Problem Set 5

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Problem 1

Let M be the open submanifold of $\mathbb{R}^2$ with both coordinates positive, and define $F : M \to M$ as $F(x,y) = (xy, \frac{y}{x})$. Show that F is a diffeomorphism, and compute dFX and dFY for

$$\begin{aligned} X &= x\partial_x + y\partial_y \\ Y &= y\partial_x \end{aligned}$$

Proof. To begin with, we observe that so long as $x \neq 0$, F is in fact smooth. Furthermore, it has an inverse

$$F^{-1}(x,y) = (\sqrt{\frac{x}{y}}, \sqrt{xy})$$

which is also smooth on M, and defined for all of M.

Furthermore, we can calculate its Jacobian:

$$J(F) = dF = \begin{bmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{bmatrix}$$

which expresses dF in the coordinates ∂_x, ∂_y at every point.

Now, we calculate $dF(X)$:

$$\begin{aligned} dF(X) &= dF(x\partial_x + y\partial_y) \\ &= xdF(\partial_x) + ydF(\partial_y) \\ &= x(y\partial_x - \frac{y}{x^2}\partial_y) + y(x\partial_x + \frac{1}{x}\partial_y) \\ &= xy\partial_x - \frac{y}{x}\partial_y + yx\partial_x + \frac{y}{x}\partial_y \\ &= 2xy\partial_x \end{aligned}$$

and $dF(Y)$:

$$\begin{aligned} dF(Y) &= dF(x\partial_y) \\ &= xdF(\partial_y) \\ &= x(x\partial_x + \frac{1}{x}\partial_y) \\ &= x^2\partial_x + \partial_y \end{aligned}$$

□

Problem 2

Let M be a smooth manifold, $S \subseteq M$ an embedded submanifold. Given $X \in \mathcal{X}(S)$, show that there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$. Show that every such vector field extends to all of M if and only if S is properly embedded.

Proof. Recall that a vector field $X \in (X)(S)$ is a linear derivation of the algebra $C^\infty(M)$ over $\mathbb{R}$. That is, X is a linear map $X : C^\infty(M) \to C^\infty(M)$ such that

$$X(fg) = X(f)g + fX(g)$$

Now, from an earlier assignment, we know that for $S \subseteq M$ an embedded submanifold of a manifold M, we have the existence of extensions of C^∞ functions on S to C^∞ functions on the neighborhood $U \subseteq M$ of S. That is, the restriction function $r : C^\infty(U) \to C^\infty(S)$ has a section $e : C^\infty(S) \to C^\infty(U)$ such that $r \circ e = id$. Thus, we have the following diagram:

$$\begin{tikzcd} C^\infty(S) \arrow[r, "X"] & C^\infty(S) \arrow[d, "e"] \\ C^\infty(U) \arrow[u, "r"] & C^\infty(U) \end{tikzcd}$$

Now, we define Y to be the linear map $Y : C^\infty(U) \to C^\infty(U)$ that makes the diagram commute. That is:

$$\begin{tikzcd} C^\infty(S) \arrow[r, "X"] & C^\infty(S) \arrow[d, "e"] \\ C^\infty(U) \arrow[u, "r"] \arrow[r, dashed, "Y"] & C^\infty(U) \end{tikzcd}$$

We note that such a Y is not unique, since the extension e is not uniquely defined. In fact, all extensions differ by an element of $C^\infty(U)/C^\infty(S)$.

Y is clearly linear, since it is the composition of linear arrows, so all that we need to show is that Y is a derivation. First, we observer that since e is a section of r, it follows that $X \circ r = r \circ Y$. That is, the diagram

$$\begin{tikzcd} C^\infty(S) \arrow[r, "X"] & C^\infty(S) \arrow[d, shift left, "e"] \\ C^\infty(U) \arrow[u, "r"] \arrow[r, dashed, "Y"] & C^\infty(U) \arrow[u, shift left, "r"] \end{tikzcd}$$

commutes. We note also that the restriction map r is multiplicative. That is, $r(fg) = r(f)r(g)$.

