Chapter 3

Determinants

Section 3.1

Introduction to Determinants

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

We have said most of what we are going to say about the first problem. We are now aiming towards the second problem.

Outline

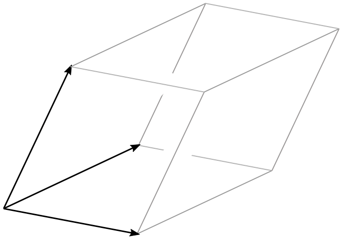
- The idea of the determinant
- A formula for the determinant
- More formulas for the determinant
- Determinants of triangular matrices
- A formula for the inverse of a matrix

The idea of determinant

Let A be an $n \times n$ matrix.

$\rightsquigarrow$ n vectors in $\mathbb{R}^n$

$\rightsquigarrow$ a parallelepiped P:



$\rightsquigarrow$ volume

Idea: A is invertible $\Leftrightarrow$ the volume of P is...

The idea of determinant

Idea: A is invertible $\Leftrightarrow$ the volume of P is nonzero

The determinant is a number $\det(A)$ whose absolute value is the volume of P.

For 2×2 matrices we already have a formula:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

This is the (signed) area of the parallelogram spanned by the columns. Try it!

(What does the sign of the determinant mean?)

The idea of determinant

Let's do a reality check. We wanted:

$$A \text{ is invertible} \Leftrightarrow \det(A) \neq 0$$

Let's row reduce:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

A formula for the determinant

We will give a recursive formula.

First some terminology:

$$\begin{aligned} A_{ij} &= ij\text{th minor of } A \\ &= (n-1)\times(n-1) \text{ matrix obtained by deleting the } i\text{th row and } j\text{th column} \end{aligned}$$

$$\begin{aligned} C_{ij} &= (-1)^{i+j}\det(A_{ij}) \\ &= ij\text{th cofactor of } A \end{aligned}$$

Finally:

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{1j}C_{1j} \\ &= \end{aligned}$$

A formula for the determinant

The recursive formula:

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$$

Need to start somewhere...

1×1 matrices

$$\det(a_{11}) =$$

2×2 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

A formula for the determinant

3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

Determinants

Compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

A formula for the determinant

Another formula for 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Use this formula to compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Expanding across other rows and columns

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{ij} C_{ij} \text{ for any fixed } i \\ &= \sum_{i=1}^{n} a_{ij} C_{ij} \text{ for any fixed } j \end{aligned}$$

Compute:

$$\det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute:

$$\det \begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

A formula for the inverse

(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \ldots & C_{nn} \end{pmatrix} \\ &= \frac{1}{\det(A)} (C_{ij})^T \end{aligned}$$

Check that these agree!

The proof uses Cramer's rule (see the notes on the course home page).

A formula for the inverse

(from Section 3.3)

$n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix}$$

$$= \frac{1}{\det(A)} (C_{ij})^T$$

Compute:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$