Math 1553

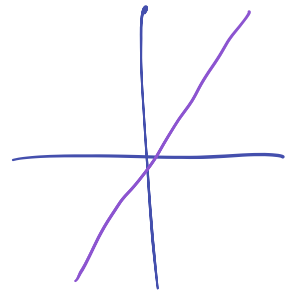
Section M (M01-M05)

Georgia Tech Fall 2021

Dan Margalit

Linear. Algebra.

What is Linear Algebra?



Linear

$y = 2x$

$3x + 2y + z = 1$

$\cancel{x^2 + \sin x + xy + e^x}$

no ~~exponents~~

no trig etc.

Algebra

- from al-jebr (Arabic), meaning reunion of broken parts
- 9^{th} century Abu Ja'far Muhammad ibn Muso al-Khwarizmi

Why a whole course?

Engineers need to solve *lots* of equations in *lots* of variables.

$$\begin{array}{rcrcrcrcrcrcr}
3x_1 & + & 4x_2 & + & 10x_3 & + & 19x_4 & - & 2x_5 & - & 3x_6 & = & 141 \\
7x_1 & + & 2x_2 & - & 13x_3 & - & 7x_4 & + & 21x_5 & + & 8x_6 & = & 2567 \\
-x_1 & + & 9x_2 & + & \frac{3}{2}x_3 & + & x_4 & + & 14x_5 & + & 27x_6 & = & 26 \\
\frac{1}{2}x_1 & + & 4x_2 & + & 10x_3 & + & 11x_4 & + & 2x_5 & + & x_6 & = & -15
\end{array}$$

Often, it's enough to know some information about the set of solutions without having to solve the equations at all!

In real life, the difficult part is often in recognizing that a problem can be solved using linear algebra in the first place: need *conceptual* understanding.

Linear Algebra in Engineering

Almost every engineering problem, no matter how huge, can be reduced to linear algebra:

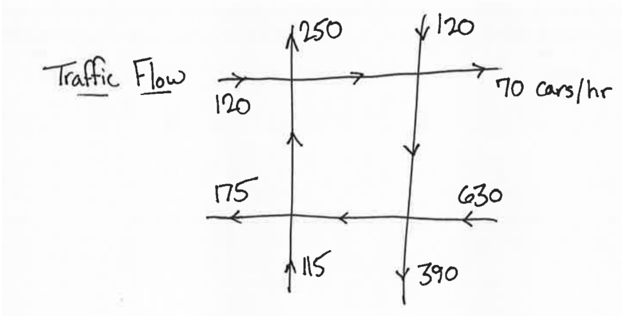
$$Ax = b \qquad \text{or}$$

$$Ax = \lambda x \qquad \text{or}$$

$$Ax \approx x$$

Applications of Linear Algebra

Civil Engineering: How much traffic lies in the four unlabeled segments?



Applications of Linear Algebra

Chemistry: Balancing reaction equations

$$\underline{\quad x \quad}\ C_2H_6 + \underline{\ \ y\ \ }\ O_2 \rightarrow \underline{\quad z \quad}\ CO_2 + \underline{\quad w \quad}\ H_2O$$

Applications of Linear Algebra

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If the numbers of first, second, and third year rabbits in 2021 are 10, 4, and 5, then what are they in 2022?

$$F_{2022} = 6 \cdot 4 + 8 \cdot 5 = 64$$
$$S_{2022} = 1/2 \cdot 10 = 5$$
$$T_{2022} = 1/2 \cdot 4 = 2.$$

If the numbers of first, second, and third year rabbits in year n are F_n, S_n, and T_n, what are the numbers in year $n+1$?

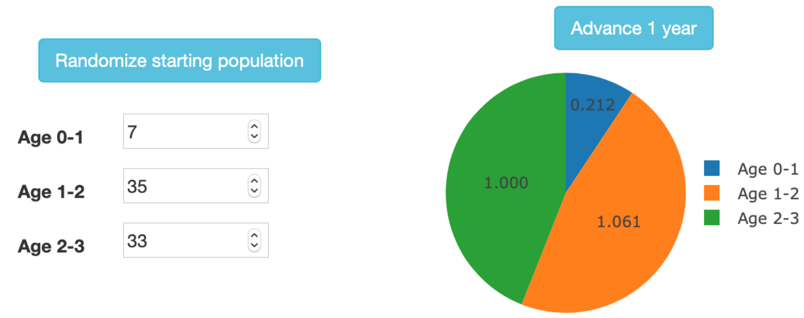
$$F_{n+1} = 6 \cdot S_n + 8 \cdot T_n$$
$$S_{n+1} = 1/2\, F_n$$
$$T_{n+1} = 1/2\, S_n$$

linear eqns

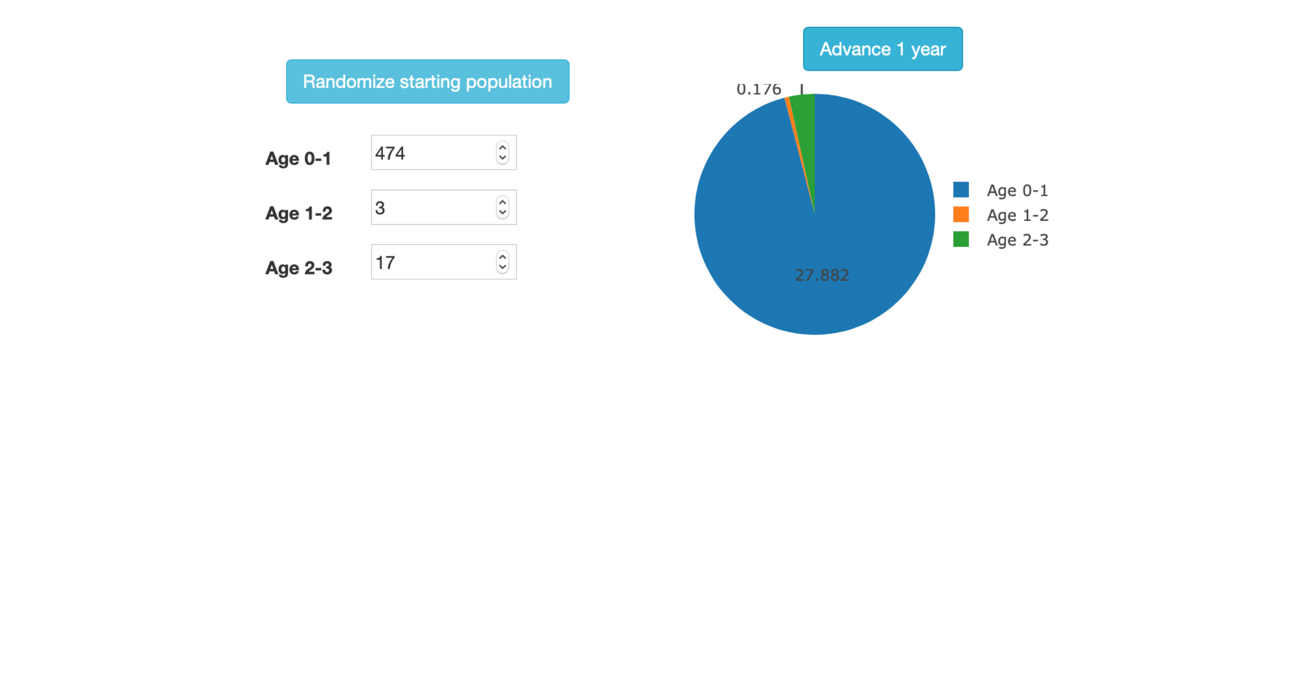
Demo

What happens in long term?

Rabbit populations



Rabbit populations



Rabbit populations

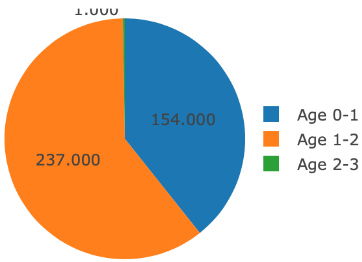
Randomize starting population

Age 0-1 154

Age 1-2 237

Age 2-3 1

Advance 1 year



Rabbit populations

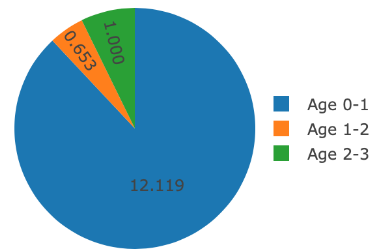
Randomize starting population

Age 0-1 1430

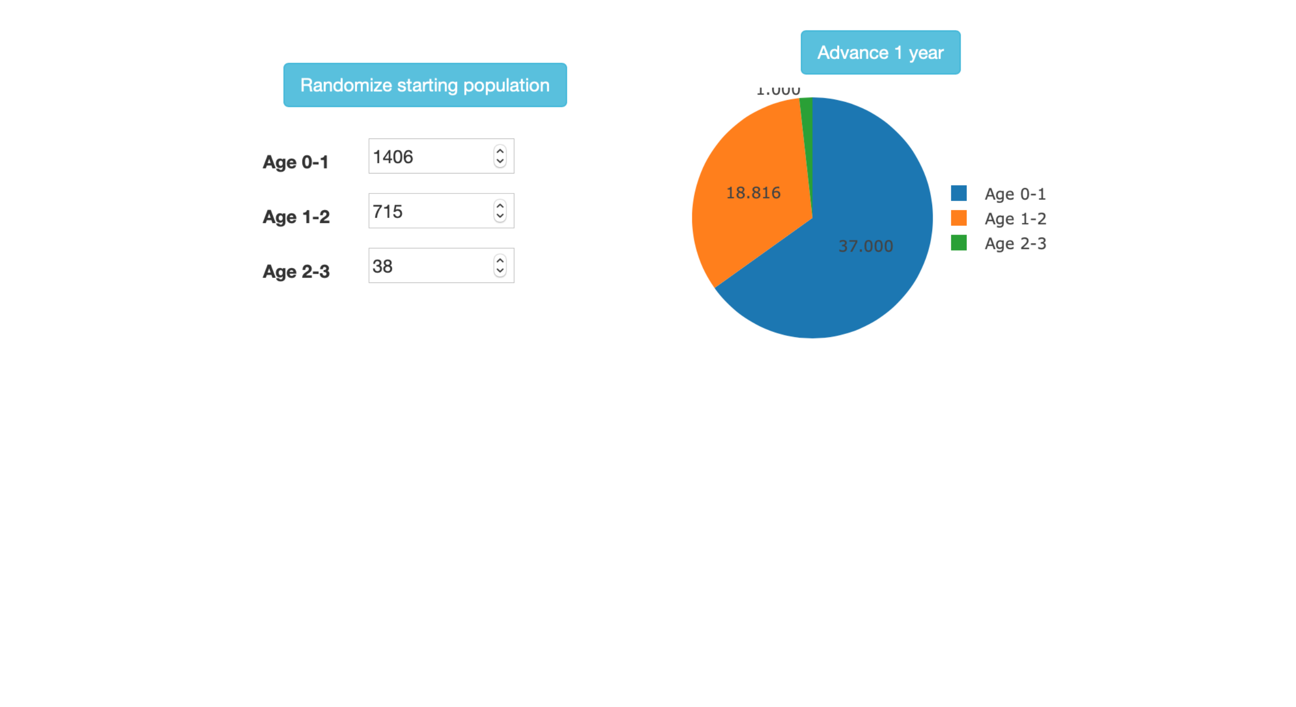
Age 1-2 77

Age 2-3 118

Advance 1 year



Rabbit populations

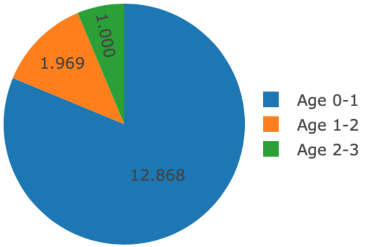


Rabbit populations

Randomize starting population

Age 0-1	4594
Age 1-2	703
Age 2-3	357

Advance 1 year



Rabbit populations

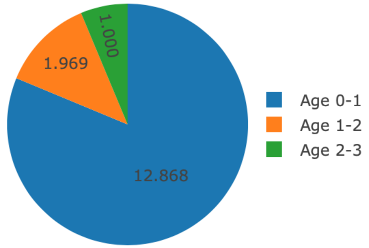
Randomize starting population

Age 0-1 4594

Age 1-2 703

Age 2-3 357

Advance 1 year



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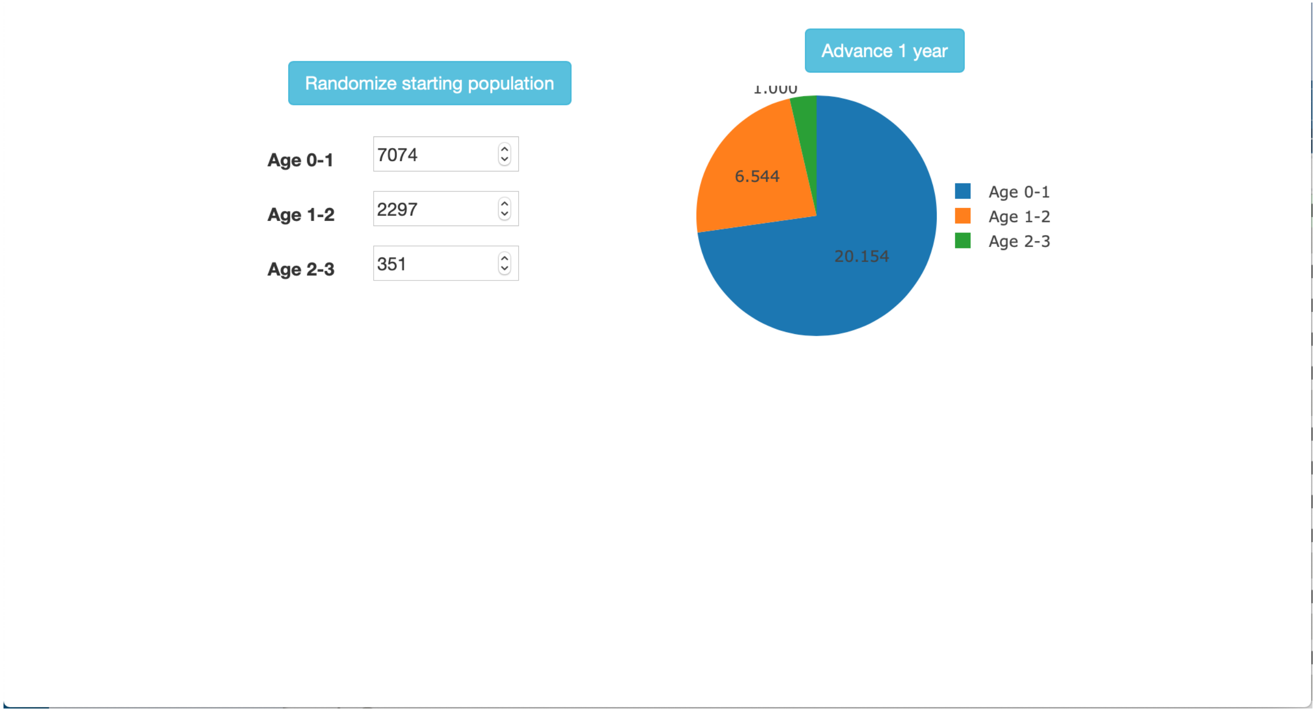


Rabbit populations

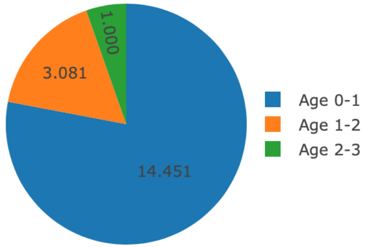
Randomize starting population

Age 0-1 16590

Age 1-2 3537

Age 2-3 1148

Advance 1 year

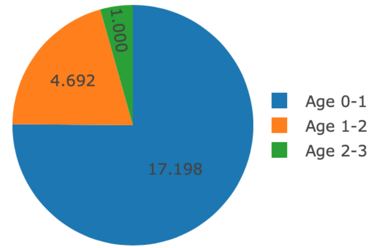


Rabbit populations

Advance 1 year

Randomize starting population

Age 0-1	30406
Age 1-2	8295
Age 2-3	1768



Rabbit populations

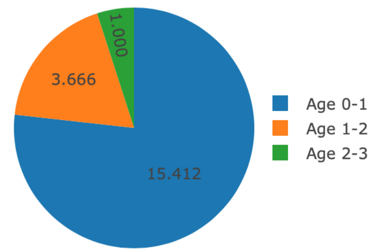
Randomize starting population

Age 0-1 63914

Age 1-2 15203

Age 2-3 4147

Advance 1 year



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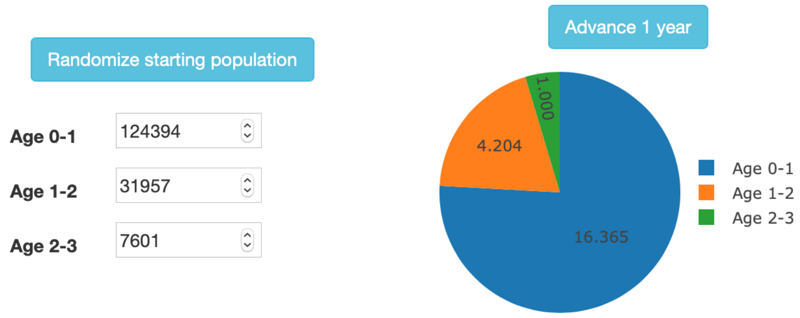
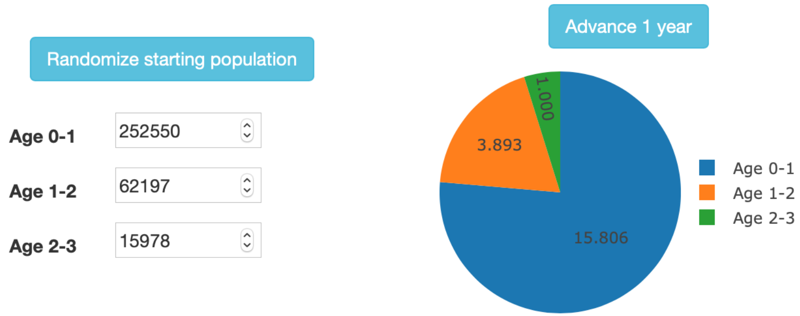
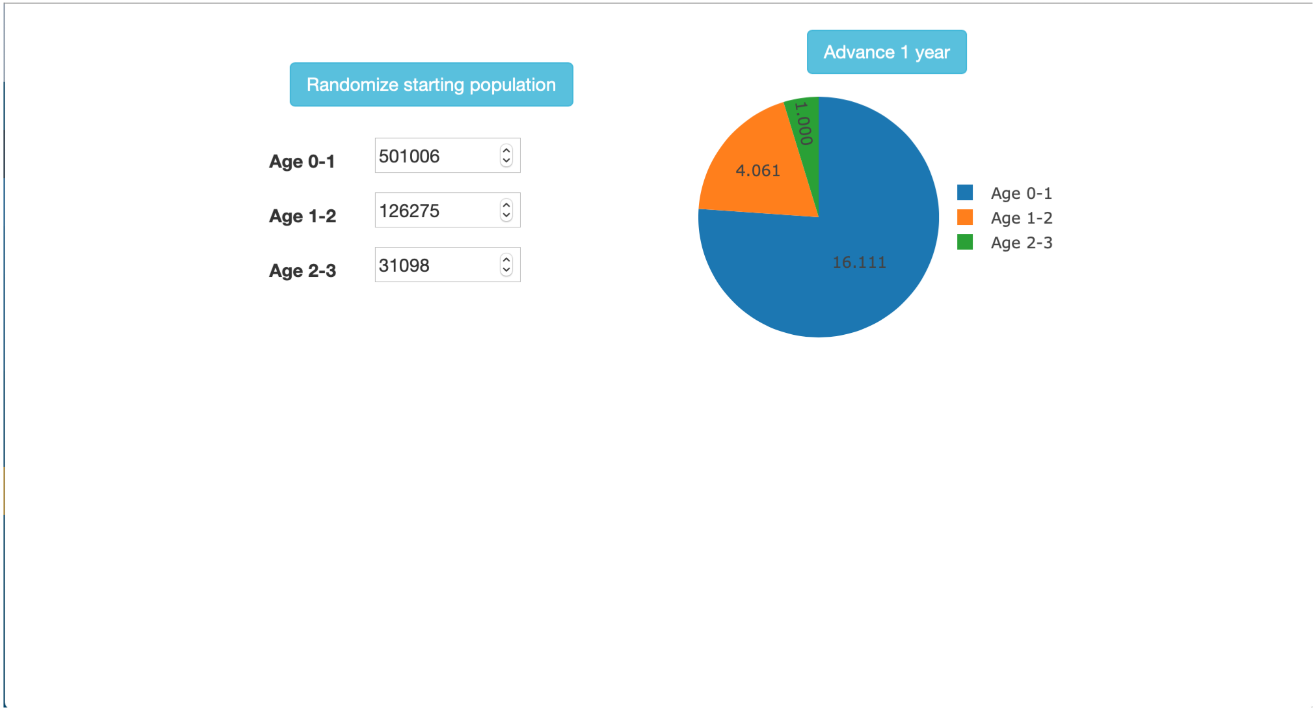


Rabbit populations



Rabbit populations



Rabbit populations

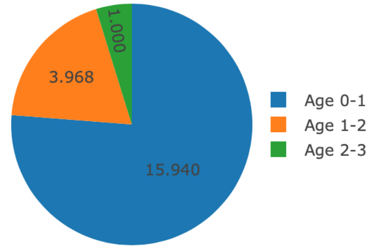
Randomize starting population

Age 0-1 1006434

Age 1-2 250503

Age 2-3 63137

Advance 1 year



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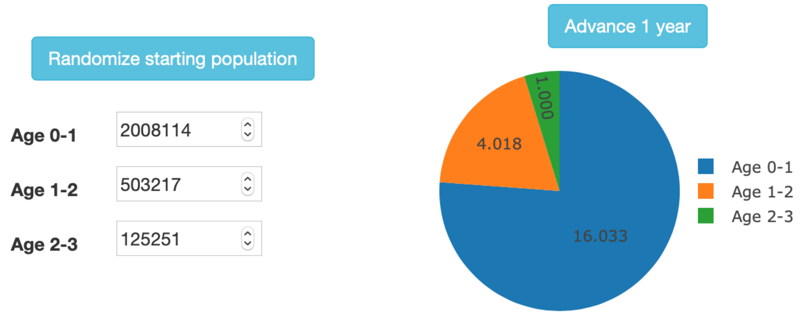

eigenvector.

Demo

Applications of Linear Algebra

Geometry and Astronomy: Find the equation of a circle passing through 3 given points, say (1,0), (0,1), and (1,1). The general form of a circle is $a(x^2 + y^2) + bx + cy + d = 0 \rightsquigarrow$ system of linear equations.

Very similar to: compute the orbit of a planet: $a(x^2 + y^2) + bx + cy + d = 0$

Applications of Linear Algebra

Google: "The 25 billion dollar eigenvector." Each web page has some importance, which it shares via outgoing links to other pages $\rightsquigarrow$ system of linear equations. Stay tuned!

Overview of the course

- Solve the matrix equation $Ax = b$

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  - Understand the dynamics of a linear transformation via the computation of eigenvalues, eigenvectors, and diagonalization
- Almost solve the equation $Ax = b$
  - Find best-fit solutions to systems of linear equations that have no actual solution using least squares approximations

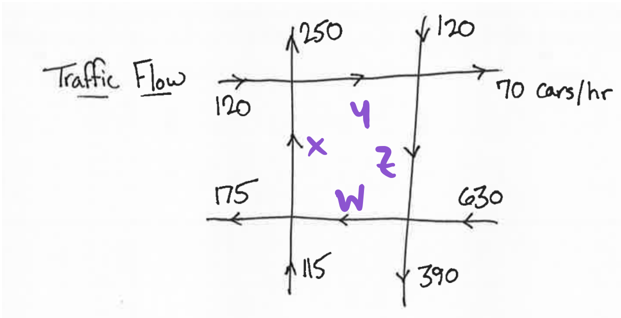
Announcements Aug 25

- Please turn on your camera if you are able and comfortable doing so
- Mathematical autobiography due on Friday on Piazza (tag: margalit-autbio): picture, name/pronunciation, major, hobbies, math you've taken, feelings about math and/or class, plus anything else!
- Use Piazza for general questions
- WeBWorK Warmup due Fri (not for a grade). WeBWorK 1.1 due Tuesday
- Office hours for this week only: Thu 1-2 Skiles courtyard
- TA Office Hours
  - Ian tba
  - Patrick tba
  - Joseph tba
  - Ivan tba
  - Jieun tba
- Studio on Friday in person; Studio for M02 will be recorded/streamed
- Section M web site: Google me, click on Teaching, Math 1553
  - future blank slides, past lecture slides
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`

Canvas → Assignments → Webwork

Applications of Linear Algebra

Civil Engineering: How much traffic lies in the four unlabeled segments?



For each intersection

cars in = # cars out

$\rightsquigarrow \; x + 120 = y + 250$

3 other eqns...

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- each rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If the numbers of first, second, and third year rabbits in 2021 are 10, 4, and 5, then what are they in 2022?

$$F_{2022} = 6 \cdot S_{2021} + 8 \cdot T_{2021} = 6 \cdot 4 + 8 \cdot 5$$
$$S_{2022} = 1/2\, F_{2021} = 1/2 \cdot 10$$
$$T_{2022} = 1/2\, S_{2021} = 1/2 \cdot 4$$

If the numbers of first, second, and third year rabbits in year n are F_n, S_n, and T_n, what are the numbers in year $n+1$?

$$F_{n+1} = 6 \cdot S_n + 8 \cdot T_n$$
$$S_{n+1} = 1/2 \cdot F_n$$
$$T_{n+1} = 1/2 \cdot S_n$$

What happens in the long term?

Magical thing: population ratios $\sim 16:4:1$
& population $\sim$ doubles each year

Demo

Section 1.1

Solving systems of equations

Outline of Section 1.1

- Learn what it means to solve a system of linear equations
- Describe the solutions as points in $\mathbb{R}^n$
- Learn what it means for a system of linear equations to be inconsistent

Solving equations

Solving equations

What does it mean to solve an equation?

$2x = 10$ $x = 5$ ✓

$x + y = 1$

$x + y + z = 0$ one soln: $(0,0,0) \rightsquigarrow x=0, y=0, z=0.$

$(1,1,-2)$

$(2,-1,-1)$

Find one solution to each. Can you find all of them?

↳ yes!

A solution is a *list* of numbers (a.k.a. a *vector*). For example $(3, -4, 1)$.

Solving equations

What does it mean to solve a system of equations?

$$\begin{aligned} x + y &= 2 \\ y &= 1 \end{aligned}$$

Finding x, y that work for both eqns.

$x = 1$
$y = 1$

What about...

$$\begin{aligned} x + y + z &= 3 \\ x + y - z &= 1 \\ x - y + z &= 1 \end{aligned}$$

Harder...

Is $(1, 1, 1)$ a solution? ✓ Is $(2, 0, 1)$ a solution? ✗ Fails 3rd eqn What are all the solutions?

Soon, you will be able to see just by looking that there is exactly one solution.

$$\mathbb{R}^n$$

$\mathbb{R}^n$

$\mathbb{R}$ = denotes the set of all real numbers

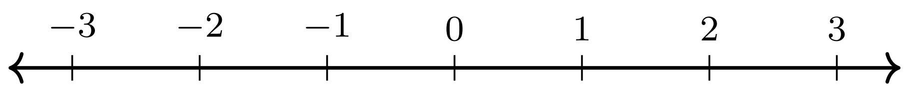
A point in $\mathbb{R}^4$:

$(0,0,0,0)$

$(1,2,3,4)$

$(\pi, e, -1, \sqrt{3})$

Geometrically, this is the *number line*.



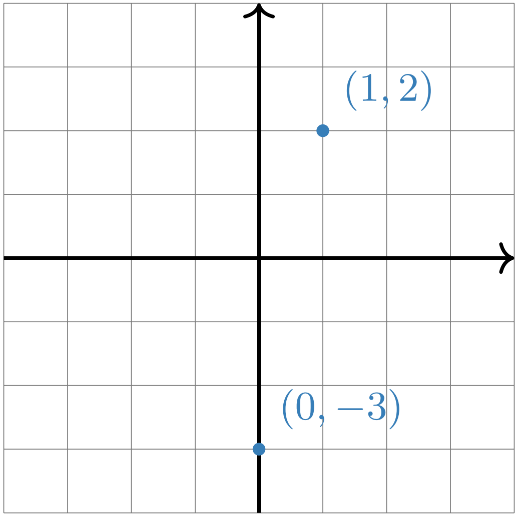
$$\mathbb{R}^n = \text{all ordered } n\text{-tuples (or lists) of real numbers } (x_1, x_2, x_3, \ldots, x_n)$$

Solutions to systems of equations are exactly points in $\mathbb{R}^n$. In other words, $\mathbb{R}^n$ is where our solutions will lie (the n depends on the system of equations).

We say $\mathbb{R}^n$ instead of $\mathbb{R}^2$ or $\mathbb{R}^3$ because many of the things we learn this semester work just as well for $\mathbb{R}^{777}$ as they do for $\mathbb{R}^2$ and $\mathbb{R}^3$. So when we say $\mathbb{R}^n$ we are talking about all of these at once. That is power!

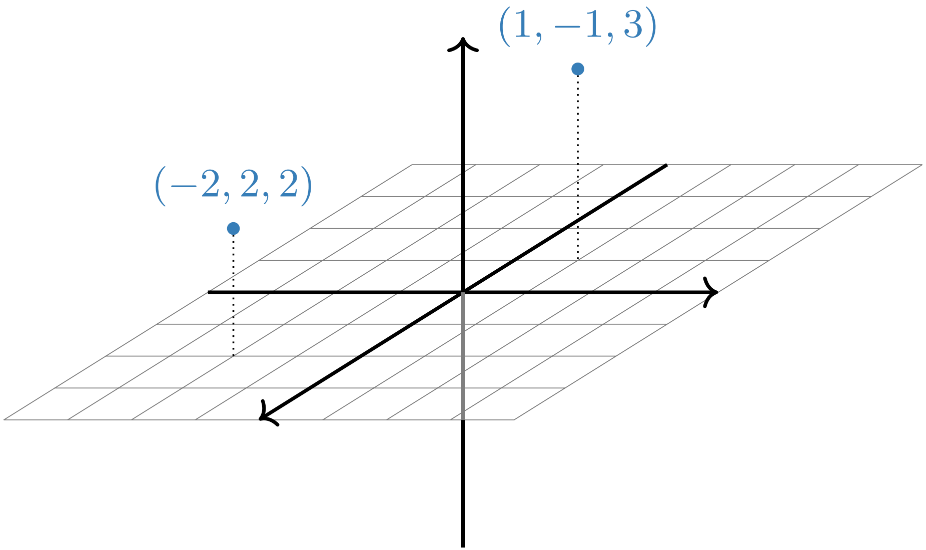
$\mathbb{R}^n$

When $n = 2$, we can visualize of $\mathbb{R}^2$ as the *plane*.



$\mathbb{R}^n$

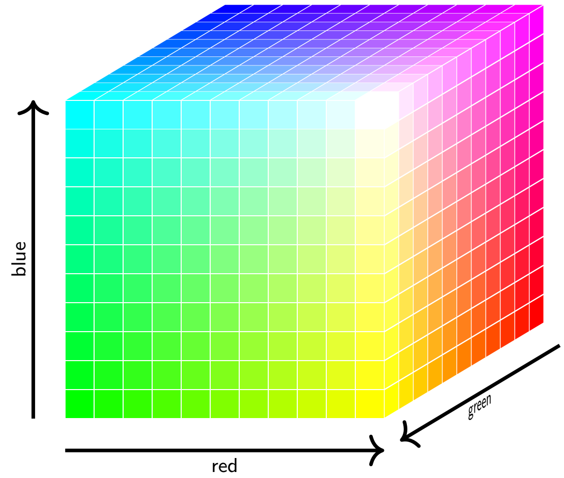
When $n = 3$, we can visualize $\mathbb{R}^3$ as the *space* we (appear to) live in.



$\mathbb{R}^3$ is 3D space

a point in $\mathbb{R}^3$ is a list of 3 numbrs

We can think of the space of all *colors* as (a subset of) $\mathbb{R}^3$:



So what is $\mathbb{R}^4$? or $\mathbb{R}^5$? or $\mathbb{R}^n$?

...go back to the *definition*: ordered n-tuples of real numbers

$$(x_1, x_2, x_3, \ldots, x_n).$$

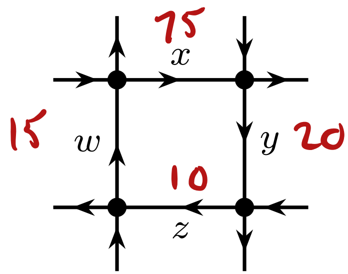
They're still "geometric" spaces, in the sense that our intuition for $\mathbb{R}^2$ and $\mathbb{R}^3$ sometimes extends to $\mathbb{R}^n$, but they're harder to visualize.

$\mathbb{R}^n$

Last time we could have used $\mathbb{R}^3$ to describe a rabbit population in a given year: (first year, second year, third year).

$$(5, 3, 7)$$

Similarly, we could have used $\mathbb{R}^4$ to label the amount of traffic (x, y, z, w) passing through four streets.

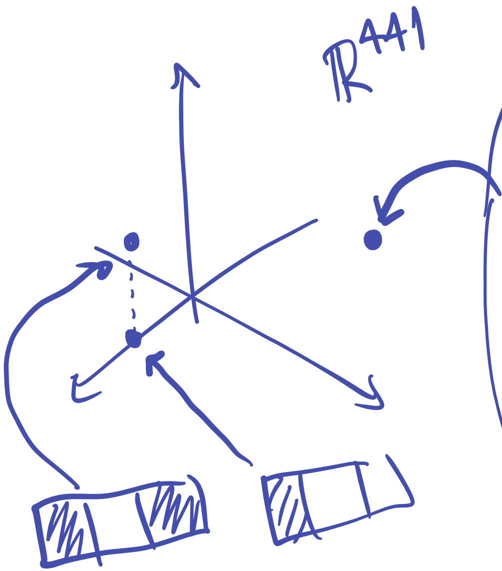


$$(75, 20, 10, 15)$$

We'll make definitions and state theorems that apply to any $\mathbb{R}^n$, but we'll only draw pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.

$\mathbb{R}^n$ and QR codes

This is a 21×21 QR code. We can also think of this as an element of $\mathbb{R}^n$.



$\mathbb{R}^2$?

$\mathbb{R}^{21}$?

$\mathbb{R}^{441}$?

How? Which n? $\mathbb{R}^{441}$ $(1,1,1,1,1,1,1,0,0,0,\ldots)$

What about a greyscale image?

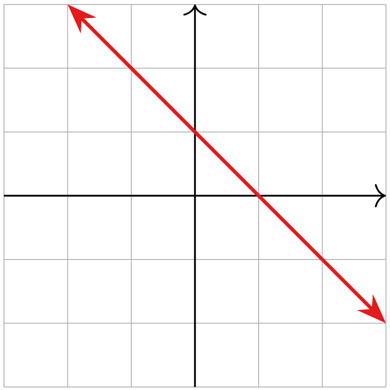
This is a powerful idea: instead of thinking of a QR code as 441 pieces of information, we think of it as one piece of information.

Visualizing solutions: a preview

One Linear Equation

What does the solution set of a linear equation look like?

$x + y = 1 \rightsquigarrow$ a line in the plane: $y = 1 - x$



Algebra

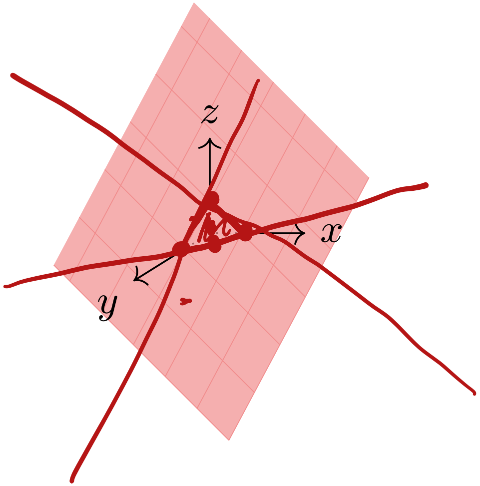
$\updownarrow$

Geometry

One Linear Equation

What does the solution set of a linear equation look like?

$x + y + z = 1 \rightsquigarrow$ a plane in space:



$x + y = 1$

$z = 0$.

One Linear Equation

Continued

What does the solution set of a linear equation look like?

$x + y + z + w = 1 \rightsquigarrow$ a "3-plane" in "4-space"...

3D plane $\mathbb{R}^4$

1 eqn in 100 vars

solns: 99-dim plane in $\mathbb{R}^{100}$

Systems of Linear Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$
$$2x + y = 8$$

Intersection of 2 lines in $\mathbb{R}^2$

In this case:
1 pt.
(lines not same/parallel)

What are the other possibilities for two equations with two variables?

Soln can be: 1 pt, ∞ solns, no solns

$x+y=1$
$2x+2y=2$

$x=0$
$x=1$

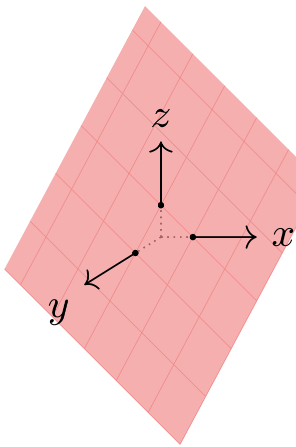
What if there are more variables? More equations?

