Homework 1

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Problem 1

Let $f : X \to X'$ and $g : Y \to Y'$ be smooth maps. Prove that the composite map $f \times g : X \times Y \to X' \times Y'$ is smooth.

Proof. Recall the definition of a smooth map. $f : X \to Y$ is called smooth if for every chart ϕ on X and ψ on Y, the composite $\psi f \phi^{-1}$ is smooth. Recall also that for manifolds M, N, the product manifold $M \times N$ is defined as the cartesian product $M \times N$ with the differentiable structure generated by products of charts on M and N.

With that aside, we proceed with the proof. Let $(x, y) \in X \times Y$. We will show that $f \times g$ is smooth at (x, y). Let ϕ be a chart around $x \in X$, ϕ' a chart around $f(x)$, ψ a chart around y, and ψ' around $g(y)$. Then, $\phi \times \psi$ is a chart around (x, y), and $\phi' \times \psi'$ is a chart around $(f \times g)(x, y)$.

Now, the composite

$$(\phi' \times \psi') \circ (f \times g) \circ (\phi \times \psi)^{-1} = (\phi' \circ f \circ \phi^{-1}) \times (\psi' \circ g \circ \psi^{-1})$$

is the product of smooth functions on Euclidean space, which is trivially seen to be smooth. Thus, $f \times g$ is smooth at every point, as desired. $\square$

Problem 2

Prove that the projection map $\pi_x : X \times Y \to X$ is smooth.

Proof. Let $(x, y) \in X \times Y$, and let ϕ be a chart around $x \in X$, ψ a chart around $y \in Y$. Now, the composite

$$\phi \circ \pi_x \circ (\phi \times \psi)^{-1}(\phi(x), \psi(y)) = \phi(x)$$

is just the standard projection operator on Euclidean space, which we know to be smooth. Thus, π_x is smooth at every point, as desired. $\square$

Problem 3

Let $U \subset X$ be open. Prove that for all $p \in U$, $T_pU = T_pX$.

Proof. First, I assert that U has a manifold structure given by $\{(V \cap U, \phi) \mid (V, \phi) \text{a chart in } X\}$. This works because $V \cap U$ is open, and thus $\phi|_{V \cap U}$ is a coordinate chart. Compatibility of the charts follows from the manifold structure on X itself, which guarantees the charts are compatible.

Let $p \in U$, and let (V, ϕ) be a chart around $p \in X$. Let's also require that $V \subset U$. We know that T_pX is the image $d\phi^{-1}(\phi(V))$ of the derivative of ϕ^{-1} on its domain. Furthermore, we know that $(V, \phi) = (V \cap U, \phi)$ is also a chart for U around p. Thus, at p, ϕ works as both a chart on X and a chart on U, and the tangent space (which is defined entirely with respect to the chart) must be the same. That is, $T_pU = T_pX$ as desired. □

Problem 4

Prove that if $f : X \to Y$ is a diffeomorphism, then df_x is an isomorphism for all $x \in X$.

Proof. Recall that the differential is functorial. That is, $d(f \circ g) = df \circ dg$ and $d(\mathbb{1}) = \mathbb{1}$ (this follows from the chain rule). Then, as a consequence, we know that $d(f^{-1}) = (df)^{-1}$.

Now, since f is a diffeomorphism, it has an inverse f^{-1} such that $f \circ f^{-1} = f^{-1} \circ f = \mathbb{1}$. Thus, we know that df_x has both a right and left inverse as df_x^{-1}. However, any linear map with both a left and right inverse is necessarily an isomorphism. Thus, df_x is an isomorphism for all x, as desired. □

Problem 5

Show that T_pX is the set of velocity vectors of curves through p.

Proof. We first show that any $v \in T_pX$ is the velocity vector of some curve through p.

To see this, let $v \in T_pX$, and choose a coordinate system (U, ϕ) centered at p such that $v = d\phi^{-1}(\partial_1)$ where ∂_1 is the first basis vector for $T_0\mathbb{R}^n$ (i.e. $\partial_1 = (1, 0, \dots, 0)$ if we identify $T_0\mathbb{R}^n$ with $\mathbb{R}^n$.)

Then, consider the curve $\gamma : (-\varepsilon, \varepsilon) \to X$ defined as

$$\gamma(t) = \phi^{-1}(t, 0, \dots, 0)$$

This has derivative

$$\begin{aligned} \gamma'(0) &= \partial_t \phi^{-1}(t, 0, \dots, 0) \\ &= \partial_t(t)\partial_1 + \partial_t(0)\partial_2 + \cdots + \partial_t(0)\partial_n \\ &= \partial_1 = v \end{aligned}$$

Thus, every $v \in T_pX$ is the derivative of some curve.

Next, we show that every velocity vector is in the tangent space. This is clear, since if $\gamma : [-1, 1] \to X$ is a curve with $\gamma(0) = p$, we can fix a coordinate system (U, ϕ) around p with coordinate functions x^i, and calculate

$$\gamma'(0) = \partial_t \gamma^i(t)|_0 \partial_i$$

where $\gamma^i(t)$ is $x^i(\gamma(t))$, and ∂_i is the basis for T_pX generated by the x^i functions. Thus, $\gamma'(0) \in T_pX$ as desired. □

Problem 6

Prove that if $f : X \to Y$ is a submersion, and $U \subset X$ is open, then $f(U) \subset Y$ is open.

Proof. Suppose $y \in f(U)$. Now, for any $x \in U$ with $f(x) = y$, we can find a neighborhood V of y for which there is a smooth section $\sigma : V \to X$ with $\sigma(y) = x$. Then, for each $z \in \sigma^{-1}(U)$, we have $z = f(\sigma(z)) \in f(U)$. So, $\sigma^{-1}(U)$ is an open neighborhood of y contained in $f(U)$. Thus, since we can do this for all $y \in f(U)$, we see that $f(U)$ is open, as desired. □