Problem Set 1

Daniel Halmrast

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PROBLEM 1

Show that the line with two origins is not Hausdorff, but is locally Euclidean and is second countable.

Proof. The line with two origins is defined as the quotient of $\mathbb{R}_a \coprod \mathbb{R}_b$ by the equivalence relation $(x, a) \sim (x, b)$ for $x \neq 0$ (the notation (x, a) indicates, for example, the point x on the line $\mathbb{R}_a$. For convenience, an interval will be represented similarly as, for example, $((c, d), a)$ for the interval (c, d) in $\mathbb{R}_a$).

To show this space is locally Euclidean, consider any point $[x]$ on the line with two origins. if $[x] \neq 0_a, 0_b$, then $[x]$ has a neighborhood $[((x - \epsilon, x + \epsilon), a)] \cong (x - \epsilon, x + \epsilon)$ which does not intersect the origins, and is homeomorphic to the same interval in $\mathbb{R}$. If $[x]$ is equal to 0_a or 0_b (without loss of generality, let $[x] = 0_a$), then the neighborhood $[(-1, 1), a)] \cong (-1, 1)$ is homeomorphic to that interval in $\mathbb{R}$. Thus, each point has a neighborhood homeomorphic to an open interval in $\mathbb{R}$ and the line with two origins is locally Euclidean.

To see this space is second countable, consider the canonical projection map $\pi : \mathbb{R} \coprod \mathbb{R} \to \mathbb{R} \coprod \mathbb{R}/\sim$ given by $\pi(x) = [x]$. This projection is an open map. To see this, consider a general basic open set $((c, d), a)$ or $((c, d), b)$. For (c, d) such that $0 \notin (c, d)$, the interval projects to (c, d), which is open. Now, without loss of generality, consider an interval $((c, d), a)$ containing 0. This projects to the set $(c, 0) \cup \{0_a\} \cup (0, d)$. This set can easily be seen to be open in the quotient by checking its inverse image is open. Thus, since π maps basic open sets to open sets, π is an open map. Noting also that π

is a surjection, we can apply Lemma 1 for quotients (defined in class) to show that the quotient is second countable.

Finally, to see this space is not Hausdorff, consider the two origins 0_a and 0_b. Each open neighborhood of 0_a contains some interval $(-\epsilon_a, 0) \cup \{0_a\} \cup (0, \epsilon_a)$. Similarly, each neighborhood of 0_b contains some interval $(-\epsilon_b, 0) \cup \{0_b\} \cup (0, \epsilon_b)$. These intervals intersect, so the neighborhoods of 0_a and 0_b intersect as well. Since this holds true for all neighborhoods of the origins, the two origins are not seperable by open sets, and the line with two origins is not Hausdorff. $\qquad\square$

Problem 2

Show that the coproduct of $\mathbb{R}_\alpha$ for $\alpha \in I$ with I uncountable is locally Euclidean and Hausdorff, but is not second countable.

Proof. To see that this space is locally Euclidean, consider a point

$$x \in \coprod_{\alpha \in I} \mathbb{R}_\alpha$$

Since x is in the coproduct, it is in exactly one of the $\mathbb{R}_\alpha$. Thus, the neighborhood $\mathbb{R}_\alpha$ of x is homeomorphic to $\mathbb{R}$. Since this holds for any x, the space is locally Euclidean.

Furthermore, consider two points x, y distinct from each other in the coproduct. In the case $x \in \mathbb{R}_\alpha$ and $y \in \mathbb{R}_\beta$ for $\alpha \neq \beta$, the open sets $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ separate x and y. If instead x and y are from the same copy $\mathbb{R}_\alpha$, then they are seperable in the subspace $\mathbb{R}_\alpha$ (since $\mathbb{R}$ is Hausdorff) and are seperable in the coproduct.

However, the coproduct is not second countable. Let $\mathscr{B}$ be a basis for the coproduct. Then, consider the set of disjoint open sets

$$\{\mathbb{R}_\alpha | \alpha \in I\}$$

By the definition of a basis, each $\mathbb{R}_\alpha$ must contain at least one basis element. Furthermore, since each are disjoint from the rest, the basis element is unique. Thus, for each $\alpha \in I$, $\mathscr{B}$ must have at least one distinct basis element $B_\alpha \subset \mathbb{R}_\alpha$. Thus, $\mathscr{B}$ must have uncountably many elements. Therefore, the coproduct does not admit a countable basis, and is not second countable. $\qquad\square$

Problem 7

Let $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ be given as

$$\sigma(x^1, x^2, \ldots, x^n) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}$$

And similarly define $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

Show that $\sigma(x)$ sends a point to the intersection of the $\mathbb{R}^n$ subspace and the line from the north pole N through the point x.

