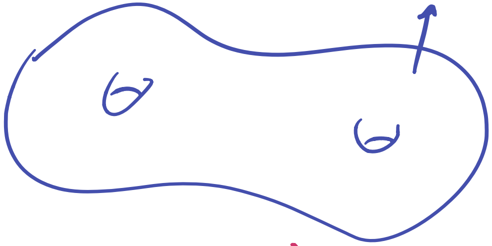
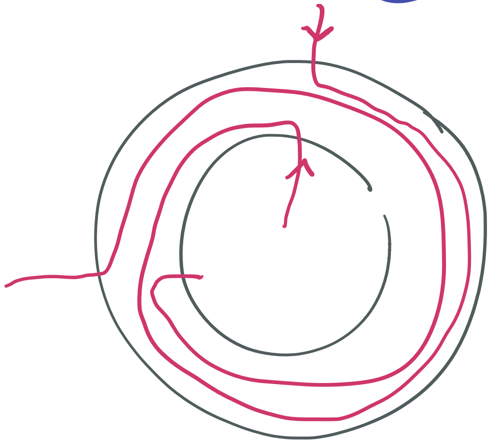
Dehn twists





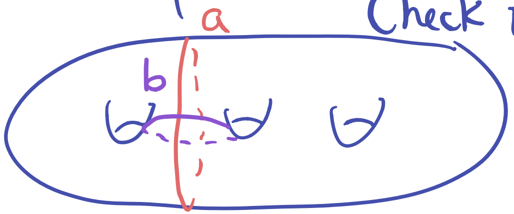
Prop. Dehn twist have ∞ order.

If a nonsep the T_a^k acts nontrivially on $H_1(S_g)$ $k \neq 0$

hence $T_a^k \neq id$.

But for a sep. T_a^k acts trivially. Can draw $T_a^k(b)$.

Check $i(T_a^k(b), b) \neq 0$



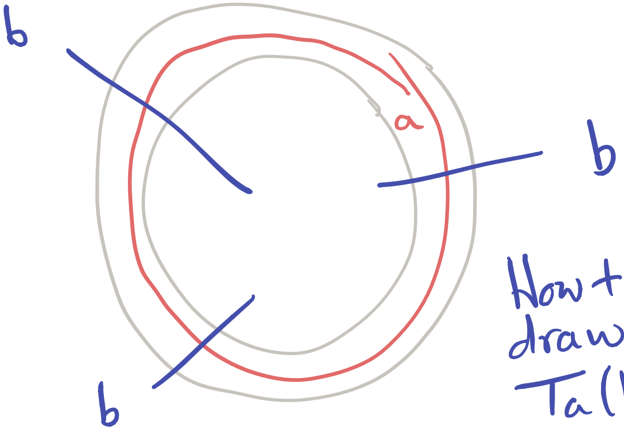
$\Rightarrow T_a^k(b) \neq b$

$\Rightarrow T_a^k \neq id$

Prop. $i(T_a^k(b), b) = |k|\, i(a,b)^2$

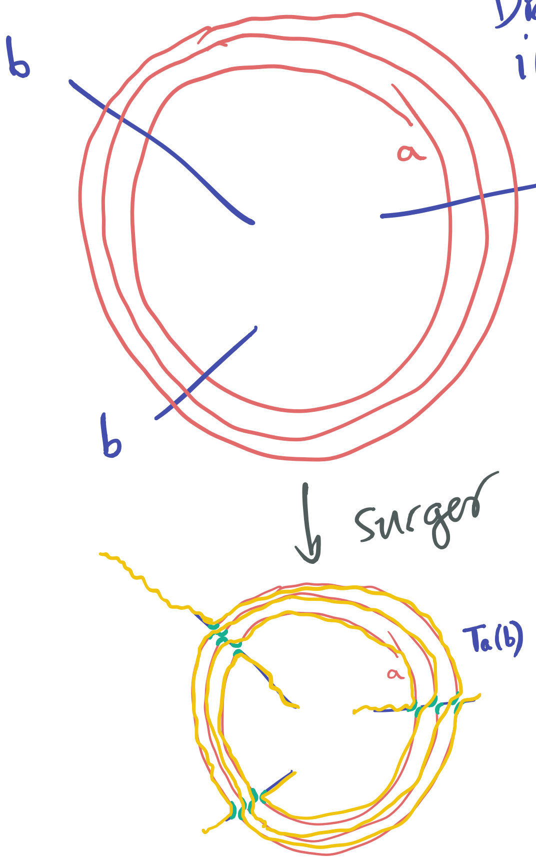
Cor. $|T_a| = \infty$.

