Problem Set 2

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PROBLEM 1

Prove that the push-forward measure is a measure.

Proof. Let $(\mathscr{A}_1, \Omega_1)$ and $(\mathscr{A}_2, \Omega_2)$ be measurable spaces, and let $\mu : \mathscr{A}_1 \to \mathbb{R}^+$ be a measure on $\mathscr{A}_1$. Furthermore, let $f : \Omega_1 \to \Omega_2$ be a measurable function. Then, the *push forward measure* $f_*\mu$ on $\mathscr{A}_2$ is defined as:

$$f_*\mu(E) = \mu(f^{-1}(E))\text{for } E \in \mathscr{A}_2$$

Now, let's verify that such a construction is actually a measure. To do this, we will check the nullity of the empty set with respect to the measure, and the σ-additivity of the measure.

To check the nullity of the empty set, we can directly compute its measure.

$$\begin{aligned} f_*\mu(\emptyset) &= \mu(f^{-1}(\emptyset)) \\ &= \mu(\emptyset) \\ &= 0 \end{aligned}$$

Thus, the measure respects the nullity of the empty set.

To check the σ-additivity of the measure, let $\{E_i\}$ be a countable collection of disjoint measurable subsets of Ω_2 (That is, $E_i \in \mathscr{A}_2 \forall i$). Then, the measure of the union can be calculated directly:

$$\begin{aligned} f_*\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} E_i\right)\right) \\ &= \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(E_i)\right) \\ &= \bigcup_{i=1}^{\infty} \mu\left(f^{-1}(E_i)\right) \\ &= \bigcup_{i=1}^{\infty} f_*\mu(E_i) \end{aligned}$$

In this calculation, we used two important facts about inverse images. First, the inverse image preserves unions, which let us pass the union through the inverse image in line 2. Second, the inverse image preserves intersections, which guaranteed that the collection $\{f^{-1}(E_i)\}$ remained disjoint to establish equality in line 3.

Thus, the push-forward measure satisfies the two measure axioms, and is a measure as desired. $\square$

Problem 2

Calculate, for measurable sets A and B with measure μ, the measures:

$$\mu(A \setminus B)$$
$$\mu(A \Delta B)$$

Proof. First, let's find $\mu(A \setminus B)$. This can be calculated directly using the identity $(A \setminus B) \cup B = A \cup B$ (this identity is easily seen to be true by chasing basic set theory definitions). Thus,

$$\begin{aligned} \mu((A \setminus B) \cup B &= \mu(A \cup B) \\ \mu(A \setminus B) + \mu(B) &= \mu(A \cup B) \\ \mu(A \setminus B) &= \mu(A \cup B) - \mu(B) \end{aligned}$$

Here, we use the fact that $A \setminus B$ and B are disjoint from each other to split the measure on line 2. Note that this identity only holds if $\mu(A \cup B)$ and $\mu(B)$ are not both of infinite measure, so that subtraction can be performed.

Second, let's calculate $\mu(A \Delta B)$. For this, we will use the identity $A \Delta B = (A \setminus B) \cup (B \setminus A)$. It should be observed that the sets $A \setminus B$ and $B \setminus A$ are disjoint from each other. Now, by direct calculation:

$$\begin{aligned} \mu(A \Delta B) &= \mu((A \setminus B) \cup (B \setminus A)) \\ &= \mu(A \setminus B) + \mu(B \setminus A) \\ &= \mu(A \cup B) - \mu(B) + \mu(A \cup B) - \mu(A) \\ &= 2\mu(A \cup B) - \mu(B) - \mu(A) \end{aligned}$$

$\square$

Problem 3

For a measure μ on $\mathscr{A}$, show that the countable union of null sets is a null set.

Proof. Let $\{E_i\}$ be a countable collection of null sets (that is, $\mu(E_i) = 0 \ \forall i$). Then we have the following:

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} E_i\right) &\leq \sum i = 1^{\infty} \mu(E_i) \\ &= \sum_{i=1}^{\infty} 0 \\ &= 0 \end{aligned}$$

Furthermore, since the measure is positive, we have $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq 0$. Thus, the measure of the union is bounded above and below by zero, so it must be equal to zero, as desired. $\square$

Problem 4

For μ a Borel measure on $\mathbb{R}$, with $\mu(\mathbb{R}) < \infty$, define the function

$$f_\mu : \mathbb{R} \to \mathbb{R}$$
$$f_\mu(t) = \mu((-\infty, t]))$$

Part a

Prove that f_μ is nondecreasing.

