

# 1 Preliminaries

**Homework 1.** Prove that  $V^{**} \cong V$  for finite-dimensional vector space  $V$ .

From this, it is clear that  $T_p^*M \otimes T_pM \cong \text{Hom}(T_pM, T_pM)$  for a manifold  $M$ .

Recall the tangent bundle  $TM$  is defined as

$$TM = \coprod_{p \in M} T_pM$$

and a vector field on the manifold  $M$  is simply a section of the tangent bundle projection  $TM \xrightarrow{\pi} M$ . In other words, a vector field is a function  $f : M \rightarrow TM$  such that  $\pi \circ f = \text{id}$ . Requiring the section to be smooth makes it into a smooth vector field.

We can also do the same thing for the cotangent bundle  $T^*M$  to obtain a covector field.

Now, we can take the tensor product of copies of  $TM$  and  $T^*M$  to obtain our tensor bundles, and tensor fields will be sections of these bundles.

Let  $(U, \phi)$  be a smooth chart on  $M$  with coordinate functions  $x^i$ , coordinate vector fields  $\partial_i$ , and coordinate one-forms  $dx^i$ . Recall that  $dx^i$  is defined to be the dual basis to  $\partial_i$ , that is,

$$dx^i(\partial_j) = \delta_j^i$$

Recall also that the exterior derivative of a function  $df$  is defined as

$$df(v) = v(f)$$

and this definition applied to the coordinate functions  $x^i$  (yielding  $dx^i$ ) coincides with the definition above. Note that  $\partial_i$  form a basis for  $T_pM$  and  $dx^i$  form a basis for  $T_p^*M$ . Tensor products of them, then, form a basis for the tensor product space.

**Homework 2.** Prove that, for a vector space  $V$  with basis  $v_i$ , dual basis  $v^i$ , the set

$$\{v^i \otimes v^j \mid 1 \leq i, j \leq n\}$$

forms a basis for  $V^* \otimes V^*$ . Here  $v^i \otimes v^j(u, v) = v^i(u)v^j(v)$ .

## 2 Affine Connections

### 2.1 The Metric

**Definition 2.1.** Let  $M^n$  be a smooth manifold of dimension  $n$ . A Riemannian Metric  $g$  on  $M$  is a rank  $(0, 2)$  tensor (a section of  $T^*M \otimes T^*M$ ) that is symmetric and positive-definite. In other words,  $g$  is a rank  $(0, 2)$  tensor that restricts to an inner product on the tangent space at every point.

We can express  $g$  in local coordinates!

$$g_{ij} = g(\partial_i, \partial_j)$$

or

$$g = g_{ij} dx^i \otimes dx^j$$

**Homework 3.** Show that the two expressions for  $dvol$ , namely

$$\begin{aligned} dvol &= \wedge_i \omega^i \\ dvol &= \sqrt{|g|} dx^n \end{aligned}$$

## 2.2 Integration of Top Degree Differential Forms

Let  $M^n$  be an orientable  $n$ -dimensional manifold, and  $\omega \in \Omega^n(M)$ . Furthermore let  $(U, \phi)$  be a positive coordinate chart. On  $U$  we have that

$$\omega = f dx^1 \wedge \dots \wedge dx^n$$

for some  $f \in C^\infty(M)$ .

Now, let  $K \subset U$  be compact. We define

$$\begin{aligned} \int_K \omega &= \int_{\phi(K)} \phi^{-1*} \omega \\ &= \int_{\phi(K)} f \circ \phi^{-1} \phi^{-1*} dx^1 \wedge \dots \wedge \phi^{-1*} dx^n \\ &= \int_{\phi(K)} f \circ \phi^{-1} dx^1 \wedge \dots \wedge dx^n \end{aligned}$$

where the last integral is just the standard integral in  $\mathbb{R}^n$ .

Is this definition independent of choice of coordinates? Let's check. Let  $(V, \psi)$  be another coordinate chart containing  $K$ . Then, the integral with respect to this coordinate system is

$$\int_K \omega = \int_{\psi(K)} g \circ \psi^{-1} dy^1 \wedge \dots \wedge dy^n$$

for  $g$  defined as

$$\omega = h dy^1 \wedge \dots \wedge dy^n$$

with coordinate functions  $y^i$ . The claim is that these integrals are equal.

Consider the change-of-coordinates map  $\psi \circ \phi^{-1}$  from the  $x^i$  to the  $y^i$  coordinate system. Since  $K$  is in both  $U$  and  $V$ , its image  $\phi(K)$  lies in the domain of  $\psi \circ \phi^{-1}$ .