Need: Surgery description of Dehn twists



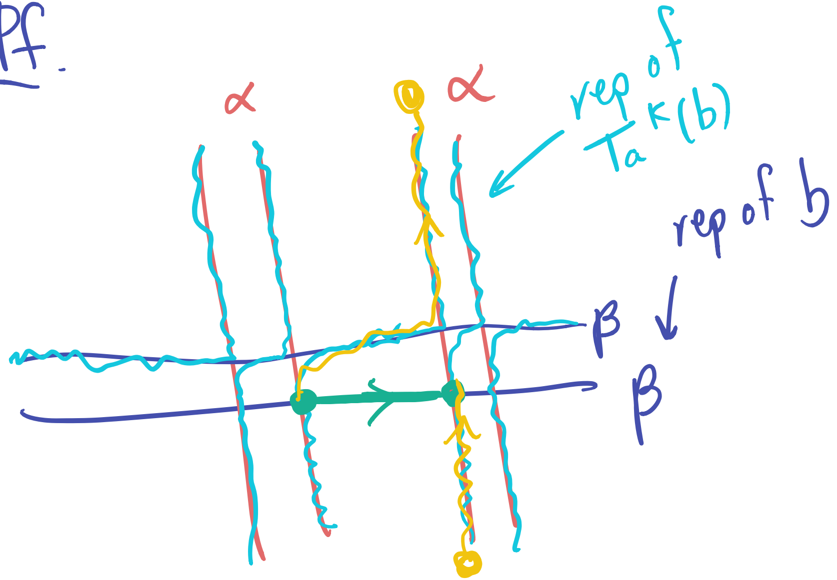
How to draw $T_a(b)$?

Draw $i(a,b)$ copies of a



Prop. $i(T_a^k(b), b) = |k|\, i(a,b)^2$

Pf.

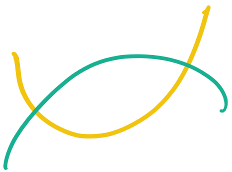


$i(a,b) = 2 \qquad k = 1.$

Our rep of $T_a^k(b)$ intersects β $|k|\, i(a,b)^2$ times.

Remains to check:

No bigons.



∎

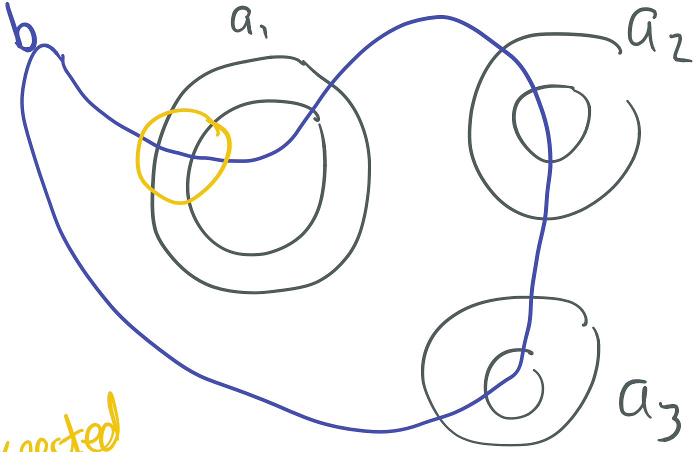
Prop. $a_1, \dots, a_n$ $\quad i(a_i, a_j) = 0$.

