TWO-GENERATOR SUBGROUPS OF THE PURE BRAID GROUP

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December 9, 2008

ABSTRACT. We show that any two elements of the pure braid group either commute or generate a free group, settling a question of Luis Paris. Our proof involves the theory of 3-manifolds and the theory of group actions on trees.

1. INTRODUCTION

Let PB_n denote the pure braid group on n strands; see [1] for the definition. During a conference at Banff in 2004, Luis Paris asked the second author whether or not two noncommuting elements of PB_n could satisfy a nontrivial relation. Our Main Theorem answers this question.

Main Theorem. *Let $n \geq 0$, and let $g, h \in PB_n$. Either g and h commute, or they generate a free group.*

Let F_k denote the free group of rank k, and let $\langle g, h \rangle$ denote the group generated by elements g and h. Since PB_n is torsion free, our Main Theorem has a sharper formulation, namely, the group $\langle g, h \rangle$ is isomorphic to the trivial group, $\mathbb{Z}$, $\mathbb{Z}^2$, or F_2.

As $PB_0 \cong PB_1 \cong 1$ and $PB_2 \cong \mathbb{Z}$, the Main Theorem is vacuous in these cases. Also, it follows from Artin's presentation for PB_n that $PB_3 \cong F_2 \times \mathbb{Z}$ [1, p. 120], and so the Main Theorem is easy to prove in that case. For $n \geq 4$, the only case where the Main Theorem was already known was the case where g and h are both positive multitwists [7, Theorem 3.2] [9].

Our Main Theorem is very specific to the pure braid group. That is, for $n \geq 3$, the subgroup of B_n generated by PB_n and a single half-twist g is an index 2 supergroup of PB_n, and does not satisfy the conclusion of the Main Theorem; indeed, there is a Dehn twist h so that g does not commute with h, but $ghgh = hghg$ (see [15, Lemma 2.3]).

In the early 1980's, Ivanov and McCarthy independently proved that, given two elements g and h of the braid group (more generally, the mapping class group of

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a surface), there is an $N > 0$ so that g^N and h^N either commute or generate a free group [10, 18]. Hamidi-Tehrani gave an algorithm, when g and h are pseudo-Anosov, for finding such a power N [8, Theorem 3.19]. Fujiwara also provided criteria for finding an appropriate N [6, Proposition 1.1 and Theorem 3.1], and has announced that one can find such an N that depends only on the surface [5]. Our Main Theorem can be rephrased as saying that for PB_n, we can take $N = 1$. Also, as an immediate consequence, for the braid group B_n, one can take $N = n!$.

Appealing to Fujiwara's work [6], Mangahas proved that any finitely generated subgroup of a mapping class group that is not virtually abelian has exponential growth, with the growth rate bounded away from 1 by a constant depending only on the surface, and not on the generating set [17, Theorem 1.1]. An immediate consequence of our Main Theorem is that any nonabelian subgroup $G < PB_n$ has growth rate at least 3, independently of the generating set, and independently of n (compare with the linear setting [3]).

Acknowledgments. We would like to thank Javier Aramayona, Mladen Bestvina, Tom Church, Nathan Dunfield, Daniel Groves, Stephen Humphries, Ilya Kapovich, Lars Louder, and Luis Paris for helpful conversations and suggestions. We would also like to thank Benson Farb for hosting a conference on Torelli groups in 2006, where this project was begun. Finally, we would especially like to thank Nathan Broaddus for discussing the problem and for writing a computer program that searched for relations in the pure braid group.

2. Reduction to mapping class groups

The center of PB_n is isomorphic to $\mathbb{Z}$, and the quotient of PB_n by its center is naturally isomorphic to $\mathrm{PMod}(S_{0,n+1})$, the *pure mapping class group* of the $(n + 1)$-times punctured sphere. This is the subgroup of the mapping class group $\mathrm{Mod}(S_{0,n+1})$ consisting of those mapping classes that fix each puncture. Moreover, PB_n splits over its center:

$$PB_n \cong \mathrm{PMod}(S_{0,n+1}) \times \mathbb{Z}$$

(see [4, §8.2]).