All that remains is to apply the change of variables to the integrals. Recall that if one has a diffeomorphism  $F : \Omega_1 \rightarrow \Omega_2$  for compact  $\Omega_i$ , one has that

$$\int_{\Omega_2} f dy^1 \dots dy^n = \int_{\Omega_1} f \circ F |J_F| dx^1 \dots dx^n$$

where  $|J_F|$  is the determinant of the Jacobian matrix for  $F$ .

**Homework 4.** Check that the two integrals claimed to be equal are actually equal.

Now we have an idea for how to integrate  $\omega$  on a single chart, let's extend this. Let  $(\eta_i, U_i)$  be a partition of unity of  $M$  where each  $U_i$  is contained in a single chart on  $M$ . Then,

$$\omega = \sum \omega \eta_i$$

and we can integrate by extending linearly

$$\int_K \omega = \sum \int_K \omega \eta_i$$

where the right hand side has integrals over functions supported in a single chart, and is well-defined. But is this independent of the choice of partition of unity? Short answer: yes (Optional homework).

## 2.3 Integration on an Orientable Smooth Riemannian Manifold

Recall that a Riemannian manifold has a volume form

$$dvol = \sqrt{|g_{ij}|} dx^1 \wedge \dots \wedge dx^n$$

which is obtained by taking an orthonormal frame  $e_i$  and considering the dual frame  $\omega^i$  defined as

$$\omega^i e_j = \delta_j^i$$

and letting

$$dvol = \omega^1 \wedge \dots \wedge \omega^n$$

This construction is independent of choice of orthonormal frame.

*Proof.* Let  $\epsilon_i$  be another orthonormal frame with dual frame  $\alpha^i$ . Then,  $\epsilon_i = a_i^j e_j$  and  $\alpha^i = b_j^i \omega^j$  and so

$$\begin{aligned} \alpha^1 \wedge \dots \wedge \alpha^n &= b_{j_1}^1 \omega^{j_1} \wedge \dots \wedge b_{j_n}^n \omega^{j_n} \\ &= \sum_{\sigma \in S_n} b_{\sigma(1)}^1 \dots b_{\sigma(n)}^n \text{sgn}(\sigma) \omega^1 \wedge \dots \wedge \omega^n \\ &= |b| \omega^1 \wedge \dots \wedge \omega^n \\ &= \omega^1 \wedge \dots \wedge \omega^n \end{aligned}$$

where the last line was obtained from the fact that  $b$  is the orthogonal change-of-basis matrix from  $e$  to  $\epsilon$ .  $\square$

Then, we define

$$\text{Vol}(K) = \int_K dvol$$

## 2.4 Integrating a Non-Orientable Manifold

How do we integrate a manifold that is not orientable? The previous construction was coordinate-independent only because we chose positive oriented coordinates...

Let  $K \subset U$  be a compact set in a single chart on the manifold. Then, we can define

$$\text{Vol}(K) = \int_K \sqrt{|g_{ij}|} dx^n$$

Now, this is independent of choice of coordinates, since if  $K$  lies in the intersection of two charts, we can use the Jacobian change-of-variables formula to show that the two calculations of the volume are equal.

The problem is that  $dy^n = \det(J_{x \rightarrow y}) dx^n$  depends also on the sign of the determinant of the Jacobian.

On an orientable Manifold, we have  $dvol \in \Omega^n(M)$  (i.e.  $dvol \in \Gamma(\Lambda^n T^*M)$ ), and in fact a manifold is orientable if and only if it admits a nowhere-vanishing top degree form.

**Homework 5.** *Prove that a manifold is orientable if and only if it admits a nowhere-vanishing top degree form.*

## 2.5 Existence of Metrics

**Theorem 1.** *On each smooth manifold  $M$  there exists smooth Riemannian metrics.*

*Proof.* Let  $(U_i, \phi_i)$  be an atlas of  $M$ , and  $\eta_j$  be a partition of unity subordinate to it. Then, on each  $U_i$  we have a smooth Riemannian metric given by

$$g_i = dx_i^1 \otimes dx_i^1 + \dots + dx_i^n \otimes dx_i^n$$

Then, we define

$$g = \sum \eta_i g_i$$

□

## 2.6 Lower-Dimensional Integration on Riemannian Manifolds

Suppose we want to find the arc length of a curve  $\gamma : I \rightarrow M$ . We can define the length of  $\gamma$  to be

$$L(\gamma) = \int_I |\gamma'| dt$$

where  $|\gamma'|$  is the length of the tangent vector with respect to the metric.

