Problem Set 5

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Problem 1

Let M be the open submanifold of $\mathbb{R}^2$ with both coordinates positive, and define $F : M \to M$ as $F(x, y) = (xy, \frac{y}{x})$. Show that F is a diffeomorphism, and compute dFX and dFY for

$$X = x\partial_x + y\partial_y$$
$$Y = y\partial_x$$

Proof. To begin with, we observe that so long as $x \neq 0$, F is in fact smooth. Furthermore, it has an inverse

$$F^{-1}(x, y) = (\sqrt{\frac{x}{y}}, \sqrt{xy})$$

which is also smooth on M, and defined for all of M.

Furthermore, we can calculate its Jacobian:

$$J(F) = dF = \begin{bmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{bmatrix}$$

which expresses dF in the coordinates ∂_x, ∂_y at every point.

Now, we calculate $dF(X)$:

$$\begin{aligned} dF(X) &= dF(x\partial_x + y\partial_y) \\ &= xdF(\partial_x) + ydF(\partial_y) \\ &= x(y\partial_x - \frac{y}{x^2}\partial_y) + y(x\partial_x + \frac{1}{x}\partial_y) \\ &= xy\partial_x - \frac{y}{x}\partial_y + yx\partial_x + \frac{y}{x}\partial_y \\ &= 2xy\partial_x \end{aligned}$$

and $dF(Y)$:

$$\begin{aligned} dF(Y) &= dF(x\partial_y) \\ &= xdF(\partial_y) \\ &= x(x\partial_x + \frac{1}{x}\partial_y) \\ &= x^2\partial_x + \partial_y \end{aligned}$$

□

Problem 2

Let M be a smooth manifold, $S \subseteq M$ an embedded submanifold. Given $X \in \mathcal{X}(S)$, show that there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$. Show that every such vector field extends to all of M if and only if S is properly embedded.

Proof. Recall that a vector field $X \in (X)(S)$ is a linear derivation of the algebra $C^\infty(M)$ over $\mathbb{R}$. That is, X is a linear map $X : C^\infty(M) \to C^\infty(M)$ such that

$$X(fg) = X(f)g + fX(g)$$

Now, from an earlier assignment, we know that for $S \subseteq M$ an embedded submanifold of a manifold M, we have the existence of extensions of C^∞ functions on S to C^∞ functions on the neighborhood $U \subseteq M$ of S. That is, the restriction function $r : C^\infty(U) \to C^\infty(S)$ has a section $e : C^\infty(S) \to C^\infty(U)$ such that $r \circ e = id$. Thus, we have the following diagram:

$$\begin{tikzcd} C^\infty(S) \arrow[r, "X"] & C^\infty(S) \arrow[d, "e"] \\ C^\infty(U) \arrow[u, "r"] & C^\infty(U) \end{tikzcd}$$

Now, we define Y to be the linear map $Y : C^\infty(U) \to C^\infty(U)$ that makes the diagram commute. That is:

$$\begin{tikzcd} C^\infty(S) \arrow[r, "X"] & C^\infty(S) \arrow[d, "e"] \\ C^\infty(U) \arrow[u, "r"] \arrow[r, dashed, "Y"] & C^\infty(U) \end{tikzcd}$$

We note that such a Y is not unique, since the extension e is not uniquely defined. In fact, all extensions differ by an element of $C^\infty(U)/C^\infty(S)$.

Y is clearly linear, since it is the composition of linear arrows, so all that we need to show is that Y is a derivation. First, we observer that since e is a section of r, it follows that $X \circ r = r \circ Y$. That is, the diagram

$$\begin{tikzcd} C^\infty(S) \arrow[r, "X"] & C^\infty(S) \arrow[d, shift left, "e"] \\ C^\infty(U) \arrow[u, "r"] \arrow[r, dashed, "Y"] & C^\infty(U) \arrow[u, shift left, "r"] \end{tikzcd}$$

commutes. We note also that the restriction map r is multiplicative. That is, $r(fg) = r(f)r(g)$.

