Problem Set 5

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Problem 1

Prove the linearity of the general Lebesgue integral for complex-valued functions.

Proof. To begin with, we prove a sequence of lemmas concerning the linearity of easier functions.

Lemma 1 (Linearity of Positive Real Functions)**.** For f, g positive, real-valued measurable functions from a measure space (Ω, μ), the integral is linear with respect to pointwise addition and positive scalar multiplication. That is, for $\alpha, \beta \geq 0$,

$$\int_\Omega \alpha f + \beta g d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu$$

Proof. This has been proven in the notes, and will not be replicated here. $\square$

Lemma 2. For f a positive real-valued measurable function from a measure space (Ω, μ), the integral respects general scalar multiplication. That is, for $\alpha \in \mathbb{R}$,

$$\int_\Omega \alpha f d\mu = \alpha \int_\Omega f d\mu$$

Proof. This lemma has already been proven for $\alpha \geq 0$. So, suppose $\alpha < 0$. In particular, $-\alpha > 0$, and thus by straightforward application of the definition of the Lebesgue integral along with lemma 1, we have

$$\begin{aligned}
\int_\Omega \alpha f d\mu &= \int_\Omega -(-\alpha f) d\mu \\
&= -\int_\Omega -\alpha f d\mu \\
&= -(-\alpha) \int_\Omega f d\mu \\
&= \alpha \int_\Omega f d\mu
\end{aligned}$$

which is the desired result. $\square$

Lemma 3. For f_1, f_2, g_1, g_2 positive real-valued measurable functions from a measure space (Ω, μ) such that $f_1 - f_2 = g_1 - g_2$,

$$\int_\Omega f_1 d\mu - \int_\Omega f_2 d\mu = \int_\Omega g_1 d\mu - \int_\Omega g_2 d\mu$$

Proof. This follows immediately from integrating $f_1 - f_2$ and $g_1 - g_2$, and applying lemma 2 and lemma 1. $\square$

We are now ready to begin the proof of the linearity of the Lebesgue integral.

Let f and g be complex-valued measurable functions from a measure space (Ω, μ), and let $\alpha, \beta \in \mathbb{C}$. We wish to compute the integral

$$\int_\Omega \alpha f + \beta g d\mu$$

Now, letting $f = u_f + iv_f$, and $g = u_g + iv_g$, we have that

$$\begin{aligned}\int_\Omega \alpha f + \beta g d\mu &= \int_\Omega (\Re(\alpha) + i\Im(\alpha))(u_f + iv_f) + (\Re(\beta) + i\Im(\beta))(u_g + iv_g) d\mu \\ &= \int_\Omega (\Re(\alpha)u_f - \Im(\alpha)v_f + \Re(\beta)u_g - \Im(\beta)v_g) + i(\Re(\alpha)v_f + \Im(\alpha)u_f + \Re(\beta)v_g + \Im(\beta)u_g) d\mu\end{aligned}$$

$\square$