Problem Set 3

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PROBLEM 1

Prove that the 1-norm on $\mathbb{R}^n$ defines a metric on $\mathbb{R}^n$ that is equivalent to the standard 2-norm metric on $\mathbb{R}^n$.

Proof. Let d_1 be the metric induced by the 1-norm on $\mathbb{R}^n$. Clearly, d_1 is positive definite, since it comes from a norm. So, let's show it satisfies the triangle inequality.

In proving the triangle inequality, we first state a general property of norms. The so-called triangle inequality of norms is given as

$$|x+y| \leq |x| + |y|$$

which is true for any normed space.

Let x, y, z be distinct points in $\mathbb{R}^n$ with coordinates x^i, y^i, z^i. Then we have that

$$\begin{aligned} d(x,z) &= \sum_i |x^i - z^i| \\ &= \sum_i |x^i - z^i + y^i - y^i| \\ &= \sum_i |(x^i - y^i) + (y^i - z^i)| \\ &\leq \sum_i |x^i - y^i| + |z^i - y^i| \\ &= \sum_i |x^i - y^i| + \sum_i |z^i - y^i| \\ &= d(x,y) + d(y,z) \end{aligned}$$

and thus the metric satisfies the axioms for a metric.

Now, let's show that the metric is equivalent to the standard 2-norm metric on $\mathbb{R}^n$. To do this, we will show that each point in a standard n-ball has a 1-norm ball contained in the n-ball, and vice versa.

So, without loss of generality (via translation) let $B_r(0)$ be the open ball of radius r around 0, and let $x \in B_r(0)$. In particular, there is some $\delta > 0$ such that $d(x, 0) < r - \delta$. Now, take C_δ to be the 1-norm ball of radius δ. Now, if $y \in C_\delta$, then we have that

$$\begin{aligned}
d(x,y) &= \sum_i |x^i - y^i| \\
&< \delta \\
\implies (\sum_i |x^i - y^i|)^2 &< \delta^2 \\
\implies \sum_i (|x^i - y^i|)^2 &< \delta^2 \\
\implies d_2(x,y) < \delta \implies d_2(y,0) &< d_2(x,0) + d_2(x,y) \\
&< r - \delta + \delta \\
&< r
\end{aligned}$$

so, the 1-ball of radius δ is contained in $B_r(0)$ as desired. Thus, since each $x \in B_r(0)$ has a neighborhood (in 1-norm) contained in $B_r(0)$, $B_r(0)$ is open in the 1-norm topology.

For the other way, we first prove the more general fact about norms on $\mathbb{R}^n$.

Lemma 1. *There exists a constant C such that for all $x \in \mathbb{R}^n$,*

$$||x||_1 \leq C||x||_2$$

Proof. We first observe the basic fact that, for $x_1, x_2 \in \mathbb{R}^+$, we have

$$2x_1x_2 \leq x_1^2 + x_2^2$$

Now, it follows quickly that

$$\begin{aligned}
||x||^2 = \left(\sum_{i=1}^n |x_i|\right) &= \sum_{i=1}^n |x_i|^2 + \sum_{i \neq j} 2|x_i||x_j| \\
&\leq \sum_{i=1}^n |x_i|^2 + (n-1)\sum_{i=1}^n |x_i|^2 \\
&= n\sum_{i=1}^n |x_i|^2
\end{aligned}$$

Thus $\sqrt{n}$ is a constant for which the lemma holds. □

Now, since we have a bound on the norms, we can prove that a 1-norm ball is open in the 2-norm. To do so, let $\Delta_r(0)$ be the 1-norm ball of radius r at zero, and let $x \in \Delta_r(0)$. In particular, we have that there exists a δ such that $d_1(x, 0) < r - \delta$. Now, let $\varepsilon = \frac{\delta}{\sqrt{n}}$, and consider the 2-norm ball $V_\varepsilon(x)$. Then, we will show that $V_\epsilon(x) \subset \Delta_r(0)$. To do so, let $y \in V_\epsilon(x)$, and observe that

$$\begin{aligned}
d_1(x,y) &< \sqrt{n} d_2(x,y) \\
&< \sqrt{n}\frac{\delta}{\sqrt{n}} \\
&= \delta
\end{aligned}$$

and

$$\begin{aligned}
d_1(0,y) &\leq d_1(0,x) + d_1(x,y) \\
&\leq r - \delta + \delta \\
&= r
\end{aligned}$$

as desired. □

Problem 2

Munkres Problem 4

Consider the box, uniform, and product topologies on $\mathbb{R}^\omega$.

Part a

In which topologies are the following functions continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \ldots) \\ g(t) &= (t, t, t, \ldots) \\ h(t) &= (t, \frac{1}{2}t, \frac{1}{3}t, \ldots) \end{aligned}$$

Proof. We first note that the universal property of product spaces guarantees that a function f is continuous in the product topology if and only if its component functions $\pi_i \circ f$ are continuous. Since this is true for all three of f, g, h, it follows that they are all continuous in the product topology.

