1 Complete Manifolds

Recall we have a distance on a manifold as

$$d(p,q) = \inf\{L(\gamma) \mid \gamma : I \to M, \gamma(0) = p, \gamma(1) = q\}$$

which metrizes the topology on M. Recall also that the Gauss lemma guarantees that for each $p \in M$, there is some $r > 0$ for which $B_r(p)$ is a normal ball (is the diffeomorphic image under exp of some ball in T_pM). We note also from before that inside a normal ball, the shortest path from p to q is achieved by the unique radial geodesic from p to q.

Now we get to the new stuff:

Theorem 1. *Let (M^n, g) be a connected Riemannian manifold, and $p \in M$. The following are equivalent:*

- $\exp_p$ *is defined on all of T_pM.*
- *The closed and bounded sets of M are compact.*
- *M is complete as a metric space.*
- *M is geodesically complete. That is, every geodesic of M can be extended for all time. Alternately,* $\exp_q$ *is defined on all of T_qM for every $q \in M$.*
- *There exists a sequence of compact subsets K_n of M such that $\{K_n\}$ is increasing, $\lim K_n = M$, and if $q_n \in M \setminus K_n$, then $d(p, q_n) \to \infty$.*

Additionally, any of these statements imply the following: For any $q \in M$, there is a geodesic from p to q such that $L(\gamma) = d(p,q)$, or the geodesic minimizes distance. This is equivalent to $B_r(p) = \exp_p(B_r(0))$ for any $r > 0$.

Proof. Equivalence of the first five is easy. So, lets prove the first one implies the last corollary. Suppose p is such that $\exp_p$ is defined on all of T_pM.

Take $\delta > 0$ such that $B_\delta(p)$ is a normal ball. Choose $x_0 \in \partial B_\delta(p)$ such that $d(x_0, q) = d(q, \partial B_\delta(p))$ (doable since $\partial B_\delta(p)$ is compact). (we assume q is not in the normal ball, since if it were the proof would be trivial).

Now, we have $x_0 = \exp_p(\delta v)$ for some $\|v\| = 1$. Set $\gamma(t)$ to be the geodesic $\gamma(t) = \exp_p(t\delta v)$. Let $r = d(p,q)$. Then, $\gamma(r) = q$ (need to prove) and γ minimizes this length.

We can prove this by showing

$$d(p,q) = d(p, \gamma(t)) + d(\gamma(t), q)$$

which for $t = r$ guarantees

$$d(p,q) = d(p, \gamma(r)) + d(\gamma(r), q) = r + 0$$

To that end, let $I = \{t \in [\delta, r] \mid d(p,q) = d(p,\gamma(t)) + d(\gamma(t), r)\}$. We claim first that this is nonempty. This is clear, since $\delta \in I$. This follows from the fact that $\gamma(\delta) = x_0$ and $d(x_0, q) = r - \delta$, so $d(p,q) = \delta + r - \delta = d(p, \gamma(\delta)) + d(\gamma(\delta), q)$.

Furthermore, we prove that for any $t \in I$, $t < r$, there is some ε for which $t + \varepsilon \in I$.

Suppose $t < r$ is in I. Take a normal ball or radius ε around $\gamma(t)$. Then, let $y_0 \in \partial B_\varepsilon(\gamma(t))$ and such that $d(\partial B_\varepsilon(\gamma(t))) = d(q, y_0)$. We want to show that $y_0 = \gamma(t + \varepsilon)$.

To see this, note that

$$d(\gamma(t), q) = r - t$$

and note that

$$L(\gamma|_{[0,t]}) = t$$

which clearly implies that γ minimizes the distance between p and $\gamma(t)$. This follows from

$$\begin{aligned} d(p, q) &= L(\gamma|_{[0,t]}) + d(\gamma(t), q) \\ &\geq d(p, \gamma(t)) + d(\gamma(t), q) \\ &\geq d(p, q) \end{aligned}$$

and so $L(\gamma) = d(p, \gamma(t))$.

