Problem Set 5

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December 1, 2017

PROBLEM 1

Let M be the open submanifold of $\mathbb{R}^2$ with both coordinates positive, and define $F : M \to M$ as $F(x, y) = (xy, \frac{y}{x})$. Show that F is a diffeomorphism, and compute dFX and dFY for

$$\begin{aligned}
X &= x\partial_x + y\partial_y \\
Y &= y\partial_x
\end{aligned}$$

Proof. To begin with, we observe that so long as $x \neq 0$, F is in fact smooth. Furthermore, it has an inverse

$$F^{-1}(x, y) = (\sqrt{\frac{x}{y}}, \sqrt{xy})$$

which is also smooth on M, and defined for all of M.

Furthermore, we can calculate its Jacobian:

$$J(F) = dF = \begin{bmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{bmatrix}$$

which expresses dF in the coordinates ∂_x, ∂_y at every point.

Now, we calculate $dF(X)$:

$$\begin{aligned}
dF(X) &= dF(x\partial_x + y\partial_y) \\
&= xdF(\partial_x) + ydF(\partial_y) \\
&= x(y\partial_x - \frac{y}{x^2}\partial_y) + y(x\partial_x + \frac{1}{x}\partial_y) \\
&= xy\partial_x - \frac{y}{x}\partial_y + yx\partial_x + \frac{y}{x}\partial_y \\
&= 2xy\partial_x
\end{aligned}$$

and $dF(Y)$:

$$\begin{aligned}
dF(Y) &= dF(x\partial_y) \\
&= xdF(\partial_y) \\
&= x(x\partial_x + \frac{1}{x}\partial_y) \\
&= x^2\partial_x + \partial_y
\end{aligned}$$

$\square$

Problem 2

Let M be a smooth manifold, $S \subseteq M$ an embedded submanifold. Given $X \in \mathcal{X}(S)$, show that there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$. Show that every such vector field extends to all of M if and only if S is properly embedded.

Proof. $\square$

Problem 3