Announcements: November 12

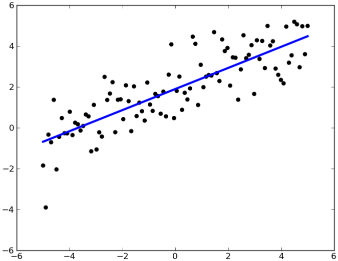
- Final Exam Dec 11 6-8:50p (cumulative!)
- No WeBWorK due this week
- No office hours this week
- Math Lab Monday-Thursday 11:15-5:15 Clough 280 Schedule
- PLUS Sessions
  - Tue/Thu 6-7 Westside Activity Room
  - Mon/Wed 6-7 GT Connector

Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Chapter 7

Orthogonality

Section 7.1

Dot products and Orthogonality

Outline

- Dot products
- Length and distance
- Orthogonality

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$
\begin{aligned}
u \cdot v &= \sum_{i=1}^{n} u_i v_i \\
&= u_1 v_1 + \cdots + u_n v_n \\
&= u^T v
\end{aligned}
$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \Leftrightarrow u = 0$

Length

Let v be a vector in $\mathbb{R}^n$

$$\begin{aligned} \|v\| &= \sqrt{v \cdot v} \\ &= \text{length of } v \end{aligned}$$

Why? Pythagorean Theorem

Fact. $\|cv\| = c\|v\|$

v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.

Distance

The distance between v and w is the length of $v - w$ (or $w - v$!).

Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$

Why? Pythagorean theorem again!

$$\begin{aligned} u \perp v &\Leftrightarrow \|u\|^2 + \|v\|^2 = \|u - v\|^2 \\ &\Leftrightarrow u \cdot u + v \cdot v = u \cdot u - 2u \cdot v + v \cdot v \\ &\Leftrightarrow u \cdot v = 0 \end{aligned}$$

Problem. Find a vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

Summary of Section 7.1

- $u \cdot v = \sum u_i v_i$
- $u \cdot u = \|u\|^2$ (length of u squared)
- The unit vector in the direction of v is $v/\|v\|$.
- The distance from u to v is $\|u - v\|$
- $u \cdot v = 0 \Leftrightarrow u \perp v$

Section 7.2

Orthogonal complements

Outline of Section 7.2

- Orthogonal complements
- Computing orthogonal complements

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?

Demo Demo

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1)\}$. Find the equation of the plane $W^\perp$.

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find the equation of the line $W^\perp$.

Orthogonal complements

Finding them

Problem. Let $W = \text{Span}\{(1,1,-1),(-1,2,1)\}$. Find the equation of the line $W^\perp$.

Theorem. $A = m \times n$ matrix

$$(\text{Row}A)^\perp = \text{Nul}\, A$$

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_w is in W and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^\perp$, so they must both be equal to 0.

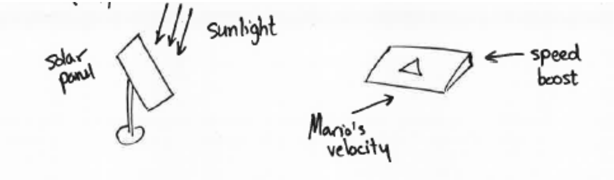
Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Summary of Section 7.2

- $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$
- Facts:
  1. $W^\perp$ is a subspace of $\mathbb{R}^n$
  2. $(W^\perp)^\perp = W$
  3. $\dim W + \dim W^\perp = n$
  4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
     $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
  5. The intersection of W and $W^\perp$ is $\{0\}$.
- $(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A$ (this is how you *find* $W^\perp$)
- Every vector v can be written uniquely as $v = w + w'$ with w in W and w' in $W^\perp$

Section 7.3

Orthogonal projection

Outline

- Orthogonal bases
- A formula for projecting onto any subspace
- Breaking a vector into components

Orthogonal Projections

The orthogonal projection of a vector v onto a subspace W is:

$$v_W = v - v_{W^\perp}$$

Fact. The orthogonal projection v_W is the closest point in W to v. The distance from v to W is $\|v_{W^\perp}\|$.

Orthogonal Projections

Theorem. If W is a subspace of $\mathbb{R}^n$ with $W = \mathrm{Span}\{v_1, \ldots, v_m\}$ and let A be the matrix whose columns are $v_1, \ldots, v_m$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Why? Notice $Av_{W^\perp} = 0$ and write $v_W = c_1 v_1 + \cdots + c_m v_m$. So:

$$A^T v = A^T (v_W + v_{W^\perp}) = A^T v_W = A^T (c_1 v_1 + \cdots + c_m v_m) = A^T A x$$

where $x = (c_1, \ldots, c_m)$.

Orthogonal Projection onto a line

Theorem. Let $L = \mathrm{Span}\{u\}$. For any vector v in $\mathbb{R}^n$ we have:

$$v_L = \frac{u \cdot v}{u \cdot u} u$$