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# Final Exam

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## PROBLEM 1

For every  $n \in \mathbb{N}$ , let  $\mu_n$  be a measure on  $(\Omega, \mathcal{A})$  with  $\mu_n(\Omega) = 1$ . For every  $E \in \mathcal{A}$ , define

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n}$$

Give a careful proof that  $\mu$  is a measure on  $(\omega, \mathcal{A})$  with  $\mu(\Omega) = 1$ .

*Proof.* We wish to prove that  $\mu$  is a measure on  $(\Omega, \mathcal{A})$ . That is, we wish to show that that  $\mu(\emptyset) = 0$ , that  $\mu(E) \geq 0$  for all  $E \in \mathcal{A}$ , and that for a countable collection of disjoint sets  $\{E_j\}_{j=1}^{\infty}$  for which  $E_j \in \mathcal{A}$  for all  $j$ ,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

To begin with, we note that since each  $\mu_n$  is a measure, we have that  $\mu_n(\emptyset) = 0$ . Thus,

$$\begin{aligned} \mu(\emptyset) &= \sum_{n=1}^{\infty} \frac{\mu_n(\emptyset)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{0}{2^n} \\ &= 0 \end{aligned}$$

as desired.

Next, we note that since each  $\mu_n$  is a measure,  $\mu_n(E) \geq 0$  for all  $E \in \mathcal{A}$ . Thus, since both  $\mu_n(E)$  and  $2^n$  are greater than zero for each  $n$ , it must be that

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n} \geq 0$$

as desired.

To show that  $\mu$  is countably additive, we first prove the following lemma:

**Lemma.** *For a doubly indexed sequence  $\{a_{ij}\}$  of positive numbers,*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

*provided that either sum converges.*

*Proof.* We note first that  $a_{ij}$  can be thought of as a function from  $\mathbb{N} \times \mathbb{N}$  to  $\mathbb{R}$ .

Now, Tonelli's theorem tells us that for any positive function  $f : \Omega \times \Sigma \rightarrow \mathbb{R}$  on the product space  $\Omega \times \Sigma$  of  $\sigma$ -finite measure spaces  $(\Omega, \mathcal{A}, \mu)$  and  $(\Sigma, \mathcal{B}, \nu)$  such that  $f$  is measurable with respect to  $\mathcal{A} \otimes \mathcal{B}$ , we have that

$$\int_{\Omega} \left( \int_{\Sigma} f(x, y) d\nu(y) \right) d\mu(x) = \int_{\Sigma} \left( \int_{\Omega} f(x, y) d\mu(x) \right) d\nu(y)$$

Now, consider the case where  $\Omega = \Sigma = \mathbb{N}$ ,  $\mathcal{A} = \mathcal{B} = 2^{\mathbb{N}}$ , and  $\mu = \nu = \mu_c$  the counting measure. The function  $a_{ij}$  from  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$  is positive (by hypothesis), and is measurable on  $2^{\mathbb{N}} \otimes 2^{\mathbb{N}} = 2^{\mathbb{N} \times \mathbb{N}}$ , since every function is measurable with respect to this  $\sigma$ -algebra. Thus, applying Tonelli's theorem yields

$$\begin{aligned} \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{ij} \right) &= \int_{\mathbb{N}} \left( \int_{\mathbb{N}} a_{ij} d\mu_c(j) \right) d\mu_c(i) \\ &= \int_{\mathbb{N}} \left( \int_{\mathbb{N}} a_{ij} d\mu_c(i) \right) d\mu_c(j) \\ &= \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} a_{ij} \right) \end{aligned}$$

as desired. □

Equipped with this result, we now prove that  $\mu$  is countably additive. To do so, let  $\{E_j\}_{j=1}^{\infty}$  be a countable collection of disjoint measurable sets. Now, we know by the fact that each  $\mu_n$  is a measure that

$$\mu_n \left( \bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu_n(E_j)$$

Thus, we have

$$\begin{aligned} \mu \left( \bigcup_{j=1}^{\infty} E_j \right) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n \left( \bigcup_{j=1}^{\infty} E_j \right) \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^n} \mu_n(E_j) \end{aligned}$$

We apply the above lemma to get

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^n} \mu_n(E_j) &= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n(E_j) \\ &= \sum_{j=1}^{\infty} \mu(E_j)\end{aligned}$$

as desired.

