1 Jacobi Fields

Consider M a Riemannian manifold, $p \in M$. Let Ω be the maximal domain of the exponential map $\exp_p$. We wish to understand what happens to the image of a ball under the exponential map (which is a diffeo for small balls, but what about larger ones?). In essence, where is the exponential map a local diffeomorphism? Or, where is $d\exp_p$ an isomorphism? For what u is $(d\exp_p)_u : T_u T_p M \cong T_p M \to T_{\exp_p(u)} M$ an isomorphism? This is true if and only if the kernel is trivial.

Suppose v is such that $d\exp_p|_u(v) = 0$. We know that

$$d\exp_p|_u(v) = \partial_s \exp_p(u + sv)|_{s=0}$$

Now, we have a family of radial geodesics $\exp(t(u + sv))$ (familial wrt s) which connect $\exp(tu)$ to $\exp(t(u + v))$, call the family $\gamma(t, s) = \exp_p(t(u + sv))$. γ is a smooth map from a disk D (or a rectangle) to M. (Horizontal slices in the rectangle are geodesics). We have two important vector fields: $\partial_s\gamma$ and $\partial_t\gamma$. $\partial_s\gamma$ is the rate of change of the deformation of the starting curve, the deformation vector field. Let $J = \partial_s\gamma$. Do we have an equation for J? What happens if we differentiate J?

$$\begin{aligned}
\nabla_{\partial_t}\nabla_{\partial_t}\partial_s\gamma &= \nabla_{\partial_t}\nabla_{\partial_s}\partial_t\gamma && \text{commutativity not proven in class} \\
&= \nabla_t\nabla_s\partial_t\gamma - \nabla_s\nabla_t\partial_t\gamma + \nabla_a s\nabla_t\partial_t\gamma + \nabla_{[\partial_s,\partial_t]}\partial_t\gamma \\
&= R(\partial_s\gamma, \partial_t\gamma)\partial_t\gamma && \text{since } \nabla_t\partial_t\gamma \text{ vanishes (geodesic)}
\end{aligned}$$

which gives us a formula for J as

$$J'' + R(\gamma_0', J)\gamma_0 = 0$$

Theorem 1. *The linear map $F(w) = Jw$ is an isomorphism.*

Proof. If $Jw = 0$, then $J'w = 0$ so $w = J'w(0) = 0$. So F is injective. Now, to show surjectivity, we let J be a Jacobi field along γ such that $J(0) = 0$, $J(1) = 0$, and set $w = J'(0)$. Then $Jw = J$ and F is surjective. $\square$

Theorem 2. *$d\exp_p|_v$ is an isomorphism if and only if 1 is not a conjugate time for $\gamma(t) = \exp_p(tv)$ i.e. $\exp_p(v)$ is not conjugate to p along γ.*

Here, $\exp_p(v)$ is conjugate to p if the Jacobi field along γ vanishes at both p and $\exp_p(v)$.

1.1 Jacobi Fields on Manifolds With Constant Curvature

Recall the Jacobi equation

$$J'' + R_m(\gamma', J')\gamma' = 0 \tag{1.1}$$

First assume $J = f\gamma'$. Then, the jacobi equation yields

$$J'' + fR(\gamma', \gamma')\gamma' = 0 \tag{1.2}$$

$$\begin{array}{ll} \kappa = 0 & J^i(t) = a^i t \\ \kappa < 0 & J^i(t) = a^i \sinh(\sqrt{-\kappa} t) \\ \kappa > 0 & J^i(t) = a^i \sin(\sqrt{\kappa} t) \end{array}$$

$$\begin{array}{ll} \kappa = 0 & J(t) = tw(t) \\ \kappa < 0 & J(t) = \frac{\sinh(\sqrt{-\kappa} t)}{\sqrt{-\kappa}} \\ \kappa > 0 & J(t) = \frac{\sin(\sqrt{\kappa} t)}{\sqrt{\kappa}} \end{array}$$

and thus $J'' = 0$ so $f = at + b$, and so $J(t) = (at + b)\gamma'$. With the initial condition $J(0) = 0$, we have $J(t) = at\gamma'(t)$, which will never be zero for positive time and $a \neq 0$. So, J as a tangential field cannot form conjugate points.

Now, let's assume J is normal to $\gamma'(t)$. Now, let's choose a parallel frame along γ the geodesic such that e_1 is parallel to $\gamma'(t)$. Then, $J = f^i e_i$ which yields the system of differential equations

$$f''^i e_i + f^i R_m(\gamma', e_i)\gamma' = 0 \tag{1.3}$$

If we assume that our manifold has constant curvature, we have

$$R(X, Y)Z = \kappa(g(Z, X)Y - g(Z, Y)X) \tag{1.4}$$

which yields the equations

$$f''^i e_i + f^i \kappa(\|\gamma'\|^2 e_i = 0 \tag{1.5}$$

or

$$f''^i + f^i \kappa \|\gamma'\|^2 = 0 \tag{1.6}$$

Furthermore, if we assume γ is unit parameterized, we have the equations

$$f''^i + f^i \kappa = 0 \tag{1.7}$$

Which has solutions (for $J(0) = 0$)

Suppose instead we wish to solve this without an orthonormal frame. Let $w = J'(0)$ with $J(0) = 0$ the initial data. Let $w(t)$ be the parallel vector field of w along γ. We assume the solution has the form $J(t) = f(t)w(t)$. Then, we have $f(0) = 0$ and $f'(0) = 1$ initial conditions. Then, we have the equation

$$f'' w + f\kappa(\|\gamma'\|^2 w - (\gamma' \cdot w)\gamma') = 0 \tag{1.8}$$

Now, we have already taken care of the tangential part, so let's assume w is orthogonal to γ'. Then, we have the solutions ¡++¿