Homework 4

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Problem 1

Let $\{H_\gamma \mid \gamma \in \Gamma\}$ be a family of Hilbert spaces, and let H be the vector space of sections of $\cup_{\gamma\in\Gamma} H_\gamma$ over Γ with

$$f \in \Gamma(\cup H_\gamma, \Gamma)$$
$$\sum_{\gamma\in\Gamma} \|f(\gamma)\|^2 < \infty$$

Show that

$$\|f\| = \left(\sum_{\gamma\in\Gamma} \|f(\gamma)\|^2\right)^{\frac{1}{2}}$$

is a norm on H, and that with this norm H is a Euclidean space. Is H necessarily a Hilbert space?

Proof. We immediately recognize this construction as the direct integral

$$\int_\Gamma^\oplus H_\gamma d\mu$$

where $\mu = \mu_c$ is the counting measure on Γ. This space is defined to be the set of all sections $\Gamma(\coprod_{\gamma\in\Gamma} H_\gamma, \Gamma)$ over Γ with the property that if f is a section, then its composition

$$g_\gamma(f(\gamma), f(\gamma)) = \|f(\gamma)\|^2 \in L^2(\Gamma, \mu_c)$$

(with g_γ the metric on H_γ) is required to be an L^2 function. For the remainder of this problem, we will let (Γ, μ) be an arbitrary measure space, and prove the more general result that

$$H = \int_\Gamma^\oplus H_\gamma d\mu(\gamma)$$

is a Hilbert space. Note that in this more general setting, H is actually a set of equivalence classes of sections where $s \sim t \iff \|s(\gamma)\|_\gamma \sim \|t(\gamma)\|_\gamma$ as functions in $L^2(\Gamma, \mu)$. That is, s is equivalent to t if and only if $s(\gamma)$ and $t(\gamma)$ differ on a set of Hilbert spaces of measure zero.

(For more general measure spaces, we might run into measurability issues, but since this example is against the counting measure, we need not worry about such technicalities.)

An inner product on the direct integral can be defined as

$$\langle s|t\rangle := \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma)$$

where it is understood that $\langle s(\gamma)|t(\gamma)\rangle = g_\gamma(s(\gamma), t(\gamma))$. We need to show that this inner product is indeed a well-defined inner product on $\int_\Gamma^\oplus H_\gamma d\mu$.

We first show that this is well-defined. That is, we need to show that the integral is finite. To see this, let $s, t \in H$. In particular, this means that

$$\begin{aligned}
\|s(\gamma)\|_\gamma &\in L^2(\Gamma, \mu) \\
\|t(\gamma)\|_\gamma &\in L^2(\Gamma, \mu)
\end{aligned}$$

We calculate the inner product as

$$\begin{aligned}
|\langle s|t\rangle| &= |\int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma)| \\
&\leq \int_\Gamma |\langle s(\gamma)|t(\gamma)\rangle| d\mu(\gamma) \\
&\leq \int_\Gamma \|s(\gamma)\|_\gamma \|t(\gamma)\|_\gamma d\mu(\gamma) && \text{By Cauchy-Schwarz inequality} \\
&= \|\left(\|s(\gamma)\|_\gamma\right)\left(\|t(\gamma)\|_\gamma\right)\|_1 && \text{By definiton of } L^1 \text{ norm} \\
&\leq \big(\|\left(\|s(\gamma)\|_\gamma\right)\|_2\big)\big(\|\left(\|t(\gamma)\|_\gamma\right)\|_2\big) && \text{By Holder's inequality with } p = q = 2 \\
&< \infty && \text{Since } \|s(\gamma)\|_\gamma \text{ and } \|t(\gamma)\|_\gamma \text{ are in } L^2(\Gamma, \mu)
\end{aligned}$$

and thus, the proposed inner product is well-defined.

