QFT

Homework 2

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Problem 1

Recall the φ^4 Lagrangian is given by

$$\mathscr{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4$$

and has an energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + g^{\mu\nu}\mathscr{L}$$

Part A

Problem. *Derive the equation of motion for φ subject to the φ^4 Lagrangian.*

To calculate the equation of motion for φ, we just have to find the stationary points of

$$S = \int d^4x\mathscr{L} = \int d^4x\left(\frac{-1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4\right)$$

That is, we find when $\delta S = 0$. To do so, we calculate

$$\begin{aligned}
\delta S &= \int d^4x\delta\mathscr{L}\\
&= \int d^4x\left(\frac{-1}{2}\delta(\partial^\mu\varphi\partial_\mu\varphi) - \frac{1}{2}m^2\delta(\varphi^2) - \frac{1}{3!}g\delta(\varphi^3) - \frac{1}{4!}\lambda\delta(\varphi^4)\right)\\
&= \int d^4x\left(\frac{-1}{2}(\partial^\mu\delta\varphi\partial_\mu\varphi + \partial^\mu\varphi\partial_\mu\delta\varphi) - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi\right)\\
&= \int d^4x\left(\partial^2\varphi\delta\varphi - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi\right)\\
&= \int d^4x\left(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3\right)\delta\varphi
\end{aligned}$$

Which is zero for arbitrary variation if $\left(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3\right) = 0$. Thus, this is the equation of motion for φ.

Part B

Problem. *Show that the energy-momentum tensor $T^{\mu\nu}$ satisfies $\partial_\mu T^{\mu\nu} = 0$.*

This is just an exercise in direct calculation:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu \left(\partial^\mu\varphi\partial^\nu\varphi\right) + \partial_\mu g^{\mu\nu}\mathscr{L} \\
&= \partial_\mu\partial^\mu\varphi\partial^\nu\varphi + \partial^\mu\varphi\partial_\mu\partial^\nu\varphi + \partial^\nu\left(\frac{-1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4\right) \\
&= \partial_\mu\partial^\mu\varphi\partial^\nu\varphi + \partial^\mu\varphi\partial_\mu\partial^\nu\varphi - \frac{1}{2}\partial^\nu\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}\partial^\mu\varphi\partial^\nu\partial_\mu\varphi - m^2\varphi\partial^\nu\varphi - \frac{1}{2}g\varphi^2\partial^\nu\varphi - \frac{1}{3!}\lambda\varphi^3\partial^\nu\varphi \\
&= \partial_\mu\partial^\mu\varphi\partial^\nu\varphi - m^2\varphi\partial^\nu\varphi - \frac{1}{2}g\varphi^2\partial^\nu\varphi - \frac{1}{3!}\lambda\varphi^3\partial^\nu\varphi \\
&= \left(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^2 - \frac{1}{3!}\lambda\varphi^3\right)\partial^\nu\varphi
\end{aligned}$$

Which is clearly zero if φ follows its equation of motion.

Problem 2

Consider a complex scalar field φ governed by the Lagrangian

$$\mathscr{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0$$

Part A

Problem. *Show φ obeys the Klein-Gordon equation.*

We calculate the variation in $S = \int d^4x \mathscr{L}$ directly:

$$\begin{aligned}
\delta S &= \int d^4x \delta \mathscr{L} \\
&= \int d^4x \left(-\delta(\partial^\mu \varphi^\dagger) \partial_\mu \varphi - \partial^\mu \phi^\dagger \delta(\partial_\mu \varphi) - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\
&= \int d^4x \left(-\partial^\mu \delta \varphi^\dagger \partial_\mu \varphi - \partial^\mu \phi^\dagger \partial_\mu \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\
&= \int d^4x \left(\delta \varphi^\dagger \partial_2 \varphi + \partial^2 \phi^\dagger \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\
&= \int d^4x \left(\left(\partial^2 \varphi - m^2 \varphi \right) \delta \varphi^\dagger + \left(\partial^2 \varphi^\dagger - m^2 \varphi^\dagger \right) \delta \varphi \right)
\end{aligned}$$

Which is zero for arbitrary variations when both φ and $\varphi^\dagger$ follow the Klein-Gordon equation.

