Problem Set 4

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Problem 1

For $E \subset \Omega$ measurable, prove the implication

$$\int_E f d\mu = 0 \ \forall f \geq 0 \implies \mu(E) = 0$$

Proof. This follows immediately by letting $f = \chi_E$, and observing that

$$\begin{aligned} \int_E \chi_E(x) d\mu(x) &= \int_\Omega \chi_E(x)\chi_E(x) d\mu(x) \\ &= \int_\Omega \chi_E(x) d\mu(x) \\ &= \mu(E) \end{aligned}$$

Which is zero by the hypothesis. Thus, $\mu(E) = 0$ as desired. □

Problem 2

For $f \geq 0$ measurable on Ω with $\mu(\Omega) > 0$, show that

$$\int_\Omega f(x) d\mu(x) = 0 \implies [f] = 0$$

(i.e. f is zero μ-almost everywhere).

Proof. Consider the equivalence class $[f]$ in $L^1(\Omega, \mu)$. In particular, since $\int_\Omega |f| d\mu = ||f||_1 = 0$, we must have that $[f] = 0$, which means f agrees with the 0 function μ-almost everywhere. It follows immediately, then, that f is zero μ-almost everywhere. $\square$

Problem 3

Use Fatou's lemma to show that for a sequence $\{f_n\}$ of positive measurable functions, the inequality

$$\int_\Omega \liminf f_n d\mu \leq \liminf \int_\Omega f_n d\mu$$

Proof. We note first that the inequality is vacuously true if $\liminf \int_\Omega f_n d\mu = \infty$.

So, assume that $\liminf \int_\Omega f_n d\mu = M$ for some positive number M. Then, consider the family of subsequences

$$\{f_{n_i}\}_\epsilon = \{f_n \mid \int_\Omega f_n d\mu < M + \epsilon\}$$

Now, for any ϵ, this defines an infinite subsequence, since if the integrals of the sequence were not frequently below $M+\epsilon$, then $M+\epsilon$ would be an eventual lower bound higher than M, which contradicts M being the $\liminf$ of the integrals.

Now, we apply Fatou's lemma by observing that for each f_{n_i} we have that

$$\int_\Omega f_{n_i} d\mu < M + \epsilon$$

which gives us the upper bound

$$\int_\Omega \liminf f_{n_i} d\mu \leq M + \epsilon$$

Now, a basic property of the $\liminf$ is that for a sequence x_n with a subsequence x_{n_i},

$$\liminf x_n \leq \liminf x_{n_i}$$

Thus, we also have that

$$\int_\Omega \liminf f_n d\mu \leq \int_\Omega \liminf f_{n_i} d\mu \leq M + \epsilon$$

However, since this is true for all $\epsilon > 0$, it must be that

$$\int_\Omega \liminf f_n d\mu \leq \int_\Omega \liminf f_{n_i} d\mu \leq M$$

And by the definition of M, we have the desired inequality

$$\int_\Omega \liminf f_n d\mu \leq M = \liminf \int_\Omega f_n d\mu$$

$\square$

Problem 4

Notes 2-13

Prove that $f \in L^1 \implies |f| < \infty$ μ-almost everywhere, and describe the spaces $L^1(\mathbb{N}, \mu_c)$ and $L^1(\Omega, \delta_p$.

Proof. Let $f \in L^1(\Omega, \mu)$. In particular, we have that $\int_\Omega |f| d\mu < \infty$. And thus, for every measurable set E, we have

$$\int_\Omega f = \int_E f + \int_{E^c} f < \infty$$

Which implies that the integral $\int_E f d\mu$ is bounded as well. Thus, $|f|$ cannot be ∞ on any set of positive measure, since if it were the case that $f = \infty$ on some set E with positive measure, the integral

$$\int_E f d\mu$$

would not be bounded.

Now, in an earlier assignment, we proved that for $f : \mathbb{N} \to \mathbb{R}$, $\int_\mathbb{N} f d\mu_c = \sum_{i=0}^\infty f(i)$. Thus, $L^1(\mathbb{N}, \mu_c)$ is just the space ^{1} of absolutely convergent sequences.

