Problem Set 6

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PROBLEM 1

Show that the weak and weak-$*$ topologies are Hausdorff.

Proof. We start by showing the weak topology is Hausdorff. Recall from the midterm that X^* separates points of X. So, for x, y distinct points in X, let ϕ separate them. Choose $U, V \subset \mathbb{R}$ disjoint open sets containing $\phi(x)$ and $\phi(y)$ respectively. Then, since ϕ is continuous in the weak topology, $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are disjoint open sets in X that contain x, y respectively, and thus X is Hausdorff.

For the weak-$*$ topology, note that all we have to show is that X separates the functionals in X^*, and we can apply the same argument as above. So, suppose ϕ, ψ are distinct elements of X^*. Then, by definition, there is some $x \in X$ for which $\phi(x) \neq \psi(x)$. Thus, X separates points in X^*, and X^* with the weak-$*$ topology is Hausdorff. $\square$

Problem 2

Find the weak closure of the unit sphere S.

Proof. Note first that we have proven already the unit ball is closed. Now, since S is contained in the unit ball B, we know that $\overline{S} \subset B$ as well.

Now, let $x \in B$. Recall that any basic open set centered at x contains a translation of a subspace through x. In particular, any open set U containing x must also contain a subspace which goes through x. Now, since x is inside B, and B is bounded, it must be that U intersects the boundary S.

Thus, for every point $x \in B$, every neighborhood of x intersects S, and thus $x \in \overline{S}$, and in particular $B \subset \overline{S}$.

Thus, $\overline{S} = B$. □

Problem 3

Show that the sets $W(\phi; x_1, \ldots, x_n)$ form a basis for a topology on X^*. Furthermore, convergence in this topology is the same as convergence in weak-$*$ topology.

Proof. First, we show that these form a basis for a topology. First, it is clear that every element $\phi \in X^*$ is contained in a basic open set. Namely, $\phi \in W(\phi; x)$ for any x.

Next, we show that every point in the intersection of two basic open sets has a basic open set containing it that is contained in the intersection.

So, let $W(\phi_1; x_1, \ldots, x_n)$ and $W(\phi_2; y_1, \ldots, y_m)$ be basic open sets, and let ψ be in their intersection. Then, the basic open set $U = W(\psi, C_1x_1, \ldots, C_nx_n, D_1y_1, \ldots, D_my_m)$ contains ψ, and for suitable choice of constants will be contained in the intersection. To see this, let ε_i be such that $\|(\phi_1 - \psi)(x_i)\| < 1 - \varepsilon_i$, and δ_i be such that $\|(\phi_2 - \psi)(y_i)\| < 1 - \delta_i$. If we set $C_i = \frac{1}{\varepsilon_i}$ and $D_i = \frac{1}{\delta_i}$, we note that for any $\varphi \in U$, we have

$$\begin{aligned}
\|(\varphi - \phi_1)(x_i)\| &= \|(\varphi - \psi + \psi - \phi_1)(x_i)\| \\
&\leq \|(\varphi - \psi)(x_i)\| + \|(\psi - \phi_1)(x_i)\| \\
&= \varepsilon_i \|(\varphi - \psi)(C_i x_i)\| + \|(\psi - \phi_1)(x_i)\| \\
&\leq \varepsilon_i(1) + (1 - \varepsilon_i) = 1
\end{aligned}$$

and similarly for ϕ_2. Thus, for any $\varphi \in U$, we have that $\varphi \in W(\phi_1; x_1, \ldots, x_n)$ and $\varphi \in W(\phi_2; y_1, \ldots, y_m)$ as desired.

Thus, these sets form a basis for a topology.

Now, let's show that convergence in this topology is equivalent to weak-$*$ convergence.

