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# Midterm

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November 8, 2017

## PROBLEM 1

### PART A

Use the standard charts on  $S^n$  to calculate the matrix representation of  $di : T_p S^n \rightarrow T_p \mathbb{R}^{n+1}$ , and show that  $di$  is injective, and thus  $i$  is an embedding.

*Proof.* For this calculation, we will use the chart given by hemisphere projection. That is, the domains for the charts will be the open sets  $U_i^\pm = \{(x^1, \dots, x^{n+1}) \mid x^i > 0 (x^i < 0 \text{ resp.})\}$  with maps

$$\phi_i^\pm(x^1, \dots, x^{n+1}) = (x^1, \dots, \hat{x}^i, \dots, x^{n+1})$$

Where a hat denotes omission of the variable.

Now, suppose  $p \in U_i^+$  (without loss of generality, we take the positive hemisphere of  $x^i$ , but the argument can be repeated exactly with the negative hemisphere as well.) and let the coordinate representation of  $p$  be

$$\phi(p) = (x^1, \dots, \hat{x}^i, \dots, x^{n+1})$$

Then, the inclusion map looks like

$$\begin{aligned} i \circ \phi^{-1}((x^1, \dots, \hat{x}^i, \dots, x^{n+1})) &= (y^1, \dots, y^{n+1}) \\ &= (x^1, \dots, x^{i-1}, \sqrt{1 - x^a x_a}, \dots, x^{n+1}) \end{aligned}$$

and the Jacobian  $di$  can be calculated directly using the identity  $di_j^k = \partial_j(y^k)$ . Which gives the matrix (for  $j = 1, \dots, i-1, i+1, \dots, n+1$  and  $k = 1, \dots, n$ )

$$\partial_j(y^k) = \delta_j^k - \frac{1}{\sqrt{1 - x^a x_a}} \delta^{ik} x_j$$

Which is clearly injective, since the rows  $k \neq i$  are the basis covectors for  $\mathbb{R}^n$ , and thus  $\partial_j(y^k)$  has rank  $n$  as desired.

Furthermore, since  $S^n$  is compact, and the inclusion is injective, it follows that  $i$  defines an embedding of  $S^n$  into  $\mathbb{R}^{n+1}$ .  $\square$

## PART B

Show that  $T_p S^n$ , when identified with  $di(T_p S^n)$  is the subspace of  $\mathbb{R}^{n+1}$  consisting of all vectors perpendicular to the radial vector to  $p$ .

*Proof.* This follows by direct calculation. To see this, let  $v \in T_p S^n$ . Then,

$$\begin{aligned} di(v) &= \partial_j y^k v^j = \delta_j^k v^j - \frac{1}{\sqrt{1 - x^a x_a}} \delta^{ik} x_j v^j \\ &= v^k - \frac{x^a v_a}{\sqrt{1 - x^a x_a}} \delta^{ik} \end{aligned}$$

where it is assumed that  $x_a v^a$  does not sum over the  $i^{th}$  component of  $v$ .

Recalling earlier that the embedding sends

$$p = (x^1, \dots, \hat{x}^i, \dots, x^{n+1})$$

to

$$(y^1, \dots, y^{n+1}) = (x^1, \dots, x^{i-1}, \sqrt{1 - x^a x_a}, \dots, x^{n+1})$$

we can compute the inner product  $g_{jk} v^j y^k$  directly.

$$\begin{aligned} g_{jk} v^j y^k &= v^k x_k + \delta_{ij}^{ik} v^j y_k \\ &= v^k x_k + v^i y_i \\ &= v^k x_k - \frac{v^a x_a}{\sqrt{1 - x^a x_a}} \sqrt{1 - x^a x_a} \\ &= v^k x_k - v^a x_a \\ &= 0 \end{aligned}$$

Thus,  $di(v)$  is perpendicular to  $p$ , as desired. □

## PART C

For  $F$  a smooth map from  $\mathbb{R}^{n+1}$  to  $\mathbb{R}^{m+1}$  such that  $F(S^n) \subset S^m$ , show that  $d(F|_{S^n}) = dF|_{T_p S^n}$ .

*Proof.* To begin with, let  $v \in T_p S^n$ . In particular, there is a curve  $\gamma$  such that  $\gamma(0) = p$  and  $\gamma'(0) = v$ . Then,

$$\begin{aligned} d(F|_{S^n})(\gamma'(0)) &= \partial_t|_0 F|_{S^n}(\gamma(t)) \\ &= \partial_t|_0 F(\gamma(t)) \\ &= dF(\gamma'(0)) \\ &= dF(\gamma'(0))|_{T_p S^n} \end{aligned}$$

□

## PROBLEM 2

Show that the tangent bundle  $TM$  is always orientable.

