# Problem Set 1

Daniel Halmrast

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## Problem 1

Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ to $S^{n-1}$.

*Proof.* The straight-line homotopy from $v$ to $\frac{v}{\|v\|}$ satisfies the criteria for a deformation retract. Namely, the retract is given by

$$\begin{aligned} r :& \mathbb{R}^n \setminus \{0\} \to S^{n-1} \\ r(v) =& \frac{v}{\|v\|} \end{aligned}$$

With homotopy

$$\begin{aligned} F :& \mathbb{R}^n \setminus \{0\} \times I \to S^{n-1} \\ F(v,t) =& (1-t)v + t\frac{v}{\|v\|} \end{aligned}$$

$\square$

# Problem 2

Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

*Proof.* Let $f : X \to Y$ be a map, which is homotopic to a homotopy equivalence $g : X \to Y$ with homotopy inverse $h : Y \to X$. That is, $g \circ h \simeq \mathbb{1}_Y$ and $h \circ g \simeq \mathbb{1}_X$. Furthermore, let $F : X \times I \to Y$ be the homotopy between $f$ and $g$.

First, let's consider the map $h \circ f : X \to X$. We wish to show $h \circ f \simeq \mathbb{1}_X$. To do so, let's consider the homotopy

$$h \circ F : X \times I \to X$$

This is the composition of two continuous functions, and so it is continuous. Furthermore, since $F(0, x) = f(x)$ and $F(1, x) = g(x)$, this is actually a homotopy between $h \circ f$ and $h \circ g$. Now, since $h \circ f \simeq h \circ g \simeq \mathbb{1}_X$ and homotopy equivalence is an equivalence relation, it follows immediately that $h \circ f \simeq \mathbb{1}_X$.

Now, consider the map $f \circ h : Y \to Y$. We wish to show $f \circ h \simeq \mathbb{1}_Y$. To do so, consider the homotopy

$$F \circ (h \times \mathbb{1}_I) : Y \times I \to Y$$

It is easy to see this is a homotopy between $f \circ h$ and $g \circ h$, and so we have that $f \circ h \simeq g \circ h \simeq \mathbb{1}_X$, and so $f \circ h \simeq \mathbb{1}_X$, as desired. □

# Problem 3

A deformation retraction in the weak sense of a space $X$ to a subspace $A$ is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}_X$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all $t$. Show that if $X$ deformation retracts onto $A$ in the weak sense, then the inclusion map $i : A \to X$ is a homotopy equivalence.

*Proof.* Let $f_t : X \to X$ be a deformation retraction in the weak sense of $X$ onto $A$, and let $i$ be the inclusion map from $A$ to $X$. We will show that $i \circ f_1 \simeq \mathbb{1}_X$ and that $f_1 \circ i \simeq \mathbb{1}_A$.

Considering $i \circ f_1$, we note that this is actually equal to $f_1$, since the inclusion map is the identity on $A$, and $f_1$ maps into $A$. Now, $f_1$ is homotopic to $f_0$ which is equal to $\mathbb{1}_X$, and so by transitivity of homotopy equivalence, $f_1 \simeq \mathbb{1}_X$.

Now, let's consider $f_1 \circ i$. We note first that this is equal to $f_1|_A$, since $i$ is the identity on $A$. Furthermore, the restrictions $f_t|_A$ define a homotopy from $f_1|_A$ to $f_0|_A$, and so we have that

$$f_1 \circ i = f_1|_A \simeq f_0|_A = \mathbb{1}_X|_A = \mathbb{1}_A$$

as desired. Thus, $i$ is a homotopy equivalence with homotopy inverse $f_1$. $\square$

# Problem 4

Show that if a space $X$ deformation retracts to a point $x \in X$, then for each neighborhood $U$ of $x$ in $X$, there exists a neighborhood $V \subset U$ of $x$ such that the inclusion map $V \to U$ is nullhomotopic.

*Proof.* Let $F : X \times I \to X$ be the deformation retraction of $X$ onto $x_0 \in X$, and let $U$ be a neighborhood of $x_0$. Now, consider the open set

$$F^{-1}(U) \subset X \times I$$

Now, since $F(x_0, t) = x_0$ for all $t$, we know that $\{x_0\} \times I$ is in $F^{-1}(U)$. Applying the tube lemma, we find an open set $V$ containing $x_0$ for which $V \times I \subset F^{-1}(U)$. In particular, this means that for all $v \in V$, we have that $F(v, t) \in U$ for all $t$.