We first make the following observation. The sets $W(\phi; x_1, \ldots, x_n)$ are generated by finite intersections of the subbasis $W(\phi; x)$. Furthermore, this is equivalent to the subbasis given by

$$\{\{\psi \in X^* \mid \|(\psi - \phi)(x) < r\}, \phi \in X^*, x \in X, r \in \mathbb{R}\}$$

(since $\|(\psi - \phi)(x)\| < r$ is the same as $\|(\psi - \phi)(\frac{x}{r}) < 1$.)

This leads to the neighborhood subbasis of zero given by

$$\{\{\psi \in X^* \mid \|\psi(x)\| < r\}, x \in X, r \in \mathbb{R}\}$$

which is equivalent to

$$\{x^{-1}(U) \mid U \subset \mathbb{R}\text{open}, 0 \in U\}$$

Now, without loss of generality, let's show that $\phi_n \to 0$ in weak-$*$ if and only if $\phi_n \to 0$ in this topology.

Now, clearly each $x : X^* \to \mathbb{R}$ is continuous at zero, and thus is continuous. So, if $\phi_n \to 0$ in the topology, then $x(\phi_n) \to x(0) = 0$ for all x (since x is continuous, it preserves limits). Thus, $\phi_n \to 0$ in weak-$*$ as well.

Suppose instead that $\phi_n \to 0$ in weak-$*$. It suffices to show that every basic open neighborhood of 0 eventually contains the sequence ϕ_n. So, let $B = \cap_{i=1}^m x_i^{-1}(U_i)$ for U_i an open neighborhood of zero. Then, it follows that since $\phi_n(x_i) \to 0$ for all i, then for each i there is an N_i such that $\phi_n(x_i) \in U_i$ for all $n > N_i$.

Set $N = \max(N_i)$. Then, for each $n > N$, we have

$$\begin{aligned}
&\phi_n \in x_i^{-1}(U_i) \\
\implies &\phi_n \in \cap_{i=1}^m x_i^{-1}(U_i) = B
\end{aligned}$$

and thus B eventually contains the sequence ϕ_n, and so $\phi_n \to 0$ in the topology as desired. ∎

Problem 4

Let X be a seperable Banach space with countable dense subset x_n. Show that the weak topology restricted to the unit ball coincides with the metric topology given by

$$d(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|(\phi - \psi)(x_n)\|}{1 + \|(\phi - \psi)(x_n)\|}$$

Proof. First, we show that the identity map from X with the weak topology to X with the metric topology is continuous. For this, it suffices to show that the ball $B_\varepsilon(0)$ contains a basic open set of the weak topology centered at zero (intersected with the ball).

To see this, we consider the set $W = W(0; y_1, \ldots, y_m)$ where $y_i = \frac{x_i}{\varepsilon}$, and m is large enough so that $2^{-m} < \varepsilon$. Then, we have for $\varphi \in W$,

$$\begin{aligned}
d(\varphi, 0) &= \sum_{n=1}^{\infty} 2^{-n} \frac{\|\varphi(x_n)\|}{1 + \|\varphi(x_n)\|} \\
&\leq \sum_{n=1}^{m} 2^{-n} \|\varphi(x_n)\| + \sum_{n=m+1}^{\infty} 2^{-m} \\
&\leq 2 \sum_{n=1}^{m} \varepsilon \|\varphi(y_n)\| + \varepsilon \\
&\leq 2m\varepsilon + \varepsilon
\end{aligned}$$

and thus (for a more suitable choice of y_i and m) the ball $B_\varepsilon(0)$ contains the basic open set W, and thus the identity map is continuous from the weak topology to the metric topology.

Now, we know two important facts. One, the unit ball in the weak topology is compact. Two, the metric topology is always Hausdorff. So, any closed set in a compact space is compact, and images of compact spaces under continuous maps are compact. Finally, compact subsets of Hausdorff spaces are always closed. Thus, the image of any closed set under the identity is closed. So, for U open in the weak topology, its complement U^c is closed, and thus is closed in the metric topology as well. Thus, U is open in the metric topology as well.

Thus, the two topologies coincide, as desired. $\square$