Problem Set 1

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PROBLEM 1

Prove that U is open if and only if for all x in U, there exists some neighborhood U_x such that $x \in U_x \subseteq U$.

Proof. (=>) Assume U is open. If $U = \emptyset$, the proposition is vacuously true. So, suppose U is nonempty. Then, for each $x \in U$, the open set U satisfies $x \in U \subseteq U$. Thus, this half of the proof is complete.

(<=) Now, assume that for each $x \in U$, there exists U_x open such that $x \in U_x \subseteq U$. Then, U can be written as the union

$$U = \bigcup_{x \in U} U_x$$

This is easily seen by verifying that, since each x is in some U_x, $U \subseteq \bigcup_x U_x$ and since each $U_x \subseteq U$, $\bigcup_x U_x \subseteq U$.

Thus, U is the union of open sets, and is open. □

PROBLEM 2

Prove that, for a product space $X \times Y$, the projection $\pi_x : X \times Y \to X$ is an open map.

Proof. Let $U = \bigcup_{\alpha \in I} V_\alpha \times W_\alpha$ be an arbitrary open set in $X \times Y$, with V_α open in X, and W_α open in Y.

Then,

$$\begin{aligned} \pi_x(U) &= \pi_x(\cup_\alpha V_\alpha \times W_\alpha \\ &= \cup_\alpha \pi_x(V_\alpha \times W_\alpha) \\ &= \cup_\alpha V_\alpha \end{aligned}$$

which is the union of open sets, and is open in X. □

Problem 3

Show that, for $A \subset X$ and $B \subset Y$, the identity $\overline{A \times B} = \overline{A} \times \overline{B}$ holds.

Proof. ($\subseteq$) Let $(a, b) \in \overline{A \times B}$ be an element of the closure. Then, by the definition of closure, there exists some net (a_α, b_α) that converges to (a, b) with each $(a_\alpha, b_\alpha) \in A \times B$. Now, since the projection maps π_x and π_y are continuous, they preserve nets. Thus, we have

$$\pi_x(a_\alpha, b_\alpha) \to \pi_x(a, b)$$
$$a_\alpha \to a$$

and since each $a_\alpha \in A$, $a \in \overline{A}$. Similarly in the second coordinate, we have $b \in \overline{B}$. Thus, $(a, b) \in \overline{A} \times \overline{B}$.

($\supseteq$) Let $(a, b) \in \overline{A} \times \overline{B}$. In particular, this means that $a \in \overline{A}$ and $b \in \overline{B}$. Then, by definition of closure, there exists a net $a_\alpha \to a$ and $b_\beta \to b$ such that for all α, $a_\alpha \in A$ and for all β, $b_\beta \in B$. Then, the net $(a_\alpha, b_\beta) \to (a, b)$ is such that each $(a_\alpha, b_\beta) \in A \times B$, so $(a, b) \in \overline{A \times B}$.

(Note that the net (a_α, b_β) is still a net, if the indexing space is taken to be the product of the indexing space of α and β with the directed order

$$(\alpha_1, \beta_1) > (\alpha_2, \beta_2) \iff \alpha_1 > \alpha_2 \text{and } \beta_1 > \beta_2$$

.) ☐

Problem 4

Prove that, for $A, B \subset X$, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. ($\subseteq$) Let $x \in \overline{A \cup B}$. Then, there is some net $x_\alpha \to x$ for $x_\alpha \in A \cup B$. Now, this net is either frequently in A, or frequently in B (or possibly both). Suppose x_α frequently be in A. Then, consider the subnet x_{α_β} of elements of x_α that are in A. This subnet also converges to x, so $x \in \overline{A}$. Similarly, if x_α is frequently in B, then $x \in \overline{B}$. Thus, x is either in $\overline{A}$ or $\overline{B}$, that is, $x \in \overline{A} \cup \overline{B}$.

($\supseteq$) Let $x \in \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A}$. Then, there exists a net $x_\alpha \to x$ such that $x_\alpha \in A$ for all α. In particular, $x_\alpha \in A \cup B$ for all α, so $x \in \overline{A \cup B}$, since it is the limit of a net in $A \cup B$. By exactly the same argument, if $x \in \overline{B}$, then $x \in \overline{A \cup B}$. ☐

Problem 5

Show that for a collection A_α of subsets of X, then $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$, but equality does not necessarily hold.

