Problem Set 3

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Problem 1

Part a

Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ is a surjective smooth submersion.

Proof. Surjectivity follows almost immediately from the definition of the map, since any equivalence class $[z]$ is mapped to by $\pi(z)$. Now, we must show it is a smooth submersion.

We will do this using the Global Rank Theorem (Theorem 4.14), which states that a surjection with constant rank is a smooth submersion, as well as Corollary 4.13, which states that a smooth map from a connected space has constant rank if and only if there exist local charts at each point in the manifold for which the coordinate representation of the smooth map is linear.

That is, if we can show that $\forall z \in \mathbb{C}^{n+1} \setminus \{0\}$, there exist charts around z and $\pi(z)$ for which π has a linear coordinate representation, then π is a smooth submersion.

Now, the coordinate charts on $\mathbb{C}^{n+1}$ are just the standard rectangular coordinates

$$(z^1, \dots, z^{n+1}) \mapsto (\Re(z^1), \Im(z^1), \dots, \Re(z^{n+1}), \Im(z^{n+1}))$$

And for each $i \in \{1, \dots, n+1\}$, we have a local coordinate chart on $\mathbb{C}P^n$ around where z^i is not zero, given as

$$(z^1 : \dots : z^i : \dots : z^{n+1}) \mapsto \frac{1}{z^i}(z^1, \dots, z^{i-1}, z^{i+1}, \dots, z^{n+1})$$

(Here, this coordinate chart actually maps to $\mathbb{C}^n$, which has a global coordinate chart as defined above, and the composition then defines a coordinate chart to $\mathbb{R}^{2n}$.)

So now we consider the coordinate representation of π around a point z_0 for which z_0^i is not zero, which (as complex coordinates) is given as

$$\tilde{\pi}(z^1, \dots, z^i, \dots, z^{n+1}) = \frac{1}{z^i}(z^1, \dots, z^{i-1}, z^{i+1}, \dots, z^{n+1})$$

□

Part b

Show that $\mathbb{C}P^1 \cong S^2$.

Proof. $\square$

Problem 2

For M a nonempty smooth compact manifold, show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Proof. $\square$

Problem 3

Use the covering map $\varepsilon^2 : \mathbb{R}^2 \to \mathbb{T}^2$ to show that the immersion $\chi : \mathbb{R}^2 \to \mathbb{R}^3$ descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that χ passes to the quotient to define a smooth map $\tilde{\chi} : \mathbb{T}^2 \to \mathbb{R}^3$, then show that $\tilde{\chi}$ is a smooth embedding whose image is the given surface of revolution.

Proof. $\square$