Problem Set 4

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Problem 1

Show that if a smooth function f is such that for all x, there is some N_x such that the nth derivative of f vanishes at x for all $n > N_x$, then f is a polynomial.

Proof. Let G be the set of all $x \in \mathbb{R}$ such that there exists some neighborhood of x for which f is a polynomial on that neighborhood, and let $F = G^c$.

Now, we first observe that F has no isolated points. Indeed, suppose $c \in \mathbb{R}$ were such that f was a polynomial on (a, c) and (c, b) for some $a, b \in \mathbb{R}$. Then, by observation 2 made in class, since f at c has derivatives vanishing beyond N_c and f is a polynomial on (a, c) and (c, b), we know that f is the same polynomial on $(a, c]$ and $[c, b)$ and so c cannot be in F (since it is now in G).

Now, we wish to show that $F = \emptyset$. So, suppose for contradiction F is nonempty. Now, since G is open (by definition) F is closed, and is thus a complete metric space as a closed subset of $\mathbb{R}$.

Define $E_n = \{x \in F \mid f^{(j)}(x) = 0 \forall j > n\}$. Now, $F = \cup E_n$ and so at least one of them E_{n_0} must contain an interval $(x_0 - r, x_0 + r) \cap F$ for $x_0 \in F$ (by Baire's lemma).

Now, for all $x \in (x_0 - r, x_0 + r) \cap F$, we have that $f^{(j)}(x) = 0$ for all $j > n_0$. If $(x_0 - r, x_0 + r)$ does not intersect G, then $f^{(j)}(x) = 0$ for all $j > n_0$ and all $x \in (x_0 - r, x_0 + r)$ and so $x_0 \in G$, a contradiction.

So, if $I = (a, b)$ is an interval in G contained in $(x_0 - r, x_0 + r)$, then $b \in F$. But b is such that $f^{(j)}(b) = 0$, and so f is a polynomial on $[a, b]$, and this leads to a contradiction on $(x_0 - r, x_0 + r)$ containing points of F.

Thus, F is empty, and f is a polynomial on all of $\mathbb{R}$. □

Problem 2

Prove that if a vector space is Banach with respect to two norms then the topologies induced by the norms are either equivalent or incomparable.

Proof. We will show that for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V that is complete with respect to the induced topologies τ_1, τ_2, if $\tau_1 \subset \tau_2$, then $\tau_1 = \tau_2$ which completes the proof.

To see this, suppose $\tau_1 \subset \tau_2$. Then, the ball $B^1(0,1)$ with respect to the 1 norm is open in τ_2, and thus 0 has a ball $B^2(0,\epsilon) \subset B^1(0,1)$. Thus, by homogeneity of vector spaces, every 1 norm ball contains a 2 norm ball centered at the same point.

Now, let V_1 denote V under the $\|\cdot\|_1$ norm. Now, we know that $V = \cup_{x\in V_1} B^2(x,1)$ for B^2 the balls in the 2 norm. Now, Baire's lemma guarantees that for some x_0, there is a ball $B^1(x_0,\delta) \subset B^2(x_0,1)$. Then, translating these to the origin yields $B^1(0,\delta) \subset B^2(0,1)$. Thus, every 2 norm ball contains a 1 norm ball centered at the same point.

Thus, the two topologies are equivalent. □

Problem 3

Let $A \in B(X,Y)$ for X, Y Banach spaces. Suppose that for every $f \in Y$, the equation $Au = f$ is solvable. Prove that there exists $C < \infty$ such that for every f in y, one can find a solution to $Au = f$ with $\|u\|_X < C\|f\|_Y$.

Proof. Since $Au = f$ is solvable for any f, A is injective and satisfies the hypotheses for the open mapping theorem. Thus, A is an open map. This means that $A(B_X(0,1))$ is open in Y, and contains zero. Thus, there is some ϵ for which $B_Y(0,\epsilon) \subset A(B_X(0,1))$. Thus, for f such that $\|f\| < \epsilon$, and for u such that $Au = f$, we have $\|u\| < 1$.

So, let f be arbitrary. Then, let $\tilde{f} = \frac{\epsilon f}{2\|f\|}$ with solution $\tilde{u} = \frac{\epsilon u}{2\|f\|}$. Now, $\|\tilde{f}\| < \epsilon$ and so $\|\tilde{u}\| < 1$. Thus, $\|u\| < \frac{2}{\epsilon}\|f\|$ as desired. □

Problem 4

Let X be a normed space, $E \subset X$. Suppose that for every $\phi \in X^*$ the set $\{\phi(x) \mid x \in E\}$ is bounded. Prove that E is strongly bounded in X.

Proof. Let's start by embedding E into X^{**} via the canonical mapping. Then, we notice that the hypothesis states that for all $\phi \in X^*$, the set $\{x(\phi) \mid x \in E\}$ is bounded. In particular, this means that for all $\phi \in X^*$, we have that $\sup_{x\in E} |x(\phi)|$ is finite. Since X^* is complete, we can apply the uniform boundedness principle to the collection E of linear functionals on X^* (as operators from X^* to $\mathbb{R}$) to get the result

$$\sup_{x\in E} \|x\| < \infty$$

and thus since E was isometrically embedded into X^{**}, it follows that E is bounded as desired. □

Problem 5

Let X be a Banach space, and let $E \subset X^*$. Suppose also that for every $x \in X$, the set $\{\phi(x) \mid \phi \in E\}$ is bounded. Prove that E is strongly bounded in X^*.

Proof. We note first that the condition that $\{\phi(x) \mid \phi \in E\}$ is bounded means that $\sup_{\phi \in E} |\phi(x)|$ is bounded for each x. Thus, we can apply the uniform boundedness principle to the family of operators E from X to $\mathbb{R}$ to get the result

$$\sup_{\phi \in E} \|\phi\| < \infty$$

as desired.

If X is not complete, then we cannot use the uniform boundedness principle, and the proof falls apart. □

Problem 6

Let X be a Banach space decomposed as $X = X_1 \oplus X_2$ with closed subspaces X_i. Define the projection operators $P_i : X \to X_i$ and prove they are bounded.

Proof. Without loss of generality, we will define P_1 and show it is bounded. The argument works the same for P_2.

Define $P_1(x) = P_1(x_1 + x_2) = x_1$ for the (unique) decomposition $x = x_1 + x_2$. Now, clearly P_1 is linear, so all we have to do is show it is bounded.

Since $P_1 : X \to X_1$ is a map between Banach spaces, we only need to show its graph is closed. So, let $(x_k, P_1(x_k)) \to (x, y)$ for some $x \in X$ and $y \in X_1$. Now, for each x_k, we can decompose it as

$$x_k = (x_k)_1 + (x_k)_2$$

for $(x_k)_i \in X_i$. Now, since $x_k \to x$ and $P_1(x_k) = (x_k)_1 \to y$, it must be that $(x_k)_2 \to z$ for some $z \in X_2$. Thus, $x = y + z$ and $P_1(x) = y$ as desired. Thus, the graph of P_1 is closed, and P_1 is bounded as desired. □