Scores: 1 2 3 4 5 6 7 8 9 10 Name ______________________

Section K___

Mathematics 2602

Final Exam

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2 May 2013

1. State the principle of mathematical induction.

Give the definition of a one-to-one correspondence.

2. Determine the truth value of the following proposition:

$$(\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x = y^2)) \to (1 + 1 = 3)$$

A. True

B. False

C. Inconclusive

D. The statement is not a proposition.

Define an equivalence relation on $\mathbb{R}^2 \setminus 0$ where $\vec{v} \sim \vec{w}$ if there is a positive real number so that $t\vec{v} = \vec{w}$. What is the quotient set?

3. Show that $(\neg p) \to (p \to q)$ is a tautology.

Find a one-to-one correspondence between $\mathbb{R}$ and $(1, \infty)$.

4. The third matrix found in an application of the Floyd–Warshall algorithm is:

$$M_2 = \begin{pmatrix} 0 & \infty & 2 \\ \infty & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

What is the distance between vertices 1 and 2?

Find all solutions to the system of congruences

$$\begin{aligned} x &\equiv 2 \mod 4 \\ x &\equiv 6 \mod 7 \end{aligned}$$

5. Recall that the Fibonacci numbers are defined by the recursion relation

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

Use the principle of mathematical induction to show that

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$$

for all $n \geq 1$.

6. Arrange the following functions in a list so that each function is big-O of the next:

$$n^{3/2}\ ,\ 99999\log n\ ,\ n\log n\ ,\ (n!)^2\ ,\ 2^n\ ,\ n^2\ ,\ 3^{n-1}$$

Show that $\log(n!) = \mathcal{O}(n\log n)$.

7. Solve the recurrence relation given by $a_0 = 4$, $a_1 = 10$ and

$$a_n = 6a_{n-1} - 9a_{n-2} + 4n, \quad n \geq 2.$$

8. A bagel shop has 8 kinds of bagels. How many ways are there to choose two dozen bagels with at least one of each kind if there are only two sesame bagels (and many of the other kinds)? Do not simplify your answer.

How many integers from 1 to 100 are either odd, a square, divisible by 7, or some combination of these? Do not simplify.

9. Suppose you arrange the numbers 1 though 9 around a circle. Explain why there must be three consecutive numbers whose sum is at least 12.

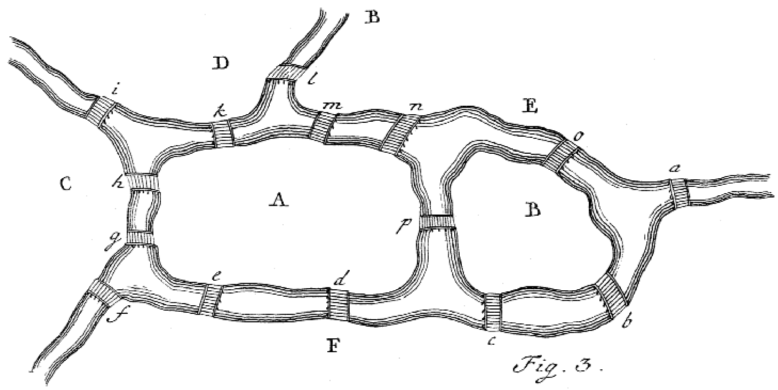
A candy store has bubble gum balls in 7 colors. How many ways are there to buy 15 gum balls if you buy at least one of each color?

There are four committees, each with five people. For every pair of distinct committees, there is one person that is on both committees. Exactly one person is on three committees and nobody is on four committees. How many people are on at least one committee?

10. From a standard deck of 52 cards, you deal 13 cards each to Alice, Bob, Charley, and Daisy. What is the probability that Alice gets 3 hearts, given that Alice and Bob together get 3 hearts?

Expand and simplify the expression $\left(x + \dfrac{2}{x}\right)^5$.

11. Is it possible to take a walk that crosses each bridges exactly once, if one is not required to return to the starting point? Explain your answer.



What if one is required to return to the starting point? Explain your answer.

Which of the following graphs are Hamiltonian? Select all that apply.

