Problem Set 5

Daniel Halmrast

February 20, 2018

Problem 1

Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$ so that f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. We begin by explicitly constructing the homotopy that is stationary on $S^1 \times \{0\}$. This is done via the homotopy

$$f_t(\theta, s) = (\theta + t2\pi s, s)$$

Clearly, $f_0 = \mathbb{1}$ and $f_1 = f$. Furthermore, $f_t(\theta, 0) = (\theta, 0)$ and so f_t is stationary on $S^1 \times \{0\}$

However, there is no homotopy that is stationary on both boundary circles. This is made clear by considering what f does to the path $s \mapsto (\theta_0, s)$, namely f wraps the path around the cylinder once. Thus, the projection of this path onto the space $s^1 \times \{1\}$ is a loop homotopic to the counterclockwise generator for $\pi_1(S^1)$.

If such a homotopy existed between the identity and f that was stationary on both boundary circles, it would induce a homotopy between the straight line $s \mapsto (\theta_0, s)$ and the image of this path under f that fixes the endpoints. By composing with the projection onto $S^1 \times \{1\}$, we would obtain a homotopy between the constant path $s \mapsto \theta_0$ and the projection of the image $s \mapsto \theta_0 + 2\pi s$ which fixes the endpoints (which coincide). Thus, this would be a homotopy between a constant path and a path homotopic to [1], which cannot happen.

□

Problem 2

Does the Borsuk-Ulam theorem hold for the torus? That is, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ does there exist a point (x, y) for which $f(x, y) = f(-x, -y)$?

Proof. I assert that the Borsuk-Ulam theorem does not hold for the torus. To see this, we construct an explicit function from T^2 to $\mathbb{R}^2$ which does not have any antipodal points with the same value.

Consider the function $f : T^2 \to \mathbb{R}^2$ given as follows. First, let T^2 be embedded in $\mathbb{R}^3$. Then, consider the vector field $\frac{\partial}{\partial \phi}$ where ϕ runs parallel to the $x - y$ plane. Since T^2 is embedded in $\mathbb{R}^3$, these vectors can be thought of as living in the tangent bundle to $\mathbb{R}^3$. Thus, for each vector, it makes sense to take its projection onto the $x - y$ plane. Now, since the original vector field $\frac{\partial}{\partial \phi}$ is smooth, and projection is a continuous operation, this defines a continuous map from T^2 to $\mathbb{R}^2$.

In coordinates, this map is given as

$$f(\theta, \phi) = (\cos(\phi), \sin(\phi)) \tag{0.1}$$

and clearly, $f(\theta, \phi) \neq f(-\theta, -\phi)$ as desired. $\square$

Problem 3

From the isomorphism $\pi_1(X\times Y,(x_0,y_0)) \cong \pi_1(X,x_0)\times\pi_1(Y,y_0)$ it follows that loops in $X\times\{y_0\}$ and $\{x_0\}\times Y$ represent commuting elements of $\pi_1(X\times Y,(x_0,y_0))$. Construct an explicit homotopy demonstrating this.

Proof. Let f be a loop in $X\times\{y_0\}$, and let g be a loop in $\{x_0\}\times Y$. We wish to show that $f\cdot g \simeq g\cdot f$. We can construct this homotopy explicitly by "sliding" f along g. Concretely the homotopy is as follows.

First, define $g_t = g\Big|_{[0,t]}$ and $g_{t'} = g\Big|_{[t,1]}$ so that $g_{t'}\cdot g_t = g$. That is, g_t is the segment of g from $g(0)$ to $g(t)$. Furthermore, let f_y be the loop f in the subspace $X\times\{y\}$.

Now, define a homotopy h as

$$h_t = g_{t'}\cdot f_{\pi_y(g(t))}\cdot g_t \tag{0.2}$$

Now, this is well-defined, since g_t has an endpoint at $(x_0,\pi_y(g(t)))$, and f starts and ends at $(x_0,\pi_y(g(t)))$, and $g_{t'}$ begins at $(x_0,\pi_y(g(t))$ and ends at (x_0,y_0). We just have to make sure this is continuous. Since h is a map into $X\times Y$, we just have to check that the corresponding map $H : S^1\times I \to X\times Y$ is continuous onto both X and Y. That is, H is continuous if and only if $\pi_X H$ and $\pi_Y H$ are continuous.

However, it should be clear that $\pi_X H = f$, since $\pi_X(g(t)) = x_0$ for all t. Thus, the path $g_{t'}\cdot f\cdot g_t$ projects down to just $x_0\cdot f\cdot x_0$, which is clearly continuous.

Similarly, $\pi_Y H = g_{t'}\cdot y_0\cdot g_t$, which is (after reparameterization) equal to $g_{t'}\cdot g_t = g$, which is clearly continuous for all t.

