Announcements Oct 20

- Masks $\rightsquigarrow$ Thank you!
- Midterm 2 Tonite! 8–9:15p on Teams (2 channels). Sec. 2.5–3.4 (not 2.8)
- No quiz Friday
- Thu office hour cancelled
- Review session: Prof. M Today 4:30–5:15 Howey L1

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Review for Midterm 2

Important terms

- linearly independent
- subspace
- column space
- null space
- basis
- dimension
- one-to-one
- onto
- linear transformation
- ~~inverse~~
- rank-nullity theorem

Summary of Section 2.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

  has only the trivial solution. It is linearly dependent otherwise.
- The cols of A are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ A has a pivot in each column
- The number of pivots of A equals the dimension of the span of the columns of A
- The set $\{v_1, \ldots, v_k\}$ is linearly independent $\Leftrightarrow$ they span a k-dimensional plane
- The set $\{v_1, \ldots, v_k\}$ is linearly dependent $\Leftrightarrow$ some v_i lies in the span of $v_1, \ldots, v_{i-1}$.
- To find a collection of linearly independent vectors among the $\{v_1, \ldots, v_k\}$, row reduce and take the (original) v_i corresponding to pivots.

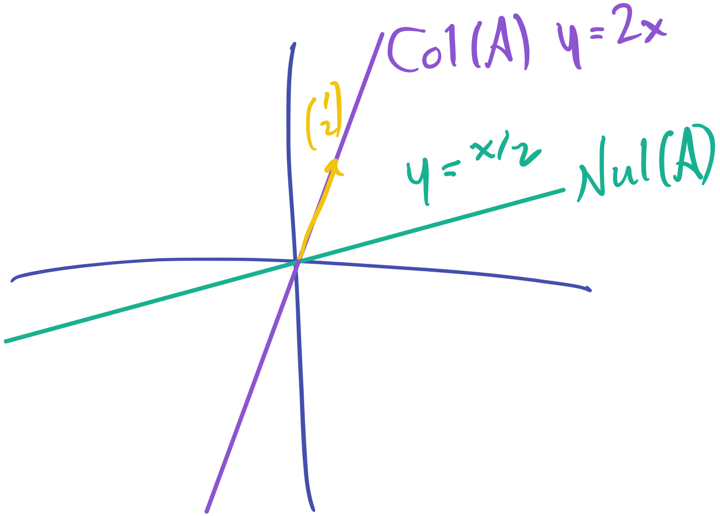
Typical exam questions 2.5

- State the definition of linear independence.
- *Always/sometimes/never.* A collection of 99 vectors in $\mathbb{R}^{100}$ is linearly dependent.
- *Always/sometimes/never.* A collection of 100 vectors in $\mathbb{R}^{99}$ is linearly dependent.
- Find all values of h so that the following vectors are linearly independent:

$$\left\{ \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ h \end{pmatrix} \right\}$$

- *True/false.* If A has a pivot in each column, then the rows of A are linearly independent.
- *True/false.* If u and v are vectors in $\mathbb{R}^5$ then $\{u, v, \sqrt{2}u - \pi v\}$ is linearly independent.
- If you have a set of linearly independent vectors, and their span is a line, how many vectors are in the set?

Find A with



$$\begin{pmatrix}1 & -2\\ 2 & -4\end{pmatrix}$$

$$\begin{pmatrix}1 & a\\ 2 & 2a\end{pmatrix}$$

$$y = x/2$$

$$y - x/2 = 0$$

$$2y - x = 0$$

$$\rightsquigarrow \begin{pmatrix}1 & a\\ 0 & 0\end{pmatrix} \rightsquigarrow \text{Null space}$$

$$x + ay = 0$$

$$\rightarrow a = -2$$

Section 2.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u + v$ is also in V. closure under add
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$. closure under scalar mult.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Four things are the same: subspaces, spans, planes through 0, null spaces

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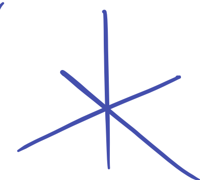
Let V be the subset of $\mathbb{R}^3$ consisting of the x-axis, the y-axis, and the z-axis. Which properties of a subspace does V fail?

