TOPOLOGY

Problem Set 2

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PROBLEM 1

Prove that there is an embedding of X into $X \times Y$.

Proof. For this proof, $\{\bullet\}$ will represent the one-point set.

To start with, we will prove the following lemma:

Lemma. *For X any topological space, $X \cong X \times \{\bullet\}$.*

Proof. By the definition of the product space, the projection maps

$$\begin{tikzcd}
& X \times \{\bullet\} \arrow[dl, "\pi_x"] \arrow[dr, "\pi_\bullet"'] & \\
X & & \{\bullet\}
\end{tikzcd}$$

exist and are continuous open maps. Now, all we need to show is that π_x is injective, and it will follow immediately that it is a homeomorphism.

To see this, let $x \in X$ and consider $\pi_x^{-1}(\{x\}) = \{(x, \bullet)\}$. Since the inverse image of a singleton is again a singleton, the function is injective.

Thus, X is homeomorphic to $X \times \{\bullet\}$. $\square$

Now, let $f : \{\bullet\} \to Y$ be a continuous function. Consider the diagram:

$$\begin{tikzcd}
& X & \\
& X \times \{\bullet\} \arrow[u, leftrightarrow, "\cong"'] \arrow[d, dashed, "id \times f"] \arrow[ddl, bend right, "id"'] \arrow[ddr, bend left, "f"] & \\
& X \times Y \arrow[dl, "\pi_x"] \arrow[dr, "\pi_y"'] & \\
X & & Y
\end{tikzcd}$$

where id and f are the obvious extensions $id(x, \bullet) = id(x) = x$ and $f(x, \bullet) = f(\bullet)$. Here, the product map $id \times f$ is continuous by the universal property of products. Now, we just need to show that $id \times f$ is injective with a continuous inverse on its image.

To see that $id \times f$ is injective, consider a point $(id(x), f(\bullet))$ in the image of $id \times f$, and consider its preimage:

$$(id \times f)^{-1}(\{(id(x), f(\bullet))\}) = \{(x, \bullet)\}$$

Since the preimage of any singleton is again a singleton, the function $id \times f$ is injective.

Now, lets consider the diagram

$$\begin{tikzcd}
 & X \times Y \arrow[d, dashed, "\pi_x \times c"] \arrow[ddl, bend right, "\pi_x"'] \arrow[ddr, bend left, "c"] & \\
 & X \times \{\bullet\} \arrow[dl, "\pi_x"] \arrow[dr, "pi_\bullet"'] & \\
X & & \{\bullet\}
\end{tikzcd}$$

where c is unique constant function from Y to the terminal object $\{\bullet\}$.

Here, the dashed arrow $\pi_x \times c$ is continuous by the universal property of products. It is easy to see that $\pi_x \times c|_{(id \times f)(X \times \{\bullet\})}$ is the inverse of $id \times f$ on the image of $id \times f$.

Hence, since the inverse of $id \times f$ is continuous, $id \times f$ is an embedding of $X \cong X \times \{\bullet\}$ into $X \times Y$. $\square$

Problem 2

Prove that every open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Proof. Consider an open interval $(a, b) \subset \mathbb{R}$. It is easy to see that $(a, b) \cong (-1, 1)$, since the operations of scaling and translation are continuous functions with continuous inverses.

Thus, all we need to prove is that $(-1, 1) \cong \mathbb{R}$. To see this, consider the function

$$\tan(\frac{\pi}{2} x)$$

defined on $(-1, 1)$, which is a continuous bijection with continuous inverse. (proofs for the continuity of tan and arctan are easily given by basic analysis arguments, and will not be reproduced here.) $\square$

Problem 3

Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous at exactly one point.

Proof. The function

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = x\chi_{\mathbb{Q}}(x)$$

is continuous only at zero. To see this, we will use the neighborhood definition of continuity. That is, f is continuous at x if for each neighborhood of $f(x)$, its preimage contains a neighborhood of x.

First, we will prove that f is continuous at zero. It suffices to show that each basic open neighborhood of $f(x)$ has a preimage that contains an open neighborhood of x. So, let $(-\varepsilon, \varepsilon)$ be a basic neighborhood of $f(0) = 0$. Then,

$$f^{-1}((-\varepsilon, \varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon, \varepsilon)$$

which contains $(-\varepsilon, \varepsilon)$ an open neighborhood of 0 as desired.

Now, let $x \neq 0$. We will show that f is not continuous at x. If x is irrational, then $f(x) = 0$. Now, choose ε so that $x \notin (-\varepsilon, \varepsilon)$. Then, by the above calculation, we have

$$f^{-1}((-\varepsilon, \varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon, \varepsilon)$$

which does not contain any neighborhood of x.

If x is rational, then $f(x) = x$. Choose ε such that $0 \notin V_\varepsilon(x)$. Then,

$$f^{-1}(V_\varepsilon(x)) = V_\varepsilon(x) \cap \mathbb{Q}$$

which does not contain any open neighborhood of x (This is easily seen by observing that any neighborhood of x must intersect $\mathbb{R} \setminus \mathbb{Q}$, but the inverse image contains only rational points). $\square$

Problem 4

Suppose Y is Hausdorff, and $X \underset{g}{\overset{f}{\rightrightarrows}} Y$