Final Exam

Daniel Halmrast

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Problem 1

Show that hyperbolic space H^n is complete.

Proof. We will first show that H^n is homogeneous, and then appeal to the next problem to conclude H^n is complete.

To see that H^n is homogeneous, we consider two families of isometries. For simplicity, we will write points in H^n as (x, y) with $x \in \mathbb{R}^{n-1}$ the first $n-1$ coordinates, and $y \in \mathbb{R}$ the last coordinate. The first isometry we consider is

$$\begin{aligned} T_a : H^n &\to H^n \\ (x, y) &\mapsto (x + a, y) \end{aligned}$$

for any $a \in \mathbb{R}^{n-1}$. To see this is an isometry, we just need to compute dT_a and show it preserves the metric. So, let $v \in T_pH^n$ for some $p \in H^n$, $p = (x_p, y_p)$, and take $\gamma(t) = p + vt = (x_p + v_xt, y_p + v_yt)$ a curve in H^n. Note that $\gamma'(0) = v$. Now, we have that

$$\begin{aligned} dT_a(v) &= dT_a(\gamma'(0)) \\ &= \partial_t T_a(\gamma(t))|_{t=0} \\ &= \partial_t(x_p + v_xt + a, y_p + v_yt)|_{t=0} \\ &= (v_x, v_y) = v \end{aligned}$$

Thus, $dT_a(v) = v$. Furthermore, since the metric at $(x + a, y)$ is the same as at (x, y) (since the scaling factor only depends on y) we have that for $u, v \in T_pM$,

$$g(u, v)_{(x,y)} = g(dT_au, dT_av)_{(x+a,y)}$$

and thus T_a is an isometry (I suppose you'd have to check that T_a is a diffeomorphism as well, but this is obvious. Clearly T_a is smooth, and it has a smooth inverse T_{-a}).

Secondly, we consider the isometry

$$\begin{aligned} M_\alpha : H^n &\to H^n \\ (x, y) &\mapsto (\alpha x, \alpha y) \end{aligned}$$

for $\alpha > 0$. This maps H^n into H^n, since it keeps the y coordinate positive. Furthermore, it is a diffeomorphism (it is clearly smooth, and $M_{\frac{1}{\alpha}}$ acts as an inverse). I also claim it is an isometry. Again letting $\gamma = (x_p + v_x, y_p + v_y)$ for $(v_x, v_y) \in T_{(x,y)}H^n$ we note that

$$\begin{aligned} dM_\alpha(v) &= dM_\alpha(\gamma'(0)) \\ &= \partial_t M_\alpha(\gamma(t))|_{t=0} \\ &= \partial_t(\alpha(x_p + v_x), \alpha(y_p + v_y))|_{t=0} \\ &= \alpha V \end{aligned}$$

Finally, we compute the metric

$$\begin{aligned} g(u,v)_(x,y) &= g_{ab}u^a v^b \\ &= \frac{1}{y^2}u_b v^b \end{aligned}$$

$$\begin{aligned} g(dM_\alpha u, dM_\alpha v)_{(\alpha x, \alpha y)} &= g_{ab}\alpha u^a \alpha v^b \\ &= \frac{1}{(\alpha y)^2}\alpha^2 u_b v^b \\ &= \frac{1}{y^2}u_b v^b \end{aligned}$$

Where $u_b = \eta_{ab}u^a$ and so $u_b v^b$ is the standard inner product on $\mathbb{R}^n$. Thus, M_α is an isometry.

I assert that the action of these two isometries is transitive. Indeed, given (x, y) and (x', y') in H^n, we construct the isometry as follows. First, apply T_{-x} to map (x, y) to $(0, y)$. Then, apply $M_{\frac{y'}{y}}$ to map $(0, y)$ to $(0, y')$. Finally, apply $T_{x'}$ to map $(0, y')$ to (x', y').

Thus, for any two points (x, y) and (x', y') in H^n, there is an isometry connecting them. Thus, by the result of the next problem, H^n is complete. $\square$

Problem 2

Show that a homogeneous space is complete.

Proof. Let M be a homogeneous manifold. We will show that M is geodesically complete.

