Homework 2

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Problem 1

Characterize all the norm-closed faces of the unit ball in $C([0,1])$ under the sup norm.

Proof. □

Problem 2

Characterize all the extreme points of the set

$$K = \{f \in \ell^1 \mid 0 \leq f(n) \leq 1 \forall n \in \mathbb{N}, \int_{\mathbb{N}} f d\mu = 1\}$$

Proof. I assert that all the extreme points of K are the basis vectors $e_n = (0, \dots, 0, 1, 0, \dots)$ where 1 is in the nth position.

First, we observe that these points are indeed extreme points. Suppose that $f, g \in K$ with

$$e_n = tf + (1-t)g$$

for some $t \in (0,1)$. Now, for all $i \neq n$, we have

$$0 = tf(i) + (1-t)g(i)$$

but $f(i)$ and $g(i)$ are both in $[0,1]$, and t and $1-t$ are both positive and nonzero, which forces $f(i) = g(i) = 0$. The normalization condition on K forces $\int_{\mathbb{N}} f d\mu = \int_{\mathbb{N}} g d\mu = 1$ which implies that $f(n) = g(n) = 1$, and so $f = g = e_n$. Thus, each e_n is indeed an extreme point of K.

Next, we observe that these are all the extreme points. Suppose f is not e_n for any n. In particular, this means that there are at least two integers $n_0 1, n_2$ for which $f(n_1) \in (0,1)$ and $f(n_2) \in (0,1)$. Now, let $\varepsilon > 0$ be such that $f(n_i) \pm \varepsilon \in (0,1)$.

Now, define g and h as

$$g(i) = \begin{cases} f(n_1) + \varepsilon, & i = n_1 \\ f(n_2) - \varepsilon, & i = n_2 \\ f(i), & \text{else} \end{cases}$$

$$h(i) = \begin{cases} f(n_1) - \varepsilon, & i = n_1 \\ f(n_2) + \varepsilon, & i = n_2 \\ f(i), & \text{else} \end{cases}$$

By our choice of ε, $g(\mathbb{N}), h(\mathbb{N}) \in [0, 1]$ and by construction

$$\int_{\mathbb{N}} g d\mu = \int_{\mathbb{N}} h d\mu = 1$$

since we have only moved ε from one element of the sum to another. Thus, $g, h \in K$. Furthermore, for all $i \in \mathbb{N}$, $i \neq n_1, n_2$,

$$\frac{1}{2} g(i) + \frac{1}{2} h(i) = \frac{1}{2} f(i) + \frac{1}{2} f(i) = f(i)$$

and

$$\begin{aligned} \frac{1}{2} g(n_1) + \frac{1}{2} h(n_1) &= \frac{1}{2}(f(n_1) + \varepsilon) + \frac{1}{2}(f(n_1) - \varepsilon) \\ &= \frac{1}{2}(2f(n_1)) = f(n_1) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} g(n_2) + \frac{1}{2} h(n_2) &= \frac{1}{2}(f(n_2) - \varepsilon) + \frac{1}{2}(f(n_2) + \varepsilon) \\ &= \frac{1}{2}(2f(n_2)) = f(n_2) \end{aligned}$$

which verifies that $f = \frac{1}{2} g + \frac{1}{2} h$, and thus g and h belong to the same face as f, and in particular, f is not an extreme point. $\square$