Problem Set 5

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Problem 1

Prove the linearity of the general Lebesgue integral for complex-valued functions.

Proof. To begin with, we prove a sequence of lemmas concerning the linearity of easier functions.

Lemma 1 (Linearity of Positive Real Functions)**.** For f, g positive, real-valued measurable functions from a measure space (Ω, μ), the integral is linear with respect to pointwise addition and positive scalar multiplication. That is, for $\alpha, \beta \geq 0$,

$$\int_\Omega \alpha f + \beta g d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu$$

Proof. This has been proven in the notes, and will not be replicated here. □

Lemma 2. For f a positive real-valued measurable function from a measure space (Ω, μ), the integral respects complex scalar multiplication. That is, for $\alpha \in \mathbb{C}$,

$$\int_\Omega \alpha f d\mu = \alpha \int_\Omega f d\mu$$

Proof. We note first that for $\alpha = a + ib$, we have

$$\begin{aligned} \int_\Omega \alpha f d\mu &= \int_\Omega (a + ib) f d\mu \\ &= \int_\Omega (af + ibf) d\mu \\ &= \int_\Omega af d\mu + i \int_\Omega bf d\mu \end{aligned}$$

So, it suffices to show that the integral of a real-valued function respects real scalar multiplication. Without loss of generality, let α be real.

This lemma has already been proven for $\alpha \geq 0$. So, suppose $\alpha < 0$. In particular, $-\alpha > 0$, and thus by straightforward application of the definition of the Lebesgue integral along with lemma 1, we have

$$\begin{aligned}\int_\Omega \alpha f d\mu &= \int_\Omega -(-\alpha f) d\mu \\ &= -\int_\Omega -\alpha f d\mu \\ &= -(-\alpha) \int_\Omega f d\mu \\ &= \alpha \int_\Omega f d\mu\end{aligned}$$

which is the desired result. $\square$

Lemma 3. For f_1, f_2, g_1, g_2 positive real-valued measurable functions from a measure space (Ω, μ) such that $f_1 - f_2 = g_1 - g_2$,

$$\int_\Omega f_1 d\mu - \int_\Omega f_2 d\mu = \int_\Omega g_1 d\mu - \int_\Omega g_2 d\mu$$

Proof. This follows immediately from integrating $f_1 - f_2$ and $g_1 - g_2$, and applying lemma 2 and lemma 1. $\square$

We are now ready to begin the proof of the linearity of the Lebesgue integral.

To do this proof, we will consider two cases. The first case will handle additivity, and the second case will handle scalar multiplication.

For the first case, let $f = u_f + iv_f$, and $g = u_g + iv_g$ be complex-valued measurable functions. Now, it follows immediately that

$$\begin{aligned}\int_\Omega (f+g) d\mu &= \int_\Omega ((u_f + iv_f) + (u_g + iv_g)) d\mu \\ &= \int_\Omega (u_f + u_g) d\mu + i \int_\Omega (v_f + v_g) d\mu\end{aligned}$$

for real-valued functions u_f, v_f, u_g, v_g. So, it suffices to show the integral is additive for real-valued functions, and the complex case will follow immediately.

So, assume without loss of generality that f and g are real. It follows directly that, since $f = f^+ - f^-$ and $g = g^+ - g^-$, the equalities

$$\begin{aligned}f + g &= (f+g)^+ - (f+g)^- \\ &= f^+ + g^+ - f^- - g^-\end{aligned}$$

hold. Thus, lemma 3 guarantees that

$$\int_\Omega (f+g)^+ d\mu - \int_\Omega (f+g)^- d\mu = \int_\Omega (f^+ + g^+) d\mu - \int_\Omega (f^- + g^-) d\mu$$

Putting it all together, we have

$$\begin{aligned}\int_\Omega (f+g) d\mu &= \int_\Omega (f+g)^+ d\mu - \int_\Omega (f+g)^- d\mu \\ &= \int_\Omega (f^+ + g^+) d\mu - \int_\Omega (f^- + g^-) d\mu \\ &= \int_\Omega f^+ d\mu + \int_\Omega g^+ d\mu - \int_\Omega f^- d\mu - \int_\Omega g^- d\mu \\ &= \int_\Omega f d\mu + \int_\Omega g d\mu\end{aligned}$$

where the equality from the second to the third line is done using lemma 1. Thus, the integral respects additivity of complex-valued functions.

For the second case, let f be a complex-valued measurable function from a measure space (Ω, μ), and let $\alpha \in \mathbb{C}$. We wish to evaluate

$$\int_\Omega \alpha f d\mu = \int_\Omega \alpha \Re(f) d\mu + i \int_\Omega \alpha \Im(f) d\mu$$

for real-valued functions $\Re(f)$ and $\Im(f)$.