Part B

Problem. *Find the conjugate momenta for φ and $\varphi^\dagger$, and write down the Hamiltonian in terms of these.*

We can read off the conjugate momenta easily:

$$\begin{aligned}\pi(x) &= \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} \\ &= \frac{\partial}{\partial \dot{\varphi}}(\partial_\mu \varphi^\dagger(x)\partial_\mu\varphi(x) - m^2\varphi^\dagger(x)\varphi(x) + \Omega_0) \\ &= \dot{\varphi}^\dagger(x)\end{aligned}$$

and similarly

$$\pi^\dagger(x) = \dot{\varphi}(x)$$

We write down the Hamiltonian density as

$$\mathscr{H} = \pi(x)\dot{\varphi}(x) + \pi^\dagger(x)\dot{\varphi}^\dagger(x) - \mathscr{L}$$

and calculate

$$\begin{aligned}\mathscr{H} &= \pi(x)\dot{\varphi}(x) + \pi^\dagger(x)\dot{\varphi}^\dagger(x) + \Big(\partial^\mu\varphi^\dagger(x)\partial_\mu\varphi(x) + m^2\varphi^\dagger(x)\varphi(x) - \Omega_0\Big) \\ &= \pi(x)\pi^\dagger(x) + \pi^\dagger(x)\pi(x) + \partial^0\varphi^\dagger(x)\partial_0\varphi(x) + \partial^i\varphi^\dagger(x)\partial_i\varphi(x) + m^2\varphi^\dagger(x)\varphi(x) - \Omega_0 \\ &= \pi(x)\pi^\dagger(x) + \pi^\dagger(x)\pi(x) - \pi(x)\pi^\dagger(x) + \partial^i\varphi^\dagger(x)\partial_i\varphi(x) + m^2\varphi^\dagger(x)\varphi(x) - \Omega_0 \\ &= \pi^\dagger(x)\pi(x) + \partial^i\varphi^\dagger(x)\partial_i\varphi(x) + m^2\varphi^\dagger(x)\varphi(x) - \Omega_0\end{aligned}$$

Which gives the Hamiltonian for the system as

$$H = \int d^4x\, \Big(\pi^\dagger(x)\pi(x) + \partial^i\varphi^\dagger(x)\partial_i\varphi(x) + m^2\varphi^\dagger(x)\varphi(x) - \Omega_0\Big)$$

Part C

Problem. *Expanding φ as*

$$\varphi(x) = \int \tilde{d^3k}\, \Big(a(k)\exp(ikx) + b^\dagger(k)\exp(-ikx)\Big)$$

solve for expressions of $a(k)$ and $b(k)$ in terms of $\varphi(x)$, $\varphi^\dagger(x)$ and their time derivatives.

We'll evaluate the integrals

$$\int d^3x\exp(-ikx)\varphi(x)$$

$$\int d^3x\exp(-ikx)\partial_t\varphi(x)$$

$$\int d^3x\exp(-ikx)\varphi^\dagger(x)$$

$$\int d^3x\exp(-ikx)\partial_t\varphi^\dagger(x)$$

So we first derive an expression for $\partial_t\varphi(x)$ and its conjugate.

$$\begin{aligned}\partial_t\varphi(x) &= \int \tilde{d^3k}\partial_t\, \Big(a(k)\exp(ikx) + b^\dagger(k)\exp(-ikx)\Big) \\ &= \int \tilde{d^3k}\, \Big(a(k)(-i\omega)\exp(ikx) - b^\dagger(k)(-i\omega)\exp(-ikx)\Big) \\ &= \int \tilde{d^3k}(-i\omega)\, \Big(a(k)\exp(ikx) - b^\dagger(k)\exp(-ikx)\Big)\end{aligned}$$

