Problem Set 6

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PROBLEM 1

Prove that in a normed space, a sequence can have at least one strong limit. Prove that a strongly convergent sequence is Cauchy.

Proof. To begin with, we note that every normed space is necessarily Hausdorff, and that every Hausdorff space has the property that sequences have at most one limit.

To see that a normed space is Hausdorff, consider an arbitrary normed space X, and two distinct points x, y. Now, by the definition of a norm, it must be that $||x - y|| > 0$. So, let

$$||x - y|| = \varepsilon$$

Then, the open sets $B(x, \frac{\varepsilon}{2})$ and $B(Y, \frac{\varepsilon}{2})$ separate x and y. Thus, normed spaces are Hausdorff.

Furthermore, it is clear that a sequence converges to at most one limit in Hausdorff spaces. To see this, we recall the topological definition of convergence, which states that a sequence (x_n) converges to x if and only if every neighborhood of x eventually contains the sequence.

Suppose for a contradiction that (x_n) had two limits, x, and y. Since x and y are distinct points in a Hausdorff space, there must be some neighborhood V_x of x and V_y of y such that $V_x \cap V_y = \emptyset$.

However, this contradicts the sequence converging to both x and y, since for (x_n) to converge to x, it must eventually be in V_x, which means it is eventually outside V_y, and thus cannot converge to y as well.

Now, let (x_n) be a convergent sequence in a normed space X, and let x be its limit. We will show that this sequence is Cauchy.

To do this, let $\varepsilon > 0$ be arbitrary. Now, we know that there exists some N such that for any $n > N$,

$$||x_n - x|| < \frac{\varepsilon}{2}$$

Furthermore, for any $m, n > N$ we have that

$$\begin{aligned}
||x_m - x_n|| &= ||x_m - x + x - x_n|| \\
&\leq ||x_m - x|| + ||x_n - x|| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

and thus the sequence is Cauchy. □

PROBLEM 2

Show that the closure of the ball $B(a,r)$ is the closed ball $\overline{B}(a,r) = \{x \mid |x-a| \leq r\}$

Proof. Suppose x is in $\overline{B}(a,r)$. Then, consider the sequence

$$(x_n) = (x-a)(1-\frac{1}{n}) + a$$

Clearly, this sequence converges to x, and each term is in $B(a,r)$. To see this, we observe that

$$\begin{aligned} ||x_n - a|| &= ||(x-a)(1-\frac{1}{n}) + a - a|| \\ &= ||x-a||(1-\frac{1}{n} \\ &\leq r(1-\frac{1}{n} \\ &\leq r \end{aligned}$$

Thus, each point in $\overline{B}(a,r)$ is a limit point of $B(a,r)$, and since $\overline{B}(a,r)$ is closed, it follows that it is the closure of $B(a,r)$ (since the closure of $B(a,r)$ is the smallest closed set containing it.) $\square$

Problem 3

Prove that for a linear operator A between normed spaces V, W, the following are equivalent:

1. A is continuous at every $p \in V$.
2. A is continuous at 0_V.
3. A is bounded in the sense that
$$\sup_{||x||=1} ||Ax|| < \infty$$
4. A is bounded in the sense that for some $M > 0$,
$$||Ax|| \leq M||x||$$
for all $x \in V$.

Furthermore, prove that the set $\mathscr{B}(V, W)$ of the bounded linear operators from V to W is a normed space.

Proof. $(1 \implies 2)$ This follows immediately from the statement of 1.

$(2 \implies 3)$ Suppose A is continuous at zero. Then, choose $\varepsilon = 1$. We must have some $\delta > 0$ such that $||x|| \leq \delta$ implies $||Ax|| < \varepsilon = 1$. Then, we have

$$||Ax|| = ||A\left(\frac{||x||}{\delta}\frac{x\delta}{||x||}\right)|| = \frac{||x||}{\delta}||A\left(\frac{x\delta}{||x||}\right)|| \leq \frac{||x||}{\delta}(1)$$

and thus A is bounded by $\frac{1}{\delta}$. In particular,

$$\sup \frac{||Ax||}{||x||} \leq \frac{\frac{||x||}{\delta}}{||x||} = \frac{1}{\delta} < \infty$$

as desired.

$(3 \implies 4)$ Suppose A is bounded in the sense of statement 3. In particular, let

$$\sup_{x \neq 0} \frac{||Ax||}{||x||} = M$$

for some positive M. Then, for all $x \in V$ with $x \neq 0$,

$$\begin{aligned} \frac{||Ax||}{||x||} &\leq M \\ \implies ||Ax|| &\leq M||x|| \end{aligned}$$

as desired. Note that if $x = 0$, then $Ax = 0$ as well and the statement is vacuously true.

