Homework 1

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Problem A

Let V be the real vector space $\mathrm{Set}(\mathbb{R},\mathbb{R})$ of all functions from $\mathbb{R}$ to itself, and let $K = \{f \in V \mid \mathrm{im}(f) \subset [0,1]\}$. Prove that K is convex and find all extreme points and finite-dimensional faces.

Proof. To start with, we show that K is convex. Let $f,g \in K$ be arbitrary. We will show that the function $h = \lambda f + (1-\lambda)g$ is in K for all $\lambda \in [0,1]$. This is clear, however, since for all $x \in \mathbb{R}$,

$$\begin{aligned} h(x) &= \lambda f(x) + (1-\lambda)g(x) \\ &\leq \lambda(1) + (1-\lambda)(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} h(x) &= \lambda f(x) + (1-\lambda)g(x) \\ &\geq \lambda(0) + (1-\lambda)(0) \\ &= 0 \end{aligned}$$

and so $\mathrm{im}(h) \subset [0,1]$ as desired.

Next, we wish to find the finite dimensional faces of K. I assert that the finite dimensional faces of K are defined as follows. First, partition $\mathbb{R}$ into three sets A, N, P such that $\|A\| < \infty$. Then, define a face

$$F = \{f \in K \mid f^{-1}(\{0\}) \supset N, f^{-1}(\{1\}) \supset P\}$$

To see that this is a face, we have to check that it is convex, and that it contains its linear interpolations.

So, let $f,g \in F$, and let $\lambda \in [0,1]$. We need to show that

$$h(x) = \lambda f(x) + (1-\lambda)g(x)$$

is in F. Clearly $h \in K$ as a convex linear combination of elements of K, so we only need to examine two cases: $x \in N$ and $x \in P$.

If $x \in N$, then

$$h(x) = \lambda(0) + (1-\lambda)(0) = 0$$

and so $h^{-1}(\{0\}) \supset N$.

If $x \in P$, then

$$h(x) = \lambda(1)_(1-\lambda)(1) = 1$$

and so $h^{-1}(\{1\}) \supset P$.

Finally, we show that F contains all its linear interpolations. That is, for any $h \in F$, if there exists $f, g \in K$ and $t \in (0,1)$ with $h = tf + (1-t)g$, then $f, g \in F$ as well. So, suppose $h \in F$ and f, g and t are as described. We examine again two cases.

If $x \in N$, then $h(x) = 0$ and so

$$0 = tf(x) + (1-t)g(x)$$

but $t \in (0,1)$ and $f, g \geq 0$, so it must be that $f(x) = g(x) = 0$ as desired. Thus, $f^{-1}(\{0\}) \supset N$ (and similarly for g).

If $x \in P$, then $h(x) = 1$ and so

$$1 = tf(x) + (1-t)g(x)$$

but $t \in (0,1)$ and $f(x), g(x) \leq 1$, so it must be that $f(x) = g(x) = 1$ as desired. Thus $f^{-1}(\{1\}) \supset P$ (and similarly for g).

Thus, we have shown that the set F defined this way is a face. Next, we show it is finite-dimensional. In particular, we show that the dimension of F is $\|A\|$.

Recall that the dimension of a face is defined as the dimension of $\operatorname{span}\{g - f \mid g \in F\}$ for some fixed $f \in F$. So, let $f = \chi_P$. I assert that the span of $\{g - f \mid g \in F\}$ has a basis given by $g_i = \chi_{\{a_i\}}$ for $a_i \in A$.

First, observe that $\{g_i\}$ is clearly a linearly independent set. Next, we observe that any function of the form $g - f$ for $g \in F$ can be written as a finite linear combination of the g_i basis functions. This is clear, since for any $g \in F$, we know that

$$(g-f)(x) = \begin{cases} f(x), & x \in A \\ 0, & x \notin A \end{cases}$$

and so

$$(g-f)(x) = \sum_{a_i \in A} f(a_i) g_i(x)$$

as desired.

I now assert that this describes all the finite-dimensional faces. To see this, let G be a face that cannot be described using the construction above. In particular, there is some $g \in G$ and some subset $E \subset \mathbb{R}$ with $\|E\| = \infty$ and $g(E) \in (0,1)$. Since G is a face, this means that G contains all functions which agree with g outside E.

Now, fix

$$f(x) = \begin{cases} g(x), & x \notin E \\ 0, & x \in E \end{cases}$$

Then, for every $e \in E$, the function $h_e = \chi_{\{e\}}$ is in $\{h - f \mid h \in G\}$. Moreover, the collection $\{h_e\}$ is clearly linearly independent. Thus, we have found an infinite linearly independent subset of $\{h - f \mid h \in G\}$, and so G is infinite dimensional.

Finally, we observe that the extreme points of K are simply the faces defined as above with $A = \emptyset$. That is, the extreme points of K are the functions $f \in K$ with $f(\mathbb{R}) \in \partial[0,1]$. $\square$

Problem B

Do the same, replacing the domain $\mathbb{R}$ with $\mathbb{N}$.

Proof. Note that the construction for part A generalizes to functions from arbitrary sets into $\mathbb{R}$, and so the finite-dimensional faces and extreme points are described in exactly the same way. $\square$

Problem C

Let X be the real vector space $\text{Set}(\mathbb{N}, \mathbb{C})$ and let $E = \{f \in X \mid |f|(\mathbb{N}) \in [0,1]\}$. Prove E is convex, and find all extreme points and finite-dimensional faces.

Proof. Convexity of E follows almost immediately from the fact that the unit disk is convex. That is, for $f, g \in E$ and $\lambda \in [0,1]$, we know that

$$\begin{aligned}
\|\lambda f(x) + (1-\lambda)g(x)\| &\leq \lambda\|f(x)\| + (1-\lambda)\|g(x)\| \\
&\leq \lambda(1) + (1-\lambda)(1) \\
&= 1
\end{aligned}$$

as desired.

