Problem Set 1

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Problem 2

Part a

Give an example of a sequence of sets where $\liminf_{j\to\infty} E_j \subsetneq \limsup_{j\to\infty} E_j$.

Proof. Consider the sequence of sets

$$E_j = \begin{cases} \{1\} & \text{if } j \in 2\mathbb{Z} \\ \{0\} & \text{else} \end{cases}$$

For this sequence, since both 1 and 0 are in infinitely many E_j,

$$\limsup_{j\to\infty} E_j = \{0, 1\}$$

However, there are infinitely many E_j that do not contain 1, and similarly there are infinitely many E_j that do not contain 0. Therefore,

$$\liminf_{j\to\infty} E_j = \emptyset$$

□

Part b

Show the lim sup and lim inf can be defined using set theory operations.

Proof. The $\limsup E_j$ is defined as the set of all points which belong to infinitely many E_j. That is, $x \in \limsup E_j \iff \forall k > 0, x \in \cup_j E_j$. Or, formally,

$$\limsup E_j = \bigcap_{k=0}^{\infty} (\bigcup_{j>k} E_j)$$

as desired. (i.e. x is in the lim sup if for all $k > 0$, there is some $j > k$ for which $x \in E_j$.)

Similarly, the lim inf is defined as the set of all points which belong to all but finitely many E_j. That is, for each x in the lim inf, there exist some $k > 0$ such that $x \in E_j \forall j > k$. Formally,

$$\liminf E_j = \bigcup_{k=1}^{\infty} (\bigcap_{j>k} E_j)$$

as desired. (i.e. x is in the lim inf if there exists some $k > 0$ such that $x \in E_j$ for all $j > k$) □

Part c

Show that if all E_j are in a σ -algebra $\mathcal{A}$, then both limits of E_j are in $\mathcal{A}$.

Proof. Recall that $\mathcal{A}$ is closed under countable unions. Furthermore, we know that

$$\left(\bigcup_{j=1}^{\infty} E_j^c \right)^c = \bigcap_{j=1}^{\infty} (E_j^c)^c$$

by DeMorgan's Law. Since each E_j^c is in $\mathcal{A}$ (since $\mathcal{A}$ is closed under complements), The infinite intersection is also in $\mathcal{A}$.

Therefore, since $\mathcal{A}$ is closed under both countable unions and intersections, the lim sup and lim inf, which are built from countable unions and intersections, are both in $\mathcal{A}$. □

Part d

Suppose that $E_1 \subset E_2 \subset \ldots$. Prove that

$$\limsup_{j\to\infty} E_j = \liminf_{j\to\infty} E_j = \bigcap_j E_j$$

Proof. Observe first that for this particular sequence,

$$\bigcup_{j=n}^{\infty} E_j = E_n$$

and for all n,

$$\bigcap_{i=n}^{\infty} E_i = \bigcap_{i=1}^{\infty} E_i$$

Now, chasing definitions yields

$$\begin{aligned}
\limsup E_j &= \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right) \\
&= \bigcap_{j=1}^{\infty} (E_j)
\end{aligned}$$

as desired.

Similarly,

$$\begin{aligned}
\liminf E_j &= \bigcup_{j=1}^{\infty} \left(\bigcap_{k=j}^{\infty} E_k \right) \\
&= \bigcup_{j=1}^{\infty} \left(\bigcap_{k=1}^{\infty} E_k \right) \\
&= \bigcap_{k=1}^{\infty} E_k
\end{aligned}$$

as desired. The last equality is attained by observing that the components of the union are constant with respect to j, so the union is just the constant element itself. □

Part e

Develop a similar formula for the limits of $E_1 \supset E_2 \supset \ldots$.

Proof. Consider the sequence of complements of E_j, $E_1^c \subset E_1^c \subset \ldots$. By the above, this sequence has a limit

$$\limsup E_j^c = \liminf E_j^c = \bigcap_{j=1}^{\infty} E_j^c$$

Taking complements of everything yields

$$\liminf E_j = \limsup E_j = \bigcup_{j=1}^{\infty} E_j$$

as desired.

□

Problem 3

Part d

Let $\{D_1, D_2, \ldots\}$ be a countable disjoint partition of a set Ω. Show that the set of countable unions of D_j is a σ-algebra.

Proof. Let $\mathcal{A}$ be the described set of countable unions of D_j, along with $\emptyset$. The countable union $\bigcup_{j=1}^{\infty} D_j = \Omega$ is in $\mathcal{A}$, along with $\emptyset$. This fulfills axiom 1.

