Final Exam

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Problem 1

For every $n \in \mathbb{N}$, let μ_n be a measure on $(\Omega, \mathscr{A})$ with $\mu_n(\Omega) = 1$. For every $E \in \mathscr{A}$, define

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n}$$

Give a careful proof that μ is a measure on $(\Omega, \mathscr{A})$ with $\mu(\Omega) = 1$.

Proof. We wish to prove that μ is a measure on $(\Omega, \mathscr{A})$. That is, we wish to show that that $\mu(\emptyset) = 0$, that $\mu(E) \geq 0$ for all $E \in \mathscr{A}$, and that for a countable collection of disjoint sets $\{E_j\}_{j=1}^{\infty}$ for which $E_j \in \mathscr{A}$ for all j,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

To begin with, we note that since each μ_n is a measure, we have that $\mu_n(\emptyset) = 0$. Thus,

$$\begin{aligned}\mu(\emptyset) &= \sum_{n=1}^{\infty} \frac{\mu_n(\emptyset)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{0}{2^n} \\ &= 0\end{aligned}$$

as desired.

Next, we note that since each μ_n is a measure, $\mu_n(E) \geq 0$ for all $E \in \mathscr{A}$. Thus, since both $\mu_n(E)$ and 2^n are greater than zero for each n, it must be that

$$\mu(E) = \sum_{n=1}^{\infty} \frac{\mu_n(E)}{2^n} \geq 0$$

as desired.

To show that μ is countably additive, we first prove the following lemma:

Lemma. *For a doubly indexed sequence $\{a_{ij}\}$ of positive numbers,*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

Proof. We note first that a_{ij} can be thought of as a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$.

Now, Tonelli's theorem tells us that for any positive function $f : \Omega \times \Sigma \to \mathbb{R}$ on the product space $\Omega \times \Sigma$ of σ-finite measure spaces $(\Omega, \mathscr{A}, \mu)$ and $(\Sigma, \mathscr{B}, \nu)$ such that f is measurable with respect to $\mathscr{A} \otimes \mathscr{B}$, we have that

$$\int_{\Omega}\left(\int_{\Sigma} f(x,y)d\nu(y)\right)d\mu(x) = \int_{\Sigma}\left(\int_{\Omega} f(x,y)d\mu(x)\right)d\nu(y)$$

Now, consider the case where $\Omega = \Sigma = \mathbb{N}$, $\mathscr{A} = \mathscr{B} = 2^{\mathbb{N}}$, and $\mu = \nu = \mu_c$ the counting measure. The function a_{ij} from $\mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is positive (by hypothesis), and is measurable on $2^{\mathbb{N}} \otimes 2^{\mathbb{N}} = 2^{\mathbb{N}\times\mathbb{N}}$, since every function is measurable with respect to this σ-algebra. Thus, applying Tonelli's theorem yields

$$\begin{aligned}
\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) &= \int_{\mathbb{N}}\left(\int_{\mathbb{N}} a_{ij} d\mu_c(j)\right)d\mu_c(i) \\
&= \int_{\mathbb{N}}\left(\int_{\mathbb{N}} a_{ij} d\mu_c(i)\right)d\mu_c(j) \\
&= \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)
\end{aligned}$$

as desired. ☐

Equipped with this result, we now prove that μ is countably additive. To do so, let $\{E_j\}_{j=1}^{\infty}$ be a countable collection of disjoint measurable sets. Now, we know by the fact that each μ_n is a measure that

$$\mu_n\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu_n(E_j)$$

Thus, we have

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{\infty} E_j\right) &= \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n\left(\bigcup_{j=1}^{\infty} E_j\right) \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{2^n}\mu_n(E_j)
\end{aligned}$$

We apply the above lemma to get

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2^n}\mu_n(E_j) &= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2^n}\mu_n(E_j)\\ &= \sum_{j=1}^{\infty}\mu(E_j)\end{aligned}$$

as desired.

