Generating Torelli

Goal: $\mathcal{I}(S_g)$ is gen. by BP maps (and Dehn twists about sep curves)

Original proof: 1971 Birman gives presentation for $Sp_{2g}(\mathbb{Z})$
1978 Powell interprets relations
1980 Johnson, lantern relation

Want a proof analogous to $\text{Mod}(S_g)$ case.

Complex of homologous curves

Fix (primitive) $x \in H_1(S_g;\mathbb{Z})$

$C_x(S_g)$ = subgraph of $C(S_g)$ spanned by (unoriented) reps of x.

goal: Connected.

"borrowing complex"

Will use auxilliary complex $B_x(S_g)$, the complex of cycles. Points of $B_x(S_g)$ are simple, irredundant reps of x.

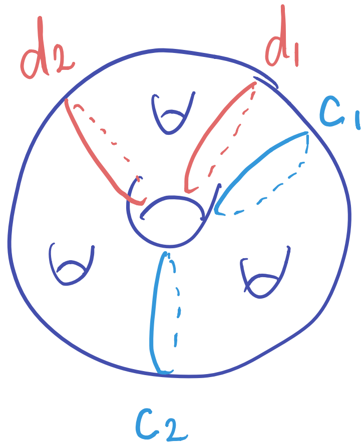
The Complex of Cycles

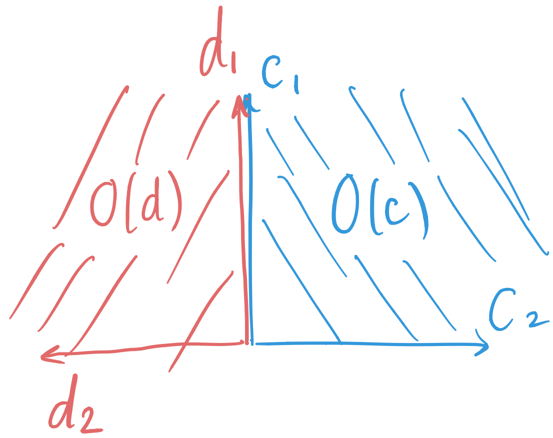
c = oriented multicurve, n components

$\rightsquigarrow \quad [0,\infty)^n \longrightarrow H_1(S_g;\mathbb{Z})$ orthant $O(c)$

$$A(S_g) = \coprod_c O(c) \;/\sim$$

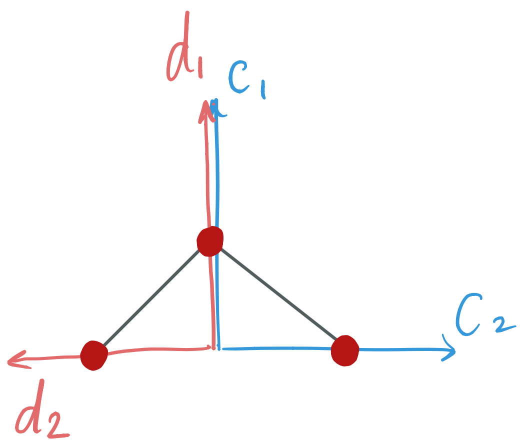
example.



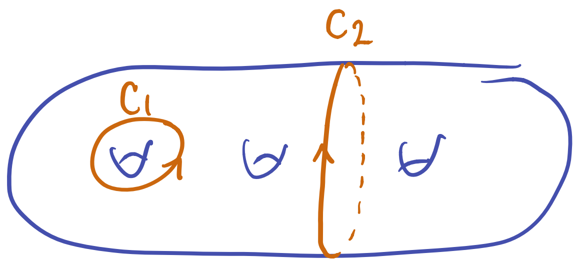


$$A_x(S_g) \subseteq A(S_g) \quad \text{reps of } x.$$

Say $x = [c_1]$

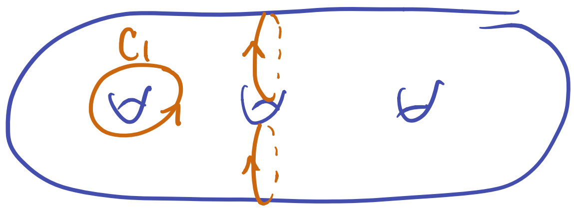


The cells of $A_x(S_g)$ are not necessarily compact:



