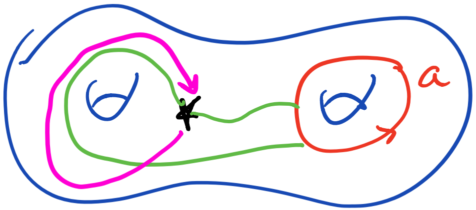
1.2.1 Closed curves & geodesics

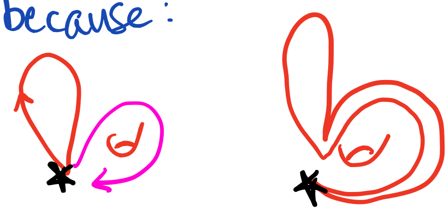
$$\left\{ \begin{array}{c} \text{conj classes} \\ \text{in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{free hom. classes} \\ \text{of oriented } \cancel{\text{s}}\text{.c.c.} \end{array} \right\}$$

$\boxed{\leftarrow}$



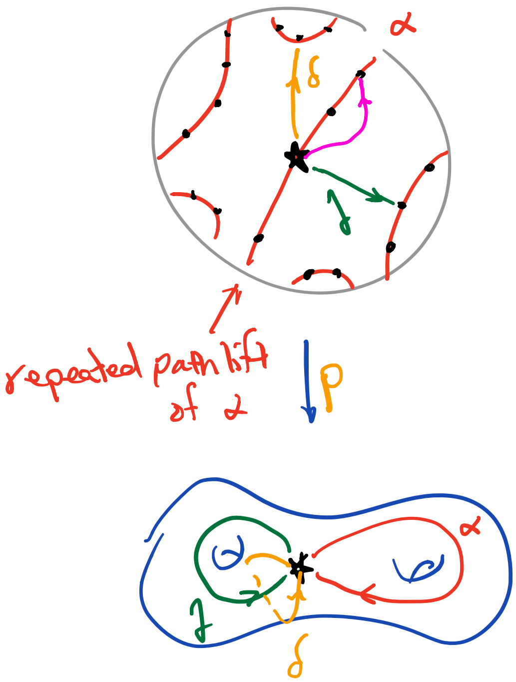
The two elts of π_1 you get differ by a point push $\longleftrightarrow$ conj.

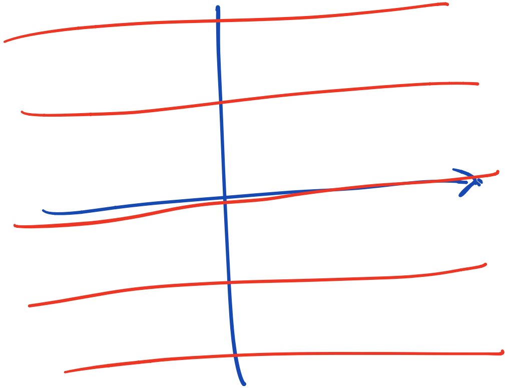
because:

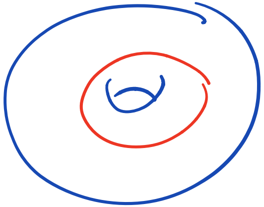


$$\left\{\begin{matrix}\text{elts of conj}\\ \text{class } a\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{lifts to } \mathbb{H}^2\\ \text{of } a\end{matrix}\right\}$$

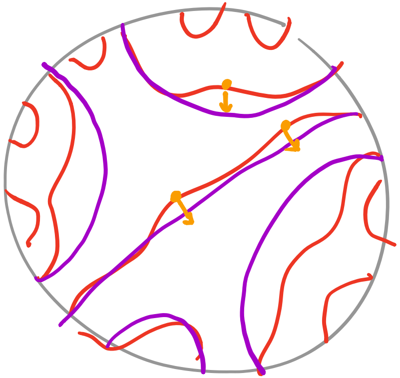
lift = component of $p^{-1}(\alpha)$

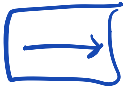




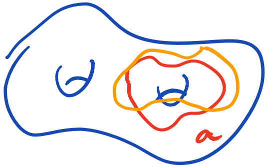


$\left\{ \begin{array}{l} \text{free hom. classes} \\ \text{of (simple) curves} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{(simple)} \\ \text{geodesics} \end{array} \right\}$



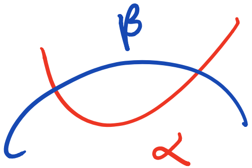


straight line homotopy to closest projection



injectivity: homotopies can't change endpts at $\partial_\infty \mathbb{H}^2$.

Bigon criterion α, β are in min pos. $\Longleftrightarrow$ α, β form no bigons.



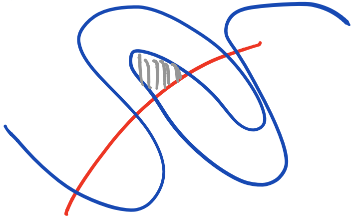
Bigon.

$\boxed{\Rightarrow}$ easy.

