1 Preliminaries

Homework 1. *Prove that $V^{**} \cong V$ for finite-dimensional vector space V.*

From this, it is clear that $T_p^*M \otimes T_pM \cong \mathrm{Hom}(T_pM, T_pM)$ for a manifold M.

Recall the tangent bundle TM is defined as

$$TM = \coprod_{p \in M} T_pM$$

and a vector field on the manifold M is simply a section of the tangent bundle projection $TM \xrightarrow{\pi} M$. In other words, a vector field is a function $f : M \to TM$ such that $\pi \circ f = id$. Requiring the section to be smooth makes it into a smooth vector field.

We can also do the same thing for the cotangent bundle T^*M to obtain a covector field.

Now, we can take the tensor product of copies of TM and T^*M to obtain our tensor bundles, and tensor fields will be sections of these bundles.

Let (U, ϕ) be a smooth chart on M with coordinate functions x^i, coordinate vector fields ∂_i, and coordinate one-forms dx^i. Recall that dx^i is defined to be the dual basis to ∂_i, that is,

$$dx^i(\partial_j) = \delta^i_j$$

Recall also that the exterior derivative of a function df is defined as

$$df(v) = v(f)$$

and this definition applied to the coordinate functions x^i (yielding dx^i) coincides with the definition above. Note that ∂_i form a basis for T_pM and dx^i form a basis for T_p^*M. Tensor products of them, then, form a basis for the tensor product space.

Homework 2. *Prove that, for a vector space V with basis v_i, dual basis v^i, the set*

$$\{v^i \otimes v^j \mid 1 \leq i, j \leq n\}$$

forms a basis for $V^* \otimes V^*$. Here $v^i \otimes v^j(u, v) = v^i(u)v^j(v)$.

2 Affine Connections

2.1 The Metric

Def. 2.1. *Let M^n be a smooth manifold of dimension n. A* Riemannian Metric *g on M is a rank $(0, 2)$ tensor (a section of $T^*M \otimes T^*M$) that is symmetric and positive-definite. In other words, g is a rank $(0, 2)$ tensor that restricts to an inner product on the tangent space at every point.*

We can express g in local coordinates!

$$g_{ij} = g(\partial_i, \partial_j)$$

or

$$g = g_{ij} dx^i \otimes dx^j$$

2.2 Integration of Top Degree Differential Forms

Let M^n be an orientable n-dimensional manifold, and $\omega \in \Omega^n(M)$. Furthermore let (U, ϕ) be a positive coordinate chart. On U we have that

$$\omega = f dx^1 \wedge \ldots \wedge dx^n$$

for some $f \in C^\infty(M)$.

Now, let $K \subset U$ be compact. We define

$$\begin{aligned}\int_K \omega &= \int_{\phi(K)} \phi^{-1*}\omega \\ &= \int_{\phi(K)} f \circ \phi^{-1} \phi^{-1*} dx^1 \wedge \ldots \wedge \phi^{-1*} dx^n \\ &= \int_{\phi(K)} f \circ \phi^{-1} dx^1 \wedge \ldots \wedge dx^n\end{aligned}$$

where the last integral is just the standard integral in $\mathbb{R}^n$.

Is this definition independent of choice of coordinates? Let's check. Let (V, ψ) be another coordinate chart containing K. Then, the integral with respect to this coordinate system is

$$\int_K \omega = \int_{\psi(K)} g \circ \psi^{-1} dy^1 \wedge \ldots \wedge dy^n$$

for g defined as

$$\omega = h dy^1 \wedge \ldots \wedge dy^n$$

with coordinate functions y^i. The claim is that these integrals are equal.

Consider the change-of-coordinates map $\psi \circ \phi^{-1}$ from the x^i to the y^i coordinate system. Since K is in both U and V, its image $\phi(K)$ lies in the domain of $\psi \circ \phi^{-1}$.

All that remains is to apply the change of variables to the integrals. Recall that if one has a diffeomorphism $F : \Omega_1 \to \Omega_2$ for compact Ω_i, one has that

$$\int_{\Omega_2} f dy^1 \ldots dy^n = \int_{\Omega_1} f \circ F |J_F| dx^1 \ldots dx^n$$

where $|J_F|$ is the determinant of the Jacobian matrix for F.

Homework 3. *Check that the two integrals claimed to be equal are actually equal.*

Now we have an idea for how to integrate ω on a single chart, let's extend this. Let (η_i, U_i) be a partition of unity of M where each U_i is contained in a single chart on M. Then,

$$\omega = \sum \omega \eta_i$$

and we can integrate by extending linearly

$$\int_K \omega = \sum \int_K \omega \eta_i$$

where the right hand side has integrals over functions supported in a single chart, and is well-defined. But is this independent of the choice of partition of unity? Short answer: yes (Optional homework).

2.3 Integration on an Orientable Smooth Riemannian Manifold

Recall that a Riemannian manifold has a volume form

$$dvol = \sqrt{|g_{ij}|} dx^1 \wedge \ldots \wedge dx^n$$

which is obtained by taking an orthonormal frame e_i and considering the dual frame ω^i defined as

$$\omega^i e_j = \delta^i_j$$

and letting

$$dvol = \omega^1 \wedge \ldots \wedge \omega^n$$

This construction is independent of choice of orthonormal frame.

Proof. Let ϵ_i be another orthonormal frame with dual frame α^i. Then, $\epsilon_i = a^j_i e_j$ and $\alpha^i = b^i_j \omega^j$ and so

$$\begin{aligned}
\alpha^1 \wedge \ldots \wedge \alpha^n &= b^1_{j_1} \omega^{j_1} \wedge \ldots \wedge b^n_{j_n} \omega^{j_n} \\
&= \sum_{\sigma \in S_n} b^1_{\sigma(1)} \ldots b^n_{\sigma(n)} sgn(\sigma) \omega^1 \wedge \ldots \wedge \omega^n \\
&= |b| \omega^1 \wedge \ldots \wedge \omega^n \\
&= \omega^1 \wedge \ldots \wedge \omega^n
\end{aligned}$$

where the last line was obtained from the fact that b is the orthogonal change-of-basis matrix from e to ϵ. □

Then, we define

$$\mathrm{Vol}(K) = \int_K dvol$$