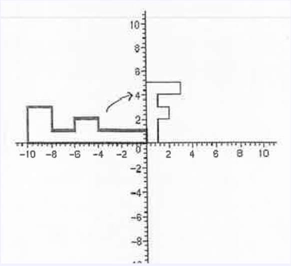
Discussion

Discussion Question

Find a matrix that does this.



▸ Transformation Challenge

Announcements Oct 13

- Masks $\rightsquigarrow$ Thank you!
- Quiz 3.2-3.3 Friday
- WeBWorK 3.2 & 3.3 due tonite!
- Special office hr: Thu 11-12 Teams (special time!)
- Midterm 2 Oct 20 8–9:15p on Teams

- Use Piazza for general questions
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 3.3

Linear Transformations

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

$T(u+v) = T(u)+T(v)$
$T(cv) = cT(v)$

$T(v) = Av$

harder: these are all matrix transf.

easier: these are linear transf.
$A(v+w) = Av+Aw$
$T(v+w) = T(v)+T(w)$

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

RECIPE

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

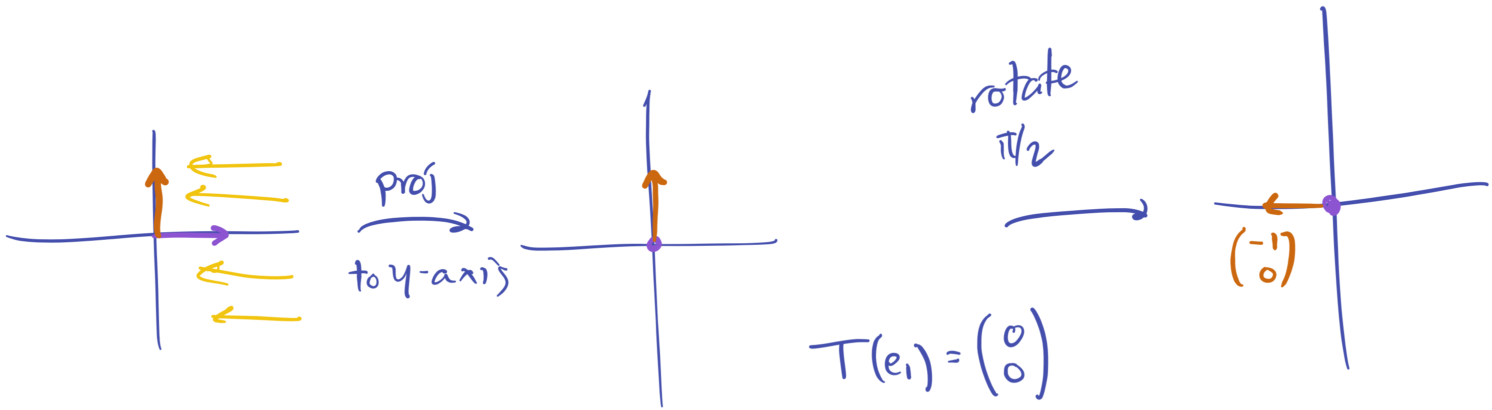
$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

Find $T(e_1)$, $T(e_2)$



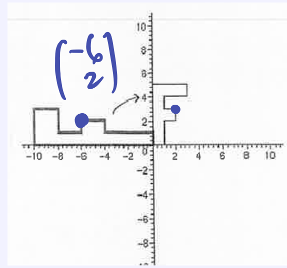
$$T(e_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Discussion

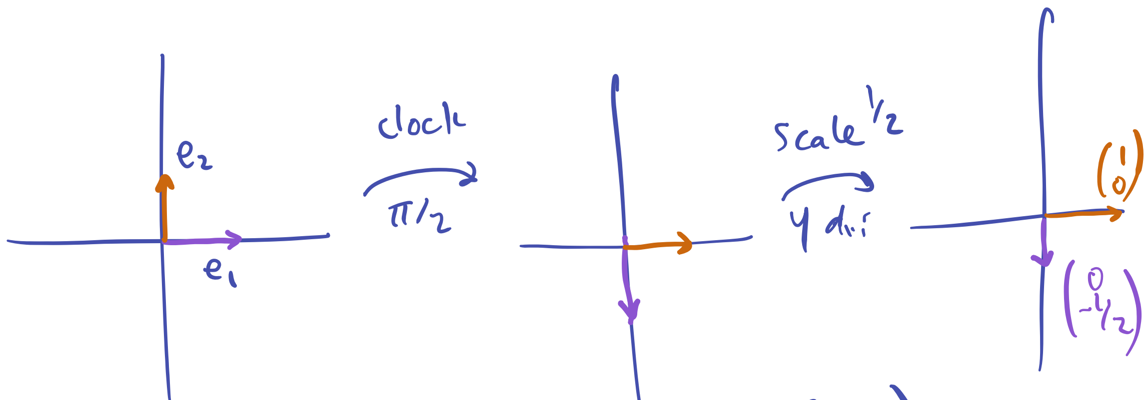
Discussion Question

Find a matrix that does this.



