Problem Set 2

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Preliminaries

Before the homework begins, we prove a few useful lemmas.

Lemma 1. *Let $g, h : X \to Y$ be homotopic maps. Then, for any function $f : Y \to Z$ for arbitrary Z, the push-forwards f_*g and f_*h are also homotopic. Similarly, if instead we have $f : Z \to X$, then the pullbacks f^*g and f^*h are homotopic.*

Proof. Let $g_t : X \to Y$ be a homotopy with $g_0 = g$ and $g_1 = h$. Then, for $f : Y \to Z$, the homotopy $f_*g_t : X \to Z$ yields

$$f_*g_0 = f_*g$$
$$f_*g_1 = f_*h$$

and so f_*g is homotopic to f_*h.

Similarly, if $f : Z \to X$, the homotopy $f^*g_t : Z \to Y$ yields

$$f^*g_0 = f^*g$$
$$f^*g_1 = f^*h$$

and so f^*g and f^*h are homotopic. □

Lemma 2. *Let X be a contractible space. Then, there exists a homotopy $f_t : X \to X$ with $f_0 = \mathbb{1}_X$ and $f_1 = x_0$ the constant function to some $x_0 \in X$.*

Proof. Since X is contractible, it has the homotopy type of a point. Specifically, there exists a homotopy equivalence $g : X \to \{\cdot\}$ with homotopy inverse $h : \{\cdot\} \to X$.

Now, we will call $h(\{\cdot\}) = x_0$, since the image of h has only one point. Since h is the homotopy inverse of g, it follows that $hg \simeq \mathbb{1}_X$ with homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}_X$ and $f_1 = hg$. But hg is easily verified to be the constant map x_0, and the homotopy f satisfies the conditions desired. □

Problem 1

Show that the retract of a contractible space is contractible.

Proof. Let X retract onto A via $r : X \to A$, with X a contractible space. We wish to show A is contractible. To do so, we will show that every map $g : A \to Y$ for arbitrary Y is nullhomotopic (see next problem).

Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "\mathbb{1}_A"'] & X \arrow[r, "r"] & A \arrow[r, "g"] & Y \end{tikzcd}$$

Now, gr defines a map from X to Y, which we know is nullhomotopic by the next problem. In particular, this means that $gr \simeq y_0$ for some constant y_0, and by Lemma 1, we know that $i^*(gr) \simeq i^*(y_0) = y_0$. But $i^*(gr) = gri = g$, and so $g \simeq y_0$.

Thus, since every map from A to Y is nullhomotopic, it follows that A is contractible.

$\square$

Problem 2

Show that a space X is contractible if and only if every map $f : X \to Y$ for any Y is nullhomotopic. Similarly, show that X is contractible if and only if every map $f : Y \to X$ for arbitrary Y is nullhomotopic.

Proof. This proof is broken into two parts, one for each iff statement.

For the first statement, we first assume X is contractible, and let $f : X \to Y$ be arbitrary. Now, since x is contractible, we know that $\mathbb{1}_X \simeq x_0$ for some constant function x_0 (Lemma 2). Thus, it follows that $f_*\mathbb{1}_X \simeq f_*x_0 = f(x_0)$ where $f(x_0)$ is the constant function from X to the point $f(x_0)$. Thus, since $f_*\mathbb{1}_X = f$, we have that f is homotopic to a constant map, and is nullhomotopic.

Now, suppose for every space Y and every map $f : X \to Y$, f is nullhomotopic. In particular, take $Y = X$ and $f = \mathbb{1}_X$. Then, it follows immediately that $\mathbb{1}_X$ is nullhomotopic, and X is contractible.

For the second statement, we first assume X is contractible, and let $f : Y \to X$ be arbitrary. Now, since x is contractible, we know that $\mathbb{1}_X \simeq x_0$ for some constant function x_0 (Lemma 2). Thus, it follows that $f^*\mathbb{1}_X \simeq f^*x_0 = x_0$. Since $f^*\mathbb{1}_X = f$, we have that f is homotopic to a constant map, and is nullhomotopic.

Conversely, suppose every map $f : Y \to X$ is nullhomotopic. Taking $Y = X$ and $f = \mathbb{1}_X$, we see that $\mathbb{1}_X$ is homotopic to the constant map, and thus X is contractible. $\square$

Problem 3

Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \mathbb{1}_Y$ and $hf \simeq \mathbb{1}_X$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Proof. Suppose $f : X \to Y$ and there exist maps $g, h : Y \to X$ such that $fg \simeq \mathbb{1}_X$ and $hf \simeq \mathbb{1}_Y$. Now, from Lemma 1 we have that

$$
\begin{aligned}
gfh &= h^*(gf) \simeq h^*(\mathbb{1}_X) = h \\
gfh &= g_*(fh) \simeq g_*(\mathbb{1}_Y) = g
\end{aligned}
$$

and so $g \simeq h$. Then, again by Lemma 1, we have

$$
\begin{aligned}
&fg \simeq \mathbb{1}_Y && \text{by hypothesis} \\
&gf = f^*g \simeq f^*h = hf \simeq \mathbb{1}_X && \text{by Lemma 1}
\end{aligned}
$$

and so f is a homotopy equivalence with homotopy inverse g.

