

# PLANAR GRAPHS

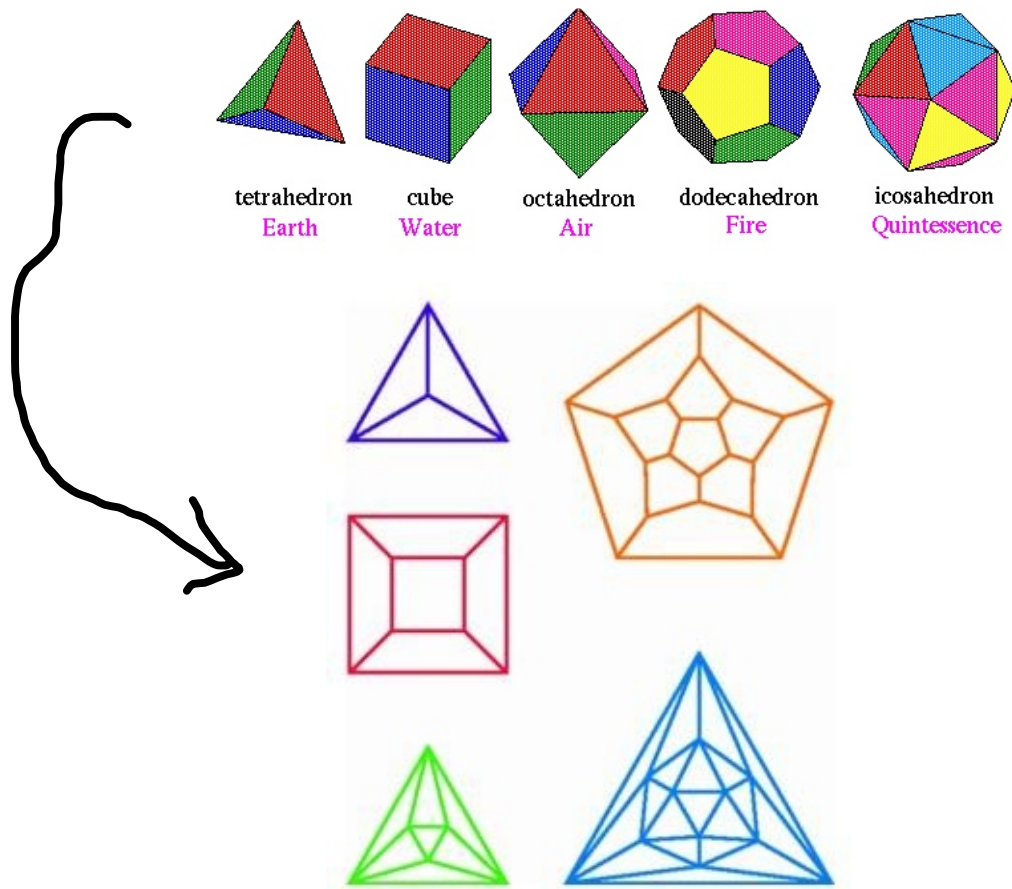
A graph is *planar* if it can be drawn in the plane so that no two edges cross.

The Three House-Three Utility Problem asks whether or not  $K_{3,3}$  is planar.



# PLATONIC SOLIDS

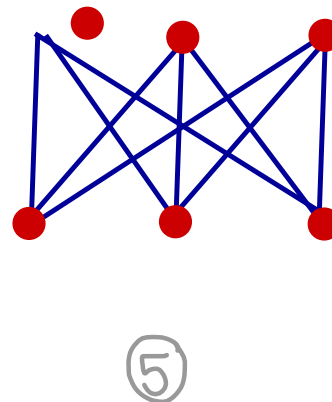
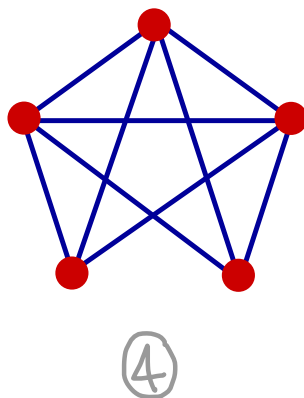
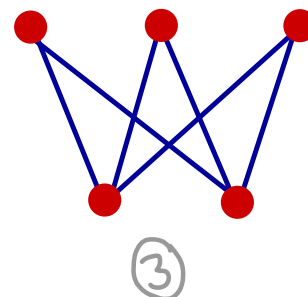
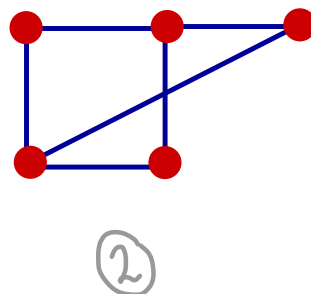
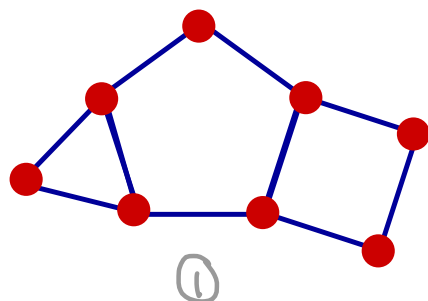
One collection of interesting planar graphs comes from the five Platonic solids:



Plato

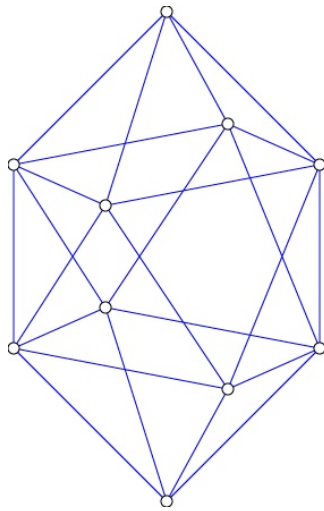
# PLANAR GRAPHS

Which of the following graphs are planar?



# PLANAR GRAPHS

Is this graph planar?



# VERTICES, EDGES, AND FACES

A planar drawing of a planar graph divides the plane into distinct regions, or *faces*.

	vertices	edges	faces
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			



What is the pattern?

# EULER'S THEOREM

**THEOREM.** Any planar drawing of a graph with  $V$  vertices,  $E$  edges, and  $F$  faces satisfies

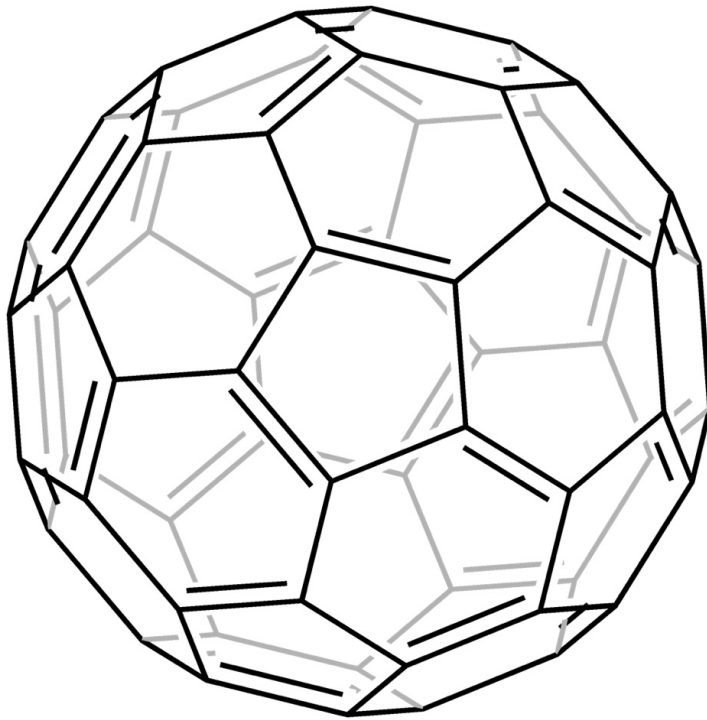
$$V - E + F = 2$$

In 1988, the Mathematical Intelligencer ran a survey. It was decided that the 5 most beautiful results in mathematics were:

- (i) Euler's identity  $e^{ix} = \cos x + i \sin x$
- (ii) Euler's polyhedral formula  $V - E + F = 2$
- (iii) Euclid's proof of the infinitude of the primes
- (iv) Euclid's proof that there are only 5 regular solids
- (v) Euler's summation  $\sum 1/n^2 = \pi^2/6$

# EULER'S THEOREM

Does the Buckeyball satisfy  $V-E+F=2$ ?



# EULER'S THEOREM

**THEOREM.** Any planar drawing of a connected graph with  $V$  vertices,  $E$  edges, and  $F$  faces satisfies

$$V - E + F = 2$$



$K_{3,3}$  IS NOT PLANAR

THEOREM.  $K_{3,3}$  is not planar.

