Problem Set 6

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March 6, 2018

Problem 1

Show that M_g does not retract onto C the circle that separates M_g into two compact surfaces M'_h and M'_k with $M'_i = M_i \setminus D^2$. In particular, show M'_h does not retract onto its boundary C. However, show M_g does retract onto C' the nonseparating circle.

Proof. Suppose for a contradiction that M'_h did retract onto C. Then, we would have the following diagram:

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[rr, bend right, "\mathbb{1}"'] & M'_h \arrow[r, "r"] & C \end{tikzcd}$$

Now, let's calculate some fundamental groups. We know that $\pi_1(C) \cong \mathbb{Z}$. Furthermore, if we regard M'_h by its cell structure, we see that $M'_h = M_h \setminus D^2$ is homotopic to its 1-skeleton, which is just a wedge of $2h$ circles. Thus, $\pi_1(M'_h) \cong \star_{i=1}^{2h}\mathbb{Z}$.

Applying the π_1 functor to the diagram above, we reach the conclusion that

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}"'] & \star^{2h}\mathbb{Z} \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$$

Now, we let $[f]$ be the generator for $\pi_1(C)$. By considering again M'_h as (homotopic to) the 1-skeleton of M_h (which has a cell structure of a regular $2h$-gon with opposite sides identified), we see that $i_*([f])$ gets sent to a loop that travels exactly once counterclockwise around the $2h$-gon. Reading off this loop as the concatenation of generating loops $a_1, a_2, \dots, a_{2h}$ we see that

$$i_*([f]) = [a_1][a_2]\dots[a_{2h}][a_1]^{-1}[a_2]^{-1}\dots[a_{2h}]^1$$

Now, this is a nontrivial map, but if we abelianize the diagram, we see that i_{*Ab} is actually the trivial map! Thus, if a retract were to exist, we would have

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "0"] \arrow[rr, bend right, "\mathbb{1}"'] & \mathbb{Z}^{2h} \arrow[r, "r_*"] & \mathbb{Z} \end{tikzcd}$$

which cannot happen. Thus, it follows that M_g as well cannot retract onto C.

Now, M_g can retract onto the nonseparating circle C'. Consider again M_g as a cell complex, namely a $2g$-gon with opposite sides identified. The loop C' in this complex is a single pair of opposite sides. We can explicitly construct a retract of this complex by deforming it into a rectangle whose horizontal edges are C', then sending a point x to its projection onto C'. $\square$

Problem 2

Consider the two arcs α, β embedded in $D^2 \times I$ as shown in the book. Prove that the loop γ going once around D^2 is not nullhomotopic in $X = D^2 \times I \setminus (\alpha \cup \beta)$.

Proof. We proceed by explicit calculation of the fundamental group of X.

To use van Kampen's theorem, we need to cover X with two open sets. Let A be the open set covering X up to γ plus a little bit more, so that A deformation retracts onto the closed space of X to the left of γ. Similarly, B is the open set covering the right half of X up to γ. Their intersection $A \cap B$ deformation retracts onto the slice of X that γ outlines.

Now, we need to calculate the fundamental groups of A, B and $A \cap B$. We begin with the fundamental group of A. Careful inspection of the space X reveals that A is a cylinder $D^2 \times I$ with two lines (the two halves of α) removed, as well as an arc (the slice of β). If we homotope the arc so that one of its endpoints stays fixed while the other goes to the other side of the cylinder, we end up with a cylinder with three parallel lines removed. This deformation retracts via the straight line homotopy to $D^2 \times \{1\} \setminus \{a_1, a_2, b\}$ where a_1, a_2, b are the three endpoints of the deleted lines corresponding to the two halves of α and β respectively.. We've already calculated the fundamental group of this space, however. $\pi_1(A)$ is the free product on three generators. Specifically, by letting $[a_1], [a_2], [b]$ be the generators of loops around a_1, a_2, b respectively, $\pi_1(A) = \langle [a_1], [a_2], [b] | \rangle$.

We can do the same thing to B, noting that here we have $\pi_1(B) = \langle [b_1], [b_2], [a] | \rangle$ for b_1, b_2, a being the endpoints of the two halves of β and α respectively.

Now, we need to calculate the fundamental group of $A \cap B$. This space, however, is clearly just a disk with the four points a_1, a_2, b_1, b_2 removed. Thus, $\pi_1(A \cap B) = \langle [a_1], [a_2], [b_1], [b_2] | \rangle$.

