Last time: $H_1(\mathrm{Mod}(S_g);\mathbb{Z}) = 0$. $g \geq 3$

Today (and next time?):

$$H_2(\mathrm{Mod}(S_g);\mathbb{Z}) = \mathbb{Z}$$

$$H_2(\mathrm{Mod}(S_g^1);\mathbb{Z}) = \mathbb{Z}$$

$$H_2(\underbrace{\mathrm{Mod}(S_{g,1})}_{\text{surface bundles over surfaces}};\mathbb{Z}) = \mathbb{Z}^2$$

$g \geq 4$

Upshot: $\exists$ alg. top which tells us a surf. bundle over surf is nontrivial.

Univ. coeff thm: Same answers for H^2 since

$$1 \to \cancel{\mathrm{Ext}(H_1(\mathrm{Mod}(S_g)))}^{\,0} \to H^2(\mathrm{Mod}(S_g)) \to \mathrm{Hom}(H_2(\mathrm{Mod}(S_g)),\mathbb{Z}) \to 1$$

$$\mathrm{Hom}(H_2(\mathrm{Mod}(S_g)),\mathbb{Z}) = H_2(\mathrm{Mod}(S_g);\mathbb{Z})/\text{torsion}$$

Overall Strategy

① Upper bounds on H_2 using Hopf formula à la Pitsch

② Lower bounds on H^2 by constructing two indep. classes: Meyer sig cocycle, Euler class.

Hopf Formula

Recall: $H_1(G) = G/[G,G]$ $\quad H_2(G) = H_2(K(G,1))$ by defn

$G = \langle F \mid R \rangle \cong F/K \quad K = \langle\langle R \rangle\rangle$

$$H_2(G;\mathbb{Z}) \cong \underbrace{K \cap [F,F]}_{\text{relns that are prod's of commutators}} / [K,F]$$

conjugate relations are equivalent.

surfaces

e.g. commuting elts $\leftrightarrow T^2$

$\pi_1(S_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle$

Given

$\pi_1(S_g) \longrightarrow G$

$\rightsquigarrow S_g \longrightarrow K(G,1)$

$S_g = K(\pi_1(S_g),1)$

So: $H_2(G) \leq K/[K,F] \longleftarrow$ abelian, gen. by relations R.

So: an elt of $H_2(G)$ looks like $r_1^{n_1} r_2^{n_2} \cdots r_N^{n_N}$

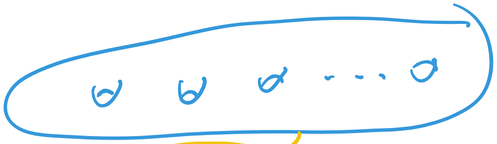
Pitsch: For $G = MCG$, at most one choice of $(n_1, \ldots, n_N)$.

Hopf formula and MCG

For $\mathrm{Mod}(S_g^1)$ an elt of H_2 is of form

$$\left(\prod D_{ij}^{n_{ij}}\right)\left(\prod B_i^{n_i}\right) C^{n_0} L^{n_L}$$

disjointness   braid   chain   lantern.



chain, lantern

braid, disj.

Will show: $n_{ij} = 0$, $n_i = 0$ i large

Hopf formula & commuting elts

For $g, h \in G$ $\quad g \leftrightarrow h$

$\rightsquigarrow \{g,h\}$ = class of $[g,h]$ in H_2 (think torus)

Fact 1. If $g \leftrightarrow h, k$ then

$$\{g, hk\} = \{g,h\} + \{g,k\}$$

since $[x, yz] = [x,y][x,z]^y$ conj. by y.

Fact 2. $\{g, h^{-1}\} = -\{g,h\}$

Back to MCG

Lemma. $T_a \leftrightarrow T_b$

$\Rightarrow \{T_a, T_b\} = 0$ in $H_2(MCG)$

Pf. Cut S along a

$H_1(Mod(S \setminus a)) = 0$,

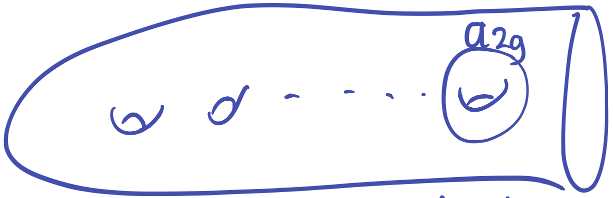
So $T_b = \prod [x_i, y_i]$

with $x_i, y_i \leftrightarrow T_a$ in $Mod(S_g)$

Apply Facts 1 & 2:

$\{T_a, \prod [x_i, y_i]\} = 0$. $\square$

Eliminating more relations



The MCG gen $T_{a_{2g}}$

only appears in ~~disjointness relns &~~ one braid rel.

In that braid reln it appears with exponent sum $= 1$.

Q. Can we show this class is nonzero?

But... elts of $[F,F]$, hence H_2, have all exp sums $= 0$.

So $n_{2g} = 0$.

Q. Can we show H_3 is stable using similar idea?

Now have a finite lin. alg problem involving chain reln, lantern reln, a few braid relns:

Which choices of $\underbrace{n_0, n_1, n_2, n_3, n_4}_{\text{exps on braid relns}}, n_c, n_L$ make it so each MCG gen appears with exp sum 0?

Answer: 1 choice!

Lower bound: Constructing nonzero elts of H^2

Fact. A short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

with $\mathbb{Z}$ central gives $e \in H^2(G;\mathbb{Z})$

and $e = 0 \iff$ seq. is split
$\iff \tilde{G} \cong G \times \mathbb{Z}$

But we have:

$$1 \longrightarrow \underbrace{\langle T_\partial \rangle}_{\cong\, \mathbb{Z}} \longrightarrow \mathrm{Mod}(S_g^1) \xrightarrow{\text{cap } \partial} \mathrm{Mod}(S_{g,1}) \longrightarrow 1$$

Non-split since $\mathrm{Mod}(S_{g,1})$ has torsion $\rightsquigarrow \underset{\text{Euler class}}{e} \in H^2(\mathrm{Mod}(S_{g,1});\mathbb{Z})$.

Meyer Signature Cocycle

Still need an elt of $H^2(\mathrm{Mod}(S_g);\mathbb{Z})$.

Q. What is an elt of H^2? A. $\mathrm{Hom}(H_2(\mathrm{Mod}(S_g)),\mathbb{Z})$.

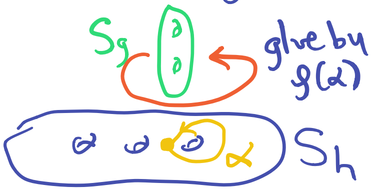
So, given elt of $H_2(\mathrm{Mod}(S_g))$, need a number.

Q. What is an elt of $H_2(\mathrm{Mod}(S_g))$? A. Surface in $K(\mathrm{Mod}(S_g),1)$

$$S_h \to K(\mathrm{Mod}(S_g),1)$$

$$\rho: \pi_1(S_h) \to \mathrm{Mod}(S_g).$$

The latter gives S_g-bundle over S_h (4-manifold):



4-manifolds have signature (describes intersection form on $H_2(M^4)$).

Signature is the desired number!