

# GENERATING TORELLI

Goal:  $I(S_g)$  is gen. by BP maps (and Dehn twists about sep curves)

Original proof: 1971 Birman gives presentation for  $Sp_{2g}(\mathbb{Z})$

1978 Powell interprets relations

1980 Johnson, lantern relation

Want a proof analogous to  $Mod(S_g)$  case.

Complex of homologous curves

Fix (primitive)  $x \in H_1(S_g; \mathbb{Z})$

$C_x(S_g)$  = subgraph of  $C(S_g)$  spanned by  
(unoriented) reps of  $x$ .

goal: Connected.

“borrowing complex”

Will use auxilliary complex  $B_x(S_g)$ , the  
complex of cycles. Points of  $B_x(S_g)$   
are simple, irredundant reps of  $x$ .

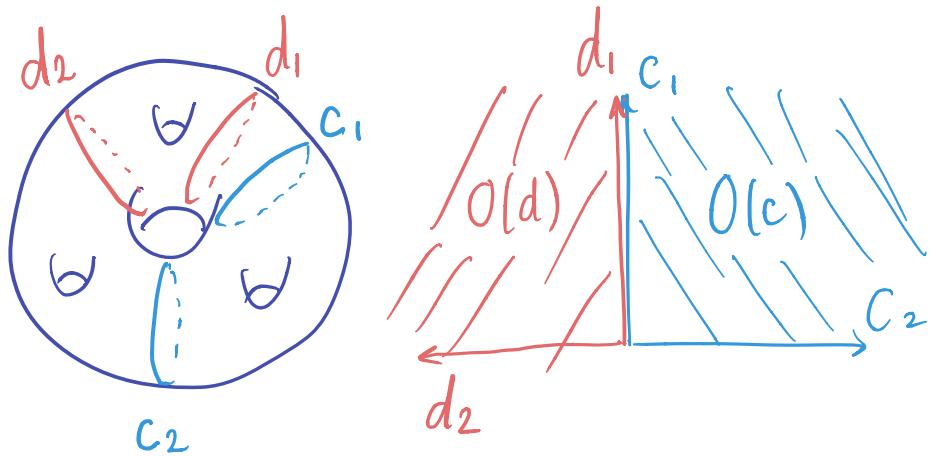
# The Complex of Cycles

$C = \text{oriented multicurve, } n \text{ components}$

$$\rightsquigarrow [0, \infty)^n \rightarrow H_1(S_g; \mathbb{Z}) \text{ orthant } O(c)$$

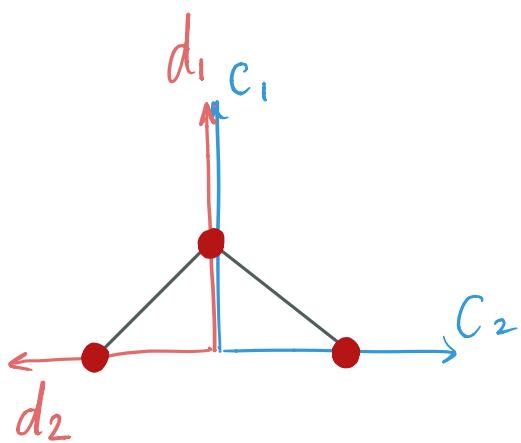
$$A(S_g) = \coprod_c O(c) / \sim$$

example.

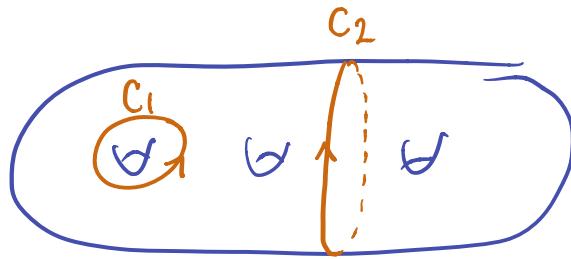


$$A_x(S_g) \subseteq A(S_g) \text{ reps of } x.$$

Say  $x = [c_1]$

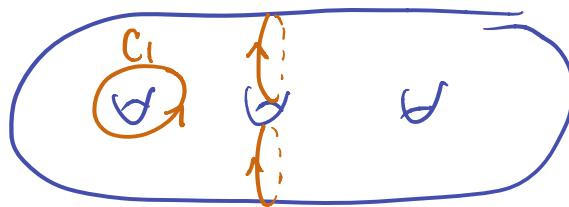


The cells of  $A_x(S_g)$  are not necessarily compact:



If  $[c_1] = x$  then  $[c_1 + bc_2] = x \quad \forall b \in \mathbb{R}$

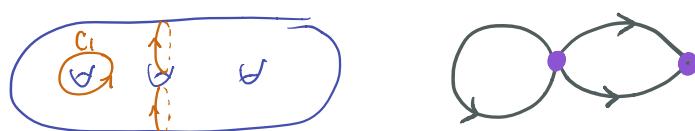
Or:



An oriented multicurve is reduced if

- (1) the corresponding cell is compact
- $\iff$  (2) it has no homologically trivial subset
- $\iff$  (3) the dual directed graph is recurrent

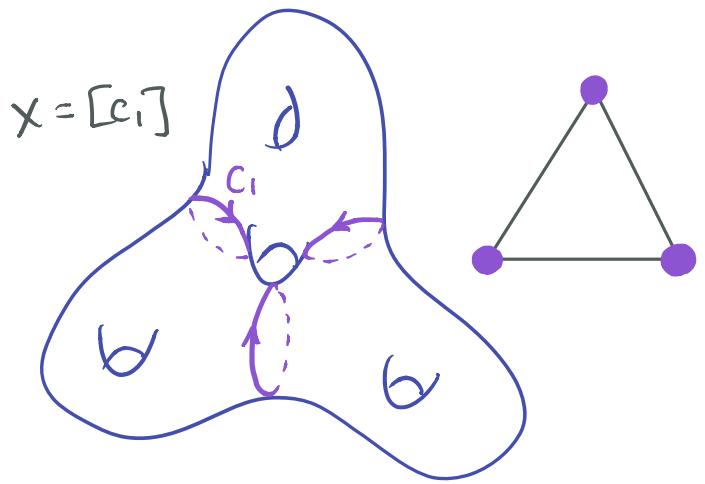
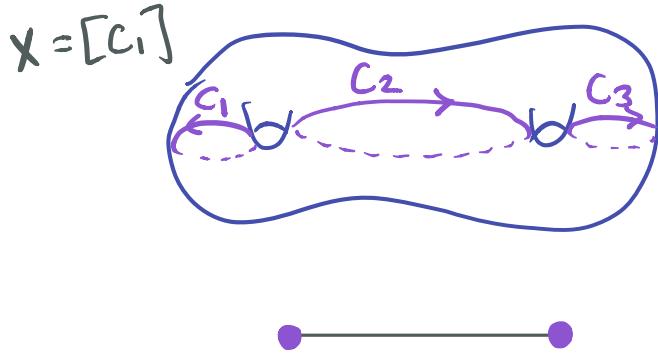
Dual graphs:



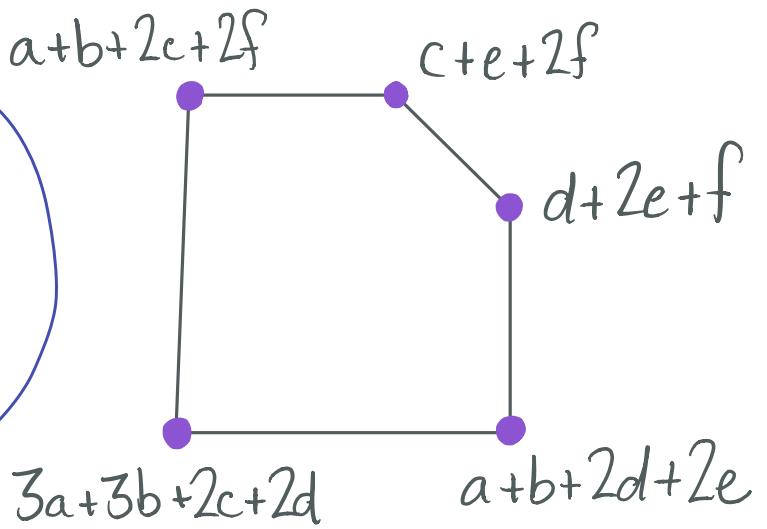
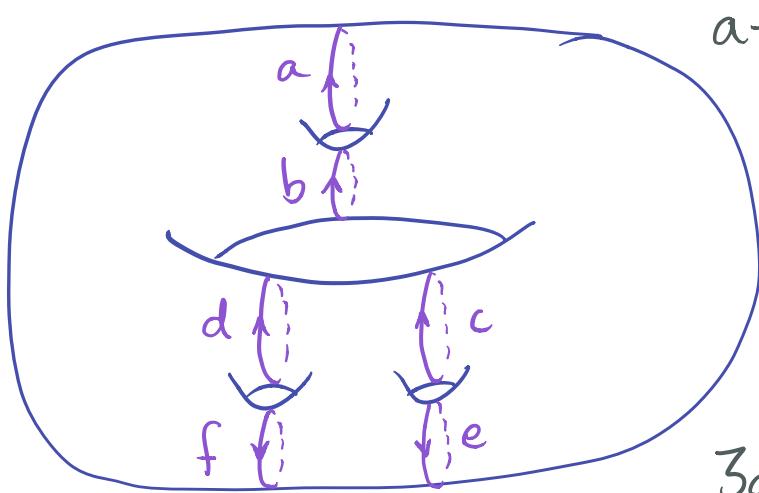
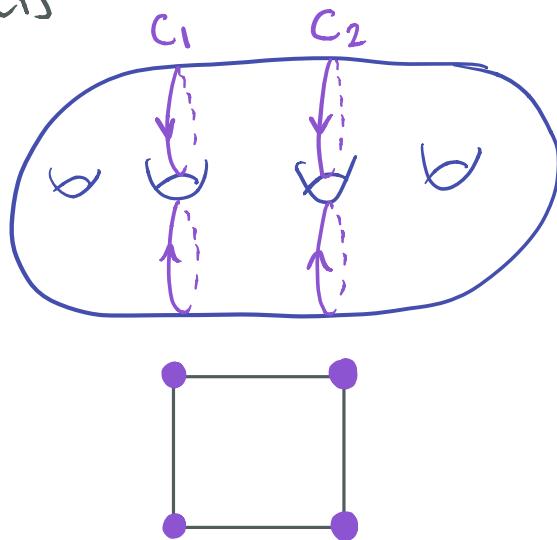
The complex of cycles  $B_x(S_g)$  is the subcomplex of  $A_x(S_g)$  whose cells correspond to reduced oriented multicurves.

