

GENERATING TORELLI

Goal: $I(S_g)$ is gen. by BP maps (and Dehn twists about sep curves)

Original proof: 1971 Birman gives presentation for $Sp_{2g}(\mathbb{Z})$

1978 Powell interprets relations

1980 Johnson, lantern relation

Want a proof analogous to $Mod(S_g)$ case.

Complex of homologous curves

Fix (primitive) $x \in H_1(S_g; \mathbb{Z})$

$C_x(S_g)$ = subgraph of $C(S_g)$ spanned by
(unoriented) reps of x .

goal: Connected.

“borrowing complex”

Will use auxilliary complex $B_x(S_g)$, the
complex of cycles. Points of $B_x(S_g)$
are simple, irredundant reps of x .

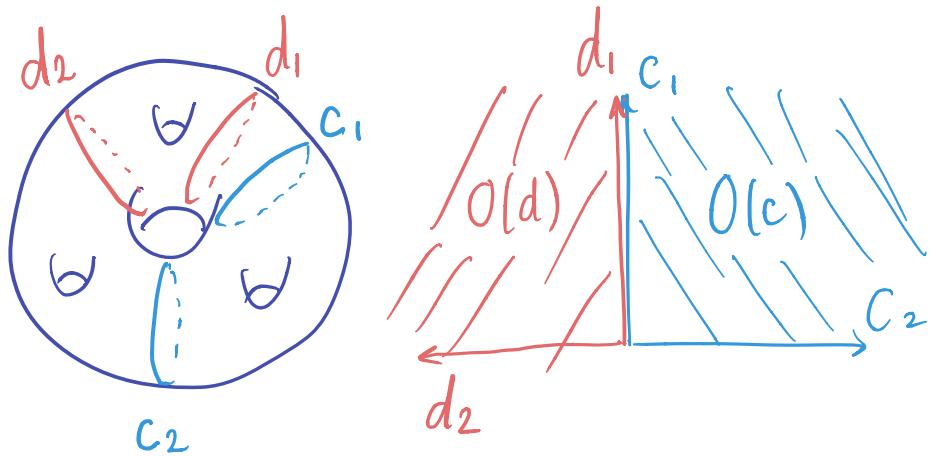
The Complex of Cycles

$C = \text{oriented multicurve, } n \text{ components}$

$$\rightsquigarrow [0, \infty)^n \rightarrow H_1(S_g; \mathbb{Z}) \text{ orthant } O(c)$$

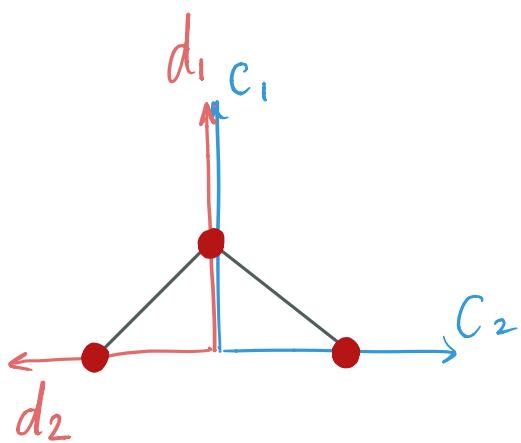
$$A(S_g) = \coprod_c O(c) / \sim$$

example.

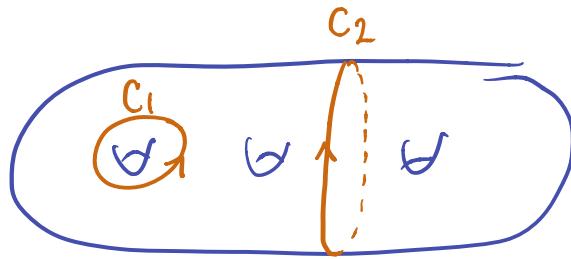


$$A_x(S_g) \subseteq A(S_g) \text{ reps of } x.$$

Say $x = [c_1]$

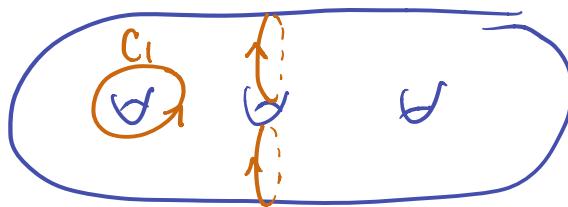


The cells of $A_x(S_g)$ are not necessarily compact:



If $[c_1] = x$ then $[c_1 + bc_2] = x \quad \forall b \in \mathbb{R}$

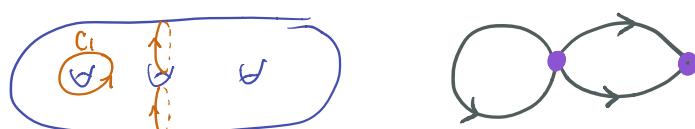
Or:



An oriented multicurve is reduced if

- (1) the corresponding cell is compact
- \iff (2) it has no homologically trivial subset
- \iff (3) the dual directed graph is recurrent

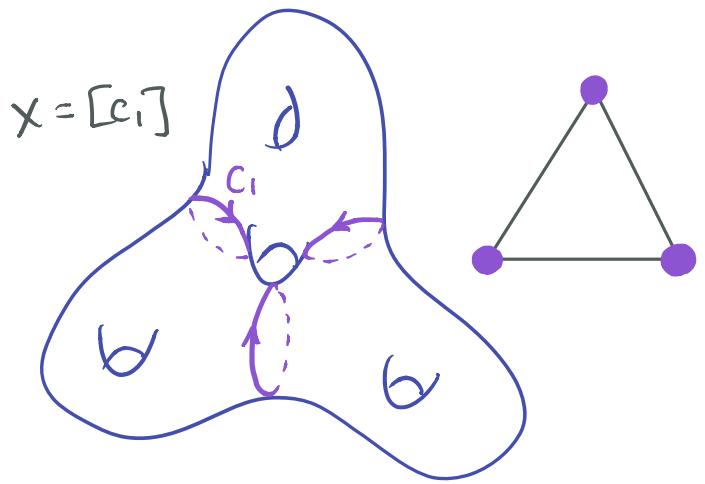
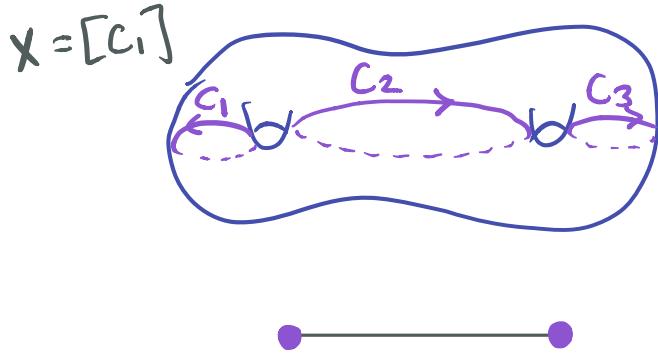
Dual graphs:



