Homotopy Theory

Problem Set 4

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Preliminaries

Lemma 1. *For two paths $f, g : I \to X$ with $f \simeq g$ relative to ∂I, and $h : I \to X$ with $f(1) = g(1) = h(0)$, then $hf \simeq hg$, where hf is the path that first traverses f first, and then h (similarly for hg).*

Similarly, if $h : I \to X$ with $h(1) = f(0) = g(0)$, then $fh \simeq gh$.

Proof. Let $F : I \times I \to X$ be the homotopy from f to g relative to ∂I, and let $h : I \to X$ be such that $f(1) = g(1) = h(0)$. The homotopy between hf and hg is given by

$$H : I \times I \to X$$
$$H(t, s) = \begin{cases} F(2t, s), & \text{if } t < \frac{1}{2} \\ h(2(t - \frac{1}{2})), & \text{else} \end{cases}$$

That is, run the homotopy on the f section of the path, and leave h alone. Since the homotopy F fixes the endpoints, it follows that $F(1, s) = h(0)$ for all s, and the homotopy H is well-defined. H is clearly continuous, then, by the pasting lemma. Therefore, $H(t, 0) = hf \simeq H(t, 1) = hg$ as desired.

Suppose instead that $h : I \to X$ is such that $h(1) = f(0) = g(0)$. Then, it follows that $\overline{h}(0) = \overline{f}(1) = \overline{g}(1)$ which satisfies the hypotheses for the previous result, and so $\bar{h}\bar{f} \simeq \bar{h}\bar{g}$, and so $\overline{fh} \simeq \overline{gh}$, which immediately implies that $fh \simeq gh$ as desired. $\square$

Problem 1

Show that the composition of paths satisfies the following property: if $f_0 \cdot g_0 \simeq f_1 \cdot g_1$, and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

Proof. Since $g_0 \simeq g_1$ (assumed to be relative to ∂I), we have that

$$f_0 g_0 \simeq f_0 g_1$$

by Lemma 1. Now, since $f_1 g_1 \simeq f_0 g_0$, it follows that $f_1 g_1 \simeq f_0 g_1$ by transitivity of $\simeq$.

Now, letting $\overline{g}$ be the inverse path of g, we have that

$$\begin{aligned} f_1 g_1 &\simeq f_0 g_1 \\ f_1 g_1 \overline{g_1} &\simeq f_0 g_1 \overline{g_1} && \text{by Lemma 1} \\ f_1 &\simeq f_0 && \text{by } g_1 \overline{g_1} \simeq 0 \text{ and Lemma 1} \end{aligned}$$

as desired. □

Problem 2

Show that the change of basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. Let g, h be paths in a space X with the same starting and ending points, and such that $g \simeq h$. We will show that $\beta_h = \beta_g$. In particular, we will show that conjugating by h is homotopic to conjugating by g.

So, let f be a loop based at the endpoint of g, h. We will show that $\bar{g}fg \simeq \bar{h}fh$. This, however, is just a straightforward application of Lemma 1.

To see this, we note that since $g \simeq h$, we have that $fg \simeq fh$. Now, since $barg \simeq \bar{h}$, we can also write

$$\bar{g}fg \simeq \bar{g}fh = \bar{g}(fh) \simeq \bar{h}(fh) = \bar{h}fh$$

as desired □

Problem 3

For a path-connected space X, show that $\pi_1(X)$ is Abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of h.

Proof. ($\implies$) Suppose that for a path-connected space X, we have that $\pi_1(x)$ is Abelian. Furthermore, let h, g be two paths in X such that $g(0) = h(0) = x_0$ and $g(1) = h(1) = x_1$. Furthermore, let f be a loop based at x_1. We wish to show that $\beta_h([f]) = \beta_g([f])$ which is equivalent to showing that $\beta_{\bar{g}}\beta_h([f]) = [f]$. Now, $\beta_{\bar{g}}\beta_h([f])$ is just $[g\bar{h}fh\bar{g}]$. Note however that $h\bar{g}$, $g\bar{h}$, and f are loops based at x_1. Since $\pi_1(X)$ is Abelian, it follows that

$$\begin{aligned}
&= [g\bar{h}][f][h\bar{g}] \\
&= [f][g\bar{h}][h\bar{g}] \\
&= [f][g\bar{h}h\bar{g}] \\
&= [f]
\end{aligned}$$

as desired.

