Midterm

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Problem 1

Consider a Riemannian manifold (M, g) of dimension n.Let $\{e_i\}$ be a local orthonormal frame on $u \subset M$, and let $\{\omega_i\}$ be the dual basis. Prove that there is a unique set of smooth 1-forms ω_i^j such that

$$d\omega^i = \omega^k \wedge \omega_k^i$$

and

$$\omega_i^j + \omega_i^j = 0$$

without appealing to the Levi-Civita connection.

Proof. We note first that the set $\{\omega^i \wedge \omega^j\}_{i<j}$ forms a basis for the second exterior power $\Lambda^2 U$. Thus, we can express

$$d\omega^i = \frac{1}{2} a_{\alpha\beta}{}^i \omega^\alpha \wedge \omega^\beta$$

For convenience in notation, we will let α, β both run up to n, and require that $a_{\alpha\beta}{}^i$ be anti-symmetric in its lower two indices. That is,

$$\begin{aligned} d\omega^i &= \sum_{\alpha<\beta} (\frac{1}{2} a_{\alpha\beta}{}^i - \frac{1}{2} a_{\beta\alpha}{}^i) \omega^\alpha \wedge \omega^\beta \\ &= \sum_{\alpha<\beta} a_{\alpha\beta}{}^i \omega^\alpha \wedge \omega^\beta \end{aligned}$$

Now, we define ω_i^j as follows.

$$\omega_{ij} = -\frac{1}{2}(a_{ij\alpha} + a_{i\alpha j} - a_{j\alpha i})\omega^\alpha$$

Now, a is antisymmetric in its first two indices, so

$$\begin{aligned}\omega_{ij} &= -\frac{1}{2}(a_{ij\alpha} + a_{i\alpha j} - a_{j\alpha i})\omega^\alpha \\ &= -\frac{1}{2}(-a_{ji\alpha} - a_{j\alpha i} + a_{i\alpha j})\omega^\alpha \\ &= \frac{1}{2}(a_{ji\alpha} + a_{j\alpha i} - a_{i\alpha j})\omega^\alpha \\ &= -\omega_{ji}\end{aligned}$$

and so the family of one-forms ω_{ij} is antisymmetric in its indices.

Next, we wish to show that the equation

$$d\omega^i = \omega^k \wedge \omega_k^i$$

holds for this definition of ω_k^i. Allowing for somewhat haphazard placement of indices, we know that

$$\begin{aligned}\omega^j \wedge \omega_{ij} &= -\frac{1}{2}a_{ij\alpha}\omega^j \wedge \omega^\alpha - \frac{1}{2}a_{i\alpha j}\omega^j \wedge \omega^\alpha + \frac{1}{2}a_{j\alpha i}\omega^j \wedge \omega^\alpha \\ &= -\frac{1}{2}a_{ij\alpha}\omega^j \wedge \omega^\alpha + \frac{1}{2}a_{i\alpha j}\omega^\alpha \wedge \omega^j + \frac{1}{2}a_{j\alpha i}\omega^j \wedge \omega^\alpha \\ &= \frac{1}{2}a_{j\alpha i}\omega^j \wedge \omega^\alpha \\ &= \frac{1}{2}a_{j\alpha}{}^i\omega^j \wedge \omega^\alpha\end{aligned}$$

which, by the definition of $a_{j\alpha}{}^i$, is equal to $d\omega^i$, as desired.

Now, to see that these are unique, we can simply count the degrees of freedom. To start with, ω_i^j specifies n^2 one-forms, each with n degrees of freedom, for a total of n^3 degrees of freedom. However, requiring ω_i^j to be antisymmetric means that we only need to specify $\frac{n(n-1)}{2}$ one-forms, leading to $\frac{n^2(n-1)}{2}$ degrees of freedom.

Now, the condition that $d\omega^i = \omega^k \wedge \omega_k^i$ for all i imposes more restrictions. We expand the left hand side as

$$d\omega^i = \sum_{\alpha<\beta} a_{\alpha\beta}{}^i\omega^\alpha \wedge \omega^\beta$$

and let

$$\omega_i^j = b_{i\alpha}^j\omega^\alpha$$

Then, we can expand the right hand side as

$$\begin{aligned}\omega^k \wedge \omega_k^i &= \omega^k \wedge (b_{k\alpha}^i\omega^\alpha) \\ &= b_{k\alpha}^i\omega^k \wedge \omega^\alpha \\ &= \sum_{k<\alpha}(b_{k\alpha}^i - b_{\alpha k}^i)\omega^k \wedge \omega^\alpha\end{aligned}$$

equating both sides to each other for each of the $\frac{n(n-1)}{2}$ terms in the sum yields $\frac{n(n-1)}{2}$ equations for each i. Furthermore, since this equality must hold for all i, we have a total of $\frac{n^2(n-1)}{2}$ constraints.

