Problem Set 8

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Problem 1

Show that $\frac{\sin(x)}{x}$ is not in $L^1((0,\infty),\lambda^1)$.

Proof. We wish to evaluate

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x)$$

and show that it diverges. To do so, we split the integral into half-cycles

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x) = \sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{x} d\lambda^1(x)$$

Now, we know that on each half-cycle,

$$\frac{|\sin(x)|}{x} \geq \frac{|\sin(x)|}{(n+1)\pi}$$

so we have a lower bound for the integral:

$$\sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{x} d\lambda^1(x) \geq \sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{(n+1)\pi} d\lambda^1(x)$$

Now, since each half-cycle is either entirely positive or entirely negative, we know that

$$\int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{(n+1)\pi} d\lambda^1(x) = \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right|$$

And finally, we can evaluate the integral directly:

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right| &= \sum_{n=0}^{\infty} \left| \frac{1}{(n+1)\pi} [\cos(x)]_{n\pi}^{(n+1)\pi} \right| \\ &= \sum_{n=0}^{\infty} \frac{2}{(n+1)\pi} \\ &= \infty \end{aligned}$$

Thus, since

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x) \geq \sum_{n=0}^{\infty} \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right| = \infty$$

the integral diverges, and $\frac{\sin(x)}{x}$ is not in $L^1((0,\infty),\lambda^1)$. $\square$

Problem 2

Prove that L^p for $1 \leq p \leq \infty$ is complete. Note that case $p = 1$ has already been covered in class.

Proof. To begin with, let $p < \infty$. For this proof, we will show that an absolutely convergent series will converge to a function in L^p. So, let $\{f_n\}$ be such that

$$\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty$$

So, let $g_n = \sum_{k=1}^{n} |f_k|$. We know by Minkowski's inequality that

$$\|g_n\|_p \leq \sum_{k=1}^{n} \|f_n\|_p \leq M$$

Now, g_n converges pointwise to some g, since it's a monotonically increasing bounded sequence.

Now, we know that

$$\int g_n^p \leq M^p$$

and by Fatou's lemma,

$$\int g^p \leq M^p$$

as well.

Thus, the series

$$\sum_{n=1}^{\infty} |f_n(x)|$$

converges pointwise for μ-almost every x, so the series

$$\sum_{n=1}^{\infty} f_n$$

converges pointwise as well. Call the limit f.

Now, let $s_n(x)$ denote the n^{th} partial sum of $\sum_{n=1}^{\infty} f_n(x)$. We know that $|s_n(x)| \leq g_n(x)$ and thus $|f|(x) \leq g(x)$.

So, by the dominated convergence theorem, we have that

$$\lim \|s_n\|_p = \|\lim s_n\|_p = \|f\|_p$$

as desired.

Now, suppose $p = \infty$. Let $\{f_n\}$ be Cauchy in L^∞. Then, we know that for μ-almost every x, the sequence $\{f_n(x)\}$ is Cauchy as well, and by the completeness of $\mathbb{R}$, has a limit. Call this pointwise limit $f(x)$.

Now, we know that $\forall \varepsilon > 0$, we have some N_ε such that $\forall x$ (μ-almost all x),

$$|f_k(x) - f_j(x)| < \varepsilon$$

for $j, k > N_\varepsilon$. Thus, taking a limit as j goes to infinity, we have that

$$|f_k(x) - f(x)| < \varepsilon$$

for all x, as desired. $\square$

Problem 3

Part 1: Notes 3.11

Prove that ℓ_n^p, ℓ^p, $1 \le p \le \infty$ are Banach. Prove that $\ell^{p_1} \subset \ell^{p_2}$ for $1 \le p_1 \le p_2 \le \infty$, and that

$$\|x\|_{p_2} \le \|x\|_{p_1}$$

Proof. We first note that ℓ_n^p is isomorphic (as vector spaces) with $\mathbb{R}^n$, by the canonical identification $(x^i) \mapsto (x^i)$ (where x^i is the i^{th} point in the sequence, and the i^{th} component of the vector). Furthermore, since all norms on a finite dimensional vector space are equivalent, the ℓ^p norm applied to $\ell_n^p \cong \mathbb{R}^n$ is equivalent to the standard 2-norm on $\mathbb{R}^n$. Now, since $\mathbb{R}^n$ is complete with this norm, it follows that ℓ_n^p is complete as well.

