# Homework 1

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October 11, 2018

## Problem 1

Show that multiparticle nonrelativistic quantum can be recovered from QFT. Namely, define

$$H = \int d^3x a^\dagger(x) \left( \frac{-\hbar^2}{2m} \nabla^2 + U(x) \right) a(x) + \int d^3x d^3y V(x-y) a^\dagger(x) a^\dagger(y) a(y) a(x)$$

and

$$|\psi, t\rangle = \int d^3x_1 \dots d^3x_n \psi(x_1, \dots, x_n; t) a^\dagger(x_1) \dots a^\dagger(x_n) |0\rangle$$

We will show that $|\psi, t\rangle$ satisfies the abstract Schrodinger equation if and only if $\psi$ satisfies the Schrodinger equation

$$i\partial_t \psi = H\psi$$

for

$$H = \sum_{i=1}^{n} \frac{-\hbar^2}{2m} \nabla_i^2 + U(x_i) + \sum_{j=1}^{n} \sum_{i=1}^{j-1} V(x_i - x_j)$$

We calculate $H|\psi, t\rangle$ directly in three parts. That is:

$$\begin{aligned} H|\psi, t\rangle &= \int d^3 a^\dagger(x) \frac{-\hbar^2}{2m} \nabla^2 a(x) |\psi, t\rangle \\ &+ \int d^3x a^\dagger(x) U(x) a(x) |\psi, t\rangle \\ &+ \int d^3x d^3y V(x-y) a^\dagger(x) a^\dagger(y) a(y) a(x) |\psi, t\rangle \end{aligned}$$

For this calculation, we'll use the fact that

$$a(x) a^\dagger(x_1) \dots a^\dagger(x_n) = \sum_{i=1}^{n} (-1)_\pm^{i-1} \delta(x - x_i) a^\dagger(x_1) \dots \hat{a^\dagger(x_i)} \dots a^\dagger(x_n) + (-1)_\pm^n a^\dagger(x_1) \dots a^\dagger(x_n) a(x)$$

where $(-1)_\pm^n$ indicates the $(-1)^n$ only appears in the fermionic calculation, and $\hat{a^\dagger(x_i)}$ indicates that the $i$th creation operator is omitted. This is calculated by iterated application of the rule

$$a(x) a^\dagger(y) = \pm a^\dagger(y) a(x) + \delta(x-y)$$

with a plus sign for bosonic calculations, and a minus sign for fermionic calculations.

We can now calculate directly the three components of $H|\psi,t\rangle$ as

$$
\begin{aligned}
\int d^3x a^\dagger(x)\frac{-\hbar^2}{2m}\nabla^2 a(x)|\psi,t\rangle &= \int d^3x_1\dots d^3x_n d^3x a^\dagger(x)\frac{-\hbar^2}{2m}\nabla^2 a(x)\psi a^\dagger(x_1)\dots a^\dagger(x_n)|0\rangle \\
&= \int d^3x_1\dots d^3x_n d^3x a^\dagger(x)\frac{-\hbar^2}{2m}\nabla^2\psi \\
&\left(\sum_{i=1}^n(-1)^{i-1}_\pm\delta(x-x_i)a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)+(-1)^n_\pm a^\dagger(x_1)\dots a^\dagger(x_n)a(x)\right)|0\rangle \\
&= \int d^3x_1\dots d^3x_n d^3x a^\dagger(x)\frac{-\hbar^2}{2m}\nabla^2\psi \\
&\left(\sum_{i=1}^n(-1)^{i-1}_\pm\delta(x-x_i)a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle \\
&= \int d^3x_1\dots d^3x_n\sum_{i=1}^n a^\dagger(x_i)\frac{-\hbar^2}{2m}\nabla_i^2\psi \\
&\left((-1)^{i-1}_\pm a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle \\
&= \int d^3x_1\dots d^3x_n\sum_{i=1}^n\frac{-\hbar^2}{2m}\nabla_i^2\psi\left((-1)^{i-1}_\pm a^\dagger(x_i)a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle \\
&= \int d^3x_1\dots d^3x_n\sum_{i=1}^n\frac{-\hbar^2}{2m}\nabla_i^2\psi\left(a^\dagger(x_1)\dots a^\dagger(x_i)\dots a^\dagger(x_n)\right)|0\rangle
\end{aligned}
$$

Note that we integrated over $x$, and the $\delta$ factors in the sum mean that integrating over $x$ just swaps $x$ with $x_i$. We also used integration by parts implicitly to go from line 4 and 5 to line 6 and 7 by shifting over the $\nabla^2$ onto $a^\dagger(x)$ before integrating out the $x$, changing it to a $\nabla_i^2 a^\dagger(x_i)$, then integrating by parts again to move it back to $\psi$. Finally, in the fermionic calculation, we note that the factor of $(-1)^{i-1}$ disappears when we move $a^\dagger(x_i)$ across to the $i$th position, as it will pick up an extra factor of $(-1)^{i-1}$.

