Problem Set 3

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Problem 1

Part a

Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ is a surjective smooth submersion.

Proof. Surjectivity follows almost immediately from the definition of the map, since any equivalence class $[z]$ is mapped to by $\pi(z)$. Now, we must show it is a smooth submersion.

We will do this using the Global Rank Theorem (Theorem 4.14), which states that a surjection with constant rank is a smooth submersion.

Now, the coordinate charts on $\mathbb{C}^{n+1}$ are just the standard rectangular coordinates

$$(z^1, \ldots, z^{n+1}) \mapsto (\Re(z^1), \Im(z^1), \ldots, \Re(z^{n+1}), \Im(z^{n+1}))$$

And for each $i \in \{1, \ldots, n+1\}$, we have a local coordinate chart on $\mathbb{C}P^n$ around where z^i is not zero, given as

$$(z^1 : \ldots : z^i : \ldots : z^{n+1}) \mapsto \frac{1}{z^i}(z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^{n+1})$$

(Here, this coordinate chart actually maps to $\mathbb{C}^n$, which has a global coordinate chart as defined above, and the composition then defines a coordinate chart to $\mathbb{R}^{2n}$.)

So now we consider the coordinate representation of π around a point z_0 for which z_0^i is not zero, which (as complex coordinates) is given as

$$\tilde{\pi}(z^1, \ldots, z^i, \ldots, z^{n+1}) = \frac{1}{z^i}(z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^{n+1})$$

This is clearly smooth from $\mathbb{C}^{n+1} \to \mathbb{C}^n$, so π is smooth as well. Now, let's find what the rank at this point is.

To do so, we compute the Jacobian of $d\pi$, $J_j^k = \partial_j(\tilde{\pi}^k)$. This is a somewhat complicated Jacobian, so we consider first the minor to the left of the i^{th} column. On this minor, we have that

$$J_j^k = \frac{1}{z^i}\delta_j^k$$

Which is just $\frac{1}{z^i} * I$ for the $i-1 \times i-1$ identity matrix.

Now, to the right of the i^{th} column, we have

$$J^k_j = \frac{1}{z^i}\delta^k_{(j+1)}$$

And finally, for the i^{th} column itself, we have that

$$J^k_i = \partial_i(\tilde{\pi}^k) = \frac{-1}{z^i}\tilde{\pi}^k$$

So the whole matrix looks like

$$\begin{bmatrix} \frac{1}{z^i} & 0 & \dots & 0 & \frac{-z^1}{(z^i)^2} & 0 & \dots & 0 \\ 0 & \frac{1}{z^i} & \dots & 0 & \frac{-z^2}{(z^i)^2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{z^i} & \frac{-z^{i-1}}{(z^i)^2} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{-z^{i+1}}{(z^i)^2} & \frac{1}{z^i} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \frac{-z^{n+1}}{(z^i)^2} & 0 & \dots & \frac{1}{z^i} \end{bmatrix}$$

Which, since z^i is not zero, has full rank. Thus, since this is true for any point in $\mathbb{C}^{n+1}$, it follows that π has full rank at every point, and is a smooth submersion. □

Part B

Show that $\mathbb{C}P^1 \cong S^2$.

Proof. To do this, we will consider the two projection maps

$$\begin{tikzcd} & S^3 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C}P^1 & & S^2 \end{tikzcd}$$

Given by π_1 the canonical projection map from S^3 to $S^3/U(1)$, and π_2 is the Hopf fibration, which was shown to be a smooth surjection in the last homework assignment. (Here, we will use the fibration map $\pi_2(z_1, z_2) = (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2)$).

A small calculation shows that the Hopf fibration is, in fact, a surjective smooth submersion, since in rectangular coordinates, we have

$$\pi_2(x, y, w, v) = (2(xw + yv), 2(yw - xv), X^2 + y^2 - v^2 - w^2)$$

Which has a differential of

$$d\pi_2 = 2\begin{bmatrix} w & v & y & x \\ -v & w & -x & y \\ x & y & -v & -w \end{bmatrix}$$

Which is clearly full rank, and thus surjective.

We note first the fibers of these maps coincide. This is clear since the fiber associated with $\pi_1([(z_1, z_2)]) = \{\lambda(z_1, z_2)\}$ for all $\lambda \in U(1)$, which all map to the same point via π_2, since

$$\begin{aligned} \pi_2(\lambda(z_1, z_2)) &= (2\lambda z_1\overline{\lambda z_2}, |\lambda z_1|^2 - |\lambda z_2|^2) \\ &= (2\lambda\overline{\lambda}z_1\overline{z_2}, |z_1|^2 - |z_2|^2) \\ &= (2z_1\overline{z_2}, |z_1|^2 - |z_2|^2) \end{aligned}$$

Thus, a fiber of π_1 is a fiber of π_2.

