MATH 220A: MIDTERM EXAMINATION
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DANIEL HALMRAST
TIME TAKEN: 9 HOURS

Problem 1

Let G be a non-abelian group of order 26, and let $H \leq G$ with $|H| = 13$. Prove that each element of G which is not in H is of order 2. Deduce that if a, b are elements of G of orders 2 and 13 respectively, then $aba = b^{-1}$.

Proof. We will first prove that any element in G that is not in H has order 2. We will do this by careful use of the class equation. In particular, letting G act on itself by conjugation (the group action $(g, x) \mapsto gxg^{-1}$), we have that

$$|G| = |Z(G)| + \sum |Orb_G(x)|$$

Where $Z(G)$ is the center of G, and the sum is taken over representatives of the distinct conjugacy classes (orbits) of G.

Since G is non-abelian, we know that $|G| \neq |Z(G)|$. Furthermore, since $Z(G) \leq G$, we know that $|Z(G)|$ divides $|G| = 26$. Thus, either $|Z(G)| = 13$, $|Z(G)| = 2$, or $|Z(G)| = 1$.

Suppose that $|Z(G)| = 13$. Then, the class equation reads

$$|G| = 26 = 13 + \sum |Orb_G(x)|$$

Now, the orbit-stabilizer theorem tells us that $|Orb_G(x)| = [G : c_G(x)]$, and Lagrange's theorem tells us that this index divides the order of the group. Thus, we have that for each x, $|Orb_G(x)|$ is either 13 or 2. However, since the sum must equal 13, at least one term in the sum must be odd. Therefore, the only term in the sum is the single orbit of order 13.

This implies that for any $x \notin Z(G)$ we have that $c_G(x)$ has index 13, implying that $|c_G(x)| = 2$. This cannot be, however, since each element of the center $Z(G)$ clearly centralizes x, so $|c_G(x)| \geq 13$.

Thus, $|Z(G)|$ cannot be 13. So, assume $|Z(G)| = 2$.

Now, we have a class equation that reads

$$|G| = 26 = 2 + \sum |Orb_G(x)|$$

In particular, $\sum |Orb_G(x)| = 24$. By the same argument as above, we must have that each orbit is of order 2 or 13. However, we cannot have an orbit of order 13, since that would leave $24 - 13 = 11$ for the sum of the orders of the rest of the orbits, and each orbit must have order 2 or 13, which clearly cannot sum to 11.

Thus, we have that each orbit has order 2, with a stabilizer of index 2. Now, a basic argument will show that, for $g \in G$, $\mathrm{Stab}_G(g) = c_G(g)$, and for $g' = kgk^{-1}$ for any $k \in g$, $c_G(g') = kc_G(g')k^{-1}$. That is, $c_G(kgk^{-1}) = kc_G(g)k^{-1}$ for any $g, k \in G$.

Since each orbit has only two elements in it (say, g and $g' = kgk^{-1}$), it must be that the conjugcacy class of centralizers of g has at most two elements. Since each centralizer has order 13, it is conjugate to H, but since the number of conjugates of H must be equal to $1 \mod 13$, it must be that H is the only element of this conjugacy class. Thus, for each g in some orbit, we have that $c_G(g) = H$.

Suppose that $x \in G$ is in some orbit (with elements x and x'). Then, we have that the centralizer of x is H. Now, let $k \notin H$ be arbitrary. Since $k \notin H$, $kxk^{-1} = x'$. And since k doesn't centralize x' either, we have that $kx'k^{-1} = k^2x(k^{-1})^2 = x$.

Now, the order of k is either trivial, 2, 13, or 26. If the order of k is 1, then $k = e \in H$ which contradicts $k \notin H$. If the order of k is 13, then $\langle k \rangle$ has order 13, and is then equal to H, which contradicts $k \notin H$. If k has order 26, then $\langle k \rangle = G$, which cannot be since G is non-abelian.

Thus, $|k| = 2$ as desired.

Suppose instead that $|Z(G)| = 1$. In this case, we must have some orbit of order 13, along with some orbits of order 2. In particular, applying the above argument to an orbit of order 2 completes the argument that each $k \notin H$ has order 2 as desired.

