

AC Circuits Homework 2

Daniel Halmrast

September 28, 2016

Problem (1.a). Consider an RLC circuit with source voltage $V_s = V_0 \sin(\omega t)$. Determine the current $I = I_0 \sin(\omega t + \phi)$.

Solution (1.a). Our defining differential equation is

$$V_S = \frac{1}{C}Q(t) + R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2}$$

Or, in more familiar terms of I ,

$$V_S = \frac{1}{C} \int_0^t I dt + RI + L\frac{dI}{dt}$$

Substituting $I = I_0 \sin(\omega t + \phi)$ and $V_s = V_0 \sin(\omega t)$ we get

$$V_0 \sin(\omega t) = \frac{1}{C} \int_0^t I_0 \sin(\omega t + \phi) dt + RI_0 \sin(\omega t + \phi) + L\frac{d}{dt}I_0 \sin(\omega t + \phi)$$

$$= I_0 \left[\frac{-1}{C\omega} \cos(\omega t + \phi) + R \sin(\omega t + \phi) + L\omega \cos(\omega t + \phi) \right]$$

Using sum of angles identities, we split the RHS into $\sin(\omega t)$ and $\cos(\omega t)$ parts.

$$\begin{aligned} V_0 \sin(\omega t) &= \left[I_0 R \cos(\phi) + \left(\frac{I_0}{C\omega} - I_0 L\omega \right) \sin(\phi) \right] \sin(\omega t) \\ &\quad + \left[I_0 R \sin(\phi) + \left(I_0 L\omega - \frac{I_0}{C\omega} \right) \cos(\phi) \right] \cos(\omega t) \end{aligned}$$

Equating the coefficients yields two equations.

$$\begin{aligned} V_0 &= I_0 R \cos(\phi) + \left(\frac{I_0}{C\omega} - I_0 L\omega \right) \sin(\phi) \\ 0 &= I_0 R \sin(\phi) + \left(I_0 L\omega - \frac{I_0}{C\omega} \right) \cos(\phi) \end{aligned}$$

The second equation simplifies to

$$R \sin(\phi) = \left(\frac{1}{C\omega} - L\omega \right) \cos(\phi)$$

$$\boxed{\tan(\phi) = \frac{\frac{1}{C\omega} - L\omega}{R}} \quad (1)$$

This immediately yields the following useful identities

$$\sin(\phi) = \frac{\frac{1}{C\omega} - L\omega}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2}} \quad (2)$$

$$\cos(\phi) = \frac{R}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2}} \quad (3)$$

Going back to the first coefficient equality, and using results 2 and 3, we see that

$$\begin{aligned} V_0 &= I_0 R \cos(\phi) + \left(\frac{I_0}{C\omega} - I_0 L\omega \right) \sin(\phi) \\ &= I_0 \left[R \frac{R}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2}} + \left(\frac{I_0}{C\omega} - I_0 L\omega \right) \frac{\frac{1}{C\omega} - L\omega}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2}} \right] \\ &= I_0 \left[\frac{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2}} \right] \\ &= I_0 \sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2} \end{aligned}$$

Therefore

$$\boxed{I_0 = \frac{V_0}{\left(R^2 + \left(\frac{1}{C\omega} - L\omega \right)^2 \right)^{\frac{1}{2}}}} \quad (4)$$

Problem (1.b). Graph $\phi(\omega)$.

Solution (1.b). ϕ is defined implicitly by

$$\tan(\phi) = \frac{\frac{1}{C\omega} - L\omega}{R} \quad (1)$$

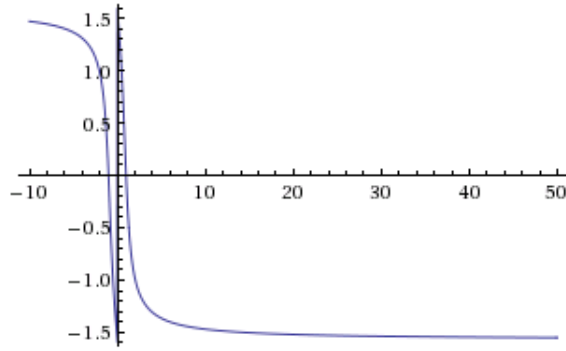
Which we will explicitly write as

$$\phi(\omega) = \arctan\left(\frac{\frac{1}{C\omega} - L\omega}{R}\right) \quad (2)$$

Clearly, as $\omega \rightarrow 0$, we have $\phi \rightarrow \arctan(\infty) = \frac{\pi}{2}$. Also, when $\omega = \frac{1}{\sqrt{LC}}$, $\phi = 0$.

A graph of $\phi(\omega)$ for $L = C = 1$ is shown.

Plot:



Problem (1.c). Plot $V_{R_0}(\omega)$.

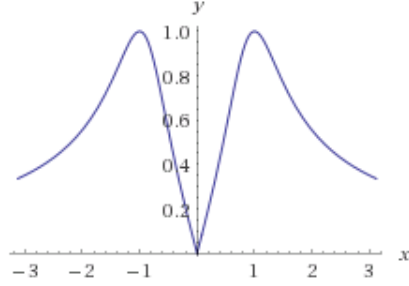
Solution (1.c). By the relation $V_R = IR$, V_{R_0} is RI_0 Thus

$$V_{R_0}(\omega) = \frac{V_0 R}{(R^2 + (\frac{1}{C\omega} - L\omega)^2)^{\frac{1}{2}}} \quad (1)$$

It is easy to see that when $\omega \rightarrow 0$, $V_R \rightarrow 0$. Also, V_R attains a maximum at the characteristic frequency $\omega = \frac{1}{\sqrt{LC}}$.

A graph of $V_{R_0}(\omega)$ for $V_0 = L = C = 1$ is shown.

Plots:



Problem (1.d). Solve for the voltage across the capacitor, using $V_C = \frac{Q(t)}{C}$

Solution (1.d). Rewriting V_C in a more familiar form, we have

$$\begin{aligned} V_C &= \frac{1}{C} \int_0^t I dt \\ &= \frac{1}{C} \int_0^t I_0 \sin(\omega t + \phi) dt \end{aligned}$$

Which yields

$$V_C = \frac{-I_0}{C\omega} \cos(\omega t + \phi)$$

With

$$I_0 = \frac{V_0}{(R^2 + (\frac{1}{C\omega} - L\omega)^2)^{\frac{1}{2}}} \quad (1)$$

$$\tan(\phi) = \frac{\frac{1}{C\omega} - L\omega}{R} \quad (2)$$

The reader is now encouraged to recall the trigonometry identity relating sin and cos

$$\cos(x) = \sin(x + \frac{\pi}{2})$$

Thus, V_C can be expressed in terms of the sin function as

$$\begin{aligned} V_C &= \frac{-I_0}{C\omega} \sin(\omega t + \phi + \frac{\pi}{2}) \\ &= \frac{I_0}{C\omega} \sin(\omega t + \phi - \frac{\pi}{2}) \end{aligned}$$