
Homework 2

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PROBLEM 1

Recall the φ^4 Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4$$

and has an energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + g^{\mu\nu}\mathcal{L}$$

PART A

Problem. *Derive the equation of motion for φ subject to the φ^4 Lagrangian.*

To calculate the equation of motion for φ , we just have to find the stationary points of

$$S = \int d^4x \mathcal{L} = \int d^4x \left(-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{3!}g\varphi^3 - \frac{1}{4!}\lambda\varphi^4 \right)$$

That is, we find when $\delta S = 0$. To do so, we calculate

$$\begin{aligned} \delta S &= \int d^4x \delta\mathcal{L} \\ &= \int d^4x \left(-\frac{1}{2}\delta(\partial^\mu\varphi\partial_\mu\varphi) - \frac{1}{2}m^2\delta(\varphi^2) - \frac{1}{3!}g\delta(\varphi^3) - \frac{1}{4!}\lambda\delta(\varphi^4) \right) \\ &= \int d^4x \left(-\frac{1}{2}(\partial^\mu\delta\varphi\partial_\mu\varphi + \partial^\mu\varphi\partial_\mu\delta\varphi) - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi \right) \\ &= \int d^4x \left(\partial^2\varphi\delta\varphi - m^2\varphi\delta\varphi - \frac{1}{2}g\varphi^3\delta\varphi - \frac{1}{3!}\lambda\varphi^3\delta\varphi \right) \\ &= \int d^4x \left(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3 \right) \delta\varphi \end{aligned}$$

Which is zero for arbitrary variation if $(\partial^2\varphi - m^2\varphi - \frac{1}{2}g\varphi^3 - \frac{1}{3!}\lambda\varphi^3) = 0$. Thus, this is the equation of motion for φ .

PART B

Problem. Show that the energy-momentum tensor $T^{\mu\nu}$ satisfies $\partial_\mu T^{\mu\nu} = 0$.

This is just an exercise in direct calculation:

$$\begin{aligned}
 \partial_\mu T^{\mu\nu} &= \partial_\mu (\partial^\mu \varphi \partial^\nu \varphi) + \partial_\mu g^{\mu\nu} \mathcal{L} \\
 &= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi + \partial^\mu \varphi \partial_\mu \partial^\nu \varphi + \partial^\nu \left(-\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{3!} g \varphi^3 - \frac{1}{4!} \lambda \varphi^4 \right) \\
 &= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi + \partial^\mu \varphi \partial_\mu \partial^\nu \varphi - \frac{1}{2} \partial^\nu \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} \partial^\mu \varphi \partial^\nu \partial_\mu \varphi - m^2 \varphi \partial^\nu \varphi - \frac{1}{2} g \varphi^2 \partial^\nu \varphi - \frac{1}{3!} \lambda \varphi^3 \partial^\nu \varphi \\
 &= \partial_\mu \partial^\mu \varphi \partial^\nu \varphi - m^2 \varphi \partial^\nu \varphi - \frac{1}{2} g \varphi^2 \partial^\nu \varphi - \frac{1}{3!} \lambda \varphi^3 \partial^\nu \varphi \\
 &= \left(\partial^2 \varphi - m^2 \varphi - \frac{1}{2} g \varphi^2 - \frac{1}{3!} \lambda \varphi^3 \right) \partial^\nu \varphi
 \end{aligned}$$

Which is clearly zero if φ follows its equation of motion.

PROBLEM 2

Consider a complex scalar field φ governed by the Lagrangian

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0$$

PART A

Problem. *Show φ obeys the Klein-Gordon equation.*

We calculate the variation in $S = \int d^4x \mathcal{L}$ directly:

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \left(-\delta(\partial^\mu \varphi^\dagger) \partial_\mu \varphi - \partial^\mu \varphi^\dagger \delta(\partial_\mu \varphi) - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\ &= \int d^4x \left(-\partial^\mu \delta \varphi^\dagger \partial_\mu \varphi - \partial^\mu \varphi^\dagger \partial_\mu \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\ &= \int d^4x \left(\delta \varphi^\dagger \partial^2 \varphi + \partial^2 \varphi^\dagger \delta \varphi - m^2 \varphi \delta \varphi^\dagger - m^2 \varphi^\dagger \delta \varphi \right) \\ &= \int d^4x \left((\partial^2 \varphi - m^2 \varphi) \delta \varphi^\dagger + (\partial^2 \varphi^\dagger - m^2 \varphi^\dagger) \delta \varphi \right) \end{aligned}$$

Which is zero for arbitrary variations when both φ and φ^\dagger follow the Klein-Gordon equation.

PART B

Problem. Find the conjugate momenta for φ and φ^\dagger , and write down the Hamiltonian in terms of these.

We can read off the conjugate momenta easily:

$$\begin{aligned}\pi(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \\ &= \frac{\partial}{\partial \dot{\varphi}} (\partial_\mu \varphi^\dagger(x) \partial_\mu \varphi(x) - m^2 \varphi^\dagger(x) \varphi(x) + \Omega_0) \\ &= \dot{\varphi}^\dagger(x)\end{aligned}$$

and similarly

$$\pi^\dagger(x) = \dot{\varphi}(x)$$

We write down the Hamiltonian density as

$$\mathcal{H} = \pi(x) \dot{\varphi}(x) + \pi^\dagger(x) \dot{\varphi}^\dagger(x) - \mathcal{L}$$

and calculate

$$\begin{aligned}\mathcal{H} &= \pi(x) \dot{\varphi}(x) + \pi^\dagger(x) \dot{\varphi}^\dagger(x) + \left(\partial^\mu \varphi^\dagger(x) \partial_\mu \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \right) \\ &= \pi(x) \pi^\dagger(x) + \pi^\dagger(x) \pi(x) + \partial^0 \varphi^\dagger(x) \partial_0 \varphi(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \\ &= \pi(x) \pi^\dagger(x) + \pi^\dagger(x) \pi(x) - \pi(x) \pi^\dagger(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \\ &= \pi^\dagger(x) \pi(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0\end{aligned}$$

