Problem Set 5

Daniel Halmrast

February 26, 2018

1 Problem 1

Show that for a Banach space X, X^* is weak-$*$ sequentially complete. Formulate an analagous statement for X being weak sequentially complete, and show where the proof collapses.

Proof. We first show that X^* is weak-$*$ sequentially complete. So, let ϕ_j be a sequence for which $\langle \phi_j, x\rangle$ converges for all $x \in X$. We can define the limit to be $\phi := x \mapsto \lim \phi_j(x)$. Now, we just have to show that ϕ is linear and bounded.

Linearity of ϕ follows directly from its definition, and the linearity of limits. That is, for $x, y \in X$,

$$\begin{aligned}\phi(x+y) &= \lim \phi_j(x+y)\\ &= \lim(\phi_j(x) + \phi_j(y))\\ &= \lim \phi_j(x) + \lim \phi_j(y)\\ &= \phi(x) + \phi(y)\end{aligned}$$

and for $\alpha \in \mathbb{C}$,

$$\begin{aligned}\phi(\alpha x) &= \lim \phi_j(\alpha x)\\ &= \lim \alpha\phi_j(x)\\ &= \alpha \lim \phi_j(x)\\ &= \alpha\phi(x)\end{aligned}$$

and thus ϕ is linear.

Now, we just need to show that ϕ is bounded. This is clear from the uniform boundedness principle applied to the family ϕ_j of operators from X to $\mathbb{C}$. That is, since for each $x \in X$, we have that

$$\sup_j \|\phi_j(x)\| < \infty$$

(because $\phi_j(x)$ converges for each fixed x), it follows then by the principle of uniform boundedness that

$$\sup_j\{\|\phi_j\|\} < \infty$$

as well. Let C be the uniform bound on the norms of ϕ_j. Then, it follows that for all x and for all j,

$$\|\phi_j(x)\| \leq C\|x\|$$

Thus, it follows that the limit is also bound by this inequality

$$\|\phi(x)\| = \|\lim \phi_j(x)\| = \lim \|\phi_j(x)\| \leq C\|x\|$$

and ϕ is bounded as desired.

Thus, ϕ_j converges in weak-$*$ to ϕ, which is in X^* as we just proved.

Now, the analagous statement for X being weak sequentially complete is as follows: Let x_j be a sequence in X for which $\phi(x_j)$ converges for all $\phi \in X^*$. For X to be weak sequentially complete, there must be some $x \in X$ for which x_j converges weakly to x.

The proof fails when we attempt to apply PUB. In particular, we can define the limit of x_j by treating it as a sequence of linear functionals on X^*, but we are not guaranteed that the limit of this sequence stays in the embedding of X to X^{**}. For example, taking $X = L^1(\mathbb{R})$, and $x_j = j\chi_{[-\frac{1}{j},\frac{1}{j}]}$, this converges weakly to the delta distribution, which is not an L^1 function. □

2 Problem 2

Let X be Banach. Prove that a sequence ϕ_j in X^* converges weak-$*$ if and only if it is strongly bounded and there exists a dense subset $E \subset X$ for which $\phi_j(u)$ converges for all $u \in X$.

Proof. ($\Longrightarrow$) Suppose first that $\phi_j \to \phi$ in weak-$*$. Then, the principle of uniform boundedness guarantees that the set $\{\phi_j\}$ is strongly bounded. Furthermore, since $\phi_j(x) \to \phi(x) < \infty$ for all x, setting $E = X$ finishes the proof.

($\Longleftarrow$) Suppose ϕ_j is such that it is strongly bounded, and there exists a dense subset $E \subset X$ for which $\phi_j(u)$ converges for all $u \in E$. All we need to show is that $\phi(x)$ converges for all $x \in X$, and by problem 1 we will know that ϕ_j converges weakly to some $\phi \in X^*$.

So, we will show that $\phi_j(x)$ is a Cauchy sequence for all x. So, fix $x \in X$, and let u_n be a sequence in E converging to x. Then, for fixed $\epsilon > 0$, fix m so that $|u_m - x| < \epsilon$, and fix N so that for all $j, k > N$, $|\phi_j(u_m) - \phi_k(u_m)| < \epsilon$. Finally, let C be the uniform bound on $\|\phi_j\|$. Now, we have that

$$\begin{aligned}
|\phi_j(x) - \phi_k(x)| &= |\phi_j(x) - \phi_j(u_m) + \phi_j(u_m) - \phi_k(x) + \phi_k(u_m) - \phi_k(u_m)| \\
&\leq |\phi_j(x) - \phi_j(u_m)| + |\phi_k(x) - \phi_k(u_m)| + |\phi_j(u_m) - \phi_k(u_m)| \\
&= |\phi_j(x - u_m)| + |\phi_k(x - u_m)| + |\phi_j(u_m) - \phi_k(u_m)| \\
&\leq (\|\phi_j\| + \|\phi_k\|)|x - u_m| + \epsilon \\
&\leq (\|\phi_j\| + \|\phi_k\|)\epsilon + \epsilon \\
&\leq 2C\epsilon + \epsilon
\end{aligned}$$

and so the sequence is Cauchy, as desired. Thus, $\phi_j(x)$ converges for all x, and so ϕ_j converges in weak-$*$ as desired. □

Problem 3

Let $d\phi_n(x) = \cos(\pi n x) d\lambda^1(x)$. Prove that

$$\int_I g d\phi_n \to 0$$

for all $g \in C^1(I)$. Prove that ϕ_n converges in weak-$*$ as measures in $C(I)^*$.

