# Problem Set 1

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## PROBLEM 1

Let $\mathscr{F}$ be the set of all measurable functions which are finite $\mu$-a.e. on $\Omega$.

### PART A

Prove $\mathscr{F}$ is a vector space.

*Proof.* We first note that $\mathscr{F}$ is a subset of the vector space of all measurable functions (modulo functions zero $\mu$-a.e.). We just need to show, then, that $\mathscr{F}$ is closed under addition and scalar multiplication.

So, let $f$ and $g$ be measurable functions that are finite $\mu$-a.e. on $\Omega$. Now, let's consider $f+g$. If $x \in \Omega$ is such that $f(x)$ and $g(x)$ are finite, then the sum $f(x)+g(x)$ is finite. Let $E$ be the set of all such $x$. We will show that $\Omega \setminus E$ has measure zero, so that $f+g$ is finite $\mu$-a.e.

To see that $\Omega \setminus E$ has measure zero, we note that $\Omega \setminus E = \{|f| = \infty\} \cup \{|g| = \infty\}$. Now, since $f$ and $g$ are finite $\mu$-a.e., we know that each of these sets has measure zero, and so the union $\Omega \setminus E$ has measure zero as well. Thus, $f+g$ is finite $\mu$-a.e.

It is immediately clear as well that for arbitrary scalar $\alpha$, we have that $\alpha f$ is also finite $\mu$-a.e., since for $x \in \{|f| < \infty\}$, we have that $|f(x)| < \infty$, which implies that $|\alpha f(x)| < \infty$ as well.

Thus, $\mathscr{F}$ is an algebraically closed subspace of a vector space, and is a vector space itself. $\square$

### PART B

Prove that $\mathscr{F}$ is a metric space with the metric

$$d(f,g) = \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$$

*Proof.* To show that $d$ is a metric, we need to show that $d(f,f) = 0$, $d(f,g) > 0$ for $f \neq g$, and the triangle inequality $d(f,h) \leq d(f,g) + d(g,h)$.

It is clear that $d(f,f)=0$, since this amounts to

$$\begin{aligned} d(f,f) &= \int_\Omega \frac{|f-f|}{1+|F-f|}d\mu \\ &= \int_\Omega \frac{0}{1}d\mu \\ &= 0 \end{aligned}$$

Now, suppose $f$ and $g$ differ on a positive-measure set $E$. In other words, $|f-g| \neq 0$ on $E$. Now, since $|f-g|$ is positive on $E$, $\frac{|f-g|}{1+|f-g|}$ is as well. Thus, $\frac{|f-g|}{1+|f-g|} \neq 0$ in $L^1$, and so $\|\frac{|f-g|}{1+|f-g|} - 0\|_{L^1} = \int_\Omega \frac{|f-g|}{1+|f-g|}d\mu > 0$ as desired.

Finally, we wish to prove the triangle inequality. This will follow from the convexity and monotonicity of the function $\rho(x) = \frac{x}{1+x}$. Recall that $\rho$ being convex means

$$\rho(x+y) \leq \rho(x) + \rho(y)$$

and $\rho$ being monotonic means

$$x \leq y \implies \rho(x) \leq \rho(y)$$

With these equipped, we are ready to prove the triangle inequality.

We wish to show that

$$\rho(|f-g|) \leq \rho(|f-h|) + \rho(|h-g|)$$

To do so, we observe that

$$\begin{aligned} \rho(|f-h|) + \rho(|h-g|) &\leq \rho(|f-h| + |h-g|) \\ &\leq \rho(|f-g|) \end{aligned}$$

where the last line followed by monotonicity of $\rho$ and the triangle inequality for the norm. □

## Part c

Show that $d$ metrizes the convergence in measure.

