Problem Set 4

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Problem 1

Prove that if any of the following hold for S an immersed submanifold of a manifold M, then S is embedded:

- S has codimension 0 in M.
- The inclusion map $S \subseteq M$ is proper.
- S is compact.

Proof. Suppose S has codimension 0 in M. Then, we must show that S is open with respect to M, and since every open subset of a manifold with codimension zero is an embedded submanifold, S will be embedded.

From the previous homework, we know that an immersion between two manifolds of the same dimension is an open mapping, so the image of S under the inclusion map must be open. Thus, S is open with respect to M, and is an embedded submanifold.

Suppose the inclusion map is proper. Then, by Proposition 4.22, since the inclusion map is an injective smooth immersion which is proper, S is embedded.

Suppose S is compact. Then, by Proposition 4.22, since the inclusion map is an injective smooth immersion from a compact set, S is embedded. $\square$

Problem 2

Show that the image of the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ given by $\beta(t) = (\sin(2t), \sin(t))$ is not an embedded submanifold.

Proof. To show that this is not an embedded submanifold, we will show that it does not get the subspace topology from $\mathbb{R}^2$. To see this, we consider the neighborhood $(-1, 1)$ of zero, which is open in $\mathrm{im}(\beta)$. In particular, this neighborhood cannot be an intersection of an open set in $\mathbb{R}^2$ with $\mathrm{im}(\beta)$. This is evident, since any neighborhood of zero in $\mathbb{R}^2$ must intersect $\beta(-\pi + \varepsilon)$ and $\beta(\pi - \varepsilon)$ for sufficiently small ε.

So, since any open set in $\mathbb{R}^2$ containing zero also contains $\beta(\pi - \varepsilon)$ for all ε sufficiently small, every open neighborhood of $\beta(0)$ in the subspace topology must contain $\beta(\pi - \varepsilon)$ for all sufficiently small ε. Since the neighborhood $\beta(-1, 1)$ does not contain $\beta(\pi - \varepsilon)$, it is not open in the subspace topology, and since $(-1, 1)$ is open with respect to the manifold structure on $\mathrm{im}(\beta)$, it must be that $\mathrm{im}(\beta)$ is not an embedded submanifold. $\square$

PROBLEM 3

Show that the boundary of the unit square does not have a topology and a smooth structure for which it is an immersed submanifold of $\mathbb{R}^2$.

Proof. Suppose for a contradiction that the boundary ∂I^2 did admit a smooth structure for which it is an immersed submanifold. In particular, we know that at the corner $(0,0)$, the tangent space $T_0 \partial I^2$ is a one-dimensional subspace of $T_0\mathbb{R}^2 \cong \mathbb{R}^2$. So, by Proposition 5.35, there must be a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \partial I^2$ such that $\gamma(0) = (0,0)$ and $\gamma'(0) \neq 0$. Now, if we write $\gamma(t) = (x(t), y(t))$ we know that $x(t)$ takes a global minimum at $(0,0)$, and thus $x'(0) = 0$. Similarly, $y(t)$ attains a global minimum at $(0,0)$, so $y'(0) = 0$. However, this contradicts the assertion that $\gamma'(0) \neq 0$, so no such submanifold structure can exist. $\square$

Problem 4

Show that for any Lie group G, the multiplication map $\mu : G \times G \to G$ is a smooth submersion.

Proof. Since G is a Lie group, it follows that μ is smooth. Now, all that is needed to show is that μ is a submersion.

From the next problem, we know that

$$dm|_{(e,e)}(X, Y) = X + Y$$

Now, since every vector can be written as the sum of two vectors (trivially $X = X + 0$), it follows that dm is surjective at the identity. But what about elsewhere? Consider that

$$\begin{aligned} dm|_{(\sigma,\sigma)}(X, Y) &= X_\sigma + Y_\sigma \\ &= dl_\sigma(X_e + Y_e) \\ &= dl_\sigma(dm|_{(e,e)}(X, Y)) \end{aligned}$$

Thus, $dm|_{(\sigma,\sigma)} = dl_\sigma \circ dm|_{(e,e)}$ is the composition of a surjection and a bijection from $T_{(\sigma,\sigma)}G \times G$ to $T_\sigma G$, and is therefore surjective onto $T_\sigma G$.

