Thm (Dehn '22)

$\mathrm{PMod}(S_{g,n})$ is fin. gen. by Dehn twists.

Lickorish '60s: nonsep. curves.

Humphries '70s: $2g+1$ curves. (minimal)

Pf sketch. Let $f \in \mathrm{PMod}(S_{g,n})$

Choose some curve a.

Step 1 Find $\prod T_{c_i}$ s.t.

$C(S)$ is conn.

$$\prod T_{c_i}\, f(a) = a$$

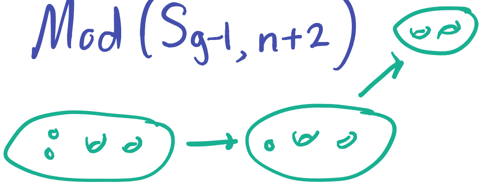
(with orientation)

how to get fin. gen. here?

Step 2. $\mathrm{Stab}(a)$ fin. gen. by Dehn tw's.

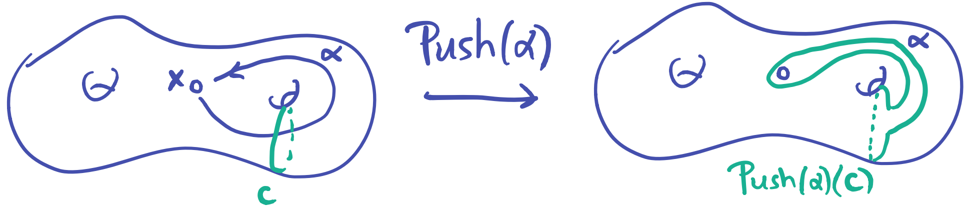
$\xrightarrow{\text{cutting homom.}} \mathrm{Mod}(S_{g-1,\,n+2})$

Birman exact seq.



Towards Birman ex. seq

Push map: $\pi_1(S,x) \longrightarrow \mathrm{Mod}(S,x)$ isotopy extension



Not obviously well def.

Forgetful map: $\mathrm{Mod}(S,x) \longrightarrow \mathrm{Mod}(S)$

Note: $\mathrm{Push}(\pi_1(S,x)) \subseteq \ker(\mathrm{Forget})$

Birman: this is $=$

$\ker(\mathrm{Forget}) \longrightarrow \pi_1(S,x)$

Given φ choose a homotopy to id, so x traces a loop.

Thm (Birman '69) $\chi(S)<0$ This is exact:

$$1 \longrightarrow \pi_1(S,x) \xrightarrow{\text{Push}} \mathrm{Mod}(S,x) \xrightarrow{\text{Forget}} \mathrm{Mod}(S) \longrightarrow 1$$

Pf. This is a fiber bundle:

$$\begin{array}{c} \mathrm{Homeo}^+(S,x) \longrightarrow \mathrm{Homeo}^+(S) \\ \qquad\qquad\qquad\qquad \Big\downarrow \varepsilon = \text{eval at } x \\ \qquad\qquad\qquad\qquad U \subseteq S \end{array}$$

Choose $U \in S$, $x \in U$

$\forall\, u \in U$, choose φ_u s.t. $\varphi_u(x) = u$

vary continuously wrt u.

$$U \times \mathrm{Homeo}^+(S,x) \longrightarrow \varepsilon^{-1}(U)$$
$$(u, \psi) \longmapsto \varphi_u \circ \psi$$

What is a fiber bundle?

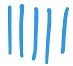
$$\begin{array}{ccl} F & \rightarrow & E = \text{total sp.} \\ & & \downarrow p \\ \text{"fiber"} & & U \subset B = \text{base sp.} \end{array}$$

Locally: $p^{-1}(U) = U \times F$

examples:

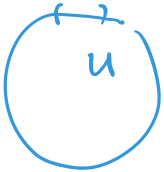
E = cylinder or Möbius band

$B = S^1$, $F = I$



(u,x)

$x \in I$



S^1

$$\boxed{\begin{array}{l}\text{For} \\ E = \text{cov space}, \\ F = \text{discrete set}\end{array}}$$

This is a fiber bundle:

$$\begin{tikzcd} \mathrm{Homeo}^+(S,x) \arrow[r] & \mathrm{Homeo}^+(S) \arrow[d, "\varepsilon \,=\, \text{eval at } x"] \\ & S \end{tikzcd}$$

$\rightsquigarrow$ LES for fiber bundles.

$$\begin{tikzcd} \cdots \arrow[r] & \pi_1 \mathrm{Homeo}^+(S,x) \arrow[r] & \cancel{\pi_1 \mathrm{Homeo}^+(S)} \arrow[r] & \pi_1 S \arrow[dll, "\text{this is the push map}"] \\ & \pi_0 \mathrm{Homeo}^+(S,x) \arrow[r, "\text{this is the forgetful map.}"'] & \pi_0 \mathrm{Homeo}^+(S) \arrow[r] & \cancel{\pi_0 S} \end{tikzcd}$$

$\pi_1 \mathrm{Homeo}^+(S) = 1$ (Hamstrom) $\chi(S) < 0$.

