Which are one to one / onto?

Poll

Which give one to one-to-one / onto matrix transformations?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

Demo

Demo

Demo

Announcements Oct 6

- Masks $\rightsquigarrow$ Thank you!
- Quiz 2.5-3.1 (not 2.8) Friday
- No class Monday!
- WeBWorK 3.2 & 3.3 due Wednesday nite
- Midterm 2 Oct 20 8–9:15p

- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 3.2

One-to-one and onto transformations

Which are one to one / onto?

$f(x)=x^2$ not 1-1 b/c

$-5, 5$

have same output

pivot in every...

Poll

Which give one to one-to-one / onto matrix transformations?

...col ...row

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

2 pivots — not 1-1, onto

2 pivots — one-to-one, not onto

1 pivot — neither.

Demo

Demo

Demo

Section 3.3

Linear Transformations

Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation

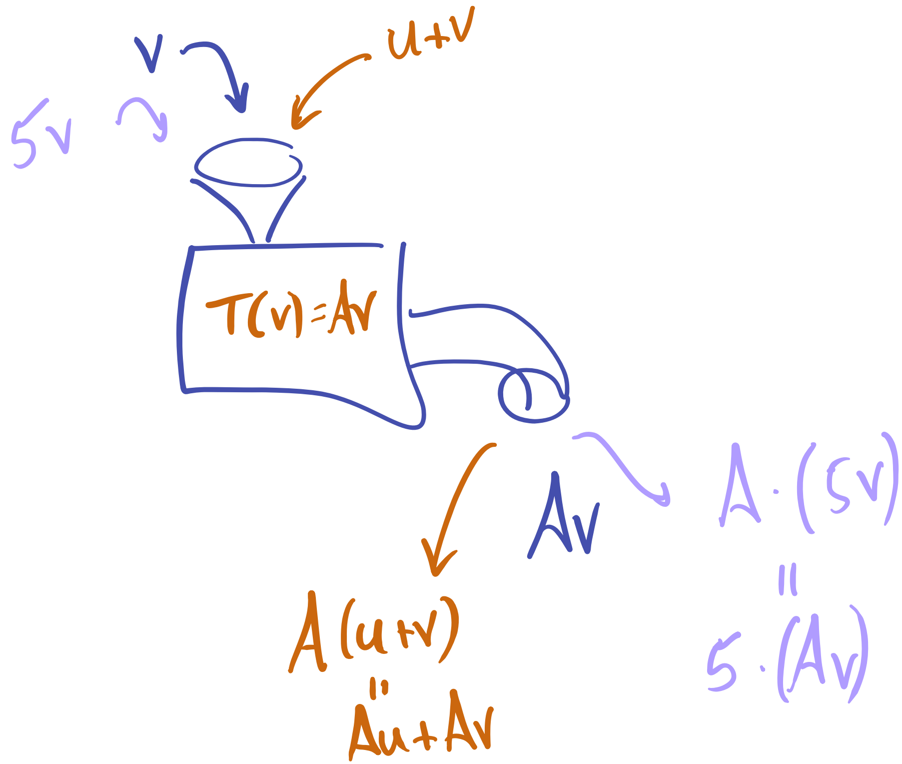
Spoiler

Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

First examples: matrix transformations.



Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

Notice that $T(0) = 0$. *Why?* $T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0$

We have the standard basis vectors for $\mathbb{R}^n$:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \end{aligned}$$

Every vector is a lin combo of the e_i.

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. *Why?*

What is $T(5e_1 - 7e_2) = T(5e_1) - T(7e_2) = 5\,T(e_1) - 7\,T(e_2)$

In engineering, this is called the principle of superposition.

If $T(e_1) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $T(e_2) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ then $T\begin{pmatrix} 5 \\ -7 \end{pmatrix} = 5\begin{pmatrix} 3 \\ -1 \end{pmatrix} - 7\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

Which are linear transformations?

