Problem Set 6

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November 13, 2017

Problem 1

Prove that in a normed space, a sequence can have at least one strong limit. Prove that a strongly convergent sequence is Cauchy.

Proof. To begin with, we note that every normed space is necessarily Hausdorff, and that every Hausdorff space has the property that sequences have at most one limit.

To see that a normed space is Hausdorff, consider an arbitrary normed space X, and two distinct points x, y. Now, by the definition of a norm, it must be that $||x - y|| > 0$. So, let

$$||x - y|| = \varepsilon$$

Then, the open sets $B(x, \frac{\varepsilon}{2})$ and $B(Y, \frac{\varepsilon}{2})$ separate x and y. Thus, normed spaces are Hausdorff.

Furthermore, it is clear that a sequence converges to at most one limit in Hausdorff spaces. To see this, we recall the topological definition of convergence, which states that a sequence (x_n) converges to x if and only if every neighborhood of x eventually contains the sequence.

Suppose for a contradiction that (x_n) had two limits, x, and y. Since x and y are distinct points in a Hausdorff space, there must be some neighborhood V_x of x and V_y of y such that $V_x \cap V_y = \emptyset$.

However, this contradicts the sequence converging to both x and y, since for (x_n) to converge to x, it must eventually be in V_x, which means it is eventually outside V_y, and thus cannot converge to y as well.

Now, let (x_n) be a convergent sequence in a normed space X, and let x be its limit. We will show that this sequence is Cauchy.

To do this, let $\varepsilon > 0$ be arbitrary. Now, we know that there exists some N such that for any $n > N$,

$$||x_n - x|| < \frac{\varepsilon}{2}$$

Furthermore, for any $m, n > N$ we have that

$$\begin{aligned} ||x_m - x_n|| &= ||x_m - x + x - x_n|| \\ &\leq ||x_m - x|| + ||x_n - x|| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and thus the sequence is Cauchy. $\square$

Problem 2

Show that the closure of the ball $B(a, r)$ is the closed ball $\overline{B}(a, r) = \{x \mid |x - a| \leq r\}$

Proof. Suppose x is in $\overline{B}(a, r)$. Then, consider the sequence

$$(x_n) = (x - a)(1 - \frac{1}{n}) + a$$

Clearly, this sequence converges to x, and each term is in $B(a, r)$. To see this, we observe that

$$\begin{aligned} ||x_n - a|| &= ||(x - a)(1 - \frac{1}{n}) + a - a|| \\ &= ||x - a||(1 - \frac{1}{n} \\ &\leq r(1 - \frac{1}{n} \\ &\leq r \end{aligned}$$

Thus, each point in $\overline{B}(a, r)$ is a limit point of $B(a, r)$, and since $\overline{B}(a, r)$ is closed, it follows that it is the closure of $B(a, r)$ (since the closure of $B(a, r)$ is the smallest closed set containing it.) $\square$

Problem 3

Prove that for a linear operator A between normed spaces V, W, the following are equivalent:

1. A is continuous at every $p \in V$.
2. A is continuous at 0_V.
3. A is bounded in the sense that
$$\sup_{||x||=1} ||Ax|| < \infty$$
4. A is bounded in the sense that for some $M > 0$,
$$||Ax|| \leq M||x||$$
for all $x \in V$.

Furthermore, prove that the set $\mathscr{B}(V, W)$ of the bounded linear operators from V to W is a normed space.

Proof. ($1 \implies 2$) This follows immediately from the statement of 1.

($2 \implies 3$) Suppose A is continuous at zero. Then, choose $\varepsilon = 1$. We must have some $\delta > 0$ such that $||x|| \leq \delta$ implies $||Ax|| < \varepsilon = 1$. Then, we have

$$||Ax|| = ||A\left(\frac{||x||}{\delta}\frac{x\delta}{||x||}\right)|| = \frac{||x||}{\delta}||A\left(\frac{x\delta}{||x||}\right)|| \leq \frac{||x||}{\delta}(1)$$

and thus A is bounded by $\frac{1}{\delta}$. In particular,

$$\sup \frac{||Ax||}{||x||} \leq \frac{\frac{||x||}{\delta}}{||x||} = \frac{1}{\delta} < \infty$$

as desired.

