Problem Set 3

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Problem 1

Enumerate all subcomplexes of S^∞ with the cell structure on S^∞ that has S^n as the n-skeleton.

Proof. We notice at first that each n-skeleton is a subcomplex, and so S^n is a subcomplex of S^∞ for each n.

There is another subcomplex in each dimension. Namely, by omitting one of the n-cells attaching to the $n-1$-skeleton, we obtain another subcomplex in the nth dimension that is the $n-1$ skeleton along with a single n-cell attached in the usual way. In fact, depending on which n-cell we omit, we can obtain two different subcomplexes.

So far, we have three subcomplexes in each dimension. I assert that this is all the subcomplexes. Suppose there existed a subcomplex in n dimensions that did not contain the entire $n-1$-skeleton. In particular, this means that the attaching map of the n-cell, which is bijective from ∂D^n onto the entire $n-1$-skeleton, is not well-defined, and so no such subcomplex can be constructed. □

Problem 2

Show S^∞ is contractible.

Proof. We will show that the n-skeleton of $X = S^\infty$ is contractible in X^{n+1}. To see this, consider the subcomplex X^n along with a single disk D^{n+1} attached in the usual way. In particular, X^n is identified with ∂D^{n+1}, and since D^{n+1} is contractible, it follows that ∂D^{n+1} contracts to a point in D^{n+1}.

Thus, each X^n is contractible in X, and by running all of them sequentially (say, running the nth homotopy in time $[2^{-n}, 2^{-(n+1)}]$) we obtain a contraction of S^∞. □

Problem 3

Show that $S^1 \star S^1 = S^3$. In general, show $S^m \star S^n = S^{m+n+1}$.

Proof. We will prove the more general result that

$$\star_{i=1}^{n} S^0 = S^{n-1}$$

along with the associativity of $\star$.

To see the first result, we will use the interpretation of the star product as the set of all convex formal linear combinations of the two spaces. In particular, interpreting S_i^0 to be the two points $1, -1$ on the ith coodinate axis in $\mathbb{R}^n$, we see that join across all n copies of S^0 is really all points $x = (x^1, \dots, x^n)$ satisfying

$$x^i \leq 1 \text{ and } x^i \geq -1$$
$$\sum_i x^i = 1$$

which is just the convex hull of the $2n$ unit vectors along the coordinate axes of $/R^n$. In other words, it is the ball of radius 1 in the 1-norm. However, this is obviously homeomorphic to $S^n - 1$ as desired.

Now, we will show that $\star$ is associative. However, this falls almost immediately from the definition of $\star$ in terms of formal linear combinations.

Thus, $S^n = \star_{i=1}^{n+1} S^0$, and so

$$\begin{aligned} S^m \star S^n &= (\star_{i=1}^{m+1} S^0) \star (\star_{i=1}^{n+1} S^0) \\ &= \star_{i=1}^{m+n+2} S^0 \\ &= S^{m+n+1} \end{aligned}$$

as desired. □

Problem 4

Show that the space obtained by attaching n 2-cells along any collection of n circles in S^2 is homotopy equivalent to the wedge sum of $n+1$ 2-spheres.

Proof. Consider the (potentially disconnected) graph formed by the n circles on the surface S^2. This graph in particular can be homotoped to a tree relative to the points of intersection (the nodes of the graph). This is done by taking any loop (two distinct intervals connected on both endpoints) and sliding the two intervals to meet each other. After obtaining a tree, the tree can be homotoped to a point,

Now, the key thing about these homotopies is that they define a homotopy on the space obtained by attaching 2-cells along the n circles on S^2 as well. This is clear, since the first homotopy does not move points of intersection (so as not to have discontinuities). Thus, this space obtained by attaching the n 2-cells to S^2 is homotopy equivalent to the space attaching n 2-cells to a point, which is just the wedge sum of $n+1$ copies of S^2 as desired. □

Problem 5

Show that the subspace X of $\mathbb{R}^3$ formed by a self-intersecting Klein bottle is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

Problem 6

Show that a CW complex is contractible if its the union of contractible spaces whose intersection is contractible.

