Problem Set 8

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Problem 1

Show that $\frac{\sin(x)}{x}$ is not in $L^1((0,\infty),\lambda^1)$.

Proof. We wish to evaluate

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x)$$

and show that it diverges. To do so, we split the integral into half-cycles

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x) = \sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{x} d\lambda^1(x)$$

Now, we know that on each half-cycle,

$$\frac{|\sin(x)|}{x} \geq \frac{|\sin(x)|}{(n+1)\pi}$$

so we have a lower bound for the integral:

$$\sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{x} d\lambda^1(x) \geq \sum_{n=0}^{\infty} \int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{(n+1)\pi} d\lambda^1(x)$$

Now, since each half-cycle is either entirely positive or entirely negative, we know that

$$\int_{(n\pi,(n+1)\pi)} \frac{|\sin(x)|}{(n+1)\pi} d\lambda^1(x) = \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right|$$

And finally, we can evaluate the integral directly:

$$\begin{aligned}
\sum_{n=0}^{\infty} \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right| &= \sum_{n=0}^{\infty} \left| \frac{1}{(n+1)\pi} [\cos(x)]_{n\pi}^{(n+1)\pi} \right| \\
&= \sum_{n=0}^{\infty} \frac{2}{(n+1)\pi} \\
&= \infty
\end{aligned}$$

Thus, since

$$\int_{(0,\infty)} \frac{|\sin(x)|}{x} d\lambda^1(x) \geq \sum_{n=0}^{\infty} \left| \int_{(n\pi,(n+1)\pi)} \frac{\sin(x)}{(n+1)\pi} d\lambda^1(x) \right| = \infty$$

the integral diverges, and $\frac{\sin(x)}{x}$ is not in $L^1((0,\infty),\lambda^1)$. $\square$

Problem 2

Prove that L^p for $1 \leq p \leq \infty$ is complete. Note that case $p = 1$ has already been covered in class.

Proof. To begin with, let $p < \infty$. $\square$

Problem 3

Part 1: Notes 3.11

Prove that ℓ^p_n, ℓ^p, $1 \leq p \leq \infty$ are Banach. Prove that $\ell^{p_1} \subset \ell^{p_2}$ for $1 \leq p_1 \leq p_2 \leq \infty$, and that

$$\|x\|_{p_2} \leq \|x\|_{p_1}$$

Proof. We first note that ℓ^p_n is isomorphic (as vector spaces) with $\mathbb{R}^n$, by the canonical identification $(x^i) \mapsto (x^i)$ (where x^i is the i^{th} point in the sequence, and the i^{th} component of the vector). Furthermore, since all norms on a finite dimensional vector space are equivalent, the ℓ^p norm applied to $\ell^p_n \cong \mathbb{R}^n$ is equivalent to the standard 2-norm on $\mathbb{R}^n$. Now, since $\mathbb{R}^n$ is complete with this norm, it follows that ℓ^p_n is complete as well.

For the case of ℓ^p, we note that $\ell^p = L^p(\mathbb{N}, 2^{\mathbb{N}}, \mu_c)$. By Problem 2, we know that L^p spaces are complete, so it follows that ℓ^p is complete as well.

Now, we will prove the norm inequality. Without loss of generality, we will let $(x_n) \in \ell^{p_1}$ such that $\|(x_n)\|_{p_1} = 1$ (i.e. scale the sequence by its norm, which will not change the inequality).

Now, we wish to show that

$$\left(\sum_{n=1}^{\infty} |x_n|^{p_2}\right)^{\frac{1}{p_2}} \leq \left(\sum_{n=1}^{\infty} |x_n|^{p_1}\right)^{\frac{1}{p_1}} (= 1)$$

We observe first that since $\|(x_n\|_{p_1} = 1$ and $p_1 \geq 1$, it must be that each x_n is less than 1. Thus, we have that for each n,

$$|x_n|^{p_2} \leq |x_n|^{p_1}$$

since $p_2 \geq p_1$, and each term $x_n < 1$.

Thus, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n|^{p_2} &\leq \sum_{n=1}^{\infty} |x_n|^{p_1} \\ &= 1 \\ \implies \left(\sum_{n=1}^{\infty} |x_n|^{p_2}\right)^{\frac{1}{p_2}} &\leq 1 \end{aligned}$$

as desired.

Thus, if $(x_n) \in \ell^{p_1}$, we have that $\|(x_n)\|_{p_2} \leq \|(x_n)\|_{p_1} < \infty$, and so (x_n) is in ℓ^{p_2} as well. Thus, $\ell^{p_1} \subset \ell^{p_2}$ as desired. $\square$

Part 2: Notes 3.13

Prove that $L^{p_1}(\Omega, \mu) \subset L^{p_2}(\Omega, \mu)$ when $\mu(\Omega) < \infty$. To do so, establish the inequality for the average integral

$$\|f\|_{\bar{p_1}} \leq \|f\|_{\bar{p_2}}$$

where the barred norm is the average norm defined in the notes.

Furthermore, prove that this does not hold in the case $\mu(\Omega) = \infty$.

Proof. □