Problem Set 4

Daniel Halmrast

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Problem 1

For $E \subset \Omega$ measurable, prove the implication

$$\int_E f d\mu = 0 \; \forall f \geq 0 \implies \mu(E) = 0$$

Proof. This follows immediately by letting $f = \chi_E$, and observing that

$$\begin{aligned}
\int_E \chi_E(x) d\mu(x) &= \int_\Omega \chi_E(x)\chi_E(x) d\mu(x) \\
&= \int_\Omega \chi_E(x) d\mu(x) \\
&= \mu(E)
\end{aligned}$$

Which is zero by the hypothesis. Thus, $\mu(E) = 0$ as desired. $\square$

Problem 2

For $f \geq 0$ measurable on Ω with $\mu(\Omega) > 0$, show that

$$\int_\Omega f(x)d\mu(x) = 0 \implies [f] = 0$$

(i.e. f is zero μ-almost everywhere).

Proof. Consider the equivalence class $[f]$ in $L^1(\Omega, \mu)$. In particular, since $\int_\Omega |f|d\mu = ||f||_1 = 0$, we must have that $[f] = 0$, which means f agrees with the 0 function μ-almost everywhere. It follows immediately, then, that f is zero μ-almost everywhere. $\square$

Problem 3

Use Fatou's lemma to show that for a sequence $\{f_n\}$ of positive measurable functions, the inequality

$$\int_\Omega \liminf f_n d\mu \leq \liminf \int_\Omega f_n d\mu$$

Proof. We note first that the inequality is vacuously true if $\liminf \int_\Omega f_n d\mu = \infty$.

So, assume that $\liminf \int_\Omega f_n d\mu = M$ for some positive number M. Then, consider the family of subsequences

$$\{f_{n_i}\}_\epsilon = \{f_n \mid \int_\Omega f_n d\mu < M + \epsilon\}$$

Now, for any ϵ, this defines an infinite subsequence, since if the integrals of the sequence were not frequently below $M+\epsilon$, then $M+\epsilon$ would be an eventual lower bound higher than M, which contradicts M being the $\liminf$ of the integrals.

Now, we apply Fatou's lemma by observing that for each f_{n_i} we have that

$$\int_\Omega f_{n_i} d\mu < M + \epsilon$$

which gives us the upper bound

$$\int_\Omega \liminf f_{n_i} d\mu \leq M + \epsilon$$

Now, a basic property of the $\liminf$ is that for a sequence x_n with a subsequence x_{n_i},

$$\liminf x_n \leq \liminf x_{n_i}$$

Thus, we also have that

$$\int_\Omega \liminf f_n d\mu \leq \int_\Omega \liminf f_{n_i} d\mu \leq M + \epsilon$$

However, since this is true for all $\epsilon > 0$, it must be that

$$\int_\Omega \liminf f_n d\mu \leq \int_\Omega \liminf f_{n_i} d\mu \leq M$$

And by the definition of M, we have the desired inequality

$$\int_\Omega \liminf f_n d\mu \leq M = \liminf \int_\Omega f_n d\mu$$

$\square$

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