Let ε be such that $B_\varepsilon(p) \subset M$ is a normal ball at $p \in M$. Since M is homogeneous, this implies that $B_\varepsilon(q)$ is a normal ball at $q \in M$ for any other q. To see this, we note that for ϕ the isometry sending p to q,

$$\phi \circ \exp_p \circ d\phi^{-1}$$

defines a diffeomorphism between $B_\varepsilon(0) \subset T_qM$ and the image $B_\varepsilon(q)$. This is well-defined, since ϕ is an isometry, so $\|v\| = \|d\phi^{-1}v\|$. Furthermore, we can see that $\exp_q = \phi \circ \exp_p \circ d\phi^{-1}$. Obesrve that $\gamma(t) = \exp_q(tv)$ is the unique geodesic through q with tangent vector v. However,

$$\tilde{\gamma}(t) = \phi \circ \exp_p \circ d\phi^{-1}(tv)$$

has the same properties. Namely $\tilde{\gamma}(0) = \phi(p) = q$, and $\tilde{\gamma}'(0) = d\phi(d\phi^{-1}(v)) = v$. Thus, $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in [0,1]$, and so $\exp_q$ and $\phi \circ \exp_p \circ d\phi^{-1}$ agree at all points in the normal ball. Thus, $B_\varepsilon(q)$ is a normal ball, as desired.

Recall that in a normal ball at p, any geodesic going through p can be extended throughout the entire normal ball. This follows from the fact that if γ is a geodesic passing through p at some time t_p with $\gamma'(t_p) = v$, it is the unique geodesic (up to reparameterization) with $\gamma(t_p) = p$ and $\gamma'(t_p) = v$. Now, since radial geodesics through p are defined on the entire normal ball, the radial geodesic starting at p with tangent vector v is defined throughout the normal ball, and is an extension of γ. Thus, γ can be extended through the normal ball.

It follows immediately, then, that any geodesic γ (with unit speed, without loss of generality) defined on some interval (a,b) can be extended to a geodesic defined on $(a, b+\frac{\varepsilon}{2})$ by observing that γ passes through $\gamma(b-\frac{\varepsilon}{2})$, and since $\gamma(b-\frac{\varepsilon}{2})$ has a normal ball of radius ε around it, we know that γ can be extended through this normal ball to be defined on $(a, b-\frac{\varepsilon}{2}+\varepsilon) = (a, b+\frac{\varepsilon}{2})$.

Thus, it follows immediately that geodesics can be extended indefinitely (the symmetric argument works to show γ can be extended the other way) and thus M is geodesically complete. $\square$

Problem 3

Part a

Let v be a linear field on $\mathbb{R}^n$. That is, v is a vector field, and v is linear when thought of as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Show that a linear field given by a matrix A is a killing field if and only if A is antisymmetric.

Proof. Let X be a linear vector field. Then, X is expressible as a matrix A. That is, $X(f(x_1, \ldots, x_n)) = Af(x_1, \ldots, x_n)$. In order for X to be a killing field, we must have that its local flow around each point is an isometry. That is, for $\phi : (-\varepsilon, \varepsilon) \times U \to M$ the flow of X around a point p, $d\phi(t, \cdot)$ preserves inner products.

Now, the flow of X is the solution to

$$\partial_t \phi^a = A\phi^a$$

which is solved by setting $\phi = \exp(At)$. Now, let's calculate the differential. For $v \in T_p\mathbb{R}^n$, let $\gamma(s) = p + sv$. Then

$$\begin{aligned} d\phi(v) &= \partial_s \phi(\gamma(s))|_0 \\ &= \partial_s \exp(At)(p + sv) \\ &= \exp(At)v \end{aligned}$$

and so $d\phi = \phi$. We require that

$$\langle u, v\rangle = \langle \exp(At)u, \exp(At)v\rangle$$

which amounts to requiring

$$\langle u, v\rangle = \langle u, \exp(A^T t)\exp(At)v\rangle$$

Now, this happens for all u, v if and only if $\exp(A^T t)\exp(At) = I$, which holds for all t if and only if $A^T = -A$. Thus, in order for X to be a killing field, A must be antisymmetric, and vice versa. □

Part b

Let X be a killing field on M with $p \in M$, and let U be a normal neighborhood of p in M. Assume that p is a unique point of U with $X_p = 0$. Show that in U, X is tangent to the geodesic spheres centered at p.

Proof. Let $\phi_q : (-\varepsilon, \varepsilon) \times V_q \to M$ denote the local flow of X around any point q. Since $X_p = 0$, we know that $\phi(t, p) = p$. That is, p is fixed by the flow of X.