The Main Theorem follows easily from the next theorem, whose proof will occupy most of the paper.

Theorem 2.1. *Let $n \geq 0$ and let $g, h \in \mathrm{PMod}(S_{0,n})$. The group $\langle g, h \rangle$ is either abelian or is isomorphic to F_2.*

Proof of the Main Theorem assuming Theorem 2.1. Let G denote $\langle g, h \rangle$, and let $\bar{G}$ denote the image of G under the map $PB_n \to PB_n/\mathbb{Z} \cong \mathrm{PMod}(S_{0,n+1})$. By Theorem 2.1, $\bar{G}$ is either abelian or is isomorphic to F_2. If $\bar{G}$ is abelian, then, as G is a split central extension of $\bar{G}$ by an abelian group, it follows that G is abelian. If $\bar{G} \cong F_2$, then we have a sequence of maps $F_2 \to G \to \bar{G} \cong F_2$ where the first map sends the generators of F_2 to g and h. Since free groups are Hopfian [16, Theorem 2.13], and since both maps are surjective, it follows that both maps are isomorphisms, and so $G \cong F_2$. $\square$

3. Filling a puncture

In Section 4, we will prove Theorem 2.1 by means of induction. The inductive step is realized by "filling in" one of the punctures of $S_{0,n}$. In this section, we recall some group theoretic, combinatorial, and geometric consequences of the filling operation.

3.1. The Birman exact sequence.

The surface $S_{0,n}$ is obtained from the 2-sphere by removing a set of n points, say $p_1, \ldots, p_n$. Filling the kth puncture back in, we obtain $S_{0,n-1}$, so that $S_{0,n} = S_{0,n-1} - \{p_k\}$. Via the inclusion $i_k : S_{0,n} \to S_{0,n-1}$, any homeomorphism of $S_{0,n}$ that induces the trivial permutation of the punctures extends uniquely to a homeomorphism of $S_{0,n-1}$, and so induces a surjective homomorphism $\rho_k : \mathrm{PMod}(S_{0,n}) \to \mathrm{PMod}(S_{0,n-1})$. The Birman exact sequence [2, §1] describes the kernel of this map for $n \geq 4$:

$$1 \to \pi_1(S_{0,n-1}, p_k) \to \mathrm{PMod}(S_{0,n}) \xrightarrow{\rho_k} \mathrm{PMod}(S_{0,n-1}) \to 1.$$

3.2. Curve complexes and trees.

The complex of curves of a surface S is the simplicial complex whose vertices are isotopy classes of essential simple closed curves in S, and whose k-simplices are sets of $k+1$ isotopy classes of pairwise disjoint curves. We denote this complex $\mathcal{C}(S)$. Here, a curve is *essential* if it is neither homotopic to a point nor a puncture.

One sometimes modifies the definition of the curve complex for $S_{0,4}$ so as to obtain the Farey graph. We do not do this, and hence $\mathcal{C}(S_{0,4})$ is a countable set of vertices.

The inclusion $i_k : S_{0,n} \to S_{0,n-1}$ induces a map Π_k from the set of isotopy classes of curves in $S_{0,n}$ to the set of isotopy classes of curves in $S_{0,n-1}$. There is a well defined subcomplex $\Pi_k^{-1}(v)$ of $\mathcal{C}(S_{0,n})$, which consists of the union of all simplices of $u \subset \mathcal{C}(S_{0,n})$, such that upon filling in the kth puncture, all vertices of u represent v in $S_{0,n-1}$. Via the inclusion $\pi_1(S_{0,n-1}) \to \mathrm{PMod}(S_{0,n})$ from the Birman exact sequence, we obtain an action of $\pi_1(S_{0,n-1})$ on $\Pi_k^{-1}(v)$. With Kent and Schleimer, the first author showed that $\Pi_k^{-1}(v)$ is $\pi_1(S_{0,n-1})$–equivariantly isomorphic to the Bass–Serre tree T_v associated to the splitting of $\pi_1(S_{0,n-1})$ defined by v [14, Theorem 7.1].

