Problem Set 3

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February 13, 2018

Problem 1

Show that for $A \in \mathscr{B}(X, Y)$, $\|A\| = \|A^*\|$. Furthermore, if A is invertible, show that A^* is invertible with inverse $(A^*)^{-1} = (A^{-1})^*$.

Proof. We know that

$$\begin{aligned}\|A^*\| &= \sup_{\phi \in Y^*, \|\phi\|=1} \|A^*\phi\| \\ &= \sup_{\phi \in Y^*, \|\phi\|=1} \sup_{x \in X, \|x\|=1} \|A^*\phi(x)\|\end{aligned}$$

Now, by the definition of the norm, we have that

$$\phi(Ax) \leq \|\phi\|\|Ax\|$$

and so

$$\begin{aligned}\|A^*\| = \sup_{\phi \in Y^*, \|\phi\|=1} \sup_{x \in X, \|x\|=1} \|A^*\phi(x)\| \quad &\leq \sup_{\phi \in Y^*, \|\phi\|=1} \sup_{x \in X, \|x\|=1} \|\phi\|\|Ax\| \\ = \sup_{x \in X, \|x\|=1} \|Ax\| \\ = \|A\|\end{aligned}$$

and so $\|A^*\| \leq \|A\|$ as desired.

For the other inequality, we note that $\|A\| = \sup_{x \in X, \|x\|=1} \|Ax\|$, and since $\|Ax\| = \sup_{\phi \in Y^*, \|\phi\|=1} |\phi(Ax)|$ (proved in HW2 by explicit construction of ϕ that attains the norm), we have that

$$\|A\| = \sup_{x \in X, \|x\|=1} \sup_{\phi \in Y^*, \|\phi\|=1} |\phi(Ax)|$$

The definition of the norm $\|x\| = \|x\|_{X^{**}}$ shows that

$$x(A^*\phi) \leq \|x\|\|A^*\phi\|$$

Now, by a similar argument to the last inequality, we have

$$\begin{aligned}
\|A\| &= \sup_{x\in X, \|x\|=1} \sup_{\phi\in Y^*, \|\phi\|=1} |\phi(Ax)| \\
&= \sup_{x\in X, \|x\|=1} \sup_{\phi\in Y^*, \|\phi\|=1} |x(A^*\phi)| \\
&\leq \sup_{x\in X, \|x\|=1} \sup_{\phi\in Y^*, \|\phi\|=1} \|x\|\|A^*\phi\| \\
&= \sup_{\phi\in Y^*, \|\phi\|=1} \|A^*\phi\| \\
&= \|A^*\|
\end{aligned}$$

and so $\|A\| \leq \|A^*\|$. Combining both inequalities, we have that

$$\|A\| = \|A^*\|$$

as desired.

if A is invertible, it is easily shown that $(A^{-1})^*$ inverts A^*. That is, we wish to show that for all $\phi \in Y^*$, we have that

$$(A^{-1})^* A^* \phi = \phi$$

In particular, we wish to show that for all $y \in Y$,

$$(A^{-1})^* A^* \phi(y) = \phi(y)$$

This is clear, however, since

$$\begin{aligned}
(A^{-1})^* A^* \phi(y) &= A^* \phi(A^{-1} y) \\
&= \phi(AA^{-1} y) \\
&= \phi(y)
\end{aligned}$$

as desired. □

Problem 2

Prove the Fredholm Theorem.

Proof. Recall from the previous homework that for a subspace V of a Banach space X, we have that

$$\overline{V} = \cap_{\phi \text{s.t. } V \subset \ker \phi} \ker \phi$$

Letting $V = \text{im} A$, we see that

$$\overline{\text{im} A} = \cap_{\phi \text{s.t. } \text{im} A \subset \ker \phi} \ker \phi$$

Now, the right hand side is just the set of all $y \in Y$ for which $\phi(y) = 0$ for any ϕ such that $\phi(Ax) = 0$ for all x. That is, ϕ is such that $A^* \phi(x) = 0$ for all x, so $A^* \phi = 0$. That is, the right hand side is the set of all $y \in Y$ for which $\phi(y) = 0$ for all ϕ in $\ker A^*$, as desired. $\square$

Problem 3

Explain the difference between the weak-$*$ convergence of a sequence (ϕ_j) in X^* and the weak convergence of (ϕ_j) in $Y = X^*$. State the relations between the strong, weak, and weak-$*$ convergences on X^*.