Finally, we wish to show that $\mu(\Omega) = 1$. This follows from direct computation (observing that $\mu_n(\Omega) = 1$ for all n):

$$\begin{aligned}\mu(\Omega) &= \sum_{n=1}^{\infty}\frac{\mu_n(\Omega)}{2^n}\\ &= \sum_{n=1}^{\infty}\frac{1}{2^n}\\ &= \frac{1}{1-\frac{1}{2}} - 1\\ &= 1\end{aligned}$$

as desired. Here, we used the standard formula for a geometric series

$$\sum_{n=1}^{\infty} a^n = \frac{1}{1-a} - 1$$

for $0 < a < 1$. $\square$

Problem 2

Suppose $\mu(\Omega) < \infty$. Prove that

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

Proof. We note first that the trivial case of $\|f\|_{L^\infty} = 0$ is clear, since

$$\begin{aligned}\|f\|_{L^\infty} = 0 &\implies f = 0 \ \mu-\text{almost everywhere}\\ &\implies \|f\|_{L^p} = 0 \ \forall p\\ &\implies \lim_{p\to\infty} \|f\|_{L^p} = 0\end{aligned}$$

Therefore, for the rest of this problem, it is assumed that $\|f\|_{L^\infty} > 0$.

Suppose first that $\|f\|_{L^\infty} < \infty$. Then, we are free to scale f so that $\|f\|_{L^\infty} = 1$. (This is clear, since

$$\lim_{p\to\infty} \|cf\|_{L^p} = c \lim_{p\to\infty} \|f\|_{L^p}$$

so

$$\lim_{p\to\infty} \|cf\|_{L^p} = \|cf\|_{L^\infty} \iff \lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

and so multiplying f by a constant will not change the equality.)

So, without loss of generality, let $\|f\|_{L^\infty} = 1$. We will show first that

$$\lim_{p\to\infty} \|f\|_{L^p} \leq 1$$

and then that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq 1$$

First, we prove that $\lim_{p\to\infty} \|f\|_{L^p} \leq 1$. To do so, we consider the altered function

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } f(x) \leq \|f\|_{L^\infty} \\ 0, & \text{if } f(x) > \|f\|_{L^\infty} \end{cases}$$

Now, we know that $\mu\{|f| > \|f\|_{L^\infty}\} = 0$ by the definition of the L^∞ norm, so it follows that $\tilde{f}$ and f differ only on a set of measure zero, and thus are in the same equivalence class in L^p for all p.

So, we have that $\tilde{f} \leq \|f\|_{L^\infty} = 1$, and thus $\tilde{f}^p \leq 1$ for all $p \geq 1$. Therefore,

$$\begin{aligned}\int_\Omega |\tilde{f}(x)|^p d\mu &\leq \int_\Omega 1 d\mu\\ &= \mu(\Omega)\end{aligned}$$

which implies that

$$\begin{aligned}\|f\|_{L^p} &= \left(\int_\Omega |\tilde{f}(x)|^p d\mu\right)^{\frac{1}{p}}\\ &\leq (\mu(\Omega))^{\frac{1}{p}}\end{aligned}$$

and for $\mu(\Omega) < \infty$, we have that $\lim_{p\to\infty}(\mu(\Omega))^{\frac{1}{p}} = 1$. Thus,

$$\lim_{p\to\infty} \|f\|_{L^p} \leq 1$$

as desired.

Next, we prove that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq 1$$

To do so, we consider the set $\{|f| > 1-\epsilon\}$, which has positive measure for every $\epsilon > 0$ by the fact that $\|f\|_{L^\infty} = 1$. Thus, we know that

$$\begin{aligned}
\|f\|_{L^p} &= \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} \\
&\geq \left(\int_{\{|f|>1-\epsilon\}} |1-\epsilon|^p d\mu\right)^{\frac{1}{p}} \\
&= ((1-\epsilon)^p \mu(\{|f| > 1-\epsilon\}))^{\frac{1}{p}} \\
&= (1-\epsilon)(\mu(\{|f| > 1-\epsilon\}))^{\frac{1}{p}}
\end{aligned}$$

Since $\lim_{p\to\infty}(\mu(\{|f| > 1-\epsilon\}))^{\frac{1}{p}} = 1$, it follows that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq (1-\varepsilon)(1)$$

and since this holds for any $\epsilon > 0$, it follows that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq 1$$

as desired.

Thus, for $\|f\|_{L^\infty} < \infty$, we have that

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

as desired.

