1 Preliminaries

Homework 1. *Prove that $V^{**} \cong V$ for finite-dimensional vector space V.*

From this, it is clear that $T_p^*M \otimes T_pM \cong \mathrm{Hom}(T_pM, T_pM)$ for a manifold M.

Recall the tangent bundle TM is defined as

$$TM = \coprod_{p \in M} T_pM$$

and a vector field on the manifold M is simply a section of the tangent bundle projection $TM \xrightarrow{\pi} M$. In other words, a vector field is a function $f : M \to TM$ such that $\pi \circ f = id$. Requiring the section to be smooth makes it into a smooth vector field.

We can also do the same thing for the cotangent bundle T^*M to obtain a covector field.

Now, we can take the tensor product of copies of TM and T^*M to obtain our tensor bundles, and tensor fields will be sections of these bundles.

Let (U, ϕ) be a smooth chart on M with coordinate functions x^i, coordinate vector fields ∂_i, and coordinate one-forms dx^i. Recall that dx^i is defined to be the dual basis to ∂_i, that is,

$$dx^i(\partial_j) = \delta^i_j$$

Recall also that the exterior derivative of a function df is defined as

$$df(v) = v(f)$$

and this definition applied to the coordinate functions x^i (yielding dx^i) coincides with the definition above. Note that ∂_i form a basis for T_pM and dx^i form a basis for T_p^*M. Tensor products of them, then, form a basis for the tensor product space.

Homework 2. *Prove that, for a vector space V with basis v_i, dual basis v^i, the set*

$$\{v^i \otimes v^j \mid 1 \leq i, j \leq n\}$$

forms a basis for $V^* \otimes V^*$. Here $v^i \otimes v^j(u, v) = v^i(u)v^j(v)$.

2 Affine Connections

2.1 The Metric

Definition 2.1. *Let M^n be a smooth manifold of dimension n. A* Riemannian Metric *g on M is a rank $(0, 2)$ tensor (a section of $T^*M \otimes T^*M$) that is symmetric and positive-definite. In other words, g is a rank $(0, 2)$ tensor that restricts to an inner product on the tangent space at every point.*

We can express g in local coordinates!

$$g_{ij} = g(\partial_i, \partial_j)$$

or

$$g = g_{ij} dx^i \otimes dx^j$$

Homework 3. *Show that the two expressions for dvol, namely*

$$dvol = \wedge_i \omega^i$$
$$dvol = \sqrt{|g|} dx^n$$

2.2 Integration of Top Degree Differential Forms

Let M^n be an orientable n-dimensional manifold, and $\omega \in \Omega^n(M)$. Furthermore let (U, ϕ) be a positive coordinate chart. On U we have that

$$\omega = f dx^1 \wedge \ldots \wedge dx^n$$

for some $f \in C^\infty(M)$.

Now, let $K \subset U$ be compact. We define

$$\begin{aligned} \int_K \omega &= \int_{\phi(K)} \phi^{-1*} \omega \\ &= \int_{\phi(K)} f \circ \phi^{-1} \phi^{-1*} dx^1 \wedge \ldots \wedge \phi^{-1*} dx^n \\ &= \int_{\phi(K)} f \circ \phi^{-1} dx^1 \wedge \ldots \wedge dx^n \end{aligned}$$

where the last integral is just the standard integral in $\mathbb{R}^n$.

Is this definition independent of choice of coordinates? Let's check. Let (V, ψ) be another coordinate chart containing K. Then, the integral with respect to this coordinate system is

$$\int_K \omega = \int_{\psi(K)} g \circ \psi^{-1} dy^1 \wedge \ldots \wedge dy^n$$

for g defined as

$$\omega = h dy^1 \wedge \ldots \wedge dy^n$$

with coordinate functions y^i. The claim is that these integrals are equal.