If $[c_1] = x$ then $[c_1 + bc_2] = x$ $\forall\, b \in \mathbb{R}$

Or:



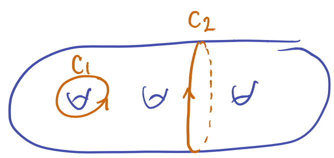
An oriented multicurve is <u>reduced</u> if

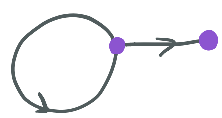
(1) the corresponding cell is compact

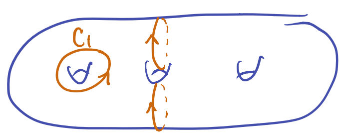
$\iff$ (2) it has no homologically trivial subset

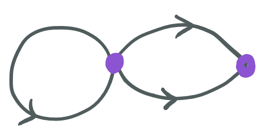
$\iff$ (3) the dual directed graph is recurrent

Dual graphs:







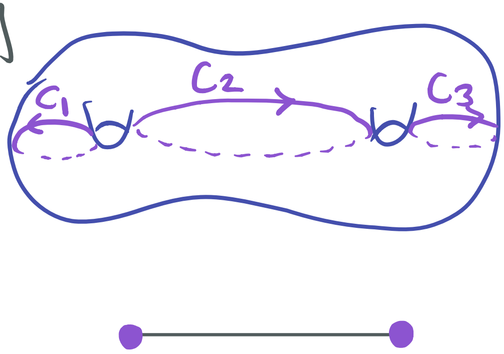


The complex of cycles $B_x(S_g)$ is the subcomplex of $A_x(S_g)$ whose cells correspond to reduced oriented multicurves.

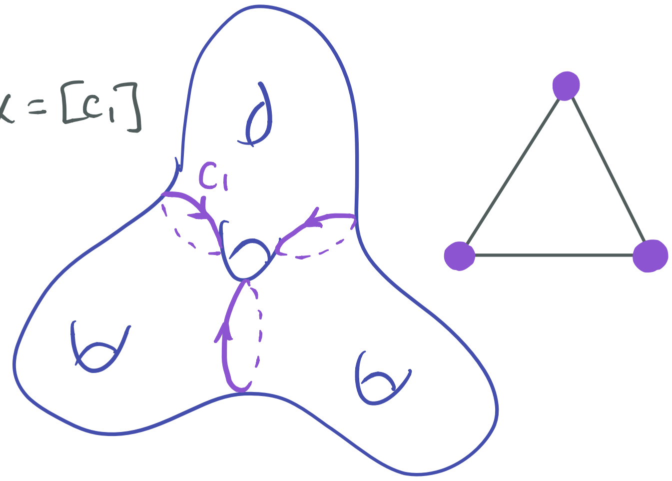
We'll show $B_x(S_g)$ is contractible.

