Math 240B Notes
Differential Geometry Quarter 2

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1 Preliminaries

Homework 1. *Prove that $V^{**} \cong V$ for finite-dimensional vector space V.*

From this, it is clear that $T_p^*M \otimes T_pM \cong \mathrm{Hom}(T_pM, T_pM)$ for a manifold M.

Recall the tangent bundle TM is defined as

$$TM = \coprod_{p \in M} T_pM$$

and a vector field on the manifold M is simply a section of the tangent bundle projection $TM \xrightarrow{\pi} M$. In other words, a vector field is a function $f : M \to TM$ such that $\pi \circ f = id$. Requiring the section to be smooth makes it into a smooth vector field.

We can also do the same thing for the cotangent bundle T^*M to obtain a covector field.

Now, we can take the tensor product of copies of TM and T^*M to obtain our tensor bundles, and tensor fields will be sections of these bundles.

Let (U, ϕ) be a smooth chart on M with coordinate functions x^i, coordinate vector fields ∂_i, and coordinate one-forms dx^i. Recall that dx^i is defined to be the dual basis to ∂_i, that is,

$$dx^i(\partial_j) = \delta^i_j$$

Recall also that the exterior derivative of a function df is defined as

$$df(v) = v(f)$$

and this definition applied to the coordinate functions x^i (yielding dx^i) coincides with the definition above. Note that ∂_i form a basis for T_pM and dx^i form a basis for T_p^*M. Tensor products of them, then, form a basis for the tensor product space.

Homework 2. *Prove that, for a vector space V with basis v_i, dual basis v^i, the set*

$$\{v^i \otimes v^j \mid 1 \le i, j \le n\}$$

forms a basis for $V^* \otimes V^*$. Here $v^i \otimes v^j(u, v) = v^i(u)v^j(v)$.

2 Affine Connections

2.1 The Metric

Definition 2.1. *Let M^n be a smooth manifold of dimension n. A* Riemannian Metric *g on M is a rank $(0,2)$ tensor (a section of $T^*M \otimes T^*M$) that is symmetric and positive-definite. In other words, g is a rank $(0,2)$ tensor that restricts to an inner product on the tangent space at every point.*

We can express g in local coordinates!

$$g_{ij} = g(\partial_i, \partial_j)$$

or

$$g = g_{ij} dx^i \otimes dx^j$$

Homework 3. *Show that the two expressions for dvol, namely*

$$dvol = \wedge_i \omega^i$$
$$dvol = \sqrt{|g|} dx^n$$

2.2 Integration of Top Degree Differential Forms

Let M^n be an orientable n-dimensional manifold, and $\omega \in \Omega^n(M)$. Furthermore let (U, ϕ) be a positive coordinate chart. On U we have that

$$\omega = f dx^1 \wedge \ldots \wedge dx^n$$

for some $f \in C^\infty(M)$.

Now, let $K \subset U$ be compact. We define

$$\begin{aligned}
\int_K \omega &= \int_{\phi(K)} \phi^{-1*} \omega \\
&= \int_{\phi(K)} f \circ \phi^{-1} \phi^{-1*} dx^1 \wedge \ldots \wedge \phi^{-1*} dx^n \\
&= \int_{\phi(K)} f \circ \phi^{-1} dx^1 \wedge \ldots \wedge dx^n
\end{aligned}$$

where the last integral is just the standard integral in $\mathbb{R}^n$.

Is this definition independent of choice of coordinates? Let's check. Let (V, ψ) be another coordinate chart containing K. Then, the integral with respect to this coordinate system is

$$\int_K \omega = \int_{\psi(K)} g \circ \psi^{-1} dy^1 \wedge \ldots \wedge dy^n$$

for g defined as

$$\omega = h dy^1 \wedge \ldots \wedge dy^n$$

with coordinate functions y^i. The claim is that these integrals are equal.

Consider the change-of-coordinates map $\psi \circ \phi^{-1}$ from the x^i to the y^i coordinate system. Since K is in both U and V, its image $\phi(K)$ lies in the domain of $\psi \circ \phi^{-1}$.

All that remains is to apply the change of variables to the integrals. Recall that if one has a diffeomorphism $F : \Omega_1 \to \Omega_2$ for compact Ω_i, one has that

$$\int_{\Omega_2} f dy^1 \dots dy^n = \int_{\Omega_1} f \circ F |J_F| dx^1 \dots dx^n$$

where $|J_F|$ is the determinant of the Jacobian matrix for F.

Homework 4. *Check that the two integrals claimed to be equal are actually equal.*

Now we have an idea for how to integrate ω on a single chart, let's extend this. Let (η_i, U_i) be a partition of unity of M where each U_i is contained in a single chart on M. Then,

$$\omega = \sum \omega \eta_i$$

and we can integrate by extending linearly

$$\int_K \omega = \sum \int_K \omega \eta_i$$

where the right hand side has integrals over functions supported in a single chart, and is well-defined. But is this independent of the choice of partition of unity? Short answer: yes (Optional homework).

2.3 Integration on an Orientable Smooth Riemannian Manifold

Recall that a Riemannian manifold has a volume form

$$dvol = \sqrt{|g_{ij}|} dx^1 \wedge \dots \wedge dx^n$$

which is obtained by taking an orthonormal frame e_i and considering the dual frame ω^i defined as

$$\omega^i e_j = \delta^i_j$$

and letting

$$dvol = \omega^1 \wedge \dots \wedge \omega^n$$

This construction is independent of choice of orthonormal frame.

Proof. Let ϵ_i be another orthonormal frame with dual frame α^i. Then, $\epsilon_i = a_i^j e_j$ and $\alpha^i = b_j^i \omega^j$ and so

$$\begin{aligned}
\alpha^1 \wedge \ldots \wedge \alpha^n &= b_{j_1}^1 \omega^{j_1} \wedge \ldots \wedge b_{j_n}^n \omega^{j_n} \\
&= \sum_{\sigma \in S_n} b_{\sigma(1)}^1 \ldots b_{\sigma(n)}^n sgn(\sigma) \omega^1 \wedge \ldots \wedge \omega^n \\
&= |b| \omega^1 \wedge \ldots \wedge \omega^n \\
&= \omega^1 \wedge \ldots \wedge \omega^n
\end{aligned}$$

where the last line was obtained from the fact that b is the orthogonal change-of-basis matrix from e to ϵ. □

Then, we define

$$\mathrm{Vol}(K) = \int_K dvol$$

2.4 Integrating a Non-Orientable Manifold

How do we integrate a manifold that is not orientable? The previous construction was coordinate-independent only because we chose positive oriented coordinates...

Let $K \subset U$ be a compact set in a single chart on the manifold. Then, we can define

$$\mathrm{Vol}(K) = \int_K \sqrt{|g_{ij}|} dx^n$$

Now, this is independent of choice of coordinates, since if K lies in the intersection of two charts, we can use the Jacobian change-of-variables formula to show that the two calculations of the volume are equal.

The problem is that $dy^n = det(J_{x \to y}) dx^n$ depends also on the sign of the determinant of the Jacobian.

On an orientable Manifold, we have $dvol \in \Omega^n(M)$ (i.e. $dvol \in \Gamma(\Lambda^n T^* M)$), and in fact a manifold is orientable if and only if it admits a nowhere-vanishing top degree form.