Now, let q be any point in U the normal neighborhood of p. We know that there is a unique radial geodesic from p to q defined as $\gamma(t) = \exp_p(tv)$ for some v. Now, ϕ is defined across all of $\gamma(t)$ for $t \in [0, 1]$ since $\gamma([0, 1])$ is a compact set, and thus can be covered by a finite number of sets V_q on which the flow is defined.

Now, since $\phi(t, \cdot)$ is an isometry, it maps geodesics to geodesics. Thus, the image $\phi(t, \gamma([0, 1]))$ is a geodesic from $\phi(t, p) = p$ to $\phi(t, q)$. Furthermore, this geodesic is defined by $\tilde{\gamma(t)} = \exp_p(tu)$ for some u. Now, we know that

$$\begin{aligned} d\phi(t, v) &= d\phi(t, \gamma'(0)) \\ &= \partial_s \phi(t, \gamma(s)) \\ &= \partial_s(\tilde{\gamma}(s)) \\ &= u \end{aligned}$$

Thus u and v have the same norm, and so $\gamma(1) = q$ and $\tilde{\gamma}(1) = \phi(t, q)$ are the same distance from q.

Thus, ϕ moves points along the geodesic spheres, and so X is tangential to the geodesic spheres, as desired. $\square$

Part c

Let X be a smooth vector field on M and let $f : M \to N$ be an isometry. Let Y be a vector field on N defined by $Y(f(p)) = df_p(X(p))$. Prove that Y is a killing field if and only if X is.

Proof. Suppose X is a killing field. That is, the local flow ϕ is an isometry. Now, we can push forward a local flow on X to a local flow on Y. That is,

$$\psi(t, x) = f(\phi(t, f^{-1}(x)))$$

defines a flow on Y. This is clear, since

$$\begin{aligned}
\partial_t \psi(t, x) &= \partial_t f(\phi(t, f^{-1}(x))) \\
&= df(\partial_t \phi(t, f^{-1}(x))) \\
&= df(X(\phi(t, f^{-1}(x)))) \\
&= Y(f(\phi(t, f^{-1}(x)))) \\
&= Y(\psi(t, x))
\end{aligned}$$

as desired. Now, for any fixed t, $\psi(t, \cdot) = f \circ \phi(t, \cdot) \circ f^{-1}$ is the composition of isometries, and is therefore an isometry as desired. Thus, Y is a killing field if X is.

By symmetry of the problem, this implies that Y is a killing field if and only if X is. $\square$

Part d

Show that X is a killing field if and only if

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$$

for all X, Y, Z.

Proof. Recall the definition of the Lie derivative of a tensor field T along a vector field v with flow ϕ

$$\mathfrak{L}_v(T)(p) = \lim_{t \to 0} \left\{ \frac{\phi^*(-t, T(\phi(t, p))) - T(p)}{t} \right\}$$

We first establish that the Lie derivative along a vector field v of the metric g is zero if and only if v is a killing field. To see this, let $p \in M$, and choose a coordinate system x_i around p that is compatible with the flow of v. That is,

$$\phi(t, (x_1, \ldots, x_n)) = \phi(x_1 + t, x_2, \ldots, x_n)$$

Which can be done by setting x_1 so that $v = \partial_1$ around p. (I suppose we are implicitly assuming $v \neq 0$ around and at p. We will handle the case $v = 0$ around p later).

In this coordinate system, we have

$$\mathfrak{L}_v(g)(p) = \partial_1(g)|_p$$

which is zero if and only if g does not vary along the flow ϕ. In particular, this means that v is a killing field at p (its flow is a local isometry) if and only if $\mathfrak{L}_v(g)(p) = 0$.

This condition is exactly the killing equation. To see this, we first establish a different form of the Lie derivative. Recall that for any vector field u we have

$$\mathfrak{L}_v(u) = [v, u]$$

and for any function f we have

$$\mathfrak{L}_v(f) = v(f)$$

These conditions along with the Leibniz rule allow us to characterize $\mathfrak{L}_v$ for more arbitrary tensors. Let ω be a one-form, and u a vector field. We have

$$\mathfrak{L}_v(\omega_i u^i) = v(\omega_i u^i)$$

and

$$\mathfrak{L}_v(\omega_i u^i) = u^i \mathfrak{L}_v(\omega_i) + \omega_i [v, u]^i$$

