«CLASS»

«TITLE»

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Problem 1

Show that hyperbolic space H^n is complete.

Proof. We will first show that H^n is homogeneous, and then appeal to the next problem to conclude H^n is complete.

To see that H^n is homogeneous, we consider two families of isometries. For simplicity, we will write points in H^n as (x, y) with $x \in \mathbb{R}^{n-1}$ the first $n-1$ coordinates, and $y \in \mathbb{R}$ the last coordinate. The first isometry we consider is

$$\begin{aligned} T_a : H^n &\to H^n \\ (x, y) &\mapsto (x + a, y) \end{aligned}$$

for any $a \in \mathbb{R}^{n-1}$. To see this is an isometry, we just need to compute dT_a and show it preserves the metric. So, let $v \in T_pH^n$ for some $p \in H^n$, $p = (x_p, y_p)$, and take $\gamma(t) = p + vt = (x_p + v_xt, y_p + v_yt)$ a curve in H^n. Note that $\gamma'(0) = v$. Now, we have that

$$\begin{aligned} dT_a(v) &= dT_a(\gamma'(0)) \\ &= \partial_t T_a(\gamma(t))|_{t=0} \\ &= \partial_t(x_p + v_xt + a, y_p + v_yt)|_{t=0} \\ &= (v_x, v_y) = v \end{aligned}$$

Thus, $dT_a(v) = v$. Furthermore, since the metric at $(x + a, y)$ is the same as at (x, y) (since the scaling factor only depends on y) we have that for $u, v \in T_pM$,

$$g(u, v)_{(x,y)} = g(dT_au, dT_av)_{(x+a,y)}$$

and thus T_a is an isometry (I suppose you'd have to check that T_a is a diffeomorphism as well, but this is obvious. Clearly T_a is smooth, and it has a smooth inverse T_{-a}).

Secondly, we consider the isometry

$$\begin{aligned} M_\alpha : H^n &\to H^n \\ (x, y) &\mapsto (\alpha x, \alpha y) \end{aligned}$$

for $\alpha > 0$. This maps H^n into H^n, since it keeps the y coordinate positive. Furthermore, it is a diffeomorphism (it is clearly smooth, and $M_{\frac{1}{\alpha}}$ acts as an inverse). I also claim it is an isometry. Again letting $\gamma = (x_p + v_x, y_p + v_y)$ for $(v_x, v_y) \in T_{(x,y)}H^n$ we note that

$$\begin{aligned} dM_\alpha(v) &= dM_\alpha(\gamma'(0)) \\ &= \partial_t M_\alpha(\gamma(t))|_{t=0} \\ &= \partial_t(\alpha(x_p + v_x), \alpha(y_p + v_y))|_{t=0} \\ &= \alpha V \end{aligned}$$

Finally, we compute the metric

$$\begin{aligned} g(u,v)_(x,y) &= g_{ab}u^a v^b \\ &= \frac{1}{y^2}u_b v^b \end{aligned}$$

$$\begin{aligned} g(dM_\alpha u, dM_\alpha v)_{(\alpha x, \alpha y)} &= g_{ab}\alpha u^a \alpha v^b \\ &= \frac{1}{(\alpha y)^2}\alpha^2 u_b v^b \\ &= \frac{1}{y^2}u_b v^b \end{aligned}$$

Where $u_b = \eta_{ab}u^a$ and so $u_b v^b$ is the standard inner product on $\mathbb{R}^n$. Thus, M_α is an isometry.

I assert that the action of these two isometries is transitive. Indeed, given (x, y) and (x', y') in H^n, we construct the isometry as follows. First, apply T_{-x} to map (x, y) to $(0, y)$. Then, apply $M_{\frac{y'}{y}}$ to map $(0, y)$ to $(0, y')$. Finally, apply $T_{x'}$ to map $(0, y')$ to (x', y').

Thus, for any two points (x, y) and (x', y') in H^n, there is an isometry connecting them. Thus, by the result of the next problem, H^n is complete. $\square$

Problem 2

Show that a homogeneous space is complete.

Proof. Let M be a homogeneous manifold. We will show that M is geodesically complete.

Let ε be such that $B_\varepsilon(p) \subset M$ is a normal ball at $p \in M$. Since M is homogeneous, this implies that $B_\varepsilon(q)$ is a normal ball at $q \in M$ for any other q. To see this, we note that for ϕ the isometry sending p to q,

$$\phi \circ \exp_p \circ d\phi^{-1}$$

defines a diffeomorphism between $B_\varepsilon(0) \subset T_qM$ and the image $B_\varepsilon(q)$. This is well-defined, since ϕ is an isometry, so $\|v\| = \|d\phi^{-1}v\|$. Furthermore, we can see that $\exp_q = \phi \circ \exp_p \circ d\phi^{-1}$. Obesrve that $\gamma(t) = \exp_q(tv)$ is the unique geodesic through q with tangent vector v. However,

$$\tilde{\gamma}(t) = \phi \circ \exp_p \circ d\phi^{-1}(tv)$$

has the same properties. Namely $\tilde{\gamma}(0) = \phi(p) = q$, and $\tilde{\gamma}'(0) = d\phi(d\phi^{-1}(v)) = v$. Thus, $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in [0,1]$, and so $\exp_q$ and $\phi \circ \exp_p \circ d\phi^{-1}$ agree at all points in the normal ball. Thus, $B_\varepsilon(q)$ is a normal ball, as desired.

Recall that in a normal ball at p, any geodesic going through p can be extended throughout the entire normal ball. This follows from the fact that if γ is a geodesic passing through p at some time t_p with $\gamma'(t_p) = v$, it is the unique geodesic (up to reparameterization) with $\gamma(t_p) = p$ and $\gamma'(t_p) = v$. Now, since radial geodesics through p are defined on the entire normal ball, the radial geodesic starting at p with tangent vector v is defined throughout the normal ball, and is an extension of γ. Thus, γ can be extended through the normal ball.

