FACTORING IN THE HYPERELLIPTIC TORELLI GROUP

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Abstract. The hyperelliptic Torelli group is the subgroup of the mapping class group consisting of elements that act trivially on the homology of the surface and that also commute with some fixed hyperelliptic involution. Hain conjectured that this group is generated by Dehn twists about symmetric separating curves, that is, separating curves fixed by the hyperelliptic involution. We describe two basic types of elements of the hyperelliptic Torelli group, and give explicit factorizations into Dehn twists about symmetric separating curves. Our factorizations correspond to new, intricate relations in the pure braid group.

1. Introduction

Let $s : S_g \to S_g$ be a hyperelliptic involution of a closed, connected, orientable surface of genus g; see Figure 1. The hyperelliptic Torelli group $\mathcal{SI}(S_g)$ is the group of isotopy classes of homeomorphisms of S_g that commute with S_g and act trivially on $H_1(S_g)$.

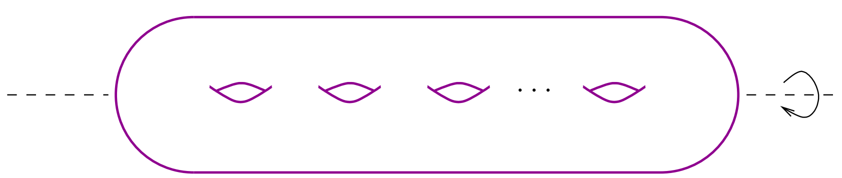

Figure 1. Rotation by π about the indicated axis is a hyperelliptic involution.

The first example of a nontrivial element of $\mathcal{SI}(S_g)$ is a Dehn twist about a *symmetric* separating curve in S_g, by which we mean a separating curve that is fixed by s. Indeed, any Dehn twist about a separating

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curve acts trivially on $H_1(S_g)$, and the Dehn twist about a symmetric curve commutes with s.

It follows from the work of Johnson that each element of $\mathcal{SI}(S_g)$ can be written as a product of Dehn twists about (possibly asymmetric) separating curves ([9, Lemma 2D] plus the main result of [10]).

Hain has conjectured that $\mathcal{SI}(S_g)$ is in fact generated by Dehn twists about symmetric separating curves [7, Conjecture 1]; see also Morifuji [13, Section 4]. In this paper, we give evidence for the conjecture by factoring two particular basic types of elements of $\mathcal{SI}(S_g)$ into Dehn twists about symmetric separating curves. Along the way, we will uncover several other useful relations in $\mathcal{SI}(S_g)$.

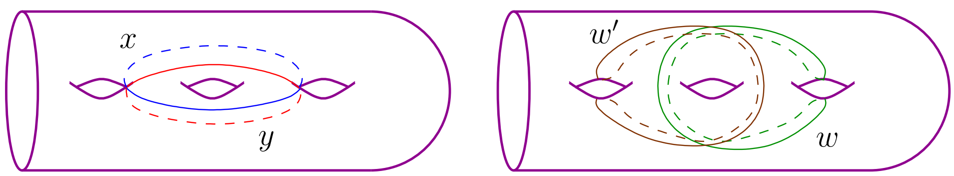

FIGURE 2. The curves x and y form a symmetric simply intersecting pair.

Symmetric simply intersecting pair maps. The first type of element of $\mathcal{SI}(S_g)$ we will factor is a *symmetric simply intersecting pair map*. This is a commutator of two Dehn twists

$$[T_x, T_y]$$

where x and y are symmetric nonseparating curves in S_g that intersect twice and have trivial algebraic intersection; see the left-hand side of Figure 2 (we should imagine this surface as a subsurface of the surface in Figure 1). Since x and y are symmetric, each twist commutes with s, and hence so does the commutator. Also, whenever two curves in S_g have trivial algebraic intersection, the commutator of their Dehn twists acts trivially on $H_1(S_g)$.

We prove the following theorem in Section 2.

Theorem 1.1. *Each symmetric simply intersecting pair map in S_g is equal to a product of two Dehn twists about symmetric separating curves. In particular, if x, y, w, and w' are the simple closed curves shown in Figure 2, we have:*

$$[T_x, T_y] = T_w^{-1} T_{w'}.$$

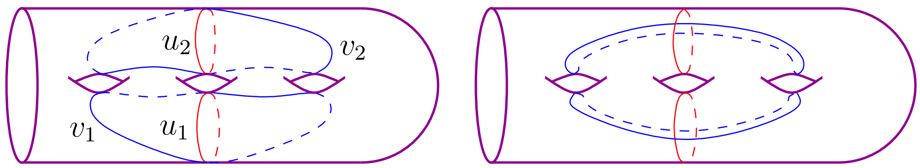

FIGURE 3. Left: the curves u_1, v_1, u_2, and v_2 form a symmetrized simply intersecting pair. Right: an alternate configuration giving a symmetrized simply intersecting pair.

Symmetrized simply intersecting pair maps. The second type of element of $\mathcal{SI}(S_g)$ we will factor is a *symmetrized simply intersecting pair map*, by which we mean a commutator

$$[T_{u_1}T_{u_2}, T_{v_1}T_{v_2}],$$

where u_1, u_2, v_1, and v_2 are configured as in the left-hand side Figure 3 (an alternate configuration is shown on the right-hand side). Precisely, u_1, v_1, u_2, and v_2 are nonseparating curves with $|u_1 \cap v_1| = 2$, $\widehat{i}(u_1, v_1) = 0$, $s(u_1) = u_2$, $s(v_1) = v_2$, and $(u_1 \cup v_1) \cap (u_2 \cap v_2) = \emptyset$. We learned about symmetrized simply intersecting pair maps from Andy Putman.

A *bounding pair map on symmetric separating curves* is a product $T_x T_y^{-1}$ where x and y are disjoint symmetric separating curves. Such a bounding pair map can be realized by rotating the subsurface between x and y by 2π.

Theorem 1.2. *Each symmetrized simply intersecting pair map in S_g is equal to a product of three bounding pair maps on symmetric separating curves. Specifically, for curves shown in Figure 4, we have*

$$[T_{u_1}T_{u_2}, T_{v_1}T_{v_2}] = \left(T_{c_1}T_{c_2}^{-1}\right)\left(T_{c_3}T_{c_4}^{-1}\right)\left(T_{c_5}T_{c_6}^{-1}\right).$$

The configuration of curves in Theorem 1.2 is really rather simple. If we reorder the twists so that the inverse twists come first in each pair, then each twist commutes with the twists immediately preceding and following. In other words, in the list $c_2, c_1, c_4, c_3, c_6, c_5$, each curve is disjoint from the next.

We prove Theorem 1.2 in Section 3. While the proof of Theorem 1.1 is quite simple, the proof of Theorem 1.2 is more involved. As intermediate steps, we prove two auxiliary relations: first a relation in $\mathcal{SI}(S_2)$, and then a lift of this relation to $\mathcal{SI}(S_g)$. For the moment, we content ourselves to explain the relation in $\mathcal{SI}(S_2)$.