Homework 5. *Prove that a manifold is orientable if and only if it admits a nowhere-vanishing top degree form.*

2.5 Existence of Metrics

Theorem 1. *On each smooth manifold M there exists smooth Riemannian metrics.*

Proof. Let (U_i, ϕ_i) be an atlas of M, and η_j be a partition of unity subordinate to it. Then, on each U_i we have a smooth Riemannian metric given by

$$g_i = dx_i^1 \otimes dx_i^1 + \ldots + dx_i^n \otimes dx_i^n$$

Then, we define

$$g = \sum \eta_i g_i$$

□

2.6 Lower-Dimensional Integration on Riemannian Manifolds

Suppose we want to find the arc length of a curve $\gamma : I \to M$. We can define the length of γ to be

$$L(\gamma) = \int_I |\gamma'| dt$$

where $|\gamma'|$ is the length of the tangent vector with respect to the metric.

Definition 2.2. *Let $p, q \in M$ be points in a connected manifold M. We define the distance between p and q to be*

$$\inf_{\gamma \in C^\infty(I,M)} \{L(\gamma) \mid \gamma(0) = p, \gamma(1) = q\}$$

Note that we can relax the condition that γ be smooth to γ being only piecewise smooth, since any piecewise smooth curve is uniformly approximated by smooth curves.

This distance, denoted $d(p, q)$, turns out to metrize the manifold.

Theorem 2. *$d(\cdot, \cdot)$ is a metric on M, and the metric topology generated by d coincides with the topology of M.*

Proof. First, we show that d is a metric. Symmetry of d should be obvious, since $L(\gamma) = L(-\gamma)$ and the curves from p to q directly coincide with curves from q to p via the map $\gamma \mapsto -\gamma$.

Now, d is also clearly positive-definite, since the length functional is positive-definite.

It should also be clear that $d(p, q) = 0$ if and only if $p = q$. Clearly, if $p = q$, then the constant curve $\gamma(t) = p$ has length zero, so $d(p, p) = 0$. Now, if $p \neq q$, then since M is Hausdorff, they must have positive distance from each other. This follows from the second claim that the topologies coincide.

The triangle inequality follows from the fact that given three points p, q, m, the curve going from p to m, and then from m to q, is a curve from p to q, and so $d(p, q) \leq d(p, m) + d(m, q)$ (since it is part of the infimum).

Now, we show that the topologies coincide... □

Homework 6. *Show that the topology on M coincides with the metric topology from d.*

Homework 7. *Show that for $(\mathbb{R}^n, g_{\mathbb{R}^n})$, $d(p, q) = \|p - q\|$.*

2.7 Connections on a Riemannian Manifold

Let (M^n, g) be a smooth Riemannian Manifold, $X \in \mathfrak{X}(M)$. We wish to take the derivative of this vector field. Recall that the Lie derivative allows us to take the derivative of X along another vector field Y, however this operation is not linear with respect to the module of smooth functions. That is,

$$L_X(fY) = fL_XY + (Xf)Y$$

Also, the Lie derivative is not defined for a single point, since it takes into account the motion of X around any particular point.

What we really want is ∇_v, the covariant derivative.

Definition 2.3. *A* Connection *is a map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \quad (X, Y) \mapsto \nabla_X Y$$

such that $\nabla_X Y$ is linear in both X with respect to the module $C^\infty(M)$, scalar linear in Y and satisfies the Leibniz rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

Definition 2.4. *A* Connection *is the following: for each $p \in M$, we have a map $\nabla : T_pM \times C^\infty(TM) \to T_pM$ that sends (v, Y) to $\nabla_v Y$. Such that ∇ is linear in v, linear in Y, and satisfies the Leibniz rule*

$$\nabla_v(fY) = (vf)Y_p + f(p)\nabla_v(Y)$$

and, for all X, Y in $\mathfrak{X}(M)$, $\nabla_X Y \in \mathfrak{X}(M)$ where

$$(\nabla_X Y)_p = \nabla_{X_p} Y$$

Interpreting ∇ as an operator from $\mathfrak{X}(M)$, we see that it actually adds a covariant index. That is,

$$\nabla_\mu v^\nu$$

takes in a vector, and outputs a $(1, 1)$ tensor.

Example. *The directional derivative in $\mathbb{R}^n$ yields a connection. For $v \in T_x\mathbb{R}^n$, and X a smooth vector field on $\mathbb{R}^n$, we have*

$$D_{(x,v)}X = \partial_t X(x + tv)|_{t=0}$$

and we define $\nabla_v X = (x, D_{(x,v)}X)$

Now, on TM for a general Riemannian manifold, there are many different connections. However, given a metric, we have a unique metric compatible, torsion-free connection called the *Levi-Civita Connection.*

Theorem 3. *For M a smooth Riemannian manifold, then there exists a unique connection ∇ on TM such that*

- ∇ *is symmetric i.e.*
$$\nabla_X Y - \nabla_Y X = [X, Y]$$
(The Christoffel symbols are symmetric in lower indices)

- ∇ *is metric-compatible. That is,*
$$\nabla g = 0$$
or
$$\nabla_\gamma g_{\mu\nu} v^\nu = g_{\mu\nu} \nabla_\gamma v^\nu$$
or
$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Proof. See Carroll (p.99) for an explicit construction of the torsion free, metric compatible connection in terms of the components of g. The formula is

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

□

Homework 8. *Prove that the resulting connection is indeed a connection.*

Now to prove that the two definitions of a connection coincide.

From the local to the global definition is trivial, so we wish to prove that we can localize the global definition.

Proof. Consider a smooth connection ∇ on M. Let $U \subset M$ be open, and Y a smooth vector field on M, X a smooth vector field on U.

Now, for $p \in U$, choose a smooth function η on M such that $\eta = 1$ in a neighborhood V_1 of p, and $\eta = 0$ on $M \setminus V_2$ with $\overline{V_1} \subset V_2$, $\overline{V_2} \subset U$ and $\overline{V_i}$ compact.

Homework 9. *Construct a one-dimensional smooth bump function on* $\mathbb{R}$

Now, set $\tilde{X} = \eta X$, which is defined globally on M. We can now define

$$\nabla_X Y|_{V_1} = \nabla_{\tilde{X}} Y|_{V_1}$$

and we can do this for every point $p \in M$. Now, we must show that such a construction is unique.

Suppose instead that we chose a different V_1', V_2', η'. We have a new globally-defined vector field $X' = \eta' X$, and we wish to show that $\nabla_{\tilde{X}} Y = \nabla_{X'} Y$ at p.

So, we construct

$$\nabla_{\tilde{X}}(Y) - \nabla_{X'}(Y) = \nabla_{\tilde{X} - X'} Y$$

Now, we know that $\tilde{X} - X'$ is zero at (and nearby) p, so

$$\tilde{X} - X' = \zeta(\tilde{X} - X')$$

So, we have that

$$\begin{aligned}\nabla_{\tilde{X}-X'} &= \nabla_{\zeta(\tilde{X}-X')} \\ &= \zeta\nabla_{\tilde{X}-X'} \\ &= 0\end{aligned}$$

and so they agree around p.

Next, consider $p \in M$, with Y a smooth vector field. Choose a coordinate chart (U, ϕ) around p, with $v \in T_pM$, $v = v^i\partial_i$.

Then, we set $\nabla_v Y = \nabla_{v^i\partial_i} Y = v^i\nabla_{\partial_i} Y$, where we have already defined what ∇_{∂_i} should be, since ∂_i is a locally defined vector field.

Now, we need to show this is independent of coordinate charts. Let (V, ψ) be another coordinate chart, with $v = v^j\partial'_j$ for coordinate field ∂'_i. The claim is that

$$v^i\nabla_{\partial_i} Y = v^j\nabla_{\partial'_j} Y$$

which is easily verified, since $J(\partial \to \partial')\nabla_{\partial_i} = \nabla_{\partial'_j}$, and so

$$v^j\nabla_{\partial'_j} = v^j\nabla_{J(\partial\to\partial')^j_i\partial_j}$$

but $v^i = J^i_j b^j$, and so they agree. □

2.8 The Levi-Cevita Connection

Recall that we have a unique torsion-free, metric compatible connection ∇ for any Riemannian manifold. We wish to localize this ∇ further.

