# RESEARCH STATEMENT

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My research interests lie within mathematical analysis, including geometric measure theory, partial differential equations (PDEs) – especially elliptic PDEs, and real analysis. As a PhD student at Courant Institute, advised by Fanghua Lin, I have been working mostly on studying the eigenfunctions of the Dirichlet-to-Neumann operator, in particular on estimating the size of their nodal sets.

In my Master's thesis, under supervision of Jan Malý, I studied Sobolev $W^{1,n-1}$ homeomorphisms from a subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$. I proved a formula for the derivative of the inverse, which has to be in the space of functions with bounded variation ($BV$).

My statement is organized as follows. Section 1 discusses my work as a PhD student. It describes the motivation for my work, my results and main tools used, and possible future research directions. Section 2 discusses my Master's Thesis work.

## 1. Nodal Sets of Steklov eigenfunctions

1.1. **Introduction.** Many classical results in linear elliptic partial differential equations are motivated by complex analysis: the maximum principle (for modulus of holomorphic functions), the unique continuation properties, the Cauchy integral representation formula, and the interior gradient estimates of holomorphic functions were all generalized to the solutions of linear second order elliptic PDEs with suitably smooth coefficients. A lot is known about the nodal (zero) sets of holomorphic functions in complex plane, and it is an important general research topic to find analogues for the nodal and critical point sets of solutions to PDEs. In some cases, properties of nodal sets of solutions are themselves the primary concern: in a study of moving defects in nematic liquid crystals by Lin [32], the singular set of optical axes (i. e. defects) of liquid crystals in motion can be described precisely by the nodal set of solutions to certain parabolic equations. In other cases, nodal sets provide an important information in the study of other properties of solutions.

In my PhD Thesis, I am studying the nodal set of the Steklov eigenfunctions on the boundary of a smooth bounded domain in $\mathbb{R}^n$ – the eigenfunctions of the Dirichlet-to-Neumann map $\Lambda$. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, this map associates to each function $u$ defined on the boundary $\partial\Omega$, the normal derivative of the harmonic function on $\Omega$ with boundary data $u$. More generally, one can consider an $n$-dimensional smooth Riemannian manifold $(M, g)$ instead of $\Omega$, and replace the Laplacian in the following by the Laplace-Beltrami operator $\Delta_g$. My methods, which build on the work of Lin, Garofalo and Han ([20, 33, 24]), can be used also for solutions to more general elliptic equations and/or boundary conditions.

The Steklov eigenfunctions were introduced by Steklov [38] in 1902 for bounded domains in the plane. He was motivated by physics – the functions represent the steady state temperature on $\Omega$ such that the flux on the boundary is proportional to the temperature. The problem can also be interpreted as vibration of a free membrane with the mass uniformly distributed on the boundary. Note that the eigenfunction's nodal set represents the stationary points on the boundary.

Among others, the Steklov problem is also important in conformal geometry, Sobolev trace inequalities, and inverse problems. I will present some of these connections in section 1.5.

1.2. **Problem setting and main results.** Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. The Dirichlet-to-Neumann operator $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is defined as follows. For $f \in H^{1/2}(\partial\Omega)$, we solve the Laplace equation

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

This gives a solution $u \in H^1(\Omega)$, and we set $(\Lambda f)$ to be the trace of $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$, where $\nu$ is the exterior unit normal. We obtain a bounded self-adjoint operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. It has a discrete spectrum $\{\lambda_j\}_{j=0}^{\infty}$, $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots$, $\lim_{j\to\infty} \lambda_j = \infty$. The eigenfunctions of $\Lambda$ (called Steklov eigenfunctions) can be identified with the trace on $\partial\Omega$ of their harmonic extensions to $\Omega$, which satisfy

$$\begin{aligned} \Delta u_j &= 0 \qquad \text{in } \Omega, \\ \frac{\partial u_j}{\partial \nu} &= \lambda_j u_j \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The main goal of my research is to estimate the size (the Hausdorff $\mathcal{H}^{n-2}$-measure) of the nodal set of $u_j|_{\partial\Omega}$ in terms of $\lambda_j$ as $\lambda_j$ grows to infinity, provided $\Omega$ is fixed. In the following, we will omit the index $j$ in $u_j$, $\lambda_j$: $\lambda$ will denote an eigenvalue of the Dirichlet-to-Neumann map and $u$ the corresponding eigenfunction (harmonically extended to $\Omega$). I have proved the following bound in case that $\Omega$ has analytic boundary:

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^n$ be an analytic set. Then there exists a constant $C$ depending only on $\Omega$ and $n$ such that for any $\lambda > 0$ and $u$ which is a (classical) solution to* (1) *there holds*

$$\mathcal{H}^{n-2}(\{x \in \partial\Omega : u(x) = 0\}) \le C\lambda^5. \tag{2}$$

The scaling $\lambda^5$ in (2) is not optimal. Actually, my proof gives approximately $\lambda^{4.6}$, but I believe that the optimal scaling is $\lambda$. However, even this polynomial bound is valuable in a problem like this – the main difficulty in the estimate is to avoid an exponential bound $e^{C\lambda}$.

In case that $\Omega$ is smooth (and not necessarily analytic), I have obtained some partial results. See section 1.3 for details.

To put my work into context, note that this specific problem is similar in nature to a classical question of estimating the size of nodal sets of eigenfunctions of the Laplace operator in a compact manifold. The following conjecture was proposed by Yau in [43]:

**Conjecture 1.2.** *Suppose $(M^n, g)$ is a smooth $n$-dimensional connected and compact Riemannian manifold without boundary. Consider an eigenfunction $u$ corresponding to the eigenvalue $\lambda$, i.e. ,*

$$\Delta_g u + \lambda u = 0 \quad \text{on } M.$$

*Then there holds*

$$c_1\sqrt{\lambda} \le \mathcal{H}^{n-1}(\{x \in M; u(x) = 0\}) \le c_2\sqrt{\lambda},$$

*where $c_1$ and $c_2$ are positive constants depending only on $(M, g)$.*

This conjecture was proved in case that $(M^n, g)$ is analytic by Donnelly and Fefferman in [12]. It is still open whether Conjecture 1.2 holds if $(M^n, g)$ is only smooth. The known results for the smooth case are far from optimal (see [25]).

Another similar problem has been studied for the Neumann eigenfunctions on a piecewise analytic plane domain $\Omega \subset \mathbb{R}^2$ in [40]. They are concerned about the asymptotics of the number of nodal points of the eigenfunctions on the boundary $\partial\Omega$, as the eigenvalue $\lambda$ increases to infinity. They prove that this number is bounded above by $C_\Omega\lambda$.

1.3. **Main tools.** In this section I describe the main tools of the proof of Theorem 1.1, as well as my partial results in the non-analytic case. I discuss the following:

- A preliminary step: extension of the eigenfunctions beyond the boundary $\partial\Omega$;
- The main task: a doubling lemma, and further steps;
- Frequency as the main tool.

Consider a $C^2$ domain $\Omega \subset \mathbb{R}^n$ and $\lambda$, $u$ satisfying $\Delta u = 0$ in $\Omega$, $\frac{\partial u}{\partial \nu} = \lambda u$ on $\partial\Omega$.

**Extension of $u$:** First, we extend the function $u$ defined on $\overline{\Omega}$ to an open set $D$ containing $\partial\Omega$, so that the boundary $\partial\Omega$ becomes a hypersurface in $D$. For a suitably small $\delta > 0$ and $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$, where $\mathrm{dist}(x, \partial\Omega) = d(x)$ is the distance function, we define

$$v(x) := u(x)e^{\lambda d(x)} \quad \text{for } x \in \Omega_\delta \cup \partial\Omega.$$

Note that $v(x) = 0$ if and only if $u(x) = 0$. Then $v$ satisfies an elliptic equation

$$\begin{aligned} \mathrm{div}(A(x)\nabla v) + b(x) \cdot \nabla v + c(x)v &= 0 \quad \text{in } \Omega_\delta, \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3}$$

where

$$\begin{aligned} ||A||_{L^\infty(\Omega_\delta)} &\leq C, \\ ||b||_{L^\infty(\Omega_\delta)} &\leq C\lambda, \\ ||c||_{L^\infty(\Omega_\delta)} &\leq C\lambda^2, \end{aligned} \tag{4}$$

with the constant $C$ depending only on the domain $\Omega$.