Next, we show sesquilinearity. We adopt the mathematics convention that the inner product $\langle s|t\rangle$ is linear in the first term, and conjugate linear in the second term. Let $s, t \in H$, and let $\alpha \in \mathbb{C}$. Then,

$$\begin{aligned}
\langle \alpha s|t\rangle &= \int_\Gamma \langle \alpha s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\
&= \int_\Gamma \alpha\langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\
&= \alpha \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\
&= \alpha\langle s|t\rangle
\end{aligned}$$

Furthermore, with $r \in H$ as well, we have

$$\begin{aligned}
\langle r + s|t\rangle &= \int_\Gamma \langle r(\gamma) + s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\
&= \int_\Gamma \langle r(\gamma)|t(\gamma)\rangle + \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\
&= \int_\Gamma \langle r(\gamma)|t(\gamma)\rangle d\mu(\gamma) + \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\
&= \langle r|t\rangle + \langle s|t\rangle
\end{aligned}$$

and so the proposed inner product is linear in the first term. Furthermore, we see that

$$\begin{aligned}
\langle s|t\rangle &= \int_\Gamma \langle s(\gamma)|t(\gamma)\rangle d\mu(\gamma) \\
&= \int_\Gamma \overline{\langle t(\gamma)|s(\gamma)\rangle} d\mu(\gamma) \\
&= \overline{\int_\Gamma \langle t(\gamma)|s(\gamma)\rangle d\mu(\gamma)} \\
&= \overline{\langle t|s\rangle}
\end{aligned}$$

and so the proposed inner product is conjugate linear in the second term.

Finally, we need to show this inner product is positive-definite. That is, we need to show

$$\langle s|s\rangle \geq 0$$

with equality if and only if $s = 0$. So, let $s \in H$. Trivially, if $s = 0$, then

$$\langle s|s\rangle = \int_\Gamma \langle s(\gamma)|s(\gamma)\rangle d\mu(\gamma) = \int_\Gamma 0 = 0$$

So, let $s \neq 0$. Then, in particular, $\|s(\gamma)\|_\gamma$ differs from zero on a set $E \subset \Gamma$ of positive measure. Thus,

$$\begin{aligned}
\langle s|s\rangle &= \int_\Gamma \langle s(\gamma)|s(\gamma)\rangle d\mu(\gamma) \\
&= \int_\Gamma \|s(\gamma)\|_\gamma^2 d\mu(\gamma) \\
&\geq \int_E \|s(\gamma)\|_\gamma^2 d\mu(\gamma) \\
&> 0
\end{aligned}$$

as desired. Thus, this proposed inner product is indeed an inner product on H.

Observe that this completes the first two parts of this problem. We defined

$$\|s\|^2 = \langle s|s\rangle = \int_\Gamma \|s(\gamma)\|_\gamma^2 d\mu(\gamma)$$

and setting $\mu = \mu_c$ the counting measure, we get

$$\|s\|^2 = \sum_{\gamma\in\Gamma} \|s(\gamma)\|_\gamma^2$$

which is a norm, since it is induced by an inner product.

Furthermore, we have defined an inner product on H, which makes it a Euclidean space. Now, we just have to show that H is actually a Hilbert space.

We need to show that H is complete with respect to its norm. So, suppose $s_n \in H$ is a sequence of elements (sections) in H such that $\{s_n\}$ is a Cauchy sequence with respect to the norm.

In particular, this means that for $\varepsilon > 0$ there is an N such that for all $n, m > N$,

$$\|s_n - s_m\|^2 < \varepsilon$$

which translates to

$$\|s_n - s_m\|^2 = \int_\Gamma \|s_n(\gamma) - s_m(\gamma)\|_\gamma^2 d\mu(\gamma) < \varepsilon$$

and so μ-almost every term $\|s_n(\gamma) - s_m(\gamma)\|_\gamma^2 < \varepsilon$ as well. Thus, each term is Cauchy, and converges to some limit $s(\gamma)$ pointwise μ-a.e.(note that this convergence is uniform over γ, since the sequence is uniformly Cauchy and pointwise convergent). It is quickly verified that $s(\gamma) \in H$, since

$$\begin{aligned}
\|s\|^2 &= \int_\Gamma \|s(\gamma)\|_\gamma^2 d\mu(\gamma) \\
&= \int_\Gamma \| \lim_{n\to\infty} s_n(\gamma)\|_\gamma^2 d\mu(\gamma) \\
&= \lim_{n\to\infty} \int_\Gamma \|s_n(\gamma)\|_\gamma^2 d\mu(\gamma) \qquad \text{by continuity of } \|\cdot\|_\gamma \text{ and } s_n(\gamma) \to s(\gamma) \text{ uniformly} \\
&= \lim_{n\to\infty} \|s_n\|^2 < \infty
\end{aligned}$$

