Math 240B Notes
Differential Geometry Quarter 2

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1 Preliminaries

Homework 1. *Prove that $V^{**} \cong V$ for finite-dimensional vector space V.*

From this, it is clear that $T_p^*M \otimes T_pM \cong \mathrm{Hom}(T_pM, T_pM)$ for a manifold M.

Recall the tangent bundle TM is defined as

$$TM = \coprod_{p \in M} T_pM$$

and a vector field on the manifold M is simply a section of the tangent bundle projection $TM \xrightarrow{\pi} M$. In other words, a vector field is a function $f : M \to TM$ such that $\pi \circ f = id$. Requiring the section to be smooth makes it into a smooth vector field.

We can also do the same thing for the cotangent bundle T^*M to obtain a covector field.

Now, we can take the tensor product of copies of TM and T^*M to obtain our tensor bundles, and tensor fields will be sections of these bundles.

Let (U, ϕ) be a smooth chart on M with coordinate functions x^i, coordinate vector fields ∂_i, and coordinate one-forms dx^i. Recall that dx^i is defined to be the dual basis to ∂_i, that is,

$$dx^i(\partial_j) = \delta^i_j$$

Recall also that the exterior derivative of a function df is defined as

$$df(v) = v(f)$$

and this definition applied to the coordinate functions x^i (yielding dx^i) coincides with the definition above. Note that ∂_i form a basis for T_pM and dx^i form a basis for T_p^*M. Tensor products of them, then, form a basis for the tensor product space.

Homework 2. *Prove that, for a vector space V with basis v_i, dual basis v^i, the set*

$$\{v^i \otimes v^j \mid 1 \leq i, j \leq n\}$$

forms a basis for $V^* \otimes V^*$. Here $v^i \otimes v^j(u, v) = v^i(u)v^j(V)$.

2 Affine Connections

Def. 2.1. *Let M^n be a smooth manifold of dimension n. A* Riemannian Metric g *on M is a rank $(0, 2)$ tensor (a section of $T^*M \otimes T^*M$) that is symmetric and positive-definite. In other words, g is a rank $(0, 2)$ tensor that restricts to an inner product on the tangent space at every point.*

We can express g in local coordinates!

$$g_{ij} = g(\partial_i, \partial_j)$$

or

$$g = g_{ij} dx^i \otimes dx^j$$