Problem Set 2

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Problem 2-1

For f the Heaviside step function (with $f(0) = 1$), show that $\forall x \in \mathbb{R}$, there exist smooth charts (U, ϕ) around x and (V, ψ) around $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from its domain to its image, but f is not smooth in a smooth manifold sense.

Proof. For $x \neq 0$, neighborhoods avoiding zero can be chosen, and identity charts make f locally smooth. For $x = 0$, set $U = (-\epsilon, \epsilon)$, $V = (1 - \epsilon, 1 + \epsilon)$ and have $\phi_U = \psi_V = \text{id}$. Then, on $U \cap f^{-1}(V) = [0, \epsilon)$ we have $\psi \circ f \circ \phi^{-1}(x) = 1$ which is smooth. But this fails the test in proposition 2.5, so f is not smooth in a manifold sense. $\square$

Problem 2-3

For each of the following maps, show that the map is smooth via computation through coordinate representations.

Part A

The power map $p_n : S^1 \to S^1$ defined as $p_n(z) = z^n$.

Proof. For this problem, we will use two coordinate charts on S^1. First, let's parameterize the circle by θ, so that the point θ is identified with $\exp(i\theta)$ in the standard embedding of the circle into $\mathbb{C}$. Then, the first coordinate chart will be for $\theta \in (0, 2\pi)$ given as $\phi(\theta) = \theta$. The second coordinate chart will be for $\theta \in (-\pi, \pi)$ (where $2\pi\theta \sim \theta$) given as $\psi(\theta) = \theta$.

Now, the transition maps can easily be verified to be smooth. To see this, let θ_0 be a point in the intersection of the two charts. Then, if $\theta \in (0, \pi)$, we have

$$\phi(\theta) = \theta$$
$$\psi(\theta) = \theta$$

Which are easily verified to be smooth and compatible with each other.

Suppose, then, that $\theta \in (\pi, 2\pi)$. Then, we have that

$$\phi(\theta) = \theta$$
$$\psi(\theta) = \theta - 2\pi$$

With transition charts

$$\phi \circ \psi^{-1}(\theta) = \theta + 2\pi$$
$$\psi \circ \phi^{-1}(\theta) = \theta - 2\pi$$

which are clearly smooth.

Now, we just have to check that the power function, which can be thought of in terms of our parameterization as $p_n(\theta) = n\theta(\mod 2\pi)$, is smooth.

So, let's compute some coordinate representations. We have a total of four to check.

$$\phi \circ p_n \circ \phi^{-1}(\theta) = n\theta(\mod 2\pi)$$
$$\psi \circ p_n \circ \psi^{-1}(\theta) = n(\theta + 2\pi)(\mod 2\pi) - 2\pi$$
$$\phi \circ p_n \circ \psi^{-1}(\theta) = n(\theta + 2\pi)(\mod 2\pi)$$
$$\psi \circ p_n \circ \phi^{-1}(\theta) = n\theta(\mod 2\pi) - 2\pi$$

Now, addition of a scalar is a smooth operation, so we just have to check that the function p_n is smooth as a function of θ.

Now, we observe that p_n is continuous as a function of θ by viewing $p_n : [0, 2\pi) \to \mathbb{R}$ as a continuous function $\theta \mapsto n\theta$, and passing through the quotient $\mathbb{R}/2\pi\mathbb{Z}$. Since the derivative $p_n' = np_{n-1}$ is also of the same form, it is continuous as well, and by induction each derivative of p_n is continuous, so p_n is smooth.

Thus, the composition maps defined above are smooth, and p_n is a smooth function from S^1 to itself. □

Part b

The antipodal map $\alpha : S^n \to S^n$ by $\alpha(x) = -x$.

