Final Exam

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Problem 1

Let M be a complete Riemannian manifold with sectional curvature $K, K \geq k > 0$. Let γ be a nontrivial closed geodesic in M. Show that for any $p \in M$,

$$d(p,\gamma) \leq \frac{\pi}{2\sqrt{k}}$$

Proof. For simplicity, we normalize our space so that $k = 1$. Now, suppose for a contradiction that there is some $p \in M$ with $d(p,\gamma) > \frac{\pi}{2}$. Denote by $\sigma : [0,1] \to M$ a minimizing geodesic from p to a closest point q on γ, which we know satisfies $\ell(\sigma) = l > \frac{\pi}{2}$. Observe also that since q is the closest point to p on γ, we have that

$$\langle \sigma', \gamma' \rangle = 0$$

at the intersection point (that is, they intersect orthogonally). Finally, let t_0 be such that $\gamma(t_0) = q$ so that $\sigma(1) = \gamma(t_0)$.

Now, let $v = \frac{\gamma'(t_0)}{\|\gamma'(t_0)\|}$ be the normalized tangent vector to γ at t_0. Let $v(t)$ be the vector field along σ generated by parallel transport of v, and define a variation field V along σ as

$$V(t) = \sin(\frac{\pi t}{2})v(t)$$

so that $V(0) = 0$ at p and $V(1) = v = \frac{\gamma'(t_0)}{\|\gamma'(t_0)\|}$ at $q = \gamma(t_0)$. Finally, we define a variation on σ with V as its variational field by setting

$$h(t,s) = \exp_{\sigma(t)}(sV(t))$$

so that $h(l,s) = \gamma(s)$. We will show this variation has negative second variation of energy, which will contradict $\gamma(t_0) = q$ being the closest point to p.

We directly calculate the second variation of energy of this variational field as

$$\frac{1}{2}E''(0) = -\int_0^1 \langle V, V'' + R(\sigma', V)\sigma' \rangle dt - \langle \nabla_s \partial_s h, \sigma' \rangle(0,0) + \langle \nabla_s \partial_s h, \sigma' \rangle(0,l)$$

Now, since $V(0) = 0$, the first boundary term vanishes ($\nabla_s(0) = 0$ and so $\langle \nabla_s \partial_s h(0,s), \sigma'(0)\rangle|_{s=0} = 0$), and since the variation at l goes along γ a geodesic, $\nabla_s \partial_s h(l,s) = \nabla_s \partial_s \gamma(l) = \nabla_s \gamma'(l) = 0$ (by the geodesic equation) and so the second boundary term vanishes.

Thus, we only need calculate

$$\begin{aligned}
\frac{1}{2}E''(0) &= -\int_0^1 \langle V, V'' + R(\sigma', V)\sigma'\rangle dt \\
&= -\int_0^1 \langle \sin(\frac{\pi t}{2})v(t), \partial_t^2 \sin(\frac{\pi t}{2})v(t) + R(\sigma', \sin(\frac{\pi t}{2})v(t))\sigma'\rangle dt \\
&= -\int_0^1 \sin^2(\frac{\pi t}{2})(\langle v(t), \frac{-\pi^2}{4}v(t)\rangle + \langle v(t), R(\sigma', v(t))\sigma'\rangle)dt \\
&= \int_0^1 \sin^2(\frac{\pi t}{2})(\frac{\pi^2}{4} - \langle v(t), R(\sigma', v(t))\sigma'\rangle)dt
\end{aligned}$$