Take this class!

Poll

Is the plane $x + y + z = 1$ in $\mathbb{R}^3$ equal to $\mathbb{R}^2$? What about the xy-plane in $\mathbb{R}^3$?

1. yes + yes
2. yes + no
3. no + yes
4. no + no

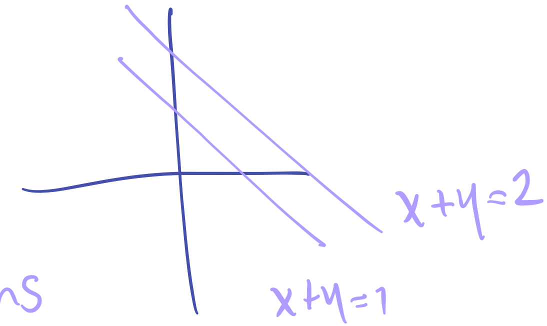


Consistent versus Inconsistent

possibly ∞ many

We say that a system of linear equations is consistent if it has a solution and inconsistent otherwise.

$$\begin{aligned} x + y &= 1 \\ x + y &= 2 \end{aligned}$$



don't intersect

Why is this inconsistent?

Can't have a pair of nums that sum to 1 & 2.

No point (x,y) in $\mathbb{R}^2$ satisfies both eqns.

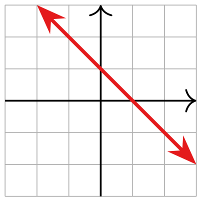
What are other examples of inconsistent systems of linear equations?

e.g. $x+y+z=1$ (same as $2x+2y+2z=2$)

$2x+2y+2z=1$

Parametric form

The equation $2x + 2y = 2$ is an implicit equation for the line in the picture.



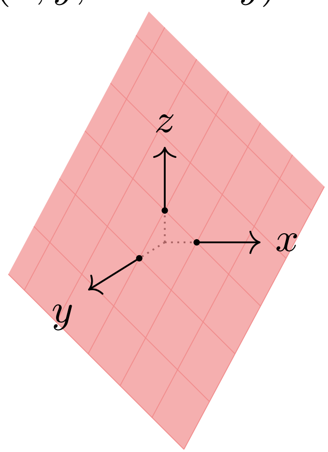
$x+y=1$

It also has a parametric form: $(x, 1-x)$.

put in anything for x to get all solns. $(e, 1-e)$ $(7, 1-7)$

The difference is that in the parametric form you get to plug in whatever you want for all variables. There's no guesswork, and no solving of anything.

Similarly the equation $x + y + z = 1$ is an implicit equation. One parametric form is: $(x, y, 1-x-y)$.

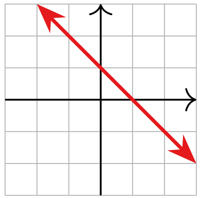


$x=1, y=0 \rightsquigarrow (1,0,0)$

$x=5, y=7 \to (5,7,-11)$

Parametric form

The equation $y = 1 - x$ is an implicit equation for the line in the picture.



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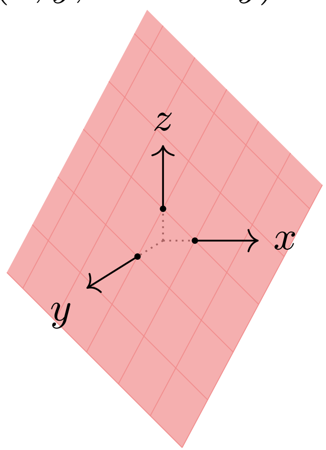
The difference is that in the parametric form you get to plug in whatever you want for all variables. There's no guesswork, and no solving of anything.

Similarly the equation $x + y + z = 1$ is an implicit equation. One parametric form is: $(x, y, 1 - x - y)$.



What is an implicit equation and a parametric form for the xy-plane in $\mathbb{R}^3$?

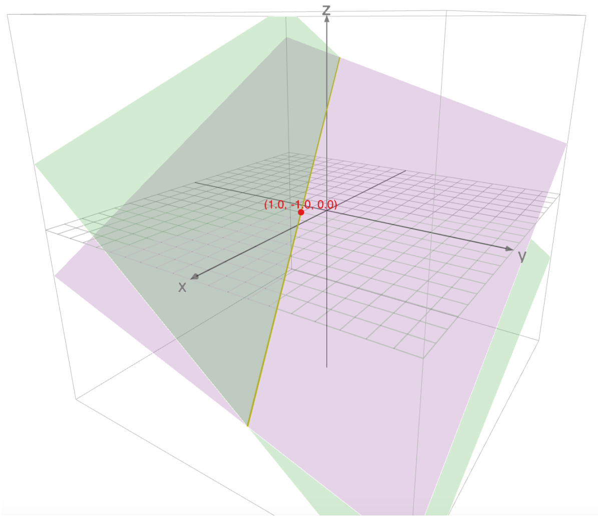
Parametric form

The system of equations

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned}$$

is the implicit form for the line of intersection in the picture.

↳ unsolved system of lin eqns.



$z=0$: $(1,-1,0)$
$z=5$: $(-24,-11,5)$

explicit/

The line of intersection also has a parametric form: $(1-5z, -1-2z, z)$

We think of the former as being the problem and the latter as being the explicit solution. One of our first tasks this semester is to learn how to go from the implicit form to the parametric form.

Summary of Section 1.1

- A solution to a system of linear equations in n variables is a point in $\mathbb{R}^n$.
- The set of all solutions to a single equation in n variables is an $(n-1)$-dimensional plane in $\mathbb{R}^n$
- The set of solutions to a system of m linear equations in n variables is the intersection of m of these $(n-1)$-dimensional planes in $\mathbb{R}^n$.
- A system of equations with no solutions is said to be inconsistent.
- Line and planes have implicit equations and parametric forms.

Typical exam questions

Write down and example of a point in $\mathbb{R}^7$.

Find all values of h so that the following system of linear equations is consistent:

$$\begin{aligned} x + y + z &= 2 \\ 2x + 2y + 2z &= h \end{aligned}$$

True/False: Points in $\mathbb{R}^3$ are also points in $\mathbb{R}^4$.

Find two different parametric solutions to the equation $x - 3y = 5$.

True/False: the set of solutions to $x_1 = 1$ in $\mathbb{R}^5$ is a line.

Announcements Aug 30

- Please turn on your camera if you are able and comfortable doing so
- Use Piazza for general questions
- WeBWorK 1.1 due Tuesday nite!
- Quiz on 1.1 due Friday. Open 6:30 AM - 8 PM on Canvas, have 15 mins. Quizzes
- Office hrs: Tue 4-5 Teams, Thu 1-2 Skiles courtyard/Teams, + Pop-ups
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- Counseling center: `https://counseling.gatech.edu`
- You can do it!

HW #3 ~~4~~ 3 vars, ~~4~~ 2 eqns

$$x+y+z=1$$
$$x+y+z=2$$

Section 1.2

Row reduction

Outline of Section 1.2

- Solve systems of linear equations via elimination
- Solve systems of linear equations via matrices and row reduction
- Learn about row echelon form and reduced row echelon form of a matrix
- Learn the algorithm for finding the (reduced) row echelon form of a matrix
- Determine from the row echelon form of a matrix if the corresponding system of linear equations is consistent or not.

Solving systems of linear equations by elimination

Example

Solve:

$$-y + 8z = 10$$
$$5y + 10z = 0$$

How many ways can you do it?

Substitution... not the best for many eqns vars.

elimination: $5(-y+8z=10)$

$+ \quad 5y + 10z = 0$

$50z = 50$

$z = 1$

back substitute: $y = -2$

Example

Solve:

$$\begin{aligned} -x + y + 3z &= -2 \\ 2x - 3y + 2z &= 14 \\ 3x + 2y + z &= 6 \end{aligned}$$

Hint: Eliminate x!

multiplying is ok

$$\begin{array}{rl} & 2(-x+y+3z=-2) \\ + & 2x-3y+2z=14 \\ \hline & -y+8z=10 \end{array}$$

adding eqns is ok.

$$\begin{array}{l} 3(-x+y+3z=-2) \\ 3x+2y+z=6 \\ \hline 5y+10z=0 \end{array}$$

elim. vars is good!

Many other ways. It's an art!

By last page: $z=1$, $y=-2$

Back subst: $x = y+3z+2 = -2+3+2 = 3$

Solving systems of linear equations with matrices

Example

Solve:

$$-y+8z=10$$
$$5y+10z=0$$

It is redundant to write $\overset{y}{\cancel{x}}$ and $\overset{z}{\cancel{y}}$ again and again, so we rewrite using (augmented) *matrices*. In other words, just keep track of the coefficients, drop the $+$ and $=$ signs. We put a vertical line where the equals sign is.

$$\begin{array}{c} \begin{matrix} y & \quad z & = & \text{const} \end{matrix} \\ \left(\begin{array}{cc|c} -1 & 8 & 10 \\ 5 & 10 & 0 \end{array}\right) \end{array} \xrightarrow[\text{by } 5]{\text{mult top}} \left(\begin{array}{cc|c} -5 & 40 & 50 \\ 5 & 10 & 0 \end{array}\right) \xrightarrow[\text{top to bot}]{\text{add}} \left(\begin{array}{cc|c} -5 & 40 & 50 \\ 0 & 50 & 50 \end{array}\right)$$

$$\xrightarrow[\text{div. bot by } 50]{\text{div top by } 5} \left(\begin{array}{cc|c} -1 & 8 & 10 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[\text{from top}]{\text{Subtract } 8\times\text{bot}} \left(\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[\text{by } -1]{\text{mult top}} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right)$$

$z=1$

can back subst now or...

$y=-2$

$z=1$

Example

Solve:

$$-x + y + 3z = -2$$
$$2x - 3y + 2z = 14$$
$$3x + 2y + z = 6$$

Again we rewrite using augmented matrices...

Making 0's is elimination

R2 → R2 − 2R1

$$\left(\begin{array}{ccc|c} -1 & 1 & 3 & -2 \\ 2 & -3 & 2 & 14 \\ 3 & 2 & 1 & 6 \end{array}\right) \xrightarrow[\text{by } -1]{\text{mult top}} \left(\begin{array}{ccc|c} +1 & -1 & -3 & +2 \\ 2 & -3 & 2 & 14 \\ 3 & 2 & 1 & 6 \end{array}\right) \xrightarrow[\text{from mid}]{\text{sub. } 2\times \text{top}} \left(\begin{array}{ccc|c} +1 & -1 & -3 & +2 \\ 0 & -1 & 8 & 10 \\ 3 & 2 & 1 & 6 \end{array}\right)$$

$$\xrightarrow[\text{from bot}]{\text{sub } 3\times\text{top}} \left(\begin{array}{ccc|c} +1 & -1 & -3 & +2 \\ 0 & -1 & 8 & 10 \\ 0 & 5 & 10 & 0 \end{array}\right) \xrightarrow[\text{slide on bot two rows}]{\text{do the last}} \left(\begin{array}{ccc|c} 1 & -1 & -3 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

can back subst to find x.

$z = 1$
$y = -2$

$$\xrightarrow[\text{3R3 to R1}]{\text{add } \cancel{2}\text{R2 to R1}} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$x = 3$
$y = -2$
$z = 1$

Row operations

Our manipulations of matrices are called row operations:

row swap, row scale, row replacement

R1 ↔ R2

R1 → 7R1 — mult top by 7

R1 → R1 + 5R2 — add 5 times middle to top.

If two matrices differ by a sequence of these three row operations, we say they are row equivalent.

→ same solns!

Goal: Produce a system of equations like:

$$\begin{aligned} x \quad\quad\quad &= 2 \\ y \quad\quad &= 1 \\ z &= 5 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

What does this look like in matrix form?

Row operations

Why do row operations not change the solution?

Solve:

$$\begin{aligned} x + y &= 2 \\ -2x + y &= -1 \end{aligned}$$

System has one solution, $x = 1, y = 1$.

What happens to the two lines as you do row operations?

$$\left(\begin{array}{rr|r} 1 & 1 & 2 \\ -2 & 1 & -1 \end{array} \right) \rightsquigarrow$$

They pivot around the solution!

Row Reduction and Echelon Forms

Row echelon form

Remember our goal.

Goal: Produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= 2 \\ y \qquad &= 1 \\ z &= 5 \end{aligned}$$

Or at least...

Easier goal: Produce a system of equations like

$$\begin{aligned} x + 5y - 3z &= 2 \\ y + 7z &= 1 \\ z &= 5 \end{aligned}$$

can back subst.

Row Reduction and Echelon Forms

Zero row: row of all 0's

(junk)

A matrix is in row echelon form if

1. all zero rows are at the bottom, and
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above.

leading entries/pivots

$$\begin{pmatrix} \boxed{1} & 5 & 7 & 9 & 2 \\ 0 & \boxed{3} & 2 & -1 & 2 \\ 0 & 0 & 0 & \boxed{7} & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\leftarrow$ zero row

This system is easy to solve using back substitution.

The pivot positions are the leading entries in each row.

Reduced Row Echelon Form

A system is in reduced row echelon form if also:

3 ~~1~~. the leading entry in each nonzero row is 1

4 ~~2~~. each leading entry of a row is the only nonzero entry in its column

For example:

$$\begin{pmatrix} \boxed{1} & 0 & \star & 0 & \star \\ 0 & \boxed{1} & \star & 0 & \star \\ 0 & 0 & 0 & \boxed{1} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is even easier to solve.

Important. In any discussion of row echelon form, we ignore any vertical lines!

Can every matrix be put in reduced row echelon form?

Reduced Row Echelon Form

Poll

Which are in reduced row echelon form?

ignore

No! $\left(\begin{array}{c|c} 1 & 0 \\ 0 & \boxed{2} \end{array}\right)$ $\qquad$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

not a 1 (rule 3)

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$$

$\left(\begin{array}{cc|c} \boxed{1} & 17 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right)$ $\qquad$ $\begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Yes! $\qquad$ Yes!

REF:

1. all zero rows are at the bottom, and — if there are any!
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above.

RREF:

3 ~~1.~~ the leading entry in each nonzero row is 1

4 ~~2.~~ each leading entry of a row is the only nonzero entry in its column

Row Reduction

Theorem. Each matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm. That shows a matrix is equivalent to at least one matrix in reduced row echelon form.

Row Reduction Algorithm

To find row echelon form:

Step 1 Swap rows so a leftmost nonzero entry is in 1st row (if needed)

Step 2 Scale 1st row so that its leading entry is equal to 1

Step 3 Use row replacement so all entries below this 1 (or, pivot) are 0

Then cover the first row and repeat the three steps.

To then find reduced row echelon form:

- Use row replacement so that all entries above the pivots are 0.

Examples.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right) \left(\begin{array}{ccc|c} 4 & -5 & 3 & 2 \\ 1 & -1 & -2 & -6 \\ 4 & -4 & -14 & 18 \end{array}\right)$$

Interactive Row Reducer

Solutions of Linear Systems

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

What are the solutions? Say the variables are x and y.

Solutions of Linear Systems: Consistency

Solve the linear system associated to:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Say the variables are x, y, and z.

$0 = 1 \implies$ inconsistent

A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

Example with a parameter

For which values of h does the following system have a solution?

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= h \end{aligned}$$

Solve this by row reduction and also solve it by thinking geometrically.

Summary of Section 1.2

- To solve a system of linear equations we can use the method of elimination.
- We can more easily do elimination with matrices. The allowable moves are row swaps, row scales, and row replacements. This is called row reduction.
- A matrix in row echelon form corresponds to a system of linear equations that we can easily solve by back substitution.
- A matrix in reduced row echelon form corresponds to a system of linear equations that we can easily solve just by looking.
- We have an algorithm for row reducing a matrix to row echelon form.
- The reduced row echelon form of a matrix is unique.
- Two matrices that differ by row operations are called row equivalent.
- A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

Reduced Row Echelon Form

Poll

Which are in reduced row echelon form?

$$\left(\begin{array}{c|c} 1 & 0 \\ 0 & 2 \end{array}\right) \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

REF:

1. all zero rows are at the bottom, and
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above.

RREF:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

Announcements Sep 1

- Please turn on your camera if you are able and comfortable doing so
- Quiz on 1.1 Friday. Open 6:30a–8p on Canvas/Assignments, 15 mins
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- You can do it!

1.3 Parametric Form

Outline of Section 1.3

- Find the parametric form for the solutions to a system of linear equations.
- Describe the geometric picture of the set of solutions.

Free Variables

We know how to understand the solution to a system of linear equations when every column to the left of the vertical line has a pivot. For instance:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

If the variables are x and y what are the solutions?

$x = 5$

$y = 2$

Free Variables

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? For instance:

no pivot in x_3 column.

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 5 & 0 \\ 0 & \boxed{1} & 2 & 1 \end{array}\right)$$

pivot: for matrix in REF, first non-0 entry in each row.

represents two equations:

$$\begin{aligned} x_1 \qquad + 5x_3 &= 0 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

There is one free variable x_3, corresponding to the non-pivot column. To solve, we move the free variable to the right:

$$\begin{aligned} x_1 \qquad &= -5x_3 \\ x_2 &= 1 - 2x_3 \\ x_3 &= x_3 \text{ (free; any real number)} \end{aligned}$$

∞ many solns.

This is the parametric solution. We can also write the solution as:

1 free var

3 total vars

$$(-5x_3, 1 - 2x_3, x_3)$$

If you choose $x_3 = 4$ get:

$(-20, -7, 4)$

line in $\mathbb{R}^3$

What is one particular solution? What does the set of solutions look like?

Free Variables

Williams example $\left(\begin{array}{ccccc|c} 1 & 0 & 5 & 6 & 4 & 7 \\ 0 & 1 & 7 & 2 & 3 & 8 \end{array}\right)$

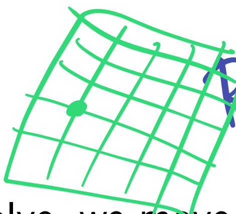
solution: 3 dim plane in $\mathbb{R}^5$

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 \qquad + 5x_3 \qquad &= 0 \\ x_4 &= 0 \end{aligned}$$

no pivot $\rightsquigarrow$ free vars

So the associated matrix is:



RREF!

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right).$$

$x_1 \quad + 5x_3 = 0$

$x_4 = 0$

To solve, we move the free variable to the right:

free variable = parameter,

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= \quad x_2 \quad \text{(free)} \\ x_3 &= \quad x_3 \quad \text{(free)} \\ x_4 &= 0 \end{aligned}$$

2 free vars

4 total vars

Or: $(-5x_3, x_2, x_3, 0)$. This is a plane in $\mathbb{R}^4$.

These are all solutions. One solution is: $(-5, 2, 1, 0)$ $\quad x_3 = 1$, $x_2 = 2$

The original equations are the implicit equations for the solution. The answer to this question is the parametric solution.

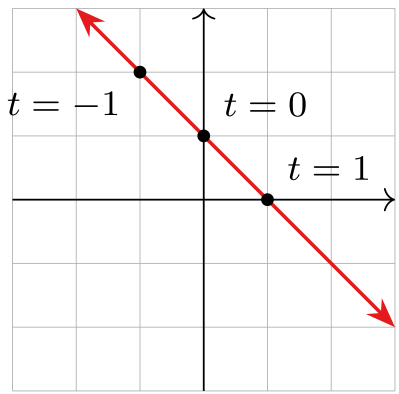
Free variables

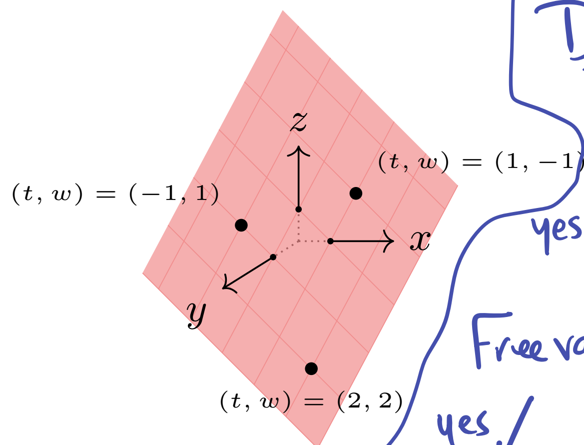
Geometry

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional plane in $\mathbb{R}^n$.

Why does this make sense?

$x+y+z=1$





Decision tree

Consistent?

yes → Free vars?

no → 0 solns

Free vars? yes → ∞ many slns, k-dim plane where k = # free vars.

Free vars? no → 1 soln

Inconsistent $\Longleftrightarrow$ pivot on right hand side

free vars = # cols on left hand side w/o pivots.

Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. ~~two points~~
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane $= \mathbb{R}^4$

$x+y+z+w=1$
$x+y+z+w=0$

$$\left(\begin{array}{cccc|c} 1&1&1&1&1 \\ 1&1&1&1&0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1&1&1&1&1 \\ 0&0&0&0&-1 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} \square&\square&\square&\square&\square&\square \\ \square&\square&\square&\square&\square&\square \\ \square&\square&\square&\square&\square&\square \end{array}\right)$$

3×5

# pivots	:	# nonpivots	
3 or	line	1	line
2 or	plane	2	plane
1 or	3D plane	3	3D plane
0	4D plane $= \mathbb{R}^4$	4	4D plane

Implicit versus parametric equations of planes

Find a parametric description of the plane

free

$$x + y + z = 1$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \end{array}\right)$$

$x = 1 - y - z$

$y = y$

$z = z$

$(1 - y - z, y, z)$

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

implicit

$x+y+z=1$

$x+3y-z=5$

explicit / parametric

$(1-z, 3+5z, z)$

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Typical exam questions

True/False: If a system of equations has 100 variables and 70 equations, then there must be infinitely many solutions.

True/False: If a system of equations has 70 variables and 100 equations, then it must be inconsistent.

How can we tell if an augmented matrix corresponds to a consistent system of linear equations?

If a system of linear equations has finitely many solutions, what are the possible numbers of solutions?

Announcements Sep 8

- Please turn on your camera if you are able and comfortable doing so
- Current plan: Class on Monday in Howey L1/Teams (email forthcoming)
  - attendance optional
  - masks + distance from me expected
  - testing encouraged
- Quiz on 1.2 & 1.3 Friday. Open 6:30a–8p on Canvas/Assignments, 15 mins
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Chapter 2

System of Linear Equations: Geometry

Where are we?

In Chapter 1 we learned to solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution. In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning. There are three main points:

Sec 2.3: $Ax = b$ is consistent $\Leftrightarrow$ b is in the span of the columns of A.

Sec 2.4: The solutions to $Ax = b$ are parallel to the solutions to $Ax = 0$.

Sec 2.9: The dim's of $\{b : Ax = b$ is consistent$\}$ and $\{$solutions to $Ax = b\}$ add up to the number of columns of A.

Section 2.1

Vectors

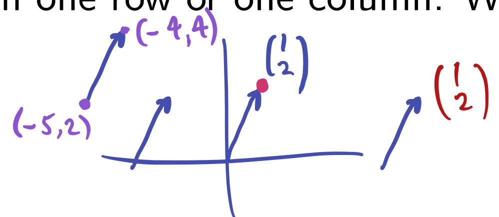
Outline

- Think of points in $\mathbb{R}^n$ as vectors.
- Learn how to add vectors and multiply them by a scalar
- Understand the geometry of adding vectors and multiplying them by a scalar
- Understand linear combinations algebraically and geometrically

Vectors

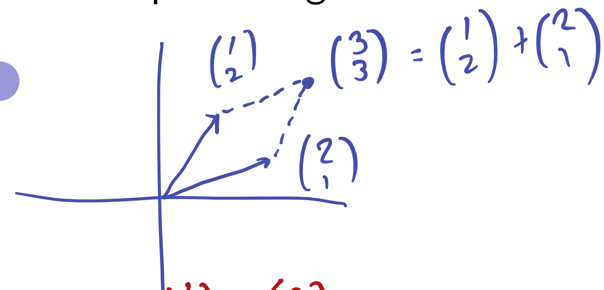
A vector is a matrix with one row or one column. We can think of a vector with n rows as:

- a point in $\mathbb{R}^n$
- an arrow in $\mathbb{R}^n$



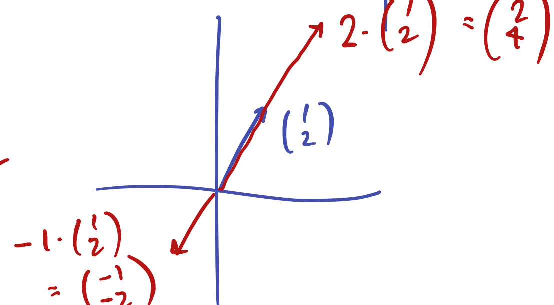
To go from an arrow to a point in $\mathbb{R}^n$, we subtract the tip of the arrow from the starting point. Note that there are many arrows representing the same vector.

Adding vectors / parallelogram rule Demo



Scaling vectors Demo

↳ multiply by real number



A scalar is just a real number. We use this term to indicate that we are scaling a vector by this number.

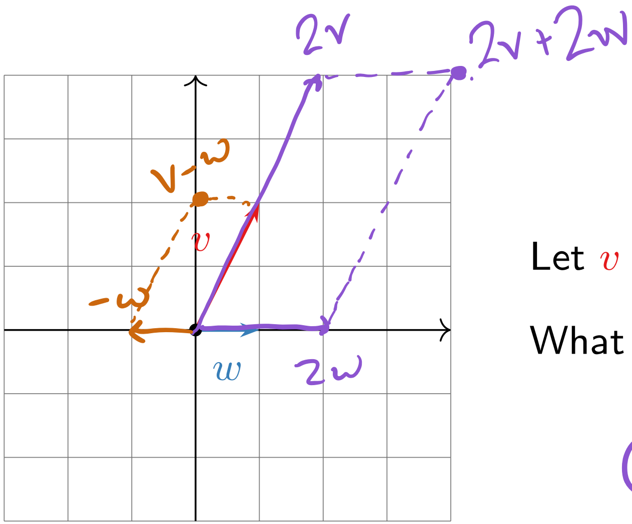
Linear Combinations

A linear combination of the vectors $v_1, \ldots, v_k$ is any vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.

then
Scale & add the vectors v_i



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

① $2v + 2w = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$1 \cdot v + (-1) \cdot w$

② $v - w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

③ $2v - w = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ etc.

④ $\pi \cdot v + 0 \cdot w = \begin{pmatrix} \pi \\ 2\pi \end{pmatrix}$

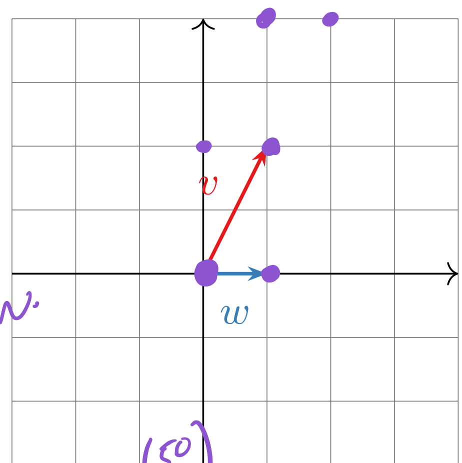
Poll

Is there a vector in $\mathbb{R}^2$ that is not a linear combination of v and w?

- yes
- $\boxed{\text{no}}$

Every pt in $\mathbb{R}^2$ is a lin combo of v, w.

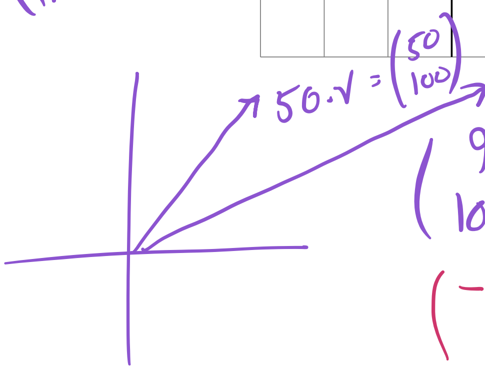
In language of Sec 2.2: the span of $\begin{pmatrix}1\\2\end{pmatrix}$ & $\begin{pmatrix}1\\0\end{pmatrix}$ is $\mathbb{R}^2$.



$v = \begin{pmatrix}1\\2\end{pmatrix}$

$w = \begin{pmatrix}1\\0\end{pmatrix}$

Secretly: solving a linear system.



$$\begin{pmatrix}98\\100\end{pmatrix} = \underline{50} \cdot \begin{pmatrix}1\\2\end{pmatrix} + \underline{48} \begin{pmatrix}1\\0\end{pmatrix}$$

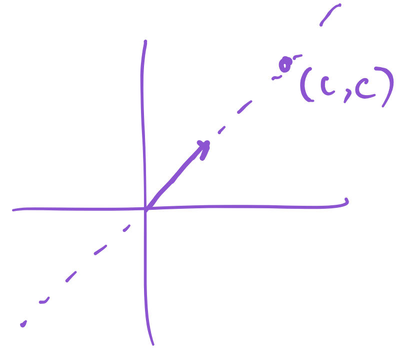
$$\begin{pmatrix}-1\\0\end{pmatrix} = \underline{0} \cdot \begin{pmatrix}1\\2\end{pmatrix} + \underline{-1} \begin{pmatrix}1\\0\end{pmatrix}$$

Linear Combinations

What are some linear combinations of $(1, 1)$?

line $y=x$

$c\cdot(1,1)=(c,c)$

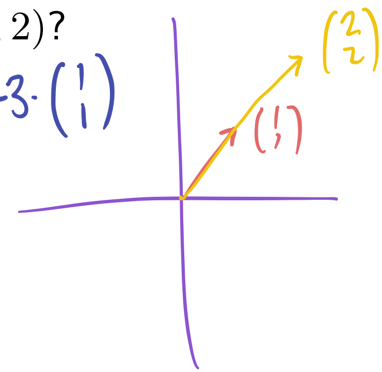


What are some linear combinations of $(1, 1)$ and $(2, 2)$?

$15\cdot\binom{1}{1} + -9\cdot\binom{2}{2} = -3\cdot\binom{1}{1}$

$-18\cdot\binom{1}{1}$

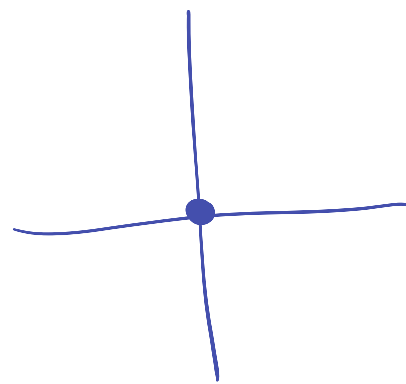
line $y=x$



What are some linear combinations of $(0, 0)$?

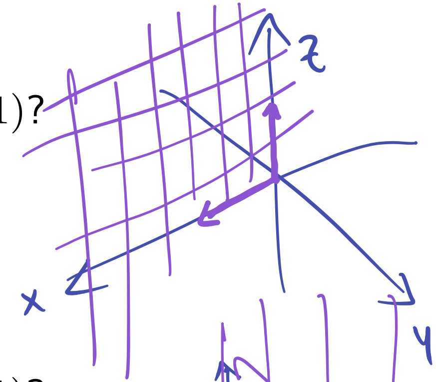
$c\cdot\binom{0}{0}$

just the origin.



Linear Combinations

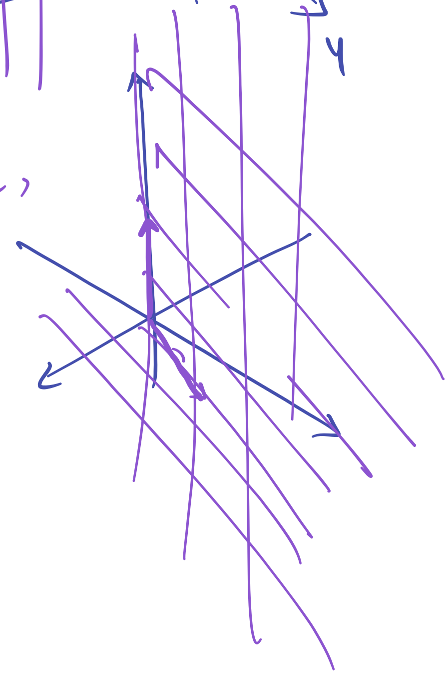
What are all linear combinations of $(1, 0, 0)$ and $(0, 0, 1)$?



XZ-plane

$$\begin{pmatrix}77\\0\\-19\end{pmatrix} = 77\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + (-19)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

What are all linear combinations of $(1, 1, 0)$ and $(0, 0, 1)$?



Take plane from last example, rotate by... 45° about... z-axis

What are all linear combinations of $(3, 2, 4)$ and $(-4, 2, 1)$?

It's a plane in $\mathbb{R}^3$

Summary of Section 2.1

- A vector is a point/arrow in $\mathbb{R}^n$
- We can add/scale vectors algebraically & geometrically (parallelogram rule)
- A linear combination of vectors $v_1, \ldots, v_k$ is a vector

$$c_1 v_1 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.

Typical exam questions

True/False: For any collection of vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, the zero vector in $\mathbb{R}^n$ is a linear combination of $v_1, \ldots, v_k$.

True/False: The vector $(1, 1)$ can be written as a linear combination of $(2, 2)$ and $(-2, -2)$ in infinitely many ways.

Describe geometrically the set of linear combinations of the vectors $(1, 0, 0)$ and $(1, 2, 3)$.

Suppose that v is a vector in $\mathbb{R}^n$, and consider the set of all linear combinations of v. What geometric shape is this?

Section 2.2

Vector Equations and Spans

Outline of Section 2.2

- Learn the equivalences:

$$\text{vector equations} \leftrightarrow \text{augmented matrices} \leftrightarrow \text{linear systems}$$

- Learn the definition of span
- Learn the relationship between spans and consistency

in the span of

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Super important!

yes $\Longleftrightarrow$ system below is consistent.

Write down an equation in order to solve this problem. This is called a vector equation.

$$x \cdot \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \cdot \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Notice that the vector equation can be rewritten as a system of linear equations. Solve it!

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned} \quad \longleftrightarrow \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

Now answer by row reduction. Is there a pivot on RHS? You check!!

Linear combinations, vector equations, and linear systems

Repeat of last slide in general language.

In general, asking:

$$\text{Is } b \text{ a linear combination of } v_1, \ldots, v_k?$$

is the same as asking if the vector equation

$$x_1 v_1 + \cdots + x_k v_k = b$$

is consistent, which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

Span

Essential vocabulary word!

$\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} = \mathbb{R}^3$

$\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}9\\8\\1\end{pmatrix}\right\} = \mathbb{R}^3$

$$\begin{aligned}\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1v_1 + x_2v_2 + \cdots + x_kv_k \mid x_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of two vectors in $\mathbb{R}^2$? $\boxed{1}$ $(0,0)$ (both vectors are zero)

or $\boxed{2}$ line if one is a multiple of other

or $\boxed{3}$ all of $\mathbb{R}^2$

Demo

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

$(0,0,0)$, line, plane, all of $\mathbb{R}^3$

all three vectors are multiples of each other

3rd vector is in plane spanned by first 2

e.g. $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}9\\8\\0\end{pmatrix}$

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Span

Essential vocabulary word!

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 0 & 0 & 0 \\ 0 & 5 & -45 \end{array}\right)$$

CONSISTENT!

$$\begin{aligned} \operatorname{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1v_1 + x_2v_2 + \cdots x_kv_k \mid x_i \text{ in } \mathbb{R}\} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k. \end{aligned}$$

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ in Span of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

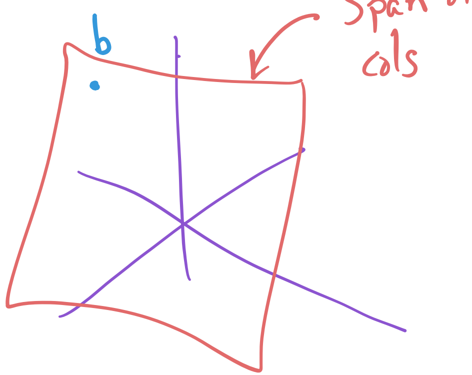
Four ways of saying the same thing:

beginning of 2.2

- b is in $\operatorname{Span}\{v_1, v_2, \ldots, v_k\}$ ← geometry
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $x_1v_1 + \cdots + x_kv_k = b$ has a solution ← algebra
- the system of linear equations corresponding to

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

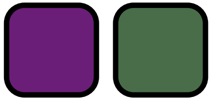


Demo Demo

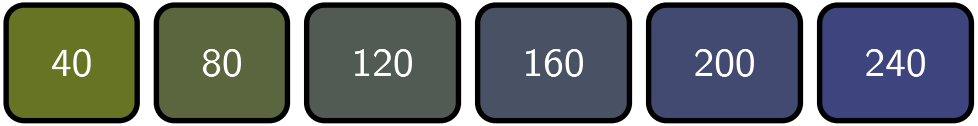
Application: Additive Color Theory

Consider now the two colors

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$$



For which h is $(116, 130, h)$ in the span of those two colors?



Summary of Section 2.2

- vector equations $\leftrightarrow$ augmented matrices $\leftrightarrow$ linear systems
- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

Typical exam questions

Is $\begin{pmatrix} 8 \\ 16 \\ 1 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Write down the vector equation for the previous problem.

True/False: The vector equation $x_1 v_1 + \cdots + x_k v_k = 0$ is always consistent.

True/False: It is possible for the span of 3 vectors in $\mathbb{R}^3$ to be a line.

True/False: the plane $z = 1$ in $\mathbb{R}^3$ is a span.

