Midterm

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Problem 1

Part a

Use the standard charts on S^n to calculate the matrix representation of $di : T_pS^n \to T_p\mathbb{R}^{n+1}$, and show that di is injective, and thus i is an embedding.

Proof. For this calculation, we will use the chart given by hemisphere projection. That is, the domains for the charts will be the open sets $U_i^{\pm} = \{(x^1 \dots, x^{n+1}) \mid x^i > 0 (x^i < 0 \text{resp.})\}$ with maps

$$\phi_i^{\pm}(x^1, \dots, x^{n+1}) = (x^1, \dots, \hat{x^i}, \dots, x^{n+1})$$

Where a hat denotes omission of the variable.

Now, suppose $p \in U_i^+$ (without loss of generality, we take the positive hemisphere of x^i, but the argument can be repeated exactly with the negative hemisphere as well.) and let the coordinate representation of p be

$$\phi(p) = (x^1, \dots, \hat{x^i}, \dots, x^{n+1})$$

Then, the inclusion map looks like

$$\begin{aligned} i \circ \phi^{-1}((x^1, \dots, \hat{x^i}, \dots, x^{n+1}) &= (y^1, \dots, y^{n+1}) \\ &= (x^1, \dots, x^{i-1}, \sqrt{1 - x^a x_a}, \dots, x^{n+1}) \end{aligned}$$

and the Jacobian di can be calculated directly using the identity $di_j^k = \partial_j(y^k)$. Which gives the matrix (for $j = 1, \dots, i-1, i+1, \dots, n+1$ and $k = 1, \dots, n$)

$$\partial_j(y^k) = \delta_j^k - \frac{1}{\sqrt{1 - x^a x_a}} \delta^{ik} x_j$$

Which is clearly injective, since the rows $k \neq i$ are the basis covectors for $\mathbb{R}^n$, and thus $\partial_j(y^k)$ has rank n as desired.

Furthermore, since S^n is compact, and the inclusion is injective, it follows that i defines an embedding of S^n into $\mathbb{R}^{n+1}$. □

Part B

Show that T_pS^n, when identified with $di(T_pS^n)$ is the subspace of $\mathbb{R}^{n+1}$ consisting of all vectors perpendicular to the radial vector to p.

Proof. This follows by direct calculation. To see this, let $v \in T_pS^n$. Then,

$$\begin{aligned} di(v) = \partial_j y^k v^j &= \delta_j^k v^j - \frac{1}{\sqrt{1 - x^a x_a}} \delta^{ik} x_j v^j \\ &= v^k - \frac{x^a v_a}{\sqrt{1 - x^a x_a}} \delta^{ik} \end{aligned}$$

where it is assumed that $x_a v^a$ does not sum over the i^{th} component of v.

Recalling earlier that the embedding sends

$$p = (x^1, \ldots, \hat{x^i}, \ldots, x^{n+1})$$

to

$$(y^1, \ldots, y^{n+1}) = (x^1, \ldots, x^{i-1}, \sqrt{1 - x^a x_a}, \ldots, x^{n+1})$$

we can compute the inner product $g_{jk} v^j y^k$ directly.

$$\begin{aligned} g_{jk} v^j y^k &= v^k x_k + \delta_{ij}^{ik} v^j y_k \\ &= v^k x_k + v^i y_i \\ &= v^k x_k - \frac{v^a x_a}{\sqrt{1 - x^a x_a}} \sqrt{1 - x^a x_a} \\ &= v^k x_k - v^a x_a \\ &= 0 \end{aligned}$$

Thus, $di(v)$ is perpendicular to p, as desired. $\square$

Part C

For F a smooth map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{m+1}$ such that $F(S^n) \subset S^m$, show that $d(F|_{S^n}) = dF|_{T_pS^n}$.

Proof. To begin with, let $v \in T_pS^n$. In particular, there is a curve γ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then,

$$\begin{aligned} d(F|_{S^n})(\gamma'(0)) &= \partial_t|_0 F|_{S^n}(\gamma(t)) \\ &= \partial_t|_0 F(\gamma(t)) \\ &= dF(\gamma'(0)) \\ &= dF(\gamma'(0))|_{T_pS^n} \end{aligned}$$

$\square$

Problem 2

Show that the tangent bundle TM is always orientable.

Proof. For this problem, we will use that fact that a manifold is orientable if there exist charts such that the coordinate transition maps have a Jacobian of positive determinant.