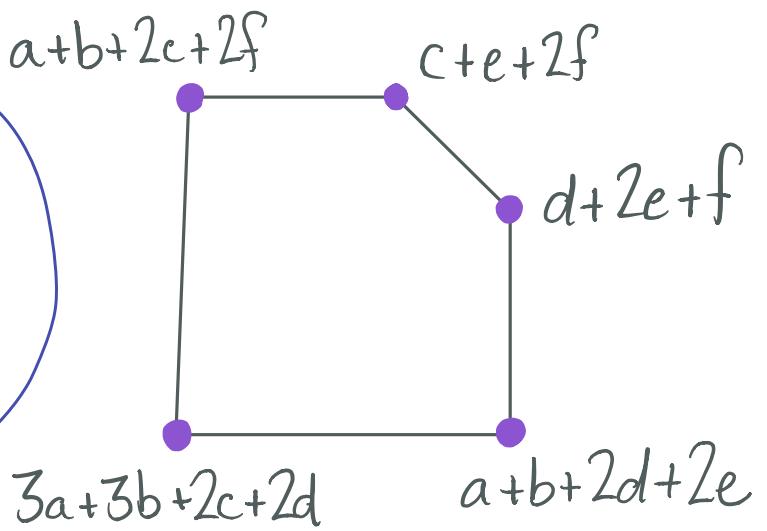
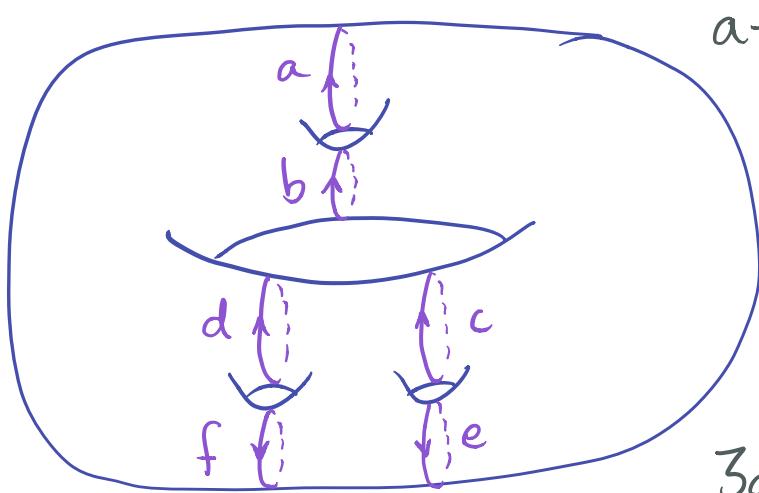
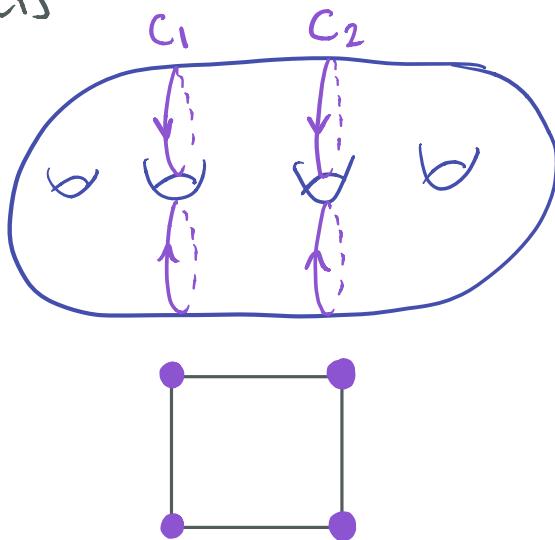
The complex of cycles $B_x(S_g)$ is the subcomplex of $A_x(S_g)$ whose cells correspond to reduced oriented multicurves.

We'll show $B_x(S_g)$ is contractible.

Examples of cells



$$x = [c_1] + [c_2]$$



Q. Which polytopes arise?

Properties of Cells

Prop. The dim. of a cell = # compl. comp.'s - 1 .

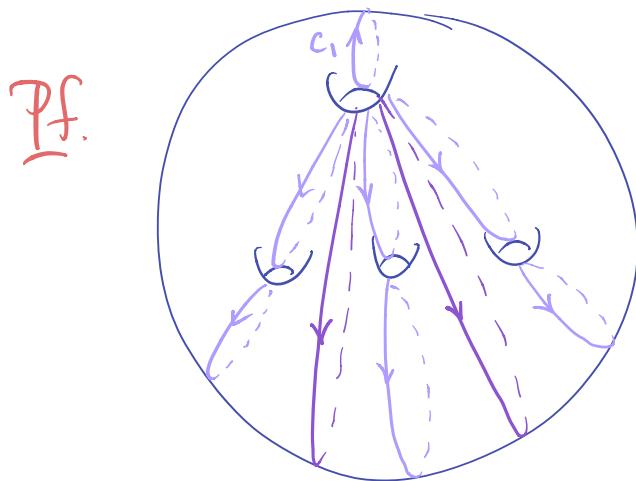
Pf. Defn of homology.