setting these equal, we have

$$\begin{aligned}
v(\omega_i u^i) &= u^i \mathfrak{L}_v(\omega_i) + \omega_i [v, u]^i \\
\nabla_v(\omega_i u^i) &= u^i \mathfrak{L}_v(\omega_i) + \omega_i [v, u]^i \\
u^i \nabla_v \omega_i + \omega_i \nabla_v u^i &= u^i \mathfrak{L}_v(\omega_i) + \omega_i (\nabla_v u^i - \nabla_u v^i) \\
u^i \nabla_v \omega_i &= u^i \mathfrak{L}_v(\omega_i) - \omega_i \nabla_u v^i \\
\mathfrak{L}_v(\omega_i) u^i &= u^i v^j \nabla_j \omega_i + \omega_i u^j \nabla_j v^i \\
\mathfrak{L}_v(\omega_i) &= v^j \nabla_j \omega_i + \omega_j \nabla_i v^j
\end{aligned}$$

and we extend this definition inductively to get

$$\mathfrak{L}_v(g_{ab}) = v^c \nabla_c g_{ab} + g_{ac} \nabla_b v^c + g_{cb} \nabla_a v^c$$

and since ∇ is metric compatible, $\nabla_c g_{ab} = 0$. Then,

$$\mathfrak{L}_v(g_{ab}) = g_{ac} \nabla_b v^c + g_{cb} \nabla_a v^c = \nabla_b v_a + \nabla_a v_b$$

Thus, v is a killing field if and only if

$$\nabla_b v_a + \nabla_a v_b = 0$$

which is the local coordinate version of

$$g(\nabla_Y v, Z) + g(Y, \nabla_Z v) = 0$$

To see this, set $Y = \partial_a$, $Z = \partial_b$, and compute

$$\begin{aligned}
g(\nabla_Y v, Z) + g(Y, \nabla_Z v) &= g_{cd} \nabla_a v^c (\partial_b)^d + g_{cd} (\partial_a)^c, \nabla_b v^d \\
&= g_{cb} \nabla_a v^c + g_{ad} \nabla_b v^d \\
&= \nabla_a v_b + \nabla_b v_a
\end{aligned}$$

as desired. Thus, v is a killing field if and only if it satisfies the killing equation.

Note that in the case $v = 0$ around p, then v trivially satisfies the killing equation, and v is a killing field around p, since the flow of v is stationary. $\square$

Part E

Let X be a killing field with $X(q) \neq 0$ for some $q \in M$. Prove that there exist coordinates around q for which the coefficients g_{ij} of the metric do not depend on one of the coordinates.

Proof. We assumed this in the previous part, but we will prove it more rigorously now. Let $v = X(q)$, and form a coordinate system ∂_i with $\partial_n = v$ in T_qM. This can be done via a coordinate mapping $\psi : M \to \mathbb{R}^n$ which sends the integral curve of X through q to the nth coordinate axis.

Now, since X is a killing field, its flow is a local isometry. So, take a small neighborhood U of q such that our coordinate system is defined on U and the flow of X is an isometry on U. Now, by our choice of coordinate system, the flow ϕ of X is defined in these coordinates as

$$\phi(t, (x_1, \dots, x_n)) = (x_1, \dots, x_n + t)$$

Thus, since ϕ is an isometry, g at $(x_1, \dots, x_n)$ is the same as g at $(x_1, \dots, x_n + t)$ for $t \in (-\varepsilon, \varepsilon)$ as desired. □

Problem 4

We can define a metric on TM by the following: Let $(p,v) \in TM$ and $W, V \in T_vTM$. Let α, β be curves in TM with $\alpha(0) = \beta(0) = (p,v)$ and $\alpha'(0) = V, \beta'(0) = W$. Define the metric on TM as

$$g_T(V,W)_{(p,v)} = g(d\pi(V), d\pi(W))_p + g(\nabla_t\alpha|_0, \nabla_s\beta|_0)_p$$

where by $\nabla_t\alpha$ we mean the covariant derivative along $\pi(\alpha)$ of the vector field $\alpha(t)$.

0.1 Part A

Prove that this metric is well-defined as a Riemannian metric.

Proof. We first show that this definition is independent of choice of curves α, β. This follows from the fact that $\nabla_t\alpha|_0$ only depends on $\alpha(0)$ and $\alpha'(0)$, which follows immediately from the definition of ∇_t.

Now, all we need to show is that this is a metric. That is, we need to show g_T is smooth, symmetric, bilinear, and positive-definite.

