Problem Set 3

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Problem 1

Enumerate all subcomplexes of S^∞ with the cell structure on S^∞ that has S^n as the n-skeleton.

Proof. We notice at first that each n-skeleton is a subcomplex, and so S^n is a subcomplex of S^∞ for each n.

There is another subcomplex in each dimension. Namely, by omitting one of the n-cells attaching to the $n-1$-skeleton, we obtain another subcomplex in the nth dimension that is the $n-1$ skeleton along with a single n-cell attached in the usual way. In fact, depending on which n-cell we omit, we can obtain two different subcomplexes.

So far, we have three subcomplexes in each dimension. I assert that this is all the subcomplexes. Suppose there existed a subcomplex in n dimensions that did not contain the entire $n-1$-skeleton. In particular, this means that the attaching map of the n-cell, which is bijective from ∂D^n onto the entire $n-1$-skeleton, is not well-defined, and so no such subcomplex can be constructed. $\square$

Problem 2

Show S^∞ is contractible.

Proof. We will show that the n-skeleton of $X = S^\infty$ is contractible in X^{n+1}. To see this, consider the subcomplex X^n along with a single disk D^{n+1} attached in the usual way. In particular, X^n is identified with ∂D^{n+1}, and since D^{n+1} is contractible, it follows that ∂D^{n+1} contracts to a point in D^{n+1}.

Thus, each X^n is contractible in X, and by running all of them sequentially (say, running the nth homotopy in time $[2^{-n}, 2^{-(n+1)}]$) we obtain a contraction of S^∞. $\square$

Problem 3

Show that $S^1 \star S^1 = S^3$. In general, show $S^m \star S^n = S^{m+n+1}$.

Proof. We will prove the more general result that

$$\star_{i=1}^{n} S^0 = S^{n-1}$$

along with the associativity of $\star$.

To see the first result, we will use the interpretation of the star product as the set of all convex formal linear combinations of the two spaces. In particular, interpreting S_i^0 to be the two points $1, -1$ on the ith coodinate axis in $\mathbb{R}^n$, we see that join across all n copies of S^0 is really all points $x = (x^1, \ldots, x^n)$ satisfying

$$x^i \leq 1 \text{ and } x^i \geq -1$$
$$\sum_i x^i = 1$$

which is just the convex hull of the $2n$ unit vectors along the coordinate axes of $/R^n$. In other words, it is the ball of radius 1 in the 1-norm. However, this is obviously homeomorphic to $S^n - 1$ as desired.

Now, we will show that $\star$ is associative. However, this falls almost immediately from the definition of $\star$ in terms of formal linear combinations.

Thus, $S^n = \star_{i=1}^{n+1} S^0$, and so

$$\begin{aligned} S^m \star S^n &= (\star_{i=1}^{m+1} S^0) \star (\star_{i=1}^{n+1} S^0) \\ &= \star_{i=1}^{m+n+2} S^0 \\ &= S^{m+n+1} \end{aligned}$$

as desired. □

Problem 4

Show that the space obtained by attaching n 2-cells along any collection of n circles in S^2 is homotopy equivalent to the wedge sum of $n+1$ 2-spheres.

Proof. Consider the (potentially disconnected) graph formed by the n circles on the surface S^2. This graph in particular can be homotoped to a tree relative to the points of intersection (the nodes of the graph). This is done by taking any loop (two distinct intervals connected on both endpoints) and sliding the two intervals to meet each other. After obtaining a tree, the tree can be homotoped to a point,

Now, the key thing about these homotopies is that they define a homotopy on the space obtained by attaching 2-cells along the n circles on S^2 as well. This is clear, since the first homotopy does not move points of intersection (so as not to have discontinuities). Thus, this space obtained by attaching the n 2-cells to S^2 is homotopy equivalent to the space attaching n 2-cells to a point, which is just the wedge sum of $n+1$ copies of S^2 as desired. □

Problem 5

Show that the subspace X of $\mathbb{R}^3$ formed by a self-intersecting Klein bottle is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.