More generally, suppose fg and hf are homotopy equivalences. That is, that there exist functions $\gamma : Y \to Y$ and $\delta : X \to X$ such that $fg\gamma \simeq \gamma fg \simeq \mathbb{1}_Y$ and $hf\delta \simeq \delta hf \simeq \mathbb{1}_X$.

Now, since $fg\gamma \simeq \mathbb{1}_Y$, we have that (using Lemma 1)

$$
\begin{aligned}
h_* fg\gamma &\simeq h_* \mathbb{1}_Y \\
hfg\gamma &\simeq h \\
\delta_* hfg\gamma &\simeq \delta_* h \\
\delta hfg\gamma &\simeq \delta h \\
g\gamma &\simeq \delta h
\end{aligned}
$$

Now, it is easily verified that $\delta hfg\gamma$ is a homotopy inverse for f. To see this, note that

$$
\begin{aligned}
\delta hfg\gamma f &\simeq \delta hf && \simeq \mathbb{1}_X \\
f\delta hfg\gamma &\simeq f\delta h && \simeq fg\gamma \simeq \mathbb{1}_Y
\end{aligned}
$$

and so f is a homotopy equivalence. $\square$

Problem 4

Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path components of X and the set of path components of Y, and that f restricts to a homotopy equivalence on each path component. Prove the same for components instead of path components. Deduce that if the components of X coincide with the path components of X, then the same is true for Y homotopy equivalent to X.

Proof. There is a functor $Path : Top \to Set$ which takes a space X to the set of its path components, and takes maps $f : X \to Y$ to induced maps $\tilde{f} : Path(X) \to Path(Y)$ by mapping a path component X_p to the path component containing $f(X_p)$. This function is well-defined, since the image of a path connected space is path connected.

This is functorial, since it sends the identity map in Top to the identity in Set (since the identity in Top will necessarily map path components to themselves), and furthermore it respects composition. That is, for $g : X \to Y$ and $f : Y \to Z$, we have

$$\widetilde{fg} = \tilde{f}\tilde{g}$$

This is easily verified by considering a path-component X_p of X. Now, we have that $\widetilde{fg}(X_p)$ is the path component containing $fg(X_p)$, and $\tilde{f}\tilde{g}(X_p)$ is the path component containing the image under f of the path component containing $g(X_p)$. But this is simply the path component containing $fg(X_p)$, and thus $\widetilde{fg} = \tilde{f}\tilde{g}$ as desired.

Now we just have to show that such a functor is homotopy invariant. That is, we wish to show that for $f \simeq g$, we have that $\tilde{f} = \tilde{g}$. So, let $f, g : X \to Y$ be two homotopic maps, and let X_p be a path component of X. We wish to show that the path component containing $f(X_p)$ also contains $g(X_p)$. That is, we wish to find a path from $f(x_p)$ to $g(x_p)$ for all $x_p \in X_p$.

Let $F : X \times I \to Y$ be the homotopy from f to g. Fixing x_p, we have a function $F_{x_p} : I \to Y$ defined by $F_{x_p}(t) = F(x_p, t)$. This function has the property that $F_{x_p}(0) = f(x_p)$ and $F_{x_p}(1) = g(x_p)$ since F is the homotopy from f to g. Thus, F_{x_p} defines a path from $f(x_p)$ to $g(x_p)$ and therefore $\tilde{f}(X_p) = \tilde{g}(X_p)$. Since X_p was arbitrary, we have that $\tilde{f} = \tilde{g}$ as desired.

Thus, the *Path* functor is homotopy invariant. In particular, if X is homotopy equivalent to Y via a function $f : X \to Y$ with homotopy inverse $g : Y \to X$ (that is, $fg \simeq \mathbb{1}_Y$ and $gf \simeq \mathbb{1}_X$), we can apply *Path* to find that

$$\tilde{f}\tilde{g} = \widetilde{fg} = \mathbb{1}_{Path(Y)}$$

and

$$\tilde{g}\tilde{f} = \widetilde{gf} = \mathbb{1}_{Path(X)}$$

and so $\tilde{f}$ is a bijection with inverse $\tilde{g}$ between the set of path components of X and the set of path components of Y.