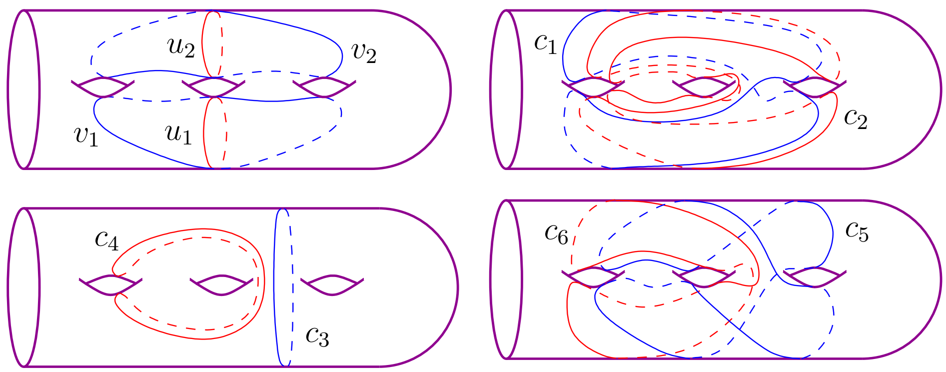

FIGURE 4. The curves used in Theorem 1.2.

A surprising relation in genus two. Consider the product of Dehn twists

$$[T_b T_d^{-1}, T_a T_c^{-1}] T_e^2$$

in S_2, where the curves are as shown in the left hand side of Figure 5. This product acts trivially on $H_1(S_2)$. Since it is also the case that each curve involved is fixed by s, this product lies in $\mathcal{SI}(S_2)$.

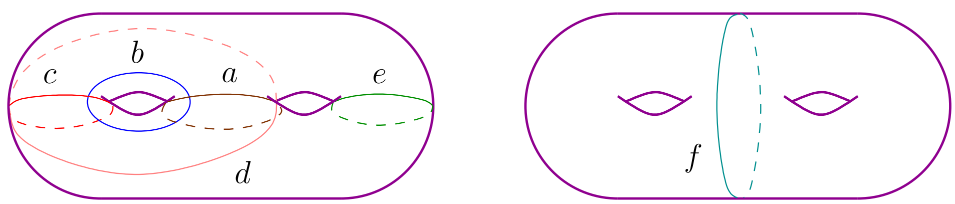

FIGURE 5. The curves a, b, c, d, e, and f from Theorem 1.3.

As above, each element of $\mathcal{SI}(S_g)$ is equal to a product of Dehn twists about separating curves (not necessarily symmetric). So, for the given commutator, which separating curves are they? We will prove the following theorem in Section 4.

Theorem 1.3. *Let a, b, c, d, e, and f be as in Figure 5. We have:*

$$[T_b T_d^{-1}, T_a T_c^{-1}] T_e^2 = T_f.$$

In Section 4, we explain how Theorem 1.3 gives rise to an analogous relation in $\mathcal{SI}(S_g)$ for $g \geq 2$.

Relations in the braid group. By the Birman–Hilden theorem (Section 3.1 below), the relations from Theorems 1.1 and 1.2 immediately give relations in the mapping class group of the disk with seven marked

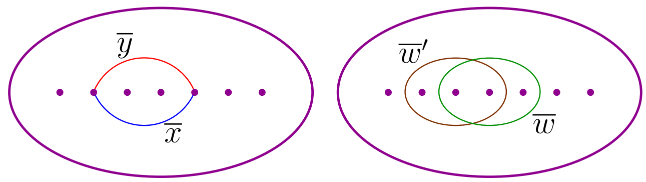

FIGURE 6. The curves used to translate the relation from Theorem 1.1 into a relation in B_7.

points, or, what is the same thing, the braid group B_7. The relations are

$$[T_{\overline{x}}^{1/2}, T_{\overline{y}}^{1/2}] = T_{\overline{w}}^{-2} T_{\overline{w}'}^{2}$$

and

$$[T_{\overline{u}}, T_{\overline{v}}] = T_{\overline{c}_1}^{2} T_{\overline{c}_2}^{-2} T_{\overline{c}_3}^{2} T_{\overline{c}_4}^{-2} T_{\overline{c}_5}^{2} T_{\overline{c}_6}^{-2},$$

where $T_{\overline{x}}^{1/2}$ and $T_{\overline{y}}^{1/2}$ denote the half-twists about the arcs $\overline{x}$ and $\overline{y}$, and where all curves and arcs are as shown in Figures 6 and 7.

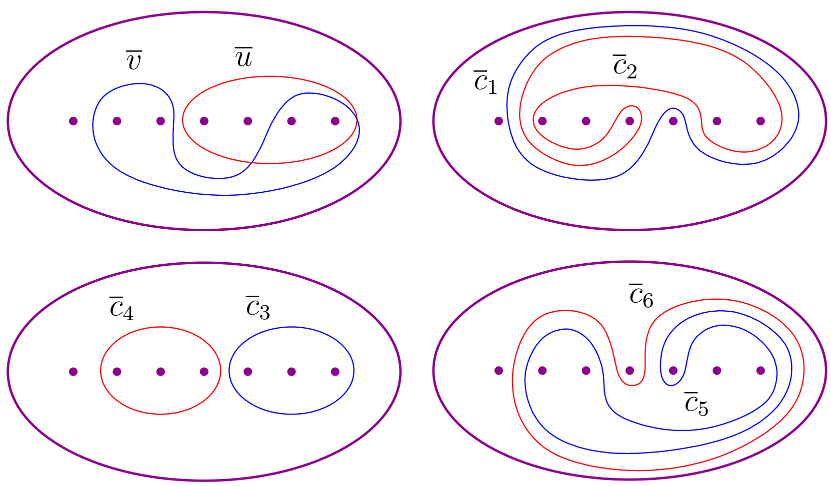

FIGURE 7. The curves used to translate the relation from Theorem 1.2 into a relation in B_7.

On the origin of relations. Most known natural relations between Dehn twists—for instance the braid relation [6, Proposition 3.11], the chain relation [6, Proposition 4.12], the lantern relation [6, Proposition 5.1], the hyperelliptic relation [6, Section 5.1.4], the star relation [6, Section 5.2.3], the crossed lantern relation [15, Section 1], the generalized lantern relations [16], the daisy relation [5, Section 1] and [14, Lemma 3.5], the Matsumoto fibration relation [1, Example 3.9], etc.—are easily obtained by considering the relationships between mapping class groups on the one hand and braid groups, Artin groups, fundamental groups of configuration spaces, and the fundamental group of the unit tangent bundle of a surface on the other hand; see, e.g., [11, Section 4], [6, Section 5.1], [15, Section 1], [2, Section 7], and [8] for more discussion. Matsumoto in fact gives a complete presentation of the mapping class group in terms of Artin group relations [12].

Theorem 1.1 is rather easily obtained from the lantern relation. At first, the relation from Theorem 1.2 did not seem to have such a simple origin. However, in the end, we did find a relatively simple interpretation in terms of push maps; see the second proof of that theorem. The simplicity of the latter approach suggests that the group $\mathcal{SI}(S_g)$ is more tractable than previously thought.

Further remarks. Any relation between Dehn twists, once discovered, can be verified using the Alexander method of [6, Proposition 2.8] or, in the case of a punctured disk, using any of the solutions to the word problem for the braid group. More interesting are the methods for discovering relations. In this paper, we explain how to derive all of our relations from a few general principles, namely, the Birman exact sequence, the lantern relation, the Witt–Hall identity, and the Birman–Hilden theorem.

