Problem Set 1

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Problem 1

Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}.

Proof. The straight-line homotopy from v to $\frac{v}{\|v\|}$ satisfies the criteria for a deformation retract. Namely, the retract is given by

$$\begin{aligned} r :& \mathbb{R}^n \setminus \{0\} \to S^{n-1} \\ r(v) =& \frac{v}{\|v\|} \end{aligned}$$

With homotopy

$$\begin{aligned} F :& \mathbb{R}^n \setminus \{0\} \times I \to S^{n-1} \\ F(v,t) =& (1-t)v + t\frac{v}{\|v\|} \end{aligned}$$

$\square$

Problem 2

Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof. Let $f : X \to Y$ be a map, which is homotopic to a homotopy equivalence $g : X \to Y$ with homotopy inverse $h : Y \to X$. That is, $g \circ h \simeq \mathbb{1}_Y$ and $h \circ g \simeq \mathbb{1}_X$. Furthermore, let $F : X \times I \to Y$ be the homotopy between f and g.

First, let's consider the map $h \circ f : X \to X$. We wish to show $h \circ f \simeq \mathbb{1}_X$. To do so, let's consider the homotopy

$$h \circ F : X \times I \to X$$

This is the composition of two continuous functions, and so it is continuous. Furthermore, since $F(0, x) = f(x)$ and $F(1, x) = g(x)$, this is actually a homotopy between $h \circ f$ and $h \circ g$. Now, since $h \circ f \simeq h \circ g \simeq \mathbb{1}_X$ and homotopy equivalence is an equivalence relation, it follows immediately that $h \circ f \simeq \mathbb{1}_X$.

Now, consider the map $f \circ h : Y \to Y$. We wish to show $f \circ h \simeq \mathbb{1}_Y$. To do so, consider the homotopy

$$F \circ (h \times \mathbb{1}_I) : Y \times I \to Y$$

It is easy to see this is a homotopy between $f \circ h$ and $g \circ h$, and so we have that $f \circ h \simeq g \circ h \simeq \mathbb{1}_X$, and so $f \circ h \simeq \mathbb{1}_X$, as desired. □

Problem 3

A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}_X$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts onto A in the weak sense, then the inclusion map $i : A \to X$ is a homotopy equivalence.

Proof. Let $f_t : X \to X$ be a deformation retraction in the weak sense of X onto A, and let i be the inclusion map from A to X. We will show that $i \circ f_1 \simeq \mathbb{1}_X$ and that $f_1 \circ i \simeq \mathbb{1}_A$.

Considering $i \circ f_1$, we note that this is actually equal to f_1, since the inclusion map is the identity on A, and f_1 maps into A. Now, f_1 is homotopic to f_0 which is equal to $\mathbb{1}_X$, and so by transitivity of homotopy equivalence, $f_1 \simeq \mathbb{1}_X$.

Now, let's consider $f_1 \circ i$. We note first that this is equal to $f_1|_A$, since i is the identity on A. Furthermore, the restrictions $f_t|_A$ define a homotopy from $f_1|_A$ to $f_0|_A$, and so we have that

$$f_1 \circ i = f_1|_A \simeq f_0|_A = \mathbb{1}_X|_A = \mathbb{1}_A$$

as desired. Thus, i is a homotopy equivalence with homotopy inverse f_1. □

Problem 4

Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X, there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \to U$ is nullhomotopic.

Proof. Let $F : X \times I \to X$ be the deformation retraction of X onto $x_0 \in X$, and let U be a neighborhood of x_0. Now, consider the open set

$$F^{-1}(U) \subset X \times I$$

Now, since $F(x_0, t) = x_0$ for all t, we know that $\{x_0\} \times I$ is in $F^{-1}(U)$. Applying the tube lemma, we find an open set V containing x_0 for which $V \times I \subset F^{-1}(U)$. In particular, this means that for all $v \in V$, we have that $F(v, t) \in U$ for all t.