Thus, there are no degrees of freedom for ω_i^j satisfying the equations, and the solution we found is unique. $\square$

Problem 2

Use the curvature form method to calculate the sectional curvature for the n-sphere S^n with the induced metric from $\mathbb{R}^{n+1}$.

Proof. We will induct on the dimension n. The base case will be $n = 2$.

Now, for $n = 2$ we choose the orthonormal coframe $\omega^\theta = d\theta$ and $\omega^\phi = \sin(\theta)d\phi$ so that

$$ds^2 = d\theta^2 + \sin^2(\theta)d\phi^2 = (\omega^\theta)^2 + (\omega^\phi)^2$$

(which verifies that these form an orthonormal coframe).

Now, we can calculate the connection 1-forms. By antisymmetry, $\omega^\theta_\theta = \omega^\phi_\phi = 0$. Furthermore,

$$d\omega^\theta = d^2\theta = 0$$

implies that

$$\omega^\theta_\phi = f\omega^\phi$$

for some scalar function f.

We also know that

$$d\omega^\phi = \cos(\theta)d\theta \wedge d\phi$$

and so $\omega^\phi_\theta = \cos(\theta)d\phi$, forcing $\omega^\theta_\phi = -\cos(\theta)d\phi$. This completely specifies the connection 1-forms.

Now, let's calculate the curvature 2-forms. By definition,

$$\Omega^i_j = -(d\omega^i_j + \omega^i_k \wedge \omega^k_j)$$

Clearly, Ω is antisymmetric, so we only have to find Ω^ϕ_θ.

$$\begin{aligned}
\Omega^\phi_\theta &= -(d\omega^\phi_\theta + \omega^\phi_\theta \wedge \omega^\theta_\phi) \\
&= -(d(\cos(\theta)d\phi) - \omega^\phi_\theta \wedge \omega^\phi_\theta \\
&= -(d(\cos(\theta)d\phi) + 0) \\
&= \sin(\theta)d\theta \wedge d\phi \\
&= \omega^\theta \wedge \omega^\phi \\
&= -\omega^\phi \wedge \omega^\theta
\end{aligned}$$

and so $\Omega^i_j = -\omega^i \wedge \omega^j$.

Finally, we observe that

$$R(X,Y)e_i = \Omega^j_i(X,Y)e_j$$

and so, letting $Z = \xi^i e_i$, we have

$$\begin{aligned}
R(X,Y)Z &= R(X,Y)e_i\xi^i \\
&= \Omega^j_i(X,Y)e_j\xi^i \\
&= (-\omega^j \wedge \omega^i)(X,Y)e_j\xi^i \\
&= (\xi^i\omega^i \wedge \omega^j)(X,Y)e_j \\
&= (Z^\flat \wedge \omega^j)(X,Y)e_j \\
&= (Z^\flat \wedge (\mathrm{Id}))(X,Y) \\
&= g(Z,X)Y - g(Z,Y)X
\end{aligned}$$

Recalling that for a space of constant sectional curvature, $R(X,Y)Z = \kappa(g(Z,X)Y - g(Z,Y)X)$ we observe that $\kappa = 1$ and so the sphere S^2 has constant sectional curvature 1.

Now, let's prove the inductive step. Suppose that the sphere S^{n-1} has a local orthonormal frame such that $\Omega^i_j = -\omega^i \wedge \omega^j$. We will show that there is a local orthonormal frame on S^n such that $\Omega^i_j = -\omega^i \wedge \omega^j$ as well.

For ease of notation, we will denote all objects from S^{n-1} with tildes (e.g. the local orthonormal frame is $\{\tilde{\omega}^i\}$).