We'll show  $B_x(S_g)$  is contractible.

# Examples of cells



$$x = [c_1] + [c_2]$$



Q. Which polytopes arise?

# Properties of Cells

Prop. The dim. of a cell = # compl. comp.'s - 1 .

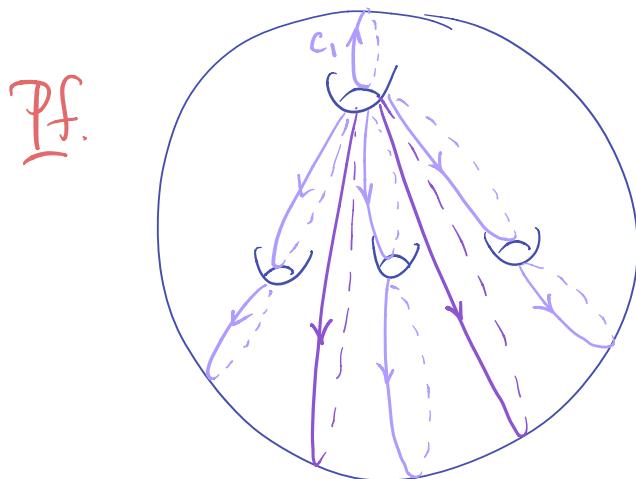
Pf. Defn of homology.

$\Rightarrow$  vertices  $\longleftrightarrow$  nonsep. multicurves.

Prop. Vertices of  $B_x(S_g)$  are oriented multicurves with integral weights.

Pf. Given a vertex, consider a loop intersecting in one point.

Prop.  $\dim B_x(S_g) = 2g-3$ .



$$x = [c_1].$$

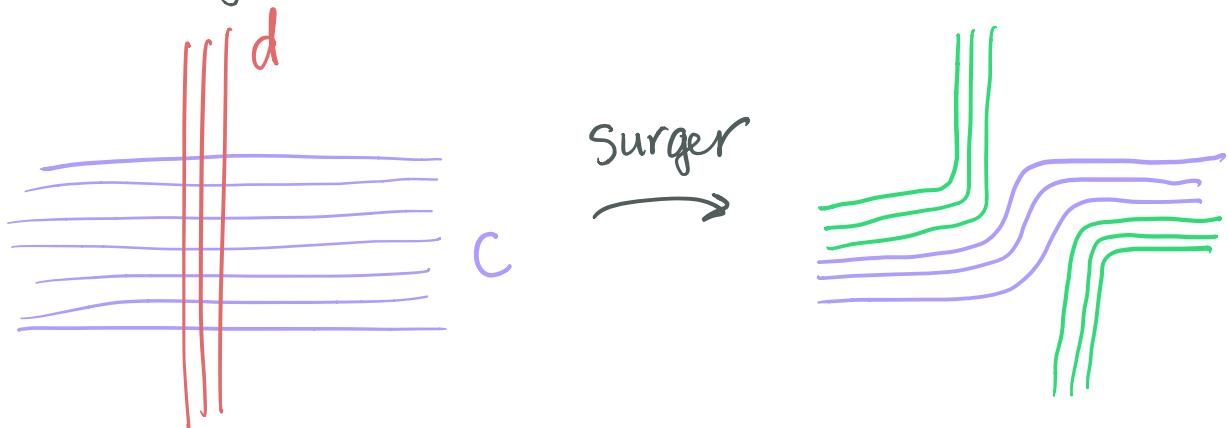
$\leadsto B_x(S_2)$  is a graph.

