Topology

Problem Set 2

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Problem 1

Prove that there is an embedding of X into $X \times Y$.

Proof. For this proof, $\{\bullet\}$ will represent the one-point set.

To start with, we will prove the following lemma:

Lemma. *For X any topological space, $X \cong X \times \{\bullet\}$.*

Proof. By the definition of the product space, the projection maps

$$\begin{tikzcd}
 & X \times \{\bullet\} \arrow[dl, "\pi_x"'] \arrow[dr, "\pi_\bullet"] & \\
X & & \{\bullet\}
\end{tikzcd}$$

exist and are continuous open maps. Now, all we need to show is that π_x is injective, and it will follow immediately that it is a homeomorphism.

To see this, let $x \in X$ and consider $\pi_x^{-1}(\{x\}) = \{(x, \bullet)\}$. Since the inverse image of a singleton is again a singleton, the function is injective.

Thus, X is homeomorphic to $X \times \{\bullet\}$. □

Now, let $f : \{\bullet\} \to Y$ be a continuous function. Consider the diagram:

$$\begin{tikzcd}
 & X \arrow[d, leftrightarrow, "\cong"] & \\
 & X \times \{\bullet\} \arrow[d, dashed, "id \times f"] \arrow[ddl, bend right, "id"'] \arrow[ddr, bend left, "f"] & \\
 & X \times Y \arrow[dl, "\pi_x"'] \arrow[dr, "\pi_y"] & \\
X & & Y
\end{tikzcd}$$

where id and f are the obvious extensions $id(x, \bullet) = id(x) = x$ and $f(x, \bullet) = f(\bullet)$. Here, the product map $id \times f$ is continuous by the universal property of products. Now, we just need to show that $id \times f$ is injective with a continuous inverse on its image.

To see that $id \times f$ is injective, consider a point $(id(x), f(\bullet))$ in the image of $id \times f$, and consider its preimage:

$$(id \times f)^{-1}(\{(id(x), f(\bullet))\}) = \{(x, \bullet)\}$$

Since the preimage of any singleton is again a singleton, the function $id \times f$ is injective.

Now, lets consider the diagram

$$\begin{tikzcd}
& X \times Y \arrow[ddl, bend right, "\pi_x"'] \arrow[d, dashed, "\pi_x \times c"] \arrow[ddr, bend left, "c"] & \\
& X \times \{\bullet\} \arrow[dl, "\pi_x"] \arrow[dr, "pi_\bullet"'] & \\
X & & \{\bullet\}
\end{tikzcd}$$

where c is unique constant function from Y to the terminal object $\{\bullet\}$.

Here, the dashed arrow $\pi_x \times c$ is continuous by the universal property of products. It is easy to see that $\pi_x \times c|_{(id \times f)(X \times \{\bullet\})}$ is the inverse of $id \times f$ on the image of $id \times f$.

Hence, since the inverse of $id \times f$ is continuous, $id \times f$ is an embedding of $X \cong X \times \{\bullet\}$ into $X \times Y$. □

Problem 2

Prove that every open interval in $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Proof. Consider an open interval $(a, b) \subset \mathbb{R}$. It is easy to see that $(a, b) \cong (-1, 1)$, since the operations of scaling and translation are continuous functions with continuous inverses.

Thus, all we need to prove is that $(-1, 1) \cong \mathbb{R}$. To see this, consider the function

$$\tan(\frac{\pi}{2}x)$$

defined on $(-1, 1)$, which is a continuous bijection with continuous inverse. (proofs for the continuity of tan and arctan are easily given by basic analysis arguments, and will not be reproduced here.) □

Problem 3

Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous at exactly one point.

Proof. The function

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = x\chi_{\mathbb{Q}}(x)$$

is continuous only at zero. To see this, we will use the neighborhood definition of continuity. That is, f is continuous at x if for each neighborhood of $f(x)$, its preimage contains a neighborhood of x.

First, we will prove that f is continuous at zero. It suffices to show that each basic open neighborhood of $f(x)$ has a preimage that contains an open neighborhood of x. So, let $(-\varepsilon, \varepsilon)$ be a basic neighborhood of $f(0) = 0$. Then,

$$f^{-1}((-\varepsilon, \varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon, \varepsilon)$$

which contains $(-\varepsilon, \varepsilon)$ an open neighborhood of 0 as desired.

Now, let $x \neq 0$. We will show that f is not continuous at x. If x is irrational, then $f(x) = 0$. Now, choose ε so that $x \notin (-\varepsilon, \varepsilon)$. Then, by the above calculation, we have

$$f^{-1}((-\varepsilon, \varepsilon)) = \mathbb{R} \setminus \mathbb{Q} \cup (-\varepsilon, \varepsilon)$$

which does not contain any neighborhood of x.

