Announcements Apr 15

- Cameras on
- Peer evaluations due ~~Fri~~ Sun
- Presentations next week ~20 mins
- Final draft due Apr 27 3:30.
- Office Hours Fri - postponed.
- Makeup problems
- CIOS

Today

- Ends of groups:
  Freudenthal - Hopf Thm
- Summary

Ends of Groups

Freudenthal-Hopf Thm

G = fin gen gp

$\Longrightarrow$ G has 0, 1, 2, or (∞ many) ends

Some defns:

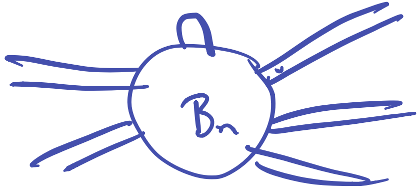
Γ = connected graph, locally finite.

v = base vertex.

B_n = ball of radius n around v.

$\|\Gamma \setminus B_n\|$ = # unbounded pieces of $\Gamma \setminus B_n$.

Γ

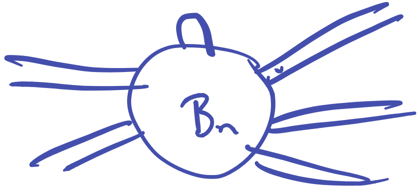


$\|\Gamma \setminus B_n\|$
$= 5$

$$e(\Gamma) = \lim_{n\to\infty} \|\Gamma \setminus B_n\|.$$

$\|\Gamma \setminus B_n\| =$ # unbounded pieces of $\Gamma \setminus B_n$.

Γ



$\|\Gamma \setminus B_n\| = 5$

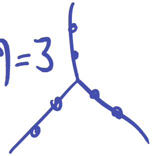
$$e(\Gamma) = \lim_{n\to\infty} \|\Gamma \setminus B_n\|.$$

Examples ① Γ finite.

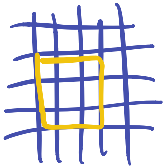
$\Rightarrow e(\Gamma) = 0$

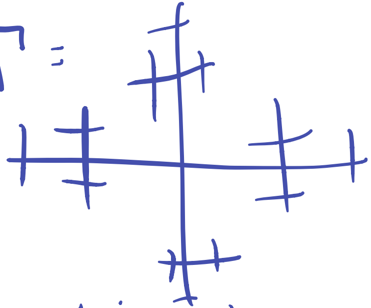
② $\Gamma =$ 

$e(\Gamma) = 2$

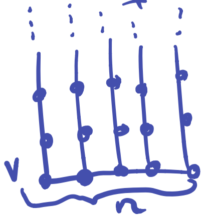
2.5 $\Gamma =$ 

$e(\Gamma) = 3$

③ $\Gamma =$  $e(\Gamma) = 1$

④ $\Gamma =$ 

$e(\Gamma) = \infty$

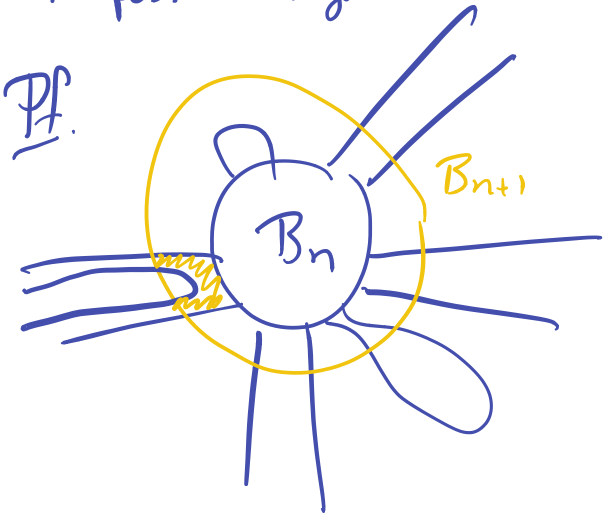
⑤ $\Gamma =$ 

$e(\Gamma) = n$.

$e(\Gamma)$ is well defined

Lemma. $\|\Gamma \setminus B_n\|$ is a non-decreasing seq. of pos. integers.

Pf.



When you take a unbdd subgraph and remove a bounded subgraph $(B_{n+1} \setminus B_n)$ it becomes ≥ 1 unbounded piece

(also, pieces can't merge when you remove stuff.)

