Homework 3

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Problem 1

Prove the following inequality. Let $f : [0,\pi] \to \mathbb{R}$ be a real C^2 function such that $f(0) = f(\pi) = 0$. Then

$$\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt$$

with equality if and only if $f(t) = c\sin(t)$.

Proof. Let γ be a normalized geodesic joint the antipodal points $p, -p$ of S^2. Let $v(t)$ be a parallel field along γ with $g(v,\gamma') = 0$, $\|v\| = 1$. Let $V = fv$. We calculate

$$\begin{aligned}
I_\pi(V,V) &= \int_0^\pi g(V',V') - g(R(\gamma',V)\gamma',V)dt \\
&= \int_0^\pi g(f'v,f'v) - f^2K(\gamma',v)dt \\
&= \int_0^\pi (f')^2\|v\|^2 - f^2(1)dt \\
&= \int_0^\pi (f')^2 dt - \int_0^\pi f^2 dt \qquad\qquad \geq 0
\end{aligned}$$

where the last line follows from the Morse index theorem. This establishes the inequality.

Note that equality holds if and only if $I_\pi(V,V) = 0$, which implies that V is a Jacobi field. Thus, it must satisfy the Jacobi equation

$$f''(t) + K(\gamma',v)f(t) = 0$$

which, on S^2, is just

$$f''(t) + f(t) = 0$$

with Dirichlet boundary conditions. This is solved only when $f(t) = c\sin(t)$ for some constant c, as desired. ∎

Problem 2

Let M^2 be a complete simply connected 2-dimensional Riemannian manifold. Suppose that for each point $p \in M$, the locus $C(p)$ of first conjugate points to p reduces to a unique $q \neq p$ and that $d(p, C(p)) = \pi$. Prove that if the sectional curvature K of M satisfies $K \leq 1$, then M is isometric to the sphere S^2 with $K = 1$.

Proof. Let J be a Jacobi field along a normalized geodesic γ joining p to q with $J(0) = J(\pi) = 0$ and $g(J, \gamma') = 0$. Let $\{e_i, \gamma'\}$ be an orthonormal parallel frame to γ, and write

$$J = a^i e_i$$

Define $K(t) = K(\gamma', J)$. We calculate

$$\begin{aligned}
0 = I_\pi(J, J) &= -\int_0^\pi g(J'' + R(\gamma', J)\gamma', J)dt \\
&= -\int_0^\pi g(J'', J)dt - \int_0^\pi K(t)\|J\|^2 dt \\
&= -\int_0^\pi a''^i a_i dt - \int_0^\pi K(t) a^i a_i dt \\
&= \int_0^\pi a'^i a'_i dt - \int_0^\pi K(t) a^i a_i dt && \text{using integration by parts} \\
&\geq \int_0^\pi a^i a_i dt - \int_0^\pi K(t) a^i a_i dt && \text{by problem 1} \\
&= \int_0^\pi a^i a_i (1 - K(t))dt \geq 0
\end{aligned}$$

and thus $K(t) = 1$ for all t.

Next, we show that every point in M is an interior point for some geodesic of the form above. To that end, let $p \in M$, and let $v \in T_pM$, $\|v\| = 1$. Define $\gamma : [0, \pi] \to M$ as the unique normalized geodesic with $\gamma(\frac{\pi}{2}) = p$ and $\gamma'(\frac{\pi}{2}) = v$. Now, let σ be a geodesic from $\gamma(0)$ to some conjugate point q, concatenated with the minimizing geodesic from q to $\gamma(\pi)$. That is, we've formed the triangle from $\gamma(0)$ to $\gamma(\pi)$ and q. σ has a Jacobi field on it that vanishes at $\gamma(0)$ and q. Furthermore, since M is simply connected, there is a homotopy between σ and γ. Pushing the Jacobi field from σ to γ along the homotopy parallel transport, we construct a Jacobi field on γ vanishing at $\gamma(0)$ and some other point $\gamma(t)$. Since the conjugate locus of $\gamma(0)$ is a distance π away, it must be that $t = \pi$, and $\gamma(0)$ is conjugate to $\gamma(\pi)$, which has p as an interior point, as desired. $\square$

Problem 3

Let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $a(t) \geq 0$ for all t, and $a(0) > 0$. Prove that the solution to the differential equation

$$(\partial_t^2 + a)\phi = 0$$

with initial conditions $\phi(0) = 1, \phi'(0) = 0$ has at least one positive zero and one negative zero.