Thus, the multiplication map defines a smooth map whose differential is everywhere surjective, and is thus a submersion. □

Problem 5

Let G be a Lie group.

Part A

for $m : G \times G \to G$ the multiplication map, show that the differential $dm_{(e,e)}$ is given by

$$dm_{(e,e)}(X,Y) = X + Y$$

Proof. Let s_1 be the section $s_1 : G \to G \times G$ given by $s_1(\tau) = (\tau, e)$, and s_2 the section given by $s_2(\tau) = (e, \tau)$. Furthermore, define the isomorphism

$$\begin{aligned} \psi :& (T_eG)^2 \to T_e(G \times G) \\ & (X,Y) \mapsto ds_1|_e X + ds_2|_e Y \end{aligned}$$

We note finally that $m \circ s_1 = m \circ s_2 = id_G$, by the definition of the section maps.

Now, let's calculate $dm|_{(e,e)}(\psi(X,Y))(f)$ for some test function f.

$$\begin{aligned} dm|_{(e,e)}(\psi(X,Y))(f) &= \psi(X,Y)|_e(f \circ m) \\ &= (ds_1|_e X + ds_2|_e Y)(f \circ m) \\ &= ds_1|_e X(f \circ m) + ds_2|_e Y(f \circ m) \\ &= X(f \circ m \circ s_1) + Y(f \circ m \circ s_2) \\ &= Xf + Yf \\ &= (X+Y)f \end{aligned}$$

Thus, $dm|_{(e,e)}(X,Y) = X + Y$ as desired. □

Part B

Show that the inversion map $i : G \to G$ given by $i(\tau) = \tau^1$ has a differential $di|_e(X) = -X$.

Proof. We note first that $m(\tau, i(\tau)) = e$ for all τ by the definition of inversion. Thus, differentiating both sides (at the identity) yields

$$d(m(\bullet, i(\bullet))) = 0$$

but what is the left hand side? Lets denote the function $\tau \mapsto (\tau, i(\tau))$ as $id \times i$. Then, we can use the chain rule to evaluate (for $X \in T_eG$)

$$\begin{aligned} d(m \circ id \times i)X &= dm \circ d(id \times i)X \\ &= dm \circ (id \times di)X \\ &= dm(X, di(X)) \\ &= X + di(X) \end{aligned}$$

Now, since it must hold for all X that $d(m \circ id \times i)X = 0$, we must have that $X + di(X) = 0$, or $di(X) = -X$.

In this problem, we used the fact that the differential distributes over products of maps, which has been proven in an earlier homework, but follows from the functoriality of the differential and the universal property of products. □

Problem 6

Show that if G is a smooth manifold with a group structure so that the multiplication $m : G \times G \to G$ is smooth, then G is a Lie group.

Proof. For this problem, we only have to show that the inversion map is smooth. To do so, we first show that the map $F : G \times G \to G \times G$ given by

$$F(\tau, \sigma) = (\tau, \tau\sigma)$$

is a bijective smooth local diffeomorphism, and is thus a diffeomorphism.

To see this, we note first that F is clearly bijective, since for any $(x, y) \in G \times G$, we have a unique point $(x, x^{-1}y)$ such that $F(x, x^{-1}y) = (x, y)$. Thus, F is bijective.

Now, we need to show F is smooth. To do this, we observe that F is the composition of smooth functions. Specifically,

$$F = (id \times m) \circ (\Delta \times id)$$

Where $\Delta : G \to G \times G$ is given by $\Delta(\tau) = (\tau, \tau)$ and is clearly smooth.

Thus, since m, id, and Δ are all smooth, so are their products and compositions, and so F is smooth as well.

Now, let's show that dF is bijective at the identity. To do this, we will use the earlier decomposition of F to calculate its differential.

$$\begin{aligned} dF &= d(id \times m) \circ d(\Delta \times id) \\ &= (id \times dm) \circ (d\Delta \times id) \\ dF(X, Y) &= (X, X + Y) \end{aligned}$$

where the last equality was obtained by observing that $dm(X, Y) = X + Y$ at the identity. Since the map $(X, Y) \mapsto (X, X + Y)$ is bijective, it must be that dF is bijective at the identity.