*Proof.* Suppose first that $f_n \to f$ in $\mu$. That is, for all $t > 0$,

$$\mu(\{|f_n - f| > t\}) \to 0$$

Now, consider the integral

$$\begin{aligned} \int_\Omega \frac{|f_n-f|}{1+|f_n-f|}d\mu &= \int_{\{|f_n-f|\leq t\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu + \int_{\{|f_n-f|>t\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu \\ &\leq \int_{\{|f_n-f|\leq t\}} \frac{t}{1+t}d\mu + \int_{\{|f_n-f|>t\}} 1 d\mu \end{aligned}$$

Now, the first term goes to zero as $t$ goes to zero, and the second term is just $\mu(\{|f_n - f| > t\})$, which goes to zero as $n$ goes to infinity. Thus, the expression goes to zero, and $d(f_n, f)$ goes to zero, as desired.

Now, suppose $d(f_n, f)$ goes to zero. We wish to prove that for all $t > 0$, $\mu(\{|f_n - f| > t\}) \to 0$. To do so, we consider

$$\begin{aligned}
d(f_n, f) &= \int_\Omega \frac{|f_n - f|}{1 + |f_n - f|} d\mu \\
&= \int_{\{|f_n - f| > t\}} \frac{|f_n - f|}{1 + |f_n - f|} d\mu + \int_{\{|f_n - f| \le t\}} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \\
&\ge \int_{\{|f_n - f| > t\}} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \\
&\ge \int_{\{|f_n - f| > t\}} \frac{t}{1 + t} d\mu \qquad = \frac{t}{1+t}\mu(\{|f_n - f| > t\})
\end{aligned}$$

and since $d(f_n, f)$ goes to zero, so does $\frac{t}{1+t}\mu(\{|f_n - f| > t\})$, which implies that for fixed $t > 0$,

$$\mu(\{|f_n - f| > t\}) \to 0$$

as well. ☐

# Problem 2

## Part A

Prove that

$$\limsup_j \{\phi_j \ge t\} \subset \{x \mid \limsup_j \phi_j(x) \ge t\}$$

and show that the set containment is proper with a counterexample.

*Proof.* Let $x \in \Omega$ such that $x \in \limsup_j \{\phi_j \ge t\}$. That is, $x$ is in infinitely many sets $\{\phi_j \ge t\}$. This implies that

$$\begin{aligned}
&\forall m \in \mathbb{N}, \exists n > m \text{ s.t. } \phi_n(x) \ge t \\
\implies &\forall m \in \mathbb{N}, \sup_{n > m} \phi_n(x) \ge t \\
\implies &\inf_m \sup_{n > m} \phi_n(x) \ge t \\
\implies &\limsup_n \phi_n(x) \ge t
\end{aligned}$$

and thus $x \in \{x \mid \limsup_j \phi_j(x) \ge t\}$ as desired.

To show containment is proper, consider the sequence of constant functions on $[0, 1]$ defined by $\phi_n(x) = 1 - \frac{1}{n}$.

Now, we set $t = 1$. For each $x \in [0, 1]$, we have that

$$\limsup_n \phi_n(x) = \limsup_n 1 - \frac{1}{n} = 1$$

and so $\{x \mid \limsup_j \phi_j(x) \ge 1\} = [0, 1]$. However, for each $n$, we have that

$$\{\phi_n \ge 1\} = \emptyset$$

and so

$$\limsup_n \{\phi_n \ge 1\} = \emptyset$$

which is clearly not $[0, 1]$ as desired. ☐

## Part B

Prove that

$$\limsup_j \{\phi_j > t\} \supset \{x \mid \limsup_j \phi_j(x) > t\}$$

and show that the set containment is proper with a counterexample.

*Proof.* Suppose $x \in \Omega$ with $\limsup_j \phi_j(x) > t$. In particular, this means that

$$\begin{aligned}
&\inf_m \sup_{n>m} \phi_n(x) > t \\
\implies &\forall m \in \mathbb{N}, \sup_{n>m} \phi_n(x) > t \\
\implies &\forall m \in N, \exists n > m, \phi_n(x) > t \\
\implies &\phi_n(x) > t \text{ for infinitely many } \phi_n \\
\implies &x \in \limsup_j \{\phi_j > t\}
\end{aligned}$$

as desired.