\Rightarrow vertices \longleftrightarrow nonsep. multicurves.

Prop. Vertices of $B_x(S_g)$ are oriented multicurves with integral weights.

Pf. Given a vertex, consider a loop intersecting in one point.

Prop. $\dim B_x(S_g) = 2g-3$.



$$x = [c_1].$$

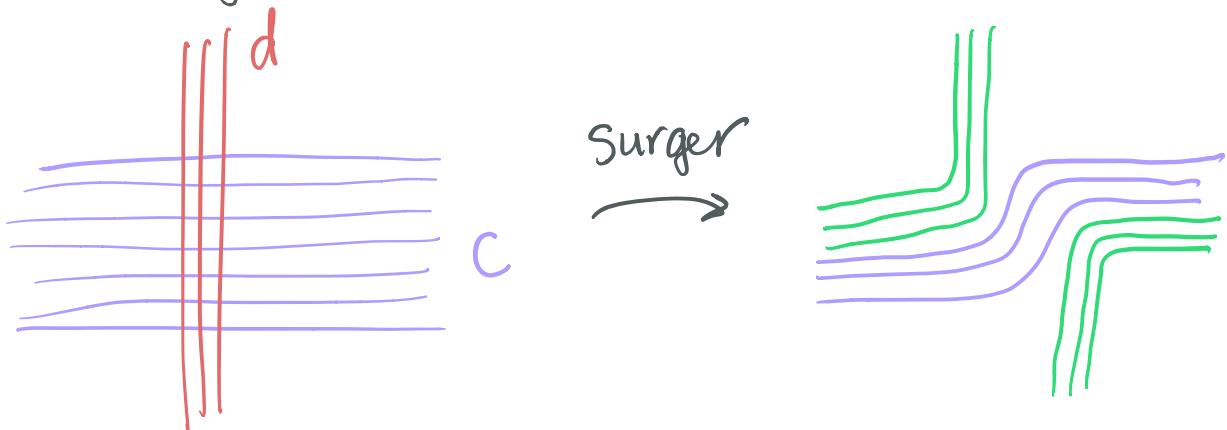
$\leadsto B_x(S_2)$ is a graph.

CONTRACTIBILITY

Theorem. $B_x(S_g)$ is contractible.

Surgery on 1-cycles

Say $c, d \in A_x(S_g)$. Thicken c, d according to weights and then:



If $[c] = [d] = x$, this procedure will result in a 1-cycle rep'ing x . Why?

$$H_1(S_g; \mathbb{Z}) \cong H^1(S_g; \mathbb{Z}) \cong \text{Hom}(H_1(S_g; \mathbb{Z}), \mathbb{Z}) \leftrightarrow [S_g, S^1]$$

The original c, d give maps $S_g \rightarrow S^1$ by integrating against width of annuli. The surgered picture corresponds to the map $S_g \rightarrow S^1$ obtained by integrating against both widths.

Prop. $A_x(S_g)$ is contractible

Pf. Fix some $c \in A_x(S_g)$. Consider:

$$F_t(d) = \text{Surger}(tc + (1-t)d)$$

□

Draining 1-cycles

Suppose $c \in A_x(S_g)$ is not reduced.

↪ $\{R_i\}$ subsurfaces with $\partial R_i \subseteq c$

$$\text{Drain}_t(c) = c - t \sum \partial R_i$$

Prop. $A_x(S_g)$ def. retracts to $B_x(S_g)$.

In partic. $B_x(S_g)$ is contractible.

Pf. Drain

□

In particular, $B_x(S_2)$ is a tree.