We carry the exact same calculations for the second integral, without the integration by parts, to find

$$
\int d^3x a^\dagger(x)U(x)a(x)|\psi,t\rangle = \int d^3x_1\dots d^3x_n\sum_{i=1}^n U(x_i)\psi a^\dagger(x_1)\dots a^\dagger(x_n)
$$

And finally, we calculate the third integral as

$$\int d^3xd^3yV(x-y)a^\dagger(x)a^\dagger(y)a(y)a(x)|\psi,t\rangle = \int d^3x_1\dots d^3x_nd^3xd^3y\psi V(x-y)a^\dagger(x)a^\dagger(y)a(y)$$

$$\left(\sum_{i=1}^n(-1)_\pm^{i-1}\delta(x-x_i)a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle$$

$$= \int d^3x_1\dots d^3x_nd^3y\sum_{i=1}^n\psi V(x_i-y)a^\dagger(x_i)a^\dagger(y)a(y)$$

$$\left((-1)_\pm^{i-1}a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle$$

$$= \int d^3x_1\dots d^3x_nd^3y\sum_{i=1}^n\psi V(x_i-y)(-1)_\pm a^\dagger(y)a^\dagger(x_i)a(y)$$

$$\left((-1)_\pm^{i-1}a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle$$

$$= \int d^3x_1\dots d^3x_nd^3y\sum_{i=1}^n\psi V(x_i-y)a^\dagger(y)(a(y)a^\dagger(x_i)\pm\delta(x_i-y)$$

$$\left((-1)_\pm^{i-1}a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle$$

The first calculation was done identical to the other two integrals, and the second calculation was done by direct evaluation of the commutators. Now, if we integrate out $y$ in the part of the integral with $\delta(x_i-y)$, we get a factor of $V(0)=0$, and so that integral goes to zero. Thus, we have

$$= \int d^3x_1\dots d^3x_nd^3y\sum_{i=1}^n\psi V(x_i-y)a^\dagger(y)a(y)a^\dagger(x_i)$$

$$\left((-1)_\pm^{i-1}a^\dagger(x_1)\dots a^\dagger\hat{(x_i)}\dots a^\dagger(x_n)\right)|0\rangle$$

$$= \int d^3x_1\dots d^3x_nd^3y\sum_{i=1}^n\psi V(x_i-y)a^\dagger(y)a(y)$$

$$\left(a^\dagger(x_1)\dots a^\dagger(x_i)\dots a^\dagger(x_n)\right)|0\rangle$$

We then carry the exact same calculation out for $y$, integrating out the factors of $\delta$ to get

$$= \int d^3x_1\dots d^3x_n\sum_{i=1}^n\sum_{j=1}^n\psi V(x_i-x_j)$$

$$\left(a^\dagger(x_1)\dots a^\dagger(x_n)\right)|0\rangle$$

as desired. Thus, since

$$i\partial_t|\psi,t\rangle = \int d^3x_1\dots d^3x_ni\partial_t\psi a^\dagger(x_1)\dots a^\dagger(x_n)$$

we equate both sides to get

$$\int d^3x_1\dots d^3x_ni\partial_t\psi a^\dagger(x_1)\dots a^\dagger(x_n)$$

$$= \int d^3x_1\dots d^3x_n\left(\sum_{i=1}^n\frac{-\hbar^2}{2m}\nabla_i^2+U(x_i)+\sum_{j=1}^n\sum_{i=1}^{j-1}V(x_i-x_j)\right)\psi a^\dagger(x_1)\dots a^\dagger(x_n)$$

Equating integrands yields the desired result. Thus, the abstract Schrodinger equation is solved if and only if $\psi$ solves the regular Schrodinger equation.

# Problem 2

Show that the infinitesimal Lorentz transformations are antisymmetric.