If x is rational, then $f(x) = x$. Choose ε such that $0 \notin V_\varepsilon(x)$. Then,

$$f^{-1}(V_\varepsilon(x)) = V_\varepsilon(x) \cap \mathbb{Q}$$

which does not contain any open neighborhood of x (This is easily seen by observing that any neighborhood of x must intersect $\mathbb{R} \setminus \mathbb{Q}$, but the inverse image contains only rational points). $\square$

Problem 4

Suppose Y is Hausdorff, and $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are continuous. If $f|_A = g|_A$ for a dense subset $A \subset Y$, prove that $f = g$.

Proof. Let f and g be parallel morphisms that satisfy the assumptions.

Now, let $y \in Y$. Since A is dense, $y \in \overline{A}$, so there exists some net $\{y_\alpha\}$ such that $y_\alpha \in A$ for all α and $y_\alpha \to y$. In particular, since Y is Hausdorff, this net converges to the unique limit y.

By the hypothesis, $f(y_\alpha) = g(y_\alpha)$ $\forall \alpha$, and since both f and g are continuous, they preserve limits. That is $f(y_\alpha) \to f(y)$ and $g(y_\alpha) \to g(y)$. Since $f(y_\alpha) = g(y_\alpha)$ for all α and limits of nets in Y are unique, they must converge to the same element, and $f(y) = g(y)$.

Since this works for all $y \in Y$, $f = g$. $\square$

Problem 5

Prove that if A_α is a closed subset of X_α for all α, then $\prod A_\alpha$ is closed in $\prod X_\alpha$.

Proof. To show that $\prod A_\alpha$ is closed, we need to show that it contains its limit points. To do so, let $\{a_\gamma\}$ be a convergent net in the product $\prod A_\alpha$. In particular, each of its projections $\pi_\alpha(a_\gamma)$ is also a net in A_α, and since A_α is closed, this net converges to elements in A_α.

Thus, each coordinate α of the net $\{a_\gamma\}$ converges in A_α, so any limit point must have coordinates in the A_α as well. That is, if a is a limit point of $\{a_\gamma\}$, then for each α, $\pi_\alpha(a) \in A_\alpha$, which means that $a \in \prod A_\alpha$ as desired.

Since $\prod A_\alpha$ contains all its limit points, it is closed. $\square$

Problem 6

Let $y \in \prod X_\alpha$, and $\{x_n\}$ a sequence of points in $\prod X_\alpha$. Show that $x_n \to y$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(y)$ for all α.

Proof. (=>) For the first direction, assume that $x_n \to y$. Since each π_α is continuous, they preserve limits. Thus, for each α, $\pi_\alpha(x_n) \to \pi_\alpha(y)$ as desired.

(<=) For the other direction, let $\{x_n\}$ be such that for all α, $\pi_\alpha(x_n) \to \pi_\alpha(y)$. In particular, this means that the filter $\mathscr{F} = \{A \subset \prod X_\alpha \mid \exists n \in \mathbb{N} : \; x_m \in A \; \forall m > n\}$ pushes forward along each π_α to a filter that converges to $\pi_\alpha(y)$.

Now, we just need to show that $\mathscr{F}$ converges to y (Equivalently, that $\mathscr{F}$ contains each neighborhood of y). To do so, we will show that each neighborhood of y contains an element of $\mathscr{F}$, then since $\mathscr{F}$ is a filter, it is closed under supersets and contains each neighborhood of y.

So, let U be a neighborhood of y. In particular, there exists a basis element

$$B = V_1 \times V_2 \times \ldots \times V_n \times X \times X \ldots \subset U$$

Now, since the push-forward of $\mathscr{F}$ along each projection is a convergent filter, $N_\alpha \in \pi_{\alpha *}(\mathscr{F})$ for each neighborhood $N_\alpha$of $\pi_\alpha(y)$.

In particular, $V_i \in \pi_{\alpha *}(\mathscr{F})$, which means that $\pi_\alpha^{-1}(V_i) \in \mathscr{F}$. Now, we can write B as

$$B = \bigcap_{i=1}^{n} \pi_\alpha^{-1}(V_i)$$

which is a finite intersection of elements of $\mathscr{F}$, so $B \in \mathscr{F}$. Thus, $U \supset B$ is in $\mathscr{F}$ as well. Since U was any neighborhood of y, the neighborhood filter $\mathscr{N}_y \subset \mathscr{F}$ and $\mathscr{F} \to y$ as desired. □

Problem 7

Let $\mathbb{R}^\omega$ be the space of sequences of real numbers, and let $\mathbb{R}^\infty$ be the space of sequences that are eventually zero. What is $\overline{\mathbb{R}^\infty} \subset \mathbb{R}^\omega$?