It follows immediately, then, that any geodesic γ (with unit speed, without loss of generality) defined on some interval (a, b) can be extended to a geodesic defined on $(a, b + \frac{\varepsilon}{2})$ by observing that γ passes through $\gamma(b - \frac{\varepsilon}{2})$, and since $\gamma(b - \frac{\varepsilon}{2})$ has a normal ball of radius ε around it, we know that γ can be extended through this normal ball to be defined on $(a, b - \frac{\varepsilon}{2} + \varepsilon) = (a, b + \frac{\varepsilon}{2})$.

Thus, it follows immediately that geodesics can be extended indefinitely (the symmetric argument works to show γ can be extended the other way) and thus M is geodesically complete. $\square$

Problem 3

Part a

Let v be a linear field on $\mathbb{R}^n$. That is, v is a vector field, and v is linear when thought of as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Show that a linear field given by a matrix A is a killing field if and only if A is antisymmetric.

Proof. Let X be a linear vector field. Then, X is expressible as a matrix A. That is, $X(f(x_1, \dots, x_n)) = Af(x_1, \dots, x_n)$. In order for X to be a killing field, we must have that its local flow around each point is an isometry. That is, for $\phi : (-\varepsilon, \varepsilon) \times U \to M$ the flow of X around a point p, $d\phi(t, \cdot)$ preserves inner products.

Now, the flow of X is the solution to

$$\partial_t \phi^a = A\phi^a$$

which is solved by setting $\phi = \exp(At)$. Now, let's calculate the differential. For $v \in T_p\mathbb{R}^n$, let $\gamma(s) = p + sv$. Then

$$\begin{aligned} d\phi(v) &= \partial_s \phi(\gamma(s))|_0 \\ &= \partial_s \exp(At)(p + sv) \\ &= \exp(At)v \end{aligned}$$

and so $d\phi = \phi$. We require that

$$\langle u, v\rangle = \langle \exp(At)u, \exp(At)v\rangle$$

which amounts to requiring

$$\langle u, v\rangle = \langle u, \exp(A^T t)\exp(At)v\rangle$$

Now, this happens for all u, v if and only if $\exp(A^T t)\exp(At) = I$, which holds for all t if and only if $A^T = -A$. Thus, in order for X to be a killing field, A must be antisymmetric, and vice versa. $\square$

Part b

Let X be a killing field on M with $p \in M$, and let U be a normal neighborhood of p in M. Assume that p is a unique point of U with $X_p = 0$. Show that in U, X is tangent to the geodesic spheres centered at p.

Proof. Let $\phi_q : (-\varepsilon, \varepsilon) \times V_q \to M$ denote the local flow of X around any point q. Since $X_p = 0$, we know that $\phi(t, p) = p$. That is, p is fixed by the flow of X.

Now, let q be any point in U the normal neighborhood of p. We know that there is a unique radial geodesic from p to q defined as $\gamma(t) = \exp_p(tv)$ for some v. Now, ϕ is defined across all of $\gamma(t)$ for $t \in [0, 1]$ since $\gamma([0, 1])$ is a compact set, and thus can be covered by a finite number of sets V_q on which the flow is defined.

Now, since $\phi(t, \cdot)$ is an isometry, it maps geodesics to geodesics. Thus, the image $\phi(t, \gamma([0, 1]))$ is a geodesic from $\phi(t, p) = p$ to $\phi(t, q)$. Furthermore, this geodesic is defined by $\tilde{\gamma(t)} = \exp_p(tu)$ for some u. Now, we know that

$$\begin{aligned} d\phi(t, v) &= d\phi(t, \gamma'(0)) \\ &= \partial_s \phi(t, \gamma(s)) \\ &= \partial_s(\tilde{\gamma}(s)) \\ &= u \end{aligned}$$

Thus u and v have the same norm, and so $\gamma(1) = q$ and $\tilde{\gamma}(1) = \phi(t, q)$ are the same distance from q.

Thus, ϕ moves points along the geodesic spheres, and so X is tangential to the geodesic spheres, as desired. □

Part c

Let X be a smooth vector field on M and let $f : M \to N$ be an isometry. Let Y be a vector field on N defined by $Y(f(p)) = df_p(X(p))$. Prove that Y is a killing field if and only if X is.

Proof. Suppose X is a killing field. That is, the local flow ϕ is an isometry. Now, we can push forward a local flow on X to a local flow on Y. That is,

$$\psi(t, x) = f(\phi(t, f^{-1}(x)))$$

defines a flow on Y. This is clear, since

$$\begin{aligned}
\partial_t \psi(t, x) &= \partial_t f(\phi(t, f^{-1}(x))) \\
&= df(\partial_t \phi(t, f^{-1}(x))) \\
&= df(X(\phi(t, f^{-1}(x)))) \\
&= Y(f(\phi(t, f^{-1}(x)))) \\
&= Y(\psi(t, x))
\end{aligned}$$

as desired. Now, for any fixed t, $\psi(t, \cdot) = f \circ \phi(t, \cdot) \circ f^{-1}$ is the composition of isometries, and is therefore an isometry as desired. Thus, Y is a killing field if X is.

By symmetry of the problem, this implies that Y is a killing field if and only if X is. □

Part d

Show that X is a killing field if and only if

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$$

for all X, Y, Z.

Proof. Recall the definition of the Lie derivative of a tensor field T along a vector field v with flow ϕ

$$\mathfrak{L}_v(T)(p) = \lim_{t \to 0} \left\{ \frac{\phi^*(-t, T(\phi(t, p))) - T(p)}{t} \right\}$$

□

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