
Homework 5

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PROBLEM 1

Let E, F be closed subspaces of a Hilbert space. Prove that $P_E P_F = P_E$ if and only if $E \subseteq F$.

Proof. Suppose first that $P_E P_F = P_E$. Then, in particular,

$$P_E^* = (P_E P_F)^* = P_F^* P_E^* = P_F P_E = P_E$$

by self-adjointness of projections. Thus, $P_E P_F = P_F P_E = P_E$, and the projections commute. Thus, the von Neumann algebra $W^*(P_E, P_F, I)$ is Abelian, and is isometrically $*$ -isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ) . In particular, the projections P_E, P_F get sent to self-adjoint idempotents $P_E \mapsto M_{\chi_S}$ and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$.

Now, the requirement $P_F P_E = P_E$ corresponds to the requirement

$$M_{\chi_{S'}} M_{\chi_S} = M_{\chi_S}$$

which means that $S \subset S'$. This, in turn, implies that $E \subset F$.

Indeed, E is the subspace of H on which P_E is the identity, which corresponds to the subspace

$$\tilde{E} = \int_S^\oplus H(x) d\mu(x)$$

on which M_{χ_S} is the identity. Similarly,

$$\tilde{F} = \int_{S'}^\oplus H(x) d\mu(x)$$

Clearly, $\tilde{E} \subset \tilde{F}$ (since $S \subset S'$) and so $E \subset F$ as well.

For the converse direction, assume that $E \subset F$. Then, on F , $P_F = I_F$, and $P_F P_E = I_F P_E = P_E$. Furthermore, on F^\perp , $P_F P_E = 0 = P_E$. Thus, on all of $H = F \oplus F^\perp$, $P_F P_E = P_E$ as desired. \square

PROBLEM 2

Characterize the closed subspaces E, F of a Hilbert space H satisfy $P_F P_E = P_E P_F$.

Proof. I assert that P_E and P_F commute if and only if H can be decomposed into the four orthogonal components

$$H = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp$$

To see this, suppose first that P_E and P_F commute. Then, consider the Abelian von Neumann algebra $W^*(P_E, P_F, I)$. Like before, we use the Borel functional calculus to identify $W^*(P_E, P_F, I)$ with $L^\infty(X, \mu)$ acting on $\tilde{H} = \int_X^\oplus H(x) d\mu(x)$, which is unitarily equivalent to H (in the sense that there is a unitary transformation U such that $\tilde{H} = UH$ and $UW^*(P_E, P_F, I)U^* = L^\infty(X, \mu)$).

Under this identification, $P_E \mapsto M_{\chi_S}$, and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$. In particular, this decomposes \tilde{H} into four orthogonal components

$$\begin{aligned} \tilde{H} &= \int_X^\oplus H(x) d\mu(x) \\ &= \int_{S \cap S'}^\oplus H(x) d\mu(x) \oplus \int_{S \cap S'^c}^\oplus H(x) d\mu(x) \oplus \int_{S' \cap S^c}^\oplus H(x) d\mu(x) \oplus \int_{S^c \cap S'^c}^\oplus H(x) d\mu(x) \end{aligned}$$

which translates into the decomposition on H as

$$H = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp$$

as desired.

Conversely, suppose H can be decomposed this way. Then, let $v \in H$ be decomposed as

$$v = v_1 + v_2 + v_3 + v_4$$

where $v_1 \in E \cap F$, $v_2 \in E \cap F^\perp$, $v_3 \in E^\perp \cap F$ and $v_4 \in E^\perp \cap F^\perp$.

We compute the effect of $P_E P_F$ and $P_F P_E$ directly.

$$\begin{aligned} P_E P_F(v) &= P_E P_F(v_1 + v_2 + v_3 + v_4) \\ &= P_E(v_1 + v_3) \\ &= v_1 \\ P_F P_E(v) &= P_F P_E(v_1 + v_2 + v_3 + v_4) \\ &= P_F(v_1 + v_2) \\ &= v_1 \end{aligned}$$

and thus $P_E P_F = P_F P_E$ as desired. □

PROBLEM 3

Let E, F be closed subspaces of a Hilbert space H . An operator U is said to be a partial isometry from E to F if $U|_E$ is an isometry onto F , and $U|_{E^\perp} = 0$. Prove that U is a partial isometry $\iff U^*U$ is a projection $\iff UU^*$ is a projection.

