MATH 220B: FINAL EXAMINATION
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Problem 1

Part i. Prove that the nilpotent elements of a ring R form an ideal N.

Proof. Let N be the collection of all nilpotent elements of R. We will show this collection is closed under addition, and is stable with respect to multiplication in R.

First, suppose $a, b \in N$. That is, there exist integers $m, n \geq 1$ such that $a^m = b^n = 0$. We wish to show that $a+b$ is nilpotent. This is clear, however, since

$$(a+b)^{nm} = \sum_{k=1}^{n} \binom{n}{k} a^{nm-k} b^k$$

and each term in the sequence has either $nm - k > m$ or $k > n$, and so a^{nm-k} or b^k is zero. Thus, $(a+b)^{nm} = 0$ and $a+b$ is nilpotent as desired.

Now, we show that N is stable with respect to multiplication. That is, for any $r \in R$, $rN = N$. This amounts to showing that $ra \in N$ for any $a \in N$. So, let a be such that $a^n = 0$. Then,

$$(ra)^n = r^n a^n = 0$$

as well, so $ra \in N$ as desired.

Thus, N is an ideal. $\square$

Part ii. Show further that if R is Noetherian, then N is a nilpotent ideal.

Proof. Since R is Noetherian, we know that all ideals of R are finitely generated. In particular, N is finitely generated by some finite set $S = \{n_1, \cdots, n_k\}$. Then, every element $x \in N$ is expressible as

$$x = \sum_{i=1}^{k} a_i n_i$$

Now, let $m_1, \cdots, m_k$ be such that $n_i^{m_i} = 0$ (since n_i is nilpotent). I assert that $N^{m_1 m_2 \ldots m_k} = \{0\}$. This follows from the fact that for $x_i \in N$,

$$\begin{aligned} \prod_{i=1}^{m_1 m_2 \ldots m_k} x_i &= \prod_{i=1}^{m_1 m_2 \ldots m_k} \left(\sum_{l=1}^{k} a_{il} n_i \right) \\ &= \sum_j c_j n_1^{p_{1_j}} n_2^{p_{2_j}} \cdots n_k^{p_{k_j}} \end{aligned}$$

(for some constants c_j) where for each j, $\sum_{i=1}^{k} p_{i_j} = m_1 m_2 \cdots m_k$. This implies that in each term, at least one exponent p_{i_j} is greater than m_i, and so $n_i^{p_{i_j}} = 0$, and thus the term is zero. Since each term in the expansion is zero, this implies that

$$\prod_{i=1}^{m_1 m_2 \ldots m_k} x_i = 0$$

In particular, this holds for all $x_i \in N$. This implies that $N^{m_1 m_2 \ldots m_k} = \{0\}$ as desired. $\square$

Part iii. Give an example of a ring R for which N is not a niplotent ideal.

Proof. Let

$$G = \langle a_i | a_i^i = 0 \rangle$$

be the (infinitely) presented group, and let $R = \mathbb{Z}[G]$ be the group ring. Note that for each i, $a_i \in N$ the nilpotent ideal. However, for any fixed integer $m > 0$, N^m contains the element a_{m+1}^m which is not zero. Thus, for any m, $N^m \neq \{0\}$ and N is not nilpotent, as desired. $\square$

Problem 2

Part i. State the Hilbert Basis Theorem.

Theorem. *For R a Noetherian ring, $R[x_1, .., x_k]$ is Noetherian.*

Part ii. let k be a field. Show that every algebraic set can be defined by a finite system of equations.

Proof. Let S be an algebraic set of k^n. That is, S is the set of all points α such that $f_\lambda(\alpha) = 0$ for each f_λ in a family $\mathcal{P} = \{f_\lambda \mid f_\lambda \in k[x_1, \cdots, x_n]\}$ of polynomials (indexed by λ).

Consider the subspace generated by $\mathcal{P}$. That is, consider

$$V = \operatorname{span}(\{f_\lambda \mid f_\lambda \in \mathcal{P}\})$$

This is (by definition) a linear subspace of $k[x_1, \cdots, x_n]$. Since k is Noetherian, it follows that $k[x_1, \cdots, x_n]$ is Noetherian as well. Thus, V is finitely generated as a submodule of the Noetherian module $k[x_1, \cdots, x_n]$. Let $g_1, \cdots, g_m$ be the generators of V. I claim that these define the algebraic set S.

Let S' be the set of all $\alpha \in k^n$ for which $g_i(\alpha) = 0$ for all generators g_i. That is, S' is the algebraic set corresponding to $g_1, \cdots, g_m$.

First observe that $S' \subseteq S$. To see this, suppose $\alpha \in S'$. For $f_\lambda \in \mathcal{P}$, we know that

$$f_\lambda = \sum_{i=1}^{m} a_i g_i$$

since $g_1, \cdots, g_n$ generate V. Thus,

$$f_\lambda(\alpha) = \sum_{i=1}^{n} a_i g_i(\alpha) = \sum_{i=1}^{n} a_i(0) = 0$$

and so $\alpha \in S$ as well.

