Final Exam

Daniel Halmrast

June 12, 2018

Problem 1

Let M be a complete Riemannian manifold with sectional curvature $K, K \geq k > 0$. Let γ be a nontrivial closed geodesic in M. Show that for any $p \in M$,

$$d(p, \gamma) \leq \frac{\pi}{2\sqrt{k}}$$

Proof. For simplicity, we normalize our space so that $k = 1$. Now, suppose for a contradiction that there is some $p \in M$ with $d(p, \gamma) > \frac{\pi}{2}$. Denote by σ a minimizing geodesic from p to a closest point q on γ, which we know satisfies $\ell(\sigma) > \frac{\pi}{2}$. Finally, denote by q' the point opposite q on γ (that is, parameterizing γ with $\gamma(0) = q$, let $q' = \gamma(\frac{\ell(\gamma)}{2})$).

We split this proof in two cases based on the length of γ.

Suppose first that $\ell(\gamma) \geq \pi$. We now invoke the Toponogov comparison theorem on the triangle formed by the geodesics σ from p to q and γ from q to q'. We will label the edges of the triangle as:

Side	Label
$\overline{pq'}$	a
$\overline{pq}(= \sigma)$	b
$\overline{qq'}(= \gamma)$	c

Since σ minimizes the distance from p to γ, we know that the angle formed between σ' and γ' is exactly $\frac{\pi}{2}$. That is, σ and γ intersect orthogonally.

Now, we form the associated triangle in S^2 with constant curvature 1, labeling the sides in the same way as $\tilde{a}, \tilde{b}, \tilde{c}$, and requiring that $b = \tilde{b}$ and $c = \tilde{c}$ in length, and the angle opposite a is equal to the angle opposite $\tilde{a}$. In particular, we know that $\tilde{b} > \frac{\pi}{2}$, and the angle opposite $\tilde{a}$ (labeled $\tilde{A}$) is exactly $\frac{\pi}{2}$. By simple spherical geometry, we know that

$$\begin{aligned} \cos(\tilde{a}) &= \cos(\tilde{b})\cos(\tilde{c}) + \sin(\tilde{b})\sin(\tilde{c})\cos(\tilde{A}) \\ &= \cos(\tilde{b})\cos(\tilde{c}) + 0 \end{aligned}$$

Now, since $\tilde{b} > \frac{\pi}{2}$, $\cos(\tilde{b}) < 0$. Furthermore, we know that $\cos(\tilde{c}) < 0$ as well, since $\tilde{c} > \frac{\pi}{2}$ (by assumption, $\ell(\gamma) \geq \pi$, so $d(q, q') = \frac{\ell(\gamma)}{2} \geq \frac{\pi}{2}$).

However, since both $\cos(\tilde{b})$ and $\cos(\tilde{c})$ are negative, their product is positive. This implies that $\cos(\tilde{a})$ is positive, and so $\tilde{a} < \frac{\pi}{2}$. By the Toponogov comparison theorem, $a \leq \tilde{a} < \frac{\pi}{2}$. However, this contradicts q being a closest point to p, since $d(p, q') \leq a < \frac{\pi}{2}$ but $d(p, q) > \frac{\pi}{2}$. Thus, no such p can exist.

Now, suppose $\ell(\gamma) < \pi$. Since we assumed $d(p, \gamma) > \frac{\pi}{2}$, we know that the diameter $d(M) > \frac{\pi}{2}$ as well...

If we accept the hint in problem 5, this situation cannot occur, and the proof is finished. $\square$

Problem 2

Let M be a compact n-dimensional manifold of positive sectional curvature, and A, B two closed totally geodesic submanifolds. Show that A and B must intersect if $\dim(A) + \dim(B) \geq n$.

Proof. Suppose for a contradiction that A and B do not intersect. Then, I claim there is a point $a \in A$ and $b \in B$ such that $d(a, b) = d(A, B)$ and a geodesic γ from a to b which realizes this distance. Furthermore, γ is orthogonal to A and B. We will show that such a γ has a variation whose second variation of energy is negative, leading to a contradiction of γ being minimizing.

