Problem Set 3

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PROBLEM 1

Enumerate all subcomplexes of S^∞ with the cell structure on S^∞ that has S^n as the n-skeleton.

Proof. We notice at first that each n-skeleton is a subcomplex, and so S^n is a subcomplex of S^∞ for each n.

There is another subcomplex in each dimension. Namely, by omitting one of the n-cells attaching to the $n-1$-skeleton, we obtain another subcomplex in the nth dimension that is the $n-1$ skeleton along with a single n-cell attached in the usual way. In fact, depending on which n-cell we omit, we can obtain two different subcomplexes.

So far, we have three subcomplexes in each dimension. I assert that this is all the subcomplexes. Suppose there existed a subcomplex in n dimensions that did not contain the entire $n-1$-skeleton. In particular, this means that the attaching map of the n-cell, which is bijective from ∂D^n onto the entire $n-1$-skeleton, is not well-defined, and so no such subcomplex can be constructed. $\square$

PROBLEM 2

Show S^∞ is contractible.

Proof. We will show that the n-skeleton of $X = S^\infty$ is contractible in X^{n+1}. To see this, consider the subcomplex X^n along with a single disk D^{n+1} attached in the usual way. In particular, X^n is identified with ∂D^{n+1}, and since D^{n+1} is contractible, it follows that ∂D^{n+1} contracts to a point in D^{n+1}.

Thus, each X^n is contractible in X, and since $X = \cup_{n=0}^{\infty} X^n$, it follows that X is contractible as well. $\square$

Problem 3

Show that $S^1 \star S^1 = S^3$. In general, show $S^m \star S^n = S^{m+n+1}$.

Proof. Recall the definition of a join $X \star Y$ $\square$

Problem Extra

In the proof of the Brouwer fixed point theorem, construct explicitly the retraction from D to ∂D by assuming $f(x) \neq x$.