Geometry

Homework

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Problem 1

Prove that $V^{**} \cong V$ for a finite dimensional vector space V.

Proof. The isomorphism is given by

$$\begin{aligned} \Phi &: V \to V^{**} \\ \Phi(v) &= ev_v = (\phi \mapsto \phi(v)) \end{aligned}$$

First, we observe that this map is linear. Indeed, for $v_1, v_2 \in V$ and α, β a scalar, we have

$$\begin{aligned} \Phi(\alpha v_1 + \beta v_2)(\phi) &= \phi(\alpha v_1 + \beta v_2) \\ &= \alpha\phi(v_1) + \beta\phi(v_2) \\ &= \alpha\Phi(v_1)(\phi) + \beta\Phi(v_2)(\phi) \end{aligned}$$

as desired.

We need to show this is surjective and injective. Injectivity of Φ is easily shown by examining the kernel of Φ. Suppose v is such that $ev_v(\phi) = \phi(v) = 0$ for all $\phi \in V^*$. Then, since V^* separates points of V, it follows that $v = 0$. Thus, the kernel is trivial, as desired.

Now, we need to show this map is surjective. To do so, we appeal to v being finite-dimensional, and let $\{e_i\}$ be a basis for V with dual basis $\{\omega^i\}$. Then, let $x \in V^{**}$. Define a corresponding vector $\tilde{x} = \sum_i x(\omega^i)e_i$. Then

$$\begin{aligned} \Phi(\tilde{x})(\phi) &= \phi(\tilde{x}) \\ &= \sum_i x(\omega^i)\phi(e_i) \\ &= \sum_i x(\omega^i\phi(e_i)) \\ &= x(\phi) \end{aligned}$$

as desired. Thus, Φ is a linear isomorphism. $\square$

Problem 2

Prove that for V a vector space with basis v_i and dual basis v^i, the set

$$\{v^i \otimes v^j \mid 1 \leq i, j \leq n\}$$

forms a basis for $V^* \otimes V^*$.

Proof. For this, we will show that the vector space

$$W = \operatorname{span}(\{v^i \otimes v^j \mid 1 \leq i, j \leq n\})$$

satisfies the universal property of tensor products. That is, we wish to show that for each bilinear map $h : V^* \times V^* \to U$ for some vector space U, there is a unique linear map $\tilde{h} : W \to U$ such that the diagram

$$\begin{tikzcd} V^* \times V^* \arrow[r, "h"] \arrow[d, "\otimes"] & U \\ W \arrow[ur, dotted, "\tilde{h}"] & \end{tikzcd}$$

commutes. We will guess that $\otimes : V^* \times V^* \to W$ is given as

$$\otimes(a_i v^i, b_j v^j) = a_i b_j v^i \otimes v^j$$

So, let $h : V^* \times V^* \to U$ be a bilinear map. Define $\tilde{h} : W \to U$ as

$$\tilde{h}(\sum_{i,j} a_{ij} v^i \otimes v^j) = a_{ij} h(v^i, v^j)$$

Then it is clear that the diagram

$$\begin{tikzcd} V^* \times V^* \arrow[r, "h"] \arrow[d, "\otimes"] & U \\ W \arrow[ur, dotted, "\tilde{h}"] & \end{tikzcd}$$

commutes, since

$$\begin{aligned} h(a_i v^i, b_j v^j) &= a_i b_j h(v^i, v^j) \\ \tilde{h} \circ \otimes(a_i v^i, b_j v^j) &= \tilde{h}(a_i b_j v^i \otimes v^j) \\ &= a_i b_j h(v^i, v^j) \end{aligned}$$

It should be clear from construction that $\tilde{h}$ is unique.

Thus, since every bilinear map from $V^* \times v^*$ factors through W, W satisfies the universal property of tensor products, and is isomorphic to $V^* \otimes V^*$. Thus, the set $\{v^i \otimes v^j\}$ forms a basis for $V^* \otimes V^*$ as desired. $\square$

Problem 3

Show that $dvol = \wedge_i \omega^i = \sqrt{|g|} dx^n$.