To see that such a γ exists, recall that for any closed submanifold N of M, and point $p \notin N$, there exists a point $q \in N$ with $d(p, q) = d(p, N)$ and a minimizing geodesic γ from p to q which realizes this distance, and is orthogonal to N. Letting $p \in A$ and $N = B$, we define the function $f : A \to \mathbb{R}$ as $f(p) = d(p, B)$. Since A is compact (as a closed subset of M a compact space), this function achieves a minimum. Call the point which achieves such a minimum a. Clearly, a and the corresponding close point $b \in B$ are such that $d(a, b) = d(A, B)$, and by construction the minimizing geodesic γ is orthogonal to B. By symmetry, γ is also orthogonal to A as well.

Now, I assert that the second variation of energy for any orthogonal variational field V with associated variation $h(t, s)$ of γ with $h(0, s) \in A$ and $h(l, s) \in B$ is given by

$$\frac{1}{2}E''(0) = I_l(V, V) + \langle V(l), S^{(2)}_{\gamma'(l)}V(l)\rangle - \langle V(l), S^{(1)}_{\gamma'(0)}V(l)\rangle$$

where $S^{(i)}_{\gamma'}$ is the linear map associated to the second fundamental form of A, B in the direction of γ'.

To see this, we calculate directly

$$\begin{aligned}\frac{1}{2}E''(0) &= I_l(V, V) + \langle \nabla_s V, \gamma'\rangle(0, l) - \langle \nabla_s V, \gamma'\rangle(0, 0)\\ &= I_l(V, V) + \langle B(V, V), \gamma'\rangle(0, l) - \langle B(V, V), \gamma'\rangle(0, 0)\\ &= I_l(V, V) + \langle V(l), S^{(2)}_{\gamma'(l)}V(l)\rangle - \langle V(l), S^{(1)}_{\gamma'(0)}V(l)\rangle\end{aligned}$$

as desired.

Now, since A and B are totally geodesic submanifolds, the second fundamental form vanishes, and we are left with

$$\frac{1}{2}E''(0) = I_l(V, V)$$

which will be important later.

Let $\{a_i\}$ be a basis for $T_a(A)$. We can parallel transport this basis along γ to obtain a set of $\dim(A)$ linearly independent vectors orthogonal to γ at $T_b(M)$. Denote by $\{b_i\}$ a basis for $T_b(B) \subset T_b(M)$. Since $\dim(A) + \dim(B) \geq n$, there must be some linear combination of the parallel transports of $\{a_i\}$ that sum to some vector $v \in T_b(B)$ orthogonal to γ (since the dimension of the subspace of $T_b(M)$ orthogonal to γ is $n - 1$, and there are $\dim(A) + \dim(B) \geq n > n-1$ vectors in the set $\{a_i\} \cup \{b_i\}$, this set cannot be linearly independent. Since each basis set $\{a_i\}$ and $\{b_i\}$ are linearly independent, there must be some linear combination of $\{a_i\}$ that sums to some linear combination of $\{b_i\}$). That is to say, there is a vector $v \in T_a(A)$ such that the parallel transport of v into $T_b(M)$ lies in $T_b(B)$.

Consider a variational field generated by parallel transport of such a vector v (call this variational field V). In particular, this variation satisfies the hypotheses for the calculation in the previous paragraph. So, we calculate:

$$\begin{aligned}\frac{1}{2}E''(0) &= I_l(V, V)\\ &= \int_0^l \langle V', V'\rangle - K(V, \gamma')dt\end{aligned}$$

Now, since V is parallel to γ, $V' = 0$, and so

$$\frac{1}{2}E''(0) = -\int_0^l K(V, \gamma')dt$$

and since K is always positive, the second variation of energy of V is negative, contradicting γ being a minimal geodesic from a to b. Thus, it cannot be that A and B do not intersect, as desired. $\square$

Problem 3

Let M^2 be a complete simply connected 2-dimensional Riemannian manifold. Suppose that for each point $p \in M$, the locus $C(p)$ of first conjugate points to p reduces to a unique $q \neq p$ and that $d(p, C(p)) = \pi$. Prove that if the sectional curvature K of M satisfies $K \leq 1$, then M is isometric to the sphere S^2 with $K = 1$.