Announcements Sep 13

- Please wear a mask. FOR EXTRA CREDIT
- Quiz 2.1 & 2.2 Friday. Open 6:30a–8p on Canvas/Assignments, 15 mins
- WeBWorK 2.1 & 2.2 due Tuesday nite
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, old quizzes/exams, advice
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

M-Th 11-6
F 11-3

Section 2.1

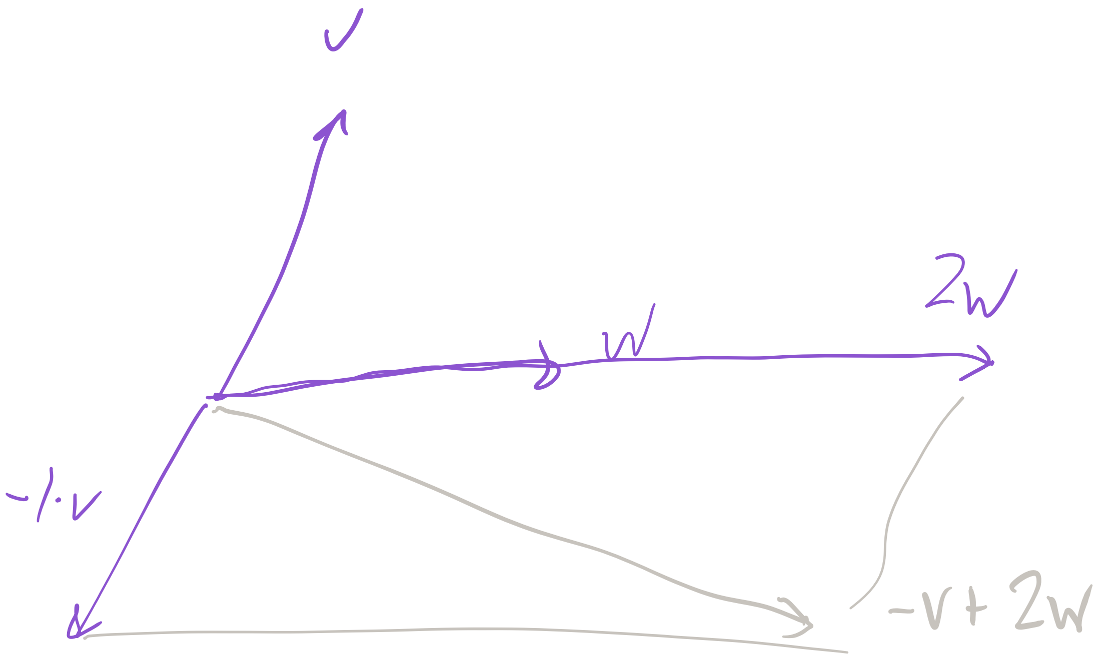
Vectors

Linear Combinations

A linear combination of the vectors $v_1, \ldots, v_k$ is any vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.



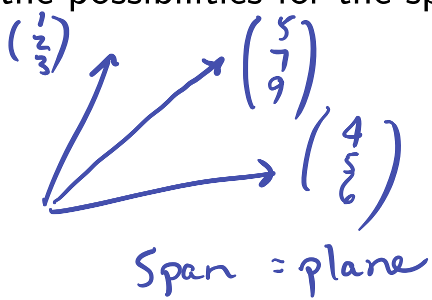
Span

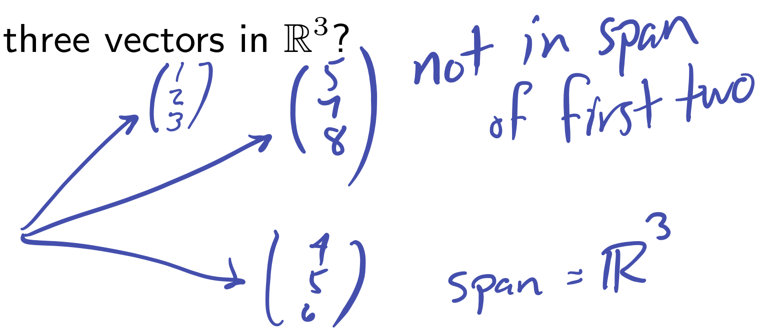
Essential vocabulary word!

$$\begin{aligned}\operatorname{Span}\{v_1, v_2, \ldots, v_k\} &= \overset{\text{set of}}{\{} x_1v_1 + x_2v_2 + \cdots + x_kv_k \overset{\text{so that}}{\mid} x_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)}\\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k\\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo





Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Section 2.2

Vector Equations and Spans

The four ways

Four ways of saying the same thing:

- b is in Span$\{v_1, v_2, \ldots, v_k\}$ $\leftarrow$ geometry
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $x_1 v_1 + \cdots + x_k v_k = b$ has a solution $\leftarrow$ algebra
- the system of linear equations corresponding to

$$\underbrace{\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right)}_{A},$$

is consistent.

- $Ax = b$ has a soln

Demo Demo

Section 2.3

Matrix equations

Multiplying Matrices by column vectors

numbers

$$\text{matrix} \times \text{column} : \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 \begin{pmatrix} | \\ x_1 \\ | \end{pmatrix} + \cdots + b_n \begin{pmatrix} | \\ x_n \\ | \end{pmatrix}$$

columns

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = 7 \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 8 \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 21 \\ 35 \end{pmatrix} + \begin{pmatrix} 16 \\ 32 \\ 48 \end{pmatrix} = \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Multiplying Matrices by column vectors

Another way to multiply

row vector $\times$ column vector : $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$

numbers

example.

$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 4$
$= 11$

matrix $\times$ column vector : $\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$

as above

rows

col vector.

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 7 + 2 \cdot 8 \\ 3 \cdot 7 + 4 \cdot 8 \\ 5 \cdot 7 + 6 \cdot 8 \end{pmatrix} = \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Linear Systems vs Augmented Matrices vs Matrix Equations vs Vector Equations

A matrix equation is an equation $Ax = b$ where A is a matrix and b is a vector. So x is a vector of variables.

A is an $m \times n$ matrix if it has m rows and n columns. What sizes must x and b be?

Solving is answering: Is $\begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$ in span of $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$?

Example:

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3 \times \cancel{2}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\cancel{2} \times 1} = \underbrace{\begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}}_{3 \times 1}$$

matrix eqn

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

vector eqn

$$x\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$$

linsys

$$x + 2y = 9$$
$$3x + 4y = 10$$
$$5x + 6y = 11$$

aug mat

you do.

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.

Solving matrix equations

Solve the matrix equation

\# of babies for 2nd & 3rd year rabbits

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix}$$

current population

$$\left(\begin{array}{ccc|c} 0 & 6 & 8 & 20 \\ 1/2 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 6 & 8 & 20 \end{array}\right) \xrightarrow{R3 \to R3 - 6R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 8 & 8 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Solution: $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

population the year before

What does this mean about rabbits?

They are so cute!

Solutions to Linear Systems vs Spans

columns of A

Say that

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

algebra $\iff$ geometry

Why?

Look back at "The Four Ways" slide.

Again this is a basic fact we will use over and over and over.

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

last slide.

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{5} \end{array}\right)$$

inconsistent

So:

$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ not in span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

solution: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

So $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ is in span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Is a given vector in the span?

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

algebra $\iff$ geometry

Is $(9, 10, 11)$ in the span of $(1, 3, 5)$ and $(2, 4, 6)$?

$$\left(\begin{array}{cc|c} 1 & 2 & 9 \\ 3 & 4 & 10 \\ 5 & 6 & 11 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & -2 & -17 \\ 0 & -4 & -34 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} \boxed{1} & 2 & 9 \\ 0 & \boxed{-2} & -17 \\ 0 & 0 & 0 \end{array}\right)$$

yes – no pivot on RHS.

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 17/2 \\ 0 & 0 & 0 \end{array}\right) \quad \text{So: } -8\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 17/2 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ 11 \end{pmatrix}$$

soln: $\begin{pmatrix} -8 \\ 17/2 \end{pmatrix}$

Pivots vs Solutions

Rephrasing what we know, with the "all"

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

(so never a pivot on RHS)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} x = b$$

has a soln when b is xy-plane.

More generally, if you have some vectors and you want to know the dimension of the span, you should row reduce and count the number of pivots.

Properties of the Matrix Product Ax

$c =$ real number, $u, v =$ vectors,

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

Application. If u and v are solutions to $Ax = 0$ then so is every element of $\mathrm{Span}\{u, v\}$.

Guiding questions

Here are the guiding questions for the rest of the chapter:

1. What are the solutions to $Ax = 0$?

2. For which b is $Ax = b$ consistent?

These are two separate questions!

Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 10, 20)$, $(0, -1, -2)$
2. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 1, 0)$, $(0, 0, \sqrt{2})$
3. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 5, 7)$, $(0, 6, 8)$
4. $(0, 1, 2)$ is in the span of $(5, 7, 0)$, $(6, 8, 0)$, $(3, 3, 4)$

Summary of Section 2.3

- Two ways to multiply a matrix times a column vector:

$$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

  OR

$$\begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_{\color{red}n} \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{\color{red}n} \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 x_1 & \cdots & b_n x_n \\ | & & | \end{pmatrix}$$

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.
- Fact. $Ax = b$ has a solution $\Leftrightarrow$ b is in the span of columns of A
- Theorem. Let A be an $m \times n$ matrix. The following are equivalent.
  1. $Ax = b$ has a solution for all b
  2. The span of the columns of A is $\mathbb{R}^m$
  3. A has a pivot in each row

Typical exam questions

- If A is a 3×5 matrix, and the product Ax makes sense, then which $\mathbb{R}^n$ does x lie in?

- Rewrite the following linear system as a matrix equation and a vector equation:

$$x + y + z = 1$$

- Multiply:

$$\begin{pmatrix} 0 & 2 \\ 0 & 4 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

- Which of the following matrix equations are consistent?

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

(And can you do it without row reducing?)

Announcements Sep 15

- Masks $\rightsquigarrow$ extra credit.
- Quiz 2.1 & 2.2 Friday. Open 6:30a–8p on Canvas/Assignments, 15 mins
- WeBWorK 2.3 & 2.4 due Tuesday nite
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, old quizzes/exams, advice
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Chapter 2

System of Linear Equations: Geometry

Where are we?

In Chapter 1 we learned to solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution. In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning. There are three main points:

Sec 2.3: $Ax = b$ is consistent $\Leftrightarrow$ b is in the span of the columns of A.

Sec 2.4: The solutions to $Ax = b$ are parallel to the solutions to $Ax = 0$.

Sec 2.9: The dim's of $\{b : Ax = b \text{ is consistent}\}$ and $\{\text{solutions to } Ax = b\}$ add up to the number of columns of A.

Announcements Sep 20

- Masks $\rightsquigarrow$ extra credit.
- WeBWorK 2.3 & 2.4 due Tuesday nite
- Midterm Wednesday 8–9:15p on Teams ~~in your Studio channel~~, up to Sec 2.4
- No quiz Friday
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
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- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 2.4

Solution Sets

Homogeneous vs. Nonhomogeneous Systems

Key realization. Set of solutions to $Ax = b$ obtained by taking one solution and adding all possible solutions to $Ax = 0$.

$$Ax = 0 \text{ solutions} \rightsquigarrow Ax = b \text{ solutions}$$

$$x_k v_k + \cdots + x_n v_n \rightsquigarrow p + x_k v_k + \cdots + x_n v_n$$

So: set of solutions to $Ax = b$ is parallel to the set of solutions to $Ax = 0$. It is a translate of a plane through the origin. (Again, we are using geometry to understand algebra!)

So by understanding $Ax = 0$ we gain understanding of $Ax = b$ for all b. This gives structure to the set of equations $Ax = b$ for all b.

Demo

Demo

Parametric Vector Forms for Solutions

Nonhomogeneous case

homog.

Find the parametric vector forms for $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{matrix} x_1 = 3x_2 \\ x_2 = x_2 \end{matrix} \rightsquigarrow x_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

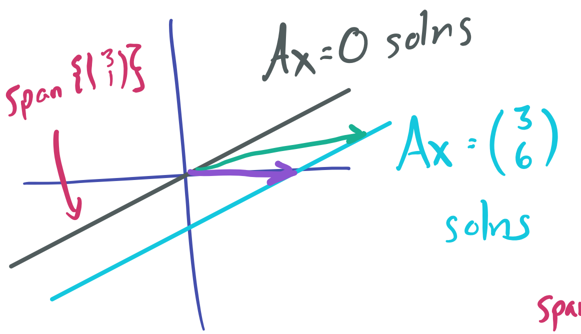
param form

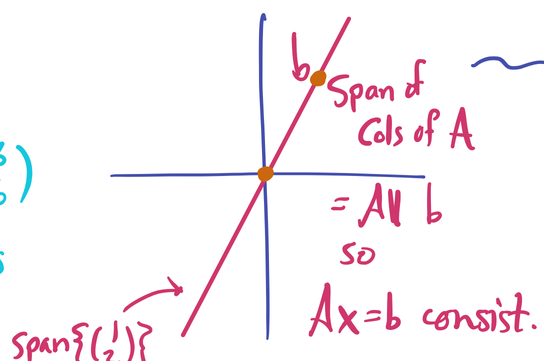
param vect form

non-homog

...and $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 2 & -6 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{matrix} x_1 = 3 + 3x_2 \\ x_2 = x_2 \end{matrix}$$





$$\rightsquigarrow \begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

one partic soln

Solving matrix equations

The matrix equation

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution.

3 pivots.

What does this mean about the matrix equation

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix}?$$

one solution.

What does this mean about rabbits?

There is exactly one rabbit population that results in (20,1,1) the following year.

Section 2.5

Linear Independence

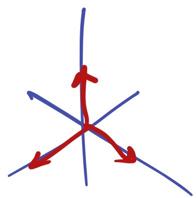
Section 2.5 Outline

- Understand what is means for a set of vectors to be linearly independent
- Understand how to check if a set of vectors is linearly independent

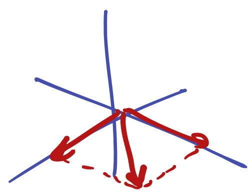
Linear Independence

The idea of linear independence: a collection of vectors $v_1, \ldots, v_k$ is linearly independent if they are all pointing in truly different directions. Precisely, this means that none of the v_i is in the span of the others.

For example, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent.



Also, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$ are linearly dependent.



What is this good for? A basic question we can ask about solving linear equations is: What is the smallest number of vectors needed in the parametric solution to a linear system? We need linear independence to answer this question. See the last slide in this section.

Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$\underline{x_1} v_1 + \underline{x_2} v_2 + \cdots + \underline{x_k} v_k = 0$$

has only the trivial solution. It is linearly dependent otherwise.

indep

$$\underline{0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underline{0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underline{0} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, linearly dependent means there are $x_1, x_2, \ldots, x_k$ not all zero so that

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

This is a *linear dependence* relation.

$$\underline{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underline{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \underline{1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

dependent.

Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$ homogen.

has only the trivial solution. no free vars

A 4x3 matrix is 3 vectors in $\mathbb{R}^4$. Those can be indep.

Fact. The columns of A are linearly independent
$\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
$\Leftrightarrow$ A has a pivot in each column

Why?

$$A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

A 3x4 matrix can't have a pivot in each col so....

4 vectors in $\mathbb{R}^3$ can't be independent

Linear Independence

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

No! dep.

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 4 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{-2} & -2 \\ 0 & 6 & \boxed{6} \end{pmatrix}$$

Indep.

Linear Independence

When is $\{v\}$ is linearly dependent?

When V = zero vector.

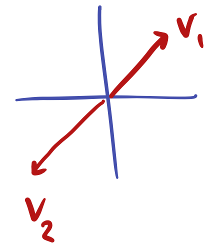
$$\_\_\, V = 0$$

When is $\{v_1, v_2\}$ is linearly dependent?

When v_1 is a multiple of v_2 (or vice versa)

or: they lie on same line thru 0.

$$\_\_\, v_1 + \_\_\, v_2 = 0.$$



or: one in span of other.

When is the set $\{v_1, v_2, v_3\}$ linearly dependent?

example

dep $\quad 2\cdot\begin{pmatrix}3\\7\end{pmatrix} - 1\begin{pmatrix}6\\14\end{pmatrix} = 0.$

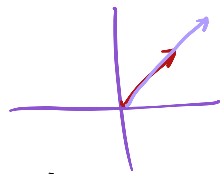
dep $\quad 5\cdot\begin{pmatrix}0\\0\end{pmatrix} + 0\cdot\begin{pmatrix}6\\14\end{pmatrix} = 0$

▸ Demo

Linear Independence

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if they span a k-dimensional plane. (algebra $\leftrightarrow$ geometry)

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if we can remove a vector from the set without changing (the dimension of) the span.



Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of $v_1, \ldots, v_{i-1}$.

Span and Linear Independence

Is $\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Try using the last fact: the set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of $v_1, \ldots, v_{i-1}$.

- $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$ not in span of previous vectors (there are none)
- $\begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}$ not in span of $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$ (not a multiple)
- $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ not in span of $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}$ because of the 4

So <u>indep</u>.

Linear independence and free variables

Theorem. Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and consider the vector equation

$$x_1 v_1 + \cdots + x_k v_k = 0.$$

The set of vectors corresponding to non-free variables are linearly independent.

So, given a bunch of vectors $v_1, \ldots, v_k$, if you want to find a collection of v_i that are linearly independent, you put them in the columns of a matrix, row reduce, find the pivots, and then take the original v_i corresponding to those columns.

Example. Try this with $(1,1,1)$, $(2,2,2)$, and $(1,2,3)$.

Linear independence and coordinates

Fact. If $v_1, \ldots, v_k$ are linearly independent vectors then we can write each element of

$$\mathrm{Span}\{v_1, \ldots, v_k\}$$

in exactly one way as a linear combination of $v_1, \ldots, v_k$.

Fact 1. Say $v_1, \ldots, v_k$ are in $\mathbb{R}^n$. If $k > n$, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

Fact 2. If one of $v_1, \ldots, v_k$ is 0, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

Parametric vector form and linear independence

Poll

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

Example. In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Parametric Vector Forms and Linear Independence

In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of x_3 and x_4 that gives that solution.

Summary of Section 2.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

  has only the trivial solution. It is linearly dependent otherwise.
- The cols of A are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ A has a pivot in each column
- The number of pivots of A equals the dimension of the span of the columns of A
- The set $\{v_1, \ldots, v_k\}$ is linearly independent $\Leftrightarrow$ they span a k-dimensional plane
- The set $\{v_1, \ldots, v_k\}$ is linearly dependent $\Leftrightarrow$ some v_i lies in the span of $v_1, \ldots, v_{i-1}$.
- To find a collection of linearly independent vectors among the $\{v_1, \ldots, v_k\}$, row reduce and take the (original) v_i corresponding to pivots.

Typical exam questions

- State the definition of linear independence.
- *Always/sometimes/never.* A collection of 99 vectors in $\mathbb{R}^{100}$ is linearly dependent.
- *Always/sometimes/never.* A collection of 100 vectors in $\mathbb{R}^{99}$ is linearly dependent.
- Find all values of h so that the following vectors are linearly independent:

$$\left\{ \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ h \end{pmatrix} \right\}$$

- *True/false.* If A has a pivot in each column, then the rows of A are linearly independent.
- *True/false.* If u and v are vectors in $\mathbb{R}^5$ then $\{u, v, \sqrt{2}u - \pi v\}$ is linearly independent.
- If you have a set of linearly independent vectors, and their span is a line, how many vectors are in the set?

Announcements Sep 27

- Masks ⇝ Music?
- Mid-semester survey in Canvas → Quizzes
- WeBWorK 2.5 & 2.6 due Tuesday nite
- Midterm 2 Oct 20 8–9:15p
- No quiz Friday(?!)
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Many TA office hours listed on Canvas
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- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Find a lin ind span set for Col(A)

$$A = \begin{pmatrix} 1 & \cancel{2} & \cancel{3} & 7 & \cancel{8} \\ 2 & \cancel{4} & \cancel{6} & 8 & \cancel{10} \end{pmatrix}$$

mult of 1st

sum of 1st & 4th

$$\rightsquigarrow \begin{pmatrix} \boxed{1} & 2 & 3 & 7 & 8 \\ 0 & 0 & 0 & \boxed{-6} & -6 \end{pmatrix}$$

x_1 in Nul(A)

x_2 is soln to $Ax=b$

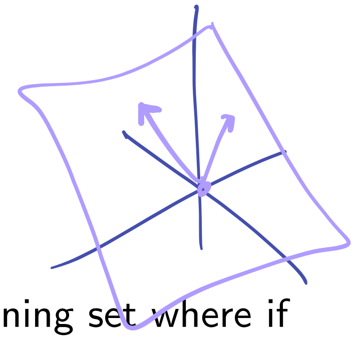
then $A(17x_1+x_2)=b$

Section 2.7

Bases

Bases

V = subspace of $\mathbb{R}^n$ (possibly $V = \mathbb{R}^n$)   plane thru 0,

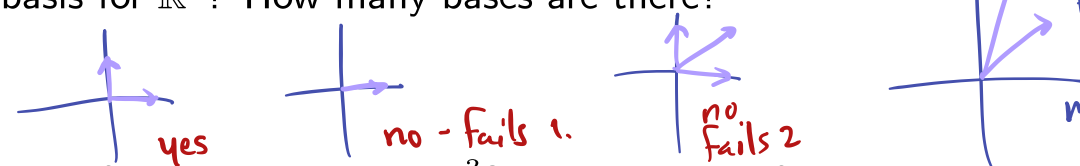


A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \ldots, v_k\}$
2. $v_1, \ldots, v_k$ are linearly independent

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

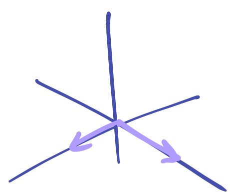
Q. What is one basis for $V = \mathbb{R}^2$? How many bases are there?



yes   no - fails 1.   no fails 2

any 2 vectors that are not mult. of each other is a basis

Q. What is one basis for the xy-plane in $\mathbb{R}^3$? Find all bases for the xy-plane.



Any 2 vectors in xy-plane. not multiples of each other

Dimension

$V =$ subspace of $\mathbb{R}^n$

$\dim(V) =$ dimension of $V = k =$the number of vectors in the basis

(What is the problem with this definition of dimension?)

How do we know for a fixed V. all bases have same number of vectors?

It turns out: it's ok. All bases have same dim.

Bases for $\mathbb{R}^n$

Let us consider the special case where V is equal to all of $\mathbb{R}^n$.

What are all bases for $V = \mathbb{R}^n$? Or, if we have a set of vectors $\{v_1, \ldots, v_k\}$, how do we check if they form a basis for $\mathbb{R}^n$? First, we make them the columns of a matrix....

- For the vectors to be linearly independent we need a pivot in every column.
- For the vectors to span $\mathbb{R}^n$ we need a pivot in every row.

Conclusion: $k = n$ and the matrix has n pivots.

$$\begin{pmatrix} \square & & \\ & \square & \\ & & \square \end{pmatrix}$$

So $\begin{pmatrix} \boxed{5} & 7 & 10 \\ 0 & \boxed{5} & 11 \\ 0 & 0 & \boxed{12} \end{pmatrix}$

cols form a basis for $\mathbb{R}^3$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Too tall $\qquad$ Too wide

The standard basis for $\mathbb{R}^n$

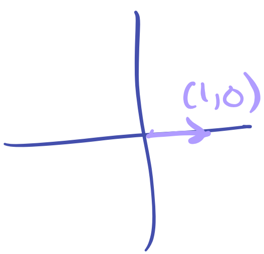
We have the standard basis vectors for $\mathbb{R}^n$:

$$
\begin{aligned}
e_1 &= (1, 0, 0, \ldots, 0) \\
e_2 &= (0, 1, 0, \ldots, 0) \\
&\vdots
\end{aligned}
$$

$\mathbb{R}^2$

$e_1 = (1, 0)$

$e_2 = (0, 1)$



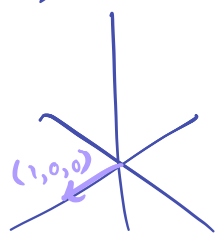
$\mathbb{R}^3$

$e_1 = (1, 0, 0)$

$e_2 = (0, 1, 0)$

$e_3 = (0, 0, 1)$

$$\begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$



etc.

Who cares about bases?

A basis $\{v_1, \ldots, v_k\}$ for a subspace V of $\mathbb{R}^n$ is useful because:

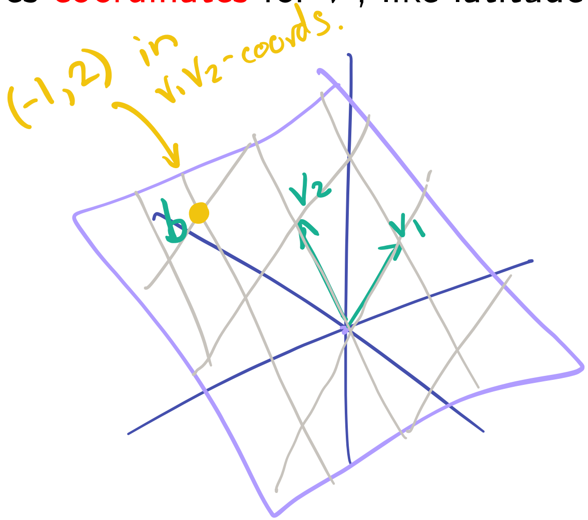
lin ind

Every vector v in V can be written in exactly one way:

$$v = c_1 v_1 + \cdots + c_k v_k$$

span

So a basis gives coordinates for V, like latitude and longitude. See Section 2.8.



Solve $x \cdot v_1 + y \cdot v_2 = b$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Col(A)

$$\sim \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Pivot in 1st col $\sim$ So take 1st col of orig. matrix A

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

are pivot cols

Nul(A)

$x = -y - z$

$y = y$

$z = z$

corr to non-pivot cols

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is the basis.

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Oopsies!

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\underline{\mathrm{Nul}(A)}$

$x = z$

$y = -2z$

$z = z$

basis: $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

corr to non-pivot cols

$\underline{\mathrm{Col}(A)}$

basis $\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$

pivot cols.

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\mathrm{Nul}(A)$
- the pivot columns of A form a basis for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

What should you do if you are asked to find a basis for Span$\{v_1, \ldots, v_k\}$?

Find a basis for span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}, \ldots \right.$

Make a matrix where those are the cols. Now it's a Col(A) problem.

Bases for planes

Find a basis for the plane $2x + 3y + z \boxed{= 0}$ in $\mathbb{R}^3$.

makes it a Nul(A) problem

Nul(A) $\quad A = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$

$x = -\frac{3}{2}y - \frac{1}{2}z$

$y = y$

$z = z$

basis for Nul(A) $\quad \underbrace{\left\{ \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \right\}}$

So, these span Nul(A) & are linind

Basis theorem

Basis Theorem

If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

So in a 2D plane, a span set with 2 vectors must be lin ind (so we have a basis)

So in a 3D plane a lin ind set of 3 vectors spans the 3D plane (so we have a basis)

We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

Typical exam questions

- Find a basis for the yz-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in R^7?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of A is the number of pivots of A.
- True/false: If b lies in the column space of A, and the columns of A are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span R^3 must be linearly independent.

Announcements Sep 29

Null space problem

Q. Find a basis for the intersection of the planes

$$x+y+z+w=0$$
$$x+2y+3z+4w=0$$

in $\mathbb{R}^4$

A. Find vector param form

$$A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

- Masks ⇝ Music!
- Mid-semester survey in Canvas → Quizzes
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- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 2.9

The rank theorem

cols with pivots

+ # cols without pivots

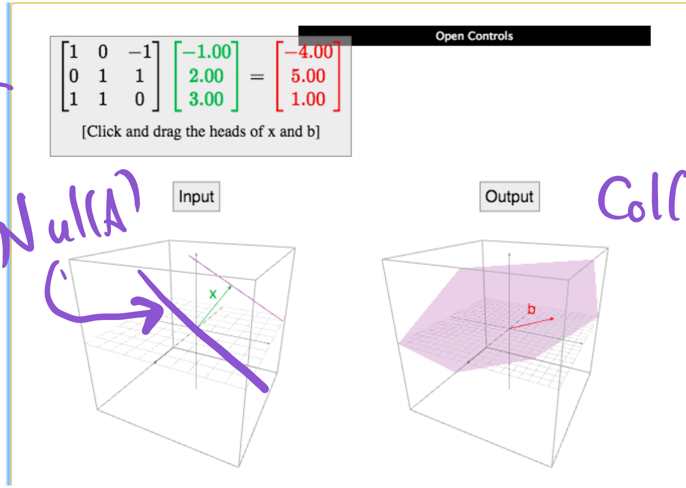
= # cols

Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:

2 pivots

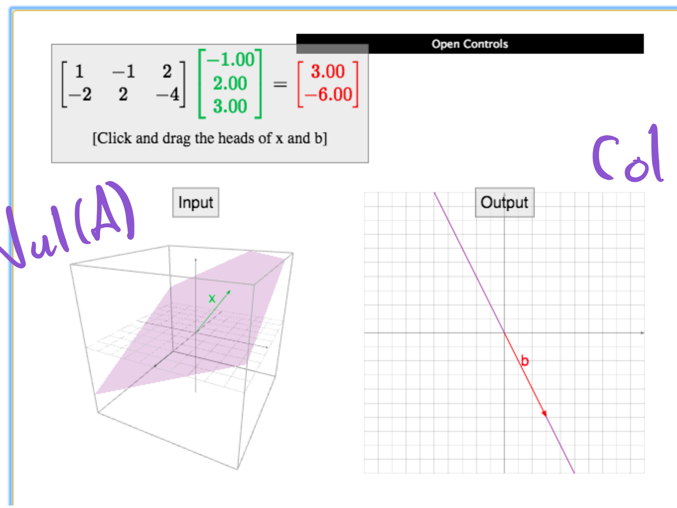
Nul(A)



Col(A)

$1 + 2 = 3$

Nul(A)



Col(A)

$2 + 1 = 3.$

Rank Theorem

#basis vecs.

numbers

$$\text{rank}(A) = \overset{\shortparallel}{\dim}\,\text{Col}(A) = \#\text{ pivot columns}$$

b/c basis vecs for Col(A) are pivot cols.

$$\text{nullity}(A) = \dim \text{Nul}(A) = \#\text{ nonpivot columns}$$

$= \#$ free vars.

Rank Theorem. $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $Ax = 0$. So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ 1 & \cancel{1} & \cancel{1} \end{pmatrix}$

rank(A) = 1.

Nul (A) = 2

About names

Again, why did we need all these vocabulary words? One answer is that the rank theorem would be harder to understand if it was:

The size of a minimal spanning set for the set of solutions to $Ax = 0$ plus the size of a minimal spanning set for the set of b so that $Ax = b$ has a solution is equal to the number of columns of A.

Compare to: $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$

"A common concept in history is that knowing the name of something or someone gives one power over that thing or person." –Loren Graham
`http://philoctetes.org/news/the_power_of_names_religion_mathematics`

Section 2.9 Summary

- **Rank Theorem.** $\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = \#\operatorname{cols}(A)$

Typical exam questions

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?

Chapter 3

Linear Transformations and Matrix Algebra

Where are we?

In Chapter 1 we learned to solve all linear systems algebraically.

In Chapter 2 we learned to think about the solutions geometrically.

In Chapter 3 we continue with the algebraic abstraction. We learn to think about solving linear systems in terms of inputs and outputs. This is similar to control systems in AE, objects in computer programming, or hot pockets in a microwave.

More specifically, we think of a matrix as giving rise to a function with inputs and outputs. Solving a linear system means finding an input that produces a desired output. We will see that sometimes these functions are invertible, which means that you can reverse the function, inputting the outputs and outputting the inputs.

The invertible matrix theorem is the highlight of the chapter; it tells us when we can reverse the function. As we will see, it ties together everything in the course so far.

Sections 3.1

Matrix Transformations

Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$

From matrices to functions

Let A be an $m \times n$ matrix.

A function is a rule with inputs & outputs:

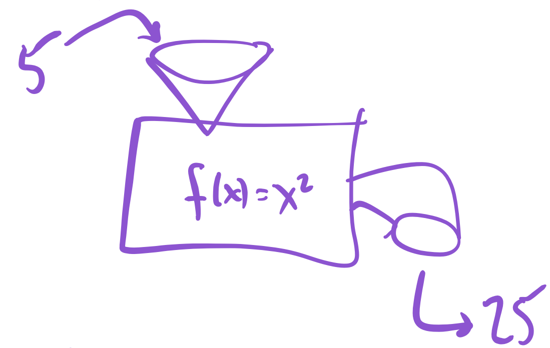
We define a function

inputs outputs

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(v) = Av$$

vector



This is called a matrix $\underbrace{\text{transformation}}_{\text{another word for function}}$.

The domain of T is $\mathbb{R}^n$.
inputs

The co-domain of T is $\mathbb{R}^m$.
potential outputs

Important: Every input has one output

The range of T is the set of outputs: $\mathrm{Col}(A)$
actual outputs, always contained in codomain

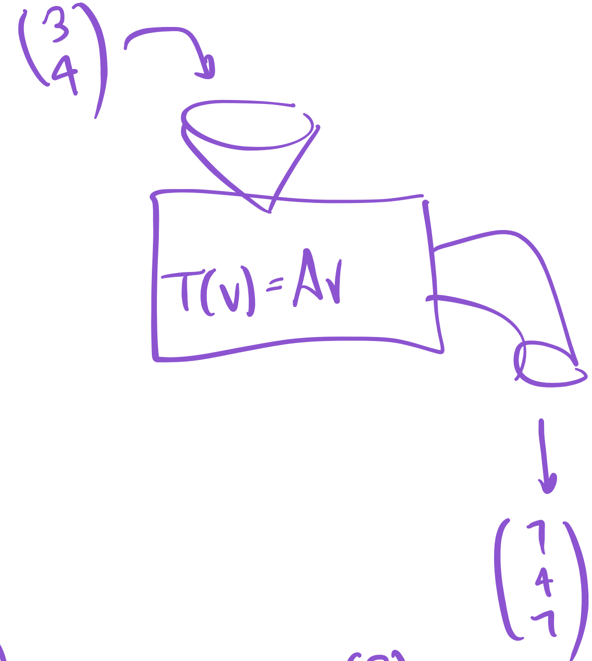
This gives us a*nother* point of view of $Ax = b$

input output

Demo

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.



What is $T(u)$? $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$

Find v in $\mathbb{R}^2$ so that $T(v) = b$

input

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$ answer: $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

or

$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right)$

Find a vector in $\mathbb{R}^3$ that is not in the range of T.

Any vector not in Col(A)

Any vector where top & bottom numbers not same. eg. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Square matrices

$n \times n$

For a square matrix we can think of the associated matrix transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

as doing something to $\mathbb{R}^n$.

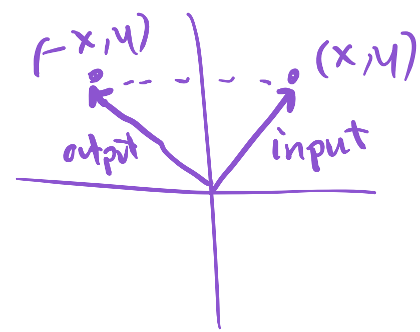
input

Example. The matrix transformation T for

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

What does T do to $\mathbb{R}^2$?

~~Reflection~~ Flip over y-axis

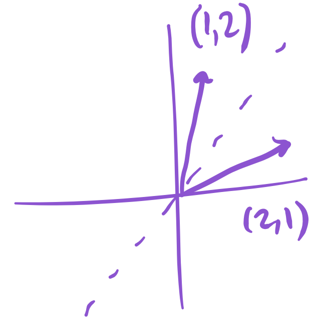


Square matrices

What does each matrix do to $\mathbb{R}^2$?

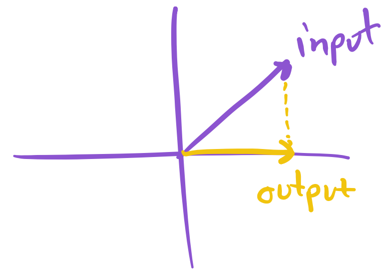
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

flip/reflect over $y=x$



$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

(orthogonal) projection to x-axis



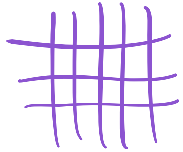
$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

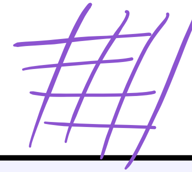
dilating by 3

What is the range in each case?

$\mathbb{R}^2$, x-axis, $\mathbb{R}^2$

Poll



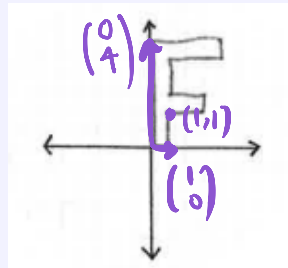


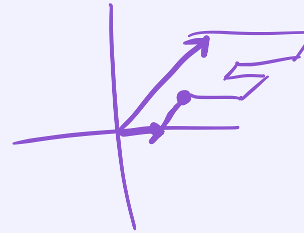
Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?

italicize!

or shear





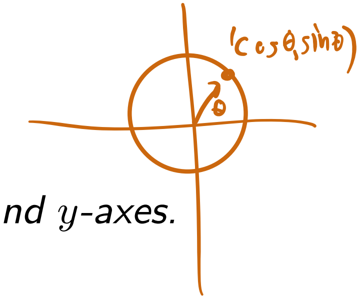
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

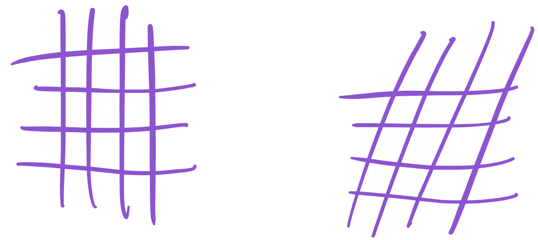
Square matrices



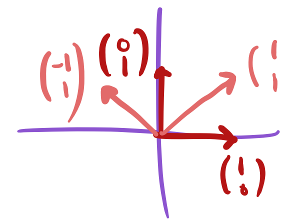
What does each matrix do to $\mathbb{R}^2$?