By lemma 2, we have

$$\begin{aligned}\int_\Omega \alpha f d\mu &= \int_\Omega \alpha \Re(f) d\mu + i \int_\Omega \alpha \Im(f) d\mu \\ &= \alpha \left(\int_\Omega \Re(f) d\mu + i \int_\Omega \Im(f) d\mu \right) \\ &= \alpha \int_\Omega f d\mu\end{aligned}$$

Thus, the integral is linear, as desired. □

Problem 2

Part 1

Prove the statement in part 3.

Proof. In particular, we have ξ_k a sequence of partition refinements for which

$$\phi_k = \sum_{I_j} \sup_{x \in I_j} f(x)\chi_{I_j}$$

and

$$\psi_k = \sum_{I_j} \inf_{x \in I_j} f(x)\chi_{I_j}$$

(where I_j is the j^{th} interval in the partition ξ_k) converge to the upper and lower sums (respectively) with respect to the partition ξ_k as $k \to \infty$. We also define $P = \bigcup_{k=1}^{\infty} \xi_k$ to be the set of all endpoints of the partitions ξ_k.

Now, we need to show that f continuous at $x_0 \iff \phi(x_0) = f(x_0) = \psi(x_0)$ where $\phi_k \to \phi$ and $\psi_k \to \psi$.

($\Longrightarrow$) To see the first implication, we observe that f being continuous at x_0 means that

$$\lim_{\delta \to 0} \sup_{x \in V_\delta(x_0)} f(x) - \inf_{x \in (V_\delta(x_0))} f(x) = 0$$

Now, fix $\epsilon > 0$. Since f is continuous, we can choose some δ such that on the interval $(x_0 - \delta, x_0 + \delta)$, $\sup f - \inf f < \epsilon$. Furthermore, we can choose some k for which $x_0 \in I_j$ for some interval in the partition ξ_k, and such that $I_j \subseteq (x_0 - \delta, x_0 + \delta)$.

It follows immediately, then, that $\phi_k(x_0) - \psi_k(x_0) < \epsilon$, and thus as k goes to infinity, we have $\phi(x_0) = \psi(x_0)$. And since $\psi_k \leq f \leq \phi_k \; \forall k$, we have $f(x_0) = \psi(x_0) = \phi(x_0)$ as desired.

($\Longleftarrow$) For the other direction, assume $f(x_0) = \psi(x_0) = \phi(x_0)$. Now, fix $\epsilon > 0$. In particular, we can choose an $N > 0$ such that for all $k > N$, $\phi(x_0) - \psi(x_0) < \epsilon$.

By the definition of ϕ_k and ψ_k, this means that

$$\sup_{I_j} f - \inf_{I_j} < \epsilon$$

where I_j is the interval of ξ_k that contains x_0. So, let V be a δ-neighborhood of x_0 contained in I_j. Then, the variation of f on V is bounded by ϵ, and f is continuous at x_0. $\square$

Part 2

Prove the statement in part 5.

Proof. In particular, we need to prove that for a bounded function f with a set of discontinuities E with measure zero, f is both Riemann-integrable and Lebesgue-integrable, and that the integrals coincide.

Notice that the previous step in the notes establishes that for f a bounded Riemann-integrable function, f is also Lebesgue-integrable, and the integrals coincide. Thus, we only have to show that f is Riemann-integrable.

To do so, let M be such that $|f| < M$, and let E be the set of discontinuities of f. Since $\lambda^1(E) = 0$, we can choose a set of disjoint open sets $\{U_i\}$ that cover E with total measure $\lambda^1(\bigcup_i U_i) < \frac{\epsilon}{4M}$ (This follows from the Borel-regularity of the Lebesgue measure). In particular, since the region of integration is compact, we can make $\{U_i\}$ finite.

Now, let's attempt to estimate the difference in the upper and lower sums of $\int f$.

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) = \sum_{i=1}^{n} (\sup_{x\in I_i} f - \inf_{x\in I_i} f)\Delta x_i$$

where the sum is being taken over the subintervals I_i of the partition P.

Now, let's split the sum into the intervals that do not intersect E, and the intervals that do. In particular, we take as our initial partition the set of intervals U_i, along with the intervals in between them, so that the U_i part of the partition contains every element of E, and the other part does not.