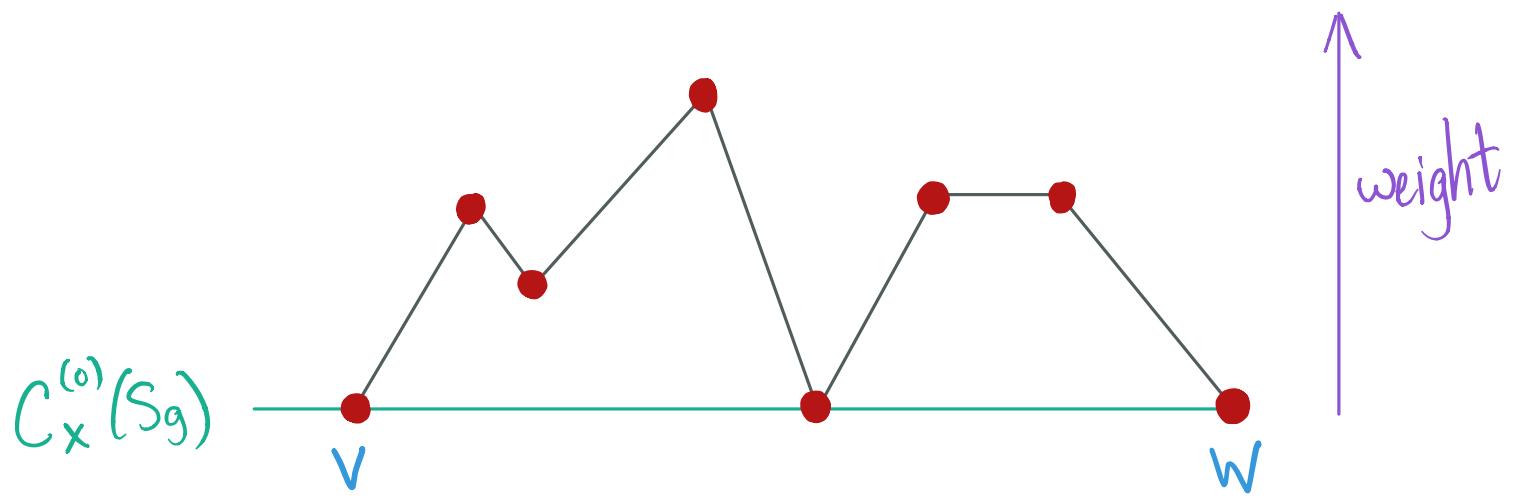
CONNECTIVITY OF $C_x(S_g)$

Basic strategy

Define weight : $B_x(S_g) \rightarrow \mathbb{Z}$
 $\sum w_i c_i \mapsto \sum w_i$

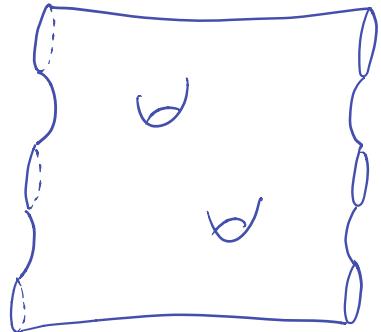
Note: $C_x(S_g) = \text{weight}^{-1}(1)$.

Now, given $v, w \in C_x^{(0)}(S_g)$, we connect them
in $B_x^{(1)}(S_g)$:



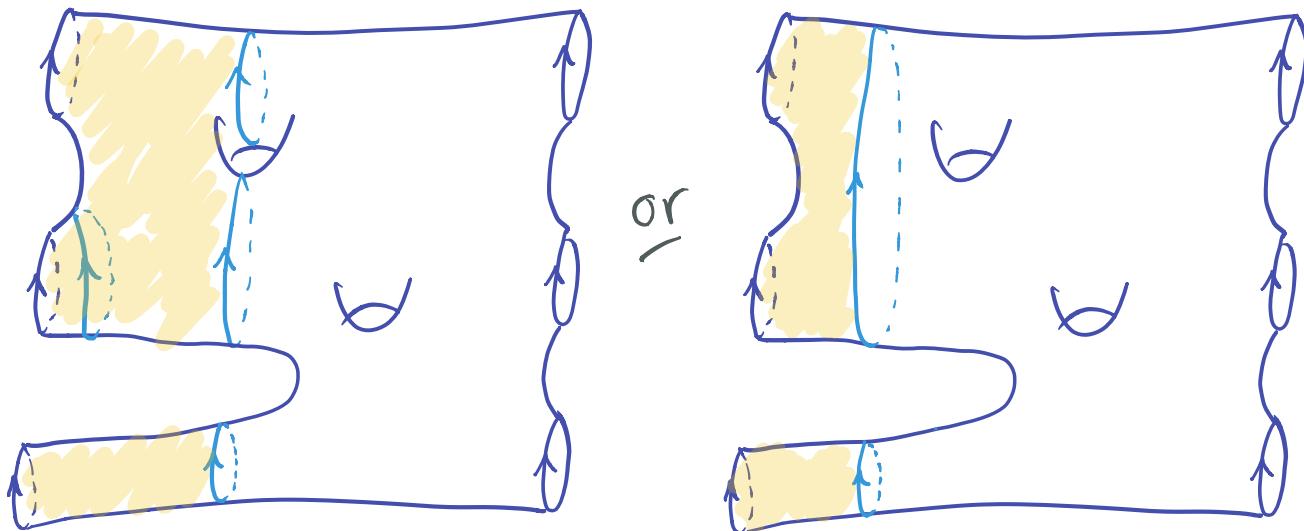
We then push the highest vertex down inductively until the path lies in $C_x(S_g)$.

Key idea: If we cut along a vertex of $B_x^{(0)}(Sg)$ we get



"cobordism"

What does an edge in $B_x(Sg)$ look like?

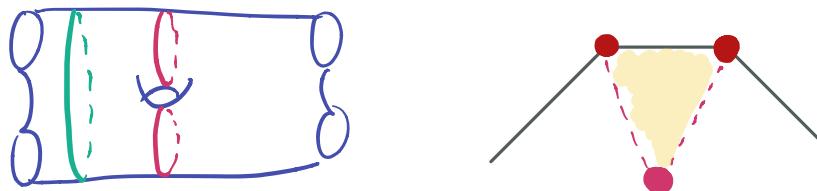


The region between transfers weight from one side to the other \Rightarrow the new vertex will have smaller weight iff there are fewer interior curves than boundary curves.

Call the edge on the right a pants edge.
This is the simplest way to reduce weight.

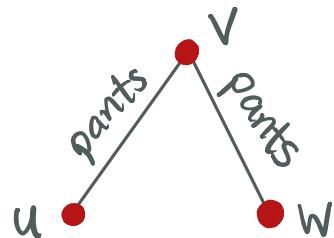
PROOF THAT $C_x(S_g)$ is connected

Step 1. Make maxima isolated, by making pants edges/triangles.

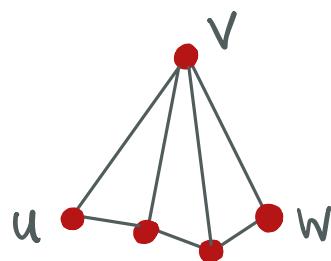


Step 2. Make highest edges into pants edges in same way

Step 3. Given



Connect uv to v by a seq of pants triangles

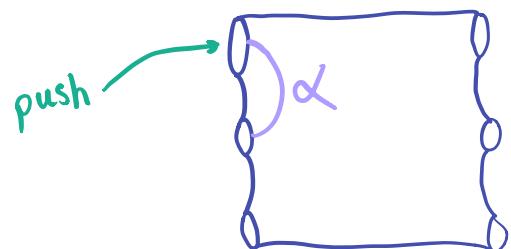


Can then push v down. Apply this process inductively.

