«CLASS»

«TITLE»

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Problem 1

Show that hyperbolic space H^n is complete.

Proof. We will first show that H^n is homogeneous, and then appeal to the next problem to conclude H^n is complete.

To see that H^n is homogeneous, we consider two families of isometries. For simplicity, we will write points in H^n as (x, y) with $x \in \mathbb{R}^{n-1}$ the first $n-1$ coordinates, and $y \in \mathbb{R}$ the last coordinate. The first isometry we consider is

$$\begin{aligned} T_a : H^n &\to H^n \\ (x, y) &\mapsto (x + a, y) \end{aligned}$$

for any $a \in \mathbb{R}^{n-1}$. To see this is an isometry, we just need to compute dT_a and show it preserves the metric. So, let $v \in T_pH^n$ for some $p \in H^n$, $p = (x_p, y_p)$, and take $\gamma(t) = p + vt = (x_p + v_xt, y_p + v_yt)$ a curve in H^n. Note that $\gamma'(0) = v$. Now, we have that

$$\begin{aligned} dT_a(v) &= dT_a(\gamma'(0)) \\ &= \partial_t T_a(\gamma(t))|_{t=0} \\ &= \partial_t(x_p + v_xt + a, y_p + v_yt)|_{t=0} \\ &= (v_x, v_y) = v \end{aligned}$$

Thus, $dT_a(v) = v$. Furthermore, since the metric at $(x + a, y)$ is the same as at (x, y) (since the scaling factor only depends on y) we have that for $u, v \in T_pM$,

$$g(u, v)_{(x,y)} = g(dT_au, dT_av)_{(x+a,y)}$$

and thus T_a is an isometry (I suppose you'd have to check that T_a is a diffeomorphism as well, but this is obvious. Clearly T_a is smooth, and it has a smooth inverse T_{-a}).

Secondly, we consider the isometry

$$\begin{aligned} M_\alpha : H^n &\to H^n \\ (x, y) &\mapsto (\alpha x, \alpha y) \end{aligned}$$

for $\alpha > 0$. This maps H^n into H^n, since it keeps the y coordinate positive. Furthermore, it is a diffeomorphism (it is clearly smooth, and $M_{\frac{1}{\alpha}}$ acts as an inverse). I also claim it is an isometry. Again letting $\gamma = (x_p + v_x, y_p + v_y)$ for $(v_x, v_y) \in T_{(x,y)}H^n$ we note that

$$\begin{aligned}
dM_\alpha(v) &= dM_\alpha(\gamma'(0)) \\
&= \partial_t M_\alpha(\gamma(t))|_{t=0} \\
&= \partial_t(\alpha(x_p + v_x), \alpha(y_p + v_y))|_{t=0} \\
&= \alpha V
\end{aligned}$$

Finally, we compute the metric

$$\begin{aligned}
g(u, v)_(x, y) &= g_{ab}u^a v^b \\
&= \frac{1}{y^2}u_b v^b
\end{aligned}$$

$$\begin{aligned}
g(dM_\alpha u, dM_\alpha v)_{(\alpha x, \alpha y)} &= g_{ab}\alpha u^a \alpha v^b \\
&= \frac{1}{(\alpha y)^2}\alpha^2 u_b v^b \\
&= \frac{1}{y^2}u_b v^b
\end{aligned}$$

Where $u_b = \eta_{ab}u^a$ and so $u_b v^b$ is the standard inner product on $\mathbb{R}^n$. Thus, M_α is an isometry.

I assert that the action of these two isometries is transitive. Indeed, given (x, y) and (x', y') in H^n, we construct the isometry as follows. First, apply T_{-x} to map (x, y) to $(0, y)$. Then, apply $M_{\frac{y'}{y}}$ to map $(0, y)$ to $(0, y')$. Finally, apply $T_{x'}$ to map $(0, y')$ to (x', y').

Thus, for any two points (x, y) and (x', y') in H^n, there is an isometry connecting them. Thus, by the result of the next problem, H^n is complete. $\square$

Problem 2

Show that a homogeneous space is complete.

Proof. Let M be a homogeneous manifold. We will show that M is geodesically complete.

Let ε be such that $B_\varepsilon(p) \subset M$ is a normal ball at $p \in M$. Since M is homogeneous, this implies that $B_\varepsilon(q)$ is a normal ball at $q \in M$ for any other q. To see this, we note that for ϕ the isometry sending p to q,

$$\phi \circ \exp_p \circ d\phi^{-1}$$

defines a diffeomorphism between $B_\varepsilon(0) \subset T_qM$ and the image $B_\varepsilon(q)$. This is well-defined, since ϕ is an isometry, so $\|v\| = \|d\phi^{-1}v\|$. Furthermore, we can see that $\exp_q = \phi \circ \exp_p \circ d\phi^{-1}$. Obesrve that $\gamma(t) = \exp_q(tv)$ is the unique geodesic through q with tangent vector v. However,

$$\tilde{\gamma}(t) = \phi \circ \exp_p \circ d\phi^{-1}(tv)$$

has the same properties. Namely $\tilde{\gamma}(0) = \phi(p) = q$, and $\tilde{\gamma}'(0) = d\phi(d\phi^{-1}(v)) = v$. Thus, $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in [0,1]$, and so $\exp_q$ and $\phi \circ \exp_p \circ d\phi^{-1}$ agree at all points in the normal ball. Thus, $B_\varepsilon(q)$ is a normal ball, as desired.

Recall that in a normal ball at p, any geodesic going through p can be extended throughout the entire normal ball. This follows from the fact that if γ is a geodesic passing through p at some time t_p with $\gamma'(t_p) = v$, it is the unique geodesic (up to reparameterization) with $\gamma(t_p) = p$ and $\gamma'(t_p) = v$. Now, since radial geodesics through p are defined on the entire normal ball, the radial geodesic starting at p with tangent vector v is defined throughout the normal ball, and is an extension of γ. Thus, γ can be extended through the normal ball.

It follows immediately, then, that any geodesic γ (with unit speed, without loss of generality) defined on some interval (a, b) can be extended to a geodesic defined on $(a, b + \frac{\varepsilon}{2})$ by observing that γ passes through $\gamma(b - \frac{\varepsilon}{2})$, and since $\gamma(b - \frac{\varepsilon}{2})$ has a normal ball of radius ε around it, we know that γ can be extended through this normal ball to be defined on $(a, b - \frac{\varepsilon}{2} + \varepsilon) = (a, b + \frac{\varepsilon}{2})$.

Thus, it follows immediately that geodesics can be extended indefinitely (the symmetric argument works to show γ can be extended the other way) and thus M is geodesically complete. $\square$

Problem 3