Before proceeding further, we prove the following lemma:

Lemma. *Suppose M and N are smooth manifolds. In particular, their product $M \times N$ is a smooth manifold. Furthermore, for pairs of coordinates x_m, y_m on $U_m \subset M$ and x_n, y_n on $U_n \subset N$, the product coordinates $x = (x_m, x_n)$ and $y = (y_m, y_n)$ are smooth coordinates on $U_m \times U_n$, and the Jacobian $J(x \to y)$ is given componentwise. That is,*

$$J(x \to y) = (J(x_m \to y_m), J(x_n \to y_n))$$

Proof. That $M \times N$ is a smooth manifold follows almost immediately by taking products of coordinate charts on M and N. Now, we have the following diagram:

$$\begin{tikzcd}
 & & M \times N & & \\
 & & U_m \times U_n \arrow[u, hook] \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, dashed, shift right, "x"'] \arrow[dd, dashed, shift left, "y"] & & \\
M & U_m \arrow[l, hook'] \arrow[d, shift right, "x_m"'] \arrow[d, shift left, "y_m"] & & U_n \arrow[r, hook] \arrow[d, shift right, "x_n"'] \arrow[d, shift left, "y_n"] & N \\
 & \mathbb{R}^m \arrow[dr, hook] & \arrow[d, dashed, shift right] \arrow[d, dashed, shift left] & \mathbb{R}^n \arrow[dl, hook'] & \\
 & & \mathbb{R}^{m+n} & &
\end{tikzcd}$$

which implies that the induced coordinates x and y are smooth. Now, let's expand the lower half of the commutative diagram to get the transition maps $x_m \to y_m$ and $x_n \to y_n$:

$$\begin{tikzcd}
\mathbb{R}^m \arrow[dd, "x_m \to y_m"'] \arrow[dr, hook] & & \mathbb{R}^n \arrow[dd, "x_n \to y_n"] \arrow[dl, hook'] \\
 & \mathbb{R}^{m+n} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, dashed, "x \to y"] & \\
\mathbb{R}^m \arrow[dr, hook] & & \mathbb{R}^n \arrow[dl, hook'] \\
 & \mathbb{R}^{m+n} &
\end{tikzcd}$$

Differentiating this diagram (applying the differential functor) yields:

$$\begin{tikzcd}
\mathbb{R}^m \arrow[dd, "J(x_m \to y_m)"'] \arrow[dr, hook] & & \mathbb{R}^n \arrow[dd, "J(x_n \to y_n)"] \arrow[dl, hook'] \\
 & \mathbb{R}^{m+n} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, dashed, "J(x \to y)"] & \\
\mathbb{R}^m \arrow[dr, hook] & & \mathbb{R}^n \arrow[dl, hook'] \\
 & \mathbb{R}^{m+n} &
\end{tikzcd}$$

Thus, $J(x \to y) = (J(x_m \to y_m), J(x_n \to y_n))$ as desired. $\square$

We are now ready to prove the general result.

So, let M be a smooth manifold with tangent bundle TM. Furthermore, for a point p, suppose there are two coordinate charts x^i and y^i on a neighborhood of p. We wish to calculate the Jacobian of the induced coordinate transformations on the tangent bundle.

To do so, we first appeal to the fact that TM is locally trivializable. That is, on some neighborhood U containing p, $\pi^{-1}(U) \cong M \times T_pM$, where π is the canonical projection of TM onto M. In particular, this means that in $\pi^{-1}(U)$, we have the coordinate charts $x^i \times dx^i$ and $y^i \times dy^i$.

Thus, the transition map is just $(x \to y, dx \to dy)$, where $x \to y$ is the transition map from the x coordinate system to the y coordinate system, and $dx \to dy$ is the transition map from the $\partial_x|_p$ coordinate system to the $\partial_y|_p$ coordinate system.

Recall that $dx \to dy$ is simply the Jacobian of the original coordinate transform. That is, $dx \to dy = J(x \to y)$. Now, from the above lemma,

$$J(x \to y, dx \to dy) = J(x \to y, J(x \to y)) = (J(x \to y), J(J(x \to y)))$$

It should be clear that $J^2 = J$, since the Jacobian of a transformation is linear. Thus, we have that

$$J(x \to y, dx \to dy) = (J(x \to y), J(x \to y))$$

To calculate the determinant of this, we appeal to the fact that the determinant of a linear transformation of the form (A, B) is the product of the determinants of A and B. Thus,

$$\begin{aligned} \det J(x \to y, dx \to dy) &= \det(J(x \to y)) \det(J(x \to y)) \\ &= (\det(J(x \to y)))^2 \end{aligned}$$

Which is always positive.

Since this can be done at any point p in the manifold, we have an atlas for TM where the determinant of the coordinate transforms is always positive, and thus TM is orientable. $\square$