Homework 5

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Problem 1

Let E, F be closed subspaces of a Hilbert space. Prove that $P_E P_F = P_E$ if and only if $E \subseteq F$.

Proof. Suppose first that $P_E P_F = P_E$. Then, in particular,

$$P_E^* = (P_E P_F)^* = P_F^* P_E^* = P_F P_E = P_E$$

by self-adjointness of projections. Thus, $P_E P_F = P_F P_E = P_E$, and the projections commute. Thus, the von Neumann algebra $W^*(P_E, P_F, I)$ is Abelian, and is isometrically $*$-isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ). In particular, the projections P_E, P_F get sent to self-adjoint idempotents $P_E \mapsto M_{\chi_S}$ and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$.

Now, the requirement $P_F P_E = P_E$ corresponds to the requirement

$$M_{\chi_{S'}} M_{\chi_S} = M_{\chi_S}$$

which means that $S \subset S'$. This, in turn, implies that $E \subset F$.

Indeed, E is the subspace of H on which P_E is the identity, which corresponds to the subspace

$$\tilde{E} = \int_S^\oplus H(x) d\mu(x)$$

on which M_{χ_S} is the identity. Similarly,

$$\tilde{F} = \int_{S'}^\oplus H(x) d\mu(x)$$

Clearly, $\tilde{E} \subset \tilde{F}$ (since $S \subset S'$) and so $E \subset F$ as well.

For the converse direction, assume that $E \subset F$. Then, on F, $P_F = I_F$, and $P_F P_E = I_F P_E = P_E$. Furthermore, on $F^\perp$, $P_F P_E = 0 = P_E$. Thus, on all of $H = F \oplus F^\perp$, $P_F P_E = P_E$ as desired. $\square$

Problem 2

Characterize the closed subspaces E, F of a Hilbert space H satisfy $P_F P_E = P_E P_F$.

Proof. I assert that P_E and P_F commute if and only if H can be decomposed into the four orthogonal components

$$H = E \cap F \oplus E \cap F^{\perp} \oplus E^{\perp} \cap F \oplus E^{\perp} \cap F^{\perp}$$

To see this, suppose first that P_E and P_F commute. Then, consider the Abelian von Neumann algebra $W^*(P_E, P_F, I)$. Like before, we use the Borel functional calculus to identify $W^*(P_E, P_F, I)$ with $L^{\infty}(X, \mu)$ acting on $\tilde{H} = \int_X^{\oplus} H(x) d\mu(x)$, which is unitarily equivalent to H (in the sense that there is a unitary transformation U such that $\tilde{H} = UH$ and $UW^*(P_E, P_F, I)U^* = L^{\infty}(X, \mu)$).

Under this identification, $P_E \mapsto M_{\chi_S}$, and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$. In particular, this decomposes $\tilde{H}$ into four orthogonal components

$$\begin{aligned}
\tilde{H} &= \int_X^{\oplus} H(x) d\mu(x) \\
&= \int_{S \cap S'}^{\oplus} H(x) d\mu(x) \oplus \int_{S \cap S'^c}^{\oplus} H(x) d\mu(x) \oplus \int_{S' \cap S^c}^{\oplus} H(x) d\mu(x) \oplus \int_{S^c \cap S'^c}^{\oplus} H(x) d\mu(x)
\end{aligned}$$

which translates into the decomposition on H as

$$H = E \cap F \oplus E \cap F^{\perp} \oplus E^{\perp} \cap F \oplus E^{\perp} \cap F^{\perp}$$

as desired.

Conversely, suppose H can be decomposed this way. Then, let $v \in H$ be decomposed as

$$v = v_1 + v_2 + v_3 + v_4$$

where $v_1 \in E \cap F$, $v_2 \in E \cap F^{\perp}$, $v_3 \in E^{\perp} \cap F$ and $v_4 \in E^{\perp} \cap F^{\perp}$.