Examples of cells

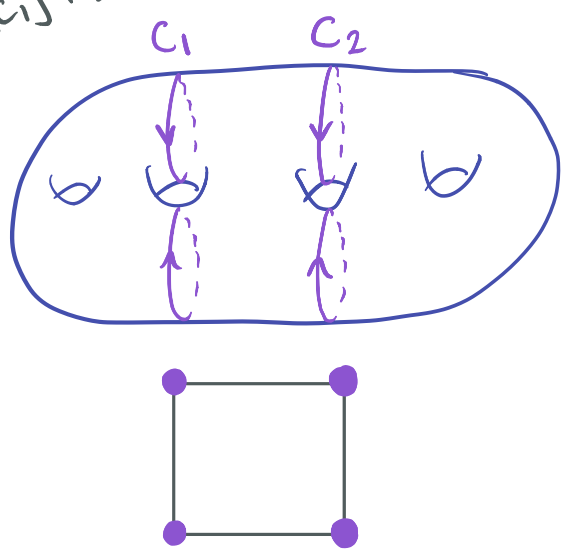
$x = [c_1]$

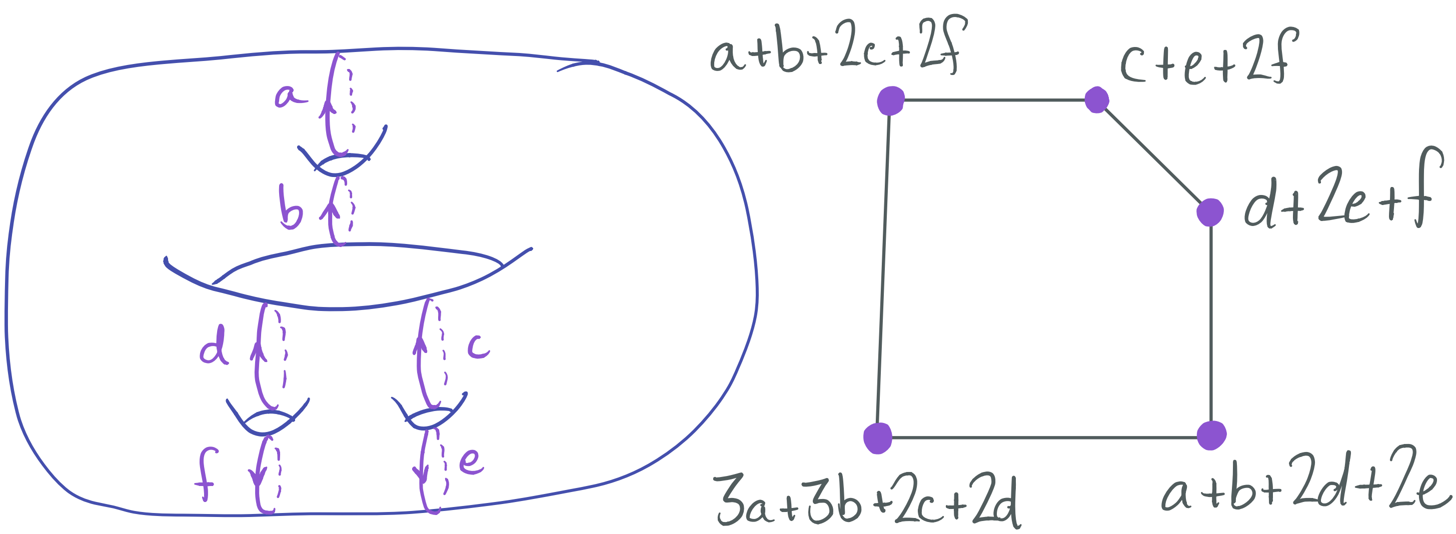


$x = [c_1]$



$x = [c_1] + [c_2]$





Q. Which polytopes arise?

Properties of Cells

Prop. The dim. of a cell = # compl. comp.'s -1.

Pf. Defn of homology.

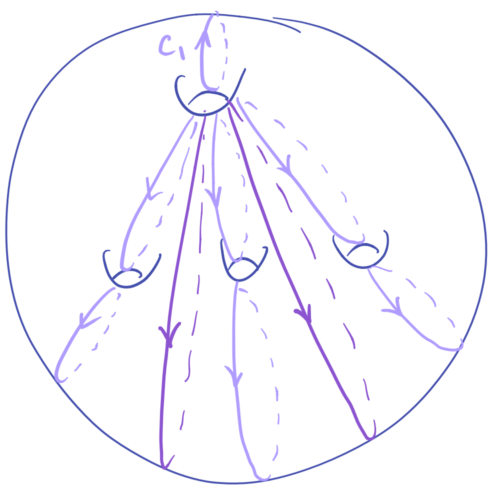
$\Longrightarrow$ vertices $\longleftrightarrow$ nonsep. multicurves.

Prop. Vertices of $B_x(S_g)$ are oriented multicurves with integral weights.

