Problem Set 6

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Problem 2

Show that the closure of the ball $B(a,r)$ is the closed ball $\overline{B}(a,r) = \{x \mid |x-a| \leq r\}$

Proof. Suppose x is in $\overline{B}(a,r)$. Then, consider the sequence

$$(x_n) = (x-a)(1-\frac{1}{n}) + a$$

Clearly, this sequence converges to x, and each term is in $B(a,r)$. To see this, we observe that

$$\begin{aligned} ||x_n - a|| &= ||(x-a)(1-\frac{1}{n}) + a - a|| \\ &= ||x-a||(1-\frac{1}{n} \\ &\leq r(1-\frac{1}{n} \\ &\leq r \end{aligned}$$

Thus, each point in $\overline{B}(a,r)$ is a limit point of $B(a,r)$, and since $\overline{B}(a,r)$ is closed, it follows that it is the closure of $B(a,r)$ (since the closure of $B(a,r)$ is the smallest closed set containing it.) □

Problem 10

Show that ℓ^1 is not complete in the ℓ^∞ norm.

Proof. We will show that the sequence $(x_m)_k = (\frac{1}{m^{1+\frac{1}{k}}})$ converges to the function $x_m = \frac{1}{m}$, which is not in ℓ^1, even though each term in the sequence is in ℓ^1.

To show convergence, we wish to show that the functions

$$f_k(n) = \frac{1}{n^{1+\frac{1}{k}}}$$

converges uniformly to $f(n) = \frac{1}{n}$.

To do so, we will consider instead the extended functions

$$f_k(x) = x^{1+\frac{1}{k}},\ x \in [0,1]$$

which clearly converges pointwise to $f(x) = x$. Now, since f is defined on a compact domain, it must also uniformly converge to $f(x) = x$. Thus, the restriction $f_k(x)|_{\{\frac{1}{n}\}}$ also converges uniformly to $f(n) = \frac{1}{n}$.

Thus,the sequence of sequences $(x_m)_k$ converges uniformly (in ℓ^∞) to (x_m) as desired. Furthermore, since each $(x_m)_k$ is in ℓ^1, but (x_m) is not, it follows that ℓ^1 is not complete in the ℓ^∞ norm. $\square$