Midterm

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Problem 1

Define carefully what it means for a map $f : X \to Y$ to be transverse to a submanifold Z of Y.

Suppose that X and Z are smooth and transverse submanifolds of Y. Prove that if $y \in X \cap Z$ then

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z)$$

Proof. We begin by defining transversality. Suppose $f : X \to Y$ is a smooth map, and $Z \subset Y$ a submanifold of Y. We say that f is transverse to Z if

$$T_{f(x)}(Y) = T_f(x)(Z) + df_x(T_x(X))$$

for all $x \in f^{-1}(Z)$. That is, the tangent space at $f(x)$ in Y is spanned by the tangent space of Z and the push-forward of the tangent space of X.

Now, suppose X and Z are smooth and transverse submanifolds of Y. That is, $T_y(Y) = T_y(X)+T_y(Z)$ for all $y \in X\cap Z$. Suppose first that $v \in T_y(X\cap Z)$. In particular, this means there is a curve $\gamma_v : I \to X\cap Z$ with $\gamma_v(0) = y$ and $\gamma_v'(0) = v$. Clearly, γ_v is also a curve in X and in Z, and so $v \in T_y(X)$ and $v \in T_y(Z)$ as desired. Thus, $T_y(X\cap Z) \subset T_y(X)\cap T_y(Z)$. Call this inclusion Φ. Clearly, Φ is injective. Thus, all we need to show is that $\dim(T_y(X)\cap T_y(Z)) = \dim(T_y(X\cap Z))$ to establish equality.

So, let (U, ϕ) be a slice chart of Z at y. That is, U is a neighborhood of Y, and $\phi : U \to \mathbb{R}^n$ is a coordinate chart such that $\phi(Z) \subset \mathbb{R}^k \times \{0\}^{n-k}$. In particular, we consider the augmented "height" function $\psi : U \to \mathbb{R}^{n-k}$ for which $\psi(Z) = \{0\}$. Thus, $Z = \psi^{-1}(\{0\})$. Let i be the inclusion of X into Y, and observe that $X \cap Z = (\psi \circ i)^{-1}(\{0\})$. We will show that $\{0\}$ is a regular value for $\psi \circ i$.

To that end, we wish to show that $d(\psi \circ i)_y$ is surjective. So, let $v \in \mathbb{R}^{n-k}$. Now, since ψ is part of a coordinate chart, $d\psi_y$ is surjective, and its kernel is $T_y(Z)$. Write a generic element of the fiber of v as $w + v_z$ for $v_z \in T_y(Z)$. Since this is in Y, and X and Z are transverse, $w + v_z = v_x + v_z'$ for some $v_x \in T_y(X)$ and $v_z' \in T_y(Z)$. Absorbing v_z' into v_z, we see that $w + v_z = v_x$. So, thinking of v_x as an element of $T_y(X)$, we see that

$$d(\psi \circ i)_y(v_x) = d\psi_{i(y)} \circ di_y(v_x) = d\psi_y(v_x) = v$$

and so $d(\psi \circ i)$ is surjective as desired.

Thus, the codimension of $X \cap Z$ in X is $n-k$, which is the codimension of Z in Y. That is,

$$\dim(T_y(X)) - \dim(T_y(X \cap Z)) = \dim(T_y(Y)) - \dim(T_y(Z))$$

or

$$\dim(T_y(X \cap Z)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim(T_y(Y))$$

However, since $T_y(Y) = T_y(X) + T_y(Z)$, we know that

$$\dim(T_y(Y)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim(T_y(X) \cap T_y(Z))$$

or

$$\dim(T_y(X) \cap T_y(Z)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim(T_y(Y))$$

and so

$$\dim(T_y(X \cap Z)) = \dim(T_y(X) \cap T_y(Z))$$

and since the inclusion $\Phi : T_y(X \cap Z) \to T_y(X) \cap T_y(Z)$ is injective, the spaces are equal, as desired. $\square$

Problem 2

Suppose $f : X \to Y$ is a smooth map between compact manifolds of the same dimension. Suppose $y \in Y$ is a regular value of f.

Part i

Prove that $f^{-1}(\{y\})$ is a finite set.

Proof. Since y is a regular value, we know that $f^{-1}(\{y\})$ is a submanifold of X with dimension

$$\dim(f^{-1}(\{y\}) = \dim(X) - \dim(Y) = 0$$

Since the only manifolds of dimension zero are countable discrete sets, $f^{-1}(\{y\})$ is an (at most) countable collection of points with the discrete topology. Since Y is compact, this automatically implies that $f^{-1}(\{y\})$ is finite. This follows from the fact that every infinite set in a compact space has an accumulation point, and discrete sets have no accumulation points. $\square$

Part ii

Problem 3

Show that the set of rank 1 matrices in $M(2,\mathbb{R})$ is a 3-dimensional submanifold of $M(2,\mathbb{R})$.

Proof. A 2×2 rank-1 matrix is a nonzero matrix with nontrivial kernel. This set is exactly specified as the set of all nonzero 2×2 matrices with determinant zero. That is, letting R denote the set of rank-1 matrices,

$$R = \det^{-1}(0) \setminus \{0\}$$

In particular, since $M(2,\mathbb{R}) \setminus \{0\}$ is an open subset of $M(2,\mathbb{R})$, it is a manifold, and so we only need to consider $\det^{-1}(\{0\})$ in $M(2,\mathbb{R}) \setminus \{0\}$. To show this is a manifold, we will show it is a submanifold of $M(2,\mathbb{R}) \setminus \{0\}$ by showing 0 is a regular value of det.

To show this, we need to show that for every nonzero matrix $A \in \det^{-1}(\{0\})$, $d(\det)_A$ is surjective. Since the codomain of det is $\mathbb{R}$, it suffices to show that there is at least one vector in $T_A(R(2,\mathbb{R}) \setminus \{0\})$ which does not map to zero.

Observe first that A is always of the form

$$A = \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix}$$

for real numbers a, b, λ such that they are not all identically zero (I suppose this is up to a similarity transformation, but the determinant is invariant under similarity transformations, so we won't worry about it).

Let $\gamma(t) = A + tI$ so that $\gamma(0) = A$ and $\gamma'(0) = I$. We will show that $d(\det)_A(I) \neq 0$. We calculate

$$\begin{aligned}
d(\det)_A(I) &= \partial_t(\det(\gamma(t)))|_0 \\
&= \partial_t(\det(A + tI))|_0 \\
&= \partial_t \left(\det \left(\begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) |_0 \\
&= \partial_t \left((a + t)(\lambda b + t) - \lambda ab \right) |_0 \\
&= \partial_t \left(t^2 + at + \lambda bt + \lambda ab - \lambda ab \right) |_0 \\
&= a + \lambda b
\end{aligned}$$

Thus for all $A = \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix}$ with $a \neq -\lambda b$, $d(\det)_A(I) \neq 0$.

Defining $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and repeating the calculation for $d(\det)_A(B)$, we see that $d(\det)_A(B) = a - \lambda b$ which is nonzero for $a \neq \lambda b$.

Finally, defining $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we calculate $d(\det)_A(C) = -b - \lambda a$, which is nonzero when $b \neq -\lambda a$. This exhausts all possible forms for A, and so the determinant is surjective at each point $A \in \det^{-1}(\{0\})$, as desired.

In particular, this means that the codimension of R is 1, making it a 3 dimensional submanifold of $M(2,\mathbb{R})$ as desired. $\square$