AC Circuits Homework 2

Daniel Halmrast

September 28, 2016

Problem (1.a)**.** *Consider an RLC circuit with source voltage* $V_s = V_0 \sin(\omega t)$*. Determine the current* $I = I_0 \sin(\omega t + \phi)$*.*

Solution (1.a)**.** Our defining differential equation is

$$V_S = \frac{1}{C}Q(t) + R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2}$$

Or, in more familiar terms of I,

$$V_S = \frac{1}{C}\int_0^t I dt + RI + L\frac{dI}{dt}$$

Substituting $I = I_0 \sin(\omega t + \phi)$ and $V_s = V_0 \sin(\omega t)$ we get

$$V_0 \sin(\omega t) = \frac{1}{C}\int_0^t I_0 \sin(\omega t + \phi)dt + RI_0 \sin(\omega t + \phi) + L\frac{d}{dt}I_0 \sin(\omega t + \phi)$$

$$= I_0\Big[\frac{-1}{C\omega}\cos(\omega t + \phi) + R\sin(\omega t + \phi) + L\omega\cos(\omega t + \phi)\Big]$$

Using sum of angles identites, we split the RHS into $\sin(\omega t)$ and $\cos(\omega t)$ parts.

$$\begin{aligned} V_0 \sin(\omega t) = &\Big[I_0 R\cos(\phi) + \big(\frac{I_0}{C\omega} - I_0 L\omega\big)\sin(\phi)\Big]\sin(\omega t) \\ &+ \Big[I_0 R\sin(\phi) + \big(I_0 L\omega - \frac{I_0}{C\omega}\big)\cos(\phi)\Big]\cos(\omega t) \end{aligned}$$

Equating the coefficients yields two equations.

$$V_0 = I_0 R\cos(\phi) + \big(\frac{I_0}{C\omega} - I_0 L\omega\big)\sin(\phi)$$
$$0 = I_0 R\sin(\phi) + \big(I_0 L\omega - \frac{I_0}{C\omega}\big)\cos(\phi)$$

The second equation simplifies to

$$R\sin(\phi) = \big(\frac{1}{C\omega} - L\omega\big)\cos(\phi)$$
$$\boxed{\tan(\phi) = \frac{\frac{1}{C\omega} - L\omega}{R}} \tag{1}$$

This immediately yields the following useful identities

$$\sin(\phi) = \frac{\frac{1}{C\omega} - L\omega}{\sqrt{R^2 + (\frac{1}{C\omega} - L\omega)^2}} \tag{2}$$

$$\cos(\phi) = \frac{R}{\sqrt{R^2 + (\frac{1}{C\omega} - L\omega)^2}} \tag{3}$$

Going back to the first coefficient equality, and using results 2 and 3, we see that

$$\begin{aligned} V_0 &= I_0 R\cos(\phi) + \big(\frac{I_0}{C\omega} - I_0 L\omega\big)\sin(\phi) \\ &= I_0\Big[R\frac{R}{\sqrt{R^2 + (\frac{1}{C\omega} - L\omega)^2}} + \big(\frac{I_0}{C\omega} - I_0 L\omega\big)\frac{\frac{1}{C\omega} - L\omega}{\sqrt{R^2 + (\frac{1}{C\omega} - L\omega)^2}}\Big] \\ &= I_0\Big[\frac{R^2 + \big(\frac{1}{C\omega} - L\omega\big)^2}{\sqrt{R^2 + (\frac{1}{C\omega} - L\omega)^2}}\Big] \\ &= I_0\sqrt{R^2 + (\frac{1}{C\omega} - L\omega)^2} \end{aligned}$$

Therefore

$$\boxed{I_0 = \frac{V_0}{(R^2 + (\frac{1}{C\omega} - L\omega)^2)^{\frac{1}{2}}}} \tag{4}$$

Problem (1.b)**.** *Graph $\phi(\omega)$.*

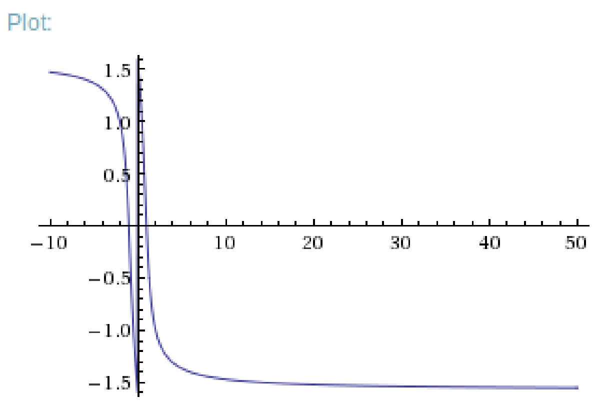
Solution (1.b)**.** ϕ is defined implicitly by

$$\tan(\phi) = \frac{\frac{1}{C\omega} - L\omega}{R} \tag{1}$$

Which we will explicitly write as

$$\phi(\omega) = \arctan\left(\frac{\frac{1}{C\omega} - L\omega}{R}\right) \tag{2}$$

Clearly, as $\omega \to 0$, we have $\phi \to \arctan(\infty) = \frac{\pi}{2}$. Also, when $\omega = \frac{1}{\sqrt{LC}}$, $\phi = 0$.
A graph of $\phi(\omega)$ for $L = C = 1$ is shown.



Problem (1.c)**.** *Plot $V_{R_0}(\omega)$.*

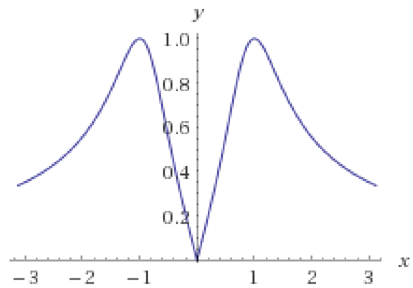
Solution (1.c)**.** By the relation $V_R = IR$, V_{R_0} is RI_0 Thus

$$V_{R_0}(\omega) = \frac{V_0 R}{(R^2 + (\frac{1}{C\omega} - L\omega)^2)^{\frac{1}{2}}} \tag{1}$$

It is easy to see that when $\omega \to 0$, $V_R \to 0$. Also, V_R attains a maximum at the characteristic frequency $\omega = \frac{1}{\sqrt{LC}}$.

A graph of $V_{R_0}(\omega)$ for $V_0 = L = C = 1$ is shown.

Plots:



Problem (1.d)**.** *Solve for the voltage across the capacitor, using* $V_C = \frac{Q(t)}{C}$

Solution (1.d)**.** Rewriting V_C in a more familiar form, we have

$$\begin{aligned} V_C &= \frac{1}{C}\int_0^t I dt \\ &= \frac{1}{C}\int_0^t I_0 \sin(\omega t + \phi) dt \end{aligned}$$

Which yields

$$V_C = \frac{-I_0}{C\omega}\cos(\omega t + \phi)$$

With

$$I_0 = \frac{V_0}{(R^2 + (\frac{1}{C\omega} - L\omega)^2)^{\frac{1}{2}}} \tag{1}$$

$$\tan(\phi) = \frac{\frac{1}{C\omega} - L\omega}{R} \tag{2}$$

The reader is now encouraged to recall the trigonometry identity relating sin and cos

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

Thus, V_C can be expressed in terms of the sin function as

$$\begin{aligned} V_C &= \frac{-I_0}{C\omega}\sin(\omega t + \phi + \frac{\pi}{2}) \\ &= \frac{I_0}{C\omega}\sin(\omega t + \phi - \frac{\pi}{2}) \end{aligned}$$