For the case of ℓ^p, we note that $\ell^p = L^p(\mathbb{N}, 2^{\mathbb{N}}, \mu_c)$. By Problem 2, we know that L^p spaces are complete, so it follows that ℓ^p is complete as well.

Now, we will prove the norm inequality. Without loss of generality, we will let $(x_n) \in \ell^{p_1}$ such that $\|(x_n)\|_{p_1} = 1$ (i.e. scale the sequence by its norm, which will not change the inequality).

Now, we wish to show that

$$\left(\sum_{n=1}^{\infty} |x_n|^{p_2}\right)^{\frac{1}{p_2}} \le \left(\sum_{n=1}^{\infty} |x_n|^{p_1}\right)^{\frac{1}{p_1}} (= 1)$$

We observe first that since $\|(x_n\|_{p_1} = 1$ and $p_1 \ge 1$, it must be that each x_n is less than 1. Thus, we have that for each n,

$$|x_n|^{p_2} \le |x_n|^{p_1}$$

since $p_2 \ge p_1$, and each term $x_n < 1$.

Thus, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n|^{p_2} &\le \sum_{n=1}^{\infty} |x_n|^{p_1} \\ &= 1 \\ \implies \left(\sum_{n=1}^{\infty} |x_n|^{p_2}\right)^{\frac{1}{p_2}} &\le 1 \end{aligned}$$

as desired.

Thus, if $(x_n) \in \ell^{p_1}$, we have that $\|(x_n)\|_{p_2} \le \|(x_n)\|_{p_1} < \infty$, and so (x_n) is in ℓ^{p_2} as well. Thus, $\ell^{p_1} \subset \ell^{p_2}$ as desired. □

Part 2: Notes 3.13

Prove that $L^{p_1}(\Omega, \mu) \subset L^{p_2}(\Omega, \mu)$ when $\mu(\Omega) < \infty$. To do so, establish the inequality for the average integral

$$\|f\|_{\bar{p_1}} \le \|f\|_{\bar{p_2}}$$

where the barred norm is the average norm defined in the notes.

Furthermore, prove that this does not hold in the case $\mu(\Omega) = \infty$.

Proof. We first prove the average norm inequality.

$$\int |u|^{p_1}\frac{1}{\mu(\Omega)}d\mu \leq \left(\int (|u|^{p_1})^{\frac{p_2}{p_1}}d\mu\right)^{p_1} p_2 \left(\int (\frac{1}{\mu(\Omega)})^{\frac{p_2}{p_2-p_1}}d\mu\right)^{\frac{p_2-p_1}{p_2}}$$
$$= \|u\|_{p_2}^{p_1}((\frac{1}{\mu(\Omega)})^{\frac{p_2}{p_2-p_1}}\mu(\Omega))^{\frac{p_2-p_1}{p_2}}$$
$$\Longrightarrow (\int |u|^{p_1}\frac{1}{\mu(\Omega)}d\mu)^{\frac{1}{p}} \leq \|u\|_{p_2}(\frac{1}{\mu(\Omega)}\mu(\Omega)^{1-\frac{p_1}{p_2}})^{\frac{1}{p_1}}$$
$$= \|u\|_{p_2}(\frac{1}{\mu(\Omega)})^{\frac{1}{p_2}}$$

as desired.

Now, it follows immediately from this inequality that if $\|f\|_{p_2} < \infty$, it must be that $\|f\|_{p_1} < \infty$ as well, and thus $f \in L^{p_2}$ implies $f \in L^{p_1}$.

However, for the measure space $\Omega = (0, \infty)$ with the Lebesgue measure, this fails. This is clear by considering the function $f(x) = \frac{1}{x^{\frac{1}{p_1}}}$, which is in L^{p_2} since $\frac{1}{x^{\frac{p_2}{p_1}}}$ is integrable ($\frac{p_2}{p_1} > 1$), but $\frac{1}{x}$ is not. □

Part 3: Notes 3.15

Prove that ess sup$f = \inf\{C : f \leq C\mu$-a.e.. Prove that inf can be replaced with min. Prove that ess sup$|f| = M \iff |f| \leq M\mu$-a.e.$, \mu\{f > M - \varepsilon\} \neq 0 \forall \varepsilon > 0$.

Proof. The first statement to be proved follows immediately from the definitions, since the set definition is just a restatement of $\mu\{f > C\} = 0$. Thus, the sets the inf is taken over are the same, so the inf itself is the same.

