
Final Exam

Daniel Halmrast

March 18, 2018

PROBLEM 1

Carefully prove a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is contractible.

Proof. For this proof, we will begin by proving a couple of useful lemmas.

Lemma 1. *The wedge sum is the coproduct in the category Top_* of pointed topological spaces.*

Proof. To show this is the coproduct, we need to show that it satisfies the universal property for coproducts. That is for $(X, x_0), (Y, y_0)$ pointed topological spaces, and (U, u_0) any other pointed topological space with arrows $f : (X, x_0) \rightarrow (U, u_0)$ and $g : (Y, y_0) \rightarrow (U, u_0)$,

$$\begin{array}{ccc}
 (X, x_0) & & (Y, y_0) \\
 \searrow i_{x*} & & \swarrow i_{y*} \\
 & (X, x_0) \vee (Y, y_0) & \\
 \swarrow f & \downarrow \exists!(f, g) & \searrow g \\
 & (U, u_0) &
 \end{array}$$

where (f, g) is the unique arrow that makes the diagram commute. Now, recall that the wedge sum is defined as the quotient

$$(X, x_0) \vee (Y, y_0) = X \amalg Y / x_0 \sim y_0$$

so we can bootstrap by using the fact that the disjoint union is the coproduct in Top . That is,

we define i_{x*} and i_{y*} to be the unique maps given by the universal property

$$\begin{array}{ccc}
 X & & Y \\
 \downarrow i_x & & \downarrow i_y \\
 X \amalg Y & & \\
 \downarrow q & & \\
 X \vee Y & &
 \end{array}$$

(Dashed curved arrows from X and Y to $X \vee Y$ are labeled i_{x*} and i_{y*} respectively.)

Note that these maps preserve basepoints, since

$$i_{x*}(x_0) = q \circ i_x(x_0) = q(x_0)$$

and similarly for y_0 .

Now, suppose we have maps $f : (X, x_0) \rightarrow (U, u_0)$ and $g : (Y, y_0) \rightarrow (U, u_0)$. Define (f, g) to be the unique map

$$(f, g) : X \amalg Y \rightarrow U$$

that makes the diagram

$$\begin{array}{ccc}
 X & & Y \\
 \downarrow i_x & & \downarrow i_y \\
 X \amalg Y & & \\
 \downarrow (f, g) & & \\
 U & &
 \end{array}$$

(Curved arrows from X and Y to U are labeled f and g respectively.)

commute. Now, since $f(x_0) = g(y_0) = u_0$, it follows that (f, g) is constant on the fibers of $q : X \amalg Y \rightarrow X \vee Y$ and thus by the universal property of quotient maps factors through $X \vee Y$. That is,

$$\begin{array}{ccc}
 & X \amalg Y & \\
 (f, g) \swarrow & & \searrow q \\
 U & \xleftarrow{(f, g)_*} & X \vee Y
 \end{array}$$

commutes for a unique $(f, g)_*$. Thus, we have for f, g arrows from (X, x_0) and (Y, y_0) to (U, u_0) a unique arrow $(f, g)_*$ from $X \vee Y$ which makes

$$\begin{array}{ccc}
 (X, x_0) & & (Y, y_0) \\
 \downarrow i_{x*} & & \downarrow i_{y*} \\
 (X, x_0) \vee (Y, y_0) & & \\
 \downarrow (f, g)_* & & \\
 (U, u_0) & &
 \end{array}$$

(Curved arrows from (X, x_0) and (Y, y_0) to (U, u_0) are labeled f and g respectively.)

commute as desired. □

Lemma 2. *The wedge product is stable with respect to (basepoint-preserving) homotopy. That is, if $X_1 \simeq X_2$ and $Y_1 \simeq Y_2$ relative to basepoints, then*

$$X_1 \vee Y_1 \simeq X_2 \vee Y_2$$

Proof. Since the wedge sum is the coproduct of pointed topological spaces, we construct the homotopy equivalence as follows:

Let $f_x : X_1 \rightarrow X_2$ and $g_x : X_2 \rightarrow X_1$ be homotopy equivalences of X_2 and X_1 . That is, f_x and g_x are homotopy inverses of each other. Furthermore, let $f_y : Y_1 \rightarrow Y_2$ and $g_y : Y_2 \rightarrow Y_1$ be homotopy inverses as well. Then, by the universal property of coproducts, we have the commutative diagram

$$\begin{array}{ccccc}
 & & X_2 \vee Y_2 & & \\
 & \nearrow & \uparrow \text{dashed} & \nwarrow & \\
 X_2 & & & & Y_2 \\
 f_x \uparrow \downarrow g_x & & \tilde{g} \uparrow \downarrow \tilde{f} & & f_y \uparrow \downarrow g_y \\
 X_1 & & & & Y_1 \\
 & \nwarrow & \downarrow \text{dashed} & \nearrow & \\
 & & X_1 \vee Y_1 & &
 \end{array}$$

where \tilde{f} is defined in terms of the universal property of coproducts with respect to the compositions $X_1 \xrightarrow{f_x} X_2 \longrightarrow X_2 \vee Y_2$ and $Y_1 \xrightarrow{f_y} Y_2 \longrightarrow X_2 \vee Y_2$ (and similarly for \tilde{g}).

