Homework 3

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April 24, 2018

Problem 1

Prove that for X a compact metric space, the multiplicative linear functionals on $C(X)$ are exactly the point evaluation functionals

$$\delta_x(f) = f(x)$$

Proof. We first establish the following result:

Lemma 1. *For every multiplicative linear functional ϕ on a unital Banach algebra $\mathscr{A}$, the kernel of ϕ is a maximal ideal in $\mathscr{A}$. Conversely, every maximal ideal in $\mathscr{A}$ is the kernel of some multiplicative linear functional.*

Proof. Let ϕ be a multiplicative linear functional on $\mathscr{A}$. We know that $\ker(\phi)$ is a closed ideal in $\mathscr{A}$, since it is the kernel of an algebra homomorphism. Furthermore, this ideal is maximal. This follows from the fact that $\operatorname{im}(\phi) = \mathbb{C} \cong \mathscr{A}/\ker(\phi)$, which has dimension one (here, we used the fact that $\phi \neq 0$, since the zero functional is not multiplicative since it has to send I to 1).

That is, we have shown that for ϕ a multiplicative linear functional on $\mathscr{A}$, $\ker(\phi)$ is a maximal ideal in $\mathscr{A}$.

Conversely, suppose $\mathscr{M}$ is a maximal ideal of $\mathscr{A}$. We examine the space $\mathscr{A}/\mathscr{M}$. Specifically, we show that for each nonzero $X + \mathscr{M} \in \mathscr{A}/\mathscr{M}$, $X + \mathscr{M}$ is invertible. This follows from the fact that the ideal

$$\mathscr{J}_X = \{AX + Y \mid A \in \mathscr{A}, Y \in \mathscr{M}\}$$

properly contains

$$\mathscr{M} = \{0X + Y \mid Y \in \mathscr{M}\}$$

and so $\mathscr{J}_X = \mathscr{A}$ by maximality of $\mathscr{M}$. Thus, there is some $A \in \mathscr{A}$ and $Y \in \mathscr{M}$ with

$$AX + Y = I$$

and so $X + \mathscr{M}$ is invertible. We finally observe that this implies that $\mathscr{A}/\mathscr{M} \cong \mathbb{C}$ isometrically. This can be seen directly. For ease of notation, we denote $X := X + \mathscr{M} \in \mathscr{A}/\mathscr{M}$. Now, we know that

$$\sigma(X) \neq \emptyset$$

However, since each $X \in \mathscr{A}/\mathscr{M}$ that is nonzero is invertible, the spectrum can contain at most one element. This is because at most one of

$$X - \lambda_1 I$$
$$X - \lambda_2 I$$

is zero, and the other must be invertible. Thus, $\sigma(X) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$. The map $\Phi : \mathscr{A}/\mathscr{M} \to \mathbb{C}$ given by

$$\Phi(X) = \lambda \in \sigma(X)$$

is easily seen to be a bijective multiplicative linear isometry.

Putting it all together, let $q : \mathscr{A} \to \mathscr{A}/\mathscr{M}$ be the canonical quotient map. Then, the map

$$\Phi \circ q : \mathscr{A} \to \mathbb{C}$$

is a multiplicative linear functional with kernel $\mathscr{M}$, as desired. □

With this lemma, the problem is easy. To characterize the multiplicative linear functionals on $C(X)$, we just need to characterize its maximal ideals. Specifically, we will show that the maximal ideals of $C(X)$ are

$$\mathscr{M}_x = \{f \in C(X) \mid f(x) = 0\}$$

That is, $\mathscr{M}_x$ is the set of functions that vanish at x.

We first show that $\mathscr{M}_x$ is maximal (the fact that it is an ideal is clear). To see this, suppose $\mathscr{I}$ is another ideal containing $\mathscr{M}_x$ with $\mathscr{I} \neq \mathscr{M}_x$. Then, there is some $f \in \mathscr{I}$ with $f(x) > 0$. Now, we also know that there is some $g \in C(X)$ with $g^{-1}(\{0\}) = \{x\}$. That is, g vanishes only at x. We can also force $g(y) > 0$ for all $y \neq x$.

