
Homework 2

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PROBLEM 1

Recall the φ^4 Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4$$

and has an energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + g^{\mu\nu}\mathcal{L}$$

PART A

Problem. *Derive the equation of motion for φ subject to the φ^4 Lagrangian.*

To calculate the equation of motion for φ , we just have to find the stationary points of

$$S = \int d^4x \mathcal{L} = \int d^4x \left(-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4 \right)$$

That is, we find when $\delta S = 0$. To do so, we calculate

$$\begin{aligned} \delta S &= \int d^4x \delta\mathcal{L} \\ &= \int d^4x \left(-\frac{1}{2}\delta(\partial^\mu\varphi\partial_\mu\varphi) - \frac{1}{2}m^2\delta(\varphi^2) - \frac{1}{3!}g\delta(\varphi^3) - \frac{1}{4!}\lambda\delta(\varphi^4) \right) \\ &= \int d^4x \left(-\frac{1}{2}(\partial^\mu\delta\varphi\partial_\mu\varphi + \partial^\mu\varphi\partial_\mu\delta\varphi) - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi \right) \\ &= \int d^4x \left(\partial^2\varphi\delta\varphi - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi \right) \\ &= \int d^4x \left(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3 \right) \delta\varphi \end{aligned}$$

Which is zero for arbitrary variation if $(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3) = 0$. Thus, this is the equation of motion for φ .

PART B

Problem. Show that the energy-momentum tensor $T^{\mu\nu}$ satisfies $\partial_\mu T^{\mu\nu} = 0$.

This is just an exercise in direct calculation:

$$\begin{aligned}
 \partial_\mu T^{\mu\nu} &= \partial_\mu (\partial^\mu \varphi \partial^\nu \varphi) + \partial_\mu g^{\mu\nu} \mathcal{L} \\
 &= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi + \partial^\mu \varphi \partial_\mu \partial^\nu \varphi + \partial^\nu \left(-\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{3!} g \varphi^3 - \frac{1}{4!} \lambda \varphi^4 \right) \\
 &= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi + \partial^\mu \varphi \partial_\mu \partial^\nu \varphi - \frac{1}{2} \partial^\nu \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} \partial^\mu \varphi \partial^\nu \partial_\mu \varphi - m^2 \varphi \partial^\nu \varphi - \frac{1}{2} g \varphi^2 \partial^\nu \varphi - \frac{1}{3!} \lambda \varphi^3 \partial^\nu \varphi \\
 &= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi - m^2 \varphi \partial^\nu \varphi - \frac{1}{2} g \varphi^2 \partial^\nu \varphi - \frac{1}{3!} \lambda \varphi^3 \partial^\nu \varphi \\
 &= \left(\partial^2 \varphi - m^2 \varphi - \frac{1}{2} g \varphi^2 - \frac{1}{3!} \lambda \varphi^3 \right) \partial^\nu \varphi
 \end{aligned}$$

Which is clearly zero if φ follows its equation of motion.

PROBLEM 2

Consider a complex scalar field φ governed by the Lagrangian

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0$$

PART A

Problem. *Show φ obeys the Klein-Gordon equation.*

We calculate the variation in $S = \int d^4x \mathcal{L}$ directly:

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \left(-\delta(\partial^\mu \varphi^\dagger) \partial_\mu \varphi - \partial^\mu \varphi^\dagger \delta(\partial_\mu \varphi) - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\ &= \int d^4x \left(-\partial^\mu \delta \varphi^\dagger \partial_\mu \varphi - \partial^\mu \varphi^\dagger \partial_\mu \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\ &= \int d^4x \left(\delta \varphi^\dagger \partial^2 \varphi + \partial^2 \varphi^\dagger \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\ &= \int d^4x \left((\partial^2 \varphi - m^2 \varphi) \delta \varphi^\dagger + (\partial^2 \varphi^\dagger - m^2 \varphi^\dagger) \delta \varphi \right) \end{aligned}$$

Which is zero for arbitrary variations when both φ and φ^\dagger follow the Klein-Gordon equation.

