Homework

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Problem 1

Prove that $V^{**} \cong V$ for a finite dimensional vector space V.

Proof. The isomorphism is given by

$$\begin{aligned} \Phi &: V \to V^{**} \\ \Phi(v) &= ev_v = (\phi \mapsto \phi(v)) \end{aligned}$$

First, we observe that this map is linear. Indeed, for $v_1, v_2 \in V$ and α, β a scalar, we have

$$\begin{aligned} \Phi(\alpha v_1 + \beta v_2)(\phi) &= \phi(\alpha v_1 + \beta v_2) \\ &= \alpha\phi(v_1) + \beta\phi(v_2) \\ &= \alpha\Phi(v_1)(\phi) + \beta\Phi(v_2)(\phi) \end{aligned}$$

as desired.

We need to show this is surjective and injective. Injectivity of Φ is easily shown by examining the kernel of Φ. Suppose v is such that $ev_v(\phi) = \phi(v) = 0$ for all $\phi \in V^*$. Then, since V^* separates points of V, it follows that $v = 0$. Thus, the kernel is trivial, as desired.

Now, we need to show this map is surjective. To do so, we appeal to v being finite-dimensional, and let $\{e_i\}$ be a basis for V with dual basis $\{\omega^i\}$. Then, let $x \in V^{**}$. Define a corresponding vector $\tilde{x} = \sum_i x(\omega^i)e_i$. Then

$$\begin{aligned} \Phi(\tilde{x})(\phi) &= \phi(\tilde{x}) \\ &= \sum_i x(\omega^i)\phi(e_i) \\ &= \sum_i x(\omega^i\phi(e_i)) \\ &= x(\phi) \end{aligned}$$

as desired. Thus, Φ is a linear isomorphism. □

Problem 2

Prove that for V a vector space with basis v_i and dual basis v^i, the set

$$\{v^i \otimes v^j \mid 1 \leq i, j \leq n\}$$

forms a basis for $V^* \otimes V^*$.

Proof. For this, we will show that the vector space

$$W = \operatorname{span}(\{v^i \otimes v^j \mid 1 \leq i, j \leq n\})$$

satisfies the universal property of tensor products. That is, we wish to show that for each bilinear map $h : V^* \times V^* \to U$ for some vector space U, there is a unique linear map $\tilde{h} : W \to U$ such that the diagram

$$\begin{tikzcd} V^* \times V^* \arrow[r, "h"] \arrow[d, "\otimes"] & U \\ W \arrow[ur, dotted, "\tilde{h}"] & \end{tikzcd}$$

commutes. We will guess that $\otimes : V^* \times V^* \to W$ is given as

$$\otimes(a_i v^i, b_j v^j) = a_i b_j v^i \otimes v^j$$

So, let $h : V^* \times V^* \to U$ be a bilinear map. Define $\tilde{h} : W \to U$ as

$$\tilde{h}(\sum_{i,j} a_{ij} v^i \otimes v^j) = a_{ij} h(v^i, v^j)$$

Then it is clear that the diagram

$$\begin{tikzcd} V^* \times V^* \arrow[r, "h"] \arrow[d, "\otimes"] & U \\ W \arrow[ur, dotted, "\tilde{h}"] & \end{tikzcd}$$

commutes, since

$$\begin{aligned} h(a_i v^i, b_j v^j) &= a_i b_j h(v^i, v^j) \\ \tilde{h} \circ \otimes(a_i v^i, b_j v^j) &= \tilde{h}(a_i b_j v^i \otimes v^j) \\ &= a_i b_j h(v^i, v^j) \end{aligned}$$

It should be clear from construction that $\tilde{h}$ is unique.

Thus, since every bilinear map from $V^* \times v^*$ factors through W, W satisfies the universal property of tensor products, and is isomorphic to $V^* \otimes V^*$. Thus, the set $\{v^i \otimes v^j\}$ forms a basis for $V^* \otimes V^*$ as desired. $\square$

Problem 3

Show that $dvol = \wedge_i \omega^i = \sqrt{|g|} dx^n$.

Proof. Recall that $dx^n = \wedge_{i=1}^n dx^i$, and our manifold is n-dimensional.

Recall that for an n-fold wedge product $\wedge_{i=1}^n v^i$, we have (for $v^i = a^i_j \omega^j$ in an orthonormal frame)

$$\wedge_{i=1}^n v^i = |\det(a^1_j, \ldots, a^n_j)| \wedge_{i=1}^n \omega^i = \sqrt{\det(A^t A)} \wedge_{i=1}^n \omega^i$$

for A the matrix with columns a^i. We can apply this to $v^i = dx^i$ to get the desired result. $\square$

Problem 4

Show that the definition of the integral of a top degree form on a single chart is independent of choice of coordinates.

Proof. Recall the definition of the integral of a top degree differential form on a compact set K in a single coordinate frame is

$$\int_K \omega = \int_{\phi(K)} f \circ \phi^{-1} dx^n$$

for $\omega = f dx^1 \wedge \cdots \wedge dx^n$.