Let a face F be defined as follows. Partition $\mathbb{N}$ into two sets A, P with $\|A\| < \infty$. Then, fix a function $g_0 : \mathbb{N} \to \partial\mathbb{D}$ and define

$$F = \{f \in E \mid f(x) = g_0(x), x \in P\}$$

I assert that F is a face. To see this, note that F is clearly convex, since for any $f, g \in F$ and $\lambda \in [0,1]$ the function

$$h(x) = \lambda f(x) + (1-\lambda)g(x)$$

is such that for any $x \in P$

$$\begin{aligned}
h(x) &= \lambda f(x) + (1-\lambda)g(x) \\
&= (\lambda + 1 - \lambda)g_0(x) \\
&= g_0(x)
\end{aligned}$$

as desired.

Next, we show E contains things that linearly interpolate to it. Suppose $h \in F$ with $f, g \in E$ and $t \in (0,1)$ with $h = tf + (1-t)g$. Then, we know that for $x \in P$, $h(x) = g_0(x)$ is in $\partial\mathbb{D}$. In particular, $g_0(x)$ is an extreme point, which forces $f(x) = g(x) = g_0(x)$ as desired.

Thus, F is a face. Next, we observe it is finite-dimensional. To see this, we fix $f \in F$ as

$$f(x) = \begin{cases} g_0(x), & x \in P \\ 0, & x \notin P \end{cases}$$

and note that the functions

$$g_i(x) = \begin{cases} 1, & x = a_i \\ 0 & x \neq a_i \end{cases}$$

for each $a_i \in A$ form the set $\{g_i, ig_i\}$ of linearly independent functions that span the set

$$\text{span}(\{g - f \mid g \in F\})$$

Thus, F has dimension $2\|A\|$ as desired.

By a similar argument to the one used in part A, any face not defined this way must be infinite-dimensional. Finally, we note the extreme points are found by setting $A = \emptyset$. □

Problem D

Let Y be the real vector space $\text{Top}([0,1], \mathbb{R})$ and let $D = \{f \in Y \mid f([0,1]) \subset [0,1]\}$. Prove that D is convex, and find all extreme points.

Proof. We begin by proving that D is convex. So, let $f, g \in D$, and let $\lambda \in [0,1]$. Then, we know from part A that $h(x) = \lambda f(x) + (1-\lambda)g(x)$ is such that $h([0,1]) \subset [0,1]$. Furthermore, since h is a linear combination of continuous functions, it is continuous as well. Thus, D is convex.

I assert that the only extreme points of D are the constant functions $f_1(x) = 1$ and $f_0(x) = 0$. Now, these are clearly extreme points, since 1 and 0 are extreme points of $[0,1]$. That is, if f, g are such that for some $t \in (0,1)$, $f_1(x) = 1 = tf(x) + (1-t)g(x)$, then $f(x) = g(x) = 1$ for all x (and similarly for f_0).

Now, suppose $f \in D$ with f not equal to the constant functions f_0 and f_1. Then, there is some x for which $f(x) \neq 1$ and $f(x) \neq 0$. Furthermore, by continuity there is some interval (a,b) around x for which f is not 1 or 0. Clearly, there exists functions $g, h \in D$ and $f(x) = tg(x) + (1-t)h(x)$, with $g, h \neq f$ (take, for example, functions g and h that agree with f outside (a,b), and vary an equal distance away from f in the positive and negative directions. Perhaps one could take ε such that $f(x) \pm \varepsilon \in (0,1)$ for all $x \in (a,b)$, and take

$$g(x) = f(x) + \varepsilon \sin(\frac{\pi}{b-a}x)$$

and

$$h(x) = f(x) - \varepsilon \sin(\frac{\pi}{b-a}x)$$

for $x \in (a,b)$).

Thus, f is not an extreme point, as desired. $\square$

Problem E

Let U be the real vector space $\mathrm{Top}([0,1],\mathbb{C})$ and let $F = \{f \in U \mid |f|([0,1]) \subset [0,1]\}$. Prove F is convex, and find all extreme points.

Proof. The fact that F is convex follows from the triangle inequality. That is, for any $f, g \in F$ and $\lambda \in [0,1]$ we know that for $h(x) = \lambda f(x) + (1-\lambda)g(x)$ we have

$$\begin{aligned} \|h(x)\| &= \|\lambda f(x) + (1-\lambda)g(x)\| \\ &\leq \lambda\|f(x)\| + (1-\lambda)\|g(x)\| \\ &\leq \lambda(1) + (1-\lambda)(1) \\ &= 1 \end{aligned}$$

as desired. Furthermore, h is indeed continuous since it is a linear combination of continuous functions.

Furthermore, I assert that the extreme points are the functions in F whose image lies entirely in $\partial\mathbb{D}$. To see these are extreme points, observe that if $h \in F$ is such that the image of h lies inside $\partial\mathbb{D}$, then for each $x \in [0,1]$, $h(x)$ is an extreme point of $\mathbb{D}$. Thus, if

$$h(x) = tf(x) + (1-t)g(x)$$

for some $f, g \in F$ and $t \in (0,1)$, then $f(x) = g(x) = h(x)$ (since $h(x)$ is an extreme point). Thus, h is an extreme point of F.

Furthermore, these exhaust the extreme points. To see this, suppose h is such that $h(x_0) \notin \partial\mathbb{D}$ for some x_0. By continuity, we know that $h(x) \notin \partial\mathbb{D}$ for all x in some interval (a,b) around x_0. Thus, by a similar construction to the one in part D, we can find f and g in F which linearly interpolate to h. Thus, h cannot be an extreme point. □