$(4 \implies 1)$ Suppose A is bounded in the sense that there is some $M > 0$ such that

$$||Ax|| \leq M||x||$$

for all x in V. Now, let $\varepsilon > 0$ be arbitrary. Then, the bound $\delta = \frac{\varepsilon}{M}$ on $||x - p||$ forces

$$\begin{aligned} ||Ax - Ap|| = ||A(x - p)|| &\leq M||x - p|| \\ &\leq M\frac{\varepsilon}{M} = \varepsilon \end{aligned}$$

Thus, A is continuous at p.

Finally, we will prove that the space $\mathscr{B}(V, W)$ is a normed vector space under the operator norm $||A|| = \sup_{||x||=1} ||Ax||$.

To see this, we need to check that the norm is positive definite and satisfies the triangle inequality.

The norm is clearly positive definite, since it is taken as the sup of a set of nonnegative numbers, and if $||A|| = 0$, then (by the definition of operator norm from statement 4)

$$||Ax|| \leq (0)||x|| \ \forall x$$

which forces $||Ax|| = 0$ for all $x \in V$. Since the vector norm is positive definite, it follows that $Ax = 0$ for all $x \in V$, and thus $A = 0$.

Now, we need to show that

$$||A + B|| \leq ||A|| + ||B||$$

To see this, let $||A|| = M_A$ and $||B|| = M_B$. Then, we have that

$$\begin{aligned} ||(A + B)(x)|| &= ||Ax + Bx|| \\ &\leq ||Ax|| + ||Bx|| \end{aligned}$$

Since this holds for all x, it follows that

$$||A + B|| \leq ||A|| + ||B||$$

as desired. Thus, $\mathscr{B}(V, W)$ is a normed vector space. □

Problem 4

Suppose that V, W are normed vector spaces, and that W is Banach. Prove that $\mathscr{B}(V, W)$ is Banach as well.

Proof. To begin with, let (A_n) be a Cauchy sequence in $\mathscr{B}(V, W)$. In particular, for any $\varepsilon > 0$, we can find an $N > 0$ such that for all $n, m > N$,

$$||A_n - A_m|| < \varepsilon$$

So, let $x \in V$, and consider the sequence $(A_n x)$. We will show this sequence is Cauchy, and by completeness of W, has a limit.

To see how this sequence is Cauchy, let $\varepsilon > 0$, and choose an N such that for all $n, m > N$, $||A_n - A_m|| < \frac{\varepsilon}{||x||}$. Then, we have

$$\begin{aligned} ||(A_n - A_m)(x)|| &= ||A_n x - A_m x|| \\ &< \frac{\varepsilon}{||x||} ||x|| \\ &= \varepsilon \end{aligned}$$

Thus, the sequence is Cauchy, and has a limit. Since this can be done for every $x \in V$, we can define $Ax = \lim_{n \to \infty} A_n x$.

Clearly, A is linear, since the limit respects scalar multiplication and addition. Furthermore, it can be seen that A is bounded. To see this, we observe that

$$\begin{aligned} ||Ax|| &= || \lim_{n \to \infty} A_n x|| \\ &= \lim_{n \to \infty} ||A_n x|| \\ &\leq \lim_{n \to \infty} ||A_n|| ||x|| \end{aligned}$$

Now, $\lim_{n \to \infty} ||A_n||$ is a sequence of real numbers, and can be seen to be Cauchy, since

$$|||A_n|| - ||A_m||| \leq ||A_n - A_m||$$

and since A_n is Cauchy in the operator norm, $||A_n - A_m||$ can be bounded, and thus $||A_n||$ is a Cauchy sequence of real numbers, which converges to some $K > 0$.

So, we have

$$\begin{aligned} ||Ax|| &\leq \lim_{n \to \infty} ||A_n|| ||x|| \\ &= K ||x|| \end{aligned}$$

and thus A is bounded.

Now, all we have to show is that $\lim A_n = A$. To do this, we fix $\varepsilon > 0$, and consider

$$||A - A_n|| = \sup_{||x||=1} ||(A - A_n)x||$$

Now, we will show that $||(A - A_n)x||$ is bounded above by ε when n is sufficiently large. To see this, let $\varepsilon > 0$, and choose N large enough so that $\forall n, m > N$, $||A_m - A_n|| < \frac{\varepsilon}{2}$. Now, we have

$$\begin{aligned} ||(A - A_n)x|| &\leq ||(A - A_m)x|| + ||(A_m - A_n)x|| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

where m was chosen via pointwise convergence to make $||(A - A_m)x|| < \frac{\varepsilon}{2}$.

Thus, $\lim_{n \to \infty} A_n = A$ which is in $\mathscr{B}(V, W)$. □

Problem 5

Show that a subspace W of a Banach space V is Banach if and only if its closed.