Let $e(t) = \frac{\sigma'(t)}{l}$ so that $\|e(t)\| = 1$ for all t. Then, we further calculate

$$\begin{aligned}
\frac{1}{2}E''(0) &= \int_0^1 \sin^2(\frac{\pi t}{2})(\frac{\pi^2}{4} - \langle v(t), R(\sigma', v(t))\sigma'\rangle)dt \\
&= \int_0^1 \sin^2(\frac{\pi t}{2})(\frac{\pi^2}{4} - l^2 K(e(t), v(t)))dt \\
&\leq \int_0^1 \sin^2(\frac{\pi t}{2})(\frac{\pi^2}{4} - l^2(1))dt
\end{aligned}$$

and since $l > \frac{\pi}{2}$, we know that $l^2 > \frac{\pi^2}{4}$ and so

$$\begin{aligned}
\frac{1}{2}E''(0) &\leq \int_0^1 \sin^2(\frac{\pi t}{2})(\frac{\pi^2}{4} - l^2(1))dt \\
&< \int_0^1 \sin^2(\frac{\pi t}{2})(\frac{\pi^2}{4} - \frac{\pi^2}{4})dt \\
&= 0
\end{aligned}$$

and thus the second variation of energy of this variational field is negative. This implies that there is a curve near σ which has endpoints at p and $\gamma(t_0 \pm \varepsilon)$ for some $\varepsilon > 0$ with strictly smaller length. This contradicts the fact that σ realizes the minimum distance from p to γ, and so it cannot be that $\ell(\sigma) > \frac{\pi}{2}$. $\square$

Problem 2

Let M be a compact n-dimensional manifold of positive sectional curvature, and A, B two closed totally geodesic submanifolds. Show that A and B must intersect if $\dim(A) + \dim(B) \geq n$.

Proof. Suppose for a contradiction that A and B do not intersect. Then, I claim there is a point $a \in A$ and $b \in B$ such that $d(a,b) = d(A,B)$ and a geodesic γ from a to b which realizes this distance. Furthermore, γ is orthogonal to A and B. We will show that such a γ has a variation whose second variation of energy is negative, leading to a contradiction of γ being minimizing.

To see that such a γ exists, recall that for any closed submanifold N of M, and point $p \notin N$, there exists a point $q \in N$ with $d(p,q) = d(p,N)$ and a minimizing geodesic γ from p to q which realizes this distance, and is orthogonal to N. Letting $p \in A$ and $N = B$, we define the function $f : A \to \mathbb{R}$ as $f(p) = d(p,B)$. Since A is compact (as a closed subset of M a compact space), this function achieves a minimum. Call the point which achieves such a minimum a. Clearly, a and the corresponding close point $b \in B$ are such that $d(a,b) = d(A,B)$, and by construction the minimizing geodesic γ is orthogonal to B. By symmetry, γ is also orthogonal to A as well.

Now, I assert that the second variation of energy for any orthogonal variational field V with associated variation $h(t,s)$ of γ with $h(0,s) \in A$ and $h(l,s) \in B$ is given by

$$\frac{1}{2}E''(0) = I_l(V,V) + \langle V(l), S^{(2)}_{\gamma'(l)}V(l)\rangle - \langle V(l), S^{(1)}_{\gamma'(0)}V(l)\rangle$$

where $S^{(i)}_{\gamma'}$ is the linear map associated to the second fundamental form of A, B in the direction of γ'.

To see this, we calculate directly

$$\begin{aligned}\frac{1}{2}E''(0) &= I_l(V,V) + \langle \nabla_s V, \gamma'\rangle(0,l) - \langle \nabla_s V, \gamma'\rangle(0,0)\\ &= I_l(V,V) + \langle B(V,V), \gamma'\rangle(0,l) - \langle B(V,V), \gamma'\rangle(0,0)\\ &= I_l(V,V) + \langle V(l), S^{(2)}_{\gamma'(l)}V(l)\rangle - \langle V(l), S^{(1)}_{\gamma'(0)}V(l)\rangle\end{aligned}$$

as desired.

Now, since A and B are totally geodesic submanifolds, the second fundamental form vanishes, and we are left with

$$\frac{1}{2}E''(0) = I_l(V,V)$$

which will be important later.