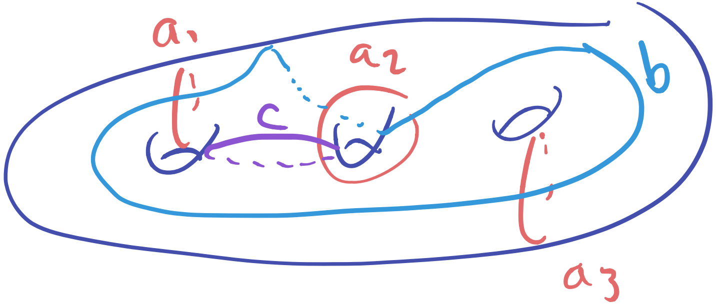
$e_i \geq 0$.

$M = \prod T_{a_i}^{e_i}$ multitwist

$$\left| i(M(b), c) - \boxed{\sum_{i=1}^{n} e_i\, i(a_i, b)\, i(a_i, c)} \right| \leq i(b, c)$$

expected # of intersections





Note: for $b = c$ and $n = 1$ get last prop.

Q Example where expected value is not right.

Prop. $a_1, \dots, a_n$ $\quad i(a_i, a_j) = 0$.

$e_i \geq 0$.

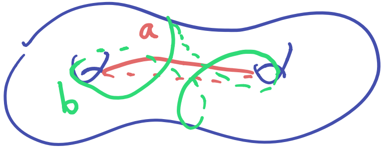
$M = \prod T_{a_i}^{e_i}$ multitwist

$$\left| i(M(b), c) - \boxed{\sum_{i=1}^{n} e_i \, i(a_i, b) \, i(a_i, c)} \right| \leq i(b, c)$$

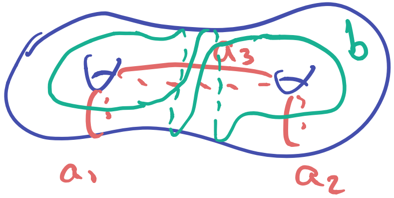
Cor. $\exists$ pair of filling curves on any S. with $\chi(S) < 0$.

$\{a, b\}$ filling if $\max\{i(a,c), i(b,c)\} > 0$ $\forall c$

example



Pf of Cor. Choose pants decomp. of S



* Find a b s.t. $i(a_i, b) > 0$ $\forall i$

Let $a = \left(\prod T_{a_i}\right)(b)$

By prop, a & b are filling.

Basic Facts

Fact 1 $T_a = T_b \iff a = b$

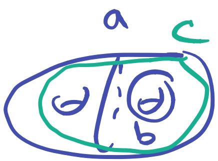
Pf. Find c s.t. $i(a,c) \neq 0$

$i(b,c) = 0$

Then $i(T_a(c),c) = i(a,c)^2 \neq 0$

$i(T_b(c),c) = i(c,c)^2 = 0.$

How to find c?



Case 1. $i(a,b) > 0$ take $c = b$

Case 2 $i(a,b) = 0$. Use change of coords.

Fact 2 $f T_a f^{-1} = T_{f(a)}$

commutes

Fact 3 $(f \leftrightarrow T_a) \iff f(a) = a.$

Pf. $\Longrightarrow$ Fact 1 + Fact 2.

$\Longleftarrow$ Fact 2

Fact 4. a,b non sep

Then T_a conj to T_b in MCG

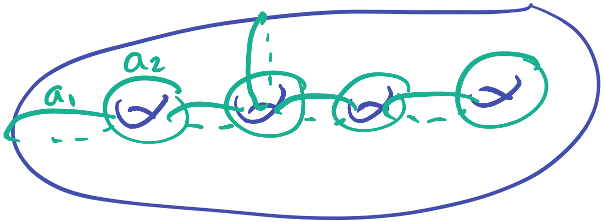
Pf. Fact 2 + Change of coords.

Fact 5. $i(a,b) = 0 \iff (T_a \leftrightarrow T_b)$

Pf. Use first Prop & Fact 2

Thm. For $g \geq 3$, $Z(\mathrm{Mod}(S_g)) = 1$.

Pf. Use the Alex. system



The graph $\Gamma = \cup a_i$ has no nontrivial autos.