Recall that for spherical coordinates, the metric is given inductively as

$$ds^2 = d\chi^2 + \sin^2(\chi) d\Phi^2$$

where $d\Phi^2$ is the metric of S^{n-1}. Thus, we can choose a local orthonormal frame as

$$\omega^\chi = d\chi$$

and

$$\omega^i = \sin(\chi)\tilde{\omega}^i$$

for all $\tilde{\omega}^i$.

Now, let's calculate the connection forms. For this calculation, greek letters will be used to denote indices coming from the orthonormal frame on S^{n-1}.

$$\begin{aligned} d\omega^\alpha &= \cos(\chi) d\chi \wedge \tilde{\omega}^\alpha + \sin(\chi) d\tilde{\omega}^\alpha \\ &= \cot(\chi)\sin(\chi) d\chi \wedge \tilde{\omega}^\alpha + \sin(\chi)(\tilde{\omega}^\beta \wedge \tilde{\omega}^\alpha_\beta) \\ &= \cot(\chi) d\chi \wedge \omega^\alpha + \omega^\beta \wedge \tilde{\omega}^\alpha_\beta \\ &= \cot(\chi)\omega^\chi \wedge \omega^\alpha + \omega^\beta \wedge \tilde{\omega}^\alpha_\beta \end{aligned}$$

and so it follows that

$$\begin{aligned} \omega^\alpha_\chi &= -\omega^\chi_\alpha = \cot(\chi)\omega^\alpha \\ \omega^\alpha_\beta &= \tilde{\omega}^\alpha_\beta \end{aligned}$$

these are easily verifed to solve the constraint equations for ω^i_j.

Now, we calculate the curvature 2-forms, using the inductive hypothsesis $\tilde{\Omega}^\alpha_\beta = -\tilde{\omega}^\alpha \wedge \tilde{\omega}^\beta$ to get

$$\begin{aligned} -\Omega^\alpha_\beta &= d\omega^\alpha_\beta + \omega^\alpha_k \wedge \omega^k_\beta \\ &= -\tilde{\Omega}^\alpha_\beta + \omega^\alpha_\chi \wedge \omega^\chi_\beta \\ &= \tilde{\omega}^\alpha \wedge \tilde{\omega}^\beta + (\cot(\chi)\omega^\alpha) \wedge (-\cot(\chi)\omega^\beta) \\ &= \csc^2(\chi)\omega^\alpha \wedge \omega^\beta - \cot^2(\chi)\omega^\alpha \wedge \omega^\beta \\ &= \omega^\alpha \wedge \omega^\beta \end{aligned}$$

as desired.

All that remains is to calculate Ω^α_χ and show that it is equal to $-\omega^\alpha \wedge \omega^\chi$.

$$\begin{aligned} -\Omega^\alpha_\chi &= d\omega^\alpha_\chi + \omega^\alpha_k \wedge \omega^k_\chi \\ &= d(\cot(\chi)\omega^\alpha) + \omega^\alpha_k \wedge \omega^k_\chi \\ &= -\csc^2(\chi) d\chi \wedge \omega^\alpha + \cot(\chi) d\omega^\alpha + \tilde{\omega}^\alpha_\beta \wedge (\cot(\chi)\omega^\beta) \\ &= -\csc^2(\chi)\omega^\chi \wedge \omega^\alpha + \cot(\chi)(\omega^\chi \wedge (\cot(\chi)\omega^\alpha) + \omega^\beta \wedge \tilde{\omega}^\alpha_\beta) - (\cot(\chi)\omega^\beta) \wedge \tilde{\omega}^\alpha_\beta \\ &= -\csc^2(\chi)\omega^\chi \wedge \omega^\alpha + \cot(\chi)\omega^\chi \wedge (\cot(\chi)\omega^\alpha) + \cot(\chi)\omega^\beta \wedge \tilde{\omega}^\alpha_\beta - \cot(\chi)\omega^\beta \wedge \tilde{\omega}^\alpha_\beta \\ &= -\csc^2(\chi)\omega^\chi \wedge \omega^\alpha + \cot^2(\chi)\omega^\chi \wedge \omega^\alpha) + 0 \\ &= -\omega^\chi \wedge \omega^\alpha \\ &= \omega^\alpha \wedge \omega^\chi \end{aligned}$$

as desired.

Thus, following the exact same argument made in the case of S^2, we see that the sectional curvature on the sphere is 1. $\square$

Problem 3

Let c be an arbitrary parallel of latitude on S^2, with V_0 a tangent vector to S^2 at some point on c. Describe geometrically the parallel transport of V_0 along c.

