Problem Set 2

Daniel Halmrast

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Problem 1

Show that the function sending ϕ to $\phi^{-1}(\{1\})$ is a bijection between nonzero bounded linear functionals and hyperplanes not containing 0.

Proof. We first show that for arbitrary bounded linear functional ϕ, the set $\phi^{-1}(\{1\})$ is a closed hyperplane.

To see this, let $\phi \in X^* \setminus \{0\}$. In particular, this means that $H := \phi^{-1}(\{1\})$ is nonempty. So, let $x_0 \in H$. Then, we have $\phi(x_0) = 1$. Now, let $x \in X$ be arbitrary, and consider

$$\phi(x - \phi(x)x_0) = \phi(x) - \phi(x)\phi(x_0) = 0$$

This implies that $y := x - \phi(x)x_0$ is in the kernel of ϕ. Solving for x yields

$$x = y + \phi(x)x_0$$

and so $X = \ker \phi \oplus \operatorname{span}(x_0)$. □