Suppose instead that $\|f\|_{L^\infty} = \infty$. That is, for each $M > 0$, $\mu(\{|f| > M\}) > 0$. It follows that

$$\begin{aligned}
\|f\|_{L^p} &= \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} \\
&\geq \left(\int_{|f|>M} M^p d\mu\right)^{\frac{1}{p}} \\
&= (M^p \mu(\{f > M\}))^{\frac{1}{p}} \\
&= M(\mu(\{|f| > M\}))^{\frac{1}{p}}
\end{aligned}$$

We know already that $\lim_{p\to\infty}(\mu(\{|f| > M\}))^{\frac{1}{p}} = 1$, so it follows that

$$\lim_{p\to\infty} \|f\|_{L^p} \geq M(1) = M$$

Since M was arbitrary, it follows that

$$\lim_{p\to\infty} \|f\|_{L^p} = \infty$$

as desired. $\square$

Problem 3

For $n \in \mathbb{N}$, define the n^{th} truncation of a positive measurable function f to be

$$f_n(x) = \begin{cases} f(x), & f(x) \le n \\ n, & f(x) \ge n \end{cases}$$

Prove that

$$f \in L^1 \iff \sup_n \|f_n\|_{L^1} < \infty$$

Proof. We observe first that f_n converges pointwise to f. This is clear, since for each x such that $f(x) < \infty$, there is some N for which $f(x) < N$, and thus $\forall m > N, f_m(x) = f(x)$ by the definition of $f_n(x)$. Furthermore, for each x such that $f(x) = \infty$, it follows that $\forall n, f(x) > n$ and thus $f_n(x) = n$, which tends to infinity as n goes to infinity. Thus,

$$\lim_{n\to\infty} f_n(x) = f(x) \ \forall x$$

The sequence $\{f_n\}$ is also monotonically increasing. To see this, we consider three cases:

First, suppose x is such that $f(x) \le n$. In that case,

$$f_n(x) = f_{n+1}(x) = f(x)$$

and thus $f_n(x) \le f_{n+1}(x)$.

Second, suppose x is such that $n < f(x) \le n+1$. In this case, we have that

$$\begin{aligned} f_n(x) &= n \\ &< f(x) \\ &= f_{n+1}(x) \end{aligned}$$

and so $f_n(x) \le f_{n+1}(x)$.

Finally, suppose x is such that $n+1 < f(x)$. In this case, we have that

$$\begin{aligned} f_n(x) &= n \\ &< n+1 \\ &= f_{n+1}(x) \end{aligned}$$

and so $f_n(x) < f_{n+1}(x)$ as desired.

It follows, then, that

$$\|f_n\|_{L^1} \le \|f_{n+1}\|_{L^1}$$

for all n, since the L^1 norm (the integral) preserves orders on positive functions. Therefore,

$$\sup_n \|f_n\|_{L^1} = \lim_{n\to\infty} \|f_n\|_{L^1} = \lim_{n\to\infty} \int_\Omega f_n d\mu$$

applying the monotone convergence theorem yields

$$\begin{aligned} \sup_n \|f_n\|_{L^1} &= \lim_{n\to\infty} \int_\Omega f_n d\mu \\ &= \int_\Omega \lim_{n\to\infty} f_n d\mu \\ &= \int_\Omega f d\mu \\ &= \|f\|_{L^1} \end{aligned}$$

as desired. Note that this holds even if $\|f\|_{L^1} = \infty$ by the monotone convergence theorem.

Thus, $f \in L^1$ precisely when $\sup_n \|f_n\|_{L^1} < \infty$, as desired. $\square$

Problem 4

Let f be a measurable function on $(\Omega, \mathscr{A}, \mu)$ which is finite μ-almost everywhere, and let $\mu(\Omega) < \infty$. Define

$$E_n = \{x \in \Omega \mid n-1 \leq |f(x)| < n\}$$

for $n \in \mathbb{N}$.

Prove that $f \in L^1 \iff \sum_{n=1}^{\infty} n\mu(E_n) < \infty$.

Proof. To begin with, we alter the function f on the set $\{x \mid f(x) = \infty\}$ to be identically zero on that set. Thus, the new altered function (which we will continue to denote as f) is finite everywhere. Since f was only altered on a set of measure zero, this will not affect either side of the equality we are trying to prove.

