Problem Set 5

Daniel Halmrast

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PROBLEM 1

Show that for $(X, \mathscr{T})$ a compact Hausdorff space, any topology $\mathscr{T} \subsetneq \mathscr{T}'$ makes $(X, \mathscr{T}')$ no longer compact Hausdorff. Furthermore, for $\mathscr{T}' \subsetneq \mathscr{T}$, the space $(X, \mathscr{T}')$ is not compact Hausdorff.

Proof. For this proof, we will show that for $\mathscr{T}, \mathscr{T}'$ topologies on X with $\mathscr{T} \subset \mathscr{T}'$ and the property that X is compact Hausdorff under both these topologies, then they must be equal.

To do so, we consider the identity function $id : (X, \mathscr{T}') \to (X, \mathscr{T})$, which is continuous since $\mathscr{T} \subset \mathscr{T}'$. Now, for any C closed in $(X, \mathscr{T}')$, since $(X, \mathscr{T}')$ is compact and id is continuous, $id(C) = C$ is compact in $(X, \mathscr{T})$. However, since $(X, \mathscr{T})$ is Hausdorff, C must also be closed. Thus, id is a closed map, and thus a homeomorphism, and the two topologies must be equal. □

PROBLEM 2

Prove that for compact subsets A, B of X, their union $A \cup B$ is compact as well.

Proof. Let $\mathscr{O}$ be an open cover of $A \cup B$. In particular, $\mathscr{O}$ covers A, and has a finite subset $\mathscr{O}_A$ that covers A. Similarly, there is a finite subset $\mathscr{O}_B$ that covers B. Their union $\mathscr{O}_A \cup \mathscr{O}_B$ covers $A \cup B$, and is finite, as it is the union of finite sets. It is also a subcover, since both $\mathscr{O}_A$ and $\mathscr{O}_B$ are subsets of $\mathscr{O}$. Thus. $\mathscr{O}$ has a finite subcover, and $A \cup B$ is compact as desired. □

PROBLEM 3

Suppose A is a subset of a metric space. Show that A is compact implies that A is closed and bounded. Give an example where the converse does not hold.

Proof. Suppose A is a compact subset of a metric space (X, d). Since all metric spaces are Hausdorff, it follows immediately that A is closed (since it is a compact subset of a Hausdorff space).

Now, fix $x \in A$, and consider the open cover $\mathscr{O} = \{V_n(x) \mid n \in \mathbb{N}\}$ of balls centered at x with radius n. This has a finite subcover, by compactness of A. So, let N be the largest radius in the finite subcover. For each $V_m(x)$ in the finite subcover, then, we have that $V_m(x) \subset V_N(x)$. Thus, since the subcover covers A, it follows that $V_N(x)$ covers A, and thus A is bounded.

For an example where the converse fails, consider the metric space $\square$