Final

Daniel Halmrast

June 14, 2018

Problem 1

Part i

Give the definition of degree for a smooth map $f : A \to B$ between closed oriented manifolds of the same dimension.

Show that if $g : B \to C$ is another such map, then

$$\deg(g \circ f) = \deg(f)\deg(g)$$

Proof. We assume here that the manifolds A, B, and C are all connected.

Consider a regular value $y \in B$ of f. The inverse image $f^{-1}(\{y\})$ is a finite set of points (since A is compact, and $f^{-1}(\{y\})$ is of dimension zero). For each point $x \in f^{-1}(\{y\})$, we say the sign of df_x at x (denoted $\operatorname{sgn}(df_x)$) is $+1$ if df_x preserves orientation, and -1 if df_x reverses orientation. Then, the degree of f is defined as the sum

$$\deg(f) = \sum_{x \in f^{-1}(\{y\})} \operatorname{sgn}(df_x)$$

Recall that this definition is well-defined, as it is independent of choice of regular value.

Now, we turn our attention to the composition $g \circ f$. Recall that $y \in C$ is called a regular value of $g \circ f$ if for every $x \in (g \circ f)^{-1}(\{y\})$, the differential $d(g \circ f)_x$ is surjective. Now, by Sard's theorem, the set of critical values for $g \circ f$ has measure zero, as well as the set of critical values for g, in C. Therefore, on any chart (U, ϕ) in C, there must exist a point which is regular for both g and $g \circ f$. To see this, let R denote the set of regular values of $g \circ f$ in U, and R' the set of regular values of g in U. If R and R' were disjoint, then we would have

$$\begin{aligned} \mu(R) = \mu(U) &= \mu(R') && \text{By Sard's Theorem} \\ \mu(R \cup R') = \mu(R) + \mu(R') &= 2\mu(U) > \mu(U) \end{aligned}$$

which is a contradiction. From here on out, let $y \in C$ be a regular value of both $f \circ g$ and g.

Now, we will show that for all $x \in (g \circ f)^{-1}(\{y\})$, $f(x)$ is a regular value for f. so, let x be as specified. This means that the differential $d(g \circ f)_x = dg_{f(x)} \circ df_x$ is surjective. In particular, since the dimensions of $T_x A$ and $T_y C$ are equal, $d(g \circ f)_x$ is an isomorphism. Furthermore, since $T_{f(x)}B$

also has the same dimension, it must be that df_x and $dg_{f(x)}$ are both isomorphisms as well. This follows from the fact that $dg_{f(x)} \circ df_x$ is surjective, so $dg_{f(x)}$ is surjective onto a space of the same dimension, and hence is an isomorphism. Similarly, $dg_{f(x)} \circ df_x$ is injective, so df_x is injective into a space of the same dimension, and is thus an isomorphism. From all this, we conclude that df_x is surjective for all $x \in (g \circ f)^{-1}(\{y\})$. In particular, since $f^{-1}(\{f(x)\}) \subset (g \circ f)^{-1}(\{y\})$, we have that df_x is surjective for all x in the preimage of $f(x)$, and so $f(x)$ is a regular value of f.

Finally, we show that $\deg(g \circ f) = \deg(g)\deg(f)$. $\square$

<++>