Now we are ready to show that the inclusion map from $V$ to $U$ is nullhomotopic. We note that $F \circ (i \times \mathbb{1}_I) : V \times I \to U$ defines a homotopy from $f_0 \circ i = i$ to $f_1 \circ i = c_{x_0}$, where $c_{x_0}$ is the constant function to $x_0$. This is clear, since the image of $i$ is $V \subset U$, and the image of $V$ under $F$ is always in $U$, as proved above. Thus, since the domain of $F$ matches the image of $i$, and the image of $F$ stays inside $U$, this is a well-defined homotopy.

Therefore, $i \simeq c_{x_0}$ as desired. $\square$

# Problem 5

Consider the subspace $X \subset \mathbb{R}^2$ defined as

$$X = [0,1] \times \{0\} \cup \bigcup_{r \in \mathbb{Q}} \{r\} \times [0, 1-r]$$

## Part a

Show that $X$ deformation retracts to any point in $[0,1] \times \{0\}$, but not any other point.

*Proof.* To construct the deformation retraction of $X$ to a point in the interval $[0,1] \times \{0\}$, we first note that $X$ deformation retracts onto $[0,1] \times \{0\}$ via the straight-line homotopy along the $y$-axis. It is clear also that the unit interval deformation retracts to any point on it via the straight-line homotopy along the $x$-axis. Running the first homotopy for the first half time, and running the second homotopy for the second half time yields a deformation retraction of $X$ onto a point in the interval $[0,1] \times \{x\}$.

Now, consider a point $x$ not in the base interval. Consider also a neighborhood $U$ of $x$ that does not intersect the base interval. Any neighborhood $V \subset U$ containing $x$ will necessarily intersect at least one other stalk than the one $x$ is in (since the rationals are dense in $\mathbb{R}$), and these stalks will not be connected, since $V$ does not intersect the base interval. Thus, the inclusion map of $V$ into $U$ cannot be nullhomotopic, and by problem 4, we know that $X$ therefore cannot deformation retract onto $x$. □

## Part b

Let $Y$ be the subset of $\mathbb{R}^2$ that is the union of infinite copies of $X$ in a zigzag pattern. Show that $Y$ is contractible, but does not deformation retract onto any point.

*Proof.* To show that $Y$ is contractible, we reference part c of this problem, which asserts the existence of a deformation retraction in the weak sense of $Y$ onto the zigzag subspace $Z$. Now, problem 3 guarantees that if $Y$ deformation retracts onto $Z$ in the weak sense, then the inclusion $i : Z \to Y$ is a homotopy equivalence, and thus $Y$ and $Z$ have the same homotopy type. However, $Z$ is homeomorphic to $\mathbb{R}$, which has the homotopy type of a point. Therefore, by transitivity of the homotopy equivalence, $Y$ has the homotopy type of a point as well.

Now, we must show that $Y$ does not deformation retract onto any point. To do so, we look at any point $x$ in $Y$. If $x$ is not in $Z$, the same argument from part a can be applied to show that $Y$ cannot deformation retract onto $x$. If $x$ is in $Z$, we observe that $x$ is actually in a stalk of the copy of $X$ running parallel to the line segment of $Z$ that $x$ is on. Noting then that $x$ is on a stalk, we apply the same argument as the one in part a to see that $Y$ cannot deformation retract onto $x$. □

## Part c

Let $Z$ be the zigzag subspace of $Y$ homeomorphic to $\mathbb{R}$. Show that there is a deformation retraction in the weak sense of $Y$ onto $Z$, but no true deformation retraction.

*Proof.* We can construct a deformation retraction in the weak sense explicitly for $Y$ onto $Z$. For each stalk, we define its "direction of motion" to be towards $Z$, and on $Z$ we define its "direction of motion" to be towards the right. Now, the deformation retraction in the weak sense sends points at constant velocity 1 along the direction of motion. Away from $Z$, this is clearly continuous, and on $Z$, we see that all points are moving at the same speed, so whatever stalks $Z$ is close to are retracting at the same speed $Z$ itself is moving. Thus, points stay close to each other, and the motion is continuous. This is a weak deformation retraction, since it does not fix any point in $Z$, but every point in $Z$ gets mapped to (for example, by the point exactly one unit of length before it on $Z$).