Clearly, g_T is smooth as the composition of a bunch of smooth functions ($\pi : TM \to M$, g, and ∇ are all smooth, and g_T is the sum of compositions of these)

Furthermore, g_T is symmetric, since each component in its sum is symmetric.

g_T is bilinear, since g is bilinear, and $d\pi$ and ∇_t are both linear.

Finally, we observe that g_T is positive definite. We know that g_T is always positive (or zero) since it is the sum of positive terms. So, we only have to check that

$$g_T(V,V) = 0$$

implies that $V = 0$. Note that if $g_T(V,V) = 0$, we know that

$$g(d\pi(V), d\pi(V)) = 0 \implies d\pi(V) = 0$$
$$g(\nabla_t\alpha, \nabla_t\alpha) = 0 \implies \nabla_t\alpha|_0 = 0$$

The first statement means that $\partial_t\pi(\alpha(t)) = \partial_t(p(t))$ is zero, and so V must be a vertical vector. However, the second statement implies that the vertical part of V is zero, and so V itself must be zero. Thus, g_T is positive-definite as desired.

Thus, g_T is a well-defined metric on TM. $\square$

Part B

Prove that the curve $t \mapsto (p(t), v(t))$ is horizontal if and only if $v(t)$ is parallel along $p(t)$ in M.

Proof. Let $\gamma(t) = (p(t), v(t))$ be a horizontal curve. Since this is a horizontal curve, we know that $\gamma'(t)$ is orthogonal to the fiber $\pi^{-1}(p(t))$. That is, we know that for any vertical vector W, we have

$$g_T(\gamma'(t), W) = 0$$

Since W is vertical, this implies that $d\pi(W) = 0$ and thus

$$g_T(\gamma'(t), W) = g(\nabla_t\gamma(0), \nabla_s\beta(0))$$

for some curve $\beta(s)$ with $\beta(0) = \gamma(0)$ and $\beta'(0) = W$.

Now, since W is vertical and nonzero, we know that $\nabla_s\beta(0)$ is nonzero (this follows from positive-definiteness of g_T and by the fact that $d\pi(W) = 0$, which means that $g_T(W) = g(\nabla_s\beta(0), \nabla_s\beta(0))$ which is nonzero).

Thus, the only way that $g(\nabla_t\gamma(0), \nabla_s\beta(0))$ is zero for all t is if $\nabla_t\gamma(0)$ is zero. However, this is just the statement that the vector field $v(t)$ is parallel along $p(t)$ as desired.

Conversely, let $\gamma(t) = (p(t), v(t))$ be such that $v(t)$ is parallel along $p(t)$, and let W be a vertical vector at $p(t)$. Then, we have

$$g_T(\gamma'(t), W) = g(d\pi(\gamma'(t)), d\pi(W)) + g(\nabla_t\gamma(0), \nabla_s\beta(0))$$

where β is defined in the same way as before. Since W is vertical, $d\pi(W) = 0$, and since $v(t)$ is parallel to $p(t)$, $\nabla_t\gamma(0) = 0$. Thus,

$$g_T(\gamma'(t), W) = 0$$

and γ is a horizontal curve as desired. □

0.2 Part c

Prove that the geodesic field is a horizontal vector field.

Proof. Let G be the geodesic field on TM. That is, G has trajectories $t \mapsto (\gamma(t), \gamma'(t))$ for geodesics γ. We will prove that this field is parallel. However, this follows almost immediately from the definition of a geodesic. By definition, γ is a geodesic if and only if $\nabla_t\gamma'(s) = 0$ for all s. This means that $\gamma'(t)$ is parallel along γ, and by the result in part b, we know that G is a horizontal vector field. □

Part d

Prove that the trajectories of the geodesic field are geodesics on TM with the metric g_T.

Proof. Let $\alpha(t) = (\gamma(t), \gamma'(t))$ be an integral curve of G. We wish to show this is a geodesic with the metric g_T. This amounts to showing

$$\nabla_t\alpha' = 0$$

under the induced metric. Now, since α is an integral curve of G, by part c we know that α is a horizontal curve. This implies that

$$g_T(\cdot, \alpha'(t)) = g(d\pi(\cdot), d\pi(\alpha'(t)))$$

(since the vertical part is zero). Thus, along α, $g_T = \pi^*(g)$.