Thus $f, g \in \mathscr{I}$, and thus so is $f + g$. Furthermore, by construction $f + g \neq 0$, and so $\frac{1}{f+g}$ is well-defined. Thus, $f + g$ has an inverse in $C(X)$, and since $f + g \in \mathscr{I}$, $\mathscr{I} = C(X)$ and $\mathscr{M}_x$ is a maximal ideal, as desired.

We can also show that these are the only maximal ideals. Suppose $\mathscr{I}$ is a maximal ideal such that for each $x \in X$, there is some $f_x \in \mathscr{I}$ with $f_x(x) = 0$. Since each f_x is continuous, there is a neighborhood U_x around x for which f_x is nonzero in U_x. This forms an open cover of X, which has a finite subcover indexed by x_i. Now, take the function

$$F = \sum_{i=1}^{n} (f_{x_i})^2$$

which is a finite sum and product of things in $\mathscr{I}$, and is thus in $\mathscr{I}$. However, $F(y) \neq 0$ for all $y \in X$, and so $F(y)$ is invertible. Thus, $\mathscr{I} = C(X)$.

Thus, all maximal ideals of $C(X)$ are of the form $\mathscr{M}_x$. Each multiplicative linear functional ϕ_x, then, has kernel $\mathscr{M}_x$ and is thus of the form

$$\phi_x(f) = f(x)$$

as desired. □

Problem 2

Prove that these functionals are exactly the extreme points of K, the positive part of the unit ball in $C(X)^*$.

Proof. We first show that these are extreme points of K. To see this, suppose $\psi_1, \psi_2 \in K$ with

$$\mathrm{ev}_x = \phi_x = t\psi_1 + (1-t)\psi_2$$

We wish to show $\psi_1 = \psi_2 = \phi_x$. To do so, we invoke the Riesz-Markov theorem to translate into a statement about measures. That is, the statement above is equivalent to

$$\delta_x = t\mu_1 + (1-t)\mu_2$$

where we know that $\|\mu_1(X)\| = \|\mu_2(X)\| = 1$. However, this means that for all $E \subset X$,

$$\delta_x(E) = t\mu_1(E) + (1-t)\mu_2(E)$$

which, when considering the cases $x \in E$ and $x \notin E$, we see that $\mu_1 = \mu_2 = \delta_x$, and thus ev_x is an extreme point.

Next, we show that these are all the extreme points. To see this, suppose $\mu \in C(X)^*$ with $\mu \neq \delta_x$ for any x. In particular, we know that we can find $S_1, S_2 \subset X$ such that $X = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$, and $\mu(S_1), \mu(S_2) > 0$. Then, we have

$$\mu = \frac{\mu(S_1)}{\mu(X)} \left(\frac{\mu(X)}{\mu(S_1)} \chi_{S_1} \mu \right) + \frac{\mu(S_2)}{\mu(X)} \left(\frac{\mu(X)}{\mu(S_2)} \chi_{S_2} \mu \right)$$

where $\frac{\mu(S_1)}{\mu(X)} + \frac{\mu(S_2)}{\mu(X)} = 1$, and each term in the convex linear combination is in K. Thus, μ is not an extreme point, as desired. $\square$

Problem 3

Find all the two-dimensional faces of K.

Proof. I assert that the two-dimensional faces of K are the convex linear combinations of any three extreme points.

First, we observe that

$$F = \{a\delta_x + b\delta_y + c\delta_z \mid a + b + c = 1\}$$

is indeed a face. To see this, note that by definition F is convex, and with three degrees of freedom and one constraint, it is two-dimensional. Now, we just need to show it is closed under linear interpolation. So, suppose

$$a\delta_x + b\delta_y + c\delta_z = t\phi + (1-t)\psi$$

for $\phi, \psi \in K$. By utilizing Riesz-Markov theorem, we know that this equation must hold for the induced measures as well. So, we have that for all measurable E,

$$a\delta_x(E) + b\delta_y(E) + c\delta_z(E) = t\phi(E) + (1-t)\psi(E)$$

Let E be such that $x, y, z \notin E$. Then,

$$0 + 0 + 0 = t\phi(E) + (1-t)\psi(E)$$

which forces $\phi(E) = \psi(E) = 0$. Finally, observing that since $\phi, \psi \in K$, we must have $|\phi| = |\psi| = 1$ and so ϕ and ψ must be linear combinations of $\delta_x, \delta_y, \delta_z$ whose coefficients add up to 1, and are thus in F as desired. Thus, F is a face.

