Homework 2

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PROBLEM 1

Let M be a complete Riemannian manifold with non-positive sectional curvature. Prove that

$$|(d\exp_p)_v(w)| \geq |w|$$

for all $p \in M$, $v \in T_pM$, and $w \in T_v(T_pM)$.

Proof. We compare M to Euclidean space $\mathbb{R}^n$, noting that $K = K_M \leq \tilde{K} = K_{\mathbb{R}^n} = 0$. Now, let $\gamma(t) = \exp_p(tv)$ be the geodesic generated by v in M, and observe that

$$(d\exp_p)_{tv}(tw)$$

is a Jacobi field along γ (in fact, this is the first Jacobi field Do Carmo studies in chapter 5). Since M is complete, such a field is defined, as $\exp_p(tv)$ is defined.

Now, in $\mathbb{R}^n$, we construct $\tilde{\gamma}(t) = \exp_0(tv) = tv$ and note that

$$\tilde{J}(t) = (d\exp_0)_{tv}(tw) = tw$$

Now, clearly

$$\|\gamma'(t)\| = \|\tilde{\gamma}'(t)\| = v$$

and

$$J(0) = \tilde{J}(0) = 0$$

and

$$g(J'(0), \gamma'(0)) = g(\tilde{J}'(0), \tilde{\gamma}'(0)) = g(w, v)$$

and

$$\|J'(0)\| = \|\tilde{J}'(0)\| = \|w\|$$

and since M has non-positive sectional curvature, γ has no conjugate points. Thus, the hypotheses for the Rauch comparison theorem are satisfied, and immediately we know that

$$|\tilde{J}(t)| \leq |J(t)|$$

plugging in $t = 1$, we observe that

$$|(d\exp_p)_v(w)| \geq |w|$$

as desired. $\square$

Problem 2

Let

$$f''(t) + K(t)f(t) = 0, f(0) = 0, t \in [0, \ell]$$
$$\tilde{f}''(t) + \tilde{K}(t)\tilde{f}(t) = 0, \tilde{f}(0) = 0, t \in [0, \ell]$$

be two ODEs. Suppose $\tilde{K}(t) \geq K(t)$ for $t \in [0, \ell]$ and that $f'(0) = \tilde{f}'(0) = 1$.

Part A

Show that for all $t \in [0, \ell]$,

$$\begin{aligned} 0 &= \int_0^t \left\{ \tilde{f}(f'' + Kf) - f(\tilde{f}'' + \tilde{K}\tilde{f}) \right\} dt \\ &= [\tilde{f}f' - f\tilde{f}']_0^t + \int_0^t (K - \tilde{K}) f\tilde{f} dt \end{aligned}$$

and conclude that the first zero of f does not occur before the first zero of $\tilde{f}$.

Proof. The first equality follows immediately from the differential equation. That is,

$$\int_0^t \left\{ \tilde{f}(f'' + Kf) - f(\tilde{f}'' + \tilde{K}\tilde{f}) \right\} dt = \int_0^t \tilde{f}(0) - f(0) dt = 0$$

for the second equality, we integrate by parts. Noticing that

$$\int_0^t \tilde{f}f'' dt = [\tilde{f}f']_0^t - \int_0^t \tilde{f}'f' dt$$

we see that

$$\begin{aligned} &\int_0^t \left\{ \tilde{f}(f'' + Kf) - f(\tilde{f}'' + \tilde{K}\tilde{f}) \right\} dt \\ &\qquad = \int_0^t \tilde{f}f'' dt - \int_0^t f\tilde{f}'' dt + \int_0^t K\tilde{f}f dt - \int_0^t \tilde{K}\tilde{f}f dt \\ &\qquad = [\tilde{f}f']_0^t - \int_0^t \tilde{f}'f' dt - [f\tilde{f}']_0^t + \int_0^t \tilde{f}'f' dt + \int_0^t (K - \tilde{K}) f\tilde{f} dt \\ &\qquad = [\tilde{f}f' - f\tilde{f}']_0^t + \int_0^t (K - \tilde{K}) f\tilde{f} dt \end{aligned}$$

as desired.

