$

Now, since  $\sigma_i$  are all self-adjoint, any (real) linear combination of them will be as well. Thus,

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \|(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2\|$$

We compute the right-hand side directly

$$(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2 = \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma^2\sigma_3^2 + \alpha\beta\{\sigma_1, \sigma_2\} + \alpha\gamma\{\sigma_1, \sigma_3\} + \beta\gamma\{\sigma_2, \sigma_3\}$$

where  $\{\sigma_i, \sigma_j\}$  is the anticommutator of  $\sigma_i$  and  $\sigma_j$ . Now, it is well-known that the Pauli matrices satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

as well as the relation

$$\sigma_i^2 = I$$

(where  $\varepsilon_{ijk}$  is the levi-civita symbol) which forces

$$\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}$$

and so

$$\begin{aligned} \{\sigma_i, \sigma_j\} &= \sigma_i\sigma_j + \sigma_j\sigma_i \\ &= i\varepsilon_{ijk}\sigma_k + \delta_{ij} + i\varepsilon_{jik}\sigma_k + \delta_{ji} \\ &= i\varepsilon_{ijk}\sigma_k - i\varepsilon_{ijk}\sigma_k + 2\delta_{ij} \\ &= 2\delta_{ij} \end{aligned}$$

Thus,

$$\begin{aligned} (\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2 &= \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma^2\sigma_3^2 + \alpha\beta\{\sigma_1, \sigma_2\} + \alpha\gamma\{\sigma_1, \sigma_3\} + \beta\gamma\{\sigma_2, \sigma_3\} \\ &= \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma^2\sigma_3^2 \\ &= (\alpha^2 + \beta^2 + \gamma^2)I \end{aligned}$$

and so

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \|(\alpha^2 + \beta^2 + \gamma^2)I\| = (\alpha^2 + \beta^2 + \gamma^2)\|I\| = (\alpha^2 + \beta^2 + \gamma^2)$$

as desired. Thus, the pullback metric coincides with the operator norm, and  $\Phi$  is a linear isometric isomorphism, as desired.  $\square$

## PROBLEM 5

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