# Parts II & III

$\mathrm{Mod}(S) \circlearrowright \mathrm{Teich}(S)$

$\parallel$

space of hyp metrics on $S$ / isotopy

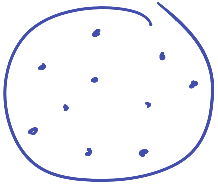
This action tells us about **both** $\mathrm{Mod}(S)$ & $\mathrm{Teich}(S)$

for example:

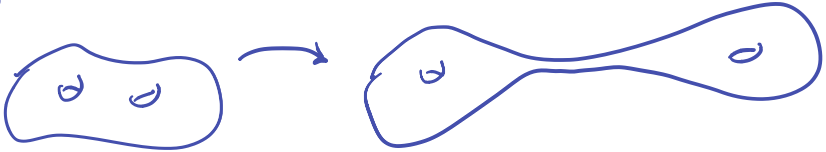
- $\mathrm{Isom}\,\mathrm{Teich}(S) \cong \mathrm{Mod}^{\pm}(S)$
- Nielsen-Thurston classification for elements of $\mathrm{Mod}(S)$

This is geometric gp thy.

Teich(S)



Teich metric.

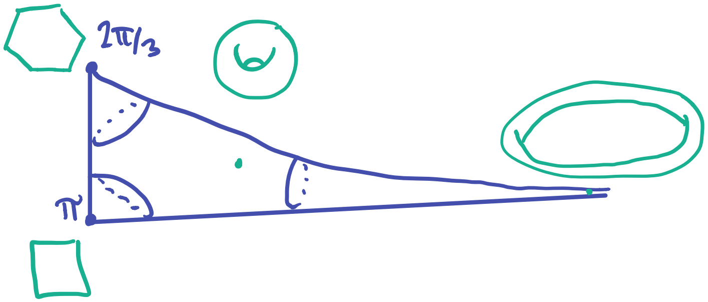


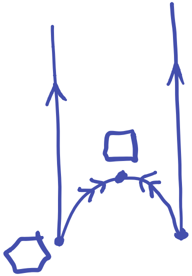
# Moduli space

$\chi(S) < 0$

$$\mathcal{M}(S) = \{\text{hyp metrics}\}/\text{isometry}.$$
$$= \{\text{complex str's}\}/\sim$$
$$= \{\text{algebraic strs}\}/\sim$$
$$= \{\text{conformal strs}\}/\sim$$

$$\mathcal{M}(T^2) = \{\text{unit area Eucl. metrics}\}/_{\text{isom.}}$$

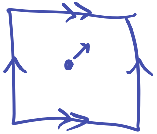


$SL_2\mathbb{Z} \curvearrowright \mathbb{H}^2$

# Teichmüller space

(orbifold) univ. cover of $\mathcal{M}(S)$.



$\mu$ = Eucl. metric

$\varphi \in \mathrm{Diff}_0(T^2)$

$\varphi^*(\mu)$ is a different Eucl. metric on $T^2$, isometric to $\mu$ via $\varphi$.

$$\mathrm{Teich}(S) = \{\text{hyp. metrics}\}/\text{isotopy}$$

$$= \{\text{hyp. metrics}\}/\mathrm{Diff}_0(S)$$

isotopic to id.

(action is pullback)

$$= \{(X,\varphi) : X \text{ hyp surf. } \varphi : S \to X \text{ diffeo}\}/\sim$$

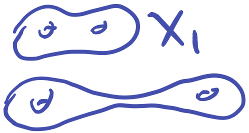
marked surface

$(X_1,\varphi_1) \sim (X_2,\varphi_2)$ if $\exists$ isometry $I : X_1 \to X_2$ s.t

$$\begin{tikzcd} & X_1 \arrow[d, "I"] \\ S \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"'] & X_2 \end{tikzcd}$$

commute up to isotopy.

