Problem Set 4

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Preliminaries

Lemma 1. *For two paths $f, g : I \to X$ with $f \simeq g$ relative to ∂I, and $h : I \to X$ with $f(1) = g(1) = h(0)$, then $hf \simeq hg$, where hf is the path that first traverses f first, and then h (similarly for hg).*

Similarly, if $h : I \to X$ with $h(1) = f(0) = g(0)$, then $fh \simeq gh$.

Proof. Let $F : I \times I \to X$ be the homotopy from f to g relative to ∂I, and let $h : I \to X$ be such that $f(1) = g(1) = h(0)$. The homotopy between hf and hg is given by

$$H : I \times I \to X$$
$$H(t, s) = \begin{cases} F(2t, s), & \text{if } t < \frac{1}{2} \\ h(2(t - \frac{1}{2})), & \text{else} \end{cases}$$

That is, run the homotopy on the f section of the path, and leave h alone. Since the homotopy F fixes the endpoints, it follows that $F(1, s) = h(0)$ for all s, and the homotopy H is well-defined. H is clearly continuous, then, by the pasting lemma. Therefore, $H(t, 0) = hf \simeq H(t, 1) = hg$ as desired.

Suppose instead that $h : I \to X$ is such that $h(1) = f(0) = g(0)$. Then, it follows that $\overline{h}(0) = \overline{f}(1) = \overline{g}(1)$ which satisfies the hypotheses for the previous result, and so $\bar{h}\bar{f} \simeq \bar{h}\bar{g}$, and so $\overline{fh} \simeq \overline{gh}$, which immediately implies that $fh \simeq gh$ as desired. □

Problem 1

Show that the composition of paths satisfies the following property: if $f_0 \cdot g_0 \simeq f_1 \cdot g_1$, and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

Proof. Since $g_0 \simeq g_1$ (assumed to be relative to ∂I), we have that

$$f_0 g_0 \simeq f_0 g_1$$

by Lemma 1. Now, since $f_1 g_1 \simeq f_0 g_0$, it follows that $f_1 g_1 \simeq f_0 g_1$ by transitivity of $\simeq$.

Now, letting $\overline{g}$ be the inverse path of g, we have that

$$\begin{aligned} f_1 g_1 &\simeq f_0 g_1 \\ f_1 g_1 \overline{g_1} &\simeq f_0 g_1 \overline{g_1} && \text{by Lemma 1} \\ f_1 &\simeq f_0 && \text{by } g_1\overline{g_1} \simeq 0 \text{ and Lemma 1} \end{aligned}$$

as desired. $\square$

Problem 2

Show that the change of basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. Let g, h be paths in a space X with the same starting and ending points, and such that $g \simeq h$. We will show that $\beta_h = \beta_g$. In particular, we will show that conjugating by h is homotopic to conjugating by g.

So, let f be a loop based at the endpoint of g, h. We will show that $\bar{g} f g \simeq \bar{h} f h$. This, however, is just a straightforward application of Lemma 1.

To see this, we note that since $g \simeq h$, we have that $fg \simeq fh$. Now, since $barg \simeq \bar{h}$, we can also write

$$\bar{g} f g \simeq \bar{g} f h = \bar{g}(fh) \simeq \bar{h}(fh) = \bar{h} f h$$

as desired $\square$

Problem 3

For a path-connected space X, show that $\pi_1(X)$ is Abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of h.

Proof. ($\Longrightarrow$) Suppose that for a path-connected space X, we have that $\pi_1(x)$ is Abelian. Furthermore, let h, g be two paths in X such that $g(0) = h(0) = x_0$ and $g(1) = h(1) = x_1$. Furthermore, let f be a loop based at x_1. We wish to show that $\beta_h([f]) = \beta_g([f])$ which is equivalent to showing that $\beta_{\bar{g}}\beta_h([f]) = [f]$. Now, $\beta_{\bar{g}}\beta_h([f])$ is just $[g\bar{h}fh\bar{g}]$. Note however that $h\bar{g}$, $g\bar{h}$, and f are loops based at x_1. Since $\pi_1(X)$ is Abelian, it follows that

$$\begin{aligned} &= [g\bar{h}][f][h\bar{g}] \\ &= [f][g\bar{h}][h\bar{g}] \\ &= [f][g\bar{h}h\bar{g}] \\ &= [f] \end{aligned}$$

as desired.

($\Longleftarrow$) Suppose that X is such that for any two paths h, g with $h(0) = g(0) = x_0$ and $h(1) = g(1) = x_1$, we have that $\beta_h = \beta_g$. We wish to show that for any two elements $[f_1], [f_2] \in \pi_1(X)$, we have that $[f_1][f_2] = [f_2][f_1]$. Alternately, we can show that $[f_1][f_2][\bar{f}_1] = [f_2]$. This is obvious, though, since f_1 and f_2 satisfy the hypotheses for h, g, which implies that $\beta_{f_1} = \beta_{f_2}$. So,

$$\begin{aligned} [f_2][\bar{f}_1] &= \beta_{\bar{f}_1}[f_2] \\ &= \beta_{\bar{f}_2}[f_2] \\ &= [f_2][f_2][f_2^{-1}] \\ &= [f_2] \end{aligned}$$

as desired. $\square$

Problem 4

Show that if a subspace $x \subset \mathbb{R}^n$ is locally star-shaped, then every path in X is homotopic in X to a piecewise linear path. Show specifically this holds when X is open, and when X is a union of finitely many closed convex sets.