Consider the change-of-coordinates map $\psi \circ \phi^{-1}$ from the x^i to the y^i coordinate system. Since K is in both U and V, its image $\phi(K)$ lies in the domain of $\psi \circ \phi^{-1}$.

All that remains is to apply the change of variables to the integrals. Recall that if one has a diffeomorphism $F : \Omega_1 \to \Omega_2$ for compact Ω_i, one has that

$$\int_{\Omega_2} f dy^1 \ldots dy^n = \int_{\Omega_1} f \circ F |J_F| dx^1 \ldots dx^n$$

where $|J_F|$ is the determinant of the Jacobian matrix for F.

Homework 4. *Check that the two integrals claimed to be equal are actually equal.*

Now we have an idea for how to integrate ω on a single chart, let's extend this. Let (η_i, U_i) be a partition of unity of M where each U_i is contained in a single chart on M. Then,

$$\omega = \sum \omega \eta_i$$

and we can integrate by extending linearly

$$\int_K \omega = \sum \int_K \omega \eta_i$$

where the right hand side has integrals over functions supported in a single chart, and is well-defined. But is this independent of the choice of partition of unity? Short answer: yes (Optional homework).

2.3 Integration on an Orientable Smooth Riemannian Manifold

Recall that a Riemannian manifold has a volume form

$$dvol = \sqrt{|g_{ij}|} dx^1 \wedge \ldots \wedge dx^n$$

which is obtained by taking an orthonormal frame e_i and considering the dual frame ω^i defined as

$$\omega^i e_j = \delta^i_j$$

and letting

$$dvol = \omega^1 \wedge \ldots \wedge \omega^n$$

This construction is independent of choice of orthonormal frame.

Proof. Let ϵ_i be another orthonormal frame with dual frame α^i. Then, $\epsilon_i = a^j_i e_j$ and $\alpha^i = b^i_j \omega^j$ and so

$$\begin{aligned}
\alpha^1 \wedge \ldots \wedge \alpha^n &= b^1_{j_1} \omega^{j_1} \wedge \ldots \wedge b^n_{j_n} \omega^{j_n} \\
&= \sum_{\sigma \in S_n} b^1_{\sigma(1)} \ldots b^n_{\sigma(n)} sgn(\sigma) \omega^1 \wedge \ldots \wedge \omega^n \\
&= |b| \omega^1 \wedge \ldots \wedge \omega^n \\
&= \omega^1 \wedge \ldots \wedge \omega^n
\end{aligned}$$

where the last line was obtained from the fact that b is the orthogonal change-of-basis matrix from e to ϵ. □

Then, we define

$$\mathrm{Vol}(K) = \int_K dvol$$

2.4 Integrating a Non-Orientable Manifold

How do we integrate a manifold that is not orientable? The previous construction was coordinate-independent only because we chose positive oriented coordinates...

Let $K \subset U$ be a compact set in a single chart on the manifold. Then, we can define

$$\mathrm{Vol}(K) = \int_K \sqrt{|g_{ij}|} dx^n$$

Now, this is independent of choice of coordinates, since if K lies in the intersection of two charts, we can use the Jacobian change-of-variables formula to show that the two calculations of the volume are equal.

The problem is that $dy^n = det(J_{x \to y}) dx^n$ depends also on the sign of the determinant of the Jacobian.

On an orientable Manifold, we have $dvol \in \Omega^n(M)$ (i.e. $dvol \in \Gamma(\Lambda^n T^* M)$), and in fact a manifold is orientable if and only if it admits a nowhere-vanishing top degree form.