Now we are ready to show that the inclusion map from V to U is nullhomotopic. We note that $F \circ (i \times \mathbb{1}_I) : V \times I \to U$ defines a homotopy from $f_0 \circ i = i$ to $f_1 \circ i = c_{x_0}$, where c_{x_0} is the constant function to x_0. This is clear, since the image of i is $V \subset U$, and the image of V under F is always in U, as proved above. Thus, since the domain of F matches the image of i, and the image of F stays inside U, this is a well-defined homotopy.

Therefore, $i \simeq c_{x_0}$ as desired. □

Problem 5

Consider the subspace $X \subset \mathbb{R}^2$ defined as

$$X = [0,1] \times \{0\} \cup \bigcup_{r \in \mathbb{Q}} \{r\} \times [0, 1 - r]$$

Part a

Show that X deformation retracts to any point in $[0,1] \times \{0\}$, but not any other point.

Proof. To construct the deformation retraction of X to a point in the interval $[0,1] \times \{0\}$, we first note that X deformation retracts onto $[0,1] \times \{0\}$ via the straight-line homotopy along the y-axis. It is clear also that the unit interval deformation retracts to any point on it via the straight-line homotopy along the x-axis. Running the first homotopy for the first half time, and running the second homotopy for the second half time yields a deformation retraction of X onto a point in the interval $[0,1] \times \{x\}$.

Now, consider a point x not in the base interval. Consider also a neighborhood U of x that does not intersect the base interval. Any neighborhood $V \subset U$ containing x will necessarily intersect at least one other stalk than the one x is in (since the rationals are dense in $\mathbb{R}$), and these stalks will not be connected, since V does not intersect the base interval. Thus, the inclusion map of V into U cannot be nullhomotopic, and by problem 4, we know that X therefore cannot deformation retract onto x. $\square$

Part b

Let Y be the subset of $\mathbb{R}^2$ that is the union of infinite copies of X in a zigzag pattern. Show that Y is contractible, but does not deformation retract onto any point.

Proof. To show that Y is contractible, we reference part c of this problem, which asserts the existence of a deformation retraction in the weak sense of Y onto the zigzag subspace Z. Now, problem 3 guarantees that if Y deformation retracts onto Z in the weak sense, then the inclusion $i : Z \to Y$ is a homotopy equivalence, and thus Y and Z have the same homotopy type. However, Z is homeomorphic to $\mathbb{R}$, which has the homotopy type of a point. Therefore, by transitivity of the homotopy equivalence, Y has the homotopy type of a point as well.

Now, we must show that Y does not deformation retract onto any point. To do so, we look at any point x in Y. If x is not in Z, the same argument from part a can be applied to show that Y cannot deformation retract onto x. If x is in Z, we observe that x is actually in a stalk of the copy of X running parallel to the line segment of Z that x is on. Noting then that x is on a stalk, we apply the same argument as the one in part a to see that Y cannot deformation retract onto x. $\square$

Part c

Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$. Show that there is a deformation retraction in the weak sense of Y onto Z, but no true deformation retraction.

Proof. We can construct a deformation retraction in the weak sense explicitly for Y onto Z. For each stalk, we define its "direction of motion" to be towards Z, and on Z we define its "direction of motion" to be towards the right. Now, the deformation retraction in the weak sense sends points at constant velocity 1 along the direction of motion. Away from Z, this is clearly continuous, and on Z, we see that all points are moving at the same speed, so whatever stalks Z is close to are retracting at the same speed Z itself is moving. Thus, points stay close to each other, and the motion is continuous. This is a weak deformation retraction, since it does not fix any point in Z.

However, there is no true deformation retraction of Y onto Z, since if there were, it could be concatenated with a deformation retraction of Z onto a point in Z to yield a deformation retraction of Y onto a point in $Z \subset Y$. However, this would contradict part b, and so no such deformation retraction can exist. $\square$