Here, the limit $\lim_n \|s_n\|^2 < \infty$ follows from the fact that the sequence $\|s_n\|^2$ is a Cauchy sequence of real numbers, and thus has a limit.

Now, we need to show that s_n converges to s in norm. To show this, we just need to show that

$$\|s_n - s\|^2 = \int_\Gamma \|s_n(\gamma) - s(\gamma)\|_\gamma^2 d\mu(\gamma) \to 0$$

However, this follows immediately from the uniform convergence observed earlier. In particular, we consider the sequence $f_n(\gamma) = \|s_n(\gamma) - s(\gamma)\|_\gamma^2$, which tends uniformly to zero. Then, we have that

$$\lim_{n\to\infty} \int_\Gamma f_n(\gamma) d\mu(\gamma) = \int_\Gamma 0 d\mu(\gamma) = 0$$

and so $\lim_{n\to\infty} \|s_n - s\| = 0$ as desired. Thus, H is complete.

(This construction was first presented to me in Hall's *Quantum Theory for Mathematicians*, and much of the proof technique mirrors what I remember from the text.) $\square$

Problem 2

Prove that if $f \in C(\mathbb{T})$, then
$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt \to 0$$
as $|n| \to \infty$.

Proof. This result follows immediately from the earlier observation that the set $\{|e^{int}\rangle\}_{n=-\infty}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{T})$. In particular, this means that
$$\sum_n |e^{int}\rangle\langle e^{int}| = I$$
This follows from the fact that each projection $|e^{int}\rangle\langle e^{int}|$ is orthogonal to the others, and thus the sum is also a projection. So, if the sum was not the identity, there would be some subspace on which the sum acts as the zero operator. This would imply that a unit vector in this subspace is orthogonal to each e^{int}, which cannot happen since e^{int} is an orthonormal basis.

Applying this to f, we see that
$$I|f\rangle = |f\rangle = \sum_n \langle e^{int}|f\rangle|e^{int}\rangle$$
but since $f \in L^2(\mathbb{T})$, $\|f\|^2 < \infty$, and thus
$$\|f\|^2 = \langle f|f\rangle = \sum_n \langle e^{int}|f\rangle^2 < \infty$$
Thus, it must be that
$$\langle e^{int}|f\rangle^2 = \int_{\mathbb{T}} e^{-int}f(t)dt \to 0$$
to keep the sum finite, as desired. □

Problem 3

Show that if $T \in B(H)$ is hermitian, then $\exp(iT)$ is unitary.

Proof. We only need to show that $\exp(iT)^* = \exp(iT)^{-1}$. To do so, we will make use of the *Continuous Functional Calculus* given by the Gelfand transform. Recall that there exists a unique $*$-isomorphism $\gamma : C^*(A) \to C(\sigma(A))$ between the unital C-$*$ algebra generated by a normal operator A and the algebra of continuous functions on the spectrum of A under the sup norm.

The construction of this isomorphism is done in two steps. First, recall that we have already shown that there is a unique isometric $*$-isomorphism between $C^*(A)$ and $C(\mathscr{M}_A)$ the continuous functions on the multiplicative linear functionals on $C^*(A)$ given by the Gelfand map Γ defined as

$$\Gamma(A)(\phi) = \phi(A)$$

Now, we next apply the fact that $\sigma(A) = \{\phi(A) \mid \phi \in \mathscr{M}_A\}$. This follows from an easy argument: suppose $\lambda \in \sigma(A)$. Then $A - \lambda I$ is not invertible, and generates a proper ideal in $C^*(A)$. This is contained in a maximal ideal, and is thus in the kernel of some multiplicative linear functional ϕ_λ. This implies that