Proof. Suppose first that U is a partial isometry from E to F . I assert that $U^*U = P_E$. To see this, suppose $e \in E$, $v \in H$ and let $v = e' + v'$ where $e' \in E$ and $v' \in E^\perp$. Then,

$$\begin{aligned}\langle U^*Ue|v \rangle &= \langle U^*Ue|e' + v' \rangle \\ &= \langle Ue|Ue' \rangle + \langle Ue|Uv' \rangle \\ &= \langle e|e' \rangle + 0 \\ \implies \langle U^*Ue - e|v \rangle &= 0\end{aligned}$$

and since this holds for all $v \in H$, $U^*Ue - e = 0$ and thus $U^*Ue = e$ and U^*U is the identity on E .

Furthermore, for $v' \in E^\perp$,

$$U^*Uv' = U^*(0) = 0$$

and so U^*U is the zero map on E^\perp . Thus, U^*U agrees with P_E at all points, so $U^*U = P_E$ as desired.

Conversely, suppose U^*U is a projection P_E for some closed subspace E . Define $F = U(E)$. We will first show that U is an isometry of E onto F . To see this, suppose $e, e' \in E$. We calculate directly

$$\begin{aligned}\langle Ue|Ue' \rangle &= \langle U^*Ue|e' \rangle \\ &= \langle P_Ee|e' \rangle \\ &= \langle e|e' \rangle\end{aligned}$$

and thus U is an isometry from E to F . Note that this immediately implies that F is a closed subspace. Finally, we show that $U|_{E^\perp} = 0$. Let $v' \in E^\perp$, Then,

$$\begin{aligned}\|Uv'\|^2 &= \langle Uv'|Uv' \rangle \\ &= \langle U^*Uv'|v' \rangle \\ &= \langle 0|v' \rangle = 0\end{aligned}$$

and so $U^*U|_{E^\perp} = 0$ as desired. Thus, U is a partial isometry.

Finally, we show that if U is a partial isometry, then U^* is a partial isometry, which will complete the proof. So, suppose U is a partial isometry. We will show that U^* is a partial isometry from F to E . First, we check that $U^*|_F$ is an isometry. Now, for each $f \in F$, there is some $e \in E$ with $Ue = f$. Thus, for $e, e' \in E$, $f, f' \in F$ with $Ue = f, Ue' = f'$,

$$\begin{aligned}\langle U^*f|U^*f' \rangle &= \langle U^*Ue|U^*Ue' \rangle \\ &= \langle P_Ee|P_Ee' \rangle \\ &= \langle e|e' \rangle \\ &= \langle Ue|Ue' \rangle \\ &= \langle f|f' \rangle\end{aligned}$$

and so U^* is an isometry from F to E . Finally, we show that $U^*|_{F^\perp} = 0$. This is immediate, since

$$\ker U^* = U(H)^\perp = F^\perp$$

where we used the identity $\ker A^* = A(H)^\perp$ for all bounded operators A .

Note that this completes the proof. If U is a partial isometry, then U^* is a partial isometry, which implies that $U^{**}U^* = UU^*$ is a projection. Similarly, if $UU^* = U^{**}U^*$ is a projection, then U^* is a partial isometry, and thus $U^{**} = U$ is a partial isometry as well. \square

PROBLEM 4

Prove that the subspace of self-adjoint operators on $B(\mathbb{C}^2)$ is a subset of the linear subspace spanned (over \mathbb{R}) by I and the three Pauli matrices

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and prove that the span of $\{\sigma_i\}_{i=1}^3$ is isometrically isomorphic to a three-dimensional real Hilbert space under the operator norm.

