Final Exam

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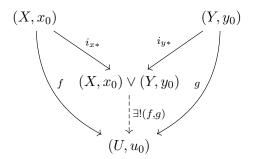
Problem 1

Carefully prove a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is contractible.

Proof. For this proof, we will begin by proving a couple of useful lemmas.

Lemma 1. The wedge sum is the coproduct in the category Top_* of pointed topological spaces.

Proof. To show this is the coproduct, we need to show that it satisfies the universal property for coproducts. That is for $(X, x_0), (Y, y_0)$ pointed topological spaces, and (U, u_0) any other pointed topological space with arrows $f: (X, x_0) \to (U, u_0)$ and $g: (Y, y_0) \to (U, u_0)$,

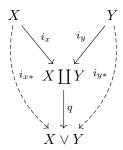


where (f,g) is the unique arrow that makes the diagram commute. Now, recall that the wedge sum is defined as the quotient

$$(X, x_0) \lor (Y, y_0) = X \prod Y/x_0 \sim y_0$$

so we can bootstrap by using the fact that the disjoint union is the coproduct in Top. That is,

we define i_{x*} and i_{y*} to be the unique maps given by the universal property



Note that these maps preserve basepoints, since

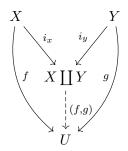
$$i_{x*}(x_0) = q \circ i_x(x_0) = q(x_0)$$

and similarly for y_0 .

Now, suppose we have maps $f:(X,x_0)\to (U,u_0)$ and $g:(Y,y_0)\to (U,u_0)$. Define (f,g) to be the unique map

$$(f,g): X\coprod Y \to U$$

that makes the diagram



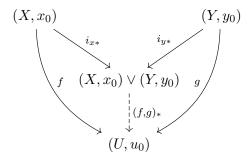
commute. Now, since $f(x_0) = g(y_0) = u_0$, it follows that (f, g) is constant on the fibers of $q: X \coprod Y \to X \vee Y$ and thus by the universal property of quotient maps factors through $X \vee Y$. That is,

$$X \coprod Y$$

$$(f,g) \qquad q$$

$$U \leftarrow \cdots \qquad X \lor Y$$

commutes for a unique $(f,g)_*$. Thus, we have for f,g arrows from (X,x_0) and (Y,y_0) to (U,u_0) a unique arrow $(f,g)_*$ from $X\vee Y$ which makes



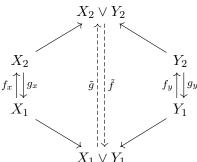
commute as desired.

Lemma 2. The wedge product is stable with respect to (basepoint-preserving) homotopy. That is, if $X_1 \simeq X_2$ and $Y_1 \simeq Y_2$ relative to basepoints, then

$$X_1 \vee Y_1 \simeq X_2 \vee Y_2$$

Proof. Since the wedge sum is the coproduct of pointed topological spaces, we construct the homotopy equivalence as follows:

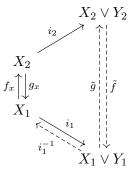
Let $f_x: X_1 \to X_2$ and $g_x: X_2 \to X_1$ be homotopy equivalences of X_2 and X_1 . That is, f_x and g_x are homotopy inverses of each other. Furthermore, let $f_y: Y_1 \to Y_2$ and $g_y: Y_2 \to Y_1$ be homotopy inverses as well. Then, by the universal property of coproducts, we have the commutative diagram



where \tilde{f} is defined in terms of the universal property of coproducts with respect to the compositions $X_1 \xrightarrow{f_x} X_2 \longrightarrow X_2 \vee Y_2$ and $Y_1 \xrightarrow{f_y} Y_2 \longrightarrow {}_2 \vee Y_2$ (and similarly for \tilde{g}).

I assert that \tilde{f} and \tilde{g} are homotopy inverses. It should be clear that by symmetry of the problem, I need only check that $\tilde{f} \circ \tilde{g} \simeq \mathbb{1}$.

Suppose $x \in X_1 \vee Y_1$, and without loss of generality let x be in the inclusion of X_1 to $X_1 \vee Y_1$. Then, we have the diagram



(where i_1^{-1} is only defined on the image of i_1 , but we are assuming that $x \in i_1(X_1)$ for this diagram chase, so this arrow exists). From here, it is clear that $\tilde{g}(x) = i_2 \circ f_x(i_1^{-1}(x))$, and if we identify X_1 and X_2 as subspaces of their wedge product, we have $\tilde{g}(x) = f_x(x)$. Similarly, we have $\tilde{f}(x) = g_x(x)$. Thus,

$$\tilde{f} \circ \tilde{g}(x) = g \circ f(x) \simeq \mathbb{1}(x)$$

as desired.

Thus, \tilde{f} and \tilde{g} are homotopy inverses, and $X_1 \vee Y_1 \simeq X_2 \vee Y_2$ as desired.

Now, we prove the main result.

