Homework

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March 18, 2018

Problem 1

Prove that $V^{**} \cong V$ for a finite dimensional vector space V.

Proof. The isomorphism is given by

$$\begin{aligned} \Phi : V &\to V^{**} \\ \Phi(v) = ev_v &= (\phi \mapsto \phi(v)) \end{aligned}$$

First, we observe that this map is linear. Indeed, for $v_1, v_2 \in V$ and α, β a scalar, we have

$$\begin{aligned} \Phi(\alpha v_1 + \beta v_2)(\phi) &= \phi(\alpha v_1 + \beta v_2) \\ &= \alpha\phi(v_1) + \beta\phi(v_2) \\ &= \alpha\Phi(v_1)(\phi) + \beta\Phi(v_2)(\phi) \end{aligned}$$

as desired.

We need to show this is surjective and injective. Injectivity of Φ is easily shown by examining the kernel of Φ. Suppose v is such that $ev_v(\phi) = \phi(v) = 0$ for all $\phi \in V^*$. Then, since V^* separates points of V, it follows that $v = 0$. Thus, the kernel is trivial, as desired.

Now, we need to show this map is surjective. To do so, we appeal to v being finite-dimensional, and let $\{e_i\}$ be a basis for V with dual basis $\{\omega^i\}$. Then, let $x \in V^{**}$. Define a corresponding vector $\tilde{x} = \sum_i x(\omega^i)e_i$. Then

$$\begin{aligned} \Phi(\tilde{x})(\phi) &= \phi(\tilde{x}) \\ &= \sum_i x(\omega^i)\phi(e_i) \\ &= \sum_i x(\omega^i\phi(e_i)) \\ &= x(\phi) \end{aligned}$$

as desired. Thus, Φ is a linear isomorphism. $\square$

Problem 2

Prove that for V a vector space with basis v_i and dual basis v^i, the set

$$\{v^i \otimes v^j \mid 1 \leq i, j \leq n\}$$

forms a basis for $V^* \otimes V^*$.

Proof. For this, we will show that the vector space

$$W = \operatorname{span}(\{v^i \otimes v^j \mid 1 \leq i, j \leq n\})$$

satisfies the universal property of tensor products. That is, we wish to show that for each bilinear map $h : V^* \times V^* \to U$ for some vector space U, there is a unique linear map $\tilde{h} : W \to U$ such that the diagram

$$\begin{tikzcd} V^* \times V^* \arrow[r, "h"] \arrow[d, "\otimes"] & U \\ W \arrow[ur, dotted, "\tilde{h}"] & \end{tikzcd}$$

commutes. We will guess that $\otimes : V^* \times V^* \to W$ is given as

$$\otimes(a_i v^i, b_j v^j) = a_i b_j v^i \otimes v^j$$

So, let $h : V^* \times V^* \to U$ be a bilinear map. Define $\tilde{h} : W \to U$ as

$$\tilde{h}(\sum_{i,j} a_{ij} v^i \otimes v^j) = a_{ij} h(v^i, v^j)$$

Then it is clear that the diagram

$$\begin{tikzcd} V^* \times V^* \arrow[r, "h"] \arrow[d, "\otimes"] & U \\ W \arrow[ur, dotted, "\tilde{h}"] & \end{tikzcd}$$

commutes, since

$$\begin{aligned} h(a_i v^i, b_j v^j) &= a_i b_j h(v^i, v^j) \\ \tilde{h} \circ \otimes(a_i v^i, b_j v^j) &= \tilde{h}(a_i b_j v^i \otimes v^j) \\ &= a_i b_j h(v^i, v^j) \end{aligned}$$

It should be clear from construction that $\tilde{h}$ is unique.

Thus, since every bilinear map from $V^* \times v^*$ factors through W, W satisfies the universal property of tensor products, and is isomorphic to $V^* \otimes V^*$. Thus, the set $\{v^i \otimes v^j\}$ forms a basis for $V^* \otimes V^*$ as desired. □

Problem 3

Show that $dvol = \wedge_i \omega^i = \sqrt{|g|} dx^n$.