Pf. Given a vertex, consider a loop intersecting in one point.

Prop. $\operatorname{Dim} B_x(S_g) = 2g-3$.

Pf.



$x = [c_1]$.

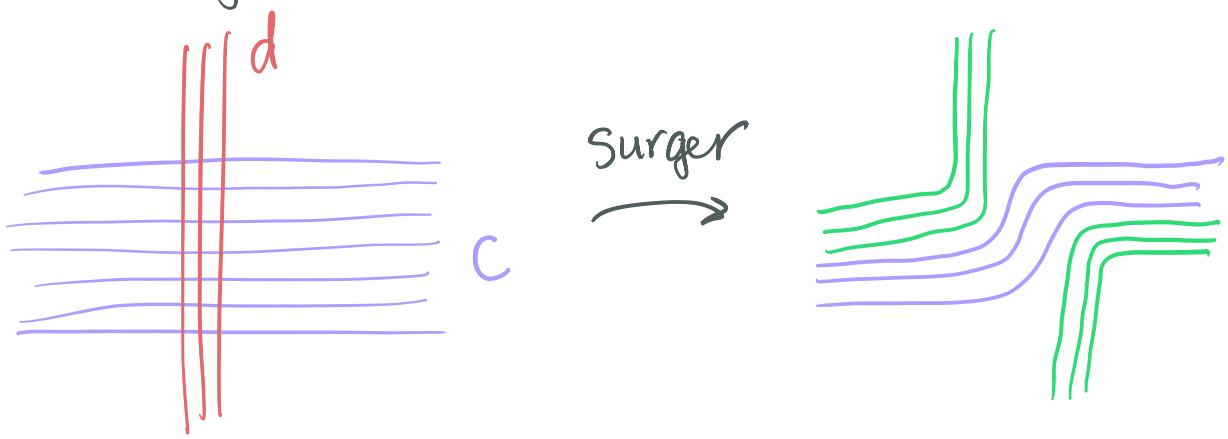
$\rightsquigarrow B_x(S_2)$ is a graph.

CONTRACTIBILITY

Theorem. $B_x(S_g)$ is contractible.

Surgery on 1-cycles

Say $c, d \in A_x(S_g)$. Thicken c, d according to weights and then:



If $[c] = [d] = x$, this procedure will result in a 1-cycle rep'ing x. Why?

$$H_1(S_g;\mathbb{Z}) \cong H^1(S_g;\mathbb{Z}) \cong \mathrm{Hom}(H_1(S_g;\mathbb{Z}),\mathbb{Z}) \longleftrightarrow [S_g, S^1]$$

The original c, d give maps $S_g \to S^1$ by integrating against width of annuli. The surgered picture corresponds to the map $S_g \to S^1$ obtained by integrating against both widths.

Prop. $Ax(S_g)$ is contractible

Pf. Fix some $c \in Ax(S_g)$. Consider:

$$F_t(d) = \operatorname{Surger}(tc + (1-t)d) \qquad \square$$

Draining 1-cycles

Suppose $c \in Ax(S_g)$ is not reduced.
$\rightsquigarrow \{R_i\}$ subsurfaces with $\partial R_i \subseteq c$
$\operatorname{Drain}_t(c) = c - t \sum \partial R_i$

Prop. $Ax(S_g)$ def. retracts to $Bx(S_g)$.
In partic. $Bx(S_g)$ is contractible.

Pf. Drain $\square$

In particular, $Bx(S_2)$ is a tree.