Now, we claim that these are the only faces. This follows from the Krein-Milman theorem, which states that a compact and convex set is the convex hull of its extreme points. Now, under the weak-$*$ topology, the unit ball is compact, and since K is a closed subset of the unit ball, it is compact as well. Furthermore, any closed face will also be compact.

So, suppose F is a closed face of K. Then, F is a convex hull of its extreme points, which will be some collection of point-mass measures. However, if this collection has $n > 3$ measures, the resulting space will be $n - 1$ dimensional, which is greater than two. Thus, the only two-dimensional faces of K are the convex hulls of three extreme points of K.

$\square$

Problem 4

Let B_1^+ be the set $\{f \in L^1(\mathbb{R}) \mid f(x) \geq 0 \forall x, \int_{\mathbb{R}} f = 1\}$. Find the extreme points of B_1^+.

Proof. I claim that there are no extreme points of B_1^+. To see this, suppose $f \in B_1^+$. In particular, this means that for some positive-measure set E, $f|_E > 0$. Now, split E into two sets E_1, E_2 with equal positive measure. Set $\varepsilon > 0$ such that $f - \varepsilon > 0$ on E.

Define

$$g_\pm(x) = \begin{cases} f(x), & x \in E^c \\ f(x) \pm \varepsilon, & x \in E_1 \\ f(x) \mp \varepsilon, & x \in E_2 \end{cases}$$

clearly, $\int g_\pm = 1$, and furthermore,

$$f(x) = \frac{1}{2} g_+(x) + \frac{1}{2} g_-(x)$$

so f is not an extreme point. $\square$

Problem 5

Find the extreme points of F_1^+, the set of all positive $n \times n$ self-adjoint complex matrices with trace 1.

Proof. I assert that all the extreme points of F_1^+ are the one-dimensional projection operators.

First, observe that $F_1^+ \subset M_1^+$, where M_1^+ from last homework is the set of all positive, self-adjoint $n \times n$ matrices less than I. This is clear, since for $M \in F_1^+$, we know that $\sigma(M) \subset [0, 1]$ since the trace of M (the sum of the eigenvalues) is 1, and since M is positive, it has all positive eigenvalues. Thus, its eigenvalues are positive and sum to one, and thus must be between zero and one. This is the condition necessary to be in M_1^+, as desired.

Thus, if M is an extreme point of M_1^+, and $M \in F_1^+$, then M is an extreme point of F_1^+ as well. We noted that the projections are extreme points in M_1^+, and specifically the one-dimensional projections are in F_1^+. Thus, they are extreme points.

We next observe that these are the only extreme points. To see this, suppose $M \in F_1^+$ with M not a one-dimensional projection. Since the eigenvalues of M add up to one, and M is not a one-dimensional projection, M must have at least two eigenvalues less than one. So, write M as

$$M = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2| + \sum_{i=3}^{n} \lambda_i |e_i\rangle\langle e_i|$$

and let $\varepsilon > 0$ be such that $0 < \lambda_1 \pm \varepsilon < 1$ and $0 < \lambda_2 \pm \varepsilon < 1$. Then, define

$$M_\pm = (\lambda_1 \pm \varepsilon)|e_1\rangle\langle e_1| + (\lambda_2 \mp \varepsilon)|e_2\rangle\langle e_2| + \sum_{i=3}^{n} \lambda_i |e_i\rangle\langle e_i|$$

which are clearly in F_1^+. We finally observe that

$$M = \frac{1}{2}M_+ + \frac{1}{2}M_-$$

and so M is not an extreme point, as desired. $\square$