Let $f, g \in C^\infty(U)$. We calculate

$$\begin{aligned} Y(fg) &= e \circ X \circ r(fg) \\ &= e \circ X(r(f)r(g)) \\ &= e(X(r(f))r(g) + r(f)X(r(g))) \\ &= e(r(Y(f))r(g) + r(f)r(Y(g)) \\ &= (e \circ r)(Y(f)g + fY(g)) \end{aligned}$$

so if $e \circ r$ is the identity on $Y(f)g + fY(g)$, then Y is a derivation. However, we recall that e is only unique up to a factor of $C^\infty(U)/C^\infty(S)$. So, for each $f \in C^\infty(U)$, we define e_f... $\square$

Problem 3

Problem 4

Show that $\mathbb{R}^3$ is a Lie algebra with the cross product.

Proof. The cross product is, by definition, bilinear, so it suffices to check that the cross product satisfies the Jacobi identity.

We proceed to calculate the Jacobi identity directly. Now, we know that

$$((A \times B) \times C)^i = \epsilon^i_{jk}\epsilon^j_{mn} A^m B^n C^k$$

where ϵ^i_{jk} is the Levi-Civita symbol. So

$$\begin{aligned}((A \times B) \times C + (B \times C) \times A + (C \times A) \times B)^i &= \epsilon^i_{jk}\epsilon^j_{mn}(A^k B^m C^n + B^k C^m A^n + C^k A^m B^n) \\ &= T^i_{kmn} V^{kmn}\end{aligned}$$

where $T^i_{kmn} = \epsilon^i_{jk}\epsilon^j_{mn}$ and $V^{kmn} = A^k B^m C^n + B^k C^m A^n + C^k A^m B^n$.

Now, $\epsilon^i_{jk}\epsilon^j_{mn} = T^i_{kmn}$is a symbol of rank $(1,3)$ whose components can be calculated directly. It is easy to see that this symbol is antisymmetric in the first and last two components. Furthermore, the only nonzero terms (up to antisymmetry) is $T^i_{kki} = 1$. Thus

$$\begin{aligned}((A \times B) \times C + (B \times C) \times A + (C \times A) \times B)^i &= T^i_{kmn} V^{kmn} \\ &= T^i_{kki} V^{kki} + T^i_{kik} V^{kik} && \text{for fixed } i, k \\ &= T^i_{kki} V^{kki} - T^i_{kki} V^{kki} && \text{by antisymmetry of } T \\ &= 0\end{aligned}$$

as desired.

Thus, the cross product is a bilinear map that satisfies the Jacobi identity, and $\mathbb{R}^3$ with this product is a Lie algebra. □

Problem 5

Let $A \subseteq \mathcal{X}(\mathbb{R}^3)$ be the subspace spanned by the vector fields

$$\begin{aligned} X &= y\partial_z - z\partial_y \\ Y &= z\partial_x - x\partial_z \\ Z &= x\partial_y - y\partial_x \end{aligned}$$

Show that A is a Lie subalgebra of $\mathcal{X}(\mathbb{R}^3)$.

Proof. We note first that this defines a two-dimensional distribution on $\mathbb{R}^3$, since for all points (x, y, z) for which $z \neq 0$, we have that $Z = \frac{x}{z}X + \frac{y}{z}Y$, and similarly if $z = 0$, we have that $-X = \frac{y}{x}Y + \frac{z}{x}Z$.

So, all we need to check is that this distribution is involutive. Furthermore, since $[X, X] = 0$ for all X, it suffices to only check that for X, Y that span the distribution, $[X, Y]$ is in the distribution.

So, since X and Y are linearly independent for $z \neq 0$, they span the distribution for all $z \neq 0$. Now, we can calculate

$$\begin{aligned}
[X,Y]^i &= X^j \partial_j Y^i - Y^j \partial_j X^i \\
[X,Y]^x &= X^j \partial_j Y^x - Y^j \partial_j X^x \\
&= X^j \partial_j z - 0 \\
&= X^z = y
\end{aligned}$$

$$[X,Y]^y =$$

□