Section 2.4

Solution Sets

Outline

- Understand the geometric relationship between the solutions to $Ax = b$ and $Ax = 0$
- Understand the relationship between solutions to $Ax = b$ and spans
- Learn the parametric vector form for solutions to $Ax = b$

Homogeneous systems

Solving $Ax = b$ is easiest when $b = 0$. Such equations are called homogeneous.

Homogenous systems are always consistent. *Why?*

When does $Ax = 0$ have a nonzero/nontrivial solution?

If there are k-free variables and n total variables, then the solution is a k-dimensional plane through the origin in $\mathbb{R}^n$. In particular it is a span.

Parametric Vector Forms for Solutions

Homogeneous case

Solve the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We already know the parametric form:

$$\begin{aligned} x_1 &= 8x_3 + 7x_4 \\ x_2 &= -4x_3 - 3x_4 \\ x_3 &= x_3 \quad \text{(free)} \\ x_4 &= x_4 \quad \text{(free)} \end{aligned}$$

We can also write this in parametric vector form:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Or we can write the solution as a span: $\mathrm{Span}\{(8, -4, 1, 0), (7, -3, 0, 1)\}$.

Parametric Vector Forms for Solutions

Homogeneous case

Find the parametric vector form of the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

Variables, equations, and dimension

Poll

For $b \neq 0$, the solutions to $Ax = b$ are...

1. always a span
2. sometimes a span
3. never a span

Nonhomogeneous Systems

Suppose $Ax = b$ and $b \neq 0$.

As before, we can find the parametric vector form for the solution in terms of free variables.

What is the difference?

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form of the solution to $Ax = b$ where:

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 3 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & -13 \\ 0 & 1 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We already know the parametric form:

$$\begin{aligned} x_1 &= -13 + 8x_3 + 7x_4 \\ x_2 &= 8 - 4x_3 - 3x_4 \\ x_3 &= x_3 \quad \text{(free)} \\ x_4 &= x_4 \quad \text{(free)} \end{aligned}$$

We can also write this in parametric vector form:

$$\begin{pmatrix} -13 \\ 8 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

This is a translate of a span: $(-13, 8, 0, 0) + \mathrm{Span}\{(8, -4, 1, 0), (7, -3, 0, 1)\}$.

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form for the solution to $Ax = (9)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 9 \end{array}\right)$$

Homogeneous vs. Nonhomogeneous Systems

Key realization. Set of solutions to $Ax = b$ obtained by taking one solution and adding all possible solutions to $Ax = 0$.

$$Ax = 0 \text{ solutions} \rightsquigarrow Ax = b \text{ solutions}$$

$$x_k v_k + \cdots + x_n v_n \rightsquigarrow p + x_k v_k + \cdots + x_n v_n$$

So: set of solutions to $Ax = b$ is parallel to the set of solutions to $Ax = 0$. It is a translate of a plane through the origin. (Again, we are using geometry to understand algebra!)

So by understanding $Ax = 0$ we gain understanding of $Ax = b$ for all b. This gives structure to the set of equations $Ax = b$ for all b.

Demo

Demo

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector forms for $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

...and $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

Solving matrix equations

The matrix equation

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution.

What does this mean about the matrix equation

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix}?$$

What does this mean about rabbits?

Two different things

Suppose A is an $m \times n$ matrix. Notice that if $Ax = b$ is a matrix equation then x is in $\mathbb{R}^n$ and b is in $\mathbb{R}^m$. There are two different problems to solve.

1. If we are given a specific b, then we can solve $Ax = b$. This means we find all x in $\mathbb{R}^n$ so that $Ax = b$. We do this by row reducing, taking free variables for the columns without pivots, and writing the (parametric) vector form for the solution.

2. We can also ask for which b in $\mathbb{R}^m$ does $Ax = b$ have a solution? The answer is: when b is in the span of the columns of A. So the answer is "all b in $\mathbb{R}^m$" exactly when the span of the columns is $\mathbb{R}^m$ which is exactly when A has m pivots.

If you go back to the ▸ Demo from earlier in this section, the first question is happening on the left and the second question on the right.

Example. Say that $A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$. We can ask: (1) Does $Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ have a solution? and (2) For which b does $Ax = b$ have a solution?

Summary of Section 2.4

- The solutions to $Ax = 0$ form a plane through the origin (span)
- The solutions to $Ax = b$ form a plane not through the origin
- The set of solutions to $Ax = b$ is parallel to the one for $Ax = 0$
- In either case we can write the parametric vector form. The parametric vector form for the solution to $Ax = 0$ is obtained from the one for $Ax = b$ by deleting the constant vector. And conversely the parametric vector form for $Ax = b$ is obtained from the one for $Ax = 0$ by adding a constant vector. This vector translates the solution set.

Typical exam questions

- Suppose that the set of solutions to $Ax = b$ is the plane $z = 1$ in $\mathbb{R}^3$. What is the set of solutions to $Ax = 0$?
- Suppose that the set of solutions to $Ax = 0$ is the line $y = x$ in $\mathbb{R}^2$. Is it possible that there is a b so that the set of solutions to $Ax = b$ is the line $x + y = 1$?
- Suppose that the set of solutions to $Ax = b$ is the plane $x + y = 1$ in $\mathbb{R}^3$. Is is possible that there is a b so that the set of solutions to $Ax = b$ is the z-axis?
- Suppose that the set of solutions to $Ax = 0$ is the plane $x + 2y - 3z = 0$ in $\mathbb{R}^3$ and that the vector $(1, 3, 5)$ is a solution to $Ax = b$. Find one other solution to $Ax = b$. Find all of them.
- Is there a 2×2 matrix so that the set of solutions to $Ax = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ is the line $y = x + 1$? If so, find such an A. If not, explain why not.