Midterm

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Problem 1

Part a

Use the standard charts on S^n to calculate the matrix representation of $di : T_pS^n \to T_p\mathbb{R}^{n+1}$, and show that di is injective, and thus i is an embedding.

Proof. For this calculation, we will use the chart given by hemisphere projection. That is, the domains for the charts will be the open sets $U_i^\pm = \{(x^1 \dots, x^{n+1}) \mid x^i > 0 (x^i < 0 \text{resp.})\}$ with maps

$$\phi_i^\pm(x^1, \dots, x^{n+1}) = (x^1, \dots, \hat{x^i}, \dots, x^{n+1})$$

Where a hat denotes omission of the variable.

Now, suppose $p \in U_i^+$ (without loss of generality, we take the positive hemisphere of x^i, but the argument can be repeated exactly with the negative hemisphere as well.) and let the coordinate representation of p be

$$\phi(p) = (x^1, \dots, \hat{x^i}, \dots, x^{n+1})$$

Then, the inclusion map looks like

$$\begin{aligned} i \circ \phi^{-1}((x^1, \dots, \hat{x^i}, \dots, x^{n+1}) &= (y^1, \dots, y^{n+1}) \\ &= (x^1, \dots, x^{i-1}, \sqrt{1 - x^a x_a}, \dots, x^{n+1}) \end{aligned}$$

and the Jacobian di can be calculated directly using the identity $di_j^k = \partial_j(y^k)$. Which gives the matrix (for $j = 1, \dots, i-1, i+1, \dots, n+1$ and $k = 1, \dots, n$)

$$\partial_j(y^k) = \delta_j^k - \frac{1}{\sqrt{1 - x^a x_a}} \delta^{ik} x_j$$

Which is clearly injective, since the rows $k \neq i$ are the basis covectors for $\mathbb{R}^n$, and thus $\partial_j(y^k)$ has rank n as desired.

Furthermore, since S^n is compact, and the inclusion is injective, it follows that i defines an embedding of S^n into $\mathbb{R}^{n+1}$. $\square$

Part B

Show that T_pS^n, when identified with $di(T_pS^n)$ is the subspace of $\mathbb{R}^{n+1}$ consisting of all vectors perpendicular to the radial vector to p.

Proof. This follows by direct calculation. To see this, let $v \in T_pS^n$. Then,

$$\begin{aligned} di(v) = \partial_j y^k v^j &= \delta^k_j v^j - \frac{1}{\sqrt{1 - x^a x_a}} \delta^{ik} x_j v^j \\ &= v^k - \frac{x^a v_a}{\sqrt{1 - x^a x_a}} \delta^{ik} \end{aligned}$$

where it is assumed that $x_a v^a$ does not sum over the i^{th} component of v.

Recalling earlier that the embedding sends

$$p = (x^1, \ldots, \hat{x^i}, \ldots, x^{n+1})$$

to

$$(y^1, \ldots, y^{n+1}) = (x^1, \ldots, x^{i-1}, \sqrt{1 - x^a x_a}, \ldots, x^{n+1})$$

we can compute the inner product $g_{jk} v^j y^k$ directly.

$$\begin{aligned} g_{jk} v^j y^k &= v^k x_k + \delta^{ik}_{ij} v^j y_k \\ &= v^k x_k + v^i y_i \\ &= v^k x_k - \frac{v^a x_a}{\sqrt{1 - x^a x_a}} \sqrt{1 - x^a x_a} \\ &= v^k x_k - v^a x_a \\ &= 0 \end{aligned}$$

Thus, $di(v)$ is perpendicular to p, as desired. □

Part C

For F a smooth map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{m+1}$ such that $F(S^n) \subset S^m$, show that $d(F|_{S^n}) = dF|_{T_pS^n}$.

Proof. To begin with, let γ be a curve in S^n such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then,

$$\begin{aligned} d(F|_{S^n})(\gamma'(0)) &= \partial_t|_0 F|_{S^n}(\gamma(t)) \\ &= \partial_t|_0 F(\gamma(t)) \\ &= dF(\gamma'(0)) \\ &= dF(\gamma'(0))|_{T_pS^n} \end{aligned}$$

Here, the equality from line 2 to line 3 comes from the fact that F maps S^n into S^m, and the equality from line 3 to line 4 comes from the fact that $\gamma'(0)$ started in T_pS^n to begin with. □

Problem 2

Show that the tangent bundle TM is always orientable.