Next, we observe that $S \subseteq S'$. To see this, let $\alpha \in S$. We note first that for any $f \in V$, $f(\alpha) = 0$. This follows from the fact that f is in the span of $\mathcal{P}$, and so

$$f = \sum_{i=1}^{k} a_i f_{\lambda_i}$$

for $f_{\lambda_i} \in \mathcal{P}$. Thus,

$$f(\alpha) = \sum_{i=1}^{k} a_i f_{\lambda_i}(\alpha) = 0$$

In particular, since V is generated by $g_1, \cdots, g_m$, we know that $g_i \in V$ for all i, and so $g_i(\alpha) = 0$ as well. Thus $\alpha \in S'$, and $S \subseteq S'$

We have shown that $S' = S$, and so S is determined by the finite set of polynomials $\{g_1, \cdots, g_n\}$. $\square$

Problem 3

Give examples to show each of the following might occur.

Part a. $M \otimes_R N \neq M \otimes_{\mathbb{Z}} N$.

Proof. Let $M = N = \mathbb{Z}^2$, and $R = \mathbb{Z}^2$. Then

$$\mathbb{Z}^2 \otimes_{\mathbb{Z}^2} \mathbb{Z}^2 \cong \mathbb{Z}^2$$

This is clear, since for any simple tensor $a \otimes_{\mathbb{Z}^2} b$, we have

$$a \otimes b = a(1 \otimes b) = ab(1 \otimes 1)$$

Thus, every simple tensor is a multiple of $1 \otimes 1$, and thus every tensor is a multiple of $1 \otimes 1$. So, we can define an isomorphism $\Phi : \mathbb{Z}^2 \to \mathbb{Z}^2 \otimes_{\mathbb{Z}^2} \mathbb{Z}^2$ as

$$\Phi(a, b) = ab(1 \otimes 1)$$

with inverse $\Phi^{-1}(a \otimes b) = ab$ extended linearly.

However,

$$\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}^2 = (\mathbb{Z} \times \mathbb{Z}) \otimes (\mathbb{Z} \times \mathbb{Z}) \cong \mathbb{Z}^4$$

(easily verified using the property $M \otimes (N_1 \times N_2) = (M \otimes N_1) \times (M \otimes N_2)$). Since $\mathbb{Z}^2 \neq \mathbb{Z}^4$, the two tensor products are not equal, as desired. $\square$

Part b. $u \in M \otimes_R N$ but $u \neq m \otimes_R n$ for any $m \in M$ and $n \in N$.

Proof. Let $M = N = \mathbb{R}^n$, with $R = \mathbb{R}$, and $n \geq 3$. Recall that for finite-dimensional vector spaces, $\mathrm{Hom}(V, \mathbb{R}) = V^* \cong V$ (an elementary result not proven here). Recall also that $\mathrm{Hom}(V, \cdot)$ is right-adjoint to $\cdot \otimes V$. The proof for this is easy: let X, Z be real vector spaces, and let $f \in \mathrm{Hom}(X \otimes_{\mathbb{R}} V, Z)$. Then, we can define an isomorphism Φ as

$$\Phi(f)(x) = \tilde{f}(x) = (v \mapsto f(x \otimes v))$$

clearly, this is linear in x and v, and so $\tilde{f}$ is an element of $\mathrm{Hom}(X, \mathrm{Hom}(V, Z))$. Furthermore, Φ itself is linear, since

$$\Phi(f+g)(x) = (v \mapsto f(x \otimes v) + g(x \otimes v)) = \Phi(f)(x) + \Phi(g)(x)$$

and

$$\begin{aligned}\Phi(\alpha f)(x) &= (v \mapsto \alpha f(x \otimes v)) \\ &= \alpha(v \mapsto f(x \otimes v)) = \alpha\Phi(f)(x)\end{aligned}$$

as desired.

Φ also has trivial kernel, since if $\Phi(f)(x) = 0$ for all x, then

$$(v \mapsto f(x \otimes v)) = 0$$

which implies that $f(x \otimes v) = 0$ for all x and v, and so $f = 0$.

Finally, we note that Φ is surjective, since for any $g \in \mathrm{Hom}(X, \mathrm{Hom}(V, Z))$, we can define $\tilde{g}$ to be the map

$$\tilde{g}(x \otimes v) = g(x)(v)$$

and

$$\Phi(\tilde{g})(x) = (v \mapsto \tilde{g}(x \otimes v)) = v \mapsto g(x)(v) = g(x)$$

thus, Φ is an isomorphism as desired.

We now combine these facts to show that for $V = \mathbb{R}^n$, $V \otimes V \cong \mathrm{Hom}(V, V)$.

Note that

$$\begin{aligned} V \otimes V &\cong \operatorname{Hom}(V \otimes V, \mathbb{R}) \\ &\cong \operatorname{Hom}(V, \operatorname{Hom}(V, \mathbb{R})) \\ &\cong \operatorname{Hom}(V, V) \end{aligned}$$

as desired.