Definition 2.5. *Let γ be a smooth curve in M. A vector field X* along *γ is an assignment $X : I \to TM$ with $X(t) \in T_{\gamma(t)}M$ where X is called smooth if its coordinate decomposition*

$$X = \xi^i(t)\partial_i$$

is smooth in each component.

Definition 2.6. *$\nabla_{\partial_t} X$ is define along γ as follows: Let I_{t_0} be an open interval around t_0, which maps into chart (U, ϕ). Then,*

$$\begin{aligned}\nabla_{\partial_t} X &= \nabla_{\partial_t}\xi^i(t)\partial_i \\ &= \partial_t\xi^i(t)\partial_i + \xi^i(t)\nabla_{\partial_t}\partial_i \\ &= \partial_t\xi^i(t)\partial_i + \xi^i(t)\nabla_{\partial_t\gamma}\partial_i\end{aligned}$$

which is already defined.

The second term in this expansion turns into

$$\xi^i(t)\nabla_{\partial_t\gamma}\partial_i = \xi^i\partial_t x^j\nabla_{\partial_j}\partial_i$$

and we define

$$\Gamma^k_{ij}\partial_k = \nabla_{\partial_j}\partial_i$$

Where Γ^k_{ij} is the Christoffel symbol (of the first kind) for the connection.

Homework 10. *Show that for the Levi-Civita connection,*

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l})$$

2.9 The Connection in Local Coordinates

Definition 2.7. *The connection forms of the connection ω^j_i associated with an orthonormal frame e_i is defined as*

$$\nabla e_i = \omega^j_i \otimes e_j$$

Knowing that the frame is orthonormal and the connection is metric compatible, we get

$$\begin{aligned}
\langle e_i, e_j \rangle &= \delta_{ij} \\
\langle \nabla_X e_i, e_j \rangle + \langle e_i, \nabla_X e_j \rangle &= 0 \\
\langle \omega^k_i(X) e_k, e_j \rangle + \langle e_i, \omega^l_j(X) e_l \rangle &= 0 \\
\omega^j_i(X) + \omega^i_j(X) &= 0
\end{aligned}$$

and so ω^i_j is antisymmetric.

Theorem 4. *The following holds for the connection forms:*

- *ω is antisymmetric*
- *$d\omega^i = \omega^i_j \wedge \omega^j$*

To prove this, we can use the identity

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$$

for one-forms α.

2.10 Parallel Transport

Let X be a vector field on M along γ.

Definition 2.8. *X is called parallel if*

$$\nabla_{\partial_t} X = 0$$

Theorem 5. *For each $v \in T_{\gamma(0)}M$, there is a unique solution to the initial value problem*

$$\begin{aligned}
\nabla_{\partial_t} X &= 0 \\
X(0) &= v
\end{aligned}$$

Proof. Let (U, ϕ) be a coordinate chart around $\gamma(0)$. Then, in U,

$$\nabla_{\partial_t} X = 0$$

is the same as

$$\partial_t \xi^k + \Gamma_{ij}^k \xi^i \partial_t x^j$$

which is a first order linear ODE with smooth coefficients, and so it has unique solutions for the initial value $X(0) = v$, or $\xi^i(0) = v^i$. $\square$

Now, since the ODE is linear, there is a linear map between initial values and solutions, that is we have a linear map from $T_{\gamma(0)}M$ to $T_{\gamma(1)}M$ by evaluating X at 1. This map is the parallel transport map, and it is invertible by running the curve backwards. Thus, this map is an isomorphism. Even better...

Proposition 1. *The parallel transport map is an isometry.*

Homework 11. *Prove that*

$$\partial_t g(X,Y) = g(\nabla_{\partial_t} X, Y) + g(X, \nabla_{\partial_t} Y)$$

Definition 2.9. *The* Holonomy Group *of a Riemannian manifold* (M,g) *based at a point* $p \in M$*, denoted* $H_{g,p}$ *is defined to be*

$$H_{g,p} = \{P_\gamma : \ \gamma \text{ a smooth loop at } p\}$$

where P_γ *is the parallel transport along* γ*, with the group structure of loop concatenation.*

Definition 2.10. *The* Reduced Holonomy Group *is the subgroup of* $H_{g,p}$ *consisting of parallel propogators whose loops are homotopic to the identity.*

3 Geodesics and Curvature

3.1 Geodesics

Definition 3.1. *Let* (M^n, g) *be a Riemannian manifold, and let* $\gamma : I \to M$ *a smooth curve.* γ *is called a* geodesic *if its second derivative vanishes. That is, if it solves the geodesic equation*

$$\nabla_{\partial_t} \partial_t \gamma = 0$$

Now, let's examine the geodesic equation further. In local coordinates, we have

$$\begin{aligned}
\nabla_{\partial_t} \partial_t \gamma &= \nabla_{\partial_t} \partial_t x^i \partial_i \\
&= \partial_t \partial_t x^k \partial_k + \partial_t x^k \nabla_{\partial_t} \partial_k \\
&= (\partial_t \partial_t x^k + \Gamma_{ij}^k \partial_t x^i \partial_t x^j) \partial_k
\end{aligned}$$

and so the local coordinate version of the differential equation is the system of equations

$$(\partial_t)^2 x^k + \gamma_{ij}^k \partial_t x^i \partial_t x^j = 0$$

which are guaranteed local unique solutions for initial conditions of γ and γ'.

Let's look at properties of geodesics. In particular, we can look at

$$\partial_t |\gamma'|^2 = \partial_t (g(\gamma', \gamma')) = 2g(\nabla_{\partial_t} \gamma', \gamma') = 0$$

and so the velocity of the geodesic does not change.

3.2 The Exponential Map

Let $p \in M$. We can define an exponential map $\exp : T_pM \to M$ via the following:

Definition 3.2. *The* exponential map $\exp : T_pM \to M$ *is defined as* $\exp(v) = \gamma(1)$ *where* γ *is a geodesic with* $\gamma(0) = p$ *and* $\gamma'(0) = v$.

Why do we insist that $\exp_p(v) = \gamma(1)$? Consider

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

where $t \in \mathbb{R}$. The last equality is obtained in the following way:

Lemma 1. $\gamma_{tv}(1) = \gamma_v(t)$ *for all* t.

Proof. Consider $\gamma(t) = \gamma_{sv}(t)$. This is the geodesic such that $\gamma(0) = p$ and $\gamma'(0) = sv$. Now, notice that $\tilde{\gamma}(t) = \gamma_v(st)$ is defined so that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}'(0) = \partial_t \gamma_v(st) = \gamma_v'(0)\partial_t(st)|_{t=0} = sv$ and by uniqueness of geodesics, $\gamma = \tilde{\gamma}$ as desired. $\square$

Let's examine the domain for the exponential map. With no assumptions on the structure of the manifold, what can we say about solutions to the geodesic equation?

Recall the escape lemma for flows along vector fields. If γ is a maximal integral curve of a vector field X whose domain J has a least upper bound b, then for each $t_0 < b$, $\gamma([t_0, b))$ is not contained in any compact subset of the manifold. That is, if γ goes into a compact subset of the manifold, it will not die in the interior of the compact subset.

We also have the uniform time lemma, which guarantees that for U open with compact closure, any $K > 0$, there is some $\epsilon > 0$ such that the geodesic $\gamma(t)$ with $\gamma(t_0) = p$ $\gamma'(t_0) = v$ exists for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ and the map

$$\gamma : U^* \times (t_0 - \epsilon, t_0 + \epsilon) \to M$$
$$\gamma(v, t) = \gamma(t)$$

and here $U^* = \{v \in TM, \|v\| < K, \pi(v) \in U\}$.