We also have the change-of-coordinates formula for a diffeomorphism $F : \Omega_1 \to \Omega_2$ as

$$\int_{\Omega_2} f dy^n = \int_{\Omega_1} f \circ F |J_F| dx^n$$

where J_F is the jacobian of F.

So, let $\phi : M \to \mathbb{R}^n$ be the original coordinate system, and let $\psi : M \to \mathbb{R}^n$ be another coordinate system covering K. Then, we have the diffeomorphism $F = \psi \circ \phi^{-1}$ and we can apply this to get

$$\int_{\psi(K)} g \circ psi^{-1} dy^n = \int_{\phi(K)} g \circ \psi^{-1} \circ F |J_F| dx^n$$

which is just

$$\begin{aligned}
\int_{\phi(K)} g \circ \psi^{-1} \circ F |J_F| dx^n &= \int_{\phi(K)} g \circ \psi^{-1} \circ \psi \circ \phi^{-1} |J_F| dx^n \\
&= \int_{\phi(K)} g \circ \phi^{-1} |J_F| dx^n \\
&= \int_K g |J_F| dx^n \\
&= \int_K \omega
\end{aligned}$$

where we used the fact that $\omega = g dy^1 \wedge \cdots \wedge dy^n = g|J_F| dx^1 \wedge \cdots \wedge dx^n$ since $|J_F| = \det(J_F)$ on a two positively oriented charts.

Thus, the two integrals agree. □

Problem 5

Prove that a manifold is orientable if and only if it admits a nowhere vanishing top degree form.

Proof. Suppose M is an orientable manifold. That is, there exists an atlas $\{U_\alpha, \phi_\alpha\}$ of M for which the Jacobian of each transition map has positive determinant. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to the atlas U_α. Then, define

$$\omega = \sum_\alpha \psi_\alpha dx_\alpha^i \wedge \cdots \wedge dx_\alpha^n$$

where x_α^i are the coordinate functions on U_α. We claim that ω is a nowhere-vanishing form. Clearly, if p is a point in M contained in only one chart, then $\omega_p = dx_p^1 \wedge \cdots \wedge dx_p^n$ and does not vanish. If p is such that it is contained in more than one chart, then ω at p is the sum of positive terms (since each coordinate system is positive, we have $dx^n = \det(J)dy^n$ and $\det(J)$ is always positive) and does not vanish.

Suppose instead that M admits a nowhere-vanishing top degree form ω. Then, let $\{U_\alpha, \phi_\alpha\}$ be an atlas of M. For x^i_α the coordinate functions for ϕ_α, define a new coordinate system to be such that if ω is expressed as

$$\omega = f dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$$

then f is positive. This is done by setting x^1_α to its negative if f is negative on that chart. Note that since ω never vanishes, we know that f will be either entirely positive or entirely negative on a chart. Thus, such a choice can be made consistently.

Then, the modified atlas is a positive coordinate chart for M, which is easily verified, since

$$\omega = f dx^n = f \det(J) dy^n$$

and ω always has positive coefficient, so $\det(J)$ is positive. □

Problem 6

Show that the topology of M coincides with the metric topology

$$d_g(x,y) = \inf_{\gamma \in C^\infty(I,M)} \{L(\gamma) \mid \gamma(0) = x, \gamma(1) = y\}$$

where $L(\gamma)$ is the total length of γ defined by

$$L(\gamma) = \int_I g(\gamma', \gamma') dt$$

Proof. First, we observe the following: for g a Riemannian metric, and γ a curve contained entirely in a single coordinate chart ϕ, there exist constants c, C such that

$$cL_{\mathbb{R}^n}(\gamma) \leq L_g(\gamma) \leq CL_{\mathbb{R}^n}(\gamma)$$

where $L_{\mathbb{R}^n}(\gamma)$ is the length of $\phi \circ \gamma$ using the euclidean metric on $\mathbb{R}^n$. This follows from the fact that the metric induced by g along ϕ^{-1} defines a norm on $\mathbb{R}^n$, and all norms are equivalent. That is,

$$k\|v\|_{\mathbb{R}^n} \leq \|v\|_{\phi^{-1*}g} \leq K\|v\|_{\mathbb{R}^n}$$

for constants k, K. Thus, the lengths (defined in terms of integrals of the metric) follow the same inequality.

It should also be clear that the metrics induced by $\mathbb{R}^n$ and g are equivalent as well. To see this, note that for any $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} cd_{\mathbb{R}^n}(x,y) &= \inf_{\gamma(0)=x,\gamma(1)=y} cL_{\mathbb{R}^n}(\gamma) \\ &\leq \inf_{\gamma(0)=x,\gamma(1)=y} L_g(\gamma) \\ &= d_g(x,y) \end{aligned}$$

(the first equality is proved in the next problem) and similarly for $d_g(x,y) \leq Cd_{\mathbb{R}^n}(x,y)$. This shows that the two topologies induced by the two metrics are equal.