Also, it follows from the results of our earlier paper [3] that any element of $\mathcal{SI}(S_g)$ that is supported on a subsurface of genus three and fixes an essential curve in that subsurface is a product of Dehn twists about symmetric separating curves. The point of this paper is to give explicit factorizations.

Finally, while the goal in this paper is to factor certain elements of $\mathcal{SI}(S_g)$ into Dehn twists about symmetric separating curves, it does not appear to be an easier problem to factor them into Dehn twists about arbitrary separating curves.

Conventions. In this paper, Dehn twists are to the left and a product of Dehn twists is applied right to left. Elements of the fundamental group are multiplied left to right. This may lead to some confusion, since we will often relate elements of the fundamental group to elements in the mapping class group. As such, we point out the places where we need to switch from left-to-right multiplication to right-to-left.

Also, we define the *mapping class group* of a surface S to be the group $\mathrm{Mod}(S)$ of isotopy classes of orientation-preserving homeomorphisms of S that restrict to the identity on ∂S and preserve the set of marked points on S.

Acknowledgments. First, we would like to thank Andy Putman for suggesting the problem of factoring the symmetrized simply intersecting pair maps. We would also like to thank Joan Birman and Leah Childers for helpful conversations. We would not have been able to discover Theorem 1.2 in its present form without the aid of a Maple computer program β-Twister, written by Marta Aguilera and Juan González-Meneses, that allows the user to compute the image of a simple closed curve under a braid. This program made it possible to not only check our relations, but to simplify them tremendously.

2. Factoring symmetric simply intersecting pair maps

In this section we prove Theorem 1.1. The theorem will follow from the lantern relation, which we now recall.

The lantern relation. Let x and y be simple closed curves in a surface S. Assume that x and y intersect in two points and have algebraic intersection zero. A closed regular neighborhood of $x \cup y$ is a sphere with four boundary components, say d_1, d_2, d_3, and d_4. Within this sphere, we can find a curve w as shown in Figure 8. The *lantern relation*, due to Dehn [6, Section 5.1.1], states that

$$T_w T_x T_y = T_{d_1} T_{d_2} T_{d_3} T_{d_4}$$

in the mapping class group $\mathrm{Mod}(S)$. If we choose the curve w' instead of w, we obtain $T_{w'} T_y T_x = T_{d_1} T_{d_2} T_{d_3} T_{d_4}$ (this follows from the previous relation, since there is an orientation-preserving homeomorphism of the sphere taking the triple (w, x, y) to (w', y, x), and the T_{d_i} commute pairwise).

Proof of Theorem 1.1. Consider the configuration of curves shown in Figure 9. By the lantern relation, we have that $T_w T_x T_y = T_{d_1}^2 T_{d_2} T_{d_3}$ and that $T_{w'} T_y T_x = T_{d_1}^2 T_{d_2} T_{d_3}$.

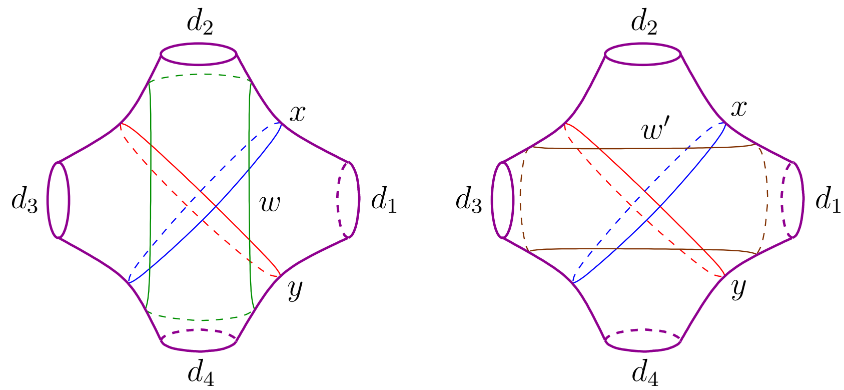

FIGURE 8. The curves used in the lantern relation.

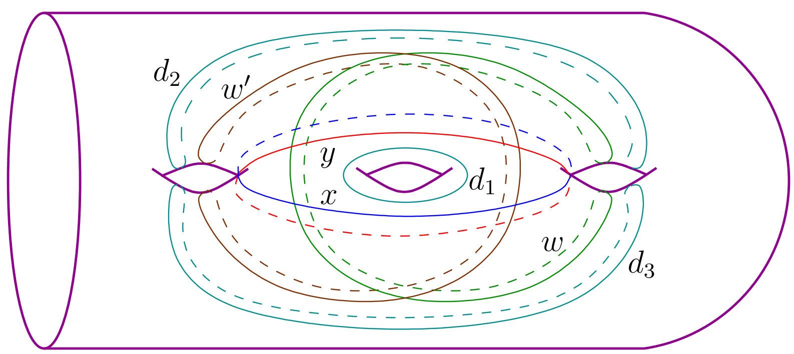

FIGURE 9. The curves used in writing $[T_x, T_y]$ as $T_w^{-1}T_{w'}$.

Thus, we have:

$$\begin{aligned} [T_x, T_y] &= (T_xT_y)(T_x^{-1}T_y^{-1}) \\ &= (T_w^{-1}T_{d_1}^2T_{d_2}T_{d_3})(T_{d_3}^{-1}T_{d_2}^{-1}T_{d_1}^{-2}T_{w'}) \\ &= T_w^{-1}T_{w'} \end{aligned}$$

This completes the proof. □

3. Factoring symmetrized simply intersecting pair maps

In this section we will give two proofs of Theorem 1.2. In particular, we will factor the symmetrized simply intersecting pair map $[T_{u_1}T_{u_2}, T_{v_1}T_{v_2}]$ from the left hand side of Figure 3 into a product of six Dehn twists about symmetric separating curves. Both proofs use the Birman–Hilden theorem and the Birman exact sequence, which we now recall.

3.1. **The hyperelliptic mapping class group and the Birman–Hilden theorem.** Let S_g^1 denote the surface obtained from S_g by removing an s-invariant open neighborhood of one of the fixed points of s. There is an induced hyperelliptic involution s of S_g^1, and we can define $\mathrm{SMod}(S_g^1)$ as the group of homotopy classes of homeomorphisms of S_g^1 that commute with s and fix the boundary pointwise.

A special case of the Birman–Hilden theorem gives that isotopic s-equivariant homeomorphisms of S_g^1 are s-equivariantly isotopic [6, Section 9.4.1]. It follows that there is a well-defined map $\mathrm{SMod}(S_g^1) \to \mathrm{Mod}(S_g^1/\langle s\rangle)$; in our situation, this map turns out to be an isomorphism, and so $\mathrm{SMod}(S_g^1)$ is isomorphic to the mapping class group of $S_g^1/\langle s\rangle$, a disk D_{2g+1} with $2g+1$ cone points (for our purposes, cone points should be thought of as marked points). The mapping class group $\mathrm{Mod}(D_{2g+1})$ is isomorphic to the braid group on $2g+1$ strands.