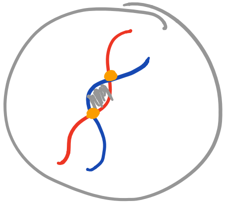
$\boxed{\Leftarrow}$ two proofs.

Lemma. α, β form no bigons
$\Longleftrightarrow$ any two lifts intersect 0,1 times.

Pf of Lemma. $\boxed{\Rightarrow}$ Lift the bigon to $\mathbb{H}^2$ (lifting criterion)
$\pi_1(\text{bigon}) = 1$,



$\boxed{\Leftarrow}$

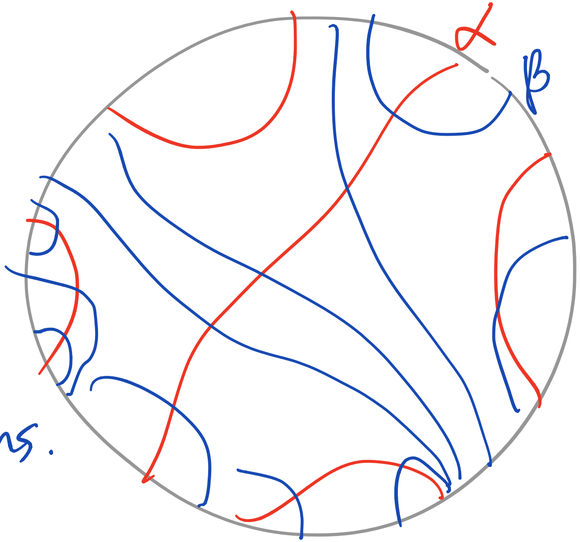


Check this bigon in $\mathbb{H}^2$ descends to bigon in S.
(check inj).

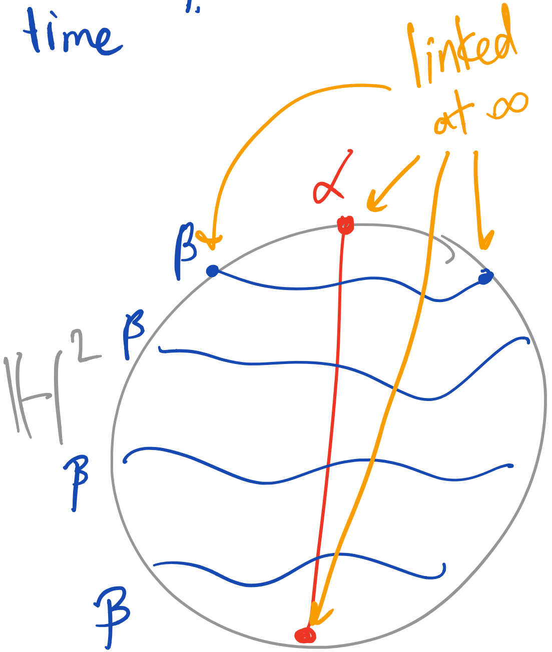
To prove Big. Crit, need to show:

If all lifts of α, β intersect ≤ 1 time

then α, β min. pos.



Homotopies is S can't change linking at ∞ and so can't remove intersections.

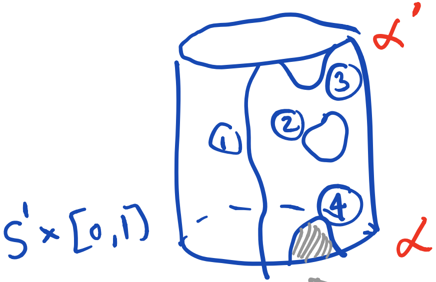


Proof #2.

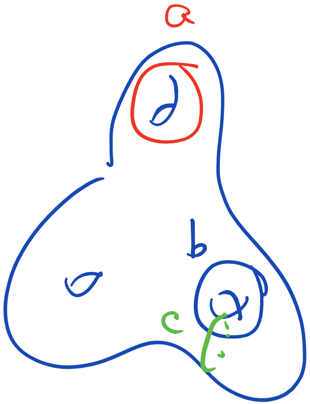
Suppose α, β not in min. pos.

Want to find a bigon.

Let $H: S^1 \times [0,1] \longrightarrow S$ be a homotopy of α that reduces intersection.



maps to bigon in S.



Chapter 2

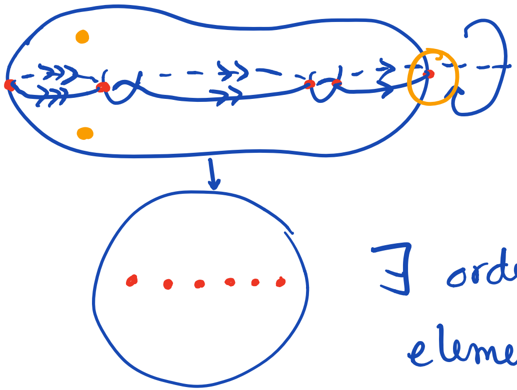
$$\mathrm{Mod}(S) = \pi_0\big(\mathrm{Homeo}^+(S,\partial S)\big)$$

& marked pts fixed as a set.

$$\cong \mathrm{Homeo}^+(S,\partial S)\,/\,\text{homotopy}.$$

order 5 or 10 in $\mathrm{Mod}(S_2)$?