Let's examine how each of these include into A and B. Clearly, $i_A([a_1]) = [a_1]$, $i_A([a_2]) = [a_2]$, and similarly for B. Furthermore, $i_A([b_1]) = [b]$ (if we let b_1 be the endpoint fixed by the homotopy of A), and careful inspection reveals that $i_A([b_2]) = -[b]$. The results for B follows similarly. That is

$$\begin{aligned} i_A([a_1]) &= [a_1] \\ i_A([a_2]) &= [a_2] \\ i_A([b_1]) &= [b] \\ i_A([b_2]) &= -[b] \\ i_B([a_1]) &= [a] \\ i_B([a_2]) &= -[a] \\ i_B([b_1]) &= [b_1] \\ i_B([b_2]) &= [b_2] \end{aligned}$$

Thus, we have a full presentation of the fundamental group.

$$\pi_1(X) = \langle [a_1], [a_2], [b], [b_1], [b_2], [a] | [a_1] = [a], [a_2] = -[a], [b] = [b_1], -[b] = [b_2] \rangle$$

which reduces to

$$\pi_1(X) = \langle [a], [b] | \rangle$$

Now, careful inspection of γ reveals that it is homotopic to

$$[\gamma] = [b_1][a_1][b_2][a_2] = [b][a][b]^{-1}[a]^{-1}$$

which is clearly nontrivial. Thus, $[\gamma] \neq 0$ as desired. $\square$

Problem 3

Show that $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ is uncountable.

Proof. Recall from an earlier homework we proved that for two points x, y in $\mathbb{R}^2 \setminus \mathbb{Q}^2$, there are uncountably many paths from x to y. I assert that none of these paths are homotopic. To see this, suppose γ_1, γ_2 are two distinct paths from x to y. Then, $\gamma_1\bar{\gamma}_2$ is a loop based at x. In particular, this loop defines the boundary of a disk D^2 in $\mathbb{R}^2$. Since $\gamma_1 \neq \gamma_2$, this disk has some interior, and since $\mathbb{Q}^2$ is dense, the interior must contain some point in $\mathbb{Q}^2$. Therefore, $\gamma_1\bar{\gamma}_2$ cannot be nullhomotopic, and thus $\gamma_1 \not\simeq \gamma_2$ as desired.

Thus, for any choice of two paths from x to y,we obtain a loop based at x which is not homotopic to any other loop of paths from x to y. This defines an injection from the set of (distinct) pairs of paths from x to y (which is uncountable) into $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$, which implies that $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ is uncountable as desired. $\square$

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Problem 4

Let $p : \tilde{X} \to X$ be a covering space, with subspace $A \subset X$, and let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p : \tilde{A} \to A$ is a covering space.

Proof. We wish to show that A is evenly covered by p. So, let $x \in A$, and let $U \subset X$ be an open neighborhood such that $p^{-1}(U)$ evenly covers U. In particular, $U \cap A$ is open in A, and $p^{-1}(U \cap A) = p^{-1}(U) \cap p^{-1}(A)$ evenly covers $U \cap A$. This follows, since $p^{-1}(U)$ is a union of disjoint open sets, each of which is homeomorphic to U. Thus, $p^{-1}(U \cap A) = p^{-1}(U) \cap \tilde{A}$ is a union of disjoint open sets in $\tilde{A}$ (with the subspace topology) which are homeomorphic to $U \cap A$.

Thus, $p : \tilde{A} \to A$ is a covering space, as desired. $\square$

Problem 5

Show that if $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering spaces, so is their product.

Proof. Let $(x_1, x_2) \in X_1 \times X_2$, and let U_1, U_2 be open neighborhoods of x_1, x_2 which are evenly covered by p_1, p_2 respectively. I assert that the product $U_1 \times U_2$ is evenly covered by $p_1 \times p_2$.

Let $\tilde{U_{i\alpha}}$ be an open set in $\tilde{X}_1$ homeomorphic to U_i $(p_i(\tilde{U_{i\alpha}}) = U_i)$. Now, since each $\tilde{U_{i\alpha}}$ is disjoint from the others, we know that the products will be disjoint as well. That is, the subset

$$\bigcup_{\alpha,\beta} \tilde{U_{1\alpha}} \times \tilde{U_{2\beta}}$$

is a union of disjoint products $\tilde{U}_{1\alpha} \times \tilde{U}_{2\beta}$. Furthermore, each of these is homeomorphic to $U_1 \times U_2$. This is clear, since $p_i|_{\tilde{U}_{i\alpha}}$ is a homeomorphism, and thus the product $p_1 \times p_2$ restricted to $\tilde{U}_{1\alpha} \times \tilde{U}_{2\beta}$ is a homeomorphism as well.

Thus, $U_1 \times U_2$ is evenly covered by $p_1 \times p_2$, and $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2$ is a covering space. $\square$

Problem 6

Let $p : \tilde{X} \to X$ be a covering space with finite fibers. Show that $\tilde{X}$ is compact Hausdorff if and only if X is.

Proof. ($\Longrightarrow$) Suppose $\tilde{X}$ is compact Hausdorff. Then, since X is the image of $\tilde{X}$ under the continuous map p, it follows immediately that X is compact as well. Now we just need to show X is Hausdorff. Now, if $x \in X$, then there exists a neighborhood U of x which is evenly covered by p. In particular, this means that U is homeomorphic to a subset $\tilde{U}$ of $\tilde{X}$, which is Hausdorff. Thus, it follows that U is Hausdorff, and thus X is locally Hausdorff.

