1 Curvature

Let's just straight-up define the curvature:

Definition 1.1. *Consider a Riemannian manifold (M, g), with smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$. We define*

$$R_m(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

Alternately,

$$R_{abc}^{d}\omega_d = \nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c$$

(Wald, p. 37)

Now, we need to establish that this is a tensor by showing it is function linear in each component.

Observe that

$$\begin{aligned} R_m(X,Y)fZ &= -\nabla_X \nabla_Y fZ + \nabla_Y \nabla_X fZ + \nabla_{[X,Y]} fZ \\ &= -X(Yf)Z - (Yf)\nabla_X Z - (Xf)\nabla_Y Z - f\nabla_X \nabla_Y Z + Y(Xf)Z + (Xf)\nabla_Y Z + Yf\nabla_X Z + f\nabla_Y \\ &= -f\nabla_X \nabla_Y Z + f\nabla_Y \nabla_X Z + f\nabla_{[X,Y]} Z \end{aligned}$$

as desired

Homework 1. *Show this is function-linear in other components.*

Note you can lower the contravariant index by applying g_{ab} i.e.

$$R_{abcd} = g_{dd'} R_{abc}^{d'}$$

Calculating Curvature

We can calculate the Riemann curvature tensor in coordinates by using the definitions of the covariant derivative.

$$\mathbb{R}_{abc}^{d} = \partial_b \Gamma_{ac}^{d} - \partial_a \Gamma_{bc}^{d} + \sum_{\alpha} (\Gamma_{ac}^{\alpha} \Gamma_{\alpha b}^{d} - \Gamma_{bc}^{\alpha} \Gamma_{\alpha a}^{d})$$

To make things easier, we can use local Riemannian normal coordinates by pushing the coordinates from T_pM to M via the exponential map.

Homework 2. *Show that in Riemannian normal coordinates,*

$$\Gamma_{ij}^{k} = 0 \text{ at } p$$

and

$$\partial_k g_{ij} = 0 \text{ at } p$$

Definition 1.2. *an orthonormal frame $\{e_i\}$ on an open neighborhood of a point $p \in M$ is called normal around p if*

$$\nabla_a e_i = 0$$

at p.

The curvature follows the Bianchi Identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

In general, we have four important properties of the metric:

- $R^d_{abc} = R^d_{[ab]c}$ antiymmetry of the first two components
- $R^d_{[abc]} = 0$ the Bianchi identity
- $R_{abcd} = R_{ab[cd]}$ antiymmetry of the second two components
- $R_{abcd} = R_{cdab}$ symmetry in the first and second half components.

Note that item 4 can be derived from the other three.

An important concept not covered in Do Carmo:

Definition 1.3. *Given a finite dimensional vector space (over $\mathbb{R}$) V, consider the tensor C of rank $(0,4)$ (4 covariant indices). C is called an algebraic curvature tensor on V if it satisfies the above four properties (with appropriate index lowering).*

Sectional Curvature

Let $p \in M$ and let σ be a $2-$dimensional subspace of T_pM.

Definition 1.4. *The sectional curvature $K(\sigma)$ is defined to be*

$$K(\sigma) = R_m(e_1, e_2, e_1, e_2)$$

for $\{e_1, e_2\}$ an orthonormal basis for σ.

This definition is independent of choice of orthonormal basis by exploiting linearity of R_m.

This can also be expressed in an arbitrary basis u, v by

$$K(\sigma) = \frac{R_m(u,v,u,v)}{\|u \wedge v\|^2} \tag{1.1}$$

Where $\|u \wedge v\|^2$ is calculated from the inner product induced by the metric. That is, for $\{e_i\}$ an orthonormal basis for V, we declare $\{e_i \wedge e_j\}\, i < j$ to be orthonormal.

