A Counterexample to Chuck's Conjecture

Perhaps the simplest counterexample to Chuck's Conjecture is the following:

Example 1. Let X be a compact metric space, equipped with the Borel σ-algebra, and fix $x \in X$. Suppose μ is a probability measure on $(X, \mathscr{B}(X))$ for which $\{x\}$ is the only atom, with $\mu(\{x\}) = 0.1$. We define a second measure ν on X as

$$\nu(\{x\}) = 0.05$$
$$\nu(A) = \frac{0.9}{0.95}\mu(A)$$

for all $A \in \mathscr{B}(X)$ measurable subsets of X with $x \notin A$. For example, we can set $X = I$, $\mu = 0.9\lambda^1 + 0.1\delta_x$ and $\nu = 0.95\lambda^1 + 0.05\delta_x$ so that $\mu(X) = \nu(X) = 1$.

With such measures μ and ν, there is no perfect sharing. That is, there is no measurable subset $S \in \mathscr{B}(X)$ with $\mu(S) = \nu(S) = \frac{1}{2}$.

To see this, suppose there did exist a perfect sharing set S, and without loss of generality take $x \in S$ (if $x \notin S$, we consider S^c which is also a perfect sharing set containing x). Now, let $A = S \setminus \{x\}$. Then,

$$\frac{1}{2} = \mu(S) = \mu(A) + \mu(\{x\}) = \mu(A) + 0.1$$

and so

$$\mu(A) = 0.4$$

However, we also know that

$$\frac{1}{2} = \nu(S) = \nu(A) + \nu(\{x\}) = \nu(A) + 0.05$$

and so it must be that

$$\nu(A) = 0.45$$

By definition of ν, we have that

$$\nu(A) = \frac{0.9}{0.95}\mu(A) = \frac{0.9}{0.95}(0.4) \approx 0.38 \neq 0.45$$

so no such perfect sharing can exist for these two measures.

The previous example generalizes easily to measures with a finite number of atoms.

Example 2. let X be as before, and let μ be a probability measure on X with finitely many atoms $P = \{x_i\}_{i=1}^n$ which satisfies the hypotheses of the conjecture. Let $c = \mu(P)$, observing that $c < \frac{1}{2}$. Define ν a second probability measure on X with

$$\nu(\{x_i\}) = \frac{1}{2}\mu(\{x_i\})$$

so that $\nu(P) = \frac{c}{2}$. Then, for all $A \in \mathscr{B}(X)$ with $A \cap P = \emptyset$, define

$$\nu(A) = \frac{1 - \frac{c}{2}}{1 - c}\mu(A)$$

so that ν is a constant proportion of μ on $X \setminus P$, and $\nu(X) = 1$. Again, there is no perfect sharing set on X with respect to these two measures.

Suppose such a perfect sharing S existed, and without loss of generality suppose $P \cap S \neq \emptyset$. Like before, we set $A = S \setminus P$, and calculate

$$\frac{1}{2} = \mu(S) = \mu(A) + \mu(P) = \mu(A) + c$$

and so

$$\mu(A) = \frac{1}{2} - c$$

However, we calculate

$$\frac{1}{2} = \nu(S) = \nu(A) - \nu(P) = \nu(A) - \frac{c}{2}$$

and so it must be that

$$\nu(A) = \frac{1 - c}{2}$$

But by definition of ν, we have

$$\nu(A) = \frac{1 - \frac{c}{2}}{1 - c}\mu(A) = \frac{1 - \frac{c}{2}}{1 - c}(\frac{1}{2} - c) = \frac{(2 - c)(1 - 2c)}{4(1 - c)}$$

which can only be equal to $\frac{1-c}{2}$ for $c = 0$. So, for positive c (which corresponds to μ having positive-mass singletons) there is no perfect sharing between μ and ν.