Final Exam

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Problem 1

Carefully prove a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is contractible.

Proof. For this proof, we will begin by proving a couple of useful lemmas.

Lemma 1. *The wedge sum is the coproduct in the category Top_* of pointed topological spaces.*

Proof. To show this is the coproduct, we need to show that it satisfies the universal property for coproducts. That is for $(X, x_0), (Y, y_0)$ pointed topological spaces, and (U, u_0) any other pointed topological space with arrows $f : (X, x_0) \to (U, u_0)$ and $g : (Y, y_0) \to (U, u_0)$,

$$\begin{tikzcd}
(X,x_0) \arrow[dr, "i_{x*}"] \arrow[ddr, bend right, "f"'] & & (Y,y_0) \arrow[dl, "i_{y*}"'] \arrow[ddl, bend left, "g"] \\
& (X,x_0) \vee (Y,y_0) \arrow[d, dashed, "\exists!(f,g)"] & \\
& (U,u_0) &
\end{tikzcd}$$

where (f, g) is the unique arrow that makes the diagram commute. Now, recall that the wedge sum is defined as the quotient

$$(X, x_0) \vee (Y, y_0) = X \coprod Y / x_0 \sim y_0$$

so we can bootstrap by using the fact that the disjoint union is the coproduct in Top. That is, we define i_{x*} and i_{y*} to be the unique maps given by the universal property

$$\begin{tikzcd}
X \arrow[dr, "i_x"] \arrow[ddr, dashed, bend right, "i_{x*}"'] & & Y \arrow[dl, "i_y"'] \arrow[ddl, dashed, bend left, "i_{y*}"] \\
& X \coprod Y \arrow[d, "q"] & \\
& X \vee Y &
\end{tikzcd}$$

Note that these maps preserve basepoints, since

$$i_{x*}(x_0) = q \circ i_x(x_0) = q(x_0)$$

and similarly for y_0.

Now, suppose we have maps $f : (X, x_0) \to (U, u_0)$ and $g : (Y, y_0) \to (U, u_0)$. Define (f, g) to be the unique map

$$(f, g) : X \coprod Y \to U$$

that makes the diagram

$$\begin{tikzcd}
X \arrow[dr, "i_x"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "i_y"'] \arrow[ddl, bend left, "g"] \\
& X \coprod Y \arrow[d, dashed, "(f,g)"] & \\
& U &
\end{tikzcd}$$

commute. Now, since $f(x_0) = g(y_0) = u_0$, it follows that (f, g) is constant on the fibers of $q : X \coprod Y \to X \vee Y$ and thus by the universal property of quotient maps factors through $X \vee Y$. That is,

$$\begin{tikzcd}
& X \coprod Y \arrow[dl, "(f,g)"'] \arrow[dr, "q"] & \\
U & & X \vee Y \arrow[ll, dashed, "(f,g)_*"]
\end{tikzcd}$$

commutes for a unique $(f, g)_*$. Thus, we have for f, g arrows from (X, x_0) and (Y, y_0) to (U, u_0) a unique arrow $(f, g)_*$ from $X \vee Y$ which makes

$$\begin{tikzcd}
(X, x_0) \arrow[dr, "i_{x*}"] \arrow[ddr, bend right, "f"'] & & (Y, y_0) \arrow[dl, "i_{y*}"'] \arrow[ddl, bend left, "g"] \\
& (X, x_0) \vee (Y, y_0) \arrow[d, dashed, "(f,g)_*"] & \\
& (U, u_0) &
\end{tikzcd}$$

commute as desired. □

Lemma 2. *The wedge product is stable with respect to (basepoint-preserving) homotopy. That is, if* $X_1 \simeq X_2$ *and* $Y_1 \simeq Y_2$ *relative to basepoints, then*

$$X_1 \vee Y_1 \simeq X_2 \vee Y_2$$

Proof. Since the wedge sum is the coproduct of pointed topological spaces, we construct the homotopy equivalence as follows:

Let $f_x : X_1 \to X_2$ and $g_x : X_2 \to X_1$ be homotopy equivalences of X_2 and X_1. That is, f_x and g_x are homotopy inverses of each other. Furthermore, let $f_y : Y_1 \to Y_2$ and $g_y : Y_2 \to Y_1$ be homotopy inverses as well. Then, by the universal property of coproducts, we have the commutative diagram

$$\begin{tikzcd}
 & X_2 \vee Y_2 \arrow[ddd, dashed, shift left, "\tilde{f}"] & \\
X_2 \arrow[ur] \arrow[d, shift left, "g_x"] & & Y_2 \arrow[ul] \arrow[d, shift left, "g_y"] \\
X_1 \arrow[u, shift left, "f_x"] \arrow[dr] & & Y_1 \arrow[u, shift left, "f_y"] \arrow[dl] \\
 & X_1 \vee Y_1 \arrow[uuu, dashed, shift left, "\tilde{g}"] &
\end{tikzcd}$$

where $\tilde{f}$ is defined in terms of the universal property of coproducts with respect to the compositions $X_1 \xrightarrow{f_x} X_2 \longrightarrow X_2 \vee Y_2$ and $Y_1 \xrightarrow{f_y} Y_2 \longrightarrow {}_2 \vee Y_2$ (and similarly for $\tilde{g}$).

I assert that $\tilde{f}$ and $\tilde{g}$ are homotopy inverses. It should be clear that by symmetry of the problem, I need only check that $\tilde{f} \circ \tilde{g} \simeq \mathbb{1}$.

Suppose $x \in X_1 \vee Y_1$, and without loss of generality let x be in the inclusion of X_1 to $X_1 \vee Y_1$. Then, we have the diagram

$$\begin{tikzcd}
 & X_2 \vee Y_2 \arrow[ddd, dashed, shift left, "\tilde{f}"] \\
X_2 \arrow[ur, "i_2"] \arrow[d, shift left, "g_x"] & \\
X_1 \arrow[u, shift left, "f_x"] \arrow[dr, shift left, "i_1"] & \\
 & X_1 \vee Y_1 \arrow[uuu, dashed, shift left, "\tilde{g}"] \arrow[ul, dashed, shift left, "i_1^{-1}"]
\end{tikzcd}$$

(where i_1^{-1} is only defined on the image of i_1, but we are assuming that $x \in i_1(X_1)$ for this diagram chase, so this arrow exists). From here, it is clear that $\tilde{g}(x) = i_2 \circ f_x(i_1^{-1}(x))$, and if we identify X_1 and X_2 as subspaces of their wedge product, we have $\tilde{g}(x) = f_x(x)$. Similarly, we have $\tilde{f}(x) = g_x(x)$. Thus,

$$\tilde{f} \circ \tilde{g}(x) = g \circ f(x) \simeq \mathbb{1}(x)$$

as desired.

Thus, $\tilde{f}$ and $\tilde{g}$ are homotopy inverses, and $X_1 \vee Y_1 \simeq X_2 \vee Y_2$ as desired. $\square$

Now, we prove the main result.

Let Z be a CW complex which satisfies the hypotheses. In particular, let X and Y be such that $Z = X \cup Y$, X and Y are both contractible, and their intersection $A = X \cap Y$ is contractible as well.

Since A is a contractible subcomplex of Z, we know (via Hatcher prop 0.16 and 0.17) that Z is homotopy equivalent to Z/A. Thus, we only need to show that Z/A is contractible.

Let $q : Z \to Z/A$ be the canonical quotient map. Since $Z = X \cup Y$, we know that $q(Z) = Z/A = q(X) \cup q(Y)$. In particular, $q(X) = X/A$ and $q(Y) = Y/A$ (this follows from the definition of the quotient). Thus, $Z/A = X/A \cup Y/A$. Since the quotient map is the identity on $Z \setminus A$ and collapses A to a point, it follows that $X/A \cap Y/A = A/A = \{a_0\}$ where a_0 is the point $q(A)$. Thus, Z/A is actually the wedge sum

$$Z/A = X/A \vee Y/A$$

Now, since A is contractible, we know that $X/A \simeq X$ and $Y/A \simeq Y$. In particular, since X and Y are contractible, so is X/A and Y/A. Thus, $X/A \simeq Y/A \simeq \{\cdot\}$. By lemma 2, this implies that

$$X/A \vee Y/A \simeq \{\cdot\} \vee \{\cdot\} = \{\cdot\}$$

and thus, $X/A \vee Y/A$ is contractible. Thus, $Z/A = X/A \vee Y/A$ is contractible as well, and Z itself is contractible as desired. $\square$

Problem 2

Prove that there is no retraction $r : D^n \to \partial D^n$ using the fact that $\pi_n(S^n) = \mathbb{Z}$.