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) = \sum_{i=1}^{m} (\sup_{x\in I_i} f - \inf_{x\in I_i} f)\Delta x_i + \sum_{i=1}^{m'} (\sup_{x\in U_i} f - \inf_{x\in U_i} f)\Delta x_i$$

Since f is continuous in the first sum, we can refine the partition so that the upper and lower sums are within $\frac{\epsilon}{2}$ of each other. Then, using the fact that the measure of the union of the $\{U_i\}$ is less than $\frac{\epsilon}{4M}$, we obtain the bound

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) \leq \frac{\epsilon}{2} + 2M\frac{\epsilon}{4M} = \epsilon$$

Thus, the upper and lower sums converge to each other, and f is Riemann-integrable as desired.

□

Problem 3

Prove that the support of f for $f \in L^1(\Omega, \mu)$ is σ-finite.

Proof. We shall observe that the sets $E_n = \{f > \frac{1}{n}\}$ are of finite measure, and union to the support of f.

The fact that $\bigcup_n E_n = \operatorname{supp}(f)$ is immediate. All that remains is to show that the measure of each of these is finite.

Suppose for a contradiction that for some n, E_n has infinite measure. Then, we can establish a lower bound for the integral

$$\int_\Omega |f| d\mu \geq \int_{E_n} \frac{1}{n} d\mu$$

but since $\mu(E_n) = \infty$, we have that

$$\int_{E_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(E_n) = \infty$$

which cannot be a lower bound for $\int |f|$, since $f \in L^1$.

Thus, each E_n is finite, and the support of f is σ-finite as desired. □

Problem 4

Show that if $f \in L^1$, then f is finite almost everywhere. Provide an example where the converse fails.

Proof. Suppose for a contradiction that there was a set E of positive measure for which $f = \infty$ on E. Then,

$$\begin{aligned}\int_\Omega |f| d\mu &= \int_E |f| d\mu + \int_{E^c} |f| d\mu \\ &= \infty\mu(E) + \int_{E^c} |f| d\mu \\ &= \infty\end{aligned}$$

which is a clear contradiction.

However, the function $f(x) = 1$ is finite everywhere, but is not in L^1. Hence, the contrapositive fails. □

Problem 5

Compute

$$\lim_{n\to\infty}\left(\int_0^n (1+\frac{x}{n})^n \exp(-2x)dx\right)$$

Proof. For this proof, we will first show that the doubly-indexed limit

$$\lim_{n\to\infty}\lim_{i\to\infty}\left(\int_0^n (1+\frac{x}{i})^i \exp(-2x)dx\right)$$

converges. This follows by first observing that inside the integral, we have a sequence of functions approaching $\exp(x)\exp(-2x)=\exp(-x)$, which is clearly (eventually) dominated by $\frac{1}{x^2}$. Thus, the dominated convergence theorem kicks in, and

$$\begin{aligned}
\lim_{n\to\infty}\lim_{i\to\infty}\left(\int_0^n (1+\frac{x}{i})^i \exp(-2x)dx\right) &= \lim_{n\to\infty}\left(\int_0^n \lim_{i\to\infty}(1+\frac{x}{i})^i \exp(-2x)dx\right)\\
&= \lim_{n\to\infty}\left(\int_0^n \exp(-x)dx\right)\\
&= \lim_{n\to\infty}(1-\exp(-n))\\
&= 1
\end{aligned}$$

Now, since this doubly-indexed limit converges, it must be the case that the singly-indexed limit converges to the same thing. To see this, we note that any doubly-indexed sequence (n,i) has the diagonal (n,n) as a subsequence. Thus, since every doubly-indexed sequence converges, every singly-indexed sequence converges, so the limit converges as well. □

Problem 6

Suppose $f_n \geq 0$ is measurable for all $n \in \mathbb{N}$. Furthermore, assume that $f_1 \leq f_2 \leq \ldots \leq 0$, and $f_n(x) \to f(x)$ for each $x \in \Omega$, and that $f_1 \in L^1$. Prove that

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$$

and show that if the condition $f_1 \in L^1$ is dropped, that the theorem does not hold in general.

Proof. We note that every f_n is dominated by $f_1 \in L^1$, so it follows immediately from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$$

as desired.

If one drops the condition that $f_1 \in L^1$, then the sequence $f_n = \frac{1}{n}$ does not satisfy this theorem.

In particular,

$$\int_\mathbb{R} f_n d\mu = \frac{1}{n}\mu(\mathbb{R}) = \infty$$

but

$$f_n(x) \to 0 \ \forall x$$

and thus

$$\int_\mathbb{R} \lim_{n\to\infty} f_n d\mu = \int_\mathbb{R} 0 d\mu = 0$$

which does not agree with $\lim_{n\to\infty} \int_\mathbb{R} f_n d\mu = \infty$. □