Now, we can see that $\exp_p$ is defined on a closed ball $\overline{B}_\epsilon(0) \subset T_pM$ for some $\epsilon > 0$, and furthermore for any compact set K, there is some $\epsilon > 0$ such that $\exp_p$ is defined on $\overline{B}_\epsilon(0)$ for all $p \in K$.

In $\mathbb{R}^n$, we have geodesics as linear affinely parameterized curves i.e. $\gamma(t) = \vec{a}t + \vec{b}$.

Example. *Consider the sphere* S^n *with the induced metric from* $\mathbb{R}^{n+1}$*. Then, the Levi-Civita connection is given as*

$$\nabla^{S^n}_{\partial_t} \gamma' = \left(\nabla^{\mathbb{R}^{n+1}}_{\partial_t} \gamma'\right)^T$$

Where T *is the tangential projection onto* S^n.

More generally, for N a submanifold of M, and X a vector field on N, we can extend X to a neighborhood in M, and take $\nabla^N_{\partial_t} X = (\nabla^M_{\partial_t} X)^T$.

3.3 Further properties of geodesics

In $\mathbb{R}^n$, it is clear that the straight line geodesic between two points is the path of shortest length between them. Is this the case in general? Do geodesics exist between any two points?

Obviously there are not geodesics between arbitrary points in a general connected Riemannian manifold (motivating example: Schwarzschield geometry). However, if a Riemannian manifold is complete, then it has geodesics between all points.

Does there always exist a geodesic of minimal length? And are such geodesics unique? No. On S^2, we have an infinite number of geodesics from the north to the south pole. Suppose instead, however, that we restrict to anything but the south pole. Then, there exist unique geodesics of minimal length from the north pole to any point. This has to do with the fact that the geodesics from the north pole do not cross until the south pole.

We can prove that *locally*, points are connected by a minimal geodesic, and that open balls around a point correspond to exponential projections of open balls in the tangent space.

Now, an important lemma:

Lemma 2. Gauss Lemma: *Let M be a manifold, and $p \in M$ with exponential $\exp_p$. We wish to understand $(d\exp_p)_v : T_pM \to T_{\exp_p(v)}M$. It is true that*

$$g(v, w) = g(d(\exp_p)_v(v), d(\exp_p)_v(w))$$

Proof. We begin by calculating $(d\exp_p)_v(V)$. Note that here, the v in the parentheses is actually in T_vT_pM, which is canonically identified with T_pM.

Specifically, we wish to show $\|(d\exp_p)_v(v)\| = \|v\|$, or that $g((d\exp_p)_v(v), (d\exp_p)_v(v)) = g(v, v)$.

To see this, consider the geodesic $c(t) = \exp_p(tv)$. Now, c is affinely parameterized, so it has constant speed (magnitude of tangent vector). Now, $c'(0) = v$, and $c'(1) = (d\exp_p)_v(\partial_t(tv)|_{t=1}) = (d\exp_p)_v(v)$ and since c is affinely parameterized, these two have the same magnitude.

Now, let $w \in T_pM$ such that w is perpendicular to v. We can choose a path $\tau(s) = v + sw$ such that $\tau(0) = v$ and $\tau'(0) = w$. Consider

$$F(t, s) = \exp_p(t(v + sw))$$

where, by varying s, we get a family of geodesics from the tangent vectors $v + sw$. Now, for $t \in [0, 1]$ (actually $(-\epsilon, 1 + \epsilon)$), $s \in (-\epsilon, \epsilon)$, we have a smooth map $F : [0, 1] \times (-\epsilon, \epsilon) \to M$.

Lemma 3. *For a smooth map $F : [a, b] \times [c, d] \to M$ with first coordinate t and second coordinate s,*

$$\nabla_{\partial_s}\partial_t F = \nabla_{\partial_t}\partial_s F$$

Homework 12. *Prove this lemma, using the fact that $[\partial_s, \partial_t] = 0$.*

Now, we have that

$$\partial_t F(t,0) = c'(t)$$

since $F(t,0) = \exp_p(tv) = c(t)$. We also have

$$\partial_s F(1,0) = (d\exp_p)_v(\partial_s(t(v+sw))|_{t=1,s=0}) = (d\exp_p)_v(w)$$

Now, we wish to show that

$$g((d\exp_p)_v(v), (d\exp_p)_v(w)) = 0$$

which is clear, since

$$\begin{aligned} g((d\exp_p)_v(v), (d\exp_p)_v(w)) &= g(\partial_t F(1,0), \partial_s F(1,0)) \\ &= g(\partial_t F, \partial_s F)|_{t=1,s=0} \end{aligned}$$

Now,

$$\begin{aligned} \partial_t g(\partial_t F, \partial_s F)|_{s=0} &= g(\nabla_{\partial_t}\partial_t F, \partial_s F) + g(\partial_t F, \nabla_{\partial_t}\partial_s F) \\ &= g(\partial_t F, \nabla_{\partial_t}\partial_s F) && \text{since } F \text{ is along a geodesic, second derivatives vanish} \\ &= g(\partial_t F, \nabla_{\partial_s}\partial_t F) \\ &= \frac{1}{2}\partial_s g(\partial_t F, \partial_t F) && \text{By symmetry of the metric} \end{aligned}$$

Now, suppose instead that we use a circular arc in T_pM between v and w so that $\|\partial_t F\|$ is independent of s. Then, it follows that $\partial_s g(\partial_t F, \partial_t F) = 0$ as desired.

Now, let's calculate $g(\partial_t F, \partial_s F)|_{t=0,s=0}$. We have

$$\begin{aligned} \partial_t F|_{t=0,s=0} &= c'(0) = v \\ \partial_s F|_{t=0,s=0} &= \partial_s \exp_p(t\tau(s))|_{t=0,s=0} = 0 \end{aligned}$$

and so $g((d\exp_p)_v(v), (d\exp_p)_v(w)) = 0$.

These two facts then prove the Gauss lemma by writing $u = \alpha v + \beta w$ for w perpendicular to v. $\square$

If we were to follow the proof through using the straight line in T_pM instead of a circular arc, we would need to use the following lemma

Lemma 4. $(d\exp_p)_0 = id$

Proof. Let $v \in T_pM$, and let $\gamma(t) = tv$ be a curve in T_pM with tangent vector v at zero. Then,

$$(d\exp_p)_0(v) = \partial_t \exp_p(tv) = v$$

as desired. $\square$

Proposition 2. *For U_p an open set in T_pM, $0 \in U_p$, the exponential map $\exp_p|_{U_p}$ is a diffeomorphism onto its image, and $\exp_p(U_p)$ is open in M.*

$B_r(0) \subset T_pM$ is called a normal ball if $\exp_p$ restricts to a diffeomorphism from $B_r(0)$ to its image.

Theorem 6. *Let $B_{r_0}(0)$ be a normal ball. Then, for each $v \in B_{r_0}(0)$, the radial geodesic $c(t) = \exp_p(tv)$ for $t \in [0,1]$ is the unique shortest smooth curve up to reparameterization from p to $\exp_p(v)$.*

A corollary of this is that $\exp_p(B_r(0)) = B_r(p)$.

Proof. Let $v \in B_{r_0}(0)$ as described in the hypothesis. Let $c(t) = \exp_p(v)$, with $c(0) = p$ and $c(1) = q$. Furthermore, let γ be any curve from p to q.

