MATH 220A: FINAL EXAMINATION
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Problem 1

Part i. Show that a finite group G with a non-trivial cyclic Sylow-2 subgroup has a subgroup of index 2.

Proof. Let P_2 be a non-trivial cyclic Sylow-2 subgroup of G, with generator x. Since P_2 is a cyclic Sylow-2 subgroup of G, there exists some natural number α such that $|x| = 2^\alpha$.

From Cayley's theorem, we know that there exists an embedding ρ of G into the symmetric group $S_{|G|}$. Furthermore, it is easily shown that $\rho(x)$ is an odd permutation. This is evident, since the order of $\rho(x)$ is 2^α, and $\rho(x)$ consists of exactly $\frac{|G|}{2^\alpha} = r$ disjoint 2^α-cycles. Since r is odd (since α is the largest power such that 2^α divides $|G|$), it follows that $\rho(x)$ is the product of r disjoint odd cycles, and is odd itself. Thus, $\rho(x) \notin A_{|G|}$, and therefore $\rho(G) \not\leq A_{|G|}$.

I assert that exactly half of $\rho(G)$ is contained in $A_{|G|}$. To see this, consider the sets $T_1 = \rho(G) \cap A_{|G|}$ and $T_2 = \rho(G) \setminus T_1$. Clearly, all elements of T_1 are even permutations, and all elements of T_2 are odd permutations. Now, consider the function $l_x : T_1 \to T_2$ given by $l_x(\tau) = \rho(x)\tau$. This function is well-defined, since multiplying an even permutation (an element of T_1) by an odd permutation (namely, $\rho(x)$) yields an odd permutation (an element of T_2). Furthermore, l_x is clearly invertible (via left-multiplication by $\rho(x)^{-1}$) to a function from T_2 to T_1, and thus is a bijection between T_1 and T_2

Therefore, $|\rho(G) \cap A_{|G|}| = \frac{|\rho(G)|}{2}$, and since ρ is injective, it follows that

$$|\rho^{-1}(\rho(G) \cap A_{|G|})| = |\rho(G) \cap A_{|G|}| = \frac{|\rho(G)|}{2} = \frac{|G|}{2}$$

And thus, Lagrange's theorem guarantees that the index of $\rho^{-1}(\rho(G) \cap A_{|G|})$ is 2, as desired. $\square$

Part ii. Suppose in addition that $|G| = 2^m r$, where 2 does not divide r. By induction on m, show that G contains a normal subgroup of order r.

Proof. This proof will induct on m. The base case of $m = 1$ follows immediately. From part i, we know that G has a subgroup of index 2, with order $\frac{|G|}{2} = r$. Since the index of this subgroup is 2, it is normal in G, and is a normal subgroup of order r, as desired.

Now, suppose the theorem holds for all $k \leq m$, and let $|G| = 2^{m+1} r$. We know that G has a (normal) subgroup H of index 2 from part i, and so H has order $2^m r$. By induction, then, H has a normal subgroup M of order r. Since H has index 2, it follows that M is normal in G as well, as desired. $\square$

Problem 2

Part i. Give an example of a group that is not soluble.

Proof. The simple group of order 168 is not Abelian, and is thus not soluble. □

Part ii. Give an example with proof of a soluble group which is not nilpotent.

Proof. The dihedral group D_3 of the triangle is soluble, but not nilpotent. To see that this group is soluble, we consider the normal chain

$$D_3 \geq R_3 \geq 1$$

where R_3 is the subgroup of rotations of the triangle. Since D_3 has six elements, and R_3 has three, R_3 has index 2, and is thus normal in D_3. Furthermore, the group D_3/R_3 has order $\frac{6}{3} = 2$, and is therefore Abelian. Finally, we observe that $R_3/1 = R_3$ is Abelian as well. Clearly, all the factor groups of this normal chain are Abelian, and so D_3 is soluble.