Proof. Let $x \in S^n \setminus \{N\}$, and consider the line through N and x parameterized as

$$f(t) = tx + (1-t)N$$

This function intersects the $\mathbb{R}^n$ subspace when the $n+1$ coordinate is zero. Now, the $n+1$ coordinate is parameterized as

$$f^{n+1}(t) = tx^{n+1} + (1-t)(1)$$

This is zero when $t = i = t_0 = \frac{1}{1-x^{n+1}}$. At this point, the other n coordinates of x are given by

$$f^i(t) = tx^i + (1-t)(0) = tx^i$$

so

$$f^i(t_0) = \frac{x^i}{1-x^{n+1}}$$

Thus, we have recovered $\sigma(x)$ as desired.

Following similar logic for the $\widetilde{\sigma}(x)$ map, we see that the line connecting S and x is given by the equation

$$f(t) = tx + (1-t)S$$

Similarly, the point where the $n+1$ coordinate is zero in this equation is given by

$$0 = f^{n+1}(t) = tx^{n+1} + (1-t)(-1)$$

which is zero when $t = t_0 = \frac{1}{x+1}$. Thus, the point of intersection is

$$f^i(t_0) = \frac{x^i}{1+x^{n+1}}$$

Since $\sigma_i(x) = \frac{x^i}{1-x^{n+1}}$,

$$-\sigma^i(-x) = -\frac{-x^i}{1-(-x^{n+1})} = \frac{x^i}{1+x^{n+1}}$$

which is just $\widetilde{\sigma}(x)$ as desired.

$\square$

Show the map σ is bijective and compute its inverse.

Proof. To see that σ is injective, observe that σ was defined qualitatively as the intersection of the target subspace $\mathbb{R}^n$ with the line uniquely determined by the points x and N. So, let x_1 and x_2 be elements of S^n such that $\sigma(x_1) = \sigma(x_2)$. Then, consider the line defined through the points $\sigma(x_1)$ and N. This line crosses the punctured sphere $S^n \setminus \{N\}$ in exactly one location, which must be x_1, since $\sigma(x_1)$ determined the line, but it also must be x_2 by a similar argument. Thus, $x_1 = x_2$.

Surjectivity will be proved by showing that the inverse map has no domain restrictions on $\mathbb{R}^n$.

The inverse is given as

$$\sigma^{-1}(u^1, u^2, \ldots, u^n) = \frac{(2u^1, 2u^2, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}$$

This is easily seen to be the inverse of σ, since

$$\sigma(\sigma^{-1}(u^1, \ldots, u^n)) = \sigma(\frac{(2u^1, 2u^2, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1})$$

$$= \left(\frac{1}{|u|^2 + 1} \frac{1}{1 - \frac{|u|^2 + 1}{|u|^2 - 1}} \right) (2u^1, \ldots, 2u^n)$$

$$= (u^1, \ldots, u^n)$$

Which has no domain restrictions, so σ is surjective. $\qquad\square$

Compute the transition map, and verify it is C^∞.

Proof. The transition map $\widetilde{\sigma} \circ \sigma^{-1}$ can be seen to be

$$\widetilde{\sigma} \circ \sigma^{-1} = \widetilde{\sigma} \left(\frac{(2u^1, 2u^2, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \right)$$

$$= -\sigma \left(\frac{(-2u^1, -2u^2, \ldots, -2u^n, -|u|^2 + 1)}{|u|^2 + 1} \right)$$

$$= -\frac{1}{1 - \frac{-|u|^2 + 1}{|u|^2 + 1}} \left(-2u^1, \ldots, -2u^n \right)$$

which is clearly smooth, since it is a quotient of smooth functions. $\qquad\square$

Part d

Show this structure is compatible with the projection structure defined in the book.

Proof. To show this, consider the compatibility maps

$$\phi_i \circ \sigma^{-1} = \frac{(2u^1, 2u^2, \ldots \hat{u_i}, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}$$

which is clearly smooth for any i. The reverse $\sigma \circ \phi_i^{-1}$ is also clearly smooth, so the charts are compatible with each other (Repeat the argument for the south pole stereographic projection to attain full compatibility at all points, but the argument is the same). $\qquad\square$

Problem "Batch 1 Extra"

Show that the graph of $|x|$ with the subspace topology from $\mathbb{R}^2$ is a smooth manifold.

Proof. Consider the projection mapping:

$$\phi(x, |x|) = x$$

This is clearly defined for all points in the graph, and is easily seen to be a homeomorphism of the graph with $\mathbb{R}$. Furthermore, since this is a global atlas, no transition maps need to be calculated, and the manifold is trivially smooth. $\qquad\square$

Batch 2 Problem 1

Show that a manifold M with a smooth structure has infinitely many such structures.

Proof. Let (U_α, ϕ_α) be an atlas of M such that for all U_α, U_β distinct pairs, the coordinate charts on the intersection $U_\alpha \cap U_\beta$ do not contain the origin of $\mathbb{R}^n$, Let there also be some U_0 such that the coordinate chart on U_0 does contain the origin. Furthermore, we require that each U_α map by ϕ_α into the unit ball $\mathbb{B}^n$.

With this construction in place, we construct for each $s > 0$ a new differentiable structure on M by composing the coordinate charts with the homeomorphism

$$F_s(x) = |x|^{s-1}x$$

from $\mathbb{R}^n$ to itself. Clearly, this is a homeomorphism, but at the origin it fails to be smooth in the standard differential structure on $\mathbb{R}^n$.

