1 Curvature

Let's just straight-up define the curvature:

Definition 1.1. *Consider a Riemannian manifold (M, g), with smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$. We define*

$$R_m(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$$

Alternately,

$$R_{abc}^{d}\omega_d = \nabla_a\nabla_b\omega_c - \nabla_b\nabla_a\omega_c$$

(Wald, p. 37)

Now, we need to establish that this is a tensor by showing it is function linear in each component.

Observe that

$$\begin{aligned} R_m(X,Y)fZ &= -\nabla_X\nabla_Y fZ + \nabla_Y\nabla_X fZ + \nabla_{[X,Y]}fZ \\ &= -X(Yf)Z - (Yf)\nabla_X Z - (Xf)\nabla_Y Z - f\nabla_X\nabla_Y Z + Y(Xf)Z + (Xf)\nabla_Y Z + Yf\nabla_X Z + f\nabla_Y \\ &= -f\nabla_X\nabla_Y Z + f\nabla_Y\nabla_X Z + f\nabla_{[X,Y]}Z \end{aligned}$$

as desired

Homework 1. *Show this is function-linear in other components.*

Note you can lower the contravariant index by applying g_{ab} i.e.

$$R_{abcd} = g_{dd'}R_{abc}^{d'}$$

Calculating Curvature

We can calculate the Riemann curvature tensor in coordinates by using the definitions of the covariant derivative.

$$\mathbb{R}_{abc}^{d} = \partial_b\Gamma_{ac}^{d} - \partial_a\Gamma_{bc}^{d} + \sum_{\alpha}(\Gamma_{ac}^{\alpha}\Gamma_{\alpha b}^{d} - \Gamma_{bc}^{\alpha}\Gamma_{\alpha a}^{d})$$

To make things easier, we can use local Riemannian normal coordinates by pushing the coordinates from T_pM to M via the exponential map.

Homework 2. *Show that in Riemannian normal coordinates,*

$$\Gamma_{ij}^{k} = 0 \text{ at } p$$

and

$$\partial_k g_{ij} = 0 \text{ at } p$$

Definition 1.2. *an orthonormal frame $\{e_i\}$ on an open neighborhood of a point $p \in M$ is called normal around p if*

$$\nabla_a e_i = 0$$

at p.

The curvature follows the Bianchi Identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

In general, we have four important properties of the metric:

- $R^{d}_{abc} = R^{d}_{[ab]c}$ antiymmetry of the first two components
- $R^{d}_{[abc]} = 0$ the Bianchi identity
- $R_{abcd} = R_{ab[cd]}$ antiymmetry of the second two components
- $R_{abcd} = R_{cdab}$ symmetry in the first and second half components.