1 Geodesics and Curvature

1.1 Geodesics

Definition 1.1. *Let* (M^n, g) *be a Riemannian manifold, and let* $\gamma : I \to M$ *a smooth curve.* γ *is called a* geodesic *if its second derivative vanishes. That is, if it solves the geodesic equation*

$$\nabla_{\partial_t} \partial_t \gamma = 0$$

Now, let's examine the geodesic equation further. In local coordinates, we have

$$\begin{aligned} \nabla_{\partial_t} \partial_t \gamma &= \nabla_{\partial_t} \partial_t x^i \partial_i \\ &= \partial_t \partial_t x^k \partial_k + \partial_t x^k \nabla_{\partial_t} \partial_k \\ &= (\partial_t \partial_t x^k + \Gamma^k_{ij} \partial_t x^i \partial_t x^j) \partial_k \end{aligned}$$

and so the local coordinate version of the differential equation is the system of equations

$$(\partial_t)^2 x^k + \gamma^k_{ij} \partial_t x^i \partial_t x^j = 0$$

which are guaranteed local unique solutions for initial conditions of γ and γ'.

Let's look at properties of geodesics. In particular, we can look at

$$\partial_t |\gamma'|^2 = \partial_t (g(\gamma', \gamma')) = 2g(\nabla_{\partial_t} \gamma', \gamma') = 0$$

and so the velocity of the geodesic does not change.

1.2 The Exponential Map

Let $p \in M$. We can define an exponential map $\exp : T_p M \to M$ via the following:

Definition 1.2. *The* exponential map $\exp : T_p M \to M$ *is defined as* $\exp(v) = \gamma(1)$ *where* γ *is a geodesic with* $\gamma(0) = p$ *and* $\gamma'(0) = v$*.*

Why do we insist that $\exp_p(v) = \gamma(1)$? Consider

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

where $t \in \mathbb{R}$. The last equality is obtained in the following way:

Lemma 1. $\gamma_{tv}(1) = \gamma_v(t)$ *for all* t*.*

Proof. Consider $\gamma(t) = \gamma_{sv}(t)$. This is the geodesic such that $\gamma(0) = p$ and $\gamma'(0) = sv$. Now, notice that $\tilde{\gamma}(t) = \gamma_v(st)$ is defined so that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}'(0) = \partial_t \gamma_v(st) = \gamma'_v(0) \partial_t (st)|_{t=0} = sv$ and by uniqueness of geodesics, $\gamma = \tilde{\gamma}$ as desired. $\square$

Let's examine the domain for the exponential map. With no assumptions on the structure of the manifold, what can we say about solutions to the geodesic equation?

Recall the escape lemma for flows along vector fields. If γ is a maximal integral curve of a vector field X whose domain J has a least upper bound b, then for each $t_0 < b$, $\gamma([t_0, b))$ is not contained in any compact subset of the manifold. That is, if γ goes into a compact subset of the manifold, it will not die in the interior of the compact subset.

We also have the uniform time lemma, which guarantees that for U open with compact closure, any $K > 0$, there is some $\epsilon > 0$ such that the geodesic $\gamma(t)$ with $\gamma(t_0) = p$ $\gamma'(t_0) = v$ exists for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ and the map

$$\begin{aligned} \gamma : U^* \times (t_0 - \epsilon, t_0 + \epsilon) &\to M \\ \gamma(v, t) &= \gamma(t) \end{aligned}$$

and here $U^* = \{v \in TM, \|v\| < K, \pi(v) \in U\}$.

Now, we can see that $\exp_p$ is defined on a closed ball $\overline{B}_\epsilon(0) \subset T_pM$ for some $\epsilon > 0$, and furthermore for any compact set K, there is some $\epsilon > 0$ such that $\exp_p$ is defined on $\overline{B}_\epsilon(0)$ for all $p \in K$.

In $\mathbb{R}^n$, we have geodesics as linear affinely parameterized curves i.e. $\gamma(t) = \vec{a}t + \vec{b}$.

Example. *Consider the sphere S^n with the induced metric from $\mathbb{R}^{n+1}$. Then, the Levi-Civita connection is given as*

$$\nabla^{S^n}_{\partial_t} \gamma' = \left(\nabla^{\mathbb{R}^{n+1}}_{\partial_t} \gamma' \right)^T$$

Where T is the tangential projection onto S^n.

More generally, for N a submanifold of M, and X a vector field on N, we can extend X to a neighborhood in M, and take $\nabla^N_{\partial_t} X = (\nabla^M_{\partial_t} X)^T$.