Proof. For this problem, we will use that fact that a manifold is orientable if there exist charts such that the coordinate transition maps have a Jacobian of positive determinant.

Before proceeding further, we prove the following lemma:

Lemma. *Suppose M and N are smooth manifolds. In particular, their product $M \times N$ is a smooth manifold. Furthermore, for pairs of coordinates x_m, y_m on $U_m \subset M$ and x_n, y_n on $U_n \subset N$, the product coordinates $x = (x_m, x_n)$ and $y = (y_m, y_n)$ are smooth coordinates on $U_m \times U_n$, and the Jacobian $J(x \to y)$ is given componentwise. That is,*

$$J(x \to y) = (J(x_m \to y_m), J(x_n \to y_n))$$

Proof. That $M \times N$ is a smooth manifold follows almost immediately by taking products of coordinate charts on M and N. Now, we have the following diagram:

$$\begin{tikzcd}
 & & M \times N & & \\
 & & U_m \times U_n \arrow[u, hook] \arrow[dl] \arrow[dr] \arrow[ddd, dashed, shift right, "x"'] \arrow[ddd, dashed, shift left, "y"] & & \\
M & U_m \arrow[l, hook'] \arrow[d, shift right, "x_m"'] \arrow[d, shift left, "y_m"] & & U_n \arrow[r, hook] \arrow[d, shift right, "x_n"'] \arrow[d, shift left, "y_n"] & N \\
 & \mathbb{R}^m \arrow[dr, hook] & & \mathbb{R}^n \arrow[dl, hook'] & \\
 & & \mathbb{R}^{m+n} & &
\end{tikzcd}$$

which implies that the induced coordinates x and y are smooth. Now, let's expand the lower half of the commutative diagram to get the transition maps $x_m \to y_m$ and $x_n \to y_n$:

$$\begin{tikzcd}
\mathbb{R}^m \arrow[dd, "x_m \to y_m"'] \arrow[dr, hook] & & \mathbb{R}^n \arrow[dl, hook'] \arrow[dd, "x_n \to y_n"] \\
 & \mathbb{R}^{m+n} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, dashed, "x \to y"] & \\
\mathbb{R}^m \arrow[dr, hook] & & \mathbb{R}^n \arrow[dl, hook'] \\
 & \mathbb{R}^{m+n} &
\end{tikzcd}$$

Differentiating this diagram (applying the differential functor) yields:

$$\begin{tikzcd}
\mathbb{R}^m \arrow[dd, "J(x_m \to y_m)"'] \arrow[dr, hook] & & \mathbb{R}^n \arrow[dl, hook'] \arrow[dd, "J(x_n \to y_n)"] \\
 & \mathbb{R}^{m+n} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, dashed, "J(x \to y)"] & \\
\mathbb{R}^m \arrow[dr, hook] & & \mathbb{R}^n \arrow[dl, hook'] \\
 & \mathbb{R}^{m+n} &
\end{tikzcd}$$

Thus, $J(x \to y) = (J(x_m \to y_m), J(x_n \to y_n))$ as desired. $\square$

We are now ready to prove the general result.

So, let M be a smooth manifold with tangent bundle TM. Furthermore, for a point p, suppose there are two coordinate charts x^i and y^i on a neighborhood of p. We wish to calculate the Jacobian of the induced coordinate transformations on the tangent bundle.

To do so, we first appeal to the fact that TM is locally trivializable. That is, on some neighborhood U containing p, $\pi^{-1}(U) \cong M \times T_pM$, where π is the canonical projection of TM onto M. In particular, this means that in $\pi^{-1}(U)$, we have the coordinate charts $x^i \times dx^i$ and $y^i \times dy^i$.

Thus, the transition map is just $(x \to y, dx \to dy)$, where $x \to y$ is the transition map from the x coordinate system to the y coordinate system, and $dx \to dy$ is the transition map from the $\partial_x|_p$ coordinate system to the $\partial_y|_p$ coordinate system.

