Midterm

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PROBLEM 1

Define carefully what it means for a map $f : X \to Y$ to be transverse to a submanifold Z of Y.

Suppose that X and Z are smooth and transverse submanifolds of Y. Prove that if $y \in X \cap Z$ then

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z)$$

Proof. We begin by defining transversality. Suppose $f : X \to Y$ is a smooth map, and $Z \subset Y$ a submanifold of Y. We say that f is transverse to Z if

$$T_{f(x)}(Y) = T_f(x)(Z) + df_x(T_x(X))$$

for all $x \in f^{-1}(Z)$. That is, the tangent space at $f(x)$ in Y is spanned by the tangent space of Z and the push-forward of the tangent space of X.

Now, suppose X and Z are smooth and transverse submanifolds of Y. That is, $T_y(Y) = T_y(X)+T_y(Z)$ for all $y \in X \cap Z$. Suppose first that $v \in T_y(X \cap Z)$. In particular, this means there is a curve $\gamma_v : I \to X \cap Z$ with $\gamma_v(0) = y$ and $\gamma_v'(0) = v$. Clearly, γ_v is also a curve in X and in Z, and so $v \in T_y(X)$ and $v \in T_y(Z)$ as desired. Thus, $T_y(X \cap Z) \subset T_y(X) \cap T_y(Z)$. Call this inclusion Φ. Clearly, Φ is injective. Thus, all we need to show is that $\dim(T_y(X) \cap T_y(Z)) = \dim(T_y(X \cap Z))$ to establish equality.

So, let (U, ϕ) be a slice chart of Z at y. That is, U is a neighborhood of Y, and $\phi : U \to \mathbb{R}^n$ is a coordinate chart such that $\phi(Z) \subset \mathbb{R}^k \times \{0\}^{n-k}$. In particular, we consider the augmented "height" function $\psi : U \to \mathbb{R}^{n-k}$ for which $\psi(Z) = \{0\}$. Thus, $Z = \psi^{-1}(\{0\})$. Let i be the inclusion of X into Y, and observe that $X \cap Z = (\psi \circ i)^{-1}(\{0\})$. We will show that $\{0\}$ is a regular value for $\psi \circ i$.

To that end, we wish to show that $d(\psi \circ i)_y$ is surjective. So, let $v \in \mathbb{R}^{n-k}$. Now, since ψ is part of a coordinate chart, $d\psi_y$ is surjective, and its kernel is $T_y(Z)$. Write a generic element of the fiber of v as $w + v_z$ for $v_z \in T_y(Z)$. Since this is in Y, and X and Z are transverse, $w + v_z = v_x + v_z'$ for some $v_x \in T_y(X)$ and $v_z' \in T_y(Z)$. Absorbing v_z' into v_z, we see that $w + v_z = v_x$. So, thinking of v_x as an element of $T_y(X)$, we see that

$$d(\psi \circ i)_y(v_x) = d\psi_{i(y)} \circ di_y(v_x) = d\psi_y(v_x) = v$$

and so $d(\psi \circ i)$ is surjective as desired.

Thus, the codimension of $X \cap Z$ in X is $n - k$, which is the codimension of Z in Y. That is,

$$\dim(T_y(X)) - \dim(T_y(X \cap Z)) = \dim(T_y(Y)) - \dim(T_y(Z))$$

or

$$\dim(T_y(X \cap Z)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim(T_y(Y))$$

However, since $T_y(Y) = T_y(X) + T_y(Z)$, we know that

$$\dim(T_y(Y)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim(T_y(X) \cap T_y(Z))$$

or

$$\dim(T_y(X) \cap T_y(Z)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim(T_y(Y))$$

and so

$$\dim(T_y(X \cap Z)) = \dim(T_y(X) \cap T_y(Z))$$

and since the inclusion $\Phi : T_y(X \cap Z) \to T_y(X) \cap T_y(Z)$ is injective, the spaces are equal, as desired. $\square$

Problem 2

Suppose $f : X \to Y$ is a smooth map between compact manifolds of the same dimension. Suppose $y \in Y$ is a regular value of f.

Part i

Prove that $f^{-1}(\{y\})$ is a finite set.

Proof. Since y is a regular value, we know that $f^{-1}(\{y\})$ is a submanifold of X with dimension

$$\dim(f^{-1}(\{y\}) = \dim(X) - \dim(Y) = 0$$

Since the only manifolds of dimension zero are countable discrete sets, $f^{-1}(\{y\})$ is an (at most) countable collection of points with the discrete topology. Since Y is compact, this automatically implies that $f^{-1}(\{y\})$ is finite. This follows from the fact that every infinite set in a compact space has an accumulation point, and discrete sets have no accumulation points. $\square$

Part ii

Prove that there is an open neighborhood U of y so that $f^{-1}(U)$ is a finite disjoint union of open sets $\{V_i\}$ so that each V_i is a neighborhood of $x_i \in f^{-1}(\{y\})$ and each V_i maps diffeomorphically onto U by f.