$$\phi_\lambda(A - \lambda I) = 0$$

and since ϕ_λ is linear, and $\phi_\lambda(I) = 1$ (true of every multiplicative linear functional), we have that

$$\phi_\lambda(A) = \lambda$$

as desired. Conversely, if $\lambda \notin \sigma(A)$, then there is some $B \in C^*(A)$ for which $(A - \lambda I)B = I$. Applying an arbitrary nontrivial multiplicative linear functional yields

$$\phi(A - \lambda I)\phi(B) = 1$$

and so $\phi(A - \lambda I) \neq 0$ for any multiplicative linear functional ϕ as desired.

With this result in hand, we proceed with the construction of the continuous functional calculus. Define $\tau : \mathscr{M}_A \to \sigma(A)$ by

$$\tau(\phi) = \phi(A)$$

This is a continuous bijection between $\mathscr{M}_A$ with the weak-$*$ topology and $\sigma(A)$, since τ is simply evaluation at A, which is one of the generators of the weak-$*$ topology and is automatically continuous. Since $\mathscr{M}_A$ is compact (in the weak-$*$ topology, as a closed subset of the unit ball) and $\sigma(A)$ is Hausdorff (as a subset of $\mathbb{C}$ a Hausdorff space) τ is a homeomorphism. Thus, it induces a continuous algebra $*$-isomorphism h between $C(\mathscr{M}_A)$ and $C(\sigma(A))$ (both with the sup norm) given by

$$h(f) = f \circ \tau^{-1}$$

the pullback of f along τ^{-1}.

Composing these two yields the desired $*$-isomorphism

$$\gamma : C^*(A) \to C(\sigma(A))$$
$$\gamma = h \circ \Gamma$$

Now, it is easy to see that $\gamma(A) = z$ the identity on $\mathbb{C}$. This follows by direct calculation:

$$\begin{aligned}\gamma(A)(\lambda) &= h \circ \Gamma(A)(\lambda) \\ &= h(\mathrm{ev}_A)(\lambda) \\ &= \mathrm{ev}_A(\tau^{-1}(\lambda)) \\ &= \mathrm{ev}_A(\phi) \\ &= \phi(A) = \lambda\end{aligned}$$

Since γ is an algebra isomorphism, we have immediately that if p is a polynomial in A, then $\gamma(p(A)) = p(z)$. Taking norm closures, we see that if f is a continuous function (approximated by polynomials), then

$$f(A) = \gamma^{-1}(f)$$

since γ preserves convergences, as it and its inverse are continuous as the composition of Γ and τ, both of which (along with their inverses) are continuous.

With all this machinery, we are ready to prove that $\exp(iT)$ is unitary. Let γ be the unique $*$-isomorphism between $C^*(T)$ and $C(\sigma(T))$. Then, we have that

$$\begin{aligned}\gamma(\exp(iT)) &= \exp(it) \\ \gamma((\exp(iT))^*) &= \overline{(\exp(it))} = \exp(-it) \quad \text{since } \sigma(T) \subset \mathbb{R}\end{aligned}$$

Thus,

$$\gamma(\exp(iT)(\exp(iT))^*) = \exp(it)\exp(-it) = 1$$

and since $\gamma^{-1}(1) = I$, we have that

$$\exp(iT)(\exp(iT))^* = I$$

as desired.

(This argument comes mainly from what I remember of MacCluer's *Elementary Functional Analysis*, which I studied from in my undergraduate studies.) $\square$

Problem 4

Show that every nonempty compact subset $K \subset \mathbb{C}$ is the spectrum of some operator.