And similarly,

$$\partial_t \varphi^\dagger(x) = \int \tilde{d^3k}(-i\omega)\left(b(k)\exp(ikx) - a^\dagger(k)\exp(-ikx)\right)$$

Now, we can calculate those four integrals. One will be done explicitly, and the other three are done using the exact same calculation.

$$\begin{aligned}
\int d^3x \exp(-ikx)\varphi(x) &= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega} \exp(-ikx)\left(a(k')\exp(ik'x) + b^\dagger(k')\exp(-ik'x)\right) \\
&= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega}\left(a(k')\exp(i(k'-k)x) + b^\dagger(k')\exp(-i(k'+k)x)\right) \\
&= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega}(a(k')\exp(i(k'-k)^i x_i + i(\omega'-\omega)t) \\
&+ b^\dagger(k')\exp(-i(k'+k)^i x_i + -i(\omega'+\omega_t)) \\
&= \int \frac{d^3k'}{(2\pi)^3 2\omega}(a(k')(2\pi)^3\delta(k'-k)\exp(i(\omega'-\omega)t) \\
&+ b^\dagger(k')(2\pi)^3\delta(k'+k)\exp(-i(\omega'+\omega)t)) \\
&= \frac{1}{2\omega}(a(k) + b^\dagger(-k)\exp(-i(2\omega)t))
\end{aligned}$$

Following the same tactic, we find

$$\begin{aligned}
\int d^3x \exp(-ikx)\varphi(x) &= \frac{1}{2\omega}(a(k) + b^\dagger(-k)\exp(-i(2\omega)t)) \\
\int d^3x \exp(-ikx)\partial_t\varphi(x) &= \frac{-i}{2}(a(k) - b^\dagger(-k)\exp(-i(2\omega)t)) \\
\int d^3x \exp(-ikx)\varphi^\dagger(x) &= \frac{1}{2\omega}(b(k) + a^\dagger(-k)\exp(-i(2\omega)t)) \\
\int d^3x \exp(-ikx)\partial_t\varphi^\dagger(x) &= \frac{-i}{2}(b(k) - a^\dagger(-k)\exp(-i(2\omega)t))
\end{aligned}$$

So,

$$a(k) = \int d^3x \exp(-ikx)\left(\omega\varphi(x) + i\partial_t\varphi(x)\right)$$

and

$$b(k) = \int d^3x \exp(-ikx)\left(\omega\varphi^\dagger(x) + i\partial_t\varphi^\dagger(x)\right)$$

Part D

Problem. *Derive the commutation relations for $a(k)$ and $b(k)$ and their conjugates.*

By conjugating, we find that

$$a^\dagger(k) = \int d^3x \exp(ikx) \left(\omega\varphi^\dagger(x) - i\partial_t\varphi^\dagger(x)\right)$$
$$b^\dagger(k) = \int d^3x \exp(ikx) \left(\omega\varphi(x) - i\partial_t\varphi(x)\right)$$

So now we can calculate the commutators directly. Let's first rewrite the creation and annihilation operators in terms of φ and π instead:

$$a(k) = \int d^3x \exp(-ikx) \left(\omega\varphi(x) + i\pi^\dagger(x)\right)$$
$$b(k) = \int d^3x \exp(-ikx) \left(\omega\varphi^\dagger(x) + i\pi(x)\right)$$
$$a^\dagger(k) = \int d^3x \exp(ikx) \left(\omega\varphi^\dagger(x) - i\pi(x)\right)$$
$$b^\dagger(k) = \int d^3x \exp(ikx) \left(\omega\varphi(x) - i\pi^\dagger(x)\right)$$