Homework 3. *Show that the induced inner product is independent of choice of orthonormal basis.*

Lemma 1. *Let V be a vector space (finite dimensional, real) of dimension at least 2 with an inner product. Consider two algebraic curvature tensors C_1 and C_2. Let K_1, K_2 denote the sectional curvatures of C_1 and C_2. $K_1 = K_2$ if and only if $C_1 = C_2$.*

Suppose C is such that $K(\sigma) = \kappa$ for all σ. Then,

$$C(x,y,z,w) = \kappa\left(g(x,z)g(y,w) - g(x,w)g(y,z)\right) \tag{1.2}$$

Ricci Curvature

Let R_m be a Riemannian curvature tensor, with components R^d_{abc}. We can take the trace over the first and third components to get

$$R_{ac} = Rabc^b \tag{1.3}$$

Geometrically, this is defined as

Definition 1.5. $R_{C_p}(u, w) = trace(R_{m_p}(u, \cdot)w)$.

In an orthonormal frame with $g(e_j, e_k) = \delta_{jk}$, we have

$$R_{ij} = R^k_{ikj} = R_{ikjk} \tag{1.4}$$

We can also define the Ricci scalar

Definition 1.6. *$R = R_c(u, u)$ for unit vector u.*

This can be given in coordinates as

$$R = R^i_i \tag{1.5}$$

Theorem 1. *The Ricci curvature tensor is symmetric*

Proof. We know that

$$R_{ac} = R^b_{abc}$$

But by symmetry of the Riemann curvature tensor, we have

$$\begin{aligned} R_{ac} &= R^b_{abc} \\ &= R^b_{cba} \\ &= R_{ca} \end{aligned}$$

as desired ☐

Now, let u be a unit vector, and build an orthonormal basis around u. Then,

$$R_c(u, u) = \sum R(e_1, e_i, e_1, e_i) = \sum K(e_1, e_i)$$

and

$$R = \sum R_c(e_i, e_i) = \sum K(e_i, e_j)$$

We also have the following identity for the Riemann curvature tensor R:

$$R(u \wedge v, w \wedge z) = R(u, v, w, z) \tag{1.6}$$

This relies on the antisymmetry of R, since R has to be linear.

Thus, interpreting R as a map from $\Lambda^2 T_p M \times \Lambda^2 T_p M \to \mathbb{R}$ we have that R is a symmetric bilinear map.

1.1 Riesz Representation and Tangent/Cotangent isomorphism

Given a metric, we have a natural isomorphism between T_pM and T_p^*M, denoted $\flat : T_pM \to T_p^*M$ and $\sharp : T_p^*M \to T_pM$ is given by

$$\flat(v) - v^\flat = g(v, \cdot) \tag{1.7}$$

This isomorphism extends also to exterior products of tangent spaces, allowing us to raise and lower indices at will.

1.2 Constant Curvature Spaces

Recall that if a space has constant sectional curvature κ, then

$$R(x, y, z, w) = \kappa(g(x, z)g(y, w) - g(x, w)g(y, z)) \tag{1.8}$$

Examples of such spaces are

1. Euclidean flat space $\mathbb{R}^n$: $\kappa = 0$.
2. Spherical space S^n with the pullback metric from $\mathbb{R}^{n+1}$: $\kappa > 0$.
3. Hyperbolic space with the metric $\frac{ds^2}{(x^n)^2}$: $\kappa < 0$.

Calculating the curvature for $\mathbb{R}^n$ is easy: we can always find an orthonormal frame that is parallel (covariant derivative is zero). Then, since R is defined in terms of the covariant derivatives, R must be zero.

Homework 4. *Prove that $R^d_{abc} = 0$ on the product manifold $S^1 \times S^1$ with the standard product metric.*

Now, let's calculate the curvature for the other two spaces.

Let M be our manifold, and let e_i be a local orthonormal frame on $U \subset M$ with dual ω^i. Then, we know that

$$d\omega^i = \omega^j \wedge \omega^i_j \tag{1.9}$$

with

$$\omega^i_j + \omega^j_i = 0 \tag{1.10}$$

Now, recall that

$$R(e_i, e_j) = \nabla_{e_i}\nabla_{e_j} + \nabla_{e_j}\nabla_{e_i} + \nabla_{[e_i, e_j]} \tag{1.11}$$

and $\nabla_{e_j} e_k = \omega^l_k(e_j)e_l$ for connection forms ω^l_k.

