Announcements: Sep 6

- Midterm 1 on Sep 21
- Quiz 2 Friday in recitation
- WeBWorK 2.2 and 2.3 due Wednesday (tonite!)
- My office hours today 2:00-3:00 and Friday 9-10? in Skiles 234
- I hope you come to office hours
- TA Office Hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Fri 12-1 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 9:30-10:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday - Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123

2.3 Parametric Form

Free Variables

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 \quad + 5x_3 \quad &= 0 \\ x_4 &= 0 \end{aligned}$$

So the associated matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

To solve, we move the free variable to the right:

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= \quad x_2 \quad \text{(free)} \\ x_3 &= \quad x_3 \quad \text{(free)} \\ x_4 &= 0 \end{aligned}$$

Or: $(-5x_3, x_2, x_3, 0)$. This is a plane in $\mathbb{R}^4$.

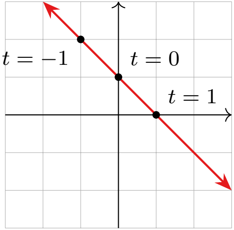
The original equations are the implicit equations for the solution. The answer to this question is the parametric solution.

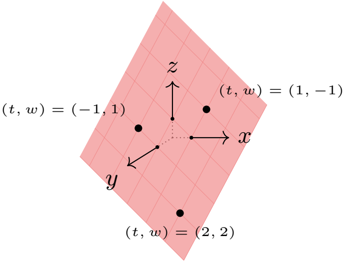
Free variables

Geometry

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional plane in $\mathbb{R}^n$.

Why does this make sense?





Implicit versus parametric equations of planes

Find a parametric description of the plane

$$x + y + z = 1$$

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Chapter 3

System of Linear Equations: Geometry

Section 3.1

Vectors

Outline

- Think of points in $\mathbb{R}^n$ as vectors.
- Learn how to add vectors and multiply them by a scalar
- Understand the geometry of adding vectors and multiplying them by a scalar
- Understand linear combinations algebraically and geometrically

Vectors

A vector is a matrix with one row or one column. We can think of a vector with n rows as:

- a point in $\mathbb{R}^n$
- an arrow in $\mathbb{R}^n$

To go from an arrow to a point in $\mathbb{R}^n$, we subtract the tip of the arrow from the starting point. Note that there are many arrows representing the same vector.

Adding vectors / parallelogram rule ▸ Demo

Scaling vectors ▸ Demo

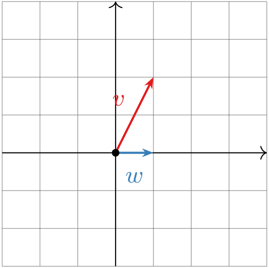
A scalar is just a real number. We use this term to indicate that we are scaling a vector by this number.

Linear Combinations

A linear combination of the vectors $v_1, \ldots, v_k$ is any vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.



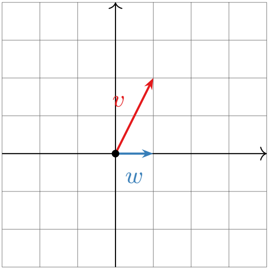
Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

Poll

Is there a vector in $\mathbb{R}^2$ that is not a linear combination of v and w?

- true
- false



Linear Combinations

What are some linear combinations of $(1, 1)$?

What are some linear combinations of $(1, 1)$ and $(2, 2)$?

What are some linear combinations of $(0, 0)$?

Linear Combinations

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$?

Write down an equation in order to solve this problem. This is called a vector equation.

Notice that the vector equation can be rewritten as a system of linear equations. Solve it!

Summary of Section 3.1

- A vector is a point/arrow in $\mathbb{R}^n$
- We can add/scale vectors algebraically & geometrically (parallelogram rule)
- A linear combination of vectors $v_1, \ldots, v_k$ is a vector

$$c_1 v_1 + \cdots + c_k v_k$$

  where $c_1, \ldots, c_k$ are real numbers.
- Asking the question of whether a certain vector is a linear combination of certain other vectors gives us a vector equation.
- Vector equations are the same as linear systems.

Section 3.2

Vector Equations and Spans

Outline of Section 3.2

- Learn the equivalences:

$$\text{vector equations} \leftrightarrow \text{augmented matrices} \leftrightarrow \text{linear systems}$$

- Learn the definition of span
- Learn the relationship between spans and consistency

Linear combinations, vector equations, and linear systems

We just saw the following question:

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$?

And saw it was the same as a vector equation:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

which is the same as the system of linear equations:

$$\begin{aligned} c_1 - c_2 &= 8 \\ 2c_1 - 2c_2 &= 16 \\ 6c_1 + c_2 &= 3 \end{aligned}$$

which we solve by row reducing, and we get $(c_1, c_2) = (5 - 3)$.

Linear combinations, vector equations, and linear systems

In general, asking if b is a linear combination of $v_1, \ldots, v_k$ is the same as solving the vector equation

$$c_1 v_1 + \cdots c_k v_k = b$$

which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

Span

Essential vocabulary word!

$$
\begin{aligned}
\mathrm{Span}\{v_1, v_2, \ldots, v_k\} &= \{c_1v_1 + c_2v_2 + \cdots c_kv_k \mid c_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\
&= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\
&= \text{plane through the origin and } v_1, v_2, \ldots, v_k.
\end{aligned}
$$

Four ways of saying the same thing:

- b is in Span$\{v_1, v_2, \ldots, v_k\}$
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $c_1v_1 + \cdots + c_kv_k = b$ has a solution
- the system of linear equations corresponding to

$$
\left(\begin{array}{cccc|c}
| & | & & | & | \\
v_1 & v_2 & \cdots & v_k & b \\
| & | & & | & |
\end{array}\right),
$$

is consistent.

Demo Demo

Pictures for spans

What are the possibilities for the span of one vector in $\mathbb{R}^2$?

What are the possibilities for the span of two vectors in $\mathbb{R}^2$?

Demo

What are the possibilities for the span of one vector in $\mathbb{R}^3$?

What are the possibilities for the span of two vectors in $\mathbb{R}^3$?

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

Conclusion: Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

Summary of Section 3.2

- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

Application

Consider the production costs:

	Materials	Labor	Overhead
Widget	\$1	\$2	\$3
Gadget	\$4	\$5	\$6

Q. What are possible expenditures on materials, labor, and overhead?

Q. If we have a budget of \$11 for materials, \$16 for labor, and \$20 for overhead, can we spend our entire budget by making widgets and gadgets?