
Homework 5

Daniel Halmrast

May 10, 2018

PROBLEM 1

Let E, F be closed subspaces of a Hilbert space. Prove that $P_E P_F = P_E$ if and only if $E \subseteq F$.

Proof. Suppose first that $P_E P_F = P_E$. Then, in particular,

$$P_E^* = (P_E P_F)^* = P_F^* P_E^* = P_F P_E = P_E$$

by self-adjointness of projections. Thus, $P_E P_F = P_F P_E = P_E$, and the projections commute. Thus, the von Neumann algebra $W^*(P_E, P_F, I)$ is Abelian, and is isometrically $*$ -isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ) . In particular, the projections P_E, P_F get sent to self-adjoint idempotents $P_E \mapsto M_{\chi_S}$ and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$.

Now, the requirement $P_F P_E = P_E$ corresponds to the requirement

$$M_{\chi_{S'}} M_{\chi_S} = M_{\chi_S}$$

which means that $S \subset S'$. This, in turn, implies that $E \subset F$.

Indeed, E is the subspace of H on which P_E is the identity, which corresponds to the subspace

$$\tilde{E} = \int_S^\oplus H(x) d\mu(x)$$

on which M_{χ_S} is the identity. Similarly,

$$\tilde{F} = \int_{S'}^\oplus H(x) d\mu(x)$$

Clearly, $\tilde{E} \subset \tilde{F}$ (since $S \subset S'$) and so $E \subset F$ as well.

For the converse direction, assume that $E \subset F$. Then, on F , $P_F = I_F$, and $P_F P_E = I_F P_E = P_E$. Furthermore, on F^\perp , $P_F P_E = 0 = P_E$. Thus, on all of $H = F \oplus F^\perp$, $P_F P_E = P_E$ as desired. \square

PROBLEM 2

Characterize the closed subspaces E, F of a Hilbert space H satisfy $P_F P_E = P_E P_F$.

Proof. I assert that P_E and P_F commute if and only if H can be decomposed into the four orthogonal components

$$H = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp$$

To see this, suppose first that P_E and P_F commute. Then, consider the Abelian von Neumann algebra $W^*(P_E, P_F, I)$. Like before, we use the Borel functional calculus to identify $W^*(P_E, P_F, I)$ with $L^\infty(X, \mu)$ acting on $\tilde{H} = \int_X^\oplus H(x) d\mu(x)$, which is unitarily equivalent to H (in the sense that there is a unitary transformation U such that $\tilde{H} = UH$ and $UW^*(P_E, P_F, I)U^* = L^\infty(X, \mu)$).

Under this identification, $P_E \mapsto M_{\chi_S}$, and $P_F \mapsto M_{\chi_{S'}}$ for some measurable subsets $S, S' \subset X$. In particular, this decomposes \tilde{H} into four orthogonal components

$$\begin{aligned} \tilde{H} &= \int_X^\oplus H(x) d\mu(x) \\ &= \int_{S \cap S'}^\oplus H(x) d\mu(x) \oplus \int_{S \cap S'^c}^\oplus H(x) d\mu(x) \oplus \int_{S' \cap S^c}^\oplus H(x) d\mu(x) \oplus \int_{S^c \cap S'^c}^\oplus H(x) d\mu(x) \end{aligned}$$

which translates into the decomposition on H as

$$H = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp$$

as desired.

Conversely, suppose H can be decomposed this way. Then, let $v \in H$ be decomposed as

$$v = v_1 + v_2 + v_3 + v_4$$

where $v_1 \in E \cap F$, $v_2 \in E \cap F^\perp$, $v_3 \in E^\perp \cap F$ and $v_4 \in E^\perp \cap F^\perp$.

We compute the effect of $P_E P_F$ and $P_F P_E$ directly.

$$\begin{aligned} P_E P_F(v) &= P_E P_F(v_1 + v_2 + v_3 + v_4) \\ &= P_E(v_1 + v_3) \\ &= v_1 \\ P_F P_E(v) &= P_F P_E(v_1 + v_2 + v_3 + v_4) \\ &= P_F(v_1 + v_2) \\ &= v_1 \end{aligned}$$

and thus $P_E P_F = P_F P_E$ as desired. □

PROBLEM 3

Let E, F be closed subspaces of a Hilbert space H . An operator U is said to be a partial isometry from E to F if $U|_E$ is an isometry onto F , and $U|_{E^\perp} = 0$. Prove that U is a partial isometry $\iff U^*U$ is a projection $\iff UU^*$ is a projection.

Proof.

□

<++>