We expand directly using $\Lambda^\mu_\nu = \delta^\mu_\nu + \delta\omega^\mu_\nu$ and omitting products of infinitesimals as

$$\begin{aligned} g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma &= g_{\rho\sigma} \\ g_{\mu\nu}\left(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\rho\delta\omega^\nu_\sigma + \delta^\nu_\sigma\delta\omega^\mu_\rho\right) &= g_{\rho\sigma} \\ g_{\rho\sigma} + g_{\rho\nu}\delta\omega^\nu_\sigma + g_{\mu\sigma}\delta\omega^\mu_\rho &= g_{\rho\sigma} \\ g_{\rho\sigma} + \delta\omega_{\rho\sigma} + \delta\omega_{\sigma\rho} &= g_{\rho\sigma} \end{aligned}$$

and so it must be that $\delta\omega_{\rho\sigma} + \delta\omega_{\sigma\rho} = 0$ as desired.

# Problem 3

Derive the commutation relations on momentum creation and annihilation operators from the canonical commutation relations.

We know

$$a(k) = \int d^3x \exp(-ikx) \left(i\partial_t\phi(x) + \omega\phi(x)\right)$$

and

$$\Pi(x) = \partial_t\phi(x)$$

along with the canonical commutation relations on $\phi$ and $\Pi$.

So, we calculate directly

$$\begin{aligned}
a(k)a(k') &= \int d^3x \exp(-ikx) \left(i\partial_t\phi(x) + \omega\phi(x)\right) \int d^3x \exp(-ik'x) \left(i\partial_t\phi(x) + \omega\phi(x)\right) \\
&= \int d^3x d^3x' \exp(-ikx)(i\Pi(x) + \omega\phi(x)) \exp(-ik'x')(i\Pi(x') + \omega\phi(x')) \\
&= \int d^3x d^3x' \exp(-ikx) \exp(-ik'x')(i\Pi(x)i\Pi(x') + i\Pi(x)\omega'\phi(x') + \omega\phi(x)i\Pi(x') + \omega\omega'\phi(x)\phi(x')) \\
&= \int d^3x d^3x' \exp(-ikx) \exp(-ik'x')(i\Pi(x')i\Pi(x) + i\Pi(x')\omega\phi(x) + i\omega\delta(x-x') \\
&+ \omega'\phi(x')i\Pi(x) + i\omega'\delta(x-x') + \omega\omega'\phi(x')\phi(x)) \\
&= a(k')a(k) + \int d^3x d^3x' \exp(-ikx) \exp(-ik'x')(i(\omega+\omega')\delta(x-x')) \\
&= a(k')a(k) + \int d^3x \exp(-ikx) \exp(-ik'x) i(\omega+\omega') \\
&= a(k')a(k)
\end{aligned}$$

where the last integral disappeared because $\exp(i(k+k')x)$ disappears on a symmetric domain. The exact same argument is made for $a^\dagger(k)a^\dagger(k')$ just by complex conjugation.

Thus, all we have to show is the final commutator. We calculate

$$\begin{aligned}
a(k)a^\dagger(k') &= \int d^3x d^3x' \exp(-ikx) \exp(ik'x')(i\Pi(x) + \omega\phi(x))(-i\Pi(x') + \omega'\phi(x')) \\
&= \int d^3x d^3x' \exp(-ikx) \exp(ik'x')(i\Pi(x)(-i\Pi(x')) \\
&+ i\Pi(x)\omega'\phi(x') + \omega\phi(x)(-i\Pi(x')) + \omega\phi(x)\omega'\phi(x')) \\
&= \int d^3x d^3x' \exp(-ikx) \exp(ik'x')(-i\Pi(x')i\Pi(x) \\
&+ \omega'\phi(x')i\Pi(x) - i\omega' i\delta(x-x') + (-i\Pi(x'))\omega\phi(x) - i\omega i\delta(x-x') + \omega'\phi(x')\omega\phi(x)) \\
&= \int d^3x d^3x' \exp(-ikx) \exp(ik'x')(-i\Pi(x')i\Pi(x) \\
&+ \omega'\phi(x')i\Pi(x) + \omega'\delta(x-x') + (-i\Pi(x'))\omega\phi(x) + \omega\delta(x-x') + \omega'\phi(x')\omega\phi(x)) \\
&= a^\dagger(k')a(k) + \int d^3x d^3x' \exp(-ikx) \exp(ik'x')(\omega+\omega')\delta(x-x') \\
&= a^\dagger(k')a(k) + \int d^3x \exp(i(k'-k)x)(\omega'+\omega) \\
&= a^\dagger(k')a(k) + (2\pi)^3 2\omega\delta(k-k')
\end{aligned}$$

as desired.

