THE SYMPLECTIC REPRESENTATION OF MCG

The symplectic group

Consider $\mathbb{R}^{2g}$ with basis $(x_1,\ldots,x_g,y_g,\ldots,y_1)$ and standard symplectic form

$$\omega = \sum_{i=1}^{g} dx_i \wedge dy_i$$

Think of ω as a <u>pairing</u> on $\mathbb{R}^{2g}$ e.g.

$$\omega(x_1+2y_2,\ x_1+y_1+x_2) = 1-2 = -1$$

This is the <u>unique</u> nondegenerate, alternating bilinear form on $\mathbb{R}^{2g}$ up to change of basis.

Connection to surfaces:

$$(\mathbb{R}^{2g}, \omega) \cong (H_1(S_g;\mathbb{R}), \hat{i})$$

$Sp_{2g}(\mathbb{R})$ = subgp of $GL_{2g}\mathbb{R}$ preserving ω:

$$\omega(u,v) = \omega(Mu, Mv)$$

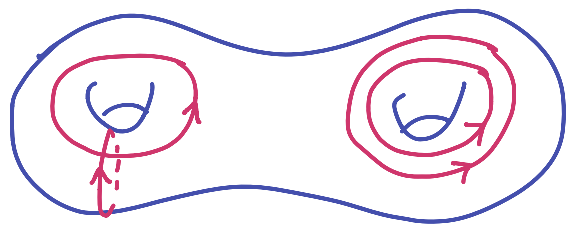
Similar with $\mathbb{Z}$.

Realizing H_1-classes by curves.

<u>Prop.</u> If $v \in H_1(S_g;\mathbb{Z})$ is primitive then $v = [c]$ where c is an oriented simple closed curve.

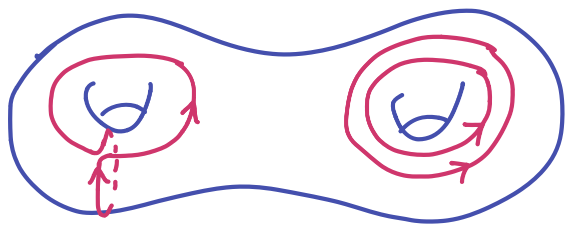
<u>Pf</u> (Meeks-Patrusky). Euclidean algorithm for scc's.

Step 1. Draw v naïvely:

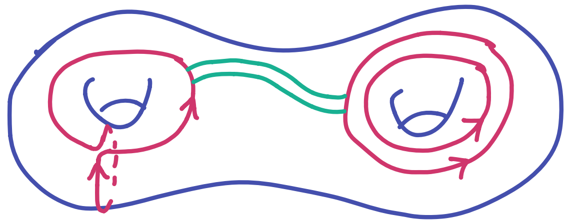


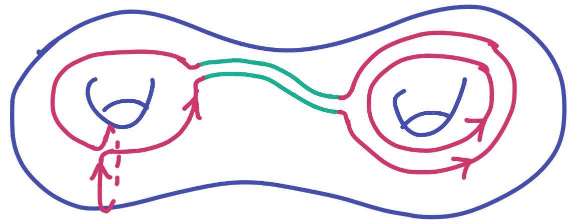
$v = x_1 + y_1 + 2x_2$

Step 2. Surger to remove crossings.



Step 3. Band surgeries to reduce the number of components





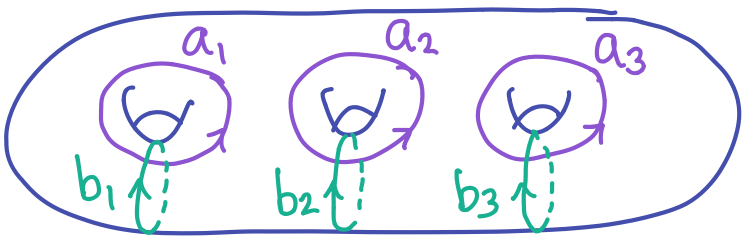
By Euclidean algorithm, this terminates in a connected curve! $\square$

Action of a Dehn Twist

<u>Prop.</u> Say a, b = oriented curves
Then $T_b^k([a]) = [a] + k\,\hat{i}(a,b)[b]$

N.B. Indep of orientation of b!.

A geometric symplectic basis:



<u>Proof.</u> Case 1. b separating

Choose <u>geometric</u> symplectic basis for $H_1(S_g)$ disjoint from b.

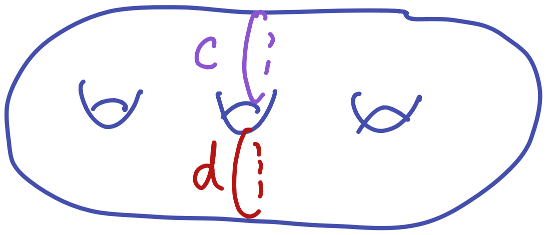
Case 2. b nonseparating.