Hint: if you can't see it all at once, see what happens to the x- and y-axes.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ shear



$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

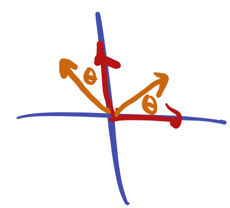
rotate by $\pi/4$ & scale by $\sqrt{2}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

rotation by θ



Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

(orthogonal) projection to xy-plane.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!

shear

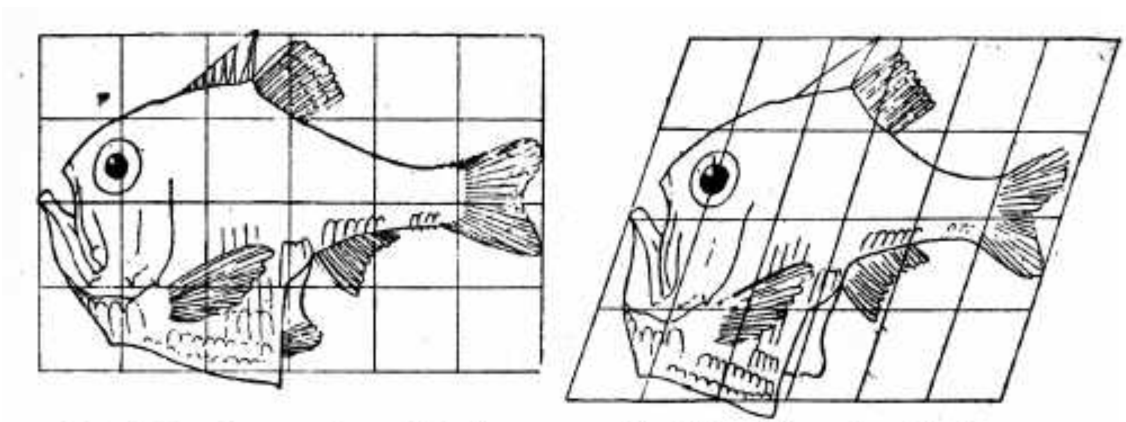


Fig. 517. *Argyropelecus Olfersi.* Fig. 518. *Sternoptyx diaphana.*

Applications of Linear Algebra

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

These relations can be represented using a matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

How does this relate to matrix transformations?

Demo

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Typical exam questions

- What does the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ do to $\mathbb{R}^3$?
- What does the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- True/false. If A is a matrix and T is the associated matrix transformation, then the statement $Ax = b$ is consistent is equivalent to the statement that b is in the range of T.
- True/false. There is a matrix A so that the domain of the associated matrix transformation is a line in $\mathbb{R}^3$.

Applications of Linear Algebra

Biology: In a population of rabbits...

RECORD!

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

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$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

How does this relate to matrix transformations?

Demo

Announcements Oct 4

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- WeBWorK 2.7+2.9 & 3.1 due Tuesday nite
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- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Sections 3.1

Matrix Transformations

From matrices to functions

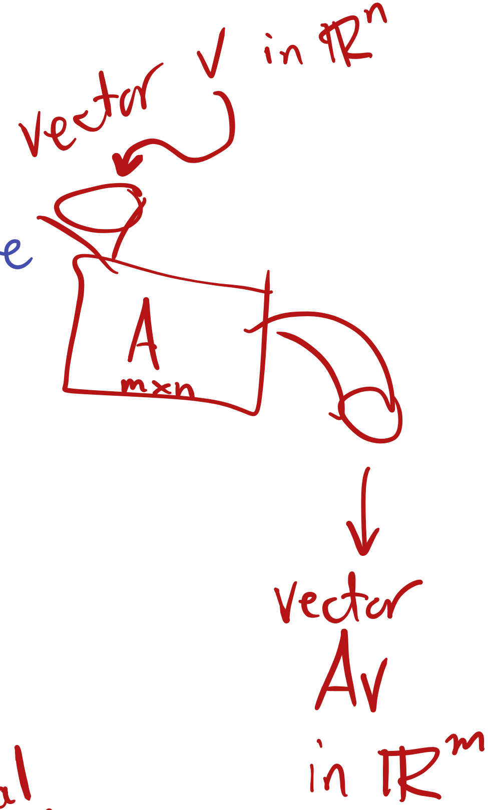
Let A be an $m \times n$ matrix.

We define a function (map)

from here to here

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(v) = Av$$



This is called a matrix transformation.

The domain of T is $\mathbb{R}^n$. inputs

The co-domain of T is $\mathbb{R}^m$. potential outputs

The range of T is the set of outputs: $\mathrm{Col}(A)$ actual outputs

This gives us a*nother* point of view of $Ax = b$

input output

Demo

Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!

"shear"

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Input

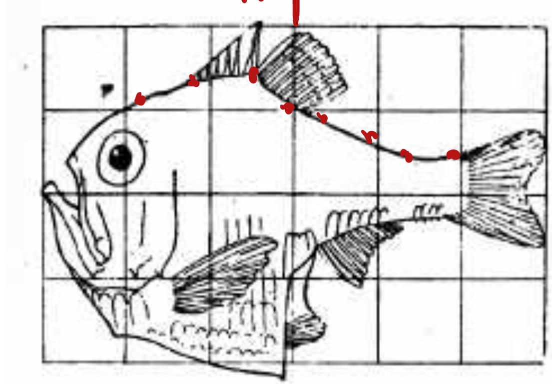


Fig. 517. *Argyropelecus Olfersi.*

Output

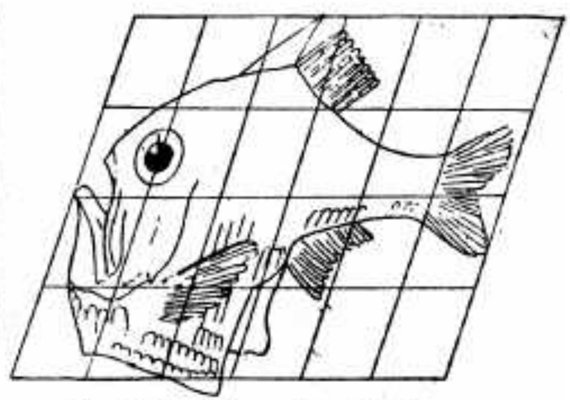


Fig. 518. *Sternoptyx diaphana.*

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These relations can be represented using a matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Input: population in year N

Output: population in year N+1

How does this relate to matrix transformations?

input

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 28 \\ 1 \\ 1 \end{pmatrix}$$

output

Demo

Section 3.2

One-to-one and onto transformations

One-to-one and onto in calculus

What do one-to-one and onto mean for a function $f : \mathbb{R} \to \mathbb{R}$?

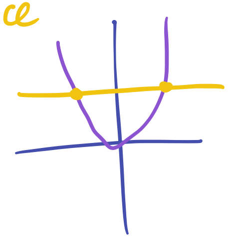
Examples

One-to-one: Different inputs have diff outputs.

x, x^3

$3x+5, e^x$

horizontal line test: each hor. line hits graph at most once



Not one-to-one: There are two inputs with same outputs

$|x|$

x^2

Onto: Range = codomain.

All vectors in codomain are outputs

each horiz line hits graph at least once.

Example

x, x^3

$3x+5$

Not onto:

x^2, e^x

$|x|$

One-to-one

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs. Same as prev. slide.

Do not confuse this with the definition of a function, which says that for each input x in $\mathbb{R}^n$ there is ~~at most~~ exactly one output b in $\mathbb{R}^m$.

Basic examples: 1 $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ all inputs have same output $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ not one-to-one

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ 2 $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ diff inputs $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}$ have diff outputs $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}$

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution ← only one input x gives the output 0.
- A has a pivot in each column
- the range of T has dimension n

We know these are same

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

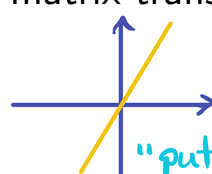
What can we say about the relative sizes of m and n if T is one-to-one?

answer: $m \geq n$ so A is tall or square.

Draw a picture of the range of a one-to-one matrix transformation $\mathbb{R} \to \mathbb{R}^{\cancel{3}2}$.

col(A)

$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



$\begin{pmatrix} 1 \\ 2 \end{pmatrix}(3) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

"putting $\mathbb{R}^1$ inside $\mathbb{R}^2$"

Onto

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

We knew these were same

- T is onto
- the columns of A span $\mathbb{R}^m$ ← $\mathrm{Col}(A) = \text{range } T$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

What can we say about the relative sizes of m and n if T is onto?

$m \leq n$ A is wide.

Give an example of an onto matrix transformation $\mathbb{R}^3 \to \mathbb{R}$.

$A = (1\ 0\ 0)$ $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x)$ (projecting) squashing $\mathbb{R}^3$ to x-axis.

One-to-one and Onto

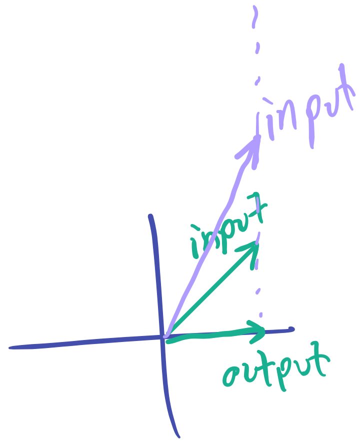
Do the following give matrix transformations that are one-to-one? onto?

$$\begin{pmatrix} \boxed{1} & 0 & 7 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{9} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1-1	✓	✓	✗ (wide)	✓
onto	✓	✗ (tall)	✓	✓

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?



		one-to-one	onto
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	reflection	✓	✓
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	projection	✗	✗
$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	scaling	✓	✓
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	shear	✓	✓
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	rotation	✓	✓

Any vec. not on x-ax is not an output.

Which are one to one / onto?

Poll

Which give one to one-to-one / onto matrix transformations?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

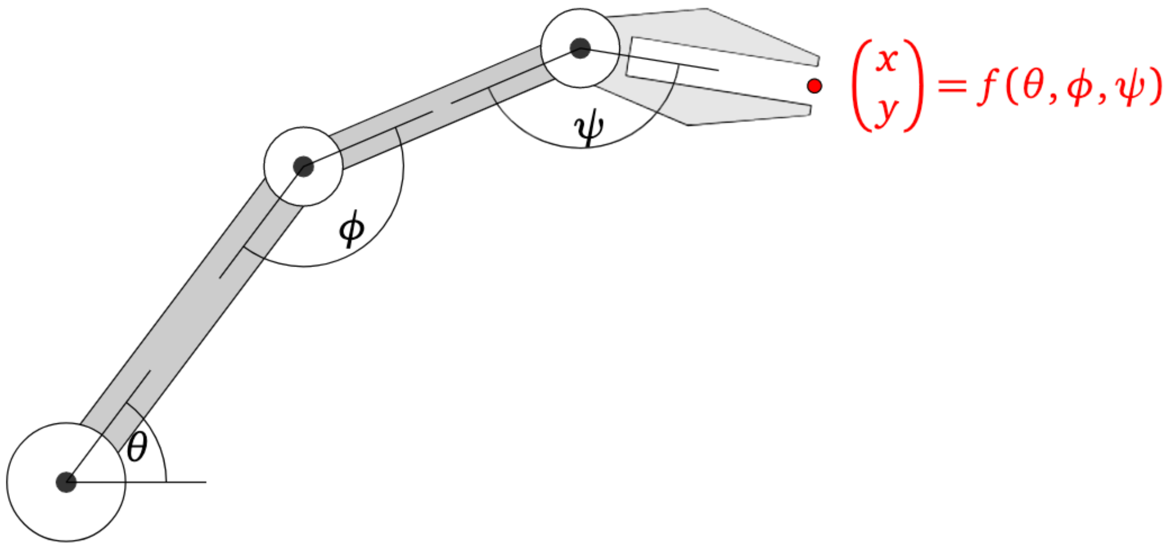
Demo

Demo

Demo

Robot arm

Consider the robot arm example from the book.



There is a natural function f here (not a matrix transformation). The input is a set of three angles and the co-domain is $\mathbb{R}^2$. Is this function one-to-one? Onto?

The geometry

Say that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

The geometry of one-to-one:

The range has dimension n (and the null space is a point).

The geometry of onto:

The range has dimension m, so it is all of $\mathbb{R}^m$ (and the null space has dimension $n - m$).

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
  - A has a pivot in each column
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m

Typical exam questions

- True/False. It is possible for the matrix transformation for a 5×6 matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the yz-plane is onto.
- True/False. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by π is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.

Which are one to one / onto?

Poll

Which give one to one-to-one / onto matrix transformations?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

Demo

Demo

Demo

Announcements Oct 6

- Masks $\rightsquigarrow$ Thank you!
- Quiz 2.5-3.1 (not 2.8) Friday
- No class Monday!
- WeBWorK 3.2 & 3.3 due Wednesday nite
- Midterm 2 Oct 20 8–9:15p

- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 3.2

One-to-one and onto transformations

Which are one to one / onto?

$f(x)=x^2$ not 1-1 b/c

-5, 5

have same output

pivot in every...

Poll

Which give one to one-to-one / onto matrix transformations?

...col ...row

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

2 pivots — not 1-1, onto

2 pivots — one-to-one, not onto

1 pivot — neither.

Demo

Demo

Demo

Section 3.3

Linear Transformations

Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation

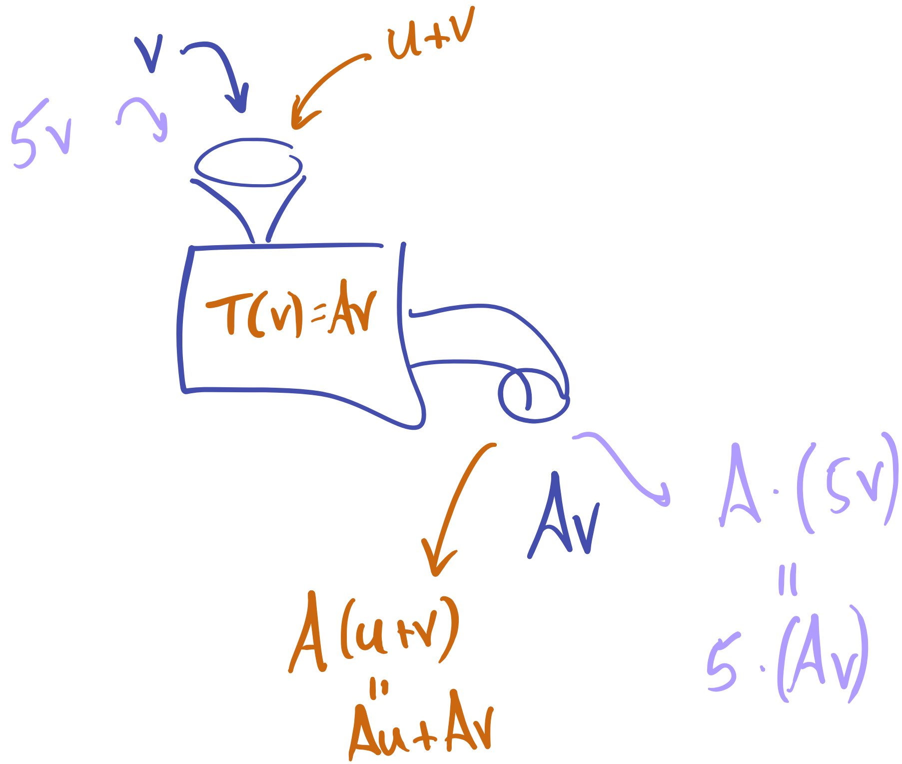
Spoiler

Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

First examples: matrix transformations.



Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

Notice that $T(0) = 0$. *Why?* $T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0$

We have the standard basis vectors for $\mathbb{R}^n$:

$$e_1 = (1, 0, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$

Every vector is a lin combo of the e_i.

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. *Why?*

What is $T(5e_1 - 7e_2) = T(5e_1) - T(7e_2) = 5T(e_1) - 7T(e_2)$

In engineering, this is called the principle of superposition.

If $T(e_1) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $T(e_2) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ then $T\begin{pmatrix} 5 \\ -7 \end{pmatrix} = 5\begin{pmatrix} 3 \\ -1 \end{pmatrix} - 7\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

Which are linear transformations?

And why?

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$ Yes. (See below)

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}\right) = T\begin{pmatrix} x+u \\ y+v \end{pmatrix} = \begin{pmatrix} x+u+y+v \\ y+v \\ (x+u)-(y+v) \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y+1 \\ y \\ x-y \end{pmatrix}$$ No

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y \\ x-y \end{pmatrix}$$ No

$$T\overset{v}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \overset{T(v)}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$T(cv)$ $\quad c\,T(v)$

$$T\left(\overset{c}{2}\overset{v}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\right) = T\underset{cv}{\begin{pmatrix} 2 \\ 2 \end{pmatrix}} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \neq 2 \cdot T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).

Linear transformations

Which properties of a linear transformation fail for this function $T : \mathbb{R}^2 \to \mathbb{R}^2$?

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ |y| \end{pmatrix}$$

- $T(cv) = cT(v)$? No

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T\left(-1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = T\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \neq -1 \cdot T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- $T(u+v) = T(u) + T(v)$? No.

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} + T\begin{pmatrix} 1 \\ 2 \end{pmatrix} \stackrel{?}{=} T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

not equal

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix A so that

exactly one

$$T(v) = Av$$

for all v in $\mathbb{R}^n$.

The matrix for a linear transformation is called the standard matrix.

How to find it?

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

The identity

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$

What is the standard matrix?

This standard matrix is called I_n or I.

Like $f(x) = x$ from calc.

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$e_1 = T(e_1)$ $\quad e_2 \quad e_3 \quad e_4$

Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

What is the standard matrix for T?

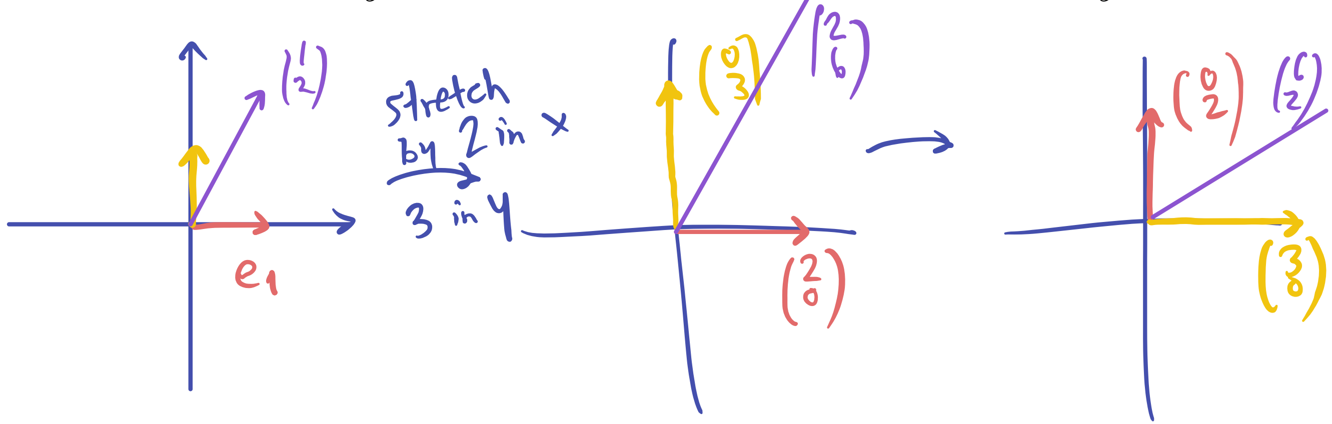
$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.



$$\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

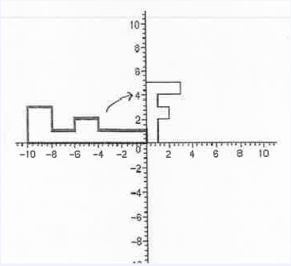
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion

Discussion Question

Find a matrix that does this.



▸ Transformation Challenge

Summary of Section 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.

Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

  What is

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix}?$$

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the z-axis by π and then scales by 2.
- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

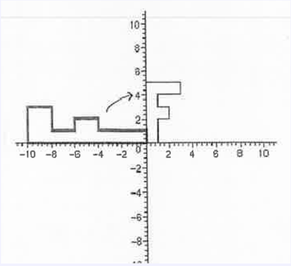
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$$

  Is this a linear transformation? If so, what is the standard matrix for T?
- Is the identity transformation one-to-one?

Discussion

Discussion Question

Find a matrix that does this.



Transformation Challenge

Announcements Oct 13

- Masks $\rightsquigarrow$ Thank you!
- Quiz 3.2-3.3 Friday
- WeBWorK 3.2 & 3.3 due tonite!
- Special office hr: Thu 11-12 Teams (special time!)
- Midterm 2 Oct 20 8–9:15p on Teams

- Use Piazza for general questions
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
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- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 3.3

Linear Transformations

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

$T(u+v) = T(u)+T(v)$

$T(cv) = cT(v)$

$T(v) = Av$

harder: these are all matrix transf.

easier: these are linear transf.

$A(v+w) = Av+Aw$

$T(v+w) = T(v)+T(w)$

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

RECIPE

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

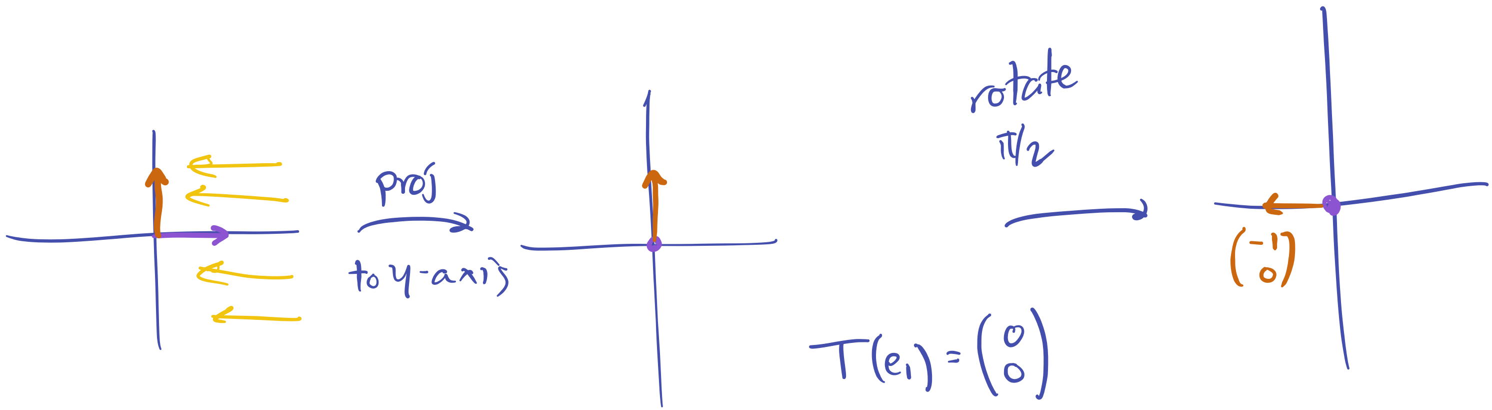
$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

Find $T(e_1)$, $T(e_2)$



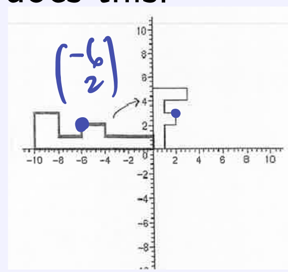
$$T(e_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Discussion

Discussion Question

Find a matrix that does this.

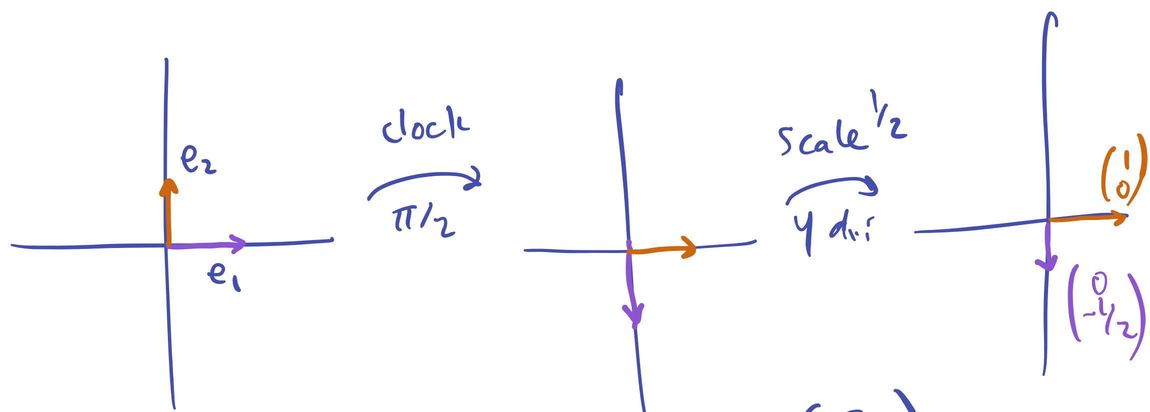


or

scale by $1/2$ in x dir

rotate by $\pi/2$ clockwise

rotate $\pi/2$ clockwise then scale by $1/2$ in y-dir.



$$\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}\begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Transformation Challenge

Section 3.4

Matrix Multiplication

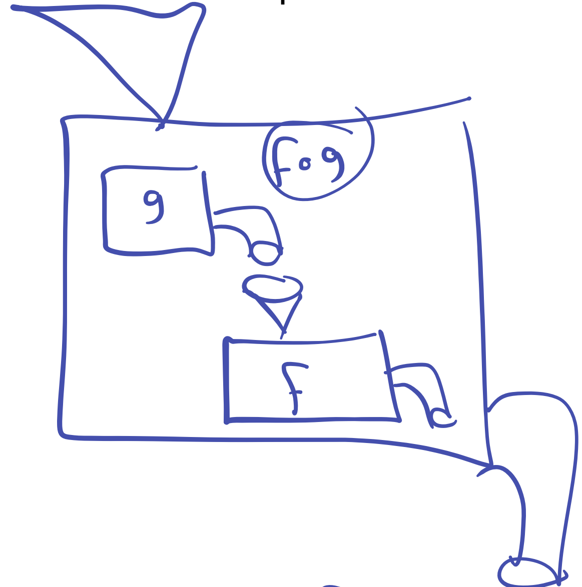
Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things

Function composition

Remember from calculus that if f and g are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$



In words: first apply g, then f.

Example: $f(x) = x^2$ and $g(x) = x + 1$.

Note that $f \circ g$ is usually different from $g \circ f$.

$f \circ g(x) = (x+1)^2$

$g \circ f(x) = x^2 + 1$

$f(g(x))$

Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^{\boxed{p}} \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^{\boxed{p}}$ and make the composition $T \circ U$.

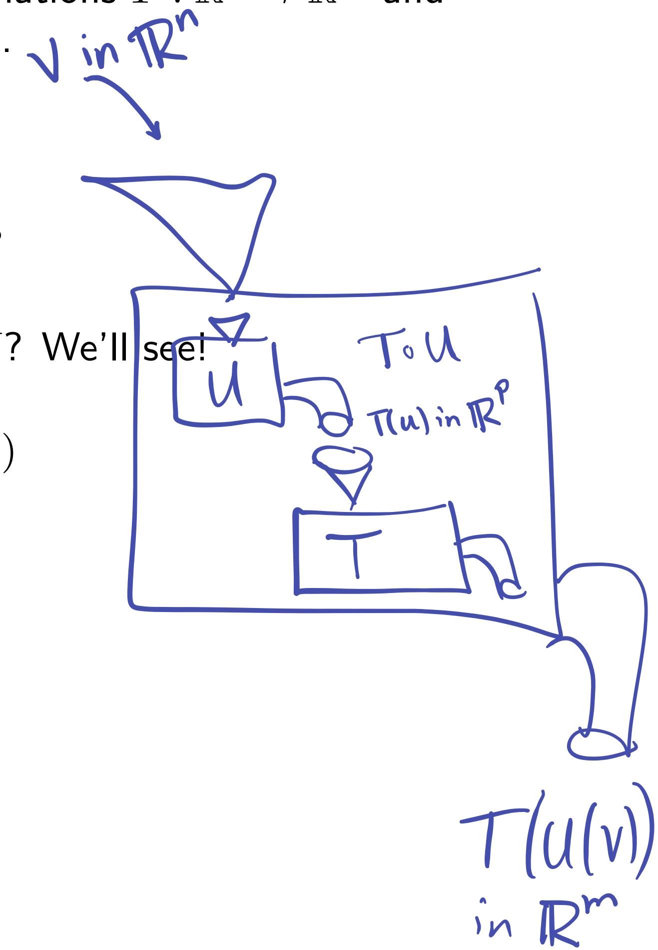
Notice that both have an p. Why?

What are the $\underbrace{\text{domain}}_{\mathbb{R}^n}$ and $\underbrace{\text{codomain}}_{\mathbb{R}^m}$ for $T \circ U$?

Natural question: What is the matrix for $T \circ U$? We'll see!

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?



Composition of linear transformations

Example. $T =$ ~~projection to y-axis and~~ scale y-dir by $1/2$
$U =$ ~~reflection about $y = x$ in $\mathbb{R}^2$~~ rotate clock by $\pi/2$

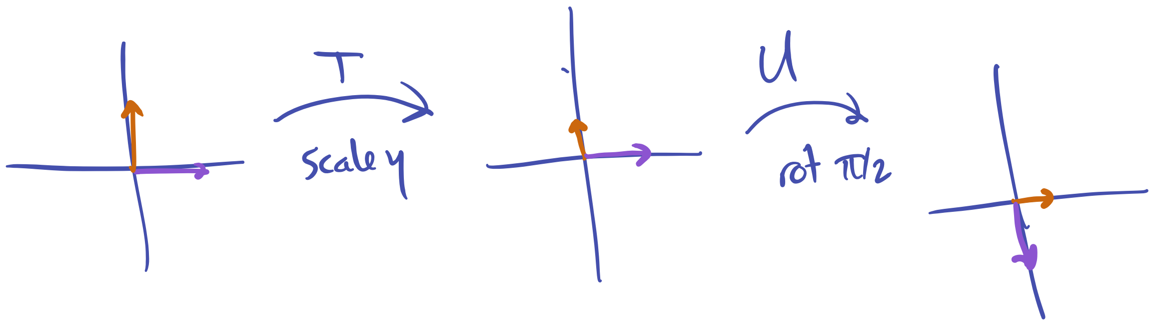
$$T \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

What is the standard matrix for $T \circ U$?

What about $U \circ T$? $\quad T \circ U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$

usual recipe



$$U \circ T \leftrightarrow \begin{pmatrix} 0 & 1/2 \\ -1 & 0 \end{pmatrix}$$

Matrix Multiplication

And now for something completely different (not really!)

Suppose A is an $m \times n$ matrix. We write a_{ij} or A_{ij} for the ijth entry.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$ and

2nd way

$$(AB)_{ij} = r_i \cdot b_j$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$$

where r_i is the ith row of A, and b_j is the jth column of B.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -13 \end{pmatrix}$$

Or: the jth column of AB is A times the jth column of B.

Multiply these matrices (both ways):

1st way

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ \boxed{17} & -13 \end{pmatrix}$$

$2 \times \cancel{3}$ $\quad \cancel{3} \times 2$ $\quad 2 \times 2$

Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do U then do T

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

composing transf's $\longleftrightarrow$ multiplying matrices.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv) = (AB)(v)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case Bv is the ith column of B. So the left-hand side is A times the ith column of B. The right-hand side is the ith column of AB which we already said was A times the ith column of B. It works!

Matrix Multiplication and Linear Transformations

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^p \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^p$. The standard matrix for $T \circ U$ is AB.

scale by 1/2 in y dir   rot. clockwise $\pi/2$

Example. $T =$ ~~projection to y-axis~~ and $U =$ ~~reflection about $y = x$ in $\mathbb{R}^2$~~

What is the standard matrix for $T \circ U$?

$$\boxed{\begin{array}{l} T \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \\ U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}}$$

$$T \circ U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$$

rot. clock $\pi/2$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} \boxed{0} & 1 \\ -1/2 & 0 \end{pmatrix}}_{T \circ U}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1/2 \\ -1 & 0 \end{pmatrix}}_{U \circ T}$$

$T \circ U$ same as:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{rot clock } \pi/2} \underbrace{\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{scale in x dir by } 1/2} = \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$$

Linear transformations are matrix transformations

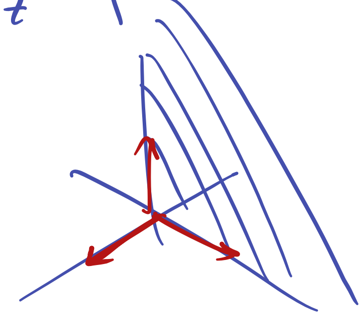
Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} & & \boxed{0} \\ \boxed{0} & & \\ & & \boxed{-1} \end{pmatrix}$$

proj to yz-plane

refl. in xy plane

multiply



Discussion Question

Are there nonzero matrices A and B with $AB = 0$?

1. Yes
2. No

Properties of Matrix Multiplication

- $A(BC) = (AB)C$ assoc.
- $A(B + C) = AB + AC$ distrib.
- $(B + C)A = BA + CA$ distrib.
- $r(AB) = (rA)B = A(rB)$
- $(AB)^T = B^T A^T$
- $I_m A = A = AI_n$, where I_k is the $k \times k$ identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- AB is not always equal to BA
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that A or B is 0

More rabbits

Recall that the following matrix describes the change in our rabbit population from this year to the next:

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

What matrix should we use if we want to describe the change in the rabbit population from this year to two years from now? Or 10 years from now?

v = this year's popul.

Av = next year's popul.

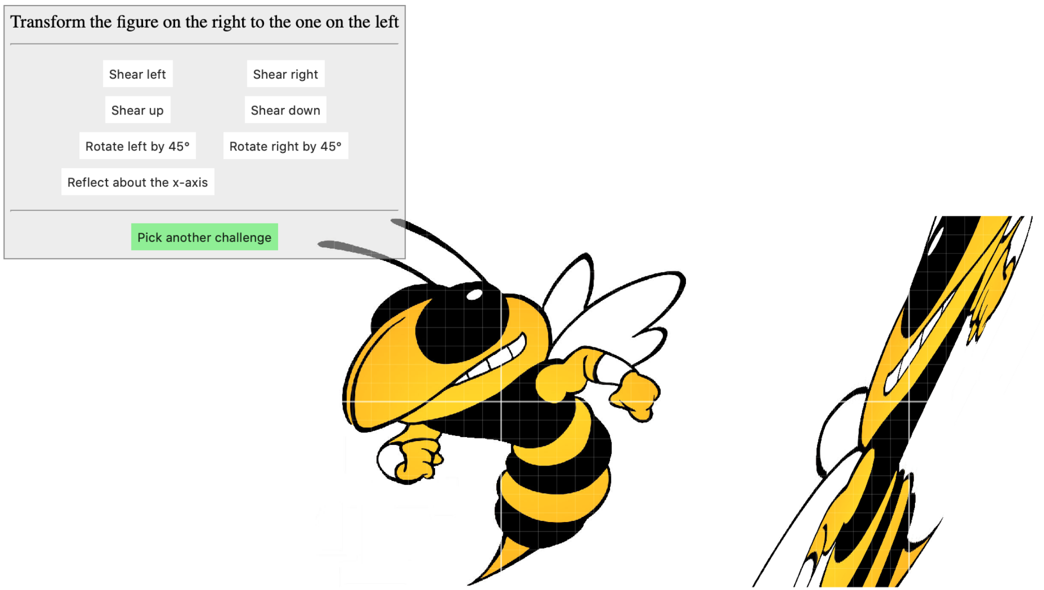
$A^2v = AAv$ = year after that

$A^{100}v$ = 100 years after 1st year

Fun with matrix multiplication

Play the Buzz game!

http://textbooks.math.gatech.edu/ila/demos/transform_game.html



In the rotation game, you need to find a composition of shears that gives a rotation!

Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0

Typical Exam Questions 3.4

- True/False. If A is a 3×4 matrix and B is a 4×3 matrix, then it makes sense to multiply A and B in both orders.
- True/False. If it makes sense to multiply a matrix A by itself, then A must be a square matrix.
- True/False. If A is a non-zero square matrix, then A^2 is a non-zero square matrix.
- True/False. If $A = -I_n$ and B is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the xy-plane and the yz-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix A for projection to the xy-plane in $\mathbb{R}^3$. What is A^2?
- Find the standard matrix A for reflection in the xy-plane in $\mathbb{R}^3$. Is there a matrix B so that $AB = I_3$?

Announcements Oct 18

- Masks ⇝ Thank you!
- Midterm 2 Wed 8–9:15p on Teams (2 channels). Sec. 2.5–3.4 (not 2.8)
- No quiz Friday
- WeBWorK 3.4 due ~~Tue @ midnight~~ tonight!
- Usual office hour: Tue 4-5 Teams (Thu office hour cancelled)
- Review Session: Joseph Cochran Tue 7–9 Skiles 230 (backup: Clough 315)
- Review session: Prof. M Wed 4:30–5:15 Howey L1

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 3.4

Matrix Multiplication

Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do U then do T

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case Bv is the ith column of B. So the left-hand side is A times the ith column of B. The right-hand side is the ith column of AB which we already said was A times the ith column of B. It works!