We note that if G is a subgroup of $\mathrm{PMod}(S_{0,n-1})$ that fixes a vertex v of $\mathcal{C}(S_{0,n-1})$, then $\rho_k^{-1}(G)$ preserves $\Pi_k^{-1}(v)$, that is, $\rho_k^{-1}(G)$ acts on the tree $\Pi_k^{-1}(v) \cong T_v$.

3.3. Brunnian subgroups.

Whittlesey considered the *Brunnian subgroup* of the group $\mathrm{PMod}(S_{0,n})$, which is defined as

$$W_n = \bigcap_{k=1}^{n} \ker(\rho_k) < \mathrm{PMod}(S_{0,n}),$$

and she proved that for $n \geq 5$ each nontrivial element of W_n is irreducible, which means that it does not fix any simplex of $\mathcal{C}(S_{0,n})$ [22, Theorem 4.1]. Since the group $\mathrm{PMod}(S_{0,n})$ is torsion free, the Nielsen–Thurston classification gives that irreducible elements of $\mathrm{PMod}(S_{0,n})$—in particular, elements of W_n—are pseudo-Anosov [21, §5]

3.4. **Surface bundles.** Any nontrivial element f of $\mathrm{PMod}(S_{0,n-1})$ generates an infinite cyclic group $\langle f \rangle < \mathrm{PMod}(S_{0,n-1})$. The mapping torus M_f of f is an $(S_{0,n-1})$–bundle over the circle, and

$$\pi_1(M_f) \cong \rho_k^{-1}(\langle f \rangle).$$

Since the universal cover of M_f is homeomorphic to $\mathbb{R}^3$, it follows that M_f is irreducible, that is, every 2-sphere bounds a 3-ball.

If f is pseudo-Anosov, then any conjugacy class of $\pi_1(S_{0,n-1})$ fixed by a nontrivial power of f is peripheral. It follows that M_f is atoroidal, that is, any immersed, π_1–injective torus in M_f is peripheral.

When f is pseudo-Anosov, the fact that M_f is irreducible and atoroidal also follows immediately from the theorem of Thurston that states that M_f admits a complete finite-volume hyperbolic metric; see [20, Theorem 0.1] and [19].

4. Proof of Theorem 2.1

Since $\mathrm{PMod}(S_{0,n})$ is trivial for $n \leq 3$, and $\mathrm{PMod}(S_{0,4}) \cong F_2$ by the Birman exact sequence, there is nothing to do for these cases. We proceed by induction. Say that $n \geq 5$, and assume that Theorem 2.1 holds for $\mathrm{PMod}(S_{0,n-1})$. In other words, assume that if $\bar{g}$ and $\bar{h}$ are any two elements of $\mathrm{PMod}(S_{0,n-1})$, then $\langle \bar{g}, \bar{h} \rangle$ is either abelian or is isomorphic to F_2.

Let g and h be two fixed elements of $\mathrm{PMod}(S_{0,n})$. As in Section 3.1, let $\rho_1, \ldots, \rho_n$ be the maps $\mathrm{PMod}(S_{0,n}) \to \mathrm{PMod}(S_{0,n-1})$ obtained by filling a puncture. We consider three possibilities for the subgroups $\bar{G}_k = \langle \rho_k(g), \rho_k(h) \rangle$. The division into cases arises as follows: either $\bar{G}_k$ is isomorphic to F_2 for some k (Case 1 below), or $\bar{G}_k$ is abelian for all k. In the latter situation, we consider any particular $\bar{G}_k$ and appeal to following dichotomy of abelian subgroups of $\mathrm{PMod}(S_{0,n})$, due to Ivanov [11, Corollary 1.8 plus Corollary 8.5].

