Topology

Problem Set 4

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PROBLEM 1

Prove that [0,1) is not homeomorphic to (0,1).

Proof. Let f be a bijective function from [0,1) to (0,1). Since f is bijective, it follows that $f^{-1}((0,1)) = [0,1)$. However, (0,1) is open, but [0,1) is not. Thus, f is not continuous. Since no continuous bijections exist from [0,1), to (0,1), they are not homeomorphic. \square

PROBLEM 2

Prove that \mathbb{R} is not homeomorphic to \mathbb{R}^2 .

Proof. Suppose there existed a homeomorphism $f: \mathbb{R} \to \mathbb{R}^2$. Now, consider restricting the domain of f to $\mathbb{R} \setminus \{0\}$. This yields a homeomorphism between $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{f(0)\}$. However, such a homeomorphism cannot exist, since $\mathbb{R} \setminus \{0\}$ is not connected, but $\mathbb{R}^2 \setminus \{f(0)\}$ is connected, and connectedness is a topological property.

PROBLEM 3

Prove that every continuous function $f:[0,1]\to[0,1]$ has a fixed point.

Proof. Suppose there existed a function $f: I \to I$ (with I = [0,1]) such that f has no fixed points. In particular, this defines a (continuous) retract $r: I \to \partial I$ given by

$$r(x) = \begin{cases} 1, & \text{if } x > f(x) \\ 0, & \text{if } x < f(x) \end{cases}$$

Now, we see clearly that $r^{-1}(1) \neq \emptyset$, since at x = 1, f(1) cannot be greater than 1, and thus must be less than 1, forcing r(1) = 1. Similarly, $r^{-1}(0) \neq \emptyset$, since at x = 0, f(0) cannot be less than 0, and thus must be greater than 0, forcing r(0) = 0.

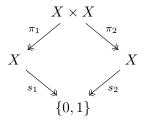
Therefore, r is a continuous function from I to the two-point set, and defines a separation of I. But I is connected, so no such separation can exist. Thus, such a retract cannot exist, and f must have a fixed point.

Problem 4

Prove that $X \times X$ is connected if and only if X is.

Proof. (\Longrightarrow) Suppose X is not connected. In particular, there exists a continuous surjection from X to the two-point set.

Thus, we have the diagram



In particular, the composition $s_1 \circ \pi_1$ is a surjection from $X \times X$ onto $\{0,1\}$, and defines a separation of $X \times X$. Thus, $X \times X$ is separated.

(\Leftarrow) Suppose $X \times X$ is not connected. In particular, there exists a continuous surjection $s: X \times X \to \{0,1\}$. Now, for each $x_{\alpha} \in X$, we know that the map $i_{\alpha}: X \to X \times X$ given by $i(x) = (x, x_{\alpha})$ is an embedding of X into $X \times X$. I assert that there exists some x_0 for which $s^{-1}(\{0\}) \cap X \times \{x_0\} \neq \emptyset$ and $s^{-1}(\{1\}) \cap X \times \{x_0\} \neq \emptyset$.