Let $\{a_i\}$ be a basis for $T_a(A)$. We can parallel transport this basis along γ to obtain a set of $\dim(A)$ linearly independent vectors orthogonal to γ at $T_b(M)$. Denote by $\{b_i\}$ a basis for $T_b(B) \subset T_b(M)$. Since $\dim(A) + \dim(B) \geq n$, there must be some linear combination of the parallel transports of $\{a_i\}$ that sum to some vector $v \in T_b(B)$ orthogonal to γ (since the dimension of the subspace of $T_b(M)$ orthogonal to γ is $n-1$, and there are $\dim(A)+\dim(B) \geq n > n-1$ vectors in the set $\{a_i\} \cup \{b_i\}$, this set cannot be linearly independent. Since each basis set $\{a_i\}$ and $\{b_i\}$ are linearly independent, there must be some linear combination of $\{a_i\}$ that sums to some linear combination of $\{b_i\}$). That is to say, there is a vector $v \in T_a(A)$ such that the parallel transport of v into $T_b(M)$ lies in $T_b(B)$.

Consider a variational field generated by parallel transport of such a vector v (call this variational field V). In particular, this variation satisfies the hypotheses for the calculation in the previous paragraph. So, we calculate:

$$\begin{aligned}\frac{1}{2}E''(0) &= I_l(V,V)\\ &= \int_0^l \langle V', V'\rangle - K(V, \gamma')dt\end{aligned}$$

Now, since V is parallel to γ, $V' = 0$, and so

$$\frac{1}{2}E''(0) = -\int_0^l K(V, \gamma')dt$$

and since K is always positive, the second variation of energy of V is negative, contradicting γ being a minimal geodesic from a to b. Thus, it cannot be that A and B do not intersect, as desired. $\square$

PROBLEM 3

Let M^2 be a complete simply connected 2-dimensional Riemannian manifold. Suppose that for each point $p \in M$, the locus $C(p)$ of first conjugate points to p reduces to a unique $q \neq p$ and that $d(p, C(p)) = \pi$. Prove that if the sectional curvature K of M satisfies $K \leq 1$, then M is isometric to the sphere S^2 with $K = 1$.

Proof. Let J be a Jacobi field along a normalized geodesic γ joining p to q with $J(0) = J(\pi) = 0$ and $g(J, \gamma') = 0$. Let $\{e_i, \gamma'\}$ be an orthonormal parallel frame to γ, and write

$$J = a^i e_i$$

Define $K(t) = K(\gamma', J)$. We calculate

$$\begin{aligned}
0 = I_\pi(J, J) &= -\int_0^\pi g(J'' + R(\gamma', J)\gamma', J)dt \\
&= -\int_0^\pi g(J'', J)dt - \int_0^\pi K(t)\|J\|^2 dt \\
&= -\int_0^\pi a''^i a_i dt - \int_0^\pi K(t) a^i a_i dt \\
&= \int_0^\pi a'^i a'_i dt - \int_0^\pi K(t) a^i a_i dt && \text{using integration by parts} \\
&\geq \int_0^\pi a^i a_i dt - \int_0^\pi K(t) a^i a_i dt && \text{by homework 3 problem 1} \\
&= \int_0^\pi a^i a_i (1 - K(t))dt \geq 0
\end{aligned}$$

and thus $K(t) = 1$ for all t. Thus, in the interior of any geodesic connecting a point p to its first conjugate point, the sectional curvature of the space spanned by γ' and J for J a Jacobi field along γ vanishing at the endpoints is identically 1.

Now, all that remains is to show every point on the manifold is an interior point of some geodesic as described above, and that for such a geodesic, there exists a Jacobi field vanishing at the endpoints in each direction orthogonal to γ'.

We handle the second assertion first. To prove that there exist Jacobi fields vanishing at the endpoints of γ in each direction orthogonal to γ', we show that the multiplicity of q as a conjugate point to p is exactly $n - 1$. We appeal to proposition 3.5 of chapter 5 of Do Carmo, which states that the multiplicity of q is exactly the dimension of the kernel of $(d \exp_p)_{v_0}$ where $v_0 = \gamma'(0)$.