Cor. $e(\Gamma)$ is well-def.

Next goal: $e(\Gamma)$ is a QI invariant

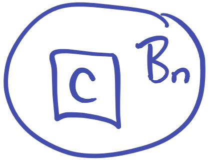
Alternate defn

$$e_c(\Gamma) = \sup\{\|\Gamma\setminus C\| : C\subseteq\Gamma \text{ finite}\}$$

Lemma. $e_c(\Gamma) = e(\Gamma)$.

Pf. $\geq$ sup over bigger set.

$\leq$ Any such C is contained in a B_n. Use argument from last slide.



$\|\Gamma\setminus B_n\| \geq \|\Gamma\setminus C\|$.

Ends is a QI invariant

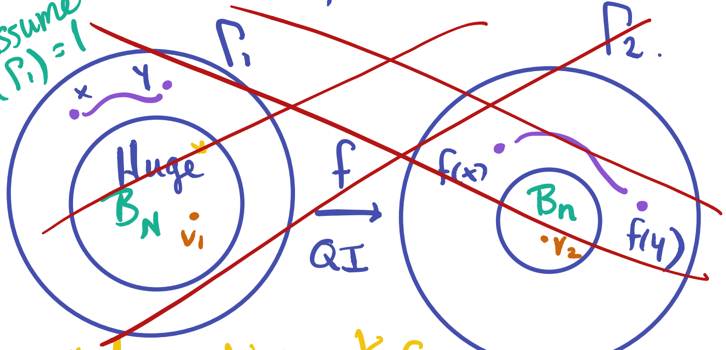
Prop. If $\Gamma_1 \underset{QI}{\simeq} \Gamma_2$ then

$$e(\Gamma_1) = e(\Gamma_2).$$

Pf. Let's convince ourselves

that $e(\Gamma_1) = 1 \Leftrightarrow e(\Gamma_2) = 1.$

Assume $e(\Gamma_1) = 1$



* depending on K, C.

~~Want: x, y connected outside B_N.~~

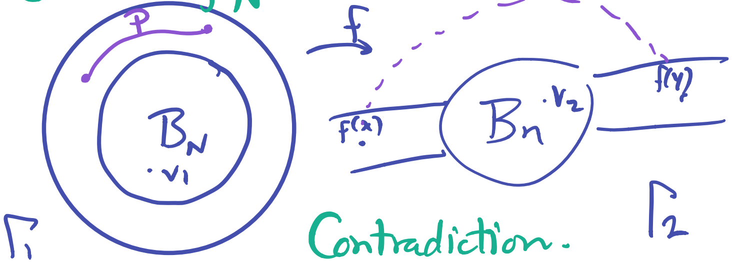
~~Note: $f(x), f(y)$ outside B_n.~~

~~Thus they are connected by path P outside B_n.~~

Assume $e(\Gamma_1) = 1$

Assume B_n cuts Γ_2 in two unbdd pieces.

Choose big N.



Contradiction.

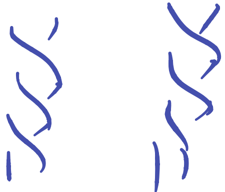
Poll

Q. How many ends does braid gp B_n have?

$B_1 \cong 1 \Rightarrow e(B_1) = 0$

$B_2 \cong \mathbb{Z} \Rightarrow e(B_2) = 2$

B_3 ???



A. $e(B_n) = 1 \quad n \geq 3.$

Step 1.

