Problem Set 2

Daniel Halmrast

October 18, 2017

PROBLEM 1

Prove that there is an embedding of X into $X \times Y$.

Proof. For this proof, $\{\bullet\}$ will represent the one-point set.

To start with, we will prove the following lemma:

Lemma. For X any topological space, $X \cong X \times \{\bullet\}$.

Proof. By the definition of the product space, the projection maps

$$X \times \{\bullet\}$$

$$x \qquad \qquad \pi_x \qquad \pi_{\bullet}$$

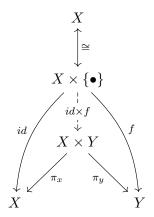
$$X \qquad \qquad \{\bullet\}$$

exist and are continuous open maps. Now, all we need to show is that π_x is injective, and it will follow immediately that it is a homeomorphism.

To see this, let $x \in X$ and consider $\pi_x^{-1}(\{x\}) = \{(x, \bullet)\}$. Since the inverse image of a singleton is again a singleton, the function is injective.

Thus, X is homeomorphic to $X \times \{\bullet\}$.

Now, let $f: \{\bullet\} \to Y$ be a continuous function. Consider the diagram:

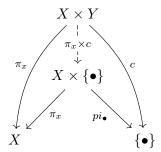


where id and f are the obvious extensions $id(x, \bullet) = id(x) = x$ and $f(x, \bullet) = f(\bullet)$. Here, the product map $id \times f$ is continuous by the universal property of products. Now, we just need to show that $id \times f$ is injective with a continuous inverse on its image.

To see that $id \times f$ is injective, consider a point $(id(x), f(\bullet))$ in the image of $id \times f$, and consider its preimage:

$$(id \times f)^{-1}(\{(id(x), f(\bullet))\}) = \{(x, \bullet)\}$$

Since the preimage of any singleton is again a singleton, the function $id \times f$ is injective. Now, lets consider the diagram



where c is unique constant function from Y to the terminal object $\{\bullet\}$.

Here, the dashed arrow $\pi_x \times c$ is continuous by the universal property of products. It is easy to see that $\pi_x \times c|_{(id \times f)(X \times \{\bullet\})}$ is the inverse of $id \times f$ on the image of $id \times f$.

Hence, since the inverse of $id \times f$ is continuous, $id \times f$ is an embedding of $X \cong X \times \{\bullet\}$ into $X \times Y$.

PROBLEM 2

Prove that every open interval in \mathbb{R} is homeomorphic to \mathbb{R} .

Proof. Consider an open interval $(a,b) \subset \mathbb{R}$. It is easy to see that $(a,b) \cong (-1,1)$, since the operations of scaling and translation are continuous functions with continuous inverses.

Thus, all we need to prove is that $(-1,1) \cong \mathbb{R}$. To see this, consider the function

$$\tan(\frac{\pi}{2}x)$$

defined on (-1,1), which is a continuous bijection with continuous inverse. (proofs for the continuity of tan and arctan are easily given by basic analysis arguments, and will not be reproduced here.)

Problem 3

Give an example of a function from \mathbb{R} to \mathbb{R} that is continuous at exactly one point.

Proof. The function

$$f: \mathbb{R} \to \mathbb{R}$$
$$f(x) = x\chi_{\mathbb{Q}}(x)$$

is continuous only at zero. To see this, we will use the neighborhood definition of continuity. That is, f is continuous at x if for each neighborhood of f(x), its preimage contains a neighborhood of x.

First, we will prove that f is continuous at zero. It suffices to show that each basic open neighborhood of f(x) has a preimage that contains an open neighborhood of x. So, let $(-\varepsilon, \varepsilon)$ be a basic neighborhood of f(0) = 0. Then,

$$f^{-1}((-\varepsilon,\varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon,\varepsilon)$$

which contains $(-\varepsilon, \varepsilon)$ an open neighborhood of 0 as desired.

Now, let $x \neq 0$. We will show that f is not continuous at x. If x is irrational, then f(x) = 0. Now, choose ε so that $x \notin (-\varepsilon, \varepsilon)$. Then, by the above calculation, we have

$$f^{-1}((-\varepsilon,\varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon,\varepsilon)$$

which does not contain any neighborhood of x.

If x is rational, then f(x) = x. Choose ε such that $0 \notin V_{\varepsilon}(x)$. Then,

$$f^{-1}(V_{\varepsilon}(x)) = V_{\varepsilon}(x) \cap \mathbb{Q}$$

which does not contain any open neighborhood of x (This is easily seen by observing that any neighborhood of x must intersect $\mathbb{R} \setminus \mathbb{Q}$, but the inverse image contains only rational points). \square

Problem 4

Suppose Y is Hausdorff, and $X \xrightarrow{g} Y$ are continuous. If $f|_A = g|_A$ for a dense subset $A \subset Y$, prove that f = g.