We compute the effect of $P_E P_F$ and $P_F P_E$ directly.

$$\begin{aligned}
P_E P_F(v) &= P_E P_F(v_1 + v_2 + v_3 + v_4) \\
&= P_E(v_1 + v_3) \\
&= v_1 \\
P_F P_E(v) &= P_F P_E(v_1 + v_2 + v_3 + v_4) \\
&= P_F(v_1 + v_2) \\
&= v_1
\end{aligned}$$

and thus $P_E P_F = P_F P_E$ as desired. □

Problem 3

Let E, F be closed subspaces of a Hilbert space H. An operator U is said to be a partial isometry from E to F if $U|_E$ is an isometry onto F, and $U|_{E^\perp} = 0$. Prove that U is a partial isometry $\iff$ U^*U is a projection $\iff$ UU^* is a projection.

Proof. Suppose first that U is a partial isometry from E to F. I assert that $U^*U = P_E$. To see this, suppose $e \in E$, $v \in H$ and let $v = e' + v'$ where $e' \in E$ and $v' \in E^\perp$. Then,

$$\begin{aligned} \langle U^*Ue|v\rangle &= \langle U^*Ue|e' + v'\rangle \\ &= \langle Ue|Ue'\rangle + \langle Ue|Uv'\rangle \\ &= \langle e|e'\rangle + 0 \\ \implies \langle U^*Ue - e|v\rangle &= 0 \end{aligned}$$

and since this holds for all $v \in H$, $U^*Ue - e = 0$ and thus $U^*Ue = e$ and U^*U is the identity on E.

Furthermore, for $v' \in E^\perp$,

$$U^*Uv' = U^*(0) = 0$$

and so U^*U is the zero map on $E^\perp$. Thus, U^*U agrees with P_E at all points, so $U^*U = P_E$ as desired.

Conversely, suppose U^*U is a projection P_E for some closed subspace E. Define $F = U(E)$. We will first show that U is an isometry of E onto F. To see this, suppose $e, e' \in E$. We calculate directly

$$\begin{aligned} \langle Ue|Ue'\rangle &= \langle U^*Ue|e'\rangle \\ &= \langle P_Ee|e'\rangle \\ &= \langle e|e'\rangle \end{aligned}$$

and thus U is an isometry from E to F. Note that this immediately implies that F is a closed subspace. Finally, we show that $U|_{E^\perp} = 0$. Let $v' \in E^\perp$, Then,

$$\begin{aligned} \|Uv'\| &= \langle Uv'|Uv'\rangle \\ &= \langle U^*Uv'|v'\rangle \\ &= \langle 0|v'\rangle = 0 \end{aligned}$$

and so $U^*U|_{E^\perp} = 0$ as desired. Thus, U is a partial isometry.

Finally, we show that if U is a partial isometry, then U^* is a partial isometry, which will complete the proof. So, suppose U is a partial isometry. We will show that U^* is a partial isometry from F to E. First, we check that $U^*|_F$ is an isometry. Now, for each $f \in F$, there is some $e \in E$ with $Ue = f$. Thus, for $e, e' \in E$, $f, f' \in F$ with $Ue = f, Ue' = f'$,

$$\begin{aligned} \langle U^*f|U^*f'\rangle &= \langle U^*Ue|U^*Ue\rangle \\ &= \langle P_Ee|P_Ee'\rangle \\ &= \langle e|e'\rangle \\ &= \langle Ue|Ue'\rangle \\ &= \langle f|f'\rangle \end{aligned}$$

and so U^* is an isometry from F to E. Finally, we show that $U^*|_{F^\perp} = 0$. This is immediate, since

$$\ker U^* = U(H)^\perp = F^\perp$$

where we used the identity $\ker A^* = A(H)^\perp$ for all bounded operators A.

Note that this completes the proof. If U is a partial isometry, then U^* is a partial isometry, which implies that $U^{**}U^* = UU^*$ is a projection. Similarly, if $UU^* = U^{**}U^*$ is a projection, then U^* is a partial isometry, and thus $U^{**} = U$ is a partial isometry as well. $\square$

Problem 4

Prove that the subspace of self-adjoint operators on $B(\mathbb{C}^2)$ is a subset of the linear subspace spanned (over $\mathbb{R}$) by I and the three Pauli matrices

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and prove that the span of $\{\sigma_i\}_{i=1}^3$ is isometrically isomorphic to a three-dimensional real Hilbert space under the operator norm.