1 ok
2 fails $e_1 + e_2$ not in V
3 ok



Find a basis ~~spanning set~~ for the plane in $\mathbb{R}^3$ defined by $x + y - 2z = 0$.

param vect form. Null space

$(\boxed{1}\ \ 1\ \ -2)$

$x = -y + 2z$
$y = y$
$z = z$

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

Typical exam questions

- Consider the set $\{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$. Is it a subspace? If not, which properties does it fail?
- Consider the x-axis in $\mathbb{R}^3$. Is it a subspace? If not, which properties does it fail?
- Consider the set $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$. Is it a subspace? If not, which properties does it fail?
- Find spanning sets for the column space and the null space of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

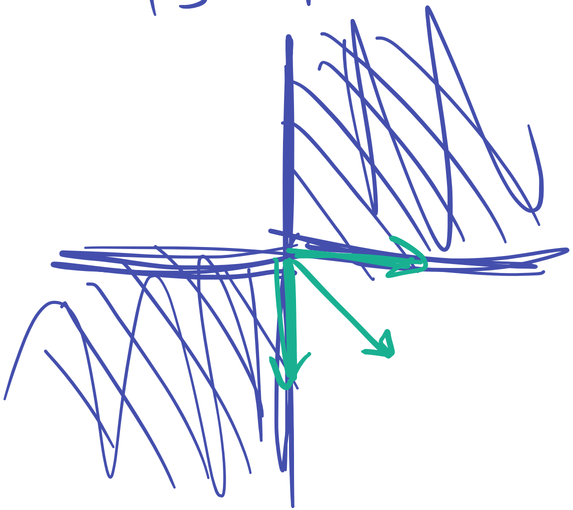
- True/False: The set of solutions to a matrix equation is always a subspace.
- True/False: The zero vector is a subspace.

$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : ab \geq 0 \right\}$$

$\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ in V

$\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ not in V

Is this a subspace?



addition?

$$\underset{\text{in } V}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \underset{\text{in } V}{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} = \underset{\text{not in } V}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Section 2.7 Summary

- A basis for a subspace V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{Span}\{v_1, \ldots, v_k\}$
  2. $v_1, \ldots, v_k$ are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\text{Col}(A)$ by taking pivot columns of A (not reduced A)
- Basis Theorem. Suppose V is a k-dimensional subspace of $\mathbb{R}^n$. Then
  - Any k linearly independent vectors in V form a basis for V.
  - Any k vectors in V that span V form a basis.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\3\\1\end{pmatrix} \begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \begin{pmatrix}-1\\1\\1\end{pmatrix} \begin{pmatrix}1\\-1\\1\end{pmatrix} \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Find a basis $\{u, v, w\}$ for $\mathbb{R}^3$ where no vector has a zero entry.

$$\begin{pmatrix}1\\7\\11\end{pmatrix} \begin{pmatrix}7\\91\\118\end{pmatrix} \begin{pmatrix}1\\2\\3\end{pmatrix}$$

Typical exam questions

- Find a basis for the yz-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in R^7?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of A is the number of pivots of A.
- True/false: If b lies in the column space of A, and the columns of A are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span R^3 must be linearly independent.

Section 2.9 Summary

- **Rank-Nullity Theorem.** $\underbrace{\mathrm{rank}(A)}_{\text{dim Col}(A)} + \overbrace{\dim \mathrm{Nul}(A)}^{\text{Nullity}} = \#\mathrm{cols}(A)$

Let A be an 4×6 nonzero matrix and suppose the columns of A are all the same. What is $\dim \mathrm{Nul}(A)$?

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ 2 & 2 & 2 & \\ 3 & 3 & 3 & \\ 4 & 4 & 4 & \end{pmatrix}$$

rank = 1

$\boxed{5}$

Typical exam questions

$$A\underset{\text{in}}{x} = \underset{\text{out}}{b}$$

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a $\overset{\text{2D}}{\text{plane}}$. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?

rank

$= \dim \mathrm{Col}(A) = 5$

$$\left(\begin{array}{ccccccc|c} \square & & & & & & & 1 \\ & \square & & & & & & 2 \\ & & \square & & & & & 3 \\ & & & \square & & & & 4 \\ & & & & \square & & & 5 \end{array}\right)$$

no pivot on RHS YES!

$$\begin{pmatrix} \square & \\ & \square \\ & \\ & \\ & \\ & \end{pmatrix}$$

one-to-one

or no free vars

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

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Find a matrix A so that the range of the matrix transformation $T(v) = Av$ is the line $y = 2x$ in $\mathbb{R}^2$.

Typical exam questions

- What does the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ do to $\mathbb{R}^3$?
- What does the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- True/false. If A is a matrix and T is the associated matrix transformation, then the statement $Ax = b$ is consistent is equivalent to the statement that b is in the range of T.
- True/false. There is a matrix A so that the domain of the associated matrix transformation is a line in $\mathbb{R}^3$.

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are indep
  - $Ax = 0$ has only 0 soln
  - A has a pivot in every col
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in every row
  - $Ax = b$ is consistent for every b in $\mathbb{R}^m$
  - the range of T has dimension m

one-to-one $\Leftrightarrow$ onto

Let A be an 5×5 matrix. Suppose that $\dim \mathrm{Nul}(A) = 0$. Must it be true that $Ax = e_1$ is consistent?