More formally, α lies in the submanifold of TM consisting of all points horizontal to $\alpha(0)$. On this submanifold, the metric looks like $g_T = \pi^*(g)$, which is well-defined, since π is bijective on this submanifold. Thus, geodesics in this submanifold are precisely the image under π^{-1} of geodesics in M. Since $\alpha = \pi^{-1}(\gamma)$ which is a geodesic, it follows that α is a geodesic as well. □

Problem 5

Let M be a Riemannian manifold of dimension 2. Let $B_\delta(p)$ be a normal ball around $p \in M$, and consider the parameterized surface

$$f(\rho, \theta) = \exp_p(\rho v(\theta))$$

where $v(\theta)$ is a circle in $B_\delta(0)$ parameterized by the central angle θ.

Part A

Show that (ρ, θ) are coordinates in an open set $U \subset M$ formed by the open ball $B_\delta(p)$ minus the ray $\exp_p(-\rho v(0))$ for $\rho \in (0, \delta)$.

Proof. The fact that this is a coordinate system is clear. Since U is contained in a normal ball, $\exp_p$ is a diffeomorphism, and thus the inverse $\exp_p^{-1}$ is a diffeomorphism into $B_\delta(0) \subset T_pM \cong \mathbb{R}^2$.

Since ρ, θ define a coordinate system on $B_\delta(0)$ (as polar coordinates in $\mathbb{R}^2$), their image under $\exp_p$ is also a coordinate system of M, as desired. □

Part B

Show that the coefficients of the metric g_{ij} are

$$g_{12} = 0, \quad g_{11} = \|\partial_\rho f\|^2 = \|v(\theta)\|^2 = 1, \quad g_{22} = \|\partial_\theta\|^2$$

Proof. We first observe that ∂_ρ is a geodesic, and so $\nabla_\rho \partial_\rho = 0$. In particular, this means that

$$\partial_\rho g(\partial_\rho, \partial_\rho) = 2g(\partial_\rho, \nabla_\rho \partial_\rho) = 0$$

and so $g_{\rho,\rho}$ does not change along ρ. Since $\|\partial_r\| = g_{\rho,\rho} = 1$ at the origin, this implies that $g_{\rho,\rho} = 1$ everywhere.

Note also that by the Gauss lemma, geodesics from p are orthogonal to the geodesic spheres centered at p, and in particular this means that $g(\partial_\rho, \partial_\theta) = 0$. Thus, $g_{12} = g_{21} = 0$ as desired.

Finally, we wish to compute $g(\partial_\theta, \partial_\theta)$. Since $\exp_p$ is an isometry around p, we can pull back g to find $g(\partial_\theta, \partial_\theta) = f^*(g(\partial_\theta, \partial_\theta))$ which yields

$$f^*(g(\partial_\theta, \partial_\theta)) = g(\partial_\theta f, \partial_\theta f) = \|\partial_{theta} f\|^2$$

as desired. □

Part C

Show that along the geodesic $f(\rho, 0)$ we have

$$(\sqrt{g_{22}})_{\rho\rho} = -K(p)\rho + R(\rho)$$

where $R(\rho) = 0 + O(\rho^2)$.

Proof. This will follow immediately from the fact that $\partial_\theta f$ is a Jacobi field of the geodesic $f(\rho, 0)$. However, this is clear from the original motivating definition of a Jacobi field in Do Carmo chapter 5. Namely, for a family of geodesics $t \mapsto \exp_p(tv(s))$ the field $\partial_s \exp_p(tv(s))$ is a Jacobi field along $\exp_p(tv(0))$.

Thus, we can apply corollary 2.10 from chapter 5 to note that

$$\sqrt{g_{22}} = \|\partial_\theta f\| = \rho - \frac{1}{6}K(p)\rho^3 + \tilde{R}(\rho)$$

with $\tilde{R} = 0 + O(\rho^4)$.

Differentiating this result twice with respect to ρ yields the desired result. □

Part D

Show that

$$\lim_{\rho \to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(\rho)$$

Proof. We utilize the same expression for g_{22} as in the last one to calculate

$$\begin{aligned}
\lim_{\rho \to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} &= \lim_{\rho \to 0} \frac{-K(p)\rho + R(\rho)}{\rho - \frac{1}{6}K(p)\rho^3 + \tilde{R}(\rho)} \\
&= \lim_{\rho \to 0} \frac{-K(\rho)\rho + R(\rho)}{\rho} \\
&= -K(\rho)
\end{aligned}$$

as desired. □