Now, suppose x, y are distinct points in X, with Hausdorff neighborhoods U_x and U_y. If U_x and U_y are disjoint, we have separated x and y with open sets, and are done. So, suppose U_x and U_y have nonempty intersection. Since X is compact, it is possible to construct an open set $V_x \subset U_x$ containing x such that V_x does not intersect $U_x \cap U_y$. Thus, V_x and U_y are disjoint, and x and y are separated by open sets as desired.

($\Longleftarrow$) Suppose X is compact Hausdorff. Now, it follows immediately that $\tilde{X}$ is Hausdorff. To see this, let x, y be distinct points in $\tilde{X}$. Now, if $p(x) = p(y)$, then x and y are in the same fiber, and since fibers of covering maps are discrete, there exist open sets U and V that separate x and y.

Now, suppose x and y are not in the same fiber. Then, let U, V be open sets in X separating $p(x)$ and $p(y)$. Then, the inverse images $p^{-1}(U)$ and $p^{-1}(V)$ contain x and y respectively, and are disjoint, since U and V are disjoint. Thus, x and y are separated, and $\tilde{X}$ is Hausdorff as desired.

Now, we finally prove that $\tilde{X}$ is compact. Let $\mathscr{O} = \{U_\alpha\}$ be an open cover of $\tilde{X}$. Furthermore, let U_x be an evenly covered neighborhood of x for each $x \in X$. Since p is a finite covering, this lifts to a finite collection $\{V_x^i\}_{i=1}^n$ of subsets of $\tilde{X}$, where each V_x^i is homeomorphic to U_x via p. In particular, we can refine $\mathscr{O}$ to $\mathscr{O}' = \{U'_\alpha\}$ a cover for which every open set is contained in some V_x^i (Let $U_\alpha = \cup_{x,i}(U_\alpha \cap V_x^i)$). Now, all we need to show is that this refinement has a finite subcover.

To do so, consider the projection of the refinement $\{p(U'_\alpha)\}$, which forms an open cover of X. This has a finite subcover $\{p(U'_j)\}_{j=1}^m$. Now, since each $p(U'_j)$ is an evenly covered neighborhood, it follows that $p^{-1}(p(U'_j)$ is a disjoint union of a finite number of elements of $\mathscr{O}'$. Thus, letting $\{W_{j_k}\}$ be these elements, we see that

$$\bigcup_{j,k} W_{j_k} = \bigcup_j p^{-1}(p(U_j)) = p^{-1}(X) = \tilde{X}$$

as desired. $\square$

Problem 7

Construct a simply connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of the sphere and a diameter. Do the same when X is a sphere and a circle intersecting at two points.

Proof. For the first space, we can have $\tilde{X}$ be an infinite line of spheres joined by line segments. If x is the point where the diameter of S^2 meets S^2, then $p^{-1}(x)$ is the right endpoints of each of the line segments. This space is simply connected (as it is homotopic to the infinite wedge of S^2, which is simply connected), and covers X as desired.

For the second space, we can have $\tilde{X}$ be the same space. This is easily verified to be a covering space of X by noting that for x, y the intersection of the circle with S^2, we let $p^{-1}(x)$ be the left endpoints of the line segments, and $p^{-1}(y)$ be the right endpoints of the line segments. □

Problem 8

Let $X = \partial I^2 \cup \{\frac{1}{n}\}_{n=1}^{\infty} \times I$. Show that for every covering space $p : \tilde{X} \to X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Conclude that X has no simply connected covering space.

Proof. We note first that by the definition of a covering space, for x on the left edge of X there is a neighborhood U of x which lifts homeomorphically to some $\tilde{U}$ in $\tilde{X}$. I didn't have time to prove that the entire left side $\{0\} \times I$ has a neighborhood that maps homeomorphically to $\tilde{X}$, so we will assume it does.

Let U be the neighborhood of $\{0\} \times I$ that maps homeomorphically into $\tilde{X}$. Then, in particular there are nontrivial loops in U that get lifted to nontrivial loops in $\tilde{X}$ (since p_* is injective). Thus, $\tilde{X}$ cannot be simply connected. □

Problem 9

Let X be the Hawaiian earring, and let $\tilde{X}$ be its covering space defined in the book. Find a two-sheeted covering space Y of $\tilde{X}$ such that the composition $Y \to \tilde{X} \to X$ is not a covering space.

Proof. Let Y be the space obtained by taking two copies of $\tilde{X}$ with the loops facing each other, and joining them by splitting the outermost loop at the peak and joining the edges. Now, let the covering map p be as follows. From the center moving outward (symmetrical in both directions) the nth copy of the Hawaiian earring has the nth nested loop mapped to the loop joining the two copies of $\tilde{X}$.

Now, this is clearly a two-sheeted covering, so all that remains is to show that Y is not a cover of X as a composition. To see this, note that any open neighborhood of the origin (the intersection of the circles) necessarily contains some loop, which eventually gets mapped to the loop joining the two copies of the Hawaiian earring. But the interior of the loop in X contains a single copy of X, whereas the interior of the loop in Y contains two copies! Thus, Y cannot be a covering, since it cannot evenly cover the origin. □