Topology

Homework 2

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Problem 1

Suppose $f : X \to Y$ is a submersion. Prove that if X is compact and Y is connected, then f is surjective.

Proof. Recall from the earlier homework that f is an open map. Thus, the image $f(X)$ is open. Furthermore, since X is compact, $f(X)$ is compact as well. Since Y is Hausdorff, $f(X)$ is closed, and so $f(X)$ is a nonempty clopen set. Since Y is connected, $f(X) = Y$ as desired. □

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Problem 2

Part a

Calculate the Lie algebra of $SO(n)$.

Proof. Consider the smooth function

$$\begin{aligned} \Phi : GL(n) &\to GL(n) \\ \Phi(A) &= AA^T \end{aligned}$$

Now, $O(n)$ is defined to be $\Phi^{-1}(I)$. We calculate the differential directly. Let $B \in T_A(O(n))$, and let $\gamma : [0, 1] \to O(n)$ be

$$\gamma(t) = A + tB$$

Then,

$$\begin{aligned} d\Phi_A(B) &= d\Phi_A(\gamma'(0)) \\ &= \partial_t(\Phi(\gamma(t)))|_0 \\ &= \partial_t((A+tB)(A+tB)^T)|_0 \\ &= \partial_t(AA^T + tAB^T + tBA^T + t^2BB^T)|_0 \\ &= AB^T + BA^T \end{aligned}$$

Now, to show that I is a regular value, we need to show that for all $A \in O(n)$, and for all $C \in T_{\Phi(A)}(GL(n))$, there is some $B \in T_A(O(n))$ with $d\Phi(B) = C$.

Take $B = \frac{1}{2}CA$, we see that

$$d\Phi_A(B) = \frac{1}{2}(A(CA)^T + CAA^T) = C$$

as desired. Thus, I is a regular value.

We next appeal to the fact that the tangent space of a level curve is the kernel of the differential. Thus, $T_I(O(n))$ is the set of all matrices for which

$$d\Phi_I(B) = B + B^T = 0$$

which is exactly $\mathfrak{o}(n)$ the set of all skew-symmetric matrices.

Now, we will observe that $SO(n)$ is open. Consider the determinant map, a continuous map from $O(n)$ to the two-point set $\{-1, 1\}$ with the discrete topology. This defines a separation of $O(n)$ into connected components. Specifically, the inverse image of 1 is $SO(n)$, and thus $SO(n)$ is both open and closed.

Thus, since $SO(n)$ is open, $T_I(SO(n)) = T_I(O(n)) = \mathfrak{o}(n)$ as desired. □

Part B

Show $SO(n)$ is compact.

Proof. To show $SO(n)$ is compact, we will show it is a closed subspace of $O(n)$, and show that $O(n)$ is compact. We have already observed before that $SO(n)$ is clopen in $O(n)$, so all we need to show is that $O(n)$ is compact. First, we recall that $O(n)$ is a level set, and thus is closed. Next, we show it is bounded. Recall that in finite-dimensional normed spaces, all norms are equivalent. So, we just need to show $O(n)$ is bounded with respect to some norm.

Take the operator norm on $M(n)$. Then, for any $A \in O(n)$,

$$\|Ax\|^2 = g(Ax, Ax) = g(A^TAx, x) = g(x, x) = \|x\|^2$$

and so $\|A\| = 1$. Thus, $O(n)$ is bounded by 1 in operator norm. So, $O(n)$ is compact. Since $SO(n)$ is a closed subgroup of $O(n)$, $SO(n)$ is compact as well, as desired. □

Problem 3

Part A

Let G be a subgroup of $\mathrm{Diff}(M)$, and suppose p is fixed by G. Show the map

$$g \mapsto dg_p$$

is a group homomorphism $G \to GL(T_pM)$.

Proof. We just need to show that this map respects the group operation. That is, we need to show that

$$d(gh)_p = dg_p \circ dh_p$$

but this is just a restatement of the functoriality of the differential, which has already been proven. $\square$

Part B

Find a basis for $\mathfrak{su}(2)$ and hence compute the dimension of $SU(2)$. Prove that for $x, y \in \mathfrak{su}(2)$,

$$\mathrm{trace}(x^* y)$$

is a nondegenerate inner product. Deduce that there is a homomorphism

$$\pi : SU(2) \to SO(3)$$

Proof. We first calculate $\mathfrak{u}(2)$. This is just the level set of I under the function

$$\Phi(A) = AA^*$$

and thus $\mathfrak{u}(2)$ is the kernel of $d\Phi_I$. However, we've already calculated what $d\Phi_I$ does in problem 2, so

$$d\Phi_I(A) = A + A^*$$

which has kernel $\mathfrak{u}(2) = \{A \in M(2, \mathbb{C}) \mid A^* = -A\}$.

Now, we observe that $SU(2)$ is the level set of 1 under the determinant map. Now, we observe that

$$d(\det_I(A) = \mathrm{tr}(A)$$

We begin by noting that the determinant can be expressed as

$$\det(I + tA) = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} (I + tA)_i^{\sigma(i)}$$

Now, if we single out the linear term in the product by multiplying by tA once and then by I the rest of the time, we end up with

$$\begin{aligned}
\mathrm{lin}(\det(I + tA)) &= \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{i=1}^{m} (\prod_{j \neq i} I_j^{\sigma(j)}) A_i^{\sigma(i)} t \\
&= \sum_{i=1}^{n} A_i^i t \\
&= t\mathrm{tr}(A)
\end{aligned}$$

and thus, the derivative at zero is $\mathrm{tr}(A)$, as desired. Here, the equality from line 1 to line 2 is made by observing that $I_j^{\sigma(j)}$ is nonzero only when $\sigma(j) = j$, or when $\sigma = id$.