Connectivity of $C_x(S_g)$

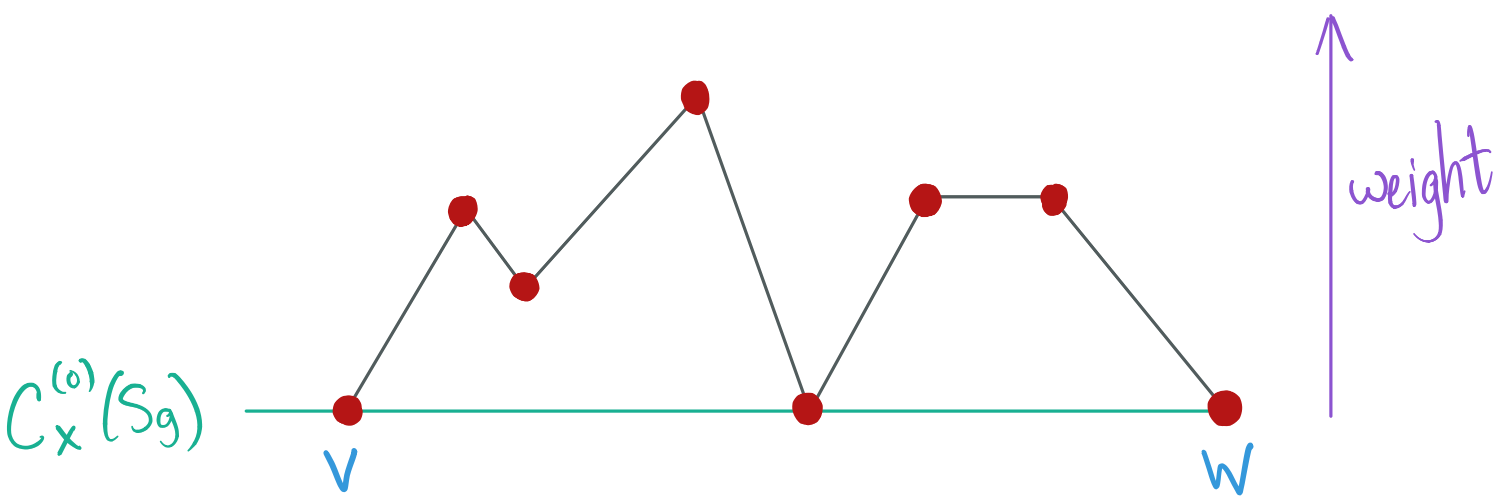
Basic strategy

Define weight : $B_x(S_g) \longrightarrow \mathbb{Z}$

$$\sum w_i c_i \longmapsto \sum w_i$$

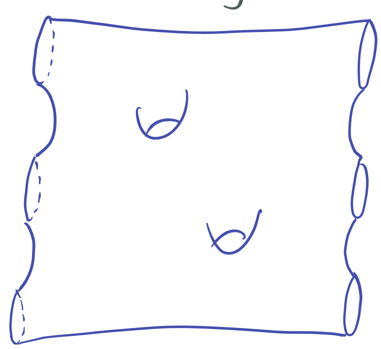
Note: $C_x(S_g) = \text{weight}^{-1}(1)$.

Now, given $v, w \in C_x^{(0)}(S_g)$, we connect them in $B_x^{(1)}(S_g)$:



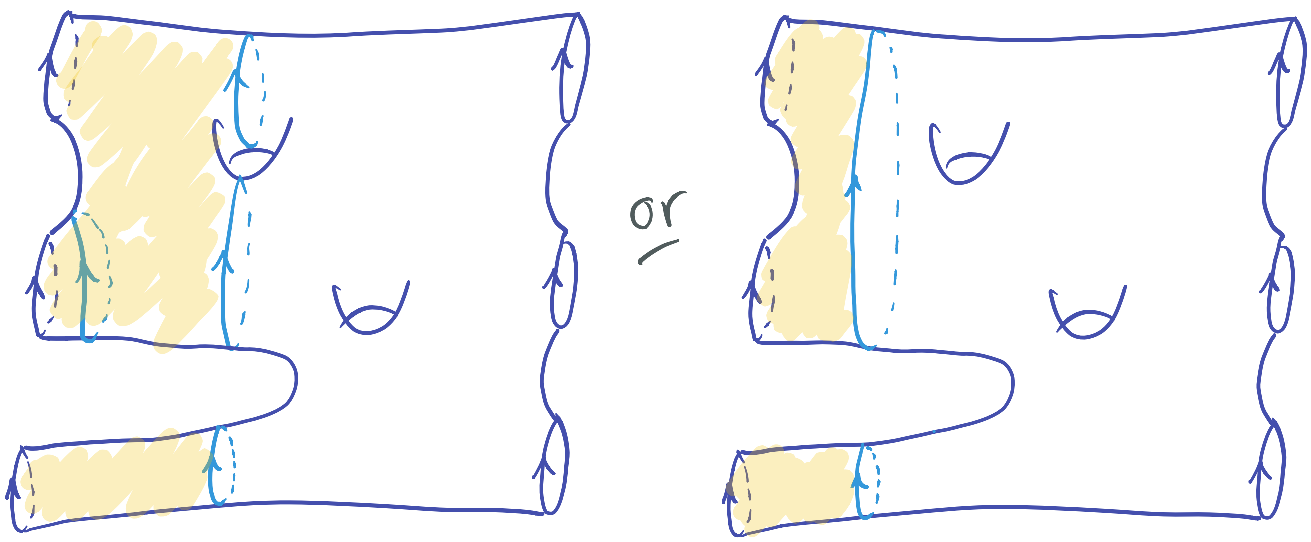
We then push the highest vertex down inductively until the path lies in $C_x(S_g)$.

