Scores: 1 2 3 4 5 6 7 8 9 10 Name Prof. M

Section K___

Mathematics 2602

Midterm 3

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1. Match each phrase to the corresponding formula.

The number of ways of putting r indistinguishable marbles into n boxes, with as many marbles per box as you like.

G

The number of ways of putting r distinguishable marbles into n boxes, with at most one marble per box.

A

The number of ways of arranging r distinguishable marbles in a line.

J

The number of ways of putting r indistinguishable marbles into n boxes, with at most one marble per box.

F

The number of ways of putting r distinguishable marbles into n boxes, with as many marbles per box as you like.

B

A.	$P(n,r)$	*F.*	$\binom{n}{r}$
B.	n^r	*G.*	$\binom{n+r-1}{r}$
C.	r^n	*H.*	$\binom{n-r+1}{n}$
D.	rn	*I.*	$n!$
E.	$\binom{r}{n}$	*J.*	$r!$

2. How many numbers from 1 to 1,000 are divisible by by 3 but not by 5 or 6? Your answer should be a number.

$$|A_3| - |(A_3 \cap A_5) \cup (A_3 \cap A_6)|$$

$$= |A_3| - |A_3 \cap A_5| - |A_3 \cap A_6| + |A_3 \cap A_5 \cap A_6|$$

$$= |A_3| - |A_{15}| - |A_6| + |A_{30}|$$

$$= \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor$$

$$= 333 - 66 - 166 + 33$$

$$= 134$$

3. Anna, Elsa, Hans, Kristoff, Olaf, Sven, and the Duke of Weselton are sitting down for dinner at a round table. How many ways are there for them to sit if Anna insists on sitting next to Kristoff, Anna also refuses to sit next to Hans or the Duke, and Olaf insists on sitting with Sven? As usual, if we rotate the table we consider that to be the same configuration.

A

Put Anna on top — 1 choice.

Kristoff to her left/right — 2 choices.

Put Olaf to left/right of Sven — 2 choices.

Put Olaf/Sven, or Elsa next to Anna — 2 choices.

Treating Olaf/Sven as one person, there are three people left to seat: Duke, Hans, and one of Elsa and Olaf/Sven — $3!$ choices

$$2^3 \cdot 3!$$

4. How many different 5-card poker hands are there with two pairs (but not four of a kind or a full house)?

Remember that a deck of cards has thirteen faces $2, 3, \ldots, 10$, J, Q, K, A and four suits $\diamondsuit$, $\heartsuit$, $\spadesuit$, and $\clubsuit$. A pair is a set of two cards with the same face, and we say a hand has two pairs if it has two of one face, two of a different face, and one more card of yet another face.

Choose two faces $\binom{13}{2}$

Choose two pairs of suits $\binom{4}{2}\binom{4}{2}$

Choose a card of another face $44 = \binom{11}{1}\binom{4}{1}$

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

5. Consider the set of numbers

$$\{2, 3, 5, 7, 11, 13, 17, 19\}.$$

Argue that there are at least four nonempty subsets with the same sum.

Smallest sum: 2

Largest sum: 77

$\rightsquigarrow$ 76 possible sums

and $2^8 - 1 = 255$ nonempty subsets

$\left\lceil \frac{255}{76} \right\rceil = 4$ sums must

be same by pigeonhole principle.

6. Factories A and B make skittles. Factory A produces four times as many skittles as Factory B. One out of every ten skittles produced by Factory A is defective and half of the skittles produced by Factory B are defective. A skittle is selected at random and is found to be defective. What is the probability the defective skittle came from Factory A?

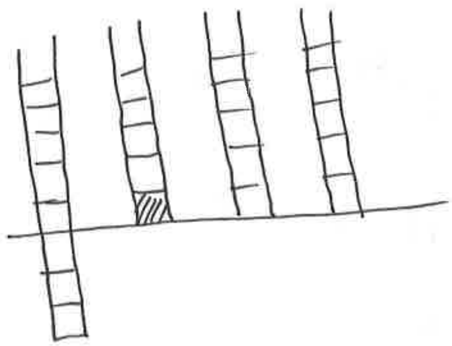
$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{P(D|A)P(A)}{P(D|B)P(B) + P(D|A)P(A)}$$

$$= \frac{\frac{1}{10} \cdot \frac{4}{5}}{\frac{1}{2} \cdot \frac{1}{5} + \frac{1}{10} \cdot \frac{4}{5}}$$

$$= \frac{\frac{4}{50}}{\frac{1}{10} + \frac{4}{50}}$$

$$= 4/9.$$

7. In how many ways can four integers a, b, c, and d be chosen so that
$a \geq -3, \quad b \geq 1, \quad c \geq 0, \quad d \geq 0, \quad \text{and} \quad a+b+c+d=23?$



26 marbles,
one already used.

$$\binom{25+4-1}{25} = \binom{28}{3}$$

8. Find the constant term in the expansion of

$$\left(-3x^7 + \frac{2}{x}\right)^{48}.$$

Need $(x^7)^k \left(\frac{1}{x}\right)^{48-k} = x^0$

$$7k - (48-k) = 0$$

$$k = 6$$

$$\binom{48}{6}(-3x^7)^6\left(\frac{2}{x}\right)^{42}$$

$$\binom{48}{6}(-3)^6\, 2^{42}$$

$$\binom{48}{6}3^6\, 2^{42}$$

9. Use induction to prove either the identity

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

where $n \geq 0$ or the hockey stick identity

$$\sum_{i=0}^{n} \binom{m+i}{i} = \binom{m+n+1}{n+1}$$

where $m, n \geq 0$. You may use Pascal's identity without proof: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Induction on n.

Base case $n=0$: $\sum_{i=0}^{0} \binom{0}{i} = \binom{0}{0} = 1 = 2^0$.

Inductive hypothesis: $\sum_{i=0}^{k} \binom{k}{i} = 2^k$.

Inductive step:

$$\sum_{i=0}^{k+1} \binom{k+1}{i} = \sum_{i=0}^{k+1} \binom{k}{i} + \binom{k}{i-1}$$

$$= \sum_{i=0}^{k+1} \binom{k}{i} + \sum_{i=0}^{k+1} \binom{k}{i-1}$$

$$= \sum_{i=0}^{k} \binom{k}{i} + \cancel{\binom{k}{k+1}}$$

$$+ \cancel{\binom{k}{-1}} + \sum_{i=1}^{k+1} \binom{k}{i-1}$$

$$= 2\sum_{i=0}^{k} \binom{k}{i}$$

$$= 2 \cdot 2^k = 2^{k+1}$$

$\boxed{}$

10. Let n be a nonnegative integer. Explain why

$$\left(\sqrt{2}-1\right)^n = a_n + b_n\sqrt{2}$$

for some integers a_n and b_n.

Each term in the binomial expansion

is $\binom{n}{k}(\sqrt{2})^k(-1)^{n-k}$

k even $\rightsquigarrow$ integer

k odd $\rightsquigarrow$ integer $\cdot \sqrt{2}$

add like terms.

Use the previous part and the fact that $\lim_{n\to\infty}(\sqrt{2}-1)^n = 0$ to show that $\sqrt{2}$ is irrational.

Assume $\sqrt{2} = p/q$.

By first part $a_n + b_n(p/q) \longrightarrow 0$

$$\frac{qa_n + pb_n}{q} \longrightarrow 0$$

But rational nums with fixed denom. can't go to zero unless they eventually are zero. Contradiction since $(\sqrt{2}-1)^n > 0$ for all n.