However, D_3 is not nilpotent. To see this, we calculate the lower central series for D_3. Recursively, this is defined as

$$\gamma_1 = D_3$$

$$\gamma_{i+1} = [\gamma_i, D_3]$$

Now, γ_2 clearly contains R_3. To see this, we note that

$$(123) = (132)(12)(132)^{-1}(12)^{-1}$$

and since (123) generates R_3, it follows that $R_3 \leq \gamma_2$. Since $\gamma_3 = [\gamma_2, D_3]$, and since $(132) \in \gamma_2$, it follows that

$$(123) = (132)(12)(132)^{-1}(12)^{-1}$$

is an element of $[\gamma_2, D_3]$, and thus $R_3 \leq \gamma_3$. Continuing this argument shows that for any γ_i, we have that

$$R_3 \leq \gamma_i$$

and since the lower central series never terminates in a 1, it must be that D_3 is not nilpotent. $\square$

Part iii. Suppose that a group G has a composition series, and that H is normal in G. Show that G has a composition series one of whose terms is H.

Proof. We know that any two normal chains of a group G have isomorphic refinements. Thus, the normal chain

$$G \geq H \geq 1$$

has a refinement isomorphic with a refinement of the composition series for G. But since the composition series for G is a composition series, it is isomorphic with its refinements. Therefore, the normal chain $G \geq H \geq 1$ has a refinement isomorphic to the composition series for G. Such a refinement, then, is a composition series for G.

Thus, $G \geq H \geq 1$ can be refined to a composition series for G, which necessarily has H as one of its terms. $\square$

Problem 3

Suppose that a group G has three distinct composition series $G \geq H_1 \geq 1$, $G \geq H_2 \geq 1$, and $G \geq H_3 \geq 1$.

Part i. Show that the six groups $H_1, H_2, H_3, G/H_1, G/H_2$, and G/H_3 are all isomorphic.

Proof. Since each series given is a composition series, it follows that the quotient groups G/H_i are simple for $i \in \{1, 2, 3\}$. Now, consider the canonical quotient map $q : G \to G/H_i$. Since G/H_i is simple, and H_j is normal in G, it follows that $q(H_j)$ is normal in G/H_i, and thus $q(H_j)$ is either 1 or G/H_i.

Suppose $q(H_j) = 1$. Then, it must be that $H_j \leq H_i$. Since H_j is normal in G, it must also be normal in H_i. However, $H_i/1 = H_i$ is a factor group of the composition series, and is simple. So, $H_j = H_i$ or $H_j = 1$. Clearly, $H_j \neq 1$, so it must be that $H_j = H_i$. Conversely, if $H_j = H_i$, then trivially $q(H_j) = 1$. Thus, $q(H_j) = 1$ precisely when $H_i = H_j$ (i.e. when $i = j$).

Now, suppose $q(H_j) = G/H_i$. This defines a surjection from H_j to G/H_i. Since H_j is simple, the kernel of this surjection must be trivial. Therefore, q restricted to H_j is actually an isomorphism between H_j and G/H_i. Thus, $H_j \cong G/H_i$ for $i \neq j$. It follows immediately, then, that each of the six groups are isomorphic to each other. □

Part ii. Show that $G = \langle H_1, H_2 \rangle$.

Proof. Observe first that H_1 and H_2 are both simple. Now, the subgroup $H_1 \cap H_2$ is normal in H_1, since H_2 is normal in G. Thus, $H_1 \cap H_2 = 1$. The isomorphism theorems tell us, then, that

$$H_1H_2/H_2 \cong H_1$$

and from part i, we have that $H_1 \cong G/H_2$. Thus, $G/H_2 \cong H_1H_2/H_2$, and the correspondence theorem guarantees that $G \cong H_1H_2$. Now, since $H_1H_2 \leq \langle H_1, H_2\rangle$, it follows that $\langle H_1, H_2\rangle = G$ as desired. $\square$

Part iii. Show H_3 is Abelian.

Proof. Consider the center $Z(H_3)$. We know that

$$Z(H_3) = Z(G/H_3) = Z(G)/H_3$$

and so it must be that $Z(G) \geq H_3$. However, since $Z(G)$ is normal, and G/H_3 is simple, either $Z(G) = H_3$ or $Z(G) = G$. Clearly, $Z(G)$ cannot equal H_3, since the argument can be repeated for H_1 to show that $Z(G) = H_1$ or $Z(G) = G$. Since $H_1 \neq H_3$, it must be that $Z(G) = G$, and thus G is Abelian. This clearly implies that H_3 is Abelian as well. $\square$

Problem 4

Part a. Define the notion of a Hall subgroup of a finite group G.

