SECTION 8.1
ALGORITHMS

ALGORITHMS

Algorithm: A clearly specified method (or procedure for solving a problem.

Etymology: Abu Ja'far Muḥammad ibn Mûsâ al-Khwârizmî

PROBLEMS

We distinguish between a problem and an instance of a problem.

Problem	Instance
Matrix multiplication	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
Traveling salesman	What is the most efficient route for my mail carrier?
Sudoku	
You name it	

COMPLEXITY

Given an algorithm, we can ask about its cost.

The cost will be a function of the size of the input.
e.g. multiplying two numbers.

Complexity $f: \mathbb{N} \longrightarrow \mathbb{R}$

size of input $\rightsquigarrow$ cost for running the algorithm on input of that size. (worst case scenario)

For this to make sense, we need to specify what we mean by size and cost. Our answer will depend on the particular problem, as well as our particular needs.

COMPLEXITY

Example: What is the cost of adding two n digit numbers in terms of the number of single digit additions?

THE BIG QUESTION

Is there a better way to do things?

Given two algorithms, which is more efficient?

Given one algorithm for a problem, is there a better one out there?

What do we mean by better?

$$7n^2 - 52 \sim n^2$$

but $n < n^2$

better than

EXAMPLE: MULTIPLYING TWO NUMBERS

Problem: What is the complexity of multiplying two n digit numbers in terms of the number of single digit multiplications?

(What about additions??)

Grade school algorithm:

Is there a better way?

DIVIDE AND CONQUER

Idea: Break the problem into more manageable subproblems.

This usually leads to a recursive relation for the complexity, since the complexity of the bigger problem is given in terms of the complexity of the smaller problems.

Is there such an algorithm for multiplying two numbers?

MULTIPLYING TWO NUMBERS

Divide and Conquer Algorithm: The idea is to break up both n digit numbers into two $n/2$ digit numbers and multiply those:

$$a = \boxed{a_1 \mid a_2} = a_1 \cdot 10^{n/2} + a_2$$
$$b = \boxed{b_1 \mid b_2} = b_1 \cdot 10^{n/2} + b_2$$

$ab =$

Example: $1011 \cdot 1213 =$

$=$

Complexity: $f(n) = 4 f(n-1)$

Actually, can improve from 4 to 3, since $(a_1b_2 + a_2b_1) = (a_1+a_2)(b_1+b_2) - a_1b_1 - a_2b_2$

$\rightsquigarrow f(n) = 3 f(n-1)$.

MULTIPLYING TWO NUMBERS

So we find the complexity of the divide and conquer algorithm by solving the recurrence relation

$$a_n = 3a_{n/2}$$

We solve this recursion relation by working backward (see last page of Lecture 3).

Assume here $n = 2^k$:

This is better than n^2!

COMPARISON

#digits	#multiplications: Grade school algorithm	#multiplications: Divide & conquer algorithm	factor
10	100	39	2.5
100	10,000	1479	6.76
1,000	1,000,000	56,871	17.58
10,000	100,000,000	2,187,007	45.72
100,000	10,000,000,000	84,103,197	118.90
1,000,000	1×10^{12}	3,234,260,557	309.19

MORE EXAMPLES

① Matrix multiplication

Usual algorithm: n^2 multiplications
Divide & conquer: $n^{\log_2 7}$

Idea: Divide matrices into 4 submatrices, then do 7 multiplications.

② Evaluation of polynomials

Usual algorithm: $2n-1$ multiplications

Horner's method: Write $p(x)$ as $x \cdot q(x) + c$

$\rightsquigarrow$ recursive relation for # of multiplications:

$f(n) = f(n-1) + 1 \rightsquigarrow f(n) = n$

③ Greatest common divisor

④ Searching a list

BIG O

O for "order" (of magnitude)

Let f and g be functions $\mathbb{N} \to \mathbb{R}$.
We say that "f is big O of g" and write