Suppose γ leaves $\exp_p(B_{\|v\|}(0))$ at some timet_1. That is, $\gamma([0,t_1)) \subset \exp(B_{\|v\|})$ and $\gamma(t_1)$ is in the boundary. Then, we know that

$$L_\gamma \geq L_{\gamma|_{[0,t_1)}}$$

so all we need to show is that

$$L_c \leq L_{\gamma|_{[0,t_1)}}$$

Now, this reduces to the second case. Namely, suppose γ is entirely contained in $\exp(B_{\|v\|}(0))$, and $\gamma(1) = q_1$ is on the boundary. Let $\tilde{\gamma}(t) = \exp_p|^{-1}_{B_{\|v\|}(0)} \circ \gamma(t)$ be the corresponding curve in T_pM. Now, all we have to do is calculate the length of γ.

$$L_\gamma = \int_I g(\gamma', \gamma')dt$$

Now, $\gamma'(t) = (d\exp_p)_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t))$ and if we assume $\tilde{\gamma}(t)$ is not zero, we can calculate the magnitude of $\gamma'(t)$. Let $\tilde{\gamma}'(t)$ be decomposed into a radial and normal part $r(t)$ and $n(t)$ with respect to the vector $\tilde{\gamma}$. Then,

$$\begin{aligned} g((d\exp_p)_{\tilde{\gamma}(t)}(r(t)+n(t)), (d\exp_p)_{\tilde{\gamma}(t)}(r+n)) &= \|r(t)\|^2 + \|d\exp_p(n(t))\|^2 \\ &\geq \|r(t)\|^2 \end{aligned}$$

Homework 13. *prove that equality is met in the previous inequality if and only if $\tilde{\gamma}(t)$ is radial.*

Then,

$$\begin{aligned} L_\gamma &= \int_{0+}^1 \|\gamma'\| dt \\ &\geq \int_{0+}^1 \|r(t)\| dt \end{aligned}$$

switching to polar coordinates, we denote $R(v) = \|v\|$, and we can calculate

$$\begin{aligned} \partial_t R(\tilde{\gamma}(t)) &= \nabla(R) \cdot \tilde{\gamma}'(t) \\ &= \frac{\tilde{\gamma}(t)}{\|\tilde{\gamma}(t)\|} \cdot \tilde{\gamma}'(t) \\ &= r(t) \end{aligned}$$

and so

$$\begin{aligned} L_\gamma &\geq \int_{0+}^{1} \|r(t)\| dt \\ &= \int_{0+}^{1} \partial_t R(\tilde{\gamma}(t)) dt \quad = L(c) \end{aligned}$$

as desired. $\square$

The corollary follows immediately.

Proof. Let $U_r = \exp_p(B_r(0))$. It should be clear that $U_r \subset B_r(p)$ since the radial geodesic $\exp_p(tv)$ for $v \in B_r(0), t \in [0,1]$ has length $\|v\| < r$. Since this geodesic is the minimal path from p to $\exp_p(v)$, it follows that $d(p, \exp_p(v)) < r$ as well.

Now, let $q \in B_r(p)$. That is, $d(p,q) < r$. Hence, we can find a smooth curve $c : I \to M$ with $L(c) < r$ and $c(0) = p, c(1) = q$. Let t_1 be such that $c([0,t_1)) \subset U_r$ (since c is continuous, and $c(0) \subset U_r$, and U_r is open). Thus, for all $t \in [0,t_1)$, we have $c(t) = \exp_p(v(t))$ for some $v(t) \in B_r(0)$. In particular, we can find an $r_1 < r$ such that $v(t) \in B_{r_1}(0)$.

Thus, we know that

$$L(c|_{[0,t_1)}) \leq L(c) \leq r_1$$

and so

$$L(\exp_p(sv(t))|_{s\in[0,1]}) = \|v(t)\| \leq L(c) < r_1$$

Now, consider the supremum of such t_1. We claim that $\sup t_1 = 1$, which implies that $L(c) \leq r_1 < r$ as desired. To see that $\sup t_1 = 1$, suppose instead that $\sup t_1 = T < 1$. This means that for all $t < T$, $c(t) = \exp_p(v(t)), t \in [0,T)$. However, taking a limit of such $v(t)$ yields some $V = v(T)$ for which $c(T) = \exp_p(V)$. However, this means that $c(T) \in U_r$ as well, and thus there is some $t' > T$ for which $c(t) = \exp_p(v(t)), t \in [0,t')$, which contradicts T being the supremum.

Thus, $c(t) = \exp_p(v(t)), t \in I$, and so $q \in U_r$ as desired. $\square$

Homework 14. *Find a counterexample to* $\exp_p(B_r(0)) = B_r(p)$ *for arbitrary* r.

(Note, for compact Riemannian manifolds, this is actually true! So S^n or T^n won't be a good counterexample...)

Corollary 1. *If a piecewise differentiable curve* $\gamma(t)$ *affinely parameterized minimizes the length between* $\gamma(0)$ *and* $\gamma(1)$ *(that is,* $L(\gamma) = d(\gamma(0), \gamma(1))$*), then* γ *is a smooth geodesic.*

Proof. (Easy proof) Apply variational calculus to the arc length formula to see that minimal paths satisfy the geodesic equation, and apply uniqueness. $\square$

4 Curvature

Let's just straight-up define the curvature:

Definition 4.1. *Consider a Riemannian manifold* (M, g)*, with smooth vector fields* $X, Y, Z \in \mathfrak{X}(M)$*. We define*

$$R_m(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

Alternately,

$$R^d_{abc}\omega_d = \nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c$$

(Wald, p. 37)

Now, we need to establish that this is a tensor by showing it is function linear in each component.

Observe that

$$\begin{aligned} R_m(X,Y)fZ &= -\nabla_X \nabla_Y fZ + \nabla_Y \nabla_X fZ + \nabla_{[X,Y]} fZ \\ &= -X(Yf)Z - (Yf)\nabla_X Z - (Xf)\nabla_Y Z - f\nabla_X \nabla_Y Z + Y(Xf)Z + (Xf)\nabla_Y Z + Yf\nabla_X Z + f\nabla_Y \\ &= -f\nabla_X \nabla_Y Z + f\nabla_Y \nabla_X Z + f\nabla_{[X,Y]} Z \end{aligned}$$

as desired

Homework 15. *Show this is function-linear in other components.*

Note you can lower the contravariant index by applying g_{ab} i.e.

$$R_{abcd} = g_{dd'} R^{d'}_{abc}$$

Calculating Curvature

We can calculate the Riemann curvature tensor in coordinates by using the definitions of the covariant derivative.

$$\mathbb{R}^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \sum_\alpha (\Gamma^\alpha_{ac}\Gamma^d_{\alpha b} - \Gamma^\alpha_{bc}\Gamma^d_{\alpha a})$$

To make things easier, we can use local Riemannian normal coordinates by pushing the coordinates from T_pM to M via the exponential map.

Homework 16. *Show that in Riemannian normal coordinates,*

$$\Gamma^k_{ij} = 0 \text{ at } p$$

and

$$\partial_k g_{ij} = 0 \text{ at } p$$

Definition 4.2. *an orthonormal frame* $\{e_i\}$ *on an open neighborhood of a point* $p \in M$ *is called normal around* p *if*

$$\nabla_a e_i = 0$$

at p*.*

The curvature follows the Bianchi Identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

In general, we have four important properties of the metric:

- $R^d_{abc} = R^d_{[ab]c}$ antiymmetry of the first two components
- $R^d_{[abc]} = 0$ the Bianchi identity
- $R_{abcd} = R_{ab[cd]}$ antiymmetry of the second two components
- $R_{abcd} = R_{cdab}$ symmetry in the first and second half components.

Note that item 4 can be derived from the other three.

