10. Teich Space.

$$\mathrm{Teich}(S) = \{\text{hyp metrics}\}/\text{isotopy}$$

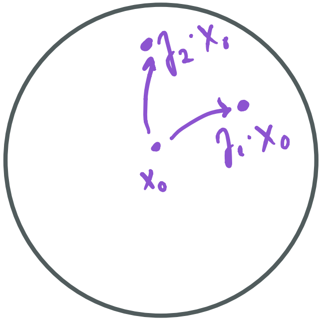
$$= \{(X, \varphi)\}/\sim$$

↑ hyp surf

$$\varphi : S \longrightarrow X$$

$$= DF(\pi_1(S_g), PSL_2\mathbb{R})/PGL_2\mathbb{R}$$

$\rightsquigarrow$ topology



Note: Teich(S) can intuitively be seen to be a manifold. Which is it?

<u>Dimension count</u>

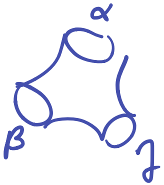
$+6g$: choosing $\rho(\gamma_1), \dots, \rho(\gamma_{2g})$ in $PSL_2\mathbb{R}$

-3 : surface relation.

-3 : conjugation

$6g-6$

Pants

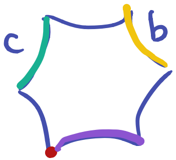
$P =$ 

Thm. The map

$$\mathrm{Teich}(P) \longrightarrow \mathbb{R}^3$$
$$X \longmapsto (\ell_X(\alpha), \ell_X(\beta), \ell_X(\gamma))$$

is a homeo.

Setup: A marked hyp. right-angled hexagon



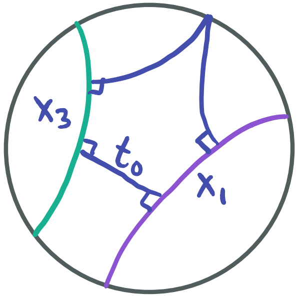
a, b, c counterclockwise

$\mathcal{H}$ = set of these / marked isometry.

Lemma. The map $\mathcal{H} \to \mathbb{R}^3$

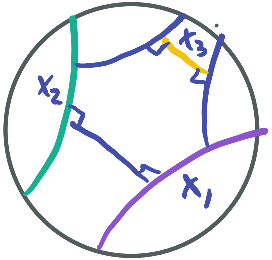
$$H \longmapsto (\ell_H(a), \ell_H(b), \ell_H(c))$$

is a bijection.

Pf. 

Start with (x_1, x_2, x_3) in $\mathbb{R}^3$

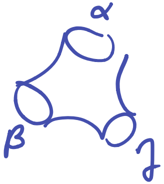
Increase t_0 until get right hexagon



(IVT)

$\square$

Pants

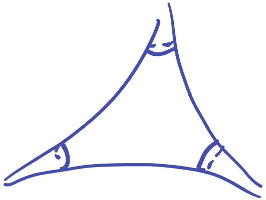
$P =$ 

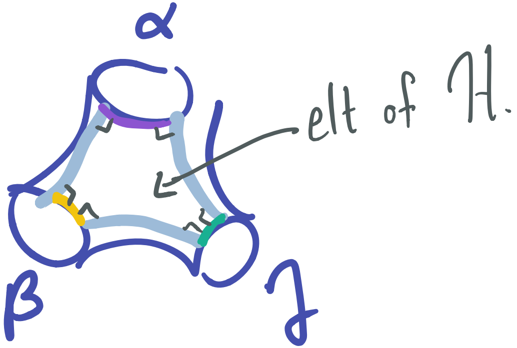
Thm. The map

$$\mathrm{Teich}(P) \longrightarrow \mathbb{R}^3$$
$$X \longmapsto (\ell_X(\alpha), \ell_X(\beta), \ell_X(\gamma))$$

is a homeo.

Pf. Draw the geodesics connecting components of P





Also, components of ∂P are cut exactly in half. (by Lemma).

Continuity ✓ $\square$.

Also: $\mathrm{Teich}(S_{0,3}) = *$

Fenchel-Nielsen Coords

Thm $\mathrm{Teich}(S_g) \cong \mathbb{R}^{6g-6}$

$3g-3$ length params

$3g-3$ twist params.

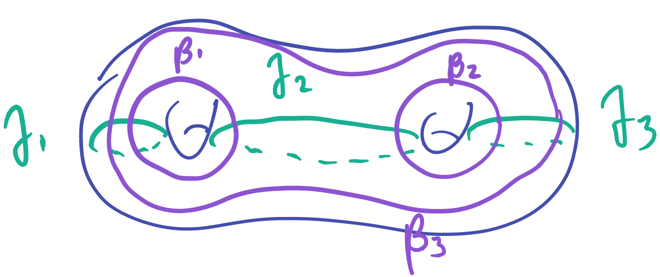
Setup:

$\gamma_1, \dots, \gamma_{3g-3}$ pants decomp

$\beta_1, \dots, \beta_n$ seams:

$(\cup \beta_i) \cap$ one pants

$= 3$ distinct arcs



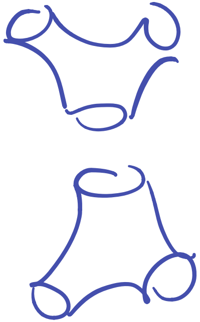
Length params: $\ell_X(\gamma_i)$

these tell us the metric on each pants (by last Thm)

Twist params: harder. how the pants are glued together.

