## Moduli space

$\mathcal{M}(S) = \{\text{hyp. str}\}/\text{isometry.}$

Also: $\mathcal{M}(S) = \text{Teich}(S)/\text{Mod}(S)$

Mod(S) acts by pullback:

$$[\varphi]\cdot X = (\varphi^{-1})^* X$$

## Torus case

Action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} = \text{Mod}(T^2)$

on marked lattice $\Lambda \cong \mathbb{Z}^2 \in \text{Teich}(T^2)$

$\updownarrow$

matrix $M \in M_2\mathbb{R}$

"A"

Action is by Möbius trans

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az-b}{-cz+d}$$

# The torus

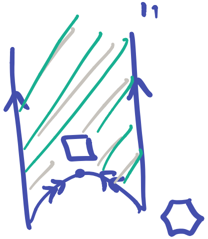
$M(T^2) =$

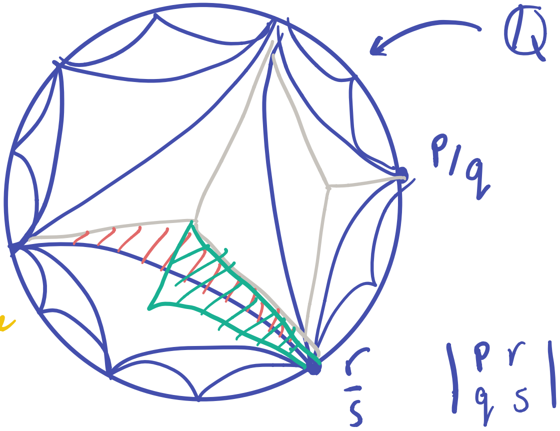
Möb action $\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$

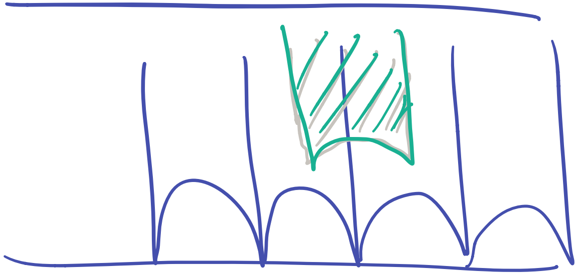
lin alg action.

in upper half-plane






$\begin{vmatrix} p & r \\ q & s \end{vmatrix} = 1$



$\mathbb{R}$ Mobius trans act on this 2-complex.

$SL_2\mathbb{Z}$ acts trans. on $\Delta$'s

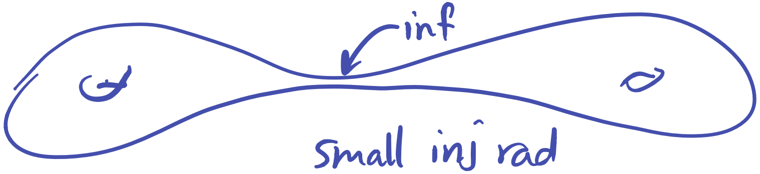
Stab of $\Delta$ rotates

# Mumford's Compactness Criterion

$\ell : \mathcal{M}(S) \longrightarrow \mathbb{R}_+$

$X \longmapsto$ length of shortest curve in $X$.

$$\ell(X) = 2 \underbrace{\operatorname{inj\,rad}(X)}_{\inf_{x \in X}\{\text{largest embedded disk at } x\}}$$




$\ell$ is continuous.

$\mathcal{M}(S)$ is not compact because $\ell$ has no minimum. (keep pinching)

Define $\mathcal{M}_\varepsilon(S) = \{X \in \mathcal{M}(S) : \ell(X) \geq \varepsilon\}$

"$\varepsilon$-thick part"

Thm. $\forall\, \varepsilon$, $\mathcal{M}_\varepsilon(S)$ compact.

So: only way to go to $\infty$ is to pinch curves.


Torus case: evident from picture

# Bers constant

**Thm.** $\chi(S) < 0$ Maybe $\partial S \neq \emptyset$.

① $\exists\, L_0 = L_0(S)$ s.t.

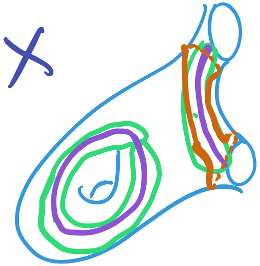
$\forall X \in \mathcal{M}(S)$ $\exists$ curve in $X$

of length $\leq L_0$.