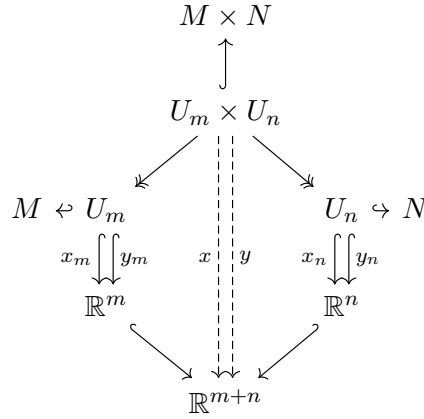
*Proof.* For this problem, we will use that fact that a manifold is orientable if there exist charts such that the coordinate transition maps have a Jacobian of positive determinant.

Before proceeding further, we prove the following lemma:

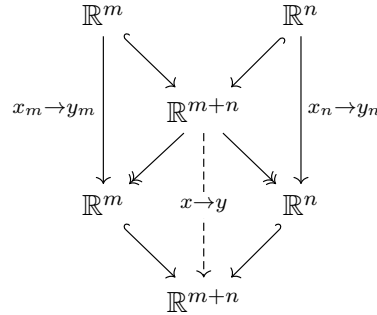
**Lemma.** Suppose  $M$  and  $N$  are smooth manifolds. In particular, their product  $M \times N$  is a smooth manifold. Furthermore, for pairs of coordinates  $x_m, y_m$  on  $U_m \subset M$  and  $x_n, y_n$  on  $U_n \subset N$ , the product coordinates  $x = (x_m, x_n)$  and  $y = (y_m, y_n)$  are smooth coordinates on  $U_m \times U_n$ , and the Jacobian  $J(x \rightarrow y)$  is given componentwise. That is,

$$J(x \rightarrow y) = (J(x_m \rightarrow y_m), J(x_n \rightarrow y_n))$$

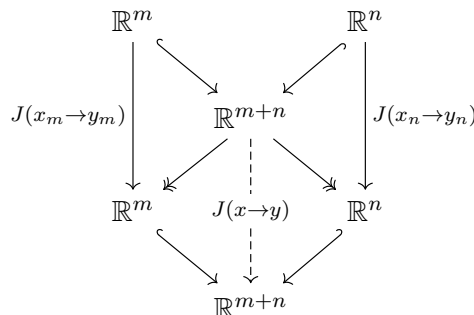
*Proof.* That  $M \times N$  is a smooth manifold follows almost immediately by taking products of coordinate charts on  $M$  and  $N$ . Now, we have the following diagram:



which implies that the induced coordinates  $x$  and  $y$  are smooth. Now, let's expand the lower half of the commutative diagram to get the transition maps  $x_m \rightarrow y_m$  and  $x_n \rightarrow y_n$ :



Differentiating this diagram (applying the differential functor) yields:



Thus,  $J(x \rightarrow y) = (J(x_m \rightarrow y_m), J(x_n \rightarrow y_n))$  as desired.  $\square$

We are now ready to prove the general result.

So, let  $M$  be a smooth manifold with tangent bundle  $TM$ . Furthermore, for a point  $p$ , suppose there are two coordinate charts  $x^i$  and  $y^i$  on a neighborhood of  $p$ . We wish to calculate the Jacobian of the induced coordinate transformations on the tangent bundle.

To do so, we first appeal to the fact that  $TM$  is locally trivializable. That is, on some neighborhood  $U$  containing  $p$ ,  $\pi^{-1}(U) \cong M \times T_p M$ , where  $\pi$  is the canonical projection of  $TM$  onto  $M$ . In particular, this means that in  $\pi^{-1}(U)$ , we have the coordinate charts  $x^i \times dx^i$  and  $y^i \times dy^i$ .

Thus, the transition map is just  $(x \rightarrow y, dx \rightarrow dy)$ , where  $x \rightarrow y$  is the transition map from the  $x$  coordinate system to the  $y$  coordinate system, and  $dx \rightarrow dy$  is the transition map from the  $\partial_x|_p$  coordinate system to the  $\partial_y|_p$  coordinate system.

Recall that  $dx \rightarrow dy$  is simply the Jacobian of the original coordinate transform. That is,  $dx \rightarrow dy = J(x \rightarrow y)$ . Now, from the above lemma,

$$J(x \rightarrow y, dx \rightarrow dy) = J(x \rightarrow y, J(x \rightarrow y)) = (J(x \rightarrow y), J(J(x \rightarrow y)))$$

It should be clear that  $J^2 = J$ , since the Jacobian of a transformation is linear. Thus, we have that

$$J(x \rightarrow y, dx \rightarrow dy) = (J(x \rightarrow y), J(x \rightarrow y))$$

To calculate the determinant of this, we appeal to the fact that the determinant of a linear transformation of the form  $(A, B)$  is the product of the determinants of  $A$  and  $B$ . Thus,

$$\begin{aligned} \det J(x \rightarrow y, dx \rightarrow dy) &= \det(J(x \rightarrow y)) \det(J(x \rightarrow y)) \\ &= (\det(J(x \rightarrow y)))^2 \end{aligned}$$

Which is always positive.

Since this can be done at any point  $p$  in the manifold, we have an atlas for  $TM$  where the determinant of the coordinate transforms is always positive, and thus  $TM$  is orientable.  $\square$