THE 27 LINES THM

$k = \mathbb{C}$

A cubic surface is the zero set in $\mathbb{P}^3$ of a homog. poly. in 4 vars.

Thm. A smooth cubic surface contains exactly 27 lines.

Basic strategy: Show that some cubic has 27 lines, then show the number of lines is locally constant in moduli space.

The Fermat cubic is

$$X = Z_p(x_0^3 + x_1^3 + x_2^3 + x_3^3)$$

(related to Fermat's last thm).

Lemma. The Fermat cubic contains exactly 27 lines.

Pf. Let X = Fermat cubic.

Observe X invariant under permutation of coords.

Up to permutation of coords, any line is the intersection of two planes of the form

$$x_0 = a_2 x_2 + a_3 x_3$$
$$x_1 = b_2 x_2 + b_3 x_3$$

(i.e. permute coords so pivots lie in first two cols.)

Such a line lies in $X \iff$

$$(a_2 x_2 + a_3 x_3)^3 + (b_2 x_2 + b_3 x_3)^3 + x_2^3 + x_3^3 = 0.$$

as a polynomial in $\mathbb{C}[x_2, x_3]$

Comparing coefficients:

$$a_2^3 + b_2^3 = -1 \qquad (1)$$
$$a_3^3 + b_3^3 = -1 \qquad (2)$$
$$a_2^2 a_3 = -b_2^2 b_3 \qquad (3)$$
$$a_2 a_3^2 = -b_2 b_3^2 \qquad (4)$$

If $a_2, b_2, a_3, b_3 \neq 0$ then $(3)^2/(4)$ gives

$$a_2^3 = -b_2^3$$

contradicting (1).

So at least one is zero. WLOG $a_2 = 0$.

$$(1) \implies b_2^3 = -1$$
$$(3) \implies b_3 = 0$$
$$(2) \implies a_3^3 = -1$$

Conversely, any such values give a line in X.
There are 9 choices, since -1 has 3 cube roots.
Permuting coords, get 27 lines. $\square$

Cor. Let X = Fermat cubic

(a) Given any line L in X, there are exactly 10 other lines in X that intersect L.

(b) Given any two disjoint lines L_1, L_2 in X there are exactly 5 other lines in X meeting both.

Pf. We have a list of all the lines, so check.
For example, consider L given by

$$x_0 + x_3 = 0$$
$$x_1 + x_2 = 0.$$

One example of a line that intersects it is

$$x_0 + x_1 = 0$$
$$x_2 + x_3 = 0.$$

(row reduce, get a free var) $\square$

The incidence graph is the complement of the Schläfli graph.

Moduli Spaces

Consider now the moduli space of all cubic surfaces, that is, the space of homog deg 3 polys in x_0, x_1, x_2, x_3 up to scale:

$$\mathbb{P}^{19} = \mathbb{P}^{\binom{3+3}{3}-1}$$

The set U of smooth cubic surfaces is dense and open (the open-ness comes from the fact that non-smoothness is characterized by the rank of the Jacobian, and the density comes from the fact that all nonempty Zariski opens are dense in Eucl. topology, hence dense in Zar. top.)

Notation: Write an elt as $f_c = \sum c_\alpha x^\alpha$ ← multi-index

The corresponding point in $\mathbb{P}^{19}$ is $c = (c_\alpha)$

Lemma. U is connected in classical topology.

Pf. It is the complement of a Zariski closed subset, which has real codim ≥ 2 $\square$

The set of lines in $\mathbb{P}^3$ corresponds to $G(2,4)$, the Grassmannian of 2-planes in k^4. This is another moduli space.

The Incidence Correspondence

There is an incidence correspondence

$$M = \{(X,L) : L \subseteq X\} \subseteq U \times G(2,4)$$

There is a projection map

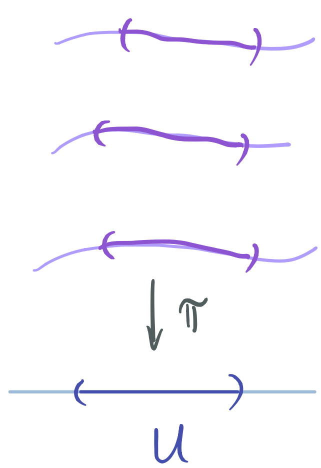
$$\pi : M \longrightarrow U$$
$$(X,L) \longmapsto X$$

The number of lines in X is $|\pi^{-1}(X)|$.
Want to show this is constant on U.