Proof. First, let $x \in \bigcup_\alpha \overline{A_\alpha}$. In particular, x is in some $\overline{A_x}$. So, let $x_\gamma \to x$ be a net in A_x. Then, $x_\gamma \in \bigcup_\alpha A_\alpha$ for all γ. So, since $x_\gamma \to x$, $x \in \overline{\bigcup_\alpha A_\alpha}$.

Now, for a counterexample to equality, consider the collection

$$A_n = \{\frac{1}{n}\} \subset \mathbb{R}$$

Now, $\overline{A_n} = A_n$ for each n, since they are all closed, so $\bigcup_n \overline{A_n} = \bigcup_n A_n$. However, 0 is a limit point of $\bigcup_n A_n$, so it is in the closure $\overline{\bigcup_n A_n}$. But 0 is not in $\bigcup_n \overline{A_n}$. Thus, $\bigcup_\alpha \overline{A_\alpha} \neq \overline{\bigcup_\alpha A_\alpha}$, ☐

Problem 6

Show that the product $X \times Y$ of two Hausdorff spaces X and Y is Hausdorff.

Proof. For this proof, we will use the unique limits definition of the Hausdorff condition. That is, a space is Hausdorff if and only if convergent nets in the space have unique limits.

Suppose for a contradiction that $X \times Y$ is not Hausdorff. Then, there exists some net (x_α, y_α) that has two or more limits. Let $(x_1, y_1), (x_2, y_2)$ be two such distinct limits.

Since these points are distinct, either $x_1 \neq x_2$, or $y_1 \neq y_2$. Suppose $x_1 \neq x_2$. Then, the projection map gives us

$$\pi_x(x_\alpha, y_\alpha) \to (x_1, y_1) \implies x_\alpha \quad \to x_1$$

However, by a similar argument, it can be shown that $x_\alpha \to x_2$. Since $x_1 \neq x_2$, the net x_α does not have unique limits in X, which is a contradiction to X being Hausdorff.

If $x_1 = x_2$, then it must be that $y_1 \neq y_2$, and by a similar argument to the one above, one reaches a contradiction on Y being Hausdorff. $\square$

Problem 7

Show that X is Hausdorff if and only if the diagonal in $X \times X$ is closed.

Proof. For this proof, we will use the separation definition of the Hausdorff condition.

$(=>)$ Suppose X is Hausdorff, and let $x, y \in X$ be distinct points. Now, let $U, V \subset X$ open such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Then, the open set $U \times V \subset X \times X$ does not intersect the diagonal, but contains (x, y).

Since this can be done for any point (x, y) not on the diagonal, the complement of the diagonal is open (by the result from problem 1) and thus the diagonal is closed.

$(<=)$ Suppose X is such that the diagonal is closed. In particular, this means that the complement to the diagonal is open. Let $x, y \in X$ with $x \neq y$. Then, the point (x, y) is not on the diagonal, and there must be some basic open set $U \times V$ containing (x, y) that does not intersect the diagonal. In particular, this means that U and V are disjoint, since if they shared an element $x_0 \in U$ and $x_0 \in V$, then $(x_0, x_0) \in U \times V$ which is a point on the diagonal contained in $U \times V$, a contradiction.

Since $(x, y) \in U \times V$, this means that $x \in U$ and $y \in V$ for disjoint open U and V.

This separation can be done for any pair of points in X, so X is Hausdorff. $\square$

Problem 8

Define $\partial A = \overline{A} \cap \overline{A^c}$. Show that $\overline{A} = \mathrm{Int}(A) \cup \partial A$.

Proof. This proof will be done by simple equation chasing.

$$\begin{aligned}
\mathrm{Int}(A) \cup \partial A &= \mathrm{Int}(A) \cup (\overline{A} \cap \overline{A^c}) \\
&= (\mathrm{Int}(A) \cup \overline{A}) \cap (\mathrm{Int}(A) \cup \overline{A^c}) \\
&= \overline{A} \cap X \\
&= \overline{A}
\end{aligned}$$

Now, the second line is obtained by simple set theory logic. The third line equality can be seen to be true by an easy argument. First, observe that $\mathrm{Int}(A) \subset \overline{A}$ since $\mathrm{Int}(A) \subset A \subset \overline{A}$ by definition. Thus,

$$\mathrm{Int}(A) \cup \overline{A} = \overline{A}$$

For the second half, observe that

$$\mathrm{Int}(A)^c = (\bigcup_{U \subset A} U)^c = \bigcap_{A^c \subset U^c} U^c = \overline{A^c}$$

so that

$$\mathrm{Int}(A) \cup \overline{A^c} = \overline{A^c}^c \cup \overline{A^c} = X$$

The result follows immediately. $\square$