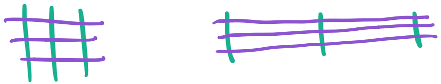
Chap 11. Teich geom.

Teich thms: Given $X, Y \in \mathrm{Teich}(S)$
$\exists$ unique map $h: X \to Y$
homot. to id that minimizes
dilatation K

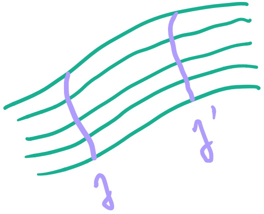
The map is locally:

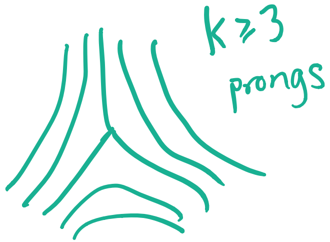


need to make sense of horiz/vert. on S.

$$\begin{pmatrix} \sqrt{K} & 0 \\ 0 & 1/\sqrt{K} \end{pmatrix}$$

Measured foliations



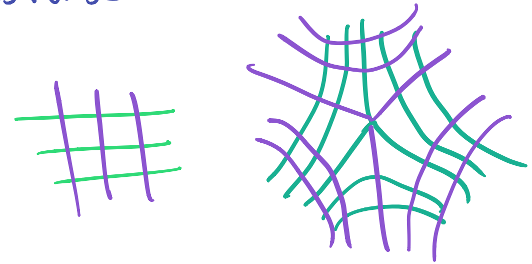


$\mu = \text{transverse measure} \geq 0.$

$\mu(\gamma) = \mu(\gamma')$

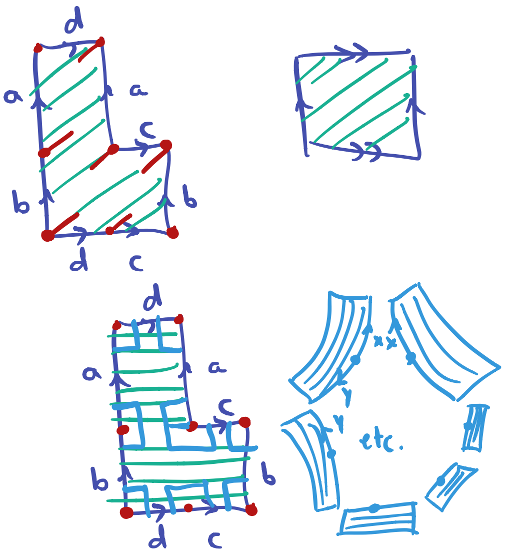
These allow us to do Teich maps as above.

