Problem Set 4

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PROBLEM 1

Prove that $[0,1)$ is not homeomorphic to $(0,1)$.

Proof. Let f be a bijective function from $[0,1)$ to $(0,1)$. Since f is bijective, it follows that $f^{-1}((0,1)) = [0,1)$. However, $(0,1)$ is open, but $[0,1)$ is not. Thus, f is not continuous.

Since no continuous bijections exist from $[0,1)$, to $(0,1)$, they are not homeomorphic. □

PROBLEM 2

Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Proof. Suppose there existed a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$. Now, consider restricting the domain of f to $\mathbb{R} \setminus \{0\}$. This yields a homeomorphism between $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{f(0)\}$. However, such a homeomorphism cannot exist, since $\mathbb{R} \setminus \{0\}$ is not connected, but $\mathbb{R}^2 \setminus \{f(0)\}$ is connected, and connectedness is a topological property. □

Problem 3

Prove that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

Proof. Suppose there existed a function $f : I \to I$ (with $I = [0, 1]$) such that f has no fixed points. In particular, this defines a (continuous) retract $r : I \to \partial I$ given by

$$r(x) = \begin{cases} 1, & \text{if } x > f(x) \\ 0, & \text{if } x < f(x) \end{cases}$$

Now, we see clearly that $r^{-1}(1) \neq \emptyset$, since at $x = 1$, $f(1)$ cannot be greater than 1, and thus must be less than 1, forcing $r(1) = 1$. Similarly, $r^{-1}(0) \neq \emptyset$, since at $x = 0$, $f(0)$ cannot be less than 0, and thus must be greater than 0, forcing $r(0) = 0$.

Therefore , r is a continuous function from I to the two-point set, and defines a separation of I. But I is connected, so no such separation can exist. Thus, such a retract cannot exist, and f must have a fixed point. □

Problem 4

Prove that $X \times X$ is connected if and only if X is.

Proof. ($\implies$) Suppose X is not connected. In particular, there exists a continuous surjection from X to the two-point set.

Thus, we have the diagram

$$\begin{tikzcd} & X \times X \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & \\ X \arrow[dr, two heads, "s_1"'] & & X \arrow[dl, two heads, "s_2"] \\ & \{0, 1\} & \end{tikzcd}$$

In particular, the composition $s_1 \circ \pi_1$ is a surjection from $X \times X$ onto $\{0, 1\}$, and defines a separation of $X \times X$. Thus, $X \times X$ is separated.

($\impliedby$) Suppose X is connected. We wish to show that any function $f : X \times X \to \{0, 1\}$ is constant.

So, let $f : X \times X \to \{0, 1\}$ be a continuous function, Now, since X is connected, and for all $x \in X$, $X \cong \{x\} \times X \cong X \times \{x\}$, it must be that f is constant on each of these fibers.

So, for arbitrary, (x_1, y_1) and (x_2, y_2), we have that

$$f((x_1, y_1)) = f((x_1, y_2)) = f((x_2, y_2))$$

where in each equality we used that f is constant on the fibers $\pi_1^{-1}(x_1)$ and $\pi_2^{-1}(y_2)$.

Thus, f is constant, and $X \times X$ cannot be separated. □

Problem 5

Suppose that A_α are all path connected subspaces of a space X, and that $\cap A_\alpha \neq \emptyset$. Prove that $\cup A_\alpha$ is path-connected.

Proof. Let x, y be distinct points in $\cup A_\alpha$. In particular, $x \in A_x$ for some A_x, and $y \in A_y$ for some A_y. Now, let $x_0 \in \cap A_\alpha$. Then, since A_x is path-connected, there is a path γ_x from x to x_0. Similarly for y, there is a path γ_y from y to x_0. Then, the path $\gamma_x - \gamma_y$ is a continuous path from x to y.

Since this can be done for any $x, y \in \cup A_\alpha$, it follows that $\cup A_\alpha$ is path-connected. $\square$

Problem 6

Prove that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path-connected.

Proof. Let $(x_1, y_1), (x_2, y_2)$ be distinct points in $\mathbb{R}^2 \setminus \mathbb{Q}^2$, and without loss of generality, let $x_1 \neq x_2$. Now, fix an irrational point x_0 such that $x_1 < x_0 < x_2$, and consider the set $\{x_0\} \times \mathbb{R} \subset \mathbb{R}^2 \setminus \mathbb{Q}^2$. Now, this set is clearly uncountable, since it is homeomorphic to $\mathbb{R}$. Furthermore, each point y in $\{x_0\} \times \mathbb{R}$ defines a path in $\mathbb{R}^2$ by taking the straight line from (x_1, y_1) to (x_0, y), then the straight line from (x_0, y) to (x_2, y_2).

Now, each path must necessarily intersect unique rationals, since if two paths intersected the same rational point (and since they originated from the same position) by linearity they must be the same line.

So, since there are only countably many rationals, but uncountably many paths defined in this manner, there must be a path that does not intersect any rationals.

Thus, there is a path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ that connects (x_1, y_1) and (x_2, y_2), and since these points were arbitrary, it must be that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path-connected, as desired. $\square$

Problem 7

Prove that every connected open subset of $\mathbb{R}^2$ is path-connected.

Proof. Suppose that an open subset $U \subset \mathbb{R}^2$ is not path-connected. In particular, let A_α be the path-components of U. Since U is open as a subset of $\mathbb{R}^2$, it follows that every point of U has an ε-ball around it contained in U. Since ε-balls are path-connected in $\mathbb{R}^2$, it follows that for $a \in A_\alpha$, the ε-ball around a is contained in A_α (since A_α is a path-component). Thus, since each point in A_α has a basic open neighborhood in A_α, it follows that A_α is open.

Since each path-component is disjoint and open, and they union to U, it follows that they form a separation of U, and thus U is not connected.

Therefore, if U is connected, then U is path-connected. $\square$