($\impliedby$) Suppose that X is such that for any two paths h, g with $h(0) = g(0) = x_0$ and $h(1) = g(1) = x_1$, we have that $\beta_h = \beta_g$. We wish to show that for any two elements $[f_1], [f_2] \in \pi_1(X)$, we have that $[f_1][f_2] = [f_2][f_1]$. Alternately, we can show that $[f_1][f_2][\bar{f}_1] = [f_2]$. This is obvious, though, since f_1 and f_2 satisfy the hypotheses for h, g, which implies that $\beta_{f_1} = \beta_{f_2}$. So,

$$\begin{aligned}
[f_2][\bar{f}_1] &= \beta_{\bar{f}_1}[f_2] \\
&= \beta_{\bar{f}_2}[f_2] \\
&= [f_2][f_2][f_2^{-1}] \\
&= [f_2]
\end{aligned}$$

as desired. $\square$

Problem 4

Show that if a subspace $x \subset \mathbb{R}^n$ is locally star-shaped, then every path in X is homotopic in X to a piecewise linear path. Show specifically this holds when X is open, and when X is a union of finitely many closed convex sets.

Proof. To begin with, let γ be a path in X. At each point $\gamma(t) \in X$, let S_t be a star-shaped neighborhood around $\gamma(t)$. In particular, $\{S_t\}$ is an open cover of $\gamma(I)$, and since $\gamma(I)$ is compact, it follows that there is some finite subcover $\{S_i\}_{i=1}^n$. In particular, we can take a finite subcover such that each point $\gamma(t)$ is in at most two open sets in the subcover, and the preimages $\gamma^{-1}(S_i)$ and $\gamma^{-1}(S_j)$ are distinct (neither contains the other). We further require that each open set in the subcover be an interval. Finally, order the subcover sequentially. That is, let S_1 be the open set containing $\gamma(0)$, and let S_i be the open set that overlaps with S_{i-1}. Define a set of partition points $\{t_i\}_{i=1}^{n-1}$ such that t_i lies in the intersection of S_i and S_{i+1}.

Now, for each S_i, let x_i be the distinguished point in the star-shaped neighborhood. That is, for each $x \in S_i$, the line segment from x to x_i is in S_i.

We are finally ready to describe the homotopy from γ to a piecewise linear function. We note first that each S_i is simply connected. In particular, all paths in S_i with fixed endpoints are homotopic to each other.

On S_1, we can homotope the segment of the path $\gamma([0, t_1])$ to the path obtained by taking the line segment from $\gamma(0)$ to x_1 and then the line segment from x_1 to $\gamma(t_1)$. Generally, on S_i, homotope the path $\gamma([t_{i-1}, t_i])$ to the path from $\gamma(t_{i-1})$ to x_i, then from x_i to $\gamma(t_i)$. Finally, in S_n, we homotope the path $\gamma([t_n, 1])$ to the path from $\gamma(t_n)$ to $\gamma(1)$.

On each open set, we homotoped to a piecewise linear path, and they agree on the intersection, and so the resulting path is piecewise linear as desired. $\square$

Problem 5

Show that for every space X, the following are equivalent:

(a) Every map $S^1 \to X$ is homotopic to the constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi(X, x_0) = 0$ for all $x_0 \in X$.

Proof. ((a) $\implies$ (b)) Suppose $f : S^1 \to X$ is such that f is homotopic to a constant map x_0. In particular, we have a homotopy $F : S^1 \times I \to X$ with $F(x, 0) = f(x)$ and $F(x, 1) = x_0$.

Now, the disk D^2 is homeomorphic to the quotient $(S^1 \times I)/S^1 \times \{1\}$ via the homeomorphism

$$\phi : D^2 \to (S^1 \times I)/S^1 \times \{0\} \phi(r, \theta) \quad = [(\theta, r)]$$

Where the coordinates on D^2 are polar coordinates with $r \leq 1$, $\theta \in [0, 2\pi)$, and $[(\theta, r)]$ is the equivalence class of the point $(\theta, r) \in S^1 \times I$.

Now, we can define $\tilde{F} : (S^1 \times I)/S^1 \times \{1\} \to X$ to be the unique map that makes the diagram

$$\begin{tikzcd}
S^1 \times I \arrow[rr, "F"] \arrow[dr, two heads, "q"'] & & X \\
& (S^1 \times I)/S^1 \times \{1\} \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

commute. Here, q is the canonical quotient map. $\tilde{F}$ is well-defined, since F is constant on the fibers of q. This is clear, since the only nontrivial fiber of q is the subspace $S^1 \times \{1\}$, which F sends identically to x_0.