Now we can use the canonical commutation relations to derive expressions for $[a(k), a^\dagger(k')]$ and $[b(k), b^\dagger(k')$. We calculate:

$$\begin{aligned}
a(k)a^\dagger(k') &= \int d^3x d^3y \exp(-ikx)\exp(ik'y) \left(\omega\varphi(x) + i\pi^\dagger(x)\right)\left(\omega'\varphi^\dagger(y) - i\pi(y)\right) \\
&= \int d^3x d^3y \exp(-ikx)\exp(ik'y) \left(\omega\varphi(x)\omega'\varphi^\dagger(y) - \omega\varphi(x)i\pi(y) + i\pi^\dagger(x)\omega'\varphi^\dagger(y) - i\pi^\dagger(x)i\pi(y)\right)
\end{aligned}$$

At this point, we invoke the rules

$$\begin{aligned}
\left[\varphi(x), \varphi^\dagger(y)\right] = \left[\pi(x), \pi^\dagger(y)\right] &= 0 && \text{Independence of fields} \\
[\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] &= 0 && \text{Canonical commutation relation} \\
[\varphi(x), \pi(y)] = [\varphi^\dagger(x), \pi^\dagger(y)] &= i\delta(x-y) && \text{Canonical commutation relation}
\end{aligned}$$

to commute the φ and π fields (and their conjugates) past each other. Thus, we find that

$$\begin{aligned}
a(k)a^\dagger(k') &= \int d^3x d^3y \exp(-ikx)\exp(ik'y) \left(\omega\varphi(x)\omega'\varphi^\dagger(y) - \omega\varphi(x)i\pi(y) + i\pi^\dagger(x)\omega'\varphi^\dagger(y) - i\pi^\dagger(x)i\pi(y)\right) \\
&= \int d^3x d^3y \exp(-ikx)\exp(ik'y) \\
&(\omega'\varphi^\dagger(y)\omega\varphi(x) - (i\pi(y)\omega\varphi(x) + i\omega(i\delta(x-y))) + (\omega'\varphi^\dagger(y)i\pi^\dagger(x) - i\omega'(i\delta(x-y))) - i\pi(y)i\pi^\dagger(x)) \\
&= a^\dagger(k')a(k) + \int d^3x d^3y \exp(-ikx)\exp(ik'y)(+\omega\delta(x-y) + \omega'\delta(x-y)) \\
&= a^\dagger(k')a(k) + \int d^3x \exp(-ikx)\exp(ik'x)(\omega + \omega') \\
&= a^\dagger(k')a(k) + (2\pi)^3 2\omega\delta(k-k')
\end{aligned}$$

and so

$$\left[a(k), a^\dagger(k')\right] = (2\pi)^3 2\omega\delta(k-k')$$

Carrying out the exact same calculation for $\left[b(k), b^\dagger(k)\right]$, we find that (by interchanging $\varphi \leftrightarrow \varphi^\dagger$ and $\pi \leftrightarrow \pi^\dagger$)

$$\left[b(k), b^\dagger(k')\right] = (2\pi)^3 2\omega\delta(k-k')$$

as well (since the commutators of φ, π and their conjugates are identical).

Part E

Problem. *Express the Hamiltonian in terms of $a, a^\dagger, b, b^\dagger$.*

We just need to substitute the expressions for the creation and annihilation operators into the expression for H. Namely, we have the following substitutions:

$$\begin{aligned}
\varphi(x) &= \int \tilde{dk} \left(a(k)\exp(ikx) + b^\dagger(k)\exp(-ikx)\right) \\
\varphi^\dagger(x) &= \int \tilde{dk} \left(b(k)\exp(ikx) + a^\dagger(k)\exp(-ikx)\right) \\
\pi(x) &= \int \tilde{dk}(-i\omega) \left(b(k)\exp(ikx) - a^\dagger(k)\exp(-ikx)\right) \\
\pi^\dagger(x) &= \int \tilde{dk}(-i\omega) \left(a(k)\exp(ikx) - b^\dagger(k)\exp(-ikx)\right) \\
\partial^i\varphi(x) &= \int \tilde{dk}(ik^i) \left(b(k)\exp(ikx) - a^\dagger(k)\exp(-ikx)\right) \\
\partial^i\varphi^\dagger(x) &= \int \tilde{dk}(ik^i) \left(a(k)\exp(ikx) - b^\dagger(k)\exp(-ikx)\right)
\end{aligned}$$