Proof. For this proof, we let $\{x_i\}_{i=1}^n$ be the preimage of y under f.

Since y is a regular value for f which maps between spaces of the same dimension, df_{x_i} is an isomorphism for each x_i. Thus, by the inverse function theorem, f is a local diffeomorphism at each x_i. Choose open neighborhoods W_i of x_i such that f is a diffeomorphism onto its image when restricted to each W_i, and such that the open sets $\{W_i\}_{i=1}^n$ are all pairwise disjoint (this can be done, since X is Hausdorff).

Let $U' = \cap_i f(W_i)$ be the intersection of the images of W_i in Y. Since each $f(W_i)$ contains $f(x_i) = y$, $y \in U'$, and since f is a diffeomorphism onto its image when restricted to W_i, f is an open map on each W_i, and thus each $f(W_i)$ is open. Since U' is the finite intersection of open neighborhoods of y, U' is an open neighborhood of y itself.

Now, consider the closed set $C = X \setminus (\cup_i W_i)$ the complement of all the open neighborhoods W_i. Since f is a continuous map between compact Hausdorff spaces, it is a closed map (it takes compact sets to compact sets, which are closed in Hausdorff spaces). Thus, $f(C)$ is closed in Y. Furthermore, $y \notin f(C)$ since $f^{-1}(\{y\}) \subset \cup_i W_i$. Thus, $Y \setminus f(C)$ is an open neighborhood of y. Set $U = (Y \setminus f(C)) \cap U'$.

We are now ready to finish the construction. Since f is a diffeomorphism from each W_i onto its image, we can take V_i to be $f^{-1}|_{W_i}(U)$. In particular, each $V_i \subset W_i$, and so they are all disjoint. Furthermore, $f(V_i) = U$ for each V_i, and f is a diffeomorphism of V_i onto U.

Finally, we note that this exhausts the preimage of U. Suppose $x \in X$ with $x \notin \cup_i V_i$. If $x \in W_i$ for some W_i with $x \notin V_i$, then since f is a diffeomorphism restricted to W_i, $f(x) \notin U$ as desired. If x is not in any W_i, then $f(x) \notin U'$ by construction of U', and thus $f(x) \notin U$. Therefore, the preimage of U is exactly the union $\cup_i V_i$ as desired.

$\square$

Problem 3

Show that the set of rank 1 matrices in $M(2,\mathbb{R})$ is a 3-dimensional submanifold of $M(2,\mathbb{R})$.

Proof. A 2×2 rank-1 matrix is a nonzero matrix with nontrivial kernel. This set is exactly specified as the set of all nonzero 2×2 matrices with determinant zero. That is, letting R denote the set of rank-1 matrices,

$$R = \det^{-1}(0) \setminus \{0\}$$

In particular, since $M(2,\mathbb{R}) \setminus \{0\}$ is an open subset of $M(2,\mathbb{R})$, it is a manifold, and so we only need to consider $\det^{-1}(\{0\})$ in $M(2,\mathbb{R}) \setminus \{0\}$. To show this is a manifold, we will show it is a submanifold of $M(2,\mathbb{R}) \setminus \{0\}$ by showing 0 is a regular value of det.

To show this, we need to show that for every nonzero matrix $A \in \det^{-1}(\{0\})$, $d(\det)_A$ is surjective. Since the codomain of det is $\mathbb{R}$, it suffices to show that there is at least one vector in $T_A(R(2,\mathbb{R}) \setminus \{0\})$ which does not map to zero.

Observe first that A is always of the form

$$A = \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix}$$

for real numbers a, b, λ such that a and b are not all identically zero.

Let $\gamma(t) = A + tI$ so that $\gamma(0) = A$ and $\gamma'(0) = I$. We will show that $d(\det)_A(I) \neq 0$. We calculate

$$\begin{aligned} d(\det)_A(I) &= \partial_t(\det(\gamma(t)))|_0 \\ &= \partial_t(\det(A + tI))|_0 \\ &= \partial_t\left(\det\left(\begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix} + t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right)|_0 \\ &= \partial_t\left((a+t)(\lambda b + t) - \lambda ab\right)|_0 \\ &= \partial_t\left(t^2 + at + \lambda bt + \lambda ab - \lambda ab\right)|_0 \\ &= a + \lambda b \end{aligned}$$

Thus for all $A = \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix}$ with $a \neq -\lambda b$, $d(\det)_A(I) \neq 0$.

Defining $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and repeating the calculation for $d(\det)_A(B)$, we see that $d(\det)_A(B) = a - \lambda b$ which is nonzero for $a \neq \lambda b$.

Finally, defining $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we calculate $d(\det)_A(C) = -b - \lambda a$, which is nonzero when $b \neq -\lambda a$. This exhausts all possible forms for A, and so the determinant is surjective at each point $A \in \det^{-1}(\{0\})$, as desired.