PART B

Problem. Find the conjugate momenta for φ and φ^\dagger , and write down the Hamiltonian in terms of these.

We can read off the conjugate momenta easily:

$$\begin{aligned}\pi(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \\ &= \frac{\partial}{\partial \dot{\varphi}} (\partial_\mu \varphi^\dagger(x) \partial_\mu \varphi(x) - m^2 \varphi^\dagger(x) \varphi(x) + \Omega_0) \\ &= \dot{\varphi}^\dagger(x)\end{aligned}$$

and similarly

$$\pi^\dagger(x) = \dot{\varphi}(x)$$

We write down the Hamiltonian density as

$$\mathcal{H} = \pi(x) \dot{\varphi}(x) + \pi^\dagger(x) \dot{\varphi}^\dagger(x) - \mathcal{L}$$

and calculate

$$\begin{aligned}\mathcal{H} &= \pi(x) \dot{\varphi}(x) + \pi^\dagger(x) \dot{\varphi}^\dagger(x) + \left(\partial^\mu \varphi^\dagger(x) \partial_\mu \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \right) \\ &= \pi(x) \pi^\dagger(x) + \pi^\dagger(x) \pi(x) + \partial^0 \varphi^\dagger(x) \partial_0 \varphi(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \\ &= \pi(x) \pi^\dagger(x) + \pi^\dagger(x) \pi(x) - \pi(x) \pi^\dagger(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \\ &= \pi^\dagger(x) \pi(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0\end{aligned}$$

Which gives the Hamiltonian for the system as

$$H = \int d^4x \left(\pi^\dagger(x) \pi(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \right)$$

PART C

Problem. Expanding φ as

$$\varphi(x) = \int d^3k \left(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx) \right)$$

solve for expressions of $a(k)$ and $b(k)$ in terms of $\varphi(x)$, $\varphi^\dagger(x)$ and their time derivatives.

We'll evaluate the integrals

$$\begin{aligned}& \int d^3x \exp(-ikx) \varphi(x) \\ & \int d^3x \exp(-ikx) \partial_t \varphi(x) \\ & \int d^3x \exp(-ikx) \varphi^\dagger(x) \\ & \int d^3x \exp(-ikx) \partial_t \varphi^\dagger(x)\end{aligned}$$

So we first derive an expression for $\partial_t \varphi(x)$ and its conjugate.

$$\begin{aligned}\partial_t \varphi(x) &= \int d^3k \partial_t \left(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx) \right) \\ &= \int d^3k \left(a(k) (-i\omega) \exp(ikx) - b^\dagger(k) (-i\omega) \exp(-ikx) \right) \\ &= \int d^3k (-i\omega) \left(a(k) \exp(ikx) - b^\dagger(k) \exp(-ikx) \right)\end{aligned}$$

And similarly,

$$\partial_t \varphi^\dagger(x) = \int d^3k (-i\omega) \left(b(k) \exp(ikx) - a^\dagger(k) \exp(-ikx) \right)$$

Now, we can calculate those four integrals. One will be done explicitly, and the other three are done using the exact same calculation.

$$\begin{aligned} \int d^3x \exp(-ikx) \varphi(x) &= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega} \exp(-ikx) \left(a(k') \exp(ik'x) + b^\dagger(k') \exp(-ik'x) \right) \\ &= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega} \left(a(k') \exp(i(k' - k)x) + b^\dagger(k') \exp(-i(k' + k)x) \right) \\ &= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega} \left(a(k') \exp(i(k' - k)^i x_i + i(\omega' - \omega)t) \right. \\ &\quad \left. + b^\dagger(k') \exp(-i(k' + k)^i x_i - i(\omega' + \omega)t) \right) \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega} \left(a(k') (2\pi)^3 \delta(k' - k) \exp(i(\omega' - \omega)t) \right. \\ &\quad \left. + b^\dagger(k') (2\pi)^3 \delta(k' + k) \exp(-i(\omega' + \omega)t) \right) \\ &= \frac{1}{2\omega} (a(k) + b^\dagger(-k) \exp(-i(2\omega)t)) \end{aligned}$$