Proof. Recall that $dx^n = \wedge_{i=1}^n dx^i$, and our manifold is n-dimensional.

Recall that for an n-fold wedge product $\wedge_{i=1}^n v^i$, we have (for $v^i = a^i_j \omega^j$ in an orthonormal frame)

$$\wedge_{i=1}^n v^i = |\det(a^1_j, \ldots, a^n_j)| \wedge_{i=1}^n \omega^i = \sqrt{\det(A^t A)} \wedge_{i=1}^n \omega^i$$

for A the matrix with columns a^i. We can apply this to $v^i = dx^i$ to get the desired result. $\square$

Problem 4

Show that the definition of the integral of a top degree form on a single chart is independent of choice of coordinates.

Proof. Recall the definition of the integral of a top degree differential form on a compact set K in a single coordinate frame is

$$\int_K \omega = \int_{\phi(K)} f \circ \phi^{-1} dx^n$$

for $\omega = f dx^1 \wedge \cdots \wedge dx^n$.

We also have the change-of-coordinates formula for a diffeomorphism $F : \Omega_1 \to \Omega_2$ as

$$\int_{\Omega_2} f dy^n = \int_{\Omega_1} f \circ F |J_F| dx^n$$

where J_F is the jacobian of F.

So, let $\phi : M \to \mathbb{R}^n$ be the original coordinate system, and let $\psi : M \to \mathbb{R}^n$ be another coordinate system covering K. Then, we have the diffeomorphism $F = \psi \circ \phi^{-1}$ and we can apply this to get

$$\int_{\psi(K)} g \circ psi^{-1} dy^n = \int_{\phi(K)} g \circ \psi^{-1} \circ F |J_F| dx^n$$

which is just

$$\begin{aligned}
\int_{\phi(K)} g \circ \psi^{-1} \circ F |J_F| dx^n &= \int_{\phi(K)} g \circ \psi^{-1} \circ \psi \circ \phi^{-1} |J_F| dx^n \\
&= \int_{\phi(K)} g \circ \phi^{-1} |J_F| dx^n \\
&= \int_K g |J_F| dx^n \\
&= \int_K \omega
\end{aligned}$$

where we used the fact that $\omega = g dy^1 \wedge \cdots \wedge dy^n = g|J_F| dx^1 \wedge \cdots \wedge dx^n$ since $|J_F| = \det(J_F)$ on a two positively oriented charts.

Thus, the two integrals agree. $\square$

Problem 5

Prove that a manifold is orientable if and only if it admits a nowhere vanishing top degree form.

Proof. Suppose M is an orientable manifold. That is, there exists an atlas $\{U_\alpha, \phi_\alpha\}$ of M for which the Jacobian of each transition map has positive determinant. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to the atlas U_α. Then, define

$$\omega = \sum_\alpha \psi_\alpha dx_\alpha^i \wedge \cdots \wedge dx_\alpha^n$$

where x_α^i are the coordinate functions on U_α. We claim that ω is a nowhere-vanishing form. Clearly, if p is a point in M contained in only one chart, then $\omega_p = dx_p^1 \wedge \cdots \wedge dx_p^n$ and does not vanish. If p is such that it is contained in more than one chart, then ω at p is the sum of positive terms (since each coordinate system is positive, we have $dx^n = \det(J)dy^n$ and $\det(J)$ is always positive) and does not vanish.

Suppose instead that M admits a nowhere-vanishing top degree form ω. Then, let $\{U_\alpha, \phi_\alpha\}$ be an atlas of M. For x^i_α the coordinate functions for ϕ_α, define a new coordinate system to be such that if ω is expressed as

$$\omega = f dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$$

then f is positive. This is done by setting x^1_α to its negative if f is negative on that chart. Note that since ω never vanishes, we know that f will be either entirely positive or entirely negative on a chart. Thus, such a choice can be made consistently.