**Definition 2.2.** *Let  $p, q \in M$  be points in a connected manifold  $M$ . We define the distance between  $p$  and  $q$  to be*

$$\inf_{\gamma \in C^\infty(I, M)} \{L(\gamma) \mid \gamma(0) = p, \gamma(1) = q\}$$

Note that we can relax the condition that  $\gamma$  be smooth to  $\gamma$  being only piecewise smooth, since any piecewise smooth curve is uniformly approximated by smooth curves.

This distance, denoted  $d(p, q)$ , turns out to metrize the manifold.

**Theorem 2.**  $d(\cdot, \cdot)$  is a metric on  $M$ , and the metric topology generated by  $d$  coincides with the topology of  $M$ .

*Proof.* First, we show that  $d$  is a metric. Symmetry of  $d$  should be obvious, since  $L(\gamma) = L(-\gamma)$  and the curves from  $p$  to  $q$  directly coincide with curves from  $q$  to  $p$  via the map  $\gamma \mapsto -\gamma$ .

Now,  $d$  is also clearly positive-definite, since the length functional is positive-definite.

It should also be clear that  $d(p, q) = 0$  if and only if  $p = q$ . Clearly, if  $p = q$ , then the constant curve  $\gamma(t) = p$  has length zero, so  $d(p, p) = 0$ . Now, if  $p \neq q$ , then since  $M$  is Hausdorff, they must have positive distance from each other. This follows from the second claim that the topologies coincide.

The triangle inequality follows from the fact that given three points  $p, q, m$ , the curve going from  $p$  to  $m$ , and then from  $m$  to  $q$ , is a curve from  $p$  to  $q$ , and so  $d(p, q) \leq d(p, m) + d(m, q)$  (since it is part of the infimum).

Now, we show that the topologies coincide... □

**Homework 6.** Show that the topology on  $M$  coincides with the metric topology from  $d$ .

**Homework 7.** Show that for  $(\mathbb{R}^n, g_{\mathbb{R}^n})$ ,  $d(p, q) = \|p - q\|$ .

## 2.7 Connections on a Riemannian Manifold

Let  $(M^n, g)$  be a smooth Riemannian Manifold,  $X \in \mathfrak{X}(M)$ . We wish to take the derivative of this vector field. Recall that the Lie derivative allows us to take the derivative of  $X$  along another vector field  $Y$ , however this operation is not linear with respect to the module of smooth functions. That is,

$$L_X(fY) = fL_XY + (Xf)Y$$

Also, the Lie derivative is not defined for a single point, since it takes into account the motion of  $X$  around any particular point.

What we really want is  $\nabla_v$ , the covariant derivative.

**Definition 2.3.** A Connection is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \quad (X, Y) \mapsto \nabla_X Y$$

such that  $\nabla_X Y$  is linear in both  $X$  with respect to the module  $C^\infty(M)$ , scalar linear in  $Y$  and satisfies the Leibniz rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

**Definition 2.4.** A Connection is the following: for each  $p \in M$ , we have a map  $\nabla : T_p M \times C^\infty(TM) \rightarrow T_p M$  that sends  $(v, Y)$  to  $\nabla_v Y$ . Such that  $\nabla$  is linear in  $v$ , linear in  $Y$ , and satisfies the Leibniz rule

$$\nabla_v(fY) = (vf)Y_p + f(p)\nabla_v(Y)$$

and, for all  $X, Y$  in  $\mathfrak{X}(M)$ ,  $\nabla_X Y \in \mathfrak{X}(M)$  where

$$(\nabla_X Y)_p = \nabla_{X_p} Y$$

Interpreting  $\nabla$  as an operator from  $\mathfrak{X}(M)$ , we see that it actually adds a covariant index. That is,

$$\nabla_\mu v^\nu$$

takes in a vector, and outputs a  $(1, 1)$  tensor.

**Example.** The directional derivative in  $\mathbb{R}^n$  yields a connection. For  $v \in T_x \mathbb{R}^n$ , and  $X$  a smooth vector field on  $\mathbb{R}^n$ , we have

$$D_{(x,v)} X = \partial_t X(x + tv)|_{t=0}$$

and we define  $\nabla_v X = (x, D_{(x,v)} X)$

Now, on  $TM$  for a general Riemannian manifold, there are many different connections. However, given a metric, we have a unique metric compatible, torsion-free connection called the *Levi-Civita Connection*.