We now wish to show that f restricts to a homotopy equivalence between the corresponding path components of X and Y. To see this, we first show that a homotopy F restricts to the path components in a well-defined way. That is, if $F(x_p, 0)$ is in a path component X_p, then $F(x_p, t_0)$ is in that path component for all $t_0 \in I$.

This is obvious, since the point $F(x_p, t_0)$ has a path from $F(x_p, 0)$ to it: namely, the path given by $F(x_p, \frac{t}{t_0})$. Thus, the homotopy F is well-defined when restricted to a path component.

So, given that f is a homotopy equivalence with homotopy inverse g, we can consider the restriction of the homotopy F between gf and $\mathbb{1}$ on a path component to get a homotopy $F|_{X_p}$ from gf restricted to X_p and $\mathbb{1}$ restricted to X_p. Similarly it can be shown that fg restricts on the path component Y_p to a map homotopic to $\mathbb{1}_{Y_p}$, and thus we have that f restricted to a path component is a homotopy equivalence.

We now wish to prove the same for the connected components of the space. We can define a similar functor $Conn : Top \to Set$ taking a space X to its connected components, and a function $f : X \to Y$ mapping a connected component X_c to the connected component containing $f(X_c)$. In a similar argument to the one above, it is easy to see that this is indeed a functor. We wish to show that it is homotopy invariant.

Now, we argued in the previous part of this proof that if f and g are homotopic to each other, then $f(x_c)$ is path connected to $g(x_c)$. Since this is the case, it follows that $f(x_c)$ and $g(x_c)$ are in the same connected component, and thus for connected component X_c, $f(X_c)$ and $g(X_c)$ are in the same connected component. Thus, $\tilde{f}$ and $\tilde{g}$ are equal, and the functor is homotopy invariant.

By a similar argument to the one made in the path component case, this means that for f a homotopy equivalence between X and Y, $\tilde{f}$ defines a bijection between $Conn(X)$ and $Conn(y)$, as desired.

Now, we wish to show that f restricts to a homotopy equivalence between the corresponding path components. Again, we just need to show that a homotopy F restricts to components in a well-defined way. That is, $F(x_c, t)$ lies in the same component for all t.

However, we have already shown that $F(x_c, t)$ lies in the same path component for all t, and it follows immediately that $F(x_c, t)$ lies in the same connected component as well. Thus, F restricts to the components in a well-defined way.

By the same argument as the one made in the path connected case, this means that the homotopy equivalence f restricts to a homotopy equivalence between the connected components of X and Y, as desired.

Finally, we conclude that if X and Y are homotopy equivalent, and the components of X coincide with the path components of X, then the same is true for Y. This is immediate by considering the fact that each component X_c is homotopy equivalent to its corresponding component $\tilde{f}_c(X_c)$, and thus its path components are in bijection with each other. Since X_c has only one path component, so does $\tilde{f}_c(X_c)$ as desired. $\square$

Problem 5

Show that two deformation retractions r_t^0 and r_t^1 from X to $A \subset X$ can be joined by a continuous family of deformation retractions r_t^s, $s \in I$ of X into A.

Proof. We define r_t^s to be

$$r_t^s = \begin{cases} r^0_{(1-2s)t}, & s \leq \frac{1}{2} \\ r^1_{2(s-\frac{1}{2})t}, & s \geq \frac{1}{2} \end{cases}$$

which is clearly continuous at $s = \frac{1}{2}$ since $r^0_{(1-2(\frac{1}{2}))t} = r^0_0 = r^1_0 = r^1_{2(\frac{1}{2}-\frac{1}{2})t}$. $\square$

Problem 6

Part A

Show that for a map $f : S^1 \to S^1$, the mapping cylinder is a CW complex.

Proof. We will show that the mapping cylinder is actually the 1-skeleton $S^1 \wedge I \wedge S^1$ along with a single 2-cell attached to it. To see this, we note that the mapping cylinder is actually

$$S^1_1 \times I \coprod S^1_2/(x_1, 1) \sim f(x_1)$$

Now, $S^1 \times I$ is just $I \times I/(0,t) \sim (1,t)$ and so the total space is just

$$\frac{I \times I \coprod S^1_2}{(0,t) \sim (1,t), (x_1, 1) \sim f(x_1)}$$

Which can be further decomposed into

$$\frac{I_s \coprod S^1_1 \coprod I \times I \coprod S^1_2}{(s)_{I_s} \sim (0,s)_{I\times I}, (x)_{S^1_1} \sim (x,0)_{I\times I}, (0,t)_{I\times I} \sim (1,t)_{I\times I}, (x_1,1)_{I\times I} \sim f(x_1)_{S^1_2}}$$