For the remainder of this problem, we will use the pointwise definition of continuity. That is, given a point $x \in \mathbb{R}$, f is convergent at x if and only if for each neighborhood U of $f(x)$, we have that $f^{-1}(U)$ contains a neighborhood of x.

For $f(t)$, let's consider the basic open neighborhood U_t in $\mathbb{R}^\omega$ around $f(t)$ in the uniform topology, which looks like

$$U_t = \bigcup_{\delta<\varepsilon} \prod_i V_\delta(it)$$

which has an inverse image of

$$f_i^{-1}(U_t) = \bigcup_{\delta<\varepsilon} f_i^{-1}(V_\delta(it)) \quad = \bigcup_{\delta<\varepsilon} V_{\frac{\delta}{i}}(t)$$

which goes to $\{t\}$ as i goes to infinity. Thus, the inverse image is just $\{t\}$, which cannot contain an open set, so f is not continuous in the uniform topology. Then, since the box topology is finer than the uniform topology, f is not continuous in the box topology either.

Now, consider g, which we will show is continuous in the uniform topology, but not in the box topology.

To see this, consider in the uniform topology, the neighborhood U_t around $g(t)$, which is given as

$$U_t = \bigcup_{\delta<\varepsilon} \prod_i V_\delta(t)$$

Now, the inverse image of this is just

$$g^{-1}(U_t) = V_\varepsilon(t)$$

which is open in $\mathbb{R}$, so g is continuous in the uniform topology.

However, in the box topology, we have the neighborhood

$$U_t = \prod_i V_{\frac{\varepsilon}{i}}(t)$$

whose inverse image (as shown above) is just $\{t\}$, so it cannot contain an open set, and g is not continuous in the box topology.

Now, consider h, which is also continuous in the uniform topology, but not in the box topology.

To see this, consider again a neighborhood in the uniform topology

$$U_t = \bigcup_{\delta < \varepsilon} \prod_i V_\delta(\frac{t}{i})$$

which has an inverse image of

$$h_i^{-1}(U_t) = \bigcup_{\delta < \varepsilon} V_{i\delta}(t)$$

and a composite inverse image of

$$h^{-1}(U_t) = \bigcup_{\delta < \varepsilon} V_\delta(t)$$

Which certainly contains an open neighborhood of t, so h is open in the uniform topology.

However, in the box topology, the neighborhood

$$U_t = \prod_i V_{\frac{\varepsilon}{i^2}}(\frac{t}{i})$$

which has an inverse image of just $\{t\}$, so h is not open in the box topology □

Part B

In which topologies do the sequences w, x, y, z (definitions omitted) converge?

Proof. We note that all sequences converge in each coordinate, so they converge in the product topology.

Now, the w sequence does not converge in the uniform topology, since for $\varepsilon = 1$, the open set $\bigcup_{\delta<\varepsilon} V_\delta(0)$ never eventually contains the sequence, since the terms far enough down the coordinates keep growing. Thus, it does not converge in the box topology either.

The x sequence does converge in the uniform topology, since for every $\varepsilon > 0$, the sequence eventually gets to where every term is below $\frac{1}{n}$ for any n, so the sequence eventually fits in the neighborhood

$$U_0 = \prod_i V_\varepsilon(0)$$

However, in the box topology, this sequence does not converge. This is because the neighborhood

$$U_0 = \prod_i V_{\frac{1}{i^2}}$$

does not eventually contain the sequence. In particular, the i coordinate of the i term in the sequence is always $\frac{1}{i}$, which is never in $V_{\frac{1}{i^2}}(0)$.

The y sequence has the same properties as the x sequence described above, so it does converge in the uniform topology. However, since the diagonal elements are always $\frac{1}{i}$, the neighborhood

$$U_0 = \prod_i V_{\frac{1}{i^2}}$$

never eventually contains the sequence.

Now, the z sequence does converge in the box topology. To see this, we note that the z sequence lies in the subspace $\mathbb{R}^2 \times \prod\{0\}$, which (by an earlier homework assignment) is homeomorphic to $\mathbb{R}^2$. Now, in $\mathbb{R}^2$, the product topology and the box topology coincide, so by the observation that the z sequence converges in the product topology, it must also converge in the box topology and the uniform topology as well. □

Munkres Problem 5

What is $\overline{\mathbb{R}^\infty}$ in the uniform topology on $\mathbb{R}^\omega$?