Proof. We observe first that for any $T \in B(\mathbb{C}^2)$ self-adjoint, we have the condition

$$T^i_j = \overline{T^j_i}$$

In particular,

$$T^1_1 = \overline{T^1_1}$$
$$T^2_2 = \overline{T^2_2}$$

and so T^i_i is real. Furthermore, we have the relation

$$T^1_2 = \overline{T^2_1}$$

Finally, we observe that we can decompose T into a scalar part and a trace-free part by

$$C = T - \operatorname{tr}(T)I$$
$$T = \operatorname{tr}(T)I + C$$

Now we are ready to show that T is a linear combination of I and the Pauli matrices. In particular, we will show that C is a linear combination of the Pauli matrices.

For C, we have the relations

$$C^1_1 = -C^2_2$$
$$C^1_2 = \overline{C^2_1}$$

as well as the requirement that C^1_1 is real. Thus, we have three degrees of freedom. That is,

$$C = \begin{bmatrix} \alpha & \beta + \gamma i \\ \beta - \gamma i & -\alpha \end{bmatrix}$$

which decomposes as

$$C = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3$$

and so

$$T = \operatorname{tr}(T)I + C$$

is a linear combination of I and $\{\sigma_i\}_{i=1}^3$ as desired.

Next, we show that the span of $\{\sigma_i\}_{i=1}^3$ under the operator norm is isometrically isomorphic to $\mathbb{R}^3$. To do so, we will show that the operator norm coincides with the pullback metric from $\mathbb{R}^3$. Specifically, we define a linear isomorphism

$$\Phi : \langle\sigma_i\rangle_{i=1}^3 \to \mathbb{R}^3$$
$$\Phi(\sigma_i) = e_i$$

where $\{e_i\}$ is an orthonormal basis for $\mathbb{R}^3$ with its usual inner product. In particular, if we denote the inner product of $\mathbb{R}^3$ as η, we can define $\Phi^*(\eta)$ by

$$\Phi^*(\eta)(u,v) = \eta(\Phi(u), \Phi(v))$$

In order to show that this metric coincides with the operator norm, we have to show that

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \Phi^*(\eta)(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3, \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3) = \alpha^2 + \beta^2 + \gamma^2$$

Now, since σ_i are all self-adjoint, any (real) linear combination of them will be as well. Thus,

$$\|\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3\|^2 = \|(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2\|$$

We compute the right-hand side directly

$$(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2 = \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma\sigma_3^2 + \alpha\beta\{\sigma_1, \sigma_2\} + \alpha\gamma\{\sigma_1, \sigma_3\} + \beta\gamma\{\sigma_2, \sigma_3\}$$

where $\{\sigma_i, \sigma_j\}$ is the anticommutator of σ_i and σ_j. Now, it is well-known that the Pauli matrices satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

as well as the relation

$$\sigma_i^2 = I$$

(where ε_{ijk} is the levi-civita symbol) which forces

$$\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}$$

and so

$$\begin{aligned}
\{\sigma_i, \sigma_j\} &= \sigma_i\sigma_j + \sigma_j\sigma_i \\
&= i\varepsilon_{ijk}\sigma_k + \delta_{ij} + i\varepsilon_{jik}\sigma_k + \delta_{ji} \\
&= i\varepsilon_{ijk}\sigma_k - i\varepsilon_{ijk}\sigma_k + 2\delta_{ij} \\
&= 2\delta_{ij}
\end{aligned}$$

Thus,

$$\begin{aligned}
(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3)^2 &= \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma\sigma_3^2 + \alpha\beta\{\sigma_1, \sigma_2\} + \alpha\gamma\{\sigma_1, \sigma_3\} + \beta\gamma\{\sigma_2, \sigma_3\} \\
&= \alpha^2\sigma_1^2 + \beta^2\sigma_2^2 + \gamma\sigma_3^2 \\
&= (\alpha^2 + \beta^2 + \gamma^2)I
\end{aligned}$$

and so

$$\|\alpha\sigma_1 + \beta^2\sigma_2 + \gamma\sigma_3\|^2 = \|(\alpha^2 + \beta^2 + \gamma^2)I\| = (\alpha^2 + \beta^2 + \gamma^2)\|I\| = (\alpha^2 + \beta^2 + \gamma^2)$$

as desired. Thus, the pullback metric coincides with the operator norm, and Φ is a linear isometric isomorphism, as desired. $\square$

Problem 5

Let H be a separable Hilbert space. Prove T^*T is positive for all $T \in B(H)$. Prove that every positive operator is equal to T^*T for some $T \in B(H)$. Can we ensure T is self-adjoint?

