Some Counterexamples to Chuck's Conjecture

Perhaps the simplest counterexample to Chuck's Conjecture is the following:

Example 1. Let X be a compact metric space, equipped with the Borel σ-algebra, and fix $x \in X$. Suppose μ is a probability measure on $(X, \mathscr{B}(X))$ for which $\{x\}$ is the only atom, with $\mu(\{x\}) = 0.1$. We define a second measure ν on X as

$$\nu(\{x\}) = 0.05$$
$$\nu(A) = \frac{0.95}{0.9}\mu(A)$$

for all $A \in \mathscr{B}(X)$ measurable subsets of X with $x \notin A$. For example, we can set $X = I$, $\mu = 0.9\lambda^1 + 0.1\delta_x$ and $\nu = 0.95\lambda^1 + 0.05\delta_x$ so that $\mu(X) = \nu(X) = 1$.

With such measures μ and ν, there is no perfect sharing. That is, there is no measurable subset $S \in \mathscr{B}(X)$ with $\mu(S) = \nu(S) = \frac{1}{2}$.

To see this, suppose there did exist a perfect sharing set S, and without loss of generality take $x \in S$ (if $x \notin S$, we consider S^c which is also a perfect sharing set containing x). Now, let $A = S \setminus \{x\}$. Then,

$$\frac{1}{2} = \mu(S) = \mu(A) + \mu(\{x\}) = \mu(A) + 0.1$$

and so

$$\mu(A) = 0.4$$

However, we also know that

$$\frac{1}{2} = \nu(S) = \nu(A) + \nu(\{x\}) = \nu(A) + 0.05$$

and so it must be that

$$\nu(A) = 0.45$$

By definition of ν, we have that

$$\nu(A) = \frac{0.95}{0.9}\mu(A) = \frac{0.95}{0.9}(0.4) \approx 0.42 \neq 0.45$$

so no such perfect sharing can exist for these two measures.

The previous example generalizes easily to arbitrary measures which satisfy the hypotheses.

Example 2. Let X be as before, and let μ be a probability measure on X which satisfies the hypotheses for the conjecture, with atoms $P = \{x_i\}_{i=1}^{\infty}$ and $\mu(P) < \frac{1}{2}$.

Fix ε so that $0 < \varepsilon < 0.1$, and that $\mu(P) + \varepsilon < \frac{1}{2}$. Fix some $x \in P^c$, and define ν another probability measure on X as

$$\nu(S) = (1-\varepsilon)\mu(S) + \varepsilon\delta_x(S)$$

for all measurable subsets S, so that $\nu(X) = 1$. This satisfies the hypotheses for the conjecture, but does not allow a perfect sharing.

To see this, suppose S is a perfect sharing set of X, and without loss of generality let $x \in S$, and let $A = S \setminus \{x\}$. Then, we have that

$$\frac{1}{2} = \mu(S) = \mu(A) + \mu(\{x\}) = \mu(A)$$

and so

$$\mu(A) = \frac{1}{2}$$

Similarly, for ν we calculate

$$\frac{1}{2} = \nu(S) = \nu(A) + \nu(\{x\}) = \nu(A) + \varepsilon$$

and so

$$\nu(A) = \frac{1}{2} - \varepsilon$$

However, we defined $\nu(A)$ to be

$$\nu(A) = (1-\varepsilon)\mu(A) + \varepsilon\delta_x(A) = (1-\varepsilon)\mu(A) = \frac{1}{2} - \frac{\varepsilon}{2} \neq \frac{1}{2} - \varepsilon$$

and so such a sharing cannot exist.

Finally, we consider a specific counterexample with infinitely many atoms in one measure that are not in the other.

Example 3. This example comes from Ethan Robinett.

Let λ be the Lebesgue measure on I^2. Let $f : \mathbb{Q}^2 \cap I^2 \to \mathbb{N}_{\geq 2}$ be a bijection of the rational points in I^2 to the natural numbers above 1. Define $c = \sum_{i=2}^{\infty} \frac{2}{10^i} = 0.0\bar{2}$, and for any Lebesgue measurable set $S \subset I^2$, define

$$\mu(S) = (1-c)\lambda(S) + \sum_{q \in \mathbb{Q}^2 \cap S} \frac{2}{10^{f(q)}}$$

Now, clearly μ is also a Lebesgue measure on I^2 with each element of $\mathbb{Q}^2 \cap I^2$ an atom for μ. Furthermore, each atom q has mass $\frac{2}{10^{f(q)}} < 0.1$, and the total mass of the atoms is $\mu(\mathbb{Q}^2) = c < 0.5$. Thus, μ also satisfies the hypotheses for the conjecture. However, there is no perfect sharing between μ and λ.

Suppose there was a perfect sharing set S. Then, we would have

$$\frac{1}{2} = \mu(S) = (1-c)\frac{1}{2} + \sum_{q \in \mathbb{Q}^2 \cap S} \frac{2}{10^{f(q)}}$$

which forces

$$\sum_{q \in \mathbb{Q}^2 \cap S} \frac{2}{10^{f(q)}} = \frac{c}{2} = 0.0\bar{1}$$

which cannot be attained, since the decimal expansion of the left-hand side is made up of only 0 and 2, whereas the right-hand side has only 0 and 1 in its expansion. Thus, no such perfect sharing can exist.