Then, the modified atlas is a positive coordinate chart for M, which is easily verified, since

$$\omega = f dx^n = f \det(J) dy^n$$

and ω always has positive coefficient, so $\det(J)$ is positive. □

Problem 6

Show that the topology of M coincides with the metric topology

$$d_g(x, y) = \inf_{\gamma \in C^\infty(I,M)} \{L(\gamma) \mid \gamma(0) = x, \gamma(1) = y\}$$

where $L(\gamma)$ is the total length of γ defined by

$$L(\gamma) = \int_I g(\gamma', \gamma')dt$$

Proof. First, we observe the following: for g a Riemannian metric, and γ a curve contained entirely in a single coordinate chart ϕ, there exist constants c, C such that

$$cL_{\mathbb{R}^n}(\gamma) \leq L_g(\gamma) \leq CL_{\mathbb{R}^n}(\gamma)$$

where $L_{\mathbb{R}^n}(\gamma)$ is the length of $\phi \circ \gamma$ using the euclidean metric on $\mathbb{R}^n$. This follows from the fact that the metric induced by g along ϕ^{-1} defines a norm on $\mathbb{R}^n$, and all norms are equivalent. That is,

$$k\|v\|_{\mathbb{R}^n} \leq \|v\|_{\phi^{-1*}g} \leq K\|v\|_{\mathbb{R}^n}$$

for constants k, K. Thus, the lengths (defined in terms of integrals of the metric) follow the same inequality.

It should also be clear that the metrics induced by $\mathbb{R}^n$ and g are equivalent as well. To see this, note that for any $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} cd_{\mathbb{R}^n}(x, y) &= \inf_{\gamma(0)=x,\gamma(1)=y} cL_{\mathbb{R}^n}(\gamma) \\ &\leq \inf_{\gamma(0)=x,\gamma(1)=y} L_g(\gamma) \\ &= d_g(x, y) \end{aligned}$$

(the first equality is proved in the next problem) and similarly for $d_g(x, y) \leq Cd_{\mathbb{R}^n}(x, y)$. This shows that the two topologies induced by the two metrics are equal.

Equivalence of the two topologies follows immediately. We can show that for U open in the manifold topology, and $x \in U$, there is a neighborhood V of x in the metric topology contained in U. Simply take a coordinate ball $V_\varepsilon(x)$ of small enough radius to be contained in a single coordinate chart ϕ. That is, the domain of ϕ contains $V_\varepsilon(x)$. Then, we know from the above observation that $\phi(V_\varepsilon(x))$ is open in $\mathbb{R}^n$ in the standard topology, and thus is open with respect to the pullback of d_g along ϕ^{-1}. Thus, $V_\varepsilon(x)$ is also open in the metric topology induced by d_g on M. The same argument with the two topologies switched completes the argument that both topologies are equal. □

Problem 7

Show that $\|a-b\|_{\mathbb{R}^n}$ is $d_{\mathbb{R}^n}(a,b) = \inf_\gamma L_{\mathbb{R}^n}(\gamma)$.

Proof. This result follows from standard variational calculus on the functional $L_{\mathbb{R}^n}(\gamma)$.

Let's minimize the functional $L(\gamma)$ by varying the path γ. We do this by setting the variation to zero. That is, $L(\gamma)$ is maximized for γ that makes $\delta L = 0$. We calculate

$$\begin{aligned}\delta L &= \int \delta(\sqrt{(}g(\gamma',\gamma')))dt \\ &= \int \frac{1}{2\sqrt{g(\gamma',\gamma')}}\delta g(\gamma',\gamma')dt\end{aligned}$$

Since arc length is independent of parameterization, we can take the unit speed parameterization of γ, so that $g(\gamma',\gamma') = 0$. Then, we have

$$\begin{aligned}\delta L &= \int \delta g(\gamma',\gamma')dt \\ &= \int \delta(g_{ab}\partial_t\gamma^a\partial_t\gamma^b)dt \\ &= \int g_{ab}\delta(\partial_t\gamma^a)\partial_t\gamma^b + g_{ab}\delta(\partial_t\gamma^b)\partial_t\gamma^a dt \quad = 2\int g_{ab}\partial_t(\delta\gamma^a)\partial_t\gamma^b dt\end{aligned}$$

integrating by parts (and tossing boundary terms since the endpoints of γ do not vary) and noting $g_{ab} = \delta_{ab}$ in $\mathbb{R}^n$ yields

$$\delta L = -2\int \partial_t^2\gamma^a\delta\gamma^b dt$$

which holds only if $\partial_t^2\gamma^a = 0$ for all a. Thus, the minimal length path from a point p to a point q is the straight line from p to q.