Following the same tactic, we find

$$\begin{aligned} \int d^3x \exp(-ikx) \varphi(x) &= \frac{1}{2\omega} (a(k) + b^\dagger(-k) \exp(-i(2\omega)t)) \\ \int d^3x \exp(-ikx) \partial_t \varphi(x) &= \frac{-i}{2} (a(k) - b^\dagger(-k) \exp(-i(2\omega)t)) \\ \int d^3x \exp(-ikx) \varphi^\dagger(x) &= \frac{1}{2\omega} (b(k) + a^\dagger(-k) \exp(-i(2\omega)t)) \\ \int d^3x \exp(-ikx) \partial_t \varphi^\dagger(x) &= \frac{-i}{2} (b(k) - a^\dagger(-k) \exp(-i(2\omega)t)) \end{aligned}$$

So,

$$a(k) = \int d^3x \exp(-ikx) (\omega \varphi(x) + i \partial_t \varphi(x))$$

and

$$b(k) = \int d^3x \exp(-ikx) (\omega \varphi^\dagger(x) + i \partial_t \varphi^\dagger(x))$$

PART D

Problem. Derive the commutation relations for $a(k)$ and $b(k)$ and their conjugates.

By conjugating, we find that

$$\begin{aligned} a^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi^\dagger(x) - i \partial_t \varphi^\dagger(x) \right) \\ b^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi(x) - i \partial_t \varphi(x) \right) \end{aligned}$$

So now we can calculate the commutators directly. Let's first rewrite the creation and annihilation operators in terms of φ and π instead:

$$\begin{aligned} a(k) &= \int d^3x \exp(-ikx) \left(\omega \varphi(x) + i \pi^\dagger(x) \right) \\ b(k) &= \int d^3x \exp(-ikx) \left(\omega \varphi^\dagger(x) + i \pi(x) \right) \\ a^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi^\dagger(x) - i \pi(x) \right) \\ b^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi(x) - i \pi^\dagger(x) \right) \end{aligned}$$

Now we can use the canonical commutation relations to derive expressions for $[a(k), a^\dagger(k')]$ and $[b(k), b^\dagger(k')]$. We calculate:

$$\begin{aligned} a(k) a^\dagger(k') &= \int d^3x d^3y \exp(-ikx) \exp(ik'y) \left(\omega \varphi(x) + i \pi^\dagger(x) \right) \left(\omega' \varphi^\dagger(y) - i \pi(y) \right) \\ &= \int d^3x d^3y \exp(-ikx) \exp(ik'y) \left(\omega \varphi(x) \omega' \varphi^\dagger(y) - \omega \varphi(x) i \pi(y) + i \pi^\dagger(x) \omega' \varphi^\dagger(y) - i \pi^\dagger(x) i \pi(y) \right) \end{aligned}$$

At this point, we invoke the rules

$$\begin{aligned} [\varphi(x), \varphi^\dagger(y)] &= [\pi(x), \pi^\dagger(y)] = 0 && \text{Independence of fields} \\ [\varphi(x), \varphi(y)] &= [\pi(x), \pi(y)] = 0 && \text{Canonical commutation relation} \\ [\varphi(x), \pi(y)] &= [\varphi^\dagger(x), \pi^\dagger(y)] = i \delta(x - y) && \text{Canonical commutation relation} \end{aligned}$$

to commute the φ and π fields (and their conjugates) past each other. Thus, we find that