An important concept not covered in Do Carmo:

Definition 4.3. *Given a finite dimensional vector space (over $\mathbb{R}$) V, consider the tensor C of rank $(0,4)$ (4 covariant indices). C is called an algebraic curvature tensor on V if it satisfies the above four properties (with appropriate index lowering).*

Sectional Curvature

Let $p \in M$ and let σ be a $2-$dimensional subspace of T_pM.

Definition 4.4. *The sectional curvature $K(\sigma)$ is defined to be*

$$K(\sigma) = R_m(e_1, e_2, e_1, e_2)$$

for $\{e_1, e_2\}$ an orthonormal basis for σ.

This definition is independent of choice of orthonormal basis by exploiting linearity of R_m.

This can also be expressed in an arbitrary basis u, v by

$$K(\sigma) = \frac{R_m(u, v, u, v)}{\|u \wedge v\|^2} \tag{4.1}$$

Where $\|u \wedge v\|^2$ is calculated from the inner product induced by the metric. That is, for $\{e_i\}$ an orthonormal basis for V, we declare $\{e_i \wedge e_j\}\, i < j$ to be orthonormal.

Homework 17. *Show that the induced inner product is independent of choice of orthonormal basis.*

Lemma 5. *Let V be a vector space (finite dimensional, real) of dimension at least 2 with an inner product. Consider two algebraic curvature tensors C_1 and C_2. Let K_1, K_2 denote the sectional curvatures of C_1 and C_2. $K_1 = K_2$ if and only if $C_1 = C_2$.*

Suppose C is such that $K(\sigma) = \kappa$ for all σ. Then,

$$C(x,y,z,w) = \kappa\left(g(x,z)g(y,w) - g(x,w)g(y,z)\right) \tag{4.2}$$

Ricci Curvature

Let R_m be a Riemannian curvature tensor, with components R^d_{abc}. We can take the trace over the first and third components to get

$$R_{ac} = Rabc^b \tag{4.3}$$

Geometrically, this is defined as

Definition 4.5. $R_{C_p}(u, w) = trace(R_{m_p}(u, \cdot)w)$.

In an orthonormal frame with $g(e_j, e_k) = \delta_{jk}$, we have

$$R_{ij} = R^k_{ikj} = R_{ikjk} \tag{4.4}$$

We can also define the Ricci scalar

Definition 4.6. *$R = R_c(u, u)$ for unit vector u.*

This can be given in coordinates as

$$R = R^i_i \tag{4.5}$$

Theorem 7. *The Ricci curvature tensor is symmetric*

Proof. We know that

$$R_{ac} = R^b_{abc}$$

But by symmetry of the Riemann curvature tensor, we have

$$\begin{aligned} R_{ac} &= R^b_{abc} \\ &= R^b_{cba} \\ &= R_{ca} \end{aligned}$$

as desired ☐

Now, let u be a unit vector, and build an orthonormal basis around u. Then,

$$R_c(u, u) = \sum R(e_1, e_i, e_1, e_i) = \sum K(e_1, e_i)$$

and

$$R = \sum R_c(e_i, e_i) = \sum K(e_i, e_j)$$

We also have the following identity for the Riemann curvature tensor R:

$$R(u \wedge v, w \wedge z) = R(u, v, w, z) \tag{4.6}$$

This relies on the antisymmetry of R, since R has to be linear.

Thus, interpreting R as a map from $\Lambda^2 T_p M \times \Lambda^2 T_p M \to \mathbb{R}$ we have that R is a symmetric bilinear map.

4.1 Riesz Representation and Tangent/Cotangent isomorphism

Given a metric, we have a natural isomorphism between T_pM and T_p^*M, denoted $\flat : T_pM \to T_p^*M$ and $\sharp : T_p^*M \to T_pM$ is given by

$$\flat(v) - v^\flat = g(v, \cdot) \tag{4.7}$$

This isomorphism extends also to exterior products of tangent spaces, allowing us to raise and lower indices at will.

4.2 Constant Curvature Spaces

Recall that if a space has constant sectional curvature κ, then

$$R(x, y, z, w) = \kappa(g(x, z)g(y, w) - g(x, w)g(y, z)) \tag{4.8}$$

Examples of such spaces are

1. Euclidean flat space $\mathbb{R}^n$: $\kappa = 0$.
2. Spherical space S^n with the pullback metric from $\mathbb{R}^{n+1}$: $\kappa > 0$.
3. Hyperbolic space with the metric $\frac{ds^2}{(x^n)^2}$: $\kappa < 0$.

Calculating the curvature for $\mathbb{R}^n$ is easy: we can always find an orthonormal frame that is parallel (covariant derivative is zero). Then, since R is defined in terms of the covariant derivatives, R must be zero.

Homework 18. *Prove that $R^d_{abc} = 0$ on the product manifold $S^1 \times S^1$ with the standard product metric.*

Now, let's calculate the curvature for the other two spaces.

Let M be our manifold, and let e_i be a local orthonormal frame on $U \subset M$ with dual ω^i. Then, we know that

$$d\omega^i = \omega^j \wedge \omega^i_j \tag{4.9}$$

with

$$\omega^i_j + \omega^j_i = 0 \tag{4.10}$$

Now, recall that

$$R(e_i, e_j) = \nabla_{e_i}\nabla_{e_j} + \nabla_{e_j}\nabla_{e_i} + \nabla_{[e_i,e_j]} \tag{4.11}$$

and $\nabla_{e_j} e_k = \omega^l_k(e_j)e_l$ for connection forms ω^l_k.

Thus,

$$\begin{aligned}\nabla_{e_i}\nabla_{e_j} e_k &= \nabla_{e_i}(\omega^l_k(e_j)e_l + \omega^l_k(e_j)\nabla_{e_j} e_l \\ &= e_i\omega^l_k(e_j)e_l + \omega^l_k(e_j)\omega^m_l(e_j)e_m\end{aligned}$$

Now, if the frame is normal, and we calculate at the center, $[e_i, e_j] = 0$ and so the last term vanishes.

So, we have

$$\begin{aligned} R(e_i, e_j)e_k &= e_i\omega_k^l(e_j)e_l + \omega_k^l(e_j)\omega_l^m(e_j)e_m + e_j\omega_k^l(e_i)e_l + \omega_k^l(e_i)\omega_l^m(e_i)e_m \\ &= d\omega_k^l(e_j, e_i)e_l + \omega_k^m \wedge \omega_m^l(e_i, e_j)e_l \\ &= (d\omega_k^l + \omega_k^m \wedge \omega_m^l)(e_i, e_j)e_l \end{aligned}$$

where the form in parentheses is the curvature form. Note that this differs from the normal convention by a negative sign, because the modern definition of the Riemann curvature tensor is $R_{abc}^d\omega_d = (-\nabla_a\nabla_b\omega_c + \nabla_b\nabla_a\omega_c)$ which is the negative of the definition found in Wald.

By convention, we define the curvature 2-form Ω to be

$$\Omega_i^j = d\omega_i^j + \omega_i^k \wedge \omega_k^j \tag{4.12}$$

These, however, are frame-dependent! We can define a global curvature form Ω on the principal bundle over the manifold with structure group $O(n)$. Then, $\Omega_x \in \Lambda_x^{2*}M \otimes o(n)$ is a 2-form with values in $o(n)$. (not important for this class)

Recall our goal to calculate the curvature of hyperbolic space. We know now that

$$R(X,Y)e_i = \Omega_i^j(X,Y)e_j \tag{4.13}$$

and the hyperbolic metric is

$$\frac{ds^2}{(x^n)^2} \tag{4.14}$$

Let's find the connection 1-forms using the orthonormal coframe $\omega^i = (\frac{dx^i}{x^n})^2$

$$\begin{aligned} d\omega^i &= -\frac{1}{y^2}dy \wedge dx^i \\ &= -\omega^n \wedge \omega^i \end{aligned}$$

with $y = x^n$. The equating these with the structure equations

$$d\omega^i = \omega^j \wedge \omega_j^i \tag{4.15}$$

and

$$\omega_j^i + \omega_i^j = 0 \tag{4.16}$$

to get

$$\omega_i^n = \omega^i$$

with the other terms (not derived from antisymmetry) are zero.

