Final Exam

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Problem 1

For every $n \in \mathbb{N}$, let μ_n be a measure on $(\Omega, \mathscr{A})$ with $\mu_n(\Omega) = 1$. For every $E \in \mathscr{A}$, define

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n}$$

Give a careful proof that μ is a measure on $(\omega, \mathscr{A})$ with $\mu(\Omega) = 1$.

Proof. We wish to prove that μ is a measure on $(\Omega, \mathscr{A})$. That is, we wish to show that that $\mu(\emptyset) = 0$, that $\mu(E) \geq 0$ for all $E \in \mathscr{A}$, and that for a countable collection of disjoint sets $\{E_j\}_{j=1}^{\infty}$ for which $E_j \in \mathscr{A}$ for all j,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

To begin with, we note that since each μ_n is a measure, we have that $\mu_n(\emptyset) = 0$. Thus,

$$\begin{aligned}\mu(\emptyset) &= \sum_{n=1}^{\infty} \frac{\mu_n(\emptyset)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{0}{2^n} \\ &= 0\end{aligned}$$

as desired.

Next, we note that since each μ_n is a measure, $\mu_n(E) \geq 0$ for all $E \in \mathscr{A}$. Thus, since both $\mu_n(E)$ and 2^n are greater than zero for each n, it must be that the sum is greater than zero as well. That is,

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n} \geq 0$$

To show that μ is additive, let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of disjoint measurable sets. □