Proof. Let J be a Jacobi field along a normalized geodesic γ joining p to q with $J(0) = J(\pi) = 0$ and $g(J, \gamma') = 0$. Let $\{e_i, \gamma'\}$ be an orthonormal parallel frame to γ, and write

$$J = a^i e_i$$

Define $K(t) = K(\gamma', J)$. We calculate

$$\begin{aligned}
0 = I_\pi(J, J) &= -\int_0^\pi g(J'' + R(\gamma', J)\gamma', J)dt \\
&= -\int_0^\pi g(J'', J)dt - \int_0^\pi K(t)\|J\|^2 dt \\
&= -\int_0^\pi a''^i a_i dt - \int_0^\pi K(t)a^i a_i dt \\
&= \int_0^\pi a'^i a_i' dt - \int_0^\pi K(t)a^i a_i dt && \text{using integration by parts} \\
&\geq \int_0^\pi a^i a_i dt - \int_0^\pi K(t)a^i a_i dt && \text{by homework 3 problem 1} \\
&= \int_0^\pi a^i a_i (1 - K(t))dt \geq 0
\end{aligned}$$

and thus $K(t) = 1$ for all t. Thus, in the interior of any geodesic connecting a point p to its first conjugate point, the sectional curvature of the space spanned by γ' and J for J a Jacobi field along γ vanishing at the endpoints is identically 1.

Now, all that remains is to show every point on the manifold is an interior point of some geodesic as described above, and that for such a geodesic, there exists a Jacobi field vanishing at the endpoints in each direction orthogonal to γ'.

We handle the second assertion first. To prove that there exist Jacobi fields vanishing at the endpoints of γ in each direction orthogonal to γ', we show that the multiplicity of q as a conjugate point to p is exactly $n-1$. We appeal to proposition 3.5 of chapter 5 of Do Carmo, which states that the multiplicity of q is exactly the dimension of the kernel of $(d\exp_p)_{v_0}$ where $v_0 = \gamma'(0)$.

Recall that for $w \in T_{V_0}(T_pM)$, we have

$$(d\exp_p)_{v_0}(w) = \partial_t(\exp_p(v_0 + tw))|_{t=0}$$

Suppose w is orthogonal to v_0. Then, $v_0 + tw$ lies approximately on the sphere of radius $\|v_0\|$ for small t. In particular,

$$\begin{aligned}
\|v_0 + tw\|^2 &= \|v_0\|^2 + t^2\|w\|^2 + 2t\langle v_0, w\rangle \\
&= \|v_0\|^2 + t^2\|w\|^2 \\
\|v_0 + tw\| &\approx \|v_0\| + O(t^2)
\end{aligned}$$

and so the geodesic $\gamma_{tw}(s) = \exp_p(s(v_0 + tw))$ has to first order the same speed as $\gamma(s) = \exp_p(sv_0)$. Now, the function $f_p : T_pM \to \mathbb{R}$ taking a vector v and returning the distance along the geodesic generated by v to its first conjugate point is continuous, so $f(v_0 + tw)$ is close to $f(v_0)$. In particular, since q is the only conjugate point to p, it must be that $\gamma_{tw}(1 + \varepsilon)$ for some small ε lands at q. Thus, $\partial_t(\exp_p(v_0 + tw))|_{t=0} = 0$ and w is in the kernel of $(d\exp_p)_{v_0}$. Since we can do this for each w orthogonal to v_0, it follows that the dimension of the kernel is $n-1$, and so the multiplicity of q is $n-1$ as desired.

Finally, we show that every point in M is the interior point of some geodesic connecting two conjugate points. In fact, we only have to show that there is a dense subset of M with this property, and since the sectional curvature is continuous with respect to points in M, it will hold that $K = 1$ for all points in M as well.

