«CLASS»

«TITLE»

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Problem 1

Show that for two nontrivial groups G and H, their free product $G \star H$ has trivial center, and that the only elements of finite order are the conjugates of finite order elements in G and H.

Proof. We first show that the free product has a trivial center. To show this, we will consider two cases of elements from $G \star H$. First, observe that for any element $g \in G$ with $g \neq e$, g (or rather, the inclusion of g into $G \star H$) cannot be in the center $Z(G \star H)$. To see this, let $h \in H$ with $h \neq e$, and note that the element $ghg^{-1}h^{-1}$ is a reduced word in the free group, and cannot be equal to the identity. Thus, $gh \neq hg$, and g is not in $Z(G \star H)$. Similarly, any element $h \in H$ is not in $Z(G \star H)$.

Now, let w be a reduced word in $G \star H$ that is not in G or H. Without loss of generality, we can write

$$w = g_1 h_1 g_2 h_2 \dots g_n h_n$$

or

$$w = g_1 h_1 g_2 h_2 \dots g_n$$

for $g_i \in G$, and $h_i \in H$. Now, since G and H have no relations to each other, this word is not equal to any word beginning with an element of H. Suppose for a contradiction that

$$w = h'_1 g'_1 h'_2 g'_2 \dots h'_m g'_m$$

with g'_m possibly equal to the identity. This would imply that

$$g_1 h_1 g_2 h_2 \dots g_n h_n = h'_1 g'_1 h'_2 g'_2 \dots h'_m g'_m$$

which would imply the relation

$$g_1 h_1 g_2 h_2 \dots g_n h_n (g'_m)^{-1} (h'_m)^{-1} \dots (g'_1)^{-1} (h'_1)^{-1} = e$$

Now, if $g'_m \neq e$, the word on the left hand side is already reduced, and thus cannot be equal to e. If $g'_m = e$, then the word could reduce further. However, note that this word only reduces further if adjacent elements from the same group cancel to the identity (that is, if adjacent elements from the same group do not cancel to the identity, then the word is reduced, and in particular is not equal to the identity). Now, careful counting of the elements of the word reveals that the (unreduced) word has one more element from H than it does elements from G. So, the total number of elements in the unreduced word is odd. Clearly, then, we cannot cancel in pairs to get the identity. Therefore, this relation cannot be true, and thus w cannot be written as a word beginning with an element from H.

It follows immediately, then, that $wh \neq hw$ for $h \in H$ a nontrivial element, since wh is a word beginning with an element of G, and hw is a word beginning with an element of H.

Thus, no element in $G \star H$ is in the center, as desired.

Next, we show that every element of finite order is the conjugate of a finite order element of G or H. So, let w be a word in $G \star H$ with $w^n = e$. Without loss of generality, let the first term of w be from G.

If $w = g_1 h_1 \dots g_n h_n$, then the concatenation w^n is already reduced (since adjacent elements are from different groups) and in particular cannot be equal to the identity.

So, suppose $w = g_1 h_1 \dots h_{n-1} g_n$, with $w^n = e$. Now, for this to be true, it must be that $g_n = g_1^{-1}$. If this were not the case, the concatenated word w^n would be reduced already (by treating $g_n g_1$ as an element of G) since $g_n g_1$ is adjacent to only elements of H. Continuing the argument, we find that $h_{n-1} = h_1^{-1}$, $g_{n-1} = g_2^{1}$ and so on, save for the central term (since w has odd length). If n is even, the central term is $h_{\frac{n}{2}}$, and if n is odd, the central term is $g_{\frac{n+1}{2}}$. For the sake of simplicity, we will assume that n is odd, but the proof works the same way for n even.