Lemma. The incidence correspondence is...

(a) closed in the Zariski topology on $U \times G(2,4)$

(b) locally (in the classical topology) the graph of a continuously differentiable fn $U \to G(2,4)$



$M \subseteq U \times G(2,4)$

Think covering spaces.

Proof of the Theorem

Pf. We use the classical (Euclidean) topology.
Since U is connected, suffices to show #lines is locally const.

Fix some $X \in U$. Let $L \subseteq \mathbb{P}^3$ be an arbitrary line.

Case 1. $L \subseteq X$. In this case the second statement of the lemma gives an open nbd $V_L \times W_L$ of (X, L) in $U \times G(2,4)$ in which the incidence corresp. is the graph of a C^1 function.
$\Longrightarrow$ every pt in V_L contains exactly 1 line in W_L.

Case 2. $L \nsubseteq X$. In this case there is an open nbd $V_L \times W_L$ of (X, L) s.t. no cubic in V_L contains any line in W_L (since the incidence corresp. is closed).

Let L vary. Since $G(2,4)$ compact, there are finitely many W_L that cover $L \times G(2,4)$. Let V be corresp. intersection of V_L, which is an open nbd of X. By construction, in V all cubic surf's have same # of lines (the number of W_L from Case 1. $\square$

Many Lines from One

Say X is a smooth cubic,
L a line on X.

Consider the planes containing L.

For each such plane, the intersection with X is a conic.

When the conic degenerates, get two lines of intersection with X, L and another line.

The equation for degeneracy is a discriminant.

Can iterate this procedure to find all the lines.

The Klein Quartic

After 27 lines, we have:

Thm. A smooth quartic curve in $\mathbb{CP}^2$ has 28 bitangents.

Instead of proving this, let's look at one particularly interesting example of a smooth quartic plane curve:

$$x^3y + y^3z + z^3x = 0$$

This is the Klein quartic surface. It is the subject of *The Eightfold Way*, the sculpture by Helaman Ferguson outside MSRI.

It was discovered by Felix Klein in 1879.

It is an analogue of a Platonic solid for 7-gons, similar to how a torus is the analog for 6-gons.

The Klein quartic divides into

24 heptagons

They meet in triples at

56 vertices

There are 84 edges. So

$$V - E + F = -4 \Rightarrow g = 3.$$

Any heptagon can be taken to any other, and each heptagon can be rotated by $1/7$, giving

$$24 \times 7 = 168 \text{ symmetries}$$

(actually, 336 if flips are allowed).

<u>84(g-1) theorem</u>. A smooth curve of genus g has $\leq 84(g-1)$ symmetries.

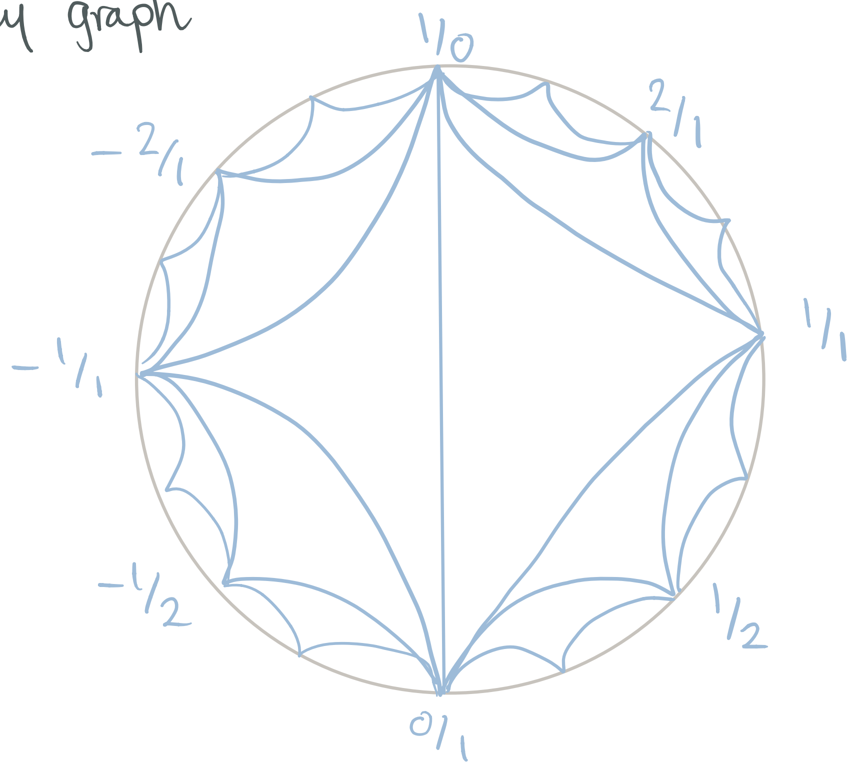
So the Klein quartic has maximal symmetry.