Proof. Let f and g be parallel morphisms that satisfy the assumptions.

Now, let $y \in Y$. Since A is dense, $y \in \overline{A}$, so there exists some net $\{y_{\alpha}\}$ such that $y_{\alpha} \in A$ for all α and $y_{\alpha} \to y$. In particular, since Y is Hausdorff, this net converges to the unique limit y.

By the hypothesis, $f(y_{\alpha}) = g(y_{\alpha}) \ \forall \alpha$, and since both f and g are continuous, they preserve limits. That is $f(y_{\alpha}) \to f(y)$ and $g(y_{\alpha}) \to g(y)$. Since $f(y_{\alpha}) = g(y_{\alpha})$ for all α and limits of nets in Y are unique, they must converge to the same element, and f(y) = g(y).

Since this works for all $y \in Y$, f = g.

PROBLEM 5

Prove that if A_{α} is a closed subset of X_{α} for all α , then $\prod A_{\alpha}$ is closed in $\prod X_{\alpha}$.

Proof. To show that $\prod A_{\alpha}$ is closed, we need to show that it contains its limit points. To do so, let $\{a_{\gamma}\}$ be a convergent net in the product $\prod A_{\alpha}$. In particular, each of its projections $\pi_{\alpha}(a_{\gamma})$ is also a net in A_{α} , and since A_{α} is closed, this net converges to elements in A_{α} .

Thus, each coordinate α of the net $\{a_{\gamma}\}$ converges in A_{α} , so any limit point must have coordinates in the A_{α} as well. That is, if a is a limit point of $\{a_{\gamma}\}$, then for each α , $\pi_{\alpha}(a) \in A_{\alpha}$, which means that $a \in \prod A_{\alpha}$ as desired.

Since $\prod A_{\alpha}$ contains all its limit points, it is closed.

PROBLEM 6

Let $y \in \prod X_{\alpha}$, and $\{x_n\}$ a sequence of points in $\prod X_{\alpha}$. Show that $x_n \to y$ if and only if $\pi_{\alpha}(x_n) \to \pi_{\alpha}(y)$ for all α .

Proof. (=>) For the first direction, assume that $x_n \to y$. Since each π_{α} is continuous, they preserve limits. Thus, for each α , $\pi_{\alpha}(x_n) \to \pi_{\alpha}(y)$ as desired.

(<=) For the other direction, let $\{x_n\}$ be such that for all α , $\pi_{\alpha}(x_n) \to \pi_{\alpha}(y)$. In particular, this means that the filter $\mathscr{F} = \{A \subset \prod X_{\alpha} \mid \exists n \in \mathbb{N} : x_m \in A \ \forall m > n\}$ pushes forward along each π_{α} to a filter that converges to $\pi_{\alpha}(y)$.

Now, we just need to show that \mathscr{F} converges to y (Equivalently, that \mathscr{F} contains each neighborhood of y). To do so, we will show that each neighborhood of y contains an element of \mathscr{F} , then since \mathscr{F} is a filter, it is closed under supersets and contains each neighborhood of y.

So, let U be a neighborhood of y. In particular, there exists a basis element

$$B = V_1 \times V_2 \times \ldots \times V_n \times X \times X \ldots \subset U$$

Now, since the push-forward of \mathscr{F} along each projection is a convergent filter, $N_{\alpha} \in \pi_{\alpha*}(\mathscr{F})$ for each neighborhood N_{α} of $\pi_{\alpha}(y)$.

In particular, $V_i \in \pi_{\alpha*}(\mathscr{F})$, which means that $\pi_{\alpha}^{-1}(V_i) \in \mathscr{F}$. Now, we can write B as

$$B = \bigcap_{i=1}^{n} \pi_{\alpha}^{-1}(V_i)$$

which is a finite intersection of elements of \mathscr{F} , so $B \in \mathscr{F}$. Thus, $U \supset B$ is in \mathscr{F} as well. Since U was any neighborhood of y, the neighborhood filter $\mathscr{N}_y \subset \mathscr{F}$ and $\mathscr{F} \to y$ as desired. \square

PROBLEM 7

Let \mathbb{R}^{ω} be the space of sequences of real numbers, and let \mathbb{R}^{∞} be the space of sequences that are eventually zero. What is $\overline{\mathbb{R}^{\infty}} \subset \mathbb{R}^{\omega}$?