Announcements: Sep 12

- Midterm 1 on Sep 21
- Quiz 3 Friday in recitation
- WeBWorK 3.1 and 3.2 due tonite!
- My office hours today 2:00-3:00 and Friday 9:30-10:30 in Skiles 234
- TA Office Hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Fri 12-1 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 9:30-10:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday - Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123
- Supplemental problems on master course web site
- Students are responsible for pressing "Save my response" on Piazza polls and having the correct email address on Piazza.
- Let's talk about efficient use of resources!


Seven Bridges of Konigsberg
Performance
12:00 Thu

Section 3.3

Matrix equations

Outline Section 3.3

- Understand the equivalences:

  linear system $\leftrightarrow$ augmented matrix $\leftrightarrow$ vector equation $\leftrightarrow$ matrix equation

- Understand the equivalence:

  $$Ax = b \text{ is consistent} \longleftrightarrow b \text{ is in the span of the columns of } A$$

  (also: what does this mean geometrically)
- Learn for which A the equation $Ax = b$ is always consistent
- Learn to multiply a vector by a matrix

Multiplying Matrices

Another way to multiply

$$\text{matrix} \times \text{column vector} : \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

OR

$$\text{matrix} \times \text{col vector} : \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 c_1 & \cdots & b_n c_n \\ | & & | \end{pmatrix}$$

Read this as: b_1 times the first column c_1 is the first column of the answer, b_2 times c_2 is the second column of the answer...

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} =$$

Linear Systems vs Augmented Matrices vs Matrix Equations vs Vector Equations

A matrix equation is an equation $Ax = b$ where A is a matrix and b is a vector. So x is a vector of variables.

A is an $m \times n$ matrix if it has m rows and n columns.
What sizes must x and b be?

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$$

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

Is a given vector in the span?

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

Is $(3, -1, 0)$ in the span of $(2, -1, 1)$ and $(1, 0, -1)$?

Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,10,20)$, $(0,-1,-2)$
2. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,5,7)$, $(0,6,8)$
3. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,1,0)$, $(0,0,\sqrt{2})$
4. $(0,1,2)$ is in the span of $(5,7,0)$, $(6,8,0)$, $(3,3,4)$

Pivots vs Solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

More generally, if you have k vectors in $\mathbb{R}^n$ and you want to know the dimension of the span, you should row reduce and count the number of pivots.

Properties of the Matrix Product Ax

$c =$ real number, $u, v =$ vectors,

- $A(u+v) =$
- $A(cv) =$

Application. If u and v are solutions to $Ax = 0$ then so is every element of $\mathrm{Span}\{u+v\}$.

Summary of Section 3.3

- Two ways to multiply a matrix times a column vector:

$$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

  OR

$$\begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_{\color{red}n} \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{\color{red}n} \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 c_1 & \cdots & b_n c_n \\ | & & | \end{pmatrix}$$

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.
- Fact. $Ax = b$ has a solution $\Leftrightarrow$ b is in the span of columns of A
- Theorem. Let A be an $m \times n$ matrix. The following are equivalent.
  1. $Ax = b$ has a solution for all b
  2. The span of the columns of A is $\mathbb{R}^m$
  3. A has a pivot in each row

Section 3.4

Solution Sets

Outline

- Understand the geometric relationship between the solutions to $Ax = b$ and $Ax = 0$
- Understand the relationship between solutions to $Ax = b$ and spans
- Learn the parametric vector form for solutions to $Ax = b$

Homogeneous systems

Solving $Ax = b$ is easiest when $b = 0$.

Homogeneous systems $\longleftrightarrow$ matrix equations $Ax = 0$.

Homogenous systems are always consistent. *Why?*

When does $Ax = 0$ have a nonzero/nontrivial solution?

If there are k-free variables and n total variables, then the solution is a k-dimensional plane in $\mathbb{R}^n$.

Homogeneous systems

Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: When solving homogeneous systems, we do need need augmented matrices. Why?

Homogeneous systems
Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Homogeneous systems

Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Homogeneous systems

Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

Dimension and Span of Homogeous Systems

If $v_1, \ldots, v_k$ are solutions to $Ax = 0$, then so is each element of $\mathrm{Span}\{v_1, \ldots, v_k\}$

Why?

$\rightsquigarrow$ set of solutions to $Ax = 0$ is a plane through the origin.

Parametric Vector Forms

Say free variables for $Ax = 0$ are $x_1, \dots, x_k$

Then the solutions to $Ax = 0$ can be written as

$$x_1 v_1 + \cdots + x_k v_k$$

for some $v_1, \dots, v_k$ (in other words, as a span!).

(The free variables are usually not the first k variables.)

This is the *parametric vector form* of the solutions.

Parametric Vector Forms for Solutions

Homogeneous case

Find the parametric vector form of the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Parametric Vector Forms for Solutions

Homogeneous case

Find the parametric vector form of the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Parametric Vector Forms for Solutions

Homogeneous case

Find the parametric vector form of the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

Variables, equations, and dimension

Poll

For $b \neq 0$, the solutions to $Ax = b$ are...

1. always a span
2. sometimes a span
3. never a span

Nonhomogeneous Systems

Suppose $Ax = b$, and $b \neq 0$.

As before, we can find the parametric vector form for the solution in terms of free variables.

What is the difference?

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form of the solution to $Ax = (5, -10)$ where

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form of the solution to $Ax = (3, 2, 6)$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form of the solution to $Ax = (3, 2, 6)$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\left(\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 3 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{rrrr|r} 1 & 0 & -8 & -7 & -13 \\ 0 & 1 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form of the solution to $Ax = (4, 2, 4)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form of the solution to $Ax = (4, 2, 4)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form for the solution to $Ax = (9)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 9 \end{array}\right)$$

Homogeneous vs. Nonhomogeneous Systems

Key realization. Set of solutions to $Ax = b$ obtained by taking one solution and adding all possible solutions to $Ax = 0$.

$$Ax = 0 \text{ solutions} \rightsquigarrow Ax = b \text{ solutions}$$

$$x_k v_k + \cdots + x_n v_n \rightsquigarrow$$

So: set of solutions to $Ax = b$ is parallel to the set of solutions to $Ax = 0$.

So by understanding $Ax = 0$ we gain understanding of $Ax = b$ for all b. This gives structure to the set of equations $Ax = b$ for all b.

Demo

Summary of Section 3.4

- The solutions to $Ax = 0$ form a plane through the origin (span)
- The solutions to $Ax = b$ form a plane not through the origin
- The set of solutions to $Ax = b$ is parallel to the one for $Ax = 0$
- In either case we can write the parametric vector form. The parametric vector form for the solution to $Ax = 0$ is obtained from the one for $Ax = b$ by deleting the constant vector. And conversely the parametric vector form for $Ax = b$ is obtained from the one for $Ax = 0$ by adding a constant vector. This vector translates the solution set.