Topology

Problem Set 3

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Problem 1

Prove that the 1-norm on $\mathbb{R}^n$ defines a metric on $\mathbb{R}^n$ that is equivalent to the standard 2-norm metric on $\mathbb{R}^n$.

Proof. Let d_1 be the metric induced by the 1-norm on $\mathbb{R}^n$. Clearly, d_1 is positive definite, since it comes from a norm. So, let's show it satisfies the triangle inequality.

In proving the triangle inequality, we first state a general property of norms. The so-called triangle inequality of norms is given as

$$|x+y| \leq |x| + |y|$$

which is true for any normed space.

Let x, y, z be distinct points in $\mathbb{R}^n$ with coordinates x^i, y^i, z^i. Then we have that

$$\begin{aligned}
d(x,z) &= \sum_i |x^i - z^i| \\
&= \sum_i |x^i - z^i + y^i - y^i| \\
&= \sum_i |(x^i - y^i) + (y^i - z^i)| \\
&\leq \sum_i |x^i - y^i| + |z^i - y^i| \\
&= \sum_i |x^i - y^i| + \sum_i |z^i - y^i| \\
&= d(x,y) + d(y,z)
\end{aligned}$$

and thus the metric satisfies the axioms for a metric.

Now, let's show that the metric is equivalent to the standard 2-norm metric on $\mathbb{R}^n$. To do this, we will show that each point in a standard n-ball has a 1-norm ball contained in the n-ball, and vice versa.

So, without loss of generality (via translation) let $B_r(0)$ be the open ball of radius r around 0, and let $x \in B_r(0)$. In particular, there is some $\delta > 0$ such that $d(x, 0) < r - \delta$. Now, take C_δ to be the 1-norm ball of radius δ. Now, if $y \in C_\delta$, then we have that

$$\begin{aligned} d(x,y) &= \sum_i |x^i - y^i| \\ &< \delta \\ \implies (\sum_i |x^i - y^i|)^2 &< \delta^2 \\ \implies \sum_i (|x^i - y^i|)^2 &< \delta^2 \\ \implies d_2(x,y) < \delta \implies d_2(y,0) &< d_2(x,0) + d_2(x,y) \\ &< r - \delta + \delta \\ &< r \end{aligned}$$

so, the 1-ball of radius δ is contained in $B_r(0)$ as desired. Thus, since each $x \in B_r(0)$ has a neighborhood (in 1-norm) contained in $B_r(0)$, $B_r(0)$ is open in the 1-norm topology.

For the other way, we first prove the more general fact about norms on $\mathbb{R}^n$.

Lemma 1. *There exists a constant C such that for all $x \in \mathbb{R}^n$,*

$$||x||_1 \leq C||x||_2$$

Proof. We first observe the basic fact that, for $x_1, x_2 \in \mathbb{R}^+$, we have

$$2x_1x_2 \leq x_1^2 + x_2^2$$

Now, it follows quickly that

$$\begin{aligned} ||x||^2 = \left(\sum_{i=1}^n |x_i|\right) &= \sum_{i=1}^n |x_i|^2 + \sum_{i \neq j} 2|x_i||x_j| \\ &\leq \sum_{i=1}^n |x_i|^2 + (n-1)\sum_{i=1}^n |x_i|^2 \\ &= n\sum_{i=1}^n |x_i|^2 \end{aligned}$$

Thus $\sqrt{n}$ is a constant for which the lemma holds. ☐

Now, since we have a bound on the norms, we can prove that a 1-norm ball is open in the 2-norm. To do so, let $\Delta_r(0)$ be the 1-norm ball of radius r at zero, and let $x \in \Delta_r(0)$. In particular, we have that there exists a δ such that $d_1(x, 0) < r - \delta$. Now, let $\varepsilon = \frac{\delta}{\sqrt{n}}$, and consider the 2-norm ball $V_\varepsilon(x)$. Then, we will show that $V_\epsilon(x) \subset \Delta_r(0)$. To do so, let $y \in V_\epsilon(x)$, and observe that

$$\begin{aligned} d_1(x,y) &< \sqrt{n} d_2(x,y) \\ &< \sqrt{n}\frac{\delta}{\sqrt{n}} \\ &= \delta \end{aligned}$$

and

$$\begin{aligned} d_1(0,y) &\leq d_1(0,x) + d_1(x,y) \\ &\leq r - \delta + \delta \\ &= r \end{aligned}$$

as desired. ☐

Problem 2

Munkres Problem 4

Consider the box, uniform, and product topologies on $\mathbb{R}^\omega$.

Part A

In which topologies are the following functions continuous?

$$f(t) = (t, 2t, 3t, \ldots)$$
$$g(t) = (t, t, t, \ldots)$$
$$h(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \ldots)$$

Proof. We first note that the universal property of product spaces guarantees that a function f is continuous in the product topology if and only if its component functions $\pi_i \circ f$ are continuous. Since this is true for all three of f, g, h, it follows that they are all continuous in the product topology.

For the remainder of this problem, we will use the pointwise definition of continuity. That is, given a point $x \in \mathbb{R}$, f is convergent at x if and only if for each neighborhood U of $f(x)$, we have that $f^{-1}(U)$ contains a neighborhood of x.

For $f(t)$, let's consider the basic open neighborhood

$$U_t = \prod_i V_{\varepsilon_i}(it)$$

Where $\varepsilon_i = \frac{1}{i^2}$. In particular, the inverse image of U_t is just $\{t\}$ itself, since

$$f_i^{-1}(V_{\varepsilon_i}(it)) = V_{\frac{\varepsilon}{i}}(t) = V_{\frac{1}{i}}(t) \to \{t\}$$

□

Munkres Problem 5

Munkres Problem 6