Let $f, g \in C^\infty(U)$. We calculate

$$\begin{aligned} Y(fg) &= e \circ X \circ r(fg) \\ &= e \circ X(r(f)r(g)) \\ &= e(X(r(f))r(g) + r(f)X(r(g))) \\ &= e(r(Y(f))r(g) + r(f)r(Y(g)) \\ &= (e \circ r)(Y(f)g + fY(g)) \end{aligned}$$

so if $e \circ r$ is the identity on $Y(f)g + fY(g)$, then Y is a derivation. However, we recall that e is only unique up to a factor of $C^\infty(U)/C^\infty(S)$. So, for each $f \in C^\infty(U)$, we define e_f... $\square$

Problem 3

Let $F : M \to N$ be a smooth submersion.

Part 1

Show that if $\dim M = \dim N$, then every smooth vector field on N has a unique lift.

Proof. Let $Y \in \mathcal{X}(N)$. Then, define $X_p = dF^{-1}_{F(p)}(Y_{F(p)})$, where $dF^{-1}_{F(p)}$ is an isomorphism, since F is a submersion to a space of the same dimension. Clearly, X and Y are F-related, as desired. □

Part b

Show that if $\dim M \neq \dim N$, then the lift is not unique.

Proof. Theorem 4.26 guarantees the existence of local sections of F, so we can define $X_p = d\pi_{F(p)}(Y_{F(p)})$, and clearly X is a lift of Y, since $d\pi \circ dF = id$. This lift will not be unique, since π is not unique. □

Part c

Suppose F is surjective. Given $X \in \mathcal{X}(M)$, show that X is the lift of a smooth vector field on N if and only if $dF_p(X_p) = dF_q(X_q)$ for all $F(p) = F(q)$.

Proof. Suppose X is the lift of some Y on N. Then, we know that for $F(p) = F(q)$,

$$dF_p(X_p) = Y_{F(p)} = Y_{F(q)} = dF_q(X_q)$$

as desired.

Suppose instead that $dF_p(X_p) = dF_q(X_q)$ for each $F(p) = F(q)$. Then, we define

$$Y_{F(p)} = dF_p(X_p)$$

Clearly, X is a lift of Y, and Y is well-defined since F is surjective, and if $F(p) = F(q)$, then $Y_{F(p)} = Y_{F(q)}$ by the condition of X. □

Part d

Suppose in addition that F has connected fibers. Show that $X \in \mathcal{X}(M)$ is a lift of a smooth vector field on N if and only if $[V, X]$ is vertical whenever V is a vertical vector field on M.

Proof. Suppose X is a lift of some vector field Y in N. Then, we know that X and Y are F-related, and V and 0 are F-related, so $[V, X]$ and $[0, Y] = 0$ are F-related, and thus $[V, X]$ is vertical.

Suppose instead that $[V, X]$ is vertical. That is $dF([V, X]) = 0$. Then,

$$\begin{aligned}
dF([V, X])(f) &= 0 \\
&= (VX - XV)(f \circ F) \\
&= VX(f \circ F) - XV(f \circ F) \\
&= VX(f \circ F) - X(0) \\
&= VX(f \circ F)
\end{aligned}$$

Thus, $V(X(f \circ F)) = 0$, which means that around a point p, $X(f \circ F) = C_p$ for some constant C_p. Now, for points p and q on the same (connected) fiber, it must be that $C_p = C_q$, and thus $dF_p(X_p) = dF_q(X_q)$ and by the previous result, X is a lift of some vector field on N. ☐

Problem 4

Show that $\mathbb{R}^3$ is a Lie algebra with the cross product.

Proof. The cross product is, by definition, bilinear and antisymmetric, so it suffices to check that the cross product satisfies the Jacobi identity.

We proceed to calculate the Jacobi identity directly. Now, we know that

$$((A \times B) \times C)^i = \epsilon^i_{jk}\epsilon^j_{mn} A^m B^n C^k$$

where ϵ^i_{jk} is the Levi-Civita symbol. So

$$\begin{aligned} ((A \times B) \times C + (B \times C) \times A + (C \times A) \times B)^i &= \epsilon^i_{jk}\epsilon^j_{mn}(A^k B^m C^n + B^k C^m A^n + C^k A^m B^n) \\ &= T^i_{kmn} V^{kmn} \end{aligned}$$

where $T^i_{kmn} = \epsilon^i_{jk}\epsilon^j_{mn}$ and $V^{kmn} = A^k B^m C^n + B^k C^m A^n + C^k A^m B^n$.

