Topology

Problem Set 2

Daniel Halmrast

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Problem 1

Prove that there is an embedding of X into $X \times Y$.

Proof. For this proof, $\{\bullet\}$ will represent the one-point set.

To start with, we will prove the following lemma:

Lemma. *For X any topological space, $X \cong X \times \{\bullet\}$.*

Proof. By the definition of the product space, the projection maps

$$
\begin{tikzcd}
& X \times \{\bullet\} \arrow[dl, "\pi_x"] \arrow[dr, "\pi_\bullet"'] & \\
X & & \{\bullet\}
\end{tikzcd}
$$

exist and are continuous open maps. Now, all we need to show is that π_x is injective, and it will follow immediately that it is a homeomorphism.

To see this, let $x \in X$ and consider $\pi_x^{-1}(\{x\}) = \{(x, \bullet)\}$. Since the inverse image of a singleton is again a singleton, the function is injective.

Thus, X is homeomorphic to $X \times \{\bullet\}$. $\square$

Now, let $f : \{\bullet\} \to Y$ be a continuous function. Consider the diagram:

$$
\begin{tikzcd}
& X \arrow[d, leftrightarrow, "\cong"] & \\
& X \times \{\bullet\} \arrow[d, dashed, "id \times f"] \arrow[ddl, bend right, "id"'] \arrow[ddr, bend left, "f"] & \\
& X \times Y \arrow[dl, "\pi_x"] \arrow[dr, "\pi_y"'] & \\
X & & Y
\end{tikzcd}
$$

where id and f are the obvious extensions $id(x, \bullet) = id(x) = x$ and $f(x, \bullet) = f(\bullet)$. Here, the product map $id \times f$ is continuous by the universal property of products. Now, we just need to show that $id \times f$ is injective with a continuous inverse on its image.

To see that $id \times f$ is injective, consider a point $(id(x), f(\bullet))$ in the image of $id \times f$, and consider its preimage:

$$(id \times f)^{-1}(\{(id(x), f(\bullet))\}) = \{(x, \bullet)\}$$

Since the preimage of any singleton is again a singleton, the function $id \times f$ is injective.

Now, lets consider the diagram

$$\begin{tikzcd}
& X \times Y \arrow[d, dashed, "\pi_x \times c"] \arrow[ddl, bend right, "\pi_x"'] \arrow[ddr, bend left, "c"] & \\
& X \times \{\bullet\} \arrow[dl, "\pi_x"'] \arrow[dr, "pi_\bullet"] & \\
X & & \{\bullet\}
\end{tikzcd}$$

where c is unique constant function from Y to the terminal object $\{\bullet\}$.

Here, the dashed arrow $\pi_x \times c$ is continuous by the universal property of products. It is easy to see that $\pi_x \times c|_{(id \times f)(X \times \{\bullet\})}$ is the inverse of $id \times f$ on the image of $id \times f$.

Hence, since the inverse of $id \times f$ is continuous, $id \times f$ is an embedding of $X \cong X \times \{\bullet\}$ into $X \times Y$. $\square$

Problem 2

Prove that every open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Proof. Consider an open interval $(a, b) \subset \mathbb{R}$. It is easy to see that $(a, b) \cong (-1, 1)$, since the operations of scaling and translation are continuous functions with continuous inverses.

Thus, all we need to prove is that $(-1, 1) \cong \mathbb{R}$. To see this, consider the function

$$\tan(\frac{\pi}{2}x)$$

defined on $(-1, 1)$, which is a continuous bijection with continuous inverse. (proofs for the continuity of tan and arctan are easily given by basic analysis arguments, and will not be reproduced here.) $\square$

Problem 3

Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous at exactly one point.

Proof. The function

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = x\chi_{\mathbb{Q}}(x)$$

is continuous only at zero. To see this, we will use the neighborhood definition of continuity. That is, f is continuous at x if for each neighborhood of $f(x)$, its preimage contains a neighborhood of x.

First, we will prove that f is continuous at zero. It suffices to show that each basic open neighborhood of $f(x)$ has a preimage that contains an open neighborhood of x. So, let $(-\varepsilon, \varepsilon)$ be a basic neighborhood of $f(0) = 0$. Then,

$$f^{-1}((-\varepsilon, \varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon, \varepsilon)$$

which contains $(-\varepsilon, \varepsilon)$ an open neighborhood of 0 as desired.

