1 Geodesics and Curvature

1.1 Geodesics

Definition 1.1. *Let (M^n, g) be a Riemannian manifold, and let $\gamma : I \to M$ a smooth curve. γ is called a* geodesic *if its second derivative vanishes. That is, if it solves the geodesic equation*

$$\nabla_{\partial_t} \partial_t \gamma = 0$$

Now, let's examine the geodesic equation further. In local coordinates, we have

$$\begin{aligned} \nabla_{\partial_t} \partial_t \gamma &= \nabla_{\partial_t} \partial_t x^i \partial_i \\ &= \partial_t \partial_t x^k \partial_k + \partial_t x^k \nabla_{\partial_t} \partial_k \\ &= (\partial_t \partial_t x^k + \Gamma^k_{ij} \partial_t x^i \partial_t x^j) \partial_k \end{aligned}$$

and so the local coordinate version of the differential equation is the system of equations

$$(\partial_t)^2 x^k + \gamma^k_{ij} \partial_t x^i \partial_t x^j = 0$$

which are guaranteed local unique solutions for initial conditions of γ and γ'.

Let's look at properties of geodesics. In particular, we can look at

$$\partial_t |\gamma'|^2 = \partial_t (g(\gamma', \gamma')) = 2g(\nabla_{\partial_t} \gamma', \gamma') = 0$$

and so the velocity of the geodesic does not change.

1.2 The Exponential Map

Let $p \in M$. We can define an exponential map $\exp : T_p M \to M$ via the following:

Definition 1.2. *The* exponential map $\exp : T_p M \to M$ *is defined as* $\exp(v) = \gamma(1)$ *where γ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$.*

Why do we insist that $\exp_p(v) = \gamma(1)$? Consider

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

where $t \in \mathbb{R}$. The last equality is obtained in the following way:

Lemma 1. *$\gamma_{tv}(1) = \gamma_v(t)$ for all t.*

Proof. Consider $\gamma(t) = \gamma_{sv}(t)$. This is the geodesic such that $\gamma(0) = p$ and $\gamma'(0) = sv$. Now, notice that $\tilde{\gamma}(t) = \gamma_v(st)$ is defined so that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}'(0) = \partial_t \gamma_v(st) = \gamma_v'(0) \partial_t (st)|_{t=0} = sv$ and by uniqueness of geodesics, $\gamma = \tilde{\gamma}$ as desired. □