Problem Set 2

Daniel Halmrast

October 18, 2017

PROBLEM 1

Prove that there is an embedding of X into $X \times Y$.

Proof. For this proof, $\{\bullet\}$ will represent the one-point set.

To start with, we will prove the following lemma:

Lemma. For X any topological space, $X \cong X \times \{\bullet\}$.

Proof. By the definition of the product space, the projection maps

$$X \times \{\bullet\}$$

$$x \qquad \qquad \pi_x \qquad \pi_{\bullet}$$

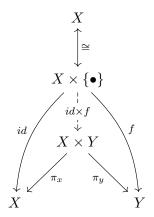
$$X \qquad \qquad \{\bullet\}$$

exist and are continuous open maps. Now, all we need to show is that π_x is injective, and it will follow immediately that it is a homeomorphism.

To see this, let $x \in X$ and consider $\pi_x^{-1}(\{x\}) = \{(x, \bullet)\}$. Since the inverse image of a singleton is again a singleton, the function is injective.

Thus, X is homeomorphic to $X \times \{\bullet\}$.

Now, let $f: \{\bullet\} \to Y$ be a continuous function. Consider the diagram:

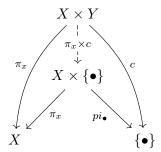


where id and f are the obvious extensions $id(x, \bullet) = id(x) = x$ and $f(x, \bullet) = f(\bullet)$. Here, the product map $id \times f$ is continuous by the universal property of products. Now, we just need to show that $id \times f$ is injective with a continuous inverse on its image.

To see that $id \times f$ is injective, consider a point $(id(x), f(\bullet))$ in the image of $id \times f$, and consider its preimage:

$$(id \times f)^{-1}(\{(id(x), f(\bullet))\}) = \{(x, \bullet)\}$$

Since the preimage of any singleton is again a singleton, the function $id \times f$ is injective. Now, lets consider the diagram



where c is unique constant function from Y to the terminal object $\{\bullet\}$.

Here, the dashed arrow $\pi_x \times c$ is continuous by the universal property of products. It is easy to see that $\pi_x \times c|_{(id \times f)(X \times \{\bullet\})}$ is the inverse of $id \times f$ on the image of $id \times f$.

Hence, since the inverse of $id \times f$ is continuous, $id \times f$ is an embedding of $X \cong X \times \{\bullet\}$ into $X \times Y$.

PROBLEM 2

Prove that every open interval in \mathbb{R} is homeomorphic to \mathbb{R} .

Proof. Consider an open interval $(a,b) \subset \mathbb{R}$. It is easy to see that $(a,b) \cong (-1,1)$, since the operations of scaling and translation are continuous functions with continuous inverses.

Thus, all we need to prove is that $(-1,1) \cong \mathbb{R}$. To see this, consider the function

$$\tan(\frac{\pi}{2}x)$$

defined on (-1,1), which is a continuous bijection with continuous inverse. (proofs for the continuity of tan and arctan are easily given by basic analysis arguments, and will not be reproduced here.)

Problem 3

Give an example of a function from \mathbb{R} to \mathbb{R} that is continuous at exactly one point.

Proof. The function

$$f: \mathbb{R} \to \mathbb{R}$$
$$f(x) = x\chi_{\mathbb{Q}}(x)$$

is continuous only at zero. To see this, we will use the neighborhood definition of continuity. That is, f is continuous at x if for each neighborhood of f(x), its preimage contains a neighborhood of x.

First, we will prove that f is continuous at zero. It suffices to show that each basic open neighborhood of f(x) has a preimage that contains an open neighborhood of x. So, let $(-\varepsilon, \varepsilon)$ be a basic neighborhood of f(0) = 0. Then,

$$f^{-1}((-\varepsilon,\varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon,\varepsilon)$$

which contains $(-\varepsilon, \varepsilon)$ an open neighborhood of 0 as desired.

Now, let $x \neq 0$. We will show that f is not continuous at x. If x is irrational, then f(x) = 0. Now, choose ε so that $x \notin (-\varepsilon, \varepsilon)$. Then, by the above calculation, we have

$$f^{-1}((-\varepsilon,\varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon,\varepsilon)$$

which does not contain any neighborhood of x.

If x is rational, then f(x) = x. Choose ε such that $0 \notin V_{\varepsilon}(x)$. Then,

$$f^{-1}(V_{\varepsilon}(x)) = V_{\varepsilon}(x) \cap \mathbb{Q}$$

which does not contain any open neighborhood of x (This is easily seen by observing that any neighborhood of x must intersect $\mathbb{R}\setminus\mathbb{Q}$, but the inverse image contains only rational points). \square

PROBLEM 4

Suppose Y is Hausdorff, and $X \xrightarrow{f \atop g} Y$