Homework 5. *Prove that a manifold is orientable if and only if it admits a nowhere-vanishing top degree form.*

2.5 Existence of Metrics

Theorem 1. *On each smooth manifold M there exists smooth Riemannian metrics.*

Proof. Let (U_i, ϕ_i) be an atlas of M, and η_j be a partition of unity subordinate to it. Then, on each U_i we have a smooth Riemannian metric given by

$$g_i = dx_i^1 \otimes dx_i^1 + \ldots + dx_i^n \otimes dx_i^n$$

Then, we define

$$g = \sum \eta_i g_i$$

□

2.6 Lower-Dimensional Integration on Riemannian Manifolds

Suppose we want to find the arc length of a curve $\gamma : I \to M$. We can define the length of γ to be

$$L(\gamma) = \int_I |\gamma'| dt$$

where $|\gamma'|$ is the length of the tangent vector with respect to the metric.

Definition 2.2. *Let $p, q \in M$ be points in a connected manifold M. We define the distance between p and q to be*

$$\inf_{\gamma \in C^\infty(I,M)} \{L(\gamma) \mid \gamma(0) = p, \gamma(1) = q\}$$

Note that we can relax the condition that γ be smooth to γ being only piecewise smooth, since any piecewise smooth curve is uniformly approximated by smooth curves.

This distance, denoted $d(p, q)$, turns out to metrize the manifold.

Theorem 2. *$d(\cdot, \cdot)$ is a metric on M, and the metric topology generated by d coincides with the topology of M.*

Proof. First, we show that d is a metric. Symmetry of d should be obvious, since $L(\gamma) = L(-\gamma)$ and the curves from p to q directly coincide with curves from q to p via the map $\gamma \mapsto -\gamma$.

Now, d is also clearly positive-definite, since the length functional is positive-definite.

It should also be clear that $d(p, q) = 0$ if and only if $p = q$. Clearly, if $p = q$, then the constant curve $\gamma(t) = p$ has length zero, so $d(p, p) = 0$. Now, if $p \neq q$, then since M is Hausdorff, they must have positive distance from each other. This follows from the second claim that the topologies coincide.

The triangle inequality follows from the fact that given three points p, q, m, the curve going from p to m, and then from m to q, is a curve from p to q, and so $d(p, q) \leq d(p, m) + d(m, q)$ (since it is part of the infimum).

Now, we show that the topologies coincide... $\square$

Homework 6. *Show that the topology on M coincides with the metric topology from d.*

Homework 7. *Show that for $(\mathbb{R}^n, g_{\mathbb{R}^n})$, $d(p, q) = \|p - q\|$.*

2.7 Connections on a Riemannian Manifold

Let (M^n, g) be a smooth Riemannian Manifold, $X \in \mathfrak{X}(M)$. We wish to take the derivative of this vector field. Recall that the Lie derivative allows us to take the derivative of X along another vector field Y, however this operation is not linear with respect to the module of smooth functions. That is,

$$L_X(fY) = fL_X Y + (Xf)Y$$

Also, the Lie derivative is not defined for a single point, since it takes into account the motion of X around any particular point.

What we really want is ∇_v, the covariant derivative.

Definition 2.3. *A* Connection *is a map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \quad (X, Y) \mapsto \nabla_X Y$$

such that $\nabla_X Y$ is linear in both X with respect to the module $C^\infty(M)$, scalar linear in Y and satisfies the Leibniz rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

Definition 2.4. *A* Connection *is the following: for each $p \in M$, we have a map $\nabla : T_pM \times C^\infty(TM) \to T_pM$ that sends (v, Y) to $\nabla_v Y$. Such that ∇ is linear in v, linear in Y, and satisfies the Leibniz rule*

$$\nabla_v(fY) = (vf)Y_p + f(p)\nabla_v(Y)$$

and, for all X, Y in $\mathfrak{X}(M)$, $\nabla_X Y \in \mathfrak{X}(M)$ where

$$(\nabla_X Y)_p = \nabla_{X_p} Y$$

Interpreting ∇ as an operator from $\mathfrak{X}(M)$, we see that it actually adds a covariant index. That is,

$$\nabla_\mu v^\nu$$

takes in a vector, and outputs a $(1, 1)$ tensor.