3.2. **The Birman exact sequence.** Let S be a surface obtained from a compact surface by removing finitely many points from the interior (or marking them). Assume that $\chi(S) < 0$. Let $p \in S$, and denote by $\mathrm{Mod}(S, p)$ the group of isotopy classes of orientation-preserving homeomorphisms of S that fix p and ∂S pointwise (and preserve the set of marked points). The forgetful map $(S, p) \to S$ induces a surjective homomorphism $\mathrm{Mod}(S, p) \to \mathrm{Mod}(S)$. The Birman exact sequence describes the kernel:

$$1 \to \pi_1(S, p)^{op} \xrightarrow{\mathcal{P}ush} \mathrm{Mod}(S, p) \to \mathrm{Mod}(S) \to 1.$$

For $\alpha \in \pi_1(S, p)$, the map $\mathcal{P}ush(\alpha)$ is the class of a homeomorphism obtained by dragging p along α; see [6, Section 4.2]. Here, $\pi_1(S, p)^{op}$ is the opposite group of $\pi_1(S, p)$, namely, the group obtained from $\pi_1(S, p)$ by reversing the order of multiplication. So for a word in $\pi_1(S, p)^{op}$, loops are concatenated right to left.

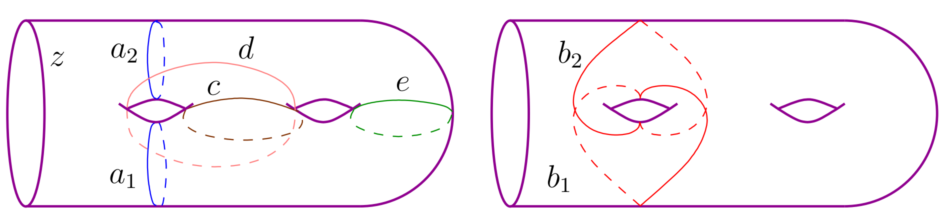

FIGURE 10. A symmetric configuration of curves in S_2^1.

3.3. **First proof.** The first proof of Theorem 1.2 proceeds by breaking the commutator $[T_{u_1}T_{u_2}, T_{v_1}T_{v_2}]$ into a product of four elements of $\mathcal{SI}(S_g)$, each supported on a genus two subsurface. The group $\mathcal{SI}(S_2)$ is simple enough to work with that we can find explicit factorizations into Dehn twists about symmetric separating curves. We will apply the following relation in $\mathcal{SI}(S_2^1)$, which we prove in Section 4.

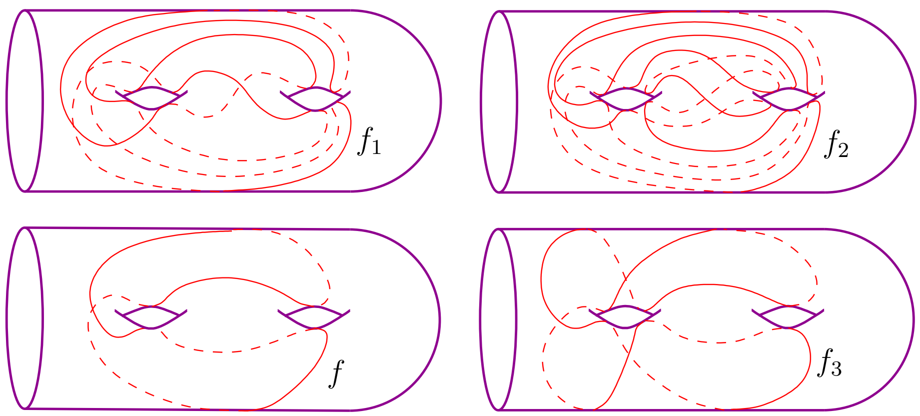

FIGURE 11. The symmetric separating curves which appear in the relation of Theorem 3.1

Theorem 3.1. *We have the following relation in $\mathcal{SI}(S_2^1)$:*

$$[(T_{b_1}T_{b_2})T_d^{-2}, (T_{a_1}T_{a_2})T_c^{-2}]T_e^8 = T_{f_1}T_{f_2}T_fT_{f_3}T_z^{-1},$$

where all curves are as shown in Figures 10 and 11.

We will now use Theorem 3.1 to prove Theorem 1.2.

First proof of Theorem 1.2. We will begin by factoring a slightly different commutator than the one in the statement of the theorem, namely, $[T_{u_1}T_{u_2}, T'_{v_1}T'_{v_2}]$, where the curves are as shown in Figure 12 (the two configurations differ by a reflection).

First of all, because all of the curves in the above commutator lie in a surface homeomorphic S_3^1, and since the inclusion $S_3^1 \to S_g$ induces a homomorphism $\mathrm{SMod}(S_3^1) \to \mathrm{SMod}(S_g)$, it is enough to factor $[T_{u_1}T_{u_2}, T_{v_1'}T_{v_2'}]$ in $\mathrm{SMod}(S_3^1)$.

Since T_{u_1}, T_{u_2}, $T_{v_1'}$, and $T_{v_2'}$ all fix the curve e, we may restrict attention to the subgroup $\mathrm{SMod}(S_3^1, e)$ of $\mathrm{SMod}(S_3^1)$ consisting of elements that stabilize the isotopy class of e.

FIGURE 12. Left: The curves u_1, v_1', u_2, and v_2' form a symmetrized simply intersecting pair. Right: the curves e, z, m, n, and o are used in the first proof of Theorem 1.2.

By the Birman–Hilden theorem, we have $\mathrm{SMod}(S_3^1) \cong \mathrm{Mod}(D_7)$, where D_7 is a disk with 7 cone points. Let $\overline{e}$ denote the arc in D_7 that is the image of e. If $\mathrm{Mod}(D_7, \overline{e})$ denotes the stabilizer of the isotopy class of $\overline{e}$, then $\mathrm{SMod}(S_3^1, e) \cong \mathrm{Mod}(D_7, \overline{e})$.

There is a homomorphism $\mathrm{Mod}(D_7, \overline{e}) \to \mathrm{Mod}(D_5, p)$ obtained by collapsing $\overline{e}$ to a marked point p. The kernel of this map is the infinite cyclic group generated by the half-twist about $\overline{e}$; see [6, Proposition 3.20]. The lift of this half-twist to $\mathrm{SMod}(S_3^1)$ is the full Dehn twist T_e.

All together, we have a homomorphism $\mathrm{SMod}(S_3^1, e) \to \mathrm{Mod}(D_5, p)$, with kernel $\langle T_e \rangle \cong \mathbb{Z}$. This kernel intersects $\mathcal{SI}(S_3^1)$ trivially. The image of $[T_{u_1}T_{u_2}, T_{v_1'}T_{v_2}']$ in $\mathrm{Mod}(D_5, p)$ is $[T_{\overline{u}}, T_{\overline{v}'}]$, where $\overline{u}$ and $\overline{v}'$ are the images of u_1 and v_1' in (D_5, p); see Figure 13.

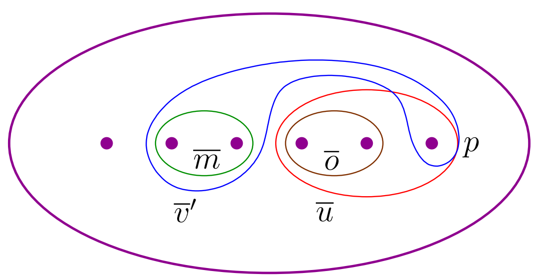

FIGURE 13. The curves $\overline{u}$ and $\overline{v}'$ in (D_5, p).

