Section 2.6

Subspaces

Outline of Section 2.6

- Definition of subspace
- Examples and non-examples of subspaces
- Spoiler alert: Subspaces are the same as spans
- Spanning sets for subspaces
- Two important subspaces for a matrix: $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$

Subspaces

A subspace of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ with:

1. The zero vector is in V.
2. If u and v are in V, then $u + v$ is also in V.
3. If u is in V and c is a scalar, then cu is in V.

The second and third properties are called "closure under addition" and "closure under scalar multiplication."

Together, the second and third properties could together be rephrased as: closure under linear combinations.

Which are subspaces?

1. the unit circle in $\mathbb{R}^2$
2. the point $(1, 2, 3)$ in $\mathbb{R}^3$
3. the xy-plane in $\mathbb{R}^3$
4. the xy-plane together with the z-axis in $\mathbb{R}^3$

Which are subspaces?

Poll

Is the first quadrant of $\mathbb{R}^2$ a subspace?

1. yes
2. no

Which are subspaces?

1. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, a + b = 0 \right\}$

2. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, a + b = 1 \right\}$

3. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, ab \neq 0 \right\}$

4. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, a, b \text{ rational} \right\}$

Spans and subspaces

Fact. Any $\mathrm{Span}\{v_1, \ldots, v_k\}$ is a subspace.

Why?

Fact. Every subspace V is a span.

Why?

So now we know that three things are the same:

- subspaces
- spans
- planes through 0

So why bother with the word "subspace"? Sometimes easier to check a subset is a subspace than to check it is a span (see null spaces, eigenspaces). Also, it makes sense (and is often useful) to think of a subspace *without a particular spanning set in mind*. Try thinking of other examples where it is useful to have two names for the same thing, like: water / $\mathrm{H_2O}$ or free throw / foul shot.

Column Space and Null Space

$A = m \times n$ matrix.

$\mathrm{Col}(A) =$ column space of $A =$ span of the columns of A

$\mathrm{Nul}(A) =$ null space of $A =$ (set of solutions to $Ax = 0$)

Example. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\mathrm{Col}(A) =$ subspace of $\mathbb{R}^m$

$\mathrm{Nul}(A) =$ subspace of $\mathbb{R}^n$

We have already been interested in both. We have been computing null spaces all semester. Also, we have seen that $Ax = b$ is consistent exactly when b is in the span of the columns of A, or, b is in $\mathrm{Col}(A)$.

Spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a spanning set for $\mathrm{Nul}(A)$
- the pivot columns of A form a spanning set for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

Notice that the columns of A form a (possibly larger) spanning set. We'll see later that the above recipe is the smallest spanning set.

Spanning sets

Find a spanning set for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

Subspaces and Null spaces

Fact. Every subspace is a null space.

Why? Given a spanning set, you can reverse engineer the A...

It's actually a little tricky to do this. Given the spanning set, you make those vectors the rows of a matrix, then row reduce and find vector parametric form, and then make those vectors the rows of a new matrix. Why does this work? Try an example!

Example. Find a matrix A whose null space is the span of $(1, 1, 1)$ and $(1, 2, 3)$. You should get the matrix $A = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$.

So now we know that four things are the same:

- subspaces
- spans
- planes through 0
- solutions to $Ax = 0$

(Make sure you understand what we mean when we say these are all the same!)

So why learn about subspaces?

If subspaces are the same as spans, planes through the origin, and solutions to $Ax = 0$, why bother with this new vocabulary word?

The point is that we have been throwing around terms like "3-dimensional plane in $\mathbb{R}^4$" all semester, but we never said what "dimension" and "plane" are. Subspaces give the proper way to define a plane. Soon we will learn the meaning of a dimension of a subspace.

All the ways

Here are all the ways we know to describe a subspace:

1. As span:

$$\mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2. As a column space:

$$\mathrm{Col}\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. As a null space:

$$\mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

4. As the set of solutions to a homogeneous linear system:

$$x + y + z = 0$$

5. Same, but in set builder notation:

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a + b + c = 0 \right\}$$

Section 2.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u + v$ is also in V.
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Four things are the same: subspaces, spans, planes through 0, null spaces

Typical exam questions

- Consider the set $\{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$. Is it a subspace? If not, which properties does it fail?
- Consider the x-axis in $\mathbb{R}^3$. Is it a subspace? If not, which properties does it fail?
- Consider the set $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$. Is it a subspace? If not, which properties does it fail?
- Find spanning sets for the column space and the null space of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- True/False: The set of solutions to a matrix equation is always a subspace.
- True/False: The zero vector is a subspace.