Announcements Mar 30

- WebWork 5.2 and 5.3 due Thursday
- Homework 8 due Friday in class
- Quiz 8 on 5.2 and 5.3 on Friday
- Homework 7 due Friday April 8
- Midterm 3 in class Friday April 8 on Chapter 5
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 5.5

Complex Eigenvalues

Outline

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

A matrix without an eigenvector

Recall the rotation matrix:

$$A = 1/\sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

This matrix has no eigenvectors. Why?

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0$$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

Now try solving these:

$$x^2 + 3 = 0$$

$$x^2 - x + 1 = 0$$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can identify $\mathbb{C}$ with $\mathbb{R}^2$: $\quad a + bi \leftrightarrow (a, b)$

We can add/multiply complex numbers:

$(2 - 3i) + (-1 + i) =$

$(2 - 3i)(-1 + i) =$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can conjugate complex numbers: $\overline{a + bi} = a - bi$

We can take absolute values of complex numbers: $|a + bi| = \sqrt{a^2 + b^2}$

We can write complex numbers in polar coordinates: $r(\cos\theta + i\,\sin\theta)$

Complex numbers and polynomials

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

Fact. Every quadratic polynomial has two complex roots.

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots.

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

Why?

A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

A 3×3 example

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

What do complex eigenvalues mean?

With n real eigenvectors, we have a picture for what the matrix does to $\mathbb{R}^n$.

What about complex eigenvectors? What does the matrix do to $\mathbb{R}^n$?

We saw that rotation matrices have complex eigenvalues. Do complex eigenvalues always correspond to rotations?

Almost...

Fact. If an $n \times n$ matrix A has a complex eigenvalue there is a 2D plane in $\mathbb{R}^n$ where A is (similar to) the product of a rotation and a dilation.

What do complex eigenvalues mean?

Fact. If an $n \times n$ matrix A has a complex eigenvalue there is a 2D plane in $\mathbb{R}^n$ where A is (similar to) the product of a rotation and a dilation.

Here is the actual statement for 2×2 matrices:

Theorem. Let A be a matrix with a complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$) and associated eigenvector v. Then

$$A = PCP^{-1}$$

where

$$P = (\operatorname{Re} v \quad \operatorname{Im} v) \quad \text{and} \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

If we write $a + bi$ as $r(\cos\theta + i\,\sin\theta)$ then C is the composition of a rotation by θ and scaling by r.

Three pictures

There are three possible pictures for the action on $\mathbb{R}^2$ of a 2×2 matrix with complex eigenvalues.

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\lambda = 1 + i$$

$$|\lambda| > 1$$

$$A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

$$\lambda = 3/5 + 4/5i$$

$$|\lambda| = 1$$

$$A = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\lambda = 1/2 + 1/2i$$

$$|\lambda| < 1$$

A 3×3 example

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 4/5 & -3/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

What does A do to $\mathbb{R}^3$?

Three pictures

Compute the decomposition PCP^{-1} for the matrix:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$