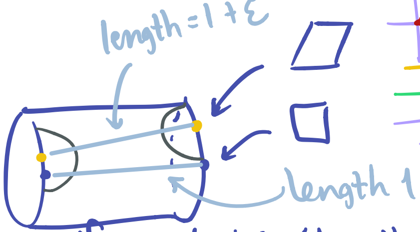
Twist parameters

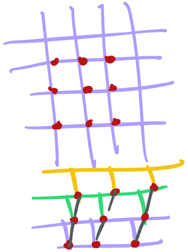
Given



If you twist before gluing, get different metrics on S_0^4

Similar:



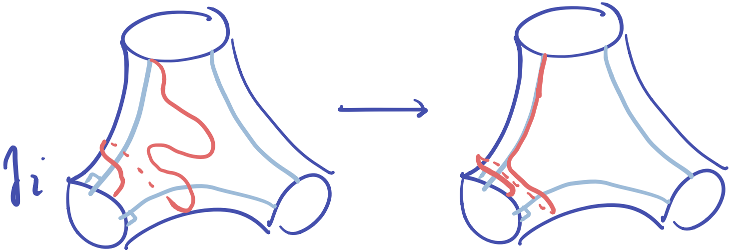


Get different tori if you twist (length spectrum)

For an arc α^* in $X \in \mathrm{Teich}(P)$

$\rightsquigarrow$ twisting about $\partial_i X$

$*$ homotopy class rel ∂X



twisting $= 2\pi + \varepsilon$

Given $X \in \mathrm{Teich}(S_g)$ & $i \in \{1, \dots, 3g-3\}$

Choose seam β_j crossing γ_i

$\rightsquigarrow$ twisting on left/right of γ_i

$$\Theta_i(X) = 2\pi \frac{t_L - t_R}{\ell(\gamma_i)}$$

Pf of Thm

Given $\ell_1, \dots, \ell_{3g-3}$
$\Theta_1, \dots, \Theta_{3g-3}$.

Want to construct unique X with those coords.

Step 1. Make disj union of pairs of pants according to ℓ_i.

Step 2. Draw seams according to Θ_i

Step 3 Glue pants so seams match up. $\rightsquigarrow X$

Step 4. Build marking $\varphi: S \to X$ by change of coords. $\square$

The $9g-9$ Thm

<u>Thm</u> $\exists \{\delta_1, \dots, \delta_{9g-9}\}$

s.t.

$$\mathrm{Teich}(S_g) \longrightarrow \mathbb{R}^{9g-9}$$
$$X \longmapsto (\ell_X(\delta_i))$$

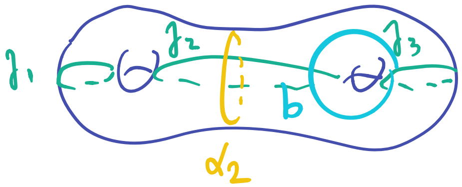
is injective.

<u>Prop</u>. Let X_s be a 1-param family in $\mathrm{Teich}(S_g)$ given by changing i^{th} twist param. & b = curve crossing γ_i

Then the fn $\mathbb{R} \to \mathbb{R}_+$

$$s \longmapsto \ell_{X_s}(b)$$

is strictly convex.



X_s obtained by varying this twist coord

<u>Pf</u>. The $9g-9$ curves are:

$\gamma_1, \dots, \gamma_{3g-3}$

$\alpha_1, \dots, \alpha_{3g-3}$ any curves with

$$i(\alpha_i, \gamma_j) \neq 0 \iff i = j$$

$\beta_1, \dots, \beta_{3g-3}$

$$\beta_i = T_{\gamma_i}(\alpha_i)$$

Pf. The $9g-9$ curves are:

$\gamma_1, \ldots, \gamma_{3g-3}$

$\alpha_1, \ldots, \alpha_{3g-3}$ any curves with

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$\beta_1, \ldots, \beta_{3g-3}$

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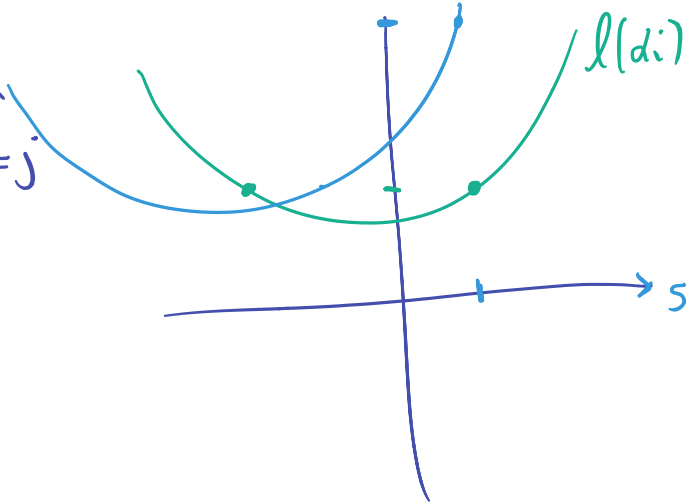
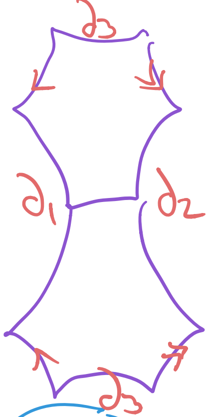


By design:

$$\ell_{X_s}(\alpha_i) = \ell_{X_{s+2\pi}}(\beta_i)$$

X_s = family corresponding to γ_i

$\square$


∂_3
∂_1
∂_2
∂_3