Proof. ($\Longrightarrow$) Suppose W is a Banach subspace of V. In particular, this means that W contains its limits, since any convergent sequence in W is Cauchy, and every Cauchy sequence in W converges in W. Then, it follows immediately that W is closed.

($\Longleftarrow$) Suppose W is a closed subspace of V. In particular, this means that every convergent sequence in W converges in W. So, for any Cauchy sequence in W, that sequence converges (since V is Banach), and since W is closed, it must converge in W. Thus, W is a Banach space. $\square$

Problem 6

Prove that ℓ^1, ℓ^∞, c_0 are all Banach spaces.

Proof. We know by Riesz-Fischer that any L^p space is a Banach space. Thus, $\ell^1 = L^1(\mathbb{N}, \mu_c)$ and $\ell^\infty = L^\infty(\mathbb{N}, \mu_c)$ are Banach spaces.

Now, $c_0 \subset \ell^\infty$, so if we show that c_0 is closed in ℓ^∞, then c_0 is Banach.

So, let (f_n) be a sequence of points in c_0 that converge in ℓ^∞, with $f = \lim f_n$.

In particular, we can choose $N_1 > 0$ such that for all $i > N_1$, $\sup |f_i(n) - f(n)| < \frac{\varepsilon}{2}$. Furthermore, we can choose $N_2 > 0$ such that for all $m > N_2$, $|f_{N_1}(m)| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
|f(m)| &= |f(m) - f_{N_1}(m) + f_{N_1}(m)| \\
&\leq |f(m) - f_{N_1}(m)| + |f_{N_1}(m)| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus, f decays to zero, and therefore $f \in c_0$ as desired. □

Problem 7

For K a compact subset of $\mathbb{R}^n$, prove that $C(K)$ under the sup norm is a Banach space.

Proof. We note first that the sum of two $C(K)$ functions is again $C(K)$, and that the scalar multiple of a $C(K)$ function is still $C(K)$. This follows almost immediately from basic undergraduate Real analysis arguments.

Since $C(K)$ is a subset of $L^\infty(K)$ (continuous functions on a compact domain are bounded), it follows that the sup norm, which coincides with the L^∞ norm, is a norm on $C(K)$.

All that is left to show is completeness. For this, we will simply show that the L^∞ limit of $C(K)$ functions is again a $C(K)$ function.

This is clear, however, since (by basic undergraduate Real analysis) the uniform limit of continuous functions is continuous. Since the sup norm defines uniform convergence, if a sequence f_n of $C(K)$ functions converges in L^∞ to f, it follows that f_n converges uniformly to f, and thus if $f_n \in C(K)$, then $f \in C(K)$ as well.

Thus, $C(K)$ contains its limits, and is a closed subspace of $L^\infty(K)$. So, $C(K)$ is Banach. $\square$

Problem 8

Let S be the set of all simple functions in the L^1 norm $||\cdot||_1$. Prove that S is a normed space. Prove that $\overline{S} = L^1$. Prove that for $\mathbb{R}^n$ with the Lebesgue measure, S is not complete. Give an example of a measure space for which S is complete.

Proof. We first show that S is a normed space. Since the L^1 norm already satisfies the axioms of a norm, it suffices to show that S is a vector subspace of L^1. This is clear, since scaling a simple function by a constant yields another simple function, and addition of two simple functions is again a simple function.

Now, we wish to show that $\overline{S} = L^1$. To see this, we will show that every $f \in L^1$ is the limit of simple functions with respect to the L^1 norm.

So, let $f \in L^1$, and let f^+, f^- be its positive and negative parts. Now, a basic theorem tells us that $f^\pm = \lim \phi_n^\pm$ for a sequence of monotonically increasing simple functions ϕ_n, and furthermore

$$\lim_{n\to\infty} \int \phi_n^\pm = \int f^\pm$$

Thus, we have that

$$\begin{aligned} f &= f^+ - f^- \\ &= \lim_{n\to\infty} (\phi_n^+ - \phi_n^-) \end{aligned}$$

where the limit is taken pointwise. So, let $\phi_n = \phi^+ - \phi^-$. Then consider

$$\begin{aligned} \int (|f - \phi_n|) &= \int (|f^+ - f^- + \phi^+ - \phi^-|) \\ &\leq \int |f^+ - \phi_n^+| - \int |f^- - \phi_n^-| \\ &= \int (f^+ - \phi_n^+) - \int (f^- - \phi_n^-) \\ &\to 0 - 0 = 0 \end{aligned}$$

where the second to last line was obtained by observing that $f^\pm \geq \phi_n^\pm \; \forall n$, and the last line was obtained by the monotone convergence theorem on the monotonic sequences $\phi_n^\pm$.