Proof. First, we show T^*T is positive. That is, we need to show that

$$\langle T^*T\eta|\eta\rangle \geq 0 \forall \eta \in H$$

This follows immediately, since

$$\begin{aligned}\langle T^*T\eta|\eta\rangle &= \langle T\eta|T\eta\rangle \\ &= \|T\eta\|^2 \geq 0\end{aligned}$$

and thus T^*T is positive.

Next, we show that every positive operator can be written this way. In particular, let A be a positive operator. We know that $\sigma(A) \subset [0, \infty)$, since A is positive. Therefore, the square root function is defined on $\sigma(A)$. Through the continuous functional calculus, we can define

$$\sqrt{A} = \int_{\sigma(A)} \sqrt{z} dE(z)$$

for E the projection-valued measure associated to A (so that $A = \int_{\sigma(A)} z dE(z)$). This satisfies

$$\begin{aligned}\sqrt{A}\sqrt{A} &= \int_{\sigma(A)} \sqrt{z} dE(z) \int_{\sigma(A)} \sqrt{z} dE(z) \\ &= \int_{\sigma(A)} \sqrt{z}\sqrt{z} dE(z) \\ &= \int_{\sigma(A)} z dE(z) = A\end{aligned}$$

and so by setting $T = T^* = \sqrt{A}$ we achieve the desired result. $\square$

Problem 6

Let H be a separable Hilbert space, and let $T \in B(H)$ be self-adjoint. Suppose $\sigma(T) = [-\varepsilon, \varepsilon] \cup [1-\varepsilon, 1+\varepsilon]$. Prove there is a projection P for which $\|T - P\| \leq \varepsilon$.

Proof. Let E be the projection-valued measure associated to T, so that

$$T = \int_{\sigma(T)} z dE(z)$$
$$I = \int_{\sigma(T)} dE(z)$$

and define

$$P = \int_{\sigma(T)} \chi_{[1-\varepsilon,1+\varepsilon]}(z) dE(z) = E([1-\varepsilon, 1+\varepsilon])$$

which is a projection. Then,

$$T - P = \int_{\sigma(T)} (z - \chi_{[1-\varepsilon,1+\varepsilon]}(z)) dE(z)$$

Let $f = z - \chi_{[1-\varepsilon,1+\varepsilon]}(z)$, defined on $\sigma(T)$. Now, $\|f\|_\infty = \varepsilon$, and so

$$\begin{aligned}
\|T - P\|^2 &= \sup_{h \in H, \|h\|=1} \langle \int f dEh | \int f dEh \rangle \\
&= \sup_{h \in H, \|h\|=1} \langle \int \bar{f} dE \int f dEh | h \rangle \\
&= \sup_{h \in H, \|h\|=1} \langle \int |f|^2 dEh | h \rangle \\
&= \sup_{h \in H, \|h\|=1} \int |f|^2 d\mu_{h,h} \\
&\leq \sup_{h \in H, \|h\|=1} \||f|^2\|_\infty \int d\mu_{h,h} \\
&\leq \sup_{h \in H, \|h\|=1} \||f|\|_\infty^2 \int d\mu_{h,h} \\
&= \sup_{h \in H, \|h\|=1} \||f|\|_\infty^2 \|h\|^2 = \||f|\|_\infty^2
\end{aligned}$$

where $\mu_{h,g}$ is the induced (complex) measure defined as

$$\mu_{h,g}(S) = \langle E(S)h | g \rangle$$

Thus,

$$\|T - P\| \leq \|f\|_\infty = \varepsilon$$

as desired. □

Problem 7

Let H be a separable complex Hilbert space, $T \in B(H)$ self-adjoint, and suppose T is invertible. Find a continuous path (with respect to the norm topology) from T to I that stays in $GL(H) \subset B(H)$ the invertible elements of $B(H)$. Can we also ensure the path stays in the self-adjoint subset of $GL(H)$?