Thus, using the homeomorphism above, we find the map $\tilde{F} \circ \phi$ to be an extension of f. This is evident, since $\tilde{F} \circ \phi|_{\partial D^2}$ is just $\tilde{F}|_{S^1 \times \{0\}}$ which is just $F(x, 0) = f(x)$ as desired.

((b) $\implies$ (c)) Suppose that any map $f : S^1 \to X$ can be extended to a map $\tilde{f} : D^2 \to X$. Now, let $x_0 \in X$ be arbitrary. We will show that $\pi_1(X, x_0) = 0$. in particular, we will show that any loop based at x_0 is homotopic to the constant loop.

So, let $f : S^1 \to X$ be a loop such that $f(0) = x_0$. By (b), we know that such an f extends to a $\tilde{f} : D^2 \to X$. Now, via the homeomorphism above, we have a map

$$\tilde{F} : (S^1 \times I)/S^1 \times \{1\} \to X$$

given by $\tilde{F} = \tilde{f} \circ \phi^{-1}$. Furthermore, for $q : S^1 \times I \to (S^1 \times I)/S^1 \times \{1\}$ the canonical quotient map, we have a map

$$F : S^1 \times I \to X$$

given by $F = \tilde{F} \circ q$. Now,

$$\begin{aligned} F|_{S^1 \times \{0\}} &= \tilde{F}|_{S^1 \times \{0\}} \\ &= \tilde{f}|_{\partial D^2} \\ &= f \end{aligned}$$

and furthermore

$$F|_{S^1 \times \{1\}} = \tilde{F}|_{[S^1 \times \{1\}]} = x_1$$

for some $x_1 \in X$. This is because $q(S^1 \times \{1\}) = [S^1 \times \{1\}]$ is just a single point.

Thus, F defines a homotopy from f to the constant map x_1 as desired.

((c) $\implies$ (a)) Suppose that $\pi_1(X, x_0) = 0$ for all $x_0 \in X$. Trivially, each loop in X is homotopic to a constant loop, as desired. $\square$

Problem 6

Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$ with no conditions on basepoints, and let $\Phi : \pi_1(X, x_0) \to [S^1, X]$ be the natural map obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X, x_0)$ if X is path-connected.

Proof. We first show that if X is path-connected, then $\Phi : \pi_1(X, x_0) \to [S^1, X]$ is onto. To see this, let $[f] \in [S^1, X]$ be some loop in X. In particular, since X is path-connected, there is a path γ from $f(0)$ to x_0. Then, $\gamma \cdot f \cdot \gamma^{-1}$ (first do γ^{-1}, then f, then γ) is a loop based at x_0, and $[\gamma f \gamma^{-1}]$ is an element of $\pi_1(X, x_0)$. Furthermore, this identification is stable with respect to homotopy. That is, for $f \simeq g$, $\gamma f \gamma^{-1} \simeq \gamma g \gamma^{-1}$ and so it follows immediately that $\Phi([\gamma f \gamma^{-1}]) = [f]$. Since $[f]$ was arbitrary, Φ is onto as desired.

Now we show that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate to each other.

($\Longrightarrow$) Suppose $\Phi([f]) = \Phi([g])$. This means that $[f] = [g]$ (as a general homotopy, not relative to the basepoint). Now, let $f_t : S^1 \to X$ be the homotopy from f to g. In particular, $f_t(0) = \gamma(t)$ is a loop based at x_0 that tracks how the basepoint moves in the homotopy. We will show that f is homotopic to $\gamma^{-1} g \gamma$ relative to the basepoint.

Now, we define a family of paths γ_s as the path γ restricted to the domain $[0, s]$. Similarly, γ_s^{-1} will be the inverse of γ_s, which is just γ^{-1} restricted to the domain $[1 - s, 1]$.

Now, we can construct the homotopy between f and $\gamma^{-1} g \gamma$. Let h_s be the homotopy defined as

$$h_s = \gamma_s^{-1} f_s \gamma_s$$

which is well-defined with respect to path concatenation, since $\gamma_s(1) = \gamma(s) = f_s(0)$ and $\gamma_s^{-1}(0) = \gamma_s(1) = \gamma(s) = f_s(1)$. Clearly this homotopy is relative to the starting point of the loop as well, since $\gamma_s(0) = x_0$ for all s, and $\gamma_s^{-1}(1) = x_0$ for all s.

Thus, $[f]$ and $[g]$ are conjugate to each other in $\pi_1(X, x_0)$.