Recall that $dx \to dy$ is simply the Jacobian of the original coordinate transform. That is, $dx \to dy = J(x \to y)$. Now, from the above lemma,

$$J(x \to y, dx \to dy) = J(x \to y, J(x \to y)) = (J(x \to y), J(J(x \to y)))$$

It should be clear that $J^2 = J$, since the Jacobian of a transformation is linear. Thus, we have that

$$J(x \to y, dx \to dy) = (J(x \to y), J(x \to y))$$

To calculate the determinant of this, we appeal to the fact that the determinant of a linear transformation of the form (A, B) is the product of the determinants of A and B. Thus,

$$\begin{aligned}\det J(x \to y, dx \to dy) &= \det(J(x \to y)) \det(J(x \to y)) \\ &= (\det(J(x \to y)))^2\end{aligned}$$

Which is always positive.

Since this can be done at any point p in the manifold, we have an atlas for TM where the determinant of the coordinate transforms is always positive, and thus TM is orientable. $\square$

Problem 3

Show that for M a smooth manifold, and $S \subset M$ a smooth submanifold, S is embedded if and only if for every $f \in C^\infty(S)$, f has a smooth extension to a neighborhood of S in M.

Proof. ($\Longrightarrow$) Suppose S is embedded. In particular, this means that each point $s \in S$ has a neighborhood U_s on which there is a slice chart ϕ of S on M. In particular, if $\dim(M) = m$ and $\dim(S) = k$, and x^i are the coordinate functions of ϕ, then there is a chart on U_s centered at s for which $(x^1, \ldots, x^m)|_S = (x^1, \ldots, x^k, 0, \ldots, 0)$.

Now, let $f \in C^\infty(S)$. Since U_s has a slice chart, it is possible to extend f locally to a function $\tilde{f}_s \in C^\infty(U_s)$ such that $\tilde{f}_s|_S = f$. This is possible through the use of smooth bump functions on the coordinates $m - k$ to k.

So, choose a countable number of such U_s, call them U_i, that cover S, and let $\Psi = \sum_{i=1}^\infty \psi_i$ be a partition of unity subordinate to $\{U_i\}$. Then, the function

$$\tilde{f}(x) = \sum_{i=1}^\infty \psi_i(x) \tilde{f}_i(x)$$

is a smooth extension of f that restricts to f.

($\Longleftarrow$) Suppose for the converse that each $f \in C^\infty(S)$ had a smooth extension to a neighborhood U of S. Now, in particular for each open set U in S, we can construct a smooth function f on S for which $\operatorname{supp}(f) = U$. This function has a smooth extension $\tilde{f}$ on some $U' \subset M$ for which $\tilde{f}|_U = f$. Furthermore, $\operatorname{supp}(\tilde{f})$ is open in M, and the intersection

$$\operatorname{supp}(\tilde{f}) \cap S = \operatorname{supp}(f) = U$$

Thus, each open set in S is the intersection of S with an open set in M, and S has the subspace topology, making S an embedding. $\square$

Problem 4

Let M be a manifold, and $p \in M$.

Part A

Show that $I_p = \{f \in C^\infty(M) \mid f(p) = 0\}$ is a maximal ideal in $C^\infty(M)$.

Proof. Suppose $I_p \subsetneq I$ for some ideal I. In particular, this means that there is some $f \in I$ with $f(p) \neq 0$. Without loss of generality, let $f(p) > 0$.

Now, by theorem 2.29 of Lee, there exists a nonnegative function g for which $g^{-1}(0) = \{p\}$. Since this function vanishes at p, it is in the ideal, as well as the function $f + g$. In particular, $f + g > 0$ at every point.

Thus, the function $\frac{1}{f+g}$ is well-defined, and by the multiplicatively absorbing property of ideals, the function $\frac{1}{f+g}(f + g) = 1$ is in the ideal as well. Since 1 is the unit, it follows that $I = C^\infty(M)$. Thus, I_p is maximal. $\square$

Part B

Show that if M is compact, any maximal ideal in $C^\infty(M)$ is of this form.