Now, let's use this result to prove that for elements a, b in G with $|a| = 2$ and $|b| = 13$, $aba = b^{-1}$.

To do so, we observe that this is equivalent to showing that $(ab)^2 = e$. So, consider the element ab. Since H is of prime order, each element of H has order 13. Thus, a cannot be in H. In particular, this means that (since H is cyclic), $ab \notin H$ as well.

By the above result, we have that $|ab| = 2$, which implies that $(ab)^2 = e$ as desired. $\square$

Problem 2

For any $\sigma \in S_n$, we say σ has a cycle type $(r_1, \ldots, r_n)$ where each $r_i \in \mathbb{N} \cup \{0\}$, and $\sum_{i=1}^{n} ir_i = n$, if $\sigma = \sigma_1 \ldots \sigma_k$ where each σ_i is disjoint from the others, and r_i of them have length i.

Part i. For $x \in G$, what is the cycle type of $\rho(X)$, where ρ is the canonical embedding of G into $S_{|G|}$?

Proof. Let $|x| = r$, and let $|G| = rs$ for some r and s. Now, we first observe that the map $x \mapsto l(x) \in S_{|G|}$ (where $l(x)$ is left-multiplication by x) has trivial kernel, and is thus an isomorphism onto its image. So, this group action is an embedding of G into $S_{|G|}$, and we will identify $l(x)$ with $\rho(x)$.

Furthermore, consider the subgroup $\langle x \rangle$ of order r. Clearly, if $l(x^n)g = g$, then $x^n = e$ and $n = r$. Therefore, for each $g \in G$, it is in a $\langle x \rangle$-orbit of order r (formed by repeated application of $l(x)$). So, each orbit of $\langle x \rangle$ has order r. Observing that the orders partition G, we have that

$$|G| = \sum |Orb(g)|$$

where the sum is taken over representatives of distinct orbits. Since each orbit has order r, and G has order rs, we must have exactly s disjoint orbits.

So, the cycle decomposition of x has S cycles of length r, and has a cycle type of $(0, \ldots, s_{(r)}, \ldots, 0)$. That is, $r_r = s$ and the rest are zero. $\square$

Part ii. Suppose that a group of order $2^r s$, with s odd, contains an element x of order 2^r. Show that for $r > 0$, G has a normal subgroup of index 2.

Proof. Since x has order 2^r, by the above argument it is represented by s disjoint 2^r-cycles, with signature

$$\varepsilon(\rho(x)) = \varepsilon(2^r\text{-cycle})^s$$

Since the signature of a 2^r-cycle is odd, and s is odd, the signature of $\rho(x)$ is odd as well.

Thus, the representation $\rho(x)$ is not an element of $A_{|G|}$, and by an argument made in homework 1, exactly half of the elements of G are in $A_{|G|}$. Therefore, the subgroup $G \cap A_{|G|}$ has order $\frac{|G|}{2}$, and has index $[G : G \cap A_{|G|}] = 2$ as desired.

Furthermore, this subgroup is normal, since conjugation of $\tau \in A_{|G|}$ with any element of G will not change the signature of τ, so it will stay in $A_{|G|}$.

(The argument for $G \cap A_{|G|}$ being exactly half of G is recreated below).

Let G be a subset of S_n, with $G \not\subset A_n$. Then, there is at least one element $\sigma \in G$ that is odd. Now, consider the sets T_1 and T_2, containing the even and odd permutations of G respectively.

Clearly, the map from T_1 to T_2 defined by $\tau \mapsto \sigma\tau$ is a bijection of sets, since it is invertible by σ^{-1}. Thus, the two sets have the same cardinality, and exactly half the elements of G are even, as desired. $\square$

Problem 3

Part i. State Sylow's theorems.