Proof. Assume for a contradiction that there exists a retraction $r : D^n \to \partial D^n$. That is, assume that there is some r such that the diagram

$$\begin{tikzcd} \partial D^n \arrow[r, "i"] \arrow[rr, bend right, "\mathbb{1}"'] & D^n \arrow[r, "r"] & \partial D^n \end{tikzcd}$$

commutes. Here i is the inclusion map of ∂D^n into D^n. (Since r restricted to ∂D^n is the identity by definition, $ri = \mathbb{1}$).

Since π_n is a covariant functor (Hatcher states this on page 342), we can apply it to this diagram to get the commutative diagram

$$\begin{tikzcd} \pi_n(\partial D^n) = \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}"'] & \pi_n(D^n) = 0 \arrow[r, "r_*"] & \pi_n(\partial D^n) = \mathbb{Z} \end{tikzcd}$$

which implies that the identity map from $\mathbb{Z}$ to itself factors through zero, which cannot be. Thus, our assumption that r exists is false, and no retract of D^n to its boundary exists. $\square$

Problem 3

In each case, either prove a covering exists, or that no such covering exists.

Part a

$p : S^2 \to T^2$.

Proof. I assert that no such covering map exists. Recall from Hatcher proposition 1.37 that for X a path-connected, locally path-connected space, and $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2$ covering spaces, then $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic if and only if $p_{1*}(\pi_1(\tilde{X}_1)) = p_{2*}(\pi_1(\tilde{X}_2))$.

Now, T^2 is path-connected and locally path-connected. Furthermore, it has a universal cover $\mathbb{R}^2$. To see this consider the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$ given by $(m, n)(x, y) = (x + m, y + n)$. Clearly, this action is properly discontinuous, since each point moves a distance of at least 1 for the action of any nontrivial element of $\mathbb{Z} \times \mathbb{Z}$, and so for any point (x, y) we can take the neighborhood $U = V_{\frac{1}{10}}((x, y))$, which easily satisfies the property $U \cap g(U) = \emptyset$ for nontrivial $g \in \mathbb{Z} \times \mathbb{Z}$.

Thus, by proposition 1.40, we know that the quotient map $p_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ is a covering space map. It is easy to see that $\mathbb{R}^2/\mathbb{Z} \times \mathbb{Z} = T^2$, and so $\mathbb{R}^2$ is a simply-connected covering space of T^2.

Now, since both S^2 and $\mathbb{R}^2$ are simply-connected, we know that

$$p_*(\pi_1(S^2)) = p_{\mathbb{R}^2*}(\pi_1(\mathbb{R}^2)) = 0$$

and so proposition 1.37 guarantees the existence of a covering space isomorphism between the two. However, $\mathbb{R}^2$ and S^2 are not homeomorphic, (S^2 is compact but $\mathbb{R}^2$ is not) and so no such isomorphism can exist. Thus, S^2 cannot cover T^2.

□

Part b

$p : T^2 \to S^2$.

Proof. Recall Hatcher proposition 1.31, which asserts that for a covering space $p : \tilde{X} \to X$, the induced map $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is injective. So, if such a covering $p : T^2 \to S^2$ existed, it would induce an injection

$$p_* : \pi_1(T^2) \to \pi_1(S^2)$$

but $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$, and $\pi_1(S^2) = 0$, so no such injection can exist! Thus, T^2 does not cover S^2. □

Part c

$p : \mathbb{R}^2 \to T^2$.

Proof. This was constructed in part a as a covering map. □

$p : \mathbb{R}^2 \to S^2$.

Proof. I assert that no such covering exists. This follows from the fact that S^2 covers itself, and simply-connected covering spaces are unique up to isomorphism.

Recall from Hatcher proposition 1.37 that for X a path-connected, locally path-connected space, and $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2$ covering spaces, then $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic if and only if $p_{1*}(\pi_1(\tilde{X}_1)) = p_{2*}(\pi_1(\tilde{X}_2))$.

