Homework 2

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Problem 1

Characterize all the norm-closed faces of the unit ball in $C([0,1])$ under the sup norm.

Proof. I assert that the norm-closed faces of the unit ball in $C([0,1])$ have the following form: Let V be a disjoint union of open subintervals of $[0,1]$, and for each closed subinterval in V^c assign it to the set N or P. Then, a face can be defined as

$$F = \{f \in B \mid f(N) = \{-1\}, f(P) = \{1\}\}$$

where B is the unit ball in $C([0,1])$. Note that if $V = [0,1]$, we recover B. So, assume from here on out that either N or P (or both) is nonempty.

First, we prove that sets of this form are faces. To do so, we must prove that F is convex, and that F is "closed under linear interpolation".

We first show that F is convex. Let $f, g \in F$, and consider the function

$$h(x) = \lambda f(x) + (1-\lambda)g(x)$$

for $\lambda \in [0,1]$, which is a linear combination of continuous functions, and is therefore continuous. Furthermore,

$$\sup_x |h(x)| \leq \lambda \sup_x |f(x)| + (1-\lambda)\sup_x |g(x)| \leq 1$$

and so $h \in B$. Furthermore, for $x \in N$,

$$h(x) = \lambda f(x) + (1-\lambda)g(x) = \lambda(-1) + (1-\lambda)(-1) = -1$$

and similarly for $x \in P$,

$$h(x) = 1$$

Thus, $h \in F$ as desired.

Next, we show that for $h \in F$, $f, g \in B$ and $t \in (0,1)$ with

$$h(x) = tf(x) + (1-t)g(x)$$

then $f(x)$ and $g(x)$ are in F as well.

This follows immediately, since for $x \in N$,

$$h(x) = -1 = tf(x) + (1-t)g(x)$$

and $f(x), g(x) \in [-1, 1]$, which forces $f(x) = g(x) = -1$. Similarly, for $x \in P$, $f(x) = g(x) = 1$. Thus, $f, g \in F$ as desired.

Thus, we have shown that F is a face of B. Next, we wish to show that all norm-closed faces of B have this form.

So, let F be an arbitrary norm-closed face of B. Define

$$N = \bigcap_{f \in F} f^{-1}(\{-1\})$$

and

$$P = \bigcap_{f \in F} f^{-1}(\{1\})$$

That is, for all $f \in F$, $f(P) = \{1\}$ and $f(N) = \{-1\}$. I assert that F is equal to the set

$$\tilde{F} = \{f \in B \mid f(N) = \{-1\}, f(P) = \{1\}\}$$

Clearly, $F \subset \tilde{F}$ by definition of N and P. So, all we need to show is that $\tilde{F} \subset F$.

So, let $g \in \tilde{F}$ be an arbitrary function. We wish to uniformly approximate g with things in F, and then use the fact that F is norm-closed to show that g is in F. Note first that g is guaranteed to match with any function in F on N and P by construction, so we only have to approximate g on $V = (P \cup N)^c$.

Now, for each $y \in V$, by definition there is a function $f_y \in F$ with $f_y(U) \in (-1, 1)$ for U a neighborhood of y. In particular, we can use the fact that F is a face (and closed under linear interpolations) to construct a $\tilde{f}_y$ such that $\tilde{f}_y(y) = g(y)$. In fact, for fixed $\varepsilon > 0$, we can find neighborhoods U_y for which $\tilde{f}_y|_{U_y}$ is within ε of $g|_{U_y}$ in supremum. Taking all such U_y across $[0, 1]$ (where trivially $f(y) = g(y)$ for $y \in N \cup P$) we get an open cover of $[0, 1]$, which has a finite subcover $\{U_{y_i}\}$.

For each $\tilde{f}_{y_i}$, change it via linear interpolation on $V \setminus U_{y_i}$ to be within ε of zero.

Finally, stitch together the finite altered $\tilde{f}_{y_i}$ to obtain a function $\tilde{f} \in F$ that approximates g within ε in sup norm. Taking limits of such $\tilde{f}$ yields g, and so $g \in F$ as desired.

$\square$

Problem 2

Characterize all the extreme points of the set

$$K = \{f \in \ell^1 \mid 0 \leq f(n) \leq 1 \forall n \in \mathbb{N}, \int_{\mathbb{N}} f d\mu = 1\}$$

Proof. I assert that all the extreme points of K are the basis vectors $e_n = (0, \dots, 0, 1, 0, \dots)$ where 1 is in the nth position.

First, we observe that these points are indeed extreme points. Suppose that $f, g \in K$ with

$$e_n = tf + (1-t)g$$

for some $t \in (0,1)$. Now, for all $i \neq n$, we have

$$0 = tf(i) + (1-t)g(i)$$

but $f(i)$ and $g(i)$ are both in $[0,1]$, and t and $1-t$ are both positive and nonzero, which forces $f(i) = g(i) = 0$. The normalization condition on K forces $\int_{\mathbb{N}} f d\mu = \int_{\mathbb{N}} g d\mu = 1$ which implies that $f(n) = g(n) = 1$, and so $f = g = e_n$. Thus, each e_n is indeed an extreme point of K.