Definition. *Let $\prod_{i=1}^{n} p_i^{\alpha_i}$ be the prime factorization of the order of G, and let π be a subset of $\{p_i\}_{i=1}^{n}$. A Hall-π subgroup of G is a subgroup of G of order $\prod_{p_i \in \pi} p_i^{\alpha_i}$.*

Part b. State Hall's criterion for finite soluble groups.

Statement. *A finite group G is soluble if and only if it has a system of Hall complements. That is, for $\prod p_i^{\alpha_i}$ the prime factorization of $|G|$, there exist subgroups of index $p_i^{\alpha_i}$ for each prime factor p_i.*

Part c. Show that all groups of order properly dividing 84 are soluble.

Proof. We first note that the prime factorization of 84 is $2^2.7.3$. Sylow's theorems guarantee that groups of order $2^2.7$, 2.7, $2^2.3$, 2.3 and 3.7 have a system of Hall complements, and thus are soluble.

So, consider the group G of order 2.3.7. Sylow's theorem guarantees that a subgroup P_7 of order 7 exists. Furthermore, the number of Sylow-7 subgroups of G is congruent to $1 \mod 7$. Since the number of Sylow-7 subgroups must also divide 2.3.7, it must be that there is only one Sylow-7 subgroup of G. In particular, this means that P_7 is normal in G. Now, since P_7 is a cyclic group (of order 7), it is soluble. Furthermore, G/P_7 is of order 2.3, and is also soluble. Thus, since both P_7 and G/P_7 are soluble, G is soluble as well. This exhausts all possible orders that properly divide 84. $\square$

Part d. Show that a group of order 84 is either soluble or simple.

Proof. Let G be a group of order 84, and suppose G is not simple. Let N be a normal subgroup of G. Now, since N is a subgroup of G, its order properly divides the order of G, and thus by part c N is soluble. Similarly, the group G/N has order properly dividing the order of G, and thus G/N is soluble as well. Since both G/N and N are soluble, G is as well. Thus, G is either simple or soluble, as desired. $\square$

Problem 5

Let G be a finite group.

Part a. Give the definitions of the commutator subgroup G' and the Frattini subgroup $\Phi(G)$ of G.

Definition. *The commutator subgroup G' of G is defined as*

$$G' = [G, G] = \{ghg^{-1}h^{-1} \mid g, h \in G\}$$

Definition. *The Frattini subgroup $\Phi(G)$ is defined to be the intersection of all maximal normal subgroups of G, or G itself if no maximal normal subgroups exist.*

Part b. Suppose that every maximal subgroup of G is normal in G. Prove that $G' \leq \Phi(G)$.

Proof. Let M be a maximal subgroup of G. Since M is normal, we may consider the quotient G/M. Now, the correspondence theorem tells us that subgroups of G/M are in bijection with subgroups of G containing M. Since M is maximal, it must be that the only subgroups of G/M are G/M and 1.

Furthermore, G/M must be a group of order p^α for some prime p. (If this were not the case, Sylow's theorems would give a proper subgroup of G/M for each prime factor of the order of G/M). Since every p-group is the direct product of cyclic p-groups, and G/M has no subgroups, it must be that G/M is cyclic of order p. Thus, G/M is Abelian.

Since G/M is Abelian, it must be that there is some group homomorphism $f : G/G' \to G/M$ such that the quotient $q_m : G \to G/M$ factors through $q_g : G \to G/G'$. Since f is a group homomorphism, it must send the coset G' into M. Thus, since $q_m = f \circ q_g$, it follows that for any $x \in G'$,

$$q_m(x) = f(q_g(x)) = f(e) = e$$

and thus $x \in M$. Thus, $G' \leq M$.

Since $G' \leq M$ for every maximal normal subgroup M in G, it follows that $G' \leq \Phi(G)$ as desired. $\square$