Announcements April 18

- CIOS open: additional dropped quiz for 85% response rate
- WebWork 6.3 and 6.4 due Thursday
- Quiz on 6.3 and 6.4 on Friday
- WebWork 6.5 due Sunday (not graded)
- Review on Monday in class; post questions on Piazza using final_exam tag
- Final Exam Wed May 4 8:00-10:50 (Sec H) and Mon May 2 2:50-5:40 (Sec J)
- Office Hours Tue 2-3 and Wed 2-3
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

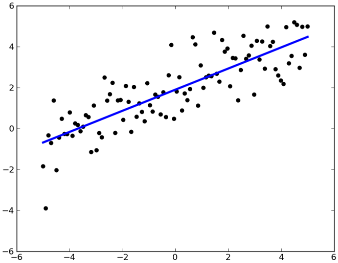
Section 6.4

The Gram–Schmidt Process

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



To solve $Ax = b$ as closely as possible, we orthogonally project b onto $\mathrm{Col}(A)$. We know how to do this if we have an orthogonal basis. But what if we don't?

Outline

- The Gram–Schmidt process: turn any basis into an orthogonal one
- QR factorization
- Application to eigenvalue computations

Gram–Schmidt Process

With two vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Gram–Schmidt Process

With three vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Theorem. Say $\{u_1, \ldots, u_k\}$ is a basis for a nonzero subspace of $\mathbb{R}^n$. Define:

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \mathrm{proj}_{\mathrm{Span}\{v_1\}(u_1)} \\
v_3 &= u_3 - \mathrm{proj}_{\mathrm{Span}\{v_1, v_2\}(u_2)} \\
&\vdots \\
v_k &= u_k - \mathrm{proj}_{\mathrm{Span}\{v_1, \ldots, v_{k-1}\}(u_k)}
\end{aligned}$$

Then $\{v_1, \ldots, v_k\}$ is an orthogonal basis for $\mathrm{Span}\{u_1, \ldots, u_k\}$.

Gram–Schmidt Process

With three vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix}$$

QR Factorization

Theorem. Say A is an $n \times n$ matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper triangular with positive diagonal entries.

Columns of Q are the vectors obtained from Gram–Schmidt, with normalized columns.

The entries of R come from the steps in the Gram–Schmidt process, with normalized rows. In our first 3×3 example:

$$\hat{Q} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} 1 & \boxed{1} & \boxed{2} \\ 0 & 1 & \boxed{1} \\ 0 & 0 & 1 \end{pmatrix}$$

The first $\boxed{1}$ comes from: $v_2 = u_2 - 1 \cdot v_1$

The other $\boxed{2}$ and $\boxed{1}$ come from $v_3 = u_3 - 2 \cdot v_1 - 1 \cdot v_2$

QR Factorization

Theorem. Say A is an $n \times n$ matrix with linearly independent columns. Then

$$A = QR$$

where Q has orthonormal columns and R is upper triangular with positive diagonal entries.
In our first 3×3 example:

$$\hat{Q} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} 1 & \boxed{1} & \boxed{2} \\ 0 & 1 & \boxed{1} \\ 0 & 0 & 1 \end{pmatrix}$$

To find Q and R, scale columns of $\hat{Q}$ to make them unit vectors and scale the corresponding rows of $\hat{R}$ by the inverse.

QR Factorization

Example

Find the QR factorization of

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

QR Factorization

What is it used for?

Say A is an $n \times n$ matrix.

Do:

$$
\begin{aligned}
A &= Q_1 R_1 && \text{QR factorization} \\
A_1 &= R_1 Q_1 && \text{swap Q and R} \\
&= Q_2 R_2 && \text{and find the QR factorization of the result} \\
A_2 &= R_2 Q_2 && \text{swap Q and R} \\
&\vdots
\end{aligned}
$$

The A_k converge to an upper triangular matrix and the diagonal entries (quickly!) converge to the eigenvalues.