1 Curvature

Let's just straight-up define the curvature:

Definition 1.1. *Consider a Riemannian manifold (M, g), with smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$. We define*

$$R_m(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

Alternately,

$$R^d_{abc} \omega_d = \nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c$$

(Wald, p. 37)

Now, we need to establish that this is a tensor by showing it is function linear in each component.

Observe that

$$
\begin{aligned}
R_m(X,Y)fZ &= -\nabla_X \nabla_Y fZ + \nabla_Y \nabla_X fZ + \nabla_{[X,Y]} fZ \\
&= -X(Yf)Z - (Yf)\nabla_X Z - (Xf)\nabla_Y Z - f\nabla_X \nabla_Y Z + Y(Xf)Z + (Xf)\nabla_Y Z + Yf\nabla_X Z + f\nabla_Y \\
&= -f\nabla_X \nabla_Y Z + f\nabla_Y \nabla_X Z + f\nabla_{[X,Y]} Z
\end{aligned}
$$

as desired

Homework 1. *Show this is function-linear in other components.*

Note you can lower the contravariant index by applying g_{ab} i.e.

$$R_{abcd} = g_{dd'} R^{d'}_{abc}$$

Calculating Curvature

We can calculate the Riemann curvature tensor in coordinates by using the definitions of the covariant derivative.

$$\mathbb{R}^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \sum_\alpha (\Gamma^\alpha_{ac} \Gamma^d_{\alpha b} - \Gamma^\alpha_{bc} \Gamma^d_{\alpha a})$$

To make things easier, we can use local Riemannian normal coordinates by pushing the coordinates from $T_p M$ to M via the exponential map.

Homework 2. *Show that in Riemannian normal coordinates,*

$$\Gamma^k_{ij} = 0 \ at \ p$$

and

$$\partial_k g_{ij} = 0 \ at \ p$$

Definition 1.2. *an orthonormal frame $\{e_i\}$ on an open neighborhood of a point $p \in M$ is called normal around p if*

$$\nabla_a e_i = 0$$

at p.

1

The curvature follows the Bianchi Identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

In general, we have four important properties of the metric:

- $R^d_{abc} = R^d_{[ab]c}$ antiymmetry of the first two components

- $R^d_{[abc]} = 0$ the Bianchi identity

- $R_{abcd} = R_{ab[cd]}$ antiymmetry of the second two components

- $R_{abcd} = R_{cdab}$ symmetry in the first and second half components.

Note that item 4 can be derived from the other three.

An important concept not covered in Do Carmo:

Definition 1.3. *Given a finite dimensional vector space (over $\mathbb{R}$) V, consider the tensor C of rank $(0,4)$ (4 covariant indices). C is called an algebraic curvature tensor on V if it satisfies the above four properties (with appropriate index lowering).*

Sectional Curvature

Let $p \in M$ and let σ be a $2-$dimensional subspace of T_pM.

Definition 1.4. *The sectional curvature $K(\sigma)$ is defined to be*

$$K(\sigma) = R_m(e_1, e_2, e_1, e_2)$$

for $\{e_1, e_2\}$ an orthonormal basis for σ.

This definition is independent of choice of orthonormal basis by exploiting linearity of R_m.

This can also be expressed in an arbitrary basis u, v by

$$K(\sigma) = \frac{R_m(u,v,u,v)}{\|u \wedge v\|^2} \tag{1.1}$$

Where $\|u \wedge v\|^2$ is calculated from the inner product induced by the metric. That is, for $\{e_i\}$ an orthonormal basis for V, we declare $\{e_i \wedge e_j\} \, i < j$ to be orthonormal.

Homework 3. *Show that the induced inner product is independent of choice of orthonormal basis.*

Lemma 1. *Let V be a vector space (finite dimensional, real) of dimension at least 2 with an inner product. Consider two algebraic curvature tensors C_1 and C_2. Let K_1, K_2 denote the sectional curvatures of C_1 and C_2. $K_1 = K_2$ if and only if $C_1 = C_2$.*

Suppose C is such that $K(\sigma) = \kappa$ for all σ. Then,

$$C(x,y,z,w) = \kappa \left(g(x,z)g(y,w) - g(x,w)g(y,z) \right) \tag{1.2}$$

Ricci Curvature

Let R_m be a Riemannian curvature tensor, with components R^d_{abc}. We can take the trace over the first and third components to get

$$R_{ac} = Rabc^b \tag{1.3}$$

Geometrically, this is defined as

Definition 1.5. $R_{C_p}(u, w) = trace(R_{m_p}(u, \cdot)w)$.

In an orthonormal frame with $g(e_j, e_k) = \delta_{jk}$, we have

$$R_{ij} = R^k_{ikj} = R_{ikjk} \tag{1.4}$$

We can also define the Ricci scalar

Definition 1.6. $R = R_c(u, u)$ *for unit vector u.*

This can be given in coordinates as

$$R = R^i_i \tag{1.5}$$

Theorem 1. *The Ricci curvature tensor is symmetric*

Proof. We know that
$$R_{ac} = R^b_{abc}$$
But by symmetry of the Riemann curvature tensor, we have

$$R_{ac} = R^b_{abc}$$
$$= R^b_{cba}$$
$$= R_{ca}$$

as desired $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, let u be a unit vector, and build an orthonormal basis around u. Then,

$$R_c(u, u) = \sum R(e_1, e_i, e_1, e_i) = \sum K(e_1, e_i)$$

and
$$R = \sum R_c(e_i, e_i) = \sum K(e_i, e_j)$$

We also have the following identity for the Riemann curvature tensor R:

$$R(u \wedge v, w \wedge z) = R(u, v, w, z) \tag{1.6}$$

This relies on the antisymmetry of R, since R has to be linear.