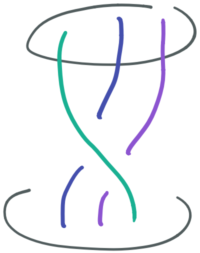
Chap 9 Braid groups

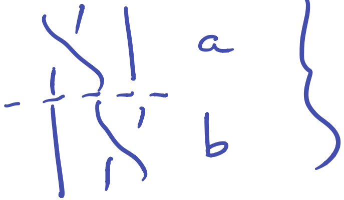
B_n = braid gp on n strands.

Def #1

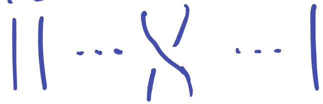


n strands in $\mathbb{R}^2 \times [0,1]$
monotonic in $[0,1]$ dir.
considered up to isotopy in $\mathbb{R}^3$

Multiplication: stack (& scale vertical)



id: 

Generators: 

σ_i

Inverses

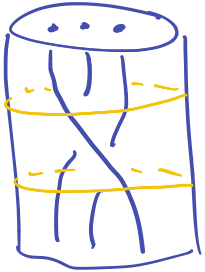
Braid closure:
braids $\longrightarrow$ knots

Defn #2

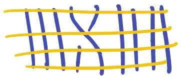
$\mathrm{Conf}_n(\mathbb{R}^2)$ = space of n un-labeled pts in $\mathbb{R}^2$

$$B_n \cong \underbrace{\pi_1 \mathrm{Conf}_n \mathbb{R}^2}_{\text{"dance"}}$$

basept: • • • • •



time

In this defn: σ_i is 

$$P\mathrm{Conf}_n \mathbb{R}^2 = (\mathbb{R}^2)^n / \text{big diagonal.}$$

$$\mathrm{Conf}_n \mathbb{R}^2 = P\mathrm{Conf}_n \mathbb{R}^2 / \Sigma_n$$

Fact. $\mathrm{Conf}_n \mathbb{R}^2$ is a $K(G,1)$

$\Rightarrow$ B_n is torsion free.

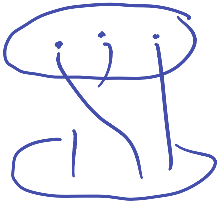
(torsion $\Rightarrow$ ∞-dim $K(G,1)$).

Defn #3

$B_n \cong \mathrm{Mod}(D_n)$ disk with n marked pts in interior.

$\mathrm{Mod}(D_n) \longrightarrow B_n$

Given $[\varphi] \in \mathrm{Mod}(D_n)$ any homotopy φ to id (ignoring marked pts) restricts to a loop in $\pi_1 \mathrm{Conf}_n \mathbb{R}^2$.

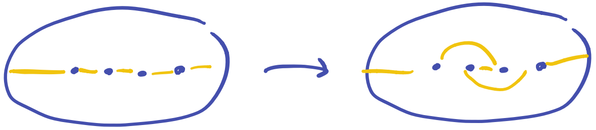


Pf of $\cong$ is BES, forgetting n pts instead of 1.

$$\begin{tikzcd} \mathrm{Homeo}^+(D^2, \{n \text{ pts}\}) \arrow[r] & \mathrm{Homeo}^+(D^2) \arrow[d] \\ & \mathrm{Conf}_n D^2 \simeq \mathrm{Conf}_n \mathbb{R}^2 \end{tikzcd}$$

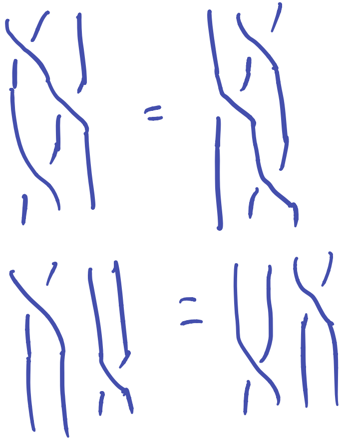
fiber bundle, $\rightsquigarrow$ LES

σ_i :



Alg. Structure

- $B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \rangle$



braid rel : R3 moves

$\sigma_i \sigma_i^{-1} = id$: R2 moves

- $B_n^{ab} = H_1(B_n;\mathbb{Z}) \cong \mathbb{Z}$

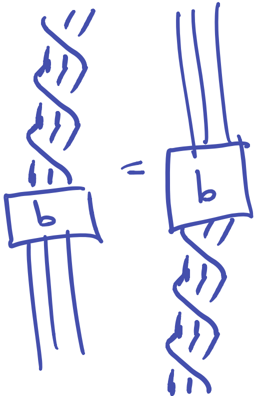
  $L : B_n \longrightarrow \mathbb{Z}$

  $\sigma_i \longmapsto 1$

  "length homom"

- $Z(B_n) = \langle T_\partial \rangle$

  $T_\partial = (\sigma_1 \cdots \sigma_{n-1})^n$



codim 1

codim 2

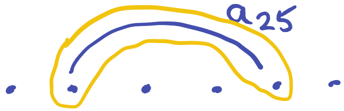
codim 2.

Travaux de Thurston. Exposé 3.

