Homework 1

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Problem 1

Suppose M is a complete Riemannian manifold of dimension n, and suppose there exist constants $a > 0$ and $c \geq 0$ such that for all pairs of points and all minimizing geodesics $\gamma(s)$ parameterized by arc length s, joining these points, we have

$$R(\gamma'(s)) \geq a + \partial_s f$$

where f is a function of s and $|f(s)| \leq c$ along γ. Prove that M is compact.

Proof. To show M is compact, we just need to show M is bounded. That is, there is some number N such that $d(p, q) < N$ for all $p, q \in M$.

Let γ be a minimizing geodesic connecting two points p, q in M. We calculate the second variation in energy along γ in a manner similar to the proof of the Bonnet-Meyers theorem in

Do Carmo. In particular, we know

$$\begin{aligned}
\frac{1}{2}E''(0) &= \int_0^1 \sin^2(\pi t)((n-1)\pi^2 - (n-1)\ell^2 R_\gamma(e_n(t)))dt \\
&= (n-1)\left[\frac{\pi^2}{2} - \int_0^1 \ell^2 R_\gamma(e_n(t))\right] \\
&\leq (n-1)\left[\frac{\pi^2}{2} - \int_0^1 \sin^2(\pi s)\ell^2 \partial_s f ds\right] \\
&= (n-1)\left[\frac{\pi^2}{2} - \int_0^1 \sin^2(\pi s)\ell^2 (a + \partial_s f) ds\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} - \int_0^1 \sin^2(\pi s)\ell^2 \partial_s f ds\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + \int_0^1 \pi \sin(\pi s)\ell^2 f ds\right] \\
&\leq (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + \int_0^1 \pi c|\sin(\pi s)|\ell^2 ds\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + \pi c\ell^2(\frac{2}{\pi}\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + 2c\ell^2\right]
\end{aligned}$$

which is negative for $\ell^2 \geq \frac{\pi^2}{a-4c}$. Since γ is assumed to be minimal, this implies that $\ell^2 < \frac{\pi^2}{a-4c}$ and thus M is bounded with the diameter of M less than $\frac{\pi}{\sqrt{a-4c}}$ as desired. $\square$

Problem 2

Let M^n be an orientable Riemannian manifold with positive curvature, and even dimension. Let γ be a closed geodesic in M^n. Prove that γ is homotopic to closed curve whose length is strictly less than γ.

Proof. We first prove the statement assumed in the book: namely, that the parallel transport map along γ leaves some vector v orthogonal to $\gamma'(0)$ invariant. To see this, note that the linear transformation

$$A : T_{\gamma(0)}M \to T_{\gamma(0)}M$$
$$A(v) = P_\gamma(v)$$

where $P_\gamma(v)$ is the parallel transport of v along γ back to itself. In particular, $A(\gamma'(0)) = 0$, and so the subspace of $T_{\gamma(0)}M$ perpendicular to $\gamma'(0)$ is invariant under A. Thus, A restricts to a map on the orthogonal complement (of odd dimension). A is just parallel transport, so it is an isometry. Furthermore, A preserves orientation (since M is orientable), so A is an orthogonal transformation with determinant 1 on an odd-dimensional subspace, and thus must have an eigenvalue of 1. Take v to be the corresponding eigenvector.

Now, let $V(t)$ be the parallel transport of v along γ. We calculate $E''(0)$ directly. Using the fact that $\gamma(0) = \gamma(a)$ and for f the variational field corresponding to V, $f(0,0) = f(0,a)$, we see

$$\frac{1}{2}E''(0) = \int_0^a g(V'.V') - g(R(\gamma', V)\gamma', V)dt$$

However, $V'(t) = \nabla_{\gamma(t)}V(t) = 0$ since V is parallel-transported. Thus,

$$\frac{1}{2}E''(0) = -\int_0^a g(R(\gamma', V)\gamma', V)dt$$

and since the space has positive curvature, $g(R(\gamma', V)\gamma', V) = K(\gamma', V) > 0$ always. Thus,

$$\frac{1}{2}E''(0) < 0$$

which implies there is another curve near to γ with smaller length, as desired. □

Problem 3

Let $\tilde{M}$ be a complete simply-connected Riemannian manifold, with curvature $K \leq 0$. Let γ be a normalized geodesic, and let $p \in \tilde{M}$ be a point which does not belong to γ. Let $d(s) = d(p, \gamma(s))$.