Statement. *Sylow's theorems are as follows:*

For a finite group G, and prime p such that p divides $|G|$ and p^α is the highest power of p that divides $|G|$, the following hold:

(1) *There exists a subgroup of G with order p^α. This subgroup is called a Sylow-p subgroup of G.*
(2) *Every Sylow-p subgroup lies in the same conjugacy class. That is, given two Sylow-p subgroups P, S, there exists $g \in G$ such that $P = gSg^{-1}$.*
(3) *The number of Sylow-p subgroups of G is equal to $1 \mod p$.*

Part ii. Let G be a finite, simple, non-abelian group. Let p be a prime number. Show that if p divides $|G|$, then $|G|$ divides $\frac{n_p!}{2}$, where n_p is the number of Sylow-p subgroups of G.

Proof. By Sylow's second theorem, we know that all of the Sylow-p subgroups of G lie in the same conjugacy class. Now, consider the action of G on this conjugacy class by conjugation

$$(g, S) \mapsto gSg^{-1}$$

for $g \in G$ and S a Sylow-p subgroup.

This action defines a map $\phi : G \to S_{n_p}$. Now, the kernel $\ker(\phi)$ is normal in G, so by the fact that G is simple, it must be trivial. (Since the orbit of the action is all of the conjugacy class, the kernel cannot be all of G, therefore, it must be just $\{e\}$.)

Therefore, ϕ is injective. In particular, G is isomorphic to a subgroup of S_{n_p}, which we will, for ease of notation, identify with G.

Suppose that G were not a subgroup of A_{n_p}. Then, we will show that the subgroup $G \cap A_{n_p}$ (which contains $\frac{|G|}{2}$ elements by the previous exercise and thus cannot be trivial or equal to G) is normal in G. To do so, let $\sigma \in A_{n_p} \cap G$. In particular, σ is even. Now, for any $\tau \in G$, the element $\tau\sigma\tau^{-1}$ has even signature, since σ is even, and the signature of τ is equal to the signature of τ^{-1}. Thus, τ normalizes σ, and since σ and τ were arbitrary, it follows that all of G normalizes all of $A_{n_p} \cap G$, and thus $A_{n_p} \cap G$ is normal in G, which is a contradiction.

So, G is a subgroup of A_{n_p}, and thus its order must divide the order of A_{n_p}. Thus, $|G|$ divides $\frac{n_p!}{2}$. □

Part iii. Let G be a group of order 48. Show that G is not simple.

Proof. To begin with, we observe that $48 = 2^4 \times 3$.

Now, let's consider the Sylow-2 subgroups of G, along with the action of G by conjugation. Now, Sylow's 2^{nd} theorem guarantees that all the Sylow-2 subgroups are in the same orbit. Thus, by the orbit-stabilizer theorem, we have that, for P a Sylow-2 subgroup,

$$\#(\text{Sylow-2 subgroups}) = |Orb_G(P)| = [G : N_G(P)]$$

Furthermore, $N_G(P)$ contains P, so it has order at least 2^4. If $|N_G(P)| = |G|$, then P would be normal and G would not be simple.

So, it follows that $|N_G(P)| = 2^4$. Then $[G : N_G(P)] = 3$, and there are 3 Sylow-2 subgroups of G. Thus, by the argument from part ii, along with the observation that 48 does not divide $\frac{3!}{2} = 3$, we know that G is not simple. □

Part iv. Find a group of order 48 that has no normal Sylow-2 subgroup.

Consider the dihedral group D_{24}, which has $24 \times 2 = 48$ elements. In particular, D_8 is a subgroup of D_{24} of order $8 \times 2 = 16$, but is not normal in D_{24}. This can be clearly illustrated by conjugation with the smallest rotation in D_{24}. If one considers conjugating a reflection by this rotation, it is clear that the rotation will set the octagon off-axis, and the reflection will be performed off the symmetry axis of the octagon, leading to a transformation not in D_8.

Thus, D_8 is not normal in D_{24}, as desired.

Problem 4

Let G be a simple group of order $168 = 2^3 \times 3 \times 7$.

part i. G has just 8 Sylow-7 subgroups, and the normalizer of each is of order 21.

Proof. We know from before that the number of Sylow-p subgroups is equal to the index of the normalizer of any particular Sylow-p subgroup. Thus, the second statement follows immediately from the first.