Proof. Consider the stereographic projection charts σ and $\tilde{\sigma}$, where $\tilde{\sigma}(x) = -\sigma(-x)$. Let's compute some coordinate representations:

$$\sigma \circ \alpha \circ \sigma^{-1}(x) = \sigma(-\sigma^{-1}(x))$$
$$\tilde{\sigma} \circ \alpha \circ \tilde{\sigma}^{-1}(x) = \tilde{\sigma}(-\tilde{\sigma}^{-1}(x))$$
$$\sigma \circ \alpha \circ \tilde{\sigma}^{-1}(x) = \sigma(-\tilde{\sigma}^{-1}(x))$$
$$\tilde{\sigma} \circ \alpha \circ \sigma^{-1}(x) = \tilde{\sigma}(-\sigma^{-1}(x))$$

Now, these are all compositions of smooth functions, which are smooth as well. Thus, the antipodal map is a smooth function. □

Part c

Show that the map $F : S^3 \to S^2$ defined as $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, is smooth.

Proof. To show that this map is smooth, we will show it is smooth in the ambient space $\mathbb{C}^2 \setminus \{0\}$ and $\mathbb{R}^3 \setminus \{0\}$.

Now, F is smooth as a map from the ambient spaces, which is clear when viewing it as a map from $\mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$. Using this, we have that

$$F(x^1, x^2, x^3, x^4) = (2(x^1x^3 + x^2x^4), 2(x^2x^3 - x^1x^4), (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)$$

which is clearly smooth. Now, since F is smooth in the ambient space, it must also be smooth when restricted to $S^3 \subset \mathbb{C}^2$. □

Problem 2-7

Show that for M a nonempty smooth n-manifold, with $n \geq 1$, the vector space $C^\infty(M)$ is infinite dimensional.

Proof. □

Problem 2-10

Consider the algebra $C(M)$ of continuous functions on M, and observe that a map$f : M \to N$ induces a map $f^* : C(N) \to C(M)$ via pre-composition.

Part a

Show that f^* is linear.

Part b

Show that f is smooth if and only if $f^*(C^\infty(N)) \subseteq C^\infty(M)$.

Part c

Given a homeomorphism $f : M \to N$, show that f is a diffeomorphism if and only if f^* restricts to an isomorphism $f^* : C^\infty(N) \to C^\infty(M)$

Problem 2-14

For A and B disjoint closed subsets of a smooth manifold M, show that there exists $f \in C^\infty$ such that $0 \leq f \leq 1$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Problem 3-5

Problem 3-6

Problem 3-7

Problem 3-8

For M a smooth manifold, and $p \in M$, let $\mathscr{V}_pM$ be the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if for all $f \in C^\infty(M)$, $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$. Show that the map $\Psi : \mathscr{V}_pM \to T_pM$ defined as $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective.

Proof. To begin with, we show that this map is well defined. To do so, let γ_1 and γ_2 be equivalent in the sense defined above. In particular, this means that $d\gamma_1(\partial_t|_0)(f) = d\gamma_2(\partial_t|_0)$ for all f in $\mathbb{C}^\infty(M)$. Thus, since the differentials are functions on $C^\infty(M)$ that are identical for all f, we have that $d\gamma_1(\partial_t|_0) = d\gamma_2(\partial_t|_0)$ which implies $\gamma_1'(0) = \gamma_2'(0)$ as desired.

Now, let's show that this is bijective. To do so, we will first show Ψ is surjective. Let v be some vector in T_pM. In particular, $v = v^i \frac{\partial}{\partial x^i}|_p$ for some coordinates x^i centered at p. Now, define a curve $\gamma : [0,1] \to M$ as $\gamma^i(t) = tv^i$. It is clear that $\gamma'(0) = v$, since $\gamma'^i(0) = v^i$, which implies $\gamma'(0) = v^i \partial_i = v$ as desired.

Second, we will show Ψ is injective. This is immediate from the definition of the equivalence relation, since by the argument for well-definedness if $\gamma_1'(0) = \gamma_2'(0)$, then $\gamma_1 \sim \gamma_2$.

Thus, Ψ is bijective, as desired. □

Problem 3-4

Show $TS^1 \cong S^1 \times \mathbb{R}$.

Proof. □