Problem Set 1

Daniel Halmrast

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Problem 1

Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}.

Proof. The straight-line homotopy from v to $\frac{v}{\|v\|}$ satisfies the criteria for a deformation retract. Namely, the retract is given by

$$\begin{aligned} r :& \mathbb{R}^n \setminus \{0\} \to S^{n-1} \\ r(v) =& \frac{v}{\|v\|} \end{aligned}$$

With homotopy

$$\begin{aligned} F :& \mathbb{R}^n \setminus \{0\} \times I \to S^{n-1} \\ F(v,t) =& (1-t)v + t\frac{v}{\|v\|} \end{aligned}$$

$\square$

Problem 2

Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof. Let $f : X \to Y$ be a map, which is homotopic to a homotopy equivalence $g : X \to Y$ with homotopy inverse $h : Y \to X$. That is, $g \circ h \simeq \mathbb{1}_Y$ and $h \circ g \simeq \mathbb{1}_X$. Furthermore, let $F : X \times I \to Y$ be the homotopy between f and g.

First, let's consider the map $h \circ f : X \to X$. We wish to show $h \circ f \simeq \mathbb{1}_X$. To do so, let's consider the homotopy

$$h \circ F : X \times I \to X$$

This is the composition of two continuous functions, and so it is continuous. Furthermore, since $F(0, x) = f(x)$ and $F(1, x) = g(x)$, this is actually a homotopy between $h \circ f$ and $h \circ g$. Now, since $h \circ f \simeq h \circ g \simeq \mathbb{1}_X$ and homotopy equivalence is an equivalence relation, it follows immediately that $h \circ f \simeq \mathbb{1}_X$.

Now, consider the map $f \circ h : Y \to Y$. We wish to show $f \circ h \simeq \mathbb{1}_Y$. To do so, consider the homotopy

$$F \circ (h \times \mathbb{1}_I) : Y \times I \to Y$$

It is easy to see this is a homotopy between $f \circ h$ and $g \circ h$, and so we have that $f \circ h \simeq g \circ h \simeq \mathbb{1}_X$, and so $f \circ h \simeq \mathbb{1}_x$, as desired. $\square$