We can now express H as

$$\begin{aligned}
H &= \int d^3x \mathscr{H} \\
&= \int d^3x \left(\pi^\dagger(x)\pi(x) + \partial^i\varphi^\dagger(x)\partial_i\varphi(x) + m^2\varphi^\dagger(x)\varphi(x) - \Omega_0\right) \\
&= \int d^3x\tilde{dk}\tilde{dk'} \\
&((-i\omega)\left(a(k)\exp(ikx) - b^\dagger(k)\exp(-ikx)\right)(-i\omega')\left(b(k')\exp(ik'x) - a^\dagger(k')\exp(-ik'x)\right) \\
&+ (ik^i)\left(a(k)\exp(ikx) - b^\dagger(k)\exp(-ikx)\right)(ik'_i)\left(b(k')\exp(ik'x) - a^\dagger(k')\exp(-ik'x)\right) \\
&+ m^2\left(a(k)\exp(ikx) + b^\dagger(k)\exp(-ikx)\right)\left(b(k')\exp(ik'x) + a^\dagger(k')\exp(-ik'x)\right))
\end{aligned}$$

We collect like terms to get

$$\begin{aligned}
H &= \int d^3x\tilde{dk}\tilde{dk'} \\
&(\exp(i(k+k')x)\left((-i\omega)a(k)(-i\omega')b(k') + (ik^i)a(k)(ik'_i)b(k') + m^2a(k)b(k')\right) \\
&+ \exp(i(k-k')x)\left((-i\omega)a(k)(i\omega')a^\dagger(k) + (ik^i)a(k)(-ik'_i)a^\dagger(k') + m^2a(k)a^\dagger(k')\right) \\
&+ \exp(i(k'-k)x)\left((i\omega)b^\dagger(k)(-i\omega')b(k') + (-ik^i)b^\dagger(k)(ik'_i)b(k') + m^2b^\dagger(k)b(k')\right) \\
&+ \exp(-i(k+k')x)\left((i\omega)b^\dagger(k)(i\omega')a^\dagger(k') + (-ik^i)b^\dagger(k)(-ik'_i)a^\dagger(k') + m^2b^\dagger(k)a^\dagger(k')\right))
\end{aligned}$$

Integrating out the d^3x yields

$$\begin{aligned}
H &= \int \tilde{dk}\tilde{dk'} \\
&(2\pi)^3(\delta(k+k')\exp(i(\omega+\omega')t)a(k)b(k')\left(-\omega\omega' - k^ik'_i + m^2\right) \\
&+ \delta(k-k')\exp(i(\omega-\omega')t)a(k)a^\dagger(k')\left(\omega\omega' + k^ik'_i + m^2\right) \\
&+ \delta(k'-k)\exp(i(\omega'-\omega)t)b^\dagger(k)b(k')\left(\omega\omega' + k^ik'_i + m^2\right) \\
&+ \delta(-k-k')\exp(-i(\omega+\omega')t)b^\dagger(k)a^\dagger(k')\left(-\omega\omega' - k^ik'_i + m^2\right)
\end{aligned}$$

We then integrate out dk' yielding

$$
\begin{aligned}
H = \int \tilde{dk} \frac{1}{2\omega} \\
&\exp(2i\omega t) a(k) b(-k) \left(-\omega^2 + k^2 + m^2\right) \\
&+ a(k) a^\dagger(k) \left(\omega^2 + k^2 + m^2\right) \\
&+ b^\dagger(k) b(k) \left(\omega^2 + k^2 + m^2\right) \\
&+ \exp(-2i\omega t) b^\dagger(k) a^\dagger(-k) \left(-\omega^2 + k^2 + m^2\right)
\end{aligned}
$$

