Final

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PROBLEM 1

PART I

Give the definition of degree for a smooth map $f : A \to B$ between closed oriented manifolds of the same dimension.

Show that if $g : B \to C$ is another such map, then

$$\deg(g \circ f) = \deg(f)\deg(g)$$

Proof. We assume here that the manifolds A, B, and C are all connected.

Consider a regular value $y \in B$ of f. The inverse image $f^{-1}(\{y\})$ is a finite set of points (since A is compact, and $f^{-1}(\{y\})$ is of dimension zero). For each point $x \in f^{-1}(\{y\})$, we say the sign of df_x at x (denoted $\text{sgn}(df_x)$) is $+1$ if df_x preserves orientation, and -1 if df_x reverses orientation. Then, the degree of f is defined as the sum

$$\deg(f) = \sum_{x \in f^{-1}(\{y\})} \text{sgn}(df_x)$$

Recall that this definition is well-defined, as it is independent of choice of regular value.

Now, we turn our attention to the composition $g \circ f$. Recall that $y \in C$ is called a regular value of $g \circ f$ if for every $x \in (g \circ f)^{-1}(\{y\})$, the differential $d(g \circ f)_x$ is surjective. By Sard's theorem, the set of critical values for $g \circ f$ has measure zero, as well as the set of critical values for g, in C. Therefore, on any chart (U, ϕ) in C, there must exist a point which is regular for both g and $g \circ f$. To see this, let R denote the set of regular values of $g \circ f$ in U, and R' the set of regular values of g in U. If R and R' were disjoint, then we would have

$$\mu(R) = \mu(U) = \mu(R') \qquad \text{By Sard's Theorem}$$
$$\mu(R \cup R') = \mu(R) + \mu(R') = 2\mu(U) > \mu(U)$$

which is a contradiction. From here on out, let $y \in C$ be a regular value of both $f \circ g$ and g.

Now, we will show that for all $x \in (g \circ f)^{-1}(\{y\})$, $f(x)$ is a regular value for f. so, let x be as specified. This means that the differential $d(g \circ f)_x = dg_{f(x)} \circ df_x$ is surjective. In particular, since the dimensions of $T_x A$ and $T_y C$ are equal, $d(g \circ f)_x$ is an isomorphism. Furthermore, since $T_{f(x)}B$

also has the same dimension, it must be that df_x and $dg_{f(x)}$ are both isomorphisms as well. This follows from the fact that $dg_{f(x)} \circ df_x$ is surjective, so $dg_{f(x)}$ is surjective onto a space of the same dimension, and hence is an isomorphism. Similarly, $dg_{f(x)} \circ df_x$ is injective, so df_x is injective into a space of the same dimension, and is thus an isomorphism. From all this, we conclude that df_x is surjective for all $x \in (g \circ f)^{-1}(\{y\})$. In particular, since $f^{-1}(\{f(x)\}) \subset (g \circ f)^{-1}(\{y\})$, we have that df_x is surjective for all x in the preimage of $f(x)$, and so $f(x)$ is a regular value of f.

Finally, we show that $\deg(g \circ f) = \deg(g)\deg(f)$. Suppose f is surjective. Then, let $z \in C$ be a regular value of $g \circ f$ and g, as before. Now, let $P = (g \circ f)^{-1}(\{y\})$, and partition P into disjoint sets $P_y = \{x \in P : f(x) = y\}$ of points which map to the same point by f. Then, we compute

$$
\begin{aligned}
\deg(g \circ f) &= \sum_{x \in P} \mathrm{sgn}(d(g \circ f)_x) \\
&= \sum_{y \in g^{-1}(\{z\})} \sum_{x \in P_y} \mathrm{sgn}(d(g \circ f)_x) \\
&= \sum_{y \in g^{-1}(\{z\})} \sum_{x \in P_y} \mathrm{sgn}(dg_y \circ df_x) \\
&= \sum_{y \in g^{-1}(\{z\})} \sum_{x \in P_y} \mathrm{sgn}(dg_y)\mathrm{sgn}(df_x) \\
&= \sum_{y \in g^{-1}(\{z\})} \mathrm{sgn}(dg_y) \sum_{x \in P_y} \mathrm{sgn}(df_x) \\
&= \sum_{y \in g^{-1}(\{z\})} \mathrm{sgn}(dg_y)\deg(f) \\
&= \deg(g)\deg(f)
\end{aligned}
$$

as desired. Here, we used the fact that each y is a regular value for f, and $\deg(f)$ is independent of choice of y. $\qquad\square$

Suppose $f : A \to B$ as in part i has degree $\deg(f) = 1$ and that the index $[\pi_1(B) : f_*\pi_1(A)] < \infty]$. Prove that $f_* : \pi_1(A) \to \pi_1(B)$ is surjective.

Proof. Let (C, p) be the covering space associated to the subgroup $H = f_*(\pi_1(A))$ over B. That is, $p_*(\pi_1(C)) = H$. Since H has finite index, (C, p) is a finite-sheeted covering space, and is thus compact. Furthermore, C can be given a manifold structure from B via pulling the differentiable structure back across p, and such a structure will be orientable, since B is orientable.