or scale by 1/2 in x dir
rotate by $\pi/2$ clockwise

rotate $\pi/2$ clockwise
then scale by 1/2
in y-dir.



$$\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}\begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

▸ Transformation Challenge

Section 3.4

Matrix Multiplication

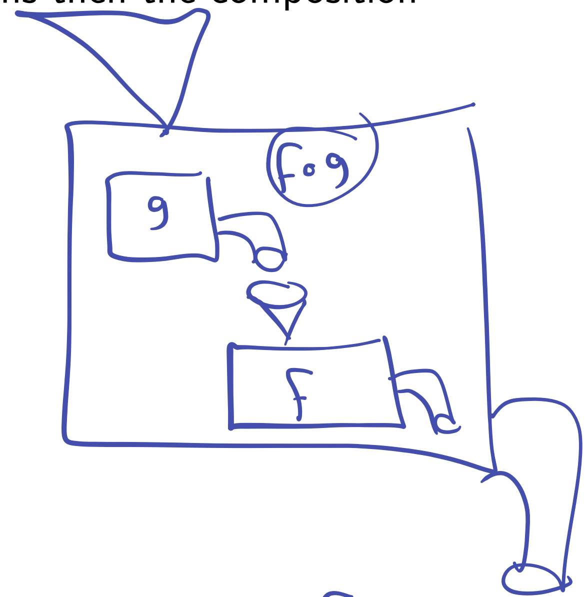
Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things

Function composition

Remember from calculus that if f and g are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$



$f(g(x))$

In words: first apply g, then f.

Example: $f(x) = x^2$ and $g(x) = x + 1$.

Note that $f \circ g$ is usually different from $g \circ f$.

$$f \circ g\,(x) = (x+1)^2$$

$$g \circ f\,(x) = x^2 + 1$$

Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^{\boxed{p}} \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^{\boxed{p}}$ and make the composition $T \circ U$.

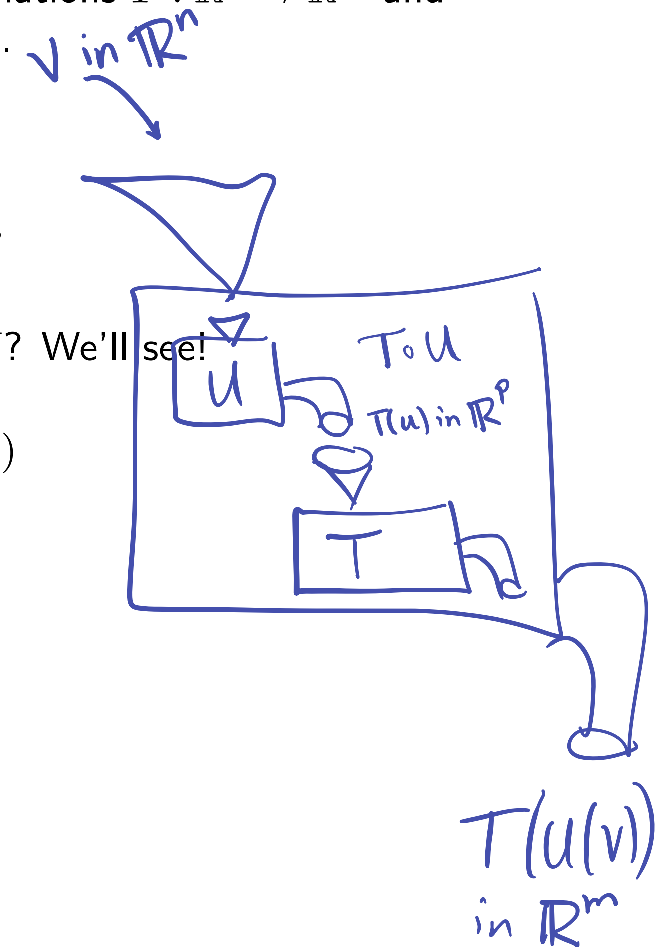
Notice that both have an p. Why?