Now, we have

$$\begin{aligned} \tilde{\Omega}_j^i &= d\omega_j^i + \omega_k^i\omega_j^k \\ \tilde{\Omega}_j^i &= 0 + \omega^i \wedge \omega^j \qquad i, j < n \\ \tilde{\Omega}_n^i &= d\omega_n^i + \omega_k^i\omega_n^k \\ &= d\omega_n^i = -d\omega^i \\ &= -\omega^i \wedge \omega^n \end{aligned}$$

So, generally, $\tilde{\Omega}^i_j = -\omega^i \wedge \omega^j$.

Now, let's calculate the whole curvature tensor. Let $Z = \xi^i e_i$. Then,

$$\begin{aligned}
R(X,Y)Z &= -\tilde{\Omega}^j_i(X,Y)e_j \\
&= -\xi^i \omega^i \wedge \omega^j(X,Y)e_j \\
&= -Z^\flat \wedge (\omega^j e_j)(X,Y) \\
&= -Z^\flat \wedge \mathrm{Id}(X,Y) \\
&= -Z^\flat(X)\mathrm{Id}(Y) + Z^\flat(Y)\mathrm{Id}(X) \quad = -g(X,Z)Y + g(Y,Z)X
\end{aligned}$$

Recall from earlier that

$$R(X,Y,Z,W) = \kappa(g(X,Z)g(Y,W) - g(X,W)g(Y,Z))$$

or

$$R(X,Y)Z = \kappa(g(X,Z)Y - g(Y,Z)X)$$

5 Isometric Immersions

5.1 Gauss Curvature Equation

Theorem 8. *Let $u, v \in T_pM$ with $p \in M$, $\|u\| = \|v\| = 1$ and $u \cdot v = 0$. Then*

$$K(u,v) = \bar{K}(u,v) + B(u,u) \cdot B(v,v) - \|B(u,v)\|^2 \tag{5.1}$$

Where $\bar{K}$ is the sectional curvature for the ambient space, and B is defined as

$$B(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y \tag{5.2}$$

The proof of this is found in Do Carmo...

Example. *Let's calculate the curvature of $S^n \subset \mathbb{R}^{n+1}$.*

Let u, v be orthogonal vectors on S^n. Now, since we are in the ambient space $\mathbb{R}^{n+1}$, $\bar{K} = 0$ everywhere. So, let's calculate the second fundamental form of the inclusion map $i : S^n \to \mathbb{R}^{n+1}$.

$$B(u,v) = (\bar{\nabla}_{\bar{X}}\bar{Y})^N$$

for $\bar{X}, \bar{Y}$ extensions of u and v into the ambient space.

So,

$$\begin{aligned}
B(u,v) &= (\bar{\nabla}_{\bar{X}}\bar{Y} \cdot \nu)\nu \qquad \nu \textit{ is unit normal away from } S^n. \\
&= (-\bar{Y} \cdot \bar{\nabla}_{\bar{X}}\nu)\nu \\
&= (-v \cdot \bar{\nabla}_u v)\nu \\
&= (-v \cdot u)\nu
\end{aligned}$$

and so using the Gauss curvature equation, we find that

$$\begin{aligned} K(u,v) &= (-u \cdot u)\nu \cdot (-v \cdot v)\nu - \|(-v \cdot u)\|^2 \\ &= (-1)(-1) - 0 = 1 \end{aligned}$$

as desired.

We can also define the mean curvature vector as $H = \frac{1}{2}\mathrm{tr}(B)$ which is just

$$H = \frac{1}{2}(\sum_i B(E_i, E_i))$$

For a 2-dimensional subspace of $\mathbb{R}^3$, the Gauss curvature and the sectional curvature are the same.

Of course, this theorem generalizes.

Definition 5.1. *An isometric immersion $f : M \to \bar{M}$ is called totally geodesic if the second fundamental form B vanishes everywhere. If $B = 0$ at a point p, then we say M is geodesic at p.*

Theorem 9. *$f : M \to \bar{M}$ (think of an embedded submanifold) is totally geodesic if and only if all geodesics of M are also geodesics of $\bar{M}$.*

Proof. ($\Longrightarrow$) Suppose M is totally geodesic. Then,

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \nabla_X Y + B(X, Y)$$

and so the connections agree, and geodesics in M are automatically geodesics in $\bar{M}$ ☐

Homework 19. *prove the reverse implication.*

As a consequence, if a submanifold is totally geodesic, then the sectional curvature of the submanifold is the same as the sectional curvature of the submanifold with respect to the ambient space.

Note that if you take a slice of a Riemannian normal coordinate frame at a point, the submanifold is geodesic at that point.

6 Complete Manifolds

Recall we have a distance on a manifold as

$$d(p, q) = \inf\{L(\gamma) \mid \gamma : I \to M, \gamma(0) = p, \gamma(1) = q\}$$

which metrizes the topology on M. Recall also that the Gauss lemma guarantees that for each $p \in M$, there is some $r > 0$ for which $B_r(p)$ is a normal ball (is the diffeomorphic image under exp of some ball in T_pM). We note also from before that inside a normal ball, the shortest path from p to q is achieved by the unique radial geodesic from p to q.

Now we get to the new stuff:

Theorem 10. *Let (M^n, g) be a connected Riemannian manifold, and $p \in M$. The following are equivalent:*

- $\exp_p$ *is defined on all of T_pM.*
- *The closed and bounded sets of M are compact.*
- *M is complete as a metric space.*
- *M is geodesically complete. That is, every geodesic of M can be extended for all time. Alternately,* $\exp_q$ *is defined on all of T_qM for every $q \in M$.*
- *There exists a sequence of compact subsets K_n of M such that $\{K_n\}$ is increasing,* $\lim K_n = M$*, and if $q_n \in M \setminus K_n$, then $d(p, q_n) \to \infty$.*

Additionally, any of these statements imply the following: For any $q \in M$, there is a geodesic from p to q such that $L(\gamma) = d(p, q)$, or the geodesic minimizes distance. This is equivalent to $B_r(p) = \exp_p(B_r(0))$ for any $r > 0$.

Proof. Equivalence of the first five is easy. So, lets prove the first one implies the last corollary. Suppose p is such that $\exp_p$ is defined on all of T_pM.

Take $\delta > 0$ such that $B_\delta(p)$ is a normal ball. Choose $x_0 \in \partial B_\delta(p)$ such that $d(x_0, q) = d(q, \partial B_\delta(p))$ (doable since $\partial B_\delta(p)$ is compact). (we assume q is not in the normal ball, since if it were the proof would be trivial).

Now, we have $x_0 = \exp_p(\delta v)$ for some $\|v\| = 1$. Set $\gamma(t)$ to be the geodesic $\gamma(t) = \exp_p(t\delta v)$. Let $r = d(p, q)$. Then, $\gamma(r) = q$ (need to prove) and γ minimizes this length.