Since $f_*(\pi_1(A)) = p_*(\pi_1(C)) = H$, the criteria for lifting f to C is satisfied, and so we have that the diagram

$$
\begin{array}{ccc}
 & & C \\
 & \tilde{f} \nearrow & \downarrow p \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

commutes. Thus, $f = p \circ \tilde{f}$.

Now, by construction p preserves orientation, and each point in B is a regular value of p with exactly n points in its preimage, where $n = [\pi_1(B) : H]$. Recall this comes from the fact that the number of sheets (sheetedness?) of (C, P) is exactly the index of $p_*(\pi_1(C)) = H$ in $\pi_1(B)$. Thus, it follows that p has degree n.

By the first part of this problem, we know that

$$1 = \deg(f) = \deg(p \circ \tilde{f}) = \deg(p)\deg(\tilde{f})$$

Since the degree of a map is always an integer, it follows that $\deg(p) = \deg(\tilde{f}) = 1$. Thus,

$$[\pi_1(B) : H] = 1$$

as desired.

$\square$

Prove Euler's identity for homogeneous polynomials

$$\sum_j x_j \partial_j p = mp(x_1, \ldots, x_n)$$

Proof. Since p is homogeneous, we have that

$$p(tx_1, \ldots, tx_n) = t^m p(x_1, \ldots, x_n)$$

Now, if we differentiate the left-hand side with respect to t, we get

$$\partial_t p(tx_1, \ldots, tx_n) = \sum_i \frac{\partial tx_i}{\partial t} \frac{\partial p(tx_1, \ldots, tx_n)}{\partial tx_i}$$

$$= \sum_i x_i \frac{\partial p(tx_1, \ldots, tx_n)}{\partial tx_i}$$

Differentiating the right-hand side yields

$$\partial_t(t^m p(x_1, \ldots, x_n)) = mt^{m-1} p(x_1, \ldots, x_n)$$

and equating the two and setting $t = 1$ yields

$$\sum_i x_i \partial_i p(x_1, \ldots, x_n) = mp(x_1, \ldots, x_n)$$

as desired. $\square$

Show that $SL(n, \mathbb{R})$ is a Lie group.

Proof. We know already that $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$, so all we need to show is that it is a submanifold of $GL(n, \mathbb{R})$. To do so, we will show it is the level set of a regular value of a smooth function. Specifically, we recall that $SL(n, \mathbb{R})$ is defined to be the set of all matrices in $GL(n, \mathbb{R})$ with determinant 1.

First, I claim the determinant map $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ is homogeneous. To see this, recall that similar matrices have the same determinant, and that every matrix is similar to a matrix in Jordan form. So, without loss of generality, we will assume our matrices are in Jordan form. Recall that in Jordan form, the determinant is given by the product of the diagonal elements. Thus, for a matrix A,

$$\det(tA) = \prod_{i=1}^n tA_{ii} = t^n \prod_{i=1}^n A_{ii} = t^n \det(A)$$

and if we think of A as a tuple of numbers, we see that the determinant map is a homogeneous polynomial.

Now, we will show that 1 is a regular value for this polynomial (call it p). For this, we have to show that the differential dp_x at each point $x \in p^{-1}(\{1\})$ is surjective. Since $\mathbb{R}$ is one-dimensional, this amounts to showing that there is some $v \in T_x(GL(n, \mathbb{R}))$ for which $dp_x(v) \neq 0$.

So, let $x \in p^{-1}(\{1\})$. Since $GL(n, \mathbb{R})$ has its differential structure from $\mathbb{R}^{n^2}$, $T_x(GL(n, \mathbb{R}))$ can be identified with $M(n, \mathbb{R})$ the set of all real $n \times n$ matrices. In particular, $x \in T_x(GL(n, \mathbb{R}))$. So, we calculate

$$dp_x(x) = x^a (dp_x)_a = x^a \partial_a p|_x = mp(x) = m$$

by using Euler's identity from part i, and the fact that $p(x) = 1$. Thus, $dp_x(x)$ is nontrivial, and so dp_x is surjective for all $x \in p^{-1}(\{1\})$. Since $SL(n, \mathbb{R}) = p^{-1}(\{1\})$, we see that $SL(n, \mathbb{R})$ is a submanifold of $GL(n, \mathbb{R})$ and is thus a Lie subgroup of $GL(n, \mathbb{R})$ as desired. $\qquad\square$

Problem 3

Part i

Suppose M is simply-connected. Prove it is orientable.

Proof. Suppose for a contradiction that M were not orientable. Then, M would have an oriented double cover $(\tilde{M}, \tilde{p})$. However, since M is simply-connected, M is its own universal cover, which would imply that M covers $\tilde{M}$ in such a way that the diagram

$$M \xrightarrow{\ p\ } \tilde{M} \xrightarrow{\ \tilde{p}\ } M$$
$$\underbrace{\qquad\qquad}_{id}$$

commutes. However, this cannot be, since $\tilde{p}$ is a two-sheeted cover of M, and is thus not invertible, so $\tilde{p} \circ p$ cannot be the identity. Thus, M is orientable. $\qquad\square$

Part ii

Prove that every Lie group is orientable.