Since the homomorphism $(\mathcal{SI}(S_3^1) \cap \mathrm{SMod}(S_3^1, e)) \to \mathrm{Mod}(D_5, p)$ is injective, it suffices to factor $[T_{\overline{u}}, T_{\overline{v}'}]$ into a product of elements where each element is the image of a Dehn twist about a symmetric separating curve in S_3^1.

The commutator $[T_{\overline{u}}, T_{\overline{v}'}]$ lies in the kernel of the forgetful homomorphism $\mathrm{Mod}(D_5, p) \to \mathrm{Mod}(D_5)$. Indeed, when we forget p, the curves

$\overline{u}$ and $\overline{v}'$ can be made disjoint (after a homotopy), and so the corresponding twists commute. Therefore, by the Birman exact sequence, the commutator $[T_{\overline{u}}, T_{\overline{v}'}]$ lies in $\pi_1(D_5) < \mathrm{Mod}(D_5, p)$.

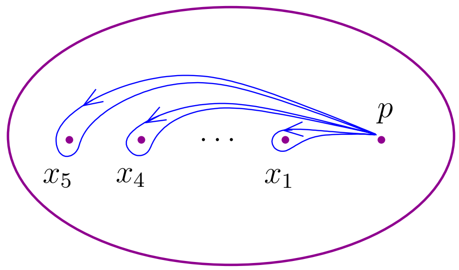

FIGURE 14. The generators x_i of $\pi_1(D_5, p)$.

We choose generators $x_1, \dots, x_5$ for $\pi_1(D_5, p)$ as shown in Figure 14. We claim that, as an element of $\pi_1(D_5, p)$, the commutator $[T_{\overline{u}}, T_{\overline{v}'}]$ is identified with the commutator $[x_4x_3, x_2x_1]$. Indeed, the mapping class corresponding to x_4x_3 is $T_{\overline{v}'}^{-1}T_{\overline{m}}$ and the mapping class corresponding to x_2x_1 is $T_{\overline{u}}^{-1}T_{\overline{o}}$, where $\overline{m}$ and $\overline{o}$ are as shown in Figure 13 (cf. [6, Section 5.1.1]). Recalling that the x_i are multiplied left to right, we obtain

$$[x_4x_3, x_2x_1] \leftrightarrow [T_{\overline{u}}T_{\overline{o}}^{-1}, T_{\overline{v}'}T_{\overline{m}}^{-1}].$$

Since $\overline{m}$ and $\overline{o}$ are disjoint from all other curves used in this commutator, those twists commute with each twist in the commutator, and so $[T_{\overline{u}}T_{\overline{o}}^{-1}, T_{\overline{v}'}T_{\overline{m}}^{-1}] = [T_{\overline{u}}, T_{\overline{v}'}]$, giving the claim.

Our goal now is to rewrite $[x_4x_3, x_2x_1]$ as a product of elements of $\pi_1(D_5, p)$ that are each supported on a disk containing p and three other marked points. Each such element corresponds to an element of $\mathcal{SI}(S_3^1)$ supported on a subsurface of genus two. We can then apply our knowledge of $\mathcal{SI}(S_2^1)$, in particular Theorem 3.1.

We do this by rewriting x_4x_3 as $(x_4x_1)(x_3x_1)^{-1}x_3^2$. Then we apply the Witt–Hall identity (see Section 4.2) several times to the commutator $[(x_4x_1)(x_3x_1)^{-1}x_3^2, x_2x_1]$ and eventually find:

$$\begin{aligned}[x_4x_3, x_2x_1] \;&= (x_4)x_3^2(x_4)^{-1}\ (x_4x_3^{-1})(x_2x_1)x_3^{-2}(x_2x_1)^{-1}(x_4x_3^{-1})^{-1} \\ &\quad (x_4x_3^{-1})[x_3x_1, x_2x_1]^{-1}(x_4x_3^{-1})^{-1}\ [x_4x_1, x_2x_1].\end{aligned}$$

This equality can be easily checked by multiplying out both sides. The right hand side is a product of four elements with the desired property.

We now must translate each of these four elements into elements of $\mathrm{SMod}(S_3^1)$.

To begin, let $\overline{s}_i$ denote the curve in D_7 obtained by pushing x_i to the right, and let s_i denote the lift to S_3^1. For example, the curve s_3 is the same as the curve f from Figure 11 (after including S_2^1 into S_3^1 in the obvious way).

Each x_i lifts to a right Dehn twist $T_{\overline{s}_i}^{-1}$ in $\mathrm{Mod}(D_7)$ and then to a right half-twist $T_{s_i}^{-1/2}$ in $\mathrm{SMod}(S_3^1)$. In particular, x_3^2 lifts to T_f^{-1}.

If we lift the commutator $[x_3x_1, x_2x_1]^{-1} = [x_2x_1, x_3x_1]$ to a mapping class in $\mathrm{Mod}(D_7)$, it has a lift in $\mathcal{SI}(S_3^1)$ which is nothing other than the product $[(T_{b_1}T_{b_2})T_d^{-2}, (T_{a_1}T_{a_2})T_c^{-2}]T_e^8$ that appears in Theorem 3.1 (after including the genus two surface there into the genus three surface here in the obvious way). Applying that theorem, we conclude that $[x_3x_1, x_2x_1]^{-1}$ is equal to a product of five Dehn twists about separating curves in S_3^1. The same is then true for $[x_4x_1, x_2x_1]$, as its image in $\mathrm{Mod}(D_7)$ is conjugate to the image of $[x_3x_1, x_2x_1]$ under $T_{\overline{m}}^{-1}$.

As the conjugate in $\mathrm{SMod}(S_3^1)$ of a Dehn twist about a symmetric separating curve is again a Dehn twist about a symmetric separating curve, we have succeeded in showing that $[T_{u_1}T_{u_2}, T_{v_1'}T_{v_2'}]$ is a product of $1+1+5+5=12$ Dehn twists about symmetric separating curves. We remark here that the lifts of the x_i were only unique up to powers of T_e; however, at this point this ambiguity has been resolved, as our product of twelve Dehn twists already acts trivially on $H_1(S_3^1)$.

It is now a matter of repeatedly applying the relation $hT_xh^{-1} = T_{h(x)}$ (and remembering to switch back to right-to-left notation) to convert the commutator $[x_4x_3, x_2x_1]$ into an explicit product of Dehn twists. Specifically, we find:

$$\begin{aligned}
[T_{u_1}T_{u_2}, T_{v_1'}T_{v_2'}] &= T_{M(z)}T_{M(f_3)}^{-1}T_{M(f)}^{-1}T_{M(f_2)}^{-1}T_{M(f_1)}^{-1} \\
&\quad T_{H(f_1)}T_{H(f_2)}T_{H(f)}T_{H(f_3)}T_{H(z)}^{-1} \\
&\quad T_{HH_2H_1(f)}T_{H_4(f)}^{-1},
\end{aligned}$$

where $H_i = T_{s_i}^{1/2}$, $H = H_4H_3^{-1}$, and $M = T_m^{-1}$, and where the curves f, f_1, f_2, and f_3 are in Figure 11, and the curves m, z, and e are in Figure 12.