Now, let's look at the fibers of π_2. We know that

$$\pi_2^{-1}(z, r) = \{(z_1, z_2) \in S^3 \mid z_1\overline{z_2} = z,\ |z_1|^2 - |z_2|^2 = r\}$$

Now, the last condition, along with the condition that $z_1, z_2 \in S^3$, fix the norms of z_1 and z_2. Thus, for any other point (z_1', z_2') in the fiber, we must have that $z_1' = \lambda_1 z_1$ and $z_2' = \lambda_2 z_2$ with $\lambda_1, \lambda_2 \in U(1)$. Furthermore, the restriction that $z_1\overline{z}_2 = z$ forces $\lambda_1 = \lambda_2$. Thus, $\pi_1(\pi_2^{-1}(z, r))$, which quotients by $\lambda \in U(1)$, sends the fiber to a single point.

Thus, each smooth surjection is constant on each other's fibers, which gives rise to the diffeomorphism

$$\begin{tikzcd}
 & S^3 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{C}P^1 \arrow[rr, hook, two heads, dashed, "F"] & & S^2
\end{tikzcd}$$

as desired. $\square$

Problem 2

For M a nonempty smooth compact manifold, show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Proof. We note two things immediately. First, since F is continuous, its image must be compact (and by the Heine-Borel theorem, must be closed). Second, since F is a smooth submersion, it is an open map. Thus, the image of F is open in $\mathbb{R}^k$. The only nonempty set in R^k that is both closed and open is $\mathbb{R}^k$ itself, but this cannot be the image of F, since the image must be compact. So, such an F cannot exist. $\square$

Problem 3

Use the covering map $\varepsilon^2 : \mathbb{R}^2 \to \mathbb{T}^2$ to show that the immersion $\chi : \mathbb{R}^2 \to \mathbb{R}^3$ descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that χ passes to the quotient to define a smooth map $\tilde{\chi} : \mathbb{T}^2 \to \mathbb{R}^3$, then show that $\tilde{\chi}$ is a smooth embedding whose image is the given surface of revolution.

Proof. We first show that χ passes smoothly through the quotient by showing it is constant on the fibers of ϵ^2. To see this, we observe that χ is periodic in integer multiples of its arguments, and that ϵ^2 is similarly periodic. That is, for $(\exp(2\pi i\theta_1), \exp(2\pi i\theta_2))$, its fiber is just $(n\theta_1, m\theta_2)$. But since χ is periodic, it must map all of this fiber to a point. Thus, χ is constant on the fibers of ϵ^2, and can be passed through the quotient to obtain $\tilde{\chi}$ which makes

$$\begin{tikzcd}
 & \mathbb{R}^2 \arrow[dl, "\chi"'] \arrow[dr, two heads, "\epsilon^2"] & \\
\mathbb{R}^3 & & \mathbb{T}^2 \arrow[ll, dashed, "\tilde{\chi}"]
\end{tikzcd}$$

commute.

Now, $\tilde{\chi}$ is clearly injective, since χ is only periodic on the fibers of ϵ^2. So, we just need to show that $\tilde{\chi}$ is an immersion, and Proposition 4.22 will give us that $\tilde{\chi}$ is an embedding.

To see that $\tilde{\chi}$ is an immersion, we observe the following diagram:

$$\begin{tikzcd}
T\mathbb{R}^2 \arrow[r, hook, two heads, "d\epsilon^2"] \arrow[rr, hook, bend right, "d\chi"'] & T\mathbb{T}^2 \arrow[r, "d\tilde{\chi}"] & T\mathbb{R}^3
\end{tikzcd}$$

Where $d\epsilon^2$ is locally bijective since ϵ^2 is a covering map, and $d\chi$ is injective since χ is an immersion. Therefore, $d\tilde{\chi}$ must be injective, and the result follows immediately. $\square$

Problem 4

Let $F : S^2 \to \mathbb{R}^4$ by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz)$$

descends along the quotient of S^2 by $O(1)$ to a smooth embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