Proof. Let X, Y be CW compexes with intersection A, and suppose these three are contractible. We wish to show that $X \cup Y$ is contractible. Now, we know that X/A and Y/A are contractible, since A is contractible and thus $X \simeq X/A$ and $Y \simeq Y/A$. So, we also have that

$$(X \cup Y)/A \simeq X \cup Y$$

and so showing $(X \cup Y)/A$ is contractible is enough to prove the conjecture.

However, $(X \cup Y)/A$ is just $X/A \vee Y/A$ where they are attached at the quotient of A. Since both X/A and Y/A are contractible, it follows immediately that their wedge is contractible as well. □

Problem 7

Let X and Y be CW complexes with 0-cells x_0 and y_0. Show that the quotient $X \star Y/(X \star \{y_0\} \cup \{x_0\} \star Y)$ and $S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\})$ are homeomorphic.

Proof. We will expand both sides of this equation, and show they are homeomorphic. The left-hand side expands as

$$\begin{aligned} X \star Y &= X \times Y \times I/(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1) \\ X \star \{y_0\} &= X \times \{y_0\} \times I/(x_1, y_0, 1) \sim (x_2, y_0, 1) \\ \{x_0\} \star Y &= \{x_0\} \times Y \times I/(x_0, y_1, 0) \sim (x_0, y_2, 0) \end{aligned}$$

Similarly, the right hand side expands as

$$\begin{aligned} \{x_0\} \wedge \{y_0\} &= \{x_0\} \times \{y_0\}/\{x_0\} \vee \{y_0\} = \{\cdot\} \\ S(X \wedge Y) &= X \times Y \times I/(X \times \{y_0\} \cup \{x_0\} \times Y), (X \wedge Y) \times \{0\}, (X \wedge Y) \times \{1\} \end{aligned}$$

and thus

$$\begin{aligned} S(X \wedge Y)/S(\{x_0\} \wedge \{y_0\}) &= (X \wedge Y) \times I/(X \wedge Y) \times \{0\}, (X \wedge Y) \times \{1\}, (\{x_0\} \wedge \{y_0\}) \times I \\ &= X \times Y \times I/(\ldots, (X \times \{y_0\} \cup \{x_0\} \times Y) \times I) \end{aligned}$$

However, the left hand side has the same space quotienting by, namely $X \times Y \times I$, and so we only have to verify that the two have the same quotient operations on them, which I did not have enough time to complete...

Suppose that the claim is true. Since both $S(X \wedge Y)$ and $X \star Y$ are being quotiented by contractible subspaces, they are homotopy equivalent to their quotient. And since the quotients are homeomorphic, it follows that $S(X \wedge Y)$ and $X \star Y$ are homotopy equivalent to each other as well.

□

Problem 8

Show that for a CW complex X with components X_α, $SX \simeq Y \vee_\alpha SX_\alpha$ for some graph Y.

Proof. It should be clear that for components X_α, the suspension SX is equal to the union of all SX_α attached at the top and bottom. That is, $SX = (\coprod SX_\alpha)/(x_\alpha, 0) \sim (x_\beta, 0), (x_\alpha, 1) \sim (x_\beta, 1)$.

So, let $Y = \vee_\alpha S^1$, and consider the space $Y \vee_\alpha X_\alpha$. We will take as the basepoint the top of the suspensions. Now, all we need to do is connect the bottom of the suspensions using Y. To do this, we first choose a distinguished component X_0, and homotope Y to the bottom of SX_0 (since X_0 is path connected, so is SX_0 so we can safely do this). Now, for each copy S^1_α in Y, fix one endpoint on the bottom of SX_0, and homotope the other endpoint to the top of SX_0. Now, send that endpoint to the bottom of SX_α (which we can do since SX_0 is connected to SX_α via the wedge, and SX_α is path connected). We now have a tree connecting all the bottom points of SX_α, which contracts to a point. The space we end up with is the union of SX_α with the top and bottom points respectively identified with each other, which is just SX as desired.

Now to show that if X is a finite graph, then SX is homotopy equivalent to the wedge sum of circles and 2-spheres. Now, each component X_α is homotopy equivalent to a wedge of circles, and the suspension of that is a wedge of 2-spheres. Thus, by the same construction as above, SX is homotopy equivalent to $Y \vee_\alpha SX_\alpha$ where Y is a wedge of circles, and each SX_α is a wedge of spheres. $\square$