So, for γ the straight line from p to q, $L(\gamma) = \|\gamma\| = \|p-q\|$ as desired. □

Problem 8

Define the Levi-Civita connection as the unique connection such that

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

and

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Show that this is indeed a connection on M.

Proof. For this proof, we will denote the Levi-Civita connection as D.

We need to show that $D_X Y$ is function linear in X, scalar linear in Y, and satisfies the Leibniz rule

$$D_X(fY) = (Xf)Y + fD_X Y$$

Recall from class that by utilizing the two properties above, we see that

$$2g(D_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y)$$

which uniquely determines the connection. Now, we just need to show that this definition satisfies the definition of a connection. To that end, let $f \in C^\infty(M)$. We calculate

$$\begin{aligned}
2g(D_{fX}Y, Z) &= fXg(Y, Z) + Yg(Z, fX) - Zg(fX, Y) \\
&\quad + g([fX, Y], Z) - g([Y, Z], fX) - g([fX, Z], Y) \\
&= fXg(Y, Z) + Yg(Z, fX) - Zg(fX, Y) \\
&\quad - g([Y, fX], Z) - g([Y, Z], fX) + g([Z, fX], Y) \\
&= fXg(Y, Z) + (Yf)g(Z, X) + fYg(Z, X) - (Zf)g(X, Y) - fZg(X, Y) \\
&\quad - g((Yf)X, Z) - g(f[Y, X], Z) - g([Y, Z], fX) + g((Zf)X, Y) + g(f[Z, X], Y) \\
&= fXg(Y, Z) + (Yf)g(Z, X) + fYg(Z, X) - (Zf)g(X, Y) - fZg(X, Y) \\
&\quad - g((Yf)X, Z) + g(f[X, Y], Z) - g([Y, Z], fX) + g((Zf)X, Y) - g(f[X, Z], Y) \\
&= (fX)g(Y, Z) + (Yf)g(X, Z) - (Yf)g(X, Z) - (Zf)g(X, Y) + (Zf)g(X, Y) \\
&\quad + fYg(X, Z) - fZg(X, Y) + g(f[X, Y], Z) - g([Y, Z], fX) - g(f[X, Z], Y) \\
&= f\left\{Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y)\right\} \\
&= fg(D_XY, Z) = g(fD_XY, Z)
\end{aligned}$$

and thus D is $C^\infty(M)$-linear in X.

Linearity in Y follows immediately from the fact that the Lie bracket and g are both scalar linear.

Finally, we show that this satisfies the Leibniz rule. This is done by direct calculation.

$$\begin{aligned}
2g(D_X(fY), Z) &= Xg(fY, Z) + fYg(Z, X) - Zg(X, fY) \\
&\quad + g([X, fY], Z) - g([fY, Z], X) - g([X, Z], fY) \\
&= (Xf)g(Y, Z) + fXg(Y, Z) + fYg(X, Z) - (Zf)g(X, Y) - fZg(X, Y) \\
&\quad + g(f[X, Y] + (Xf)Y, Z) - g(f[Y, Z] - (Zf)g(Y, X) - g([X, Z], fY) \\
&= fXg(Y, Z) + fYg(X, Z) - fZg(X, Y) \\
&\quad + fg([X, Y], Z) - fg([Y, Z], X) - fg([X, Z], Y) \\
&\quad + (Xf)g(Y, Z) + (Xf)g(Y, Z) - (Zf)g(X, Y) + (Zf)g(X, Y) \\
&= 2fg(D_XY, Z) + 2(Xf)g(Y, Z) \\
&= 2g((Xf)Y + D_XY, Z)
\end{aligned}$$

as desired.