So, let $p \in M$. Now, we know that the conjugate locus of p is a single point q. This means that there is some geodesic $\exp_p(tv)$ for a unit vector v in T_pM for which $\exp_p(\pi v) = q$ and q is conjugate to p along this geodesic. Now, since the function f_p defined above is continuous, and can only take values of π and ∞ (by the hypothesis of the problem), it follows that $f(v) = \pi$ for all v. Furthermore, since M is complete, there exists a minimizing geodesic from p to any other point on the manifold. Thus,

$$M = \exp_p(B_\pi(0)) \cup \{q\}$$

and thus every point except p and q lies in the interior of a geodesic on which p is conjugate to q. Thus, K is identically 1 on M, and since M is simply connected, it follows that $M = S^n$ for n the dimension of M. $\square$

Problem 5

Let M be a compact Riemannian manifold of dimension n with sectional curvature $K \geq 1$. Suppose $p, q \in M$ with $d(p,q) = d(M) > \frac{\pi}{2}$. Moreover, suppose $\gamma : [0,1] \to M$ is a geodesic with $\gamma(0) = \gamma(1) = p$. Show that γ has Morse index at least $n-1$.

Proof. Suppose we accept the hint that $\ell(\gamma) > \pi$. Now, fix $c \in (1, \infty)$ a constant such that $\ell(\gamma) > \pi\sqrt{c}$.

Now, in homework 3, we proved that if $\gamma : (-\infty, \infty) \to M$ is a normalized geodesic in M, then there exists a $t_0 \in \mathbb{R}$ with γ restricted to $[-t_0, t_0]$ having Morse index at least $n-1$. The proof is replicated here:

Proof. Let Y be a parallel field along γ with $g(\gamma', Y) = 0$ and $\|Y\| = 1$. Set

$$\phi_Y = g(R(\gamma', Y)\gamma', Y)$$

and

$$K(t) = \inf_Y \phi_Y(t)$$

and let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $0 \leq a(t) \leq K(t)$ with $0 < a(0) < K(0)$. Let ϕ be the solution to $\phi'' + a\phi = 0$ with $\phi(0) = 1, \phi'(0) = 0$, with $-t_1, t_2$ the two zeroes of ϕ found in the previous problem. We consider the field $X = \phi Y$, and calculate

$$\begin{aligned}
I_{[-t_1,t_2]}(X,X) &= -\int_{-t_1}^{t_2} g(X'' + R(\gamma', X)\gamma', X)dt \\
&= -\int_{-t_1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{t_1}^{t_2} g(\phi R(\gamma', Y)\gamma', \phi Y) \\
&= -\int_{-t_1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{t_1}^{t_2} \phi^2 \phi_Y dt \\
&\leq -\int_{-t-1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{-t_1}^{t_2} K(t)\phi^2(t)dt \\
&= -\int_{-t_1}^{t_2} \phi(\phi'' + K(t)\phi)dt \\
&< -\int_{-t_1}^{t_2} \phi(\phi'' + a(t)\phi)dt \\
&= 0
\end{aligned}$$

Thus, for all Y perpendicular to γ' (an $n-1$ dimensional subspace) the form $I_{[-t_1,t_2]}(Y,Y)$ is negative-definite, and so the index is greater than or equal to $n-1$. $\square$

Recall that we required the field X to vanish at the endpoints so that integration by parts works.

Now, setting $a(t) = \frac{1}{c}$ (which satisfies the hypotheses for a, since $K \geq 1$, and $\frac{1}{c} < 1$) we observe that the unique solution ϕ is

$$\phi(t) = \cos(\frac{t}{\sqrt{c}})$$

which has zeroes at $t = \pm\frac{\pi\sqrt{c}}{2}$. In particular, the distance between consecutive zeroes is $\pi\sqrt{c} < \ell(\gamma)$. So, reparameterizing γ to be unit speed, we see that $\gamma : [-\frac{\pi\sqrt{c}}{2}, \frac{\pi\sqrt{c}}{2}] \to M$ contains both zeros and does not intersect itself (since $\ell(\gamma) > \pi\sqrt{c}$). Thus, since the Morse index is an increasing function, it follows that the Morse index for one period of γ is at least $n-1$ as desired. $\square$