Homework 4

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June 1, 2018

Problem 1

Find the cut locus of a general flat torus.

Problem 2

Let M be a complete Riemannian manifold. Prove that M is compact if and only if there exists a point $p \in M$ such that every geodesic starting from p has a cut point.

Proof. ($\Longrightarrow$)
Suppose first that M is compact. In particular, we know that the diameter $d(M)$ of M is finite. Suppose for a contradiction that every point p in M has some unit speed geodesic γ starting at p with no cut point. This means that γ minimizes the distance from p to $\gamma(t)$ for all t. So, $d(p, \gamma(d(M) + 1)) = d(M) + 1$, which contradicts $d(M)$ being the diameter of M. Thus, there exists a point in M for which every geodesic starting at that point has a cut point.

($\Longleftarrow$)
This proof comes from Do Carmo, chapter 13 corollary 2.11.
Suppose M is such that there is a point $p \in M$ with every geodesic starting from p having a cut point. Since M is complete, we know that

$$M = \bigcup_{\gamma} \{\gamma(t) : \ t \leq f(p, \gamma'(0))\}$$

where the union is taken over all geodesics γ starting at p, and f is the function which takes a geodesic γ starting at a point, and returns the first t_0 for which $\gamma(t_0)$ is the cut point of $\gamma(0)$. This follows, since M being complete implies that there exists a minimizing geodesic from p to q for each pair of points $p, q \in M$.
Now, since f is continuous, and each geodesic starting at p has a cut point, it follows that f is bounded. Therefore, M is bounded, and thus M is compact, as desired. $\square$

Problem 3

Prove that for every compact, even-dimensional manifold M with sectional curvature $0 < M \leq 1$, the injectivity radius $i(M) \geq \frac{\pi}{2}$.

Proof. Suppose first M is orientable. Then, by proposition 3.4 of chapter 13 of Do Carmo, we know that $i(M) \geq \pi$, and we are done.

So, suppose instead M is not orientable. We know from problem 12 of chapter 0 of Do Carmo that there exists an orientable double cover of M. Call such a cover $(\tilde{M}, p)$. Now, since p is a local isometry, the sectional curvature of $\tilde{M}$ is equal to the sectional curvature of M. Thus, $\tilde{M}$ satisfies the hypotheses for proposition 3.4 of chapter 13, and $i(\tilde{M}) \geq \pi$.

Now, we will show that this implies that $i(M) \geq \frac{\pi}{2}$. Suppose for a contradiction there existed a point $q \in M$ with a geodesic γ starting at q (with unit speed) so that $\gamma(t_0)$ is a cut point of q with $t_0 < \frac{\pi}{2}$.

Suppose first that $\gamma(t_0)$ is conjugate to q. Let J be the Jacobi field which vanishes at $\gamma(0)$ and $\gamma(t_0)$ and is not everywhere zero. Now, fix $y_0 \in p^{-1}(q)$ the basepoint of $\tilde{M}$, and lift γ along p to a geodesic $\tilde{\gamma}$ in $\tilde{M}$ starting at y_0. Since p is a local isometry, J also lifts along p to a Jacobi field along $\tilde{\gamma}$ which vanishes at the endpoints. But this implies that $\tilde{\gamma}(0)$ and $\tilde{\gamma}(t_0)$ are conjugate to each other, which cannot happen since

$$d(\tilde{\gamma}(0), \tilde{\gamma}(t_0)) \leq t_0 < \pi$$

So, suppose instead there are two geodesics γ, σ with $\gamma(0) = \sigma(0) = q$ and $\gamma(t_0) = \sigma(t_0)$. This forms a closed geodesic loop δ with $\ell(\delta) = 2t_0 < \pi$. This loop lifts to a geodesic $\tilde{\delta}$ in $\tilde{M}$. Now, either $\tilde{\delta}$ is a closed loop, or $\tilde{\delta}(0) = y_0$ and $\tilde{\delta}(2t_0) = y_1$ with y_1 the other preimage of q.

If $\tilde{\delta}$ is a loop, then it is the concatenation of $\tilde{\gamma}$ and $\tilde{\sigma}$, each of which have length t_0 and connect y_0 to $\tilde{\gamma}(t_0)$. Thus, $\tilde{\gamma}(t)$ is in the cut locus of y_0 for some $t \in (0, t_0]$, which is a contradiction since $d(y_0, \tilde{\gamma}(t)) \leq t_0 < \pi$.

Suppose instead that $\tilde{\delta}$ is a path from y_0 to y_1. Now, let $x_0, x_1 \in \tilde{M}$ be the preimages of $\gamma(t_0)$ in $\tilde{M}$. Since δ passes through $\gamma(t_0)$ exactly once, $\tilde{\delta}$ passes through x_0 and not x_1 (without loss of generality in labeling).

However, consider a different lift of δ (say, $\hat{\delta}$) where $\delta(t_0)$ gets lifted to x_1 instead of x_0. This defines a path $\hat{\delta}$ in $\tilde{M}$ which does not pass through x_0, but does pass through x_1. Furthermore, $\hat{\delta}$ has y_0 and y_1 as its endpoints (if it were a loop, we could use the same argument as in the previous paragraph to establish a contradiction). Thus, there are two geodesics $\tilde{\delta}$ and $\hat{\delta}$ from y_0 to y_1 which are distinct, but have the same length $\ell(\tilde{\delta}) = \ell(\hat{\delta}) = 2t_0$. This implies that there is a $t \in (0, 2t_0]$ for which $\tilde{\delta}(t)$ is a cut point for y_0. This is clearly a contradiction, since

$$d(y_0, \tilde{\delta}(t)) \leq t \leq 2t_0 < \pi$$

and $i(\tilde{M}) \geq \pi$.

Thus, $i(M) \geq \frac{\pi}{2}$ as desired. □