② $\exists\, L = L(S)$ s.t. $\forall X$

$\exists$ pants decomp. of

length $\leq L$.




Pf. ① $\Rightarrow$ ② by induction

on # curves (cut open)

Given $X$ find largest radius disk $D$

with interior embedded & disjoint

from $\partial X$.

$D$ is a hyperbolic disk. radius $r$.

$$\text{Area } D = 2\pi(\cosh(r) - 1)$$
$$\leq \text{Area } X = -2\pi\chi(S)$$

If $\partial D$ touches itself $\Rightarrow$ short curve.

If $\partial D$ touches $\partial X$, it touches in at

least two points $\Rightarrow$ short arc $\Rightarrow$ short curve.

One of these 2 situations must happen $\square$

Define $\mathcal{M}_\varepsilon(S) = \{X \in \mathcal{M}(S) : \ell(X) \geq \varepsilon\}$

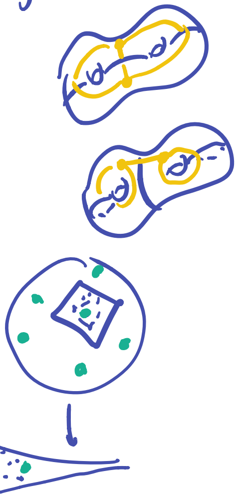
"$\varepsilon$-thick part"

Thm. $\forall\ \varepsilon$, $\mathcal{M}_\varepsilon(S)$ compact.

Pf. $\mathcal{M}(S)$ metrizable. $\Rightarrow$ enough to show seq. compact.

$(X_i) \subseteq \mathcal{M}_\varepsilon(S)$

We will find lifts to Teich(S) lying in closed cube in FN coords.



Bers: Each $X_i$ has pants decomp where curves have length in $[\varepsilon, L]$.

Pass to subseq so these pants decomps are topologically equivalent.

Choose lifts to Teich(S) where a specific pants decomp. has length in $[\varepsilon, L]$.

So length params in $[\varepsilon, L]$.

Can modify twist params to be in $[0,1]$ (Dehn twists)

$\square$

# The end of moduli space

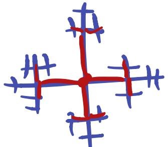
$Z$ = connected, locally compact ~~metric~~ space

$Z$ has one end if $Z \setminus K$ has one unbounded component $\forall$ compact $K$.

or if $\exists$ exhaustion $K_0 \subseteq K_1 \subseteq \ldots$ by compact sets so $Z \setminus K_i$ connected $i \gg 0$.

one end: $\mathbb{R}^n$ $n \geq 2$.

not one end: $\mathbb{R}^n$ $n \leq 1$.

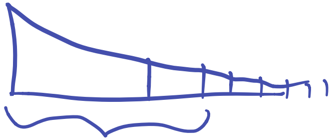


$\infty$ many ends

Cantor set.

**Thm.** $\mathcal{M}(S)$ has one end.

**Pf.** $\mathcal{M}_\varepsilon(S)$ form an exhaustion by compact sets



$\mathcal{M}_\varepsilon$

To show $\mathcal{M} \setminus \mathcal{M}_\varepsilon$ connected $\forall \varepsilon$.

Let $X, Y \in \mathcal{M} \setminus \mathcal{M}_\varepsilon$

Lift to $\tilde{X}, \tilde{Y} \in$ Teich.

short curve $c$

short curve $d$.

Connect $c, d$ in $\mathcal{C}(S)$....

pinch consec. curves one at a time... $\square$

THEOREM 2. *Modulus space is simply-connected.*

PROOF. It is proved in [4] that each element of finite order in $M(K_g)$ has a fixed point in $T(K_g)$, so that, by Theorem 1, $M(K_g)$ is generated by elements which have fixed points. Also $M(K_g)$ is a properly discontinuous group of homeomorphisms of a space homeomorphic to $\boldsymbol{R}^{6g-6}$. Furthermore, the stabiliser of a point $[\phi]$ of $T(K_g)$ is isomorphic to the group of conformal self-homeomorphisms of the compact Riemann surface $D/\phi(K_g)$ and hence is finite. Thus, applying a result of Armstrong [1] we have that $T(K_g)/M(K_g)$ has trivial fundamental group.