Proof. Recall that $dx^n = \wedge_{i=1}^n dx^i$, and our manifold is n-dimensional.

Recall that for an n-fold wedge product $\wedge_{i=1}^n v^i$, we have (for $v^i = a^i_j \omega^j$ in an orthonormal frame)

$$\wedge_{i=1}^n v^i = |\det(a^1_j, \ldots, a^n_j)| \wedge_{i=1}^n \omega^i = \sqrt{\det(A^t A)} \wedge_{i=1}^n \omega^i$$

for A the matrix with columns a^i. We can apply this to $v^i = dx^i$ to get the desired result. □

Problem 4

Show that the definition of the integral of a top degree form on a single chart is independent of choice of coordinates.

Proof. Recall the definition of the integral of a top degree differential form on a compact set K in a single coordinate frame is

$$\int_K \omega = \int_{\phi(K)} f \circ \phi^{-1} dx^n$$

for $\omega = f dx^1 \wedge \cdots \wedge dx^n$.

We also have the change-of-coordinates formula for a diffeomorphism $F : \Omega_1 \to \Omega_2$ as

$$\int_{\Omega_2} f dy^n = \int_{\Omega_1} f \circ F |J_F| dx^n$$

where J_F is the jacobian of F.

So, let $\phi : M \to \mathbb{R}^n$ be the original coordinate system, and let $\psi : M \to \mathbb{R}^n$ be another coordinate system covering K. Then, we have the diffeomorphism $F = \psi \circ \phi^{-1}$ and we can apply this to get

$$\int_{\psi(K)} g \circ psi^{-1} dy^n = \int_{\phi(K)} g \circ \psi^{-1} \circ F |J_F| dx^n$$

which is just

$$\begin{aligned}
\int_{\phi(K)} g \circ \psi^{-1} \circ F |J_F| dx^n &= \int_{\phi(K)} g \circ \psi^{-1} \circ \psi \circ \phi^{-1} |J_F| dx^n \\
&= \int_{\phi(K)} g \circ \phi^{-1} |J_F| dx^n \\
&= \int_K g |J_F| dx^n \\
&= \int_K \omega
\end{aligned}$$

where we used the fact that $\omega = g dy^1 \wedge \cdots \wedge dy^n = g|J_F| dx^1 \wedge \cdots \wedge dx^n$ since $|J_F| = \det(J_F)$ on a two positively oriented charts.

Thus, the two integrals agree. $\square$

Problem 5

Prove that a manifold is orientable if and only if it admits a nowhere vanishing top degree form.

Proof. Suppose M is an orientable manifold. That is, there exists an atlas $\{U_\alpha, \phi_\alpha\}$ of M for which the Jacobian of each transition map has positive determinant. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to the atlas U_α. Then, define

$$\omega = \sum_\alpha \psi_\alpha dx_\alpha^i \wedge \cdots \wedge dx_\alpha^n$$

where x_α^i are the coordinate functions on U_α. We claim that ω is a nowhere-vanishing form. Clearly, if p is a point in M contained in only one chart, then $\omega_p = dx_p^1 \wedge \cdots \wedge dx_p^n$ and does not vanish. If p is such that it is contained in more than one chart, then ω at p is the sum of positive terms (since each coordinate system is positive, we have $dx^n = \det(J)dy^n$ and $\det(J)$ is always positive) and does not vanish.

Suppose instead that M admits a nowhere-vanishing top degree form ω. Then, let $\{U_\alpha, \phi_\alpha\}$ be an atlas of M. For x^i_α the coordinate functions for ϕ_α, define a new coordinate system to be such that if ω is expressed as

$$\omega = f dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$$

then f is positive. This is done by setting x^1_α to its negative if f is negative on that chart. Note that since ω never vanishes, we know that f will be either entirely positive or entirely negative on a chart. Thus, such a choice can be made consistently.

Then, the modified atlas is a positive coordinate chart for M, which is easily verified, since

$$\omega = f dx^n = f \det(J) dy^n$$

and ω always has positive coefficient, so $\det(J)$ is positive. $\square$

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