Now, $\epsilon^i_{jk}\epsilon^j_{mn} = T^i_{kmn}$is a symbol of rank $(1, 3)$ whose components can be calculated directly. It is easy to see that this symbol is antisymmetric in the first and last two components. Furthermore, the only nonzero terms (up to antisymmetry) is $T^i_{kki} = 1$. Thus

$$\begin{aligned} ((A \times B) \times C + (B \times C) \times A + (C \times A) \times B)^i &= T^i_{kmn} V^{kmn} \\ &= T^i_{kki} V^{kki} + T^i_{kik} V^{kik} && \text{for fixed } i, k \\ &= T^i_{kki} V^{kki} - T^i_{kki} V^{kki} && \text{by antisymmetry of } T \\ &= 0 \end{aligned}$$

as desired.

Thus, the cross product is a bilinear map that satisfies the Jacobi identity, and $\mathbb{R}^3$ with this product is a Lie algebra. ☐

Problem 5

Let $A \subseteq \mathcal{X}(\mathbb{R}^3)$ be the subspace spanned by the vector fields

$$\begin{aligned} X &= y\partial_z - z\partial_y \\ Y &= z\partial_x - x\partial_z \\ Z &= x\partial_y - y\partial_x \end{aligned}$$

Show that A is a Lie subalgebra of $\mathcal{X}(\mathbb{R}^3)$.

Proof. We can calculate

$$\begin{aligned}
[X,Y]^i &= X^j\partial_j Y^i - Y^j\partial_j X^i \\
[X,Y]^x &= X^j\partial_j Y^x - Y^j\partial_j X^x \\
&= X^j\partial_j z - 0 \\
&= X^z = y
\end{aligned}$$

$$\begin{aligned}
[X,Y]^y &= X^j\partial_j Y^y - Y^j\partial_j X^y \\
&= 0 - Y^j\partial_j(-z) \\
&= Y^z = -x
\end{aligned}$$

$$\begin{aligned}
[X,Y]^z &= X^j\partial_j Y^z - Y^j\partial_j X^z \\
&= X^j\partial_j(-x) - Y^j\partial_j(y) \\
&= -X^x - Y^y = 0
\end{aligned}$$

Thus, $[X,Y] = y\partial_x - x\partial_y = -Z$. However, the exact same calculation on $[Y,Z]$ by permuting the indices reveals that $[Y,Z] = -X$, and $[Z,X] = -Y$.

So, the Lie algebra isomorphism will be

$$\begin{aligned}
X &\mapsto e_1 \\
Y &\mapsto -e_2 \\
Z &\mapsto e_3
\end{aligned}$$

and since this isomorphism preserves the bracket of each, it is a Lie algebra isomorphism. □

Problem 6

Construct a non-vanishing vector field on a general odd-dimensional sphere.

Proof. Let S^{2n-1} be embedded in $\mathbb{C}^n$. Furthermore, $\gamma_i(t) = \exp(it)(0,\dots,1,0,\dots,0)$ where the 1 is in the i^{th} coordinate. Then, the vector field

$$\sum_{i=1}^{n} \dot{\gamma}_i(t)$$

is a nowhere-vanishing vector field on the sphere. □

Problem 7

Suppose M is a smooth manifold, $X \in \mathcal{X}(M)$, and γ is a maximal integral curve of X.

Part A

Show that exactly one of the following holds:

- γ is constant.
- γ is injective.
- γ is periodic and nonconstant.

Proof. If γ is constant, In particular, γ cannot be injective, since its domain is uncountable, but each maps to the same point. Furthermore, clearly if γ is constant, γ cannot be nonconstant.

If γ is injective, then it cannot be constant, since the domain has more than one point. Furthermore, it cannot be periodic, since if $\gamma(t) = \gamma(t+T)$ for some $T > 0$, then γ sends t and $t + T$ to the same point.

If γ is periodic and nonconstant, then γ is clearly not constant, and γ is not injective, since $\gamma(t) = \gamma(t+T)$. ☐

Part b

Suppose γ is periodic and nonconstant. Show that there exists a unique positive number T for which $\gamma(t) = \gamma(t')$ if and only if $t' - t = kT$.

Proof. Let $P = \{T \mid \gamma(t) = \gamma(t+T)\}$ the set of all periodicity constants of γ, and let $T' = \inf P$. Now, it should be clear that P is closed, since for any $\tilde{T} \notin P$, we have that

$$\gamma(t) \neq \gamma(t + \tilde{T})$$

and since γ is continuous, this must be true on some neighborhood of t as well. Thus, there is a neighborhood of $\tilde{T}$ for which

$$\gamma(t) \neq \gamma(t + \tilde{T} - \epsilon)$$

and so P^c is open and P is closed.