Proof. Let $t \in \mathbb{R}$, and let Δt be some positive number. Then:

$$\begin{aligned} f_\mu(t + \Delta t) &= \mu((-\infty, t + \Delta t]) \\ &\geq \mu((-\infty, t]) \\ &= f_\mu(t) \end{aligned}$$

as desired. Here we used the fact that $(-\infty, t] \subset (-\infty, t + \Delta t]$ and that, for sets A and B such that $A \subset B$, the inequality

$$\mu(A) \leq \mu(B)$$

holds. □

Part b

Find $f_\mu(a) - f_\mu(b)$.

Proof. Without loss of generality, let $a > b$. (Note that $f_\mu(a) - f_\mu(b) = -(f_\mu(b) - f_\mu(a))$, so in the case $a < b$, the result will be the negative of the case for $a > b$).

With this assumption, we have:

$$\begin{aligned} f_\mu(a) - f_\mu(b) &= \mu((-\infty, a]) - \mu((-\infty, b]) \\ &= \mu((-\infty, a] \cup (-\infty, b]) - \mu((-\infty, b]) \\ &= \mu((-\infty, a] \setminus (-\infty, b]) \\ &= \mu((b, a]) \end{aligned}$$

Here, line 3 was obtained by using the identity for the measure of $A \backslash B$ from problem 2 above. □

Part c

Prove that f_μ is continuous from the right.

Proof. Let $\{x_i\}$ be a monotonically decreasing sequence with limit $x_i \to t$. In particular, since $(-\infty, t] \subset (-\infty, x_i]$ for all i, the monotonicity of the measure guarantees that $f_\mu(t) \leq f_\mu(x_i)$ for all i. Thus, the sequence $\{f_\mu(x_i)\}$ is bounded below by $f_\mu(t)$.

Now, let $\epsilon > 0$ be arbitrary, and choose an N such that for all $n > N$, $x_n < t + \epsilon$. Since $(-\infty, x_n] \subset (-\infty, t + \epsilon]$, the monotonicity of the measure gives us the fact that $f_\mu(x_n) \leq f_\mu(t)$.

Thus, we have that for all $\epsilon > 0$ and for all $n > N$, the inequality

$$f_\mu(t) \leq f_\mu(x_n) \leq f_\mu(t + \epsilon)$$

Thus, $f_\mu(x_i) \to f_\mu(t)$ as desired. □

Part d

Give an example of a μ for which f_μ is not continuous.

Proof. Consider the δ-measure centered at 0 on $\mathbb{R}$. Then for all $\epsilon > 0$, $f_\mu(0-\epsilon) = \mu((-\infty, -\epsilon]) = 0$, but $f_\mu(0) = \mu((-\infty, 0]) = 1$. Clearly, f_μ is not continuous, as desired. □

Part e

Find $\lim_{t\to-\infty} f_\mu(t)$ and $\lim_{t\to\infty} f_\mu(t)$.

Proof. For this proof, we will use the fact that every subsequence of a convergent sequence is also convergent and converges to the same limit. In particular, for a sequence $\{t_i\}$ diverging to $+\infty$ such that $f_\mu(t_i) \to C$ for some C, there exists a monotonically increasing subsequence $\{t_{i_j}\}$ such that $f_\mu(t_{i_j}) \to C$ as well. Without loss of generality, then, we will let t monotonically increase when computing the limits.

First, we compute $\lim_{t\to-\infty} f_\mu(t)$. This can be formulated as

$$\begin{aligned}\lim_{t\to\infty} f_\mu(-t) &= lim_{t\to\infty}\mu((-\infty, -t]) \\ &= \mu\left(\lim_{t\to\infty}(-\infty, -t]\right) \\ &= \mu(\emptyset) \\ &= 0\end{aligned}$$

Here, we used the fact that the measure preserves limits (Notes part 1, section 23) to establish equality in line 2, also observing that the sets $A_t = (-\infty, -t]$ are monotonic ($A_{t+\alpha} \subset A_t$ for positive α). It is also clear that the limit is the empty set, since for each $x \in \mathbb{R}$, there is some $t < x$, and $x \notin A_t$ which implies x is not in the limit.

Now, lets compute $\lim_{t\to\infty} f_\mu(t)$. This will use the same fact about measures (preserving limits).

$$\begin{aligned}\lim_{t\to\infty} f_\mu(t) &= \lim_{t\to\infty}\mu((-\infty, t]) \\ &= \mu(\lim_{t\to\infty}(-\infty, t]) \\ &= \mu(\mathbb{R})\end{aligned}$$

It is easy to see the sequence of sets $A_t = (-\infty, t]$ is a monotonically increasing sequence, so the limit preservation property of measures holds. Furthermore, their limit is easily seen to be all of $\mathbb{R}$, since for each $x \in \mathbb{R}$, there is some $t > x$, and thus $x \in A_t$, so x is also in the limit. □

Problem 5

Show that the "infinity-detecting" function defined as

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite} \\ \infty & \text{else} \end{cases}$$

on a countable set Ω is additive, but not σ-additive.

