Homework 1

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April 18, 2018

PROBLEM 1

Suppose M is a complete Riemannian manifold of dimension n, and suppose there exist constants $a > 0$ and $c \geq 0$ such that for all pairs of points and all minimizing geodesics $\gamma(s)$ parameterized by arc length s, joining these points, we have

$$R(\gamma'(s)) \geq a + \partial_s f$$

where f is a function of s and $|f(s)| \leq c$ along γ. Prove that M is compact.

Proof. To show M is compact, we just need to show M is bounded. That is, there is some number N such that $d(p,q) < N$ for all $p, q \in M$.

Let γ be a minimizing geodesic connecting two points p, q in M. We calculate the second variation in energy along γ in a manner similar to the proof of the Bonnet-Meyers theorem in

Do Carmo. In particular, we know

$$\begin{aligned}
\frac{1}{2}E''(0) &= \int_0^1 \sin^2(\pi t)((n-1)\pi^2 - (n-1)\ell^2 R_\gamma(e_n(t)))dt \\
&= (n-1)\left[\frac{\pi^2}{2} - \int_0^1 \ell^2 R_\gamma(e_n(t))\right] \\
&\leq (n-1)\left[\frac{\pi^2}{2} - \int_0^1 \sin^2(\pi s)\ell^2 \partial_s f ds\right] \\
&= (n-1)\left[\frac{\pi^2}{2} - \int_0^1 \sin^2(\pi s)\ell^2 (a + \partial_s f) ds\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} - \int_0^1 \sin^2(\pi s)\ell^2 \partial_s f ds\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + \int_0^1 \pi \sin(\pi s)\ell^2 f ds\right] \\
&\leq (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + \int_0^1 \pi c|\sin(\pi s)|\ell^2 ds\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + \pi c\ell^2 (\frac{2}{\pi}\right] \\
&= (n-1)\left[\frac{\pi^2 - \ell^2 a}{2} + 2c\ell^2\right]
\end{aligned}$$

which is negative for $\ell^2 \geq \frac{\pi^2}{a-4c}$. Since γ is assumed to be minimal, this implies that $\ell^2 < \frac{\pi^2}{a-4c}$ and thus M is bounded with the diameter of M less than $\frac{\pi}{\sqrt{a-4c}}$ as desired. □

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