The second statement follows immediately, since if $M = \inf\{C : \mu\{f > C\} = 0\}$, then there is some monotonically decreasing sequence C_n in the set that approaches M. Now, we have that $\{f > M\} = \lim\{f > C_n\}$ which is a monotonic limit of sets, which the measure preserves. So,

$$\mu\{f > M\} = \lim \mu\{f > C_n\} = \lim 0 = 0$$

and thus M is in the set the inf is taken from.]

The last statement also follows immediately, since the first property states that M is an essential upper bound for $|f|$, and the second property says that M is the least of such upper bounds. That is, M is the inf of all upper bounds on $|f|$. This characterizes the essential supremum, so the definitions are equivalent. □

Part 4: Notes 3.17

For a sequence $\{f_n\}$ of measurable functions, prove that $\|f_j - f\|_\infty \to 0$ if and only if $f_j \to f$ uniformly outside a set of measure zero.

Proof. We know that $\forall n > 0$, there is some A_n such that $\forall m > A_n$,

$$\|f_m - f\| < \frac{1}{n}$$

or,

$$\mu\{|f_m - f| > \frac{1}{n}\} = 0$$

Define $E_{mn} = \{|f_m - f| > \frac{1}{n}\}$. Now, we know that

$$\sup |f_m - f| < \frac{1}{n}$$

on $\Omega \setminus \bigcup_{m=A_n}^{\infty} E_{mn}$.

Thus, since $\mu(\bigcup_{n=1}^{\infty} \bigcup_{m=A_n}^{\infty} E_{mn}) = 0$, we know that

$$\sup |f_n - f| \to 0$$

on $\Omega \setminus \bigcup_{n=1}^{\infty} \bigcup_{m=A_n}^{\infty} E_{mn}$, as desired.

Now, for the other direction, we know that $\sup |f_n - f| \to 0$ on a Ω minus a set E of measure zero, so change f_n and f to be zero on E, and thus

$$\sup |[f_n] - [f]| \to 0$$

on all of Ω, and thus $\|f_j - f\|_\infty \to 0$ as desired. □

Part 5: Notes 3.19

Prove Holder and Minkowski inequalities. Prove that $L^{p_2} \subset L^{p_1}$ for finite measure spaces. Prove this does not hold for infinite measure spaces.

Proof. We begin by proving the lemma:

Lemma 1. *For real numbers a, b,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for $p, q > 1$.

Proof. We know that

$$\begin{aligned} ab &= \exp(\log a + \log b) \\ &= \exp\left(\frac{1}{p}\log a^p + \frac{1}{q}\log b^q\right) \end{aligned}$$

Now, by the convexity of the exponential, we have that

$$\exp\left(\frac{1}{p}\log a^p + \frac{1}{q}\log b^q\right) \leq \frac{1}{p}\exp(\log a^p)\frac{1}{q}\exp(\log b^q) = \frac{a^p}{p} + \frac{b^q}{q}$$

as desired. □

Now, we are ready to prove Holder's inequality.

Let $f, g \in L^p$, and without loss of generality let $\|f\|_p = \|g\|_q = 1$. Now, we have that

$$\begin{aligned} \int |fg| d\mu &\leq \int |\frac{f^p}{p} + \frac{g^q}{q}| d\mu \\ &\leq \frac{1}{p}\int |f^p| d\mu + \frac{1}{q}\int |g|^q d\mu \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \\ &= \|f\|_p \|g\|_q \end{aligned}$$

as desired.

Now, for the L^∞ case, we have that

$$\int |fg| d\mu \leq \|g\|_\infty \int |f| d\mu = \|g\|_\infty \|f\|_1$$

as desired.

Now, we prove the Minkowski inequality. To do so, we will use the Holder inequality.

$$\begin{aligned}\int |f+g|^p d\mu &= \int |f+g|^{p-1}|f+g| d\mu \\ &\leq (\int (|f+g|^{p-1})^{\frac{p}{p-1}} d\mu)^{\frac{p-1}{p}} (\int (|f|+|g|)^p d\mu)^{\frac{1}{p}} \\ &\leq \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p)\end{aligned}$$

so we have that

$$\|f+g\|_p^p \leq \|f+g\|_p^{p-1}(\|f\|_p + \|g\|_p)$$

and thus

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

as desired.

The case $p = 1$ falls out of the triangle inequality, and the linearity of the integral, and the case $p = \infty$ comes from the subadditivity of the supremum.

The second half of this problem is a restatement of Notes 3.13, which was proved earlier. $\square$