Thus, D is a connection. $\square$

Problem 9

Construct a one dimensional smooth bump function on $\mathbb{R}$.

Proof. Let

$$f(x) = \begin{cases} \exp(\frac{-1}{t}), & t > 0 \\ 0 & \text{else} \end{cases}$$

and let

$$g(x) = \frac{f(x)}{f(x) + f(1 - x)}$$

which is 1 for $x \geq 1$ and 0 for $x \leq 0$. Finally, set

$$h(x) = g(x + 2)g(2 - x)$$

which is zero outside of $[-2, 2]$ and 1 inside $[-1, 1]$ as desired. $\square$

Problem 10

Show that the Christoffel symbols for the Levi-Civita connection are

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(g_{il,j} + g_{jl,i} - g_{ij,l} \right)$$

Proof. Recall that a connection is completely described by the Christoffel symbols as

$$\nabla_i v^k = \partial_i v^k + \Gamma_{ij}^k v^j$$

and the conditions for a Levi-Civita connection are

$$g(\nabla_X Y, Z) = \frac{1}{2} \left(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y) \right)$$

Setting $X = \partial_i$, $Y = \partial_j$, and $Z = \partial_k$ and noting that $[\partial_i, \partial_j] = 0$ for all i and j, we see that

$$\begin{aligned} g(\nabla_i \partial_j, \partial_k) &= \frac{1}{2} \left(\partial_i g(\partial_j, \partial_k) + \partial_j g(\partial_k, \partial_i) - \partial_k g(\partial_i, \partial_j) \right) \\ &= \frac{1}{2} \left(g_{jk,i} + g_{ik,j} - g_{ij,k} \right) \\ g_{lk} (\nabla_i \partial_j)^l \partial_k^k &= \frac{1}{2} \left(g_{jk,i} + g_{ik,j} - g_{ij,k} \right) \\ (\nabla_i \partial_j)^l = \Gamma_{ij}^l &= g^{lk} \frac{1}{2} \left(g_{jk,i} + g_{ik,j} - g_{ij,k} \right) \end{aligned}$$

as desired. □

Problem 11

Prove that

$$\partial_t g(X,Y) = g(\nabla_t X, Y) + g(X, \nabla_t Y)$$

Proof. Fix an orthonormal frame P_i at $\gamma(0$ for γ the curve we are differentiating against. This frame can be parallel transported along γ to form a local orthonormal frame along γ. Then we have

$$X = x^i P_i$$

and

$$Y = y_i^P$$

from which it follows that

$$\nabla_t X = \partial_t x^i P_i$$

and similar for Y. Then

$$\begin{aligned} g(\nabla_t X, Y) + g(X, \nabla_t Y) &= \sum_i \left(\partial_t x^i y^i + \partial_t y^i x^i \right) \\ &= \partial_t (\sum_i x^i y^i) \\ &= \partial_t g(X,Y) \end{aligned}$$

as desired. □

Problem 12

Prove that for a smooth map $F : I^2 \to M$ with first coordinate t and second coordinate s,

$$\nabla_t \partial_s F = \nabla_s \partial_t F$$

Proof. This proof is given explicitly in Do Carmo Chapter 3 lemma 3.4, and will not be replicated here. $\square$

Problem 13

Prove that

$$g((d\exp_p)_{\tilde{\gamma}(t)}(r(t)+n(t)), (d\exp_p)_{\tilde{\gamma}(t)}(r+n)) = \|r(t)\|^2$$

if and only if $\tilde{\gamma}(t)$ is radial.