What are the $\underbrace{\text{domain}}_{\mathbb{R}^n}$ and $\underbrace{\text{codomain}}_{\mathbb{R}^m}$ for $T \circ U$?

Natural question: What is the matrix for $T \circ U$? We'll see!

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?



Composition of linear transformations

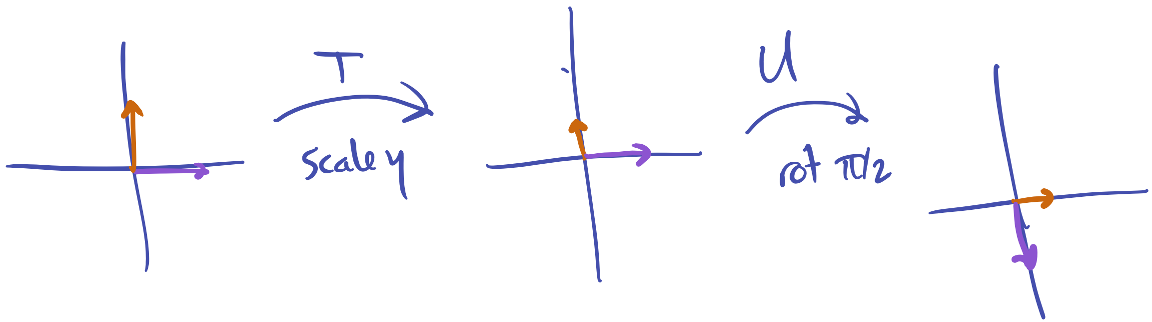
Example. $T =$ ~~projection to y-axis and~~ scale y-dir by $1/2$
$U =$ ~~reflection about $y = x$ in $\mathbb{R}^2$~~ rotate clock by $\pi/2$

$$\boxed{\begin{array}{l} T \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \\ U \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{array}}$$

What is the standard matrix for $T \circ U$?

What about $U \circ T$? $T \circ U \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$

usual recipe



$$U \circ T \longleftrightarrow \begin{pmatrix} 0 & 1/2 \\ -1 & 0 \end{pmatrix}$$

Matrix Multiplication

And now for something completely different (not really!)

Suppose A is an $m \times n$ matrix. We write a_{ij} or A_{ij} for the ijth entry.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where r_i is the ith row of A, and b_j is the jth column of B.

2nd way

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -13 \end{pmatrix}$$

Or: the jth column of AB is A times the jth column of B.

Multiply these matrices (both ways):

1st way

$$\underset{2\times \cancel{3}}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}\underset{\cancel{3}\times 2}{\begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 8 & -4 \\ \boxed{17} & -13 \end{pmatrix}}$$

Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do U then do T

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

composing transf's $\longleftrightarrow$ multiplying matrices.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv) = (AB)(v)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case Bv is the ith column of B. So the left-hand side is A times the ith column of B. The right-hand side is the ith column of AB which we already said was A times the ith column of B. It works!

Matrix Multiplication and Linear Transformations

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^p \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^p$. The standard matrix for $T \circ U$ is AB.

scale by 1/2 in y dir   rot. clockwise $\pi/2$

Example. $T =$ ~~projection to y-axis~~ and $U =$ ~~reflection about $y = x$ in $\mathbb{R}^2$~~

$$T \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

What is the standard matrix for $T \circ U$?

$$T \circ U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$$

rot. clock $\pi/2$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} \boxed{0} & 1 \\ -1/2 & 0 \end{pmatrix}}_{T \circ U}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1/2 \\ -1 & 0 \end{pmatrix}}_{U \circ T}$$

$T \circ U$ same as:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{rot clock } \pi/2} \underbrace{\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{scale in x dir by } 1/2} = \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$$

Linear transformations are matrix transformations

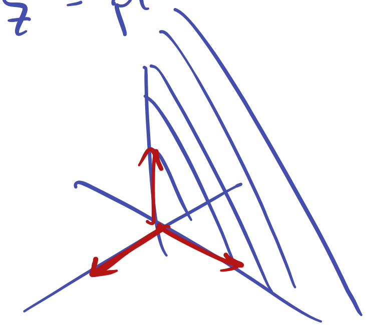
Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} & & \boxed{0} \\ \boxed{0} & & \\ & & \boxed{-1} \end{pmatrix}$$

proj to yz-plane

refl. in xy plane

multiply



Discussion Question

Are there nonzero matrices A and B with $AB = 0$?