Now, we know that $y_0 = \gamma_1(\varepsilon) = \exp_{\gamma(t)}(\varepsilon u)$ for some u. Repeating the argument from before by setting $x_0 = y_0$, $p = \gamma(t)$, and so forth. Thus, $d(\gamma(t), q) = \varepsilon + d(y_0, q)$ and so

$$\begin{aligned} d(p, q) &= d(p, \gamma(t)) + \varepsilon + d(y_0, q) \\ &= d(p, \gamma(t)) + L(\gamma_1|_{[0,\varepsilon]}) + d(y_0, q) \\ &= L(\gamma|_{[0,t]}) + L(\gamma_1|_{[0,\varepsilon]}) + d(y_0, q) \end{aligned}$$

and so $d(p, y_0) = L(\gamma|_{[0,t]}) + L(\gamma_1|_{[0,\varepsilon]})$. This implies that $\gamma|_{[0,t]} \cdot \gamma_1|_{[0,\varepsilon]}$ is a geodesic minimizing distance between p and y_0. This shows that γ joined to γ_1 at $\gamma(t)$ is smooth, and so $\gamma_1 = \gamma$, and $y_0 = \gamma(t + \varepsilon)$ as desired.

This completes the proof. To see this, note that the above implies that I contains r, and so

$$d(p, q) = d(p, \gamma(r)) + d(\gamma(r), q) = r + 0$$

and so $d(\gamma(r), q) = 0$ as desired. □

The equivalences of the statements are proved below

Proof. ($a \implies b$)

Suppose M is such that $\exp_p$ is defined on all T_pM. let $A \subset M$ be closed and bounded. Then $A \subset B_r(p)$ for some $r > 0$ (definition of boundedness). Then, by the corollary above, we have that $A \subset \exp_p(B_r(0))$. Now, $B_r(0)$ is compact, and so its image $B_r(p)$ is compact. Since A is closed and a subset of a compact set, it is compact as well.

($b \implies c$) Suppose M is such that the closed and bounded sets are the compact sets. Then, M is complete by Heine-Borel. Explicitly, let p_k be a Cauchy sequence. This sequence is bounded, so its closure is compact. Therefore, some subsequence of p_k converges. Thus, since p_k is Cauchy, it converges as well.

($c \implies d$) Let γ be a maximally extended geodesic. $\gamma : (a, b) \to M$. Assume for a contradiction that b is finite. Then, consider a sequence $t_k \to b$, and we claim that $\gamma(t_k)$ is Cauchy. This is clear, since

$$d(\gamma(t_k), \gamma(t_m)) \leq \|t_k - t_m\|$$

as desired. So, $\gamma(t_k)$ is Cauchy, and has a limit $\gamma(t_k) \to p$ by completeness of M. Now, consider a normal ball of some radius δ around $\gamma(t_k)$, enough so that δ works for all t_k. We can go far enough in the sequence such that $\gamma(t_{k+1})$ is in the normal ball around $\gamma(t_k)$. Recall that the radial geodesic from $\gamma(t_k)$ to $\gamma(t_{k+1})$ is unique, and so γ must be the radial geodesic from $\gamma(t_k)$ to $\gamma(t_{k+1})$. Furthermore, γ can be extended across the entire normal ball. Taking k large enough so that p is in the normal ball, we see that γ can be extended across p, a contradiction.

Trivially, $d \implies a$. Thus we have established equivalence of the first four.

($b \equiv e$) Suppose M satisfies the Heine Borel property. Then, take the distance balls $K_n = \overline{B}_n(p)$ which are bounded and closed, and therefore compact. Clearly, these are also increasing, and clearly satisfy the requirements for e.

Suppose instead that M is written as the union of compact sets defined in e. Then, let A be a bounded and closed set. In particular, A is contained in some K_n by boundedness, and since A is closed, it is compact as a closed subspace of a compact space. $\square$