$$\begin{aligned} a(k) a^\dagger(k') &= \int d^3x d^3y \exp(-ikx) \exp(ik'y) \left(\omega \varphi(x) \omega' \varphi^\dagger(y) - \omega \varphi(x) i \pi(y) + i \pi^\dagger(x) \omega' \varphi^\dagger(y) - i \pi^\dagger(x) i \pi(y) \right) \\ &= \int d^3x d^3y \exp(-ikx) \exp(ik'y) \\ &\quad (\omega' \varphi^\dagger(y) \omega \varphi(x) - (i \pi(y) \omega \varphi(x) + i \omega (i \delta(x - y))) + (\omega' \varphi^\dagger(y) i \pi^\dagger(x) - i \omega' (i \delta(x - y))) - i \pi(y) i \pi^\dagger(x)) \\ &= a^\dagger(k') a(k) + \int d^3x d^3y \exp(-ikx) \exp(ik'y) (\omega \delta(x - y) + \omega' \delta(x - y)) \\ &= a^\dagger(k') a(k) + \int d^3x \exp(-ikx) \exp(ik'x) (\omega + \omega') \\ &= a^\dagger(k') a(k) + (2\pi)^3 2\omega \delta(k - k') \end{aligned}$$

and so

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega \delta(k - k')$$

Carrying out the exact same calculation for $[b(k), b^\dagger(k)]$, we find that (by interchanging $\varphi \leftrightarrow \varphi^\dagger$ and $\pi \leftrightarrow \pi^\dagger$)

$$[b(k), b^\dagger(k')] = (2\pi)^3 2\omega \delta(k - k')$$

as well (since the commutators of φ, π and their conjugates are identical).

PART E

Problem. Express the Hamiltonian in terms of $a, a^\dagger, b, b^\dagger$.

We just need to substitute the expressions for the creation and annihilation operators into the expression for H . Namely, we have the following substitutions:

$$\begin{aligned}\varphi(x) &= \int \tilde{d}k \left(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx) \right) \\ \varphi^\dagger(x) &= \int \tilde{d}k \left(b(k) \exp(ikx) + a^\dagger(k) \exp(-ikx) \right) \\ \pi(x) &= \int \tilde{d}k (-i\omega) \left(b(k) \exp(ikx) - a^\dagger(k) \exp(-ikx) \right) \\ \pi^\dagger(x) &= \int \tilde{d}k (-i\omega) \left(a(k) \exp(ikx) - b^\dagger(k) \exp(-ikx) \right) \\ \partial^i \varphi(x) &= \int \tilde{d}k (ik^i) \left(b(k) \exp(ikx) - a^\dagger(k) \exp(-ikx) \right) \\ \partial^i \varphi^\dagger(x) &= \int \tilde{d}k (ik^i) \left(a(k) \exp(ikx) - b^\dagger(k) \exp(-ikx) \right)\end{aligned}$$

We can now express H as

$$\begin{aligned}H &= \int d^3x \mathcal{H} \\ &= \int d^3x \left(\pi^\dagger(x) \pi(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \right) \\ &= \int d^3x \tilde{d}k \tilde{d}k' \\ &\quad \left((-i\omega) \left(a(k) \exp(ikx) - b^\dagger(k) \exp(-ikx) \right) (-i\omega') \left(b(k') \exp(ik'x) - a^\dagger(k') \exp(-ik'x) \right) \right. \\ &\quad + (ik^i) \left(a(k) \exp(ikx) - b^\dagger(k) \exp(-ikx) \right) (ik'_i) \left(b(k') \exp(ik'x) - a^\dagger(k') \exp(-ik'x) \right) \\ &\quad \left. + m^2 \left(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx) \right) \left(b(k') \exp(ik'x) + a^\dagger(k') \exp(-ik'x) \right) \right)\end{aligned}$$