Choose a geometric symplectic basis so b is one curve. Check for a = basis elt. Apply linearity of $\psi(T_b^k)$.

Some Elements of Torelli.

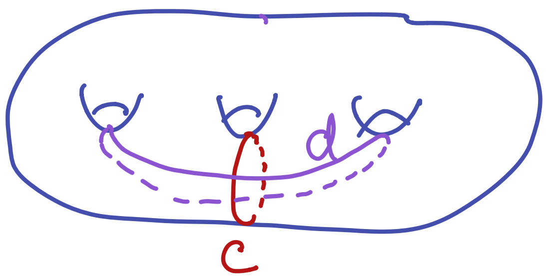
1. Dehn twists about sep curves

2. Bounding pair maps



$T_c T_d^{-1}$

3. Commutators of simply intersecting pairs



$[T_c, T_d]$

2 & 3 are special cases of:

$T_c T_d^{-1}$ where $[c] = [d]$

That all of these lie in Torelli follows immediately from the Prop.

4. $\pi_1(S_g) \trianglelefteq \mathrm{Mod}(S_{g,1})$

SURJECTIVITY OF THE Sp-REP

Thm $\Psi : \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z})$ is surjective.

$\boxed{\text{1st proof: transvections.}}$

A transvection in $Sp_{2g}(\mathbb{Z})$ is an elt whose 1-eigenspace is $(2g-1)$-dim.

$$\Updownarrow$$

$$\tau_v(u) = u + \omega(u,v)v \qquad \text{(or a power)}$$

Fact. $Sp_{2g}(\mathbb{Z})$ is gen. by transvections.

Pf of Thm. Suffices to hit τ_v, v primitive.
Prop $\rightsquigarrow$ a s.t. $[a] = v$
$\Psi(T_a) = \tau_v$ $\square$

Want a proof that does not presuppose a genset for Sp.

2^{nd} proof: geometric symplectic bases

$\mathrm{Sp}_{2g}(\mathbb{Z}) \longleftrightarrow$ symplectic bases for $\mathbb{Z}^{2g}$

$I \longleftrightarrow$ standard symplectic basis.

Proof of Thm. Given $A \in \mathrm{Sp}_{2g}(\mathbb{Z})$, realize A as a geometric symplectic basis (supe up the proof of Prop above).

Realize I by standard geometric symplectic basis.

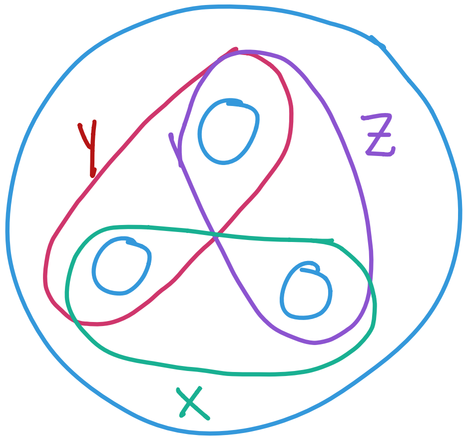
Apply change of coordinates: given two topologically equivalent configurations of curves, there is an element of $\mathrm{Mod}(S_g)$ taking one to the other. $\square$

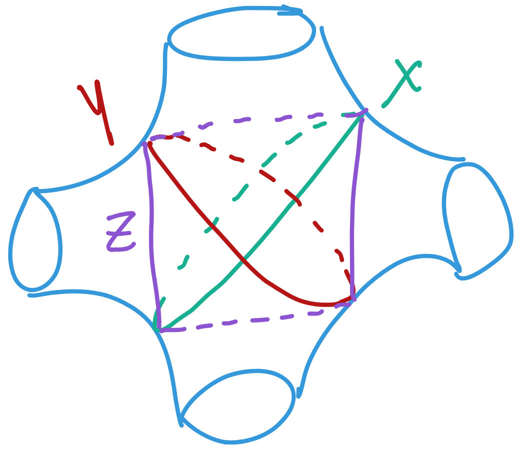
Corollaries of the proof:

1. $\mathcal{I}(S_g)$ acts transitively on s.c.c. reps of a homology class.
2. $\mathcal{I}(S_g)$ acts trans. on sep curves/BPs inducing same splitting.

$\Longrightarrow$ conjugacy classification of such elts.

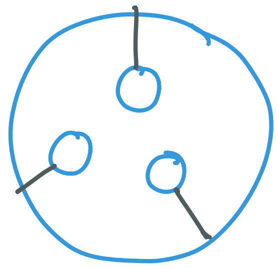
THE LANTERN RELATION.