For instance, if we replace x with y/x and y with $1/x$, get

$$y^3/x^4 + 1/x^3 + y/x = 0$$

Clearing denominators (and homogenizing) gives the original equation.

Now we'll give instructions for finding the above combinatorial description. Start with the Farey graph



There is an edge p/q ●—● r/s when $|ps - rq| = 1$.
Thus $SL_2\mathbb{Z}$ acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$$

The matrix $-I$ acts trivially since $p/q = -p/-q$ so really $PSL_2\mathbb{Z}$ acts.

Let $PSL_2\mathbb{Z}[m]$ be the kernel of

$$PSL_2\mathbb{Z} \longrightarrow PSL_2\mathbb{Z}/m$$

These subgroups also act.

For small m get familiar objects as the quotient

m	quotient
2	pillow
3	tetrahedron
4	octahedron
5	icosahedron
6	torus?
7	Klein quartic

This is the *dual* triangulation to the heptagon decomposition.

In each case, if you start in the middle of an edge and turn

LRLR....

You get back where you started. For the tetrahedron you do LR twice, for the octahedron thrice, and for Klein 4 times. For the icosahedron it is 5 times (?!).

From the description of the Klein quartic as

$$\mathbb{H}^2 / PSL_2\mathbb{Z}[7]$$

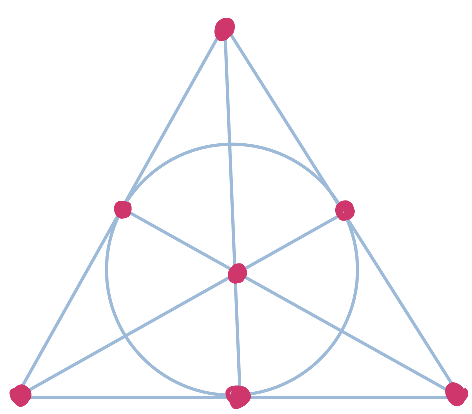
We see that $PSL_2\mathbb{Z}/7$ acts. This group has 168 elements!

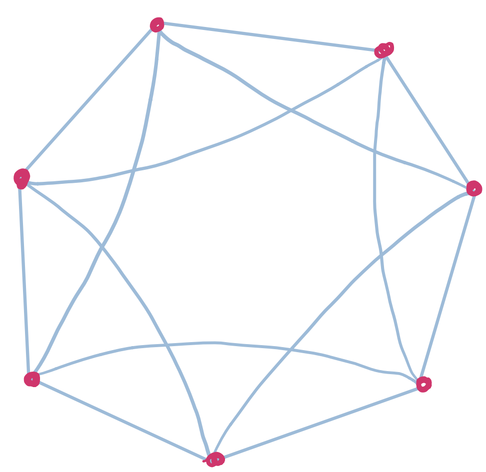
$PSL_2\mathbb{Z}/7$ is the 2^{nd} smallest nonabelian simple group, after A_5, the icosahedral group, which is also $PSL_2\mathbb{Z}/5$

There is an isomorphism

$$PSL_2\mathbb{Z}/7 \cong PSL_3\mathbb{Z}/2$$

The latter is the group of isometries of the Fano plane, $\mathbb{P}^2_{\mathbb{F}_2}$





Can permute the 7 pts. While fixing one pt can permute the 4 lines not containing that pt $\rightsquigarrow$ $7 \cdot 4! = 168$ symmetries.

What is the connection to the Fano plane?
In the Klein quartic, can group the 56 triangles into 7 groups of 8. Each group of 8 has the symmetries of a cube (24 of them).

If you draw Klein as a "tetrahedron" the 8 triangles in one group are the ones that lie at the 4 vertices of the tetrahedron, inside and out.