Final Exam

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PROBLEM 1

Carefully prove a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is contractible.

Proof. For this proof, we will begin by proving a couple of useful lemmas.

Lemma 1. *The wedge sum is the coproduct in the category Top_* of pointed topological spaces.*

Proof. To show this is the coproduct, we need to show that it satisfies the universal property for coproducts. That is for $(X, x_0), (Y, y_0)$ pointed topological spaces, and (U, u_0) any other pointed topological space with arrows $f : (X, x_0) \to (U, u_0)$ and $g : (Y, y_0) \to (U, u_0)$,

$$\begin{tikzcd}
(X,x_0) \arrow[dr, "i_{x*}"] \arrow[ddr, bend right, "f"'] & & (Y,y_0) \arrow[dl, "i_{y*}"'] \arrow[ddl, bend left, "g"] \\
& (X,x_0)\vee(Y,y_0) \arrow[d, dashed, "\exists!(f,g)"] & \\
& (U,u_0) &
\end{tikzcd}$$

where (f, g) is the unique arrow that makes the diagram commute. Now, recall that the wedge sum is defined as the quotient

$$(X, x_0) \vee (Y, y_0) = X \coprod Y/x_0 \sim y_0$$

so we can bootstrap by using the fact that the disjoint union is the coproduct in Top. That is, we define i_{x*} and i_{y*} to be the unique maps given by the universal property

$$\begin{tikzcd}
X \arrow[dr, "i_x"] \arrow[ddr, dashed, bend right, "i_{x*}"'] & & Y \arrow[dl, "i_y"'] \arrow[ddl, dashed, bend left, "i_{y*}"] \\
& X \coprod Y \arrow[d, "q"] & \\
& X \vee Y &
\end{tikzcd}$$

Note that these maps preserve basepoints, since

$$i_{x*}(x_0) = q \circ i_x(x_0) = q(x_0)$$

and similarly for y_0.

Now, suppose we have maps $f : (X, x_0) \to (U, u_0)$ and $g : (Y, y_0) \to (U, u_0)$. Define (f, g) to be the unique map

$$(f, g) : X \coprod Y \to U$$

that makes the diagram

$$\begin{tikzcd}
X \arrow[dr, "i_x"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "i_y"'] \arrow[ddl, bend left, "g"] \\
& X \coprod Y \arrow[d, dashed, "{(f,g)}"] & \\
& U &
\end{tikzcd}$$

commute. Now, since $f(x_0) = g(y_0) = u_0$, it follows that (f, g) is constant on the fibers of $q : X \coprod Y \to X \vee Y$ and thus by the universal property of quotient maps factors through $X \vee Y$. That is,

$$\begin{tikzcd}
& X \coprod Y \arrow[dl, "{(f,g)}"'] \arrow[dr, "q"] & \\
U & & X \vee Y \arrow[ll, dashed, "{(f,g)_*}"]
\end{tikzcd}$$

commutes for a unique $(f, g)_*$. Thus, we have for f, g arrows from (X, x_0) and (Y, y_0) to (U, u_0) a unique arrow $(f, g)_*$ from $X \vee Y$ which makes

$$\begin{tikzcd}
(X, x_0) \arrow[dr, "i_{x*}"] \arrow[ddr, bend right, "f"'] & & (Y, y_0) \arrow[dl, "i_{y*}"'] \arrow[ddl, bend left, "g"] \\
& (X, x_0) \vee (Y, y_0) \arrow[d, dashed, "{(f,g)_*}"] & \\
& (U, u_0) &
\end{tikzcd}$$

commute as desired. □

Lemma 2. *The wedge product is stable with respect to (basepoint-preserving) homotopy. That is, if $X_1 \simeq X_2$ and $Y_1 \simeq Y_2$ relative to basepoints, then*

$$X_1 \vee Y_1 \simeq X_2 \vee Y_2$$

Proof. Since the wedge sum is the coproduct of pointed topological spaces, we construct the homotopy equivalence as follows:

Let $f_x : X_1 \to X_2$ and $g_x : X_2 \to X_1$ be homotopy equivalences of X_2 and X_1. That is, f_x and g_x are homotopy inverses of each other. Furthermore, let $f_y : Y_1 \to Y_2$ and $g_y : Y_2 \to Y_1$ be homotopy inverses as well. Then, by the universal property of coproducts, we have the commutative diagram

$$\begin{tikzcd}
 & X_2 \vee Y_2 \arrow[dddd, dashed, shift left, "\tilde{f}"] & \\
X_2 \arrow[ur] \arrow[d, shift left, "g_x"] & & Y_2 \arrow[ul] \arrow[d, shift left, "g_y"] \\
X_1 \arrow[u, shift left, "f_x"] \arrow[dr] & & Y_1 \arrow[u, shift left, "f_y"] \arrow[dl] \\
 & & \\
 & X_1 \vee Y_1 \arrow[uuuu, dashed, shift left, "\tilde{g}"] &
\end{tikzcd}$$

where $\tilde{f}$ is defined in terms of the universal property of coproducts with respect to the compositions $X_1 \xrightarrow{f_x} X_2 \longrightarrow X_2 \vee Y_2$ and $Y_1 \xrightarrow{f_y} Y_2 \longrightarrow {}_2 \vee Y_2$ (and similarly for $\tilde{g}$).

I assert that $\tilde{f}$ and $\tilde{g}$ are homotopy inverses. It should be clear that by symmetry of the problem, I need only check that $\tilde{f} \circ \tilde{g} \simeq \mathbb{1}$.

Suppose $x \in X_1 \vee Y_1$, and without loss of generality let x be in the inclusion of X_1 to $X_1 \vee Y_1$. Then, we have the diagram

$$\begin{tikzcd}
 & X_2 \vee Y_2 \arrow[ddd, dashed, shift left, "\tilde{f}"] \\
X_2 \arrow[ur, "i_2"] \arrow[d, shift left, "g_x"] & \\
X_1 \arrow[u, shift left, "f_x"] \arrow[dr, shift left, "i_1"] & \\
 & X_1 \vee Y_1 \arrow[uuu, dashed, shift left, "\tilde{g}"] \arrow[ul, dashed, shift left, "i_1^{-1}"]
\end{tikzcd}$$

(where i_1^{-1} is only defined on the image of i_1, but we are assuming that $x \in i_1(X_1)$ for this diagram chase, so this arrow exists). From here, it is clear that $\tilde{g}(x) = i_2 \circ f_x(i_1^{-1}(x))$, and if we identify X_1 and X_2 as subspaces of their wedge product, we have $\tilde{g}(x) = f_x(x)$. Similarly, we have $\tilde{f}(x) = g_x(x)$. Thus,

$$\tilde{f} \circ \tilde{g}(x) = g \circ f(x) \simeq \mathbb{1}(x)$$

as desired.

Thus, $\tilde{f}$ and $\tilde{g}$ are homotopy inverses, and $X_1 \vee Y_1 \simeq X_2 \vee Y_2$ as desired. □

Now, we prove the main result.

Let Z be a CW complex which satisfies the hypotheses. In particular, let X and Y be such that $Z = X \cup Y$, X and Y are both contractible, and their intersection $A = X \cap Y$ is contractible as well.

Since A is a contractible subcomplex of Z, we know (via Hatcher prop 0.16 and 0.17) that Z is homotopy equivalent to Z/A. Thus, we only need to show that Z/A is contractible.

Let $q : Z \to Z/A$ be the canonical quotient map. Since $Z = X \cup Y$, we know that $q(Z) = Z/A = q(X) \cup q(Y)$. In particular, $q(X) = X/A$ and $q(Y) = Y/A$ (this follows from the definition of the quotient). Thus, $Z/A = X/A \cup Y/A$. Since the quotient map is the identity on $Z \setminus A$ and collapses A to a point, it follows that $X/A \cap Y/A = A/A = \{a_0\}$ where a_0 is the point $q(A)$. Thus, Z/A is actually the wedge sum

$$Z/A = X/A \vee Y/A$$

Now, since A is contractible, we know that $X/A \simeq X$ and $Y/A \simeq Y$. In particular, since X and Y are contractible, so is X/A and Y/A. Thus, $X/A \simeq Y/A \simeq \{\cdot\}$. By lemma 2, this implies that

$$X/A \vee Y/A \simeq \{\cdot\} \vee \{\cdot\} = \{\cdot\}$$

and thus, $X/A \vee Y/A$ is contractible. Thus, $Z/A = X/A \vee Y/A$ is contractible as well, and Z itself is contractible as desired. $\square$