$\pi_1 S$ ✓

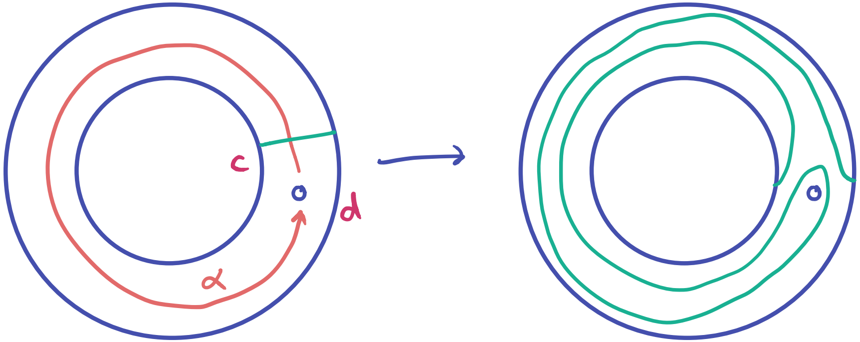
$\pi_0 S = 1$

$\pi_0 \mathrm{Homeo}^+(S,x) = \mathrm{Mod}(S,x)$

$\pi_0 \mathrm{Homeo}^+(S) = \mathrm{Mod}(S)$

$\square$

Push maps in terms of Dehn twists



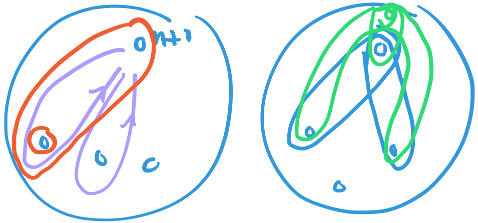
$\mathrm{Push}(\alpha)$

$= T_c T_d^{-1}$

Helps because $\pi_1(S)$ is generated by simple loops.

Special case. PMod$(S_{0,n})$ is fin. gen. by Dehn twists.

$\binom{n}{2}$ gens.



Pf. Ind. on n.

Base cases: $\mathrm{PMod}(S_{0,n}) = 1 \quad n \leq 3$.

Ind step:

fin gen

$$1 \longrightarrow \pi_1(S_{0,n}) \xrightarrow{\text{Push}} \mathrm{PMod}(S_{0,n+1}) \longrightarrow \mathrm{PMod}(S_{0,n}) \to 1$$

(image of) each gen is a product of ~~2~~ 1 Dehn twists

Fin. Gen by Dehn twists by induction. Each has a lift that is a Dehn twist $\square$

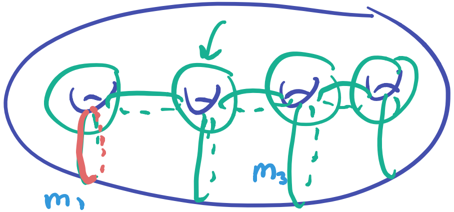
$\boxed{\text{Same argument gives step (2) for Dehn's thm}}$

In a short ex. seq, middle gp is gen by: gens on left & lifts of gen on right

Explicit sets of gens

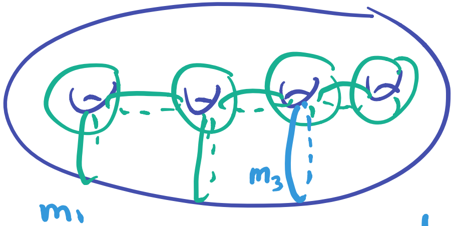
$h\,\mathrm{Push}(\alpha)\,h^{-1} = \mathrm{Push}(h\alpha)$

Lickonish:



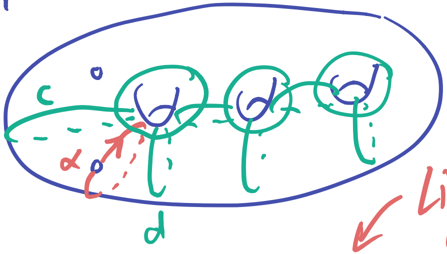
$3g-1$

Humphries:



$2g+1$

Ind step:



Lick. gens!

$\mathrm{Push}(\alpha) = T_c T_d^{-1}$

Use Lick gens to take α to other gens for π_1

Need to find product h of Hump. gens taking m_1 to m_3

$h T_{m_1} h^{-1} = T_{m_3}$

Hamidi-Tehrani

$$M \leq \frac{m^2}{6} \implies \text{free gp.}$$

100, 101, 102, 103

Q. "Effectivize" HT to make explicit F_n's.

$$1 \to \underset{\substack{\shortparallel \\ \mathbb{Z}}}{\langle T_\gamma \rangle} \to \mathrm{Mod}(S_g^1) \rightleftarrows \mathrm{Mod}(S_{g,1}) \to 1$$

$\hookrightarrow$ euler class in $H^2(\mathrm{Mod}(S_{g,1}))$

Nontriviality $\Longleftrightarrow$ non-splitting

$$S^1 \longrightarrow \pi_1 UT(S) \longrightarrow \pi_1(S) \quad \text{nonsplit.}$$