Properties of Matrix Multiplication

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$
- $(AB)^T = B^T A^T$
- $I_m A = A = AI_n$, where I_k is the $k \times k$ identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- AB is not always equal to BA
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that A or B is 0

Section 3.5

Matrix Inverses

Inverses

To solve

$$Ax = b$$

we might want to "divide both sides by A".

We will make sense of this...

$5x = 35$

$x = 35/5 = 7$

Inverses

$I_n A = A$

$I_n v = v$

$A = n \times n$ matrix.

I_3 :

A is invertible if there is a matrix B with

$$AB = BA = I_n$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

B is called the inverse of A and is written A^{-1}

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The 2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of A.

Fact. If $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then A is not invertible.

Check:

$$\frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = I_2$$

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

$\det = 1 \cdot 4 - 3 \cdot 2 = -2$

Solving Linear Systems via Inverses

Fact. If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

(multiply both sides by A^{-1})

Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{b} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = x$$

<u>But</u> How to find A^{-1}?

Solving Linear Systems via Inverses

What if we change b?

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 0 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

So finding the inverse is essentially the same as solving all $Ax = b$ equations at once (fixed A, varying b).

Some Facts

Say that A and B are invertible $n \times n$ matrices.

- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(A\cancel{B})(\cancel{B^{-1}}A^{-1})$$

$$A\,I\,A^{-1}$$

$$AA^{-1}$$

$$I$$

What is $(ABC)^{-1}$?

$C^{-1}B^{-1}A^{-1}$

etc.

A recipe for the inverse

Suppose $A = n \times n$ matrix. Only $\boxed{}$ matrices have inverses!

- Row reduce $(A \,|\, I_n)$
- If reduction has form $(I_n \,|\, B)$ then A is invertible and $B = A^{-1}$.
- Otherwise, A is not invertible.

Example. Find $\underbrace{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}}_{A}^{-1}$

$$\left(\underbrace{\begin{matrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{matrix}}_{A} \,\middle|\, \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3}\right) \rightsquigarrow \left(\begin{matrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix} \,\middle|\, \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{matrix}\right)$$

$$\rightsquigarrow \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \,\middle|\, \underbrace{\begin{matrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{matrix}}_{A^{-1}}\right)$$

What if you try this on one of our 2×2 examples, such as $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$?

Why Does This Work?

First answer: we can think of the algorithm as simultaneously solving

$$Ax_1 = e_1$$
$$Ax_2 = e_2$$

and so on. But the columns of A^{-1} are $A^{-1}e_i$, which is x_i.

Do the recipe for 2×2:

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & +1 & -1 & +2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

inverse.

Matrix algebra with inverses

We saw that if $Ax = b$ and A is invertible then $x = A^{-1}b$.

We can also, for example, solve for the matrix X, assuming that

$$AX = C + DX$$

Assume that all matrices arising in the problem are $n \times n$ and invertible.

$$AX - DX = C$$

$$\cancel{X(A-D) = C}$$

$$(A-D)X = C$$

mult. both sides ON LEFT by $(A-D)^{-1}$

$$(A-D)^{-1}(A-D)X = (A-D)^{-1}C$$

$$X = (A-D)^{-1}C.$$

Scaled vectors and invertibility

Poll

Suppose that x is a nonzero vector and $Ax = 5x$. What can you say about the matrix $A - 5I$?

- it is invertible
- it is not invertible
- we cannot tell if it is invertible or not

Invertible Functions

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a function $U : \mathbb{R}^n \to \mathbb{R}^n$, so

$$T \circ U = U \circ T = \text{identity}$$

From calculus:
$f(x) = x^3$
$g(x) = x^{1/3}$
$f \circ g(x) = x$

That is,

$$T \circ U(v) = U \circ T(v) = v \text{ for all } v \in \mathbb{R}^n$$

$T(v) = Av$

Fact. Suppose $A = n \times n$ matrix and T is the matrix transformation. Then T is invertible *as a function* if and only if A is invertible. And in this case, the standard matrix for T^{-1} is A^{-1}.

$T =$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Example. Counterclockwise rotation by $\pi/2$.

$U =$ clockwise rotation by $\pi/2$ $\longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example. T = proj to x-axis
What is inverse?
There isn't one!

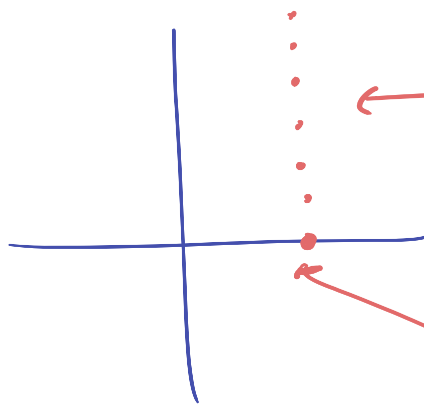
Which are invertible?

Poll

Which are invertible linear transformations of $\mathbb{R}^2$?

- reflection about the x-axis
- projection to the x-axis
- rotation by π
- reflection through the origin
- a shear
- dilation by 2

No!



all these vectors go to same place

More rabbits

$A \cdot$ current population $=$ next years population

We can use our algorithm for finding inverses to check that

$$A \quad \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1/8 & 0 & -3/2 \end{pmatrix}.$$

Recall that the first matrix tells us how our rabbit population changes from one year to the next.

If the rabbit population in a given year is $(60, 2, 3)$, what was the population in the previous year?

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1/8 & 0 & -3/2 \end{pmatrix} \begin{pmatrix} 60 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}$$

Summary of Section 3.5

- A is invertible if there is a matrix B (called the inverse) with
$$AB = BA = I_n$$
- For a 2×2 matrix A we have that A is invertible exactly when $\det(A) \neq 0$ and in this case
$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
- If A is invertible, then $Ax = b$ has exactly one solution:
$$x = A^{-1}b.$$
- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \,|\, I_n)$.
- Invertible linear transformations correspond to invertible matrices.

Typical Exam Questions 3.5

- Find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

- Find a 2×2 matrix with no zeros that is equal to its own inverse.
- Solve for the matrix X. Assume that all matrices that arise are invertible:

$$C + BX = A$$

- True/False. If A is invertible, then A^2 is invertible?
- Which linear transformation is the inverse of the clockwise rotation of $\mathbb{R}^2$ by $\pi/4$?
- True/False. The inverse of an invertible linear transformation must be invertible.
- Find a matrix with no zeros that is not invertible.
- Are there two different rabbit populations that will lead to the same population in the following year?

Section 3.6

The invertible matrix theorem

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible

(2) T is invertible

(3) The reduced row echelon form of A is I_n ← RECIPE

1-1 already said these same

(4) A has n pivots

(5) $Ax = 0$ has only 0 solution   pivot every col

(6) $\mathrm{Nul}(A) = \{0\}$

(7) $\mathrm{nullity}(A) = 0$

(8) columns of A are linearly independent

(9) columns of A form a basis for $\mathbb{R}^n$

(10) T is one-to-one

onto

(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$ pivot every row

(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$

(13) columns of A span $\mathbb{R}^n$

(14) $\mathrm{Col}(A) = \mathbb{R}^n$

(15) $\mathrm{rank}(A) = n$

(16) T is onto

new

(17) A has a left inverse $BA = I$

(18) A has a right inverse $AB = I$

The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

One way to think about the theorem is: there are lots of conditions equivalent to a matrix having a pivot in every row, and lots of conditions equivalent to a matrix having a pivot in every column, and when the matrix is a square, all of these many conditions become equivalent.

Example

Determine whether A is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn't necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 4 \\ 0 & 0 & \boxed{3} \end{pmatrix}$$

There are three pivot positions, so A is invertible by the IMT (statement c).

The Invertible Matrix Theorem

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.

12

3

o) If A is invertible then A^2 is invertible

p) If A^2 is invertible then A is invertible

True

More rabbits

Discussion Question

Recall that the following matrix describes the change in our rabbit population from this year to the next:

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which of the following statements are true?

1. There is a population of rabbits that will result in 0 rabbits in the following year.
2. There are two different populations of rabbits that result in the same population in the following year
3. For any given population of rabbits, we can choose a population of rabbits for the current year that results in the given population in the following year (this is tricky!).

Typical Exam Questions Section 3.6

In all questions, suppose that A is an $n \times n$ matrix and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. For each question, answer YES or NO.

(1) Suppose that the reduced row echelon form of A does not have any zero rows. Must it be true that $Ax = b$ is consistent for all b in $\mathbb{R}^n$?

(2) Suppose that T is one-to-one. Is is possible that the columns of A add up to zero?

(3) Suppose that $Ax = e_1$ is not consistent. Is it possible that T is onto?

(4) Suppose that $n = 3$ and that $T \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 0$. Is it possible that T has exactly two pivots?

(5) Suppose that $n = 3$ and that T is one-to-one. Is it possible that the range of T is a plane?

Announcements Oct 20

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- Midterm 2 Tonite! 8–9:15p on Teams (2 channels). Sec. 2.5–3.4 (not 2.8)
- No quiz Friday
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- Use Piazza for general questions
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Review for Midterm 2

Important terms

- linearly independent
- subspace
- column space
- null space
- basis
- dimension
- one-to-one
- onto
- linear transformation
- ~~inverse~~
- rank-nullity theorem

Summary of Section 2.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

  has only the trivial solution. It is linearly dependent otherwise.
- The cols of A are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ A has a pivot in each column
- The number of pivots of A equals the dimension of the span of the columns of A
- The set $\{v_1, \ldots, v_k\}$ is linearly independent $\Leftrightarrow$ they span a k-dimensional plane
- The set $\{v_1, \ldots, v_k\}$ is linearly dependent $\Leftrightarrow$ some v_i lies in the span of $v_1, \ldots, v_{i-1}$.
- To find a collection of linearly independent vectors among the $\{v_1, \ldots, v_k\}$, row reduce and take the (original) v_i corresponding to pivots.

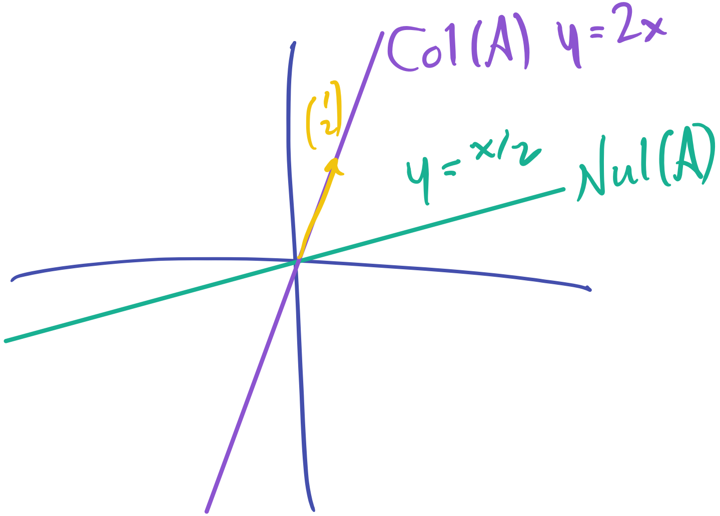
Typical exam questions 2.5

- State the definition of linear independence.
- *Always/sometimes/never.* A collection of 99 vectors in $\mathbb{R}^{100}$ is linearly dependent.
- *Always/sometimes/never.* A collection of 100 vectors in $\mathbb{R}^{99}$ is linearly dependent.
- Find all values of h so that the following vectors are linearly independent:

$$\left\{ \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ h \end{pmatrix} \right\}$$

- *True/false.* If A has a pivot in each column, then the rows of A are linearly independent.
- *True/false.* If u and v are vectors in $\mathbb{R}^5$ then $\{u, v, \sqrt{2}u - \pi v\}$ is linearly independent.
- If you have a set of linearly independent vectors, and their span is a line, how many vectors are in the set?

Find A with



$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & a \\ 2 & 2a \end{pmatrix}$$

$y = x/2$

$y - x/2 = 0$

$2y - x = 0$

$$\rightsquigarrow \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \rightsquigarrow \text{Null space}$$

$x + ay = 0$

$\rightarrow a = -2$

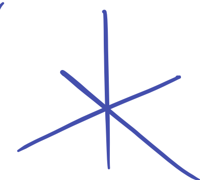
Section 2.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u+v$ is also in V. closure under add
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$. closure under scalar mult.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Four things are the same: subspaces, spans, planes through 0, null spaces

---

Let V be the subset of $\mathbb{R}^3$ consisting of the x-axis, the y-axis, and the z-axis. Which properties of a subspace does V fail?

1 ok
2 fails e_1+e_2 not in V
3 ok



Find a basis ~~spanning set~~ for the plane in $\mathbb{R}^3$ defined by $x + y - 2z = 0$.

param vect form. Null space

$(\boxed{1} \;\; 1 \;\; -2)$

$x = -y + 2z$
$y = y$
$z = z$

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

Typical exam questions

- Consider the set $\{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$. Is it a subspace? If not, which properties does it fail?
- Consider the x-axis in $\mathbb{R}^3$. Is it a subspace? If not, which properties does it fail?
- Consider the set $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$. Is it a subspace? If not, which properties does it fail?
- Find spanning sets for the column space and the null space of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

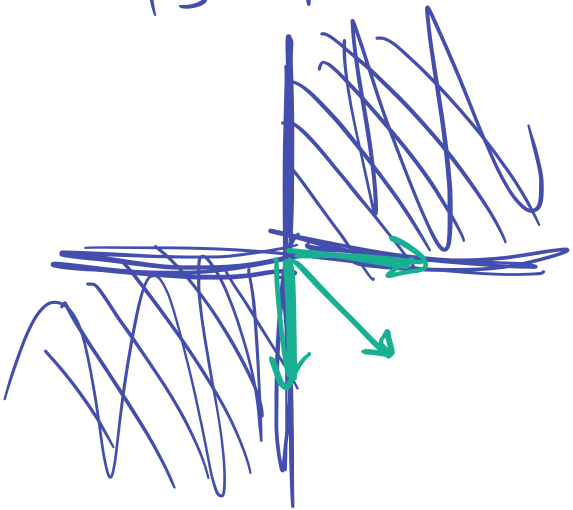
- True/False: The set of solutions to a matrix equation is always a subspace.
- True/False: The zero vector is a subspace.

$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : ab \geq 0 \right\}$$

$\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ in V

$\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ not in V

Is this a subspace?



addition?

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

in V in V not in V

Section 2.7 Summary

- A basis for a subspace V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{Span}\{v_1, \ldots, v_k\}$
  2. $v_1, \ldots, v_k$ are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\text{Col}(A)$ by taking pivot columns of A (not reduced A)
- Basis Theorem. Suppose V is a k-dimensional subspace of $\mathbb{R}^n$. Then
  - Any k linearly independent vectors in V form a basis for V.
  - Any k vectors in V that span V form a basis.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\3\\1\end{pmatrix} \begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \begin{pmatrix}-1\\1\\1\end{pmatrix} \begin{pmatrix}1\\-1\\1\end{pmatrix} \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Find a basis $\{u, v, w\}$ for $\mathbb{R}^3$ where no vector has a zero entry.

$$\begin{pmatrix}1\\7\\11\end{pmatrix} \begin{pmatrix}7\\91\\118\end{pmatrix} \begin{pmatrix}1\\2\\3\end{pmatrix}$$

Typical exam questions

- Find a basis for the yz-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in R^7?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of A is the number of pivots of A.
- True/false: If b lies in the column space of A, and the columns of A are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span R^3 must be linearly independent.

Section 2.9 Summary

Nullity

- Rank-Nullity Theorem. $\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = \#\operatorname{cols}(A)$

dim Col (A)

Let A be an 4×6 nonzero matrix and suppose the columns of A are all the same. What is $\dim \operatorname{Nul}(A)$?

$$\begin{pmatrix} 1 & 1 & 1 & \\ 2 & 2 & 2 & \cdots \\ 3 & 3 & 3 & \\ 4 & 4 & 4 & \end{pmatrix}$$

rank = 1

$$\boxed{5}$$

Typical exam questions

$Ax = b$ in out

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a 2D plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?

rank = dim Col(A) = 5

no pivot on RHS YES!

one-to-one or no free vars

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

---

Find a matrix A so that the range of the matrix transformation $T(v) = Av$ is the line $y = 2x$ in $\mathbb{R}^2$.

Typical exam questions

- What does the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ do to $\mathbb{R}^3$?
- What does the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- True/false. If A is a matrix and T is the associated matrix transformation, then the statement $Ax = b$ is consistent is equivalent to the statement that b is in the range of T.
- True/false. There is a matrix A so that the domain of the associated matrix transformation is a line in $\mathbb{R}^3$.

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are indep
  - $Ax = 0$ has only 0 soln
  - A has a pivot in every col
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in every row
  - $Ax = b$ is consistent for every b in $\mathbb{R}^m$
  - the range of T has dimension m

one-to-one $\Leftrightarrow$ onto

Let A be an 5×5 matrix. Suppose that $\dim \mathrm{Nul}(A) = 0$. Must it be true that $Ax = e_1$ is consistent?

TRUE $\begin{pmatrix} \square & & & & \\ & \square & & & \\ & & \square & & \\ & & & \square & \\ & & & & \square \end{pmatrix}$

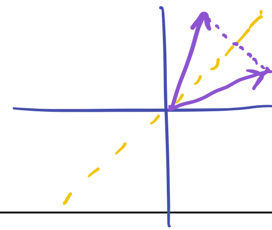
yes b/c onto

Typical exam questions

- True/False. It is possible for the matrix transformation for a 5×6 matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the yz-plane is onto. NO Range is yz-plane, not all of $\mathbb{R}^3$
- True/False. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by π is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not. NO. $\begin{pmatrix} \square & \\ & \square \end{pmatrix}$
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not. $\begin{pmatrix} \square & \\ & \square \end{pmatrix}$

Summary of Section 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.



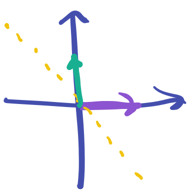
$B\ A$

$U \circ T(v) = U(T(v))$

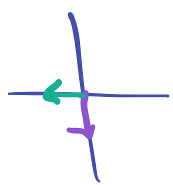
$BA\,v$

Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects over the line $y = -x$ and then rotates counterclockwise by $\pi/2$.

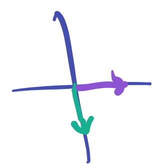
''



$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\xrightarrow{T}$



$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\xrightarrow{U}$



$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

What is

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix}? = T\begin{pmatrix} 2 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the z-axis by π and then scales by 2.
- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$$

Is this a linear transformation? If so, what is the standard matrix for T?
- Is the identity transformation one-to-one?

Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- The standard matrix for a composition of linear transformations is the product of the standard matrices.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0

Find a 2×2 matrix A so that $A^4 = I$ and $A^2 \neq I$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hint: Think about transformations.

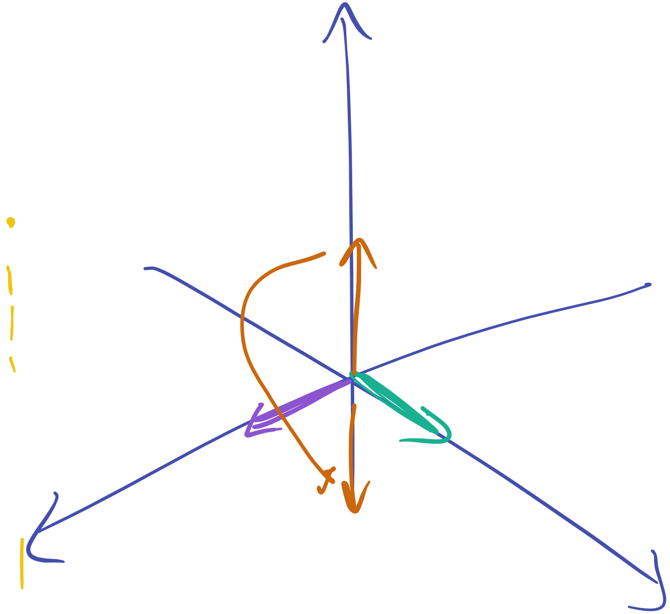
Rotation by $\pi/2$

Typical Exam Questions 3.4

$(3\times4)(4\times3)$

- True/False. If A is a 3×4 matrix and B is a 4×3 matrix, then it makes sense to multiply A and B in both orders.
- True/False. If it makes sense to multiply a matrix A by itself, then A must be a square matrix.
- True/False. If A is a non-zero square matrix, then A^2 is a non-zero square matrix.
- True/False. If $A = -I_n$ and B is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the xy-plane and the yz-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix A for projection to the xy-plane in $\mathbb{R}^3$. What is A^2?
- Find the standard matrix A for reflection in the xy-plane in $\mathbb{R}^3$. Is there a matrix B so that $AB = I_3$?

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \quad -I \cdot A = -1 \cdot I \cdot A = -A \quad \text{TRUE}$$



$$A = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{-1} \end{pmatrix}$$

Q. Is there B

so $AB = I$

Yes $B = A$

$$A^2 = I$$

Practice #18
2b $T : \mathbb{R}^4 \to \mathbb{R}^k$

- $T\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$ NO.
- For each x in $\mathbb{R}^4$ exactly one y in $\mathbb{R}^k$ so $T(x) = y$. No - Function
- Every v in $\mathbb{R}^k$ is image of at most one x in $\mathbb{R}^4$
  This is defn of 1-1
- Range of T is 4D. Yes - pivot in each col

Midterm 2b #5

$$T: \mathbb{R}^2 \to \mathbb{R}^3$$

$$T(x) = \begin{pmatrix} 2x_1 + 2x_2 \\ -x_1 + 3x_2 \\ x_1 + x_2 \end{pmatrix}$$

Describe the x's

So $T(x) = 0$.

$\mathrm{Nul}\begin{pmatrix} 2 & 2 \\ -1 & 3 \\ 1 & 1 \end{pmatrix}$ 2 pivots

$\rightsquigarrow$ pt. in $\mathbb{R}^2$

Range of T

2 pivots

$\rightsquigarrow$ plane.

2a #17

$$T: \mathbb{R}^3 \to \mathbb{R}^7$$

$$T(e_1) = T(e_2)$$

What is max poss $\underbrace{\text{dim of range}}_{\text{rank}}$.

guess: 2

$$\begin{pmatrix} 1 & 1 & * \\ 2 & 2 & * \\ 3 & 3 & * \\ 4 & 4 & * \\ 5 & 5 & * \\ 6 & 6 & * \\ 7 & 7 & * \end{pmatrix}$$

2b #15

$T: \mathbb{R}^2 \to \mathbb{R}^2$

reflect across x

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$(3\times 2)(2\times 2)$

$U: \mathbb{R}^2 \to \mathbb{R}^3$

$$U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 0 \\ y \end{pmatrix} \begin{matrix} 2x+0y \\ 0x+0y \\ 0x+1y \end{matrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find std matrix. $U \circ T$

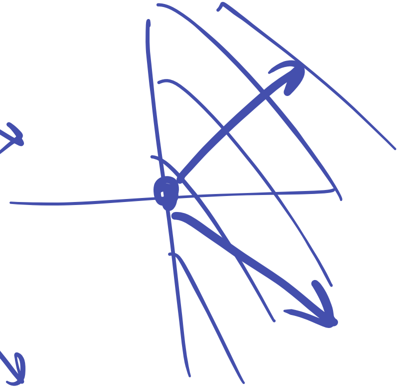
$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Practice 2a
#11b T/F $T(x) = Ax$ one-to-one.

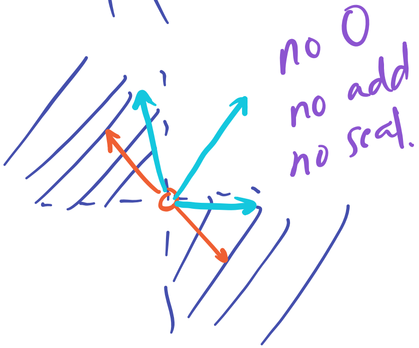
one-to-one and onto { For each b in codom. $T(x) = b$ has exactly one soln.

input

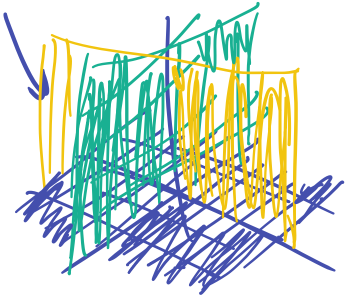
$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a \geq 0 \right\}$$



$$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : ab < 0 \right\}$$



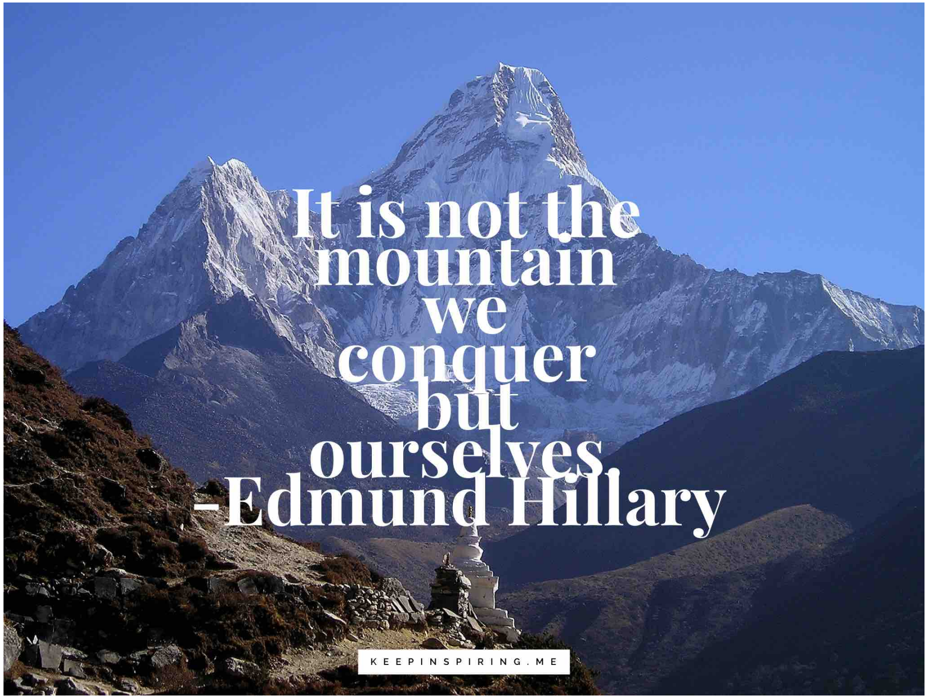
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : abc = 0 \right\}$$



$$u\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$$

2a #19

Good luck!


It is not the mountain we conquer but ourselves
-Edmund Hillary
KEEPINSPIRING.ME

The Invertible Matrix Theorem

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.

o) If A is invertible then A^2 is invertible

p) If A^2 is invertible then A is invertible

Announcements Oct 25

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 3.5 & 3.6 due Tue @ midnight
- Studio but no quiz Friday
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 3.6

The invertible matrix theorem

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible
(2) T is invertible
(3) The reduced row echelon form of A is I_n
(4) A has n pivots
(5) $Ax = 0$ has only 0 solution
(6) $\mathrm{Nul}(A) = \{0\}$
(7) $\mathrm{nullity}(A) = 0$
(8) columns of A are linearly independent
(9) columns of A form a basis for $\mathbb{R}^n$
(10) T is one-to-one
(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$
(13) columns of A span $\mathbb{R}^n$
(14) $\mathrm{Col}(A) = \mathbb{R}^n$
(15) $\mathrm{rank}(A) = n$
(16) T is onto
(17) A has a left inverse
(18) A has a right inverse

(19) A^T invertible.

T means transpose.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Span of cols of A

$=$ span of rows of A^T

The Invertible Matrix Theorem

$(AA)(A^{-1}A^{-1})$

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

Typical

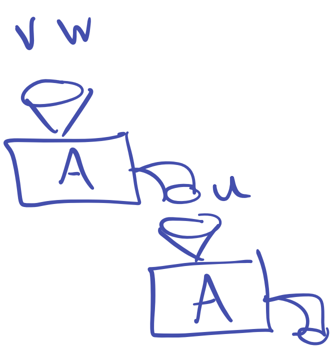
n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity. Yes

o) If A is invertible then A^2 is invertible

Harder

p) If A^2 is invertible then A is invertible

m) Yes dim of span of rows = dim of span of cols = # pivots — let me think of a better answer.



o) Yes the inverse of A^2 is: $(A^{-1})^2$

p) If A were not invertible, then T is not one-to-one then $T \circ T$ not one-to-one, so A^2 not invertible

Chapter 4

Determinants

Where are we?

- We have studied the problem $Ax = b$
- We learned to think of $Ax = b$ in terms of transformations
- We next want to study $Ax = \lambda x$ Eigenvalues
- At the end of the course we want to almost solve $Ax = b$

We need determinants for the second item.

Section 4.1

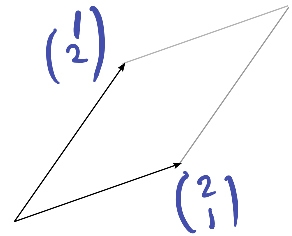
The definition of the determinant

Invertibility and volume

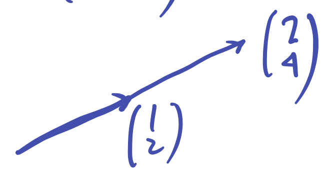
When is a 2×2 matrix invertible? $\leftarrow$ Algebra

When the rows (or columns) don't lie on a line $\Leftrightarrow$ the corresponding parallelogram has non-zero area. $\leftarrow$ Geometry

$$\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 1\cdot 1 - 2\cdot 2 = -3$$

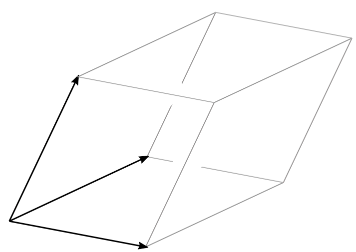


$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ one pivot



When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane $\Leftrightarrow$ the corresponding parallelepiped (3D parallelogram) has non-zero volume



Same for $n \times n$!

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

(signed)

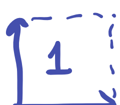
Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

④ $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

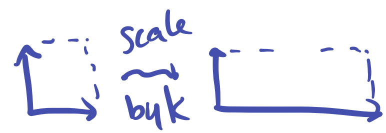
① $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[\text{repl}]{\text{row}} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

③ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow[\text{scale}]{\text{row}} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$



area 1

area doesn't change!



area scales by k.

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

Problem. Just using these rules, compute the determinants:

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

row repl

1

I

Use ①, ④

row swap

-1

I

Use ②, ④

row scale by 17

17

I

Use ③, ④

row repl

24

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ det = 24

Use ①, ④

A basic fact about determinants

Fact. If A has a zero row, then $\det(A) = 0$.

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix} = 0 \cdot \det\begin{pmatrix} a & b & c \\ d & e & f \\ 0\cdot 0 & 0\cdot 0 & 0\cdot 0 \end{pmatrix} = 0$$

Fact. If A is a diagonal matrix then $\det(A)$ is the product of the diagonal entries.

$$\det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc$$

Fact. If A is in row echelon form then $\det(A)$ is the product of the diagonal entries.

$$\det\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix} = abc$$

(row replace to get to last fact)

Why do these follow from the definition? ✓

A first formula for the determinant

Fact. Suppose we row reduce A. Then

$$\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$$

↖ if no scalings, put a 1 here

~~Use the fact to get a formula for the determinant of any 2×2 matrix.~~

$$\underset{\det -2/35}{A} \xrightarrow[\text{swap}]{\text{row}} \left(\det 2/35 \right) \xrightarrow[\text{by } 5]{\text{row scale}} \left(\det 2/7 \right) \xrightarrow[\text{swap}]{\text{row}} \left(\det -2/7 \right)$$

$$\xrightarrow[\text{by } 7]{\text{row Scale}} \left(\quad \right)_{\det -2} \xrightarrow[\text{repl.}]{\text{row}} \begin{pmatrix} 1 & 7 & 9 \\ 0 & 2 & 17 \\ 0 & 0 & -1 \end{pmatrix}_{\det = -2}$$

$$\det A = (-1)^2 \frac{1 \cdot 2 \cdot -1}{5 \cdot 7} = -2/35$$

Consequence of the above fact:

Fact. $\det A \neq 0 \Leftrightarrow A$ invertible

$$A \longrightarrow I \qquad \det = 1$$

Computing determinants

...using the definition in terms of row operations

$$\det\begin{pmatrix}0&1&0\\1&0&1\\5&7&-4\end{pmatrix} =$$

$R3 \to R3 - 5R1$

$$\begin{pmatrix}0&1&0\\1&0&1\\5&7&-4\end{pmatrix} \xrightarrow[\text{swap}]{\text{row}} \begin{pmatrix}1&0&1\\0&1&0\\5&7&-4\end{pmatrix} \xrightarrow[\text{repl.}]{\text{2 row}} \begin{pmatrix}1&0&1\\0&1&0\\0&0&-9\end{pmatrix} \xrightarrow[\text{by } -1/9]{\text{scale}} \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$$

$\boxed{\det 9}$ $\qquad$ $\det -9$ $\qquad$ $\det -9$ $\qquad$ $\det 1$

or

$$(-1)^1 \frac{1\cdot 1\cdot 1}{-1/9} = 9$$

Computing determinants

...using the definition in terms of row operations

$$\det \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \xrightarrow[\text{swaps}]{\text{2 row}} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 6 & 8 \end{pmatrix} \xrightarrow[\text{repl.}]{\text{row}} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$\det\ 8 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2$$

$$\boxed{\det = 2}$$

$$\det A = (-1)^2 \ \frac{1/2 \cdot 1/2 \cdot 8}{1.}$$

A Mathematical Conundrum

We have this definition of a determinant, and it gives us a way to compute it.

But: we don't know that such a determinant function exists.

More specifically, we haven't ruled out the possibility that two different row reductions might gives us two different answers for the determinant.

Don't worry! It is all okay.

We already gave the key idea: that determinant is just the volume of the corresponding parallelepiped. You can read the proof in the book if you want.

Fact 1. There is such a number $\det$ and it is unique.

Properties of the determinant

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions. Next time!

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$ ok, now we need to say what transpose is

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \qquad \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - cb$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

or $|\det(A)|$ is the volume/area.

If you want the proofs, see the book. Actually Fact 1 is the hardest!

Powers

$\det BA$

$\shortparallel$

Fact 5. $\det(AB) = \det(A)\det(B)$ $\longrightarrow$ $\det(A^5) = (\det A)^5$

Use this fact to compute

$$\det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix}^5\right) = 9^5$$

$\det A \det A^{-1}$

$= \det(AA^{-1}) = 1$

What is $\det(A^{-1})$? $1/9$

$\shortparallel$

$(\det A)^{-1}$

Powers

Poll

Suppose we know A^5 is invertible. Is A invertible?

1. yes
2. no
3. maybe

Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, and $(2, 5)$?

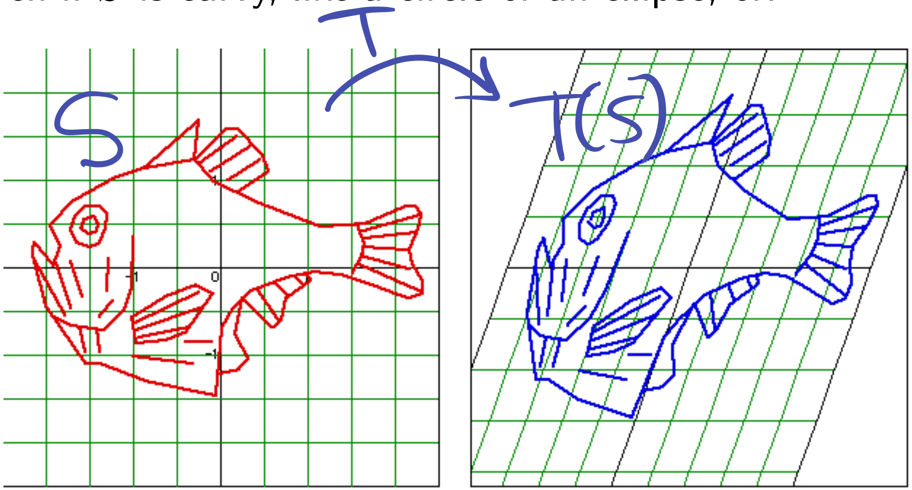
What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, $(2, 5)$, and $(5, 6)$?

Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

Summary of Sections 4.1 and 4.3

Say $\det$ is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?
- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T?
- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T?
- Find the volume of the triangular pyramid with vertices $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, and $(1,2,3)$.

Announcements Oct 27

- Masks $\rightsquigarrow$ Thank you!
- Studio but no quiz Friday
- WeBWorK 4 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard (going hybrid!)
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 4.1

The definition of the determinant

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

Today: A formula for det

like $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Properties of the determinant

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\dfrac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

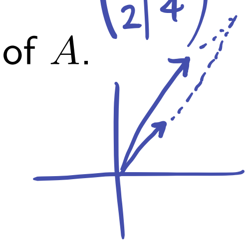
Fact 4. The function can be computed by any of the $2n$ cofactor expansions. today!

Fact 5. $\det(AB) = \det(A)\det(B)$ important! chain rule!