Proof. Recall that every operator $T \in B(H)$, T has a polar decomposition

$$T = PU$$

where P is positive and U is a partial isometry. Furthermore, if T is invertible, then U is unitary. So, let T be decomposed as such. Furthermore, we write

$$P = \exp(Q)$$
$$U = \exp(iH)$$

for self-adjoint Q and H (Q is just $\log(P)$, and H is guaranteed to exist for unitary operators as the generator for U).

Now, we can write

$$T = \exp(Q + iH)$$

and define a one-parameter group of operators

$$T_t = \exp(t(Q + iH))$$

with the property that $T_0 = I$, $T_1 = T$, and $T_t T_{-t} = I$ for all t. Thus, this one-parameter group is a subset of $GL(H)$, and has endpoints at I and T, as desired. Furthermore, it is continuous in T, since

$$T_t = \exp(tQ)\exp(tiH)$$

and $\exp(tQ), \exp(tiH)$ are both continuous with respect to t, so their product is as well.

Now, we cannot guarantee that any path will stay in the self-adjoint subset of $GL(H)$. Take for a counterexample $B(H) = H = \mathbb{C}$, $T = -1$. The self-adjoint invertible operators in $\mathbb{C}$ are $\mathbb{R} \setminus \{0\}$, which has two path components. -1 and 1 are not in the same path-component, so there cannot be a path that joins them in the self-adjoint subset of $\mathbb{C}$. $\square$

Problem 8

Let H be a separable Hilbert space, and $\mathscr{A} \subset B(H)$ a C^* algebra containing T a self-adjoint invertible operator. Prove that T^{-1} lies in $\mathscr{A}$.

Proof. Recall that $C^*(T)$ is the minimal unital C^* algebra containing T. In particular, $C^*(T) \subset \mathscr{A}$. So, we will show that $T^{-1} \in \mathscr{A}$.

Recall that the continuous functional calculus gives us a isometric $*$-isomorphism (say, γ) between $C^*(T)$ and $C(\sigma(T))$ in the sup norm. In particular, since T is invertible, $0 \notin \sigma(T)$, and so the function $f(z) = z^{-1}$ is well-defined. Furthermore, we know that

$$\gamma(T) = z$$

and so

$$\begin{aligned} I &= \gamma^{-1}(1) \\ &= \gamma^{-1}(zz^{-1}) \\ &= \gamma^{-1}(z)\gamma^{-1}(z^{-1}) \\ &= T\gamma^{-1}(z^{-1}) \end{aligned}$$

and thus $\gamma^{-1}(z^{-1})$ is the inverse of T, as desired. Thus, T^{-1} is in $C^*(T) \subset \mathscr{A}$, as desired. $\square$

Problem 9

Let H be a separable Hilbert space, and let $\mathscr{M} \subset B(H)$ be a von Neumann algebra containing T a self-adjoint operator. Let $E = \overline{T(H)}$. Prove that $P_E \in \mathscr{M}$.

Proof. Recall that $W^*(T)$ is the minimal unital von Neumann algebra containing T. Thus, $W^*(T) \subset \mathscr{M}$. We will show that $P_E \in W^*(T)$.

Recall that the spectral theorem for self-adjoint operators gives us a projection-valued measure on the spectrum $\sigma(T)$. Now, if T has trivial kernel, then $E = H$, and $P_E = I$, which is always in $W^*(T)$ by definition. If T has a kernel, then $o \in \sigma(T)$, and in particular (letting F be the projection-valued measure associated to T)

$$F(\{0\}) = P_{\ker(T)}$$

since 0 is in the point spectrum of T, and the projection-valued measure on singletons in the point spectrum yields the projection onto the associated eigenspace. Thus,

$$1 - P_{\ker(T)} = F(\chi_{\sigma(T)\setminus\{0\}})$$

is projection onto $\ker(T)^{\perp} = \overline{T(H)}$ and is in $W^*(T)$ as desired. $\square$