1.3 Further properties of geodesics

In $\mathbb{R}^n$, it is clear that the straight line geodesic between two points is the path of shortest length between them. Is this the case in general? Do geodesics exist between any two points?

Obviously there are not geodesics between arbitrary points in a general connected Riemannian manifold (motivating example: Schwarzschield geometry). However, if a Riemannian manifold is complete, then it has geodesics between all points.

Does there always exist a geodesic of minimal length? And are such geodesics unique? No. On S^2, we have an infinite number of geodesics from the north to the south pole. Suppose instead, however, that we restrict to anything but the south pole. Then, there exist unique geodesics of minimal length from the north pole to any point. This has to do with the fact that the geodesics from the north pole do not cross until the south pole.

We can prove that *locally*, points are connected by a minimal geodesic, and that open balls around a point correspond to exponential projections of open balls in the tangent space.

Now, an important lemma:

Lemma 2. Gauss Lemma: *Let M be a manifold, and $p \in M$ with exponential $\exp_p$. We wish to understand $(d\exp_p)_v : T_pM \to T_{\exp_p(v)}M$. It is true that*

$$g(v, w) = g(d(\exp_p)_v(v), d(\exp_p)_v(w))$$

Proof. We begin by calculating $(d\exp_p)_v(V)$. Note that here, the v in the parentheses is actually in T_vT_pM, which is canonically identified with T_pM.

Specifically, we wish to show $\|(d\exp_p)_v(v)\| = \|v\|$, or that $g((d\exp_p)_v(v), (d\exp_p)_v(v)) = g(v, v)$.

To see this, consider the geodesic $c(t) = \exp_p(tv)$. Now, c is affinely parameterized, so it has constant speed (magnitude of tangent vector). Now, $c'(0) = v$, and $c'(1) = (d\exp_p)_v(\partial_t(tv)|_{t=1}) = (d\exp_p)_v(v)$ and since c is affinely parameterized, these two have the same magnitude.

Now, let $w \in T_pM$ such that w is perpendicular to v. We can choose a path $\tau(s) = v + sw$ such that $\tau(0) = v$ and $\tau'(0) = w$. Consider

$$F(t, s) = \exp_p(t(v + sw))$$

where, by varying s, we get a family of geodesics from the tangent vectors $v + sw$. Now, for $t \in [0, 1]$ (actually $(-\epsilon, 1 + \epsilon)$), $s \in (-\epsilon, \epsilon)$, we have a smooth map $F : [0, 1] \times (-\epsilon, \epsilon) \to M$.

Lemma 3. *For a smooth map $F : [a, b] \times [c, d] \to M$ with first coordinate t and second coordinate s,*

$$\nabla_{\partial_s}\partial_t F = \nabla_{\partial_t}\partial_s F$$

Homework 1. *Prove this lemma, using the fact that $[\partial_s, \partial_t] = 0$.*

Now, we have that

$$\partial_t F(t, 0) = c'(t)$$

since $F(t, 0) = \exp_p(tv) = c(t)$. We also have

$$\partial_s F(1, 0) = (d\exp_p)_v(\partial_s(t(v + sw))|_{t=1,s=0}) = (d\exp_p)_v(w)$$

Now, we wish to show that

$$g((d\exp_p)_v(v), (d\exp_p)_v(w)) = 0$$

which is clear, since

$$\begin{aligned} g((d\exp_p)_v(v), (d\exp_p)_v(w)) &= g(\partial_t F(1, 0), \partial_s F(1, 0)) \\ &= g(\partial_t F, \partial_s F)|_{t=1,s=0} \end{aligned}$$

Now,

$$\begin{aligned}
\partial_t g(\partial_t F, \partial_s F)|_{s=0} &= g(\nabla_{\partial_t}\partial_t F, \partial_s F) + g(\partial_t F, \nabla_{\partial_t}\partial_s F) \\
&= g(\partial_t F, \nabla_{\partial_t}\partial_s F) && \text{since } F \text{ is along a geodesic, second derivatives vanish} \\
&= g(\partial_t F, \nabla_{\partial_s}\partial_t F) \\
&= \frac{1}{2}\partial_s g(\partial_t F, \partial_t F) && \text{By symmetry of the metric}
\end{aligned}$$

Now, suppose instead that we use a circular arc in T_pM between v and w so that $\|\partial_t F\|$ is independent of s. Then, it follows that $\partial_s g(\partial_t F, \partial_t F) = 0$ as desired.