Next, we observe that these are all the extreme points. Suppose f is not e_n for any n. In particular, this means that there are at least two integers n_01, n_2 for which $f(n_1) \in (0,1)$ and $f(n_2) \in (0,1)$. Now, let $\varepsilon > 0$ be such that $f(n_i) \pm \varepsilon \in (0,1)$.

Now, define g and h as

$$g(i) = \begin{cases} f(n_1) + \varepsilon, & i = n_1 \\ f(n_2) - \varepsilon, & i = n_2 \\ f(i), & \text{else} \end{cases}$$

$$h(i) = \begin{cases} f(n_1) - \varepsilon, & i = n_1 \\ f(n_2) + \varepsilon, & i = n_2 \\ f(i), & \text{else} \end{cases}$$

By our choice of ε, $g(\mathbb{N}), h(\mathbb{N}) \in [0,1]$ and by construction

$$\int_{\mathbb{N}} g d\mu = \int_{\mathbb{N}} h d\mu = 1$$

since we have only moved ε from one element of the sum to another. Thus, $g, h \in K$. Furthermore, for all $i \in \mathbb{N}$, $i \neq n_1, n_2$,

$$\frac{1}{2}g(i) + \frac{1}{2}h(i) = \frac{1}{2}f(i) + \frac{1}{2}f(i) = f(i)$$

and

$$\begin{aligned} \frac{1}{2}g(n_1) + \frac{1}{2}h(n_1) &= \frac{1}{2}(f(n_1) + \varepsilon) + \frac{1}{2}(f(n_1) - \varepsilon) \\ &= \frac{1}{2}(2f(n_1)) = f(n_1) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}g(n_2) + \frac{1}{2}h(n_2) &= \frac{1}{2}(f(n_2) - \varepsilon) + \frac{1}{2}(f(n_2) + \varepsilon) \\ &= \frac{1}{2}(2f(n_2)) = f(n_2) \end{aligned}$$

which verifies that $f = \frac{1}{2}g + \frac{1}{2}h$, and thus g and h belong to the same face as f, and in particular, f is not an extreme point. $\square$

Problem 3

Prove or disprove the following:

Part A

$L^1(X, \mathbb{M}, \mu)$ is an algebra under pointwise multiplication over $\mathbb{C}$, where $(X, \mathbb{M}, \mu)$ is a measure space.

Proof. This is not true in general. Take $L^1([0,1], \lambda^1)$, and let $f(x) = g(x) = \frac{1}{\sqrt{x}}$. We know that

$$\int_{[0,1]} \frac{1}{\sqrt{x}} d\lambda^1(x) = 2$$

and so $f, g \in L^1([0,1], \lambda^1)$. However, their pointwise product is

$$f(x)g(x) = \frac{1}{x}$$

with

$$\int_{[0,1]} |f(x)g(x)| d\lambda^1(x) = \int_{[0,1]} \frac{1}{x} d\lambda^1(x) = \infty$$

and so $f(x)g(x)$ is not in $L^1([0,1], \lambda^1)$ and thus $L^1([0,1], \lambda^1)$ is not closed under products, and is not an algebra. $\square$

Part B

Same question for the subspace of bounded functions in $L^1(X, \mathbb{M}, \mu)$.

Proof. Let V be the subspace of bounded functions in $L^1(X, \mathbb{M}, \mu)$. I assert that this is an algebra. First, we observe that this subspace is indeed a subspace of $L^1(X, \mathbb{M}, \mu)$, and is thus closed under addition and scalar multiplication. Thus, to prove it is an algebra, we just need to show it is closed under products.

So, let $f, g \in V$. We first show $f(x)g(x)$ is in $L^1(X, \mathbb{M}, \mu)$, then we show it is bounded.

To see that $f(x)g(x) \in L^1(X, \mathbb{M}, \mu)$, we use Holder's inequality to assert that

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty < \infty$$

since $\|f\|_1 < \infty$ and $\|g\|_\infty < \infty$. Thus, $fg \in L^1(X, \mathbb{M}, \mu)$.

Finally, we observe that fg is bounded, since both f and g are individually bounded. In particular, we know that (denoting the bound on f as $\|f\|_B$)

$$\|fg\|_B \leq \|f\|_B \|g\|_B$$

since

$$|f(x)g(x)| \leq |\|f\|_B g(x)| \leq \|f\|_B \|g\|_B$$

for each x. Thus, $\|f\|_B \|g\|_B$ is an upper bound on fg as desired. $\square$

Problem 4

Prove that the set M_1^+ of all positive self-adjoint $n \times n$ complex matrices between 0 and 1 is convex, and its extreme points are the projections.

