Final Exam

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Problem 1

For every $n \in \mathbb{N}$, let μ_n be a measure on $(\Omega, \mathscr{A})$ with $\mu_n(\Omega) = 1$. For every $E \in \mathscr{A}$, define

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n}$$

Give a careful proof that μ is a measure on $(\omega, \mathscr{A})$ with $\mu(\Omega) = 1$.

Proof. We wish to prove that μ is a measure on $(\Omega, \mathscr{A})$. That is, we wish to show that that $\mu(\emptyset) = 0$, that $\mu(E) \geq 0$ for all $E \in \mathscr{A}$, and that for a countable collection of disjoint sets $\{E_j\}_{j=1}^{\infty}$ for which $E_j \in \mathscr{A}$ for all j,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

To begin with, we note that since each μ_n is a measure, we have that $\mu_n(\emptyset) = 0$. Thus,

$$\begin{aligned}
\mu(\emptyset) &= \sum_{n=1}^{\infty} \frac{\mu_n(\emptyset)}{2^n} \\
&= \sum_{n=1}^{\infty} \frac{0}{2^n} \\
&= 0
\end{aligned}$$

as desired.

Next, we note that since each μ_n is a measure, $\mu_n(E) \geq 0$ for all $E \in \mathscr{A}$. Thus, since both $\mu_n(E)$ and 2^n are greater than zero for each n, it must be that

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n} \geq 0$$

as desired.

To show that μ is countably additive, we first prove the following lemma:

Lemma. *For a doubly indexed sequence $\{a_{ij}\}$ of positive numbers,*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

provided that either sum converges.

Proof. We note first that a_{ij} can be thought of as a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$.

Now, Tonelli's theorem tells us that for any positive function $f : \Omega \times \Sigma \to \mathbb{R}$ on the product space $\Omega \times \Sigma$ of σ-finite measure spaces $(\Omega, \mathscr{A}, \mu)$ and $(\Sigma, \mathscr{B}, \nu)$ such that f is measurable with respect to $\mathscr{A} \otimes \mathscr{B}$, we have that

$$\int_{\Omega} \left(\int_{\Sigma} f(x,y) d\nu(y) \right) d\mu(x) = \int_{\Sigma} \left(\int_{\Omega} f(x,y) d\mu(x) \right) d\nu(y)$$

Now, consider the case where $\Omega = \Sigma = \mathbb{N}$, $\mathscr{A} = \mathscr{B} = 2^{\mathbb{N}}$, and $\mu = \nu = \mu_c$ the counting measure. The function a_{ij} from $\mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is positive (by hypothesis), and is measurable on $2^{\mathbb{N}} \otimes 2^{\mathbb{N}} = 2^{\mathbb{N}\times\mathbb{N}}$, since every function is measurable with respect to this σ-algebra. Thus, applying Tonelli's theorem yields

$$\begin{aligned}
\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) &= \int_{\mathbb{N}} \left(\int_{\mathbb{N}} a_{ij} d\mu_c(j) \right) d\mu_c(i) \\
&= \int_{\mathbb{N}} \left(\int_{\mathbb{N}} a_{ij} d\mu_c(i) \right) d\mu_c(j) \\
&= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right)
\end{aligned}$$

as desired. ☐

Equipped with this result, we now prove that μ is countably additive. To do so, let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of disjoint measurable sets. Now, we know by the fact that each μ_n is a measure that

$$\mu_n\left(\bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu_n(E_j)$$

Thus, we have

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{\infty} E_j \right) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n \left(\bigcup_{j=1}^{\infty} E_j \right) \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{2^n} \mu_n(E_j)
\end{aligned}$$

We apply the above lemma to get

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2^n}\mu_n(E_j) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2^n}\mu_n(E_j)$$
$$= \sum_{j=1}^{\infty}\mu(E_j)$$

as desired.

Finally, we wish to show that $\mu(\Omega) = 1$. This follows from direct computation (observing that $\mu_n(\Omega) = 1$ for all n):

$$\begin{aligned}\mu(\Omega) &= \sum_{n=1}^{\infty}\frac{\mu_n(\Omega)}{2^n}\\ &= \sum_{n=1}^{\infty}\frac{1}{2^n}\\ &= \frac{1}{1-\frac{1}{2}} - 1\\ &= 1\end{aligned}$$

as desired. Here, we used the standard formula for a geometric series

$$\sum_{n=1}^{\infty} a^n = \frac{1}{1-a} - 1$$

for $0 < a < 1$. $\square$

Problem 2

Suppose $\mu(\Omega) < \infty$. Prove that

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

Proof. We note first that the trivial case of $\|f\|_{L^\infty} = 0$ is clear, since

$$\begin{aligned}\|f\|_{L^\infty} = 0 &\implies f = 0 \ \mu - \text{almost everywhere}\\ &\implies \|f\|_{L^p} = 0 \ \forall p\\ &\implies \lim_{p\to\infty} \|f\|_{L^p} = 0\end{aligned}$$

Therefore, for the rest of this problem, it is assumed that $\|f\|_{L^\infty} > 0$.