Now, let $x \neq 0$. We will show that f is not continuous at x. If x is irrational, then $f(x) = 0$. Now, choose ε so that $x \notin (-\varepsilon, \varepsilon)$. Then, by the above calculation, we have

$$f^{-1}((-\varepsilon, \varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon, \varepsilon)$$

which does not contain any neighborhood of x.

If x is rational, then $f(x) = x$. Choose ε such that $0 \notin V_\varepsilon(x)$. Then,

$$f^{-1}(V_\varepsilon(x)) = V_\varepsilon(x) \cap \mathbb{Q}$$

which does not contain any open neighborhood of x (This is easily seen by observing that any neighborhood of x must intersect $\mathbb{R} \setminus \mathbb{Q}$, but the inverse image contains only rational points). $\square$

PROBLEM 4

Suppose Y is Hausdorff, and $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are continuous. If $f|_A = g|_A$ for a dense subset $A \subset Y$, prove that $f = g$.

Proof. Let f and g be parallel morphisms that satisfy the assumptions.

Now, let $y \in Y$. Since A is dense, $y \in \overline{A}$, so there exists some net $\{y_\alpha\}$ such that $y_\alpha \in A$ for all α and $y_\alpha \to y$. In particular, since Y is Hausdorff, this net converges to the unique limit y.

By the hypothesis, $f(y_\alpha) = g(y_\alpha)\ \forall \alpha$, and since both f and g are continuous, they preserve limits. That is $f(y_\alpha) \to f(y)$ and $g(y_\alpha) \to g(y)$. Since $f(y_\alpha) = g(y_\alpha)$ for all α and limits of nets in Y are unique, they must converge to the same element, and $f(y) = g(y)$.

Since this works for all $y \in Y$, $f = g$. $\square$

PROBLEM 5

Prove that if A_α is a closed subset of X_α for all α, then $\prod A_\alpha$ is closed in $\prod X_\alpha$.

Proof. To show that $\prod A_\alpha$ is closed, we need to show that it contains its limit points. To do so, let $\{a_\gamma\}$ be a convergent net in the product $\prod A_\alpha$. In particular, each of its projections $\pi_\alpha(a_\gamma)$ is also a net in A_α, and since A_α is closed, this net converges to elements in A_α.

Thus, each coordinate α of the net $\{a_\gamma\}$ converges in A_α, so any limit point must have coordinates in the A_α as well. That is, if a is a limit point of $\{a_\gamma\}$, then for each α, $\pi_\alpha(a) \in A_\alpha$, which means that $a \in \prod A_\alpha$ as desired.

Since $\prod A_\alpha$ contains all its limit points, it is closed. $\square$

PROBLEM 6

Let $y \in \prod X_\alpha$, and $\{x_n\}$ a sequence of points in $\prod X_\alpha$. Show that $x_n \to y$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(y)$ for all α.

Proof. (=>) For the first direction, assume that $x_n \to y$. Since each π_α is continuous, they preserve limits. Thus, for each α, $\pi_\alpha(x_n) \to \pi_\alpha(y)$ as desired.

(<=) For the other direction, let $\{x_n\}$ be such that for all α, $\pi_\alpha(x_n) \to \pi_\alpha(y)$. In particular, this means that the filter $\mathscr{F} = \{A \subset \prod X_\alpha \mid \exists n \in \mathbb{N} : \ x_m \in A \ \forall m > n\}$ pushes forward along each π_α to a filter that converges to $\pi_\alpha(y)$.

Now, we just need to show that $\mathscr{F}$ converges to y (Equivalently, that $\mathscr{F}$ contains each neighborhood of y). To do so, we will show that each neighborhood of y contains an element of $\mathscr{F}$, then since $\mathscr{F}$ is a filter, it is closed under supersets and contains each neighborhood of y.

So, let U be a neighborhood of y. In particular, there exists a basis element

$$B = V_1 \times V_2 \times \ldots \times V_n \times X \times X \ldots \subset U$$

Now, since the push-forward of $\mathscr{F}$ along each projection is a convergent filter, $N_\alpha \in \pi_{\alpha *}(\mathscr{F})$ for each neighborhood $N_\alpha$of $\pi_\alpha(y)$.