Since S^2 is path-connected and locally path-connected, it satisfies the hypotheses for this proposition. Furthermore, the two covering space maps $p : \mathbb{R}^2 \to S^2$ and $\mathbb{1} : S^2 \to S^2$ both induce

$$p_*(\pi_1(\mathbb{R}^2)) = \mathbb{1}_*(\pi_1(S^2)) = 0$$

and thus if such a p existed, there would be a covering space isomorphism between $\mathbb{R}^2$ and S^2. However, these spaces are not homeomorphic, and thus cannot be isomorphic as covering spaces. So, no such p exists. $\square$

Problem 5

Given a sequence $f : \mathbb{N} \to (0, \infty)$ define $p_k = (0, 0, f(k))$ and C_k is the boundary of the triangle in the plane $\{(x, y, 0) \in \mathbb{R}^3\}$ with vertices $0, v_{2k}, v_{2k+1}$ where $v_n = (\frac{1}{n}, \frac{1}{n^2}, 0)$. For $k \in \mathbb{N}$ the set $A_k = \{t \cdot q + (1-t) \cdot p_k : q \in C_k, 0 \leq t \leq 1\}$ is the cone from p_k to C_k. The subspace $X \subset \mathbb{R}^3$ is $X = \bigcup A_k$.

Part a

Show X is simply-connected if $f(k) = \frac{1}{k}$.

Proof. We begin by considering a homotopy of X defined as follows. For A_1, fix the subcone, say B_1, from p_2 to C_1 (the cone inside A_1 whose face along A_2 is the entirety of $A_2 \cap A_1$), and perform a straight line homotopy of the rest of A_1 downwards into B_1.

The homotopy on A_k is defined inductively. We retract $\bigcup_{i=1}^{k} A_k$ (where each A_i has already been moved to have a vertex at p_k) onto the subcone from p_{k+1} to $\bigcup_{i=1}^{k} C_k$ using the straight line homotopy.

Finally, we stitch these homotopies together by letting the kth homotopy occur in the time $[1 - \frac{1}{k}, 1 - \frac{1}{k+1}]$. Now, for each $t \neq 1$, this action is continuous, as each individual retract of A_k is continuous as a straight line homotopy. Furthermore, this homotopy is continuous at 1. To see this, consider any point $x \in X$, Clearly, this homotopy takes x to its projection on the $x - y$ plane. So, if the height of x is not zero, then x is in some A_k (and possibly A_{k-1} if its on the boundary), and is thus at most a distance $\frac{1}{k}$ from its projection. In particular, since x moves downwards from height $\frac{1}{i}$ to height $\frac{1}{i+1}$ in time $t \in [1 - \frac{1}{i}, 1 - \frac{1}{i+1}]$, the homotopy is clearly continuous at x. Finally, if the height of x is zero, then x is fixed by the homotopy, and the homotopy is continuous at x.

Thus, X is homotopic to $\bigcup_{k=1}^{\infty} \tilde{C}_k$ for $\tilde{C}_k$ the filled in triangle with boundary C_k. This space is clearly simply connected, so X is simply connected as well. $\square$

Part B

Show X is not simply-connected if $f(k) = k$.

Proof. We proceed by constructing a nontrivial loop in X. To do so, consider the loop γ which goes around C_k in time $[1 - \frac{1}{k}, 1 - \frac{1}{k+1}]$. This loop is continuous on $[0, 1)$, since at each $t \in [0, 1)$, t is in some interval $[1 - \frac{1}{i}, 1 - \frac{1}{i+1}]$ and is thus part of the loop around C_i. Furthermore, this loop is continuous at $t = 1$. To see this, observe that any neighborhood U of $\gamma(1)$ necessarily contains all but finitely many C_k. Thus, for n the smallest integer for which C_n is completely contained in U. The image $\gamma((1 - \frac{1}{n}, 1])$ of the open set $(1 - \frac{1}{n}, 1]$ is equal to $\bigcup_{k=n}^{\infty} C_k$ which is entirely in U by definition of n.

Thus, this defines a loop in X. However, this loop is nontrivial, since the only way to homotope the part of the loop around C_k to the constant loop is by taking it over p_k. Since the heights p_k go to infinity as k does, it follows that there is no continuous way to homotope all of γ to the constant loop (the speeds of the homotopy would have to diverge, which cannot be continuous).

Thus, X is not simply connected. $\square$