Now, consider the new atlas $(U_\alpha, F_s \circ \phi_\alpha)$. The transition map between two charts is $F_s \circ \phi_\beta \circ \phi_\alpha^{-1} \circ F_s^{-1}$, which is smooth away from the origin. Since the intersections were constructed to avoid the origin, the transition map is smooth.

However, checking compatibility with some other $t \neq s$ will fail on U_0, since the compatibility map $F_t \circ \phi_0 \circ \phi_0^{-1} \circ F_s^{-1}$ which is not smooth on U_0.

Thus, for each $s > 0$, we get a distinct differentiable structure on M as desired.

$\square$

BATCH 2 PROBLEM 2

Show that $\mathbb{C}P^n$ is a compact $2n$-manifold.

Proof. Observe that $\mathbb{C}P^n$ can be constructed as the quotient $S^{2n+1}/\sim$ with $\sim$ identifying antipodal points (i.e. $x \sim y \iff x = \lambda y$ for some $\lambda \in \mathbb{C}$). Now, the quotient map from S^{2n+1} (which is compact) to $\mathbb{C}P^n$ is continuous, so the projective space $\mathbb{C}P^n$ is compact as well.

Now, let's push forward the differential structure from S^{2n+1} along the quotient map q. Since q is a local homeomorphism, we can choose a collection of open sets V_α in S^{2n+1} such that $q|_{V_\alpha}$ is a homeomorphism, $V_\alpha \subset U_\alpha$ for some U_α in the chart (U_α, ϕ_α), and $q(V_\alpha)$ covers $\mathbb{C}P^n$. Then, the charts on $\mathbb{C}P^n$ is the collection $(q(V_\alpha), \phi_\alpha \circ q|_{V_\alpha}^{-1})$, which is clearly a differential structure, as it inherits compatibility from the differential structure on S^{2n+1}. $\square$

BATCH 2 PROBLEM 3

Show the map $\widetilde{F} : \mathbb{R}P^n \to \mathbb{R}P^k$ given by $\widetilde{F}([x]) = [F(x)]$ for a homogeneous map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ is well-defined and smooth.

Proof. To show well-definedness of $\widetilde{F}$, consider $[x] = [cx]$ for some nonzero C. Now, $\widetilde{F}([x]) = [F(x)] = [c^d F(x)] = [F(cx)] = \widetilde{F}([cx])$ as desired.

Now, let's show that $\widetilde{F}$ is smooth. Now, the standard smooth structure on projective space is given by $\phi_i([x^1, \ldots, x^{n+1}]) = \frac{1}{x^i}(x^1, \ldots, \hat{x^i}, \ldots, x^{n+1})$.

The compatibility map $\phi_i \circ \widetilde{F} \circ \phi_i^{-1}$ is given as

$$\phi_i([F(u^1, \ldots, u^{i-1}, 1, u^{i+1}, \ldots, u^n)])$$

which is the composition of smooth functions, and is easily seen to be smooth. $\square$

BATCH 2 PROBLEM 4

For $\mathbb{R}$ the standard real line, and $\widetilde{\mathbb{R}}$ the real line with the structure given by the map $t \mapsto t^3$, $f : \mathbb{R} \to \mathbb{R}$ a smooth function:

Part a

Show that $f : \mathbb{R} \to \widetilde{\mathbb{R}}$ is smooth.

Proof. To show f is smooth, we consider the charts ϕ (the standard chart on $\mathbb{R}$ given by the identity) and ψ the cubic chart. Now, we check that the compatibility map $\psi \circ f \circ \phi^{-1}$ is smooth. However, this is just $\psi \circ f \circ \phi^{-1}(t) = (f(t))^3$, which, for f smooth, is a smooth function. Thus, f is smooth from $\mathbb{R} \to \widetilde{\mathbb{R}}$. $\qquad\square$

Part b

Show that for f such that $f^{(n)}(0) = 0$ for integer n not a multiple of 3, the map $f : \widetilde{\mathbb{R}} \to \mathbb{R}$ is smooth.

Proof. To show this, we must show that the compatibility map $\phi \circ f \circ \psi^{-1}$ is a smooth map, where ϕ and ψ are the same as in part a.

This map is

$$F(x) = \phi \circ f \circ \psi^{-1}(x) = f(x^{\frac{1}{3}})$$

for a smooth function f. Now, this is differentiable away from zero without issue, so consider the derivatives at zero. $F'(0) = f'(0) * \frac{1}{3} x^{\frac{-2}{3}}|_{x=0}$ which has problems at zero in the second term. But, since $f'(0) = 0$, the problems can be ignored. Similarly for higher derivatives, terms with an nth derivative of $x^{\frac{1}{3}}$ for n not a multiple of three will have problems, but those terms are the ones where $f^{(n)}(0) = 0$. So, the overall function F is smooth.

Note that the arguments above show that this derivative condition characterizes the functions that are smooth from $\widetilde{\mathbb{R}} \to \mathbb{R}$ as desired. $\qquad\square$