Since $\omega^2 = k^2 + m^2$, this simplifies greatly to

$$H = \int \tilde{dk}\omega \left(a(k) a^\dagger(k) + b^\dagger(k) b(k) \right)$$

We've omitted the offset factor Ω_0, but it can be safely added in to the final result. Of course, if we want the ground state to have zero energy, we should move the a operator to the front, yielding

$$H = \int \tilde{dk}\omega \left(a^\dagger(k) a(k) + b^\dagger(k) b(k) + (2\pi)^3 2\omega\delta(0) \right)$$

and if we set Ω_0 to cancel with this value, we get

$$H = \int \tilde{dk}\omega \left(a^\dagger(k) a(k) + b^\dagger(k) b(k) \right)$$

as desired.

Problem 3

Problem. *From the previous problem, we have a conserved current*

$$J^\mu = -i\varphi^\dagger \partial^\mu \varphi + i\varphi \partial^\mu \varphi^\dagger$$

Express the charge

$$Q = \int d^3x J^0$$

in terms of a, b and their conjugates.

We use the substitutions

$$\begin{aligned}
\varphi(x) &= \int \tilde{dk} \left(a(k)\exp(ikx) + b^\dagger(k)\exp(-ikx) \right) \\
\varphi^\dagger(x) &= \int \tilde{dk} \left(b(k)\exp(ikx) + a^\dagger(k)\exp(-ikx) \right) \\
\partial^\mu \varphi(x) &= \int \tilde{dk}(ik^\mu) \left(a(k)\exp(ikx) - b^\dagger(k)\exp(-ikx) \right) \\
\partial^\mu \varphi^\dagger(x) &= \int \tilde{dk}(ik^\mu) \left(b(k)\exp(ikx) - a^\dagger(k)\exp(-ikx) \right)
\end{aligned}$$

and calculate directly

$$\begin{aligned}
Q &= \int d^3x J^0 \\
&= \int d^3x \tilde{dk}\tilde{dk}' \\
&((-i)(b(k)\exp(ikx) + a^\dagger(k)\exp(-ikx))(i\omega')(a(k')\exp(ik'x) - b^\dagger(k')\exp(-ik'x)) \\
&+ (i)(a(k)\exp(ikx) + b^\dagger(k)\exp(-ikx))(i\omega')(b(k')\exp(ik'x) - a^\dagger(k')\exp(-ik'x))) \\
&= \int d^3x \tilde{dk}\tilde{dk}' \\
&(\omega'(b(k)\exp(ikx) + a^\dagger(k)\exp(-ikx))(a(k')\exp(ik'x) - b^\dagger(k')\exp(-ik'x)) \\
&- (\omega')(a(k)\exp(ikx) + b^\dagger(k)\exp(-ikx))(b(k')\exp(ik'x) - a^\dagger(k')\exp(-ik'x))) \\
&= \int d^3x \tilde{dk}\tilde{dk}' \\
&\omega'(\exp(i(k+k')x)(b(k)a(k') - a(k)b(k')) \\
&- \exp(i(k-k')x)(b(k)b^\dagger(k') - a(k)a^\dagger(k')) \\
&+ \exp(i(k'-k)x)(a^\dagger(k)a(k') - b^\dagger(k)b(k')) \\
&- \exp(-i(k+k')x)(a^\dagger(k)b^\dagger(k') - b^\dagger(k)a^\dagger(k')))
\end{aligned}$$