Proof. We first prove that ϕ has a positive zero. To see this, suppose for a contradiction that $\phi(t) > 0$ for all $t > 0$. From this, it follows that

$$\phi'' = -a\phi \leq 0$$

with $\phi''(0) < 0$. Thus, there is some t_0 with $\phi'(t_0) < 0$. Let $f(t) = (t - t_0)\phi'(t_0) + \phi(t_0)$ be the tangent line to ϕ at t_0. I assert that $\phi(t) \leq f(t)$ for all $t > t_0$.

Suppose that there existed some t_1 for which $\phi(t_1) > f(t_1)$. In particular, this means that the slope of the secant line

$$M = \frac{\phi(t_1) - \phi(t_0)}{t_1 - t_0}$$

is such that $M > \phi'(t_0)$. By the mean value theorem, there is some c for which $\phi'(c) = M$ with $c \in [t_0, t_1]$. However, this implies that

$$\frac{\phi'(c) - \phi'(t_0)}{c - t_0} > 0$$

and so by the mean value theorem (again) there is some $d \in [t_0, c]$ for which $\phi''(d) > 0$. This contradicts our original assumption, so ϕ must be bounded above by f. However, f has a positive zero, so it must be that ϕ has at least one positive zero as well.

Arguing by a symmetric argument for points below zero, we see that ϕ has at least one negative zero as well. $\square$

Problem 4

Suppose M^n is a complete Riemannian manifold with sectional curvature strictly positive, and let $\gamma : (-\infty, \infty) \to M$ be a normalized geodesic in M. Show that there exists $t_0 \in \mathbb{R}$ for which $\gamma([-t_o, t_o])$ has index greater or equal to $n-1$.

Proof. Let Y be a parallel field along γ with $g(\gamma', Y) = 0$ and $\|Y\| = 1$. Set

$$\phi_Y = g(R(\gamma', Y)\gamma', Y)$$

and

$$K(t) = \inf_Y \phi_Y(t)$$

and let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $0 \leq a(t) \leq K(t)$ with $0 < a(0) < K(0)$. Let ϕ be the solution to $\phi'' + a\phi = 0$ with $\phi(0) = 1, \phi'(0) = 0$, with $-t_1, t_2$ the two zeroes of ϕ found in the previous problem. We consider the field $X = \phi Y$, and calculate

$$\begin{aligned}
I_{[-t_1,t_2]}(X,X) &= -\int_{-t_1}^{t_2} g(X'' + R(\gamma', X)\gamma', X)dt \\
&= -\int_{-t_1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{t_1}^{t_2} g(\phi R(\gamma', Y)\gamma', \phi Y) \\
&= -\int_{-t_1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{t_1}^{t_2} \phi^2 \phi_Y dt \\
&\leq -\int_{-t-1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{-t_1}^{t_2} K(t)\phi^2(t)dt \\
&= -\int_{-t_1}^{t_2} \phi(\phi'' + K(t)\phi)dt \\
&< -\int_{-t_1}^{t_2} \phi(\phi'' + a(t)\phi)dt \\
&= 0
\end{aligned}$$

Thus, for all Y perpendicular to γ' (an $n-1$ dimensional subspace) the form $I_{[-t_1,t_2]}(Y,Y)$ is negative-definite, and so the index is greater than or equal to $n-1$. In particular, this holds (as the index is strictly increasing) for $[-t_0, t_0]$ for $t_0 = \max(t_1, t_2)$. $\square$

Problem 5

Show that if the sectional curvature K of M is strictly positive, M does not have any lines. Show by example this is false if $K \geq 0$.

Proof. The previous problem asserts that for any geodesic in M, there is a segment $[-t_0, t_0]$ on which it has index greater than zero. In particular, this means that $\gamma(-t_0)$ has a conjugate point. Thus, γ does not minimize the length between $-t_0$ and points past its conjugate point, so γ is not a line, as desired.

For a counterexample with $K \geq 0$, take $\mathbb{R}^n$, where the geodesics are just straight lines. These trivially minimize distance between points, and so any maximally extended geodesic in $\mathbb{R}^n$ is a line. $\square$