$S$ = top surface, fixed forever $\xrightarrow{\varphi_1}$ $X_1$, $\xrightarrow{\varphi_2}$ $X_2$



# The torus

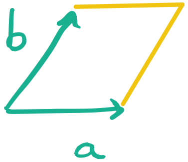
$$\mathrm{Teich}(T^2) = \{\text{Eucl. metrics}\} / \begin{smallmatrix}\text{scale}\\ \text{isometry}\end{smallmatrix}$$
$$= \{(X, \varphi)\}/\sim$$

**Prop.** $\mathrm{Teich}(T^2) \longleftrightarrow \mathbb{H}^2$

**Pf.** $\mathrm{Teich}(T^2) \longleftrightarrow$ (marked lattices in $\mathbb{R}^2$)

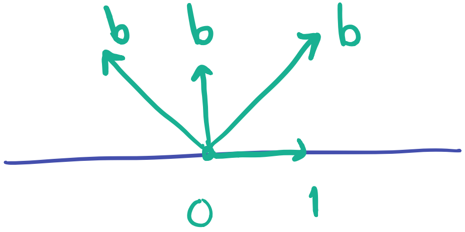
$\longleftrightarrow$ marked parallelograms / $\begin{smallmatrix}\text{scale}\\ \text{isometry}\end{smallmatrix}$

Why? $X =$ 


$\xrightarrow{\text{cut open}}$ 


Scale so $a = 1 \in \mathbb{C}$

reflect over $\mathbb{R}$ so $\mathrm{im}\, b > 0$.



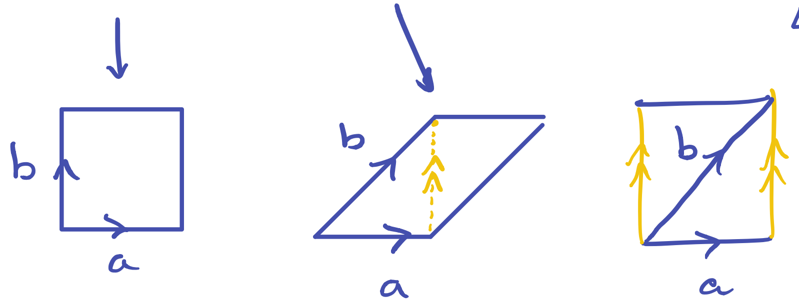

$\square$

**Prop** $\rightsquigarrow$ topology on $\mathrm{Teich}(T^2)$.

We'll see: Teich metric is hyp metric.

# Example tori

① $i$ vs. $i+1$



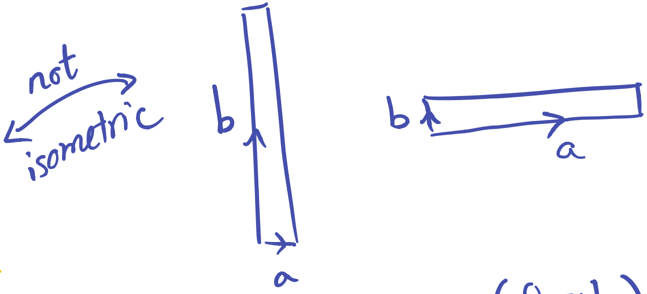

isometric! via... $T_a$

same pt in $\mathcal{M}(S)$

different in $\mathrm{Teich}(S)$

$$\ell_i(b) = 1 \qquad \ell_{i+1}(b) = \sqrt{2}$$

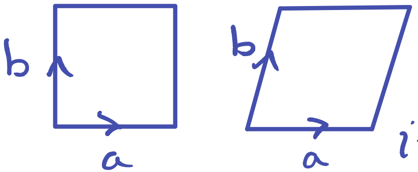
② $ni$ vs $i/n$




isometric via $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

③ $i$ vs $i+\varepsilon$

not isometric!