Equivalence of the two topologies follows immediately. We can show that for U open in the manifold topology, and $x \in U$, there is a neighborhood V of x in the metric topology contained in U. Simply take a coordinate ball $V_\varepsilon(x)$ of small enough radius to be contained in a single coordinate chart ϕ. That is, the domain of ϕ contains $V_\varepsilon(x)$. Then, we know from the above observation that $\phi(V_\varepsilon(x))$ is open in $\mathbb{R}^n$ in the standard topology, and thus is open with respect to the pullback of d_g along ϕ^{-1}. Thus, $V_\varepsilon(x)$ is also open in the metric topology induced by d_g on M. The same argument with the two topologies switched completes the argument that both topologies are equal. □

Problem 7

Show that $\|a-b\|_{\mathbb{R}^n}$ is $d_{\mathbb{R}^n}(a,b) = \inf_\gamma L_{\mathbb{R}^n}(\gamma)$.

Proof. This result follows from standard variational calculus on the functional $L_{\mathbb{R}^n}(\gamma)$.

Let's minimize the functional $L(\gamma)$ by varying the path γ. We do this by setting the variation to zero. That is, $L(\gamma)$ is maximized for γ that makes $\delta L = 0$. We calculate

$$\begin{aligned}\delta L &= \int \delta(\sqrt(g(\gamma',\gamma')))dt \\ &= \int \frac{1}{2\sqrt{g(\gamma',\gamma')}}\delta g(\gamma',\gamma')dt\end{aligned}$$

Since arc length is independent of parameterization, we can take the unit speed parameterization of γ, so that $g(\gamma',\gamma')=0$. Then, we have

$$\begin{aligned}\delta L &= \int \delta g(\gamma',\gamma')dt \\ &= \int \delta(g_{ab}\partial_t\gamma^a\partial_t\gamma^b)dt \\ &= \int g_{ab}\delta(\partial_t\gamma^a)\partial_t\gamma^b + g_{ab}\delta(\partial_t\gamma^b)\partial_t\gamma^a dt \quad = 2\int g_{ab}\partial_t(\delta\gamma^a)\partial_t\gamma^b dt\end{aligned}$$

integrating by parts (and tossing boundary terms since the endpoints of γ do not vary) and noting $g_{ab} = \delta_{ab}$ in $\mathbb{R}^n$ yields

$$\delta L = -2\int \partial_t^2\gamma^a\delta\gamma^b dt$$

which holds only if $\partial_t^2\gamma^a = 0$ for all a. Thus, the minimal length path from a point p to a point q is the straight line from p to q.

So, for γ the straight line from p to q, $L(\gamma) = \|\gamma\| = \|p-q\|$ as desired. □

Problem 8

Define the Levi-Civita connection as the unique connection such that

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

and

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Show that this is indeed a connection on M.

Proof. For this proof, we will denote the Levi-Civita connection as D.

We need to show that $D_X Y$ is function linear in X, scalar linear in Y, and satisfies the Leibniz rule

$$D_X(fY) = (Xf)Y + fD_X Y$$

Recall from class that by utilizing the two properties above, we see that

$$2g(D_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y)$$

which uniquely determines the connection. Now, we just need to show that this definition satisfies the definition of a connection. To that end, let $f \in C^\infty(M)$. We calculate

$$\begin{aligned}
2g(D_{fX}Y, Z) &= fXg(Y,Z) + Yg(Z,fX) - Zg(fX,Y) \\
&+ g([fX,Y],Z) - g([Y,Z],fX) - g([fX,Z],Y) \\
&= fXg(Y,Z) + Yg(Z,fX) - Zg(fX,Y) \\
&- g([Y,fX],Z) - g([Y,Z],fX) + g([Z,fX],Y) \\
&= fXg(Y,Z) + (Yf)g(Z,X) + fYg(Z,X) - (Zf)g(X,Y) - fZg(X,Y) \\
&- g((Yf)X,Z) - g(f[Y,X],Z) - g([Y,Z],fX) + g((Zf)X,Y) + g(f[Z,X],Y) \\
&= fXg(Y,Z) + (Yf)g(Z,X) + fYg(Z,X) - (Zf)g(X,Y) - fZg(X,Y) \\
&- g((Yf)X,Z) + g(f[X,Y],Z) - g([Y,Z],fX) + g((Zf)X,Y) - g(f[X,Z],Y) \\
&= (fX)g(Y,Z) + (Yf)g(X,Z) - (Yf)g(X,Z) - (Zf)g(X,Y) + (Zf)g(X,Y) \\
&+ fYg(X,Z) - fZg(X,Y) + g(f[X,Y],Z) - g([Y,Z],fX) - g(f[X,Z],Y) \\
&= f\left\{Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y)\right\} \\
&= fg(D_XY,Z) = g(fD_XY,Z)
\end{aligned}$$

and thus D is $C^\infty(M)$-linear in X. $\square$

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