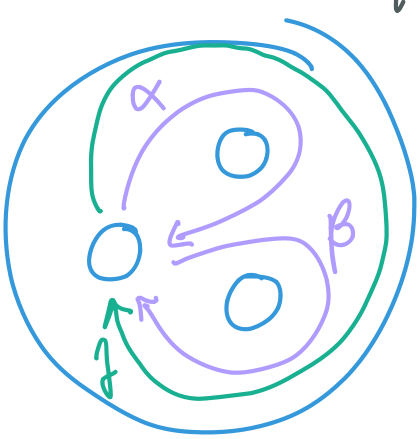
$$T_x T_y T_z = \prod_{i=1}^{4} T_{\partial_i}$$

Pf #1. Check action on



By symmetry, enough to check one (why?). Apply Alexander trick.

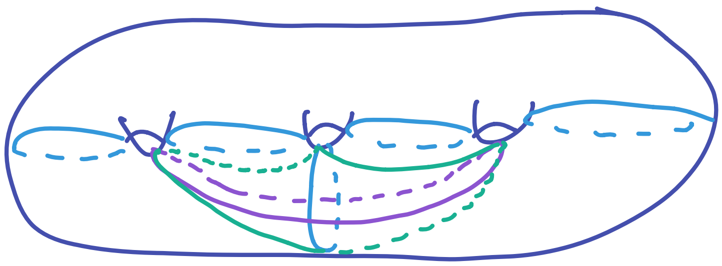
Pf #2. Consider push maps:



Note $\alpha\beta = \gamma$. Reinterpret as lantern relation.

Consequences of Lantern Relation.

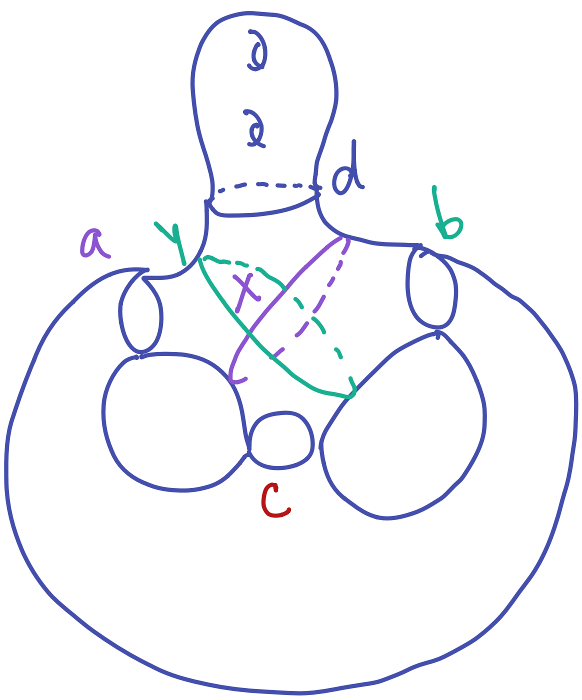
1. <u>Theorem</u> $\mathrm{Mod}(S_g)^{ab} = 1 \qquad g \geq 3$



<u>Pf.</u> All 7 twists are conjugate
$\Rightarrow$ have same image t in $\mathrm{Mod}(S_g)^{ab}$
Lantern $\Rightarrow t^3 = t^4 \Rightarrow t = 1$.

2. <u>Prop</u> Dehn twists about sep curves are products of BP maps.

<u>Pf.</u>



Lantern relation
$\Rightarrow$

$$T_d = (T_x T_a^{-1})(T_y T_b^{-1})(T_z T_c^{-1})$$

Torelli Groups Are Torsion Free

$\boxed{\text{Black Box}}$ If $f \in \mathrm{Mod}(S)$ has finite order, then $f = [\varphi]$ st φ = isometry

Theorem $\mathcal{I}(S_g)$ is torsion free

Pf. Say $f \in \mathcal{I}(S_g)$ has finite order
Black box $\rightsquigarrow$ φ = isometry.
Lefschetz FPT:

$$\sum_{p = \text{fix pt}} \mathrm{ind}_p(\varphi) = \sum (-1)^i \,\mathrm{trace}\left(\varphi_* : H_i(S_g) \circlearrowleft\right)$$

$$\#\text{ fixed pts} = 1 - 2g + 1 < 0 \qquad \square$$

Or use...

Theorem (Lanier–M) Let $g \geq 3$, $f \in \mathrm{Mod}(S_g)$ finite order.

$$\Longrightarrow \quad \langle\langle f \rangle\rangle = \begin{cases} \mathrm{Mod}(S_g) & f \neq \text{hyp. inv} \\ \text{Ext. } \mathcal{I}(S_g) & f = \text{hyp. inv} \end{cases}$$