Proof. We will show that the closure of $\mathbb{R}^\infty$ in the uniform topology is the set of all sequences which converge (in norm) to zero. This is clear, since for any sequence (x_i) which did not converge to zero, there must be some ε such that the sequence is eventually $\varepsilon - \delta$ away from zero. Then, the open neighborhood around (x_i) given as

$$U = \bigcup_{\delta<\varepsilon} \prod_i V_\delta(x_i)$$

will have infinitely many terms whose projections do not intersect zero, but any sequence in $\mathbb{R}^\infty$ must eventually be constantly zero, thus will eventually leave the neighborhood.

However, for any sequence (y_i) that converges to zero, any neighborhood of (y_i) must intersect $\mathbb{R}^\infty$. To see this, we consider that since (y_i) converges to zero, for any ε, the sequence (y_i) must eventually be within ε of zero. Thus, for any neighborhood

$$U = \bigcup_{\delta<\varepsilon} \prod V_\delta(y_i)$$

it must be that for some $N > 0$ and for all $n > N$, $V_\delta(y_n$ intersects zero. Thus, the element $(y_1, \ldots, y_N, 0, \ldots) \in \mathbb{R}^\infty$ is also in U. Therefore, the closure is all sequences that converge to zero in norm. □

Munkres Problem 6

For $x \in \mathbb{R}^\omega$, define

$$U(x, \epsilon) = \prod_i V_\epsilon(x_i)$$

Part a

Show that $U(x, \epsilon)$ is not equal to the ϵ ball centered at x in the uniform topology.

Proof. This follows immediately from part b. □

Part b

Show that $U(x, \epsilon)$ is not open in the uniform topology.

Proof. Consider, for x given above, the point

$$x' = (x_i + \epsilon - \frac{1}{i})$$

Now, $x' \in U(x, \epsilon)$, but we will show that more precisely, $x' \in \partial U$, which means that U contains part of its boundary, and cannot be open.

We note already that $x' \in U$. Thus, the constant sequence $\{x'\}$ converges to x' in u. Now, consider the sequence of points

$$f(n) = (x'_i + \frac{1}{n}) = (x_i + \epsilon - \frac{1}{i} + \frac{1}{n})$$

Which clearly converges to x', but for each n, the n term is given as $x_n + \epsilon$, which is clearly not in U for that coordinate. Thus, f is a sequence of points outside U converging to u, so x' is actually on the boundary of U.

Thus, U is not open. □

Part c

Show that the ϵ-ball around x is given as

$$\bigcup_{\delta<\epsilon} U(\delta, x)$$

Proof. To see this, suppose y is such that $d(x,y) < \epsilon$. In particular, there is some number δ such that each coordinate obeys the inequality

$$|x_i - y_i| \leq d(x,y) < \delta < \epsilon$$

So, y is in the set $U(\delta, x) \subset \bigcup_{\delta<\epsilon} U(\delta, x)$ and is thus in the ϵ-ball given. □

Problem 3

Prove that the lower limit topology on $\mathbb{R}$ is first-countable.

Proof. Let $x \in \mathbb{R}$, and let $\{q_i\}$ be an enumeration of the positive rationals. Now, a countable neighborhood basis for x can be given as

$$\mathscr{B} = \{[x - q_i, x + q_j)\}_{i,j=0}^{\infty} \cup \{[x, x + q_i\}_{i=0}^{\infty}$$

This is easily verified to be a countable neighborhood basis. To see this, let U be any basic neighborhood of x. In particular, either $U = [a, b)$ for $a < x < b$, or $U = [x, b)$ for $x < b$. We will show that U contains an element of $\mathscr{B}$.

Suppose U is of the first kind. Then, by the Archimedean property of the reals, we have that there exist two rational numbersq_1, q_2 such that $a < x - q_1 < x < +q_2 < b$. Then, immediately it follows that

$$[x - q_1, x + q_2) \subset [a, b)$$

as desired.

Now, suppose U is of the second kind. Then, similarly, there is some q_3 such that $x < x+q_3 < b$. We have immediately that

$$[x, x + q_3) \subset [x, b)$$

and thus $\mathscr{B}$ is a countable neighborhood basis, as desired. □

Problem 4

Show that if $p : X \to Y$ is split-epic, then it is a quotient map.

Proof. We note that a continuous map is a quotient map if and only if it is surjective. Since p is already assumed to be continuous, we need only show it is surjective.

To do so, let f be the right-inverse of p. In particular, we have that for every $y \in Y$, $p \circ f(y) = y$. Thus, for any $y \in Y$, $p(f(y)) = y$ and p is a surjection, as required. □

Problem 5

Show that the composition of two quotient maps is a quotient map.

Proof. Let $p : X \to Y$ and $q : Y \to Z$ be surjective morphisms. This means that for $z \in Z$, there is some $y \in Y$ such that $q(y) = z$. Furthermore, there is some $x \in X$ such that $p(x) = y$. Then $q \circ p(x) = q(p(x)) = q(y) = z$, and since every $z \in Z$ has an element of X that maps to it, $q \circ p$ is a surjection i.e. a quotient map. □