**Theorem 3.** For  $M$  a smooth Riemannian manifold, then there exists a unique connection  $\nabla$  on  $TM$  such that

- $\nabla$  is symmetric i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

(The Christoffel symbols are symmetric in lower indices)

- $\nabla$  is metric-compatible. That is,

$$\nabla g = 0$$

or

$$\nabla_\gamma g_{\mu\nu} v^\nu = g_{\mu\nu} \nabla_\gamma v^\nu$$

or

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

*Proof.* See Carroll (p.99) for an explicit construction of the torsion free, metric compatible connection in terms of the components of  $g$ . The formula is

$$\Gamma_{\mu\nu}^\gamma = \frac{1}{2} g^{\gamma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

□

**Homework 8.** *Prove that the resulting connection is indeed a connection.*

Now to prove that the two definitions of a connection coincide.

From the local to the global definition is trivial, so we wish to prove that we can localize the global definition.

*Proof.* Consider a smooth connection  $\nabla$  on  $M$ . Let  $U \subset M$  be open, and  $Y$  a smooth vector field on  $M$ ,  $X$  a smooth vector field on  $U$ .

Now, for  $p \in U$ , choose a smooth function  $\eta$  on  $M$  such that  $\eta = 1$  in a neighborhood  $V_1$  of  $p$ , and  $\eta = 0$  on  $M \setminus V_2$  with  $\overline{V_1} \subset V_2$ ,  $\overline{V_2} \subset U$  and  $\overline{V_i}$  compact.

**Homework 9.** *Construct a one-dimensional smooth bump function on  $\mathbb{R}$*

Now, set  $\tilde{X} = \eta X$ , which is defined globally on  $M$ . We can now define

$$\nabla_X Y|_{V_1} = \nabla_{\tilde{X}} Y|_{V_1}$$

and we can do this for every point  $p \in M$ . Now, we must show that such a construction is unique.

Suppose instead that we chose a different  $V'_1, V'_2, \eta'$ . We have a new globally-defined vector field  $X' = \eta' X$ , and we wish to show that  $\nabla_{\tilde{X}} Y = \nabla_{X'} Y$  at  $p$ .

So, we construct

$$\nabla_{\tilde{X}}(Y) - \nabla_{X'}(Y) = \nabla_{\tilde{X} - X'} Y$$

Now, we know that  $\tilde{X} - X'$  is zero at (and nearby)  $p$ , so

$$\tilde{X} - X' = \zeta(\tilde{X} - X')$$

So, we have that

$$\begin{aligned} \nabla_{\tilde{X} - X'} Y &= \nabla_{\zeta(\tilde{X} - X')} Y \\ &= \zeta \nabla_{\tilde{X} - X'} Y \\ &= 0 \end{aligned}$$

and so they agree around  $p$ .

Next, consider  $p \in M$ , with  $Y$  a smooth vector field. Choose a coordinate chart  $(U, \phi)$  around  $p$ , with  $v \in T_p M$ ,  $v = v^i \partial_i$ .

Then, we set  $\nabla_v Y = \nabla_{v^i \partial_i} Y = v^i \nabla_{\partial_i} Y$ , where we have already defined what  $\nabla_{\partial_i}$  should be, since  $\partial_i$  is a locally defined vector field.

Now, we need to show this is independent of coordinate charts. Let  $(V, \psi)$  be another coordinate chart, with  $v = v^j \partial'_j$  for coordinate field  $\partial'_i$ . The claim is that

$$v^i \nabla_{\partial_i} Y = v^j \nabla_{\partial'_j} Y$$

which is easily verified, since  $J(\partial \rightarrow \partial') \nabla_{\partial_i} = \nabla_{\partial'_j}$ , and so

$$v^j \nabla_{\partial'_j} = v^j \nabla_{J(\partial \rightarrow \partial')^j_i \partial_i}$$

but  $v^i = J^i_j b^j$ , and so they agree. □

## 2.8 The Levi-Cevita Connection

Recall that we have a unique torsion-free, metric compatible connection  $\nabla$  for any Riemannian manifold. We wish to localize this  $\nabla$  further.