But $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is the set of all linear maps from $\mathbb{R}^n$ to itself, which is the set of all $n \times n$ matrices over $\mathbb{R}$. This is a vector space of dimension n^2, whereas $\mathbb{R}^n \times \mathbb{R}^n$ is of dimension $2n$. Thus, for $n \geq 3$, the tensor product map $\otimes : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ cannot be surjective, and thus there exists an element u of $\mathbb{R}^n \otimes \mathbb{R}^n$ that is not in the image of $\otimes$. That is, u is not $v \otimes w$ for some $v, w \in \mathbb{R}^n$. This completes the example. □

Part c. $m \otimes_{\mathbb{Z}} n = m_1 \otimes_{\mathbb{Z}} n_1$ but $m \neq m_1$ and $n \neq n_1$.

Proof. Let $M = N = \mathbb{Z}$. Then

$$2 \otimes 1 = 1 \otimes 2$$

but $2 \neq 1$. Equality of the tensors follows immediately from bilinearity, since

$$2 \otimes 1 = 2(1 \otimes 1) = 1 \otimes 2$$

□

Part d. $M \otimes_R N = \{0\}$ but $M \neq \{0\}$ and $N \neq \{0\}$.

Proof. Let $M = 2\mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, and $R = \mathbb{Z}$. Then let $2k \otimes j$ be a simple tensor in $M \otimes_R N$. We have that

$$2k \otimes j = k \otimes 2j = k \otimes 0 = 0$$

Thus, all simple tensors are zero, and it follows that all tensors in the product are zero. Thus, $M \otimes_R N = \{0\}$ as desired. $\square$

Problem 4

Let $\mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$ be the field of integers modulo seven, and let R be any commutative ring containing $\mathbb{F}_7$ together with an element $i \notin \mathbb{F}_7$ for which $i^2 = -1$.

Part i. Show that the map $\alpha : R \to R$ given by $x \mapsto x^7 + (2+i)x$ is an endomorphism of R as an $\mathbb{F}_7$ module.

Proof. We wish to show α is linear with respect to the field $\mathbb{F}_7$. So, let $x, y \in R$. We can compute $\alpha(x+y)$ as

$$\begin{aligned}
\alpha(x+y) &= (x+y)^7 + (2+i)(x+y) \\
&= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7 \\
&+ (2+i)x + (2+i)y \\
&= x^7 + y^7 + (2+i)x + (2+i)y \\
&= \alpha(x) + \alpha(y)
\end{aligned}$$

where we used the fact that $7 = 0$ in this module. Furthermore, for $r \in \mathbb{F}_7$, we have

$$\begin{aligned}
\alpha(rx) &= (rx)^7 + (2+i)rx \\
&= r^7x^7 + r(2+i)x \\
&= rx^7 + r(2+i)x \\
&= r\alpha(x)
\end{aligned}$$

where we used the fact that $r^7 = r$ modulo 7 for $r \in \mathbb{F}_7$ (verified easily by computation). Thus α is linear over $\mathbb{F}_7$ as desired. $\square$

Part ii. Compute the effect of α on the elements 1 and i of R and on the element $1 \wedge i$ of $\wedge^2 R$.

Proof. We compute directly.

$$\alpha(1) = 1^7 + (2+i)(1) = 3 + i$$

$$\begin{aligned} \alpha(i) &= (i)^7 + (2+i)(i) \\ &= -i + 2i - 1 = -1 + i \end{aligned}$$

Furthermore, we can calculate $\alpha(1 \wedge i)$ as $\alpha(1) \wedge \alpha(i)$ by

$$\begin{aligned} \alpha(1) \wedge \alpha(i) &= (3+i) \wedge (-1+i) \\ &= (3+i) \wedge (-1) + (3+i) \wedge (i) \\ &= 3 \wedge (-1) + i \wedge (-1) + 3 \wedge i + i \wedge i \\ &= -3(1 \wedge 1) - i \wedge 1 + 3(1 \wedge i) + i \wedge i \\ &= 0 + 1 \wedge i + 3(1 \wedge i) + 0 \\ &= 4(1 \wedge i) \end{aligned}$$

□

Part iii. Suppose R has order 49. Deduce the value of $\det(\alpha)$.

Proof. Note first that if R has order 49, it must be that $R = \text{span}(1, i)$. This is true, since we know that R must contain all elements of the form $a + bi$ for a and b in $\mathbb{F}_7$, since R is an $\mathbb{F}_7$ module with i not in $\mathbb{F}_7$. However, there are 49 such elements. Thus, R must be equal to the set of elements of the form $a + bi$.

Clearly, then, R is two-dimensional. Thus, the determinant $\det(\alpha)$ can be calculated as the scalar k such that

$$\alpha(a \wedge b) = ka \wedge b$$

But we already calculated that $\alpha(1 \wedge i) = 4(1 \wedge i)$, and so $\det(\alpha) = 4$. $\square$