Problem Set 8

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Problem 1

Show that $\frac{\sin(x)}{x}$ is not in $L^1((0,\infty),\lambda^1)$.

Proof. We wish to evaluate

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x)$$

and show that it diverges. To do so, we split the integral into half-cycles

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x) = \sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{x} d\lambda^1(x)$$

Now, we know that on each half-cycle,

$$\frac{|\sin(x)|}{x} \geq \frac{|\sin(x)|}{(n+1)\pi}$$

so we have a lower bound for the integral:

$$\sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{x} d\lambda^1(x) \geq \sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{(n+1)\pi} d\lambda^1(x)$$

Now, since each half-cycle is either entirely positive or entirely negative, we know that

$$\int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{(n+1)\pi} d\lambda^1(x) = \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right|$$

And finally, we can evaluate the integral directly:

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right| &= \sum_{n=0}^{\infty} \left| \frac{1}{(n+1)\pi} [\cos(x)]_{n\pi}^{(n+1)\pi} \right| \\ &= \sum_{n=0}^{\infty} \frac{2}{(n+1)\pi} \\ &= \infty \end{aligned}$$

Thus, since

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x) \geq \sum_{n=0}^{\infty} \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right| = \infty$$

the integral diverges, and $\frac{\sin(x)}{x}$ is not in $L^1((0,\infty),\lambda^1)$.

$\square$