$$f = O(g) \quad \text{or} \quad f \in O(g)$$

if there is a natural number n_0 and a positive real number c such that

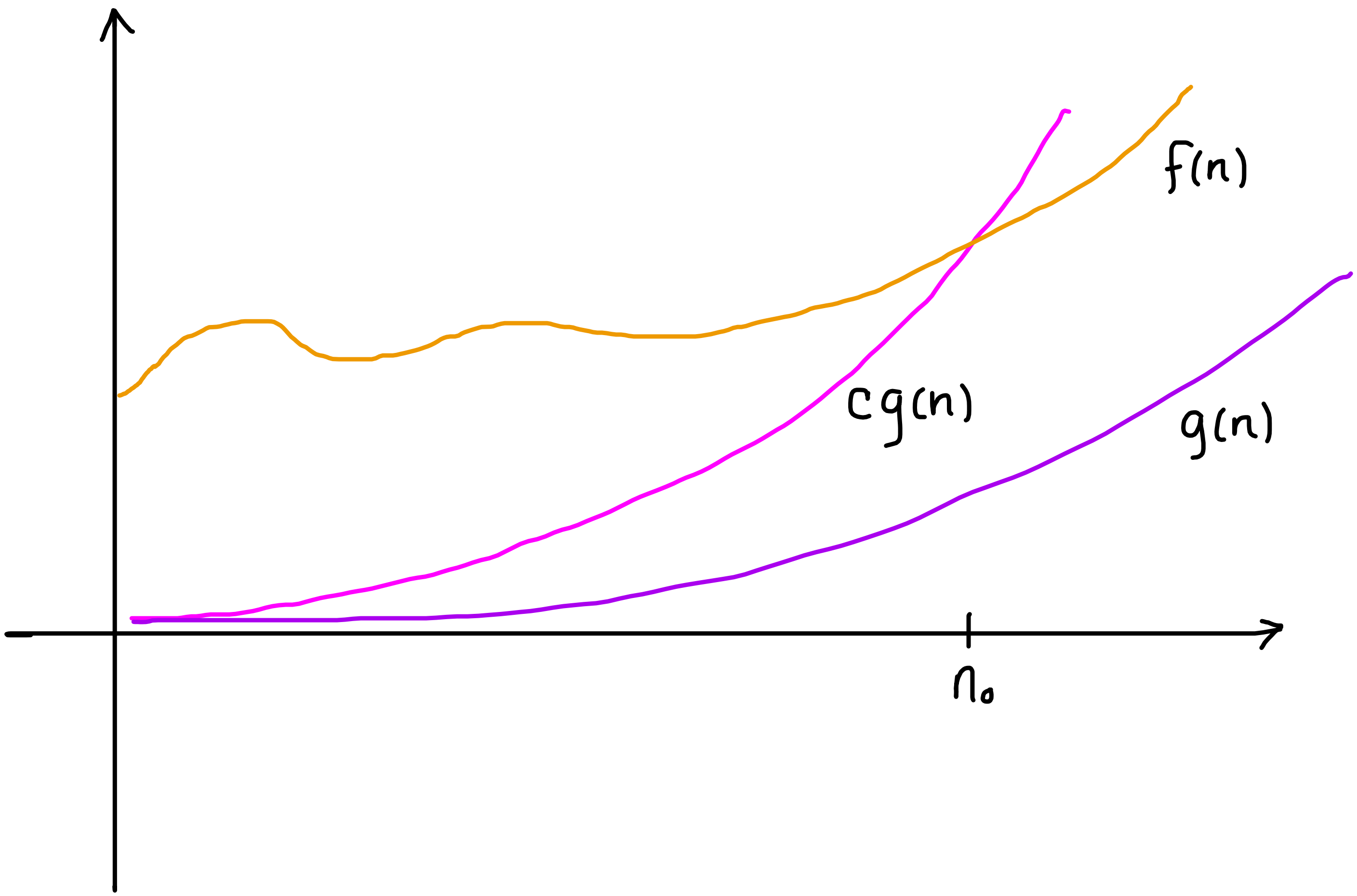
$$|f(n)| \leq c\,|g(n)|$$

for $n \geq n_0$. "for large n"

Note: If $f, g : \mathbb{N} \to [0, \infty)$ we can drop the absolute values.

Note: There are infinitely many choices for n_0 and c.

BIG O



BIG O

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for $n \geq n_0$.

First examples: 1 $f(n) = n^2$, $g(n) = 7n^2$

2 $f(n) = 4n+2$, $g(n) = n$

NOT BIG O

How do we show f is not $\mathcal{O}(g)$?

Example: $f(n) = n \quad g(n) = \sqrt{n}$

ANOTHER EXAMPLE

Example: $f(n) = n^2$, $g(n) = n^2 + n$

COMPARING FUNCTIONS

Let f and g be functions $\mathbb{N} \to \mathbb{R}$.

We say...	and write...	if...
f has smaller order than g	$f \prec g$	
f has the same order as g	$f \asymp g$	