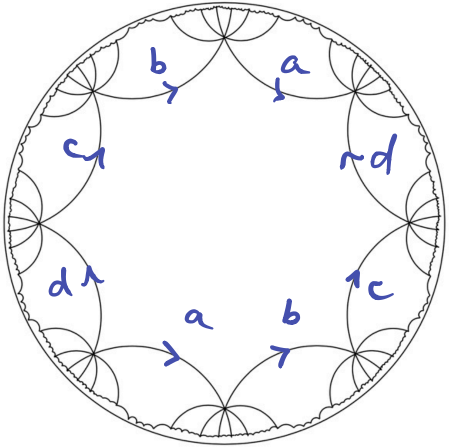

length spectra:

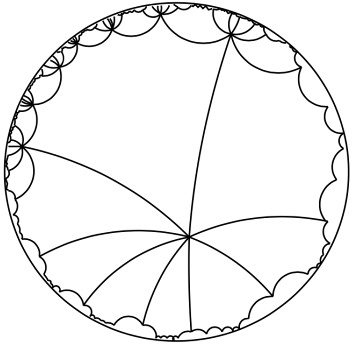
$i$: $1, 1, \sqrt{2}, \sqrt{2}, \ldots$

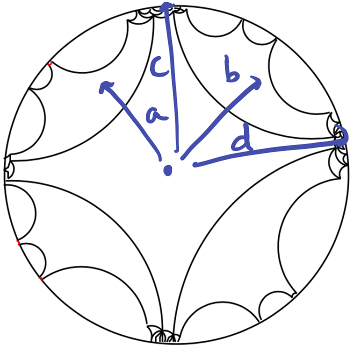
$i+\varepsilon$: $1, 1+\varepsilon', \ldots$

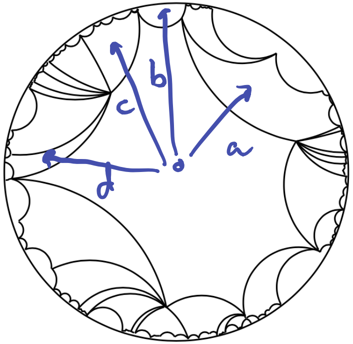
Some points in Teich$(S_2)$

Marked octagons / isometry of $\mathbb{H}^2$





# Length functions

For a curve (isotopy class) in $S$:

$$\ell_a : \mathrm{Teich}(S) \to \mathbb{R}$$
$$X \longmapsto \ell_X(a)$$

" length of $a$ in $X$-metric.

length of the geodesic

(no such map for $\mathcal{M}(S)$).

Let $\mathcal{S} = \{\text{curves in } S\}/\text{isotopy}$

Will show: $\ell : \mathrm{Teich}(S) \to \mathbb{R}^{\mathcal{S}}$ injective.

(actually: lengths of $6g-5$ curves determine the metric)

## The algebraic topology

$DF(\pi_1(S_g), PSL_2\mathbb{R})$

"

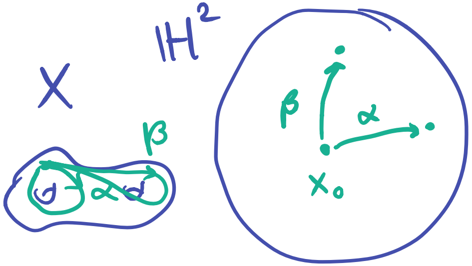
discrete faithful reps

$\pi_1(S_g) \longrightarrow PSL_2\mathbb{R}$

"

cov space actions

$\pi_1(S_g) \longrightarrow \mathrm{Isom}^+\mathbb{H}^2$




Have:

$\mathrm{Teich}(S_g) \longleftrightarrow DF(\pi_1(S_g), PSL_2\mathbb{R}) / PGL_2\mathbb{R}$

via deck gp action

conjugation.

Like torus case:

$\mathrm{Teich}(T^2) \longleftrightarrow DF(\mathbb{Z}^2, \mathrm{Isom}\,\mathbb{E}^2) / \mathrm{Isom}^{\pm}\mathbb{E}^2$

$DF(\pi_1(S_g), PSL_2\mathbb{R}) / PGL_2\mathbb{R}$ has a natural topology from $(PSL_2\mathbb{R})^{2g}$