Since $v$ satisfies a homogeneous Neumann condition on $\partial\Omega$, we can reflect it around $\partial\Omega$ and hence define an extension $v \in C^1(D)$ on an open set $D \supset \partial\Omega$ which satisfies the elliptic equation (3) a. e. in $D$, with the $L^\infty(D)$ bounds of coefficients of the same type as in (4). Importantly, $A$ will be Lipschitz in $D$. To estimate the nodal set of $u$ (resp. $v$), we use both the reformulated equation (3) for $v$ near the boundary $\partial\Omega$ and the Laplace equation for $u$ inside $\Omega$.

**Doubling lemma and further steps:** As in [33], a crucial step when estimating the nodal set is to derive a doubling condition, i. e. the control of the $L^2$-norm of $u$ on a ball $B_{2r}(x)$ by the $L^2$-norm of $u$ on a smaller ball $B_r(x)$. I proved it in the following form. We do not need analyticity of $\Omega$.

**Theorem 1.3.** *There exist constants $r_0, C$ depending only on $\Omega$ and $n$ such that $0 < r_0 < \delta$ and for any $r \leq r_0$ and $x \in \partial\Omega$,*

$$\int_{B(x,2r)\cap\partial\Omega} u^2 \leq 2^{C\lambda^5} \int_{B(x,r)\cap\partial\Omega} u^2. \tag{5}$$

From here, for analytic domains we follow the approach developed in [33] to estimate the $\mathcal{H}^{n-2}$-measure of the nodal set of $u|_{\partial\Omega}$. It involves a complexification of $u$ and the use of the theory for holomorphic functions.

For smooth, non-analytic $\Omega$, other methods including the Taylor expansion of solutions can be used (see also [24, 25]). However, the estimate involves high order derivatives of $\Omega$. I am currently working on this problem.

To prove Theorem 1.3, I derived a doubling condition analogous to (5) in the solid $D$ for $v$, and then used a quantitative Cauchy uniqueness lemma as in [33]. The main difficulty we encounter is that for a general elliptic equation, an adaptation of the results in [20] gives us a bound for the frequency only on small balls of radius of order $1/\lambda$. That globally translates into a suboptimal exponential bound. Therefore we rather use the special structure of the Steklov problem: we start in a suitable point on $\partial\Omega$, use the more general equation (3) which is satisfied by $v$ in the neighborhood of $\partial\Omega$ to obtain a local estimate, and then use the Laplace equation inside of $\Omega$ to extend the estimate globally.

**More details on the tools used:** We use the frequency function as the main tool to derive the doubling condition, as, e.g., in [19, 20, 33, 21]. For a harmonic function $u$ on ball $B_1$ centered at the origin and $r < 1$, the frequency $N(r)$ is defined as

$$N(r) = \frac{rD(r)}{H(r)},$$

where

$$D(r) = \int_{B_r} |\nabla u|^2 dx,$$

$$H(r) = \int_{\partial B_r} u^2 dx.$$

The frequency function is a tool to measure the growth of a harmonic function. If $u$ is a homogeneous harmonic polynomial, its frequency is exactly its degree. Let us list some known properties crucial for my work (see [21], [19], or [33]). The following monotonicity property of the frequency function is attributed to F. J. Almgren, Jr. ([1]).

**Proposition 1.4.** *Let $u$ be a harmonic function in $B_1$. Then $N(r)$ is a nondecreasing function of $r \in (0,1)$.*

**Corollary 1.5** (Doubling Condition)**.** *Let $u$ be a harmonic function in $B_1$. For any $R \in (0,1/2)$, there holds*

$$\fint_{B_{2R}} u^2 \leq 4^{N(1)} \fint_{B_R} u^2.$$

Next, we recall a result estimating the frequency at a given point by the frequency at a different point. This is an important property whose adaptation we use to obtain global estimates.