Proof. To begin with, let γ be a path in X. At each point $\gamma(t) \in X$, let S_t be a star-shaped neighborhood around $\gamma(t)$. In particular, $\{S_t\}$ is an open cover of $\gamma(I)$, and since $\gamma(I)$ is compact, it follows that there is some finite subcover $\{S_i\}_{i=1}^n$. In particular, we can take a finite subcover such that each point $\gamma(t)$ is in at most two open sets in the subcover, and the preimages $\gamma^{-1}(S_i)$ and $\gamma^{-1}(S_j)$ are distinct (neither contains the other). We further require that each open set in the subcover be an interval. Finally, order the subcover sequentially. That is, let S_1 be the open set containing $\gamma(0)$, and let S_i be the open set that overlaps with S_{i-1}. Define a set of partition points $\{t_i\}_{i=1}^{n-1}$ such that t_i lies in the intersection of S_i and S_{i+1}.

Now, for each S_i, let x_i be the distinguished point in the star-shaped neighborhood. That is, for each $x \in S_i$, the line segment from x to x_i is in S_i.

We are finally ready to describe the homotopy from γ to a piecewise linear function. We note first that each S_i is simply connected. In particular, all paths in S_i with fixed endpoints are homotopic to each other.

On S_1, we can homotope the segment of the path $\gamma([0,t_1])$ to the path obtained by taking the line segment from $\gamma(0)$ to x_1 and then the line segment from x_1 to $\gamma(t_1)$. Generally, on S_i, homotope the path $\gamma([t_{i-1},t_i])$ to the path from $\gamma(t_{i-1})$ to x_i, then from x_i to $\gamma(t_i)$. Finally, in S_n, we homotope the path $\gamma([t_n,1])$ to the path from $\gamma(t_n)$ to $\gamma(1)$.

On each open set, we homotoped to a piecewise linear path, and they agree on the intersection, and so the resulting path is piecewise linear as desired. $\square$

Problem 5

Show that for every space X, the following are equivalent:

(a) Every map $S^1 \to X$ is homotopic to the constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi(X, x_0) = 0$ for all $x_0 \in X$.

Proof. ((a) $\implies$ (b)) Suppose $f : S^1 \to X$ is such that f is homotopic to a constant map x_0. In particular, we have a homotopy $F : S^1 \times I \to X$ with $F(x,0) = f(x)$ and $F(x,1) = x_0$.

Now, the disk D^2 is homeomorphic to the quotient $(S^1 \times I)/S^1 \times \{1\}$ via the homeomorphism

$$\phi : D^2 \to (S^1 \times I)/S^1 \times \{0\}\phi(r,\theta) \quad = [(\theta, r)]$$

Where the coordinates on D^2 are polar coordinates with $r \leq 1$, $\theta \in [0, 2\pi)$, and $[(\theta,r)]$ is the equivalence class of the point $(\theta, r) \in S^1 \times I$.

Now, we can define $\tilde{F} : (S^1 \times I)/S^1 \times \{1\} \to X$ to be the unique map that makes the diagram

$$\begin{tikzcd}
S^1 \times I \arrow[rr, "F"] \arrow[dr, two heads, "q"] & & X \\
& (S^1 \times I)/S^1 \times \{1\} \arrow[ur, "\tilde{F}"] &
\end{tikzcd}$$

commute. Here, q is the canonical quotient map. $\tilde{F}$ is well-defined, since F is constant on the fibers of q. This is clear, since the only nontrivial fiber of q is the subspace $S^1 \times \{1\}$, which F sends identically to x_0.

Thus, using the homeomorphism above, we find the map $\tilde{F} \circ \phi$ to be an extension of f. This is evident, since $\tilde{F} \circ \phi|_{\partial D^2}$ is just $\tilde{F}|_{S^1 \times \{0\}}$ which is just $F(x, 0) = f(x)$ as desired.

((b) $\implies$ (c)) Suppose that any map $f : S^1 \to X$ can be extended to a map $\tilde{f} : D^2 \to X$. Now, let $x_0 \in X$ be arbitrary. We will show that $\pi_1(X, x_0) = 0$. in particular, we will show that any loop based at x_0 is homotopic to the constant loop.

So, let $f : S^1 \to X$ be a loop such that $f(0) = x_0$. By (b), we know that such an f extends to a $\tilde{f} : D^2 \to X$. Now, via the homeomorphism above, we have a map

$$\tilde{F} : (S^1 \times I)/S^1 \times \{1\} \to X$$

given by $\tilde{F} = \tilde{f} \circ \phi^{-1}$. Furthermore, for $q : S^1 \times I \to (S^1 \times I)/S^1 \times \{1\}$ the canonical quotient map, we have a map

$$F : S^1 \times I \to X$$

given by $F = \tilde{F} \circ q$. Now, $F|_{S^1 \times \{0\}}$ is just $\tilde{F}|_{S^1 \times \{0\}} = \tilde{f}|_{\partial D^2} = f$, and furthermore $\square$