Fact 6. $\det(A^T) = \det(A)$ ok, now we need to say what transpose is

$\left(\begin{array}{c|c} 1 & 3 \\ 2 & 4 \end{array}\right)$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.



If you want the proofs, see the book. Actually Fact 1 is the hardest!

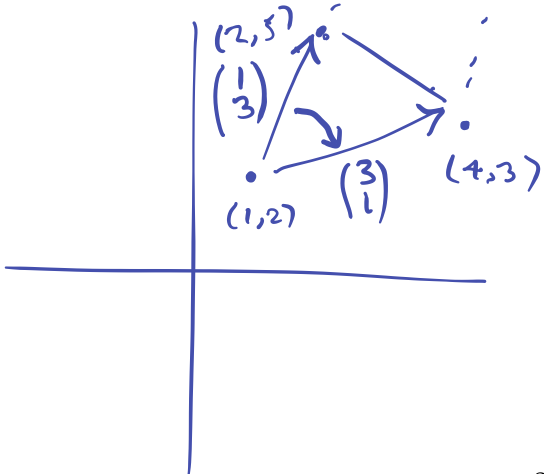
Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, and $(2,5)$?

$\det\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$



$\left|\det\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\right| = |-8| = 8$

area of parallelogram

$\boxed{4}$ area of $\triangle$

What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, $(2,5)$, and $(5,6)$?

you!

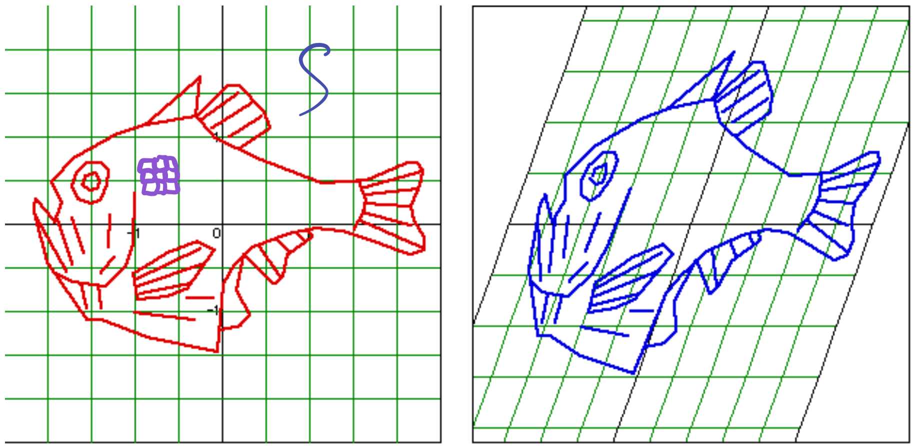
Determinants and linear transformations

apply T to every pt of S

Say A is an $n \times n$ matrix and $T(v) = Av$.

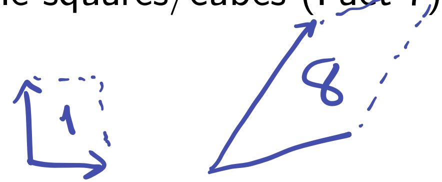
Fact 8. If S is some subset of $\mathbb{R}^n$, then $\operatorname{vol}(T(S)) = |\det(A)| \cdot \operatorname{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

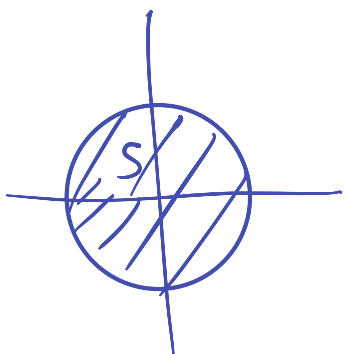


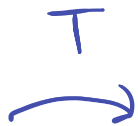
Determinants and linear transformations

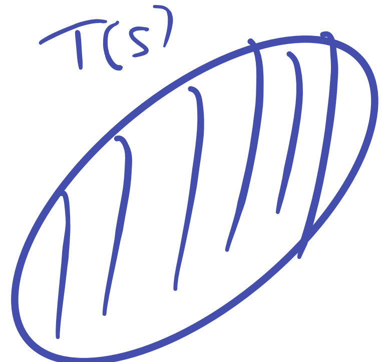
Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

Take S to be the unit disk in $\mathbb{R}^2$, that is, the set of points that have distance at most 1 from the origin. Let $A = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 2 \end{smallmatrix}\right)$, and let $T(v) = Av$ be its matrix transformation. What is the area of $\cancel{A}(S)$?
T







area $\pi \cdot 1^2 = \pi$

area: $|\det(A)|\ \mathrm{area}(S)$

$|4-8|\,\pi$

$= 4\pi$

Section 4.2

Cofactor expansions

Outline of Section 4.2

- We will give a recursive formula for the determinant of a square matrix.

A formula for the determinant

We will give a recursive formula.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

First some terminology:

$$A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$\begin{aligned} A_{ij} &= ij\text{th minor of } A \\ &= (n-1)\times(n-1) \text{ matrix obtained by deleting the } i\text{th row and } j\text{th column} \end{aligned}$$

$$\begin{aligned} C_{ij} &= (-1)^{i+j}\det(A_{ij}) \\ &= ij\text{th cofactor of } A \end{aligned}$$

$$C_{11} = (-1)^{1+1} \det A_{11} = 1 \cdot -3 = -3$$

$$C_{12} = (-1)^{1+2} \det A_{12} = -1 \cdot -6 = 6$$

$$C_{13} = (-1)^{1+3} \det A_{13} = 1 \cdot -3 = -3$$

Finally: entry of A in 1st row 1st col

all 1's

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$1 \cdot -3 + 2 \cdot 6 + 3 \cdot -3 = 0$$

(not invertible!)

Or:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

So we find the determinant of a 3×3 matrix in terms of the determinants of 2×2 matrices, etc.

Determinants

Consider

$$A = \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Compute the following:

$a_{11} = 5$ $\qquad a_{12} = 1$ $\qquad a_{13} = 0$

$A_{11} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$ $\qquad A_{12} = \begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix}$ $\qquad A_{13} = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$

$\det A_{11} = -3$ $\qquad \det A_{12} = -7$ $\qquad \det A_{13} = -12$

$C_{11} = -3$ $\qquad C_{12} = +7$ $\qquad C_{13} = -12$

$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ $\quad 5 \cdot -3 + 1 \cdot 7 + 0 \cdot$ don't care $= -8$

A formula for the determinant

We can take the recursive formula further....

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

Say that....

1×1 matrices

$$\det(a_{11}) = a_{11}$$

Now apply the formula to...

2×2 matrices

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \quad a_{11}a_{22} - a_{12}a_{21}$$

(Could also go really nuts and define the determinant of a 0×0 matrix to be 1 and use the formula to get the formula for 1×1 matrices...)

A formula for the determinant

3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \cdots$$

You can write this out. And it is a good exercise. But you won't want to memorize it.

A formula for the determinant

Another formula for 3×3 matrices

original cofactor formula.

$a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(\quad) + a_{13}(\quad)$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

(Check this is gives the same answer as before. It is a small miracle!)

This slide ONLY 3×3

Use this formula to compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$\begin{array}{cccccc} 0 & 1 & 0 & & & \\ 5 & 1 & 0 & 5 & 1 & 0 \\ -1 & 3 & 2 & -1 & 3 & 2 \\ 4 & 0 & -1 & 4 & 0 & -1 \\ & -15 & 8 & 0 & & \end{array}$$

$(-15+8+0) - (0+1+0)$

$= -7 - 1$

$= -8$

Expanding across other rows and columns

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in} \text{ for any fixed } i$$
$$\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj} \text{ for any fixed } j$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Or for odd rows and columns:

$$\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \cdots \pm a_{in}(\det(A_{in}))$$
$$\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \cdots \pm a_{nj}(\det(A_{nj}))$$

and for even rows and columns:

$$\det(A) = -a_{i1}(\det(A_{i1})) + a_{i2}(\det(A_{i2})) + \cdots \mp a_{in}(\det(A_{in}))$$
$$\det(A) = -a_{1j}(\det(A_{1j})) + a_{2j}(\det(A_{2j})) + \cdots \mp a_{nj}(\det(A_{nj}))$$

Amazingly, these are all the same!

Compute:

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

3rd col: $0 \cdot$ don't care $- \; 0 \cdot$ don't care

$+ 1 \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1 \cdot 1 = 1.$

2nd row: $-1 \cdot 1 + 1 \cdot 2 + 0 = 1$

Another!

$$\det \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} =$$

$(-1)^{i+j}$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

3rd col $\quad + 8 \cdot \frac{1}{4} = 2.$

2nd row $\quad -\frac{1}{2} \cdot \det \begin{pmatrix} 6 & 8 \\ \frac{1}{2} & 0 \end{pmatrix} = -\frac{1}{2} \cdot -4 = 2$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute with cofactor expansions (it was also easy using the definition of the determinant):

$$\det \begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$\begin{pmatrix} + & - & \\ - & + & \\ & & \end{pmatrix}$$

1st col: $+2 \cdot \det \begin{pmatrix} 1 & 2 & -3 \\ 0 & 5 & 9 \\ 0 & 0 & 10 \end{pmatrix}$

$= +2 \cdot 1 \cdot \det \begin{pmatrix} 5 & 9 \\ 0 & 10 \end{pmatrix}$

$= +2 \cdot 1 \cdot 50 = 100$

Determinants

Poll

What is the determinant?

$$\det \begin{pmatrix} 4 & 7 & 0 & 9 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 5 & 9 & 2 & 10 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

A formula for the inverse

(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix}$$

$$= \frac{1}{\det(A)} (C_{ij})^T$$

matrix of cofactors

transpose

mult. by 1/det

Check that these agree! $\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 2 \\ & & \\ & & \end{pmatrix}$ $\begin{pmatrix} 0 & & \\ 0 & & \\ 2 & & \end{pmatrix}$ $\frac{1}{2}\begin{pmatrix} \text{last} \\ \text{thing} \end{pmatrix}$

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Typical Exam Questions 4.2

- True or false. The cofactor expansion across the first row gives the negative of the cofactor expansion across the second row.
- Find the determinant of the following matrix using one of the formulas from this section:

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & 0 & 9 \end{pmatrix}$$

- Find the determinant of the following matrix using one of the formulas from this section:

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$

- Find the cofactor matrix for the above matrix and use it to find the inverse.

Announcements Nov 1

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 3.5 & 3.6 due Tue @ midnight
- ~~Studio but no~~ quiz Friday 3.5 & 3.6?
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
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- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Chapter 5

Eigenvectors and eigenvalues

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), fluid mixing, RLC circuits, clustering (data analysis), principal component analysis, Google, Netflix (collaborative prediction), infectious disease models, special relativity, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year - think of it as a vector (f, s, t) - what is the population the next year?

input output

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 28 \\ 1 \\ 1 \end{pmatrix}$$

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years N. What is the new population after N years?

Demo

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v \qquad \text{e.g. } v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}, \ \lambda = 2 \qquad \text{in last example}$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

In simpler terms: Av is a scalar multiple of v.

In other words: Av points in the same direction as v.

Think of this in terms of inputs and outputs!

eigen $=$ *characteristic (or: self)*

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

Can you find any eigenvectors/eigenvalues for the following matrix?

$v = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$ $Av = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ no!

$v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $Av = 2 \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ yes! $\lambda = 2$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$Av = 2 \cdot v$

yes $\lambda = 2$

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

$Av = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ not eigenvector

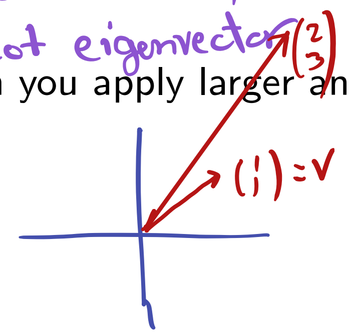
$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$Av = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

yes $\lambda = 3$

What happens when you apply larger and larger powers of A to a vector?

v
Av
A^2v
A^3v
$\vdots$



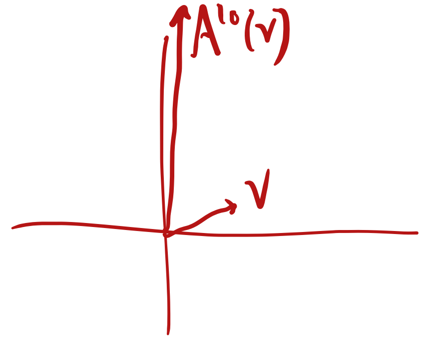
Getting pulled to y-axis.

Rabbits

What's up with them?

Eigenvectors and Eigenvalues

When we apply large powers of the matrix



$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

to a vector v not on the x-axis, we see that $A^n v$ gets closer and closer to the y-axis, and it's length gets approximately tripled each time. This is because the largest eigenvalue is 3 and its eigenspace is the y-axis.

For the rabbit matrix

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

vectors with that eigenvalue.

We will see that 2 is the largest eigenvalue, and its ~~eigenspace~~ is the span of the vector $(16, 4, 1)$. That's why all populations of rabbits tend towards the ratio 16:4:1 and why the population approximately doubles each year.

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}, \quad \lambda = 2$$

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

not eigenvector!

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

How do you check?

Eigenvectors and Eigenvalues

Confirming eigenvectors

yes! $\lambda = 2$ yes! $\lambda = 0$ Null space. no! no! 0 never is an eigenvector

Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Is $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ a multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

$\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 200 \\ 100 \end{pmatrix}$

Eigenvectors and Eigenvalues

Confirming eigenvalues

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

(Any eigenvector you find is called a 3-eigenvector.)

Never need to do again.

$$Av = 3v \rightsquigarrow Av = 3Iv \rightsquigarrow Av - 3Iv = 0$$

$$\rightsquigarrow (A - 3I)v = 0.$$

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$Ax = 0$ problem with $A - 3I$ instead of A.

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & -4 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 4 \\ 0 & 0 \end{pmatrix} \rightsquigarrow x + 4y = 0$$

$$\rightsquigarrow x = -4y, \quad y = y$$

$$\rightsquigarrow \begin{pmatrix} -4 \\ 1 \end{pmatrix} \text{ or any multiple.}$$

Check: $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix}$

What is a general procedure for finding eigenvalues?

Eigenvectors and Eigenvalues

Confirming eigenvalues

The following are equivalent:

- λ is an eigenvalue of A
- $\mathrm{Nul}(A - \lambda I)$ is nontrivial $\longleftrightarrow$ $\det A - \lambda I = 0$.

So the recipe for checking if λ is an eigenvalue of A is:

- subtract λ from the diagonal entries of A
- row reduce
- check if there are fewer than n pivots

Confirm that $\lambda = 1$ is not an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ -1 & -2 \end{pmatrix}$$

2 pivots

$\rightsquigarrow$ $\mathrm{Nul}(A - 1 \cdot I) = 0$

$\rightsquigarrow$ 1 not eigenvalue.

Eigenspaces

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Why is this a subspace?

Fact. λ-eigenspace for $A = \mathrm{Nul}(A - \lambda I)$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

2-eigenspace

$$\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

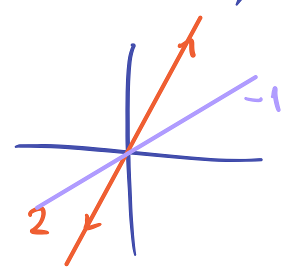
2x2 trick $\rightsquigarrow$ Span $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Swap entries in top row, negate one.

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

-1-eigenspace

$$\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$



Announcements Nov 3

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 5.1 & 5.2 due Tue @ midnight
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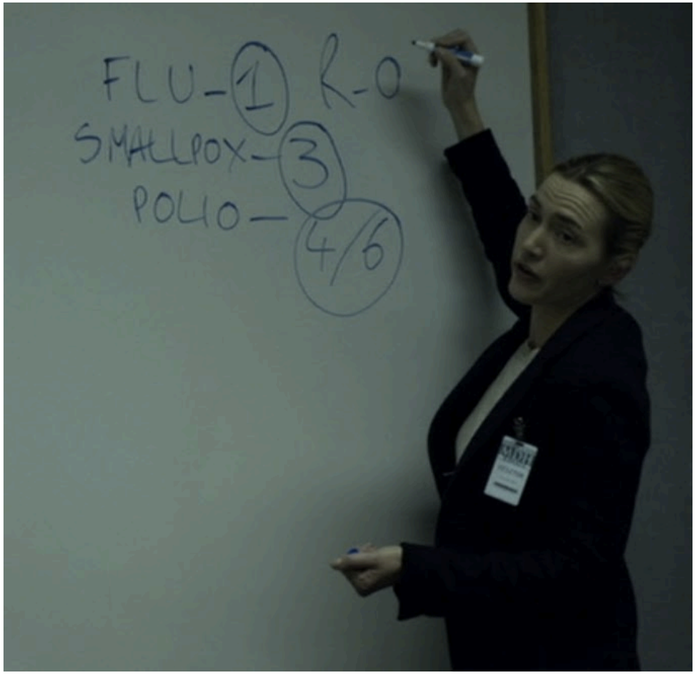
- You can do it!

$$R_0$$

R_0

For a given virus, R_0 is the average number of people that each infected person infects. If R_0 is large, that is bad. Patient zero infects R_0 people, who then infect R_0^2 people, who then infect R_0^3 people. That is exponential growth. (If R_0 is less than 1, then that's good.)



R_0

For a given virus, R_0 is the average number of people that each infected person infects. If R_0 is large, that is bad. Patient zero infects R_0 people, who then infect R_0^2 people, who then infect R_0^3 people. That is exponential growth.

Whenever we see an exponential growth rate, we should think: eigenvalue.

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment. That's a matrix. The largest eigenvalue is R_0.

R_0 is an eigenvalue

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment.

For malaria, the compartments might be mosquitoes and humans.

For a sexually transmitted disease in a heterosexual population, the compartments might be males and females.

R_0 is an eigenvalue

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment.

The SIR model has compartments for Susceptible, Infected, and Recovered.

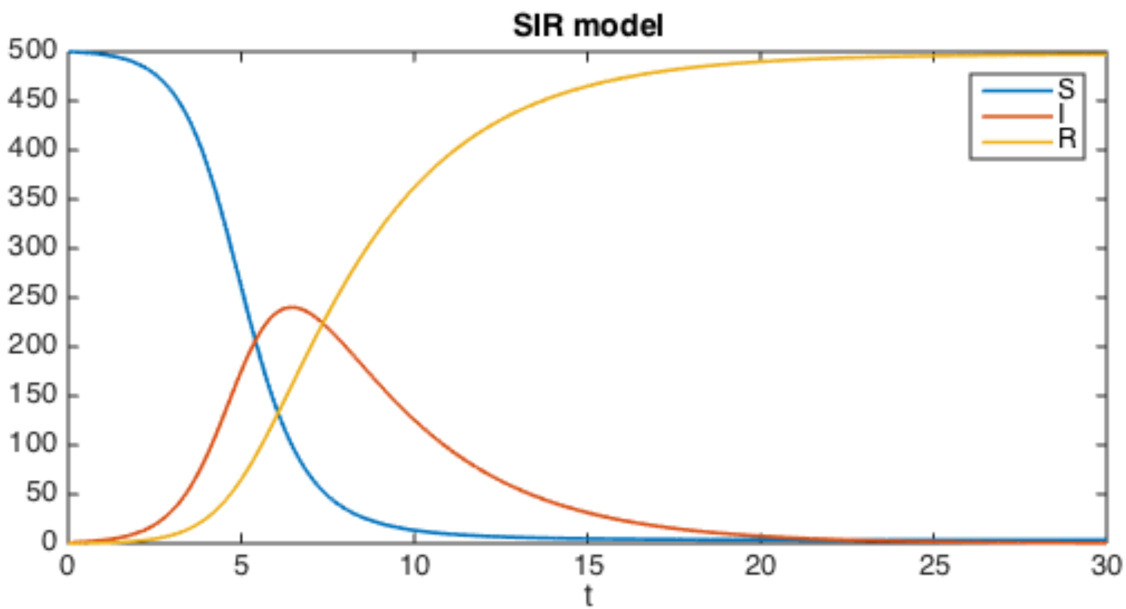
The arrows are governed by differential equations (Math 2552). Why do the labels on the arrows make sense? (The greek letters are constants).

There is a nice discussion of this by James Holland Jones (Stanford).

▸ Paper

Bell curves

The growth rate of infection does not stay exponential forever, because the recovered population has immunity. That's where you get these bell curves.



Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

In simpler terms: Av is a scalar multiple of v.

In other words: Av points in the same direction as v.

Think of this in terms of inputs and outputs!

eigen = characteristic (or: self)

This the the most important definition in the course.

Demo

Eigenspaces

Bases

Find a basis for the 2–eigenspace:

all eigenvec's with eigenval 2 plus 0 vector.

$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

$Av = 2v$ same as ... v in $\mathrm{Nul}(A-2I)$

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} \boxed{1} & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

subtract 2 from diag.

$$\rightsquigarrow \left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Check: $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Scales by 2.

$$A - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$x = x$
$y = y$

$$x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2-eigenspace: $\mathbb{R}^2$

Eigenspaces

Bases

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2-eigensp.
is xy-plane.

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{array}{l} x = x \\ y = 0 \end{array} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \checkmark$$

Fact. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A

Why?

0 is an eigenval of A

$\Longleftrightarrow Av = 0 \cdot v = 0$ some $v \neq 0$

$\Longleftrightarrow \mathrm{Nul}(A) \neq 0 \overset{\text{IVT}}{\Longleftrightarrow} A$ not invertible.

λ eigenval $\iff \mathrm{Nul}(A-\lambda I) \neq 0 \iff A-\lambda I$ fewer than n pivots.

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

$$\begin{pmatrix} 5 & 9 \\ 0 & 7 \end{pmatrix}$$

0's below diagonal

Can see 7 is an eigenval. because

$$\begin{pmatrix} -2 & 9 \\ 0 & 0 \end{pmatrix}$$

fewer than n pivots.

Same for 5

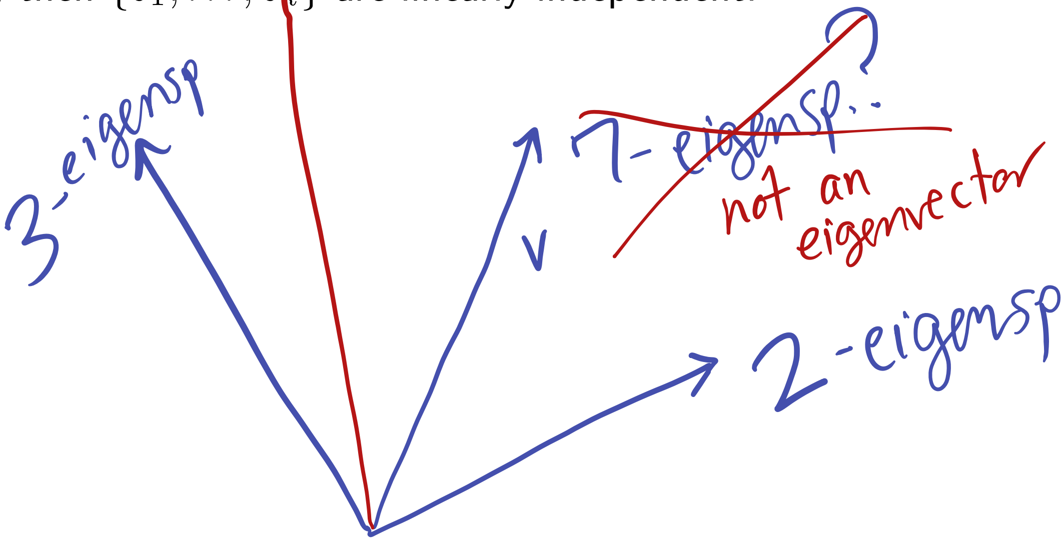
Important! You can not find the eigenvalues by row reducing first! After you find the eigenvalues, you row reduce $A - \lambda I$ to find the eigenspaces. But once you start row reducing the original matrix, you change the eigenvalues.

Eigenvalues

Distinct eigenvalues

Fact. If $v_1 \dots v_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.

Why?



Consequence. An $n \times n$ matrix has at most n distinct eigenvalues.

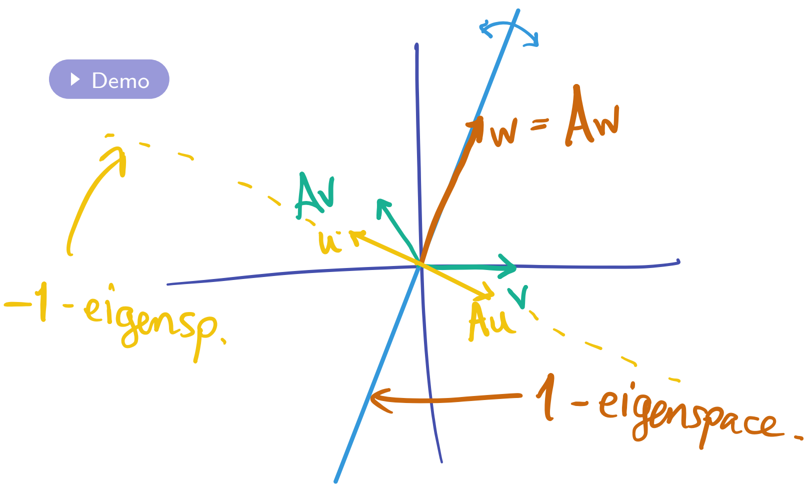
Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Orthogonal projection onto a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$
- Orthogonal projection to a plane in $\mathbb{R}^3$ (doesn't matter which plane!)

Demo



Eigenvalues for rotations?

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)?

Demo

Section 5.2

The characteristic polynomial

Outline of Section 5.2

- How to find the eigenvalues, via the characteristic polynomial
- Techniques for the 3×3 case

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \iff A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial, the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The eigenrecipe

Say you are given a square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\det(A - \lambda I) = 0$$

New

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

Sec 5.1

To find a basis, find the vector parametric solution, as usual.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - 4$$

$$= \lambda^2 - 6\lambda + 1$$

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$$

Two shortcuts for 2×2 eigenvectors

Find the eigenspaces for the eigenvalues on the last page. Two tricks.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ $\lambda = 3 \pm 2\sqrt{2}$

$(3+2\sqrt{2})$-eigenvec.

$\lambda = 3 + 2\sqrt{2}$

$$A - \lambda I = \begin{pmatrix} 2-2\sqrt{2} & 2 \\ 2 & -2-2\sqrt{2} \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 2-2\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} -2 \\ 2-\sqrt{2} \end{pmatrix}$$

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of the Fibonacci matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0 \rightsquigarrow \lambda^2 - \lambda - 1 = 0$$

$$\rightsquigarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$

golden ratio φ & $-1/\varphi$

Find eigenvecs using 2 tricks from last slide.

3×3 matrices

The 3×3 case is harder. There is a version of the quadratic formula for cubic polynomials, called Cardano's formula. But it is more complicated. It looks something like this:

$$\begin{aligned} x \quad = \quad & \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} \\ + \quad & \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} \quad - \quad \frac{b}{3a}. \end{aligned}$$

There is an even more complicated formula for quartic polynomials.

One of the most celebrated theorems in math, the Abel–Ruffini theorem, says that there is no such formula for quintic polynomials.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

What are the eigenvalues? Hint: Don't multiply everything out!

$$\det\begin{pmatrix} 7-\lambda & 0 & 3 \\ -3 & 2-\lambda & -3 \\ -3 & 0 & -1-\lambda \end{pmatrix} =$$

$$(7-\lambda)\big((2-\lambda)(-1-\lambda)\big) + 3\big(3(2-\lambda)\big)$$

$$= (2-\lambda)(\text{quadratic}) \rightsquigarrow \lambda = 2$$

two more eigenvals

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial: $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

So we plug in ± 1, ± 3, ± 5, ± 15 into the polynomial and hope for the best. Luckily we find that 1, 3, and 5 are all roots, so we found all the eigenvalues!

If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

Eigenvalues

Triangular matrices

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Warning! You cannot find eigenvalues by row reducing and then using this fact. You need to work with the original matrix. Finding eigenspaces involves row reducing $A - \lambda I$, but there is no row reduction in finding eigenvalues.

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n\lambda^n + (-1)^{n-1}\boxed{\text{trace(A)}}\lambda^{n-1} + \boxed{???}\lambda^{n-2} + \cdots \boxed{???}\lambda + \boxed{\det(\text{A})}$$

So for a 2×2 matrix:

$$\lambda^2 - \operatorname{trace}(A)\lambda + \det(A)$$

And for a 3×3 matrix:

$$-\lambda^3 + \operatorname{trace}(A)\lambda^2 - \boxed{???}\lambda + \det(A)$$

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

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Consequence 1. The constant term is zero $\Leftrightarrow$ A is not invertible

Consequence 2. The determinant is the product of the eigenvalues.

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.

Announcements Nov 8

- Masks $\rightsquigarrow$ Thank you!
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- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Eigenvalues in Structural Engineering

Watch this video about the Tacoma Narrows bridge. ▸ Watch

Here are some toy models. ▸ Check it out

The masses move the most at their natural frequencies ω. To find those, use the spring equation: $mx'' = -kx \quad \rightsquigarrow \quad \sin(\omega t)$.

With 3 springs and 2 equal masses, we get:

$$\begin{aligned} mx_1'' &= -kx_1 + k(x_2 - x_1) \\ mx_2'' &= -kx_2 + k(x_1 - x_2) \end{aligned}$$

Guess a solution $x_1(t) = A_1(\cos(\omega t) + i\sin(\omega t))$ and similar for x_2. Finding ω reduces to finding eigenvalues of $\begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}$.

Eigenvectors: $(1,1)$ & $(1,-1)$ (in/out of phase) ▸ Details

Section 5.2

The characteristic polynomial

Algebraic multiplicity

$f(x) = (x-1)^2$ roots: 1 mult 2

$g(x) = (x+1)(x-1)$ roots: ± 1 mult 1

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$A = \begin{pmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 0-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 0 & 0 & 0 & 5-\lambda \end{pmatrix}$$

$Ae_1 = 5e_1$

$Ae_4 = 5e_4$

$(5-\lambda)(-\lambda)(-1-\lambda)(5-\lambda)$

$= (5-\lambda)^2(-\lambda)(-1-\lambda)$ 5 has mult. 2.

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Section 5.4

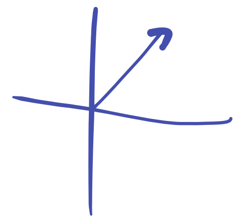
Diagonalization

We understand diagonal matrices

We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

scales x-dir by 2
y-dir by 3



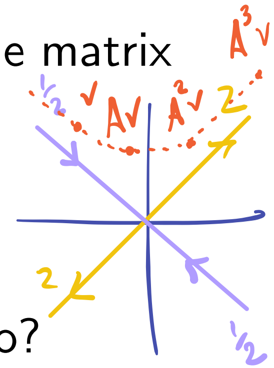
We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, powers of A are easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose want to understand the matrix



$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

2-eigenvec for A

1/2 eigenvect for A

geometrically? Or take it's 10th power? What would we do?

eigenvals of A

What if I give you the following equality:

$Ce_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$Ce_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So: $C^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e_1$, $C^{-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = e_2$

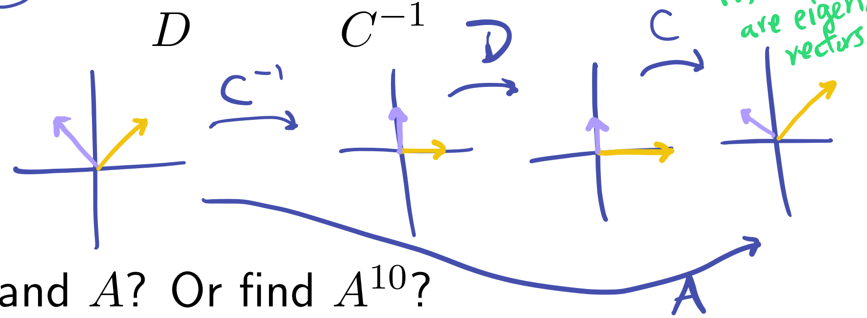
$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A = C \quad D \quad C^{-1}$$

$Av = \underline{CDC^{-1}v}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \& \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are eigen vectors!

This is called diagonalization.



How does this help us understand A? Or find A^{10}?

Powers of matrices that are similar to diagonal ones

What if I give you the following equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A \quad = \quad C \quad D \quad C^{-1}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}? Demo

$$A^2 = (CD\cancel{C^{-1}})(\cancel{C}DC^{-1}) = CD^2C^{-1}$$

$$A^{10} = CD^{10}C^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{10} & 0 \\ 0 & (1/2)^{10} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

We say that A is similar to D. Sec 5.3

How does this factorization of A help describe what A does to $\mathbb{R}^n$?
How does this help us take powers of A?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.

Diagonalization

The recipe

$A = CDC^{-1}$ where D is diagonal,

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \qquad C \qquad D \qquad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, with multiplicity, in order.

Why?

Example

Diagonalize if possible.

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

eigenvals: $2, -1$ $\rightsquigarrow$ eigenvecs must be indep.

2-eigenvecs $\underset{A-2I=}{}\begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(−1)-eigenvecs $\underset{A-(-I)=}{}\begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$A = \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1}}_{C^{-1}}$$

Variations!
① flip order of cols in C & entries of D
② scale cols of C by nonzero scalar

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

eigenvals: 3 (mult 2)

3-eigenvecs: $A-3I = \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 1D eigenspace.

Not diagonalable: don't have 2 indep eigenvectors.

Example

Diagonalize if possible.

$$A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

$\det(A - \lambda I) =$ poly in λ

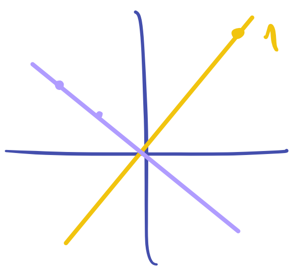
quad.

▸ Demo

Hint: the eigenvalues are 1 and $1/2$

1 eigensp: $\begin{pmatrix} -1/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1/2 eigensp: $\begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Fibonacci numbers

for fun.

Diagonalize the matrix.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues are φ *&* $-1/\varphi$, *with eigenvectors* $(\varphi, 1)$ *&* $(-1/\varphi, 1)$

What does this tell us about Fibonacci numbers? How quickly do they grow? What is the ratio between consecutive Fibonacci numbers?

Use this to give a formula for the nth Fibonacci number

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why? If eigenvals are distinct, the n eigenvecs must be lin ind.

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $1 \leq d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
$\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

So the recipe for checking diagonalizability is:

- If there are not n eigenvalues with multiplicity, then stop.
- For each eigenvalue with alg. mult. greater than 1, check if the geometric multiplicity is equal to the algebraic multiplicity. If any of them are smaller, the matrix is not diagonalizable.
- Otherwise, the matrix is diagonalizable.

More rabbits

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 4 \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: the characteristic polynomials are $-\lambda^3 + 3\lambda + 2$ *and* $-\lambda^3 + 2\lambda + 1$ *and both have rational roots.*

Interpret all of these as rabbit matrices. What can you say about the rabbit populations?

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\left(\begin{smallmatrix} 2 & 6 \\ 0 & -1 \end{smallmatrix}\right)$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Section 5.2

The characteristic polynomial

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial: $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

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If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

Characteristic polynomials, trace, and determinant

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The characteristic polynomial always looks like:

$$(-1)^n\lambda^n + (-1)^{n-1}\boxed{\text{trace(A)}}\lambda^{n-1} + \boxed{???}\lambda^{n-2} + \cdots \boxed{???}\lambda + \boxed{\det(\mathsf{A})}$$

So for a 2×2 matrix:

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A)$$

And for a 3×3 matrix:

$$-\lambda^3 + \mathrm{trace}(A)\lambda^2 - \boxed{???}\lambda + \det(A)$$

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

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Consequence 2. The determinant is the product of the eigenvalues.

Algebraic multiplicity

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Example. Find the algebraic multiplicities of the eigenvalues for

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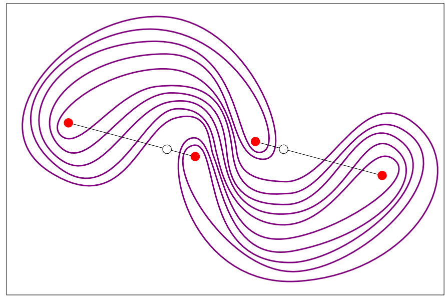
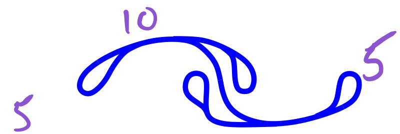
Taffy pullers

How efficient is this taffy puller?





If you run the taffy puller, the taffy starts to look like the shape on the right. Every rotation of the machine changes the number of strands of taffy by a matrix:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

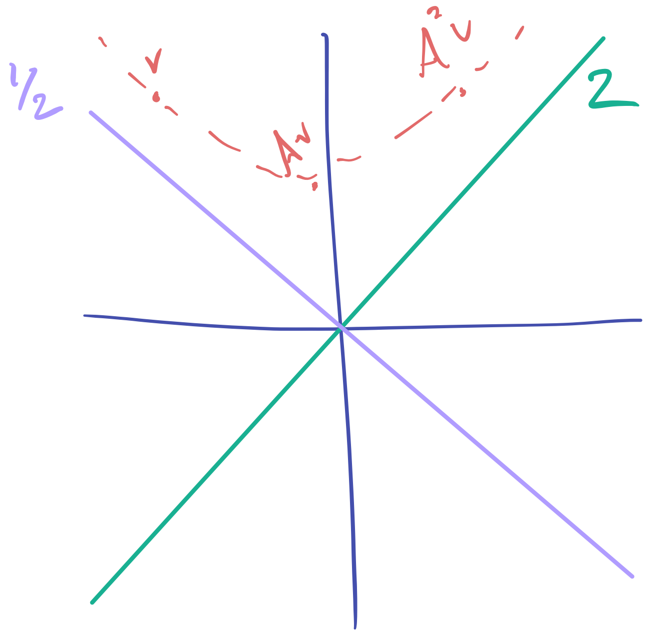
The largest eigenvalue λ of this matrix describes the efficiency of the taffy puller. With every rotation, the number of strands multiplies by λ.

