Linear Programming

Linear Programs

Example. Maximize $3x+2y = z$ ← objective function

subject to $\left.\begin{array}{l} 2x + y \leq 20 \\ x, y \geq 0 \end{array}\right\}$ ← constraints

e.g. x = widget, y = gadget   2

3 =   1

2 =   20

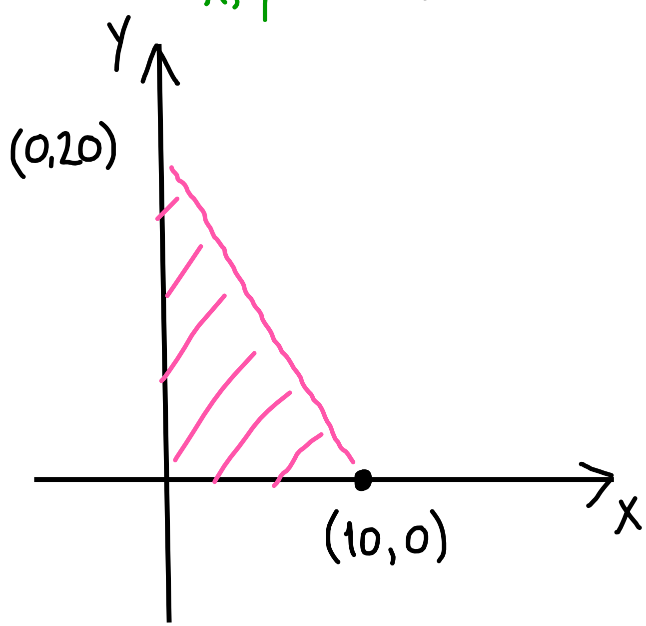
A maximization (or mimimization) problem where the objective function and constraints are all linear is called a linear programming problem.

How to solve?

LINEAR PROGRAMS

EXAMPLE. Maximize $3x+2y = z$ ← objective function

subject to $2x + y \leq 20$, $x, y \geq 0$ ← constraints



Pink triangle = Feasible region

Linear Programs

Example. Maximize $z = x + y$

subject to

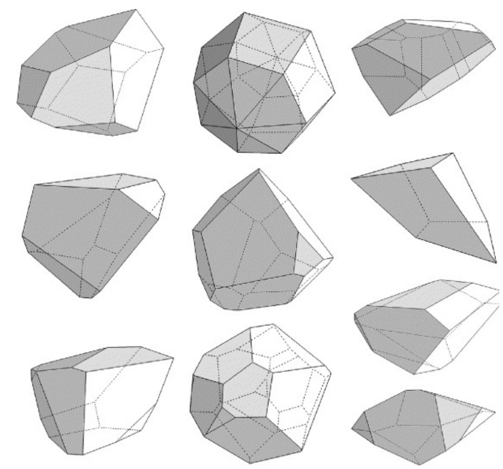
$$\begin{aligned} 2x + 3y &\le 6 \\ 4x + 2y &\le 8 \\ x, y &\ge 0 \end{aligned}$$

Notice: The optimum always occurs at a corner.

This always works! To find the optimum, we move the objective function hyperplane in the direction of its perpendicular (gradient) and observe the last point(s) of the feasible region it passes through. This will always be at a corner.

The Feasible Region

The feasible region for a linear program is the intersection of finitely many half-spaces. Thus, it is a convex (possibly infinite) polyhedron.



We deduce:

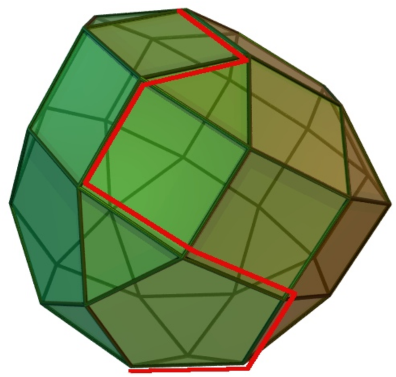
Fact. If a finite optimum exists for a linear program, then

In other words, to find optima, it is enough to look at

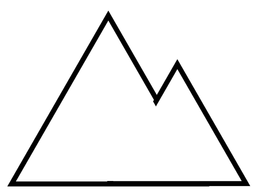
Example. Maximize $z = 3x$ s.t. $0 \leq x, y \leq 1$.

The Simplex Method

The basic idea: Start at some corner of the feasible region. See if any adjacent corners are higher (in z-value). If so, move to that corner. If not, stop.



In other words, if you always move up, you eventually get to the top. This does not work for nonconvex shapes:



Now: How to formalize this?

The Simplex Method

The simplex method was devised in 1947 by George Dantzig, of the RAND corporation.

George Dantzig

It was deemed one of the top ten algorithms of the 20th century in the Jan/Feb 2000 issue of Computing in Science and Engineering.

