Problem Set 2

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Problem 2-1

For f the Heaviside step function (with $f(0) = 1$), show that $\forall x \in \mathbb{R}$, there exist smooth charts (U, ϕ) around x and (V, ψ) around $f(x)$ such that $\psi \circ f \circ \phi^{-1}$ is smooth as a map from its domain to its image, but f is not smooth in a smooth manifold sense.

Proof. For $x \neq 0$, neighborhoods avoiding zero can be chosen, and identity charts make f locally smooth. For $x = 0$, set $U = (-\epsilon, \epsilon)$, $V = (1 - \epsilon, 1 + \epsilon)$ and have $\phi_U = \psi_V = \text{id}$. Then, on $U \cap f^{-1}(V) = [0, \epsilon)$ we have $\psi \circ f \circ \phi^{-1}(x) = 1$ which is smooth. But this fails the test in proposition 2.5, so f is not smooth in a manifold sense. $\square$

Problem 2-3

For each of the following maps, show that the map is smooth via computation through coordinate representations.

Part A

The power map $p_n : S^1 \to S^1$ defined as $p_n(z) = z^n$.

Proof. For this problem, we will use two coordinate charts on S^1. First, let's parameterize the circle by θ, so that the point θ is identified with $\exp(i\theta)$ in the standard embedding of the circle into $\mathbb{C}$. Then, the first coordinate chart will be for $\theta \in (0, 2\pi)$ given as $\phi(\theta) = \theta$. The second coordinate chart will be for $\theta \in (-\pi, \pi)$ (where $2\pi\theta \sim \theta$) given as $\psi(\theta) = \theta$.

Now, the transition maps can easily be verified to be smooth. To see this, let θ_0 be a point in the intersection of the two charts. Then, if $\theta \in (0, \pi)$, we have

$$\phi(\theta) = \theta$$
$$\psi(\theta) = \theta$$

Which are easily verified to be smooth and compatible with each other.

Suppose, then, that $\theta \in (\pi, 2\pi)$. Then, we have that

$$\begin{aligned} \phi(\theta) &= \theta \\ \psi(\theta) &= \theta - 2\pi \end{aligned}$$

With transition charts

$$\begin{aligned} \phi \circ \psi^{-1}(\theta) &= \theta + 2\pi \\ \psi \circ \phi^{-1}(\theta) &= \theta - 2\pi \end{aligned}$$

which are clearly smooth.

Now, we just have to check that the power function, which can be thought of in terms of our parameterization as $p_n(\theta) = n\theta (\mod 2\pi)$, is smooth.

So, let's compute some coordinate representations. We have a total of four to check.

$$\begin{aligned} \phi \circ p_n \circ \phi^{-1}(\theta) &= n\theta (\mod 2\pi) \\ \psi \circ p_n \circ \psi^{-1}(\theta) &= n(\theta + 2\pi)(\mod 2\pi) - 2\pi \\ \phi \circ p_n \circ \psi^{-1}(\theta) &= n(\theta + 2\pi)(\mod 2\pi) \\ \psi \circ p_n \circ \phi^{-1}(\theta) &= n\theta (\mod 2\pi) - 2\pi \end{aligned}$$

Now, addition of a scalar is a smooth operation, so we just have to check that the function p_n is smooth as a function of θ.

Now, we observe that p_n is continuous as a function of θ by viewing $p_n : [0, 2\pi) \to \mathbb{R}$ as a continuous function $\theta \mapsto n\theta$, and passing through the quotient $\mathbb{R}/2\pi\mathbb{Z}$. Since the derivative $p_n' = np_{n-1}$ is also of the same form, it is continuous as well, and by induction each derivative of p_n is continuous, so p_n is smooth.

Thus, the composition maps defined above are smooth, and p_n is a smooth function from S^1 to itself. □

Part B

The antipodal map $\alpha : S^n \to S^n$ by $\alpha(x) = -x$.

Proof. Consider the stereographic projection charts σ and $\tilde{\sigma}$, where $\tilde{\sigma}(x) = -\sigma(-x)$. Let's compute some coordinate representations:

$$\begin{aligned} \sigma \circ \alpha \circ \sigma^{-1}(x) &= \sigma(-\sigma^{-1}(x)) \\ \tilde{\sigma} \circ \alpha \circ \tilde{\sigma}^{-1}(x) &= \tilde{\sigma}(-\tilde{\sigma}^{-1}(x)) \\ \sigma \circ \alpha \circ \tilde{\sigma}^{-1}(x) &= \sigma(-\tilde{\sigma}^{-1}(x)) \\ \tilde{\sigma} \circ \alpha \circ \sigma^{-1}(x) &= \tilde{\sigma}(-\sigma^{-1}(x)) \end{aligned}$$

Now, these are all compositions of smooth functions, which are smooth as well. Thus, the antipodal map is a smooth function. □

Part C

Show that the map $F : S^3 \to S^2$ defined as $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, is smooth.

