
Problem Set 3

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PROBLEM 1

PART 1

Prove 1.18.ii from the notes.

Proof. (Copied from homework 1)

Note that if the composite function (f, g) is measurable, then this statement reduces to part i, and the proof is complete.

So, let's prove that (f, g) is measurable, given f, g are each individually measurable. (Note that this construction works for general products of measurable spaces, where the product σ -algebra is given by $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$. Generally, this says that the product measurable space has the universal property of product spaces).

Let $E, F \in B(\mathbb{R})$ be measurable sets, and consider the product $E \times F$. The inverse image $(f, g)^{-1}(E \times F) = f^{-1}(E) \cap g^{-1}(F)$ is the intersection of measurable sets (since f and g are both individually measurable), and is measurable.

Now, consider the "good set"

$$\mathcal{E} = \{G \mid G \in B(\mathbb{R}^2) \text{ and } (f, g)^{-1}(G) \in \mathcal{A}\}$$

It is clear from above that we have the inclusion relations

$$B(\mathbb{R}) \times B(\mathbb{R}) \subset \mathcal{E} \subset B(\mathbb{R}^2)$$

Now, \mathcal{E} is clearly a σ -algebra, since both conditions on \mathcal{E} preserve complements and unions. Therefore, taking σ of the inclusion relations yields:

$$\begin{aligned} \sigma(B(\mathbb{R}) \times B(\mathbb{R})) &\subset \mathcal{E} \subset B(\mathbb{R}^2) \\ \implies B(\mathbb{R}^2) &\subset \mathcal{E} \subset B(\mathbb{R}^2) \end{aligned}$$

Thus, \mathcal{E} is actually the whole Borel set $B(\mathbb{R}^2)$, and thus (f, g) is a measurable function, as desired. \square

PART 2

Complete 2.6 from the notes.

Describe all measurable functions $f : \mathbb{N} \rightarrow [0, \infty]$ that are finite μ_c -almost everywhere in the counting measure, and find

$$\int_{\mathbb{N}} f d\mu_c$$

Proof. Note that the only subset of \mathbb{N} with zero measure is \emptyset , so if f is finite μ_c -almost everywhere, then f is finite on all subsets of \mathbb{N} i.e. f is bounded.

Then, the integral becomes

$$\int_{\mathbb{N}} f d\mu_c = \sum_{i=1}^{\infty} f_i$$

This is clear to see by approximating f with simple functions that converge monotonically to f . Let

$$\phi_i = \sum_{j=1}^i f_j \chi_{\{j\}}$$

It is clear that $\phi_{i+1} \geq \phi_i$, since $\phi_i(n) = \phi_{i+1}(n)$ for all $n < i$, and for $n > i$, we have

$$\begin{aligned} \phi_i(n) &= 0 \\ &\leq \phi_{i+1}(n) \end{aligned}$$

since $\phi_i > 0$ for all i (definition of simple function).

It's also clear that $\phi_i \rightarrow f$ pointwise, since for fixed $x \in \mathbb{N}$, $\phi_{x+j}(x) = f(x)$ for all $j > 0$.

Thus, the monotone convergence theorem tells us that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{\mathbb{N}} \phi_i(x) d\mu_c(x) &= \int_{\mathbb{N}} \lim_{i \rightarrow \infty} \phi_i(x) d\mu_c(x) \\ \lim_{i \rightarrow \infty} \sum_{j=1}^i f_j \mu_c(\{j\}) &= \int_{\mathbb{N}} f(x) d\mu_c(x) \\ \sum_{j=1}^{\infty} f_j &= \int_{\mathbb{N}} f(x) d\mu_c(x) \end{aligned}$$

as desired. □

PART 3

Do the same thing for the δ -measure μ_{δ_p} on Ω .

Proof. Note that any subset that does not contain p has measure zero. Thus, f is finite μ_{δ_c} -almost everywhere if and only if $f(p)$ is finite. This is clear, since $\Omega \setminus \{p\}$ has measure zero, so f can do whatever it wants on $\Omega \setminus \{p\}$. However, the measure of $\{p\}$ is not zero, so f must be finite on $\{p\}$.

To compute the integral, we first observe the general fact that changing a function on a set of measure zero does not change the integral.

Lemma. For measurable functions f and g from a measurable space Ω such that $f = g$ μ -almost everywhere, $\int f = \int g$.

Proof.

$$\begin{aligned}
\int_{\Omega} f d\delta - \int_{\Omega} g d\delta &= \int_{\Omega} (f - g) d\delta \\
&= \int_{\{x|f(x)=g(x)\}} (f - g) d\delta + \int_{\{x|f(x) \neq g(x)\}} (f - g) d\delta \\
&= \int_{\{x|f(x)=g(x)\}} (0) d\delta + 0 \\
&= 0
\end{aligned}$$

□

Thus, we have that

$$\int_{\Omega} f d\delta = \int_{\Omega} f(p) \chi_{\{p\}} d\delta$$

and the second integral is the integral of a simple function, and is just $f(p)\delta(\{p\}) = f(p)$. Thus,

$$\int_{\Omega} f d\delta = f(p)$$

□

PART 4

Show that the Dirichlet function is measurable, and calculate its integral.

Proof. The Dirichlet function is defined as the characteristic function on \mathbb{Q} . To see this is measurable, observe that \mathbb{Q} is measurable, since it is a countable disjoint union of points, which are all measurable.

The integral can easily be seen to be zero, since

$$\begin{aligned}
\mu(\mathbb{Q}) &= \mu\left(\bigcup_{q \in \mathbb{Q}} \{q\}\right) \\
&= \sum_{q \in \mathbb{Q}} \mu(\{q\}) \\
&= 0
\end{aligned}$$

So, by the lemma of the previous problem,

$$\int_{\mathbb{R}} \chi_{\mathbb{Q}} d\mu = \int_{\mathbb{R}} 0 d\mu = 0$$

□

PART 5

Construct a sequence of functions $\{f_n\}$ satisfying the assumptions of Fatou's lemma such that

$$\lim \int f_n \neq \int \lim f_n$$

Proof. Let $f_n = \chi_{[n, n+1]}$. Then, $\int_{\mathbb{R}} f_n d\mu = 1$ for all n , but $f_n(x) \rightarrow 0$ for all x , so f_n converges pointwise to zero, and thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n d\mu &= 1 \\ &\neq 0 \\ &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} f_n d\mu \end{aligned}$$

□