# Problem 4

## Part A

Find an expression for $[\phi(x), P^i]$.

We carry out two expansions here on $a$. First:

$$\phi(x-a) = \phi(x) - \partial_\mu \phi(x) a^\mu + O(a^2)$$

and

$$T(a) = I + iP^\mu a_\mu$$

From the expression for the $T(a)$ action on $\phi(x)$ we know that (by killing $O(a^2)$ terms)

$$\begin{aligned}
\phi(x)T(a) &= T(a)\phi(x-a) \\
\phi(x)T(a) &= T(a)\left(\phi(x) - \partial_\mu \phi(x) a^\mu\right) \\
\phi(x)(I + iP^\mu a_\mu) &= (I + iP^\mu a_\mu)\phi(x) - (I + iP^\mu a_\mu)\partial_\nu \phi(x) a^\nu \\
\phi(x) iP^\mu a_\mu &= iP^\mu a_\mu \phi(x) - \partial^\mu \phi(x) a_\mu \\
\phi(x) P^\mu a_\mu &= P^\mu a_\mu \phi(x) + i\partial^\mu \phi(x) a_\mu
\end{aligned}$$

and so

$$[\phi(x), P^\mu] = i\partial^\mu \phi(x)$$

as desired.

## Part B

Show the time component of the calculation above is the Heisenberg equation of motion.

This follows immediately, noting that $P^0 = H$, and so

$$[\phi(x), H] = i\partial^0 \phi(x) = i\partial_t \phi(x)$$

as desired.

## Part C

Derive the Klein-Gordon equation from the Heisenberg equation for a free particle.

We'll use the Hamiltonian

$$H = \int \tilde{dk} \omega a^\dagger(k) a(k)$$

along with the free-field expression

$$\phi(x) = \int \tilde{dk} (a(k)\exp(ikx) + a^\dagger(k)\exp(-ikx))$$

as derived in the text. Thus,

$$\begin{aligned}
i\dot{\phi}(x) &= [\phi(x), H] \\
&= \phi(x)H - H\phi(x) \\
&= \int \tilde{d}ka(k)\exp(ikx) + a^\dagger(k)\exp(-ikx)) \int \tilde{d}k\omega a^\dagger(k)a(k) \\
&- \int \tilde{d}k\omega a^\dagger ka(k) \int \tilde{d}k(a(k)\exp(ikx) + a^\dagger(k)\exp(-ikx)) \\
&= \int \tilde{d}k\tilde{d}k'\omega\exp(ikx)(a(k)a^\dagger(k')a(k') - a^\dagger(k')a(k')a(k)) \\
&+ \omega\exp(-ikx)(a^\dagger(k)a^\dagger(k')a(k') - a^\dagger(k')a(k')a^\dagger(k)) \\
&= \int \tilde{d}k\tilde{d}k'\omega\exp(ikx)(a(k)a^\dagger(k')a(k') - a^\dagger(k')a(k)a(k')) \\
&+ \omega\exp(-ikx)(a^\dagger(k')a^\dagger(k)a(k') - a^\dagger(k')a(k')a^\dagger(k)) \\
&= \int \tilde{d}k\tilde{d}k'\omega\exp(ikx)(a(k)a^\dagger(k') - a^\dagger(k')a(k))a(k') \\
&+ \omega\exp(-ikx)a^\dagger(k')(a^\dagger(k)a(k') - a(k')a^\dagger(k)) \\
&= \int \tilde{d}k\tilde{d}k'\omega\exp(ikx)([a(k), a^\dagger(k')])a(k') \\
&+ \omega\exp(-ikx)a^\dagger(k')([a^\dagger(k), a(k')]) \\
&= \int \tilde{d}k\tilde{d}k'\omega\exp(ikx)((2\pi)^3 2\omega\delta(k-k'))a(k') \\
&+ \omega\exp(-ikx)a^\dagger(k')(-(2\pi)^3)2\omega\delta(k-k') \\
&= \int \tilde{d}k\omega\exp(ikx)((2\pi)^3 2\omega)a(k) \\
&- \omega\exp(-ikx)a^\dagger(k)((2\pi)^3)2\omega \\
&= (2\pi)^3 2\omega^2\left(\int \tilde{d}k(a(k)\exp(ikx) - a^\dagger(k)\exp(-ikx)\right)
\end{aligned}$$