Proof. To show that this map is smooth, we will show it is smooth in the ambient space $\mathbb{C}^2 \setminus \{0\}$ and $\mathbb{R}^3 \setminus \{0\}$.

Now, F is smooth as a map from the ambient spaces, which is clear when viewing it as a map from $\mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$. Using this, we have that

$$F(x^1, x^2, x^3, x^4) = (2(x^1x^3 + x^2x^4), 2(x^2x^3 - x^1x^4), (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2)$$

which is clearly smooth. Now, since F is smooth in the ambient space, it must also be smooth when restricted to $S^3 \subset \mathbb{C}^2$. $\square$

Problem 2-7

Show that for M a nonempty smooth n-manifold, with $n \geq 1$, the vector space $C^\infty(M)$ is infinite dimensional.

Proof. Let $\{U_i\}$ be a set of open subsets of M that are all pairwise disjoint, and consider the set of C^∞ functions $\{f_i\}$ on M such that $\text{supp}(f_i) \subset U_i$. Such a construction is done using partitions of unity subordinate to a carefully chosen open cover of M.

Now, it is easy to see each f_i is linearly independent of the others. To see this, suppose for a contradiction that for some $f_0 \in \{f_i\}$, $f_0 = \sum_{i \neq 0} a_i f_i$. Let $x \in \text{supp}(f_0)$. In particular, we have $f_0(x) \neq 0$. However, since the supports of $\{f_i\}$ are all pairwise disjoint, it must be that $f_i(x) = 0$ for all $f_i \neq f_0$. Thus we have

$$\begin{aligned} f_0(x) &= \sum_{i \neq 0} a_i f_i(x) \\ &= \sum_{i \neq 0} a_i(0) \\ &= 0 \end{aligned}$$

which contradicts the fact that $f_0(x) \neq 0$.

Now, since an arbitrary number of disjoint open sets can be constructed on M, it follows that there are arbitrarily many linearly independent functions in $C^\infty(M)$, so it is infinite dimensional. $\square$

Problem 2-10

Consider the algebra $C(M)$ of continuous functions on M, and observe that a map$f : M \to N$ induces a map $f^* : C(N) \to C(M)$ via pre-composition.

Part A

Show that f^* is linear.

Proof. Let $g, h \in C(N)$, and $\alpha, \beta \in \mathbb{R}$. Now,

$$\begin{aligned} f^*(\alpha g + \beta h)(x) &= (\alpha g + \beta h) \circ f(x) \\ &= \alpha g(f(x) + \beta h(f(x)) \\ &= \alpha f^*(g) + \beta f^*(h) \end{aligned}$$

Thus, f^* is linear. $\square$

Part b

Show that f is smooth if and only if $f^*(C^\infty(N)) \subseteq C^\infty(M)$.

Proof. (=>) Assume that $f : M \to N$ is smooth. Then, for any $g \in C^\infty(N)$, we have $f^*(g) = g \circ f$, which is the composition of smooth functions, and thus $f^*(g) \in C^\infty(M)$. Therefore, $f^*(C^\infty(N)) \subseteq C^\infty(M)$ as desired.

(<=) Now, suppose f is such that $f^*(C^\infty(N)) \subseteq C^\infty(M)$. In particular, for any coordinate chart ϕ on N, we have $f^*(\phi) \in C^\infty(M)$. That is, for any chart ψ on M, we have

$$\begin{aligned} &\phi \circ f \in C^\infty(M) \\ \implies &\phi \circ f \circ \psi^{-1} \in C^\infty(\mathbb{R}) \end{aligned}$$

Since this works for any ϕ on N and ψ on M, it follows that f is smooth. □

Part c

Given a homeomorphism $f : M \to N$, show that f is a diffeomorphism if and only if f^* restricts to an isomorphism $f^* : C^\infty(N) \to C^\infty(M)$

Proof. Observe first that since f is a homeomorphism, f^{-1} is well-defined and continuous.

(=>) Suppose f is a diffeomorphism. In particular, this means f and f^{-1} are smooth. By the previous result, we have that

$$\begin{aligned} f^*(C^\infty(N)) &\subseteq C^\infty(M) \\ f^{-1^*}(C^\infty(M)) &\subseteq C^\infty(N) \end{aligned}$$

In particular, we have that f^* and f^{-1^*} are surjective by the following argument.