Let Z be a CW complex which satisfies the hypotheses. In particular, let X and Y be such that $Z = X \cup Y$, X and Y are both contractible, and their intersection $A = X \cap Y$ is contractible as well.

Since A is a contractible subcomplex of Z, we know (via Hatcher prop 0.16 and 0.17) that Z is homotopy equivalent to Z/A. Thus, we only need to show that Z/A is contractible.

Let $q: Z \to Z/A$ be the canonical quotient map. Since $Z = X \cup Y$, we know that $q(Z) = Z/A = q(X) \cup q(Y)$. In particular, q(X) = X/A and q(Y) = Y/A (this follows from the definition of the quotient). Thus, $Z/A = X/A \cup Y/A$. Since the quotient map is the identity on $Z \setminus A$ and

collapses A to a point, it follows that $X/A \cap Y/A = A/A = \{a_0\}$ where a_0 is the point q(A). Thus, Z/A is actually the wedge sum

$$Z/A = X/A \vee Y/A$$

Now, since A is contractible, we know that $X/A \simeq X$ and $Y/A \simeq Y$. In particular, since X and Y are contractible, so is X/A and Y/A. Thus, $X/A \simeq Y/A \simeq \{\cdot\}$. By lemma 2, this implies that

$$X/A \vee Y/A \simeq \{\cdot\} \vee \{\cdot\} = \{\cdot\}$$

and thus, $X/A \vee Y/A$ is contractible. Thus, $Z/A = X/A \vee Y/A$ is contractible as well, and Z itself is contractible as desired.

PROBLEM 2

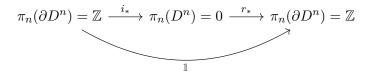
Prove that there is no retraction $r: D^n \to \partial D^n$ using the fact that $\pi_n(S^n) = \mathbb{Z}$.

Proof. Assume for a contradiction that there exists a retraction $r:D^n\to \partial D^n$. That is, assume that there is some r such that the diagram

$$\partial D^n \xrightarrow{i} D^n \xrightarrow{r} \partial D^n$$

commutes. Here i is the inclusion map of ∂D^n into D^n . (Since r restricted to ∂D^n is the identity by definition, ri = 1).

Since π_n is a covariant functor (Hatcher states this on page 342), we can apply it to this diagram to get the commutative diagram



which implies that the identity map from \mathbb{Z} to itself factors through zero, which cannot be. Thus, our assumption that r exists is false, and no retract of D^n to its boundary exists. \square

PROBLEM 3

In each case, either prove a covering exists, or that no such covering exists.

Part a

 $p:S^2\to T^2.$

Proof. I assert that no such covering map exists. Recall from Hatcher proposition 1.37 that for X a path-connected, locally path-connected space, and $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2$ covering spaces, then \tilde{X}_1 and \tilde{X}_2 are isomorphic if and only if $p_{1*}(\pi_1(\tilde{X}_1)) = p_{2*}(\pi_1(\tilde{X}_2))$.

Now, T^2 is path-connected and locally path-connected. Furthermore, it has a universal cover \mathbb{R}^2 . To see this consider the action of $\mathbb{Z} \times \mathbb{Z}$ on \mathbb{R}^2 given by (m,n)(x,y)=(x+m,y+n). Clearly, this action is properly discontinuous, since each point moves a distance of at least 1 for the action of any nontrivial element of $\mathbb{Z} \times \mathbb{Z}$, and so for any point (x,y) we can take the neighborhood $U=V_{\frac{1}{10}}((x,y))$, which easily satisfies the property $U\cap g(U)=\emptyset$ for nontrivial $g\in\mathbb{Z}\times\mathbb{Z}$.

Thus, by proposition 1.40, we know that the quotient map $p_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ is a covering space map. It is easy to see that $\mathbb{R}^2/\mathbb{Z} \times \mathbb{Z} = T^2$, and so \mathbb{R}^2 is a simply-connected covering space of T^2 .

Now, since both S^2 and \mathbb{R}^2 are simply-connected, we know that

$$p_*(\pi_1(S^2)) = p_{\mathbb{R}^2*}(\pi_1(\mathbb{R}^2)) = 0$$

and so proposition 1.37 guarantees the existence of a covering space isomorphism between the two. However, \mathbb{R}^2 and S^2 are not homeomorphic, (S^2 is compact but \mathbb{R}^2 is not) and so no such isomorphism can exist. Thus, S^2 cannot cover T^2 .

Part b

 $p: T^2 \to S^2$.

Proof. Recall Hatcher proposition 1.31, which asserts that for a covering space $p: \tilde{X} \to X$, the induced map $p_*: \pi_1(\tilde{X}) \to \pi_1(X)$ is injective. So, if such a covering $p: T^2 \to S^2$ existed, it would induce an injection

$$p_*: \pi_1(T^2) \to \pi_1(S^2)$$

but $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$, and $\pi_1(S^2) = 0$, so no such injection can exist! Thus, T^2 does not cover S^2 .

