Midterm

Daniel Halmrast

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I pledge by Frank W. Warner's "Foundations of Differentiable Manifolds and Lie Groups" this is my own work. Signed:

Problem 1

Show that for two nontrivial groups G and H, their free product $G \star H$ has trivial center, and that the only elements of finite order are the conjugates of finite order elements in G and H.

Proof. We first show that the free product has a trivial center. To show this, we will consider two cases of elements from $G \star H$. First, observe that for any element $g \in G$ with $g \neq e$, g (or rather, the inclusion of g into $G \star H$) cannot be in the center $Z(G \star H)$. To see this, let $h \in H$ with $h \neq e$, and note that the element $ghg^{-1}h^{-1}$ is a reduced word in the free group, and cannot be equal to the identity. Thus, $gh \neq hg$, and g is not in $Z(G \star H)$. Similarly, any element $h \in H$ is not in $Z(G \star H)$.

Now, let w be a reduced word in $G \star H$ that is not in G or H. Without loss of generality, we can write

$$w = g_1 h_1 g_2 h_2 \dots g_n h_n$$

or

$$w = g_1 h_1 g_2 h_2 \dots g_n$$

for $g_i \in G$, and $h_i \in H$. Now, since G and H have no relations to each other, this word is not equal to any word beginning with an element of H. Suppose for a contradiction that

$$w = h'_1 g'_1 h'_2 g'_2 \dots h'_m g'_m$$

with g'_m possibly equal to the identity. This would imply that

$$g_1 h_1 g_2 h_2 \dots g_n h_n = h'_1 g'_1 h'_2 g'_2 \dots h'_m g'_m$$

which would imply the relation

$$g_1h_1g_2h_2\dots g_nh_n(g'_m)^{-1}(h'_m)^{-1}\dots(g'_1)^{-1}(h'_1)^{-1}=e$$

Now, if $g'_m \neq e$, the word on the left hand side is already reduced, and thus cannot be equal to e. If $g'_m = e$, then the word could reduce further. However, note that this word only reduces further if adjacent elements from the same group cancel to the identity (that is, if adjacent elements from the same group do not cancel to the identity, then the word is reduced, and in particular is not equal to the identity). Now, careful counting of the elements of the word reveals that the (unreduced) word has one more element from H than it does elements from G. So, the total number of elements in the unreduced word is odd. Clearly, then, we cannot cancel in pairs to get the identity. Therefore, this relation cannot be true, and thus w cannot be written as a word beginning with an element from H.

It follows immediately, then, that $wh \neq hw$ for $h \in H$ a nontrivial element, since wh is a word beginning with an element of G, and hw is a word beginning with an element of H.

Thus, no element in $G \star H$ is in the center, as desired.

Next, we show that every element of finite order is the conjugate of a finite order element of G or H. So, let w be a word in $G \star H$ with $w^n = e$. Without loss of generality, let the first term of w be from G.

If $w = g_1h_1\dots g_nh_n$, then the concatenation w^n is already reduced (since adjacent elements are from different groups) and in particular cannot be equal to the identity.

So, suppose $w = g_1h_1\dots h_{n-1}g_n$, with $w^n = e$. Now, for this to be true, it must be that $g_n = g_1^{-1}$. If this were not the case, the concatenated word w^n would be reduced already (by treating g_ng_1 as an element of G) since g_ng_1 is adjacent to only elements of H. Continuing the argument, we find that $h_{n-1} = h_1^{-1}$, $g_{n-1} = g_2^{-1}$ and so on, save for the central term (since w has odd length). If n is even, the central term is $h_{\frac{n}{2}}$, and if n is odd, the central term is $g_{\frac{n+1}{2}}$. For the sake of simplicity, we will assume that n is odd, but the proof works the same way for n even.

