Chap 7. Torsion

__Thm.__ (Fenchel–Nielsen)
Any fin. order $f \in \mathrm{Mod}(S_{g,n})$ has a rep $\varphi \in \mathrm{Homeo}^+(S_{g,n})$ of finite order.

More: φ can be chosen to be isometry of a hyp./Eucl. metric.

__Pf.__ Later chapter.

Same true for $G \leq \mathrm{Mod}(S_{g,n})$, $|G| < \infty$ much much harder.

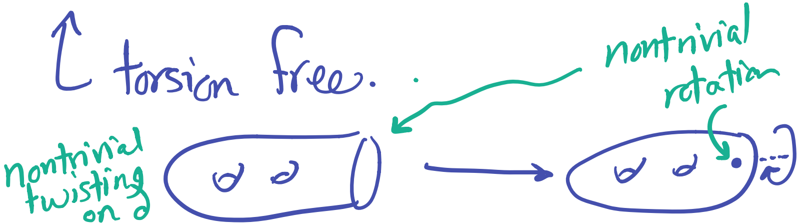
__Cor.__ $\partial S \neq \emptyset$

$\mathrm{Mod}(S)$ is torsion free.

__Pf of Cor.__

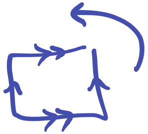
$$\mathbb{Z}^b \longrightarrow \mathrm{Mod}(S_{g,n}^b) \xrightarrow{\text{capping}} \mathrm{Mod}(S_{g,n+b})$$

$\uparrow$ torsion free.

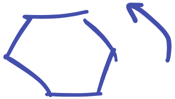


Torus case

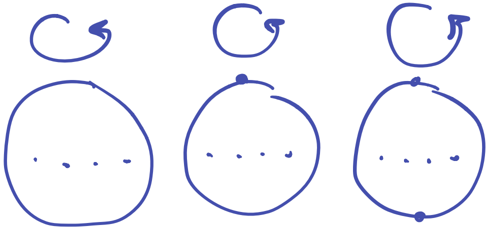
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$



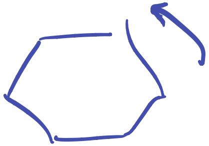
Sphere case



Brouwer: a per. homeo of S^2 is conj to Eucl. rot.

In higher genus, it is complicated to list all periodic elts (number thy).

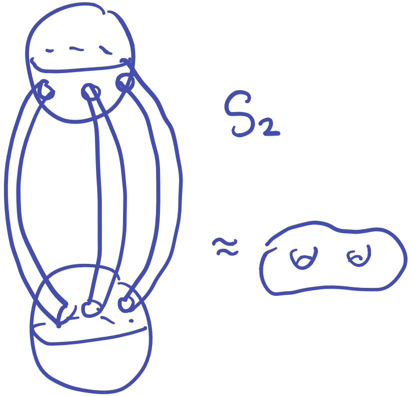
examples



$4g+2$

not realizable by rotation in $\mathbb{R}^3$

(1 fixed pt)



Torelli Ker $\mathrm{Mod}(S_g) \to Sp_{2g}\mathbb{Z}$

Thm $\mathcal{I}(S_g)$ is torsion free.

Pf. Say $f \in \mathcal{I}(S_g)$ WLOG $g \geq 2$.

$1 < |f| < \infty$.

$\rightsquigarrow$ representative φ $\rightsquigarrow$ at each fixed pt, rotation.

Apply Lefschetz f.p.t.

$$L(\varphi) = \sum_{i=0}^{2} (-1)^i \, \mathrm{tr}\left(\varphi_* : H_i(S_g) \to H_i(S_g)\right)$$

$$= 1 - 2g + 1 = 2 - 2g < 0$$

$L(\varphi)$ = #fixed pts > 0

$\blacksquare$

φ = homeo of space X with isolated fixed pts

$L(\varphi)$ = sum of degrees of fixed pts.

degree at p: deg of induced map on $S^1 \cong UT_pX$

If φ is a rotation at p then degree of φ at p is... $+1$

$84(g-1)$ Thm

Thm. $g \geq 2$, $G \leq \mathrm{Mod}(S_g)$
$|G| < \infty$

$\Longrightarrow |G| \leq 84(g-1)$.

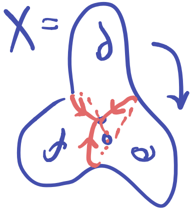
- For G abelian answer: $4g+4$
- Bound is (not) realized for ∞ many g.
- Realized for $g=3$
- Larsen. $\{g : \text{bound is realized}\}$ has same frequency in $\mathbb{N}$ as cubes.