And why?

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$ Yes. (See below)

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}\right) = T\begin{pmatrix} x+u \\ y+v \end{pmatrix} = \begin{pmatrix} x+u+y+v \\ y+v \\ (x+u)-(y+v) \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y+1 \\ y \\ x-y \end{pmatrix}$$ No

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y \\ x-y \end{pmatrix}$$ No

$$T\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{T(v)}$$

$T(cv)$ $c\,T(v)$

$$T\left(\underbrace{2}_{c}\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v}\right) = T\underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{cv} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \neq 2 \cdot T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).

Linear transformations

Which properties of a linear transformation fail for this function $T : \mathbb{R}^2 \to \mathbb{R}^2$?

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ |y| \end{pmatrix}$$

- $T(cv) = cT(v)$? No

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T\left(-1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = T\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \neq -1 \cdot T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- $T(u+v) = T(u) + T(v)$? No.

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} + T\begin{pmatrix} 1 \\ 2 \end{pmatrix} \overset{?}{=} T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

not equal

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix A so that

exactly one

$$T(v) = Av$$

for all v in $\mathbb{R}^n$.

The matrix for a linear transformation is called the standard matrix.

How to find it?

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

The identity

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$

Like $f(x) = x$ from calc.

What is the standard matrix?

This standard matrix is called I_n or I.

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$e_1 = T(e_1)$ $\quad e_2 \quad e_3 \quad e_4$

Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

What is the standard matrix for T?

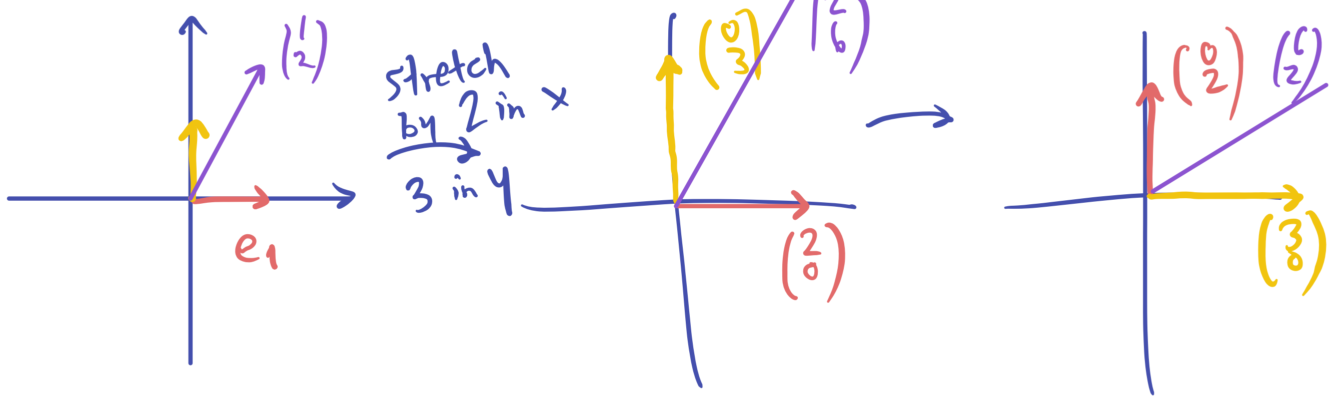
$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.



$$\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

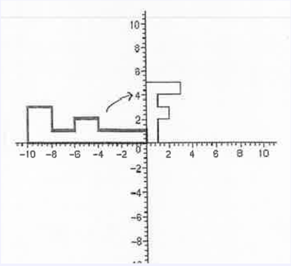
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion

Discussion Question

Find a matrix that does this.



▸ Transformation Challenge

Summary of Section 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.

Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

  What is

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix}?$$

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the z-axis by π and then scales by 2.
- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$$

  Is this a linear transformation? If so, what is the standard matrix for T?
- Is the identity transformation one-to-one?