Homework 2

Daniel Halmrast

October 13, 2018

Problem 1

Recall the φ^4 Lagrangian is given by

$$\mathscr{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4$$

and has an energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + g^{\mu\nu}\mathscr{L}$$

Part A

Problem. *Derive the equation of motion for φ subject to the φ^4 Lagrangian.*

To calculate the equation of motion for φ, we just have to find the stationary points of

$$S = \int d^4x\mathscr{L} = \int d^4x\left(\frac{-1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4\right)$$

That is, we find when $\delta S = 0$. To do so, we calculate

$$\begin{aligned}
\delta S &= \int d^4x\delta\mathscr{L}\\
&= \int d^4x\left(\frac{-1}{2}\delta(\partial^\mu\varphi\partial_\mu\varphi) - \frac{1}{2}m^2\delta(\varphi^2) - \frac{1}{3!}g\delta(\varphi^3) - \frac{1}{4!}\lambda\delta(\varphi^4)\right)\\
&= \int d^4x\left(\frac{-1}{2}(\partial^\mu\delta\varphi\partial_\mu\varphi + \partial^\mu\varphi\partial_\mu\delta\varphi) - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi\right)\\
&= \int d^4x\left(\partial^2\varphi\delta\varphi - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi\right)\\
&= \int d^4x\left(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3\right)\delta\varphi
\end{aligned}$$

Which is zero for arbitrary variation if $\left(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3\right) = 0$. Thus, this is the equation of motion for φ.

Part B

Problem. *Show that the energy-momentum tensor $T^{\mu\nu}$ satisfies $\partial_\mu T^{\mu\nu} = 0$.*

This is just an exercise in direct calculation:

$$
\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu \left(\partial^\mu \varphi \partial^\nu \varphi\right) + \partial_\mu g^{\mu\nu} \mathscr{L} \\
&= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi + \partial^\mu \varphi \partial_\mu \partial^\nu \varphi + \partial^\nu \left(\frac{-1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{3!} g \varphi^3 - \frac{1}{4!} \lambda \varphi^4 \right) \\
&= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi + \partial^\mu \varphi \partial_\mu \partial^\nu \varphi - \frac{1}{2} \partial^\nu \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} \partial^\mu \varphi \partial^\nu \partial_\mu \varphi - m^2 \varphi \partial^\nu \varphi - \frac{1}{2} g \varphi^2 \partial^\nu \varphi - \frac{1}{3!} \lambda \varphi^3 \partial^\nu \varphi \\
&= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi - m^2 \varphi \partial^\nu \varphi - \frac{1}{2} g \varphi^2 \partial^\nu \varphi - \frac{1}{3!} \lambda \varphi^3 \partial^\nu \varphi \\
&= \left(\partial^2 \varphi - m^2 \varphi - \frac{1}{2} g \varphi^2 - \frac{1}{3!} \lambda \varphi^3 \right) \partial^\nu \varphi
\end{aligned}
$$

Which is clearly zero if φ follows its equation of motion.

Problem 2

Consider a complex scalar field φ governed by the Lagrangian

$$\mathscr{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0$$

Part A

Problem. *Show φ obeys the Klein-Gordon equation.*

We calculate the variation in $S = \int d^4 x \mathscr{L}$ directly:

$$\begin{aligned}
\delta S &= \int d^4 x \delta \mathscr{L} \\
&= \int d^4 x \left(-\delta(\partial^\mu \varphi^\dagger) \partial_\mu \varphi - \partial^\mu \phi^\dagger \delta(\partial_\mu \varphi) - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\
&= \int d^4 x \left(-\partial^\mu \delta \varphi^\dagger \partial_\mu \varphi - \partial^\mu \phi^\dagger \partial_\mu \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\
&= \int d^4 x \left(\delta \varphi^\dagger \partial_2 \varphi + \partial^2 \phi^\dagger \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\
&= \int d^4 x \left(\left(\partial^2 \varphi - m^2 \varphi \right) \delta \varphi^\dagger + \left(\partial^2 \varphi^\dagger - m^2 \varphi^\dagger \right) \delta \varphi \right)
\end{aligned}$$

Which is zero for arbitrary variations when both φ and $\varphi^\dagger$ follow the Klein-Gordon equation.

Part B

Problem. *Find the conjugate momenta for φ and $\varphi^\dagger$, and write down the Hamiltonian in terms of these.*

We can read off the conjugate momenta easily:

$$\begin{aligned}
\pi(x) &= \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} \\
&= \frac{\partial}{\partial \dot{\varphi}}(\partial_\mu \varphi^\dagger(x) \partial_\mu \varphi(x) - m^2 \varphi^\dagger(x)\varphi(x) + \Omega_0) \\
&= \dot{\varphi}^\dagger(x)
\end{aligned}$$

and similarly

$$\pi^\dagger(x) = \dot{\varphi}(x)$$

We write down the Hamiltonian density as

$$\mathscr{H} = \pi(x)\dot{\varphi}(x) + \pi^\dagger(x)\dot{\varphi}^\dagger(x) - \mathscr{L}$$

and calculate

$$\begin{aligned}
\mathscr{H} &= \pi(x)\dot{\varphi}(x) + \pi^\dagger(x)\dot{\varphi}^\dagger(x) + \left(\partial^\mu \varphi^\dagger(x)\partial_\mu \varphi(x) + m^2 \varphi^\dagger(x)\varphi(x) - \Omega_0\right) \\
&= \pi(x)\pi^\dagger(x) + \pi^\dagger(x)\pi(x) + \partial^0 \varphi^\dagger(x)\partial_0 \varphi(x) + \partial^i \varphi^\dagger(x)\partial_i \varphi(x) + m^2 \varphi^\dagger(x)\varphi(x) - \Omega_0 \\
&= \pi(x)\pi^\dagger(x) + \pi^\dagger(x)\pi(x) - \pi(x)\pi^\dagger(x) + \partial^i \varphi^\dagger(x)\partial_i \varphi(x) + m^2 \varphi^\dagger(x)\varphi(x) - \Omega_0
\end{aligned}$$