Let $g \in C^\infty(M)$. Then, $f^{-1^*}(g) = g \circ f^{-1} \in C^\infty(N)$, and $f^*(f^{-1^*}(g) = g \circ f^{-1} \circ f = g$. Thus, f^* is surjective (more specifically, $(f^{-1})^* = f^{-1^*}$ on $C^\infty(N)$).

By the same argument, f^{-1^*} is surjective and the inverse of f^*. Thus, f^* is an isomorphism as desired.

(<=) Now, suppose f^* restricts to an isomorphism between $C^\infty(N)$ and $C^\infty(M)$. In particular, this means that $f^*(C^\infty(N)) \subseteq C^\infty(M)$, which implies f is smooth. Now, the above argument suggests that the same argument for $f^{-1^*} = (f^{-1})^*$ shows that f^{-1} is smooth as well. Thus, f and f^{-1} are smooth, and f is a diffeomorphism. □

Problem 2-14

For A and B disjoint closed subsets of a smooth manifold M, show that there exists $f \in C^\infty$ such that $0 \le f \le 1$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Problem 3-5

Problem 3-6

Problem 3-7

Problem 3-8

For M a smooth manifold, and $p \in M$, let $\mathscr{V}_p M$ be the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if for all $f \in C^\infty(M)$, $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$.

Show that the map $\Psi : \mathscr{V}_p M \to T_p M$ defined as $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective.

Proof. To begin with, we show that this map is well defined. To do so, let γ_1 and γ_2 be equivalent in the sense defined above. In particular, this means that $d\gamma_1(\partial_t|_0)(f) = d\gamma_2(\partial_t|_0)$ for all f in $\mathbb{C}^\infty(M)$. Thus, since the differentials are functions on $C^\infty(M)$ that are identical for all f, we have that $d\gamma_1(\partial_t|_0) = d\gamma_2(\partial_t|_0)$ which implies $\gamma_1'(0) = \gamma_2'(0)$ as desired.

Now, let's show that this is bijective. To do so, we will first show Ψ is surjective. Let v be some vector in T_pM. In particular, $v = v^i \frac{\partial}{\partial x^i}|_p$ for some coordinates x^i centered at p. Now, define a curve $\gamma : [0,1] \to M$ as $\gamma^i(t) = tv^i$. It is clear that $\gamma'(0) = v$, since $\gamma'^i(0) = v^i$, which implies $\gamma'(0) = v^i \partial_i = v$ as desired.

Second, we will show Ψ is injective. This is immediate from the definition of the equivalence relation, since by the argument for well-definedness if $\gamma_1'(0) = \gamma_2'(0)$, then $\gamma_1 \sim \gamma_2$.

Thus, Ψ is bijective, as desired. □

Problem 3-4

Show $TS^1 \cong S^1 \times \mathbb{R}$.

Proof. To prove this, we first note that there is a natural group structure on S^1 when thought of as a subset of $\mathbb{C}^*$, namely the multiplicative structure from $\mathbb{C}^*$. This is clearly a Lie group, since the map $(\theta, \phi) \mapsto \theta\phi^{-1}$ is smooth. To see this, consider the fact that, in $\mathbb{C}^*$, the map $(z_1, z_2) \mapsto z_1 z_2^{-1}$ from $\mathbb{C}^*$ to itself is clearly smooth, since multiplication, and inversion are smooth operations. Thus, S^1 is a Lie group under this operation.

Consider the space $\mathfrak{g}$, the set of all left-invariant vector fields on a Lie group G. Here, a vector field on a Lie group G is said to be *left-invariant* if for all $\sigma \in G$, we have that

$$dl_\sigma \circ X = X \circ l_\sigma$$

for l_σ the operation of left-multiplication by σ. Clearly, this forms a vector space, with addition and scalar multiplication inherited from the tangent spaces. It is clearly closed under these operations, since

$$\begin{aligned}(X+Y) \circ l_\sigma &= X \circ l_\sigma + Y \circ l_\sigma \\ &= dl_\sigma \circ X + dl_\sigma \circ Y \\ &= dl_\sigma \circ (X+Y)\end{aligned}$$

And, for $r \in \mathbb{R}$,

$$(rX) \circ l_\sigma = r(X \circ l_\sigma) = r(dl_\sigma \circ X) = dl_\sigma \circ rX$$

Thus, $\mathfrak{g}$ is a real vector space.

Now, we establish an isomorphism between $\mathfrak{g}$ and the tangent space T_eG given by $\alpha : \mathfrak{g} \to T_eG$, $\alpha(X) = X(e)$.