# CONTRACTIBILITY

Theorem.  $B_x(S_g)$  is contractible.

Surgery on 1-cycles

Say  $c, d \in A_x(S_g)$ . Thicken  $c, d$  according to weights and then:



If  $[c] = [d] = x$ , this procedure will result in a 1-cycle rep'ing  $x$ . Why?

$$H_1(S_g; \mathbb{Z}) \cong H^1(S_g; \mathbb{Z}) \cong \text{Hom}(H_1(S_g; \mathbb{Z}), \mathbb{Z}) \leftrightarrow [S_g, S^1]$$

The original  $c, d$  give maps  $S_g \rightarrow S^1$  by integrating against width of annuli. The surgered picture corresponds to the map  $S_g \rightarrow S^1$  obtained by integrating against both widths.

Prop.  $A_x(S_g)$  is contractible

Pf. Fix some  $c \in A_x(S_g)$ . Consider:

$$F_t(d) = \text{Surger}(tc + (1-t)d)$$

□

## Draining 1-cycles

Suppose  $c \in A_x(S_g)$  is not reduced.

↪  $\{R_i\}$  subsurfaces with  $\partial R_i \subseteq c$

$$\text{Drain}_t(c) = c - t \sum \partial R_i$$

Prop.  $A_x(S_g)$  def. retracts to  $B_x(S_g)$ .

In partic.  $B_x(S_g)$  is contractible.

Pf. Drain

□

In particular,  $B_x(S_2)$  is a tree.

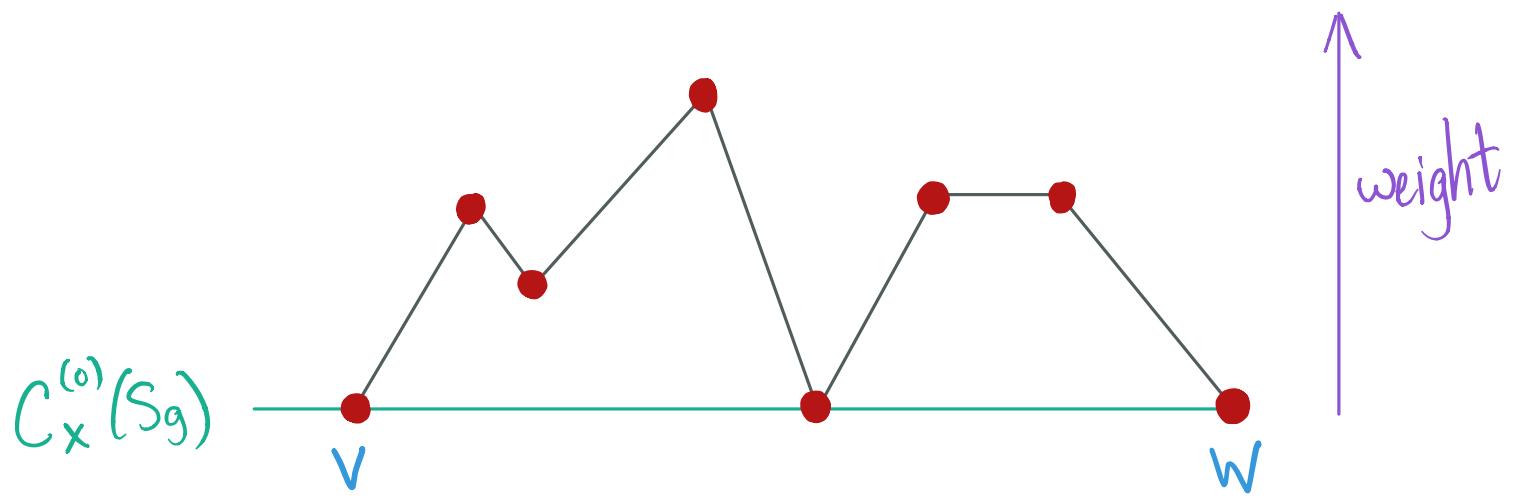
## CONNECTIVITY OF $C_x(S_g)$

Basic strategy

Define weight :  $B_x(S_g) \rightarrow \mathbb{Z}$   
 $\sum w_i c_i \mapsto \sum w_i$