Part c

 $p: \mathbb{R}^2 \to T^2$.

Proof. This was constructed in part a as a covering map.

$$p: \mathbb{R}^2 \to S^2$$
.

Proof. I assert that no such covering exists. This follows from the fact that S^2 covers itself, and simply-connected covering spaces are unique up to isomorphism.

Recall from Hatcher proposition 1.37 that for X a path-connected, locally path-connected space, and $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2$ covering spaces, then \tilde{X}_1 and \tilde{X}_2 are isomorphic if and only if $p_{1*}(\pi_1(\tilde{X}_1)) = p_{2*}(\pi_1(\tilde{X}_2))$.

Since S^2 is path-connected and locally path-connected, it satisfies the hypotheses for this proposition. Furthermore, the two covering space maps $p: \mathbb{R}^2 \to S^2$ and $\mathbb{1}: S^2 \to S^2$ both induce

$$p_*(\pi_1(\mathbb{R}^2)) = \mathbb{1}_*(\pi_1(S^2)) = 0$$

and thus if such a p existed, there would be a covering space isomorphism between \mathbb{R}^2 and S^2 . However, these spaces are not homeomorphic, and thus cannot be isomorphic as covering spaces. So, no such p exists.

PROBLEM 5

Given a sequence $f: \mathbb{N} \to (0, \infty)$ define $p_k = (0, 0, f(k))$ and C_k is the boundary of the triangle in the plane $\{(x, y, 0) \in \mathbb{R}^3\}$ with vertices $0, v_{2k}, v_{2k+1}$ where $v_n = (\frac{1}{n}, \frac{1}{n^2}, 0)$. For $k \in \mathbb{N}$ the set $A_k = \{t \cdot q + (1-t) \cdot p_k : q \in C_k, 0 \le t \le 1\}$ is the cone from p_k to C_k . The subspace $X \subset \mathbb{R}^3$ is $X = \bigcup A_k$.

Part a

Show X is simply-connected if $f(k) = \frac{1}{k}$.

Proof. We begin by considering a homotopy of X defined as follows. For A_1 , fix the subcone, say B_1 , from p_2 to C_1 (the cone inside A_1 whose face along A_2 is the entirety of $A_2 \cap A_1$), and perform a straight line homotopy of the rest of A_1 downwards into B_1 .

The homotopy on A_k is defined inductively. We retract $\bigcup_{i=1}^k A_k$ (where each A_i has already been moved to have a vertex at p_k) onto the subcone from p_{k+1} to $\bigcup_{i=1}^k C_k$ using the straight line homotopy.

Finally, we stitch these homotopies together by letting the kth homotopy occur in the time $[1-\frac{1}{k},1-\frac{1}{k+1}]$. Now, for each $t\neq 1$, this action is continuous, as each individual retract of A_k is continuous as a straight line homotopy. Furthermore, this homotopy is continuous at 1. To see this, consider any point $x\in X$, Clearly, this homotopy takes x to its projection on the x-y plane. So, if the height of x is not zero, then x is in some A_k (and possibly A_{k-1} if its on the boundary), and is thus at most a distance $\frac{1}{k}$ from its projection. In particular, since x moves downwards from height $\frac{1}{i}$ to height $\frac{1}{i+1}$ in time $t\in [1-\frac{1}{i},1-\frac{1}{i+1}]$, the homotopy is clearly continuous at x. Finally, if the height of x is zero, then x is fixed by the homotopy, and the homotopy is continuous at x.

Thus, X is homotopic to $\bigcup_{k=1}^{\infty} \tilde{C}_k$ for \tilde{C}_k the filled in triangle with boundary C_k . This space is clearly simply connected, so X is simply connected as well.

Part b

Show X is not simply-connected if f(k) = k.

Proof. We proceed by constructing a nontrivial loop in X. To do so, consider the loop γ which goes around C_k in time $[1-\frac{1}{k},1-\frac{1}{k+1}]$. This loop is continuous on [0,1), since at each $t \in [0,1)$, t is in some interval $[1-\frac{1}{i},1-\frac{1}{i+1}]$ and is thus part of the loop around C_i . Furthermore, this loop is continuous at t=1. To see this, observe that any neighborhood U of $\gamma(1)$ necessarily contains all but finitely many C_k . Thus, for n the smallest integer for which C_n is completely contained in U. The image $\gamma((1-\frac{1}{n},1])$ of the open set $(1-\frac{1}{n},1]$ is equal to $\bigcup_{k=n}^{\infty} C_k$ which is entirely in U by definition of n.

Thus, this defines a loop in X. However, this loop is nontrivial, since the only way to homotope the part of the loop around C_k to the constant loop is by taking it over p_k . Since the heights p_k go to infinity as k does, it follows that there is no continuous way to homotope all of γ to the constant loop (the speeds of the homotopy would have to diverge, which cannot be continuous).

Thus, X is not simply connected.