($\Longleftarrow$) Suppose instead that $[f]$ and $[g]$ are conjugate to each other in $\pi_1(X, x_0)$. We can write $[g] = [\gamma^{-1} f \gamma]$ for some $[\gamma]$ in $\pi_1(X, x_0)$. Let f_t be the homotopy from g to $\gamma^{-1} f \gamma$. Then, we just need to show that $\gamma^{-1} f \gamma$ is homotopic to f as a general homotopy. This is clear, however, by considering $\gamma^{-1} f \gamma$ as a loop based at $f(0)$ which is given as $\gamma \gamma^{-1} f$. This is the same loop (up to reparameterization), it just has a new base point. However, in $\pi_1(X, f(0))$, this is homotopic to f by canceling γ and γ^{-1}. Thus, $\gamma^{-1} f \gamma$ is homotopic to f (and it is assumed to be homotopic to g) and so $f \simeq g$ without regard to basepoint. $\square$

Problem 7

Let A_1, A_2, and A_3 be compact sets in $\mathbb{R}^3$. Use Borsuk-Ulam theorem to show that there is one plane $P \in \mathbb{R}^3$ such that P simultaneously divides each A_i into two pieces of equal measure.

Proof. Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Now, for $\theta \in S^1$, we can define a plane whose normal has polar coordinate θ and azimuthal coordinate ϕ which cuts A_1 in half. This is a clear application of the intermediate value theorem, by varying the translation of the plane from the origin and looking at the function $f(r)$ which measures how much of A_1 is above (in the direction of the normal) the plane.

Now, let $f : S^2 \to \mathbb{R}^2$ be the function that measures how much of A_2 and A_3 is above the plane defined above. This function is pretty clearly continuous, since a small change$d\Phi$ in the angle of the plane results in at most a change of $Rd\Phi$ in the area above the plane (for R the bound on the compact sets A_i).

Thus, there exists a point $x \in S^2$ for which $f(x) = f(-x)$. That is, there is a point for which the plane divides A_1 in half (by construction), and for which the amount of A_2 and A_3 (individually) above the plane is the same as the amount of A_2 and A_3 above the same plane with opposite orientation. However, this just means that this plane cuts A_1, A_2, and A_3 each in half, as desired. $\square$

Problem 8

If X_0 is the path component of X containing x_0, show that the inclusion $X_0 \to X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

Proof. In particular, we wish to show that the inclusion map i induces a map i^* which is surjective with trivial kernel.

The fact that i^* is surjective is trivial, since a loop f based at x_0 in X must stay in its path-component. Therefore, the image of f is completely contained in X_0, and easily factors through i. That is, there is some $\tilde{f} : S^1 \to X_0$ based at x_0 for which $i \circ \tilde{f} = f$. Clearly, then, $i^*([\tilde{f}]) = [f]$. This is a well-defined map, since if $\tilde{f} \simeq \tilde{g}$, we have that $f \simeq g$ after inclusion, and so $i^*([\tilde{f}]) = i^*[\tilde{g}])$.

Now, suppose $[f]$ is such that $i^*([f]) = [0]$. In particular, this means that $f \circ i \simeq x_0$. However, any homotopy between two maps must stay in its path component for all t (Given f_t a homotopy, $f_t(x)$ for fixed x is a continuous map from I into X, and thus must be mapped into a single path-component).

Therefore, since $f \circ i \simeq x_0$, and such a homotopy stays in X_0, we can easily restrict the homotopy to X_0 without issue, which leads to a homotopy $f \simeq x_0$.

Thus, i^* is surjective with trivial kernel, and is an isomorphism. $\square$

Problem 9

Given a space X and a path-connected subspace A containing x_0 the basepoint, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion map is surjective if and only if every path in X with endpoints in A is homotopic to a path in A.

Proof. ($\implies$) Suppose first that every path in X with endpoints in A is homotopic to a path in A. Now, let $[f] \in \pi_1(X, x_0)$. By hypothesis, $[f]$ contains a loop g in A based at x_0 (since $x_0 \in A$), and so $i^*([g]) = [f]$.

($\impliedby$) Suppose conversely that there is a path f in X with endpoints in A that is not homotopic to a path in A. Let γ be a path in A connecting $f(0)$ to $f(1)$. Then, the concatenation $f \cdot \gamma$ is a loop in X. However, since f is not homotopic to any path in A, it follows that $f \cdot \gamma$ is not homotopic to a loop in A (if it were, it would also be homotopic to a loop in A passing through $f(1)$, and either the part of the loop going to $f(1)$ or the part of the loop going from $f(1)$ would be homotopic to f, a contradiction). Therefore, $[f \cdot \gamma]$ is not in the image of i^*, and so i^* is not surjective. $\square$