However, there is no true deformation retraction of $Y$ onto $Z$, since if there were, it could be concatenated with a deformation retraction of $Z$ onto a point in $Z$ to yield a deformation retraction of $Y$ onto a point in $Z \subset Y$. However, this would contradict part b, and so no such deformation retraction can exist. $\square$

## Problem 6

Prove that the homotopy $F$ from the proof of lemma 1.2 is continuous.

*Proof.* Recall from the proof of lemma 1.2 that we have the following diagram

$$\begin{tikzcd} Z \times I \arrow[r, "\tilde{F}"] \arrow[d, "\pi"] & Z \arrow[d, "\pi"] \\ M_f \times I \arrow[r, "F"] & M_f \end{tikzcd}$$

Where $\tilde{F}$ was previously defined, and $F$ is such that the diagram commutes. Now, $F$ is unique, since the nontrivial fibers of $\pi$ are stable with respect to $\tilde{F}$. This is clear, since $\tilde{F}$ is a homotopy relative to $(X \times 1) \coprod Y$, and so it fixes that subspace.

Now, we wish to show that $F$ is continuous. That is, we wish to show that $F^{-1}(U)$ is open for every $U \in M_f$ open. Recall first that for a quotient space $X \xrightarrow{\pi} Y$, a subset $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open in $X$. Thus, to show that $F^{-1}(U)$ is open, we wish to show that $\pi^{-1} \circ F^{-1}(U)$ is open. However, by commutativity of the diagram, this amounts to showing that $\tilde{F}^{-1} \circ \pi^{-1}(U)$ is open.

Now, $\pi^{-1}(U)$ is open in $Z$ by the definition of the quotient topology. Furthermore, since $\tilde{F}$ is continuous, $\tilde{F}^{-1}(\pi^{-1}(U))$ is also open. Thus, it follows that $F^{-1}(U)$ is open as well, and $F$ is continuous. $\square$

# Problem 7

Show that for homotopies $F, G : X \times I \to Y$ such that $F(x,1) = G(x,0)$, the concatenation homotopy $H : X \times I \to Y$ given by

$$H(x,t) = \begin{cases} F(x,2t), & t \in [0, \frac{1}{2}] \\ G(x, 2(t-\frac{1}{2})), & t \in [\frac{1}{2}, 1] \end{cases}$$

is continuous.

*Proof.* Let us reinterpret the question. Consider the quotient space $(Z = X \times I_1 \coprod X \times I_2)/{\sim}$ for $(x, 1_1) \sim (x, 0_2)$ with $1_1 \in I_1, 0_2 \in I_2$. That is, glue two copies of $X \times I$ to each other along the surface $X \times \{1\}$ and $X \times \{0\}$.

Now, this space is homeomorphic to $X \times I$ via the homeomorphism that sends $(x,t)$ to $(x, \frac{t}{2})$. Furthermore, we can define $\tilde{H}$ on $Z$ as

$$\tilde{H}(x,t) = \begin{cases} F(x,t), & t \in I_1 \\ G(x,t), & t \in I_2 \end{cases}$$

This function will be well-defined with respect to the quotient, since if $(x, t_1) \sim (x, t_2)$, then it must be that $t_1 = 1, t_2 = 0$, and we are guaranteed that $F(x,1) = G(x,0)$. That is, $\tilde{H}$ is constant on the nontrivial fibers of the quotient map.

Therefore, we can construct a map $H' : Z/{\sim} \to Y$ to be the unique function that makes the diagram

$$\begin{tikzcd} Z \arrow[r, "\tilde{H}"] \arrow[d, "\pi"] & Y \\ Z/{\sim} \arrow[ur, "H'"] & \end{tikzcd}$$

commute. It is easy to see that such an $H'$ is continuous. First, observe that $\tilde{H}$ is continuous, since it maps from the coproduct space $Z$ and is continuous on each copy of $X \times I$. The universal property of coproducts then guarantees that $\tilde{H}$ itself is continuous. Then, applying a similar argument to the one in problem 6, we find that $H'$ is continuous as well.

It is clear that $H$ and $H'$ coincide up to homeomorphism. That is, the diagram

$$\begin{tikzcd} Z/{\sim} \arrow[r, "H'"] \arrow[d, "\cong"] & Y \\ X \times I \arrow[ur, "H"] & \end{tikzcd}$$

commutes. Now, since $H'$ is continuous, it follows immediately that $H$ is continuous as well. $\square$