To this end, consider the graph with
Vertices: pants edges emanating from v

\longleftrightarrow certain arcs in $S \setminus v$

edges: disjoint arcs.



To show: connected

- Notes.
- Every vertex is adjacent to one connecting first two components of $\partial(S \setminus v)$.
 - Push maps (corresponding to 1st ∂ -comp) act transitively on these.
 - π_1 (punctured sphere) has a simple gen set $\{x_i\}$

So: suffices to show that each $\text{Push}(x_i) \cdot \alpha$ lies in same component as α .

Sample case: x_i lies on LHS of $S \setminus v$.

Then if β lies on RHS we have



PROVING TORELLI IS GEN. BY BP MAPS

Ingredient #1. $C_x(S_g)$ is connected ✓

Ingredient #2. Fact. Say $G \curvearrowright X = \text{graph}$

$A \subseteq G$ s.t. \forall edges $v w$

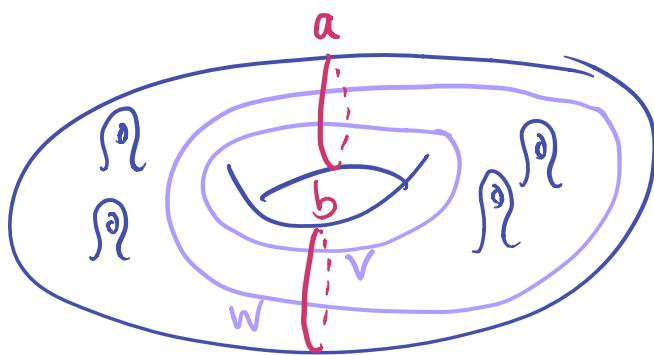
$\exists g \in A$ with $g \cdot v = w$.

Then $G = \langle A, \text{vertex stabs} \rangle$

Pf. Same as before

Ingredient #3. If $v w$ is an edge of $C_x(S_g)$

\exists BP map taking v to w .



$$T_a T_b^{-1}(v) = w$$

Thus it suffices to show $\text{Stab}_{I(S_g)}(v)$ is gen. by BP maps and Dehn twists about sep curves.

Two BIRMAN EXACT SEQUENCES FOR TORELLI

ONE MARKED POINT

$$1 \rightarrow \pi_1(S_g, p) \rightarrow I(S_g, p) \rightarrow I(S_g) \rightarrow 1$$

(restriction of usual BES)

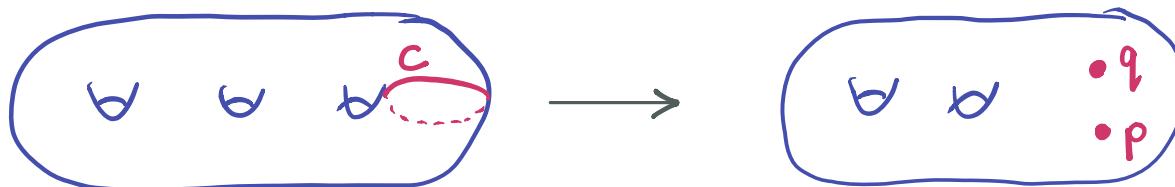
TWO MARKED POINTS

$$1 \rightarrow K \rightarrow I(S_g, \{p, q\}) \rightarrow I(S_g, p) \rightarrow 1$$

$$K = [\pi, \pi] \quad \pi = \pi_1(S_g \setminus p, q)$$

will verify
below!

What is defn of $I(S_g, \{p, q\})$? It is the image of $I(S_g, c)$ under the cutting map:



$$\leadsto I(S_g, \{p, q\}) = \ker(M_{\text{Mod}}(S_g, \{p, q\}) \rightarrow \text{Aut } H_1(S_g, \{p, q\}))$$

Can define $I(S_g, p)$ in same way, as further image under forgetting q . Get usual defn.

STABILIZERS ARE GEN. BY BP maps...