**Definition 2.5.** Let  $\gamma$  be a smooth curve in  $M$ . A vector field  $X$  along  $\gamma$  is an assignment  $X : I \rightarrow TM$  with  $X(t) \in T_{\gamma(t)}M$  where  $X$  is called smooth if its coordinate decomposition

$$X = \xi^i(t)\partial_i$$

is smooth in each component.

**Definition 2.6.**  $\nabla_{\partial_t}X$  is define along  $\gamma$  as follows: Let  $I_{t_0}$  be an open interval around  $t_0$ , which maps into chart  $(U, \phi)$ . Then,

$$\begin{aligned}\nabla_{\partial_t}X &= \nabla_{\partial_t}\xi^i(t)\partial_i \\ &= \partial_t\xi^i(t)\partial_i + \xi^i(t)\nabla_{\partial_t}\partial_i \\ &= \partial_t\xi^i(t)\partial_i + \xi^i(t)\nabla_{\partial_t\gamma}\partial_i\end{aligned}$$

which is already defined.

The second term in this expansion turns into

$$\xi^i(t)\nabla_{\partial_t\gamma}\partial_i = \xi^i\partial_t x^j \nabla_{\partial_j}\partial_i$$

and we define

$$\Gamma_{ij}^k \partial_k = \nabla_{\partial_j}\partial_i$$

Where  $\Gamma_{ij}^k$  is the Christoffel symbol (of the first kind) for the connection.

**Homework 10.** Show that for the Levi-Civita connection,

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l})$$

## 2.9 The Connection in Local Coordinates

**Definition 2.7.** The connection forms of the connection  $\omega_i^j$  associated with an orthonormal frame  $e_i$  is defined as

$$\nabla e_i = \omega_i^j \otimes e_j$$

Knowing that the frame is orthonormal and the connection is metric compatible, we get

$$\begin{aligned}\langle e_i, e_j \rangle &= \delta_{ij} \\ \langle \nabla_X e_i, e_j \rangle + \langle e_i, \nabla_X e_j \rangle &= 0 \\ \langle \omega_i^k(X)e_k, e_j \rangle + \langle e_i, \omega_j^l(X)e_l \rangle &= 0 \\ \omega_i^j(X) + \omega_j^i(X) &= 0\end{aligned}$$

and so  $\omega_j^i$  is antisymmetric.



**Theorem 4.** *The following holds for the connection forms:*

- $\omega$  is antisymmetric
- $d\omega^i = \omega_j^i \wedge \omega^j$

To prove this, we can use the identity

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$$

for one-forms  $\alpha$ .

## 2.10 Parallel Transport

Let  $X$  be a vector field on  $M$  along  $\gamma$ .

**Definition 2.8.**  $X$  is called parallel if

$$\nabla_{\partial_t} X = 0$$

**Theorem 5.** *For each  $v \in T_{\gamma(0)}M$ , there is a unique solution to the initial value problem*

$$\nabla_{\partial_t} X = 0$$

$$X(0) = v$$

*Proof.* Let  $(U, \phi)$  be a coordinate chart around  $\gamma(0)$ . Then, in  $U$ ,

$$\nabla_{\partial_t} X = 0$$

is the same as

$$\partial_t \xi^k + \Gamma_{ij}^k \xi^i \partial_t x^j$$

which is a first order linear ODE with smooth coefficients, and so it has unique solutions for the initial value  $X(0) = v$ , or  $\xi^i(0) = v^i$ .  $\square$

Now, since the ODE is linear, there is a linear map between initial values and solutions, that is we have a linear map from  $T_{\gamma(0)}M$  to  $T_{\gamma(1)}M$  by evaluating  $X$  at 1. This map is the parallel transport map, and it is invertible by running the curve backwards. Thus, this map is an isomorphism. Even better...

**Proposition 1.** *The parallel transport map is an isometry.*

**Homework 11.** *Prove that*

$$\partial_t g(X, Y) = g(\nabla_{\partial_t} X, Y) + g(X, \nabla_{\partial_t} Y)$$

**Definition 2.9.** *The Holonomy Group of a Riemannian manifold  $(M, g)$  based at a point  $p \in M$ , denoted  $H_{g,p}$  is defined to be*

$$H_{g,p} = \{P_\gamma : \gamma \text{ a smooth loop at } p\}$$

where  $P_\gamma$  is the parallel transport along  $\gamma$ , with the group structure of loop concatenation.

**Definition 2.10.** *The Reduced Holonomy Group is the subgroup of  $H_{g,p}$  consisting of parallel propagators whose loops are homotopic to the identity.*