Thus, $\mathfrak{su}(2)$ is the subspace of $\mathfrak{u}(2)$ such that $\mathrm{tr}(A) = 0$.

Next, we compute a basis. Representing an arbitrary matrix as

$$\begin{bmatrix} a + bi & c + di \\ f + gi & h + ki \end{bmatrix}$$

the trace-free requirement says that $a = -h$ and $b = -k$, and skew-symmetry says that $a = h = 0$, $c = -f$ and $d = g$. Thus, a typical matrix in $\mathfrak{su}(2)$ is

$$\begin{bmatrix} bi & c + di \\ -c + di & -bi \end{bmatrix}$$

which has basis

$$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}$$

and thus has dimension 3.

Next, we show that $\text{trace}(x^*y)$ is an inner product on this space. First, we show this is symmetric. To see this, we calculate

$$\text{trace}(x^*y) = \text{trace}(-xy) = \text{trace}(y(-x)) = \text{trace}(-yx) = \text{trace}(y^*x)$$

and so this is symmetric. Next, we show it is linear in the first term. This follows directly: let $x, y, z \in \mathfrak{su}(2)$. Then,

$$\text{trace}((x+z)^*y) = \text{trace}((-x-z)y) = \text{trace}(x^*y) + \text{trace}(z^*y)$$

and for $\alpha \in \mathbb{R}$,

$$\text{trace}((\alpha x)^*y) = \text{trace}(\alpha x^*y) = \alpha\text{trace}(x^*y)$$

and thus this form is linear in the first term.

Finally, we need to show that this is a nondegenerate form. For nonzero $x \in \mathfrak{su}(2)$, let

$$x = \begin{bmatrix} bi & c+di \\ -c+di & -bi \end{bmatrix}$$

Then,

$$\text{trace}(x^*x) = 2(b^2 + c^2 + d^2) \geq 0$$

with equality if and only if $x = 0$.

Finally, we deduce that there is a homomorphism

$$\pi : SU(2) \to SO(3)$$

which is the well-known double cover of $SO(3)$. □

Problem 4

Part a

Let p be a homogeneous polynomial. Prove that any $a \neq 0$ is a regular value of p.

Proof. We calculate the differential directly.

$$(dp)_a X^a = X^a \nabla_a p = X^a \partial_a p$$

Now, let $\beta \in p^{-1}(a)$. We wish to show dp_β is surjective. Since its codomain has dimension one, we just need to show it has nontrivial image. So, we see that

$$((dp)_\beta)_a \beta^a = \beta^a \partial_a p|_\beta = ma$$

by Euler's identity for homogeneous polynomials. Thus, if $a \neq 0$, then a is a regular value, as desired. □

Part b

Deduce that $SL(n, \mathbb{R})$ is a Lie group.

Proof. Observe that $SL(n)$ is the inverse image of 1 under the determinant map. Now, for arbitrary $n \times n$ matrices, the determinant is a homogeneous polynomial of n^2 variables (the entries of the matrix), and thus 1 is a regular value of the determinant. Thus, $SL(n)$ is a submanifold of $GL(n)$, and is a Lie group, as desired. □

Problem 5

Suppose $f : X \to Y$ is smooth, X, Y are compact with the same dimension, and $y \in Y$ is a regular value. Show that $f^{-1}(y)$ is a finite set $\{x_i\}$ and that there is an open neighborhood U of y for which $f^{-1}(U)$ is a finite disjoint union of open sets $\{V_i\}$ such that V_i is a neighborhood of x_i and each V_i is mapped diffeomorphically onto U.

Proof. We first show $f^{-1}(y)$ is a finite set. Since y is a regular value, $f^{-1}(y)$ is a submanifold of X of dimension 0 (since it has codimension $\dim(Y) = \dim(X)$). Now, the only submanifolds of dimension zero are discrete sets, so $f^{-1}(y)$ is a closed discrete subset of X. However, since X is compact, $f^{-1}(y)$ must be finite (since if $f^{-1}(y)$ were infinite, each point by itself is open, and thus $\{x_i\}$ forms an open cover of $f^{-1}(y)$ with no finite subcover, a contradiction).

Now, since df_{x_i} is surjective for each x_i (definition of y being a regular value), and $\dim(X) = \dim(Y)$, it follows that df_{x_i} is an isomorphism for each x_i. Thus there are neighborhoods W_i around each x_i such that $f|_{W_i}$ is a diffemorphism onto its image. Without loss of generality, we take the W_i to be disjoint from each other; since X is Hausdorff, we can find open sets separating x_i and x_j for $i \neq j$, and intersect them with W_i to get disjoint open sets on which f is a diffeomorphism onto.

Take $U = \bigcap_{x_i} f(W_i)$ an open neighborhood of y, and take $V_i = f|_{W_i}^{-1}(U)$. Then, $V_i \subset W_i$ open and so f is a diffeomorphism on V_i, with each V_i a neighborhood of x_i. Finally, noting that $(\cup V_i)^c$ is closed, and f is a closed map (taking compact sets to compact sets), we see that $f((\cup V_i)^c)$ is closed and avoids y. So, if we let $U' = f((\cup V_i)^c)^c$, we see that $U \cap U'$ is the desired neighborhood of y that is evenly covered. Let V_i' be the neighborhood of x_i that maps diffeomorphically to $U \cap U'$. Then, by construction we have that each V_i' is a neighborhood of x_i, all disjoin from each other, and furthermore for any $z \in (\cup V_i')^c$, either $z \in (\cup V_i^c)$ in which case $f(z) \notin U \cap U'$, or z is in some V_i but not V_i' for some i, in which case $f(z) \notin U \cap U'$ since f is a diffeomorphism from V_i to U. $\square$