Now, we know that

$$w = g_1h_1g_2h_2\dots h_{\frac{n+1}{2}-1}g_{\frac{n+1}{2}}h^{-1}_{\frac{n+1}{2}-1}\dots h_2^{-1}g_2^{-1}h_1^{-1}g_1^{-1}$$

which is just

$$w = (g_1h_1g_2h_2\dots h_{\frac{n+1}{2}-1})g_{\frac{n+1}{2}}(g_1h_1g_2h_2\dots h_{\frac{n+1}{2}-1})^{-1}$$

and thus w is a conjugate of an element of G. Now for $w^n = e$ it must be that

$$w^n = (g_1h_1g_2h_2\dots h_{\frac{n+1}{2}-1})g^n_{\frac{n+1}{2}}(g_1h_1g_2h_2\dots h_{\frac{n+1}{2}-1})^{-1} = e$$

which is only true if $g^n = e$.

Thus, the elements of finite order in $G \star H$ are all conjugates of elements of finite order in G or H.

Finally, we observe that every element that is the conjugate of an element of finite order in G (or H) is of finite order. Let $g \in G$ with $g^n = e$, and observe that for any $w \in G \star H$, we have that

$$(wgw^{-1})^n = wg^nw^{-1} = ww^{-1} = e$$

and so wgw^{-1} has finite order as well.

Thus the elements of finite order in $G\star H$ are exactly the conjugates of elements of finite order in G or H, as desired. $\square$

Problem 2

Let $X \subset \mathbb{R}^n$ be the union of convex open sets $X_1, \ldots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Prove that X is simply connected.

Proof. This proof will be done inductively on the number of convex open sets used to union to X. Now, convex sets have a distinct property which makes them nice for this problem. Namely, if X is a convex set, and γ is a path in X, γ is homotopic to the line segment from $\gamma(0)$ to $\gamma(1)$. This is clear; since convex sets are simply connected, all paths with fixed endpoints are homotopic to each other, and since each convex set is convex, it contains the straight line path from $\gamma(0)$ to $\gamma(1)$.

Now, let's first prove the base case of $n = 2$. For this, let f be a loop in X based at x_0. Without loss of generality, we let $x_0 \in X_1$. Now, if f stays in X_1, then it is nullhomotopic since X_1 is simply connected. Now, suppose f enters X_2 at some time t_1, exits X_1 at some $t > t_1$, and exits X_2 at $t_2 > t$. Then, the segment $f|_{[t_1,t_2]}$ is homotopic to the straight line from $f(t_1)$ to $f(t_2)$, which is in X_1. Repeating this for the (finite) number of times f exits X_1, we see that f is homotopic to a loop that stays in X_1, and is thus nullhomotopic.

Now suppose the theorem holds for the union of $n-1$ convex sets with the triple intersection property, and let X be the union of n convex sets with the same property. Let f be a loop in X based at $x_0 \in X_1$. Now, if f stays in $\cup_{i=1}^{n-1} X_i$ (that is, if f avoids being in only X_n), then f is a loop in the union of $n-1$ convex sets with the triple intersection property, and is nullhomotopic by the inductive hypothesis.

So, suppose f enters X_n at some time t_1 from a set X_i, leaves X_i at some $t > t_1$, then leaves X_n in $X_j \cap X_n$ at some t_2. Now, by the triple intersection property, we know that $X_i \cap X_j \cap X_n \neq \emptyset$, so let x be a point in the common intersection. Now, since $f(t_1)$, $f(t_2)$, and x are all in X_n, so is the path taking the straight line from $f(t_1)$ to x, then the straight line from x to $f(t_2)$. Then, the segment $f|_{[t_1,t_2]}$ is homotopic to this path from $f(t_1)$ through x to $f(t_2)$. However, since $f(t_1)$ and x are in X_i, so is the straight line connecting them. Furthermore, since x and $f(t_2)$ are in X_j, so is the straight line connecting them. Thus, $f|_{[t_1,t_2]}$ is homotopic to a path that stays in X_i and X_j. Repeating this process for the (finite) number of times f stays in X_n alone, we see that f is homotopic to a loop that stays in $\cup_{i=1}^{n-1} X_i$, and by the inductive hypothesis, f is nullhomotopic as desired.

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Problem 3

Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply connected if $n \geq 3$.