We collect like terms to get

$$\begin{aligned}H &= \int d^3x \tilde{d}k \tilde{d}k' \\ &\quad \left(\exp(i(k+k')x) \left((-i\omega)a(k)(-i\omega')b(k') + (ik^i)a(k)(ik'_i)b(k') + m^2a(k)b(k') \right) \right. \\ &\quad + \exp(i(k-k')x) \left((-i\omega)a(k)(i\omega')a^\dagger(k') + (ik^i)a(k)(-ik'_i)a^\dagger(k') + m^2a(k)a^\dagger(k') \right) \\ &\quad + \exp(i(k'-k)x) \left((i\omega)b^\dagger(k)(-i\omega')b(k') + (-ik^i)b^\dagger(k)(ik'_i)b(k') + m^2b^\dagger(k)b(k') \right) \\ &\quad \left. + \exp(-i(k+k')x) \left((i\omega)b^\dagger(k)(i\omega')a^\dagger(k') + (-ik^i)b^\dagger(k)(-ik'_i)a^\dagger(k') + m^2b^\dagger(k)a^\dagger(k') \right) \right)\end{aligned}$$

Integrating out the d^3x yields

$$\begin{aligned}H &= \int \tilde{d}k \tilde{d}k' \\ &\quad \left((2\pi)^3 (\delta(k+k') \exp(i(\omega+\omega')t) a(k)b(k') (-\omega\omega' - k^i k'_i + m^2) \right. \\ &\quad + \delta(k-k') \exp(i(\omega-\omega')t) a(k)a^\dagger(k') (\omega\omega' + k^i k'_i + m^2) \\ &\quad + \delta(k'-k) \exp(i(\omega'-\omega)t) b^\dagger(k)b(k') (\omega\omega' + k^i k'_i + m^2) \\ &\quad \left. + \delta(-k-k') \exp(-i(\omega+\omega')t) b^\dagger(k)a^\dagger(k') (-\omega\omega' - k^i k'_i + m^2) \right)\end{aligned}$$

We then integrate out dk' yielding

$$\begin{aligned}
H = \int \tilde{dk} \frac{1}{2\omega} & \\
& \exp(2i\omega t) a(k) b(-k) (-\omega^2 + k^2 + m^2) \\
& + a(k) a^\dagger(k) (\omega^2 + k^2 + m^2) \\
& + b^\dagger(k) b(k) (\omega^2 + k^2 + m^2) \\
& + \exp(-2i\omega t) b^\dagger(k) a^\dagger(-k) (-\omega^2 + k^2 + m^2)
\end{aligned}$$

Since $\omega^2 = k^2 + m^2$, this simplifies greatly to

$$H = \int \tilde{dk} \omega \left(a(k) a^\dagger(k) + b^\dagger(k) b(k) \right)$$

We've omitted the offset factor Ω_0 , but it can be safely added in to the final result. Of course, if we want the ground state to have zero energy, we should move the a operator to the front, yielding

$$H = \int \tilde{dk} \omega \left(a^\dagger(k) a(k) + b^\dagger(k) b(k) + (2\pi)^3 2\omega \delta(0) \right)$$

and if we set Ω_0 to cancel with this value, we get

$$H = \int \tilde{dk} \omega \left(a^\dagger(k) a(k) + b^\dagger(k) b(k) \right)$$

as desired.

PROBLEM 3

Problem. From the previous problem, we have a conserved current

$$J^\mu = -i\varphi^\dagger \partial^\mu \varphi + i\varphi \partial^\mu \varphi^\dagger$$

Express the charge

$$Q = \int d^3x J^0$$

in terms of a, b and their conjugates.

We use the substitutions

$$\begin{aligned}\varphi(x) &= \int \tilde{d}k \left(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx) \right) \\ \varphi^\dagger(x) &= \int \tilde{d}k \left(b(k) \exp(ikx) + a^\dagger(k) \exp(-ikx) \right) \\ \partial^\mu \varphi(x) &= \int \tilde{d}k (ik^\mu) \left(a(k) \exp(ikx) - b^\dagger(k) \exp(-ikx) \right) \\ \partial^\mu \varphi^\dagger(x) &= \int \tilde{d}k (ik^\mu) \left(b(k) \exp(ikx) - a^\dagger(k) \exp(-ikx) \right)\end{aligned}$$

