# Problem Set 2

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## Problem 1

Show that the function sending $\phi$ to $\phi^{-1}(\{1\})$ is a bijection between nonzero bounded linear functionals and hyperplanes not containing 0.

*Proof.* We first show that for arbitrary bounded linear functional $\phi$, the set $\phi^{-1}(\{1\})$ is a closed hyperplane.

To see this, let $\phi \in X^* \setminus \{0\}$. In particular, this means that $H := \phi^{-1}(\{1\})$ is nonempty. So, let $x_0 \in H$. Then, we have $\phi(x_0) = 1$. Now, let $x \in X$ be arbitrary, and consider

$$\phi(x - \phi(x)x_0) = \phi(x) - \phi(x)\phi(x_0) = 0$$

This implies that $y := x - \phi(x)x_0$ is in the kernel of $\phi$. Solving for $x$ yields

$$x = y + \phi(x)x_0$$

and so $X = \ker\phi \oplus \operatorname{span}(x_0)$.

Thus, we know that $\ker\phi$ has codimension 1. In particular, we know that for $y \in \ker\phi$ we have that $\phi(y + x_0) = 1$ and so the hyperplane obtained by translating $\ker\phi$ is in $\phi^{-1}(\{1\})$. Thus, since $\phi^{-1}(\{1\})$ does not contain zero, it follows that it is the translate of $\ker\phi$ by $x_0$. The hyperplane is closed because it is the inverse image of a point.

Now, let's start with a hyperplane $H$ that misses zero, and show that it is the inverse image of $\{1\}$ by some function. To begin with, we note that a hyperplane missing zero is just a hyperplane containing zero translated by some constant vector. In particular,for each hyperplane missing zero, there is a unique hyperplane through zero that is the translate of it. Let $H'$ be the hyperplane through zero that is the translate of $H$. Then, since $H'$ is closed with codimension 1, we can write $X = H' \oplus \operatorname{span}(v)$ for some $v$. Define $\phi$ to be

$$\phi(x) = \phi(h + \lambda v) = \lambda$$

for $\lambda \in \mathbb{C}$.

Now, $H$ is just $H' + c$ for some $c \in X$, and so $H'$ is the set

$$\{x : \ \phi(x - c) = 0\}$$

or,

$$\{x : \ \phi(x) = \phi(c)\}$$

letting $c = h + \eta v$ for $\eta \in \mathbb{C}$, we see that $\phi(x) = \eta$ defines the hyperplane.

Now, define $\phi'(x) = \frac{\phi(x)}{\eta}$, and note that

$$H' = \{x : \ \phi'(x) = 1$$

as desired.

Now, let's show that distinct functionals generate distinct hyperplanes. To see this, let $\phi, \psi$ be such that $\phi^{-1}(\{1\}) = \psi^{-1}(\{1\}) = H$. Now, consider the bounded linear functional defined by

$$\theta(x) = \phi(x) - \psi(x)$$

Clearly, $H \subset \ker\theta$, and so since $H$ has codimension 1 and does not contain zero, it follows immediately that $\ker\theta = X$ and so $\phi(x) = \psi(x)$ for all $x$ as desired. $\square$

## Problem 2

For $K \subset L$ an absorbing convex set, define

$$\mu_K(x) = \inf\{t > 0| \ x \in tK\}$$

Show that $\mu_K$ is convex, and show that if $\phi \in L'$, then $\phi \leq \mu_K$ implies $\phi|K \leq 1$ and vice versa.

*Proof.* We first show that $\mu_K$ is homogeneous. To see this, we note that

$$\begin{aligned}
\mu_K(ax) &= \inf\{t > 0| \ ax \in tK\} \\
&= \inf\{t > 0| \ x \in \frac{t}{a}K\} \\
&= a\inf\{t > 0| \ x \in tK\}] \\
&= a\mu_K(x)
\end{aligned}$$

as desired. Now, we show that $\mu_K$ is subadditive. To see this, consider

$$\mu_K(x) + \mu_K(y)$$

and let $t^* = \max(\mu_K(x), \mu_K(y))$. Now, we know that $d(x, t^*K) = d(y, t^*K) = 0$, and by the triangle inequality, this forces $d(x + y, t^*K)$ to be zero as well. Thus, $\mu_K(x + y) \geq t^*$. It follows immediately that

$$\mu_K(x + y) \leq \mu_K(x) + \mu_K(y)$$

since $\mu_K$ is positive.

Now to show the second statement.