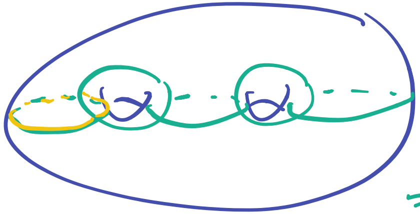
So Alex Meth $\Rightarrow f = id$.

$f \in Z(\mathrm{Mod}(S_g))$

$\Rightarrow f(a_i) = a_i \ \forall i$

(Fact 3)

What about $g = 1, 2$?



These generate. $\Rightarrow$ hyp inv. in central.

$Z(\mathrm{Mod}(S_2)) = \mathbb{Z}/2$

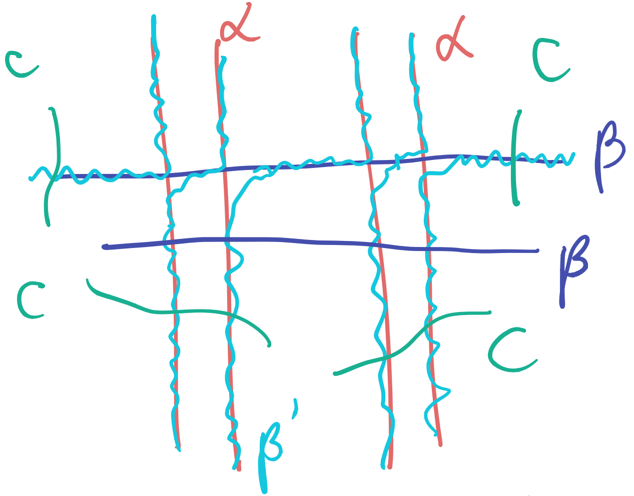
Prop. $a_1, \ldots, a_n$ $\quad i(a_i, a_j) = 0$.

$e_i \geq 0$.

