# Summary of Recent Work

Dan Margalit

In this note I'll discuss two recent projects, one on the hyperelliptic Torelli group and one on geometric models for the mapping class group.

## 1 Hain's Conjecture & Hyperelliptic Torelli Group

Hain told me the following conjecture in 2003.

**Conjecture.** *The hyperelliptic Torelli group $\mathcal{SI}(S_g)$ is generated by Dehn twists about symmetric separating curves.*

Hain's original interest in this conjecture comes from algebraic geometry: it describes the topology of the brach locus of the period mapping from Torelli space to the Siegel upper half-plane. In a series of collaborations, I resolved this conjecture in the affirmative by proving a stronger theorem: the separating curves can be taken to have genera only 1 and 2. I started working on this problem in 2005, first breaking the problem into the following three steps and then executing the steps between 2005 and 2013.

1. Show $\mathcal{SI}(S_g)$ is gen. by reducible elements (w/ Brendle & Putman [5]).
2. Find a Birman exact sequence for $\mathcal{SI}(S_g)$ (w/ Brendle [2]).
3. Reduce to separating curves of genus 1 and 2 (w/ Brendle [3]).

Step 3 was done first. We gave a general algorithm for factoring certain elements of $\mathcal{SI}(S_g)$ into Dehn twists about curves of genus 1 and 2. This was our first evidence that Hain's conjecture might be true.

Step 2 is much harder than the analogous theorem for the full Torelli group; we used a computer experiment to help us guess the right answer. This step also uses ideas from a forthcoming paper with Brendle [4], where we prove the level 4 subgroup of the braid group (as defined via the symplectic representation) is equal to the subgroup generated by squares of Dehn twists.

Our proof of Step 1 is a complete departure from the usual approaches (cf. [6]) and has applications to other contexts [7]. First, Step 1 is equivalent to:

$$\mathrm{Sp}(2g, \mathbb{Z})[2] \cong (\mathrm{PB}_{2g+1})/\mathcal{SI}(S_g)^{red}.$$

where $\mathcal{SI}(S_g)^{red}$ is the subgroup of $\mathcal{SI}(S_g)$ generated by reducible elements. To establish the isomorphism, the key is to construct an action of $\mathrm{Sp}(2g, \mathbb{Z})$ on the right-hand side that agrees with the usual action on the left. But we do not understand the right-hand side geometrically, so we construct the action generator by generator, and then check that all the relations for $\mathrm{Sp}(2g, \mathbb{Z})$ and $\mathrm{PB}_{2g+1}$ are satisfied. One should never hope to define an action this way, but we were able to push the calculation through, and finish off the conjecture.

## 2 Geometric models for the mapping class group

Ivanov proved the famous theorem that the automorphism group of the curve complex $\mathcal{C}(S_g)$ is isomorphic to the extended mapping class group $\mathrm{Mod}(S_g)$. This theorem has many applications, e.g. it implies the abstract commensurator of $\mathrm{Mod}(S_g)$ is isomorphic to $\mathrm{Mod}(S_g)$, and gives a new proof of Royden's theorem that the isometry group of the Teichmüller metric is $\mathrm{Mod}(S_g)$.

Ivanov's work inspired a number of theorems which state that the automorphism group of some particular curve complex or metric space or subgroup of $\mathrm{Mod}(S_g)$ is isomorphic to the extended mapping class group. In response, Ivanov posed the following.

**Metaconjecture.** *Every object naturally associated to a surface $S$ and having a sufficiently rich structure has $\mathrm{Mod}(S)$ as its group of automorphisms. Moreover, this can be proved by a reduction to the theorem about $\mathcal{C}(S)$.*

In a forthcoming paper with Brendle [1] we resolve Ivanov's metaconjecture for a wide class of simplicial complexes. I will concentrate here on a special case that gives the main idea. Let $A$ be a finite list of compact, connected surfaces. Let $\mathcal{D}_A(S_g)$ be the graph whose vertices are isotopy classes of nonseparating subsurfaces of $S_g$ homeomorphic to some element of $A$, and whose edges connect vertices with disjoint representatives. Define $\kappa(A)$ to be the smallest integer so that each element of $A$ can be realized as a subsurface of $S^1_{\kappa(A)}$.

**Theorem.** *Let $A$ be a set of compact, connected surfaces. The natural map*

$$\mathrm{Mod}(S_g) \to \mathrm{Aut}(\mathcal{D}_A(S_g))$$

*is an isomorphism if $g \geq 3\kappa(A) + 1$.*

Our theorem is the first of its kind to treat more than one complex with a single argument. In the spirit of Ivanov's metaconjecture, our proof passes through Ivanov's original theorem (in fact, the proof passes through a sequence of complexes interpolating between $\mathcal{D}_A(S_g)$ and $\mathcal{C}(S_g)$).

The general version of our theorem allows for connected subsurfaces of $S_g$ that are separating in $S_g$. Also, it gives a necessary *and* sufficient condition for the resulting complex to have automorphism group isomorphic to $\mathrm{Mod}(S_g)$. The condition is that the complex admits no exchange automorphisms, that is, automorphisms that swap two vertices and fix every other vertex. We also give a complete description of which complexes $\mathcal{D}_A(S_g)$ have this property; one nontrivial example that does admit exchange automorphisms is the complex $\mathcal{D}_A(S_g)$ where $A$ is the union of the set of subsurfaces of genus one with one boundary component with the set of separating curves of genus one.

Using our theorem we make progress towards the algebraic version of the metaconjecture: any sufficiently rich subgroup of $\mathrm{Mod}(S_g)$ has abstract commensurator isomorphic to $\mathrm{Mod}(S_g)$. A special case is the theorem announced by Bridson–Petttet–Souto that the abstract commensurator of each term of the Johnson filtration (or Magnus filtration) is $\mathrm{Mod}(S_g)$.

# References


[1] T. Brendle and D. Margalit, Combinatorial models for the mapping class group: complexes of connected subsurfaces, `arXiv:1110.1397`, to appear in Michigan Mathematical Journal.

[2] T. Brendle and D. Margalit, Point pushing, homology, and the hyperelliptic involution, `arXiv:1110.1397`, to appear in Michigan Mathematical Journal.

[3] T. Brendle and D. Margalit, Factoring in the hyperelliptic Torelli group, `arXiv:1202.2365`.

[4] T. Brendle and D. Margalit, Congruence subgroups for the braid group, in preparation.

[5] T. Brendle, D. Margalit, and Andrew Putman, Generators for the hyperelliptic Torelli group and the kernel of the Burau representation at $t = -1$, `arXiv:1211.4018`.

[6] A. Hatcher and D. Margalit, Generating the Torelli group, *Enseign. Math* 58 (2012), 165–188.

[7] D. Margalit and A. Putman, Natural presentations for congruence subgroups, in preparation.