Proof. If (ϕ_j) converges in weak-$*$ to ϕ, this means that for all $x \in X$, $\phi_j(x) \to \phi(x)$. That is, ϕ_j converges pointwise to ϕ. Specifically, the weak-$*$ topology is the weak topology with respect to $i_{can}X \subset X^{**}$.

On the other hand, if (ϕ_j) converges to ϕ weakly, then for all $x \in X^{**}$, $x(\phi_j) \to x(\phi)$. In particular, the weak topology is the weak topology with respect to X^{**}. That is to say, the weak topology utilizes the entirety of X^{**} to detect convergence, while the weak-$*$ convergence only uses $i_{can}(X) \subset X^{**}$.

It should be clear, however, that strong convergence implies weak convergence, which implies weak-$*$ convergence. To see this, let ϕ_j be such that $\|\phi_j - \phi\| \to 0$. Then, for any $x \in X^{**}$, since x is continuous with respect to the norm on X^*, we have that

$$x(\phi_j) \to x(0)$$

which is the condition for weak convergence. The fact that weak convergence implies weak-$*$ convergence is clear, since $i_{can}(X) \subset X^{**}$, and so if for all $x \in X^{**}$, $x(\phi_j) \to x(\phi)$, then clearly for all $x \in X$, $\phi_j(x) \to \phi(x)$. $\square$

Problem 4

Prove that the sequence of standard basis vectors $e_n \in \ell^p$, $1 < p < \infty$ converges weakly but not strongly.

Proof. We first show that (e_n) does not converge strongly. This is clear, since the sequence is not Cauchy, that is

$$\begin{aligned}\|e_n - e_m\|_p^p &= \sum_{i=1}^{\infty}(|(e_n - e_m)_i|)^p \\ &= 1^p + 1^p \neq 0\end{aligned}$$

Thus, it does not converge in norm.

Now, we show that (e_n) converges weakly. To see this, let $x \in \ell^{p*} = \ell^q$. Then, we need to show that $|x(e_n)| \to 0$. This is clear, however, since

$$|x(e_n)| = \sum_{i=1}^{\infty} |x_i(e_n)_i| = |x_n|$$

and since $x \in \ell^q$, $|x_i| \to 0$ (since $\sum_{i=1}^{\infty} |x_i|^q < \infty$), it follows that

$$|x(e_n)| \to 0$$

as desired. $\square$

Problem 5

Prove that $\frac{\epsilon}{\pi(x^2+\epsilon^2)} d\lambda^1(x) \to \delta_0$ in weak-$*$ as measures in $C([-1,1])^*$. Prove that $\frac{1}{2\epsilon}\chi_{-\epsilon,\epsilon}(x) d\lambda^1(x) \to \delta_0$ in weak-$*$ as measures in $C([-1,1])^*$.

Proof. First, we show that for all $f \in C([-1,1])$, $\int_{[-1,1]} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)} d\lambda^1(x)$ goes to $f(0)$.

To see this, we can estimate the integral away from zero as well as at zero by

$$\int_{[-1,1]} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)} d\lambda^1(x) = \int_{[-1,-\delta]\cup[\delta,1]} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx + \int_{-\delta}^{\delta} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx$$

Letting M be the bound on $|f(x)|$ (since f is on a compact set, this is defined), we can bound the first integral above and below by

$$\int_{[-1,-\delta]\cup[\delta,1]} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx \le M\frac{\epsilon}{\pi\delta^2}2(1-\delta)$$
$$\int_{[-1,-\delta]\cup[\delta,1]} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx \ge -M\frac{\epsilon}{\pi\delta^2}2(1-\delta)$$

which goes to zero as delta gets small, and can safely be ignored so long as δ shrinks slower than ϵ.