Now, let's calculate $g(\partial_t F, \partial_s F)|_{t=0,s=0}$. We have

$$\begin{aligned}
\partial_t F|_{t=0,s=0} &= c'(0) = v \\
\partial_s F|_{t=0,s=0} &= \partial_s \exp_p(t\tau(s))|_{t=0,s=0} = 0
\end{aligned}$$

and so $g((d\exp_p)_v(v), (d\exp_p)_v(w)) = 0$.

These two facts then prove the Gauss lemma by writing $u = \alpha v + \beta w$ for w perpendicular to v. $\square$

If we were to follow the proof through using the straight line in T_pM instead of a circular arc, we would need to use the following lemma

Lemma 4. $(d\exp_p)_0 = id$

Proof. Let $v \in T_pM$, and let $\gamma(t) = tv$ be a curve in T_pM with tangent vector v at zero. Then,

$$(d\exp_p)_0(v) = \partial_t \exp_p(tv) = v$$

as desired. $\square$

Proposition 1. *For U_p an open set in T_pM, $0 \in U_p$, the exponential map $\exp_p|_{U_p}$ is a diffeomorphism onto its image, and $\exp_p(U_p)$ is open in M.*

$B_r(0) \subset T_pM$ is called a normal ball if $\exp_p$ restricts to a diffeomorphism from $B_r(0)$ to its image.

Theorem 1. *Let $B_{r_0}(0)$ be a normal ball. Then, for each $v \in B_{r_0}(0)$, the radial geodesic $c(t) = \exp_p(tv)$ for $t \in [0,1]$ is the unique shortest smooth curve up to reparameterization from p to $\exp_p(v)$.*

A corollary of this is that $\exp_p(B_r(0)) = B_r(p)$.

Proof. Let $v \in B_{r_0}(0)$ as described in the hypothesis. Let $c(t) = \exp_p(v)$, with $c(0) = p$ and $c(1) = q$. Furthermore, let γ be any curve from p to q.

Suppose γ leaves $\exp_p(B_{\|v\|}(0))$ at some timet_1. That is, $\gamma([0,t_1)) \subset \exp(B_{\|v\|})$ and $\gamma(t_1)$ is in the boundary. Then, we know that

$$L_\gamma \geq L_{\gamma|_{[0,t_1)}}$$

so all we need to show is that

$$L_c \le L_{\gamma|_{[0,t_1)}}$$

Now, this reduces to the second case. Namely, suppose γ is entirely contained in $\exp(B_{\|v\|}(0))$, and $\gamma(1) = q_1$ is on the boundary. Let $\tilde{\gamma}(t) = \exp_p|^{-1}_{B_{\|v\|}(0)} \circ \gamma(t)$ be the corresponding curve in T_pM. Now, all we have to do is calculate the length of γ.

$$L_\gamma = \int_I g(\gamma', \gamma')dt$$

Now, $\gamma'(t) = (d\exp_p)_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t))$ and if we assume $\tilde{\gamma}(t)$ is not zero, we can calculate the magnitude of $\gamma'(t)$. Let $\tilde{\gamma}'(t)$ be decomposed into a radial and normal part $r(t)$ and $n(t)$ with respect to the vector $\tilde{\gamma}$. Then,

$$\begin{aligned} g((d\exp_p)_{\tilde{\gamma}(t)}(r(t)+n(t)), (d\exp_p)_{\tilde{\gamma}(t)}(r+n)) &= \|r(t)\|^2 + \|d\exp_p(n(t))\|^2 \\ &\ge \|r(t)\|^2 \end{aligned}$$

Homework 2. *prove that equality is met in the previous inequality if and only if $\tilde{\gamma}(t)$ is radial.*

Then,

$$\begin{aligned} L_\gamma &= \int_{0+}^1 \|\gamma'\|dt \\ &\ge \int_{0+}^1 \|r(t)\|dt \end{aligned}$$

switching to polar coordinates, we denote $R(v) = \|v\|$, and we can calculate

$$\begin{aligned} \partial_t R(\tilde{\gamma}(t)) &= \nabla(R) \cdot \tilde{\gamma}'(t) \\ &= \frac{\tilde{\gamma}(t)}{\|\tilde{\gamma}(t)\|} \cdot \tilde{\gamma}'(t) \\ &= r(t) \end{aligned}$$

and so

$$\begin{aligned} L_\gamma &\ge \int_{0+}^1 \|r(t)\|dt \\ &= \int_{0+}^1 \partial_t R(\tilde{\gamma}(t))dt \quad = L(c) \end{aligned}$$

as desired. $\square$