**Proposition 1.6.** *Let $u$ be a harmonic function in $B_1$. For any $R \in (0,1)$, there exists a constant $N_0 = N_0(R) \ll 1$ such that the following holds. If $N(0,1) \leq N_0$, then $u$ does not vanish in $B_R$. If $N(0,1) \geq N_0$, then there holds*

$$N\left(p, \frac{1}{2}(1-R)\right) \leq CN(0,1), \quad \text{for any } p \in B_R,$$

*where $C$ is a positive constant depending only on $n$ and $R$. In particular, the vanishing order of $u$ at any point in $B_R$ never exceeds $c(n,R)N(0,1)$.*

The frequency and all of these results can be generalized from harmonic functions to solutions of general elliptic equations, as in [19, 20].

1.4. **Future research possibilities.** There are several ways in which one can continue in my work described above. A very interesting direction is to prove an explicit estimate on the size of the nodal sets of Steklov eigenfunctions if $\Omega$ is not analytic, under minimal smoothness assumptions on $\Omega$. An important step in this way is that the doubling condition (Theorem 1.3) holds if $\Omega$ is $C^2$, and even this can probably be improved. The challenging aspect lies in the estimate of the nodal set without the use of high order derivatives.

One can also apply the developed methods to estimate the nodal sets of solutions to other problems. This topic has been extensively studied for solutions to general elliptic and parabolic PDEs, see e.g. [25, 33, 24, 23, 22]. However, we are not aware of many results treating the nodal sets on the boundary, e.g. for solutions of elliptic PDEs satisfying a Robin boundary condition.

There are also many other interesting questions one can ask about the Dirichlet-to-Neumann operator. One particular possible research problem is to use a Leibnitz-type formula for the Dirichlet-to-Neumann map of a product of two functions, $\Lambda(fg)$, to understand nonlinear problems involving the operator $\Lambda$.

1.5. **Other related problems.** In this section I will briefly present some other problems related to the Dirichlet-to-Neumann map and its eigenvalues/eigenfunctions. They are classical problems which illustrate the importance of this map.

The classical Yamabe Problem consists in showing that every Riemannian compact manifold, without boundary, admits a conformally related metric with constant scalar curvature. It was formulated by Yamabe [42] in 1960 and proved by Aubin ([3], 1976) and Schoen ([37], 1984). A related problem for a manifold $(M,g)$ with boundary asks to find a conformally related metric with zero scalar curvature on $M$ and constant mean curvature on the boundary. It was solved in most cases by Escobar ([14]) and by Marques ([34]) in the remaining ones, and can be thought of as a generalization of the Riemann mapping theorem to higher dimensions. This problem can be reformulated as finding a solution to

$$\begin{aligned} \Delta_g u &= 0 && \text{in } M, \\ \frac{du}{d\nu} + \frac{n-2}{2} h_g u &= \lambda u^{\frac{n}{n-2}} && \text{on } \partial M, \\ u &> 0 && \text{in } M, \end{aligned}$$

where $h_g$ is the mean curvature on the boundary. That corresponds to solving

$$\Lambda(u) + f(u) = 0 \tag{6}$$

on $\partial M$ for a nonlinearity $f(u) = \frac{n-2}{2} h_g u - \lambda u^{\frac{n}{n-2}}$. Equations of this type often come up when studying the geometry of a manifold. Similarly as in [7], the understanding of Steklov eigenfunctions is helpful in the study of such equations.

Another well-known example is looking for the optimal constant in the Sobolev trace inequality. The first non-zero eigenvalue for the Steklov problem, $\lambda_1$, has a variational characterization:

$$\lambda_1 = \min_{\int_{\partial M} f = 0} \frac{\int_M |\nabla f|^2}{\int_{\partial M} f^2}.$$

From this characterization follows this Sobolev trace inequality: for all functions $f \in H^1(M)$, it holds

$$\int_{\partial M} |f - \overline{f}|^2 \leq \frac{1}{\lambda_1} \int_M |\nabla f|^2, \tag{7}$$

where $\overline{f}$ is the mean value of $f$ when restricted to the boundary. The last inequality is fundamental in the study of existence and regularity of solutions of some boundary value problems. Hence it is important to know the dependence of $\lambda_1$ on the geometry of $M$. It has been studied e.g. in [41, 36, 15, 17]. Estimates for the higher eigenvalues and their distribution have also been studied, see e.g. [5, 30, 11].