Here, it is clear that I_s, S^1_1 and S^1_2 form a 1-skeleton. Now, we can take the attaching map from $I \times I \cong D^2$ to the 1-skeleton as

$$\phi^2 : \partial(I \times I) \to X^1$$
$$\phi^2(0,s) = s_{I_s}$$
$$\phi^2(1,s) = s_{I_s}$$
$$\phi^2(x,0) = x_{S^1_1}$$
$$\phi^2(x,1) = f(x)_{S^1_2}$$

Now, $I \times I$ is homeomorphic to D^2, and so we can consider $I \times I$ to be a 2-cell we attach to the 1-skeleton. Since ϕ is a valid attaching map,attaching D^2 along ϕ yields a 2-dimensional cell complex. However, this construction is identical to the construction we started with (constructing the mapping cylinder) and so the mapping cylinder is a CW complex. □

Part b

Construct a space with both the Mobius band and the annulus $S^1 \times I$ as deformation retracts.

Proof. Consider the space constructed as follows: Start with the Mobius band M, and glue a copy of $S^1 \times I$ along the map that sends $S^1 \times \{1\}$ via the identity to the equator $S^1 \subset M$ of the Mobius band. Since both the Mobius band and the annulus are CW complexes, and the gluing is done between entire subcomplexes, it follows that this construction yields a CW complex.

Now, the Mobius band deformation retracts onto its equator, and so this CW complex deformation retracts in the same way by sending M to its equator and leaving $S^1 \times I$ by itself. This yields just $S^1 \times I$, as desired.

$S^1 \times I$ also deformation retracts onto $S^1 \times \{1\}$, and so this CW complex deformation retracts in the same way by leaving M along and sending $S^1 \times I$ to $S^1 \times \{1\}$. This yields just M, as desired. □

Problem 7

Show that a CW complex is path connected if and only if its 1-skeleton is path connected.

Proof. ($\implies$) Suppose first that a CW complex X is path connected. We will induct on the dimension of X. The base case of X being one dimensional means that $X = X^1$ and thus the 1-skeleton is path connected.

Now suppose that this theorem holds for an $n-1$-dimensional CW complex. We will show that any path in an n-dimensional CW complex starting and ending on the $n-1$-skeleton is homotopic to a path in the $n-1$-skeleton rel the start and end points.

To see this, we observe two key facts about paths through n-cells. First, any path in the disk D^n for $n > 1$ can be homotoped so as to avoid passing the center. Second, any path through the disk D^n that avoids the center is homotopic to a path along the boundary ∂D^n. This homotopy is just the deformation retraction of $D^n \setminus \{0\}$ to ∂D^n given by radial projection.

Thus, any path through an n-cell is homotopic to a path through the $n-1$ skeleton. Therefore, for any two points x, y in the $n-1$-skeleton, there exists a path from x to y (since X is path connected) that stays in the $n-1$ skeleton. Thus, the $n-1$-skeleton is path-connected, and by the inductive hypothesis, the 1-skeleton is as well.

($\impliedby$) Suppose instead that for a CW complex X, its 1-skeleton is path connected. Again we will induct on the dimension of X. If X is 1-dimensional, then $X^1 = X$ and so X is path connected.

Now, suppose this holds for an $n-1$ dimensional CW complex. We will show that an n dimensional CW complex with a path-connected 1-skeleton is path connected. To see this, note that since X^1 is path connected, so is X^{n-1}. So, all we have to do to get a path from x to y (for $x, y \in X$) is find a path from x and y to the $n-1$-skeleton.

Now, if x or y are already in the $n-1$-skeleton, we are done. Suppose, however, that x is not in the $n-1$-skeleton. Then, x must be in the interior of some n-cell. However, since D^n is path connected, there always exists a path from $x \in D^n$ to the boundary ∂D^n. This path sends x to the $n-1$-skeleton, as desired.

Thus, X is path connected, as desired. $\square$

Problem 8

Show that a CW complex is locally compact if and only if each point has a neighborhood that meets only finitely many cells.

Proof. ($\Longrightarrow$) Suppose that a CW complex X is locally compact. It follows that for any point $x \in X$, there is a neighborhood U of x with compact closure. However, in proposition A1 of Hatcher, it is asserted that any compact subset of a CW complex meets only finitely many cells. Thus, the closure of U (and therefore U itself) must meet only finitely many cells. Such a neighborhood satisfies the properties desired.

($\Longleftarrow$) Suppose instead that each point $x \in X$ has a neighborhood U that meets only finitely many cells. In particular, this means that the closure of U also meets only finitely many cells. Thus, $\overline{U}$ is a subset of X^n for some n. As a closed subset of a compact Hausdorff space, $\overline{U}$ is compact as well, as desired.

So, each point has a neighborhood with compact closure, and X is locally compact. $\square$