Proof. TO begin with, we cite that the quotient map from S^2 to $\mathbb{R}P^2$ is given as a smooth covering map from problem 4-10. Thus, we have the following diagram

$$\begin{tikzcd} & S^2 \arrow[dl, "F"'] \arrow[dr, "q"] & \\ \mathbb{R}^4 & & \mathbb{R}P^2 \end{tikzcd}$$

We note that F is constant on the fibers of q (where we observe that $q^{-1}(\{[x]\}) = \{x, -x\}$) since

$$F(-x, -y, -z) = ((-x)^2 - (-y)^2, (-x)(-y), (-x)(-z), (-y)(-z)) = F(x, y, z)$$

Thus, we apply Theorem 4.30 to construct $\tilde{F}$ such that

$$\begin{tikzcd} & S^2 \arrow[dl, "F"'] \arrow[dr, "q"] & \\ \mathbb{R}^4 & & \mathbb{R}P^2 \arrow[ll, dashed, "\tilde{F}"] \end{tikzcd}$$

commutes.

Now, since $\mathbb{R}P^2$ is compact, it suffices to show that $\tilde{F}$ is an injective immersion, and Proposition 4.22 gives us that $\tilde{F}$ is a smooth embedding.

To see that $\tilde{F}$ is an injective immersion, we first show that $\tilde{F}$ is injective.

A simple diagram chase of

$$\begin{tikzcd} & S^2 \arrow[dl, "F"'] \arrow[dr, two heads, "q"] & \\ \mathbb{R}^4 & & \mathbb{R}P^2 \arrow[ll, "\tilde{F}"] \end{tikzcd}$$

Will prove this. To do so, let $[x], [y]$ be distinct points in $\mathbb{R}P^2$. Then, by injectivity of q, it follows that there are points $x, y \in S^2$ such that $q(x) = [x]$ and $q(y) = [y]$. Now, since $[x] \neq [y]$, we have that $x \neq -y$. Thus, it is clear that $F(x) \neq F(y)$. This can be seen by the following argument.

Suppose $F(x, y, z) = (q, r, s, t)$ for some constants q, r, s, t. In particular, we have that

$$\begin{aligned} x^2 &= \frac{rs}{t} \\ y^2 &= \frac{rt}{s} \\ z^2 &= \frac{st}{r} \quad xy \quad = r \\ xz &= s \\ yz &= t \end{aligned}$$

It follows immediately that if $F(x', y', z') = (q, r, s, t)$, then $(x', y', z') = (x, y, z)$ or $-(x, y, z)$.

Thus, $F(x) \neq F(y)$, and we have that

$$\begin{aligned} F(x) = \tilde{F}(q(x)) = \tilde{F}([x]) \\ F(y) = \tilde{F}([x]) \\ F(x) \neq F(y) \implies \tilde{F}([x]) \neq \tilde{F}([y]) \end{aligned}$$

which shows that $\tilde{F}$ is injective.

Similarly, it is clear that $d\tilde{F}$ is injective. To see this, we note that since q is a covering map, it is a local diffeomorphism, and thus dq is locally a bijection.

Then, observe the diagram

$$\begin{tikzcd} TS^2 \arrow[r, hook, two heads, "dq"] \arrow[rr, bend right, "dF"'] & T\mathbb{R}P^2 \arrow[r, "d\tilde{F}"] & \mathbb{R}^4 \end{tikzcd}$$

Now, a routine calculation of the Jacobian dF shows that it is injective locally, which implies by the above diagram (since dq is locally bijective) that $d\tilde{F}$ is locally injective.

Thus, $\tilde{F}$ is an injective smooth immersion from a compact manifold, and is a smooth embedding. $\square$

Problem 5

Part a

Show that an immersion between two manifolds of the same dimension is an open mapping.

Proof. To see this, let $f : M \to N$ be an immersion between M and N of the same dimension n, and let U be open in M.

Now, since f is an immersion, it must be that for each point $x \in U$, there is a neighborhood of x contained in U for which the coordinate representative of f is the identity. Thus, on that neighborhood, f is an open map. However, since U can be written as the union of such neighborhoods, and f respects unions, it follows that $f(U)$ is the union of open sets, and is open. $\square$

Part b

Use this to show there is no immersion from S^n to $\mathbb{R}^n$.

Proof. Suppose there did exist such an $f : S^n \to \mathbb{R}^n$. Since f is continuous, it must map the compact set S^n to a compact set. However, S^n is also open, and f is an open map, so the image $f(S^n)$ must also be open. Since there are no open compact sets in $\mathbb{R}^n$, such a map cannot exist. $\square$