Homework 7

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June 7, 2018

Problem 1

Suppose G is a locally compact abelian group, with μ the Haar measure on G. If G is not discrete and $\varepsilon > 0$ is fixed, find a compact neighborhood K of the identity 0 with $\mu(K) < \varepsilon$.

Proof. We first observe that $\{0\}$ has measure zero. This follows from the observation that every open set in G has infinitely many points, and that the measure μ is translation-invariant. First, we show that every open set has infinitely many points.

Suppose for a contradiction that there is an open set $U \subset G$ with finitely many points $\{x_i\}_{i=1}^n$. Then, since G is Hausdorff, we can choose neighborhoods V_i of x_1 such that $V_i \subset U$ and $x_i \notin V_i$. Taking the intersection of all V_i yields an open set with only x_1 in it, which contradicts G being non-discrete (since we could translate the open set $\{x_1\}$ to open sets around any singleton, and so every singleton would be open).

Now, if $\{0\}$ has positive measure, then every singleton would have positive measure, and since every open set is infinite, each open set would have infinite measure, which cannot happen since μ is a Borel-regular measure.

Thus, $\mu(\{0\}) = 0$. Since μ is outer-regular, we know that

$$\mu(\{0\}) = 0 = \inf_{U \supset \{0\} \text{ open}} \mu(U)$$

and so for each $\varepsilon > 0$, we can find a U open around 0 for which $\mu(U) < \varepsilon$.

Since μ is also inner-regular with respect to open sets, we know that

$$\mu(U) = \sup_{K \subset U \text{ compact}} \mu(K)$$

and so we can find a compact neighborhood K of $\{0\}$ with $\mu(K) > 0$. Since $K \subset U$, we know $\mu(K) \leq \mu(U) < \varepsilon$ as desired. $\square$

Problem 2

Suppose G is a locally compact abelian group. If G is not discrete, $\varepsilon > 0$ and G is Lindelof, prove there is an open neighborhood U of 0 for which $U + U = G$ and $\mu(U) < \varepsilon$.

Proof. For ease of notation (and without loss of generality),we will find a neighborhood U of measure less than 2ε instead.

Let U_0 be a neighborhood of 0 as in the previous problem with $\mu(U) < \varepsilon$. Then we note that

$$G = \bigcup_{g \in G} g + U_0$$

and so the set $\{g + U_0\}$ is an open cover of G. Thus, it has a finite subcover $\{g_i + U_0\}_{i=1}^{\infty}$.

For each g_i, let U_i be constructed as in problem 1 with $\mu(U_i) < \frac{\varepsilon}{2^i}$. I assert that the open set $U = U_0 \cup \bigcup_{=1}^{\infty} U_i$ is the open neighborhood of 0 we desire.

First, notice that

$$\begin{aligned} \mu(U) &\leq \mu(U_0) + \sum_{i=1}^{\infty} \mu(U_i) \\ &\leq \varepsilon + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \\ &= 2\varepsilon \end{aligned}$$

as desired. Next, note that from before

$$G = \bigcup_{i=1}^{\infty} g_i + U_0$$

and so for all $g \in G$, we have that there is some i and some $u \in U_0$ for which

$$g = g_i + u$$

and since $g_i, u \in U$, we have that $g \in U + U$ as desired.

Thus, $U + U = G$. $\square$

Problem 3

(For this problem, I collaborated with Ashwin Trisal, Daniel Epelbaum, Aaron Bagheri, Micah Pedrick, and Andre Matrins.)

Suppose G is a locally compact abelian group which is also compact, and let $F \subset G$ open such that $F + F = G$. Consider the action of G on $L^2(G)$ by

$$g(f(x)) = f(g + x)$$

for $g, x \in G$, $f \in L^2(G)$.

Part A

Show that this action is unitary.

Proof. To show this action is unitary, we need to show it is invertible, and that it preserves inner products.