Now, we know that

$$w = g_1 h_1 g_2 h_2 \dots h_{\frac{n+1}{2}-1} g_{\frac{n+1}{2}} h^{-1}_{\frac{n+1}{2}-1} \dots h_2^{-1} g_2^{-1} h_1^{-1} g_1^{-1}$$

which is just

$$w = (g_1 h_1 g_2 h_2 \dots h_{\frac{n+1}{2}-1}) g_{\frac{n+1}{2}} (g_1 h_1 g_2 h_2 \dots h_{\frac{n+1}{2}-1})^{-1}$$

and thus w is a conjugate of an element of G. Now for $w^n = e$ it must be that

$$w^n = (g_1 h_1 g_2 h_2 \dots h_{\frac{n+1}{2}-1}) g^n_{\frac{n+1}{2}} (g_1 h_1 g_2 h_2 \dots h_{\frac{n+1}{2}-1})^{-1} = e$$

which is only true if $g^n = e$.

Thus, the elements of finite order in $G \star H$ are all conjugates of elements of finite order in G or H.

Finally, we observe that every element that is the conjugate of an element of finite order in G (or H) is of finite order. Let $g \in G$ with $g^n = e$, and observe that for any $w \in G \star H$, we have that

$$(wgw^{-1})^n = wg^n w^{-1} = ww^{-1} = e$$

and so wgw^{-1} has finite order as well.

Thus the elements of finite order in $G \star H$ are exactly the conjugates of elements of finite order in G or H, as desired. $\square$

Problem 2

Let $X \subset \mathbb{R}^n$ be the union of convex open sets $X_1, \dots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Prove that X is simply connected.

Proof. This proof will be done inductively on the number of convex open sets used to union to X. Now, convex sets have a distinct property which makes them nice for this problem. Namely, if X is a convex set, and γ is a path in X, γ is homotopic to the line segment from $\gamma(0)$ to $\gamma(1)$. This is clear; since convex sets are simply connected, all paths with fixed endpoints are homotopic to each other, and since each convex set is convex, it contains the straight line path from $\gamma(0)$ to $\gamma(1)$.

Now, let's first prove the base case of $n = 2$. For this, let f be a loop in X based at x_0. Without loss of generality, we let $x_0 \in X_1$. Now, if f stays in X_1, then it is nullhomotopic since X_1 is simply connected. Now, suppose f enters X_2 at some time t_1, exits X_1 at some $t > t_1$, and exits X_2 at $t_2 > t$. Then, the segment $f|_{[t_1,t_2]}$ is homotopic to the straight line from $f(t_1)$ to $f(t_2)$, which is in X_1. Repeating this for the (finite) number of times f exits X_1, we see that f is homotopic to a loop that stays in X_1, and is thus nullhomotopic.

Now suppose the theorem holds for the union of $n-1$ convex sets with the triple intersection property, and let X be the union of n convex sets with the same property. Let f be a loop in X based at $x_0 \in X_1$. Now, if f stays in $\cup_{i=1}^{n-1} X_i$ (that is, if f avoids being in only X_n), then f is a loop in the union of $n-1$ convex sets with the triple intersection property, and is nullhomotopic.

So, suppose f enters X_n at some time t_1 from a set X_i, leaves X_i at some $t > t_1$, then leaves X_n in $X_j \cap X_n$ at some t_2. Now, by the triple intersection property, we know that $X_i \cap X_j \cap X_n \neq \emptyset$, so let x be a point in the common intersection. Now, since $f(t_1)$, $f(t_2)$, and x are all in X_n, so is the path taking the straight line from $f(t_1)$ to x, then the straight line from x to $f(t_2)$. Then, the segment $f|_{[t_1,t_2]}$ is homotopic to this path from $f(t_1)$ through x to $f(t_2)$. However, since $f(t_1)$ and x are in X_i, so is the straight line connecting them. Furthermore, since x and $f(t_2)$ are in X_j, so is the straight line connecting them. Thus, $f|_{[t_1,t_2]}$ is homotopic to a path that stays in X_i and X_j. Repeating this process for the (finite) number of times f stays in X_n alone, we see that f is homotopic to a loop that stays in $\cup_{i=1}^{n-1} X_i$, and by the inductive hypothesis, f is nullhomotopic as desired.

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Problem 3