> *Any abelian subgroup of* $\mathrm{PMod}(S_{0,n})$ *either fixes a vertex of* $\mathcal{C}(S_{0,n})$
> *or is an infinite cyclic group generated by a pseudo-Anosov.*

The two possibilities for abelian subgroups are dealt with in Case 2 and Case 3.

Case 1. $\bar{G}_k \cong F_2$ for some k.

By the Hopfian property for free groups, it follows that $\langle g, h \rangle \cong F_2$.

Case 2. $\bar{G}_k \cong \mathbb{Z}$ and is generated by a pseudo-Anosov mapping class for some k.

Let $\langle f \rangle = \langle \rho_k(g), \rho_k(h) \rangle$ with f pseudo-Anosov. In this case, we know that $\rho_k^{-1}(\langle f \rangle)$ is isomorphic to the fundamental group of the mapping torus M_f (see Section 3.4). Since M_f is atoroidal and irreducible, a theorem of Jaco–Shalen gives that $\langle g, h \rangle$ either has finite index in $\rho_k^{-1}(\langle f \rangle) \cong \pi_1(M_f)$, or it is isomorphic to one of the groups

1, $\mathbb{Z}$, $\mathbb{Z}^2$, or F_2; see [12, Theorem VI.4.1]. Thus, it suffices to show that $\langle g, h \rangle$ does not have finite index in $\pi_1(M_f)$.

Since $f \in \mathrm{PMod}(S_{0,n-1})$ acts trivially on $H_1(S_{0,n-1}, \mathbb{Z}) \cong \mathbb{Z}^{n-2}$, a Mayer–Vietoris computation shows that the first homology of M_f is isomorphic to that of $S_{0,n-1} \times S^1$, that is

$$H_1(M_f, \mathbb{Z}) \cong \mathbb{Z}^{n-1}.$$

Given a finite sheeted covering $M \to M_f$, we see that $H_1(M, \mathbb{Z}) \cong \pi_1(M)^{ab}$ contains a free abelian group of rank $n-1 \geq 4$. Therefore, any finite index subgroup of $\pi_1(M)$ is generated by no fewer than 4 elements. It follows that $\langle g, h \rangle < \rho_k^{-1}(\langle f \rangle) \cong \pi_1(M_f)$ does not have finite index, which completes the proof in Case 2.

Case 3. $\bar{G}_k$ is abelian for all k, and fixes a vertex $v \in \mathcal{C}(S_{0,n-1})$ for some k.

Set $G = \langle g, h \rangle$, and note that $[G, G]$ is a subgroup of W_n, the Brunnian subgroup of $\mathrm{PMod}(S_{0,n})$. This is because ρ_k factors through the abelianization of G for all k, and so $[G, G]$ is contained in the intersection of the kernels of all ρ_k.

Choose k so that $\bar{G}_k$ fixes a vertex $v \in \mathcal{C}(S_{0,n-1})$. Then $G < \rho_k^{-1}(\bar{G}_k)$, and, as described in Section 3.2, $\rho_k^{-1}(\bar{G}_k)$ acts on the tree $\Pi_k^{-1}(v) \cong T_v$. Note that $\rho_k^{-1}(\bar{G}_k)$ contains $\pi_1(S_{0,n-1})$, and the action of $\pi_1(S_{0,n-1})$ on $\Pi^{-1}(v)$ is equivalent to the action of $\pi_1(S_{0,n-1})$ on its Bass–Serre tree T_v. Therefore, all edge stabilizers are nontrivial. Moreover, $\rho_k^{-1}(\bar{G}_k)$ acts on $\Pi_k^{-1}(v)$ without inversions; indeed, any element of $\mathrm{Mod}(S_{0,n})$ that interchanges two disjoint isotopy classes of curves is easily seen to act nontrivially on the punctures of $S_{0,n}$.

