Announcements April 13

- WebWork 6.1 and 6.2 due Thursday
- Quiz on 6.1 and 6.2 on Friday
- Final Exam Wed May 4 8:00-10:50 (Sec H) and Mon May 2 2:50-5:40 (Sec J)
- Office Hours Tue 2-3 and Wed 2-3
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
    - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
    - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
    - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

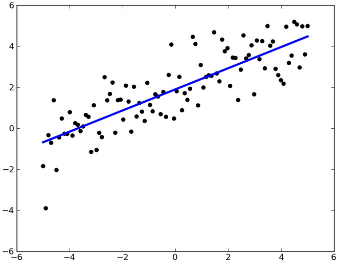
Section 6.2

Orthogonal Sets

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality. Last time we saw how to project onto a line. Now we will project onto higher-dimensional planes.

Outline

- Orthogonal bases
- A formula for projecting onto any subspace
- Breaking a vector into components

Orthogonal Sets

A set of vectors is orthogonal if each pair of vectors is orthogonal. It is orthonormal if in addition each vector is a unit vector.

Example.

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Fact. An orthogonal set of nonzero vectors is linearly independent.

Why?

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i w_i$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

In other words:

$$y = \sum_{i=1}^{k} \mathrm{proj}_{\langle u_i \rangle}(y)$$

Why?

What happens if y is not in W? The formula still works! But it gives the projection of y to W.

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i w_i$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Problem. Find the B-coordinates of $(6, 1)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}$$

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i w_i$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Problem. Find the B-coordinates of $(6, 1, -8)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

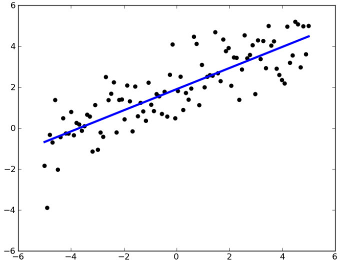
Section 6.3

Orthogonal projections

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Outline

- Projecting onto any subspace: a formula
- Projections and best possible solutions

Orthogonal projection

Projecting onto a line

Recall:

$$\mathrm{proj}_{\langle u \rangle}(v) = \frac{v \cdot u}{u \cdot u} u$$

Can use this to break v into two components:

$$v = v_L + v_{L^\perp}$$

where $L = \langle u \rangle$ and v_L is $\mathrm{proj}_{\langle u \rangle}(v)$ and $v_{L^\perp} = v - v_L$.

Problem. Let $u = (1, 2)$ and $L = \langle u \rangle$. Let $v = (1, 1)$. Write v as $v_L + v_{L^\perp}$.

Next: replace L with any subspace.

Orthogonal projection

Projecting onto any subspace

Theorem. Say W a subspace of $\mathbb{R}^n$ and y in $\mathbb{R}^n$. We can write y uniquely as:

$$y = y_W + y_{W^\perp}$$

with y_W in W and $y_{W^\perp}$ in $W^\perp$.

Moreover, if $\{u_1, \ldots, u_k\}$ is an orthogonal basis for W then

$$y_W = \sum_{i=1}^{k} \frac{y \cdot u_i}{u_i \cdot u_i} u_i$$

This y_W is $\mathrm{proj}_W(y)$.

Problem. Let $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $y = e_1$. Find y_W.

Find A so that T_A is orthogonal projection onto

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Orthogonal projection

Poll

Suppose T_A is orthogonal projection onto a plane in $\mathbb{R}^3$. What is A^2 equal to?

1. A
2. A^{-1}
3. $-A$
4. 0
5. I_n
6. $-I_n$

Best approximation

W = subspace of $\mathbb{R}^n$

Fact. The projection y_W is the point in W closest to y. In other words:

$$\|y - y_w\| < \|y - w\|$$

for any w in W other than y_w.

Why?

Best approximation

Problem. Find the distance from e_1 to $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.