2. a) Is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in V?

$$x = 2y \implies 0 = 2 \cdot 0 \implies 0 = 0 \checkmark$$

$$z \geq 0 \implies 0 \geq 0 \checkmark$$

So yes.

b) Let's say $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ and u,v in V

This means $x_1 = 2y_1$, $z_1 \geq 0$

$x_2 = 2y_2$, $z_2 \geq 0$

$u+v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$. Then $x_1 + x_2 = 2y_1 + 2y_2 = 2(y_1 + y_2)$

Thus for $x = x_1 + x_2$ and $y = y_1 + y_2$,

$x = 2y$.

Then we know $z_1, z_2 \geq 0$ which means $z_1 + z_2 \geq 0$.

So yes.

c) Let's say $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and v is in V which means

$x = 2y$ and $z \geq 0$.

Is cv in V for any real number c?

Well, if $c < 0$, let's say $c = -1$, then $cv = -1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$.

Then we see $-z < 0$ so it's not closed under scalar multiplication.

d) V is not closed under scalar multiplication so not a subspace.

3. $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & -1 & 1 \end{pmatrix}$

A is 2×3 so for $T(v) = Av$, we know v is 3×1 and the product is 2×1, i.e $T(v) = T\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

a) domain is $\vec{v}$ vectors which have 3 entries, or 3 rows so

$a = 3$.

b) codomain is $\vec{b}$ vectors which have 2 rows, so $a = 2$.

c) range corresponds to column space.

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

pivots in first 2 columns so column space is span $\left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right\}$

so range of T is $\mathbb{R}^2$

4. $A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}$

range of T means column space.

$$\begin{pmatrix} 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

so it's Span $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\right\}$

a) $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ so $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ is in the span.

c) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot$ any vector so $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in the span.

e) $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 1 \cdot \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ so $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ is in the span.

5. one-to-one means pivot in every column.

onto means pivot in every row.

So $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is one-to-one and not onto

6. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & h & h \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & h & h-2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & h-2 \end{pmatrix}$

If $h=2$ then last row would be all zeroes making this matrix not invertible.

7. a) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix}$

$T = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

pivot <u>not</u> in every row thus T is <u>not</u> onto.

b) $S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y-z \\ 0 \\ 0 \end{pmatrix}$

$S = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

S is neither one-to-one nor onto

c) $T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-1+0 \\ 0+1-1 \\ -1+0+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

So $T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

d) $S\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+0+0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

So $S\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

e) $T \circ S\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$S\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ as shown above.

$T\circ\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+0+0 \\ 0+0+0 \\ 1+0+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

So $T \circ S\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

8. $T: \underset{\text{domain}}{\mathbb{R}^{10}} \to \underset{\text{codomain}}{\mathbb{R}^{1}}$

T takes in a 10×1 vector and outputs a single number.

So T must be 1×10 matrix, so matrix w/ 1 row and 10 columns.

There can be max 1 pivot since there's only 1 row. In fact, there must be exactly 1 pivot since the question statement says that it's not the zero matrix.

1 pivot and 1 row means matrix is onto. However, 10 columns and 1 pivot means it's not one-to-one. So that tells us T isn't invertible either.

9. Linear transformations mean it has 0 vector and is closed under addition and multiplication.

a) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We can see that if $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Clearly every vector in $\mathbb{R}^2$ satisfies the transformation since the output is always $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus we can say it's closed under addition + multiplication.

b) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(x) \\ e^{y} \end{pmatrix}$

Let's say $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Then $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So not a linear transformation.

c) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$.

If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then $T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

We can see $T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} z_1 \\ y_1 \\ x_1 \end{pmatrix}$ and $T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ y_2 \\ x_2 \end{pmatrix}$.

Then $T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} z_1 + z_2 \\ y_1 + y_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ y_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} z_2 \\ y_2 \\ x_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.

Finally for $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$ and a real number c, we see that

$$T\begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} = \begin{pmatrix} cz \\ cy \\ cx \end{pmatrix} = c\begin{pmatrix} z \\ y \\ x \end{pmatrix} = cT\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

So this is a linear transformation.

d) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ y \\ 1 \end{pmatrix}$

For $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ we have $T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Not a linear transformation.

10. A is 8×12

a) dim of null space is at least 4.

A will have at most 8 pivots (since it has 8 rows), so there will be at least $12 - 8 = 4$ free variables. So this is TRUE

b) dim of column space is at least 8.

As mentioned above, A has at most 8 pivots, it can have less than 8. This is FALSE.

11. A is 5×4 matrix

a) each column has 5 rows or entries, thus the column space is a subspace of $\mathbb{R}^5$.

b) Null space is solution set to $Ax = 0$. x vectors must have dimension 4×1 so null space is a subspace of $\mathbb{R}^4$.

c) Can $\text{Col}(A) = \mathbb{R}^5$?

Since there's 4 columns and 5 rows, A will have at most 4 pivots. So in other words, col(A) will at most be the span of 4 vectors. While the vectors are in $\mathbb{R}^5$ since they have 5 entries, or rows, the linear combination of all the vectors will never be equal to $\mathbb{R}^5$ since col(A) will at most be the span of 4 vectors.