Suppose first that $\|f\|_{L^\infty} < \infty$. Then, we are free to scale f so that $\|f\|_{L^\infty} = 1$. (This is clear, since

$$\lim_{p\to\infty} \|cf\|_{L^p} = c \lim_{p\to\infty} \|f\|_{L^p}$$

so

$$\lim_{p\to\infty} \|cf\|_{L^p} = \|cf\|_{L^\infty} \iff \lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

and so multiplying f by a constant will not change the equality.)

So, without loss of generality, let $\|f\|_{L^\infty} = 1$. We will show first that

$$\lim_{p\to\infty} \|f\|_{L^p} \leq 1$$

To do so, we consider the altered function

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } f(x) \leq \|f\|_{L^\infty} \\ 0, & \text{if } f(x) > \|f\|_{L^\infty} \end{cases}$$

Now, we know that $\mu\{|f| > \|f\|_{L^\infty}\} = 0$ by the definition of the L^∞ norm, so it follows that $\tilde{f}$ and f differ only on a set of measure zero, and thus are in the same equivalence class in L^p for all p.

Now, we have that $\tilde{f} \leq \|f\|_{L^\infty} = 1$, and thus $\tilde{f}^p \leq 1$ for all $p \geq 1$. Therefore,

$$\begin{aligned}\int_\Omega |\tilde{f}(x)|^p d\mu &\leq \int_\Omega 1 d\mu\\ &= \mu(\Omega)\end{aligned}$$

which implies that

$$\begin{aligned}\|f\|_{L^p} &= \left(\int_\Omega |\tilde{f}(x)|^p d\mu\right)^{\frac{1}{p}}\\ &\leq (\mu(\Omega))^{\frac{1}{p}}\end{aligned}$$

and for $\mu(\Omega) < \infty$, we have that $\lim_{p\to\infty}(\mu(\Omega))^{\frac{1}{p}} = 1$. Thus,

$$\lim_{p\to\infty} \|f\|_{L^p} \leq 1$$

as desired.

Now, we wish to show the reverse. That is, we wish to show that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}$$

To do so, we consider the set $\{|f| > 1-\epsilon\}$, which has positive measure for every $\epsilon > 0$ by the fact that $\|f\|_{L^\infty} = 1$. Thus, we know that

$$\begin{aligned}
\|f\|_{L^p} &= \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} \\
&\geq \left(\int_{\{|f|>1-\epsilon\}} |1-\epsilon|^p d\mu\right)^{\frac{1}{p}} \\
&= ((1-\epsilon)\mu(\{|f| > 1-\epsilon\}))^{\frac{1}{p}}
\end{aligned}$$

Since $\lim_{p\to\infty}((1-\epsilon)\mu(\{|f| > 1-\epsilon\}))^{\frac{1}{p}} = 1$, it follows that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq 1$$

as desired.

Thus, for $\|f\|_{L^\infty} < \infty$, we have that

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

as desired.

So, suppose $\|f\|_{L^\infty} = \infty$. That is, for each $M > 0$, $\mu(\{|f| > M\}) > 0$. Thus,

$$\begin{aligned}
\|f\|_{L^p} &= \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} \\
&\geq \left(\int_{|f|>M} M^p d\mu\right)^{\frac{1}{p}} \\
&= (M^p\mu(\{f > M\}))^{\frac{1}{p}} \\
&= M(\mu(\{|f| > M\}))^{\frac{1}{p}}
\end{aligned}$$

We know already that $\lim_{p\to\infty}(\mu(\{|f| > M\}))^{\frac{1}{p}} = 1$, so it follows that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq M(1) = M$$

Since M was arbitrary, it follows that

$$\lim_{p\to\infty} \|f\|_{L^p} = \infty$$

as desired. $\square$

Problem 3

For $n \in \mathbb{N}$, define the n^{th} truncation of a positive measurable function f to be

$$f_n(x) = \begin{cases} f(x), & f(x) \leq n \\ n, & f(x) \geq n \end{cases}$$

Prove that

$$f \in L^1 \iff \sup_n \|f_n\|_{L^1} < \infty$$

Proof. We show first that f_n converges pointwise to f. This is clear, since for x such that $f(x) < \infty$, there is some N for which $f(x) < N$, and thus $\forall m > N, f_m(x) = f(x)$ by the definition of $f_n(x)$. Furthermore, if $f(x) = \infty$, then it follows that $\forall n, f(x) > n$ and thus $f_n(x) = n$, which tends to infinity as n goes to infinity. Thus,

$$\lim_{n\to\infty} f_n(x) = f(x) \ \forall x$$

Furthermore, the sequence $\{f_n\}$ is monotonically increasing. To see this, we consider three cases:
First, suppose x is such that $f(x) \leq n$. In that case,

$$f_n(x) = f_{n+1}(x) = f(x)$$

and thus $f_n(x) \leq f_{n+1}(x)$.
Second, suppose x is such that $n < f(x) \leq n+1$. In this case, we have that

$$\begin{aligned} f_n(x) &= n \\ &< f(x) \\ &= f_{n+1}(x) \end{aligned}$$

and so $f_n(x) \leq f_{n+1}(x)$.
Finally, suppose x is such that $n+1 < f(x)$. In this case, we have that

$$\begin{aligned} f_n(x) &= n \\ &< n+1 \\ &= f_{n+1}(x) \end{aligned}$$

and so $f_n(x) < f_{n+1}(x)$ as desired.