Proof. We first show this set is convex. To do so, let $A, B \in M_1^+$ and consider

$$C = \lambda A + (1 - \lambda)B$$

for $\lambda \in [0, 1]$. First, observe that C is self-adjoint, since

$$C^* = \lambda A^* + (1 - \lambda)B^* = \lambda A + (1 - \lambda)B = C$$

Furthermore, C is positive, since

$$\begin{aligned} \langle C\eta, \eta \rangle &= \lambda \langle A\eta, \eta \rangle + (1 - \lambda)\langle B\eta, \eta \rangle \\ &\geq 0 \end{aligned}$$

since both A and B are positive, and $\lambda, (1 - \lambda) \geq 0$.

Finally, we note that $C \leq 1$, since

$$\begin{aligned} \langle (I - C)\eta, \eta \rangle &= \langle (I - (\lambda A + (1 - \lambda)B))\eta, \eta \rangle \\ &= \langle ((\lambda + (1 - \lambda))I - (\lambda A + (1 - \lambda)B))\eta, \eta \rangle \\ &= \lambda \langle (I - A)\eta, \eta \rangle + (1 - \lambda)\langle (I - B)\eta, \eta \rangle \\ &\geq 0 \end{aligned}$$

Where we used the fact that $A, B \leq 1$, and so $I - A$ and $I - B$ are positive.

Next, we assert that the projections are the extreme points. We begin by asserting that for any $M \in M_1^+$, $\sigma(M) \subset [0, 1]$. To see this, note that M being self-adjoint implies that $\sigma(M)$ is real, and since M is positive, $\sigma(M)$ must be positive as well (if this were not the case, then there is some vector v for which $Mv = \lambda v$ with $\lambda < 0$, and $\langle Mv, v \rangle = \lambda \|v\|^2 < 0$ a contradiction). Of course, since $M \leq 1$, this implies that $\sigma(M) \leq 1$ as well (if this were not the case, then there is some vector v with $Mv = \lambda v$ and $\lambda > 1$, but then $\langle (I - M)v, v \rangle = \|v\|^2 - \lambda \|v\|^2 < 0$, a contradiction).

Thus, $\sigma(M) \subset [0, 1]$ as desired.

So, suppose P is a projection operator, and $A, B \in M_1^+$, $t \in (0, 1)$ with $P = tA + (1 - t)B$. Suppose v is such that $Pv = v$ and $\|v\| = 1$. Then, we have that

$$\langle Pv, v \rangle = 1 = t\langle Av, v \rangle + (1 - t)\langle Bv, v \rangle$$

Now, since $\langle Av, v \rangle$ and $\langle Bv, v \rangle$ are less than or equal to 1 (since $A, B \leq 1$), they must be equal to 1 to satisfy the above equation. However, the only way that $\langle Av, v \rangle = 1$ is if $Av = v$. This follows from basic linear algebra: decompose Av into a component parallel to v denoted $\lambda_1 v$ and a component orthogonal to v denoted $\lambda_2 v_c$. That is,

$$Av = \lambda_1 v + \lambda_2 v_c$$

with $\lambda_1^2 + \lambda_2^2 = \|Av\| \leq 1$. But then

$$\langle Av, v \rangle = 1 = \lambda_1 \langle v, v \rangle + \lambda_2 \langle v_c, v \rangle = \lambda_1$$

Thus, $\lambda_1 = 1$ and $\lambda_2 = 0$, and so $Av = v$. Similarly, $Bv = v$. This argument applies to all v for which $Pv = v$.

Applying the same argument to $I - P = t(I - A) + (1 - t)(I - B)$ we see that if $Pv = 0$, then $Av = Bv = 0$ as well. Thus, writing a generic vector x as $x = Px + (I - P)x$, we have

$$\begin{aligned} Ax &= A(Px + (I - P)x) \\ &= Px \end{aligned}$$

and similarly for B. Thus, P is an extreme point, as desired.

Finally, we show that these are the only extreme points. To see this, let $M \in M_1^+$ be a matrix that is not a projection. That is, M has at least one eigenvalue in $(0, 1)$.

So, let

$$M = \sum_{i=1}^{n} \lambda_i |e_j\rangle\langle e_j|$$

be its spectral decomposition, and let $\lambda_1 \in (0, 1)$. In particular, let $\varepsilon > 0$ such that $\lambda_1 \pm \varepsilon \in (0, 1)$. Then, define

$$A = (\lambda_1 + \varepsilon)|e_1\rangle\langle e_1| + \sum_{i=2}^{n} \lambda_i |e_i\rangle\langle e_i|$$

and

$$B = (\lambda_1 - \varepsilon)|e_1\rangle\langle e_1| + \sum_{i=2}^{n} \lambda_i |e_i\rangle\langle e_i|$$

which are easily verified to be in M_1^+. Then,

$$\frac{1}{2}A + \frac{1}{2}B = M$$

and thus M cannot be an extreme point. $\square$