## $K_5$ IS NOT PLANAR

**THEOREM.** If a planar graph has  $V$  vertices and  $E$  edges, then  $E \leq 3V - 6$ .

**COROLLARY.**  $K_5$  is not planar.

# DEGREES

**THEOREM.** Every planar graph has at least one vertex whose degree is less than 6.

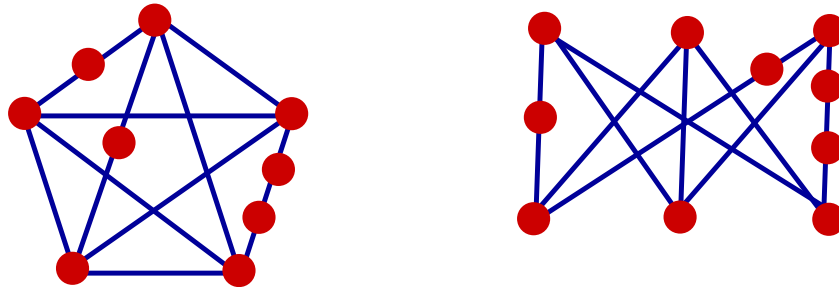
# MORE NONPLANAR GRAPHS

So far, we know  $K_5$  and  $K_{3,3}$  are not planar.  
It follows that  $K_n$  is not planar for  $n \geq 5$ ,  
 $K_{m,n}$  is not planar for  $m, n \geq 3$ .

More generally:

**PROPOSITION.** Any graph that contains  $K_5$  or  $K_{3,3}$  as a subgraph is not planar.

Note also any **subdivision** of  $K_5$  or  $K_3$  is nonplanar:



**PROPOSITION.** Any graph that contains a subdivision of  $K_5$  or  $K_{3,3}$  as a subgraph is not planar.

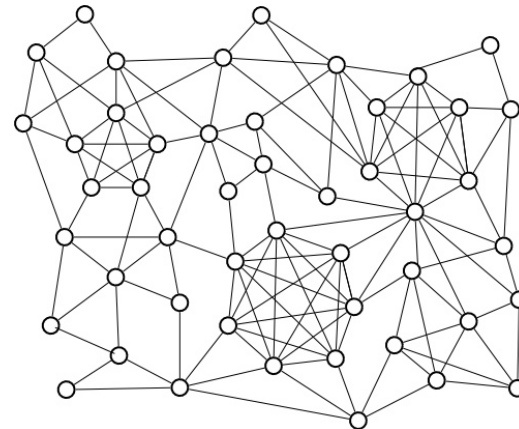
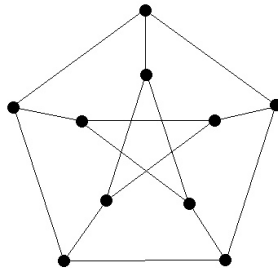
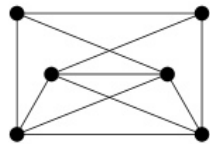
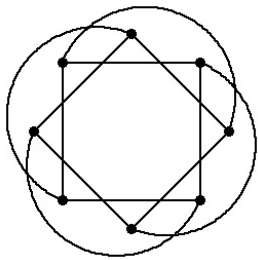
# KURATOWSKI'S THEOREM

Amazingly, the converse is also true:

**THEOREM.** A graph is planar if and only if it contains no subgraph that is a subdivision of  $K_5$  or  $K_{3,3}$ .

**PROOF.** See web site.

Which of the following graphs are planar?



# PLATONIC SOLIDS

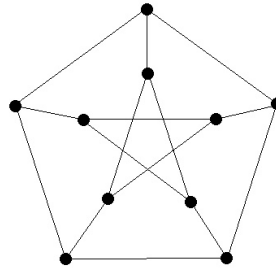
A *Platonic solid* is a 3-dimensional solid with polygonal faces, and satisfying:

- (i) The faces are regular and congruent.
- (ii) The same number of faces meet at each vertex.
- (iii) The line connecting any two points on the solid is contained in the solid.

# WAGNER'S THEOREM

A graph  $H$  is a **minor** of a graph  $G$  if  $H$  is obtained from  $G$  by taking a subgraph and collapsing some edges.

**THEOREM.** A graph is planar if and only if it does not contain  $K_5$  or  $K_{3,3}$  as a minor.



# FÄRY'S THEOREM

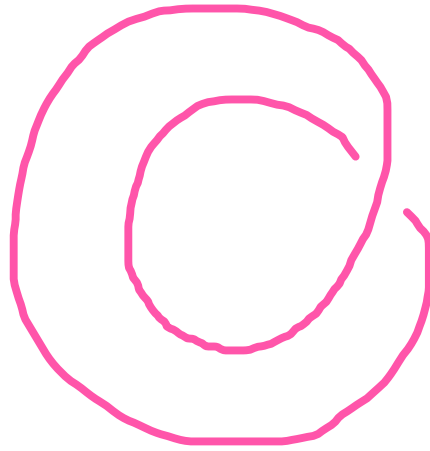
**THEOREM.** Every planar graph can be drawn in the plane using only straight lines.

The proof uses the art gallery theorem...



## OTHER SURFACES

What are the largest  $m, n$  so  $K_n$  and  $K_{m,n}$  can be drawn without crossings on a Möbius strip



or a torus?

