Problem Set 7

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Problem 1

Show that for a subset S of V, and $x \in V$, $d(x, S) = 0 \iff x \in \overline{V}$.

Proof. ($\implies$) Suppose that $d(x, S) = 0$. This means that for any $\varepsilon > 0$, there is some $s \in S$ such that $d(x, s) < \varepsilon$. So, for any ε-ball centered at x, there is some $s \in S$ in it. Thus, every neighborhood of x intersects S, and x is in the closure of S.

($\impliedby$) Suppose that $x \in \overline{S}$. Then, we know that for every $\varepsilon > 0$, the ε-ball around x intersects S. In other words, for every $\varepsilon > 0$, there is some $s \in S$ with $d(x, s) < \varepsilon$. It follows immediately, then, that $d(x, S) = \inf_{s \in S} d(x, s) = 0$ as desired. $\square$

Problem 2

Show that the ℓ^1 norm is equivalent to the norm

$$\|x\| = 2\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty}\left(1+\frac{1}{n}\right)|x_n|$$

Proof. For the first bound, we note that

$$\begin{aligned}
\|x\| &= 2\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty}\left(1+\frac{1}{n}\right)|x_n| \\
&\leq 2\sum_{n=1}^{\infty}|x_n| + \sum_{n=1}^{\infty} 2|x_n| \\
&= 2\|x_n\|_1 + 2\|x_n\|_1 \\
&= 4\|x_n\|_1
\end{aligned}$$

so the new norm is bounded above by the ℓ^1 norm.

For the other direction, we take a bit more care. We note that

$$\begin{aligned}
\|x\| &= 2\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty}\left(1+\frac{1}{n}\right)|x_n| \\
&\geq 2\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty}|x_n| \\
&\geq \frac{1}{2}\left|\sum_{n=1}^{\infty} x_n\right| + \sum_{n=2}^{\infty}|x_n| \\
&\geq \frac{1}{2}\left||x_1| - \sum_{n=2}^{\infty}|x_n|\right| + \sum_{n=2}^{\infty}|x_n| \\
&\geq \frac{1}{2}|x_1| - \frac{1}{2}\sum_{n=2}^{\infty}|x_n| + \sum_{n=2}^{\infty}|x_n| \\
&= \frac{1}{2}|x_1| + \frac{1}{2}\sum_{n=2}^{\infty}|x_n| \\
&= \frac{1}{2}\|x\|_1
\end{aligned}$$

where the inequality from line 3 to line 4 is obtained by using the reverse triangle inequality on the infinite sum.

So the new norm is bounded below by the ℓ^1 norm, and thus the norms are equivalent. □

Problem 3

Let V, W be normed vector spaces, with V Banach. Let $A \in \mathscr{B}(V, W)$. Show that $V/\ker A$ is Banach with the quotient norm.

Proof. To begin with, we observe that the short exact sequence of continuous linear functions

$$0 \longrightarrow \ker A \xhookrightarrow{i} V \xrightarrow{\pi} V/\ker A \longrightarrow 0$$

exists. This is clearly exact, since for any $a \in \ker A$, $\pi(a) = [0]$, and furthermore, for any $v \in V$ with $\pi(v) = [0]$, we have that $v \in 0 + \ker A = \ker A$. Thus, $\mathrm{im} i = \ker \pi$, and the sequence is exact.

Furthermore, this sequence splits. This follows from the general result that any short exact sequence of vector spaces splits, but a proof for this case will be replicated.

We construct a retract $r : V \to \ker A$ such that $r \circ i = id$. This is done by first fixing a basis $\mathscr{B}$ for $\ker A$. Then, a basis $\mathscr{B}'$ for V can be chosen so that $\mathscr{B}' = \mathscr{B} \coprod \mathscr{C}$ for some set of vectors $\mathscr{C}$. Then, r can be defined as

$$r\left(\sum_{b \in \mathscr{B}} v_b b + \sum_{c \in \mathscr{C}} v_c c\right) = \sum_{b \in \mathscr{B}} v_b b$$

which is clearly a linear continuous function. Furthermore, it is clear that $r \circ i = id$ by the definition of r. Thus, r defines a retract of i, and the sequence splits.

So, by the splitting lemma, we have that

$$0 \longrightarrow \ker A \underset{r}{\overset{i}{\rightleftarrows}} V \underset{s}{\overset{\pi}{\rightleftarrows}} V/\ker A \longrightarrow 0$$

where s is a section of π, and r is a retract of i. Thus, we have the continuous injective map $s : V/\ker A \to V$.

Now, let $[x]_n$ be a Cauchy sequence in $V/\ker A$. This sequence lifts along s to the sequence $s([x]_n)$ in V. Now, since V is complete, and s is continuous, it follows that $s([x]_n)$ is Cauchy in V, and we have that $s([x]_n) \to x$ for some $x \in V$.

