Topology

Problem Set 3

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Problem 1

Prove that the 1-norm on $\mathbb{R}^n$ defines a metric on $\mathbb{R}^n$ that is equivalent to the standard 2-norm metric on $\mathbb{R}^n$.

Proof. Let d_1 be the metric induced by the 1-norm on $\mathbb{R}^n$. Clearly, d_1 is positive definite, since it comes from a norm. So, let's show it satisfies the triangle inequality.

In proving the triangle inequality, we first state a general property of norms. The so-called triangle inequality of norms is given as

$$|x+y| \leq |x| + |y|$$

which is true for any normed space.

Let x, y, z be distinct points in $\mathbb{R}^n$ with coordinates x^i, y^i, z^i. Then we have that

$$\begin{aligned} d(x,z) &= \sum_i |x^i - z^i| \\ &= \sum_i |x^i - z^i + y^i - y^i| \\ &= \sum_i |(x^i - y^i) + (y^i - z^i)| \\ &\leq \sum_i |x^i - y^i| + |z^i - y^i| \\ &= \sum_i |x^i - y^i| + \sum_i |z^i - y^i| \\ &= d(x,y) + d(y,z) \end{aligned}$$

and thus the metric satisfies the axioms for a metric.

Now, let's show that the metric is equivalent to the standard 2-norm metric on $\mathbb{R}^n$. To do this, we will show that each point in a standard n-ball has a 1-norm ball contained in the n-ball, and vice versa.

So, without loss of generality (via translation) let $B_r(0)$ be the open ball of radius r around 0, and let $x \in B_r(0)$. In particular, there is some $\delta > 0$ such that $d(x,0) < r - \delta$. Now, take C_δ to be the 1-norm ball of radius δ. Now, if $y \in C_\delta$, then we have that

$$\begin{aligned} d(x,y) &= \sum_i |x^i - y^i| \\ &< \delta \\ \implies (\sum_i |x^i - y^i|)^2 &< \delta^2 \\ \implies \sum_i (|x^i - y^i|)^2 &< \delta^2 \\ \implies d_2(x,y) < \delta \implies d_2(y,0) &< d_2(x,0) + d_2(x,y) \\ &< r - \delta + \delta \\ &< r \end{aligned}$$

so, the 1-ball of radius δ is contained in $B_r(0)$ as desired. Thus, since each $x \in B_r(0)$ has a neighborhood (in 1-norm) contained in $B_r(0)$, $B_r(0)$ is open in the 1-norm topology.

For the other way, we first prove the more general fact about norms on $\mathbb{R}^n$.

Lemma 1. *There exists a constant C such that for all $x \in \mathbb{R}^n$,*

$$||x||_1 \leq C||x||_2$$

Proof. We first observe the basic fact that, for $x_1, x_2 \in \mathbb{R}^+$, we have

$$2x_1x_2 \leq x_1^2 + x_2^2$$

Now, it follows quickly that

$$\begin{aligned} ||x||^2 = \left(\sum_{i=1}^n |x_i|\right) &= \sum_{i=1}^n |x_i|^2 + \sum_{i \neq j} 2|x_i||x_j| \\ &\leq \sum_{i=1}^n |x_i|^2 + (n-1)\sum_{i=1}^n |x_i|^2 \\ &= n\sum_{i=1}^n |x_i|^2 \end{aligned}$$

Thus $\sqrt{n}$ is a constant for which the lemma holds. □

Now, since we have a bound on the norms, we can prove that a 1-norm ball is open in the 2-norm. To do so, let $\Delta_r(0)$ be the 1-norm ball of radius r at zero, and let $x \in \Delta_r(0)$. In particular, we have that there exists a δ such that $d_1(x,0) < r - \delta$. Now, let $\varepsilon = \frac{\delta}{\sqrt{n}}$, and consider the 2-norm ball $V_\varepsilon(x)$. Then, we will show that $V_\epsilon(x) \subset \Delta_r(0)$. To do so, let $y \in V_\epsilon(x)$, and observe that

$$\begin{aligned} d_1(x,y) &< \sqrt{n} d_2(x,y) \\ &< \sqrt{n}\frac{\delta}{\sqrt{n}} \\ &= \delta \end{aligned}$$

and

$$\begin{aligned} d_1(0,y) &\leq d_1(0,x) + d_1(x,y) \\ &\leq r - \delta + \delta \\ &= r \end{aligned}$$

as desired. □