Finally, we wish to show that  $\mu(\Omega) = 1$ . This follows from direct computation (observing that  $\mu_n(\Omega) = 1$  for all  $n$ ):

$$\begin{aligned}\mu(\Omega) &= \sum_{n=1}^{\infty} \frac{\mu_n(\Omega)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \frac{1}{1 - \frac{1}{2}} - 1 \\ &= 1\end{aligned}$$

as desired. Here, we used the standard formula for a geometric series

$$\sum_{n=1}^{\infty} a^n = \frac{1}{1 - a} - 1$$

for  $0 < a < 1$ . □

## PROBLEM 2

Suppose  $\mu(\Omega) < \infty$ . Prove that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

*Proof.* We note first that the trivial case of  $\|f\|_{L^\infty} = 0$  is clear, since

$$\begin{aligned} \|f\|_{L^\infty} = 0 &\implies f = 0 \text{ } \mu - \text{almost everywhere} \\ &\implies \|f\|_{L^p} = 0 \text{ } \forall p \\ &\implies \lim_{p \rightarrow \infty} \|f\|_{L^p} = 0 \end{aligned}$$

Therefore, for the rest of this problem, it is assumed that  $\|f\|_{L^\infty} > 0$ .

Suppose first that  $\|f\|_{L^\infty} < \infty$ . Then, we are free to scale  $f$  so that  $\|f\|_{L^\infty} = 1$ . (This is clear, since

$$\lim_{p \rightarrow \infty} \|cf\|_{L^p} = c \lim_{p \rightarrow \infty} \|f\|_{L^p}$$

so

$$\lim_{p \rightarrow \infty} \|cf\|_{L^p} = \|cf\|_{L^\infty} \iff \lim_{p \rightarrow \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

and so multiplying  $f$  by a constant will not change the equality.)

So, without loss of generality, let  $\|f\|_{L^\infty} = 1$ . We will show first that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} \leq 1$$

To do so, we consider the altered function

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } f(x) \leq \|f\|_{L^\infty} \\ 0, & \text{if } f(x) > \|f\|_{L^\infty} \end{cases}$$

Now, we know that  $\mu\{|f| > \|f\|_{L^\infty}\} = 0$  by the definition of the  $L^\infty$  norm, so it follows that  $\tilde{f}$  and  $f$  differ only on a set of measure zero, and thus are in the same equivalence class in  $L^p$  for all  $p$ .

Now, we have that  $\tilde{f} \leq \|f\|_{L^\infty} = 1$ , and thus  $\tilde{f}^p \leq 1$  for all  $p \geq 1$ . Therefore,

$$\begin{aligned} \int_{\Omega} |\tilde{f}(x)|^p d\mu &\leq \int_{\Omega} 1 d\mu \\ &= \mu(\Omega) \end{aligned}$$

which implies that

$$\begin{aligned} \|f\|_{L^p} &= \left( \int_{\Omega} |\tilde{f}(x)|^p d\mu \right)^{\frac{1}{p}} \\ &\leq (\mu(\Omega))^{\frac{1}{p}} \end{aligned}$$

and for  $\mu(\Omega) < \infty$ , we have that  $\lim_{p \rightarrow \infty} (\mu(\Omega))^{\frac{1}{p}} = 1$ . Thus,

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} \leq 1$$

as desired.

Now, we wish to show the reverse. That is, we wish to show that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}$$

To do so, we consider the set  $\{|f| > 1 - \epsilon\}$ , which has positive measure for every  $\epsilon > 0$  by the fact that  $\|f\|_{L^\infty} = 1$ . Thus, we know that

$$\begin{aligned}\|f\|_{L^p} &= \left( \int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} \\ &\geq \left( \int_{|f| > 1 - \epsilon} |f|^p d\mu \right)^{\frac{1}{p}} \\ &= ((1 - \epsilon)\mu(\{|f| > 1 - \epsilon\}))^{\frac{1}{p}}\end{aligned}$$

Since  $\lim_{p \rightarrow \infty} ((1 - \epsilon)\mu(\{|f| > 1 - \epsilon\}))^{\frac{1}{p}} = 1$ , it follows that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} \geq 1$$

as desired.

Thus, for  $\|f\|_{L^\infty} < \infty$ , we have that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

as desired.

So, suppose  $\|f\|_{L^\infty} = \infty$ . That is, for each  $M > 0$ ,  $\mu(\{|f| > M\}) > 0$ . Thus,

$$\begin{aligned}\|f\|_{L^p} &= \left( \int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} \\ &\geq \left( \int_{|f| > M} M^p d\mu \right)^{\frac{1}{p}} \\ &= (M^p \mu(\{|f| > M\}))^{\frac{1}{p}} \\ &= M(\mu(\{|f| > M\}))^{\frac{1}{p}}\end{aligned}$$

We know already that  $\lim_{p \rightarrow \infty} (\mu(\{|f| > M\}))^{\frac{1}{p}} = 1$ , so it follows that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} \geq M(1) = M$$

Since  $M$  was arbitrary, it follows that

$$\lim_{p \rightarrow \infty} \|f\|_{L^p} = \infty$$

as desired. □