Proof. To show additivity, let $\{E_i\}_{i=1}^n$ be a finite collection of disjoint sets of Ω. If all E_i are finite, then their union is finite, and the function μ is trivially additive.

$$
\begin{aligned}
\mu\left(\bigcup_{i=1}^n E_i\right) &= 0 \\
&= \sum_{i=1}^n 0 \\
&= \sum_{i=1}^n \mu(E_i)
\end{aligned}
$$

Now suppose there is some E_j infinite. Then,

$$
\begin{aligned}
\mu\left(\bigcup_{i=1}^n E_i\right) &= \infty \\
&= \mu(E_j) + \sum_{i\neq j}^n \mu(E_i) \\
&= \sum_{i=1}^n \mu(E_i)
\end{aligned}
$$

Where here we used the convention that $\infty + C = \infty$ for any C finite or $C = \infty$.

However, this function is not σ-additive. To see this, note that

$$\mu(\Omega) = \infty$$

But

$$\Omega = \bigcup_{x\in\Omega} \{x\}$$

and

$$\sum_{x\in\Omega} \mu(\{x\}) = \sum 0 = 0$$

Thus,

$$
\begin{aligned}
\mu\left(\bigcup_{x\in\Omega} \{x\}\right) &= \infty \\
&\neq 0 \\
&= \sum_{x\in\Omega} \mu(\{x\})
\end{aligned}
$$

and μ is not σ-additive. $\square$

Problem 6

Prove that the conditional probability measure, induced by some measure μ on Ω with $\mu(\Omega) = 1$ and for some Γ, $\mu(\Gamma)$ positive and finite, given by

$$\nu(E) = \frac{\mu(E \cap \Gamma)}{\mu(\Gamma)}$$

is a measure.

Proof. To show this is a measure, we will show it respects the nullity of the empty set, and that it is σ-additive.

To verify the nullity of the empty set, we compute $\nu(\emptyset)$ directly.

$$\begin{aligned}\nu(\emptyset) &= \frac{\mu(\emptyset \cap \Gamma)}{\mu(\Gamma)} \\ &= \frac{\mu(\emptyset)}{\mu(\Gamma)} \\ &= \frac{0}{\mu(\Gamma)} \\ &= 0\end{aligned}$$

Now, let's verify σ-additivity. Let $\{E_i\}$ be a countable collection of disjoint subsets of Ω. Then, we can directly verify that

$$\begin{aligned}\nu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \frac{\mu\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap \Gamma\right)}{\mu(\Gamma)} \\ &= \frac{\mu\left(\left(\bigcup_{i=1}^{\infty} E_i \cap \Gamma\right)\right)}{\mu(\Gamma)} \\ &= \frac{\sum_{i=1}^{\infty} \mu(E_i \cap \Gamma)}{\mu(\Gamma)} \\ &= \sum_{i=1}^{\infty} \nu(E_i)\end{aligned}$$

Thus, ν is σ-additive, and is a measure as desired. □

Problem 7

Let Ω be an uncountable set, and let the σ-algebra $\mathscr{A}$ be the set of all subsets $E \subset \Omega$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ if E^c is countable. Prove that μ is a measure.

Proof. To show this is a measure, we will show it respects the nullity of the empty set, and that it is σ-additive.

By the definition of the measure (along with the convention that $\emptyset$ is a finite set), $\mu(\emptyset) = 0$

Now, let $\{E_i\}$ be a disjoint collection of measurable subsets of Ω. Before proceeding, it is key to note that at most one of these will be uncountable. To see this, suppose both E_i and E_j are uncountable (with countable complement). In particular, $E_i \not\subset (E_j)^c$ and vice versa, since $(E_j)^c$ is countable, but E_i is uncountable. Thus, they cannot be disjoint.

So, suppose none of E_i are uncountable. Then,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= 0 \\ &= \sum_{i=1}^{\infty} 0 \\ &= \sum_{i=1}^{\infty} \mu(E_i)\end{aligned}$$

as desired.

Instead, suppose some E_j is uncountable. Then,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= 1 \\ &= 1 + \sum_{i \neq j}^{\infty} 0 \quad = \mu(E_j) + \sum_{i \neq j}^{\infty} \mu(E_i) \\ &= \sum_{i=1}^{\infty} \mu(E_i)\end{aligned}$$

as desired.

$\square$