SECTION 5.3

Solving Recurrence Relations: The Characteristic Polynomial

WHY STUDY RECURRENCE RELATIONS?

Reason #1: Sometimes a sequence of numbers is more easily described this way, e.g.: the number of moves in our solution to the Towers of Hanoi problem is $a_n = 2a_{n-1} + 1$

Also, the number of Fibonacci rabbits: $a_n = a_{n-1} + a_{n-2}$

Reason #2: They are discrete versions of differential equations:

$$a'_n = a_n - a_{n-1} \qquad a''_n = a'_n - a'_{n-1}$$

So differential equations can be approximated by a difference equation, then converted to a recurrence relation.

SOLVING RECURRENCE RELATIONS

To solve a recurrence relation means to give an explicit formula.

Example: $a_n = a_{n-1} + 2, \quad a_0 = 1$

Solution:

Can use induction to prove this is a solution:

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

$$a_n = ra_{n-1} + sa_{n-2}$$

Second order: a_n defined in terms of a_{n-1}, a_{n-2}

Linear: A linear combination of x and y is

$$5x - 2y$$

not

$$5xy \quad \text{or} \quad e^x \quad \text{or} \quad \sqrt{x+y}$$

Homogeneous: No "extra stuff" after the linear combination of a_{n-1} and a_{n-2}.

Extra stuff = function of n.

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} + a_{n-2}$, $a_0 = 0, a_1 = 1$

What is the solution?

First few terms:

What is the pattern?

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

It turns out we can solve them all!

Theorem: Consider the recurrence relation

$$a_n = r a_{n-1} + s a_{n-2}.$$

Let b_1, b_2 be the roots of

$$x^2 - rx - s$$

Then the solution to a_n is:

$$a_n = \begin{cases} c_1 b_1^n + c_2 b_2^n & \text{if } b_1 \neq b_2 \\ c_1 b_1^n + c_2 n b_2^n & \text{if } b_1 \neq b_2 \end{cases}$$

The c_i are determined by the initial conditions.

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = a_{n-2}$, $a_0 = 1$, $a_1 = 3$.

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 0$

THE CASE $b_1 = b_2$

$b_1 = b_2$

MORE PROBLEMS

① Solve $a_n = 9a_{n-2}$ where

(a) $a_0 = 6,\ a_1 = 12$

(b) $a_0 = 6,\ a_2 = 54$

(c) $a_0 = 6,\ a_2 = 10$

② Solve $a_n = 8a_{n-1} - 16a_{n-2},\ a_0 = 1,\ a_1 = 16$

③ Solve $5a_n = 11a_{n-1} - 2a_{n-2},\ a_0 = 2,\ a_1 = -8.$

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

General form: $a_n = ra_{n-1} + sa_{n-2} + f(n)$

Examples:

$$a_n = 2a_{n-1} + 1$$
$$a_n = 3a_{n-1} + 2a_{n-2} + n$$
$$a_n = 5a_{n-1} - a_{n-2} + 2^n$$
$$a_n = a_{n-1} + a_{n-2} + (n^7 + n^n + n!)$$

We do not know how to solve them all, but...

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

THEOREM: Let $a_n = ra_{n-1} + sa_{n-2} + f(n)$.
Let p_n be any particular solution to a_n.
Let q_n be the general solution to $q_n = rq_{n-1} + sq_{n-2}$.
Then $p_n + q_n$ is the general solution to a_n.

We already have a sure-fire way to find q_n.

The hard part is that we don't know how to find p_n — we have to <u>guess</u>.

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

THEOREM: Let $a_n = ra_{n-1} + sa_{n-2} + f(n)$.
Let p_n be any particular solution to a_n.
Let q_n be the general solution to $q_n = rq_{n-1} + sq_{n-2}$.
Then $p_n + q_n$ is the general solution to a_n.

Proof that $p_n + q_n$ really is a solution:

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = 2a_{n-1} + 1$

HOW TO GUESS PARTICULAR SOLUTIONS

If $f(n)$ is...	Guess p_n to be...
exponential	exponential (same base)
linear	linear
quadratic	quadratic
n^{th} degree polynomial	n^{th} degree polynomial
anything else	???

Example: Solve $a_n = 3a_{n-1} + 5\cdot 7^n$, $a_0 = 2$.

Example: $a_n = -a_{n-1} + n$, $a_0 = 1/4$.

MORE PROBLEMS

① Solve $a_n = 5a_{n-1} - 6a_{n-2} + 6 \cdot 4^n$

② Solve $a_n = a_{n-1} + 3n^2$, $a_0 = 7$

By the way, there is another method for solving #2, the method of undetermined coefficients. Idea: recursively substitute: $a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum i^2 = \cdots$