Proof. Let G be a Lie group, with identity e. Let (U, ϕ) be a coordinate chart around e. We define an oriented chart on G as follows. Recall that left multiplication is a smooth diffeomorphism of G to itself, and so for each $g \in G$, we let $(L_g(U), \phi \circ L_g^{-1})$ be a chart around g. We check that these charts are compatible. Suppose g and h are elements so that $L_g(U) \cap L_h(U)$ is nonempty. Then, the transition map for the intersection is given as

$$(\phi \circ L_g^{-1}) \circ (\phi \circ L_h^{-1})^{-1} : U \subset \mathbb{R}^n \to \mathbb{R}^n$$
$$(\phi \circ L_g^{-1}) \circ (\phi \circ L_h^{-1})^{-1} = (\phi \circ L_g^{-1}) \circ (L_h \circ \phi^{-1})$$
$$= \phi \circ L_{g^{-1}h} \circ \phi^{-1}$$

which is smooth, since $L_{g^{-1}h}$ is a smooth diffeomorphism. Finally, I assert that this atlas is oriented. We now calculate the differential of the change of coordinates from $L_g(U)$ to $L_h(U)$ for $x \in L_g(U) \cap L_h(U)$.

Let x_i be the coordinate functions $x_i : U \to \mathbb{R}$ around the identity. Then, $x_i \circ L_g^{-1}$ are the coordinate functions for $L_g(U)$ around g. In particular, ∂_i forms a basis for T_uG for $u \in U$, and $dL_g(\partial_i)$ forms a basis for T_pG for $p \in L_g(U)$. Thus, we calculate in coordinates

$$d((\phi \circ L_h^{-1})^{-1} \circ (\phi \circ L_g^{-1}))_x(v) = d(L_h)_{L_{g^{-1}(x)}} \circ d(L_g^{-1})_x(v^i dL_g \partial_i)$$
$$= v^i dL_h \partial_i = v$$

and thus the change-of-coordinates map is the identity, which has positive determinant. Thus, this atlas is indeed an orientation on G.

$\square$

Part iii

Show that every Lie group has Euler characteristic zero.

Proof. To show this, we will show that the identity function on G has Lefshetz number zero, which implies automatically that G has Euler characteristic zero.

To see this, we will construct a smooth function on G with no fixed points that is homotopic to the identity. Consider a nonzero vector $v \in \mathfrak{g}$. This generates a nowhere-vanishing vector field V by

$$V(p) = d(L_p)_e(v)$$

In particular, this vector field generates a flow

$$\theta : (-\varepsilon, \varepsilon) \times G \to G$$

which has no fixed points in positive time, since the vector field it was generated from did not vanish anywhere. Now, if we vary $t \in (-\varepsilon, \varepsilon)$, we get a homotopy from $\theta(0, \cdot) = id$ to $\theta(\frac{\varepsilon}{2}, \cdot)$, which is the map with no fixed points.

Thus, the Lefschetz number for id is the Lefschetz number for $\theta_{\frac{\varepsilon}{2}}$, which, since $\theta_{\frac{\varepsilon}{2}}$ has no fixed points, is zero. Thus, since the Lefschetz number of the identity is equal to the Euler characteristic of G, it follows that $\chi(G) = 0$ as desired. $\qquad\square$

PROBLEM 4

Give a sketch of the construction of Pontraygin manifolds. State the three main theorems and give an outline of the proof of one of them.

Proof. Let M be a manifold (smooth, compact, orientable), and f a smooth map from M to S^p. Choose a regular value $y \in S^p$ for f, and define the Pontraygin manifold associated to f as the manifold $f^{-1}(\{y\})$. We define a framing on this manifold as follows. Let $\{v_i\}$ be a basis for $T_y S^p$. Then, the pullback of this basis along f defines a trivialization of the normal bundle to $f^{-1}(\{y\})$ (since the tangent bundle to $f^{-1}(\{y\})$ is the nullspace of df, the normal bundle gets mapped isomorphically into $T_y S^p$).

The first main result is that the Pontraygin manifolds, in a sense, do not depend on choice of regular value. That is, if y, z are regular values for f and N_1 is the Pontraygin manifold from f at y, N_2 the Pontraygin manifold from f at z, then N_1 is framed cobordant to N_2.

The second main result is that if two Pontraygin manifolds $f^{-1}(\{y\})$ and $g^{-1}(\{y\})$ are equal, then f is homotopic to g.

I offer a proof sketch of the first result. Let y, z be regular values for f on S^p, and let R_t be a one-parameter family of rotations with $R_0 = id$ and R_1 taking y to z. Now, consider the map

$$H(t, x) = R_t \circ f(x)$$

which has $H(0, x) = f(x)$ and for p such that $f(p) = y$, $H(1, p) = z$. This defines a framed cobordism via $H^{-1}(y)$ from $f^{-1}(y)$ to $(R_1 \circ f)^{-1}(y) = f^{-1}(z)$ as desired. $\square$