and calculate directly

$$\begin{aligned}Q &= \int d^3x J^0 \\ &= \int d^3x \tilde{d}k \tilde{d}k' \\ &\quad ((-i)(b(k) \exp(ikx) + a^\dagger(k) \exp(-ikx))(i\omega')(a(k') \exp(ik'x) - b^\dagger(k') \exp(-ik'x)) \\ &\quad + (i)(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx))(i\omega')(b(k') \exp(ik'x) - a^\dagger(k') \exp(-ik'x))) \\ &= \int d^3x \tilde{d}k \tilde{d}k' \\ &\quad (\omega'(b(k) \exp(ikx) + a^\dagger(k) \exp(-ikx))(a(k') \exp(ik'x) - b^\dagger(k') \exp(-ik'x)) \\ &\quad - (\omega')(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx))(b(k') \exp(ik'x) - a^\dagger(k') \exp(-ik'x))) \\ &= \int d^3x \tilde{d}k \tilde{d}k' \\ &\quad \omega'(\exp(i(k+k')x)(b(k)a(k') - a(k)b(k')) \\ &\quad - \exp(i(k-k')x)(b(k)b^\dagger(k') - a(k)a^\dagger(k')) \\ &\quad + \exp(i(k'-k)x)(a^\dagger(k)a(k') - b^\dagger(k)b(k')) \\ &\quad - \exp(-i(k+k')x)(a^\dagger(k)b^\dagger(k') - b^\dagger(k)a^\dagger(k'))))\end{aligned}$$

Integrating out d^3x yields

$$\begin{aligned}Q &= \int \tilde{d}k \tilde{d}k' \\ &\quad \omega'(2\pi)^3 (\delta(k+k') \exp(i(\omega+\omega')t)(b(k)a(k') - a(k)b(k')) \\ &\quad - \delta(k-k') \exp(i(\omega-\omega')t)(b(k)b^\dagger(k') - a(k)a^\dagger(k')) \\ &\quad + \delta(k'-k) \exp(i(\omega'-\omega)t)(a^\dagger(k)a(k') - b^\dagger(k)b(k')) \\ &\quad - \delta(-k-k') \exp(-i(\omega+\omega')t)(a^\dagger(k)b^\dagger(k') - b^\dagger(k)a^\dagger(k'))))\end{aligned}$$

and finally integrating out d^3k we get

$$\begin{aligned}
Q &= \int \tilde{d}k \\
&\frac{1}{2}(\exp(i(2\omega)t)(b(k)a(-k) - a(k)b(-k)) \\
&\quad - (b(k)b^\dagger(k) - a(k)a^\dagger(k)) \\
&\quad + (a^\dagger(k)a(k) - b^\dagger(k)b(k)) \\
&\quad - \exp(-i(2\omega)t)(a^\dagger(k)b^\dagger(-k) - b^\dagger(k)a^\dagger(-k))) \\
&= \int \tilde{d}k \frac{1}{2}((a(k)a^\dagger(k) - b(k)b^\dagger(k)) + (a^\dagger(k)a(k) - b^\dagger(k)b(k))) \\
&= \int \tilde{d}k ((a^\dagger(k)a(k) - b^\dagger(k)b(k)))
\end{aligned}$$

as desired. We note that this can be expressed with the number operators N_a and N_b as

$$Q = N_a - N_b$$

and so a particles and b particles have opposite equal charges.

PROBLEM 4

Problem. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \frac{1}{2}k\partial^\nu A^\mu \partial_\mu A_\nu - \frac{1}{2}m^2 A^\nu A_\nu + J^\nu A_\nu$$

PART A

Problem. Compute the variation of the action, and find the equation of motion for A^ν .

