Homework 4

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Problem 1

Let $\{H_\gamma \mid \gamma \in \Gamma\}$ be a family of Hilbert spaces, and let H be the vector space of sections of $\cup_{\gamma\in\Gamma} H_\gamma$ over Γ with

$$f \in \Gamma(\cup H_\gamma, \Gamma)$$
$$\sum_{\gamma\in\Gamma} \|f(\gamma)\|^2 < \infty$$

Show that

$$\|f\| = \left(\sum_{\gamma\in\Gamma} \|f(\gamma)\|^2\right)^{\frac{1}{2}}$$

is a norm on H, and that with this norm H is a Euclidean space. Is H necessarily a Hilbert space?

Proof. We immediately recognize this construction as the direct integral

$$\int_\Gamma^\oplus H_\gamma d\mu$$

where $\mu = \mu_c$ is the counting measure on Γ. This space is defined to be the set of all sections $\Gamma(\coprod_{\gamma\in\Gamma} H_\gamma, \Gamma)$ over Γ with the property that if f is a section, then its composition

$$g_\gamma(f(\gamma), f(\gamma)) = \|f(\gamma)\|^2 \in L^2(\Gamma, \mu_c)$$

(with g_γ the metric on H_γ) is required to be an L^2 function. In fact, it is clear that this is the exact scenario described in the hypothesis of the problem. So, for the remainder of this problem, we will let (Γ, μ) be an arbitrary measure space, and prove the more general result that

$$H = \int_\Gamma^\oplus H_\gamma d\mu(\gamma)$$

is a Hilbert space. Note that in this more general setting, H is actually a set of equivalence classes of sections where $s\ t \iff \langle (s-t)|(s-t)\rangle = 0$. That is, s is equivalent to t if and only if $s(\gamma)$ and $t(\gamma)$ differ on a set of Hilbert spaces of measure zero.

An inner product on the direct integral can be defined as

$$\langle s|t\rangle := \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma)$$

where it is understood that $\langle s(\gamma)|t(\gamma)\rangle = g_\gamma(s(\gamma), t(\gamma))$. We need to show that this inner product is indeed a well-defined inner product on $\int_\Gamma^\oplus H_\gamma d\mu$.

We first show that this is well-defined. That is, we need to show that the integral is finite. To see this, let $s, t \in H$. In particular, this means that

$$\begin{aligned} \|s(\gamma)\|_\gamma &\in L^2(\Gamma, \mu) \\ \|t(\gamma)\|_\gamma &\in L^2(\Gamma, \mu) \end{aligned}$$

We calculate the inner product as

$$\begin{aligned} |\langle s|t\rangle| &= |\int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma)| \\ &\leq \int_\Gamma |\langle s(\gamma)|t(\gamma)\rangle| d\mu(\gamma) \\ &\leq \int_\Gamma \|s(\gamma)\|_\gamma \|t(\gamma)\|_\gamma d\mu(\gamma) && \text{By Cauchy-Schwarz inequality} \\ &= \|\left(\|s(\gamma)\|_\gamma\right)\left(\|t(\gamma)\|_\gamma\right)\|_1 && \text{By definiton of } L^1 \text{ norm} \\ &\leq \|\left(\|s(\gamma)\|_\gamma\right)\|_2 \|\left(\|t(\gamma)\|_\gamma\right)\|_2 && \text{By Holder's inequality with } p = q = 2 \\ &\leq \infty && \text{Since } \|s(\gamma)\|_\gamma \text{ and } \|t(\gamma)\|_\gamma \text{ are in } L^2(\Gamma, \mu) \end{aligned}$$

and thus, the proposed inner product is well-defined.

Next, we show sesquilinearity. We adopt the mathematics convention that the inner product $\langle s|t\rangle$ is linear in the first term, and conjugate linear in the second term. Let $s, t \in H$, and let $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned} \langle \alpha s|t\rangle &= \int_\Gamma \langle \alpha s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\ &= \int_\Gamma \alpha \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\ &= \alpha \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\ &= \alpha \langle s|t\rangle \end{aligned}$$

Furthermore, with $r \in H$ as well, we have

$$\begin{aligned} \langle r+s|t\rangle &= \int_\Gamma \langle r(\gamma) + s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\ &= \int_\Gamma \langle r(\gamma)|t(\gamma)\rangle + \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\ &= \int_\Gamma \langle r(\gamma)|t(\gamma)\rangle d\mu(\gamma) + \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\ &= \langle r|t\rangle + \langle s|t\rangle \end{aligned}$$

and so the proposed inner product is linear in the first term. Furthermore, we see that

$$\begin{aligned}\langle s|t\rangle &= \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\ &= \int_\Gamma \overline{\langle t(\gamma)|s(\gamma)\rangle} d\mu(\gamma) \\ &= \overline{\int_\Gamma \langle t(\gamma)|s(\gamma)\rangle d\mu(\gamma)} \\ &= \overline{\langle t|s\rangle}\end{aligned}$$

and so the proposed inner product is conjugate linear in the second term.

Finally, we need to show this inner product is positive-definite. That is, we need to show

$$\langle s|s\rangle \geq 0$$

with equality if and only if $s = 0$. So, let $s \in H$. Trivially, if $s = 0$, then

$$\langle s|s\rangle = \int_\Gamma \langle s(\gamma)|s(\gamma)\rangle d\mu(\gamma) = \int_\Gamma 0 = 0$$

So, let $s \neq 0$. □