Example. *The directional derivative in $\mathbb{R}^n$ yields a connection. For $v \in T_x\mathbb{R}^n$, and X a smooth vector field on $\mathbb{R}^n$, we have*

$$D_{(x,v)}X = \partial_t X(x + tv)|_{t=0}$$

and we define $\nabla_v X = (x, D_{(x,v)}X)$

Now, on TM for a general Riemannian manifold, there are many different connections. However, given a metric, we have a unique metric compatible, torsion-free connection called the *Levi-Civita Connection*.

Theorem 3. *For M a smooth Riemannian manifold, then there exists a unique connection ∇ on TM such that*

- *∇ is symmetric i.e.*

  $$\nabla_X Y - \nabla_Y X = [X, Y]$$

  (The Christoffel symbols are symmetric in lower indices)

- *∇ is metric-compatible. That is,*

  $$\nabla g = 0$$

  or

  $$\nabla_\gamma g_{\mu\nu} v^\nu = g_{\mu\nu} \nabla_\gamma v^\nu$$

  or

  $$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Proof. See Carroll (p.99) for an explicit construction of the torsion free, metric compatible connection in terms of the components of g. The formula is

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

□

Homework 8. *Prove that the resulting connection is indeed a connection.*

Now to prove that the two definitions of a connection coincide.

From the local to the global definition is trivial, so we wish to prove that we can localize the global definition.

Proof. Consider a smooth connection ∇ on M. Let $U \subset M$ be open, and Y a smooth vector field on M, X a smooth vector field on U.

Now, for $p \in U$, choose a smooth function η on M such that $\eta = 1$ in a neighborhood V_1 of p, and $\eta = 0$ on $M \setminus V_2$ with $\overline{V_1} \subset V_2$, $\overline{V_2} \subset U$ and $\overline{V_i}$ compact.

Homework 9. *Construct a one-dimensional smooth bump function on* $\mathbb{R}$

Now, set $\tilde{X} = \eta X$, which is defined globally on M. We can now define

$$\nabla_X Y|_{V_1} = \nabla_{\tilde{X}} Y|_{V_1}$$

and we can do this for every point $p \in M$. Now, we must show that such a construction is unique.

Suppose instead that we chose a different V_1', V_2', η'. We have a new globally-defined vector field $X' = \eta' X$, and we wish to show that $\nabla_{\tilde{X}} Y = \nabla_{X'} Y$ at p.

So, we construct

$$\nabla_{\tilde{X}}(Y) - \nabla_{X'}(Y) = \nabla_{\tilde{X}-X'} Y$$

Now, we know that $\tilde{X} - X'$ is zero at (and nearby) p, so

$$\tilde{X} - X' = \zeta(\tilde{X} - X')$$

Homework 10. *Construct* ζ.

So, we have that

$$\begin{aligned}\nabla_{\tilde{X}-X'} &= \nabla_{\zeta(\tilde{X}-X')} \\ &= \zeta \nabla_{\tilde{X}-X'} \\ &= 0\end{aligned}$$

and so they agree around p.

Next, consider $p \in M$, with Y a smooth vector field. Choose a coordinate chart (U, ϕ) around p, with $v \in T_pM$, $v = v^i \partial_i$.

Then, we set $\nabla_v Y = \nabla_{v^i \partial_i} Y = v^i \nabla_{\partial_i} Y$, where we have already defined what ∇_{∂_i} should be, since ∂_i is a locally defined vector field.

Now, we need to show this is independent of coordinate charts. Let (V, ψ) be another coordinate chart, with $v = v^j \partial_j'$ for coordinate field ∂_i'. The claim is that

$$v^i \nabla_{\partial_i} Y = v^j \nabla_{\partial_j'} Y$$

which is easily verified, since $J(\partial \to \partial') \nabla_{\partial_i} = \nabla_{\partial_j'}$, and so

$$v^j \nabla_{\partial_j'} = v^j \nabla_{J(\partial \to \partial')^j_i \partial_j}$$

but $v^i = J^i_j b^j$, and so they agree. $\square$

2.8 The Levi-Cevita Connection

Recall that we have a unique torsion-free, metric compatible connection ∇ for any Riemannian manifold. We wish to localize this ∇ further.