Recall that for $w \in T_{V_0}(T_p M)$, we have

$$(d \exp_p)_{v_0}(w) = \partial_t(\exp_p(v_0 + tw))|_{t=0}$$

Suppose w is orthogonal to v_0. Then, $v_0 + tw$ lies approximately on the sphere of radius $\|v_0\|$ for small t. In particular,

$$\begin{aligned}
\|v_0 + tw\|^2 &= \|v_0\|^2 + t^2\|w\|^2 + 2t\langle v_0, w\rangle \\
&= \|v_0\|^2 + t^2\|w\|^2 \\
\|v_0 + tw\| &\approx \|v_0\| + O(t^2)
\end{aligned}$$

and so the geodesic $\gamma_{tw}(s) = \exp_p(s(v_0 + tw))$ has to first order the same speed as $\gamma(s) = \exp_p(sv_0)$. Now, the function $f_p : T_p M \to \mathbb{R}$ taking a vector v and returning the distance along the geodesic generated by v to its first conjugate point is continuous, so $f(v_0 + tw)$ is close to $f(v_0)$. In particular, since q is the only conjugate point to p, it must be that $\gamma_{tw}(1 + \varepsilon)$ for some small ε lands at q. Thus, $\partial_t(\exp_p(v_0 + tw))|_{t=0} = 0$ and w is in the kernel of $(d\exp_p)_{v_0}$. Since we can do this for each w orthogonal to v_0, it follows that the dimension of the kernel is $n-1$, and so the multiplicity of q is $n-1$ as desired.

Finally, we show that every point in M is the interior point of some geodesic connecting two conjugate points. In fact, we only have to show that there is a dense subset of M with this property, and since the sectional curvature is continuous with respect to points in M, it will hold that $K = 1$ for all points in M as well.

So, let $p \in M$. Now, we know that the conjugate locus of p is a single point q. This means that there is some geodesic $\exp_p(tv)$ for a unit vector v in T_pM for which $\exp_p(\pi v) = q$ and q is conjugate to p along this geodesic. Now, since the function f_p defined above is continuous, and can only take values of π and ∞ (by the hypothesis of the problem), it follows that $f(v) = \pi$ for all v. Furthermore, since M is complete, there exists a minimizing geodesic from p to any other point on the manifold. Thus,

$$M = \exp_p(B_\pi(0)) \cup \{q\}$$

and thus every point except p and q lies in the interior of a geodesic on which p is conjugate to q. Thus, K is identically 1 on M, and since M is simply connected, it follows that $M = S^n$ for n the dimension of M. $\square$

Problem 5

Let M be a compact Riemannian manifold of dimension n with sectional curvature $K \geq 1$. Suppose $p, q \in M$ with $d(p,q) = d(M) > \frac{\pi}{2}$. Moreover, suppose $\gamma : [0,1] \to M$ is a geodesic with $\gamma(0) = \gamma(1) = p$. Show that γ has Morse index at least $n-1$.

Proof. First, we show that $\ell(\gamma) > \pi$. Suppose for a contradiction that $\ell(\gamma) \leq \pi$. We know from problem 1 that the distance from q to γ is at most $\frac{\pi}{2}$, so p is not a point on γ that is a minimal distance away from q.

So, let p' be a point on γ closest to q, and let σ_c be a minimal geodesic from q to p'. From problem 1, we know that σ_c has length $\ell(\sigma_c) \leq \frac{\pi}{2}$, and that σ_c intersects γ orthogonally.

Now, we invoke the Toponogov comparison theorem. Let γ_c be the segment of γ from p to p' with $\ell(\gamma_c) \leq \frac{\pi}{2}$ (which we can choose, since γ is a closed geodesic of length $\leq \pi$. Thus, p' splits gamma into two geodesics from p to p' whose lengths add to $\ell(\gamma)$. Choose the shortest of the two). We take as the two sides of our triangle γ_c and σ_c, intersecting at p' orthogonally. Since γ_c has length less than π and σ_c is minimizing, this satisfies the hypotheses for the TopToponogov comparison theorem. Let $a = \ell(\gamma_c)$, and $b = \ell(\sigma_c)$, with $c = d(p,q)$. Then, the associated triangle in S^2 with constant curvature $K = 1$ (labeled with corresponding letters with tildes) is such that