TRUE

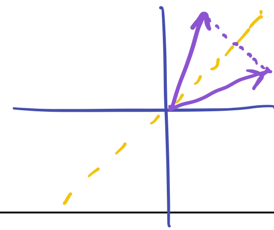
yes b/c onto

Typical exam questions

- True/False. It is possible for the matrix transformation for a 5×6 matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the yz-plane is onto. NO Range is yz-plane, not all of $\mathbb{R}^3$
- True/False. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by π is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not. NO.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.

Summary of Section 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.



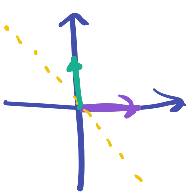
$B \quad A$

$U \circ T(v) = U(T(v))$

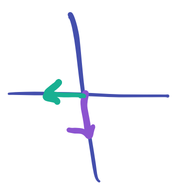
$BA\,v$

''

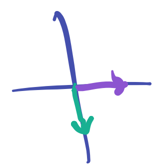
Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects over the line $y = -x$ and then rotates counterclockwise by $\pi/2$.



$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\xrightarrow{T}$



$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\xrightarrow{U}$



$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

  What is

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix}? = T\begin{pmatrix} 2 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the z-axis by π and then scales by 2.
- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$$

  Is this a linear transformation? If so, what is the standard matrix for T?
- Is the identity transformation one-to-one?

Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- The standard matrix for a composition of linear transformations is the product of the standard matrices.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0

Find a 2×2 matrix A so that $A^4 = I$ and $A^2 \neq I$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Hint: Think about transformations.

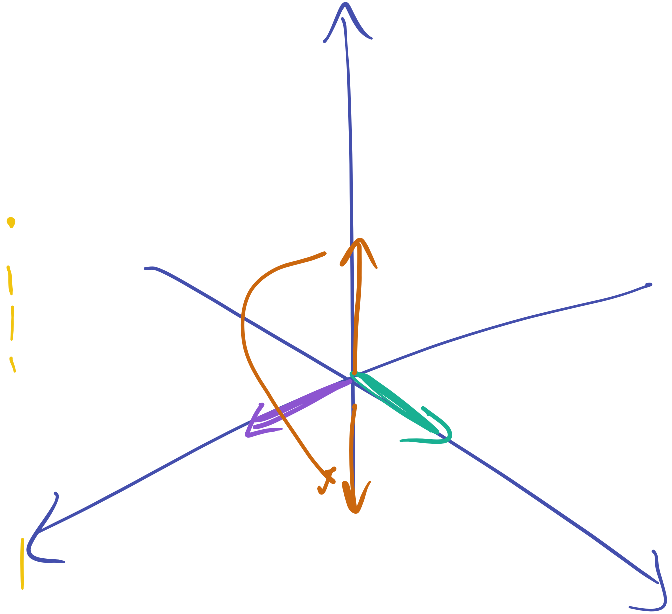
Rotation by $\pi/2$

Typical Exam Questions 3.4

$(3\times 4)(4\times 3)$

- True/False. If A is a 3×4 matrix and B is a 4×3 matrix, then it makes sense to multiply A and B in both orders.
- True/False. If it makes sense to multiply a matrix A by itself, then A must be a square matrix.
- True/False. If A is a non-zero square matrix, then A^2 is a non-zero square matrix.
- True/False. If $A = -I_n$ and B is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the xy-plane and the yz-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix A for projection to the xy-plane in $\mathbb{R}^3$. What is A^2?
- Find the standard matrix A for reflection in the xy-plane in $\mathbb{R}^3$. Is there a matrix B so that $AB = I_3$?

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ $\quad -I \cdot A = -1 \cdot I \cdot A = -A$ TRUE



$$A = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{-1} \end{pmatrix}$$

Q. Is there B

so $AB = I$

Yes $B = A$

$A^2 = I$

Practice #18
2b $T : \mathbb{R}^4 \to \mathbb{R}^k$

- $T\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$ NO.
- For each x in $\mathbb{R}^4$ exactly one y in $\mathbb{R}^k$ so $T(x) = y$. No - Function
- Every v in $\mathbb{R}^k$ is image of at most one x in $\mathbb{R}^4$
  This is defn of 1-1
- Range of T is 4D. Yes - pivot in each col

Midterm 2b #5

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$

$$T(x) = \begin{pmatrix} 2x_1 + 2x_2 \\ -x_1 + 3x_2 \\ x_1 + x_2 \end{pmatrix}$$

Describe the x's

So $T(x) = 0$.

$\mathrm{Nul}\begin{pmatrix} 2 & 2 \\ -1 & 3 \\ 1 & 1 \end{pmatrix}$ 2 pivots

$\rightsquigarrow$ pt. in $\mathbb{R}^2$

Range of T

2 pivots

$\rightsquigarrow$ plane.