Now, we know that $f(t) > 0$ on a neighborhood of zero. Let t_0 be the first zero of $\tilde{f}$ (so $\tilde{f}(t) > 0$ for $t < t_0$). Assume for contradiction that for some $t_1 < t_0$, $f(t_1) = 0$. Furthermore, suppose this is the first zero (that is $f(t) > 0$ for $t < t_1$). Then, $f'(t_1) < 0$, $\tilde{f}(t_1) > 0$, and so

$$\begin{aligned} \left[\tilde{f}f' - f\tilde{f}' \right]_0^{t_1} + \int_0^{t_1} (K - \tilde{K}) f\tilde{f} dt &= \tilde{f}(t_1)f'(t_1) - f(t_1)\tilde{f}'(t_1) + \int_0^{t_1} (K - \tilde{K} f\tilde{f} dt \\ &= \tilde{f}(t_1)f'(t_1) + \int_0^{t_1} (K - \tilde{K} f\tilde{f} dt \end{aligned}$$

now $\tilde{f}(t_1) > 0$, $f'(t_1) < 0$, $(K - \tilde{K}) < 0$ and $f(t)\tilde{f}(t) > 0$ for $t < t_1$. Thus,

$$\tilde{f}(t_1)f'(t_1) + \int_0^{t_1} (K - \tilde{K} f\tilde{f} dt < 0$$

a contradiction. $\square$

Part B

Suppose $\tilde{f}(t) > 0$ on $(0, \ell]$. Show that $f(t) \geq \tilde{f}(t)$ for $t \in [0, \ell]$, and equality holds for $t = t_1$ if and only if $K(t) = \tilde{K}(t)$ on $[0, t_1]$.

Proof. From the first equality in part a, and that $\tilde{f}(t), f(t) > 0$, we see that in order for

$$0 = [\tilde{f}f' - f\tilde{f}']_0^t + \int_0^t (K - \tilde{K}) f\tilde{f} dt$$

is satisfied only when

$$[\tilde{f}f' - f\tilde{f}']_0^t = \tilde{f}(t)f'(t) - f(t)\tilde{f}'(t) > 0$$

or,

$$\frac{f'}{f} \geq \frac{\tilde{f}'}{\tilde{f}}$$

In other words,

$$(\log f)' \geq (\log \tilde{f})'$$

Integrating this from t_0 to t, we see that

$$\log f(t) - \log f(t_0) \geq \log \tilde{f}(t) - \log \tilde{f}(t_0)$$

which implies that

$$\frac{f(t)}{\tilde{f}(t)} \geq \frac{f(t_0)}{\tilde{f}(t_0)}$$

taking the limit as $t_0 \to 0$ and noting that $f(0) = \tilde{f}(0) = 0$ (and applying l'Hopital's rule), we see that

$$\frac{f(t)}{\tilde{f}(t)} \geq \frac{f'(0)}{\tilde{f}'(0)} = 1$$

and the desired result is achieved. $\square$

Problem 3

Prove that if M is complete and the sectional curvature satisfies $K \geq \delta > 0$, then M is compact and $\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{\delta}}$.

Proof. We appeal to (half of) proposition 2.4 in chapter 10 of Do Carmo, which states that if M is such that the sectional curvature K satisfies

$$0 < \delta \leq K$$

then the distance d along two consecutive conjugate points of γ a geodesic in M satisfies

$$d \leq \frac{\pi}{\sqrt{\delta}}$$

So, if γ is a geodesic in M with $\gamma(0) = p$ for $p \in M$, then the first conjugate point to $\gamma(0)$ is at most a distance $\frac{\pi}{\sqrt{\delta}}$ away.

