Problem Set 5

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Problem 1

Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$ so that f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. We begin by explicitly constructing the homotopy that is stationary on $S^1 \times \{0\}$. This is done via the homotopy

$$f_t(\theta, s) = (\theta + t2\pi s, s)$$

Clearly, $f_0 = \mathbb{1}$ and $f_1 = f$. Furthermore, $f_t(\theta, 0) = (\theta, 0)$ and so f_t is stationary on $S^1 \times \{0\}$ □

Problem 2

Does the Borsuk-Ulam theorem hold for the torus? That is, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ does there exist a point (x, y) for which $f(x, y) = f(-x, -y)$?

Proof. I assert that the Borsuk-Ulam theorem does not hold for the torus. To see this, we construct an explicit function from T^2 to $\mathbb{R}^2$ which does not have any antipodal points with the same value.

Consider the function $f : T^2 \to \mathbb{R}^2$ given as follows. First, let T^2 be embedded in $\mathbb{R}^3$. Then, consider the vector field $\frac{\partial}{\partial \phi}$ where ϕ runs parallel to the $x - y$ plane. Since T^2 is embedded in $\mathbb{R}^3$, these vectors can be thought of as living in the tangent bundle to $\mathbb{R}^3$. Thus, for each vector, it makes sense to take its projection onto the $x - y$ plane. Now, since the original vector field $\frac{\partial}{\partial \phi}$ is smooth, and projection is a continuous operation, this defines a continuous map from T^2 to $\mathbb{R}^2$.

In coordinates, this map is given as

$$f(\theta, \phi) = (\cos(\phi), \sin(\phi)) \tag{0.1}$$

and clearly, $f(\theta, \phi) \neq f(-\theta, -\phi)$ as desired. $\square$

Problem extra

Find the standard form of $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6$, and prove or disprove:

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6 \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$