Thus, H is continuous in its projections, and by the universal property of products, is continuous in general.

Now, h_0 is just $g_{0'}\cdot f\cdot g_0 = g\cdot f$, and $h_1 = g_{1'}\cdot f\cdot g_1 = f\cdot g$, and so $f\cdot g\simeq g\cdot f$ as desired. $\square$

Problem 4

Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Proof. Throughout this problem, I will identify $\pi_1(S^1)$ with $\mathbb{Z}$ by identifying 1 with the loop that goes once around the circle counterclockwise.

Since $\pi_1(S^1) \cong \mathbb{Z}$, each homomorphism is characterized by where it sends the generator 1.

So, suppose $\phi : \pi_1(S^1) \to \pi_1(S^1)$ is such that $\phi(1) = n$ for some integer n. Define $\varphi(z) = z^n$. I claim that $\varphi_* = \phi$.

To see this, we just need to see that $\varphi_*(1) = n$. Now, the loop 1 is just the map $f(t) = \exp(2\pi it)$, and the induced loop $\varphi_*(1)$ is the composition $\phi(f(t)) = \exp(2\pi int)$ which is easily seen to be homotopic to the loop n, as desired. $\square$

Problem 5

Show that there does not exist a retraction from X to A in the following cases:

Part (a)

$X = \mathbb{R}^3$ and A is any subspace homeomorphic to S^1.

Proof. Suppose such a retraction $r : X \to A$ existed. In particular, we would have that

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "\mathbb{1}"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

commutes. Now, applying the π_1 functor to this diagram yields

$$\begin{tikzcd} \mathbb{Z} \arrow[d, leftrightarrow] & 0 \arrow[d, leftrightarrow] & \mathbb{Z} \arrow[d, leftrightarrow] \\ \pi_1(A) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

Now, this diagram implies that $\mathbb{1}$ on $\mathbb{Z}$ factors through zero, which cannot happen. Thus, no such r exists. $\square$

Part (b)

$X = S^1 \times D^2$ with A the boundary torus $S^1 \times S^1$.

Proof. Following the same diagram as the one used in the previous problem, assuming such an $r : X \to A$ exists yields the diagram

$$\begin{tikzcd} \mathbb{Z} \times \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}"'] & \mathbb{Z} \arrow[r, "r_*"] & \mathbb{Z} \times \mathbb{Z} \end{tikzcd}$$

which commutes. However, for this to commute, it must be that i_* is injective. This cannot happen, however, since $\mathbb{Z} \times \mathbb{Z}$ has two generators, but $\mathbb{Z}$ only has one. Since an injective map has to map generators to generators, i_* cannot be injective, and the diagram cannot commute, a contradiction. Thus, no such r exists. $\square$

Part (c)

$X = S^1 \times D^2$ with A the circle shown in the figure.

Proof. Again, we assume that such a retraction exists, and form the diagram

$$\begin{tikzcd} \pi_1(A) \arrow[r, "i"] \arrow[rr, bend right, "\mathbb{1}"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

which would force i_* to be injective. However, since A is contractible in X, for any loop $[f] \in \pi_1(A)$, $i_*[f]$ is homotopic to the constant loop. Thus, i_* is actually the zero map, and in particular is not injective. Thus, no such retraction r can exist. $\square$

Part (d)

$X = D^2 \vee D^2$ with $A = \partial X$.

Proof. Mirroring the approach in part (a), we show that $\pi_1(X) = 0$, with $\pi_1(A) \neq 0$. This forces i_* to fail to be injective, and thus no retraction can exist from X to A.

It should be clear that $\pi_1(D^2 \vee D^2)$ is trivial, since the space is contractible via the contraction that sends each copy of D^2 to the single point in their intersection.

Thus, since $\pi_1(X) = 0$ (and $\pi_1(A) = \mathbb{Z} \star \mathbb{Z} \neq 0$) the induced map i_* is not injective, and thus cannot have a retract. $\square$

Part (e)

X is a disk with two points on the boundary identified, and A is the boundary $S^1 \vee S^1$.

Proof. To begin with, we note that $\pi_1(X) = \mathbb{Z}$ (since $X \simeq S^1$) and that $\pi_1(A) = \mathbb{Z} \star \mathbb{Z}$ (calculated in Hatcher).

Now, we will show that the induced map i_* is not injective (for i the canonical inclusion map of A into X). Clearly, i_* cannot be injective, since there is no injective map from $\mathbb{Z} \star \mathbb{Z}$ to $\mathbb{Z}$.

To see this, note that any group homomorphism $\phi : \mathbb{Z} \star \mathbb{Z} \to \mathbb{Z}$ is completely determined by where it sends its generators. Let a and b be the generators for $\mathbb{Z} \star \mathbb{Z}$. Then, since $\mathbb{Z}$ only has one generator, it must be that $\phi(a) = n\phi(b)$ for some $n \in \mathbb{Z}$. However, this implies that $\phi(a) = \phi(nb)$ and since $\mathbb{Z} \star \mathbb{Z}$ has no relations between a and b, we know that $nb \neq a$. Thus, ϕ is not injective.

