Homework 5

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May 9, 2018

Problem 1

Let E, F be closed subspaces of a Hilbert space. Prove that $P_E P_F = P_E$ if and only if $E \subseteq F$.

Proof. Suppose first that $P_E P_F = P_E$. Then, in particular,

$$P_E^* = (P_E P_F)^* = P_F^* P_E^* = P_F P_E = P_E$$

by self-adjointness of projections. Thus, $P_E P_F = P_F P_E = P_E$, and the projections commute. Thus, the von Neumann algebra $W^*(P_E, P_F, I)$ is Abelian, and is isometrically $*$-isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ). In particular, the projections P_E, P_F get sent to self-adjoint idempotents $P_E \mapsto M_{\chi_S}$ and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$.

Now, the requirement $P_F P_E = P_E$ corresponds to the requirement

$$M_{\chi_{S'}} M_{\chi_S} = M_{\chi_S}$$

which means that $S \subset S'$. This, in turn, implies that $E \subset F$.

Indeed, E is the subspace of H on which P_E is the identity, which corresponds to the subspace

$$\tilde{E} = \int_S^\oplus H(x) d\mu(x)$$

on which M_{χ_S} is the identity. Similarly,

$$\tilde{F} = \int_{S'}^\oplus H(x) d\mu(x)$$

Clearly, $\tilde{E} \subset \tilde{F}$ (since $S \subset S'$) and so $E \subset F$ as well.

For the converse direction, assume that $E \subset F$. Then, on F, $P_F = I_F$, and $P_F P_E = I_F P_E = P_E$. Furthermore, on $F^\perp$, $P_F P_E = 0 = P_E$. Thus, on all of $H = F \oplus F^\perp$, $P_F P_E = P_E$ as desired. $\square$

Problem 2

Characterize the closed subspaces E, F of a Hilbert space H satisfy $P_F P_E = P_E P_F$.

Proof. I assert that P_E and P_F commute if and only if H can be decomposed into the four orthogonal components

$$H = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp$$

$\square$

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