Proof. For the first part, we integrate by parts to find that

$$\begin{aligned}
|\int_I g\cos(n\pi x)dx| &= |\frac{g\sin(\pi n x)}{n\pi}|_{\partial I} - \int_I \frac{\sin(\pi n x)}{n\pi} g' dx| \\
&= |\frac{1}{\pi n}\int_I \sin(\pi n x) g' dx| \\
&\leq |\frac{1}{n\pi}\int_I g' dx|
\end{aligned}$$

which clearly goes to zero, since $\int_I g' dx$ is constant.

For the second part, note that the Weierstrauss approximation theorem says that the polynomials on I are dense in $C(I)$. Now, since every polynomial is in $C^1(I)$, we know that for every polynomial,

$$\int_I p(x) d\phi_n(x) \to 0$$

from the first part of this problem. Furthermore, $\{\phi_n\}$ is strongly bounded, since

$$\|\phi_n(f)\| = |\int_I f(x)\cos(n\pi x)dx| \leq |\int_I f(x)dx| = \|f\|$$

so applying problem 2, we see that $\phi_n \to 0$ weakly, since they are strongly bounded, and converge on a dense subset of $C(I)$. □

Problem 4

Let (x_n) be a sequence in ℓ^1 such that $x_n \to x$ weakly, and $\|x_n\| \to \|x\|$. Prove that $x_n \to x$ strongly.

Proof. For notation, we will use $f_n := x_n$ and $f := x$. Now, we first show that $f_n \to f$ pointwise. This is clear, since $f_n \to f$ weakly implies that for each $y \in \ell^\infty$, $y(f_n) \to y(f)$. Specifically, by letting y be the basis vectors $e_m = \delta(x-m)$, we see that

$$e_m(f_n) = f_n(m) \to e_m(f) = f(m)$$

for all m. Thus, f_n converges pointwise to f.

Now, we let $g_n = |f| + |f_n| - |f_n - f|$. Note that g_n is positive by the triangle inequality. Now, Fatou's lemma implies that

$$\int \liminf g_n \leq \liminf \int g_n$$

and so

$$\begin{aligned}
\int \liminf(|f| + |f_n| + |f_n - f|) &\leq \liminf \int |f| + \liminf \int |f_n| + \liminf(-\int |f_n - f|) \\
\int (|f| + |f|) &\leq |f| + |f| - \limsup(\int |f_n - f|) \\
0 &\leq -\limsup \|f_n - f\|
\end{aligned}$$

and since $\|f_n - f\|$ is always positive, $-\limsup \|f_n - f\| \leq 0$ which forces $\limsup \|f_n - f\| = 0$ and thus f_n converges strongly to f. □

Problem 5

Show that the closed unit ball $\bar{B}(X)$ in a Banach space X is closed in the weak topology. Show that the closed unit ball $\bar{B}(X^*)$ in X^* is closed in the weak-$*$ topology.

Proof. In particular, we have to show that any net in $\bar{B}(X)$ that converges, converges in $\bar{B}(X)$. So, let x_α be a net in $\bar{B}(X)$, and suppose $x_\alpha \to x$ in the weak topology. We know, then, that for all $\phi \in X^*$, $\phi(x_\alpha) \to \phi(x)$. In particular,

$$|\phi(x_\alpha)| \to |\phi(x)|$$

for all ϕ. Thus,

$$|\phi(x_\alpha)| \leq \|\phi\| \|x_\alpha\| \leq \|\phi\|$$

and since $|\phi(x_\alpha)| \to |\phi(x)|$, it follows that

$$|\phi(x)| \leq \|\phi\|$$

Now, we also know that

$$\|x\| = \sup_{\|\phi\|=1} |\phi(x)|$$

and since $|\phi(x)| \leq 1$ for $\|\phi\| = 1$, we know that

$$\|x\| \leq 1$$

as well. Thus, any limit point of a net in $\bar{B}(X)$ is also in $\bar{B}(X)$, and $\bar{B}(X)$ is closed in the weak topology.

The second part follows in exactly the same way as the first, by noting that

$$\|\phi\| = \sup_{\|x\|=1} |\phi(x)|$$

and that the weak-$*$ topology is the weak topology with respect to X. $\square$