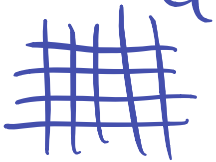
Pf. $e(B_n) = e(PB_n)$

since $[B_n : PB_n] = n! < \infty$

Step 2. $PB_n \cong PB_n/\mathbb{Z} \times \mathbb{Z}$

Fact. If G, H infinite,

$e(G \times H) = 1.$

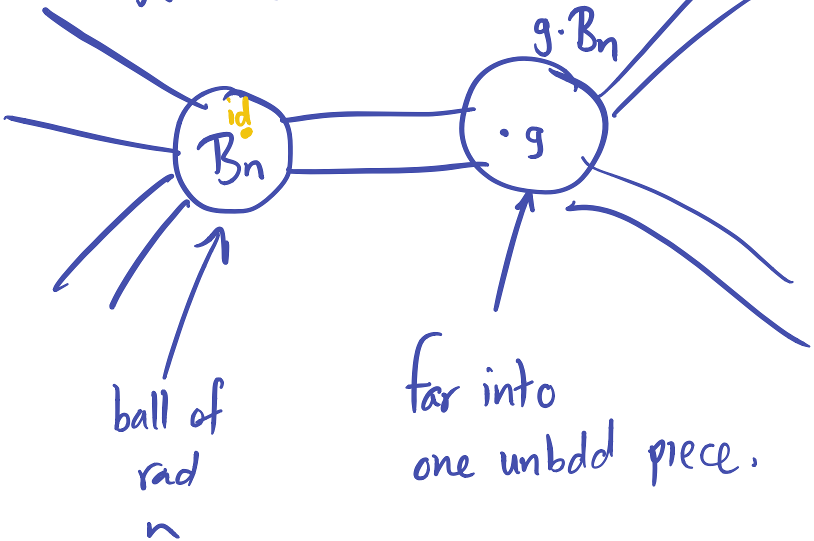


Freudenthal-Hopf Thm

G = fin gen.

Then $e(G) \in \{0,1,2,\infty\}$.

Assume 3 ends



$\Rightarrow$ 4 ends etc...

Some Ends we know

$e(\mathbb{Z}) = 2$

$e(\text{finite gp}) = 0$

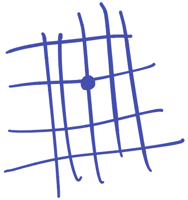
$e(\mathbb{Z}^n) = 1 \quad n \geq 2.$

$e(F_k) = \infty \quad k \geq 2.$

$e(B_n) = 1 \quad n \geq 3$

$e(SL_2\mathbb{Z}) = \infty$

$e(W_{333}) = 1$



Different # ends

$\Rightarrow$ not quasi-isometric.

Also sometimes gps with same # of ends are not QI.

example $\mathbb{Z}^m, \mathbb{Z}^n \quad m \neq n.$

(different growth rates)
$\sim r^n$

Who cares if groups are not QI?

· prop disc. ← doesn't lose info abat G.

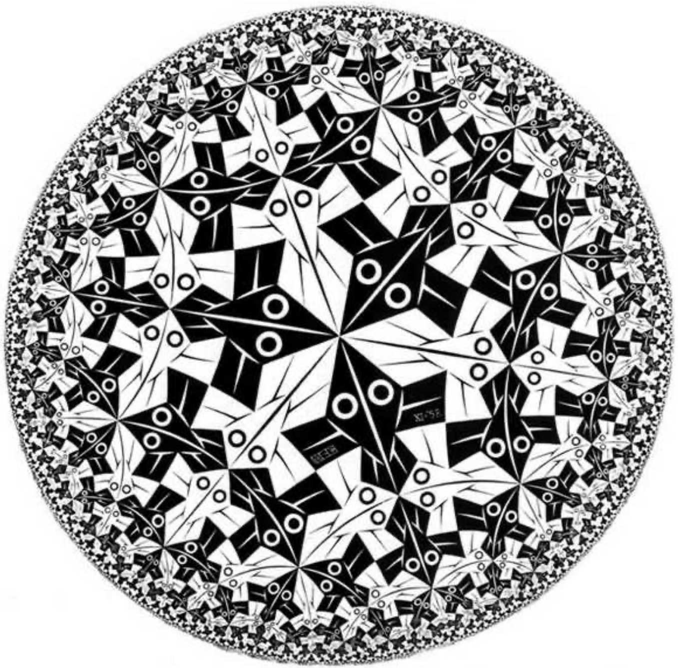
· finite fund dom. ← does lose info about Γ!

Milnor-Schwarz: If G acts geometrically on Γ then $G \underset{QI}{\cong} \Gamma$.

So. If $G, H \underset{\text{geom.}}{\circlearrowright} \Gamma$ then $G \underset{QI}{\cong} H$.

Infinitely many corollaries: B_n does not act geom. on $\mathbb{R}$, or tree.

$SL_2\mathbb{Z}$ does not act geom. on $\mathbb{R}^2$.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