Standard Form

Given a linear program, we put it in standard form by adding slack (or surplus) variables so that all inequalities become equalities.

Example. The standard form of

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 4 \\ & x_1 + 2x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

is:

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 + x_4 = 3 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Standard Form

The standard form: maximize $z = x_1 + x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

gives a system of linear equations:

$$\begin{aligned} z - x_1 - x_2 \quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

subject to the condition $x_i \geq 0$.

Standard Form

The picture of the feasible region for

$$\begin{aligned} z - x_1 - x_2 &= 0 \\ 2x_1 + x_2 + x_3 &= 4 \\ x_1 + 2x_2 + x_4 &= 3 \\ x_i \geq 0. \end{aligned}$$

is:

The Simplex Method

After putting in standard form, we want to maximize z, where:

$$\begin{aligned} z - x_1 - x_2 \qquad\quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \qquad + x_4 &= 3 \end{aligned}$$

and $x_i \geq 0$.

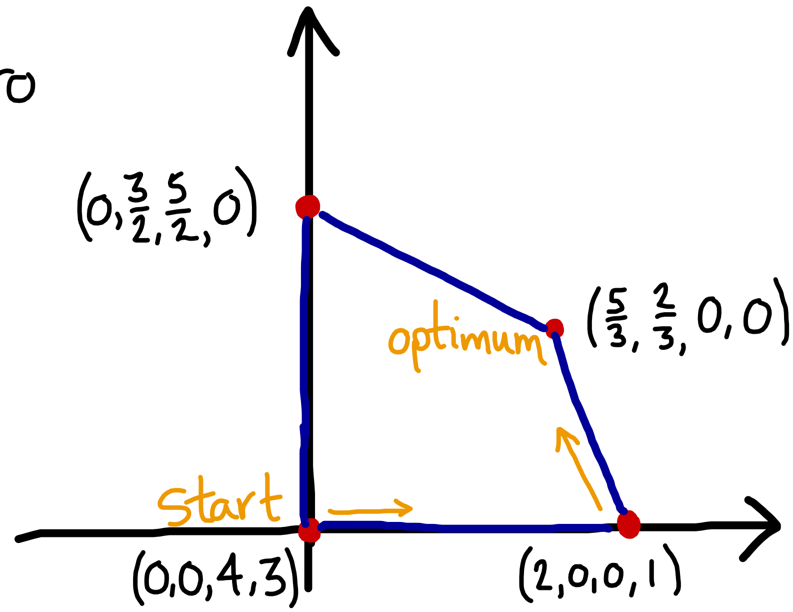
In these notes, we assume no surplus variables (i.e. all original constraints are $\leq$) and all right-hand sides are positive.

A basic variable is one that appears in only one equation.

Setting all nonbasic variables equal to 0 gives a corner of the feasible region, called a basic solution.

The Simplex Method

As we can see, by changing one nonzero coordinate to be zero, we move along an edge of the feasible region. So to make progress from our starting point to the optimal solution, we change the basic variables, one at a time.



We just need to make sure ① We are always increasing z
and ② We stay in the feasible region

Rules ① and ② below address these two points.

THE SIMPLEX METHOD

Say we have a linear program in standard form:

$$\begin{aligned} z - x_1 - x_2 \quad\quad\quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

RULE 1. The current basic solution is optimal if and only if all variables in the top row have nonnegative coefficients.

If the current basic solution is not optimal, we choose a variable with negative coefficient in the top row and make it basic using row operations.

But, we need to do this carefully!

In Rule 1, we usually choose a variable with <u>most</u> negative coeff.

The Simplex Method

Say we have a linear program in standard form:

$$
\begin{aligned}
z - x_1 - x_2 \quad\quad\quad\quad &= 0 \\
2x_1 + x_2 + x_3 \quad\;\; &= 4 \\
x_1 + 2x_2 \quad\quad + x_4 &= 3
\end{aligned}
$$

Say, by Rule 1, we decide to make x_1 basic. This means we want to use row operations to remove x_1 from all equations but one. Which to choose?

Rule 2. When making x_i basic, we leave x_i in the row where

$$\frac{\text{RHS}}{\text{coeff}(x_1)}$$

is the smallest positive number among all rows.

positive $\rightsquigarrow$ z will increase   smallest $\rightsquigarrow$ stay in feasible region

The Simplex Method

Rules 1 and 2 comprise the simplex method. Let's apply it to our example.

$$
\begin{aligned}
z - x_1 - x_2 \qquad\qquad &= 0 \\
2x_1 + x_2 + x_3 \qquad &= 4 \\
x_1 + 2x_2 \qquad + x_4 &= 3
\end{aligned}
$$

The Simplex Method

TABLEAUX

We can succinctly record (and perform) the above calculation as follows:

z	x_1	x_2	x_3	x_4	RHS	Basic Soln
1	-1	-1	0	0	0	
0	2	1	1	0	4	$(0,0,4,3)$ $z=0$
0	1	2	0	1	3	
1	0	$-1/2$	$1/2$	0	2	
0	1	$1/2$	$1/2$	0	2	$(2,0,0,1)$ $z=2$
0	0	$3/2$	$-1/2$	1	1	
1	0	0	$1/3$	$1/3$	$7/3$	
0	1	0	$2/3$	$-1/3$	$5/3$	$(5/3, 2/3, 0, 0)$ $z=7/3$
0	0	1	$-1/3$	$2/3$	$2/3$	

Note the basic solution is easily read from the RHS.

