Section 2.3

Matrix equations

Outline Section 2.3

- Understand the equivalences:

  linear system $\leftrightarrow$ augmented matrix $\leftrightarrow$ vector equation $\leftrightarrow$ matrix equation

- Understand the equivalence:

  $$Ax = b \text{ is consistent} \longleftrightarrow b \text{ is in the span of the columns of } A$$

  (also: what does this mean geometrically)

- Learn for which A the equation $Ax = b$ is always consistent
- Learn to multiply a vector by a matrix

Multiplying Matrices

$$\text{matrix} \times \text{column} : \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} | & | & & | \\ b_1 x_1 & b_2 x_2 & \cdots & b_n x_n \\ | & | & & | \end{pmatrix}$$

Read this as: b_1 times the first column x_1 is the first column of the answer, b_2 times x_2 is the second column of the answer...

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} =$$

Multiplying Matrices

Another way to multiply

$$\text{row vector} \times \text{column vector} : \begin{pmatrix} a_1 & \cdots & a_{\color{red}n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{\color{red}n} \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$$

$$\text{matrix} \times \text{column vector} : \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} =$$

Linear Systems vs Augmented Matrices vs Matrix Equations vs Vector Equations

A matrix equation is an equation $Ax = b$ where A is a matrix and b is a vector. So x is a vector of variables.

A is an $m \times n$ matrix if it has m rows and n columns.
What sizes must x and b be?

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$$

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.

Solving matrix equations

Solve the matrix equation

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix}$$

What does this mean about rabbits?

Solutions to Linear Systems vs Spans

Say that

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

algebra $\iff$ geometry

Why?

Again this is a basic fact we will use over and over and over.

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

Is a given vector in the span?

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

algebra $\iff$ geometry

Is $(9, 10, 11)$ in the span of $(1, 3, 5)$ and $(2, 4, 6)$?

Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,10,20)$, $(0,-1,-2)$
2. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,1,0)$, $(0,0,\sqrt{2})$
3. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,5,7)$, $(0,6,8)$
4. $(0,1,2)$ is in the span of $(5,7,0)$, $(6,8,0)$, $(3,3,4)$

Pivots vs Solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

More generally, if you have some vectors and you want to know the dimension of the span, you should row reduce and count the number of pivots.

Properties of the Matrix Product Ax

$c =$ real number, $u, v =$ vectors,

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

Application. If u and v are solutions to $Ax = 0$ then so is every element of $\mathrm{Span}\{u, v\}$.

Guiding questions

Here are the guiding questions for the rest of the chapter:

1. What are the solutions to $Ax = 0$?

2. For which b is $Ax = b$ consistent?

These are two separate questions!

Summary of Section 2.3

- Two ways to multiply a matrix times a column vector:

$$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

OR

$$\begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_{n} \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{n} \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 x_1 & \cdots & b_n x_n \\ | & & | \end{pmatrix}$$

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.
- Fact. $Ax = b$ has a solution $\Leftrightarrow$ b is in the span of columns of A
- Theorem. Let A be an $m \times n$ matrix. The following are equivalent.
  1. $Ax = b$ has a solution for all b
  2. The span of the columns of A is $\mathbb{R}^m$
  3. A has a pivot in each row

Typical exam questions

- If A is a 3×5 matrix, and the product Ax makes sense, then which $\mathbb{R}^n$ does x lie in?

- Rewrite the following linear system as a matrix equation and a vector equation:

$$x + y + z = 1$$

- Multiply:

$$\begin{pmatrix} 0 & 2 \\ 0 & 4 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

- Which of the following matrix equations are consistent?

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

(And can you do it without row reducing?)