We have written $[T_{u_1}T_{u_2}, T_{v_1'}T_{v_2'}]$ as a product of twelve Dehn twists about symmetric separating curves. If we draw these curves, we find that they are quite complicated. But then, using the computer program β-Twister written by Marta Aguilera and Juan González-Meneses, we notice that some of the curves are actually equal:

$$M(f_1) = H(f_1), \quad M(f_2) = H(f_2), \quad \text{and} \quad H(f) = H_4(f).$$

Thus, we can simplify the above product as follows:

$$\begin{aligned}[T_{u_1}T_{u_2}, T_{v_1'}T_{v_2'}] &= T_{M(z)}T^{-1}_{M(f_3)}T^{-1}_{M(f)}T_{H(f)}\left(T_{H(f_3)}T^{-1}_{H(z)}T_{HH_2H_1(f)}\right)T^{-1}_{H(f)} \\ &= T_{M(z)}T^{-1}_{M(f_3)}T^{-1}_{M(f)}T_{HT_f(f_3)}T^{-1}_{HT_f(z)}T_{HT_fH_2H_1(f)}\end{aligned}$$

The last product of Dehn twists is a product of three bounding pair maps, and so we have already succeeded in writing $[T_{u_1}T_{u_2}, T_{v_1'}T_{v_2'}]$ as such. These six curves are slightly more complicated than they need to be, and so at this point our only remaining goal is to simplify the picture as much as possible.

Conjugating the last relation by the map

$$h = T_nT_mT_nT_oT_nT_m$$

and using the fact that $h(\{u_1, u_2\}) = \{v_1, v_2\}$ and $h(\{v_1', v_2'\}) = \{u_1, u_2\}$, we obtain an alternate version of the above relation:

$$[T_{v_1}T_{v_2}, T_{u_1}T_{u_2}] = T_{hM(z)}T^{-1}_{hM(f_3)}T^{-1}_{hM(f)}T_{hHT_f(f_3)}T^{-1}_{hHT_f(z)}T_{hHT_fH_2H_1(f)}.$$

Now we take the inverse of both sides, and rearrange the twists in each bounding pair so that the positive twist comes first. We obtain:

$$[T_{u_1}T_{u_2}, T_{v_1}T_{v_2}] = T_{hHT_f(z)}T^{-1}_{hHT_fH_2H_1(f)}T_{hM(f)}T^{-1}_{hHT_f(f_3)}T_{hM(f_3)}T_{hM(z)}$$

Setting $c_1 = hHT_f(z)$, $c_2 = hHT_fH_2H_1(f)$, $c_3 = hM(f)$, $c_4 = hHT_f(f_3)$, $c_5 = hM(f_3)$, and $c_6 = hM(z)$ and drawing the pictures of these curves completes the proof. $\square$

3.4. **Second proof.** We will now give a quicker, alternate proof of Theorem 1.2.

Second proof of Theorem 1.2. Similar to the first proof of Theorem 1.2, the commutator $[T_{\overline{u}}, T_{\overline{v}}]$ can be identified with the element

$$[(x_2x_1)^{-1}x_3x_4(x_2x_1), x_2x_1] = [(x_2x_1)^{-1}, x_4x_3]$$

of $\pi_1(D_5, p)$ (here $\overline{v}$ is the image of v in (D_5, p)).

Our strategy is to rewrite this commutator as a product of squares of elements in $\pi_1(D_5, p)$ that are each represented by simple loops surrounding either one or three marked points. The first kind of loop corresponds to the square of a Dehn twist about a curve surrounding three points in $\mathrm{Mod}(D_7)$, and hence a Dehn twist about a symmetric separating simple closed curve in $\mathrm{SMod}(S_3^1)$. The second kind of loop will correspond to the square of a bounding pair map on separating curves in $\mathrm{Mod}(D_7)$, where the curves surround three and five punctures, respectively. The latter corresponds to a bounding pair map on symmetric separating curves in $\mathrm{SMod}(S_3^1)$.

We accomplish our goal as follows:

$[(x_2x_1)^{-1}, x_4x_3] = (x_1^{-1}x_2^{-1}x_4x_3x_2x_1^2)^2(x_1^{-2})(x_4x_3x_2)^{-2}(x_4x_3x_2x_3^{-1}x_4^{-1})^2.$

As before, this equality can be checked by multiplying out both sides.

We can then check the following correspondences between elements of $\pi_1(D_5, p)$ and $\mathrm{Mod}(D_7)$ (all curves are as shown in Figure 7):

$$\begin{aligned} x_1^{-1}x_2^{-1}x_4x_3x_2x_1^2 &\leftrightarrow T_{\overline{c}_5}T_{\overline{c}_6}^{-1} \\ x_1^{-1} &\leftrightarrow T_{\overline{c}_3} \\ (x_4x_3x_2)^{-1} &\leftrightarrow T_{\overline{c}_1}T_{\overline{c}_4}^{-1} \\ x_4x_3x_2x_3^{-1}x_4^{-1} &\leftrightarrow T_{\overline{c}_2}^{-1} \end{aligned}$$

Thus, by the above calculation, we have:

$$[T_{\overline{u}}, T_{\overline{v}}] \leftrightarrow [(x_2x_1)^{-1}, x_4x_3] \leftrightarrow T_{\overline{c}_2}^{-2}(T_{\overline{c}_1}T_{\overline{c}_4}^{-1})^2T_{\overline{c}_3}^2(T_{\overline{c}_5}T_{\overline{c}_6}^{-1})^2$$

Since $T_{\overline{c}_1}$ commutes with $T_{\overline{c}_4}$ and $T_{\overline{c}_5}$ commutes with $T_{\overline{c}_6}$,we can rewrite this as:

$$[T_{\overline{u}}, T_{\overline{v}}] \leftrightarrow T_{\overline{c}_2}^{-2}T_{\overline{c}_1}^2T_{\overline{c}_4}^{-2}T_{\overline{c}_3}^2T_{\overline{c}_5}^2T_{\overline{c}_6}^{-2}$$

Lifting back up to $\mathrm{SMod}(S_3^1)$ (or $\mathrm{SMod}(S_g)$), we have:

$$[T_u, T_v] = T_{c_2}^{-1}T_{c_1}T_{c_4}^{-1}T_{c_3}T_{c_5}T_{c_6}^{-1}$$

(a priori, the right hand side should have a power of T_e; however, since both sides already act trivially on $H_1(S_3^1)$, the power is zero). Finally, using the fact that T_{c_1} commutes with T_{c_2} and the fact that T_{c_3} commutes with T_{c_4}, we have:

$$[T_u, T_v] = T_{c_1}T_{c_2}^{-1}T_{c_3}T_{c_4}^{-1}T_{c_5}T_{c_6}^{-1},$$

as desired. □

Obviously, our second proof of Theorem 1.2 is simpler than the first in that it is shorter and does not use any auxiliary relations. On the other hand, it is hard to imagine how one would have found this proof without first knowing the exact statement of the theorem. Indeed, the factorization of $[(x_2x_1)^{-1}, x_4x_3]$ used in the second proof was only discovered by looking at the pictures of the $\overline{c}_i$ discovered via the first proof.