We begin by showing that $\|f\|_{L^1} \leq \sum_{n=1}^{\infty} n\mu(E_n)$. To do so, we construct the function

$$\phi(x) = \sum_{n=1}^{\infty} n\chi_{E_n}(x)$$

Now, it is clear by the definition of E_n that each $x \in \Omega$ belongs to exactly one E_n. Furthermore, for $x \in E_n$,

$$|f(x)| < n = \phi(x)$$

Thus, $|f(x)| < \phi(x)$ for all x.

It follows (since integrals preserve inequalities of positive functions) that

$$\begin{aligned} \|f\|_{L^1} &= \int_\Omega |f| d\mu \\ &< \int_\Omega \phi d\mu \\ &= \int_\Omega \sum_{n=1}^{\infty} n\chi_{E_n} d\mu \\ &= \sum_{n=1}^{\infty} n\mu(E_n) \end{aligned}$$

Thus, $\|f\|_{L^1} < \sum_{n=1}^{\infty} n\mu(E_n)$, and so

$$\sum_{n=1}^{\infty} n\mu(E_n) < \infty \implies \|f\|_{L^1} < \infty$$

as desired.

Now, we will show that $\|f\|_{L^1} \geq \sum_{n=1}^{\infty} (n-1)\mu(E_n)$. To do so, we define

$$\psi(x) = \sum_{n=1}^{\infty} (n-1)\chi_{E_n}(x)$$

Clearly, we have that for $x \in E_n$,

$$|f(x)| \geq n-1 = \psi(x)$$

and since this holds for all x, we have that $|f| \geq \psi$. Thus, since integrals preserve inequalities of positive functions,

$$\begin{aligned}
\|f\|_{L^1} &= \int_\Omega |f| d\mu \\
&\geq \int_\Omega \psi d\mu \\
&= \sum_{n=1}^{\infty} (n-1)\mu(E_n) \\
&= \sum_{n=1}^{\infty} n\mu(E_n) - \sum_{n=1}^{\infty} \mu(E_n)
\end{aligned}$$

Now, since the sets E_n partition Ω, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu(E_n) &= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= \mu(\Omega)
\end{aligned}$$

so it follows that

$$\begin{aligned}
\|f\|_{L^1} &\geq \sum_{n=1}^{\infty} n\mu(E_n) - \mu(\Omega) \\
\implies \|f\|_{L^1} + \mu(\Omega) &\geq \sum_{n=1}^{\infty} n\mu(E_n)
\end{aligned}$$

Now, since $\mu(\Omega)$ is finite, it is clear that

$$\|f\|_{L^1} < \infty \implies \sum_{n=1}^{\infty} n\mu(E_n) < \infty$$

as desired.

Thus, $f \in L^1 \iff \sum_{n=1}^{\infty} n\mu(E_n) < \infty$ as desired. $\square$

Problem 5

Let D_A be a dense subset of a normed space X with norm $\|\cdot\|_1$. Let Y be a Banach space with norm $\|\cdot\|_2$, and let A be a linear operator with domain D_A and codomain Y such that for some $C < \infty$,

$$\|Ax\|_2 \leq C\|x\|_1 \ \forall x \in D_A$$

Prove that there exists a unique linear bounded operator $\tilde{A} : X \to Y$ such that $\tilde{A}|_{D_A} = A$. Give an estimate for $\|\tilde{A}\|$.

Proof. To begin with, we note that for each $x \in X$, there exists some sequence $\{x_k\}$ in D_A that converges to x (since D_A is dense in X). So, define a candidate function $\tilde{A}$ to be

$$\tilde{A}(x) = \lim_{k\to\infty} A(x_k)$$

where $\{x_k\}$ is some sequence converging to x.

Now, we must prove that such an $\tilde{A}$ is well-defined, linear, bounded, and unique.

To show that $\tilde{A}$ is well-defined, we need to show that $\lim_{k\to\infty} A(x_k)$ exists for every convergent sequence $x_k \to x$, and that such a limit does not depend upon the choice of convergent sequence chosen.