1. Yes
2. No

Properties of Matrix Multiplication

- $A(BC) = (AB)C$ assoc.
- $A(B + C) = AB + AC$ distrib.
- $(B + C)A = BA + CA$ distrib.
- $r(AB) = (rA)B = A(rB)$
- $(AB)^T = B^T A^T$
- $I_m A = A = AI_n$, where I_k is the $k \times k$ identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- AB is not always equal to BA
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that A or B is 0

More rabbits

Recall that the following matrix describes the change in our rabbit population from this year to the next:

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

What matrix should we use if we want to describe the change in the rabbit population from this year to two years from now? Or 10 years from now?

v = this year's popul.

Av = next year's popul.

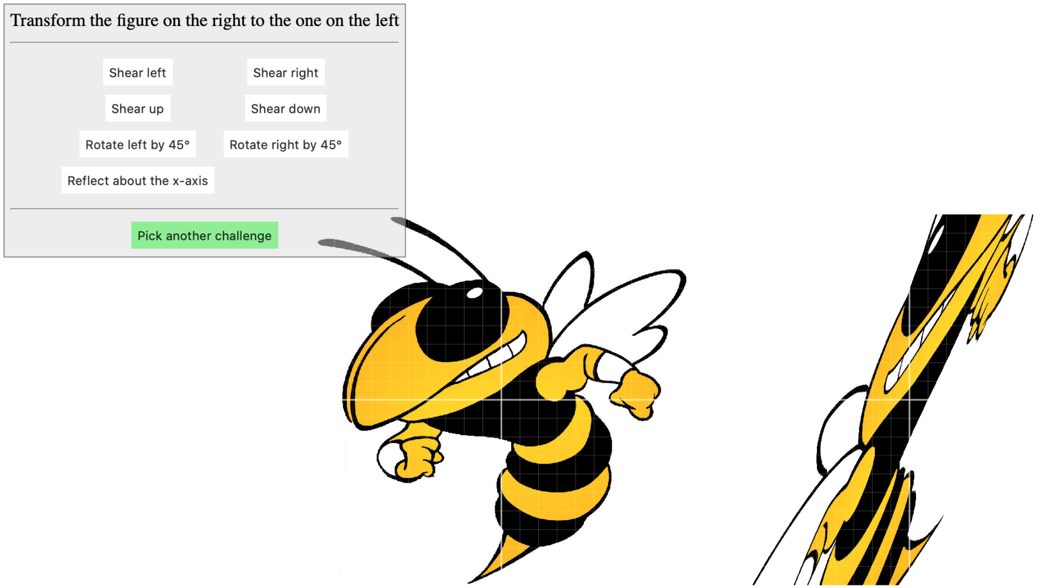
$A^2 v = AAv$ = year after that

$A^{100} v$ = 100 years after 1st year

Fun with matrix multiplication

Play the Buzz game!

http://textbooks.math.gatech.edu/ila/demos/transform_game.html



In the rotation game, you need to find a composition of shears that gives a rotation!

Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0

Typical Exam Questions 3.4

- True/False. If A is a 3×4 matrix and B is a 4×3 matrix, then it makes sense to multiply A and B in both orders.
- True/False. If it makes sense to multiply a matrix A by itself, then A must be a square matrix.
- True/False. If A is a non-zero square matrix, then A^2 is a non-zero square matrix.
- True/False. If $A = -I_n$ and B is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the xy-plane and the yz-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix A for projection to the xy-plane in $\mathbb{R}^3$. What is A^2?
- Find the standard matrix A for reflection in the xy-plane in $\mathbb{R}^3$. Is there a matrix B so that $AB = I_3$?

Section 3.5

Matrix Inverses

Inverses

To solve

$$Ax = b$$

we might want to "divide both sides by A".

We will make sense of this...

Inverses

$A = n \times n$ matrix.

A is invertible if there is a matrix B with

$$AB = BA = I_n$$

B is called the inverse of A and is written A^{-1}

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

The 2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of A.

Fact. If det$(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If det$(A) = 0$ then A is not invertible.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

Solving Linear Systems via Inverses

Fact. If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

Solving Linear Systems via Inverses

What if we change b?

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 0 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

So finding the inverse is essentially the same as solving all $Ax = b$ equations at once (fixed A, varying b).