Proof. Recall that the Gauss lemma says that $\exp_p$ is an isometry on its normal ball. Thus, recalling from the notes that

$$g((d\exp_p)_{\tilde{\gamma}(t)}(r(t)+n(t)), (d\exp_p)_{\tilde{\gamma}(t)}(r+n)) = \|r(t)\|^2 + \|d\exp_p(n(t))\|^2$$

we see that equality holds if and only if $\|d\exp_p(n(t))\| = 0$. However, since $\exp_p$ is an isometry,

$$\|d\exp_p(n(t))\| = \|n(t)\|$$

which is zero for all time if and only if $n(t)$ is zero for all time. Thus, equality holds when $\tilde{\gamma}$ is a radial geodesic. $\square$

Problem 14

Find a counterexample to $\exp_p(B_r(0)) = B_r(p)$ for arbitrary r.

Proof. $\square$

Problem 15

Show that R_m is function linear in the first two components.

Proof. It should be clear my the antisymmetry of R_m in the first two components that we only need to show the first component is function linear, and the second follows immediately.

So, we compute

$$\begin{aligned}
R(fX,Y)Z &= -\nabla_{fX}\nabla_Y Z + \nabla_Y \nabla_{fX} Z + \nabla_{[fX,Y]} Z \\
&= -f\nabla_X \nabla_Y Z + \nabla_Y(f\nabla_X Z) + \nabla_{f[X,Y]-(Yf)X} Z \\
&= -f\nabla_X\nabla_Y Z + \nabla_Y(f)\nabla_X Z + f\nabla_Y\nabla_X Z + f\nabla_{[X,Y]}Z - \nabla_{(Yf)X} Z \\
&= fR(X,Y)Z + (Yf)\nabla_X Z - (Yf)\nabla_X Z \\
&= fR(X,Y)Z
\end{aligned}$$

as desired. $\square$

Problem 16

Show that in Riemannian normal coordinates at p, the christoffel symbols vanish at p and the first derivatives of the metric vanish at p.

Proof. We first show the Christoffel symbols at p vanish. So, let γ be a geodesic with $\gamma(0) = p$. This is given in normal coordinates as

$$\gamma(t) = \exp_p(t(v_1, v_2, \cdots, v_n)) = t(x_1, x_2, \cdots, x_n)$$

Now, we know the geodesics are the solution to the geodesic equation

$$\partial_t^2 \gamma^a(t) + \Gamma^a_{bc} \partial_t \gamma^b(t) \partial_t \gamma^c(t) = 0$$

In particular, at zero we know that $\partial_t^2 \gamma^a = 0$, and so

$$\Gamma^a_{bc} \partial_t \gamma^b \partial_t \gamma^c = 0$$

But $\partial_t \gamma^c = x^c$ and so

$$\Gamma^a_{bc} x^b x^c = 0$$

and this holds for all x sufficiently small. Thus, $\Gamma^a_{bc} = 0$ for all a, as desired.

To show the first derivatives of the metric vanish, we use the metric compatibility condition of the connection. Namely,

$$\nabla_i g_{jk} = 0$$

for all i, j, k. Thus, since the Christoffel symbols vanish, we know that

$$\partial_i g_{jk} = 0$$

as desired. $\square$

Problem 17

Show that the induced inner product on two forms is independent of choice of orthonormal basis.

Proof. Suppose $\{e^j\}$ is an orthonormal basis, and $\{v^j\}$ is some other orthonormal basis. We just need to show that the matrix taking $e^i \wedge e^j$ to $v^i \wedge v^j$ is orthonormal. So, let a be the matrix such that

$$v^i = a^i_j e^j$$

Then, it is clear through rote calculation that $\langle v^i \wedge v^j, v^k \wedge v^l \rangle$ for $i < j, k < l$ is zero if $i \neq k, j \neq l$ and 1 otherwise. Thus, the inner product does not depend on choice of orthonormal basis, as desired.

$\square$

Problem 18

Prove that in the product manifold $S^1 \times S^1$ the curvature tensor R_m is identically zero.

Proof. Let θ parameterize the first S^1, and ϕ the second. The product metric is then given by

$$g = d\theta \otimes d\theta + d\phi \otimes d\phi$$

Thus $d\theta, d\phi$ form an orthonormal coframe (and $\partial_\theta, \partial_\phi$ form an orthonormal frame) that is parallel in the neighborhood of some point p. Thus, since R_m at p is defined in terms of the covariant derivative around p, it follows that $R_m = 0$ at p. Since this can be done at any p, the torus is indeed flat, as desired. $\square$

Problem 19

Prove that for $f : M \to \tilde{M}$ an isometric immersion, if all geodesics of M are also geodesics of $\tilde{M}$, then f is totally geodesic. That is, the second fundamental form vanishes.