I assert that \tilde{f} and \tilde{g} are homotopy inverses. It should be clear that by symmetry of the problem, I need only check that $\tilde{f} \circ \tilde{g} \simeq \mathbb{1}$.

Suppose $x \in X_1 \vee Y_1$, and without loss of generality let x be in the inclusion of X_1 to $X_1 \vee Y_1$. Then, we have the diagram

$$\begin{array}{ccc}
 & X_2 \vee Y_2 & \\
 i_2 \nearrow & \uparrow \text{dashed} & \\
 X_2 & & \\
 f_x \uparrow \downarrow g_x & & \tilde{g} \uparrow \downarrow \tilde{f} \\
 X_1 & & \\
 i_1 \nwarrow & \downarrow \text{dashed} & \\
 & X_1 \vee Y_1 & \\
 i_1^{-1} \nwarrow & &
 \end{array}$$

(where i_1^{-1} is only defined on the image of i_1 , but we are assuming that $x \in i_1(X_1)$ for this diagram chase, so this arrow exists). From here, it is clear that $\tilde{g}(x) = i_2 \circ f_x(i_1^{-1}(x))$, and if we identify X_1 and X_2 as subspaces of their wedge product, we have $\tilde{g}(x) = f_x(x)$. Similarly, we have $\tilde{f}(x) = g_x(x)$. Thus,

$$\tilde{f} \circ \tilde{g}(x) = g \circ f(x) \simeq \mathbb{1}(x)$$

as desired.

Thus, \tilde{f} and \tilde{g} are homotopy inverses, and $X_1 \vee Y_1 \simeq X_2 \vee Y_2$ as desired. \square

Now, we prove the main result.

Let Z be a CW complex which satisfies the hypotheses. In particular, let X and Y be such that $Z = X \cup Y$, X and Y are both contractible, and their intersection $A = X \cap Y$ is contractible as well.

Since A is a contractible subcomplex of Z , we know (via Hatcher prop 0.16 and 0.17) that Z is homotopy equivalent to Z/A . Thus, we only need to show that Z/A is contractible.

Let $q : Z \rightarrow Z/A$ be the canonical quotient map. Since $Z = X \cup Y$, we know that $q(Z) = Z/A = q(X) \cup q(Y)$. In particular, $q(X) = X/A$ and $q(Y) = Y/A$ (this follows from the definition of the quotient). Thus, $Z/A = X/A \cup Y/A$. Since the quotient map is the identity on $Z \setminus A$ and

collapses A to a point, it follows that $X/A \cap Y/A = A/A = \{a_0\}$ where a_0 is the point $q(A)$. Thus, Z/A is actually the wedge sum

$$Z/A = X/A \vee Y/A$$

Now, since A is contractible, we know that $X/A \simeq X$ and $Y/A \simeq Y$. In particular, since X and Y are contractible, so is X/A and Y/A . Thus, $X/A \simeq Y/A \simeq \{\cdot\}$. By lemma 2, this implies that

$$X/A \vee Y/A \simeq \{\cdot\} \vee \{\cdot\} = \{\cdot\}$$

and thus, $X/A \vee Y/A$ is contractible. Thus, $Z/A = X/A \vee Y/A$ is contractible as well, and Z itself is contractible as desired. \square

PROBLEM 2

Prove that there is no retraction $r : D^n \rightarrow \partial D^n$ using the fact that $\pi_n(S^n) = \mathbb{Z}$.

Proof. Assume for a contradiction that there exists a retraction $r : D^n \rightarrow \partial D^n$. That is, assume that there is some r such that the diagram

$$\begin{array}{ccccc} \partial D^n & \xrightarrow{i} & D^n & \xrightarrow{r} & \partial D^n \\ & \searrow & & \nearrow & \\ & & \mathbb{1} & & \end{array}$$

commutes. Here i is the inclusion map of ∂D^n into D^n . (Since r restricted to ∂D^n is the identity by definition, $ri = \mathbb{1}$).

Since π_n is a covariant functor (Hatcher states this on page 342), we can apply it to this diagram to get the commutative diagram

$$\begin{array}{ccccc} \pi_n(\partial D^n) = \mathbb{Z} & \xrightarrow{i_*} & \pi_n(D^n) = 0 & \xrightarrow{r_*} & \pi_n(\partial D^n) = \mathbb{Z} \\ & \searrow & & \nearrow & \\ & & \mathbb{1} & & \end{array}$$

which implies that the identity map from \mathbb{Z} to itself factors through zero, which cannot be. Thus, our assumption that r exists is false, and no retract of D^n to its boundary exists. \square

PROBLEM 3

In each case, either prove a covering exists, or that no such covering exists.