($3 \implies 4$) Suppose A is bounded in the sense of statement 3. In particular, let

$$\sup_{x \neq 0} \frac{||Ax||}{||x||} = M$$

for some positive M. Then, for all $x \in V$ with $x \neq 0$,

$$\begin{aligned} \frac{||Ax||}{||x||} &\leq M \\ \implies ||Ax|| &\leq M||x|| \end{aligned}$$

as desired. Note that if $x = 0$, then $Ax = 0$ as well and the statement is vacuously true.

($4 \implies 1$) Suppose A is bounded in the sense that there is some $M > 0$ such that

$$||Ax|| \leq M||x||$$

for all x in V. Now, let $\varepsilon > 0$ be arbitrary. Then, the bound $\delta = \frac{\varepsilon}{M}$ on $||x - p||$ forces

$$\begin{aligned} ||Ax - Ap|| = ||A(x-p)|| &\leq M||x-p|| \\ &\leq M\frac{\varepsilon}{M} = \varepsilon \end{aligned}$$

Thus, A is continuous at p.

Finally, we will prove that the space $\mathscr{B}(V, W)$ is a normed vector space under the operator norm $||A|| = \sup_{||x||=1} ||Ax||$.

To see this, we need to check that the norm is positive definite and satisfies the triangle inequality.

The norm is clearly positive definite, since it is taken as the sup of a set of nonnegative numbers, and if $||A|| = 0$, then (by the definition of operator norm from statement 4)

$$||Ax|| \leq (0)||x|| \;\forall x$$

which forces $||Ax|| = 0$ for all $x \in V$. Since the vector norm is positive definite, it follows that $Ax = 0$ for all $x \in V$, and thus $A = 0$.

Now, we need to show that

$$||A + B|| \leq ||A|| + ||B||$$

To see this, let $||A|| = M_A$ and $||B|| = M_B$. Then, we have that

$$\begin{aligned} ||(A + B)(x)|| &= ||Ax + Bx|| \\ &\leq ||Ax|| + ||Bx|| \end{aligned}$$

Since this holds for all x, it follows that

$$||A + B|| \leq ||A|| + ||B||$$

as desired. Thus, $\mathscr{B}(V, W)$ is a normed vector space. $\square$

Problem 4

Suppose that V, W are normed vector spaces, and that W is Banach. Prove that $\mathscr{B}(V, W)$ is Banach as well.

Problem 10

Show that ℓ^1 is not complete in the ℓ^∞ norm.

Proof. We will show that the sequence $(x_m)_k = (\frac{1}{m^{1+\frac{1}{k}}})$ converges to the function $x_m = \frac{1}{m}$, which is not in ℓ^1, even though each term in the sequence is in ℓ^1.

To show convergence, we wish to show that the functions

$$f_k(n) = \frac{1}{n^{1+\frac{1}{k}}}$$

converges uniformly to $f(n) = \frac{1}{n}$.

To do so, we will consider instead the extended functions

$$f_k(x) = x^{1+\frac{1}{k}},\ x \in [0,1]$$

which clearly converges pointwise to $f(x) = x$. Now, since f is defined on a compact domain, it must also uniformly converge to $f(x) = x$. Thus, the restriction $f_k(x)|_{\{\frac{1}{n}\}}$ also converges uniformly to $f(n) = \frac{1}{n}$.

Thus,the sequence of sequences $(x_m)_k$ converges uniformly (in ℓ^∞) to (x_m) as desired. Furthermore, since each $(x_m)_k$ is in ℓ^1, but (x_m) is not, it follows that ℓ^1 is not complete in the ℓ^∞ norm. $\square$