Observe first that this group action is indeed an action, in the sense that it respects the group operation. This follows immediately from the definition, since

$$(g + h)(f(x)) = f(g + h + x) = g \circ h(f(x)))$$

So, the action of g is invertible by $-g$, since

$$(g + (-g))(f(x)) = 0(f(x)) = f(x)$$

Now, we need to show that the action of g preserves inner products. So, let $f, h \in L^2(G)$, and let $g \in G$. We calculate

$$\begin{aligned} \langle g(f), g(h) \rangle &= \int_G f(g + x)\overline{h(g + x)} d\mu(x) \\ &= \int_G f(x)\overline{h(x)} d\mu(x - g) \\ &= \int_G f(x)\overline{h(x)} d\mu(x) = \langle f, h \rangle \end{aligned}$$

as desired. Here, we used the fact that the measure is translation-invariant, and so $\mu(x - g) = \mu(x)$. $\square$

Part B

Let p be the projection in $B(L^2(G))$ onto the subspace $L^2(F)$. Let

$$\begin{aligned} \phi : G &\to B(L^2(G)) \\ g &\mapsto p \circ U_g \circ p \end{aligned}$$

where U_g is the unitary operator defined by the action of g.

Part I

is $\phi(G)$ a subgroup of $B(L^2(G))$?

Proof. If $F = G$, then $\phi(g) = U_g$ and ϕ is indeed a group homomorphism, so $\phi(G)$ is a subgroup as desired. So, assume $F \neq G$.

I assert $\phi(G)$ is not a subgroup of $B(L^2(G))$. This follows from the fact that for $g = 0$, we have $\phi(0) = p \circ \mathbb{1} \circ p = p$ is not invertible. Thus, $\phi(G)$ contains a non-invertible element, and cannot be a group. $\square$

Part II

Is $\phi(G)$ compact in any of the topologies studied in class?

Proof. I assert that $\phi(G)$ is compact in the weak operator topology (WOT) as well as the strong operator topology (SOT).

To see this, we will prove that ϕ is continuous into $B(L^2(G))$ in the SOT.

Recall that the SOT is the initial topology with respect to the evaluation functions

$$\begin{aligned} \mathrm{ev}_f : B(L^2(G)) &\to L^2(G) \\ A &\mapsto A(f) \end{aligned}$$

for all $f \in L^2(G)$. So, by the definition of the initial topology, ϕ is continuous if and only if $\mathrm{ev}_f \circ \phi$ is continuous for all f.

Now,

$$\mathrm{ev}_f \circ \phi(g) = p \circ U_g \circ p(f)$$

Consider the function

$$\begin{aligned} \varphi_f : G &\to L^2(G) \\ g &\mapsto U_g \circ p(f) \end{aligned}$$

Now, $p(f)$ is a (fixed) function in $L^2(G)$, and we know that the map $g \mapsto U_g$ is continuous (Rudin's Fourier Analysis on Groups proves thisi, theorem 1.1.5), so φ_f is continuous as well.

Now, $\mathrm{ev}_f \circ \phi = p \circ \varphi_f$ is the composition of continuous functions (as projections are continuous), and so $\mathrm{ev}_f \circ \phi$ is continuous for all f. Thus, ϕ is continuous in the SOT, and $\phi(G)$ as the image of a compact set is compact.

Since the WOT is strictly weaker than the SOT, this also implies that ϕ is continuous in the WOT, and so $\phi(G)$ is compact in the WOT as well. $\square$

Part III

Is ϕ injective?

Proof. Suppose that F is such that for all $g \in G$, there is some $f \in F$ for which $g + f \in F$. Then, I assert ϕ is injective.

To see this, suppose $g_1, g_2 \in G$. Now, there is an open set $U \subset F$ containing f as specified above (so that $g_1 + f \in F$) with $g_1 + U \subset F$ and $g_2 + U \cap g_1 + U = \emptyset$. This follows from G being Hausdorff.

Now, consider $\chi_U \in L^2(G)$. We have that

$$\begin{aligned} \phi(g_1)(\chi_U) = p \circ U_g(\chi_U) = p(\chi_{g_1+U}) &= \chi_{F \cap g_1 + U} \\ phi(g_2)(\chi(U)) &= \chi_{F \cap g_2 + U} \end{aligned}$$

but $g_1 + U$ does not intersect $g_2 + U$, and $F \cap g_1 + U = g_1 + U \neq \emptyset$, so $\phi(g_1) \neq \phi(g_2)$ for all $g_1, g_2 \in G$, and ϕ is injective. $\square$

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