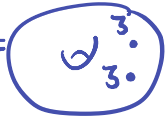
Proof uses hyp orbifolds

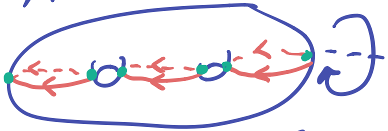
X = hyp. surface

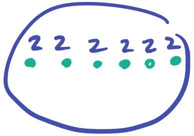
$G \leq \mathrm{Isom}^+(X)$ finite.

$\rightsquigarrow Y = X/G$ orbifold.

examples $X =$  $G \cong \mathbb{Z}/3$

$Y =$ 

$X =$  $G \cong \mathbb{Z}/2$

$Y =$ 

Riemann-Hurwitz formula

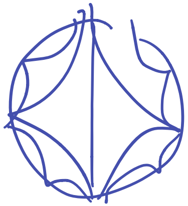
In Y, images of fixed pts are marked

and label of a marked pt is $|G|/\#\text{ preimages}$

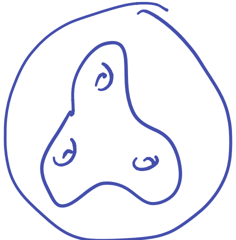
marked pts

$$\chi(Y) = (2-2g(Y)) - m + \sum_{i=1}^{m} \frac{1}{p_i}$$

labels



Fact. $\chi(X) = |G|\,\chi(Y)$.



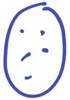
Pf of $84(g-1)$

Want to show for any $Y = X/G$, $\chi(Y)<0$

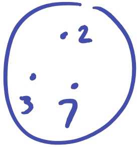
$$\chi(Y) \le -1/42.$$

$$\frac{2-2g}{84(g-1)}$$

$X =$  $2-2g$

$Y =$ 

Pf. Just check.

Only possibility is $Y =$ 

$$\chi(Y) = -1/42$$

$\blacksquare$

Realizing Finite Groups

Thm. G = finite gp

$\exists\, g$ s.t. $G \leq \mathrm{Mod}(S_g)$

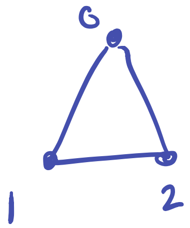
Pf #1 Build S_g from $\underbrace{\text{Cayley graph for } G}$.

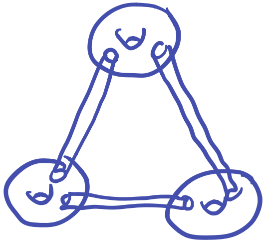
vertices: G

edges: differ by generator

$G \circlearrowright$ Cayley graph by left mult.

$\mathbb{Z}/3$





vertices $\longrightarrow$ tori

edges $\longrightarrow$ annuli

Can replace "$\exists\, g$" with "$g \gg 0$"?

Yes for cyclic groups.

Generating MCG with torsion

Thm $\mathrm{Mod}(S_g)$ is generated by elts of order 2.

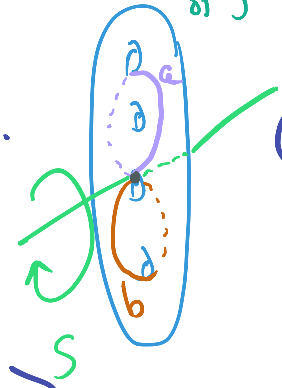
Pf. $\mathrm{Mod}(S_g)$ is perfect.

$$\mathrm{Mod}(S_g) = [\mathrm{Mod}(S_g), \mathrm{Mod}(S_g)] = \langle [T_a, T_b] : i(a,b) = 1 \rangle$$

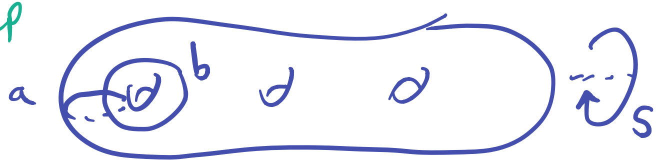
Suffices to write

$$[T_a, T_b] = \prod \text{elts of order } 2$$

Brendle-Farb: only 6 such elts are needed, indep of g.



Change of coords:



Choose involution s, $s(a) = b$.

$$[T_a, s] = T_a (s T_a^{-1} s^{-1})$$
$$= T_a T_b^{-1}$$

product of 2 elts of order 2: $T_a s T_a^{-1}$, s

Similarly $T_a^{-1} T_b$ is a product of 2 elts of order 2

$\Longrightarrow$ $[T_a, T_b] =$ prod of 4 elts of ord 2