Integrating out d^3x yields

$$\begin{aligned}
Q = &\int \tilde{dk}\tilde{dk}' \\
&\omega'(2\pi)^3(\delta(k+k')\exp(i(\omega+\omega')t)(b(k)a(k') - a(k)b(k')) \\
&- \delta(k-k')\exp(i(\omega-\omega')t)(b(k)b^\dagger(k') - a(k)a^\dagger(k')) \\
&+ \delta(k'-k)\exp(i(\omega'-\omega)t)(a^\dagger(k)a(k') - b^\dagger(k)b(k')) \\
&- \delta(-k-k')\exp(-i(\omega+\omega')t)(a^\dagger(k)b^\dagger(k') - b^\dagger(k)a^\dagger(k')))
\end{aligned}$$

and finally integrating out d^3k we get

$$
\begin{aligned}
Q = \int \tilde{dk} \\
&\frac{1}{2}(\exp(i(2\omega)t)(b(k)a(-k) - a(k)b(-k)) \\
&- (b(k)b^\dagger(k) - a(k)a^\dagger(k)) \\
&+ (a^\dagger(k)a(k) - b^\dagger(k)b(k)) \\
&- \exp(-i(2\omega)t)(a^\dagger(k)b^\dagger(-k) - b^\dagger(k)a^\dagger(-k))) \\
&= \int \tilde{dk}\frac{1}{2}((a(k)a^\dagger(k) - b(k)b^\dagger(k)) + (a^\dagger(k)a(k) - b^\dagger(k)b(k))) \\
&= \int \tilde{dk}((a^\dagger(k)a(k) - b^\dagger(k)b(k)))
\end{aligned}
$$

as desired. We note that this can be expressed with the number operators N_a and N_b as

$$Q = N_a - N_b$$

and so a particles and b particles have opposite equal charges.

Problem 4

Problem. *Consider the Lagrangian*

$$\mathscr{L} = -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \frac{1}{2}k\partial^\nu A^\mu \partial_\mu A_\nu - \frac{1}{2}m^2 A^\nu A_\nu + J^\nu A_\nu$$

Part A

Problem. *Compute the variation of the action, and find the equation of motion for A^ν.*

We'll compute the variation of the action directly, noting that

$$\delta(\partial^\mu A^\nu) = \partial^\mu \delta A^\nu$$

We find

$$\begin{aligned}
\delta S &= \int d^4x \delta\mathscr{L} \\
&= \int d^4x(-\frac{1}{2}\delta(\partial^\mu A^\nu \partial_\mu A_\nu) + \frac{1}{2}k\delta(\partial^\nu A^\mu \partial_\mu A_\nu) - \frac{1}{2}m^2\delta(A^\nu A_\nu) + J^\nu \delta A_\nu) \\
&= \int d^4x(-\frac{1}{2}\partial^\mu \delta A^\nu \partial_\mu A_\nu - \frac{1}{2}\partial^\mu A^\nu \partial_\mu \delta A_\nu \\
&+ \frac{1}{2}k\partial^\nu \delta A^\mu \partial_\mu A_\nu + \frac{1}{2}k\partial^\nu A^\mu \partial_\mu \delta A_\nu \\
&- \frac{1}{2}m^2 \delta A^\nu A_\nu - \frac{1}{2}m^2 A^\nu \delta A_\nu + J^\nu \delta A_\nu)
\end{aligned}$$

The first two terms combine if we raise/lower indices on the second term, as well as the fifth and sixth terms. We do so, and we also interchange the indices on the third term to make $\delta A^\mu \to \delta A^\nu$.

$$\delta S = \int d^4x(-\partial^\mu \delta A^\nu \partial_\mu A_\nu + \frac{1}{2}k\partial^\mu \delta A^\nu \partial_\nu A_\mu + \frac{1}{2}k\partial^\nu A^\mu \partial_\mu \delta A_\nu - m^2 \delta A^\nu A_\nu + J^\nu \delta A_\nu)$$