Which gives the Hamiltonian for the system as

$$H = \int d^4x \left(\pi^\dagger(x) \pi(x) + \partial^i \varphi^\dagger(x) \partial_i \varphi(x) + m^2 \varphi^\dagger(x) \varphi(x) - \Omega_0 \right)$$

PART C

Problem. Expanding φ as

$$\varphi(x) = \int d^3k \left(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx) \right)$$

solve for expressions of $a(k)$ and $b(k)$ in terms of $\varphi(x)$, $\varphi^\dagger(x)$ and their time derivatives.

We'll evaluate the integrals

$$\begin{aligned}& \int d^3x \exp(-ikx) \varphi(x) \\ & \int d^3x \exp(-ikx) \partial_t \varphi(x) \\ & \int d^3x \exp(-ikx) \varphi^\dagger(x) \\ & \int d^3x \exp(-ikx) \partial_t \varphi^\dagger(x)\end{aligned}$$

So we first derive an expression for $\partial_t \varphi(x)$ and its conjugate.

$$\begin{aligned}\partial_t \varphi(x) &= \int d^3k \partial_t \left(a(k) \exp(ikx) + b^\dagger(k) \exp(-ikx) \right) \\ &= \int d^3k \left(a(k) (-i\omega) \exp(ikx) - b^\dagger(k) (-i\omega) \exp(-ikx) \right) \\ &= \int d^3k (-i\omega) \left(a(k) \exp(ikx) - b^\dagger(k) \exp(-ikx) \right)\end{aligned}$$

And similarly,

$$\partial_t \varphi^\dagger(x) = \int d^3k (-i\omega) \left(b(k) \exp(ikx) - a^\dagger(k) \exp(-ikx) \right)$$

Now, we can calculate those four integrals. One will be done explicitly, and the other three are done using the exact same calculation.

$$\begin{aligned} \int d^3x \exp(-ikx) \varphi(x) &= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega} \exp(-ikx) \left(a(k') \exp(ik'x) + b^\dagger(k') \exp(-ik'x) \right) \\ &= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega} \left(a(k') \exp(i(k' - k)x) + b^\dagger(k') \exp(-i(k' + k)x) \right) \\ &= \int d^3x \frac{d^3k'}{(2\pi)^3 2\omega} \left(a(k') \exp(i(k' - k)^i x_i + i(\omega' - \omega)t) \right. \\ &\quad \left. + b^\dagger(k') \exp(-i(k' + k)^i x_i - i(\omega' + \omega)t) \right) \\ &= \int \frac{d^3k'}{(2\pi)^3 2\omega} \left(a(k') (2\pi)^3 \delta(k' - k) \exp(i(\omega' - \omega)t) \right. \\ &\quad \left. + b^\dagger(k') (2\pi)^3 \delta(k' + k) \exp(-i(\omega' + \omega)t) \right) \\ &= \frac{1}{2\omega} (a(k) + b^\dagger(-k) \exp(-i(2\omega)t)) \end{aligned}$$

Following the same tactic, we find

$$\begin{aligned} \int d^3x \exp(-ikx) \varphi(x) &= \frac{1}{2\omega} (a(k) + b^\dagger(-k) \exp(-i(2\omega)t)) \\ \int d^3x \exp(-ikx) \partial_t \varphi(x) &= \frac{-i}{2} (a(k) - b^\dagger(-k) \exp(-i(2\omega)t)) \\ \int d^3x \exp(-ikx) \varphi^\dagger(x) &= \frac{1}{2\omega} (b(k) + a^\dagger(-k) \exp(-i(2\omega)t)) \\ \int d^3x \exp(-ikx) \partial_t \varphi^\dagger(x) &= \frac{-i}{2} (b(k) - a^\dagger(-k) \exp(-i(2\omega)t)) \end{aligned}$$

So,

$$a(k) = \int d^3x \exp(-ikx) (\omega \varphi(x) + i \partial_t \varphi(x))$$

and

$$b(k) = \int d^3x \exp(-ikx) (\omega \varphi^\dagger(x) + i \partial_t \varphi^\dagger(x))$$

PART D

Problem. *Derive the commutation relations for $a(k)$ and $b(k)$ and their conjugates.*

By conjugating, we find that

$$\begin{aligned} a^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi^\dagger(x) - i \partial_t \varphi^\dagger(x) \right) \\ b^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi(x) - i \partial_t \varphi(x) \right) \end{aligned}$$

So now we can calculate the commutators directly. Let's first rewrite the creation and annihilation operators in terms of φ and π instead:

$$\begin{aligned} a(k) &= \int d^3x \exp(-ikx) \left(\omega \varphi(x) + i \pi^\dagger(x) \right) \\ b(k) &= \int d^3x \exp(-ikx) \left(\omega \varphi^\dagger(x) + i \pi(x) \right) \\ a^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi^\dagger(x) - i \pi(x) \right) \\ b^\dagger(k) &= \int d^3x \exp(ikx) \left(\omega \varphi(x) - i \pi^\dagger(x) \right) \end{aligned} \quad < ++ >$$