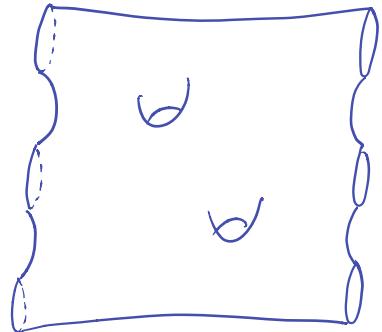
Note:  $C_x(S_g) = \text{weight}^{-1}(1)$ .

Now, given  $v, w \in C_x^{(0)}(S_g)$ , we connect them  
in  $B_x^{(1)}(S_g)$ :



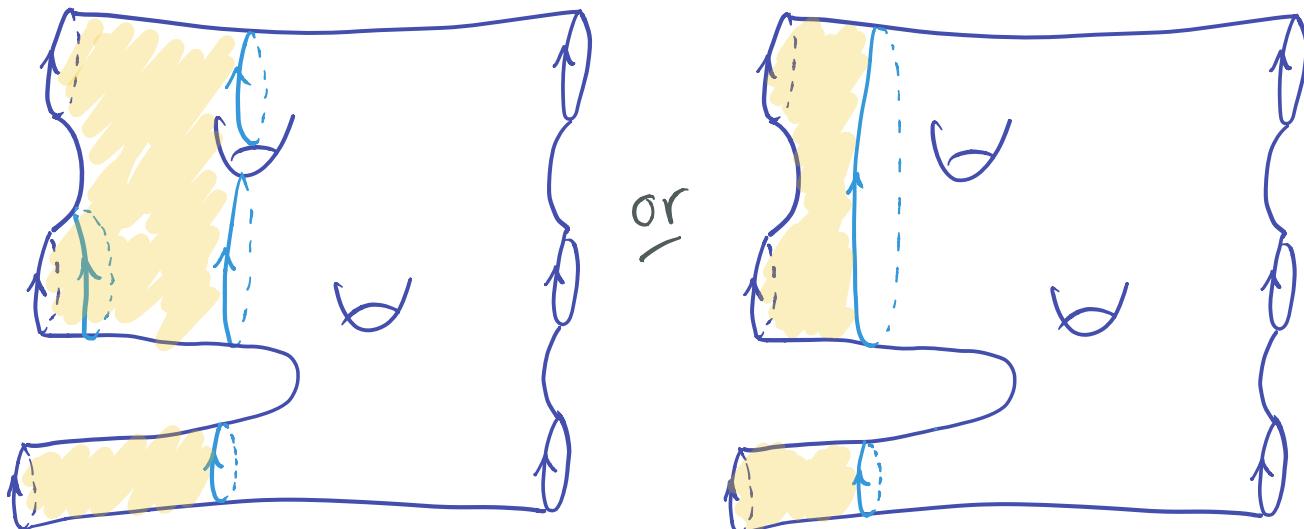
We then push the highest vertex  
down inductively until the path lies  
in  $C_x(S_g)$ .

Key idea: If we cut along a vertex of  $B_x^{(0)}(Sg)$  we get



"cobordism"

What does an edge in  $B_x(Sg)$  look like?

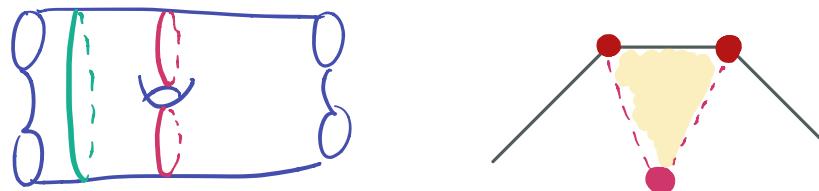


The region between transfers weight from one side to the other  $\Rightarrow$  the new vertex will have smaller weight iff there are fewer interior curves than boundary curves.

Call the edge on the right a pants edge.  
This is the simplest way to reduce weight.

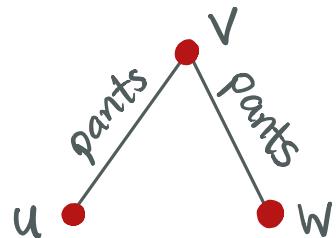
PROOF THAT  $C_x(S_g)$  is connected

Step 1. Make maxima isolated, by making pants edges/triangles.