Furthermore, for any D_n, $D_n^c = \bigcup_{j \neq n}^{\infty} D_j$ is a countable union of D_j and is in $\mathcal{A}$.

Finally, since each element of $\mathcal{A}$ is a countable union of D_j, a countable union of elements of $\mathcal{A}$ is a countable union of countable unions of D_j which is a countable union, and is in $\mathcal{A}$. □

Problem 4

Part a

Show that the union of two σ-algebras with the same unit is not necessarily a σ-algebra.

Proof. Consider the three-point set $\{1, 2, 3\}$ with $\mathcal{A}_1 = \sigma(\{2, 3\})$ and $\mathcal{A}_2 = \sigma(\{1, 2\})$. The union $\mathcal{A}_1 \cup \mathcal{A}_2$ contains $\{1\}$ and $\{3\}$ but not their union $\{1, 3\}$. □

Part b

Show the intersection of two σ-algebras is again a σ-algebra.

Proof. This is a special case of Part c, which is proved next. □

Part c

Prove that the arbitrary intersection of σ-algebras is again a σ-algebra.

Proof. Let $\mathcal{A}_\alpha$ be a collection of σ-algebras on a set Ω.

First, since each $\mathcal{A}_\alpha$ contains $\emptyset$ and Ω, their intersection does as well.

Secondly, consider an element $E \in \bigcap_\alpha \mathcal{A}_\alpha$. Since E is in each $\mathcal{A}_\alpha$, its complement E^c is in each $\mathcal{A}_\alpha$ (since $\mathcal{A}_\alpha$ is a σ-algebra) and thus E^c is in the intersection $\bigcap_\alpha \mathcal{A}_\alpha$.

Finally, consider a countable set of elements $E_i \in \bigcap_\alpha \mathcal{A}_\alpha$. Each of the E_i is in each $\mathcal{A}_\alpha$, so their union is also in $\mathcal{A}_\alpha$. Therefore, their union is also in the intersection $\bigcap_\alpha \mathcal{A}_\alpha$.

Thus, the intersection $\bigcap_\alpha \mathcal{A}_\alpha$ satisfies the axioms for a σ-algebra as desired. □

Part d

Show that the "subspace" σ-algebra given by $\mathscr{A} \cap E$ for $E \subset \Omega$ is a σ-algebra .

Proof. First, note that $\emptyset = \emptyset \cup E$ and $E = \Omega \cap E$ are in $\mathscr{A} \cap E$.

Now, let $S_E = S \cap E$ be an arbitrary measurable set. Then, the complement $S_E^c = E \setminus (S \cap E)$ is just $S^c \cap E$, which is also in $\mathscr{A} \cap E$.

Finally, let (S_i) be a sequence of measurable sets in $\mathscr{A} \cap E$. Their union is

$$\bigcup_i S_i = \bigcup_i (E_i \cap E) = (\bigcup_i E_i) \cap E$$

which is measurable.

Thus, the subspace σ-algebra satisfies the axioms for a σ-algebra as desired.

□

Part e

Show that the set $\mathscr{A} \times \Xi$ is a σ-algebra with unit $\Omega \times \Xi$.

Proof. Obviously, both $\emptyset = \emptyset \times \Xi$ and $\Omega \times \Xi$ are in the σ-algebra .

Now, it is clear that, for some measurable set $S \times \Xi$, the complement $S^c \times \Xi$ is also measurable. Similarly, unions will pass to the first component, and be preserved by the $\mathscr{A} \times \Xi$ σ-algebra .

□

Problem 6

Part a

For $E \subset \Omega$, find $\sigma_E(E)$, $\sigma_\Omega(E)$, $\sigma(\{E, E^c\})$.

$\sigma_E(E) = \{\emptyset, E\}$
$\sigma_\Omega(E) = \{\emptyset, E, E^c, \Omega\}$
$\sigma(\{E, E^c\}) = \{\emptyset, E, E^c, \Omega\}$

Part b

For a disjoint countable partition $\mathscr{D}$ of Ω, find $\sigma(\mathscr{D})$.

$$\sigma(\mathscr{D}) = \{\text{unions of elements of } \mathscr{D}\} \cup \{\emptyset\}$$

This can be seen to be a σ-algebra . By construction, the union of elements of this σ-algebra is again an element of the σ-algebra .

Furthermore, the complement of an element

$$(\bigcup_i D_i)^c = \bigcap_i D_i^c$$

And since $D_i^c = \cup_{j\neq i} D_j$ is an element of the σ-algebra , the countable intersection is as well.