Since i_* is not injective, it cannot have a retract, and so no retraction $r : X \to A$ exists. $\square$

Part (f)

X is the Mobius band, and A is the boundary circle.

Proof. We observe first that the boundary of the Mobius band loops twice around it before returning to its basepoint. Using this fact, we can calculate what the induced map i_* is.

Let $[1]_A$ be the generator of $\pi_1(A)$ going counterclockwise, and let $[1]_X$ be similarly defined for X. We only need to calculate what $i_*[1]_A$ is. However, it is clear from the observation above that $i_*[1]_A = [2]_X$.

Now, suppose a retraction $r : X \to A$ existed. since $r \circ i = \mathbb{1}$, we know that $r_* \circ i_* = \mathbb{1}$ as well. Thus, $r_*[2]_X = [1]_A$. But this is not a well-defined group homomorphism! In particular, $r_*[1]_X$ is not well-defined. Suppose $r_*[1]_X = [n]$. Then, $r_*[2]_X = 2r_*[1]_X = [2n] \neq [1]$ which contradicts $r_*[2]_X = [1]_A$. Thus, no such retraction can exist. $\square$

Problem 6

Use Lemma 1.15 to show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \to X$ induces a surjection on π_1.

Proof. Recall that Lemma 1.15 states that if a space X is the union of a collection of path-connected open sets A_α each containing the basepoint x_0, and each pairwise intersection is path-connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

Now, we can decompose X into $A \cup D^n$ where D^n is the n-cell e^n along with its boundary. Furthermore, since A is path-connected, we can take the basepoint of X without loss of generality to be on the intersection $A \cap D^n$. Thus, this decomposition satisfies the properties of Lemma 1.15. So, we just need to show that any loop in X is the inclusion of a loop in A.

Let f be any loop in X. By Lemma 1.15, we know that f is homotopic to a product of loops in A and D^n. However, each loop in D^n is nullhomotopic, and so f is homotopic to a product of loops in A, which itself is a loop in A. Call this loop $\tilde{f}$. It should be clear, then, that $i_*[\tilde{f}] = [f]$, and thus i_* is surjective. $\square$

Part (a)

Use this to show that the wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

Proof. We let $A = S^1$, and $X = S^1 \vee S^2$. Then the inclusion map i_* is a surjection. Furthermore, it should be clear that i_* is injective as well, since the inclusion of a nontrivial loop in A to X will not be nullhomotopic.

Thus, i_* is actually an isomorphism, and $\pi_1(X) = \mathbb{Z}$ as desired. $\square$

Part (b)

Use this to show that for a path-connected CW complex X the inclusion map $X^1 \to X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$.

Proof. We first show this result holds in the finite case. To do so, we induct on the number of cells attached to the 1-skeleton. In particular, we show that adding an n-cell for $n \geq 2$ keeps the inclusion map a surjection.

This should be clear. For the base case, we note that X^1 along with the next cell to be attached e_1 satisfy the hypothesis for the main proof of this problem, and so the map $i_* : \pi_1(X^1) \to \pi_1(X_1)$ is surjective (here X_1 is just $X^1 \cup e_1$, and X_k will denote the space obtained by attaching the first k cells to X^1).

Now, suppose the proposition holds for X_k. That is, the inclusion $i_* : X^1 \to X_k$ is surjective. Then, X_k along with the next cell e_{k+1} satisfy the hypotheses for the main proof of this problem, and the map $i_{k*} : \pi_1(X_k) \to \pi_1(X_{k+1})$ is surjective. Then, the composition $i_{k*} \circ i_*$ which is the inclusion of X^1 into X_{k+1} is surjective as well.

Thus, if a loop is in a finite number of cells, it is homotopic to a loop on the 1-skeleton. However, Proposition $A.1$ guarantees that any loop in X (which is compact since it is the image of the interval) only passes through finitely many cells. So, every loop in X is homotopic to a loop in X^1, and thus the inclusion map $i_* : \pi_1(X^1) \to \pi_1(X)$ is surjective as desired. $\square$

Problem extra

Find the standard form of $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6$, and prove or disprove:

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6 \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

Proof. We first find the standard form of $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6$. Now, it is a standard result that $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$, and so

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6 \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2$$

and by commutativity of the product in *Ab* we can write this in standard form as

$$\begin{aligned} \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6 &\cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \\ &\cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \\ &= \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \end{aligned}$$

Now, to show that this group is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, we just need to compute the standard form. But $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is already in standard form, as

$$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 = \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

and since both groups have different standard forms, they are not isomorphic. $\square$