So, let $\{x_k\}$ be a sequence in D_A such that $x_k \to x$ for $x \in X$. Now, we know that $\{x_k\}$ is Cauchy, since it converges. Since A is bounded, it follows that the sequence $A(x_k)$ is Cauchy as well. To see this, fix $\epsilon > 0$, and choose an N such that $\|x_m - x_n\|_1 < \frac{\epsilon}{C}$ for all $m, n > N$. Then, we have that for all $m, n > N$,

$$\begin{aligned}
\|Ax_m - Ax_n\|_2 &= \|A(x_m - x_n)\|_2 \\
&\leq C\|x_m - x_n\|_1 \\
&< C\frac{\epsilon}{C} \\
&= \epsilon
\end{aligned}$$

Thus, $\|Ax_m - Ax_n\|_2 < \epsilon$ for all $m, n > N$, so $\{Ax_k\}$ is a Cauchy sequence. Since Y is complete, this sequence must have a limit, and so the limit $\lim_{k\to\infty} Ax_k$ is well defined.

Suppose, then, that both $\{x_k\}$ and $\{y_k\}$ both converge to x. Then, we have that

$$\begin{aligned}
\|Ax_k - Ay_k\|_2 &= \|A(x_k - y_k)\|_2 \\
&\leq C\|x_k - y_k\|_1 \\
&= C\|(x_k - x) + (x - y_k)\|_1 \\
&\leq C(\|x_k - x\|_1 + \|y_k - x\|_1)
\end{aligned}$$

and since both $\|x_k - x\|_1$ and $\|y_k - x\|_1$ go to zero as k goes to infinity, it follows that $\|Ax_k - Ay_k\|_2$ goes to zero as well. Thus,

$$\lim_{k\to\infty} Ax_k = \lim_{k\to\infty} Ay_k$$

and so $\tilde{A}(x)$ is well defined.

Next, we wish to show $\tilde{A}$ is linear. To do so, consider $\tilde{A}(\alpha x + \beta y)$ for $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. Now, if $\{x_k\}$ is a sequence converging to x, and $\{y_k\}$ is a sequence converging to y, we can define the sequence $\{\alpha x_k + \beta y_k\}$, which converges to $\alpha x + \beta y$ by algebraic limit theorems. Thus, we have that

$$\begin{aligned}\tilde{A}(\alpha x + \beta y) &= \lim_{k\to\infty} A(\alpha x_k + \beta y_k) \\ &= \lim_{k\to\infty} (\alpha A x_k + \beta A y_k) \\ &= \alpha \lim_{k\to\infty} A x_k + \beta \lim_{k\to\infty} A y_k \\ &= \alpha \tilde{A} x + \beta \tilde{A} y\end{aligned}$$

and so $\tilde{A}$ is linear.

Next, we wish to show that $\tilde{A}$ is bounded. To do so, we note two important properties of limits. First, limits preserve orderings (by the Order Limit Theorem), and second, limits pass through norms (by the continuity of the norm).

Then, we have that

$$\begin{aligned}\|\tilde{A}x\|_2 &= \|\lim_{k\to\infty} A x_k\|_2 \\ &= \lim_{k\to\infty} \|A x_k\|_2 \\ &\le \lim_{k\to\infty} C\|x_k\|_1 \quad (\text{since } \|Ax_k\|_2 \le C\|x_k\|_1) \\ &= C\|\lim_{k\to\infty} x_k\|_1 \\ &= C\|x\|_1\end{aligned}$$

and so $\|\tilde{A}x\|_2 \le C\|x\|_1$ for all x, and $\tilde{A}$ is bounded, as desired.

Finally, we show that such a $\tilde{A}$ is unique. Suppose that A' were some other bounded linear extension of A. Now, since both $\tilde{A}$ and A' are linear and bounded, they are continuous. Namely, they preserve limits. So, for each $x \in X$, let $\{x_k\}$ be a sequence in D_A converging to x. Now, since A' and $\tilde{A}$ agree on D_A, we have that

$$\tilde{A}x_k = A'x_k$$

for all x_k. Then, taking the limit of both sides yields

$$\tilde{A}x = \lim_{k\to\infty} \tilde{A}x_k = \lim_{k\to\infty} A'x_k = A'x$$

and so $\tilde{A}$ and A' agree for all $x \in X$, and thus $\tilde{A} = A'$. Thus, such an extension is unique.

Finally, we observed in the proof of boundedness that $\|Ax\|_2 \le C\|x\|_1$ for all x. Thus, it follows immediately by the definition of operator norm that

$$\|A\|_{X\to Y} \le C$$

which provides an estimate for the operator norm of A. □