# Problem Set 5

Daniel Halmrast

November 29, 2017

## PROBLEM 1

Show that for $(X, \mathscr{T})$ a compact Hausdorff space, any topology $\mathscr{T} \subsetneq \mathscr{T}'$ makes $(X, \mathscr{T}')$ no longer compact Hausdorff. Furthermore, for $\mathscr{T}' \subsetneq \mathscr{T}$, the space $(X, \mathscr{T}')$ is not compact Hausdorff.

*Proof.* For this proof, we will show that for $\mathscr{T}, \mathscr{T}'$ topologies on $X$ with $\mathscr{T} \subset \mathscr{T}'$ and the property that $X$ is compact Hausdorff under both these topologies, then they must be equal.

To do so, we consider the identity function $id : (X, \mathscr{T}') \to (X, \mathscr{T})$, which is continuous since $\mathscr{T} \subset \mathscr{T}'$. Now, for any $C$ closed in $(X, \mathscr{T}')$, since $(X, \mathscr{T}')$ is compact and $id$ is continuous, $id(C) = C$ is compact in $(X, \mathscr{T})$. However, since $(X, \mathscr{T})$ is Hausdorff, $C$ must also be closed. Thus, $id$ is a closed map, and thus a homeomorphism, and the two topologies must be equal. $\square$

## PROBLEM 2

Prove that for compact subsets $A, B$ of $X$, their union $A \cup B$ is compact as well.

*Proof.* Let $\mathscr{O}$ be an open cover of $A \cup B$. In particular, $\mathscr{O}$ covers $A$, and has a finite subset $\mathscr{O}_A$ that covers $A$. Similarly, there is a finite subset $\mathscr{O}_B$ that covers $B$. Their union $\mathscr{O}_A \cup \mathscr{O}_B$ covers $A \cup B$, and is finite, as it is the union of finite sets. It is also a subcover, since both $\mathscr{O}_A$ and $\mathscr{O}_B$ are subsets of $\mathscr{O}$. Thus. $\mathscr{O}$ has a finite subcover, and $A \cup B$ is compact as desired. $\square$

## PROBLEM 3

Suppose $A$ is a subset of a metric space. Show that $A$ is compact implies that $A$ is closed and bounded. Give an example where the converse does not hold.

*Proof.* Suppose $A$ is a compact subset of a metric space $(X, d)$. Since all metric spaces are Hausdorff, it follows immediately that $A$ is closed (since it is a compact subset of a Hausdorff space).

Now, fix $x \in A$, and consider the open cover $\mathscr{O} = \{V_n(x) \mid n \in \mathbb{N}\}$ of balls centered at $x$ with radius $n$. This has a finite subcover, by compactness of $A$. So, let $N$ be the largest radius in the finite subcover. For each $V_m(x)$ in the finite subcover, then, we have that $V_m(x) \subset V_N(x)$. Thus, since the subcover covers $A$, it follows that $V_N(x)$ covers $A$, and thus $A$ is bounded.

For an example where the converse fails, consider the metric space $(\mathbb{R}, d_{LA})$ of the real numbers with the discrete (Los Angeles) metric. The set $\mathbb{R}$ itself is closed and bounded (since everything is at most distance 1 away from any particular point), but is clearly not compact, since it is an uncountable discrete set of points. □

## Problem 4

Show that for $X$ Hausdorff, $A, B$ disjoint compact subsets of $X$, then there exist disjoint open sets $U$ and $V$ containing $A$ and $B$ respectively.

*Proof.* Let $a \in A$, and $b \in B$. Since $A$ and $B$ are disjoint, it must be that $a \neq b$. Thus, there exist disjoint open sets $U_{ab}, V_{ab}$ such that $a \in U_{ab}$ and $b \in V_{ab}$. Now, since $B$ is compact, the open cover $\mathscr{O}_B = \{V_{ab} \mid b \in B\}$ has a finite subcover $\{V_{ab_i}\}_{i=1}^n$. Furthermore, the open set $U_a = \bigcap_{i=1}^n U_{ab_i}$ does not intersect the open set $V_a = \bigcup_{i=1}^n V_{ab_i}$ which covers $B$,

So, consider the open cover $\mathscr{O}_A = \{U_a \mid a \in A\}$. By compactness of $A$, this has a finite subcover $\{U_{a_j}\}_{j=1}^n$. □