Algebraic Topology

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What is algebraic topology?

$$\boxed{\text{Space}} \longrightarrow \boxed{\text{Group}}$$

$X \rightleftarrows \pi_1(X)$ fundamental group

$X \longrightarrow H_k(X)$ k-th homology group

$X \longrightarrow H^k(X)$ k-th cohomology group

What kinds of questions does it answer?

1. When are two spaces the same (or not)?

   e.g. $\mathbb{R}^m \not\cong \mathbb{R}^n$

   what about: $\mathbb{R}^3 -$  vs. $\mathbb{R}^3 -$ 

2. Embeddings

   What is smallest N s.t. a given manifold embeds in $\mathbb{R}^N$?

   Unsolved for $\mathbb{RP}^n$.

③ Fixed point theorems

Brouwer fixed pt theorem: every $D^2 \to D^2$ has a fixed pt.

Borsuk-Ulam theorem.

④ Actions

Which finite groups act freely on S^n?
(known in some cases)
Note: $\mathbb{Z}/n\mathbb{Z} \circlearrowright S^{2k-1}$ $\forall n, k$.

⑤ Sections

What is the largest k s.t. a given manifold admits a continuously varying k-plane field?

Hairy ball theorem.

⑥ Group theory

Every subgroup of a free group is free.
$[F_n, F_n]$ is not finitely generated.
Braid groups are torsion free.

⑦ Algebra

Fundamental theorem of algebra (this week!)

<u>Basic idea of homology</u>

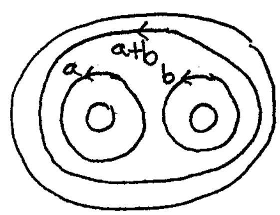
$H_k(X)$ = abelian group of k-dim holes in X

↳ prevents a k-sphere from collapsing

<u>computable</u>

example: X = pair of pants 

$H_1(X) \cong \mathbb{Z}^2$



$H^k(X)$ is dual to $H_k(X)$

$\rightsquigarrow$ consists of functions $H_k(X) \to \mathbb{Z}$

Big Goal: Poincaré Duality

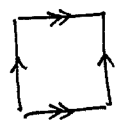
For X = n-manifold $\quad H^k(X) \cong H_{n-k}(X)$

More precisely: the functions in H^k look like "intersect with this fixed element of H_{n-k}"

What do we mean by a space?

Cell complexes aka CW complexes

C = closure finiteness (closure of open cell hits finitely many open cells)
W = weak topology

e.g. 

Quotient topology: $U \subseteq X/\sim$ is open iff its preimage in X is open.

We build CW complexes inductively

(i) Start with a discrete set of points X^0. The points are regarded as 0-cells.

(ii) Inductively form <u>n-skeleton</u> X^n from X^{n-1} by attaching n-cells D^n_α via
$$\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$$

X^n has quotient topology.

(iii) Either stop at a finite stage, or continue indefinitely.

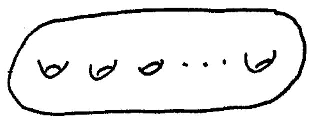
In latter case, use weak topology: a set is open iff its intersection with each cell is open.

$\dim(X)$ = sup of dim of cells

Examples of CW Complexes

① 1-dim CW complexes are graphs.

② $(4g+2)$-gon with opposite sides identified



③ $S^n = e^0 \cup e^n$ $\qquad$ $e^i = i$-cell.

④ $\mathbb{RP}^n$ = space of lines in $\mathbb{R}^{n+1}$
$= e^0 \cup e^1 \cup \cdots \cup e^n$

To see this: $\mathbb{RP}^n = S^n /$ antipodal map
$= D^n /$ antipodal map on $\partial D^n = S^{n-1}$

So on ∂D^n see $\mathbb{RP}^{n-1}$, and we glue D^n to that.

⑤ $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^n$ $\qquad$ exercise.

Subcomplexes

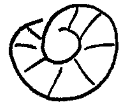
Subcomplex = closed ~~subsp~~ union of cells.

A subcomplex of a CW complex is a CW complex.

example: k-skeleton.

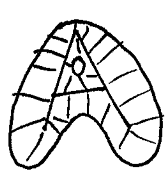
Equivalence of Spaces

Intuition: Two spaces are equivalent if one can be deformed into the other

 $\sim$ A  $\sim$ O

Special case: A <u>deformation</u> <u>retraction</u> $X \to A$ is a continuous family
$$\{ f_t : X \to X \mid t \in I \}$$
s.t.
$$f_0 = id$$
$$f_1(X) = A$$
$$f_t|_A = id \quad \forall\, t.$$
Continuous means
$$X \times I \to X$$
$$(x,t) \mapsto f_t(x)$$
is continuous.

Example: Given $f: X \to Y$, the <u>mapping</u> <u>cylinder</u> is
$$M_f = (X \times I) \amalg Y \,/\!\sim$$
where $(x,1) \sim f(x)$

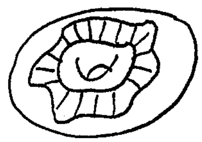
e.g.  X = boundary
Y = core

Fact: M_f deformation retracts to Y.

Homotopy Equivalence

A *homotopy* is a continuous family
$\{f_t : X \to Y \mid t \in I\}$

examples: deformation retraction



A map $f: X \to Y$ is a *homotopy equivalence*
if there is a $g: Y \to X$ such that
$fg \simeq id$ and $gf \simeq id$
$\uparrow$ homotopic

Say: X & Y are homotopy equivalent, or have the same homotopy type. $X \simeq Y$

Exercise: This is an equivalence relation.

Fact: If A is a deformation retract of X, then $X \simeq A$

Exercise:  all homotopy equiv.

Exercise: $\mathbb{R}^n \simeq *$ Say $\mathbb{R}^n$ is contractible.

Read: House with 2 rooms, Hatcher p. 4.