Section 5.4

Diagonalization

We understand matrices when we know their eigendata

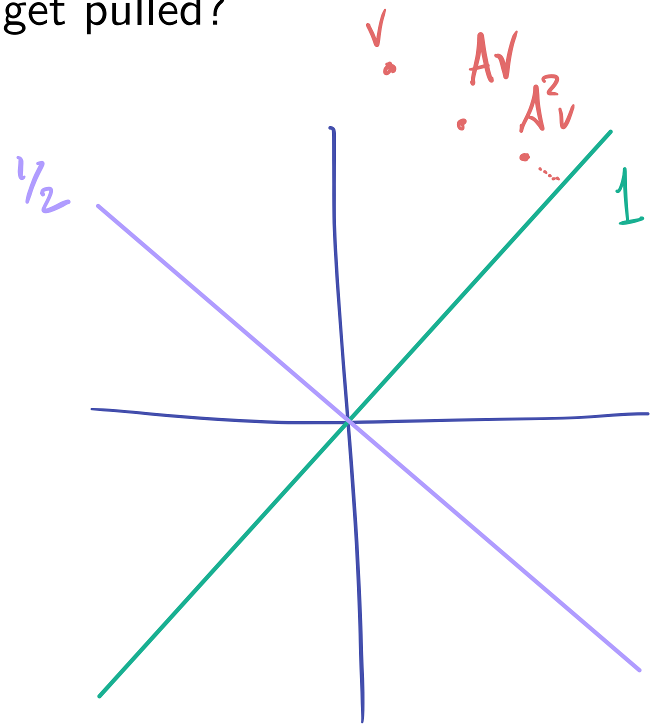
Suppose that A is a 2×2 matrix with eigenvalues 2 and $1/2$, with 2-eigenvector $(1, 1)$ and $1/2$-eigenvector $(-1, 1)$. What does A do to $\mathbb{R}^2$? Choose a vector v and find $Av, A^2v, \ldots$ In which direction do most vectors get pulled?



Demo

We understand matrices when we know their eigendata

Suppose that A is a 2×2 matrix with eigenvalues 1 and $1/2$, with 1-eigenvector $(1,1)$ and $1/2$-eigenvector $(-1,1)$. What does A do to $\mathbb{R}^2$? Choose a vector v and find $Av, A^2v, \ldots$ In which direction do most vectors get pulled?



Demo

We understand matrices when we know their eigendata

The moral of the last two examples is that if we have an $n \times n$ matrix, and if it has

1. n (real) eigenvalues and
2. n linearly independent eigenvectors

and if we know all of this data, then we have a clear picture of what the matrix does to $\mathbb{R}^n$.

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

We say that A is similar to D.

eigenvecs

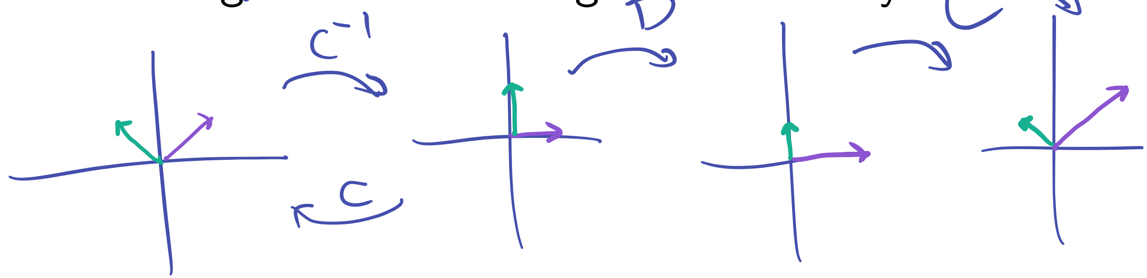
eigenvals

Example:

$$\underset{A}{\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}} = \underset{C}{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}} \underset{C^{-1}}{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}}$$

This tells us all the eigenvectors and eigenvalues! Why?

Demo



Powers of matrices that are similar to diagonal ones

Again, given the equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A = C \quad D \quad C^{-1}$$

How can we find A^{100} without doing 99 matrix calculations?

$$A^{100} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 1/2^{100} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Moral: Diagonalizable matrices behave a lot like diagonal matrices.

Diagonalization

The recipe

algebra geometry

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \qquad C \qquad\qquad D \qquad\qquad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, with multiplicity, in order.

In other words, knowing the diagonalization is the same as knowing n eigenvalues and n independent eigenvectors.

More Examples

Diagonalize if possible.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \cancel{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}}$$

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

$$A = C\,D\,C^{-1}$$

1-eigenvec

$$\begin{pmatrix} \boxed{} & \boxed{} & \boxed{} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Same matrix as on left} \end{pmatrix}^{-1}$$

3-eigenvec

-1 eigenvec

To find 3-eigenvector: $\mathrm{Null}\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \mathrm{Null}(A - 3I)$

More Examples

Diagonalize if possible.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

diag'able — not diag'able

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

Maybe diag'able, maybe not.

All hinges on 2-eigensp: must be 2D $\boxed{2D}$

2-eigensp: $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{matrix} x = -z \\ y = y \\ z = z \end{matrix} \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

So diag'ble

1-eigenv.

$$A = \begin{pmatrix} 0 & -1 & \square \\ 1 & 0 & \square \\ 0 & 1 & \square \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & \text{same} \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix}^{-1}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

Distinct eigenval's

$\rightsquigarrow$ lin ind eigenvec's

Non-Distinct Eigenvalues

Example from last time:

$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

eigenval: 2 mult 2

dim of eigensp $= 1 < 2$

$\Longrightarrow$ not diag'able.

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $1 \leq d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

Not diag'able: $d_i < a_i$ some i.

So the recipe for checking diagonalizability is:

- If there are not n eigenvalues with multiplicity, then stop.
- For each eigenvalue with alg. mult. greater than 1, check if the geometric multiplicity is equal to the algebraic multiplicity. If any of them are smaller, the matrix is not diagonalizable.
- Otherwise, the matrix is diagonalizable.

More rabbits

You do!

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 4 \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: the characteristic polynomials are $-\lambda^3 + 3\lambda + 2$ *and* $-\lambda^3 + 2\lambda + 1$ *and both have rational roots.*

Interpret all of these as rabbit matrices. What can you say about the rabbit populations?

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\left(\begin{smallmatrix} 2 & 6 \\ 0 & -1 \end{smallmatrix}\right)$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

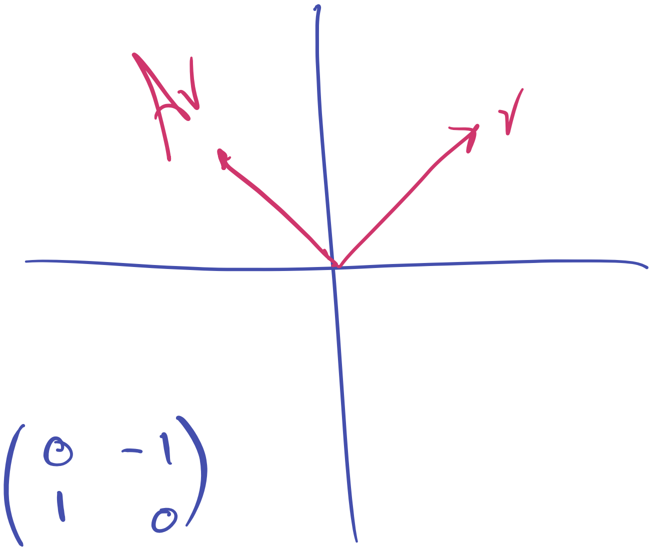
Section 5.1

Eigenvectors and eigenvalues

Eigenvalues for rotations?

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)?



none!

Demo

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 1$$

No real solns!

Section 5.5

Complex Eigenvalues

Outline of Section 5.5

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

Demo Demo

A matrix without an eigenvector

Recall that rotation matrices like

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

have no eigenvectors. Why?

2 sec's ago

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0$$

$x^2 = -1$

$x = \pm\sqrt{-1} = \pm i$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

Now try solving these:

$$x^2 + 3 = 0$$

$x = \pm\sqrt{-3} = \pm\sqrt{3}\sqrt{-1} = \pm\sqrt{3}\, i$

$$x^2 - x + 1 = 0$$

$\dfrac{+1 \pm \sqrt{1-4}}{2} = \dfrac{+1 \pm \sqrt{-3}}{2} = \dfrac{+1 \pm \sqrt{3}\, i}{2}$

Complex numbers

We can add/multiply (and divide!) complex numbers:

$(2-3i)+(-1+i) =$ $1-2i$

$(2-3i)(-1+i) =$ $-2+3i+2i-3i^2$

$-2+5i+3$

$1+5i$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can conjugate complex numbers: $\overline{a + bi} = a - bi$

$$\overline{7 - 3i} = 7 + 3i$$

$$\overline{7 + 3i} = 7 - 3i$$

Complex numbers and polynomials

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots (counted with multiplicity).

Fact. If z is a root of a real polynomial then $\overline{z}$ is also a root.

So what are the possibilities for degree 2, 3 polynomials? deg 2 : 2 real roots or 2 complex

What does this have to do with eigenvalues of matrices? deg3 : 1 or 3 real roots

$n \times n$ matrices

has n eigenvals (with multiplicity)

Complex eigenvalues

Say A is a square matrix with real entries. $A = \overline{A}$

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

Why?

$$Av = \lambda v$$

$$A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}$$

Trace and determinant

Now that we have complex eigenvalues, we have the following fact.

Fact. The sum of the eigenvalues of A (with multiplicity) is the trace of A and the product of the eigenvalues of A (with multiplicity) is the determinant.

Indeed, by the fundamental theorem of algebra, the characteristic polynomial factors as:

$$(x_1 - \lambda)(x_2 - \lambda) \cdots (x_n - \lambda).$$

From this we see that the product of the eigenvalues $x_1 x_2 \cdots x_n$ is the constant term, which we said was the determinant, and the sum $x_1 + x_2 + \cdots + x_n$ is $(-1)^{n-1}$ times the λ^{n-1} term, which we said was the trace.

Complex eigenvalues

$-i = \overline{i}$

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda^2 + 1 = 0 \rightsquigarrow \lambda = \pm i$$

i-eigenspace $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \overset{\text{trick \#1}}{\rightsquigarrow} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \underset{\#2}{\overset{\text{trick}}{\rightsquigarrow}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$-i$-eigenspace $\overset{\text{trick 3}}{}$ $\overline{\begin{pmatrix} 1 \\ -i \end{pmatrix}} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

you!

Summary of Section 5.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvalues for an $n \times n$ matrix, counting multiplicity
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

Announcements Nov 15

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 5.4 & 5.5 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Review sessions: Prof. M Mon and Wed 4:30–5:15 Howey L1
- Review session: Joe Cochran Tue 7:15–9:15 Skiles 371(???)
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 5.5

Complex Eigenvalues

Outline of Section 5.5

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

Demo Demo

Section 5.6

Stochastic Matrices (and Google!)

Outline of Section 5.6

- Stochastic matrices and applications
- The steady state of a stochastic matrix
- Important web pages

Stochastic matrices

A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.

Examples:

$$\begin{pmatrix} 1/4 & 3/5 \\ 3/4 & 2/5 \end{pmatrix} \quad \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/4 \\ 0 & 0 & 1/4 \end{pmatrix}$$

Application: Rental Cars (or Redbox...)

Say your car rental company has 3 locations. Make a matrix whose ij entry is the fraction of cars at location j that end up at location i. For example,

.3 of cars from 1 end up at 1

.4 of cars from 1 end up at 3

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} = A$$

Note the columns sum to 1. Why?

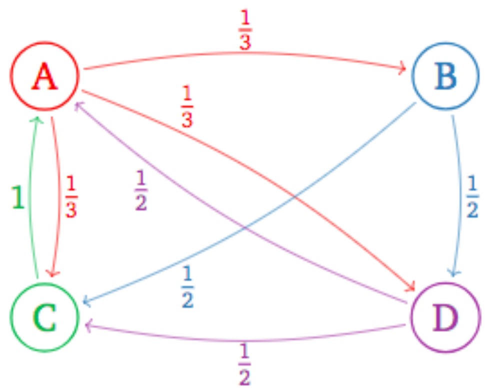
All cars end up somewhere.

If there are 100 cars at each location on the first day, and every car gets rented, how many cars are at each location on the second day? third day? nth day?

$$\underbrace{A\begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}}_{\text{after 1 day}} = \begin{pmatrix} 120 \\ 100 \\ 80 \end{pmatrix} \qquad A\begin{pmatrix} 120 \\ 100 \\ 80 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix} \text{ after 2 day}$$

Application: Web pages

Make a matrix whose ij entry is the fraction of (randomly surfing) web surfers at page j that end up at page i. If page i has N links then the ij-entry is either 0 or $1/N$.



$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Cole: C 3 in

Stefano: A 3 out

Barry: C goes all to A

Patrick: largest row.

Which web page seems most important?

Properties of stochastic matrices

Let A be a stochastic matrix.

Fact 1. One of the eigenvalues of A is 1 and all other eigenvalues have absolute value at most 1.

Why?

$$A = \begin{pmatrix} 1/4 & 3/5 \\ 3/4 & 2/5 \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} 1/4 & 3/4 \\ 3/5 & 2/5 \end{pmatrix}$$

A^T has 1 as eigenvalue: $\begin{pmatrix} 1/4 & 3/4 \\ 3/5 & 2/5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

But A^T & A have same eigenvalues!

(Second part is a little harder).

Positive stochastic matrices

Let A be a positive stochastic matrix, meaning all entries are positive. (Before, we allowed zeros).

Fact 2. The 1-eigenspace of A is 1-dimensional; it has a positive eigenvector.

The unique such eigenvector with entries adding to 1 is called the steady state vector.

Example. If $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a 1-eigenvector, what's the steady state vector?

divide by sum of entries:

$$\begin{pmatrix} 1/6 \\ 2/6 \\ 3/6 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 1/3 \\ 1/2 \end{pmatrix}$$

<u>another</u> 1-eigenvec.

Example

Find the steady state vector.

$$A = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$$

positive ✓

stochastic ✓

1 - eigenvector: $\begin{pmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{pmatrix} \xrightarrow[\text{\& scale by } -4/3]{\text{trick 1}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{trick 2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Steady state vector: $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

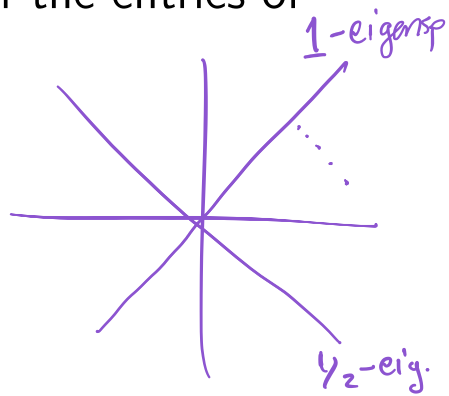
More about positive stochastic matrices

Let A be a positive stochastic matrix, meaning all entries are positive.

Fact 3. Under iteration, all nonzero vectors approach a multiple of the steady state vector. The multiple is the sum of the entries of the original vector.

On last slide:

$$\text{SSV} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$



Demo

$$A^n \begin{pmatrix} 30 \\ 42 \end{pmatrix} \longrightarrow \begin{pmatrix} 36 \\ 36 \end{pmatrix} = 72 \cdot \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

sum of entries is 72.

The last fact tells us how to distribute rental cars, and also tells us the importance of web pages!

Example

To what vector does $A^n \left(\begin{smallmatrix} 1 \\ 9 \end{smallmatrix}\right)$ approach as $n \to \infty$

$$A = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$$

$\text{SSV} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$$\begin{pmatrix} 5 \\ 5 \end{pmatrix} = (9+1) \cdot \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Application: Rental Cars

The rental car matrix is:

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Its steady state vector is:

1-eigenvector

$$\begin{pmatrix} 7/18 \\ 6/18 \\ 5/18 \end{pmatrix} \approx \begin{pmatrix} .39 \\ .33 \\ .28 \end{pmatrix}$$

If the original distribution of cars is given by $(100, 100, 100)$ what will the distribution of cars be after a very long time?

$$300 \cdot \begin{pmatrix} .39 \\ .33 \\ .28 \end{pmatrix} \approx \begin{pmatrix} 117 \\ 99 \\ 84 \end{pmatrix}$$

Application: Web pages

Damping

.15   .15   .85+.15 $.85\frac{1}{2}+.15$

.15   .15

.15

.15   .15

The web page matrix is:

$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Its steady state vector is approximately

$$\begin{pmatrix} .39 \\ .13 \\ .29 \\ .19 \end{pmatrix}$$

and so the first web page is the most important.

Fine print

There are a couple of problems with the web page matrix as given:

- What happens if there is a web page with no links? (Too many 0's)
- What if the internet graph is not connected?
- How do you find eigenvectors for a huge matrix?

Here are the solutions:

- Make a column with $1/n$ in each entry (the surfer goes to a new page randomly).
- Let B be the matrix with all entries equal to 1/n, replace A with

  $n \times n$ matrix.

  damping factor

$$.85 * A + .15 * B$$

- Approximate via iteration!

  $A^{5000} v$

Summary of Section 5.6

- A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.
- Every stochastic matrix has 1 as an eigenvalue, and all other eigenvalues have absolute value at most 1.
- A positive stochastic matrix has 1-dimensional eigenspace and has a positive eigenvector. A positive 1-eigenvector with entries adding to 1 is called a steady state vector.
- For a positive stochastic matrix, all nonzero vectors approach the steady state vector under iteration.
- Steady state vectors tell us the importance of web pages (for example).

Typical Exam Questions 5.6

- Is there a stochastic matrix where the 1-eigenspace has dimension greater than 1?
- Find the steady state vector for this matrix:

$$A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 1/3 \end{pmatrix}$$

  To what vector does $A^n \left(\begin{smallmatrix} 5 \\ 7 \end{smallmatrix}\right)$ approach as $n \to \infty$?
- Find the steady state vector for this matrix:

$$A = \begin{pmatrix} 1/3 & 1/5 & 1/4 \\ 1/3 & 2/5 & 1/2 \\ 1/3 & 2/5 & 1/4 \end{pmatrix}$$

- Make your own internet and see if you can guess which web page is the most important. Check your answer using the method described in this section.

Section 5.1

Eigenvectors and eigenvalues

Eigenvalues geometrically

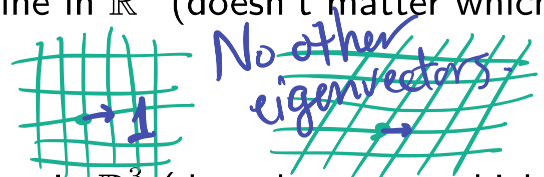
0 eigenvec $Av=0$

1 eigenvec $Av=v$

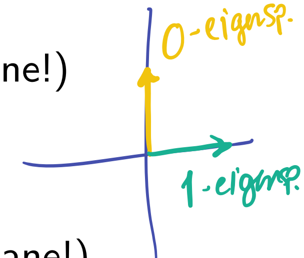
If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

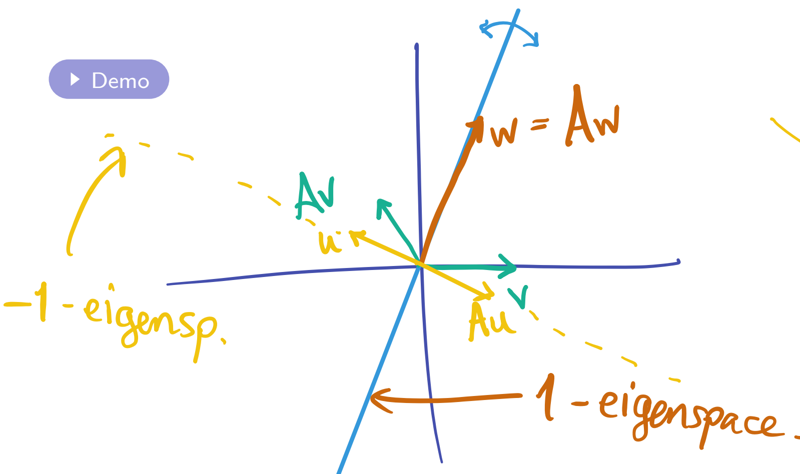
- Reflection about a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Orthogonal projection onto a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- Orthogonal projection to a plane in $\mathbb{R}^3$ (doesn't matter which plane!)

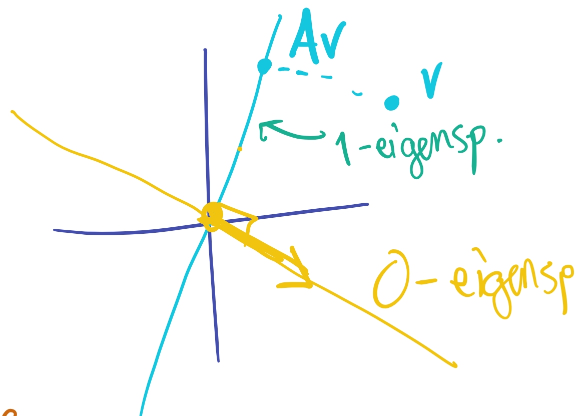


Proj to x-axis



Demo





Typical exam questions 5.1

- Find the 2-eigenvectors for the matrix

$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

- True or false: The zero vector is an eigenvector for every matrix.
- How many different eigenvalues can there be for an $n \times n$ matrix?
- Consider the reflection of $\mathbb{R}^2$ about the line $y = 7x$. What are the eigenvalues (of the standard matrix)?
- Consider the $\pi/2$ rotation of $\mathbb{R}^3$ about the z-axis. What are the eigenvalues (of the standard matrix)?

Section 5.2

The characteristic polynomial

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial: $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

So we plug in ± 1, ± 3, ± 5, ± 15 into the polynomial and hope for the best. Luckily we find that 1, 3, and 5 are all roots, so we found all the eigenvalues!

If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n\lambda^n + (-1)^{n-1}\boxed{\text{trace(A)}}\lambda^{n-1} + \boxed{???}\lambda^{n-2} + \cdots \boxed{???}\lambda + \boxed{\det(\text{A})}$$

So for a 2×2 matrix:

$$\lambda^2 - \operatorname{trace}(A)\lambda + \det(A)$$

And for a 3×3 matrix:

$$-\lambda^3 + \operatorname{trace}(A)\lambda^2 - \boxed{???}\lambda + \det(A)$$

- Sum of eigenvals is trace
- Prod. of eigenvals is det

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.

Section 5.4

Diagonalization

Fibonacci numbers

for fun.

Diagonalize the matrix.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues are φ *&* $-1/\varphi$*, with eigenvectors* $(\varphi, 1)$ *&* $(-1/\varphi, 1)$

What does this tell us about Fibonacci numbers? How quickly do they grow? What is the ratio between consecutive Fibonacci numbers?

Use this to give a formula for the nth Fibonacci number

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

More rabbits

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 4 \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: the characteristic polynomials are $-\lambda^3 + 3\lambda + 2$ *and* $-\lambda^3 + 2\lambda + 1$ *and both have rational roots.*

Interpret all of these as rabbit matrices. What can you say about the rabbit populations?

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\left(\begin{smallmatrix} 2 & 6 \\ 0 & -1 \end{smallmatrix}\right)$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

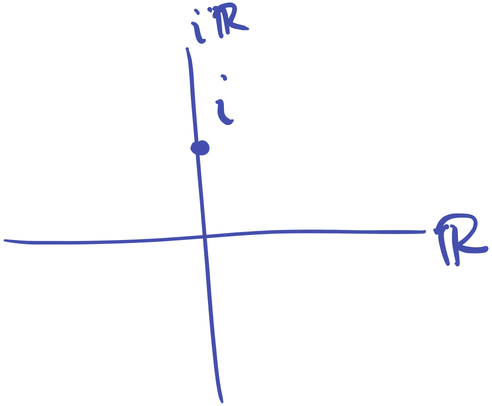
$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Section 5.5

Complex Eigenvalues

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

CCW by $\pi/2$

eigenvals $\pm i$.

i-eigenvec: $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \overset{\text{trick 1}}{\rightsquigarrow} \underline{\begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}} \xrightarrow[2]{\text{trick}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Trick3 $\quad \bar{i}$ eigenvec is $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

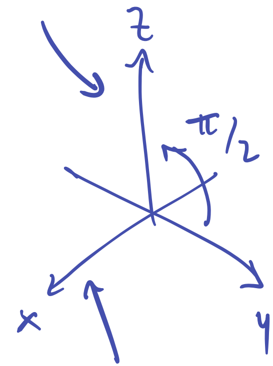
$-i$ eigenvec is $\begin{pmatrix} 1 \\ i \end{pmatrix}$

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 0 & -2 \\ 2 & 0 \end{matrix}} \end{pmatrix}$$

Guess: $\pm 2i$ - eigensp.



1-eigenspace.

eigenvals

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -2 \\ 0 & 2 & -\lambda \end{pmatrix} = (1-\lambda)(\lambda^2+4)$$

$$\rightsquigarrow 1, \pm 2i.$$

$$\lambda^2 = -4$$

$$\lambda = \sqrt{-4}$$

$$\lambda = \sqrt{4}\sqrt{-1}$$

$$= \pm 2i$$

2i-eigenvecs

$$\begin{pmatrix} 1-2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 2 & -2i \end{pmatrix} \xrightarrow{-i * \text{row } 2 \,\mapsto} \begin{pmatrix} \boxed{1-2i} & 0 & 0 \\ 0 & \boxed{-2i} & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$(1-2i)x = 0$

$-2iy - 2z = 0 \longrightarrow iy + z = 0$

$$\Rightarrow \begin{aligned} x &= 0 \\ y &= -z/i \\ z &= z \end{aligned} \rightsquigarrow \begin{pmatrix} 0 \\ -1/i \\ 1 \end{pmatrix}$$

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Spr 20

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

diag'able?

invertible?

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ alg mult of 1: 2

geom mult of 1: 1

$(1-\lambda)^2$

$(1-\lambda)^2$

dim of 1-eigsp

diagable 1 has alg mult 2 & geom mult 2.

(xy-plane)

yes

invertible det $= 1 \cdot 1 \cdot 3 \neq 0$.

5.5 WebWork.

- A real eigenval of a real matrix always has a real eigenvector. TRUE

$$\mathrm{Nul}(A-\lambda I)$$

- Every 3x3 matrix must a real eigenval.

TRUE ~~complex~~ non real ones come in pairs

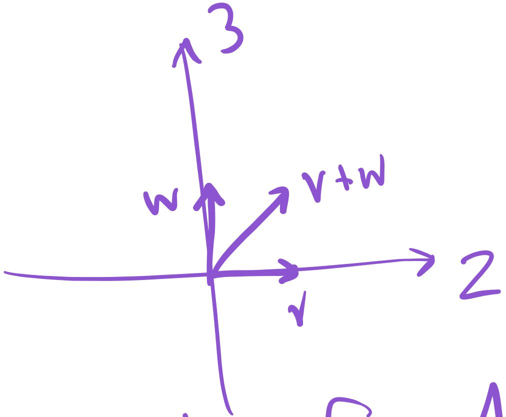
- A 5x5 real matrix has ~~even~~ odd. number of real eigenvals

Was false, but we fixed.

A $n \times n$ $\quad v, w$ eigenvectors for A

then $v+w$ is eigvec for A

False

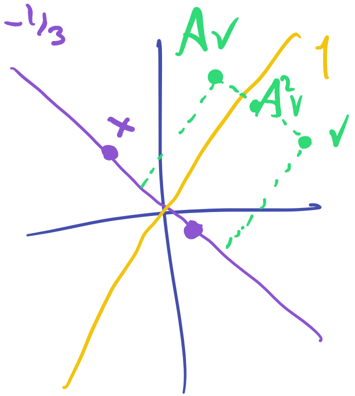


$$\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}2\\ 3\end{pmatrix}$$

A $n \times n$ $\quad v, w$ eigenvectors for A same eigenval

then $v+w$ is eigvec for A TRUE.

Spr 20
practice



$$A = CDC^{-1}$$
$$= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}^{-1}$$

F • Every nonzero vector is an eigenvector

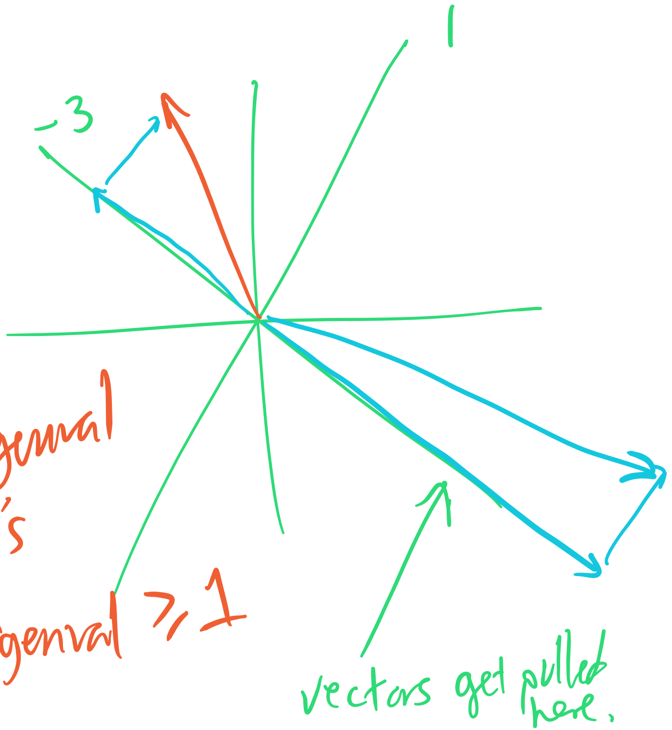
T • If $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $A^n x \longrightarrow 0$

F • Repeated mult by A pushes vectors to span $\cancel{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

biggest in abs val

$$A = CDC^{-1}$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}^{-1}$$



General rule: Vectors get pulled towards eigensp w largest eigenval in abs val unless it's on another eigensp with eigenval ≥ 1

Announcements Nov 17

- Masks $\rightsquigarrow$ Thank you!
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5
- WeBWorK 5.6 & 6.1 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups

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- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
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- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Midterm 3 Review

Summary of Section 3.5

- A is invertible if there is a matrix B (called the inverse) with

$$AB = BA = I_n$$

- For a 2×2 matrix A we have that A is invertible exactly when $\det(A) \neq 0$ and in this case

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- If A is invertible, then $Ax = b$ has exactly one solution:

- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \mid I_n)$.
- Invertible linear transformations correspond to invertible matrices.

Find the inverse of the matrix $\begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix}$

Typical Exam Questions 3.5

- Find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

- Find a 2×2 matrix with no zeros that is equal to its own inverse.
- Solve for the matrix X. Assume that all matrices that arise are invertible:

$$C + BX = A$$

- True/False. If A is invertible, then A^2 is invertible?
- Which linear transformation is the inverse of the clockwise rotation of $\mathbb{R}^2$ by $\pi/4$?
- True/False. The inverse of an invertible linear transformation must be invertible.
- Find a matrix with no zeros that is not invertible.
- Are there two different rabbit populations that will lead to the same population in the following year?

Summary of Section 3.6

- Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.
  (1) A is invertible
  (2) T is invertible
  (3) The reduced row echelon form of A is I_n
  (4) etc.

Typical Exam Questions Section 3.6

In all questions, suppose that A is an $n \times n$ matrix and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. For each question, answer YES or NO.

(1) Suppose that the reduced row echelon form of A does not have any zero rows. Must it be true that $Ax = b$ is consistent for all b in $\mathbb{R}^n$?

(2) Suppose that T is one-to-one. Is is possible that the columns of A add up to zero?

(3) Suppose that $Ax = e_1$ is not consistent. Is it possible that T is onto?

(4) Suppose that $n = 3$ and that $T \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 0$. Is it possible that T has exactly two pivots?

(5) Suppose that $n = 3$ and that T is one-to-one. Is it possible that the range of T is a plane?

Summary of Sections 4.1 and 4.3

Say det is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function det and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?
- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T?
- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T?
- Find the volume of the triangular pyramid with vertices $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, and $(1,2,3)$.

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Typical Exam Questions 4.2

- True or false. The cofactor expansion across the first row gives the negative of the cofactor expansion across the second row.
- Find the determinant of the following matrix using one of the formulas from this section:

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & 0 & 9 \end{pmatrix}$$

- Find the determinant of the following matrix using one of the formulas from this section:

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$

- Find the cofactor matrix for the above matrix and use it to find the inverse.

Summary of Section 5.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Recipe: The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

Typical exam questions 5.1

- Find the 2-eigenvectors for the matrix

$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

- True or false: The zero vector is an eigenvector for every matrix.
- How many different eigenvalues can there be for an $n \times n$ matrix?
- Consider the reflection of $\mathbb{R}^2$ about the line $y = 7x$. What are the eigenvalues (of the standard matrix)?
- Consider the $\pi/2$ rotation of $\mathbb{R}^3$ about the z-axis. What are the eigenvalues (of the standard matrix)?

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Suppose we have

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Describe the behavior of $A^n v$ for various choices of v.

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\left(\begin{smallmatrix} 2 & 6 \\ 0 & -1 \end{smallmatrix}\right)$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Summary of Section 5.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvalues for an $n \times n$ matrix, counting multiplicity
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Announcements Nov 22

next week.

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 5.6 & 6.1 due Tue @ midnight
- Office hrs: Tue 4-5 Teams
- Cumulative Final exam Tue Dec 14 6-8:50 pm on Teams.

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
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- Use Piazza for general questions
- You can do it!

Chapter 6

Orthogonality

Where are we?

Spotify!

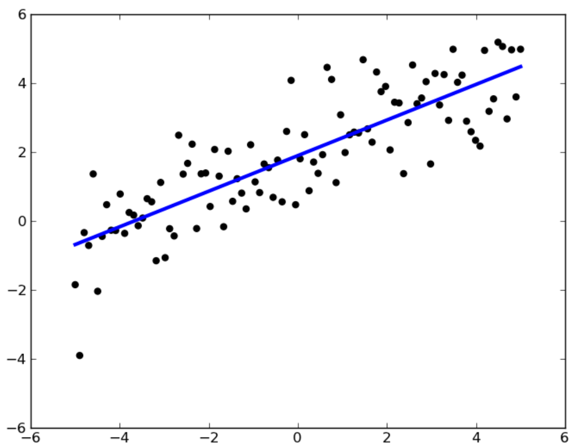
We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

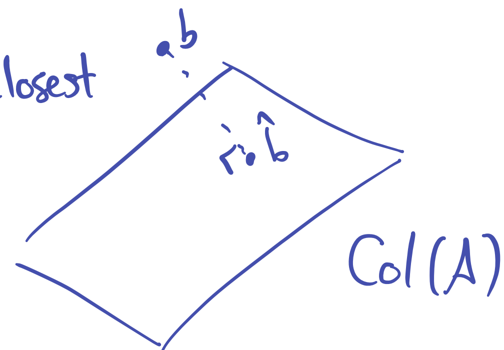
What if we can't solve $Ax = b$? How can we solve it as closely as possible?

$Ax = b$ has no soln

$\iff$ b not in $\mathrm{Col}(A)$



Find $\hat{b}$ closest to b in $\mathrm{Col}(A)$



Solve $Ax = \hat{b}$ instead.

The answer relies on orthogonality.

Section 6.1

Dot products and Orthogonality

Outline

- Dot products
- Length and distance
- Orthogonality

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$
\begin{aligned}
u \cdot v &= \sum_{i=1}^{n} u_i v_i \\
&= u_1 v_1 + \cdots + u_n v_n \\
&= u^T v
\end{aligned}
$$

Used when multiplying matrices.

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

$1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 5 + 10 + 18 = 33.$

Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \Leftrightarrow u = 0$

$(-1, -2, 3) \cdot (-1, -2, 3)$

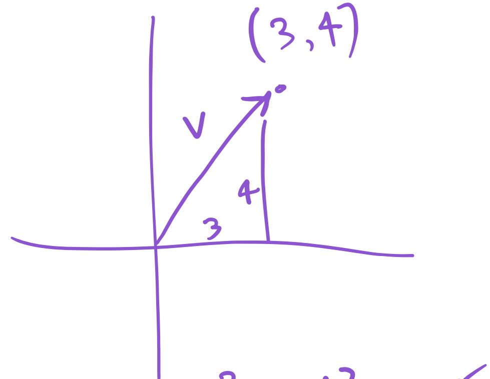
$= (-1)^2 + (-2)^2 + 3^2 \geq 0$

Length

Let v be a vector in $\mathbb{R}^n$

length

$$\|v\| = \sqrt{v \cdot v}$$
$$= \text{length of } v$$



$v \cdot v = 3^2 + 4^2 = 25$

$\sqrt{v \cdot v} = 5$

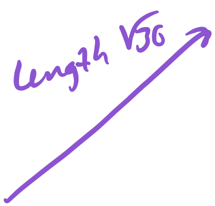
Why? Pythagorean Theorem

abs val length

Fact. $\|cv\| = |c|\|v\|$

v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $\cancel{(1, 2, 3, 4)}$. $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$



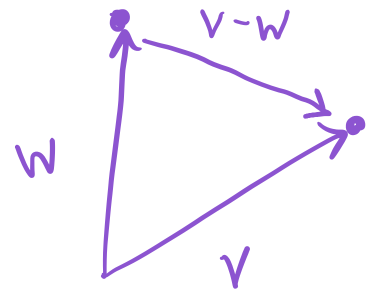
Scale so length is 1.

