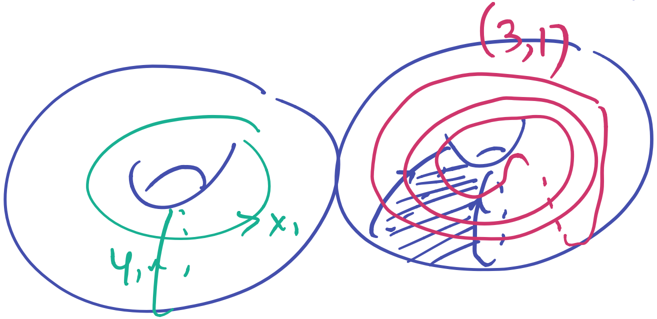
The Torus

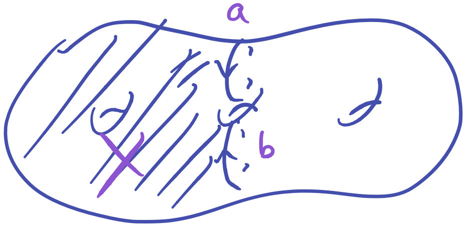
Prop. The map $\mathrm{Mod}(T^2) \to SL_2\mathbb{Z}$ given by action on $H_1(T^2;\mathbb{Z})$ is an $\cong$.

not GL because $\hat{\imath}$. and $\hat{\imath} \leftrightarrow \det$.



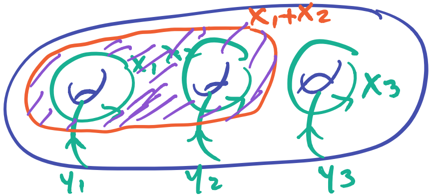
homology.

$a-b = \partial X$

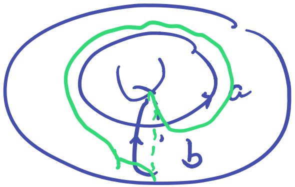


$\mathrm{Mod}(S_g) \to GL_{2g}\mathbb{Z}$

$H_1(S_g;\mathbb{Z}) = \mathbb{Z}^{2g}$



Pf. $\boxed{\text{Surjectivity}}$



Pf #1

$$T_a \longmapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad T_b \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

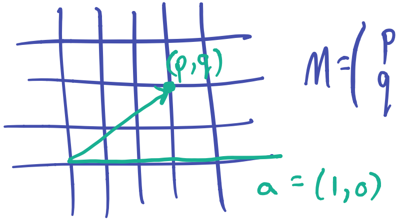
Pf # 2

Let $M \in SL_2\mathbb{Z}$, thought of as lin map $\mathbb{R}^2 \to \mathbb{R}^2$

M descends to $\varphi \in \mathrm{Homeo}^+(T^2)$

and $\varphi_* = M$

action on H_1.



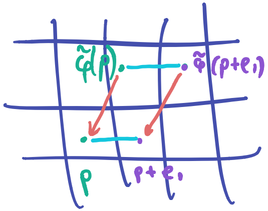
$M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$

Injectivity

Pf #1 K(G,1) theory

$$\left\{ \text{based maps } T^2 \to T^2 \right\} / \sim \quad \longleftrightarrow \quad \left\{ \text{homoms } \mathbb{Z}^2 \to \mathbb{Z}^2 \right\}$$

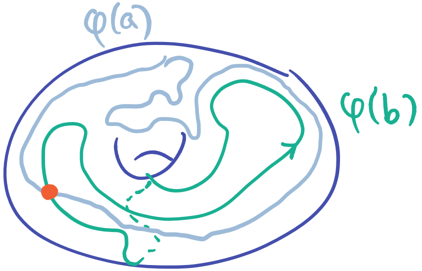
Pf #2 Straight-line homotopy.



$\varphi \in$ kernel

$\Longrightarrow$ S.L.H. equivariant w.r.t. deck trans.

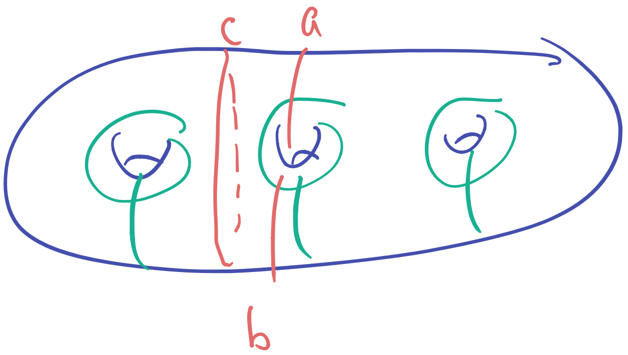
Pf #3



What about higher genus?