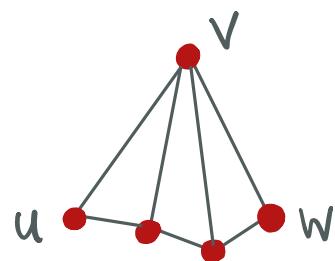


Step 2. Make highest edges into pants edges in same way

Step 3. Given



Connect  $uv$  to  $v$  by a seq of pants triangles

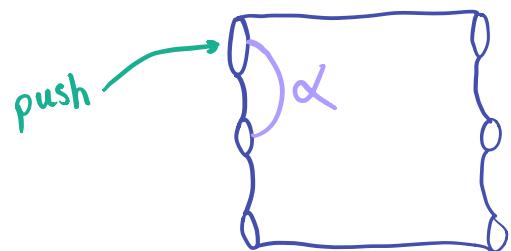


Can then push  $v$  down. Apply this process inductively.

To this end, consider the graph with  
Vertices: pants edges emanating from  $v$

$\longleftrightarrow$  certain arcs in  $S \setminus v$

edges: disjoint arcs.



To show: connected

- Notes.
- Every vertex is adjacent to one connecting first two components of  $\partial(S \setminus v)$ .
  - Push maps (corresponding to 1<sup>st</sup>  $\partial$ -comp) act transitively on these.
  - $\pi_1$  (punctured sphere) has a simple gen set  $\{x_i\}$

So: suffices to show that each  $\text{Push}(x_i) \cdot \alpha$  lies in same component as  $\alpha$ .

Sample case:  $x_i$  lies on LHS of  $S \setminus v$ .

Then if  $\beta$  lies on RHS we have



# PROVING TORELLI IS GEN. BY BP MAPS

Ingredient #1.  $C_x(S_g)$  is connected ✓

Ingredient #2. Fact. Say  $G \curvearrowright X = \text{graph}$

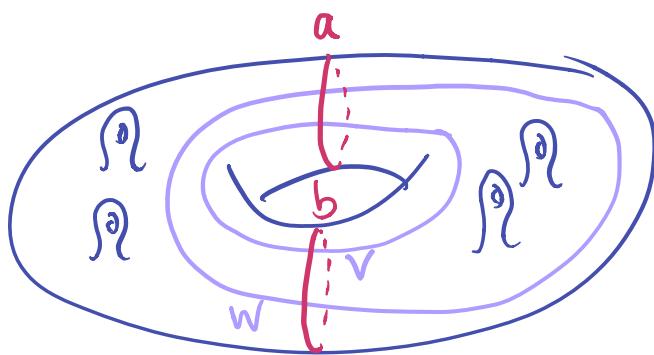
$A \subseteq G$  s.t.  $\forall$  edges  $v w$

$\exists g \in A$  with  $g \cdot v = w$ .

Then  $G = \langle A, \text{vertex stabs} \rangle$

Pf. Same as before

Ingredient #3. If  $v w$  is an edge of  $C_x(S_g)$   
 $\exists$  BP map taking  $v$  to  $w$ .



$$T_a T_b^{-1}(v) = w$$

Thus it suffices to show  $\text{Stab}_{I(S_g)}(v)$  is gen.  
 by BP maps and Dehn twists about sep curves.

# Two BIRMAN EXACT SEQUENCES FOR TORELLI

## ONE MARKED POINT

$$1 \rightarrow \pi_1(S_g, p) \rightarrow I(S_g, p) \rightarrow I(S_g) \rightarrow 1$$

(restriction of usual BES)

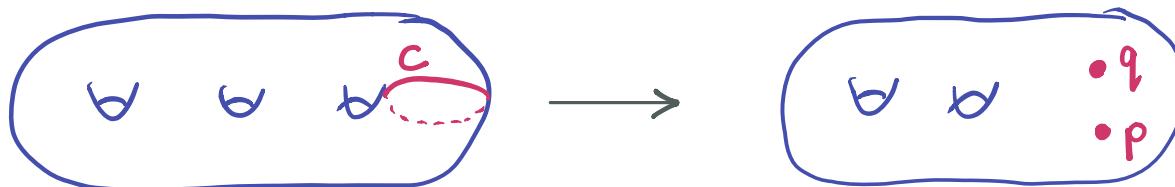
## TWO MARKED POINTS

$$1 \rightarrow K \rightarrow I(S_g, \{p, q\}) \rightarrow I(S_g, p) \rightarrow 1$$

$$K = [\pi, \pi] \quad \pi = \pi_1(S_g \setminus p, q)$$

will verify  
below!