We'll compute the variation of the action directly, noting that

$$\delta(\partial^\mu A^\nu) = \partial^\mu \delta A^\nu$$

We find

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \left(-\frac{1}{2}\delta(\partial^\mu A^\nu \partial_\mu A_\nu) + \frac{1}{2}k\delta(\partial^\nu A^\mu \partial_\mu A_\nu) - \frac{1}{2}m^2\delta(A^\nu A_\nu) + J^\nu \delta A_\nu \right) \\ &= \int d^4x \left(-\frac{1}{2}\partial^\mu \delta A^\nu \partial_\mu A_\nu - \frac{1}{2}\partial^\mu A^\nu \partial_\mu \delta A_\nu \right. \\ &\quad \left. + \frac{1}{2}k\partial^\nu \delta A^\mu \partial_\mu A_\nu + \frac{1}{2}k\partial^\nu A^\mu \partial_\mu \delta A_\nu \right. \\ &\quad \left. - \frac{1}{2}m^2\delta A^\nu A_\nu - \frac{1}{2}m^2 A^\nu \delta A_\nu + J^\nu \delta A_\nu \right) \end{aligned}$$

The first two terms combine if we raise/lower indices on the second term, as well as the fifth and sixth terms. We do so, and we also interchange the indices on the third term to make $\delta A^\mu \rightarrow \delta A^\nu$.

$$\delta S = \int d^4x \left(-\partial^\mu \delta A^\nu \partial_\mu A_\nu + \frac{1}{2}k\partial^\mu \delta A^\nu \partial_\nu A_\mu + \frac{1}{2}k\partial^\nu A^\mu \partial_\mu \delta A_\nu - m^2\delta A^\nu A_\nu + J^\nu \delta A_\nu \right)$$

Now, if we raise/lower indices on the third term, we get

$$\delta S = \int d^4x \left(-\partial^\mu \delta A^\nu \partial_\mu A_\nu + k\partial^\mu \delta A^\nu \partial_\nu A_\mu - m^2\delta A^\nu A_\nu + J^\nu \delta A_\nu \right)$$

Integrating by parts yields

$$\begin{aligned} \delta S &= \int d^4x \left(\delta A^\nu \partial^\mu \partial_\mu A_\nu - k\delta A^\nu \partial^\mu \partial_\nu A_\mu - m^2\delta A^\nu A_\nu + J^\nu \delta A_\nu \right) \\ &= \int d^4x \left(\partial^2 A_\nu - k\partial_\nu \partial^\mu A_\mu - m^2 A_\nu + J_\nu \right) \delta A^\nu \end{aligned}$$

Which holds for arbitrary variations if the term in the parentheses is always zero. Thus, we have

$$-\partial^2 A_\nu + k\partial_\nu \partial^\mu A_\mu + m^2 A_\nu = J_\nu$$

Multiplying each side by $g^{\nu\rho}$ we get

$$\begin{aligned} g^{\nu\rho}(-\partial^2 A_\nu + m^2 A_\nu) + k g^{\nu\rho} \partial_\nu \partial^\mu A_\mu &= g^{\nu\rho} J_\nu \\ g^{\nu\rho}(-\partial^2 A_\nu + m^2 A_\nu) + k g^{\mu\rho} \partial_\mu \partial^\nu A_\nu &= J^\rho \\ [g^{\nu\rho}(-\partial^2 + m^2) + k\partial^\rho \partial^\nu] A_\nu &= J^\rho \end{aligned}$$

as desired.

PART B

Problem. Find the equation of motion for $\partial_\mu A^\mu$.

We follow the hint and act on the equation of motion for A with (changing μ with ρ) ∂_μ

$$\begin{aligned}\partial_\mu (g^{\nu\mu}(-\partial^2 + m^2)A_\nu + k\partial^\mu\partial^\nu A_\nu) &= \partial_\mu J^\mu \\ (-\partial^2 + m^2)\partial^\nu A_\nu + k\partial_\mu\partial^\mu\partial^\nu A_\nu &= \partial_\mu J^\mu\end{aligned}$$

Where the last equality was obtained by observing that derivatives commute with each other and the metric, and $g^{\mu\nu}\partial_\mu = \partial^\nu$. Thus, substituting $\varphi = \partial^\nu A_\nu$, we get

$$(-\partial^2 + m^2)\varphi + k\partial^2\varphi = \partial_\mu J^\mu$$

and if we set $k = 1$, we get the equation

$$m^2\varphi = \partial_\mu J^\mu$$

as desired.