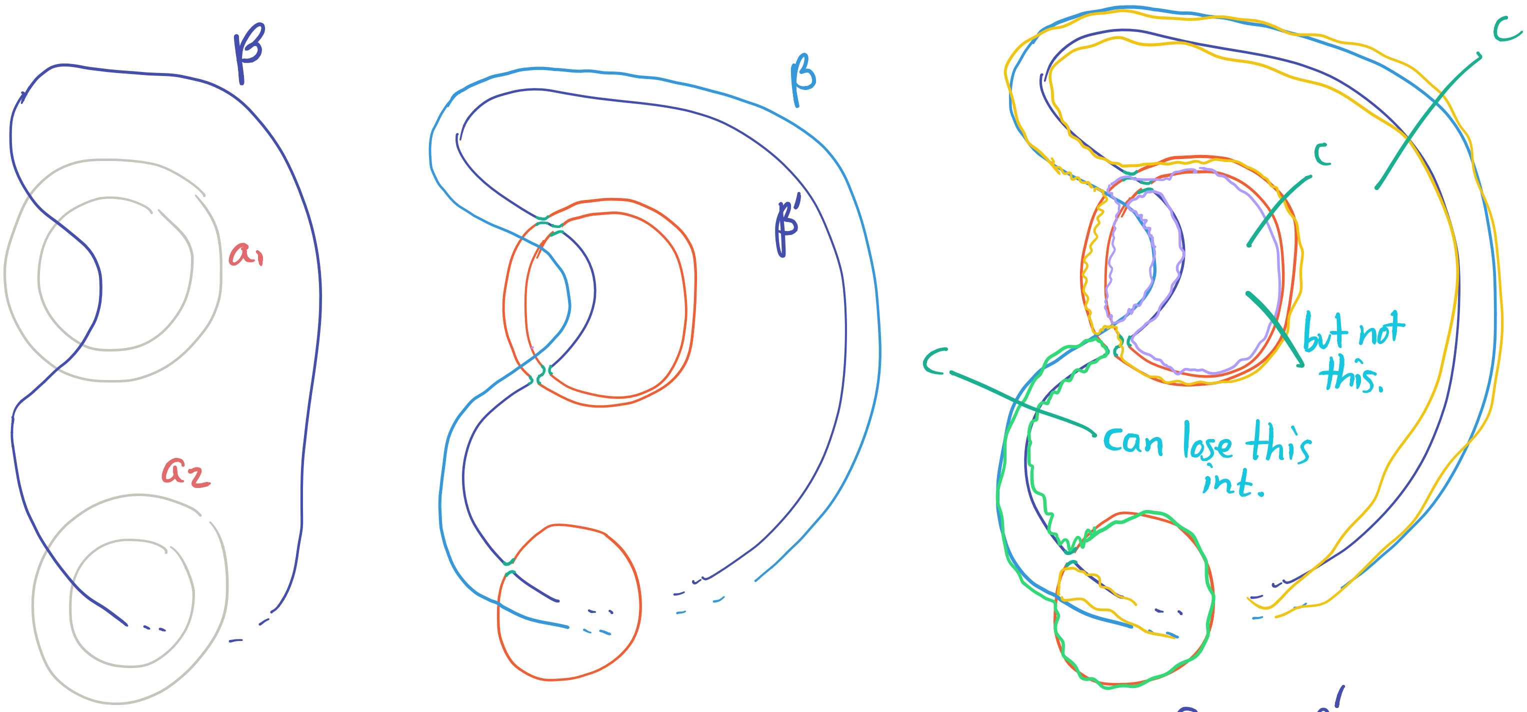
$M = \prod T_{a_i}^{e_i}$ multitwist

$$\left| i(M(b), c) - \sum_{i=1}^{n} e_i\, i(a_i, b)\, i(a_i, c) \right| \leq i(b,c)$$

Pf. Make a rep β' of $M(b)$ as before:



Key obs: $\beta \cup \beta'$ can be decomp. as $\sum e_i\, i(a_i, b)$ copies of each a_i



Zig-zag: Turn left on β'

right on β

As above: $\boxed{\beta \cup \beta' \text{ is a bunch of copies of } a_i :}$

$\forall i : \; e_i \, i(a_i, b)$ copies of a_i

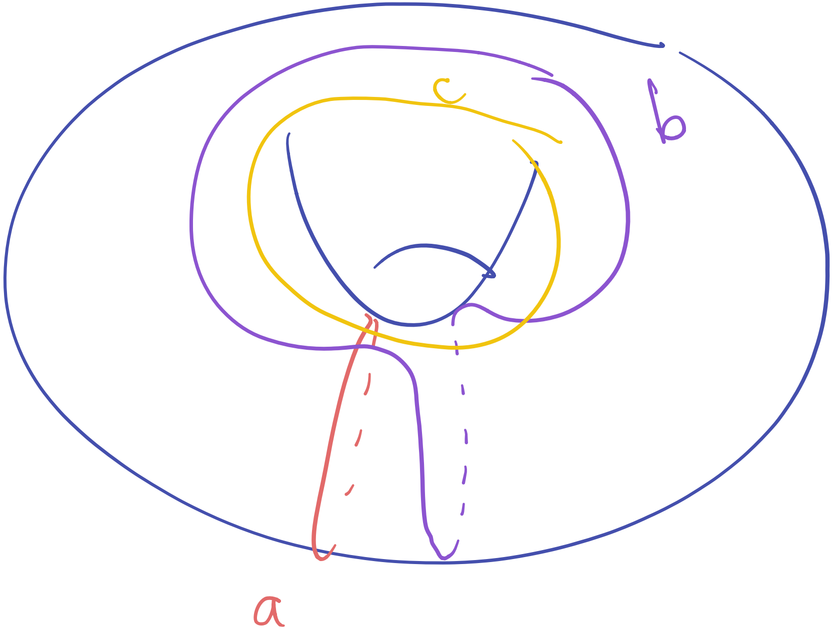
rep of c.

$$\sum e_i \, i(a_i, b)\, i(a_i, c) \leq \left| (\beta \cup \beta') \cap \gamma \right|$$

$$= \underbrace{i(M(b), c) + i(b, c)}_{\text{\# of int's you see in pic.}}$$

by fact at top

Need to prove other ineq.



$i(T_a(b), c) = 0.$

<u>expected</u>

$1 \cdot 1 \cdot 1 = 1.$