What is defn of  $I(S_g, \{p, q\})$ ? It is the image of  $I(S_g, c)$  under the cutting map:



$$\leadsto I(S_g, \{p, q\}) = \ker(M_{\text{Mod}}(S_g, \{p, q\}) \rightarrow \text{Aut } H_1(S_g, \{p, q\}))$$

Can define  $I(S_g, p)$  in same way, as further image under forgetting  $q$ . Get usual defn.

STABILIZERS ARE GEN. BY BP maps...

First,  $\text{Stab}_{I(S_g)}(c) \cong I(S_{g-1}, \{p, q\})$  since:

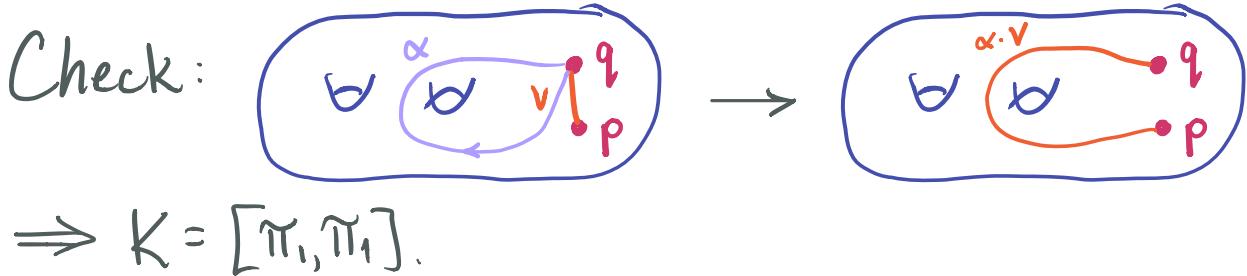
$$1 \rightarrow \langle T_c \rangle \rightarrow \text{Mod}(S_g, c) \rightarrow \text{Mod}(S_g, \{p, q\}) \rightarrow 1$$

and  $T_c \notin I(S_g)$ .

Step 1.  $K$  gen by BP maps & Dehn twists about seps.

$$\pi_1(S_g \setminus p, q) \cong H_1(S_g, \{p, q\}) \cong H_1(S_g) \oplus \mathbb{Z}$$

- trivial on first factor.
- action on 2<sup>nd</sup> factor is  $\alpha \cdot v = [\alpha] + v$ .

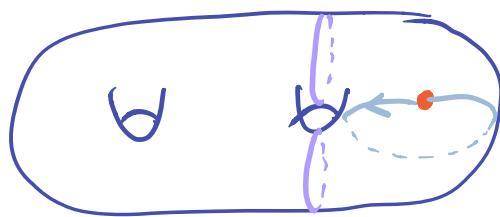


Now need:  $K$  gen. by simple (sep) loops.

Realize  $S_g$  as 4g-gon with opp sides id'd.  
and use the fact that commutator subgps  
are normally gen. by commutators of gens.

Step 2.  $\Pi_1(S_g, p) \subseteq I(S_g, p)$  gen by BP maps.

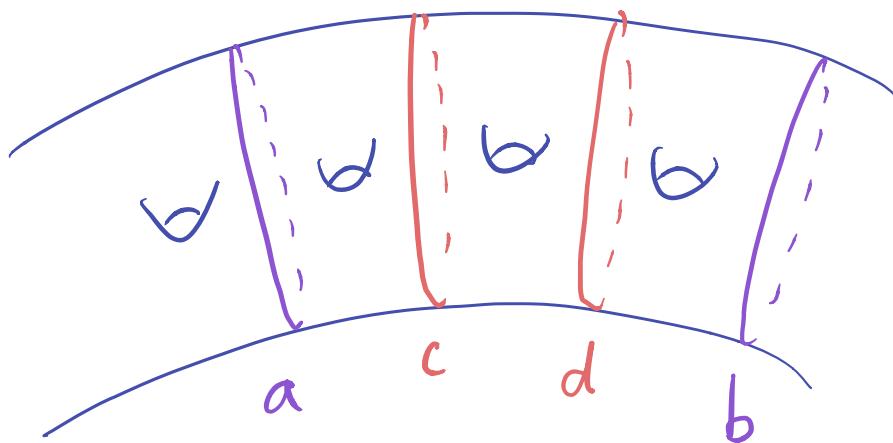
easy:



Even better:

Thm.  $I(S_g)$  is gen. by BP maps of genus one.

Pf.



$$T_a T_b^{-1} = (T_a T_c^{-1})(T_c T_d^{-1})(T_d T_b^{-1}).$$

Still need to address base case  $g=2$ !