Proof. We will show that parallel transport along c in S^2 is the same as parallel transport along c thought of as a curve in the cone C that lies tangent to S^2 at c.

In particular, note that the tangent spaces of S^2 and C coincide on c. This means that projection of a vector on c in $\mathbb{R}^3$ is the same whether it goes to TS^2 or TC. Furthermore, since the covariant derivative of a vector on c is equal to the ordinary partial derivative in $\mathbb{R}^3$ followed by projection into the tangent space, it follows that the covariant derivative of V_0 along c is the same whether taken in S^2 or C. Thus, since parallel transport is defined in terms of the covariant derivative, the parallel transport of V_0 on S^2 coincides with the parallel transport of V_0 on C.

Now, we note that C is actually flat: by making a suitable radial cut, one may flatten C so that it forms a disk with a slice missing, with the boundary of the disk coinciding with c. Here, parallel transport of V_0 along c is just ordinary translation in $C \subset \mathbb{R}^2$.

Thus, we have a complete description of the parallel transport of V_0 along c. We form the cone C tangential to c, and make a cut so that C can be isometrically embedded as a subset of $\mathbb{R}^2$. Then, identifying V_0 with its corresponding tangent vector on $c \subset \partial C$, we apply ordinary translation (parallel transport in $\mathbb{R}^2$) to V_0 along c. The result is the parallel vector field $V(t)$ along c in $\mathbb{R}^2$, which is identified with the parallel vector field $V(t) \subset TC$. Finally, noting that TC and TS^2 coincide on c, we see that $V(t) \subset TS^2$ is the parallel vector field of V_0 on c. $\square$

Problem 4

Suppose M has the following property: given any two points p, q in M, the parallel transport from p to q does not depend on the path chosen. Show that M is flat (R is identically zero).

Proof. We follow the hint outlined in the problem. Let $f : (0-\varepsilon, 1+\varepsilon)^2 \to M$ parameterize a surface in M with $f(s,0) = f(0,0)$ for all s. Let V_0 be an arbitrary vector at $f(0,0)$. Now, define a vector field V on the surface where $V(s,0) = V_0$ and $V(s,t)$ is the parallel transport of V_0 along $t \mapsto f(s,t)$.

Now, lemma 4.1 states that

$$[\nabla_{\partial_t}, \nabla_{\partial_s}]V = R(\partial_s f, \partial_t f)V$$

(for ease of notation, we denote ∇_{∂_t} as ∇_t and likewise for s).

Now, we know that $\nabla_s \nabla_t V = 0$, since V is a parallel vector field along t. Thus,

$$R(\partial_s f, \partial_t f)V - \nabla_t \nabla_s V = 0$$

However, by the hypothesis of the problem, we know that $V(s,1)$ can be thought of as the parallel transport of $V(0,1)$ along $s \mapsto f(s,1)$. This follows, since $V(s,1)$ is the parallel transport of V_0 along the path of constant s, and thus $V(s,1)$ can be thought of as the parallel transport of $V(0,1)$ backwards to $V(0,0)$, then forwards to $V(s,1)$. Since parallel transport does not depend on paths chosen, it follows that $V(s,1)$ is also the parallel transport of $V(0,1)$ along the curve $s \mapsto f(s,1)$.

Thus, $\nabla_s V(s,1) = 0$ and so we have that $\nabla_t \nabla_s V(s,1) = 0$. In particular, for $s = 0$ we have

$$R_{f(0,1)}(\partial_s f(0,1), \partial_t f(0,1))V(0,1) = 0$$

Now, f and V were arbitrary, and in particular for any vector fields X, Y, Z we can construct an f and V_0 for which $X_{f(0,1)} = \partial_s f(0,1)$ and $Y_{f(0,1)} = \partial_t f(0,1)$ and $Z_{f(0,1)} = V(0,1)$. To see this, note that we can choose an f that satisfies conditions for X and Y easily (since X and Y only specify how the parameterization should behave at $(0,1)$) and by setting V_0 as the parallel transport of $Z_{f(0,1)}$ along $f(0,t)$ to $f(0,0)$, we obtain that the parallel transport of V_0 is $Z_{f(0,1)}$ as desired.

Thus, the Riemann curvature tensor $R(X,Y)Z$ vanishes at every point for every vector field X, Y, Z as desired. $\square$