Problem Set 4

Daniel Halmrast

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Problem 1

Prove that $[0,1)$ is not homeomorphic to $(0,1)$.

Proof. Let f be a bijective function from $[0,1)$ to $(0,1)$. Since f is bijective, it follows that $f^{-1}((0,1)) = [0,1)$. However, $(0,1)$ is open, but $[0,1)$ is not. Thus, f is not continuous.

Since no continuous bijections exist from $[0,1)$, to $(0,1)$, they are not homeomorphic. $\square$

Problem 2

Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Proof. Suppose there existed a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$. Now, consider restricting the domain of f to $\mathbb{R} \setminus \{0\}$. This yields a homeomorphism between $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{f(0)\}$. However, such a homeomorphism cannot exist, since $\mathbb{R} \setminus \{0\}$ is not connected, but $\mathbb{R}^2 \setminus \{f(0)\}$ is connected, and connectedness is a topological property. $\square$

Problem 3

Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.

Proof. Suppose there existed a function $f : I \to I$ (with $I = [0,1]$) such that f has no fixed points. In particular, this defines a (continuous) retract $r : I \to \partial I$ given by

$$r(x) = \begin{cases} 1, & \text{if } x > f(x) \\ 0, & \text{if } x < f(x) \end{cases}$$

Now, we see clearly that $r^{-1}(1) \neq \emptyset$, since at $x = 1$, $f(1)$ cannot be greater than 1, and thus must be less than 1, forcing $r(1) = 1$. Similarly, $r^{-1}(0) \neq \emptyset$, since at $x = 0$, $f(0)$ cannot be less than 0, and thus must be greater than 0, forcing $r(0) = 0$.

Therefore , r is a continuous function from I to the two-point set, and defines a separation of I. But I is connected, so no such separation can exist. Thus, such a retract cannot exist, and f must have a fixed point. $\square$

Problem 4

Prove that $X \times X$ is connected if and only if X is.

Proof. ($\Longrightarrow$) Suppose X is not connected. In particular, there exists a continuous surjection from X to the two-point set.

Thus, we have the diagram

$$\begin{tikzcd} & X \times X \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & \\ X \arrow[dr, two heads, "s_1"'] & & X \arrow[dl, two heads, "s_2"] \\ & \{0,1\} & \end{tikzcd}$$

In particular, the composition $s_1 \circ \pi_1$ is a surjection from $X \times X$ onto $\{0,1\}$, and defines a separation of $X \times X$. Thus, $X \times X$ is separated.

($\Longleftarrow$) Suppose $X \times X$ is not connected. In particular, there exists a continuous surjection $s : X \times X \to \{0,1\}$. Now, for each $x_\alpha \in X$, we know that the map $i_\alpha : X \to X \times X$ given by $i(x) = (x, x_\alpha)$ is an embedding of X into $X \times X$. I assert that there exists some x_0 for which $s^{-1}(\{0\}) \cap X \times \{x_0\} \neq \emptyset$ and $s^{-1}(\{1\}) \cap X \times \{x_0\} \neq \emptyset$.

$\square$