PART A

$$p : S^2 \rightarrow T^2.$$

Proof. I assert that no such covering map exists. Recall from Hatcher proposition 1.37 that for X a path-connected, locally path-connected space, and $p_1 : \tilde{X}_1 \rightarrow X$ and $p_2 : \tilde{X}_2 \rightarrow X$ covering spaces, then \tilde{X}_1 and \tilde{X}_2 are isomorphic if and only if $p_{1*}(\pi_1(\tilde{X}_1)) = p_{2*}(\pi_1(\tilde{X}_2))$.

Now, T^2 is path-connected and locally path-connected. Furthermore, it has a universal cover \mathbb{R}^2 . To see this consider the action of $\mathbb{Z} \times \mathbb{Z}$ on \mathbb{R}^2 given by $(m, n)(x, y) = (x + m, y + n)$. Clearly, this action is properly discontinuous, since each point moves a distance of at least 1 for the action of any nontrivial element of $\mathbb{Z} \times \mathbb{Z}$, and so for any point (x, y) we can take the neighborhood $U = V_{\frac{1}{10}}((x, y))$, which easily satisfies the property $U \cap g(U) = \emptyset$ for nontrivial $g \in \mathbb{Z} \times \mathbb{Z}$.

Thus, by proposition 1.40, we know that the quotient map $p_{\mathbb{R}^2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 / \mathbb{Z} \times \mathbb{Z}$ is a covering space map. It is easy to see that $\mathbb{R}^2 / \mathbb{Z} \times \mathbb{Z} = T^2$, and so \mathbb{R}^2 is a simply-connected covering space of T^2 .

Now, since both S^2 and \mathbb{R}^2 are simply-connected, we know that

$$p_*(\pi_1(S^2)) = p_{\mathbb{R}^2*}(\pi_1(\mathbb{R}^2)) = 0$$

and so proposition 1.37 guarantees the existence of a covering space isomorphism between the two. However, \mathbb{R}^2 and S^2 are not homeomorphic, (S^2 is compact but \mathbb{R}^2 is not) and so no such isomorphism can exist. Thus, S^2 cannot cover T^2 . □

PART B

$$p : T^2 \rightarrow S^2.$$

Proof. Recall Hatcher proposition 1.31, which asserts that for a covering space $p : \tilde{X} \rightarrow X$, the induced map $p_* : \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ is injective. So, if such a covering $p : T^2 \rightarrow S^2$ existed, it would induce an injection

$$p_* : \pi_1(T^2) \rightarrow \pi_1(S^2)$$

but $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$, and $\pi_1(S^2) = 0$, so no such injection can exist! Thus, T^2 does not cover S^2 . □

PART C

$$p : \mathbb{R}^2 \rightarrow T^2.$$

Proof. This was constructed in part a as a covering map. □

PART D

$$p : \mathbb{R}^2 \rightarrow S^2.$$

Proof. I assert that no such covering exists. This follows from the fact that S^2 covers itself, and simply-connected covering spaces are unique up to isomorphism.

Recall from Hatcher proposition 1.37 that for X a path-connected, locally path-connected space, and $p_1 : \tilde{X}_1 \rightarrow X$ and $p_2 : \tilde{X}_2 \rightarrow X$ covering spaces, then \tilde{X}_1 and \tilde{X}_2 are isomorphic if and only if $p_{1*}(\pi_1(\tilde{X}_1)) = p_{2*}(\pi_1(\tilde{X}_2))$.

Since S^2 is path-connected and locally path-connected, it satisfies the hypotheses for this proposition. Furthermore, the two covering space maps $p : \mathbb{R}^2 \rightarrow S^2$ and $\mathbb{1} : S^2 \rightarrow S^2$ both induce

$$p_*(\pi_1(\mathbb{R}^2)) = \mathbb{1}_*(\pi_1(S^2)) = 0$$

and thus if such a p existed, there would be a covering space isomorphism between \mathbb{R}^2 and S^2 . However, these spaces are not homeomorphic, and thus cannot be isomorphic as covering spaces. So, no such p exists. □

PROBLEM 5

Given a sequence $f : \mathbb{N} \rightarrow (0, \infty)$ define $p_k = (0, 0, f(k))$ and C_k is the boundary of the triangle in the plane $\{(x, y, 0) \in \mathbb{R}^3\}$ with vertices $0, v_{2k}, v_{2k+1}$ where $v_n = (\frac{1}{n}, \frac{1}{n^2}, 0)$. For $k \in \mathbb{N}$ the set $A_k = \{t \cdot q + (1 - t) \cdot p_k : q \in C_k, 0 \leq t \leq 1\}$ is the cone from p_k to C_k . The subspace $X \subset \mathbb{R}^3$ is $X = \bigcup A_k$.

PART A

Show X is simply-connected if $f(k) = \frac{1}{k}$.

Proof.

□

PART B

Show X is not simply-connected if $f(k) = k$.

Proof.

□

<+++>