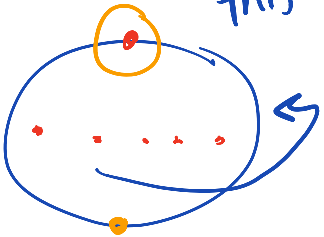
example order 5 elt in $\mathrm{Mod}(S_2)$



hyp inv.

$\exists$ order 5 element!

By lifting orbit this lifts.



Basic examples

D^2, $D^2 \setminus pt$, $S_{0,1} \cong \mathbb{R}^2$, $S_{0,0} \cong S^2$, $S_{0,3}$ maybe

A = annulus, $S_{1,0} = T^2$

marked pt

genus

Alexander Lemma

Prop. $\mathrm{Mod}(D^2) = 1$.

Pf. $\varphi \in \mathrm{Homeo}^+(D^2, \partial D^2)$

$\varphi_0 = \varphi$

$\varphi_1 = \mathrm{id}$

φ_t :

id

$1-t$

φ here

(really φ conj by scaling by $1-t$)

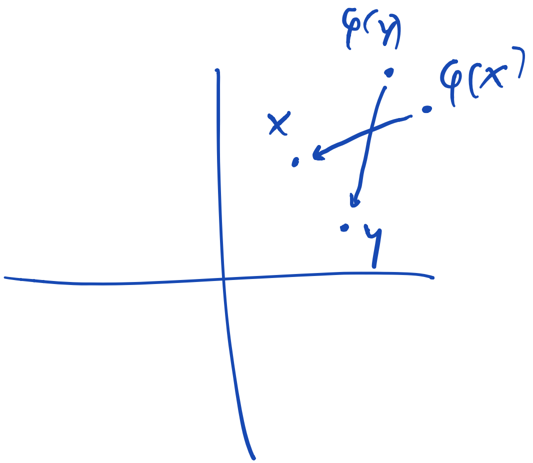
Cor of Proof

$\mathrm{Mod}(D^2 \setminus pt) = 1$.

Prop. $\mathrm{Mod}(S_{0,1}) = \mathrm{Mod}(\mathbb{R}^2) = 1$

Pf. Straight line homotopy.

$\varphi \in \mathrm{Homeo}^+(\mathbb{R}^2)$



Prop. $\mathrm{Mod}(S_{0,0}) = \mathrm{Mod}(S^2) = 1.$

Pf. First homotope so $\varphi(\text{north pole}) = \text{north pole}$

Apply prev. Prop.

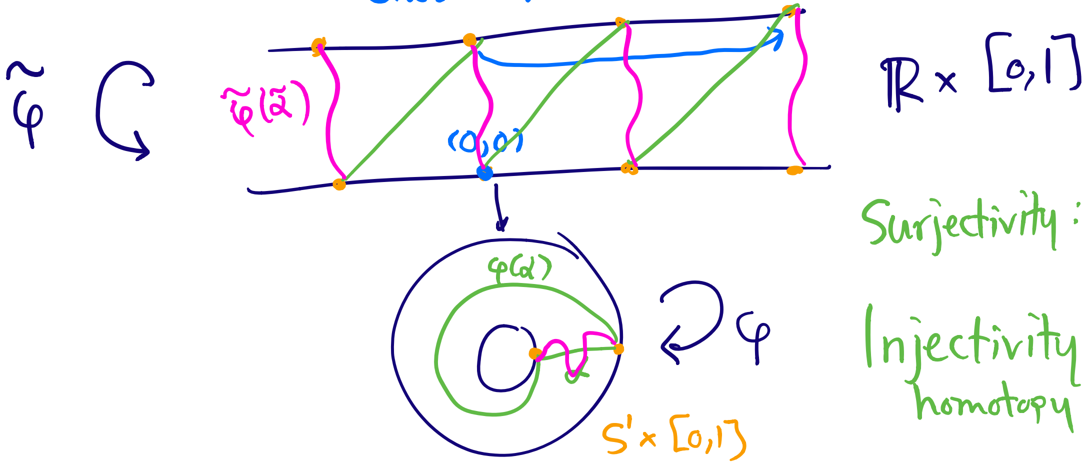
Prop. $\mathrm{Mod}(A) \cong \mathbb{Z}$.

Pf. Define $L : \mathrm{Mod}(A) \to \mathbb{Z}$.

Let $[\varphi] \in \mathrm{Mod}(A)$ univ. cover of A

Restrict $\tilde{\varphi}$ to $\mathbb{R} \times \{1\}$ in $\mathbb{R} \times [0,1]$

$\leftarrow$ chosen to fix $(0,0)$



Surjectivity: Dehn twist $\longrightarrow \pm 1$

Injectivity: Straight line homotopy $\tilde{\varphi} \to \mathrm{id}$.