For the second integral, we use the fact that $|f(x)-f(0)| < \epsilon'$ for $|x| < \delta'$ to estimate

$$\begin{aligned}\int_{-\delta}^{\delta} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx &\le \int_{-\delta}^{\delta}(f(0)+\epsilon')\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx \\ &= 2(f(0)+\epsilon')\frac{1}{\pi}\tan^{-1}(\frac{\delta}{\epsilon}) \int_{-\delta}^{\delta} f(x)\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx \quad \ge \int_{-\delta}^{\delta}(f(0)-\epsilon')\frac{\epsilon}{\pi(x^2+\epsilon^2)}dx \\ &= 2(f(0)-\epsilon')\frac{1}{\pi}\tan^{-1}(\frac{\delta}{\epsilon})\end{aligned}$$

which are both equal to $(f(0)\pm\epsilon')$ so long as $\frac{\delta}{\epsilon}$ goes to infinity, or δ shrinks slower than ϵ.

So, fixing a sequence of δ for which the integral goes to $f(0)\pm\epsilon' \to 0$, and letting ϵ go to zero faster than δ, we see that the measure converges to the delta measure centered at zero, as desired.

Thus, since this holds for any $f \in C([-1,1])$, it follows (by Riesz representation for $C([-1,1])^*$ into Borel measures on $[-1,1]$) that the measure given converges in weak-$*$ to the delta measure.

For the second statement, we wish to show that for all $f \in C([-1,1])$,

$$\int_{[-1,1]} f(x)\frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}dx = f(0)$$

This is easily done.

$$\int_{[-1,1]} f(x)\frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}dx = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} f(x)dx$$

Now, we use continuity of f to get a sequence of δ_n for which $|x| < \delta_n$ implies $|f(x)-f(0)| < \frac{1}{n}$. Then, it follows that (by setting $\epsilon = \delta_n$)

$$\begin{aligned}\int_{[-1,1]} f(x)\frac{1}{2\delta_n}\chi_{[-\delta_n,\delta_n]}dx = \frac{1}{2\delta_n}\int_{-\delta_n}^{\delta_n} f(x)dx \quad &\le \frac{1}{2\delta_n}\int_{-\delta_n}^{\delta_n}(f(0)+\frac{1}{n})dx \\ &= f(0)+\frac{1}{n}\end{aligned}$$

and similarly,

$$\begin{aligned}\int_{[-1,1]} f(x)\frac{1}{2\delta_n}\chi_{[-\delta_n,\delta_n]}dx &\geq \frac{1}{2\delta_n}\int_{-\delta_n}^{\delta_n}(f(0)-\frac{1}{n})dx \\ &= f(0)-\frac{1}{n}\end{aligned}$$

which goes to $f(0)$ as n goes to infinity. $\square$

Problem 6

Prove that a finite dimensional vector space is reflexive. Find an expression for the matrix form of A^* given the matrix form of A.

Proof. To show that V is reflexive, we only need to show that the canonical injection i is surjective. However, since V and V^* have the same dimension, so does V^* and V^{**}, and so since i is injective into a space of the same dimension, i is surjective as well, as desired.

Now, we wish to find A^* as a matrix in the dual basis. We wish to show

$$\langle Ax, y\rangle = \langle x, A^\dagger y\rangle$$

for all $x \in V$ and $y \in W$ for $A : V \to W$ (anticipating that $A^* = A^\dagger$.)

To do so, let's express A in local coordinates.

$$\begin{aligned}
\langle Ax, y\rangle &= A_{ij} x^j \overline{y^i} \\
&= \overline{\overline{A_{ij}} y^i} x^j \\
&= x^j \overline{\overline{A^T_{ji}} y^i} \\
&= \langle x, \overline{A^T} y\rangle
\end{aligned}$$

and so $A^* = A^\dagger$, as desired. $\square$