Problem Set 5

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Problem 1

Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$ so that f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. We begin by explicitly constructing the homotopy that is stationary on $S^1 \times \{0\}$. This is done via the homotopy

$$f_t(\theta, s) = (\theta + t2\pi s, s)$$

Clearly, $f_0 = \mathbb{1}$ and $f_1 = f$. Furthermore, $f_t(\theta, 0) = (\theta, 0)$ and so f_t is stationary on $S^1 \times \{0\}$ □

Problem 2

Does the Borsuk-Ulam theorem hold for the torus? That is, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ does there exist a point (x, y) for which $f(x, y) = f(-x, -y)$?

Proof. I assert that the Borsuk-Ulam theorem does not hold for the torus. To see this, we construct an explicit function from T^2 to $\mathbb{R}^2$ which does not have any antipodal points with the same value.

Consider the function $f : T^2 \to \mathbb{R}^2$ given as follows. First, let T^2 be embedded in $\mathbb{R}^3$. Then, consider the vector field $\frac{\partial}{\partial \phi}$ where ϕ runs parallel to the $x - y$ plane. Since T^2 is embedded in $\mathbb{R}^3$, these vectors can be thought of as living in the tangent bundle to $\mathbb{R}^3$. Thus, for each vector, it makes sense to take its projection onto the $x - y$ plane. Now, since the original vector field $\frac{\partial}{\partial \phi}$ is smooth, and projection is a continuous operation, this defines a continuous map from T^2 to $\mathbb{R}^2$.

In coordinates, this map is given as

$$f(\theta, \phi) = (\cos(\phi), \sin(\phi)) \tag{0.1}$$

and clearly, $f(\theta, \phi) \neq f(-\theta, -\phi)$ as desired. $\square$

Problem 3

From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

Proof. Let f be a loop in $X \times \{y_0\}$, and let g be a loop in $\{x_0\} \times Y$. We wish to show that $f \cdot g \simeq g \cdot f$. We can construct this homotopy explicitly by "sliding" f along g. Concretely the homotopy is as follows.

First, define $g_t = g\Big|_{[0,t]}$ and $g_{t'} = g\Big|_{[t,1]}$ so that $g_{t'} \cdot g_t = g$. That is, g_t is the segment of g from $g(0)$ to $g(t)$. Furthermore, let f_y be the loop f in the subspace $X \times \{y\}$.

Now, define a homotopy h as

$$h_t = g_{t'} \cdot f_{\pi_y(g(t))} \cdot g_t \tag{0.2}$$

Now, this is well-defined, since g_t has an endpoint at $(x_0, \pi_y(g(t)))$, and f starts and ends at $(x_0, \pi_y(g(t)))$, and $g_{t'}$ begins at $(x_0, \pi_y(g(t))$ and ends at (x_0, y_0). We just have to make sure this is continuous. Since h is a map into $X \times Y$, we just have to check that the corresponding map $H : S^1 \times I \to X \times Y$ is continuous onto both X and Y. That is, H is continuous if and only if $\pi_X H$ and $\pi_Y H$ are continuous.

However, it should be clear that $\pi_X H = f$, since $\pi_X(g(t)) = x_0$ for all t. Thus, the path $g_{t'} \cdot f \cdot g_t$ projects down to just $x_0 \cdot f \cdot x_0$, which is clearly continuous.

Similarly, $\pi_Y H = g_{t'} \cdot y_0 \cdot g_t$, which is (after reparameterization) equal to $g_{t'} \cdot g_t = g$, which is clearly continuous for all t.

Thus, H is continuous in its projections, and by the universal property of products, is continuous in general.

Now, h_0 is just $g_{0'} \cdot f \cdot g_0 = g \cdot f$, and $h_1 = g_{1'} \cdot f \cdot g_1 = f \cdot g$, and so $f \cdot g \simeq g \cdot f$ as desired. $\square$

Problem 4

Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Proof. Throughout this problem, I will identify $\pi_1(S^1)$ with $\mathbb{Z}$ by identifying 1 with the loop that goes once around the circle counterclockwise.

Since $\pi_1(S^1) \cong \mathbb{Z}$, each homomorphism is characterized by where it sends the generator 1.

So, suppose $\phi : \pi_1(S^1) \to \pi_1(S^1)$ is such that $\phi(1) = n$ for some integer n. Define $\varphi(z) = z^n$. I claim that $\varphi_* = \phi$.

To see this, we just need to see that $\varphi_*(1) = n$. Now, the loop 1 is just the map $f(t) = \exp(2\pi i t)$, and the induced loop $\varphi_*(1)$ is the composition $\phi(f(t)) = \exp(2\pi i n t)$ which is easily seen to be homotopic to the loop n, as desired. $\square$

Problem 5

Show that there does not exist a retraction from X to A in the following cases:

0.1 Part (a)

$X = \mathbb{R}^3$ and A is any subspace homeomorphic to S^1.

Proof. Suppose such a retraction $r : X \to A$ existed. In particular, we would have that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "\mathbb{1}"] & X \arrow[r, "r"] & A \end{tikzcd}$$

commutes. Now, applying the π_1 functor to this diagram yields

$$\begin{tikzcd} \mathbb{Z} \arrow[d, leftrightarrow] & 0 \arrow[d, leftrightarrow] & \mathbb{Z} \arrow[d, leftrightarrow] \\ \pi_1(A) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}"] & X \arrow[r, "r"] & A \end{tikzcd}$$

Now, this diagram implies that $\mathbb{1}$ on $\mathbb{Z}$ factors through zero, which cannot happen. Thus, no such r exists. □

0.2 Part (b)

$X = S^1 \times D^2$ with A the boundary torus $S^1 \times S^1$.

Proof. Following the same diagram as the one used in the previous problem, assuming such an $r : X \to A$ exists yields the diagram

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}"] & \mathbb{Z} \arrow[r, "r_*"] & \mathbb{Z} \times \mathbb{Z} \end{tikzcd}$$

□

Problem extra

Find the standard form of $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6$, and prove or disprove:

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6 \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$