Proof. Suppose f maps geodesics to geodesics. Then,

$$\nabla_t \gamma' = 0 = \tilde{\nabla}_t \gamma'$$

However, at a point, there are geodesics in every direction, and so at a point, the covariant derivatives agree. Thus, at every point, $B(X, Y) = 0$ (since $\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)$) and so f is totally geodesic. $\square$

Do Carmo Problem 1.2

Introduce a metric on $\mathbb{R}^n/\mathbb{Z}^n$ such that projection is a local isometry. Show that this torus is isometric to the flat torus.

Proof. Recall that the projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is a smooth covering map. Furthermore, note that the Euclidean metric g is invariant under the action of $\mathbb{Z}^n$. That is, for an action $f((x_1, \ldots, x_n)) = (x_1 + m_1, \ldots, x_n + m_n)$ we know that

$$f^*g = f^*(\sum_i dx^i \otimes dx^i) = \sum_i d(x^i \circ f)^2 = d(x^i + m_i)^2 = x^i \otimes x^i$$

Thus, we can define the metric as

$$\tilde{g}_q(u, v) = g(d(\pi^{-1})_p(u), d(\pi^{-1})_p(v))$$

which is clearly well-defined, since a fiber of $\pi^{-1}(p)$ consists of the orbit of p under the action of $\mathbb{Z}^n$, which g is invariant under.

Consider the diffeomorphism (with T^n parameterized as $\theta_i \in [0, 1)$)

$$\Phi : T^n \to \mathbb{R}^n/\mathbb{Z}^n$$

given by $\Phi(\theta_1, \ldots, \theta_n) = [(\theta_1, \ldots, \theta_n)]$.

This is clearly a diffeomorphism, since it has an inverse given by taking a point $[x]$ in $\mathbb{R}^n/Z^n$ to the element of its orbit in the unit square. $\square$

1 Do Carmo Problem 2.7

Let c be an arbitrary parallel of latitude on S^2, with V_0 a tangent vector to S^2 at some point on c. Describe geometrically the parallel transport of V_0 along c.

Proof. We will show that parallel transport along c in S^2 is the same as parallel transport along c thought of as a curve in the cone C that lies tangent to S^2 at c.

In particular, note that the tangent spaces of S^2 and C coincide on c. This means that projection of a vector on c in $\mathbb{R}^3$ is the same whether it goes to TS^2 or TC. Furthermore, since the covariant derivative of a vector on c is equal to the ordinary partial derivative in $\mathbb{R}^3$ followed by projection into the tangent space, it follows that the covariant derivative of V_0 along c is the same whether taken in S^2 or C. Thus, since parallel transport is defined in terms of the covariant derivative, the parallel transport of V_0 on S^2 coincides with the parallel transport of V_0 on C.

Now, we note that C is actually flat: by making a suitable radial cut, one may flatten C so that it forms a disk with a slice missing, with the boundary of the disk coinciding with c. Here, parallel transport of V_0 along c is just ordinary translation in $C \subset \mathbb{R}^2$.

Thus, we have a complete description of the parallel transport of V_0 along c. We form the cone C tangential to c, and make a cut so that C can be isometrically embedded as a subset of $\mathbb{R}^2$. Then, identifying V_0 with its corresponding tangent vector on $c \subset \partial C$, we apply ordinary translation (parallel transport in $\mathbb{R}^2$) to V_0 along c. The result is the parallel vector field $V(t)$ along c in $\mathbb{R}^2$, which is identified with the parallel vector field $V(t) \subset TC$. Finally, noting that TC and TS^2 coincide on c, we see that $V(t) \subset TS^2$ is the parallel vector field of V_0 on c. $\square$