Clearly, $\cup_i D_i = \Omega$, and the empty set is in the σ-algebra by construction. Thus, this σ-algebra satisfies the axioms for a σ-algebra , and is the smallest such one, since countable unions of elements of $\mathscr{D}$ must be included, and this σ-algebra is exactly the countable union of these elements.

Part c

Show that $\sigma(\sigma(\mathscr{C})) = \sigma(\mathscr{C})$ for any collection $\mathscr{C}$ of subsets.

Proof. By definition, $\sigma(\mathscr{C})$ is the smallest σ-algebra containing $\mathscr{C}$, Thus, $\sigma(\mathscr{A}) = \mathscr{A}$ for any σ-algebra $\mathscr{A}$.

Letting $\mathscr{A} = \sigma(\mathscr{C})$ yields the desired result. □

Part d

Show that

$$\begin{aligned}\sigma(C) &= \sigma\{F|F = E^c \text{for } E \in C\}\\ &= \sigma\{F|F = \cup_i E_i \text{for } E \in C\}\end{aligned}$$

Proof. For the first equality, notice that

$$\{F|F = E^c \text{for } E \in C\} \subset \sigma(C)$$

Since $\sigma(C)$ must contain the complements of each element of C to be a σ-algebra . Thus, taking σ of both sides yields

$$\sigma(\{F|F = E^c \text{for } E \in C\}) \subset \sigma(C)$$

For the other direction, note that similarly

$$C \subset \sigma(\{F|F = E^c \text{for } E \in C\})$$

and so

$$\sigma(C) \subset \sigma(\{F|F = E^c \text{for } E \in C\})$$

as desired.

The second equality can be argued in exactly the same way, noting that $\sigma(C)$ contains also the countable union of elements of C. □

Problem 10

Let $f : D \to \Omega$ with $(\mathscr{A}, \Omega)$ a σ-algebra. Define $f^{-1}(\mathscr{A}) := \{f^{-1}(E) | E \in \mathscr{A}\}$. Show that $f^{-1}(\mathscr{A})$ is a σ-algebra

Proof. To show that such a collection is a σ-algebra, I will show that the collection contains D and $\emptyset$, and that it is closed with respect to complements and countable unions.

First, note that $D = f^{-1}(\Omega)$ is in the collection, since Ω is in $\mathscr{A}$. Furthermore, $\emptyset = f^{-1}(\emptyset)$ is in the collection by a similar argument.

Let $A = f^{-1}(E)$ be an arbitrary element of the collection. Then, the complement

$$A^c = (f^{-1}(E))^c = f^{-1}(E^c)$$

is also in the collection, since E^c is in $\mathscr{A}$. Here, the second equality is attained by observing that the preimage map commutes with complementation.

Now, consider a countable set $\{A_i\}_{i=1}^{\infty}$ of elements of $f^{-1}(\mathscr{A})$. The union similarly commutes with the preimage map, so the relation

$$\bigcup_i A_i = \bigcup (f^{-1}(E_i)) = f^{-1}(\cup E_i)$$

which is in $f^{-1}(\mathscr{A})$ since the union $\cup E_i$ is in $\mathscr{A}$.

Thus, $f^{-1}(\mathscr{A})$ is a σ-algebra. □

Now, show that the direct image $f(\mathscr{A}) = \{f(E) | E \in \mathscr{A}\}$ is not generally a σ-algebra.

Proof. A simple counterexample is as follows:

Let $\mathscr{A}$ be the σ-algebra $\sigma_{[0,1]}\{[\frac{1}{6}, \frac{5}{6}]\}$ which is equal to $\{\emptyset, [\frac{1}{6}, \frac{5}{6}], [0, \frac{1}{6}) \cup (\frac{5}{6}, 1], [0, 1]\}$

Now, consider the mapping $\sin(\pi x)$ which sends $[0, 1]$ to itself. The forward image of $[0, \frac{1}{6}) \cup (\frac{5}{6}, 1]$ is just $(\frac{-1}{2}, \frac{1}{2})$ which does not have a complement that is the forward image of any set in $\mathscr{A}$. So $f(\mathscr{A})$ is not a σ-algebra . □

Finally, show that the push forward of a σ-algebra is a σ-algebra .

Proof. Let $S \in f_{\#}(\mathscr{A})$ for $\mathscr{A}$ a σ-algebra on Ω. Then consider S^c. $f^{-1}(S^c) = (f^{-1}(S)^c$ which is the complement of a set in $\mathscr{A}$ and is thus also in $\mathscr{A}$. Therefore, $S^c \in f_{\#}(\mathscr{A})$.