First, $\text{Stab}_{I(S_g)}(c) \cong I(S_{g-1}, \{p, q\})$ since:

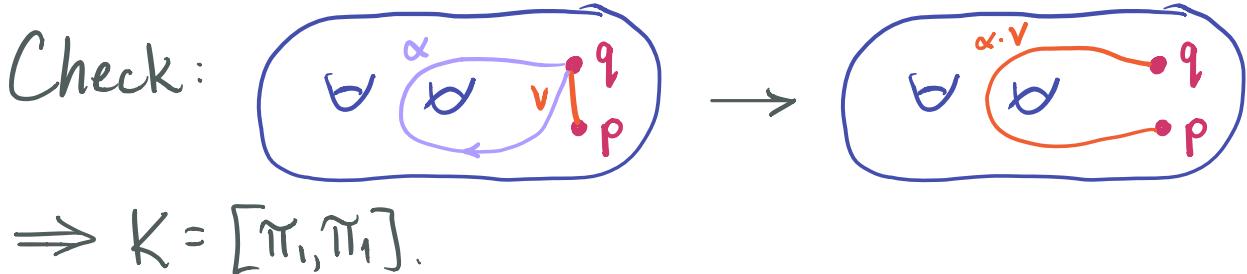
$$1 \rightarrow \langle T_c \rangle \rightarrow \text{Mod}(S_g, c) \rightarrow \text{Mod}(S_g, \{p, q\}) \rightarrow 1$$

and $T_c \notin I(S_g)$.

Step 1. K gen by BP maps & Dehn twists about seps.

$$\pi_1(S_g \setminus p, q) \cong H_1(S_g, \{p, q\}) \cong H_1(S_g) \oplus \mathbb{Z}$$

- trivial on first factor.
- action on 2nd factor is $\alpha \cdot v = [\alpha] + v$.

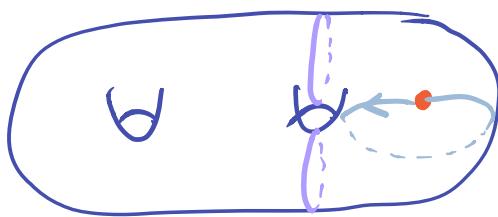


Now need: K gen. by simple (sep) loops.

Realize S_g as 4g-gon with opp sides id'd.
and use the fact that commutator subgps
are normally gen. by commutators of gens.

Step 2. $\pi_1(S_g, p) \subseteq I(S_g, p)$ gen by BP maps.

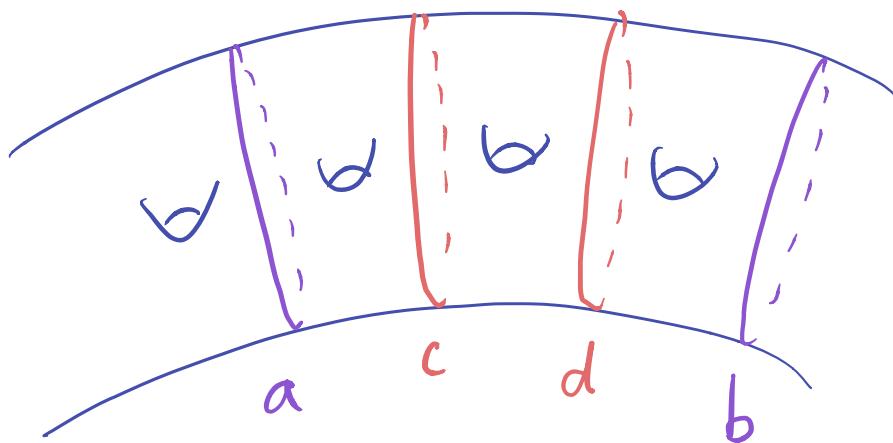
easy:



Even better:

Thm. $I(S_g)$ is gen. by BP maps of genus one.

Pf.



$$T_a T_b^{-1} = (T_a T_c^{-1})(T_c T_d^{-1})(T_d T_b^{-1}).$$