Now, since left-multiplication is a diffeomorphism, it follows (by a very similar argument to the one in problem 4) that dF is everywhere bijective. Thus, F is a local diffeomorphism, and since F is bijective, it is also a diffeomorphism.

Now, the section $s : G \to G \times G$ given by $S(\tau) = (\tau, e)$ is smooth, and thus the composition

$$\begin{gathered} \pi_2 \circ F^{-1} \circ s \\ \tau \mapsto (\tau, e) \mapsto (\tau, \tau^{-1}) \mapsto \tau^{-1} \end{gathered}$$

is smooth as well, and is equal to i, the inversion map. Thus, i is smooth. □

Problem 7

For $\det : GL_n(\mathbb{R}) \to \mathbb{R}$, compute the differential in the following steps:

Part a

Show that

$$\partial_t|_0 \det(I + tA) = \mathrm{tr}(A)$$

for any A.

Proof. We begin by noting that the determinant can be expressed as

$$\det(I + tA) = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} (I + tA)_i^{\sigma(i)}$$

Now, if we single out the linear term in the product by multiplying by tA once and then by I the rest of the time, we end up with

$$\begin{aligned}
\mathrm{lin}(\det(I + tA)) &= \sum_{\sigma \in S_n} \epsilon(\sigma) \sum_{i=1}^{m} (\prod_{j \neq i} I_j^{\sigma(j)}) A_i^{\sigma(i)} t \\
&= \sum_{i=1}^{n} A_i^i t \\
&= t\mathrm{tr}(A)
\end{aligned}$$

and thus, the derivative at zero is $\mathrm{tr}(A)$, as desired. Here, the equality from line 1 to line 2 is made by observing that $I_j^{\sigma(j)}$ is nonzero only when $\sigma(j) = j$, or when $\sigma = id$. □

Part b

For $X \in GL_n(\mathbb{R})$ and $B \in T_X GL_n(\mathbb{R})$, show that

$$d(\det)_X(B) = \det(X)\mathrm{tr}(X^{-1}B)$$

Proof. For this problem, we will identify B with the derivative of the curve

$$\gamma(t) = X + tB$$

at zero.

Then

$$\begin{aligned}
d(\det)_X(\gamma'(0)) &= \partial_t|_0 \det(\gamma(t)) \\
&= \partial_t|_0(\det(X + tB)) \\
&= \partial_t|_0(\det(X)\det(I + tX^{-1}B)) \\
&= \det(X)\partial_t|_0(\det(I + tX^{-1}B)) \\
&= \det(X)\mathrm{tr}(x^{-1}B)
\end{aligned}$$

as desired. Here we used the identity provided in the problem, stating that $\det(X + tB) = \det(X)\det(I + tX^{-1}B)$. □

Problem 8

Show that the Hopf action of S^1 on S^{2n+1} defined by

$$z \cdot w = zw$$

for $z \in S^1$ and $w \in S^{2n+1} \subset \mathbb{C}^{n+1}$ is a smooth action with orbits that are disjoint unit circles in S^{2n+1} that union to all of S^{2n+1}.

Proof. Clearly, this is a smooth map, since it is just complex multiplication, in $\mathbb{C}^{n+1} \setminus \{0\}$, which is a Lie group, and thus has smooth multiplicative structure.

Furthermore, its orbits are disjoint, since the orbits of any group action are disjoint. Now, fixing $w \in S^{2n+1}$, we have the orbit of w as the image of the map $z \mapsto wz$ for $z \in S^1$. Since left-multiplication by w is an isometric diffeomorphism (since w is of unit length), it follows that S^1 is diffeomorphic to its image, which is the orbit of w. Thus each orbit is a copy of S^1. Furthermore, since each $w \in S^{2n+1}$ is in its own orbit, it follows that the union of all the orbits is S^{2n+1} itself, as desired. $\square$