$\mathrm{Mod}(S_g) \longrightarrow \mathrm{Aut}(\mathbb{Z}^{2g})$ has a (big) kernel!

Torelli gp.



T_c

$T_a T_b^{-1}$

See Chap. 6.

Proposition 2.8 (Alexander method) *Let S be a compact surface, possibly with marked points, and let $\phi \in \mathrm{Homeo}^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S with the following properties.*

1. *The γ_i are pairwise in minimal position.*

2. *The γ_i are pairwise nonisotopic.*

3. *For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$, or $\gamma_j \cap \gamma_k$ is empty.*

(1) If there is a permutation σ of $\{1, \ldots, n\}$ so that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to ∂S.

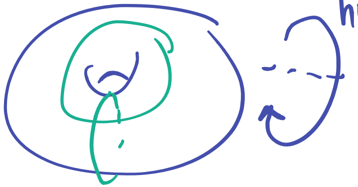
If we regard $\cup\gamma_i$ as a (possibly disconnected) graph Γ in S, with vertices at the intersection points and at the endpoints of arcs, then the composition of ϕ with this isotopy gives an automorphism ϕ_* of Γ.

(2) Suppose now that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Γ with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ has a nontrivial power that is isotopic to the identity.

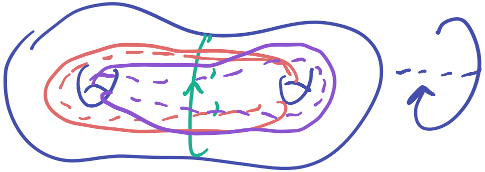
<u>Morally</u>: A mapping class is determined by its action on (finitely many) curves.

Q. Is there a version without hypoth. ?

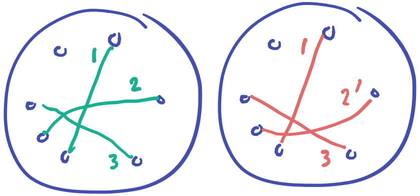
<u>Examples.</u>

① 

hyp inv

② 

Q. Is there a similar example satisfying 3. in Prop 2.8?

③ 

Is there a notion of canonical pos. for curves failing 3?

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Pf by induction on n.

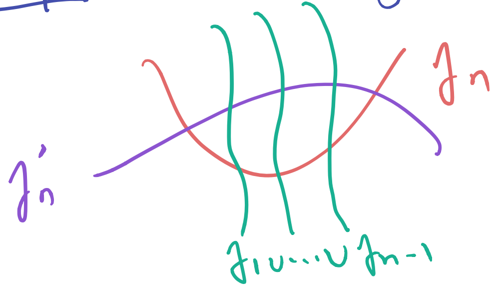
Say we modified φ by homotopy

So $\varphi(\gamma_1' \cup \cdots \cup \gamma_{n-1}') = \gamma_1 \cup \cdots \cup \gamma_{n-1}$

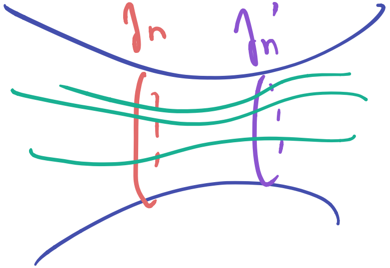
Want to isotope φ s.t. $\gamma_n' \longrightarrow \gamma_n$

& we $\boxed{\text{fix } \gamma_1 \cup \cdots \cup \gamma_{n-1}}$.

Step 1. Remove bigons



Step 2. Remove annulus



<u>Cor.</u> If $c_i = [\gamma_i]$ as in Prop.

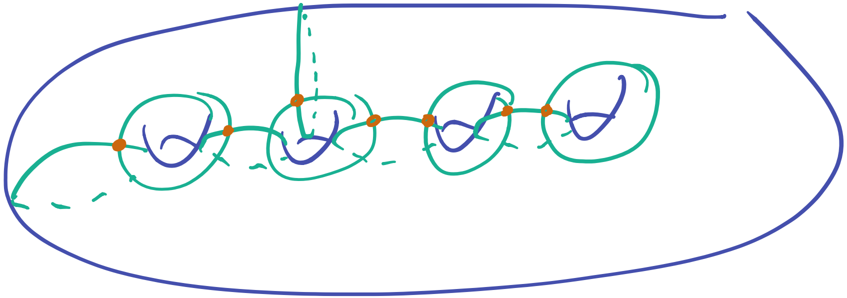
& $f \in \mathrm{Mod}(S)$ fixes $\{c_i\}$

Then f has finite order.

Moreover, f is det. by induced action on

$\cup \gamma_i$, thought of as a graph.

A good Alexander system:



This graph has no nontrivial automorphisms

So if f fixes ~~each curve~~ the $\{c_i\}$ as a set then $f = id$ in $Mod(S_4)$

filling, but not really.