Proof. Let $\{\lambda_n\}$ be a countable dense subset of K (this is possible, since $\mathbb{C}$ is seperable). Define an operator $T \in B(\ell^2)$ as

$$T(e_n) = \lambda_n e_n$$

and extend linearly. This is a bounded linear operator, since each λ_n is bounded as an element of a compact set. Furthermore, each $\lambda_n \in \sigma(T)$ by construction. Now, since the spectrum $\sigma(T)$ is closed, we have that

$$\overline{\{\lambda_n\}} = K \subset \sigma(T)$$

Now for the converse direction, suppose $\lambda \in \mathbb{C} \setminus K$. Since K is closed, there is some ε for which $\|\lambda - x\| > \varepsilon$ for all $x \in K$. Then, it is easy to see that $T - \lambda I$ is invertible. First, observe that it is bounded below, since

$$(T - \lambda I)e_n = (\lambda_n - \lambda)e_n > \varepsilon e_n$$

and thus $\|T - \lambda I\| > \varepsilon$. Secondly, notice that it has dense range, since

$$(T - \lambda I)(\frac{1}{\lambda_n - \lambda})e_n = e_n$$

and $\frac{1}{\lambda_n - \lambda}$ never diverges. Thus, λ is not in the spectrum of T. Therefore, the spectrum of T is precisely the closure of $\{\lambda_n\}$, which is K by construction. □

Problem 5

Let M be a maximal ideal of a complex unital Banach algebra B. Show that B/M is also a complex unital Banach algebra.

Proof. Recall the norm we give to a quotient space is

$$\|b+M\| = \inf_{m\in M} \|b-m\|$$

Now, this relies on M being closed, but every maximal ideal is closed, since its closure is a proper ideal containing it. This follows from the fact that the open unit ball around the identity $B(I)$ contains only invertible elements, since for T an operator with $\|T\| < 1$, $I - T$ is invertible by

$$\frac{1}{I-T} = \sum_{n=1}^{\infty} T^n$$

which converges absolutely, since $\|T\| < 1$. Thus, since M is a maximal (proper) ideal, it cannot contain invertible elements, and $M \subset B(I)^c$. Since $B(I)^c$ is closed,

$$M \subset \overline{M} \subset B(I)^c \neq B$$

and thus $\overline{M}$ is a proper ideal containing the maximal ideal M, so by definition $M = \overline{M}$ as desired.

Now, B/M is a Banach space under this norm (proven last quarter: the quotient of a Banach space by a closed linear subspace is again a Banach space), so all we have to show is that the norm is submultiplicative, and that the quotient contains the identity.

Recall that the multiplication on B/M is given by

$$(a+M)(b+M) = ab + M$$

which is easily verified to be well-defined, since

$$(a+m+M)(b+n+M) = (ab+mb+an+mn+M) = ab+M$$

To see that the norm is submultiplicative, we observe that for $a_1, a_2 \in B$ and $m_1, m_2 \in M$ we have

$$\|(a_1+m_1)(a_2+m_2)\| \leq \|a_1+m_1\|\|a_2+m_2\|$$

and taking inf over choices of m_1 and m_2 yields the desired result.

Now, the identity in B/M is just $1+M$. This is not zero, since $1 \notin M$ (otherwise M would be forced to be equal to B). Furthermore, for any $b+M \in B/M$, we have

$$(1+M)(b+M) = b+M = (b+M)(1+M)$$

as desired.

Furthermore, the identity has norm 1. To see this, note that submultiplicativity guarantees that

$$\|I+M)\| = \|(I+M)(I+M)\| \leq \|I+M\|^2$$

which forces $\|I+M\| \geq 1$. However,

$$\|I+M\| = \inf_{m\in M} \|I+m\|$$

and since $0 \in M$, we have that

$$\|I+M\| \leq \|I+0\| = \|I\| = 1$$

and so $1 \leq \|I+M\| \leq 1$ and thus $\|I+M\| = 1$ as desired.

$\square$

Problem 6

Prove that if M is a maximal ideal of A a commutative complex unital Banach algebra, then A/M is isometrically isomorphic to $\mathbb{C}$.

Proof. Recall that the quotient of a commutative ring by a maximal ideal is a field. Thus, A/M is a field. Furthermore, from the previous problem, we know that A/M is a complex unital Banach algebra. Applying the Gelfand-Mazur theorem (Theorem 10, Chapter 12 of Bolabas) we see that A/M must be (isometrically isomorphic to) $\mathbb{C}$. $\square$