In particular, this means that the codimension of R is 1, making it a 3 dimensional submanifold of $M(2,\mathbb{R})$ as desired. □

Problem 4

Describe the ingredients used in proving there is a surjective homomorphism

$$\pi : SU(2) \to SO(3)$$

Proof. This will be a sketch of the construction of the double cover of $SO(3)$ by $SU(2)$.

First, we show $SU(2)$ is a Lie group. This is done in two stages: first by showing $U(2) \subset GL(2, \mathbb{C})$ is a Lie group, then showing that $SU(2) \subset U(2)$ is a Lie group.

First, observe that $GL(2, \mathbb{C})$ is a Lie group. It is a manifold, since it is an open subset of $M(2, \mathbb{C}) \cong \mathbb{C}^4$, and it is easily verified that the group operation of matrix multiplication is smooth with respect to this structure. Now, we consider the map

$$\begin{aligned} \Phi : GL(2, \mathbb{C}) &\to GL(2, \mathbb{C}) \\ \Phi(A) &= AA^* \end{aligned}$$

and note that $U(2) = \Phi^{-1}(\{I\})$. Verifying that I is a regular value for Φ, we see that $U(2)$ is an (embedded) submanifold of $GL(2, \mathbb{C})$ and is thus a Lie subgroup of $GL(2, \mathbb{C})$.

Secondly, observe that $SU(2) = \det^{-1}(1)$ with det the determinant map from $U(2)$ to $\mathbb{C}$. Verifying that 1 is a regular value for det, we see that $SU(2)$ is an embedded submanifold of $U(2)$, and is thus a Lie subgroup, as desired.

Next, we show that the associated Lie algebra $\mathfrak{su}(2)$ has an inner product which makes it isometrically isomorphic to $\mathbb{R}^3$. First, we note that

$$\mathfrak{su}(2) = \{A \in M(2, \mathbb{C}) \mid A + A^* = 0, \operatorname{tr}(A) = 0\}$$

That is, $\mathfrak{su}(2)$ is the set of all antisymmetric trace-free matrices. This space is spanned by the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which have the algebraic properties

$$\sigma_i \sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}$$

(where ε_{ijk} is the Levi-Civita symbol).

Furthermore, an inner product can be given to $\mathfrak{su}(2)$ by

$$g(x, y) = \operatorname{tr}(x^* y)$$

Under this inner product, we have the relations

$$g(\sigma_i, \sigma_i) = \operatorname{tr}(\sigma_i^2) = \operatorname{tr}(I) = 2$$

and for $i \neq j$

$$g(\sigma_i, \sigma_j) = \operatorname{tr}(-\sigma_i \sigma_j) = \operatorname{tr}(\varepsilon_{ijk}\sigma_k) = 0$$

Thus, $\{\frac{1}{2}\sigma_i\}_{i=1}^3$ forms an orthonormal basis for $\mathfrak{su}(2)$, and under the map

$$\begin{aligned} \Phi : \mathfrak{su}(2) &\to \mathbb{R}^3 \\ \Phi(\frac{1}{2}\sigma_i) &= e_i \end{aligned}$$

we have the desired isometric isomorphism of $\mathfrak{su}(2)$ with $\mathbb{R}^3$.

Finally, we note that $SU(2)$ has a natural representation on $\mathfrak{su}(2)$ given by the adjoint representation. In the particular case of a matrix Lie group, this is given by

$$\begin{aligned} \mathrm{Ad} :& SU(2) \to \mathrm{Aut}(\mathfrak{su}(2)) \cong GL(3, \mathbb{R}) \\ \mathrm{Ad}(U) &= x \mapsto UxU^* \end{aligned}$$

We verify that $\mathrm{Ad}(U)$ preserves inner products, by noting that

$$\begin{aligned} g(\mathrm{Ad}(U)(x), \mathrm{Ad}(U)(y)) &= \mathrm{tr}((\mathrm{Ad}(U)(x))^* \mathrm{Ad}(U)(y)) \\ &= \mathrm{tr}((UxU^*)^* UyU^*) \\ &= \mathrm{tr}(Ux^*U^*UyU^*) \\ &= \mathrm{tr}(Ux^*yU^*) \\ &= \mathrm{tr}(x^*y) = g(x, y) \end{aligned}$$

and so Ad maps $SU(2)$ into $O(3)$. We verify that $\mathrm{Ad}(U)$ is orientation-preserving, so $\det(\mathrm{Ad}(U)) = 1$, and Ad maps $SU(2)$ into $SO(3)$.

Finally, to show Ad is surjective onto $SO(3)$, we argue that every rotation in $SO(3)$ is given by $\mathrm{Ad}(U)$ for some $U \in SU(2)$. This is typically done by identifying $SU(2)$ with the unit quaternions, and $\mathbb{R}^3$ with the pure imaginary quaternions. Here, a rotation in $\mathbb{R}^3$ is given by conjugation by a unit quaternion, and so each element of $SO(3)$ is $\mathrm{Ad}(U)$ for some $U \in SU(2)$, and Ad is surjective.

$\square$