A. K_2

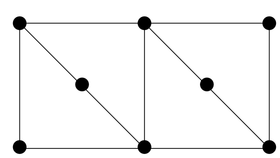
B. K_{100}

C. W_{100}

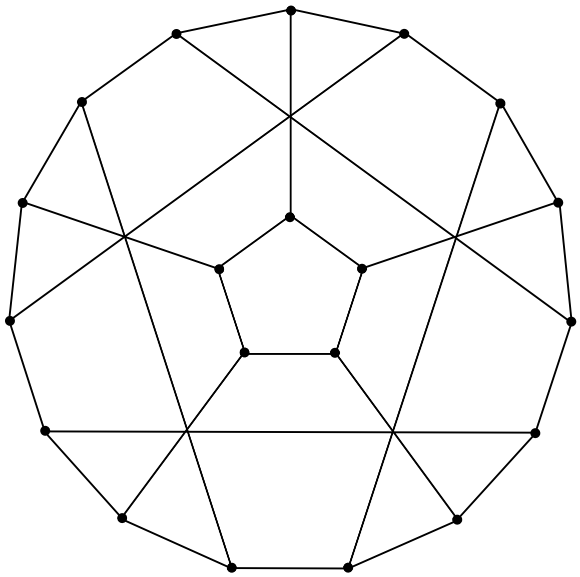
D. $K_{3,3}$

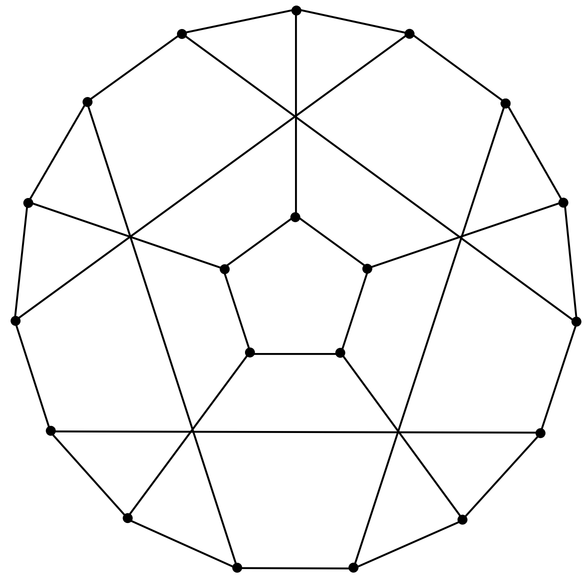
E. $K_{100,101}$

Is the following graph Hamiltonian? Explain your answer.

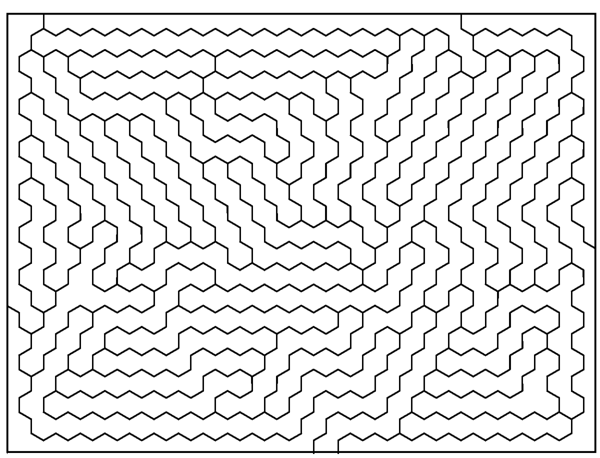


12. Is the following graph planar? Justify your answer. (Two copies provided for convenience.)

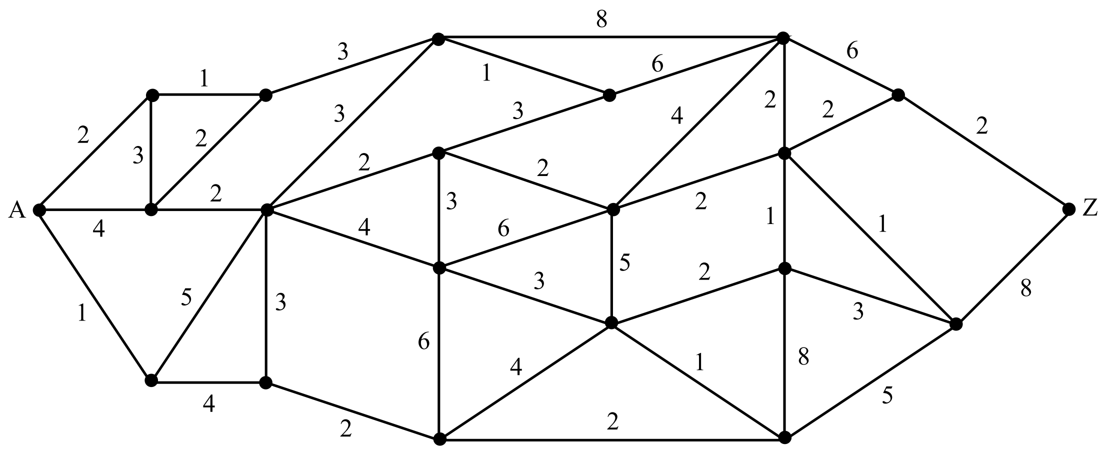




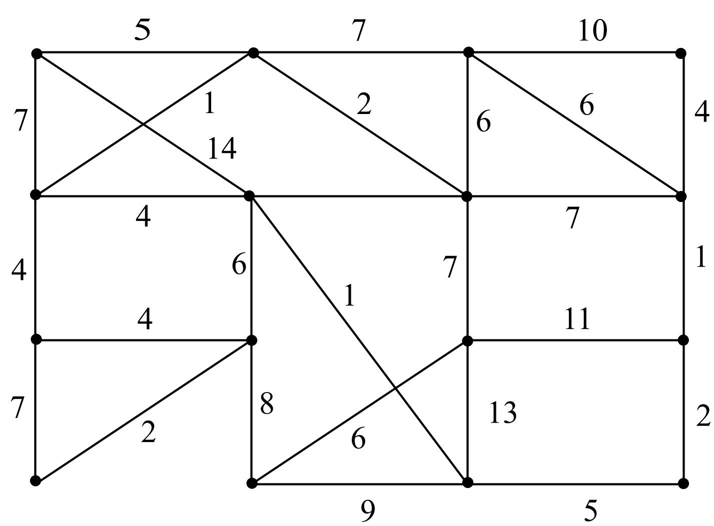
What is the chromatic number of the following map? Justify your answer.



13. Find the distance between A and Z. Shade in all shortest paths from A to Z.



Shade in a minimal spanning tree for the following graph. What is its weight?



14. Prove one of the following statements. Circle the statement you are proving.

   1. Pigeonhole principle: If m objects are in n boxes, some box has at least $\lceil m/n \rceil$ objects.
   2. The cube root of an irrational number is irrational.
   3. $\sqrt{2}$ is irrational.
   4. There are infinitely many prime numbers.
   5. There is no one-to-one correspondence between $\mathbb{N}$ and $\mathbb{R}$.
   6. If a connected graph has exactly one more vertex than edge, then it is a tree.
   7. The chromatic number of a planar graph is no more than 5.