It follows, then, that

$$\|f_n\|_{L^1} \leq \|f_{n+1}\|_{L^1}$$

for all n, since the L^1 norm (the integral) preserves orders on positive functions. Therefore,

$$\sup_n \|f_n\|_{L^1} = \lim_{n\to\infty} \|f_n\|_{L^1} = \lim_{n\to\infty} \int_\Omega f_n d\mu$$

applying the monotone convergence theorem yields

$$\begin{aligned} \sup_n \|f_n\|_{L^1} &= \lim_{n\to\infty} \int_\Omega f_n d\mu \\ &= \int_\Omega \lim_{n\to\infty} f_n d\mu \\ &= \int_\Omega f d\mu \\ &= \|f\|_{L^1} \end{aligned}$$

as desired. Note that this holds even if $\|f\|_{L^1} = \infty$ by the monotone convergence theorem.

Thus, $f \in L^1$ precisely when $\sup_n \|f_n\|_{L^1} < \infty$, as desired. □

Problem 4

Let f be a measurable function on $(\Omega, \mathscr{A}, \mu)$ which is finite μ-almost everywhere, and let $\mu(\Omega) < \infty$. Define

$$E_n = \{x \in \Omega \mid n-1 \leq |f(x)| < n\}$$

for $n \in \mathbb{N}$.

Prove that $f \in L^1 \iff \sum_{n=1}^{\infty} n\mu(E_n) < \infty$.

Proof. To begin with, we alter the function f on the set $\{x \mid f(x) = \infty\}$ to be identically zero on that set. Thus, the new altered function (which we will continue to denote as f) is finite everywhere. Since f was only altered on a set of measure zero, this will not affect either side of the equality we are trying to prove.

We begin by showing that $\|f\|_{L^1} \leq \sum_{n=1}^{\infty} E_n$. To do so, we construct the function

$$\phi(x) = \sum_{n=1}^{\infty} n\chi_{E_n}(x)$$

Now, it is clear by the definition of E_n that each $x \in \Omega$ belongs to exactly one E_n. Furthermore, for $x \in E_n$, we have that

$$|f(x)| < n = \phi(x)$$

Thus, $\phi(x) \geq |f(x)|$ for all x.

It follows (since integrals preserve inequalities of positive functions) that

$$\begin{aligned} \|f\|_{L^1} &= \int_\Omega |f| d\mu \\ &< \int_\Omega \phi d\mu \qquad = \int_\Omega \sum_{n=1}^{\infty} n\chi_{E_n} d\mu \\ &= \sum_{n=1}^{\infty} n\mu(E_n) \end{aligned}$$

Thus, $\|f\|_{L^1} < \sum_{n=1}^{\infty} n\mu(E_n)$, and so

$$\sum_{n=1}^{\infty} n\mu(E_n) < \infty \implies \|f\|_{L^1} < \infty$$

as desired.

Now, we will show that $\|f\|_{L^1} \geq \sum_{n=1}^{\infty}(n-1)\mu(E_n)$. To do so, we define

$$\psi(x) = \sum_{n=1}^{\infty}(n-1)\chi_{E_n}(x)$$

Clearly, we have that for $x \in E_n$,

$$|f(x)| \geq n-1 = \psi(x)$$

and since this holds for all x, we have that $|f| \geq \psi$. Thus, since integrals preserve inequalities of positive functions,

$$\begin{aligned}
\|f\|_{L^1} &= \int_\Omega |f| d\mu \\
&\geq \int_\Omega \psi d\mu \\
&= \sum_{n=1}^{\infty} (n-1)\mu(E_n) \\
&= \sum_{n=1}^{\infty} n\mu(E_n) - \sum_{n=1}^{\infty} \mu(E_n)
\end{aligned}$$

Now, since the E_n sets partition Ω, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu(E_n) &= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= \mu(\Omega)
\end{aligned}$$

so it follows that

$$\begin{aligned}
\|f\|_{L^1} &\geq \sum_{n=1}^{\infty} n\mu(E_n) - \mu(\Omega) \\
\implies \|f\|_{L^1} + \mu(\Omega) &\geq \sum_{n=1}^{\infty} n\mu(E_n)
\end{aligned}$$

Now, since $\mu(\Omega)$ is finite, it is clear that

$$\|f\|_{L^1} < \infty \implies \sum_{n=1}^{\infty} n\mu(E_n)$$

as desired.

Thus, $f \in L^1 \iff \sum_{n=1}^{\infty} n\mu(E_n) < \infty$ as desired. $\square$