12. $\begin{pmatrix} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 14 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

2 pivots so rank = 2.

total columns = 5 and $5-2=3$ so 3 vectors in basis for the null space.

13. $A^{-1} = \begin{pmatrix} -2 & -1 & 3 \\ 1 & 0 & -3 \\ -1 & 2 & 4 \end{pmatrix}$

$Ax = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies A^{-1}(Ax) = A^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies x = A^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

Then $x = \begin{pmatrix} -2 & -1 & 3 \\ 1 & 0 & -3 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+0+3 \\ 0+0-3 \\ 0+0+4 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$

14. $T: \mathbb{R}^2 \to \mathbb{R}^3$ T is 3×2

$U: \mathbb{R}^2 \to \mathbb{R}^4$ U is 4×2

$V: \mathbb{R}^3 \to \mathbb{R}^3$ V is 3×3

a) $T \circ U$: $(3\times 2) \circ (4 \times 2)$ not possible

b) $T \circ V$: $(3 \times 2) \circ (3 \times 3)$ not possible

c) $V \circ V$: $(3 \times 3) \circ (3 \times 3)$ possible

d) $V \circ T$: $(3 \times 3) \circ (3 \times 2)$ possible

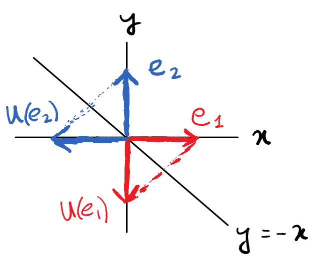
Saturday, October 17, 2020 7:00 PM

Q15) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the x-axis.

$\Rightarrow T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ $\Rightarrow$ matrix representation of T is

$$A := \left[T\begin{pmatrix}1\\0\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$U : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection across the line $y = -x$.



$U(e_1) = U\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$, $U(e_2) = U\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

$\Rightarrow$ matrix representation of U is

$$B := \left(U\begin{pmatrix}1\\0\end{pmatrix} \quad U\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}.$$

The matrix representation of $T \circ U$ is $AB = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix} = \underline{\underline{\begin{pmatrix}0 & -1\\0 & 0\end{pmatrix}}}$

Q16) $A = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$: reflection across the line $y = -x$. Call the reflection U.

Then $U\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$, $U\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$.

Q17) i) <u>True</u> : e.g $\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\end{pmatrix}$ has rank and nullity both 1.

ii) <u>True</u> : e.g. $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ has rank and nullity both 1.

iii) <u>False</u> : rank + nullity = # of columns = 3 $\Rightarrow$ rank and nullity can't both be 2.

iv) <u>False</u> : Still have 3 columns. Same reasoning as part (iii)

Q18) $AX = BX + C \Rightarrow AX - BX = C \Rightarrow (A-B)X = C$

If $A-B$ is invertible, multiplying by $(A-B)^{-1}$ on the left gives

$\underbrace{(A-B)^{-1}(A-B)}_{=\text{ identity } I.} X = (A-B)^{-1}C \quad \Rightarrow \underline{\underline{X = (A-B)^{-1}C}}$

Q19) Note that we can write $\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

So, $T\begin{pmatrix}1\\0\\0\end{pmatrix} = T\left(\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) = \frac{1}{2}T\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{2}T\begin{pmatrix}1\\0\\-1\end{pmatrix}$. by linearity of T.

$\Rightarrow T\begin{pmatrix}1\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}5\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}8\\2\end{pmatrix} = \underline{\underline{\begin{pmatrix}4\\1\end{pmatrix}}}$

Q20) i) <u>True</u> : $\dim V = 1 \Rightarrow$ any linearly indep. set consisting of one vector in V is a basis for V and $\{v\}$ is linearly indep iff $v \neq 0$

ii) <u>True</u> : v, w are linearly indep vectors so the dimension of their span is 2.

iii) <u>True</u> : Any k vectors that span $\mathbb{R}^k$ form a basis of $\mathbb{R}^k$.

Q21) i) <u>True</u> : $AB = I_n \Rightarrow B$ is invertible $\Rightarrow B$ is onto $\Rightarrow Bx = b$ is consistent for all $b \in \mathbb{R}^n$.

ii) <u>True</u> : Columns of B are linearly dep. $\Rightarrow$ there is some $v \neq 0$ for which $Bv = 0$. $\Rightarrow ABv = A(Bv) = A(0) = 0 \Rightarrow T \circ U(x) = ABx$ is not one to one. Since AB is a square matrix, $T \circ U$ is not onto either.

iii) <u>True</u> : $T \circ U(x) = ABx$. The product of invertible matrices is invertible. In particular, $(AB)^{-1} = B^{-1}A^{-1}$ and $(T \circ U)^{-1}(x) = B^{-1}A^{-1}x$.

iv) <u>False</u> : $Ax = b$ is consistent for every $b \in \mathbb{R}^n \Rightarrow A$ has a pivot in every row. Since A is a square matrix, it then has a pivot in every column so that nullity$(A) = \#$ of columns without a pivot $= 0 \Rightarrow \{x : Ax = 0\} = \{0\}$.