Similarly, we proved in an earlier assignment that for any $f : \Omega \to \mathbb{R}$ measurable, $\int_\Omega f d\delta_p = f(p)$. Thus, $[f(x)] = [f(p)(x)] = [f(p)]$. That is, the equivalence class of a function f is just all functions that are bounded at and agree at p. So, there is exactly one equivalence class for each positive real number, and the norm is just

$$||f(x)||_1 = \int_\Omega |f(x)| d\delta_p(x) = |f(p)|$$

which is the usual norm on $\mathbb{R}$. Thus, $L^1(\Omega, \delta_p) \cong \mathbb{R}$. □

Notes 2-12

Prove that the measure $\phi(E) = \int_E f d\mu$ is a measure.

Proof. First, we observe clearly two things. One, $\phi(\emptyset) = 0$, which follows directly from the definition of the integral. Second, $\phi(E) \geq 0$, which follows since f is a positive function. Now, all we need to show is σ-additivity.

To do so, we first observe that there is a monotone sequence of functions $\{\varphi_n\}$ that converge to f. Thus, it follows that

$$\phi(E) = \int_E \lim \varphi_n d\mu = \lim \int_E \varphi_n d\mu = \lim \nu_n(E)$$

Where $\nu_n(E)$ is the weighted measure of the simple function ϕ_n, given by lemma 3.

Thus, for a sequence $\{E_i\}$ of disjoint measurable subsets, we have

$$\begin{aligned}
\phi\left(\bigcup_{i=1}^{\infty} E_i\right) &= \lim_{n\to\infty} \nu_n\left(\bigcup_{i=1}^{\infty} E_i\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{\infty} \nu_n(E_i) \\
&= \sum_{i=1}^{\infty} \lim_{n\to\infty} \nu_n(E_i) \\
&= \sum_{i=1}^{\infty} \phi(E_i)
\end{aligned}$$

Here, we justify commuting the limit and the sum by observing the following:

$$\lim_n \sum_i \nu_n(E_i) = \lim_n \int_{\mathbb{N}} \nu_n(E_i) d\mu_c(i) = \int_{\mathbb{N}} \lim_n \nu_n(E_i) d\mu_c(i)$$

Which is just a quick application of the monotone convergence theorem on the sequence (in n!) of monotonic functions $\nu_n' : \mathbb{N} \to \mathbb{R}^+$ given as $\nu_n'(i) = \nu_n(E_i)$ which is monotonic by the fact that the φ_n that define it are monotonic.

Thus, ϕ has σ-additivity.

Moreover, let g be a measurable function. We can show that

$$\int_\Omega g d\phi = \int_\Omega f g d\mu$$

This follows from the fact that g can be approximated as a monotonic sequence $\{\psi_n = \sum_{i=1}^k c_i^{(n)} \phi(E_i^{(n)})\}$ of simple functions, and observing that

$$\begin{aligned}
\int_\Omega g d\phi &= \lim_{n\to\infty} \int_\Omega \psi_n d\phi \\
&= \lim_{n\to\infty} \sum_{i=1}^{k} c_i^{(n)} \phi(E_i^{(n)}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{k} c_i^{(n)} \int_\Omega f \chi_{E_i^{(n)}} d\mu \\
&= \int_\Omega f \left(\lim_{n\to\infty} \sum_{i=1}^{k} c_i^{(n)} \chi_{E_i^{(n)}} d\mu \right) \\
&= \int_\Omega f g d\mu
\end{aligned}$$

□

Problem 5

Show that for a finite measure μ and a sequence of measurable functions f_n converging to f uniformly, we have that

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$$

Proof. We observe that this equivalence is merely stating that

$$\lim_{n\to\infty}\int_\Omega |f_n - f|d\mu = 0$$

To see this is true, fix $\epsilon > 0$, and let N be large enough so that for all $i > N$, we have that $\sup_{x\in\Omega}|f_i(x) - f(x)| < \epsilon$.

Now, the integral $\int_\Omega |f_i - f|d\mu$ becomes bounded by

$$int_\Omega |f_i - f|d\mu < \int_\Omega \epsilon d\mu = \epsilon\mu(\Omega)$$

and since this holds for any epsilon, it must be that the limit of the integral $\int_\Omega |f_n - f|d\mu$ is zero as well.

However, in this proof, we used the fact that $\mu(\Omega) < \infty$. In general, this theorem will not hold. To see this, consider the sequence of functions

$$f_n(x) = \begin{cases} \frac{1}{2n}, & \text{if } |x| \le n \\ 0, & \text{else} \end{cases}$$

Now, these functions converge uniformly to the zero function, but their integral is always 1. Thus

$$\lim \int_{\mathbb{R}} f_n(x)dx = \lim 1 = 1 \neq \int_{\mathbb{R}} 0dx = 0$$

□