I now assert that $[x]_n \to \pi(x)$. To see this, we consider (since π is continuous) the fact that $\pi(s([x]_n)) \to \pi(x)$.

But since s is a section of π, it must be that $\pi \circ s = id$, so $\pi(s([x]_n)) = [x]_n \to \pi(x)$ as desired. $\square$

Problem 4

Suppose $V = X_1 \oplus X_2$ and that X_1 is finite dimensional. Show that the projection operator $P_1 : V \to X_1$ is in $\mathscr{B}(V, X_1)$.

Proof. We will first show that P_1 is bounded by using the fact that all norms on finite dimensional vector spaces are equivalent. Equipped with this, and the fact that $X_1 \cong V/X_2$, we see that

$$\begin{aligned}
\|P_1 x\| &\le C\|[x]\|_Q \\
&= C \inf_{x_2 \in X_2} \|x - x_2\| \\
&\le C\|x - 0\| \\
&= C\|x\|
\end{aligned}$$

Thus, P_1 is bounded, and is in $\mathscr{B}(V, X_1)$.

Now, since $P_2 = I - P_1$, we have that $\|P_2\| = \|I - P_1\| \le \|I\| + \|P_1\|$, and since I and P_1 are bounded, so is P_2. □

Problem 5

For V a normed space, Y a closed subspace of V with finite codimension, show that for $\phi \in V'$ with ϕ continuous on Y, $\phi \in V^*$.

Proof. Consider the decomposition $V = Y \oplus V/Y$, and let $v = y + x \in V$. Now, we have that

$$\begin{aligned}
\|\phi(y + x)\| &= \|\phi(y) + \phi(x)\| \\
&\le \|\phi(y)\| + \|\phi(x)\|
\end{aligned}$$

now, since ϕ is continuous on Y, it is bounded on Y, so $\|\phi(y)\|$ is bounded. Furthermore, since V/Y is of finite dimension, it follows that ϕ is bounded on V/Y as well, so $\|\phi(x)\|$ is bounded too.

So, we have that

$$\begin{aligned}
\|\phi(v)\| &\le \|\phi(y)\| + \|\phi(x)\| \\
&\le C_1\|y\| + C_2\|x\| \\
&= C_1\|P_1(v)\| + C_2\|P_2(v)\| \\
&\le (C_1 K_1 + C_2 K_2)\|v\|
\end{aligned}$$

where K_1 and K_2 are the bounds of the projection operators, which exist as a result of problem 4.

Thus, $\|\phi(x + y)\|$ is bounded as well, and $\phi \in V^*$ as desired. □

Problem 6

Let V_1, V_2 be subspaces of some vector space L. Prove that $(V_1 + V_2)/V_2 \cong V_1/V_1 \cap V_2$. Furthermore, prove that if $\dim L/V_1 \leq n_1$ and $\dim L/V_2 \leq n_2$, then $\dim L/V_1 \cap V_2 \leq n_1 + n_2$.

Proof. The first result is a direct restatement of the second isomorphism theorem for Abelian groups. We replicate the proof here:
Consider the surjective homomorphism $\phi : V_1 + V_2 \to V_1/V_1 \cap V_2$ as $\phi(v_1 + v_2) = [v_1]$ for $v_i \in V_i$. The kernel of this homomorphism is any vector that gets sent to $[0]$, which is precisely the vectors $0 + v_2$. It is clear that $0 + V_2 \subset \ker \phi$. Furthermore, suppose $v = v_1 + v_2$ for some $v_1 \neq 0$ ($v_1 \notin V_2$). Then, $\phi(v) = [v_1] \neq 0$ since $v_1 \notin V_2$. So, $\ker \phi = V_2$.

Thus, by the first isomorphism theorem, we have that

$$V_1 + V_2/V_2 \cong V_1/V_1 \cap V_2$$

as desired.

We observe first that since L/V_2 is finite dimensional, so is $V_1 + V_2/V_2$ since it is a subspace of L/V_2. And, if $\dim L/V_2 \leq n_2$, then $\dim V_1 + V_2/V_2 \leq n_2$ as well.

Now, consider that by the third isomorphism theorem, we have that

$$L/V_1 \cong \frac{L/(V_1 \cap V_2)}{V_1/(V_1 \cap V_2)}$$

and since both L/V_1 and $V_1/(V_1 \cap V_2) \cong V_1 + V_2/V_2$ are finite dimensional, it must be that $L/(V_1 \cap V_2)$ is as well. Furthermore, we have that

$$\begin{aligned}\dim L/V_1 &= \dim L/(V_1 \cap V_2) - \dim V_1/(V_1 \cap V_2) \\ \dim L/(V_1 \cap V_2) &= \dim L/V_1 + \dim V_1/(V_1 \cap V_2) \\ &\leq n_1 + n_2\end{aligned}$$

as desired. $\square$