So, let γ be a minimizing geodesic connecting p and q, and suppose for a contradiction that the length of γ is greater than $\frac{\pi}{\sqrt{\delta}}$. Then, by the above observation, γ has a conjugate point between p and q, which contradicts γ being a minimal geodesic.

Thus, the diameter of M is bounded above by $\frac{\pi}{\sqrt{\delta}}$ as desired. $\square$

Problem 4

Recall the definition of the Hessian as

$$\mathrm{Hess}(f)Y = \nabla_Y \mathrm{grad} f$$

Part a

Prove that the Laplacian is given as

$$\Delta f = \mathrm{trace}\mathrm{Hess} f$$

Proof. We'll proceed in Einstein index notation. Recall that

$$\nabla_Y T = Y^a \nabla_a T^{bcd\ldots}_{efg\ldots}$$

and we can raise and lower indices using

$$T_a = g_{ab} T^b$$

Now, we rewrite the Hessian as

$$(\mathrm{Hess}(f)Y)^a = Y^b \nabla_b (df)^a = Y^b \nabla_b (g^{ac}(df)_c)$$

(by observing that $(\mathrm{grad} f)^a = g^{ab}(df)_b$) or

$$\mathrm{Hess}(f)^a_b = \nabla_b (df)^a$$

and the trace is

$$\mathrm{trace}\mathrm{Hess}(f) = (\mathrm{Hess}(f))^a_a = \nabla_a (df)^a = \nabla_a (g^{ab}(df)_b)$$

We compare this to the Laplacian, defined as

$$\Delta f = \mathrm{div}\ \mathrm{grad} f$$

which can be expressed in index notation as

$$\Delta f = \nabla_a (df)^a$$

and equality is immediately observed.

□

Part b

Prove that

$$g(\mathrm{Hess}(f)Y, X) = g(Y, \mathrm{Hess}(f)X)$$

Proof. We note that this is equivalent to showing that

$$\mathrm{Hess}(f)_{ab} = g_{ac}\mathrm{Hess}(f)^c_b$$

is symmetric in its lower indices.

We proceed with index gymnastics.

$$\begin{aligned}
\mathrm{Hess}(f)_{ab} &= g_{ac}\nabla_b(df)^c \\
&= g_{ac}\nabla_b g^{cd}\nabla_d f \\
&= g_{ac}g^{cd}\nabla_b\nabla_d f \\
&= g_{ac}g^{cd}\nabla_b\partial_d f \\
&= g_{ac}g^{cd}\partial_b\partial_d f - \Gamma^e_{bd}\partial_e f \\
&= g_{ac}g^{cd}\partial_d\partial_b f - \Gamma^e_{db}\partial_e f \\
&= g_{ac}g^{cd}\nabla_d\nabla_b f \\
&= \nabla_a\nabla_b f \\
&= g_{bc}\nabla_a(df)^c &= \mathrm{Hess}(f)_{ba}
\end{aligned}$$

as desired. $\square$

Problem 5

Let G be a Lie group with trivial center. Prove that if G carries a bi-invariant metric, then both G and its universal cover are compact.

Proof. Recall from Do Carmo chapter 2 that if G has a bi-invariant metric g, then the adjoint representation

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

respects the metric. That is,

$$\mathrm{Ad}(G) \subset O(\mathfrak{g}, g)$$

Now, the kernel of Ad is the center of G, and since this is trivial, G injects into $O(\mathfrak{g}, g)$. Since $O(n)$ is compact, and $\mathrm{Ad}(G)$ is closed, it follows that $\mathrm{Ad}(G) \cong G$ is compact as well.

Now, since G admits a bi-invariant metric, it follows from Do Carmo chapter 4 exercise 1 that (since G has trivial center) the sectional curvature of G is strictly positive. Then, by corollary 3.3 in chapter 9 of Do Carmo, we see that M has a finite fundamental group. Thus, its universal cover is finite-sheeted, and a finite-sheeted cover of a compact space is compact. □