## GENUS 2

$C_x(S_2)$  is not connected ☹

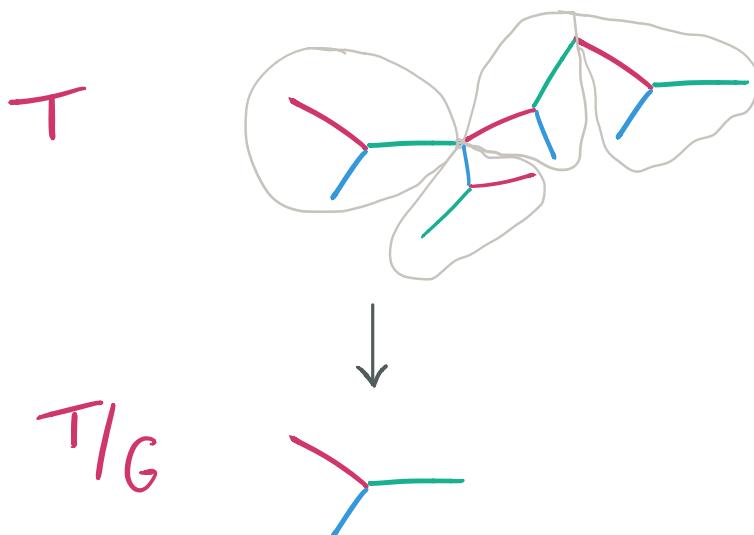
Will instead use  $I(S_2) \cup B_x(S_2)$ .

Already showed  $B_x(S_2)$  is a tree.

Will show  $B_x(S_2)/I(S_2)$  is a tree.

Fact. If  $G \cap T = \text{tree}$  and  $T/G = \text{tree}$   
then  $G$  is (freely) gen. by vertex stabilizers.

Pf. Key point:  $T/G \hookrightarrow T$   
So  $T$  covered by translates of  $T/G$



Fix  $X$ , a copy of  $T/G$  in  $T$ . ("tile")

Let  $g \in G$ . Suppose first that  $g \cdot X \cap X \neq \emptyset$

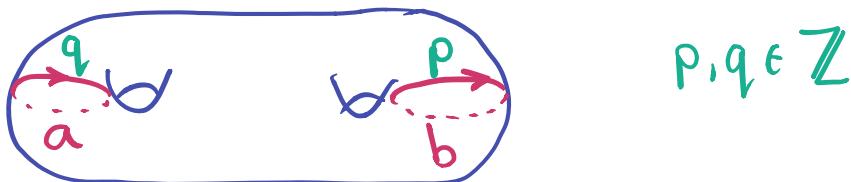
$$\Rightarrow g \cdot X \cap X = \{v\} \Rightarrow g \in \text{Stab}(v)$$

(otherwise  $T/G$  would have a loop). Induct  
on tile distance. □

Remains: ①  $Bx(S_2) / I(S_2)$  = tree  
 ② Stab's gen by Dehn twists

Proof of ① Weight fn. descends to quotient.  
 And  $\exists!$  vertex of weight 1 (in quotient).

To show: if  $\text{weight}(v) > 1 \exists! w$  with  
 $\text{weight}(w) < \text{weight}(v)$  and   
 $v$  looks like:



$w$  given by nonsep curve  $c$  in middle.  
 $\leftrightarrow$  curve in cut surface.

  
 $\circ \circ$

$c$  nonsep  $\Rightarrow c$  does not sep. left from right

$\text{weight}(w) < \text{weight}(v) \Rightarrow c$  separates top from bottom.

Can think of  $c$  as an arc connecting top two circles.

  
 $\circ \circ$

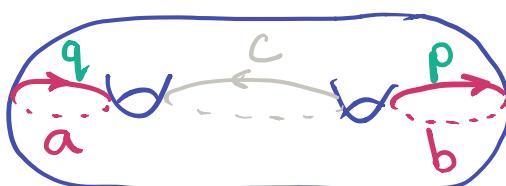
Make a graph: vertices: arcs as above  
edges: disjointness.

This connected (use usual trick with  $P\text{Mod}(S_0, 4)$  action). Adjacent vertices differ by sep twist.

## Proof of ② Two kinds of vertices

Connected multicurves: Use Birman exact seq. as before (no change).

Disconnected multicurves: By above argument can assume (up to Dehn twists about sep curves) that a



stabilizer of  $a \& b$  also fixes  $c$ .  
 $\Rightarrow$  it is trivial. □

We have shown  $I(S_2) = \langle T_c \mid c \text{ sep} \rangle$

With some more bookkeeping, we show it is freely gen. by Dehn twists, one for each symplectic splitting.