and so

$$\dot{\phi}(x) = (2\pi)^3 2\omega^2\left(\int \tilde{d}k(-ia(k)\exp(ikx) + ia^\dagger(k)\exp(-ikx)\right)$$

we can carry the exact same calculations out to find that

$$\begin{aligned}
i\ddot{\phi}(x) &= (2\pi)^3 2\omega^2(\int \tilde{d}k\tilde{d}k'\omega\exp(ikx)([-ia(k), a^\dagger(k')])a(k') \\
&+ \omega\exp(-ikx)a^\dagger(k')([ia^\dagger(k), a(k')])) \\
&= (2\pi)^3 2\omega^2(\int \tilde{d}k\tilde{d}k'\omega\exp(ikx)(-i(2\pi)^3 2\omega\delta(k-k'))a(k') \\
&+ \omega\exp(-ikx)a^\dagger(k')(-i(2\pi)^3 2\omega\delta(k-k'))) \\
&= (2\pi)^3 2\omega^2(\int \tilde{d}k\omega\exp(ikx)(-i(2\pi)^3 2\omega)a(k) \\
&+ \omega\exp(-ikx)a^\dagger(k)(-i(2\pi)^3 2\omega)) \\
&= -i(2\pi)^6 4\omega^4\phi(x)
\end{aligned}$$

and so

$$\begin{aligned}
\ddot{\phi} &= -(2\pi)^6 4\omega^4\phi(x) \\
&= -(2\pi)^6 4(k^2+m^2)^2\phi(x) \\
&= -((2\pi)^3)^2 4(k^4+m^4+2k^2m^2)\phi(x)
\end{aligned}$$

which (I think) reduces to the Klein-Gordon equation.

## Part D

We compute the commutator directly:

$$\begin{aligned}
\phi(x)\int d^3x'\Pi(x')\nabla\phi(x') &= \int d^3x'\phi(x)\Pi(x')\nabla\phi(x') \\
&= \int d^3x'(\Pi(x')\phi(x) + i\delta(x-x'))\nabla\phi(x') \\
&= \int d^3x'(\Pi(x')\phi(x)\nabla\phi(x') + i\delta(x-x')\nabla\phi(x')) \\
&= \int d^3x'\Pi(x')\nabla\phi(x')\phi(x) + i\nabla\phi(x) \\
&= P\phi(x) + i\nabla\phi(x)
\end{aligned}$$

and so

$$[\phi(x), P^i] = i\partial^i\phi(x)$$

as desired.

## Part E

Express $P$ in terms of momentum state creation and annihilation operators.

We do this directly.

$$\begin{aligned}
P &= \int d^3x\Pi(x)\nabla\phi(x) \\
&= \int \tilde{dk}\tilde{dk}'d^3x(-i\omega a(k)\exp(ikx) + i\omega a^\dagger(k)\exp(-ikx))(ik'a(k')\exp(ik'x) - ik'a^\dagger(k')\exp(-ik'x)) \\
&= \int \tilde{dk}\tilde{dk}'d^3x(\omega k a(k)a(k')\exp(i(k+k')x) - \omega k a(k)a^\dagger(k')\exp(i(k-k')x) \\
&- \omega k a^\dagger(k)a(k')\exp(i(k'-k)x) + \omega k a^\dagger(k)a^\dagger(k')\exp(-i(k+k')x)) \\
&= \int \tilde{dk}\tilde{dk}'(\omega k a(k)a(k')(2\pi)^3\delta(k+k') - \omega k a(k)a^\dagger(k')(2\pi)^3\delta(k-k') \\
&- \omega k a^\dagger(k)a(k')(2\pi)^3\delta(k-k') + \omega k a^\dagger(k)a^\dagger(k')(2\pi)^3\delta(k+k')) \\
&= \int \tilde{dk}(\omega k a(k)a(-k)(2\pi)^3 - \omega k a(k)a^\dagger(k)(2\pi)^3 \\
&- \omega k a^\dagger(k)a(k)(2\pi)^3 + \omega k a^\dagger(k)a^\dagger(-k)(2\pi)^3) \\
&= \int \tilde{dk}(-\omega k a(k)a^\dagger(k)(2\pi)^3 \\
&- \omega k a^\dagger(k)a(k)(2\pi)^3) \\
&= \int \tilde{dk}\omega k(2\pi)^3(-a(k)a^\dagger(k) + a^\dagger(k)a(k)
\end{aligned}$$