Final Exam

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Problem 1

Let $f_j \in L^1(I, \lambda^1)$ for $I = (-1, 1)$, and suppose $f_j \to f$ almost everywhere.

Part i

Prove or disprove: if

$$\|f_j\|_{L^1} < M$$

for some constant M, then $\|f_j - f\| \to 0$.

Proof. Consider the sequence $f_j = j\chi_{[0,\frac{1}{j}]}$. This converges pointwise almost everywhere to zero, but

$$\begin{aligned} \|f_j - 0\|_{L^1} &= \int_{(-1,1)} j\chi_{[0,\frac{1}{j}]} d\lambda \\ &= \int_{[0,\frac{1}{j}]} j = 1 \end{aligned}$$

which does not tend to zero. □

Part ii

Prove that if

$$\|f_j\|_{L^{1+\delta}} < M$$

for some $\delta > 0$, then $\|f_j - f\| \to 0$.

Proof. Recall theorem 4 from the notes part 4, which states that for a sequence of functions f_j in L^1 on a finite measure space converging pointwise almost everywhere to f, then $\|f_j - f\| \to 0$ if and only if the sequence $\{f_j\}$ is uniformly integrable.

So, we only need to show $\{f_j\}$ is uniformly integrable. So, fix $\varepsilon > 0$. Let q be such that $\frac{1}{1+\delta} + \frac{1}{q} = 1$ (in particular, $q > 1$). Finally, fix δ' such that $M^{\frac{1}{1+\delta}} \delta'^{\frac{1}{q}} < \varepsilon$.

Now, for any subset $E \subset I$ with $\mu(E) < \delta'$, we have by Holder's inequality

$$\begin{aligned}\int_E |f_j| d\lambda^1 &\leq \left(\int_E |f_j|^{1+\delta} d\lambda^1\right)^{\frac{1}{1+\delta}} \left(\int_E |1|^q d\lambda^1\right)^{\frac{1}{q}} \\ &\leq M^{\frac{1}{1+\delta}} \mu(E)^{\frac{1}{q}} \\ &\leq M^{\frac{1}{1+\delta}} \delta'^{\frac{1}{q}} \\ &\leq \varepsilon\end{aligned}$$

and so $\{f_j\}$ is uniformly bounded. Thus, $\|f_j - f\| \to 0$, as desired. □

Problem 2

Let $1 \le p < \infty$, and let (x_n) be a sequence in $\ell^p, x_n = (x_{n1}, x_{n2}, \dots)$. Prove that $(x_n) \to 0$ weakly if and only if (x_n) is strongly bounded and for all i, $x_{ni} \to 0$.

Proof. Recall from homework 5 that for a Banach space X, a sequence (ϕ_j) in X^* converges weak-$*$ if and only if it is strongly bounded, and there exists a dense subset $E \subset X$ for which $\phi_j(x)$ converges for all $x \in E$.

Setting q such that $(\ell^p)^* = \ell^q$, and setting $X = \ell^q$, we see that the sequence (x_n) in $\ell^p = (\ell^q)^*$ converges in weak-$*$ (which is the same as weak convergence in the reflexive space ℓ^p) if and only if it is strongly bounded, and for some dense subset $E \subset \ell^q$ for which $(x_n)(y)$ converges for all $y \in E$.

($\implies$) Suppose first that $x_n \to 0$ weakly. By definition, this means that for all $y \in \ell^q$, $y(x_n) \to y(0) = 0$. In particular, setting $y = e_i$ the standard basis sequence with all zeros except a 1 in the ith place, we see that

$$e_i(x_n) = x_{ni} \to 0$$

as desired.

Furthermore, by the theorem stated above, (x_n) being weakly convergent implies it is strongly bounded. This completes this implication.

($\impliedby$) Suppose (x_n) is such that it is strongly bounded, and $x_{ni} \to 0$ for all i. By the theorem above, we only need to show that $y(x_n) \to 0$ for all y in a dense subset $E \subset \ell^q$.

Again denoting e_i as the ith basis sequence, we see immediately that $e_i(x_n) \to 0$ for all i. Furthermore, this extends to all finite linear combinations of e_i. That is,

$$\left(\sum_{i=1}^{n} a_i e_i \right) (x_n) \to 0$$

where a_i are scalars. This is clear, since we know that $\lim(x_n + y_n) = \lim x_n + \lim y_n$ and $\lim a x_n = a \lim x_n$ for real sequences x_n. Applying this to the real sequences $e_i(x_n)$ achieves the desired result.

Thus, for any sequence y which can be written as a finite linear combination of e_i basis sequences, $y(x_n) \to 0$. We only need to show that the set of all finite linear combinations of basis sequences is dense in ℓ^q.