Pure braid gps PB_n

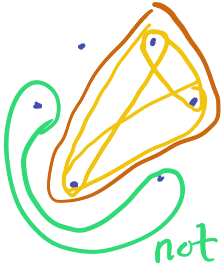
$$1 \to PB_n \to B_n \to \Sigma_n \to 1$$

- PB_n gen by a_{ij} $\binom{n}{2}$



Dehn twist

- Presentation (McCammond - M)



not convex

$PB_n = \langle$ convex Dehn twists | disjointness, lantern $\rangle$

cf. Birman-Ko-Lee

- $Z(PB_n) = Z(B_n) = \langle T_\partial \rangle$

$(a_{12} \cdots a_{1n})(a_{23} \cdots a_{2n}) \cdots (a_{n-1,n})$

- $PB_n \cong PB_n / Z(PB_n) \times \mathbb{Z}$

$$1 \to \mathbb{Z} \rightleftarrows PB_n \xrightarrow{\text{Cap}} PB_n / Z(PB_n) \to 1$$

$1 \leftarrow a_{12}$

$0 \leftarrow a_{ij}$

splitting

More on PB_n

- Combing decomp:

$$PB_n \cong F_{n-1} \rtimes PB_{n-1}$$

Iterating:

$$PB_n \cong F_{n-1} \rtimes F_{n-2} \rtimes \cdots \rtimes F_2 \rtimes \overset{\substack{PB_2\\ \downarrow}}{\mathbb{Z}}$$

- Abelianization:

$$H_1(PB_n;\mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{2}}$$

Need $\binom{n}{2}$ maps $PB_n \to \mathbb{Z}$

$\binom{n}{2}$ forget maps $PB_n \to PB_2 \cong \mathbb{Z}$

Church-Farb. $H_1(PB_n;\mathbb{Z})$ is rep. stable: As Σ_n reps

$$H_1(PB_n;\mathbb{Z}) = \underbrace{\underset{\substack{\uparrow\\ \text{trivial}\\ \text{rep.}}}{\bigcirc} \oplus \underset{\substack{\uparrow\\ \text{std}\\ \text{irrep}}}{\square}}_{\text{std rep.}} \oplus \square\square$$

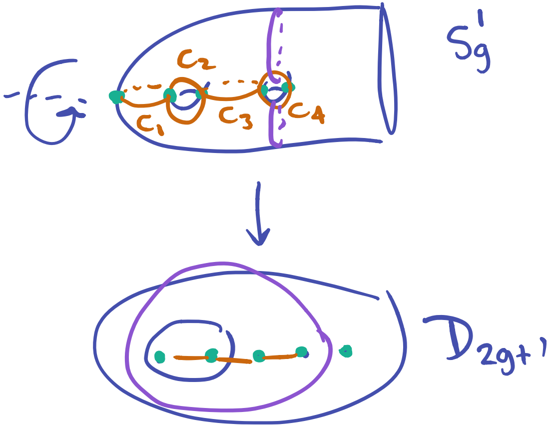
cf. Farb survey

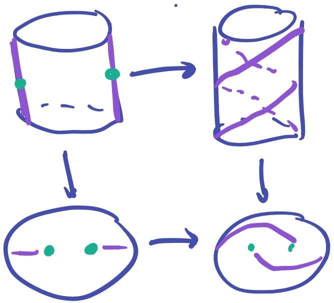
ICM

Birman-Hilden theory

Survey with Winarski

$\sigma_i \longmapsto T_{c_i}$





$$B_{2g+1} \longrightarrow \mathrm{Mod}(S_g^1)$$

$$\varphi \xrightarrow{\text{lift}} \tilde{\varphi}$$

BH thm. Injective.

Braid relns & chain relns come directly from

- braid reln in B_n
- writing center of B_n in terms of σ_i

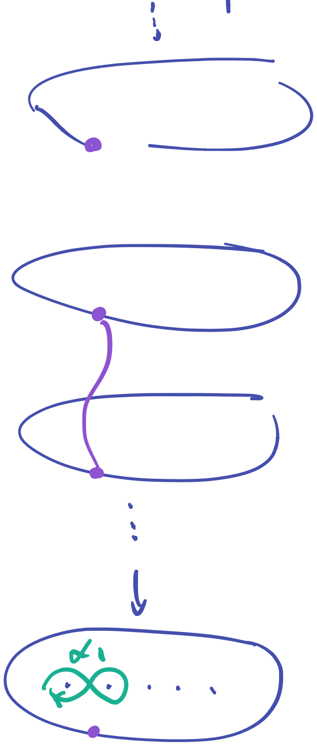
Burau rep

$H_1(X) = \langle t^i\alpha_1, \dots, t^i\alpha_{n-1} : i \in \mathbb{Z} \rangle$

f.g. as a $\langle t \rangle$-module.

∞ Parking garage

$\tilde{\varphi}$



φ

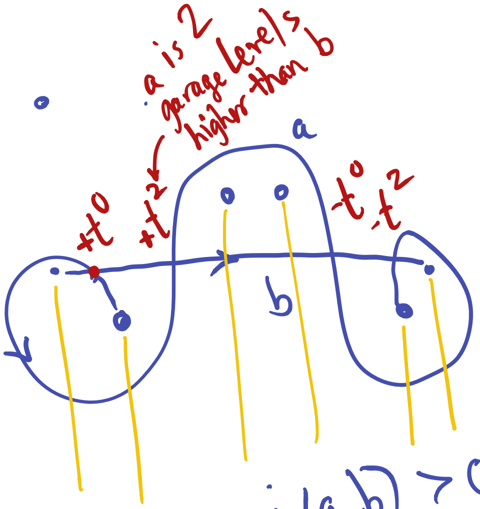
$\tilde{\varphi} \rightsquigarrow$ $n-1 \times n-1$ matrix entries in $\mathbb{Z}[t]$

action on $H_1(X)$

$$\begin{pmatrix} I & & \\ & \begin{matrix} 1-t & 1 \\ t & 0 \end{matrix} & \\ & & I \end{pmatrix}$$

$F_n \longrightarrow \mathbb{Z}$

$x_i \longmapsto 1$



$i(a,b) > 0$

"$\hat{i}$" $(a,b) = 0.$

$i(a,b) > 0$

$\hat{i}(a,b) = 0$

$\Rightarrow [T_a, T_b] \in \mathcal{I}(S_g)$

$\neq$

id