4. Genus two relations

In this section, we prove Theorem 3.1, the genus two relation used in the first proof of Theorem 1.2.

4.1. **Reduction to Theorem 1.3 and proof thereof.**

The first idea for proving Theorem 3.1 is to cap the boundary component of S_2^1 with a disk, so that we obtain a relation in $\mathcal{SI}(S_2)$. A priori, it is easier to find factorizations there.

After capping, the curves a_1 and a_2 become isotopic, as do b_1 and b_2. Call the new isotopy classes a and b. Our goal is now to factor the product:

$$[T_b^2T_d^{-2}, T_a^2T_c^{-2}]T_e^8.$$

By a direct calculation, we notice that, in the commutator, we can replace each square of a Dehn twist with a single Dehn twist, and the resulting mapping class

$$[T_bT_d^{-1}, T_aT_c^{-1}]T_e^2$$

still acts trivially on $H_1(S_2)$ (note that T_e^8 got replaced by T_e^2). We know that each element of $\mathcal{SI}(S_2)$ is a product of Dehn twists about symmetric separating curves, and so we begin here, trying to factor the last commutator into a product of such Dehn twists. This brings us directly to Theorem 1.3, which states that $[T_bT_d^{-1}, T_aT_c^{-1}]T_e^2 = T_f$, where all curves are as shown in left hand side of Figure 5.

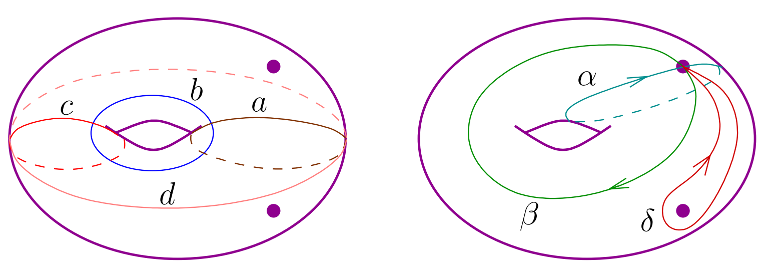

Figure 15. Realizing $[T_bT_d^{-1}, T_aT_c^{-1}]$ as a commutator of push maps.

Proof of Theorem 1.3. Each Dehn twist in the product $[T_bT_d^{-1}, T_aT_c^{-1}]T_e^2$ fixes the curve e. In other words, each twist lies in $\mathrm{Mod}(S_2, e)$, the subgroup of $\mathrm{Mod}(S_2)$ consisting of elements that preserve the isotopy class of e. There is a short exact sequence:

$$1 \to \langle T_e \rangle \to \mathrm{Mod}(S_2, e) \to \mathrm{Mod}(S_{1,2}) \to 1,$$

where $S_{1,2}$ is the twice-punctured torus obtained by deleting e from S_2; see [6, Proposition 3.20]. Since the twists all preserve the orientation of e, their images in $\mathrm{Mod}(S_{1,2})$ lie in the subgroup consisting of elements that induce the trivial permutation of the punctures. The latter is isomorphic to $\mathrm{Mod}(S_{1,1}, p)$, where $S_{1,1}$ is a once-punctured torus, and $p \in S_{1,1}$.

The image of T_e in $\mathrm{Mod}(S_{1,1}, p)$ is trivial. Under the map $S_2 - e \to (S_{1,1}, p)$, the curves a, b, c, and d map to the curves shown in the left hand side of Figure 15; the twists are mapped accordingly. We see from the picture that $(T_aT_c^{-1})^{-1}$ and $(T_bT_d^{-1})^{-1}$ are the images of the push maps $\mathcal{P}ush(\alpha)$ and $\mathcal{P}ush(\beta)$ along the loops α and β shown in the right hand side of Figure 15.

Since $\mathcal{P}ush : \pi_1(S_{1,1}, p)^{op} \to \mathrm{Mod}(S_{1,1}, p)$ is a homomorphism, we have

$$[T_bT_d^{-1}, T_aT_c^{-1}] = [\mathcal{P}ush(\beta^{-1}), \mathcal{P}ush(\alpha^{-1})] = \mathcal{P}ush([\alpha, \beta]) = \mathcal{P}ush(\delta),$$

where δ is the loop shown in the right hand side of Figure 15 and products are written in $\pi_1(S_{1,1})$, not $\pi_1(S_{1,1})^{op}$.

The mapping class $\mathcal{P}ush(\delta)$ is a Dehn twist about the curve in $(S_{1,1}, p)$ obtained by pushing δ to the left. One preimage of this Dehn twist in $\mathrm{Mod}(S_2)$ is the Dehn twist about the separating curve f shown in Figure 5.

We have shown that $[T_bT_d^{-1}, T_aT_c^{-1}]T_e^2$ and T_f have the same image in $\mathrm{Mod}(S_{1,1}, p) < \mathrm{Mod}(S_{1,2})$. But no power of T_e acts trivially on $H_1(S_2)$, so $\ker(\mathrm{Mod}(S_2, e) \to \mathrm{Mod}(S_{1,2})) \cap \mathcal{SI}(S_2) = 1$. Since $[T_bT_d^{-1}, T_aT_c^{-1}]T_e^2$ and T_f both lie in $\mathcal{SI}(S_2)$, it follows that $[T_bT_d^{-1}, T_aT_c^{-1}]T_e^2 = T_f$, as desired. $\square$

In order to obtain the factorization of the specific commutator in Theorem 3.1, we will need an alternate version of the genus two relation from Theorem 1.3, as follows.

Theorem 4.1. *Let a, b, c, d, e, and f be the simple closed curves in S_2 shown in Figure 16. We have*

$$[T_b T_d^{-1}, T_a T_c^{-1}] T_e^2 = T_f.$$

In order to check the relation from Theorem 4.1, one can either repeat the proof of Theorem 1.3 (the analog of Figure 15 is less pretty) or find a mapping class h taking the sextuple (a, b, c, d, e, f) from Figure 5 to the sextuple (a, b, c, d, e, f) from Figure 16, for instance $h = T_c T_a T_b T_c T_a T_b T_c$ where a, b, and c are as in Figure 5. Then, using the identity $hT_x h^{-1} = T_{h(x)}$, we immediately obtain the new relation.

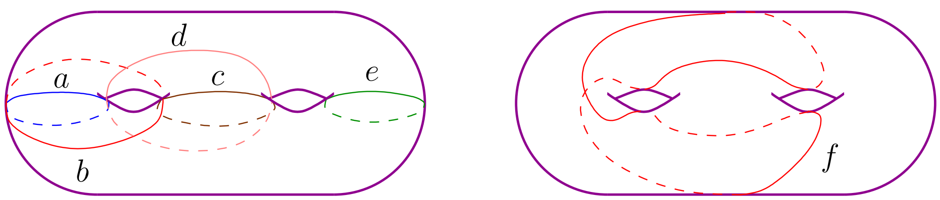

FIGURE 16. The curves used in Theorem 4.1.

We now begin the process of pushing the relation from Theorem 4.1 back up to $\mathcal{SI}(S_2^1)$.