Proof. Let X be a set of m points in $\mathbb{R}^n$. We wish to show that $\pi_1(\mathbb{R}^n \setminus X)$ is trivial. Now, since X is a discrete set of points, there is some $\varepsilon > 0$ for which the balls $B_\varepsilon(p)$ of radius ε centered at $p \in X$ do not intersect. We can deformation retract $\mathbb{R}^n \setminus X$ to $\mathbb{R}^n \setminus (\bigcup_{p \in X} B_\varepsilon(p))$ by fixing all $x \notin B_\varepsilon(p)$ and sending $x \in B_\varepsilon(p)$ along the radial line to the boundary. This is well-defined, since the central point p is not in the domain of this homotopy.

Now, we form a new space Y as follows: for each $B_\varepsilon(p)$, attach an n-cell along the boundary $\partial B_\varepsilon(p)$ (which is in $\mathbb{R}^n \setminus (\bigcup_{p \in X} B_\varepsilon(p))$). Clearly, $Y = \mathbb{R}^n$, which has trivial fundamental group. However, proposition 1.26 guarantees that there is an isomorphism between $\pi_1(\mathbb{R}^n \setminus (\bigcup_{p \in x} B_\varepsilon(p)))$ and $\pi_1(Y)$. Thus, it follows that $\pi_1(\mathbb{R}^n \setminus (\bigcup_{p \in x} B_\varepsilon(p)))$ is trivial as well, and thus since $\mathbb{R}^n \setminus X$ deformation retracts to this space, it has trivial fundamental group as well, as desired. $\square$

Problem 4

Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

Proof. To begin with, we deformation retract $\mathbb{R}^3 \setminus X$ onto $S^2 \setminus \tilde{X}$, where $\tilde{X}$ is the collection of $2n$ points $S^2 \cap X$. This deformation retraction is the standard radial retraction

$$f_t(x) = (1-t)x + t\frac{x}{\|x\|}$$

which sends the lines in X to their intersection points in $\tilde{X}$. This is well-defined, since $0 \in X$ and so $0 \notin \mathbb{R}^3 \setminus X$.

Now, we can designate one of these points as the north pole N, and use the homeomorphism from $\mathbb{R}^2$ to $S^2 \setminus \{N\}$ to construct a homeomorphism between $S^2 \setminus \tilde{X}$ and $\mathbb{R}^2 \setminus Y$ where Y is the collection of $2n-1$ points that is the image of $\tilde{X} \setminus \{N\}$ under the homeomorphism.

Finally, we draw a finite graph on $\mathbb{R}^2$ with the property that there are exactly $2n-1$ bounded complementary components, each containing exactly one point of Y. Then, since each bounded complementary component is homeomorphic to a disk with a point removed, we can deformation retract each one onto its boundary. Finally, we deformation retract the unbounded complementary component to the outside boundary of the graph. Now, we can apply problem 5 to see that the fundamental group of this space is the free group on $2n-1$ generators.

Since $\mathbb{R}^3 \setminus X$ can be homotoped to this graph via the homotopies described, we have that the fundamental group of $\mathbb{R}^3 \setminus X$ is the free group on $2n-1$ generators.

□

Problem 5

Let $X \subset \mathbb{R}^2$ be a connected graph that is the union of a finite number of straight line segments. Show that $\pi_1(X)$ is free with a basis consisting of loops formed by the boundaries of the bounded complementary regions of X, joined to a basepoint by suitably chosen paths in X.

Proof. Consider the following construction. For each bounded component of $\mathbb{R}^2 \setminus X$, attach a 2-cell e_i^2 along the boundary of the component (which is in X). Call the space obtained this way Y. Now, Y is obtained from X in the way necessary to apply proposition 1.26. Namely, we know that there is a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is the normal subgroup generated by the (conjugates of the) attaching maps.

Now, Y can easily be seen to have trivial fundamental group, since if it did have a nontrivial loop, that loop would encircle some amount of bounded components of $\mathbb{R}^2 \setminus X$. But we already attached 2-cells across each of these components, and so the loop can homotope across the bounded components, and must actually be nullhomotopic. Thus, it must be that the normal subgroup N which is the kernel of the induced surjection is the whole group $\pi_1(X, x_0)$. So it suffices to describe the subgroup N.

