Problem Set 7

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Problem 1

Show that if a path-connected, locally path-connected space X has π_1 finite, then every map $X \to S^1$ is nullhomotopic.

Proof. Let $f \in C(X, S^1)$. Since $\pi_1(X)$ is finite, we know that $f_* : \pi_1(X) \to \pi_1(S^1)$ is trivial, since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(X)$ is finite. Thus, the conditions are satisfied for f to lift to the universal cover $\mathbb{R}$ of S^1.

Thus, we have

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

Now, since $\mathbb{R}$ is simply connected, $\tilde{f}$ is nullhomotopic. Thus, the homotopy of $\tilde{f}$ with a constant map descends via p to a homotopy of f with a constant map. Thus, f is nullhomotopic as desired. □

Problem 2

Construct finite graphs X_1, X_2 which have a common finite-sheeted covering space, but such that there is no space having both X_1 and X_2 as a covering space.

Problem 3

Let a, b be the generators of $\pi_1(S^1 \vee S^1)$. Draw a picture of the covering space of $S^1 \wedge S^1$ corresponding to the normal subgroup generated by $a^2, b^2, (ab)^4$, and prove that this is the correct space.

Problem 4

Let $\tilde{X}$ be a simply connected covering space of X, and $A \subset X$ be a path-connected, locally path-connected subspace with $\tilde{A}$ a path component of $p^{-1}(A)$. Show that $p : \tilde{A} \to A$ is a covering space corresponding to the kernel of $i_* : \pi_1(A) \to \pi_1(X)$.

Proof. First, we show that $p_*\pi_1(\tilde{A}) \subseteq \ker i_*$. To see this, note that the diagram

$$\begin{tikzcd}
\tilde{A} \arrow[r, "i"] \arrow[d, "p"] & \tilde{X} \arrow[d, "p"] \\
A \arrow[r, "i"] & X
\end{tikzcd}$$

commutes. This implies that the induced diagram

$$\begin{tikzcd}
\pi_1(\tilde{A}) \arrow[r, "i_*"] \arrow[d, "p_*"] & \pi_1(\tilde{X}) = 0 \arrow[d, "p_*"] \\
\pi_1(A) \arrow[r, "i_*"] & \pi_1(X)
\end{tikzcd}$$

and a simple diagram chase shows that $p_*\pi_1(\tilde{A}) \subseteq \ker i_*$ as desired.

Now we show the reverse. Let $[f]$ be a loop in A with $i_*([f]) = 0$. Then, by the lifting property of $p : \tilde{X} \to X$, we have

$$\begin{tikzcd}
 & & \tilde{X} \arrow[d, "p"] \\
S^1 \arrow[r, "f"] \arrow[urr, dashed, "\tilde{f}"] & A \arrow[r, "i"] & X
\end{tikzcd}$$

where we fix $\tilde{f}(0)$ to be in $\tilde{A}$. Now, since this commutes, this implies that $\tilde{f}$ stays in $\tilde{A}$, and defines a loop in $\tilde{A}$ that projects to f. Thus, $p_*([\tilde{f}]) = [f]$ and thus $\ker i_* \subseteq p_*(\tilde{A})$ as desired. $\square$

Problem 5

Given covering space actions G_1, G_2 on X_1, X_2, show that the product action $G_1 \times G_2$ on $X_1 \times X_2$ is a covering space action isomorphic to $X_1/G_1 \times X_2/G_2$.

Proof. Since G_i is a covering space action on X_i, the quotient map $q_i : X_i \to X_i/G_i$ is a covering space map. Now, we proved in the last homework set that cartesian products of covering spaces are again covering spaces with the product map (the category of covering spaces is cartesian closed?). Thus,

$$p_1 \times p_2 : X_1 \times X_2 \to X_1/G_1 \times X_2/G_2$$

is a covering space. Furthermore, note that the quotient map $p_1 \times p_2$ has image

$$\frac{X_1 \times X_2}{G_1 \times G_2}$$

by the definition of the action on $X_1 \times X_2$. Thus, the two base spaces are actually isomorphic as desired. Furthermore, this action is a covering space action, since the action of $G_1 \times G_2$ is a deck transformation of $X_1 \times X_2$, and deck transformations are covering space actions. $\square$

Problem 6

Show that if a group G acts freely and properly discontinuously on a Hausdorff space X, then G is a covering space action.

Proof. Let $x \in X$, and let U be a neighborhood of x such that $H_x = \{U \cap g(U) \neq \emptyset\}$ is finite. Then for each $g \in H_x$ with $g \neq e$, we know that $gx \neq x$, and since X is Hausdorff, there exist disjoint open sets U_{gx} and V_{gx} containing x and gx respectively. Now, let

$$W = \bigcap_{g \in H_x, g \neq e} \big(U_{gx} \cap g^{-1}(V_{gx})\big)$$

This satisfies the conditions for a neighborhood of x making G act as a covering space action. To see this, let $g \in G$, and let $y \in W$. If $g \notin H_x$, then $U \cap g(U) = \emptyset$ by definition, and so $W \cap g(W) = \emptyset$ as well.

Now, suppose $g \in H_x$ with $g \neq e$. Then, we know $y \in U_{gx}$ and $y \in g^{-1}(V_{gx})$, so $gy \in V_{gx}$. In particular, this means that $gy \notin U_{gx}$, and so $gy \notin W$ as desired.

Thus, G acts as a covering space action. $\square$