We note first that the normalizer of each Sylow-7 subgroup has order at least 7, which means it has index at most 24. Furthermore, we know that the number of Sylow-7 subgroups must be congruent to $1 \mod 7$, which limits it to either 8, 15, or 22.

However, the only one of these that divides the group order is 8, and since the index of the normalizer (which is the size of the conjugacy class) of a Sylow-p subgroup must divide the order of the group, it must be that G only has 8 Sylow-7 subgroups.

As stated above, this means that each normalizer has index 8, and order 21. $\square$

Part ii. Show that G is isomorphic to a subgroup of S_8.

Proof. Consider the action of G by conjugation on the Sylow-7 subgroups of G. This defines a homomorphism $\phi : G \to S_8$. Furthermore, since G is simple, and kernels are normal in their group, $\ker \phi$ must be either trivial or the whole group. However, since G acts transitively, it must be that the kernel is trivial. Thus, ϕ is monic, and defines an isomorphism onto its image, which is a subgroup of S_8 as desired. $\square$

Part iii. Show that G has no elements of order 14 or 21.

Proof. Suppose for a contradiction that $x^{14} = e$ for some $x \in G$. Now, since x^2 generates a subgroup of order 7, $\langle x^2 \rangle = P$ is a Sylow-7 subgroup of G, and thus has a normalizer $N_G(P)$ with order 21. But, since $\langle x \rangle$ also normalizes P, we have

$$\langle x \rangle \leq N_G(P)$$

which cannot be, since $|x| = 14$ does not divide $|N_G(P)| = 21$.

Suppose, then, that there is some $x \in G$ for which $x^{21} = e$. In particular, the element x^7 generates a Sylow-3 subgroup, and all of $\langle x \rangle$ normalizes it. But this contradicts part v (proven later without using this result), so such an x cannot exist. $\square$

Part iv. Show that the normalizer of each Sylow-7 subgroup contains just 7 Sylow-3 subgroups.

Proof. To begin with, we note that $|N_G(P_7)| = 21$, where P_7 is a Sylow-7 subgroup.

Now, we know that from the perspective of $N_G(P_7)$, we must have for the Sylow-3 subgroup P_3

$$\#(\text{Sylow-3 subgroups in } N_G(P_7)) = [N_G(P_7) : N_G(P_3)]$$

which must divide 21, but must also be at least 3. That is, the number of Sylow-3 subgroups of $N_G(P_7)$ must be either 3, 7, or 21.

Since we also know that the number of Sylow-3 subgroups is congruent to $1 \mod 3$, our only option for the number of Sylow-3 subgroups is 7, as desired. $\square$

Part v. Show that G has 28 Sylow-3 subgroups, and the normalizer of each has order 6.

Proof. We know from the previous result that the number of Sylow-3 subgroups of G is at least 7. We also note that the normalizer of each must be of order at least 3, so the index of the normalizer (equal to the number of Sylow-3 subgroups) is at most $2^3 \times 7 = 56$.

We also know that this number is congruent to $1 \mod 3$, which leaves us with the options $7 + 3n$ for $n \leq 16$. Furthermore, this number must divide the order of the group, which leaves us with the options 7 and 28.

The number of Sylow-3 subgroups of G cannot be 7. To see this, assume that there are only 7 Sylow-3 subgroups. Then, for some Sylow-7 subgroup P_7, we have that all of the Sylow-3 subgroups are in $N_G(P_7)$ by the above argument. This means that for any Sylow-3 subgroup P_3, any $g \in G$, there is some $k \in N_G(P_7)$ such that

$$gP_3g^{-1} = kP_3k^{-1}$$

$$k^{-1}gP_3g^{-1}k = P_3$$

which implies that the element $k^{-1}g = n$ for some $n \in N_G(P_3)$, and $g = kn$, This means that G can be written as the product $N_G(P_3)N_G(P_7)$. However, this contradicts G being simple.

Thus, the number of Sylow-3 subgroups is 28, which implies the index of each normalizer is 28, and the order of each normalizer is $\frac{168}{28} = 6$, as desired. $\square$