Thus,

$$\begin{aligned}\nabla_{e_i}\nabla_{e_j}e_k &= \nabla_{e_i}(\omega^l_k(e_j)e_l + \omega^l_k(e_j)\nabla_{e_j}e_l \\ &= e_i\omega^l_k(e_j)e_l + \omega^l_k(e_j)\omega^m_l(e_j)e_m\end{aligned}$$

Now, if the frame is normal, and we calculate at the center, $[e_i, e_j] = 0$ and so the last term vanishes.

So, we have

$$\begin{aligned} R(e_i, e_j)e_k &= e_i\omega_k^l(e_j)e_l + \omega_k^l(e_j)\omega_l^m(e_j)e_m + e_j\omega_k^l(e_i)e_l + \omega_k^l(e_i)\omega_l^m(e_i)e_m \\ &= d\omega_k^l(e_j, e_i)e_l + \omega_k^m \wedge \omega_m^l(e_i, e_j)e_l \\ &= (d\omega_k^l + \omega_k^m \wedge \omega_m^l)(e_i, e_j)e_l \end{aligned}$$

where the form in parentheses is the curvature form. Note that this differs from the normal convention by a negative sign, because the modern definition of the Riemann curvature tensor is $R_{abc}^d\omega_d = (-\nabla_a\nabla_b\omega_c + \nabla_b\nabla_a\omega_c)$ which is the negative of the definition found in Wald.

By convention, we define the curvature 2-form Ω to be

$$\Omega_i^j = d\omega_i^j + \omega_i^k \wedge \omega_k^j \tag{1.12}$$

These, however, are frame-dependent! We can define a global curvature form Ω on the principal bundle over the manifold with structure group $O(n)$. Then, $\Omega_x \in \Lambda_x^{2*}M \otimes o(n)$ is a 2-form with values in $o(n)$. (not important for this class)

Recall our goal to calculate the curvature of hyperbolic space. We know now that

$$R(X,Y)e_i = \Omega_i^j(X,Y)e_j \tag{1.13}$$

and the hyperbolic metric is

$$\frac{ds^2}{(x^n)^2} \tag{1.14}$$

Let's find the connection 1-forms using the orthonormal coframe $\omega^i = (\frac{dx^i}{x^n})^2$

$$\begin{aligned} d\omega^i &= -\frac{1}{y^2}dy \wedge dx^i \\ &= -\omega^n \wedge \omega^i \end{aligned}$$

with $y = x^n$. The equating these with the structure equations

$$d\omega^i = \omega^j \wedge \omega_j^i \tag{1.15}$$

and

$$\omega_j^i + \omega_i^j = 0 \tag{1.16}$$

to get

$$\omega_i^n = \omega^i$$

with the other terms (not derived from antisymmetry) are zero.

Now, we have

$$\begin{aligned} \tilde{\Omega}_j^i &= d\omega_j^i + \omega_k^i\omega_j^k \\ \tilde{\Omega}_j^i &= 0 + \omega^i \wedge \omega^j \qquad i,j < n \\ \tilde{\Omega}_n^i &= d\omega_n^i + \omega_k^i\omega_n^k \\ &= d\omega_n^i = -d\omega^i \\ &= -\omega^i \wedge \omega^n \end{aligned}$$

So, generally, $\tilde{\Omega}^i_j = -\omega^i \wedge \omega^j$.

Now, let's calculate the whole curvature tensor. Let $Z = \xi^i e_i$. Then,

$$\begin{aligned}
R(X,Y)Z &= -\tilde{\Omega}^j_i(X,Y)e_j \\
&= -\xi^i \omega^i \wedge \omega^j(X,Y)e_j \\
&= -Z^\flat \wedge (\omega^j e_j)(X,Y) \\
&= -Z^\flat \wedge \mathrm{Id}(X,Y) \\
&= -Z^\flat(X)\mathrm{Id}(Y) + Z^\flat(Y)\mathrm{Id}(X) \quad = -g(X,Z)Y + g(Y,Z)X
\end{aligned}$$

Recall from earlier that

$$R(X,Y,Z,W) = \kappa(g(X,Z)g(Y,W) - g(X,W)g(Y,Z))$$

or

$$R(X,Y)Z = \kappa(g(X,Z)Y - g(Y,Z)X)$$