Now, per proposition 1.26, N is the normal subgroup generated by the loops around the bounded components of $\mathbb{R}^2 \setminus X$, joined to the basepoint by some path in X. Furthermore, these loops have no relations to each other. Consider the homotopy from X to the wedge of n circles, where n is the number of bounded components of $\mathbb{R}^2 \setminus X$ (Hatcher example 0.7 guarantees the existence of such a homotopy). The image of the loop going once around the ith bounded component is a loop going once around the ith wedge of circles. Thus, this homotopy sends the generators of N to the generators of $\pi_1(\vee_n S^1)$, which is the free group on n generators. Since the homotopy induces an isomorphism, it must be that N is free on these generators as well.

Thus, the fundamental group of X is the free group on n generators, where the generators are the loops going once around a bounded complementary region of X joined to the basepoint by a suitably chosen path, as desired. $\square$

Problem 6

Suppose a space Y is obtained from a path-connected space X by attaching n-cells for fixed $n \geq 3$. Show that the inclusion $X \to Y$ induces an isomorphism on the fundamental groups. Use Proposition 1.26 to show that the complement of a closed discrete subspace of $\mathbb{R}^n$ is simply connected if $n \geq 3$.

Proof. For the first part of this proof, we follow the proof outlined in Hatcher for proposition 1.26.

Let x_0 be the basepoint in X, and let γ_α be a path from x_0 to the basepoint of the boundary we are attaching the e^n_α cell to. Then, we construct the space Z as follows: To Y we attach a strip $S_\alpha = I \times I$ for each γ_α with $I \times \{0\}$ glued to γ_α, and $\{1\} \times I$ attached to an arc along e^n_α radially from the basepoint. Furthermore, all the strips are glued together at $\{0\} \times I$. Note that Z deformation retracts onto Y.

In each cell e^n_α, we choose a point y_α not on the arc that the strip S_α is attached to. Then, the open sets $A = Z \setminus \bigcup_\alpha \{y_\alpha\}$ and $B = Z \setminus X$ cover Z. Now, A deformation retracts onto X (since each cell e^n_α has a point removed, so there is a deformation retraction to the boundary of the cell, and the strips S_α we already observed deformation retract into X), and B is contractible, since it is homotopic to the wedge

$$B \simeq \vee_\alpha e^n_\alpha$$

by collapsing each strip S_α to a point, and this space is contractible.

Thus, van Kampen's theorem applied to the cover $\{A, B\}$ yields the following:

$$\pi_1(Z) = \pi_1(A)/N$$

where N is the normal subgroup generated by the relations $i_A([f]) = i_B([f])$ for $i_A : \pi_1(A \cap B) \to \pi_1(A)$ (and similarly for B). So, let's examine what $\pi_1(A \cap B)$ is. To do so, we apply van Kampen's theorem again to the cover built by

$$A_\alpha = (A \cap B) \setminus \cup_{\beta \neq \alpha} e^n_\beta$$

which is just all strips S_β for all β along with the single n-cell e^n_α minus a point y_α.

Now, the intersection of any two (or three) of these elements of the cover is just the strips S_β, and is path-connected. Furthermore, each A_α deformation retracts to $e^n_\alpha \setminus \{y_\alpha\}$ (plus its boundary) by contracting all the strips, and $e^n_\alpha \setminus \{y_\alpha\}$ further deformation retracts onto its boundary, which is just S^{n-1}. Now, for $n \geq 3$, S^{n-1} has trivial fundamental group, so each A_α has trivial fundamental group as well.

Thus, $\pi_1(A \cap B)$ is trivial, and the normal subgroup N is trivial as well (since the only relation that defines it is $i_A([0]) = i_B([0]) = 0$, which is a tautology). Therefore, van Kampen's theorem tells us that the homomorphism induced by the inclusion $A \to Z$ is surjective from $\pi_1(A)$ to $\pi_1(Z)$ with kernel N, but since N is trivial, this is actually an isomorphism. Furthermore, since A deformation retracts to X, and Z deformation retracts to Y, the inclusion $i : X \to Y$ induces an isomorphism between $\pi_1(X)$ and $\pi_1(Y)$, as desired.