Thus, P contains its inf, and thus $T' \in P$ and T' is the period of γ. ☐

Part c

Show that the image of γ is an immersed submanifold of M diffeomorphic to $\mathbb{R}, S^1$, or $\mathbb{R}^0$.

Proof. If γ is injective, then it is an injective smooth immersion, and thus its image is diffeomorphic to its domain (which is $\mathbb{R}$).

If γ is constant, then it is trivially diffeomorphic to $\mathbb{R}^0$.

If γ is periodic, then we can factor γ through the quotient $\mathbb{R}/T\mathbb{R} \cong S^1$, and γ is an injective smooth immersion from S^1 into M, and thus its image is diffeomorphic to its domain S^1. ☐

Problem 8

Calculate the flows of the following

Part b

$W = x\partial_x + 2y\partial_y$.

Proof. We have the differential equation

$$W = \dot{\theta}(p, t)$$

which can be written as

$$\dot{x}(t) = x(t)$$
$$\dot{y}(t) = 2y(t)$$

or (with x being the vector quantity (x, y))

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x(t)$$

Which has a general solution

$$x(t) = \begin{bmatrix} \exp(t) & 0 \\ 0 & \exp(2t) \end{bmatrix} x_0$$

and so the flow is

$$\theta(p, t) = (p_x \exp(t), p_y \exp(2t))$$

$\square$

Part B

$Y = x\partial_y + y\partial_x$.

Proof. This has the same form of solution as above, except now the equation is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t)$$

Now, we can diagonalize the matrix to get

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M = SJS^{-1}$$

for

$$S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has solution

$$x(t) = S \exp(Jt) S^{-1} x_0 = S \begin{bmatrix} \exp(-t) & 0 \\ 0 & \exp(t) \end{bmatrix} S^{-1} x_0$$

$\square$

Problem 9

Prove the escape lemma.

Proof. $\square$

Problem 10

Show that Diff(M) acts transitively on M for M a connected smooth manifold.

Proof. □

Problem 11

Give an example of smooth vector fields V, $\tilde{V}$ and W on $\mathbb{R}^2$ so that $V = \tilde{V} = \partial_x$ on the x axis, but $L_V W \neq L_{\tilde{V}} W$ at the origin.

Proof. The vector fields

$$\begin{aligned} V &= \partial_x \\ \tilde{V} &= \partial_x + y\partial_y \\ W &= \partial_y \end{aligned}$$

are such that

$$[V, W] = 0$$

but

$$\begin{aligned} \tilde{V}, Wt \qquad {}^x &= \tilde{V} W^x - W \tilde{V}^x \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\tilde{V}, W]^y &= \tilde{V} W^y - W \tilde{V}^y \\ &= -1 \neq 0 \end{aligned}$$

as desired. □

Problem 12

Let

$$\begin{aligned} X &= x\partial_x - y\partial_y \\ Y &= x\partial_y + y\partial_x \end{aligned}$$

Compute the flows and verify they do not commute.

Proof. □

Problem 13

Part A

If $z = f(x, y)$ solves the system $z_x = g, z_y = h$. Find the compatibility condition for g and h.

Proof. We know from the compatibility condition that $\partial_x \partial_y f = \partial_y \partial_x f$, which means that

$$\partial_x z_y = \partial_y z_x$$

or,

$$\partial_x h = \partial_y g$$

□

Part b

Show that this is equivalent to $[X, Y] = 0$.

Proof. The vector fields are

$$\begin{aligned} X &= \partial_x + g\partial_z \\ Y &= \partial_y + h\partial_z \end{aligned}$$

We can calculate the Lie bracket

$$\begin{aligned} [X, y] &= (XY^j - YX^j)\partial_j \\ &= 0\partial_x + 0\partial_y + (Xh - Yg)\partial_z \\ &= \partial_x h + g\partial_z h - (\partial_y g + h\partial_z g)\partial_z \\ &= \partial_x h + \partial_x z \partial_z h - (\partial_y g + \partial_y z \partial_z g) \\ &= \partial_x h - \partial_y g \end{aligned}$$

and for this to be zero, we must have that $\partial_x h = \partial_y g$, which is the compatibility condition. □