# Final Exam

Daniel Halmrast

June 11, 2018

## Problem 1

Let $M$ be a complete Riemannian manifold with sectional curvature $K, K \geq k > 0$. Let $\gamma$ be a nontrivial closed geodesic in $M$. Show that for any $p \in M$,

$$d(p,\gamma) \leq \frac{\pi}{2\sqrt{k}}$$

*Proof.* For simplicity, we normalize our space so that $k = 1$. Now, suppose for a contradiction that there is some $p \in M$ with $d(p,\gamma) > \frac{\pi}{2}$. Denote by $\sigma$ a minimizing geodesic from $p$ to a closest point $q$ on $\gamma$, which we know satisfies $\ell(\sigma) > \frac{\pi}{2}$. Finally, denote by $q'$ the point opposite $q$ on $\gamma$ (that is, parameterizing $\gamma$ with $\gamma(0) = q$, let $q' = \gamma(\frac{\ell(\gamma)}{2})$).

We now invoke the Toponogov comparison theorem on the triangle formed by the geodesics $\sigma$ from $p$ to $q$ and $\gamma$ from $q$ to $q'$. We will label the edges of the triangle as:

| Side | Label |
|---|---|
| $\overline{pq'}$ | $a$ |
| $\overline{pq}(=\sigma)$ | $b$ |
| $\overline{qq'}(=\gamma)$ | $c$ |

Since $\sigma$ minimizes the distance from $p$ to $\gamma$, we know that the angle formed between $\sigma'$ and $\gamma'$ is exactly $\frac{\pi}{2}$. That is, $\sigma$ and $\gamma$ intersect orthogonally.

Now, we form the associated triangle in $S^2$ with constant curvature 1, labeling the sides in the same way as $\tilde{a}, \tilde{b}, \tilde{c}$.. In particular, we know that $\tilde{b} > \frac{\pi}{2}$, and the angle opposite $\tilde{a}$ (labeled $\tilde{A}$) is exactly $\frac{\pi}{2}$. By simple spherical geometry, we know that

$$\begin{aligned}\cos(\tilde{a}) &= \cos(\tilde{b})\cos(\tilde{c}) + \sin(\tilde{b})\sin(\tilde{c})\cos(\tilde{A}) \\ &= \cos(\tilde{b})\cos(\tilde{c}) + 0\end{aligned}$$

Now, since $\tilde{b} > \frac{\pi}{2}$, $\cos(\tilde{b}) < 0$. Furthermore, since $\tilde{b}$ is the length of the minimal distance from $p$ to $\gamma$, we also know that $\tilde{a} \geq \tilde{b} > \frac{\pi}{2}$. Thus, $\cos(\tilde{a}) < 0$ as well. This forces $\cos(\tilde{c}) > 0$, and so $\tilde{c} < \frac{\pi}{2}$.

□

<++>