The Simplex Method

Problem. Maximize $4x_1 + x_2 - x_3 = z$

subject to

$$\begin{aligned} x_1 + \quad 3x_3 &\leq 6 \\ 3x_1 + x_2 + 3x_3 &\leq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

First we write the standard form:

THE SIMPLEX METHOD

PROBLEM. Maximize $4x_1 + x_2 - x_3 = z$

subject to
$$\begin{aligned} x_1 + \quad 3x_3 &\le 6 \\ 3x_1 + x_2 + 3x_3 &\le 9 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic Soln

The Simplex Method

Problem. Maximize $z = x_1 + \frac{1}{2}x_2$

subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

First we write the standard form:

The Simplex Method

Z	X_1	X_2	X_3	X_4	RHS	Basic Soln

The Simplex Method

Example. Maximize $z = 3x + 4y$
subject to $2x + y \leq 5$
$x, y \geq 0$

$z = 20$ at $(0, 5)$

Example. Maximize $z = 3x + 2y$
subject to $x + y \leq 4$
$2x + y \leq 5$
$x, y \geq 0$

$z = 9$ at $(1, 3)$

Example. Maximize $z = 3x + 2y$
subject to $2x + y \leq 18$
$2x + 3y \leq 42$
$3x + 2y \leq 24$
$x, y \geq 0$

$z = 24$ at $(8, 0)$

Example. Maximize $z = x_1 + 2x_2 - x_3$
subject to $2x_1 + x_2 + x_3 \leq 14$
$4x_1 + 2x_2 + 3x_3 \leq 28$
$2x_1 + 5x_2 + 5x_3 \leq 30$
$x_i \geq 0$

$z = 13$ at $(5, 4, 0)$

The Simplex Method

Tableau for last example:

Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS	Solution

The Simplex Method

Example. A company produces chairs and sofas.
A chair requires 3 hrs carpentry, 9 hrs finishing, 2 hrs upholstery.
A sofa requires 2 hrs carpentry, 4 hrs finishing, 10 hrs upholstery.
The company can afford 66 hours of carpentry, 180 hours of finishing, and 200 hours of upholstery.
The profit on a chair is \$90 and on a sofa is \$75.
How many chairs and sofas should be made to maximize profit?

Solution. Want to maximize

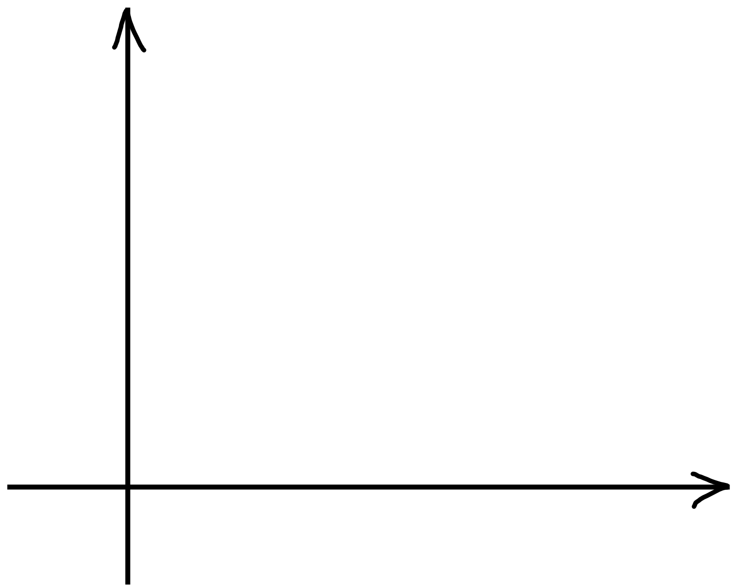
subject to

The Simplex Method

z	x_1	x_2	x_3	x_4	x_5	RHS	Solution

The Simplex Method

Picture for the last problem:



Geometry of the Simplex Method

The vertices and edges of the feasible region form a graph.

The vertices correspond to collections of x_i so that the corresponding columns are linearly independent.

Two vertices span an edge if the corresponding collections differ by one element.

Thus, swapping one basic variable for another corresponds to moving along an edge.

As we perform row operations, the feasible region (and objective function) changes, but the corresponding graph is naturally isomorphic to the original.