4.2. **The Witt–Hall identity and a doubling of the genus two relation.** The Witt–Hall identity is a group relation that serves as a distributive property for the commutator operation. Specifically, if x, y, and z are any elements of a free group (hence of any group), we have:

$$[xy, z] = x[y, z]x^{-1}[x, z]$$

and

$$[x, yz] = [x, y]y[x, z]y^{-1}.$$

It follows for example, that we can write $[x^2, y^2]$ as a product of four conjugates of $[x, y]$, as follows:

$$\begin{aligned}
[x^2, y^2] &= x[x, y^2]x^{-1}[x, y^2] \\
&= x([x, y]y[x, y]y^{-1})x^{-1}[x, y]y[x, y]y^{-1} \\
&= (x[x, y]x^{-1})\,((xy)[x, y](xy)^{-1})\,([x, y])\,(y[x, y]y^{-1})
\end{aligned}$$

We can use the Witt–Hall identity obtain a variant of the relation we gave in Theorem 4.1, as follows.

Theorem 4.2. *Let a, b, c, d, e, f, f_1, f_2, and f_3 be the curves in S_2 indicated in Figures 11 and 16 (so that the curves in the latter figure lie in S_2, we should cap the boundary component in that figure with a disk). We have the following relation in $\mathcal{SI}(S_2)$:*

$$[T_b^2T_d^{-2}, T_a^2T_c^{-2}]T_e^8 = T_{f_1}T_{f_2}T_fT_{f_3}.$$

Proof. Using the above calculation and the relation $hT_xh^{-1} = T_{h(x)}$, we find:

$$\begin{aligned}
[T_b^2T_d^{-2}, T_a^2T_c^{-2}]T_e^8 &= (T_bT_d^{-1})T_f(T_bT_d^{-1})^{-1}\ (T_bT_d^{-1}T_aT_c^{-1}) \\
&\quad T_f(T_bT_d^{-1}T_aT_c^{-1})^{-1}\ T_f\ (T_aT_c^{-1})T_f(T_aT_c^{-1})^{-1} \\
&= T_{T_bT_d^{-1}(f)}\, T_{T_bT_d^{-1}T_aT_c^{-1}(f)}\, T_f\, T_{T_aT_c^{-1}(f)} \\
&= T_{f_1}T_{f_2}T_fT_{f_3},
\end{aligned}$$

as desired. ☐

4.3. **Lifting to the surface with boundary.** We are finally ready to lift the relation in $\mathcal{SI}(S_2)$ from Theorem 4.1 to $\mathcal{SI}(S_2^1)$, and hence prove Theorem 3.1.

Proof of Theorem 3.1. If we cap the boundary of S_2^1 with a disk, then the curves from Figure 10 map to the corresponding curves in Figure 16. There is an induced map $\mathcal{SI}(S_2^1) \to \mathcal{SI}(S_2)$, and the Dehn twists about the above curves are mapped accordingly.

The kernel of the map $\mathcal{SI}(S_2^1) \to \mathcal{SI}(S_2)$ is the infinite cyclic group generated by T_z, where $z = \partial S_2^1$ [3, Theorem 4.2]. It then follows from Theorem 4.2 that there is an integer k so that

$$[(T_{b_1}T_{b_2})T_d^{-2}, (T_{a_1}T_{a_2})T_c^{-2}]T_e^8 = T_{f_1}T_{f_2}T_fT_{f_3}T_z^k.$$

It suffices to show that $k = -1$.

By the Birman–Hilden theorem, $\mathrm{SMod}(S_2^1)$ is isomorphic to $\mathrm{Mod}(D_5) \cong B_5$. The image of $\mathcal{SI}(S_2^1)$ lies in the pure braid group PB_5; see [4].

Under the monomorphism $\mathcal{SI}(S_2^1) \to PB_5$, a Dehn twist about a symmetric separating curve in S_2^1 maps to the square of a Dehn twist about the image of that curve in D_5. As such, the relation in $\mathcal{SI}(S_2^1)$ in the statement of the theorem corresponds to the relation

$$[T_{\overline{b}}T_{\overline{d}}^{-1}, T_{\overline{a}}T_{\overline{c}}^{-1}]T_{\overline{e}}^4 = T_{\overline{f}_1}^2T_{\overline{f}_2}^2T_{\overline{f}}^2T_{\overline{f}_3}^2T_{\overline{z}}^{2k}$$

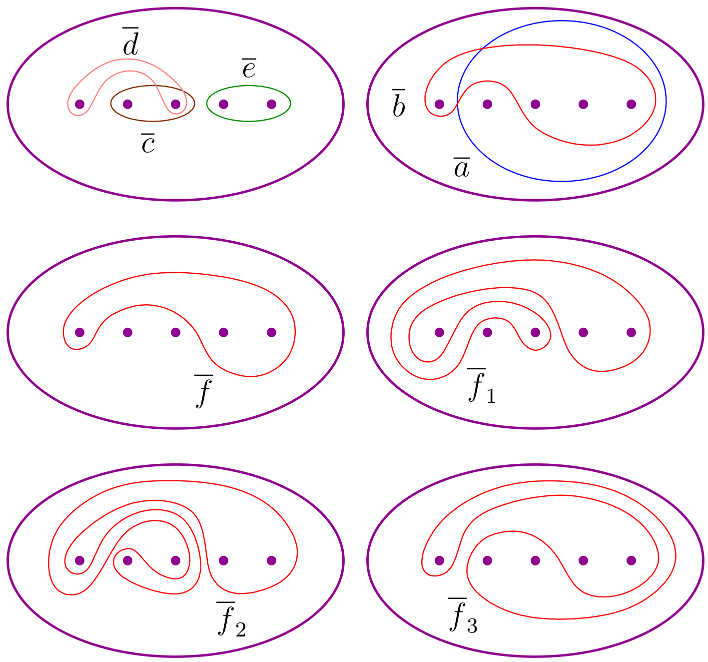

FIGURE 17. The curves in D_5 from the proof of Theorem 3.1.

in PB_5, where the curves $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{e}$, $\overline{f}$, $\overline{f}_1$, $\overline{f}_2$, and $\overline{f}_3$ are the curves shown in Figure 17; these curves are simply the images of the curves in S_2^1 under the quotient map. Thus, it suffices to show that $k = -1$ for the above relation in PB_5.

There is a homomorphism $PB_5 \to PB_2 \cong \mathbb{Z}$ obtained by forgetting the last three cone points of D_5 (the ones furthest to the right in Figure 17). Under this homomorphism, $[T_{\overline{b}}T_{\overline{d}}^{-1}, T_{\overline{a}}T_{\overline{c}}^{-1}]T_{\overline{e}}^4$ maps to 0. On the other hand, $T_{\overline{f}} \mapsto 0$, $T_{\overline{f}_1} \mapsto 0$, $T_{\overline{f}_2} \mapsto 0$, $T_{\overline{f}_3} \mapsto 1$, and $T_{\overline{z}} \mapsto 1$. Therefore,

$$T_{\overline{f}_1}^2 T_{\overline{f}_2}^2 T_{\overline{f}}^2 T_{\overline{f}_3}^2 T_{\overline{z}}^{2k} \mapsto 2 + 2k.$$

It follows that $k = -1$, as desired. $\square$

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