Inequality (7) implies the embedding $H^1(M) \hookrightarrow L^2(\partial M)$. This is not the optimal embedding. For simplicity, consider the half-space $\mathbb{R}^n_+$. For any function $f$ on $\mathbb{R}^n_+$, sufficiently smooth and decaying fast enough at infinity, there holds

$$||f||_{L^{2(n-1)/(n-2)}(\partial \mathbb{R}^n_+)} \leq C(n) ||\nabla f||_{L^2(\mathbb{R}^n_+)}, \tag{8}$$

where $C(n)$ is a known constant. The functions $f$ for which equality holds in (8) satisfy the Euler-Lagrange equation

$$\begin{aligned} \Delta f &= 0 && \text{on } \mathbb{R}^n_+, \\ \frac{\partial f}{\partial \nu} - Q(\mathbb{R}^n_+) f^{n/(n-2)} &= 0 && \text{on } \partial \mathbb{R}^n_+, \end{aligned}$$

where $Q(\mathbb{R}^n_+) = 1/C(n)^2$. This is again an equation of type (6). See [13] for details, and [31] for a generalization for manifolds.

There are many more problems related to the Steklov eigenfunctions. The understanding of the problem could have applications in inverse conductivity problems ([8, 39]), cloaking ([2]), and modeling of sloshing of a perfect fluid in a tank ([16, 6]).

## 2. Homeomorphisms in $W^{1,n-1}$

The classical inverse function theorem states that the inverse of a $C^1$-homeomorphism $f$ is again a $C^1$-homeomorphism, under the assumption that the Jacobian $J_f$ is strictly positive, and expresses the derivative of the inverse $f^{-1}$ in terms of the derivative of $f$. It is interesting to study analogues of this theorem under weaker regularity assumptions on $f$. In nonlinear elasticity, the question of invertibility and the regularity of the inverse is fundamental, and Sobolev (or $BV$) spaces are natural initial spaces for these problems, see e.g. [4, 9, 35].

Recently, Csörnyei, Hencl and Malý proved the following theorem in [10]:

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism. Then $f^{-1} \in BV_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$.*

This result is sharp for $n \geq 3$: see [29] for a construction of a homeomorphism $f \in W^{1,n-1-\epsilon}$ ($\epsilon$ arbitrarily small) such that $f^{-1}$ is pointwise differentiable a. e., but $Df^{-1} \notin L^1_{\mathrm{loc}}$ (and hence $f^{-1} \notin BV_{\mathrm{loc}}$). Moreover, even adding the assumption $|\operatorname{adj} Df| \in L^1$ (adj denoting the adjoint matrix) does not help – see the counterexample in [26].

The main result of my Master's Thesis was, under the conditions of Theorem 2.1, expressing the derivative of the inverse (the measure $Df^{-1}$) in terms of the derivative of $f$:

**Theorem 2.2.** *Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$ be a domain and $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be a homeomorphism. Assume that $f$ is positively oriented. Then for each Borel set $E \subset \Omega$ there holds*

$$Df^{-1}(f(E)) = \int_E \operatorname{adj} Df. \tag{9}$$

In dimension 2, it is enough that a homeomorphism is $BV$ for the inverse to be in $BV$ as well – see [29]. I proved the analogue of (9) also in this setting:

**Theorem 2.3.** *Let $\Omega \subset \mathbb{R}^2$ be a domain and $f \in BV(\Omega, \mathbb{R}^2)$ be a homeomorphism. Assume that $f$ is positively oriented. Then for each Borel set $E \subset \Omega$ there holds*

$$Df^{-1}(f(E)) = \operatorname{adj} Df(E).$$

(Here $Df^{-1}$ and adj $Df$ are $2 \times 2$ matrices of measures.)

Formulae like (9) were proven before under stronger assumptions on $f$ – see [27, 28, 26, 18].

The crucial tools I used in proving Theorem 2.2 were a suitable form of divergence theorem (for continuous $BV$ functions and sets of finite perimeter) and the area formula on hyperplanes. Note that the area formula fails for homeomorphisms in $W^p(\mathbb{R}^n, \mathbb{R}^n)$ if $p < n$. However, it was proven in [10] that for homeomorphisms in $W^{1,n-1}$, the area formula holds on almost all hyperplanes.

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