Denote the set of all finite linear combinations of e_i as E. Suppose $y = (y_1, y_2, \dots) \in \ell^q$. In particular, we know $\|y\|_q$ is finite, and so the tails $\sum_{i=N}^{\infty} y_i$ tend to zero. So, fix $\varepsilon > 0$, and choose N large enough so that $\sum_{i=N}^{\infty} |y_i| < \varepsilon$. Then, consider

$$y' = \sum_{i=1}^{N} y_i e_i = (y_1, y_2, \dots, y_N, 0, \dots)$$

which is clearly in E. Furthermore, we know from notes 3, part 11 that for any sequence x,

$$\|x\|_p \le \|x\|_1$$

for $p \geq 1$. Thus,

$$\begin{aligned}
\|y' - y\|_q &\leq \|y' - y\|_1 \\
&= \sum_{i=1}^{\infty} |y_1' - y_1| \\
&= \sum_{i=1}^{N} |y_i - y_i| + \sum_{i=N}^{\infty} |y_i| \\
&= 0 + \sum_{i=N}^{\infty} |y_i| \\
&< \varepsilon
\end{aligned}$$

Thus, for each $y \in \ell^q$, and each $\varepsilon > 0$, there is some $y' \in E$ such that $\|y - y'\|_q < \varepsilon$, which proves E is dense in ℓ^q.

Thus, the original sequence (x_n) converges on E a dense subset of ℓ^q, and since (x_n) is also bounded, this implies that (x_n) is weakly convergent as well.

Since $y(x_n) \to 0$ for all $y \in E$ a dense subset, by linearity we know that $y(x_n) \to 0$ for all $y \in \ell^q$, and thus $(x_n) \to 0$ weakly, as desired. $\square$

Problem 3

Prove that for a linear operator $A : X \to X$ for a Banach space X the following are equivalent

1. A is continuous. That is, $x_n \to 0$ implies $Ax_n \to 0$.
2. if $x_n \to 0$ weakly, then $Ax_n \to 0$ weakly.
3. if $x_n \to 0$, then $Ax_n \to 0$ weakly.

Proof. ($1 \implies 2$) Suppose A is continuous in the norm topology. We wish to show A is continuous with respect to the weak topology. This should be clear, however, since the weak topology is the initial topology with respect to X^*. That is, $A : Y \to X$ is continuous if and only if $\phi \circ A : Y \to \mathbb{R}$ is continuous for all ϕ.

Now, we show that $A : (X, \sigma(X^*)) \to (X, \sigma(X^*))$ is continuous. To do so, we observe that for $\phi \in X^*$,

$$\phi \circ A : (X, \sigma(X^*)) \to \mathbb{R}$$

is continuous, since

$$\phi \circ A : (X, \|\cdot\|) \to \mathbb{R}$$

is continuous as the composition of continuous functions, and thus $\phi \circ A \in X^*$, and is therefore continuous with respect to $\sigma(X^*)$ the weak topology on X.

Thus, A is continuous from $(X, \sigma(X^*))$ to itself, and therefore preserves weak limits as desired.

($2 \implies 3$) Suppose A preserves weak limits, and let $x_n \to 0$ strongly. This implies that $x_n \to 0$ weakly as well, and so by statement 2, $A(x_n) \to 0$ weakly as desired.

($3 \implies 1$) Suppose A is such that if $x_n \to 0$, then $Ax_n \to 0$ weakly. That is, for all $\phi \in X^*$,

$$\phi(Ax_n) \to 0$$

Suppose for a contradiction that A is not continuous. That is, A is not bounded. So, let x_n be a sequence tending to zero with $Ax_n \to \infty$ (by unboundedness of A). We know that Ax_n weakly converges to zero, but the uniform boundedness principle will lead us to a contradiction.

Consider Ax_n as a sequence of bounded linear operators on X^*. Since Ax_n converges weakly to zero, the sequence $\phi(Ax_n) \to 0$ for all ϕ, and thus

$$\sup_n \|Ax_n(\phi)\| < \infty$$

for each ϕ. Thus, the uniform boundedness principle implies that

$$\sup_{n, \|\phi\|=1} \|Ax_n(\phi)\| = M < \infty$$

However, since

$$\|x\| = \sup_{\|\phi\|=1} |\phi(x)|$$

(proved in an earlier homework), we know that

$$\sup_n \sup_{\|\phi\|=1} \|Ax_n(\phi)\| = \sup_n \|Ax_n\| < M$$

by properties of sup, and so the set $\{Ax_n\}$ is bounded, a contradiction. Thus, A must be continuous, as desired. $\square$