2a #17

$$T : \mathbb{R}^3 \to \mathbb{R}^7$$

$$T(e_1) = T(e_2)$$

What is max poss $\underbrace{\text{dim of range}}_{\text{rank}}$.

$$\begin{pmatrix} 1 & 1 & * \\ 2 & 2 & * \\ 3 & 3 & * \\ 4 & 4 & * \\ 5 & 5 & * \\ 6 & 6 & * \\ 7 & 7 & * \end{pmatrix}$$

guess : 2

2b #15

$T: \mathbb{R}^2 \to \mathbb{R}^2$

reflect across x

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$(3\times 2)(2\times 2)$

$U: \mathbb{R}^2 \to \mathbb{R}^3$

$$U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 0 \\ y \end{pmatrix} \begin{matrix} 2x+0y \\ 0x+0y \\ 0x+1y \end{matrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find std matrix. $U \circ T$

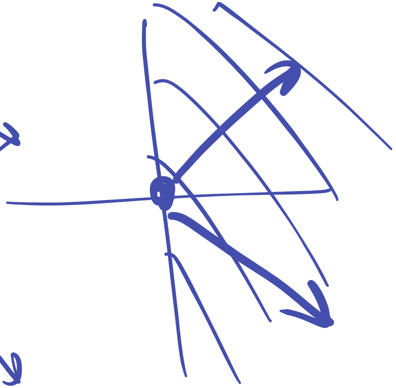
$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Practice 2a
#11 b T/F $T(x) = Ax$ one-to-one.

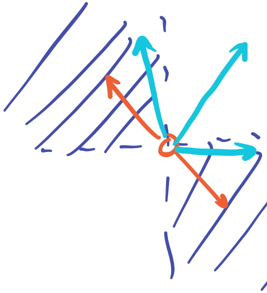
one-to-one and onto: For each b in codom. $T(x) = b$ has exactly one soln.

input

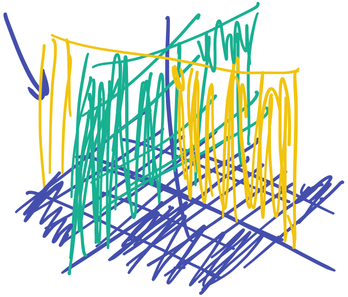
$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a \geq 0 \right\}$$



$$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : ab < 0 \right\}$$



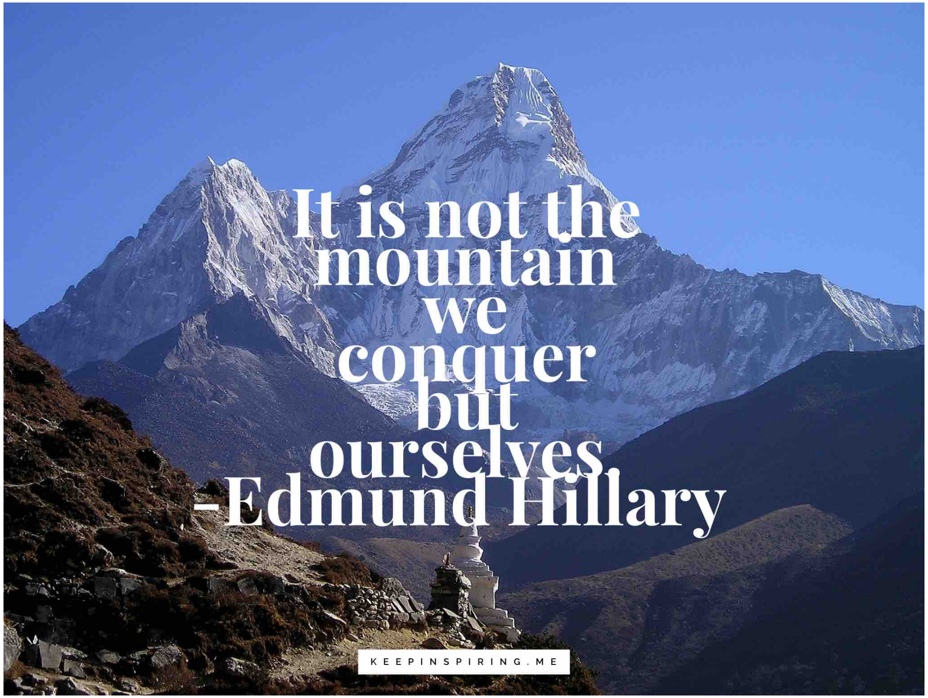
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : abc = 0 \right\}$$



$$u\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 0 \\ y \end{pmatrix}$$

2a #19

Good luck!


It is not the mountain we conquer but ourselves
-Edmund Hillary
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