Proof. Suppose for a contradiction that some maximal ideal I such that at each point $p \in M$, there is some f_p such that $f_p \neq 0$.

In particular, this means that there is a neighborhood U_p of p for which f_p is nonzero. The neighborhoods U_p form an open cover of M, of which there is a finite subcover $\{U_i\}_{i=1}^n$. Then, the function $F = \sum_{i=1}^n (f_i)^2$ is everywhere nonzero, and in the ideal since I is additively closed. Thus, the function $\frac{F}{F} = 1$ is well-defined everywhere, and in the ideal I. Thus, $I = C^\infty(M)$, which contradicts I being (nontrivially) maximal.

Thus, for each ideal I in $C^\infty(M)$, I must have all functions vanish at at least one point p. Considering only the maximal ideals, it must follow that for a maximal ideal I, the functions in I vanish at exactly one point p, so $I = I_p$. This follows by observing that if functions in I vanished at two points p and q, then I would be contained in I_p and I_q, and would not be maximal.

Thus, every maximal ideal in $C^\infty(M)$ is of the form I_p. $\square$

Problem 5

Part a

Show that, for a quaternion $q = a + bi + cj + dk$, $|q|^2 = q\bar{q}$ is equal to $a^2 + b^2 + c^2 + d^2$. Conclude that S^3 can be identified with the unit quaternions.

Proof. This follows from direct computation.

$$\begin{aligned}
q\bar{q} &= (a + bi + cj + dk)(a - bi - cj - dk) \\
&= a^2 - abi - acj - adk + abi - (bi)^2 - bcij - bdik \\
&+ acj - bcji - (cj)^2 - cdjk + adk - bdki - cdkj - (dk)^2 \\
&= a^2 + b^2 + c^2 + d^2 - bcij - bcji - bdik - bdki - cdjk - cdkj - bdki - bdik \\
&= a^2 + b^2 + c^2 + d^2
\end{aligned}$$

where the last equality was obtained by observing that $ij = -ji$ and $jk = -kj$.

Thus, the norm on $\mathbb{H}$ coincides with the norm on $\mathbb{R}^4$, and so the topologies agree. Thus, S^3 can be identified with the unit quaternions in an isometric way. □

Part b

Show that S^3 is a Lie group with quaternion multiplication.

Proof. We will first show that the operation of quaternion multiplication is smooth in the ambient space $\mathbb{R}^4 \setminus \{0\} \cong \mathbb{H} \setminus \{0\}$, and conclude that since S^3 is a closed embedded subgroup of $\mathbb{H} \setminus \{0\}$, it is a Lie group under the same operation.

We will show that the operation of multiplication is smooth by directly calculating the multiplication in the standard coordinates on $\mathbb{R}^4$.

So, let $q_1 = (x^1, x^2, x^3, x^4)$ and $q^2 = (y^1, y^2, y^3, y^4)$. Then,

$$\begin{aligned}
q_1 q_2 &= (x^1 + x^2 i + x^3 j + x^4 k)(y^1 + y^2 i + y^3 j + y^4 k) \\
&= x^1 y^1 + x^1 y^2 i + x^1 y^3 j + x^1 y^4 k \\
&+ x^2 y^1 i + x^2 y^2 ii + x^2 y^3 ij + x^2 y^4 ik \\
&+ x^3 y^1 j + x^3 y^2 ji + x^3 y^3 jj + x^3 y^4 jk \\
&+ x^4 y^1 k + x^4 y^2 ki + x^4 y^3 kj + x^4 y^4 kk \\
&= x^1 y^1 - x^2 y^2 - x^3 y^3 - x^4 y^4 \\
&+ (x^1 y^2 + x^2 y^1 + x^3 x^4 - x^4 x^3) i \\
&+ (x^1 y^3 - x^2 y^4 + x^3 y^1 + x^4 y^2) j \\
&+ (x^1 y^4 + x^2 y^3 - x^3 y^2 + x^4 y^1) k
\end{aligned}$$

which is clearly a smooth operation. Thus, S^3 is a Lie group under quaternion multiplication. □