Let D denote the discrete subspace of $\mathbb{R}^n$. Now, since D is discrete and closed, it follows that for each point p in D, there is a ball $B_{\varepsilon_p}(p)$ with radius ε_p such that $B_{\varepsilon_p}(p) \cap D = p$ (if this were not the case, then p would be a limit point of D, which cannot happen since D is discrete).

We now follow the same strategy as the one used in problem 3. Namely, we first note that $\mathbb{R}^n \setminus D$ deformation retracts onto $\mathbb{R}^n \setminus (\bigcup_{p \in D} B_{\varepsilon_p}(p))$ through the radial homotopy in each $B_{\varepsilon_p}(p)$ to the boundary.

Now, we construct the space Y by attaching an n-cell for each $B_{\varepsilon_p}(p)$ along its boundary. Clearly, $Y = \mathbb{R}^n$, which has trivial fundamental group. However, proposition 1.26 guarantees that Y and $\mathbb{R}^n \setminus (\bigcup_{p\in D} B_{\varepsilon_p}(p))$ have the same fundamental group (the trivial group). Thus, since $R^n \setminus D$ is homotopic to $\mathbb{R}^n \setminus (\bigcup_{p\in D} B_{\varepsilon_p}(p))$, they have the same (trivial) fundamental group, and so $\mathbb{R}^n \setminus D$ is simply connected, as desired. $\square$

Problem 7

Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle in the other torus.

Proof. Let T_1 and T_2 denote the tori being glued together, and let X be the resulting space. Furthermore, let U_i be a neighborhood of the circle $S^1 \times \{x_0\}$ in T_i which deformation retracts onto the circle (for example, $U_i = S^1 \times (x_0 - \varepsilon, x_0 + \varepsilon)$).

Now, X is the union of $A_1 = T_1 \cup U_2$ and $A_2 = T_2 \cup U_1$, and the intersection $(T_1 \cup U_2) \cap (T_2 \cup U_1)$ is just $U_1 \cup U_2$, which is path-connected. Letting the basepoint be x_0, which is in each U_i, we note that these two open sets satisfy the hypotheses for van Kampen's theorem. Thus, the fundamental group $\pi_1(X)$ is the pushout of

$$\begin{tikzcd} \pi_1(A_1 \cap A_2) \arrow[r, "i_2"] \arrow[d, "i_1"] & \pi_1(A_2) \\ \pi_1(A_1) & \end{tikzcd}$$

which is just the free product of $\pi_1(A_1)$ with $\pi_1(A_2)$ modded out by the normal subgroup generated by the relation $i_1([f]) = i_2([f])$ for $[f] \in \pi_1(A_1 \cap A_2)$.

In particular, A_1 deformation retracts to T_1 by construction of U_2, and similarly A_2 deformation retracts to T_2. Thus,

$$\pi_1(A_1) = \pi_1(A_2) = \mathbb{Z} \oplus \mathbb{Z}$$

In particular, A_1 is the free abelian group generated by the loops f_1 going once around $\{x_0\} \times S^1$ and g_1 going once around $S^1 \times \{x_0\}$ in T_1, and $\pi_1(A_2)$ is the free abelian group generated by the loops f_2 going once around $\{x_0\} \times S^1$ and g_2 going once around $S^1 \times \{x_0\}$ in T_2.

Now, $\pi_1(A_1 \cap A_2) = \mathbb{Z}$, since $A_1 \cap A_2$ deformation retracts to the common circle $S^1 \times \{x_0\}$. This group is generated by the loop g going once around $S^1 \times \{x_0\}$. Furthermore, $i_1([g]) = [g_1]$ and $i_2([g]) = [g_2]$. Thus, the normal subgroup we need to quotient by is the one generated by the relation $[g_1] = [g_2]$.

So, the fundamental group of X is

$$\pi_1(X) = (FrAb(f_1, g_1) \star FrAb(f_2, g_2))/(g_1 = g_2)$$

Where $FrAb(f_i, g_i)$ denotes the free abelian group on two generators f_i, g_i.

$\square$