Problem Set 1

Daniel Halmrast

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Problem 1

Let $\mathscr{F}$ be the set of all measurable functions which are finite μ-a.e. on Ω.

Part a

Prove $\mathscr{F}$ is a vector space.

Proof. We first note that $\mathscr{F}$ is a subset of the vector space of all measurable functions (modulo functions zero μ-a.e.). We just need to show, then, that $\mathscr{F}$ is closed under addition and scalar multiplication.

So, let f and g be measurable functions that are finite μ-a.e. on Ω. Now, let's consider $f+g$. If $x \in \Omega$ is such that $f(x)$ and $g(x)$ are finite, then the sum $f(x)+g(x)$ is finite. Let E be the set of all such x. We will show that $\Omega \setminus E$ has measure zero, so that $f+g$ is finite μ-a.e.

To see that $\Omega \setminus E$ has measure zero, we note that $\Omega \setminus E = \{|f| = \infty\} \cup \{|g| = \infty\}$. Now, since f and g are finite μ-a.e., we know that each of these sets has measure zero, and so the union $\Omega \setminus E$ has measure zero as well. Thus, $f+g$ is finite μ-a.e.

It is immediately clear as well that for arbitrary scalar α, we have that αf is also finite μ-a.e., since for $x \in \{|f| < \infty\}$, we have that $|f(x)| < \infty$, which implies that $|\alpha f(x)| < \infty$ as well.

Thus, $\mathscr{F}$ is an algebraically closed subspace of a vector space, and is a vector space itself. □

Part b

Prove that $\mathscr{F}$ is a metric space with the metric

$$d(f,g) = \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu$$

Proof. To show that d is a metric, we need to show that $d(f,f) = 0$, $d(f,g) > 0$ for $f \neq g$, and the triangle inequality $d(f,h) \leq d(f,g) + d(g,h)$.

It is clear that $d(f,f)=0$, since this amounts to

$$\begin{aligned} d(f,f) &= \int_\Omega \frac{|f-f|}{1+|F-f|}d\mu \\ &= \int_\Omega \frac{0}{1}d\mu \\ &= 0 \end{aligned}$$

Now, suppose f and g differ on a positive-measure set E. In other words, $|f-g|\neq 0$ on E. Now, since $|f-g|$ is positive on E, $\frac{|f-g|}{1+|f-g|}$ is as well. Thus, $\frac{|f-g|}{1+|f-g|}\neq 0$ in L^1, and so $\|\frac{|f-g|}{1+|f-g|}-0\|_{L^1}=\int_\Omega \frac{|f-g|}{1+|f-g|}d\mu>0$ as desired.

Finally, we wish to prove the triangle inequality. This will follow from the convexity of the function $\frac{x}{1+x}$. That is, for $f,g,h\in\mathscr{F}$, we have that ☐

Part c

Show that d metrizes the convergence in measure.

Proof. Suppose first that $f_n\to f$ in μ. That is, for all $t>0$,

$$\mu(\{|f_n-f|>t\})\to 0$$

Now, consider the integral

$$\begin{aligned} \int_\Omega \frac{|f_n-f|}{1+|f_n-f|}d\mu &= \int_{\{|f_n-f|\leq t\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu + \int_{\{|f_n-f|>t\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu \\ &\leq \int_{\{|f_n-f|\leq t\}} \frac{t}{1+t}d\mu + \int_{\{|f_n-f|>t\}} 1d\mu \end{aligned}$$

Now, the first term goes to zero as t goes to zero, and the second term is just $\mu(\{|f_n-f|>t\})$, which goes to zero as n goes to infinity. Thus, the expression goes to zero, and $d(f_n,f)$ goes to zero, as desired.

Now, suppose $d(f_n,f)$ goes to zero. We wish to prove that for all $t>0$, $\mu(\{|f_n-f|>t\})\to 0$. To do so, we consider

$$\begin{aligned} d(f_n,f) &= \int_\Omega \frac{|f_n-f|}{1+|f_n-f|}d\mu \\ &= \int_{\{|f_n-f|>t\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu + \int_{\{|f_n-f|\leq t\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu \\ &\geq \int_{\{|f_n-f|>t\}} \frac{|f_n-f|}{1+|f_n-f|}d\mu \\ &\geq \int_{\{|f_n-f|>t\}} \frac{t}{1+t}d\mu \qquad = \frac{t}{1+t}\mu(\{|f_n-f|>t\}) \end{aligned}$$

and since $d(f_n,f)$ goes to zero, so does $\frac{t}{1+t}\mu(\{|f_n-f|>t\})$, which implies that for fixed $t>0$,

$$\mu(\{|f_n-f|>t\})\to 0$$

as well. ☐