Proof. We observe first that for any $T \in B(\mathbb{C}^2)$ self-adjoint, we have the condition

$$T_j^i = \overline{T_i^j}$$

In particular,

$$\begin{aligned} T_1^1 &= \overline{T_1^1} \\ T_2^2 &= \overline{T_2^2} \end{aligned}$$

and so T_i^i is real. Furthermore, we have the relation

$$T_2^1 = \overline{T_1^2}$$

Finally, we observe that we can decompose T into a scalar part and a trace-free part by

$$\begin{aligned} C &= T - \text{tr}(T)I \\ T &= \text{tr}(T)I + C \end{aligned}$$

Now we are ready to show that T is a linear combination of I and the Pauli matrices. In particular, we will show that C is a linear combination of the Pauli matrices.

For C , we have the relations

$$\begin{aligned} C_1^1 &= -C_2^2 \\ C_2^1 &= \overline{C_1^2} \end{aligned}$$

as well as the requirement that C_1^1 is real. Thus, we have three degrees of freedom. That is,

$$C = \begin{bmatrix} \alpha & \beta + \gamma i \\ \beta - \gamma i & -\alpha \end{bmatrix}$$

which decomposes as

$$C = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3$$

and so

$$T = \text{tr}(T)I + C$$

is a linear combination of I and $\{\sigma_i\}_{i=1}^3$ as desired.

Next, we show that the span of $\{\sigma_i\}_{i=1}^3$ under the operator norm is isometrically isomorphic to \mathbb{R}^3 . To do so, we will show that the operator norm coincides with the pullback metric from \mathbb{R}^3 . Specifically, we define a linear isomorphism

$$\begin{aligned} \Phi : \langle \sigma_i \rangle_{i=1}^3 &\rightarrow \mathbb{R}^3 \\ \Phi(\sigma_i) &= e_i \end{aligned}$$

where $\{e_i\}$ is an orthonormal basis for \mathbb{R}^3 with its usual inner product. In particular, if we denote the inner product of \mathbb{R}^3 as η , we can define $\Phi^*(\eta)$ by

$$\Phi^*(\eta)(u, v) = \eta(\Phi(u), \Phi(v))$$

In order to show that this metric coincides with the operator norm, we have to show that

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \Phi^*(\eta)(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3, \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3) = \alpha^2 + \beta^2 + \gamma^2$$

Now, since σ_i are all self-adjoint, any (real) linear combination of them will be as well. Thus,

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \|(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2\|$$

We compute the right-hand side directly

$$(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2 = \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma^2\sigma_3^2 + \alpha\beta\{\sigma_1, \sigma_2\} + \alpha\gamma\{\sigma_1, \sigma_3\} + \beta\gamma\{\sigma_2, \sigma_3\}$$

where $\{\sigma_i, \sigma_j\}$ is the anticommutator of σ_i and σ_j . Now, it is well-known that the Pauli matrices satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2\varepsilon_{ijk}\sigma_k$$

as well as the relation

$$\sigma_i^2 = I$$

(where ε_{ijk} is the levi-civita symbol) which forces

$$\sigma_i\sigma_j = \varepsilon_{ijk}\sigma_k + \delta_{ij}$$

and so

$$\begin{aligned} \{\sigma_i, \sigma_j\} &= \sigma_i\sigma_j + \sigma_j\sigma_i \\ &= \varepsilon_{ijk}\sigma_k + \delta_{ij} + \varepsilon_{jik}\sigma_k + \delta_{ji} \\ &= \varepsilon_{ijk}\sigma_k - \varepsilon_{ijk}\sigma_k + 2\delta_{ij} \\ &= 2\delta_{ij} \end{aligned}$$

Thus,

$$\begin{aligned} (\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2 &= \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma^2\sigma_3^2 + \alpha\beta\{\sigma_1, \sigma_2\} + \alpha\gamma\{\sigma_1, \sigma_3\} + \beta\gamma\{\sigma_2, \sigma_3\} \\ &= \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma^2\sigma_3^2 \\ &= (\alpha^2 + \beta^2 + \gamma^2)I \end{aligned}$$

and so

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \|(\alpha^2 + \beta^2 + \gamma^2)I\| = (\alpha^2 + \beta^2 + \gamma^2)\|I\| = (\alpha^2 + \beta^2 + \gamma^2)$$

as desired. Thus, the pullback metric coincides with the operator norm, and Φ is a linear isometric isomorphism, as desired. \square

PROBLEM 5

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