$$\cos(\tilde{c}) = \cos(\tilde{a})\cos(\tilde{b}) + \sin(\tilde{a})\sin(\tilde{b})\cos(\tilde{C})$$

where $\tilde{C}$ is the angle opposite side c between the sides of length a and b. Since we know $C = \frac{\pi}{2}$, we see that

$$\cos(\tilde{c}) = \cos(\tilde{a})\cos(\tilde{b})$$

but since both a and b are less than or equal to $\frac{\pi}{2}$, it follows that

$$\cos(\tilde{c}) \geq 0$$

and so

$$c \leq \tilde{c} \leq \frac{\pi}{2}$$

but this contradicts the hypothesis that $c = d(p,q) > \frac{\pi}{2}$, and so the assumption that $\ell(\gamma) \leq \pi$ is false.

Now, fix $c \in (1, \infty)$ a constant such that $\ell(\gamma) > \pi\sqrt{c}$.

In homework 3, we proved that if $\gamma : (-\infty, \infty) \to M$ is a normalized geodesic in M, then there exists a $t_0 \in \mathbb{R}$ with γ restricted to $[-t_0, t_0]$ having Morse index at least $n-1$. The proof is replicated here:

Proof. Let Y be a parallel field along γ with $g(\gamma', Y) = 0$ and $\|Y\| = 1$. Set

$$\phi_Y = g(R(\gamma', Y)\gamma', Y)$$

and

$$K(t) = \inf_Y \phi_Y(t)$$

and let $a : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $0 \leq a(t) \leq K(t)$ with $0 < a(0) < K(0)$. Let ϕ be the solution to $\phi'' + a\phi = 0$ with $\phi(0) = 1, \phi'(0) = 0$, with $-t_1, t_2$ the two zeroes of ϕ

found in the previous problem. We consider the field $X = \phi Y$, and calculate

$$\begin{aligned}
I_{[-t_1,t_2]}(X,X) &= -\int_{-t_1}^{t_2} g(X'' + R(\gamma', X)\gamma', X)dt \\
&= -\int_{-t_1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{t_1}^{t_2} g(\phi R(\gamma', Y)\gamma', \phi Y) \\
&= -\int_{-t_1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{t_1}^{t_2} \phi^2 \phi_Y dt \\
&\leq -\int_{-t-1}^{t_2} g(\phi'' Y, \phi Y)dt - \int_{-t_1}^{t_2} K(t)\phi^2(t)dt \\
&= -\int_{-t_1}^{t_2} \phi(\phi'' + K(t)\phi)dt \\
&< -\int_{-t_1}^{t_2} \phi(\phi'' + a(t)\phi)dt \\
&= 0
\end{aligned}$$

Thus, for all Y perpendicular to γ' (an $n-1$ dimensional subspace) the form $I_{[-t_1,t_2]}(Y,Y)$ is negative-definite, and so the index is greater than or equal to $n-1$. $\square$

Recall that we required the field X to vanish at the endpoints so that integration by parts works.

Now, setting $a(t) = \frac{1}{c}$ (which satisfies the hypotheses for a, since $K \geq 1$, and $\frac{1}{c} < 1$) we observe that the unique solution ϕ is

$$\phi(t) = \cos(\frac{t}{\sqrt{c}})$$

which has zeroes at $t = \pm\frac{\pi\sqrt{c}}{2}$. In particular, the distance between consecutive zeroes is $\pi\sqrt{c} < \ell(\gamma)$. So, reparameterizing γ to be unit speed, we see that $\gamma : [-\frac{\pi\sqrt{c}}{2}, \frac{\pi\sqrt{c}}{2}] \to M$ contains both zeros and does not intersect itself (since $\ell(\gamma) > \pi\sqrt{c}$). Thus, since the Morse index is an increasing function, it follows that the Morse index for one period of γ is at least $n-1$ as desired. $\square$