Homework 1

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April 9, 2018

Problem A

Let V be the real vector space $\text{Set}(\mathbb{R}, \mathbb{R})$ of all functions from $\mathbb{R}$ to itself, and let $K = \{f \in V \mid \text{im}(f) \subset [0,1]\}$. Prove that K is convex and find all extreme points and finite-dimensional faces.

Proof. To start with, we show that K is convex. Let $f, g \in K$ be arbitrary. We will show that the function $h = \lambda f + (1-\lambda)g$ is in K for all $\lambda \in [0,1]$. This is clear, however, since for all $x \in \mathbb{R}$,

$$\begin{aligned} h(x) &= \lambda f(x) + (1-\lambda)g(x) \\ &\leq \lambda(1) + (1-\lambda)(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} h(x) &= \lambda f(x) + (1-\lambda)g(x) \\ &\geq \lambda(0) + (1-\lambda)(0) \\ &= 0 \end{aligned}$$

and so $\text{im}(h) \subset [0,1]$ as desired.

Next, we wish to find the finite dimensional faces of K. I assert that the finite dimensional faces of K are defined as follows. First, partition $\mathbb{R}$ into three sets A, N, P such that $\|A\| < \infty$. Then, define a face

$$F = \{f \in K \mid f^{-1}(\{0\}) \supset N, f^{-1}(\{1\}) \supset P\}$$

To see that this is a face, we have to check that it is convex, and that it contains its linear interpolations.

So, let $f, g \in F$, and let $\lambda \in [0,1]$. We need to show that

$$h(x) = \lambda f(x) + (1-\lambda)g(x)$$

is in F. Clearly $h \in K$ as a convex linear combination of elements of K, so we only need to examine two cases: $x \in N$ and $x \in P$.

If $x \in N$, then

$$h(x) = \lambda(0) + (1 - \lambda)(0) = 0$$

and so $h^{-1}(\{0\}) \supset N$.

If $x \in P$, then

$$h(x) = \lambda(1)_{(}1 - \lambda)(1) = 1$$

and so $h^{-1}(\{1\}) \supset P$.

Finally, we show that F contains all its linear interpolations. That is, for any $h \in F$, if there exists $f, g \in K$ and $t \in (0, 1)$ with $h = tf + (1 - t)g$, then $f, g \in F$ as well. So, suppose $h \in F$ and f, g and t are as described. We examine again two cases.

If $x \in N$, then $h(x) = 0$ and so

$$0 = tf(x) + (1 - t)g(x)$$

but $t \in (0, 1)$ and $f, g \geq 0$, so it must be that $f(x) = g(x) = 0$ as desired. Thus, $f^{-1}(\{0\}) \supset N$ (and similarly for g).

If $x \in P$, then $h(x) = 1$ and so

$$1 = tf(x) + (1 - t)g(x)$$

but $t \in (0, 1)$ and $f(x), g(x) \leq 1$, so it must be that $f(x) = g(x) = 1$ as desired. Thus $f^{-1}(\{1\}) \supset P$ (and similarly for g).

Thus, we have shown that the set F defined this way is a face. Next, we show it is finite-dimensional. In particular, we show that the dimension of F is $\|A\|$.

Recall that the dimension of a face is defined as the dimension of span$\{g - f \mid g \in F\}$ for some fixed $f \in F$. So, let $f = \chi_P$. I assert that the span of $\{g - f \mid g \in F\}$ has a basis given by $g_i = \chi_{\{a_i\}}$ for $a_i \in A$.

First, observe that $\{g_i\}$ is clearly a linearly independent set. Next, we observe that any function of the form $g - f$ for $g \in F$ can be written as a finite linear combination of the g_i basis functions. This is clear, since for any $g \in F$, we know that

$$(g - f)(x) = \begin{cases} f(x), & x \in A \\ 0, & x \notin A \end{cases}$$

and so

$$(g - f)(x) = \sum_{a_i \in A} f(a_i) g_i(x)$$

as desired.

I now assert that this describes all the finite-dimensional faces. To see this, let G be a face that cannot be described using the construction above. In particular, there is some $g \in G$ and some subset $E \subset \mathbb{R}$ with $\|E\| = \infty$ and $g(E) \in (0, 1)$. Since G is a face, this means that G contains all functions which agree with g outside E.

Now, fix

$$f(x) = \begin{cases} g(x), & x \notin E \\ 0, & x \in E \end{cases}$$

Then, for every $e \in E$, the function $h_e = \chi_{\{e\}}$ is in $\{h - f \mid h \in G\}$. Moreover, the collection $\{h_e\}$ is clearly linearly independent. Thus, we have found an infinite linearly independent subset of $\{h - f \mid h \in G\}$, and so G is infinite dimensional.

Finally, we observe that the extreme points of K are simply the faces defined as above with $A = \emptyset$. That is, the extreme points of K are the functions $f \in K$ with $f(\mathbb{R}) \in \partial[0, 1]$. $\square$

Problem B

Do the same, replacing the domain $\mathbb{R}$ with $\mathbb{N}$.

Proof. Note that the construction for part A generalizes to functions from arbitrary sets into $\mathbb{R}$, and so the finite-dimensional faces and extreme points are described in exactly the same way. $\square$

Part C

Let X be the real vector space $\mathrm{Set}(\mathbb{N}, \mathbb{C})$ and let $E = \{f \in X \mid |f|(\mathbb{N}) \in [0,1]\}$. Prove E is convex, and find all extreme points and finite-dimensional faces.

Proof. Convexity of E follows almost immediately from the fact that the unit disk is convex. That is, for $f, g \in E$ and $\lambda \in [0,1]$, we know that

$$\begin{aligned} \|\lambda f(x) + (1-\lambda)g(x)\| &\leq \lambda \|f(x)\| + (1-\lambda)\|g(x)\| \\ &\leq \lambda(1) + (1-\lambda)(1) \\ &= 1 \end{aligned}$$

as desired. $\square$

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