We can prove this by showing

$$d(p, q) = d(p, \gamma(t)) + d(\gamma(t), q)$$

which for $t = r$ guarantees

$$d(p, q) = d(p, \gamma(r)) + d(\gamma(r), q) = r + 0$$

To that end, let $I = \{t \in [\delta, r] \mid d(p, q) = d(p, \gamma(t)) + d(\gamma(t), r)\}$. We claim first that this is nonempty. This is clear, since $\delta \in I$. This follows from the fact that $\gamma(\delta) = x_0$ and $d(x_0, q) = r - \delta$, so $d(p, q) = \delta + r - \delta = d(p, \gamma(\delta)) + d(\gamma(\delta), q)$.

Furthermore, we prove that for any $t \in I$, $t < r$, there is some ε for which $t + \varepsilon \in I$.

Suppose $t < r$ is in I. Take a normal ball or radius ε around $\gamma(t)$. Then, let $y_0 \in \partial B_\varepsilon(\gamma(t))$ and such that $d(\partial B_\varepsilon(\gamma(t))) = d(q, y_0)$. We want to show that $y_0 = \gamma(t + \varepsilon)$.

To see this, note that

$$d(\gamma(t), q) = r - t$$

and note that

$$L(\gamma|_{[0,t]}) = t$$

which clearly implies that γ minimizes the distance between p and $\gamma(t)$. This follows from

$$\begin{aligned} d(p,q) &= L(\gamma|_{[0,t]}) + d(\gamma(t),q) \\ &\geq d(p,\gamma(t)) + d(\gamma(t),q) \\ &\geq d(p,q) \end{aligned}$$

and so $L(\gamma) = d(p,\gamma(t))$.

Now, we know that $y_0 = \gamma_1(\varepsilon) = \exp_{\gamma(t)}(\varepsilon u)$ for some u. Repeating the argument from before by setting $x_0 = y_0$, $p = \gamma(t)$, and so forth. Thus, $d(\gamma(t),q) = \varepsilon + d(y_0,q)$ and so

$$\begin{aligned} d(p,q) &= d(p,\gamma(t)) + \varepsilon + d(y_0,q) \\ &= d(p,\gamma(t)) + L(\gamma_1|_{[0,\varepsilon]}) + d(y_0,q) \\ &= L(\gamma|_{[0,t]}) + L(\gamma_1|_{[0,\varepsilon]}) + d(y_0,q) \end{aligned}$$

and so $d(p,y_0) = L(\gamma|_{[0,t]}) + L(\gamma_1|_{[0,\varepsilon]})$. This implies that $\gamma|_{[0,t]} \cdot \gamma_1|_{[0,\varepsilon]}$ is a geodesic minimizing distance between p and y_0. This shows that γ joined to γ_1 at $\gamma(t)$ is smooth, and so $\gamma_1 = \gamma$, and $y_0 = \gamma(t+\varepsilon)$ as desired.

This completes the proof. To see this, note that the above implies that I contains r, and so

$$d(p,q) = d(p,\gamma(r)) + d(\gamma(r),q) = r + 0$$

and so $d(\gamma(r),q) = 0$ as desired. □

The equivalences of the statements are proved below

Proof. ($a \implies b$)

Suppose M is such that $\exp_p$ is defined on all T_pM. let $A \subset M$ be closed and bounded. Then $A \subset B_r(p)$ for some $r > 0$ (definition of boundedness). Then, by the corollary above, we have that $A \subset \exp_p(B_r(0))$. Now, $B_r(0)$ is compact, and so its image $B_r(p)$ is compact. Since A is closed and a subset of a compact set, it is compact as well.

($b \implies c$) Suppose M is such that the closed and bounded sets are the compact sets. Then, M is complete by Heine-Borel. Explicitly, let p_k be a Cauchy sequence. This sequence is bounded, so its closure is compact. Therefore, some subsequence of p_k converges. Thus, since p_k is Cauchy, it converges as well.

($c \implies d$) Let γ be a maximally extended geodesic. $\gamma : (a,b) \to M$. Assume for a contradiction that b is finite. Then, consider a sequence $t_k \to b$, and we claim that $\gamma(t_k)$ is Cauchy. This is clear, since

$$d(\gamma(t_k),\gamma(t_m)) \leq \|t_k - t_m\|$$

as desired. So, $\gamma(t_k)$ is Cauchy, and has a limit $\gamma(t_k) \to p$ by completeness of M. Now, consider a normal ball of some radius δ around $\gamma(t_k)$, enough so that

δ works for all t_k. We can go far enough in the sequence such that $\gamma(t_{k+1})$ is in the normal ball around $\gamma(t_k)$. Recall that the radial geodesic from $\gamma(t_k)$ to $\gamma(t_{k+1})$ is unique, and so γ must be the radial geodesic from $\gamma(t_k)$ to $\gamma(t_{k+1})$. Furthermore, γ can be extended across the entire normal ball. Taking k large enough so that p is in the normal ball, we see that γ can be extended across p, a contradiction.

Trivially, $d \implies a$. Thus we have established equivalence of the first four.

($b \equiv e$) Suppose M satisfies the Heine Borel property. Then, take the distance balls $K_n = \overline{B}_n(p)$ which are bounded and closed, and therefore compact. Clearly, these are also increasing, and clearly satisfy the requirements for e.

Suppose instead that M is written as the union of compact sets defined in e. Then, let A be a bounded and closed set. In particular, A is contained in some K_n by boundedness, and since A is closed, it is compact as a closed subspace of a compact space. □

This implies the following

Theorem 11. *let M be a complete simply connected Riemannian manifold with sectional curvature $\kappa \leq 0$. Then, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for any $p \in M$.*

Proof. We have proved before that the exponential map is a local diffeomorphism since $\kappa \leq 0$. In particular, we can pull back the metric along $\exp_p$ to get a metric $\tilde{g}$ on $T_pM = \tilde{M}$. Then, $\exp_p : \tilde{M} \to M$ is a local isometry, So, radial lines in $\tilde{M}$ are geodesics. Consider $\exp_0 : T_0\tilde{M} \to \tilde{M}$, which is defined on the whole tangent space. Thus, $\tilde{M}$ is complete.

Now, all we need to show is that $\exp_p$ is a covering map. We know that $\exp_p$ is a local homeomorphism, so all we need to show is that $\exp_p$ is a covering map. This follows from $\tilde{M}$ being complete. Let $f = \exp_p$, which we know is a local isometry. We clain that f is a covering map.

Let $p \in M$. We need to find a neighborhood of p for which $f^{-1}(U)$ is a disjoint union of things homeomorphic to (diffeomorphic to) U along f. So, let U be the normal ball of radius δ around p. Let $q \in f^{-1}(p)$. Consider $f|_{B_\delta}(q)$ which by completeness (and isometry of f) maps along f into $B_\delta(p)$. To show f is a diffeomorphism in these neighborhoods, we only need to show f is one-to-one.

Suppose for contradiction that $f(q_1) = f(q_2)$ for $q_i \in B_\delta(q)$. It should be clear that this violates the uniqueness of radial geodesics from p to $f(q_i)$. Thus, f is one-to-one, and in fact is a diffeomorphism of $B_\delta(q)$ into $B_\delta(p)$.

Finally, we show the preimages of U are disjoint. So, suppose q_1, q_2 are in $f^{-1}(p)$. Suppose $B_\delta(q_1) \cap B_\delta(q_2)$ is nonempty. Then, for q in the intersection, we have two geodesics γ_1 from q_1 to q and γ_2 from q_2 to q. Thus, they project into $B_\delta(p)$ to two curves starting at p ending at $f(q)$. Thus, they must agree. That is, $f(\gamma_1) = f(\gamma_2)$. This means that γ_1 and γ_2 are the same geodesic, and so $q_1 = q_2$ and the inverse images are disjoint.

Thus, by uniqueness of covering maps, and the fact that M covers itself as a simply connected covering space, $\tilde{M}$ is diffeomorphic to M. $\square$