In particular, $V_i \in \pi_{\alpha *}(\mathscr{F})$, which means that $\pi_\alpha^{-1}(V_i) \in \mathscr{F}$. Now, we can write B as

$$B = \bigcap_{i=1}^{n} \pi_\alpha^{-1}(V_i)$$

which is a finite intersection of elements of $\mathscr{F}$, so $B \in \mathscr{F}$. Thus, $U \supset B$ is in $\mathscr{F}$ as well. Since U was any neighborhood of y, the neighborhood filter $\mathscr{N}_y \subset \mathscr{F}$ and $\mathscr{F} \to y$ as desired. □

Problem 7

Let $\mathbb{R}^\omega$ be the space of sequences of real numbers, and let $\mathbb{R}^\infty$ be the space of sequences that are eventually zero. What is $\overline{\mathbb{R}^\infty} \subset \mathbb{R}^\omega$?

Proof. We will show $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

Let U be an open neighborhood of $x \in \mathbb{R}^\omega$. In particular, there is a basis element

$$B = V_1 \times V_2 \times \ldots \times V_n \times \mathbb{R} \times \mathbb{R} \times \ldots$$

such that $x \in B$ and $B \subset U$. We will now show that B intersects nontrivially with $\mathbb{R}^\infty$.

Consider the element $(\pi_q(x), \pi_2(x), \ldots, \pi_n(x), 0, 0, \ldots) \in \mathbb{R}^\infty$. Since $x \in B$, it follows that $\pi_i(x) \in V_i$ for $i = 1 \ldots n$. Thus, since we also know that $0 \in \mathbb{R}$, it follows that $(\pi_q(x), \pi_2(x), \ldots, \pi_n(x), 0, 0, \ldots) \in$ B as well. Thus, B intersects $\mathbb{R}^\infty$ nontrivially, as desired.

Since every open neighborhood of x intersects $\mathbb{R}^\infty$ nontrivally, x is in the closure of $\mathbb{R}^\infty$ for all x in $\mathbb{R}^\omega$ as desired. □

Problem 8

Show that the metric topology is the coarsest topology for which the distance function is continuous.

Proof. First, we show that the distance function is continuous in the metric topology. Let (X, d) be a metric space, and consider $d : X \times X \to \mathbb{R}^+$. To show that d is continuous, let $V_\varepsilon(r) \subset \mathbb{R}$ be a basic open set (with r a positive number), and observe that

$$d^{-1}(V_\varepsilon(r)) = \{(x, y) \mid d(x, y) \in V_\varepsilon(r)\}$$

Now, let $(x,y) \in d^{-1}(V_\varepsilon(r))$, and choose a δ such that $d(x,y) < r + \varepsilon - \delta$. The neighborhood $V_{\frac{\delta}{2}}(x) \times V_{\frac{\delta}{2}}(y)$ contains (x,y) and is contained in $d^{-1}(V_\varepsilon(r))$. This is clear from the triangle inequality, since for $(x',y') \in V_{\frac{\delta}{2}}(x) \times V_{\frac{\delta}{2}}(y)$, we have

$$\begin{aligned} d(x,x') &< \frac{\delta}{2} \\ d(y,y') &< \frac{\delta}{2} \\ d(x,y) &< r + \varepsilon - \delta \end{aligned}$$

Thus,

$$\begin{aligned} d(x',y') &< d(x',y) + d(y,y') \\ &< d(x,y) + d(x,x') + d(y,y') \\ &< r + \varepsilon - \delta + \frac{\delta}{2} + \frac{\delta}{2} \\ &= r + \varepsilon \end{aligned}$$

as desired.

Thus, d is a continuous function in the product topology of $X \times X$.

Now, let $\mathscr{T}$ be any topology for which d is continuous. Fix$\varepsilon > 0$. Then,

$$d^{-1}(V_\varepsilon) = \{(x,y) \mid d(x,y) < \varepsilon\}$$

is open as well, by continuity of d. Then, it follows that, for fixed x, the set $\{y|d(x,y) < \varepsilon\}$ is open too. (This is clear, since $\{y \mid d(x,y) < \varepsilon\}$ is open in $\{x\} \times X$ in the subspace topology, and from a previous problem, it is clear that $\{x\} \times X$ is homeomorphic to X, and thus $V_\varepsilon(y)$ is open in X). However, since this works for all $\varepsilon > 0$ and $x \in X$, it follows that the metric topology is coarser than $\mathscr{T}$. $\square$