Part A

Consider the minimizing geodesic σ_s joining p to $\gamma(s)$, and consider the variation $h(t,s) = \sigma_s(t)$. Show that

$$\frac{1}{2}E'(s) = g(\gamma'(s), \sigma_s'(d(s)))$$

and

$$\frac{1}{2}E''(s_0) > 0$$

if $d'(s_0) = 0$.

Proof. We proceed in the same way Do Carmo does in the proof for the first variation of energy formula. Now, we know that

$$\frac{1}{2}E'(s) = g(\partial_s h, \partial_t h)|_0^{d(s)} - \int_0^{d(s)} g(\partial_s h, \nabla_t \partial_t h) dt$$

We calculate that

$$\begin{aligned} \partial_t h(t,s) &= \sigma_s'(t)) \\ \partial_s h(d(s), s) &= \gamma'(s) \\ \partial_s h(0,s) &= 0 \end{aligned}$$

which follows from the chain rule, the fact that $\sigma_s(d(s)) = \gamma(s)$, and the fact that $\sigma_s(0) = p$. Thus,

$$\begin{aligned} \frac{1}{2}E'(s) &= g(\gamma'(s), \sigma_s'(d(s))) - \int_0^{d(s)} g(\partial_s h, \nabla_t \sigma_s'(t)) dt \\ &= g(\gamma'(s), \sigma_s'(d(s))) - \int_0^{d(s)} g(\partial_s h, 0) dt \\ &= g(\gamma'(s), \sigma_s'(d(s))) \end{aligned}$$

as desired.

Next, we examine the second variation of energy. In particular, we calculate

$$\begin{aligned} \frac{1}{2}E''(s) &= \partial_s g(\gamma'(s), \sigma_s'(d(s))) \\ &= g(\nabla_s \gamma'(s), \sigma_s'(d(s))) + g(\gamma'(s), \nabla_s \sigma_s'(d(s))) \\ &= g(0, \sigma_s'(d(s))) + g(\gamma'(s), \nabla_s \sigma_s'(d(s))) \\ &= g(\gamma'(s), \nabla_s \sigma_s'(d(s))) \\ &= g(\gamma'(s), \gamma'(s)) \end{aligned}$$

where the last equality is obtained by assuming $d'(s) = 0$. This is always positive, as desired. $\square$

Part b

Show that s_0 is a critical point if and only if $g(\gamma'(s_0), \sigma_s'(d(s_0))) = 0$, and conclude that d has a unique critical point, which is a minimum.

Proof. We conclude form the first equation that s_0 is a critical point ($E'(s_0) = 0$) if and only if $g(\gamma'(s_0), \sigma_s'(d(s_0))) = 0$ by reading directly off the formula for $E'(s)$.

Furthermore, the second equation tells us that if s_0 is a critical point of d ($d'(s_0) = 0$), then $E''(s_0)$ is positive, and thus the geodesics connecting p to points around $\gamma(s_0)$ get longer, and so $d(s_0)$ is a local minimum. I assert this is a unique minimum. This, however, follows easily from the fact that d has no maxima, since each critical point is a minimum. Thus, the critical point for which $d'(s_0) = 0$ is unique. □

Part c

Provide examples where this fails if M is not simply connected, or does not have nonpositive curvature.

Proof. Consider the flat torus, which has zero curvature, but is not simply connected. In particular, coordinatize the flat torus as the unit square with the ends identified. Let γ be the geodesic running along $(\frac{1}{4}, t)$ and $(\frac{3}{4}, t)$, and let $p = (\frac{1}{2}, \frac{1}{2})$. Then, there are two points on γ that are minimally close to p, which contradicts the last conclusion of part *b*.

On the other hand, consider the sphere S^2 which has positive curvature. Let γ be the equator, and p be the north pole. p is equidistant from every point on γ, which also contradicts the conclusion in part *b*. □