($\implies$) Suppose $\phi$ is such that $\phi \leq \mu_K$. It follows immediately from the definition that $\mu_K|_K \leq 1$, and so$\phi|_K \leq 1$ as well.

($\impliedby$) Suppose $\phi|_K \leq 1$. We wish to show (by homogeneity) that $\phi(\frac{x}{\mu_K(x)}) \leq \mu_K(\frac{x}{\mu_K(x)}) = 1$. Now, since $K$ is closed, $\frac{x}{\mu_K(x)}$ is in $K$ (since $\mu_K(x)$ is the inf of all $t$ such that $\frac{x}{t}$ is in $K$). Now, it follows by the hypothesis that

$$\phi(\frac{x}{\mu_K(x)}) \leq 1 = \mu(\frac{x}{\mu_K(x)})$$

as desired. $\square$

## Problem 3

Prove that for every $e \in X$ not equal to zero, there is some $\phi \in V^*$ such that $\phi(e) = \|e\|$ and $\|\phi\|_* = 1$.

*Proof.* We note first that there is a bounded linear functional $\rho(x)$ on $\text{span}(e)$ given by $\rho(\lambda e) = \lambda\|e\|$. Hahn-Banach then guarantees the existence of $\phi \in X^*$ with $\phi|_{\text{span}(e)} = \rho$ and $\|\phi\|_* = \|\rho\|$. This is the same as saying $\phi(e) = \|e\|$ and $\|\phi\|_* = 1$ as desired. □

## Problem 4

Let $X$, $Y$ be normed spaces, and $Z = X \oplus Y$ with $\|z\| = \|x\| + \|y\|$. What is $Z^*$?

*Proof.* I assert that $Z^* = X^* \oplus Y^*$ with the (linear) isomorphism

$$\phi_x \times \phi_y \mapsto (z = x + y \mapsto \phi_x(x) + \phi_y(y))$$

Now, clearly this map is injective, and the resulting functional is bounded (since $\phi_x$ and $\phi_y$ are bounded, and projections are bounded). So, all we need to show is that this map is surjective.

However, consider some $\psi \in Z^*$. This can be written as

$$\begin{aligned}\psi(z) &= \psi(x + y)\\ &= \psi(x) + \psi(y)\\ &= \psi|_X(x) + \psi|_Y(y)\end{aligned}$$

and since $\psi|_X$ and $\psi|_Y$ are in $X^*$ and $Y^*$ respectively, it follows that the isomorphism maps $\psi|_X \times \psi|_Y$ to $\psi$, and the isomorphism is surjective.

Thus, the spaces are equal. □

## Problem 5

Show that if $f$ is a positive linear functional on $l^\infty$, then $f$ is bounded.

*Proof.* Suppose $f$ is a positive linear functional, and let $x \in l^\infty$. Then, we know that $-x + \|x\|_{l^\infty}(1, 1, \ldots) \geq 0$. So,

$$\begin{aligned}f(-x + \|x\|_\infty(1, 1, \ldots)) &\geq 0\\ f(-x) &\geq -f((1, 1, \ldots))\|x\|_\infty\\ f(x) &\leq f((1, 1, \ldots))\|x\|_\infty\\ |f(x)| &\leq f((1, 1, \ldots))\|x\|_\infty\end{aligned}$$

and thus $f$ is bounded. □

# PROBLEM 6

Let $Y$ be a subspace of $X$. Show that the closure of $Y$ is the intersection of all kernels containing $Y$.

*Proof.* We first prove the hint: for $Y$ a closed subspace and $x_0 \notin Y$, there exists a function $\phi \in X^*$ such that $\phi(x_0) = 1$ and $\phi|_Y = 0$.

This is clear by considering the bounded linear functional $\xi$ on $Y \oplus \operatorname{span}(x_0)$ given by $\xi(y + \lambda x_0) = \lambda$. Hahn-Banach guarantees this extends to a bounded linear functional on all of $X$.

Now, clearly $\overline{y} \subset \bigcap_{f \mid \ker f \supset Y} \ker f$, since $\overline{Y}$ is the intersection of all closed subsets containing it, and kernels are closed.

Now, we wish to show the other containment. To do so, we will show that for any point not in $\overline{Y}$ is outside of some kernel containing $Y$. To see this, we apply the hint on $x_0 \notin \overline{Y}$ to obtain $\phi \in X^*$ with $\phi(x_0) = 1$ and $\phi|_Y = 0$. Thus, $\ker \phi$ contains $Y$ but does not contain $x_0$. Thus, $x_0$ is not in the intersection.

Therefore, the two sets are equal, as desired. $\square$