Thus, $||f - \phi_n||_1 \to 0$, and thus f is the limit of simple functions. Therefore, each $f \in L^1$ is the limit of simple functions, and since L^1 is complete, it follows that $\overline{S} = L^1$.

However, for $L^1(\mathbb{R}^n)$, $S \neq L^1$, since the function $f(x) = \exp(-|x|^2)$ (for $x \in \mathbb{R}^n$)is in L^1, but is not a simple function. Thus, $\overline{S} \neq S$, and S is not complete.

For an example of a measure space for which $S = L^1$, consider the one-point space $\Omega = \{\bullet\}$ with the counting measure μ_c. Every function from Ω to $\mathbb{C}$ is just a choice of scalar in $\mathbb{C}$, and is a simple function. Thus, $L^1(\Omega, \mu_c) \subset \mathrm{Hom}(\Omega, C) = S$, and thus $S = L^1$, and S is complete. $\square$

Problem 9

Find the completion of $C_0(\mathbb{R})$.

Proof. We will show that

$$\overline{C_0(\mathbb{R})} = S \stackrel{\text{def}}{=} \{f \in C(\mathbb{R}) \mid \lim_{|x|\to\infty} f(x) = 0\}$$

To do so, we show that every $f \in S$ is the limit of functions $f_n \in C_0$. So, let $f \in S$, and consider the sequence of functions

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in [-n, n] \\ f(n)(n+1-x), & \text{if } x \in [n, n+1] \\ f(-n)(x+n+1), & \text{if} x \in [-n-1, -n] \\ 0, & \text{else} \end{cases}$$

which agree with f on the interval $[-n, n]$, and on $[n, n+1]$ and $[-n-1, n]$ decrease linearly to zero.

Clearly, each $f_n \in C_0$, since their support is contained in $[-n-1, n+1]$. Thus, it suffices to show that $f_n(x)$ converges to $f(x)$ uniformly.

So, let $\varepsilon > 0$, and choose n such that $|f(x)| < \frac{\varepsilon}{2}$ for all $|x| > n$.

Now, consider

$$||f - f_n||_\infty = \sup_x |f_n(x) - f(x)|$$

In particular, inside $[-n, n]$, the functions agree and $|f_n(x) - f(x)| = 0$. Outside $[-n, n]$ we know that $|f_n(x)| \le \max(|f(n)|, |f(-n)|)$ since f_n decays monotonically to zero from $f(n)$ and $f(-n)$. By our choice of n, we have for $x \notin [-n, n]$, $|f_n(x)| \le \frac{\varepsilon}{2}$. To conclude, we have

$$\begin{aligned} |f_n(x) - f(x)| &\le |f_n(x)| + |f(x)| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

Thus, $\sup_x |f_n(x) - f(x)|$ is bounded by ε, and must decay to zero. So, f_n converges to f.

So, we know that $S \subset \overline{C_0}$. Now, we must show that S itself is complete.

To see this, we observe the basic fact that the uniform limit of functions that decay to zero at infinity is also a function that decays to zero at infinity.

So, let f_n be a Cauchy sequence of functions in S that converge to some function f. We just need to show that $f \in S$. This is clear, however, since

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\le |f(x) - f_n(x)| + |f_n(x)| \end{aligned}$$

Which goes to zero as $|x| \to \infty$ since $|f(x) - f_n(x)| \to 0$ as $n \to \infty$, and $|f_n(x)| \to 0$ as $|x| \to \infty$.

Thus S is complete, and so S is the completion of C_0. $\square$

Problem 10

Show that ℓ^1 is not complete in the ℓ^∞ norm.

Proof. We will show that the sequence $(x_m)_k = (\frac{1}{m^{1+\frac{1}{k}}})$ converges to the function $x_m = \frac{1}{m}$, which is not in ℓ^1, even though each term in the sequence is in ℓ^1.

To show convergence, we wish to show that the functions

$$f_k(n) = \frac{1}{n^{1+\frac{1}{k}}}$$

converges uniformly to $f(n) = \frac{1}{n}$.

To do so, we will consider instead the extended functions

$$f_k(x) = x^{1+\frac{1}{k}},\ x \in [0,1]$$

which clearly converges pointwise to $f(x) = x$. Now, since f is defined on a compact domain, it must also uniformly converge to $f(x) = x$. Thus, the restriction $f_k(x)|_{\{\frac{1}{n}\}}$ also converges uniformly to $f(n) = \frac{1}{n}$.

Thus,the sequence of sequences $(x_m)_k$ converges uniformly (in ℓ^∞) to (x_m) as desired. Furthermore, since each $(x_m)_k$ is in ℓ^1, but (x_m) is not, it follows that ℓ^1 is not complete in the ℓ^∞ norm. $\square$