According to the work of Kapovich–Weidmann [13, Theorem 1], either G is abelian or free, or one can find two new generators s, t for G so that one of the following holds:

(1) s and t have nonzero powers that fix a common point of $\Pi_k^{-1}(v)$, or
(2) s and tst^{-1} have nonzero powers that fix a common point of $\Pi_k^{-1}(v)$.

We can strengthen the conclusion of the Kapovich–Wiedmann theorem by applying the following fact, due to Ivanov [11, Corollary 1.8 plus Corollary 3.7].

> If $f \in \mathrm{PMod}(S_{0,n})$ has $f^n(v) = v$ for some vertex $v \in \mathcal{C}(S_{0,n})$ and some nonzero n, then $f(v) = v$.

We can thus conclude that either s and t, or s and tst^{-1}, have a common fixed point in $\Pi_k^{-1}(v)$. Without loss of generality, let this fixed point be a vertex u. It follows that the commutator $[s, t]$ also fixes u. However, $[s, t] \in [G, G] < W_n$, and hence $[s, t] = 1$ (see Section 3.3). Therefore, G is abelian, completing the proof in Case 3, and thus finishing the proof of Theorem 2.1.

5. Questions and Conjectures

Mapping class groups. As we alluded to in the introduction, one interpretation of our Main Theorem is that if g and h are any two elements of $\mathrm{Mod}(S_{0,n})$, then $\langle f^{n!}, g^{n!} \rangle$ is either abelian or free.

Question 1. *Let S be a surface of positive genus. Is there a constant $N = N(S) > 0$ so that, if $f, g \in \mathrm{Mod}(S)$, then $\langle f^N, g^N \rangle$ is either abelian or free?*

It follows from [7, Theorem 3.2] and [9] that we can take $N = 2$ if g and h are Dehn twists or positive multitwists. As mentioned above, Fujiwara provides an appropriate $N = N(S)$, under the additional assumption that f and g are pseudo-Anosov [5].

Torelli groups and congruence subgroups. Let S be a closed surface. The *Torelli group* $\mathcal{I}(S)$ is the subgroup of $\mathrm{Mod}(S)$ consisting of those elements that act trivially on $H_1(S, \mathbb{Z})$.

Conjecture 5.1. *For any closed surface S, and any $g, h \in \mathcal{I}(S)$, either g and h commute, or they generate a free group.*

In the case where g and h are Dehn twists or positive multitwists in either PB_n or $\mathcal{I}(S)$, both the Main Theorem and Conjecture 5.1 have been known for some time; see [7, Theorem 3.2] and [9]. On the other hand, it is unknown whether or not two non-commuting bounding pair maps in $\mathcal{I}(S)$ necessarily generate a free group.

Our Main Theorem would follow immediately from Conjecture 5.1, since, if the genus of S is large enough, $\mathcal{I}(S)$ contains a split central extension of PB_n.

Conjecture 5.1 is not true if we replace $\mathcal{I}(S)$ with the level m congruence subgroup of the mapping class group, that is, the subgroup consisting of elements that act trivially on $H_1(S, \mathbb{Z}/m\mathbb{Z})$. Indeed, if f is any bounding pair map, then there is an isotopy class of simple closed curves a where $f(a) \neq a$ and $f(a)$ is disjoint from a. Thus, f and T_a^m fail to commute, but T_a^m commutes with the element $fT_a^mf^{-1} = T_{f(a)}^m$, and so there is a nontrivial relation in $\langle f, T_a^m \rangle$, namely, $[T_a^m, fT_a^mf^{-1}] = 1$.

Surface braid groups. Tom Church has asked us whether the Main Theorem is true for pure surface braid groups. That is, do any two elements in any pure surface braid group commute or generate a free group?

We conjecture that the answer is 'yes'. After rephrasing in terms of mapping class groups via Birman's work [2, Theorem 1], one can see that the majority of our proof applies. The only missing ingredient is Whittlesey's result—the intersection of the "fill-in-a-puncture" homomorphisms is not purely pseudo-Anosov on a general surface.

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