THE SYMPLECTIC REPRESENTATION OF MCG

The symplectic group

Consider $\mathbb{R}^{2g}$ with basis $(x_1, \ldots, x_g, y_g, \ldots, y_1)$ and standard symplectic form

$$\omega = \sum_{i=1}^{g} dx_i \wedge dy_i$$

Think of ω as a <u>pairing</u> on $\mathbb{R}^{2g}$ e.g.

$$\omega(x_1 + 2y_2,\ x_1 + y_1 + x_2) = 1 - 2 = -1$$

This is the <u>unique</u> nondegenerate, alternating bilinear form on $\mathbb{R}^{2g}$ up to change of basis.

Connection to surfaces:

$$(\mathbb{R}^{2g}, \omega) \cong (H_1(S_g; \mathbb{R}), \hat{i})$$

$Sp_{2g}(\mathbb{R})$ = subgp of $GL_{2g}\mathbb{R}$ preserving ω:

$$\omega(u, v) = \omega(Mu, Mv)$$

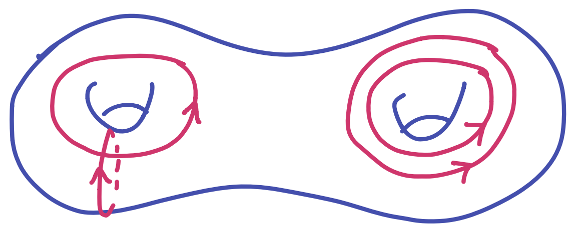
Similar with $\mathbb{Z}$.

Realizing H_1-classes by curves.

<u>Prop.</u> If $v \in H_1(S_g; \mathbb{Z})$ is primitive then $v = [c]$ where c is an oriented simple closed curve.

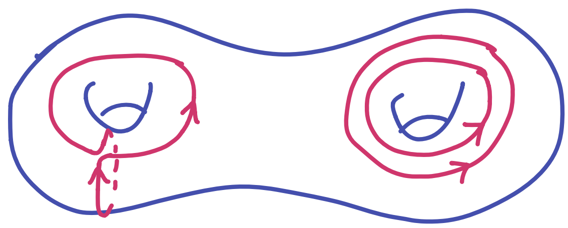
<u>Pf</u> (Meeks-Patrusky). Euclidean algorithm for scc's.

Step 1. Draw v naively:

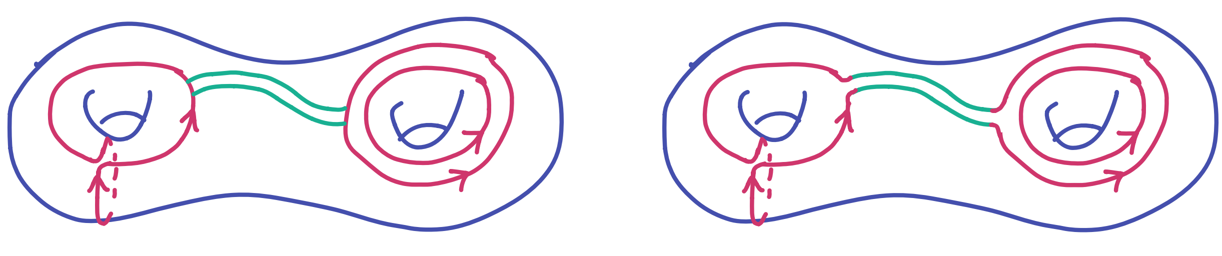


$v = x_1 + y_1 + 2x_2$

Step 2. Surger to remove crossings.



Step 3. Band surgeries to reduce the number of components

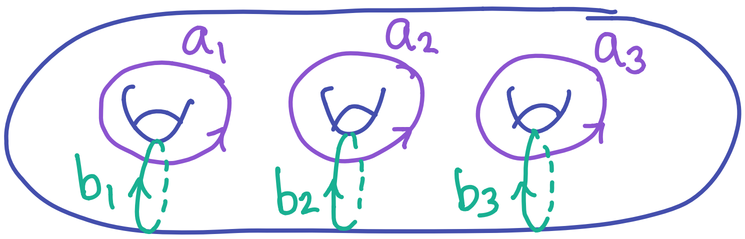


By Euclidean algorithm, this terminates in a connected curve! $\square$

Action of a Dehn Twist

Prop. Say a, b = oriented curves
Then $T_b^k([a]) = [a] + k\,\hat{i}(a,b)[b]$

A geometric symplectic basis:



Proof. Case 1. b separating

Choose geometric symplectic basis for $H_1(S_g)$ disjoint from b.

Case 2. b nonseparating.

Choose a geometric symplectic basis so b is one curve. Check for a = basis elt. Apply linearity of $\psi(T_b^k)$.

SURJECTIVITY OF THE Sp-REP

Thm $\Psi : \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z})$ is surjective.

$\boxed{\text{1st proof: transvections.}}$

A transvection in $Sp_{2g}(\mathbb{Z})$ is an elt whose 1-eigenspace is $(2g-1)$-dim.

$$\Updownarrow$$

$$\tau_v(u) = u + \omega(u,v)v \quad \text{(or a power)}$$

Fact. $Sp_{2g}(\mathbb{Z})$ is gen. by transvections.

Pf of Thm. Suffices to hit τ_v, v primitive.
Prop $\rightsquigarrow$ a s.t. $[a] = v$
$\Psi(T_a) = \tau_v$ $\square$

Want a proof that does not presuppose a genset for Sp.

2^{nd} proof: geometric symplectic bases

$Sp_{2g}(\mathbb{Z}) \longleftrightarrow$ symplectic bases for $\mathbb{Z}^{2g}$

$I \longleftrightarrow$ standard symplectic basis.

Proof of Thm. Given $A \in Sp_{2g}(\mathbb{Z})$, realize A as a geometric symplectic basis (supe up the proof of Prop above).

Realize I by standard geometric symplectic basis.

Apply change of coordinates: given two topologically equivalent configurations of curves, there is an element of $\mathrm{Mod}(S_g)$ taking one to the other. $\square$