Eigenvalues in Structural Engineering

Watch this video about the Tacoma Narrows bridge. Watch

Here are some toy models. Check it out

The masses move the most at their natural frequencies ω. To find those, use the spring equation: $mx'' = -kx \quad \rightsquigarrow \quad \sin(\omega t)$.

With 3 springs and 2 equal masses, we get:

$$\begin{aligned} mx_1'' &= -kx_1 + k(x_2 - x_1) \\ mx_2'' &= -kx_2 + k(x_1 - x_2) \end{aligned}$$

Guess a solution $x_1(t) = A_1(\cos(\omega t) + i\sin(\omega t))$ and similar for x_2. Finding ω reduces to finding eigenvalues of $\begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}$.

Eigenvectors: $(1, 1)$ & $(1, -1)$ (in/out of phase) Details

Section 5.4

Diagonalization

Section 5.4 Outline

- Diagonalization
- Using diagonalization to take powers
- Algebraic versus geometric dimension

We understand diagonal matrices

We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, powers of A are easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} =$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose want to understand the matrix

$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

geometrically? Or take it's 10th power? What would we do?

What if I give you the following equality:

$$\underset{A}{\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}} \underset{=}{=} \underset{C}{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}} \underset{C^{-1}}{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}?

Powers of matrices that are similar to diagonal ones

What if I give you the following equality:

$$\begin{array}{ccccccc} \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} & = & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \\ A & = & C & D & C^{-1} \end{array}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}? Demo

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

We say that A is similar to D.

How does this factorization of A help describe what A does to $\mathbb{R}^n$?
How does this help us take powers of A?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.

Diagonalization

The recipe

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \quad C \quad D \quad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, with multiplicity, in order.

Why?

Example

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

Demo

Hint: the eigenvalues are 1 and $1/2$

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- $a_i =$ algebraic multiplicity of λ_i
- $d_i =$ dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $1 \leq d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

So the recipe for checking diagonalizability is:

- If there are not n eigenvalues with multiplicity, then stop.
- For each eigenvalue with alg. mult. greater than 1, check if the geometric multiplicity is equal to the algebraic multiplicity. If any of them are smaller, the matrix is not diagonalizable.
- Otherwise, the matrix is diagonalizable.

More rabbits

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: the characteristic polynomials are $-\lambda^3 + 3\lambda + 2$ *and* $-\lambda^3 + 2\lambda + 1$ *and both have rational roots.*

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\left(\begin{smallmatrix} 2 & 6 \\ 0 & -1 \end{smallmatrix}\right)$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$