Announcements: Sep 6

- Midterm 1 on Sep 21
- Quiz 3 Friday in recitation
- WeBWorK 3.1 and 3.2 due Wednesday
- My office hours Wednesday 2:00-3:00 and Friday 9:30-10:30 in Skiles 234
- TA Office Hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Fri 12-1 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 9:30-10:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday - Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123
- Supplemental problems on master course web site

Section 3.2

Vector Equations and Spans

Outline of Section 3.2

- Learn the equivalences:

$$\text{vector equations} \leftrightarrow \text{augmented matrices} \leftrightarrow \text{linear systems}$$

- Learn the definition of span
- Learn the relationship between spans and consistency

Linear combinations, vector equations, and linear systems

We just saw the following question:

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$?

And saw it was the same as a vector equation:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

which is the same as the system of linear equations:

$$\begin{aligned} c_1 - c_2 &= 8 \\ 2c_1 - 2c_2 &= 16 \\ 6c_1 + c_2 &= 3 \end{aligned}$$

which we solve by row reducing, and we get $(c_1, c_2) = (5 - 3)$.

Linear combinations, vector equations, and linear systems

In general, asking if b is a linear combination of $v_1, \ldots, v_k$ is the same as solving the vector equation

$$c_1 v_1 + \cdots c_k v_k = b$$

which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

Span

Essential vocabulary word!

$$
\begin{aligned}
\mathrm{Span}\{v_1, v_2, \ldots, v_k\} &= \{c_1v_1 + c_2v_2 + \cdots c_kv_k \mid c_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\
&= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\
&= \text{plane through the origin and } v_1, v_2, \ldots, v_k.
\end{aligned}
$$

Four ways of saying the same thing:

- b is in Span$\{v_1, v_2, \ldots, v_k\}$
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $c_1v_1 + \cdots + c_kv_k = b$ has a solution
- the system of linear equations corresponding to

$$
\left(\begin{array}{cccc|c}
| & | & & | & | \\
v_1 & v_2 & \cdots & v_k & b \\
| & | & & | & |
\end{array}\right),
$$

  is consistent.

Demo Demo

Pictures for spans

What are the possibilities for the span of one vector in $\mathbb{R}^2$?

What are the possibilities for the span of two vectors in $\mathbb{R}^2$?

Demo

What are the possibilities for the span of one vector in $\mathbb{R}^3$?

What are the possibilities for the span of two vectors in $\mathbb{R}^3$?

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

Conclusion: Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

Summary of Section 3.2

- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

Section 3.3

Matrix equations

Outline Section 3.3

- Understand the equivalences:

  linear system $\leftrightarrow$ augmented matrix $\leftrightarrow$ vector equation $\leftrightarrow$ matrix equation

- Understand the equivalence:

  $$Ax = b \text{ is consistent} \longleftrightarrow b \text{ is in the span of the columns of } A$$

  (also: what does this mean geometrically)

- Learn for which A the equation $Ax = b$ is always consistent
- Learn to multiply a vector by a matrix

Multiplying Matrices

$$\text{row vector} \times \text{column vector} : \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$$

$$\text{matrix} \times \text{column vector} : \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = r_1 b + \cdots + r_m b$$

Example:

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} =$$

Multiplying Matrices

Another way to multiply

$$\text{matrix} \times \text{column vector}: \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = r_1 b + \cdots + r_m b$$

OR

$$\text{matrix} \times \text{column vector}: \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 c_1 + \cdots + b_n c_n$$

Example:

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} =$$

Linear Systems vs Augmented Matrices vs Matrix Equations vs Vector Equations

A matrix equation is an equation $Ax = b$ where A is a matrix and b is a vector. So x is a vector of variables.

A is an $m \times n$ matrix if it has m rows and n columns. What sizes must x and b be?

Example:

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \end{pmatrix}$$

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.

Solutions to Linear Systems vs Spans

Say that

$$A = \left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_n & b \\ | & | & & | & | \end{array}\right).$$

Fact. $Ax = b$ has a solution

$\iff$ there are numbers $x_1, \ldots, x_n$ with $x_1 v_1 + \cdots + x_n v_n = b$

$\iff$ b is a linear combination of the columns of A

$\iff$ b is in the span of columns of A

Why?

Again this is a basic fact we will use over and over and over.

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution

$\iff$ there are numbers $x_1, \dots, x_n$ with $x_1 v_1 + \cdots + x_n v_n = b$

$\iff$ b is a linear combination of the columns of A

$\iff$ b is in the span of columns of A

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

Is a given vector in the span?

Which of the following vectors are in the span of

$$(2, -1, 1), (1, 0, -1)?$$

- $(0, 2, 2)$
- $(3, -1, 0)$

Which of the following vectors are in the span of

$$(2, 3, 1, 4, 0), (3, 4, -1, 3, 5), (1, -1, 2, 4, 3)?$$

- $(3, 6, -5, -2, -7)$
- $(6, 19, -3, 4, -12)$

Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,10,20)$, $(0,-1,-2)$
2. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,5,7)$, $(0,6,8)$
3. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,1,0)$, $(0,0,\sqrt{2})$
4. $(0,1,2)$ is in the span of $(5,7,0)$, $(6,8,0)$, $(3,3,4)$

Pivots vs Solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

Properties of the Matrix Product Ax

$c =$ real number, $u, v =$ vectors,

- $A(u+v) =$
- $A(cv) =$

Application. If u and v are solutions to $Ax = 0$ then so is every element of $\mathrm{Span}\{u+v\}$.

Summary of Section 3.3

- Two ways to multiply a matrix times a vector:

$$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = r_1 b + \cdots + r_m b$$

$$\begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 c_1 + \cdots + b_n c_n$$

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.
- Fact. $Ax = b$ has a solution
  $\iff$ there are numbers $x_1, \ldots, x_n$ with $x_1 v_1 + \cdots + x_n v_n = b$
  $\iff$ b is a linear combination of the columns of A
  $\iff$ b is in the span of columns of A
- Theorem. Let A be an $m \times n$ matrix. The following are equivalent.
  1. $Ax = b$ has a solution for all b
  2. The span of the columns of A is $\mathbb{R}^m$
  3. A has a pivot in each row