$\|(1,2,3,4)\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$

$\frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

Distance

The distance between v and w is the length of $v - w$ (or $w - v$!).



Note: $w - v = -(v - w)$

same length!

$v =$ $\quad$ $w =$

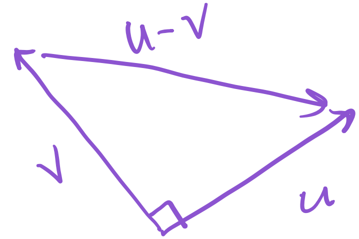
Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

$$v - w = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$$

$$\|v - w\| = \sqrt{0 + 3^2 + 4^2} = \sqrt{25} = 5$$

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$



Why? Pythagorean theorem again!

$$u \perp v \overset{\text{Pythag.}}{\Leftrightarrow} \|u\|^2 + \|v\|^2 = \|u-v\|^2 \qquad (u-v)\cdot(u-v)$$
$$\Leftrightarrow \cancel{u \cdot u} + \cancel{v \cdot v} = \cancel{u \cdot u} - 2u \cdot v + \cancel{v \cdot v}$$
$$\Leftrightarrow u \cdot v = 0$$

Problem. Find a nonzero vector in $\mathbb{R}^3$ orthogonal to $(1,2,3)$.

$$(1,2,3)\cdot(-1,-1,1) = 0$$

$$-1-2+3$$

Summary of Section 6.1

- $u \cdot v = \sum u_i v_i$
- $u \cdot u = \|u\|^2$ (length of u squared)
- The unit vector in the direction of v is $v/\|v\|$.
- The distance from u to v is $\|u - v\|$
- $u \cdot v = 0 \Leftrightarrow u \perp v$

Outline of Section 6.2

- Orthogonal complements
- Computing orthogonal complements

Orthogonal complements

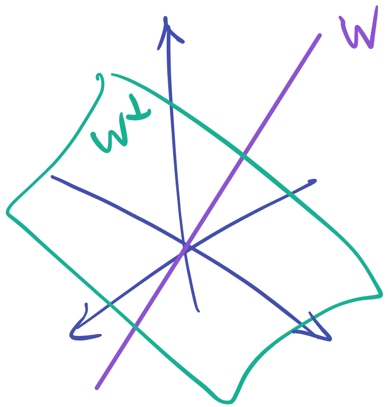
W = subspace of $\mathbb{R}^n$

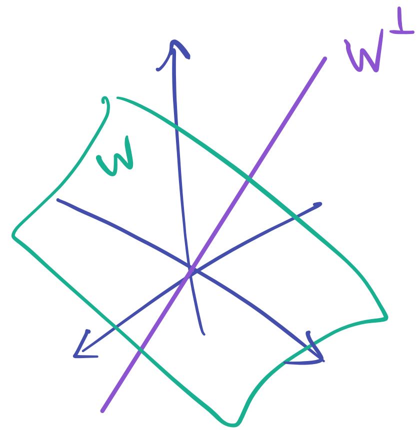
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

subspace

W perp

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?
What about the orthogonal complement of a plane in $\mathbb{R}^3$?





Demo Demo

Stefano:

$W = \mathbb{R}^n$ $\quad W = \{0\}$

$W^\perp = \{0\}$ $\quad W^\perp = \mathbb{R}^n$

Orthogonal complements

W = subspace of $\mathbb{R}^n$

$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$ (it's a null space!)
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$ (rank-nullity theorem!)
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.

k lin. eqns in n vars

$x \cdot w_i = 0$.

For items 1 and 3, which linear transformation do we use?

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1)\}$. Find the equation of the plane $W^\perp$.

line.

Which $x = (x_1, x_2, x_3)$ are perpend. to $(1,1,-1)$?

$$(x_1, x_2, x_3) \cdot (1,1,-1) = 0$$

$\mathrm{Nul}\,(1 \quad 1 \quad -1)$

$$x_1 + x_2 - x_3 = 0.$$

$W^\perp$ is the set of solns.

plane

Find a basis for $W^\perp$.

Vect param form.

1 pivot ✓

Orthogonal complements

Finding them

plane.

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find a system of equations describing the line $W^{\perp}$.

$(x,y,z)\cdot(1,1,-1) = 0$

$(x,y,z)\cdot(-1,2,1) = 0$

$x + y - z = 0$

$-x + 2y + z = 0.$

$\mathrm{Nul}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

Find a basis for $W^{\perp}$.

VPF.

1 basis vector.

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^{\perp}$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space. (VecParamForm)

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

See last 2 examples.

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$

In other words:

Theorem. $A = m \times n$ matrix

e.g. $W = \mathrm{Row}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

$$(\mathrm{Row} A)^\perp = \mathrm{Nul}\, A$$

$W^\perp = \mathrm{Nul}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

Geometry $\leftrightarrow$ Algebra

$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

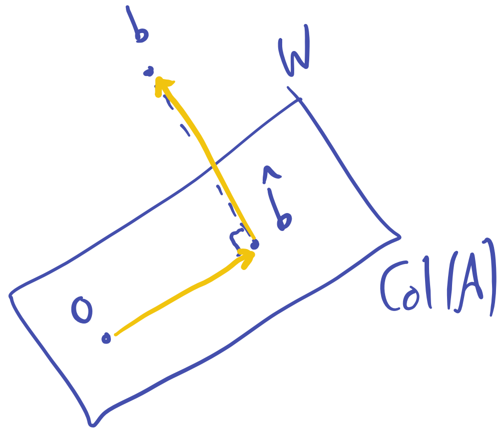
(The row space of A is the span of the rows of A.)

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.



Why?

Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$\boxed{v = v_W + v_{W^\perp}}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^\perp$, so they must both be equal to 0.

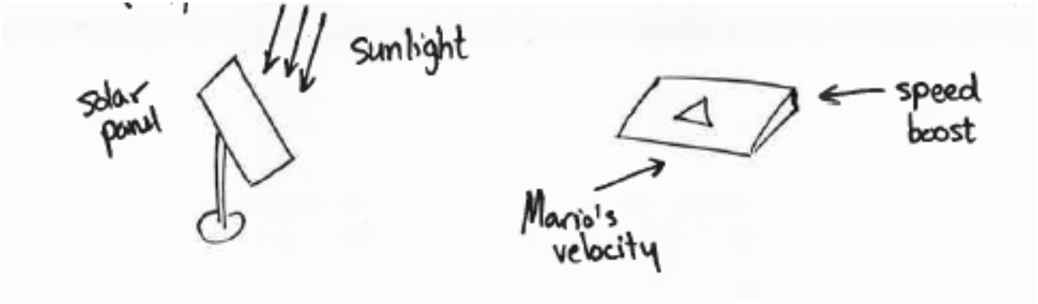
▸ Demo

▸ Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Summary of Section 6.2

- $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$
- Facts:
  1. $W^\perp$ is a subspace of $\mathbb{R}^n$
  2. $(W^\perp)^\perp = W$
  3. $\dim W + \dim W^\perp = n$
  4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
     $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
  5. The intersection of W and $W^\perp$ is $\{0\}$.
- To find $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find the null space.
- Every vector v can be written uniquely as $v = v_W + v_{W^\perp}$ with v_W in W and $v_{W^\perp}$ in $W^\perp$

Typical Exam Questions 6.2

- What is the dimension of $W^\perp$ if W is a line in $\mathbb{R}^{10}$?
- What is $W^\perp$ if W is the line $y = mx$ in $\mathbb{R}^2$?
- If W is the x-axis in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^\perp}$.
- If W is the line $y = x$ in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^\perp}$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^4$.
- What is the orthogonal complement of x_1x_2-plane in $\mathbb{R}^4$?

Announcements Nov 29

RECORD

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 5.6 & 6.1 & 6.2 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Cumulative Final exam Tue Dec 14 6-8:50 pm on Teams.

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google “Dan Margalit math”, click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google “Dan Margalit math”, click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

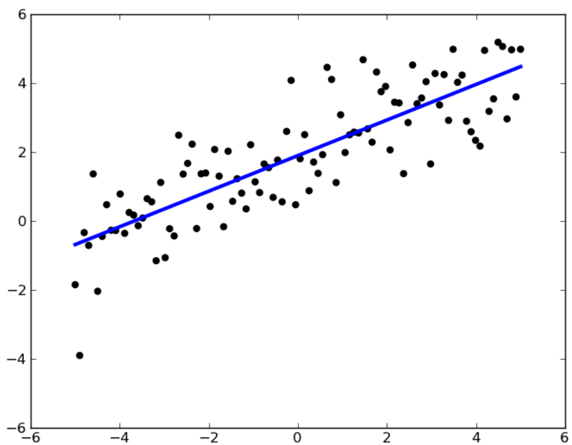
Where are we?

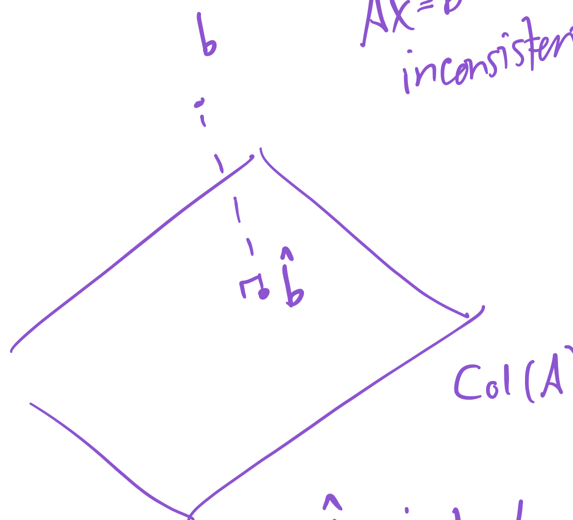
We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?

$Ax = b$ inconsistent.





dot product 0.

Solve $Ax = \hat{b}$ instead.

The answer relies on orthogonality.

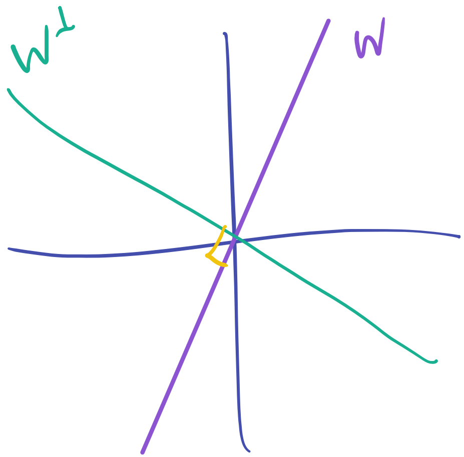
Section 6.2

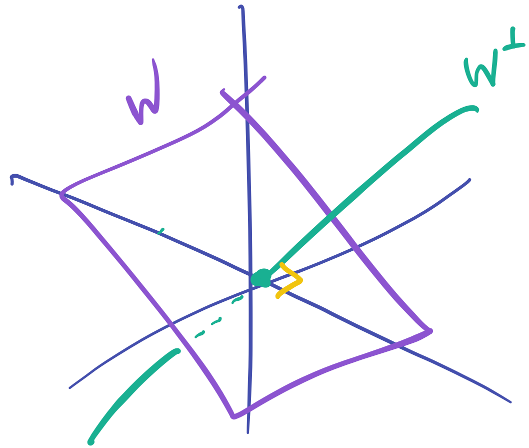
Orthogonal complements

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?
What about the orthogonal complement of a plane in $\mathbb{R}^3$?





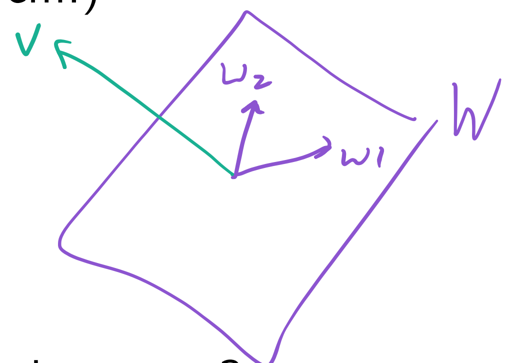
Demo Demo

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$ (it's a null space!)
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$ (rank-nullity theorem!)
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.



For items 1 and 3, which linear transformation do we use?

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

In other words:

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row} A)^\perp = \mathrm{Nul}\, A$$

Geometry $\leftrightarrow$ Algebra

$W = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

$W^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\end{pmatrix}$

$\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

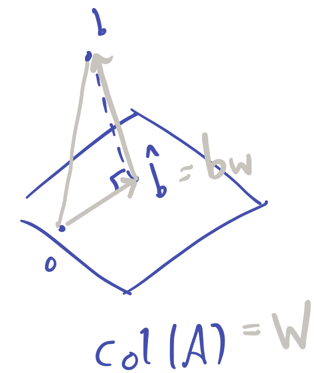
(The row space of A is the span of the rows of A.)

Orthogonal decomposition

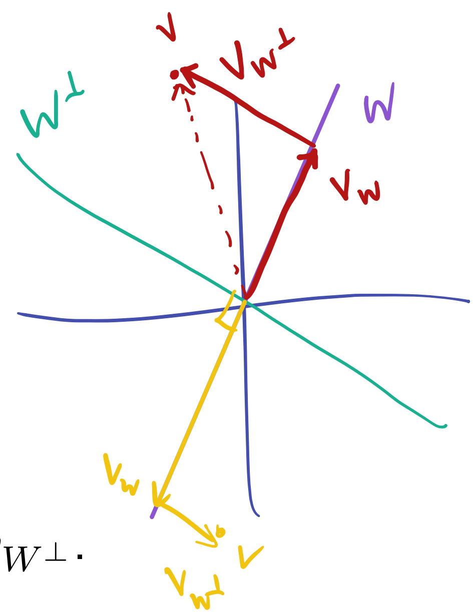
Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.



Why?



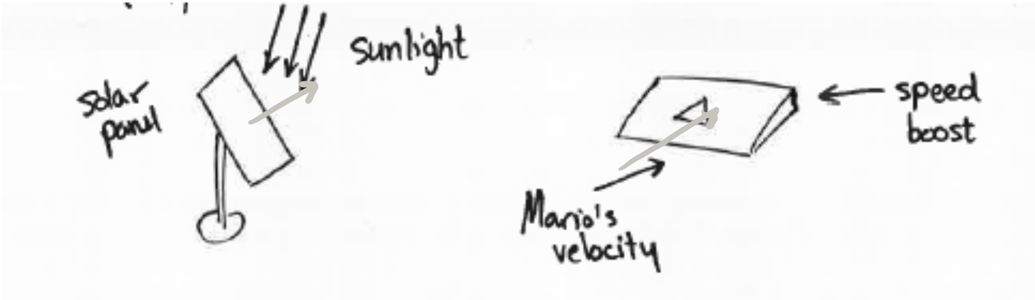
Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Also: switchbacks on hiking trails...

Section 6.3

Orthogonal projection

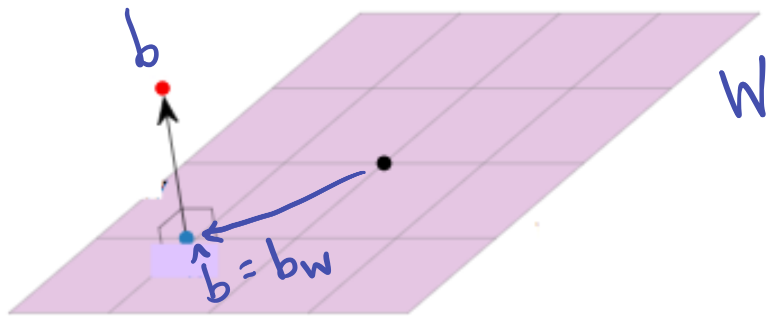
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections

Orthogonal Projections

Let b be a vector in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.

The orthogonal projection of b onto W the vector obtained by drawing a line segment from b to W that is perpendicular to W.



Fact. The following three things are all the same:

- The orthogonal projection of b onto W
- The vector b_W (the W-part of b) algebra!
- The closest vector in W to b geometry!

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

$\hat{b}$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

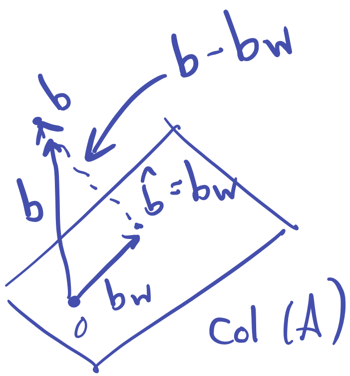
$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Why? Choose $\widehat{x}$ so that $A\widehat{x} = b_W$. We know $b - b_W = b - A\widehat{x}$ is in $W^\perp = \mathrm{Nul}(A^T)$ and so

$$0 = A^T(b - A\widehat{x}) = A^T b - A^T A\widehat{x}$$

$$\rightsquigarrow A^T A\widehat{x} = A^T b$$

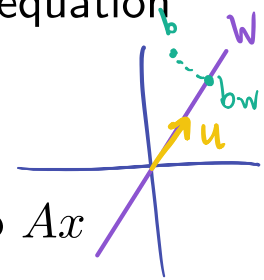


Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.



What does the theorem give when $W = \mathrm{Span}\{u\}$ is a line?

col vector.

$$A^T A x = A^T b$$

$$A = \begin{pmatrix} | \\ u \\ | \end{pmatrix}$$

$$(u \cdot u) x = u \cdot b$$

$$\rightsquigarrow \quad x = \frac{u \cdot b}{u \cdot u}$$

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$A^T A = (1 \ 2 \ 3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

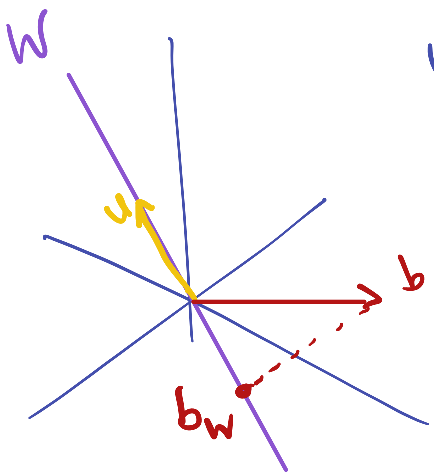
$$\rightsquigarrow \quad b_W = Ax = xu = \frac{u \cdot b}{u \cdot u} u$$

Orthogonal Projection onto a line

Special case. Let $W = \mathrm{Span}\{u\}$. For any vector b in $\mathbb{R}^n$ we have:

$$b_W = \frac{u \cdot b}{u \cdot u} u$$

Find b_W and $b_{W^\perp}$ if $b = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.



$$\frac{u \cdot b}{u \cdot u} u = \frac{-2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

$$u \cdot b = -2 \cdot -1 + -3 \cdot 1 + -1 \cdot 1 = -2$$

$$u \cdot u = -1^2 + 1^2 + 1^2 = 3$$

Orthogonal Projections

$b_W + b_{W^\perp} = b$

$b_{W^\perp} = b - b_W$

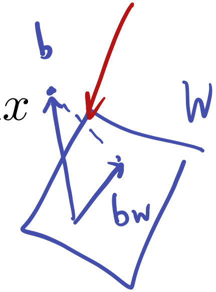
Same

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

$b_{W^\perp} = b - b_W$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.



Example. Find b_W if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T b$, then solve for x, then compute Ax.

Question. How far is b from W? $\|b - b_W\|$

Orthogonal Projections

Example. Find b_W if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T b$, then solve for x, then compute Ax.

① $A^TA = \begin{pmatrix}1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix} = \begin{pmatrix}2&1\\1&2\end{pmatrix}$

② $A^Tb = \begin{pmatrix}1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}6\\5\\4\end{pmatrix} = \begin{pmatrix}10\\11\end{pmatrix}$

③ $\left(\begin{array}{cc|c}2&1&10\\1&2&11\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1&2&11\\2&1&10\end{array}\right) \rightsquigarrow \begin{pmatrix}1&2&11\\0&-3&-12\end{pmatrix} \to \begin{pmatrix}1&2&11\\0&1&4\end{pmatrix}$

$(A^TA \mid A^Tb)$

④ $\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \underbrace{\boxed{\begin{pmatrix}7\\4\\3\end{pmatrix}}}_{b_W}$ $\qquad \rightsquigarrow \left(\begin{array}{cc|c}1&0&3\\0&1&4\end{array}\right)$

Question. How far is b from W?

$\|b_{W^\perp}\| = \|b-b_W\| = \left\|\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\| = \sqrt{-1^2+1^2+1^2}$

$= \sqrt{3}$

Orthogonal Projections

Same **Theorem.** Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Special case. If the columns of A are independent then $A^T A$ is invertible, and so

$$b_W = \underbrace{A(A^T A)^{-1}A^T}_{\text{matrix for proj to } W} b.$$

Why? The x we find tells us which linear combination of the columns of A gives us b_W. If the columns of A are independent, there's only one linear combination.

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^TA)^{-1}A^T.$$

Why? Basically the special case from last slide

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^TA)^{-1}A^T.$$

Example. Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \mathrm{Span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

$A^TA = 2 \rightsquigarrow (A^TA)^{-1} = \frac{1}{2}$

$A(A^TA)^{-1}A^T = \frac{1}{2}AA^T = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}$

The proj. of $\begin{pmatrix}5\\6\\7\end{pmatrix}$ to W is

$1/2\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}5\\6\\7\end{pmatrix} = \frac{1}{2}\begin{pmatrix}12\\0\\12\end{pmatrix} = \begin{pmatrix}6\\0\\6\end{pmatrix}$

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^TA)^{-1}A^T.$$

Example. Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

you!

Projections as linear transformations

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Then

- T is a linear transformation
- $T(b) = b$ if and only if b is in W
- $T(b) = 0$ if and only if b is in $W^\perp$
- $T \circ T = T$
- The range of T is W

Properties of projection matrices

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Let A be the standard matrix for T. Then

- The 1–eigenspace of A is W (unless $W = 0$)
- The 0–eigenspace of A is $W^\perp$ (unless $W = \mathbb{R}^n$)
- $A^2 = A$
- $\mathrm{Col}(A) = W$
- $\mathrm{Nul}(A) = W^\perp$
- A is diagonalizable; its diagonal matrix has m 1's & $n - m$ 0's where $m = \dim W$ (this gives another way to find A)

You can check these properties for the matrix in the last example. It would be very hard to prove these facts without any theory. But they are all easy once you know about linear transformations!

Summary of Section 6.3

- The orthogonal projection of b onto W is b_W
- b_W is the closest point in W to b.
- The distance from b to W is $\|b_{W^\perp}\|$.
- Theorem. Let $W = \mathrm{Col}(A)$. For any b, the equation $A^T Ax = A^T b$ is consistent and b_W is equal to Ax where x is any solution.
- Special case. If $W = \mathrm{Span}\{u\}$ then $b_W = \frac{u \cdot b}{u \cdot u} u$
- Special case. If the columns of A are independent then $A^T A$ is invertible, and so $b_W = A(A^T A)^{-1} A^T b$
- When the columns of A are independent, the standard matrix for orthogonal projection to $\mathrm{Col}(A)$ is $A(A^T A)^{-1} A^T$
- Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$. Then
  - T is a linear transformation
  - etc.
- If A is the standard matrix then
  - The 1–eigenspace of A is W (unless $W = 0$)
  - etc.

Typical Exam Questions 6.3

- True/false. The solution to $A^T Ax = A^T b$ is the point in $\mathrm{Col}(A)$ that is closest to b.
- True/false. If v and w are both solutions to $A^T Ax = A^T b$ then $v - w$ is in the null space of A.
- True/false. If A has two equal columns then $A^T Ax = A^T b$ has infinitely many solutions for every b.
- Find b_W and $b_{W^\perp}$ if $b = (1, 2, 3)$ and W is the span of $(1, 2, 1)$.
- Find b_W if $b = (1, 2, 3)$ and W is the span of $(1, 2, 1)$ and $(1, 0, 1)$. Find the distance from b to W.
- Find the matrix A for orthogonal projection to the span of $(1, 2, 1)$ and $(1, 0, 1)$. What are the eigenvalues of A? What is A^{100}?

Announcements Dec 1

87.77

76.02

- Masks ⇝ Thank you!
- CIOS ⇝ Quiz drop! 85% by Dec 6
- Remainings WeBWorKs not for a grade, for practice only
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Cumulative Final exam Tue Dec 14 6-8:50 pm on Teams.

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 6.3

Orthogonal projection

Orthogonal Projections

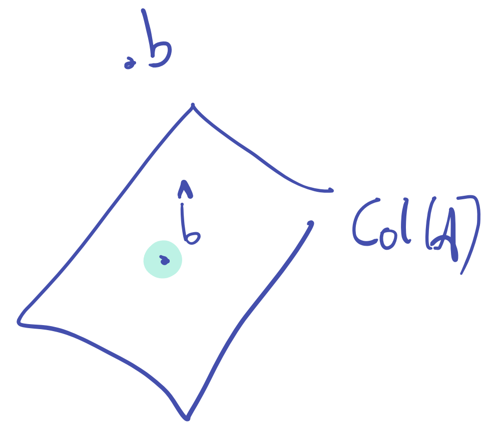
Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^TAx = A^Tb$$

$\hat{b}$

$\parallel$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.



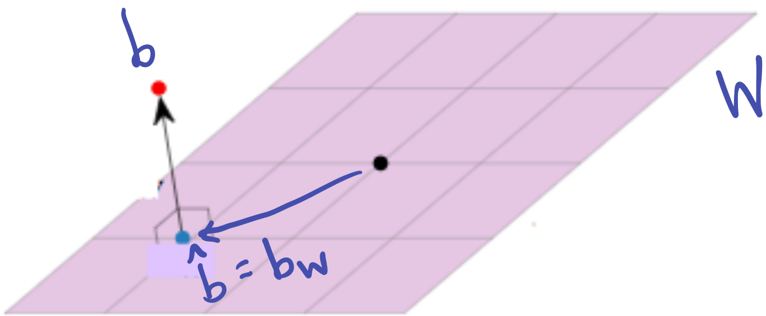
$$\hat{b} = A\hat{x}$$

where $\hat{x}$ is a soln to $A^TAx = A^Tb$

Orthogonal Projections

Let b be a vector in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.

The orthogonal projection of b onto W the vector obtained by drawing a line segment from b to W that is perpendicular to W.



Fact. The following three things are all the same:

- The orthogonal projection of b onto W
- The vector b_W (the W-part of b) algebra!
- The closest vector in W to b geometry!

Orthogonal Projections

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example. Find $\underset{\hat{b}}{\overset{}{b_W}}$ if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\hat{b} = b_W$

Steps. Find A^TA and A^Tb, then solve for x, then compute Ax.

① $A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

② $A^Tb = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$

③ $\left(\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 2 & 11 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & 1 & 10 \end{array}\right) \rightsquigarrow \begin{pmatrix} 1 & 2 & 11 \\ 0 & -3 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 11 \\ 0 & 1 & 4 \end{pmatrix}$

$\left(A^TA \mid A^Tb \right)$

④ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}}}_{b_W} \qquad \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right)$

Question. How far is b from W?

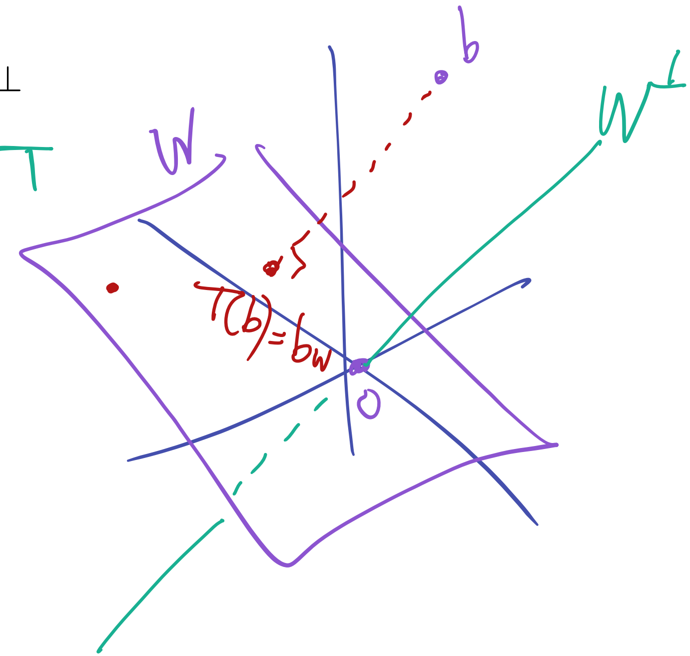
$$\|b_{W^\perp}\| = \|b - b_W\| = \left\| \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{-1^2 + 1^2 + 1^2}$$

$$= \sqrt{3}$$

Projections as linear transformations

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Then

- T is a linear transformation
- $T(b) = b$ if and only if b is in W
- $T(b) = 0$ if and only if b is in $W^\perp$
- $T \circ T = T$ $\quad T \circ T \circ T \circ T \circ T \circ T = T$
- The range of T is W



Properties of projection matrices

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Let A be the standard matrix for T. Then

- The 1–eigenspace of A is W (unless $W = 0$)
- The 0–eigenspace of A is $W^\perp$ (unless $W = \mathbb{R}^n$)
- $A^2 = A$
- $\mathrm{Col}(A) = W$
- $\mathrm{Nul}(A) = W^\perp$
- A is diagonalizable; its diagonal matrix has m 1's & $n - m$ 0's where $m = \dim W$ (this gives another way to find A)

diag'able: sum of dims of eigensp's is n.
and: $\dim W + \dim W^\perp = n$.

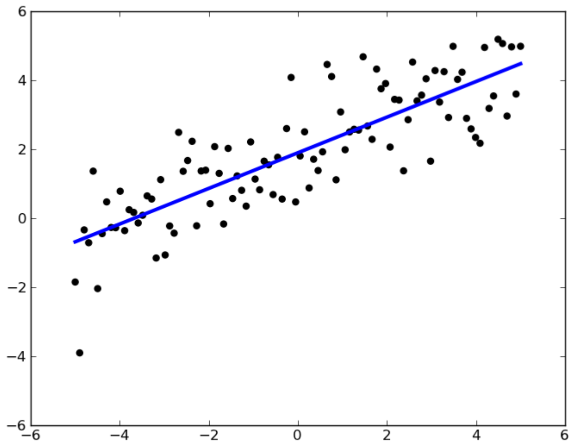
You can check these properties for the matrix in the last example. It would be very hard to prove these facts without any theory. But they are all easy once you know about linear transformations!

Section 6.5

Least Squares Problems

Least Squares problems

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



To solve $Ax = b$ as closely as possible, we orthogonally project b onto $\mathrm{Col}(A)$; call the result $\widehat{b}$. Then solve $Ax = \widehat{b}$. This is the *least squares solution* to $Ax = b$.

Outline of Section 6.5

- The method of least squares
- Application to best fit lines/planes
- Application to best fit curves

Least squares solutions

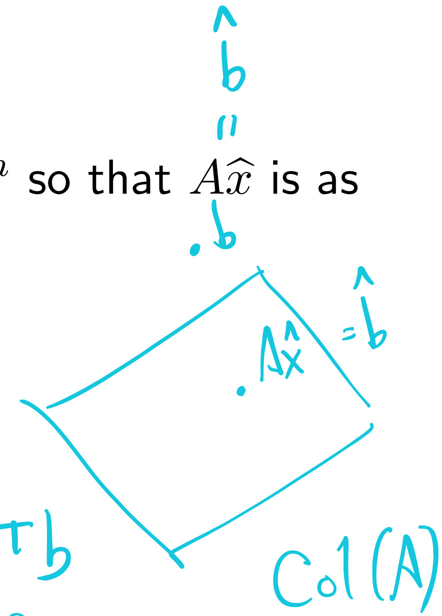
$A = m \times n$ matrix.

A least squares solution to $Ax = b$ is an $\widehat{x}$ in $\mathbb{R}^n$ so that $A\widehat{x}$ is as close as possible to b, this means $A\hat{x} = \hat{b}$.

The error is $\|A\widehat{x} - b\|$.

In last section we wanted $\hat{b}$.

Now we want $\hat{x}$.



Solve $A^TAx = A^Tb$

(like before)

but don't multiply the answer by A.

Demo

Least squares solutions

A least squares solution to $Ax = b$ is an $\widehat{x}$ in $\mathbb{R}^n$ so that $A\widehat{x}$ is as close as possible to b.

$= \|b_{W^\perp}\|$

The error is $\|A\widehat{x} - b\|$. distance from $\hat{b}$ to b.

Theorem. The least squares solutions to $Ax = b$ are the solutions to

$$(A^T A)x = (A^T b)$$

So this is just like what we did before when we were finding the projection of b onto $\mathrm{Col}(A)$. But now we just solve and don't (necessarily) multiply the solution by A.

Least squares solutions

Example

Theorem. The least squares solutions to $Ax = b$ are the solutions to

$$(A^T A)x = (A^T b)$$

Find the least squares solutions to $Ax = b$ for this A and b:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

What is the error?

Least squares solutions

Example

$b - \hat{b} = b_{W^\perp}$

⑤ $\hat{b} = A\hat{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

$\|b - \hat{b}\| = \left\|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\|$

$= \sqrt{1^2 + (-2)^2 + 1^2}$

$= \sqrt{6} \leftarrow$ error.

Formula: $(A^T A)x = (A^T b)$

Find the least squares solution/error to $Ax = b$:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

① $A^T A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$

② $A^T b = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

③ $\left(\begin{array}{cc|c} 5 & 3 & 0 \\ 3 & 3 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -10 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \end{array}\right)$

④ Answer: $\hat{x} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ (Don't multiply by A)

Least squares solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a unique least squares solution for all b in $\mathbb{R}^n$
2. The columns of A are linearly independent
3. $A^T A$ is invertible

In this case the least squares solution is $(A^T A)^{-1}(A^T b)$.

Application

Best fit lines

Eqn of line: $y = Mx + B$. Want M, B.

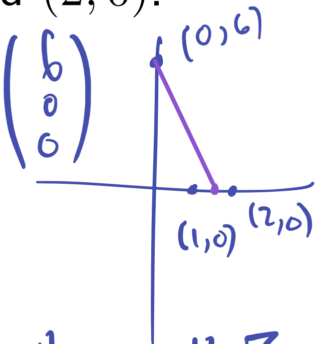
Problem. Find the best-fit line through $(0,6)$, $(1,0)$, and $(2,0)$.

$(0,6)$: $6 = M\cdot 0 + B\cdot 1$

$(1,0)$: $0 = M\cdot 1 + B\cdot 1$

$(2,0)$: $0 = M\cdot 2 + B\cdot 1$

$$\rightsquigarrow \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \quad x = \begin{pmatrix} M \\ B \end{pmatrix}$$



No soln! No line going thru all 3.

Demo

We already found the least squares soln:

$$\hat{x} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} M \\ B \end{pmatrix}$$

Best fit line: $y = -3x + 5$.

Best fit lines

Poll

What does the best fit line minimize?

1. the sum of the squares of the distances from the data points to the line
2. the sum of the squares of the vertical distances from the data points to the line
3. the sum of the squares of the horizontal distances from the data points to the line
4. the maximal distance from the data points to the line

Least Squares Problems

More applications

Determine the least squares problem $Ax = b$ to find the best parabola $y = Cx^2 + Dx + E$ for the points:

$$(0,0), (2,0), (3,0), (0,1)$$

plug in pts $\rightsquigarrow$ 4 eqns in 3 vars (C, D, E).

Demo

Least Squares Problems

More applications

Determine the least squares problem $Ax = b$ to find the best fit ellipse $Cx^2 + Dxy + Ey^2 + Fx + Gy + H = 0$ for the points:

$$(0,0),(2,0),(3,0),(0,1)$$

Gauss invented the method of least squares to predict the orbit of the asteroid Ceres as it passed behind the sun in 1801.

Least Squares Problems

Best fit plane

Determine the least squares problem $Ax = b$ to find the best fit linear function $f(x, y) = Cx + Dy + E$

x	y	$f(x,y)$
1	0	0
0	1	1
-1	0	3
0	-1	4

Summary of Section 6.5

- A least squares solution to $Ax = b$ is an $\widehat{x}$ in $\mathbb{R}^n$ so that $A\widehat{x}$ is as close as possible to b.
- The error is $\|A\widehat{x} - b\|$.
- The least squares solutions to $Ax = b$ are the solutions to $(A^T A)x = (A^T b)$.
- To find a best fit line/parabola/etc. write the general form of the line/parabola/etc. with unknown coefficients and plug in the given points to get a system of linear equations in the unknown coefficients.

Typical Exam Questions 6.5

- Find the best fit line through $(1,0)$, $(2,1)$, and $(3,1)$. What is the error?
- Find the best fit parabola through $(1,0)$, $(2,1)$, $(3,1)$, and $(3,0)$. What is the error?
- True/false. For every set of three points in $\mathbb{R}^2$ there is a unique best fit line.
- True/false. If $\widehat{x}$ is the least squares solution to $Ax = b$ for an $m \times n$ matrix A, then $\widehat{x}$ is the closest point in $\mathbb{R}^n$ to b.
- True/false. If $\widehat{x}$ and $\widehat{y}$ are both least squares solutions to $Ax = b$ then $\widehat{x} - \widehat{y}$ is in the null space of A.