Key idea: If we cut along a vertex of $B_x^{(0)}(S_g)$ we get



"cobordism"

What does an edge in $B_x(S_g)$ look like?

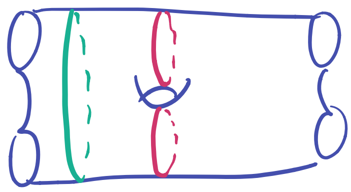


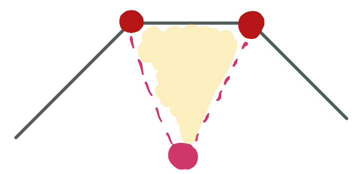
The region between transfers weight from one side to the other $\Longrightarrow$ the new vertex will have smaller weight iff there are fewer interior curves than boundary curves.

Call the edge on the right a <u>pants edge</u>.
This is the simplest way to reduce weight.

PROOF THAT $Cx(S_g)$ is connected

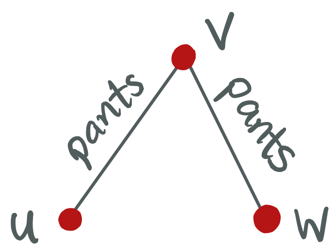
Step 1. Make maxima isolated, by making pants edges/triangles.



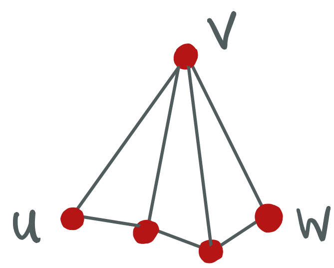


Step 2. Make highest edges into pants edges in same way

Step 3. Given



Connect uv to v by a seq of pants triangles



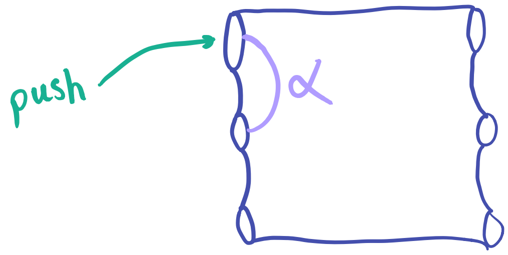
Can then push v down. Apply this process inductively.

To this end, consider the graph with

Vertices: pants edges emanating from v

$\longleftrightarrow$ certain arcs in $S \setminus v$

edges: disjoint arcs.



To show: connected

Notes.
- Every vertex is adjacent to one connecting first two components of $\partial(S \setminus v)$.
- Push maps (corresponding to 1^{st} ∂-comp act transitively on these.
- π_1(punctured sphere) has a simple gen set $\{x_i\}$

So: suffices to show that each $\mathrm{Push}(x_i) \cdot \alpha$ lies in same component as α.

Sample case: x_i lies on LHS of $S \setminus v$.
Then if β lies on RHS we have

$\square$