Now to see the containment may be strict, consider the sequence of functions on $[0, 1]$ defined by

$$\phi_n(x) = \frac{1}{n}$$

with $t = 0$. Now, clearly $\{\phi_n > 0\} = [0, 1]$ for all $n$, but for any $x \in [0, 1]$, we have that $\limsup_n \phi_n(x) = 0$ and so

$$\{x \in [0, 1] \mid \limsup_n \phi_n(x) > 0\} = \emptyset$$

and the sets are not equal. □

## Problem 3

Where in the proof of (notes 4.11) was the finiteness of $\Omega$ used? Can this hypothesis be omitted?

*Proof.* In the proof of the Borel-Cantelli lemma, we assert that for a decreasing sequence of sets $F_n$,

$$\mu(\lim F_n) = \lim(\mu(F_n))$$

which requires the space to be finite. More specifically, passing the measure through the limit requires the first set in the sequence to be finite measure. This was never asserted in the proof of 4.11.

However, it is easy to see that (after passing to a sufficient subsequence) the set $F_1 = \cup_{l>1} E_{j_{k_l}}^{\frac{1}{l}}$ is of finite measure. Just take $j_{k_l}$ to be the first term in the sequence $j_k$ such that $\{g_{j_{k_l}} < \frac{1}{l}\}$ is finite. □

## Problem 4

Where in the proof of (notes 4.13) was the finiteness of $\Omega$ used? can this hypothesis be omitted?

*Proof.* Again, in the proof (eqn 4.4) we use the fact that the limit of a monotonic sequence of sets commutes with the measure, which is only true if the space is finite measure.

However, in this case, the hypothesis that $\Omega$ is finite cannot be dropped. Consider as a counterexample the sequence of functions

$$f_n = \chi_{[n,n+1]}$$

which converge pointwise to zero. However, removing any set of finite measure does not change the fact that this sequence does not converge uniformly. $\square$

## Problem 5

Construct a sequence of $L^1$ functions that converge $\mu$-a.e. to an $L^1$ function, but do not converge in $L^1$ norm. Show that such a sequence is not uniformly integrable, and describe the concentration phenomenon.

*Proof.* Consider the sequence

$$f_n = n\chi_{[0,\frac{1}{n}]}$$

which converges $\mu$-a.e. to the zero function. However, in $L^1$ norm, we have that

$$\begin{aligned}
\|f_n - 0\|_{L^1} &= \int_{\mathbb{R}} |f_n - 0| d\lambda^1 \\
&= \int_{\mathbb{R}} n\chi_{[0,\frac{1}{n}]} d\lambda^1 \\
&= \int_{[0,\frac{1}{n}]} n d\lambda^1 \\
&= 1
\end{aligned}$$

which does not go to zero as $n$ goes to infinity.

This sequence however is not uniformly integrable. To see this, let $\epsilon = \frac{1}{2}$, and let $\delta$ be arbitrary. Choose $n$ such that $\frac{1}{n} < \delta$. Then,

$$\int_{[0,\delta]} f_n d\mu \geq \int_{[0,\frac{1}{n}]} n\chi_{[0,\frac{1}{n}]} d\mu = 1$$

which is certainly greater than $\epsilon$.

This is because the sequence of functions has a "concentration" at zero, where more and more of the mass of the function is being compressed to. $\square$

## Problem 6

Prove the dominated convergence theorem for convergence in measure.

*Proof.* Recall that a sequence of real numbers $x_n$ converges to $x$ if each subsequence of $x_n$ has a sub-subsequence that converges to $x$.

So, let $f_n$ be a sequence of functions converging to $f$ in measure, and let $g$ be such that $|f_n| < g$ $\mu$-almost everywhere.

Now, let $f_{n_j}$ be an arbitrary subsequence. We know that $f_{n_j}$ converges in measure to $f$, so we can find a subsequence $f_{n_{j_k}}$ that converges pointwise to $f$. Applying the old dominated convergence theorem to this sequence yields

$$\lim \int f_{n_{j_k}} = \int f$$

Thus since every subsequence of the sequence $\int f_n$ has a sub-subsequence that converges to $f$, it follows that

$$\lim \int f_n = \int f$$

as desired. $\square$