Now, if we raise/lower indices on the third term, we get

$$\delta S = \int d^4x(-\partial^\mu \delta A^\nu \partial_\mu A_\nu + k\partial^\mu \delta A^\nu \partial_\nu A_\mu - m^2 \delta A^\nu A_\nu + J^\nu \delta A_\nu)$$

Integrating by parts yields

$$\begin{aligned}
\delta S &= \int d^4x(\delta A^\nu \partial^\mu \partial_\mu A_\nu - k\delta A^\nu \partial^\mu \partial_\nu A_\mu - m^2 \delta A^\nu A_\nu + J^\nu \delta A_\nu) \\
&= \int d^4x(\partial^2 A_\nu - k\partial_\nu \partial^\mu A_\mu - m^2 A_\nu + J_\nu)\delta A^\nu
\end{aligned}$$

Which holds for arbitrary variations if the term in the parentheses is always zero. Thus, we have

$$-\partial^2 A_\nu + k\partial_\nu \partial^\mu A_\mu + m^2 A_\nu = J_\nu$$

Multiplying each side by $g^{\nu\rho}$ we get

$$\begin{aligned}
g^{\nu\rho}(-\partial^2 A_\nu + m^2 A_\nu) + kg^{\nu\rho}\partial_\nu \partial^\mu A_\mu &= g^{\nu\rho} J_\nu \\
g^{\nu\rho}(-\partial^2 A_\nu + m^2 A_\nu) + kg^{\mu\rho}\partial_\mu \partial^\nu A_\nu &= J^\rho \\
\left[g^{\nu\rho}(-\partial^2 + m^2) + k\partial^\rho \partial^\nu\right] A_\nu &= J^\rho
\end{aligned}$$

as desired.

Part B

Problem. *Find the equation of motion for $\partial_\mu A^\mu$.*

We follow the hint and act on the equation of motion for A with (changing μ with ρ) ∂_μ

$$\begin{aligned}\partial_\mu \left(g^{\nu\mu}(-\partial^2 + m^2)A_\nu + k\partial^\mu\partial^\nu A_\nu\right) &= \partial_\mu J^\mu \\ (-\partial^2 + m^2)\partial^\nu A_\nu + k\partial_\mu\partial^\mu\partial^\nu A_\nu &= \partial_\mu J^\mu\end{aligned}$$

Where the last equality was obtained by observing that derivatives commute with each other and the metric, and $g^{\mu\nu}\partial_\mu = \partial^\nu$. Thus, substituting $\varphi = \partial^\nu A_\nu$, we get

$$(-\partial^2 + m^2)\varphi + k\partial^2\varphi = \partial_\mu J^\mu$$

and if we set $k = 1$, we get the equation

$$m^2\varphi = \partial_\mu J^\mu$$

as desired.

Part C

Problem. *Show that for $k = 1$ and $m = 0$, we recover Maxwell's equations.*

We have the equation

$$-\partial^2 A_\nu + k\partial_\nu\partial^\mu A_\mu + m^2 A_\nu = J_\nu$$

Substituting $m = 0, k = 1$ we get

$$-\partial^2 A_\nu + \partial_\nu\partial^\mu A_\mu = J_\nu$$

and if we raise/lower indices across the board, we get

$$\begin{aligned} -\partial^\mu\partial_\mu A^\nu + \partial^\nu\partial_\mu A^\mu &= J^\nu \\ \partial_\mu\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) &= J^\nu \end{aligned}$$

Or, using the definition of the EM field strength tensor $F^{\mu\nu} = \partial^{[\mu}A^{\nu]} = \partial^\mu A^\nu - \partial^\nu A^\mu$ we obtain the expression

$$\partial_\mu F^{\nu\mu} = J^\nu$$

Which is exactly Maxwell's equations in covariant form. Notice that the definition of F guarantees it satisfies

$$F^{\mu\nu,\rho} + F^{\rho\mu,\nu} + F^{\nu\rho,\mu} = 0$$

which is the second half of Maxwell's equations in covariant form.