Now, α is clearly linear, so we just need to show it is injective and surjective. To see this, let $\alpha(X) = \alpha(Y)$. Then, for each $\theta \in G$, we have

$$\begin{aligned}X(\theta) &= dl_\theta X(e) \\ &= dl_\theta Y(e) \\ &= Y(\theta)\end{aligned}$$

Thus, $\alpha(X) = \alpha(Y)$ implies that $X = Y$, so α is injective.

To show surjectivity, let $x \in T_eG$. Then, define a vector field X to be $X(\sigma) = dl_\sigma(x)$. Clearly, X is left-invariant, since for all $\theta, \sigma \in G$, we have

$$X(l_\sigma \theta) = X(\theta\sigma) = dl_{\sigma\theta}(x) = dl_\sigma dl_\theta(x) = dl_\sigma X(\theta)$$

Here, we used the functoriality of d to split $dl_{\sigma\theta} = dl_\sigma dl_\theta$.

Now, it is clear that $\alpha(X) = X(e) = x$, so α is surjective as well. Therefore, the tangent space T_eG is isomorphic to the set $\mathfrak{g}$ of left-invariant vector fields on G.

This establishes the basic isomorphism we will use. Define $\Phi : G \times T_eG \to TG$ by

$$\Phi(\sigma, x) = dl_\sigma \alpha^{-1}(x)$$

That is, for a vector $x \in T_eG$, identify it with the left-invariant vector field $X \in \mathfrak{g}$ by $\alpha(X) = x$. Then, Φ takes the tangent vector x and sends it to the tangent vector $X(\sigma)$.

Φ can be shown to be a smooth bijection. First, we will show it is surjective and injective, then we will show it is smooth.

First, let $\Phi(\theta_1, x_1) = \Phi(\theta_2, x_2)$. Clearly, $\theta_1 = \theta_2$, since if $\Phi(\theta_1, x_1) = \Phi(\theta_2, x_2)$, then its projections back to G must be equal as well. Thus$\theta_1 = \theta_2$. Now, let $X_i = \alpha^{-1}(x_i)$. Then, we have that $X_1(\theta) = X_2(\theta)$. Since X_i is left-invariant, we must have that

$$X_1(e) = dl_{\theta^{-1}} \circ X_1(\theta) = dl_{\theta^{-1}} \circ X_2(\theta) = X_2(e)$$

So $x_1 = x_2$ and Φ is injective.

Second, let $(\sigma, x) \in TG$. Clearly, $\Phi(\sigma, x) = X(\sigma) = (\sigma, x)$ by the definition of Φ, so Φ is surjective as well.

Now we can see also that Φ is smooth. To do so, let's choose a coordinate chart (U, ϕ) centered at e given as $(x_1, \dots, x_n)$ (which naturally gives a basis for T_eG as $\{\partial_1|_e, \dots, \partial_n|_e\}$). This chart induces a chart at θ given on $l_\theta(U)$ by $\phi \circ l_{\theta^{-1}}$, and induces a basis on $T_\theta G$ by pushing forward $\partial_i|_e$ along dl_θ to get $\partial_i|_\theta$.

So, for any $(\theta, x) \in G \times T_eG$, we have the coordinate chart $(l_\theta U \times T_eG, \widetilde{\phi})$ given as

$$\widetilde{\phi}(\sigma, x^i \partial_i|_e) = (\phi(l_{\theta^{-1}}(\sigma)), x^i)$$

Recall also that we need a coordinate chart on TG, but this is induced from the coordinate chart defined above. In particular, (for π the standard projection map from TG to G) on $\pi^{-1}(l_\theta(U))$ we have the chart:

$$\widetilde{\varphi}(\sigma, x^i \partial_i|_\sigma) = (\phi(l_{\theta^{-1}}(\sigma)), x^i)$$

Now, let's compute the transition map $\widetilde{\varphi} \circ \Phi \circ \widetilde{\phi}^{-1}$.

$$\begin{aligned}
\widetilde{\varphi} \circ \Phi \circ \widetilde{\phi}^{-1}(\phi(l_{\theta^{-1}}(\sigma)), x^i) &= \widetilde{\varphi} \circ \Phi(\sigma, x^i \partial_i|_e) \\
&= \widetilde{\varphi}(\sigma, dl_\sigma(x^i \partial_i|_e)) \\
&= \widetilde{\varphi}(\sigma, dl_{\sigma\theta^{-1}}(x^i \partial_i|_\theta)) \\
&= (\phi(l_{\theta^{-1}}(\sigma)), dx^i(l_{\sigma\theta^{-1}}(x^i \partial_i|_\theta))
\end{aligned}$$

Which is a smooth function, so Φ is a diffeomorphism.

Therefore, the tangent bundle of a Lie group is trivial. Applying this to the special case of $G = S^1$, we have that $TS^1 \cong S^1 \times \mathbb{R}$ as desired. $\square$