Definition 2.5. *Let γ be a smooth curve in M. A vector field X* along γ *is an assignment $X : I \to TM$ with $X(t) \in T_{\gamma(t)}M$ where X is called smooth if its coordinate decomposition*

$$X = \xi^i(t)\partial_i$$

is smooth in each component.

Definition 2.6. *$\nabla_{\partial_t}X$ is define along γ as follows: Let I_{t_0} be an open interval around t_0, which maps into chart (U, ϕ). Then,*

$$\begin{aligned}\nabla_{\partial_t}X &= \nabla_{\partial_t}\xi^i(t)\partial_i \\ &= \partial_t\xi^i(t)\partial_i + \xi^i(t)\nabla_{\partial_t}\partial_i \\ &= \partial_t\xi^i(t)\partial_i + \xi^i(t)\nabla_{\partial_t\gamma}\partial_i\end{aligned}$$

which is already defined.

The second term in this expansion turns into

$$\xi^i(t)\nabla_{\partial_t\gamma}\partial_i = \xi^i\partial_t x^j\nabla_{\partial_j}\partial_i$$

and we define

$$\Gamma^k_{ij}\partial_k = \nabla_{\partial_j}\partial_i$$

Where Γ^k_{ij} is the Christoffel symbol (of the first kind) for the connection.

Homework 11. *Show that for the Levi-Civita connection,*

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l})$$

2.9 The Connection in Local Coordinates

Definition 2.7. *The connection forms of the connection ω^j_i associated with an orthonormal frame e_i is defined as*

$$\nabla e_i = \omega^j_i \otimes e_j$$

Knowing that the frame is orthonormal and the connection is metric compatible, we get

$$\begin{aligned}\langle e_i, e_j\rangle &= \delta_{ij} \\ \langle \nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle &= 0 \\ \langle \omega^k_i(X)e_k, e_j\rangle + \langle e_i, \omega^l_j(X)e_l\rangle &= 0 \\ \omega^j_i(X) + \omega^i_j(X) &= 0\end{aligned}$$

and so ω^i_j is antisymmetric.

Theorem 4. *The following holds for the connection forms:*

- ω *is antisymmetric*
- $d\omega^i = \omega^i_j \wedge \omega^j$

To prove this, we can use the identity

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

for one-forms α.

2.10 Parallel Transport

Let X be a vector field on M along γ.

Definition 2.8. *X is called parallel if*

$$\nabla_{\partial_t} X = 0$$

Theorem 5. *For each $v \in T_{\gamma(0)}M$, there is a unique solution to the initial value problem*

$$\nabla_{\partial_t} X = 0$$
$$X(0) = v$$

Proof. Let (U,ϕ) be a coordinate chart around $\gamma(0)$. Then, in U,

$$\nabla_{\partial_t} X = 0$$

is the same as

$$\partial_t \xi^k + \Gamma^k_{ij} \xi^i \partial_t x^j$$

which is a first order linear ODE with smooth coefficients, and so it has unique solutions for the initial value $X(0) = v$, or $\xi^i(0) = v^i$. □

Now, since the ODE is linear, there is a linear map between initial values and solutions, that is we have a linear map from $T_{\gamma(0)}M$ to $T_{\gamma(1)}M$ by evaluating X at 1. This map is the parallel transport map, and it is invertible by running the curve backwards. Thus, this map is an isomorphism. Even better...

Proposition 1. *The parallel transport map is an isometry.*

Homework 12. *Prove that*

$$\partial_t g(X,Y) = g(\nabla_{\partial_t} X, Y) + g(X, \nabla_{\partial_t} Y)$$