Now, consider a union of countable S_i for S_i in the push-forward σ-algebra . $f^{-1}(\bigcup_i S_i) = \bigcup_i f^{-1}(S_i)$ which is a union of things in $\mathscr{A}$ and is thus in $\mathscr{A}$ as desired.

Thus, the push-forward σ-algebra is closed under the operations of a σ-algebra , and is a σ-algebra itself. □

Problem 12

Show that a function $F : \Omega \to \mathbb{C}$ is measurable if and only if its projection functions $Re(F)$ and $Im(F)$ are measurable. Here the σ-algebra on $\mathbb{C}$ is the Borel σ-algebra , along with the σ-algebra on $\mathbb{R}$.

Proof. (=>) To show the forward implication, we first prove the following lemma:

Lemma. *For $f : X \to Y$ continuous, f is also measurable on the Borel σ-algebra of X and Y.*

Proof. To see this, consider the "good set"

$$E = \{G | G \in B(Y) \text{and } f^{-1}(G) \in B(X)\}$$

Clearly, $\mathscr{T}_Y \subset E$, since any open set in Y is in $B(Y)$, and since f is continuous, the inverse image of an open set is open, and is in $B(X)$. Note also that $E \subset B(Y)$ by construction.

Now, E is also a σ-algebra . To see this, note that it is closed under complements and unions, since both $B(Y)$ and the inverse image respect these operations.

So, this leads to the following relation:

$$\sigma(\mathscr{T}_Y) \subset \sigma(E) \subset \sigma(B(Y))$$

which simplifies to

$$B(Y) \subset E \subset B(Y)$$

That is, $E = B(Y)$. In words, each measurable set in Y has an inverse image under f that is measurable. Thus, f is measurable. □

Now, the projection functions $Re(z)$ and $Im(z)$ are continuous, so they are measurable. Furthermore, the composition of measurable functions is itself measurable. So, $Re(F) = Re \circ F$ is measurable, and $Im(F) = Im \circ F$ is also measurable.

(<=) To show reverse implication, note that $F(x) = Re(F)(x) + iIm(F)(x)$. Let $\Phi : \mathbb{R}^2 \to \mathbb{C}$ such that $\Phi(x, y) = x + iy$. Clearly, Φ is continuous. Therefore, by an obvious generalization of problem 18, the composition $\Phi \circ (Re(F), Im(F))$ is also measurable. Thus, $F = \Phi \circ (Re(F), Im(F))$ is measurable as desired.

□

Problem 18

Part ii

Prove that for a continuous function $\Phi : \mathbb{R}^2 \to \mathbb{R}$, the composition $\Phi \circ (f, g) : \Omega \to \mathbb{R}$ is measurable.

Proof. Note that if the composite function (f, g) is measurable, then this statement reduces to part i, and the proof is complete.

So, let's prove that (f, g) is measurable, given f, g are each individually measurable. (Note that this construction works for general products of measurable spaces, where the product σ-algebra is given by $\sigma(\mathscr{A}_1 \times \mathscr{A}_2)$. Generally, this says that the product measurable space has the universal property of product spaces).

Let $E, F \in B(\mathbb{R})$ be measurable sets, and consider the product $E \times F$. The inverse image $(f, g)^{-1}(E \times F) = f^{-1}(E) \cap g^{-1}(F)$ is the intersection of measurable sets (since f and g are both individually measurable), and is measurable.

Now, consider the "good set"

$$\mathscr{E} = \{G | G \in B(\mathbb{R}^2) \text{and } (f, g)^{-1}(G) \in \mathscr{A}\}$$

It is clear from above that we have the inclusion relations

$$B(\mathbb{R}) \times B(\mathbb{R}) \subset \mathscr{E} \subset B(\mathbb{R}^2)$$

Now, $\mathscr{E}$ is clearly a σ-algebra , since both conditions on $\mathscr{E}$ preserve complements and unions. Therefore, taking σ of the inclusion relations yields:

$$\begin{aligned} \sigma(B(\mathbb{R}) \times B(\mathbb{R})) \subset \mathscr{E} &\quad \subset B(\mathbb{R}^2) \\ \implies B(\mathbb{R}^2) \subset \mathscr{E} &\quad \subset B(\mathbb{R}^2) \end{aligned}$$

Thus, $\mathscr{E}$ is actually the whole Borel set $B(\mathbb{R}^2)$, and thus (f, g) is a measurable function, as desired. $\square$