2. The vector $\begin{pmatrix}3\\3\\6\end{pmatrix}$ is an eigenvector for the matrix $\begin{pmatrix}1&-3&3\\3&-5&3\\6&-6&4\end{pmatrix}$.

What is the corresponding eigenvalue?

We know from the definition of an eigenvalue that λ is the eigenvalue corresponding to v for the matrix A if $Av=\lambda v$. Since we already have A and v, we can just multiply them to find λ.

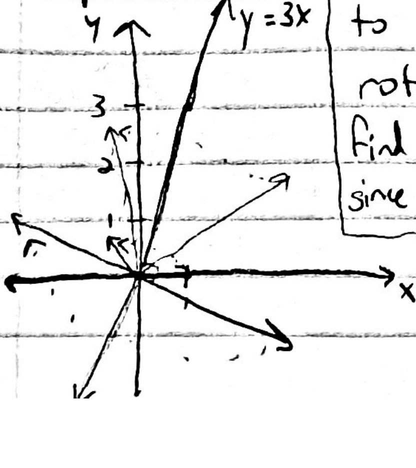
$$\begin{pmatrix}1&-3&3\\3&-5&3\\6&-6&4\end{pmatrix}\begin{pmatrix}3\\3\\6\end{pmatrix}=\begin{pmatrix}3(1)+3(-3)+6(3)\\3(3)+3(-5)+6(3)\\3(6)+3(-6)+6(4)\end{pmatrix}=\begin{pmatrix}3-9+18\\9-15+18\\18-18+24\end{pmatrix}=\begin{pmatrix}12\\12\\24\end{pmatrix}=4\cdot\begin{pmatrix}3\\3\\6\end{pmatrix}.$$

Therefore our eigenvalue is $\boxed{4}$.

3. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation given by reflection about the line $y=3x$. What are the eigenvalues of the standard matrix of T?

We can solve this problem in 2 ways, either by using the characteristic polynomial, or by inspecting the transformation. The characteristic polynomial takes longer, but it will always work, no matter how complicated the transformation is, while inspection is fast but only works when you can visualize the linear transformation.

Inspection: We can draw out our transformation and check different vectors



to try and find ones that only scale and don't rotate, since these will be eigenvectors. We only need to find up to 2 eigenvectors that are linearly independent, since that is the max we can have in $\mathbb{R}^2$.

Notice that any vector along the line $y=3x$ stays the same, and vectors along the perpindicular swap their sign.

This means that these two sets of vectors are eigenvectors. If v_1 is on the line $y=3x$, we get that $Av_1 = v_1$, so $\boxed{\lambda_1 = 1}$, and if v_2 is on the perpendicular, $Av_2 = -v_2$, so $\boxed{\lambda_2 = -1.}$

Characteristic polynomial: Our transformation brings $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}-4/5\\3/5\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}3/5\\4/5\end{pmatrix}$ which means that our transformation matrix is $A = \begin{pmatrix}-4/5 & 3/5\\3/5 & 4/5\end{pmatrix}$. You can solve this trigonometrically or with the formula for a reflection matrix: $\frac{1}{1+m^2}\begin{pmatrix}1-m^2 & 2m\\2m & m^2-1\end{pmatrix}$, where m is the slope. We then need to solve the equation $\det(A-\lambda I)=0$, which gives us

$$\det\begin{pmatrix}-\frac{4}{5}-\lambda & \frac{3}{5}\\ \frac{3}{5} & \frac{4}{5}-\lambda\end{pmatrix} = 0$$

$$\left(-\frac{4}{5}-\lambda\right)\left(\frac{4}{5}-\lambda\right) - \frac{9}{25} = 0$$

$$-\frac{16}{25} - \frac{4}{5}\lambda + \frac{4}{5}\lambda + \lambda^2 - \frac{9}{25} = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda+1)(\lambda-1) = 0$$

$$\boxed{\lambda = 1, -1}.$$

4. Projection onto the y axis sends (x,y) to $(0,y)$, so we can just try each of our options and see which get sent to a multiple of themself by the transformation. Be careful though, since by definition eigenvectors are nonzero, so $\begin{pmatrix}0\\0\end{pmatrix}$ is immediately eliminated.

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0 \cdot \begin{pmatrix}1\\0\end{pmatrix}$ ✓

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = 1 \cdot \begin{pmatrix}0\\1\end{pmatrix}$ ✓

$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \neq \lambda\begin{pmatrix}1\\1\end{pmatrix}$, since for the x coords to match, we need $\lambda=0$, but then the y coords don't match.

$T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0 \cdot \begin{pmatrix}2\\0\end{pmatrix}$ ✓

So our answers are $\boxed{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}}$. Notice that any multiple of $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ would have also worked if it had been an option.

5. Compute the characteristic polynomial of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic polynomial of a matrix A is $\det(A-\lambda I)$, so for this matrix we get

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = \boxed{\lambda^2 - 4\lambda + 3}.$$

We can also use the formula for the characteristic polynomial of a 2×2 matrix: $\lambda^2 - Tr(A)\lambda + \det(A)$.
$Tr(A) = 2 \cdot 2 = 4$; and $\det(A) = 2\cdot 2 - 1\cdot 1 = 3$, so we get the same answer.

6. The characteristic polynomial of a matrix is $-\lambda^3 - 3\lambda^2 - 2\lambda$. What are its eigenvalues?

The eigenvalues are the solutions when you set the characteristic polynomial equal to 0, so we get

$$-\lambda^3 - 3\lambda^2 - 2\lambda = 0$$

$-\lambda(\lambda^2 + 3\lambda + 2) = 0$ factoring out $-\lambda$

$\lambda(\lambda^2 + 3\lambda + 2) = 0$ multiplying by -1

$\lambda(\lambda+2)(\lambda+1) = 0$ factoring or using quadratic formula.

$$\boxed{\lambda = 0, -2, -1}$$