transverse foliations

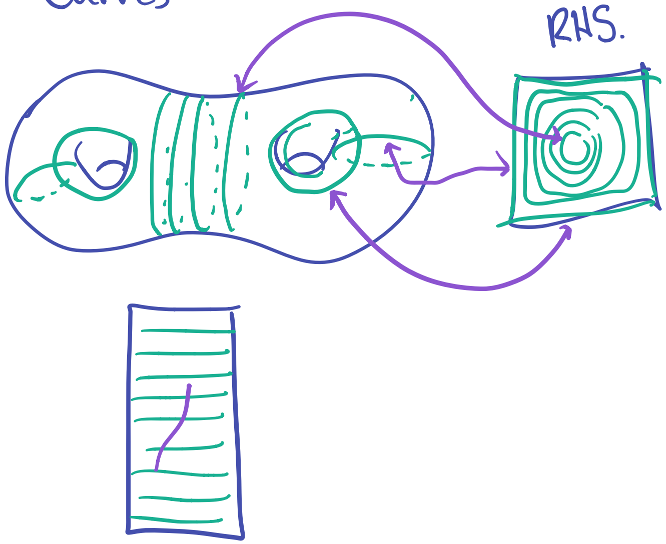


3 constructions

① Polygons

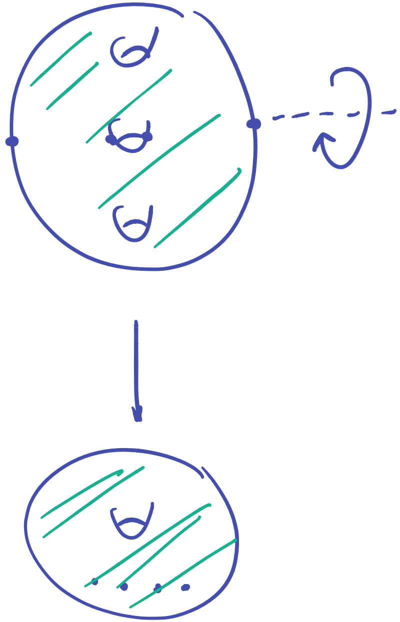


② Curves

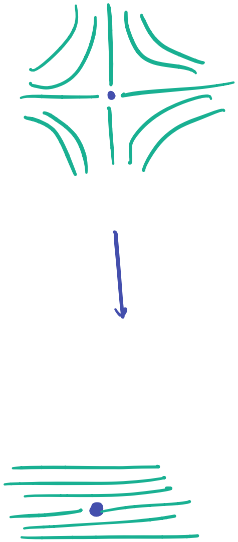


measure: Euclidean, $\perp$ to foliation

③ Branched covers



Lift a foliation from torus



Quadratic differentials

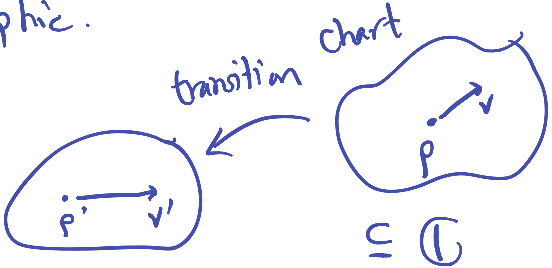
Single, complex analytic object that packages: complex str.
2 transv. foliations
with measures.

In a chart:

$$q = \varphi(z)\, dz^2$$

φ holomorphic.

So that...



Invariant under transition maps:

$z_\alpha : U_\alpha \to \mathbb{C}$ charts
$z_\beta : U_\beta \to \mathbb{C}$

$q = \varphi_\alpha(z)\, dz_\alpha^2$ or $\varphi_\beta(z)\, dz_\beta^2$ in charts

$$\varphi_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^2 = \varphi_\alpha(z_\alpha)$$

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q eats tangent vectors, gives complex number.

$$q(v) = \cancel{\varphi(v)^2}$$
$$= \varphi(p)\, v^2$$

From QD's to foliations

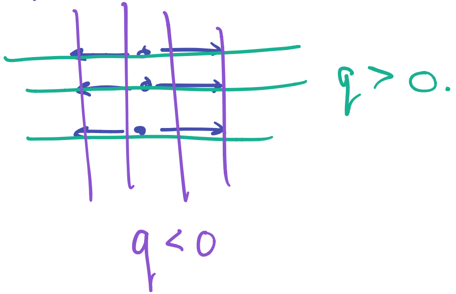
In a chart:

$$q = \varphi(z)\, dz^2$$

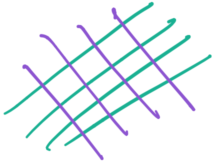
Horiz. foliation: $q > 0$

Vert. foliation: $q < 0$.

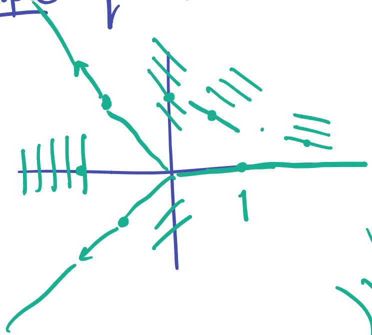
example $q = \varphi(z) dz^2 = 1 \cdot dz^2$



example $q = \alpha\, dz^2 \quad \alpha \in \mathbb{C}$

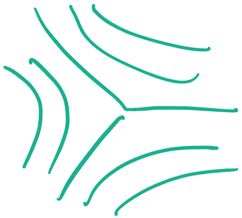


example $q = z\, dz^2$



$z^k dz^2$

$k+2$ prongs.



... and the measures

Every q has natural coords

where it is $z^k dz^2$

So: away from zeros,

measure is $|dx|$, $|dy|$

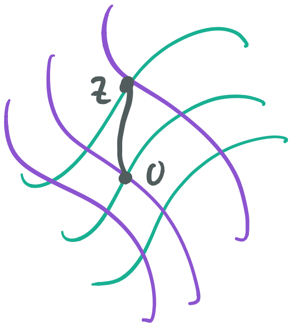
Say: $z : U \to \mathbb{C}$ chart

$$\eta(z) = \int_0^z \sqrt{\varphi(w)}\, dw$$

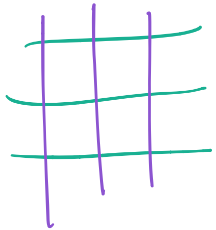
Choose a branch of $\sqrt{\ }$

dummy variable

Check: in these coords, away from zeros of q, $q = 1 \cdot dz^2$.

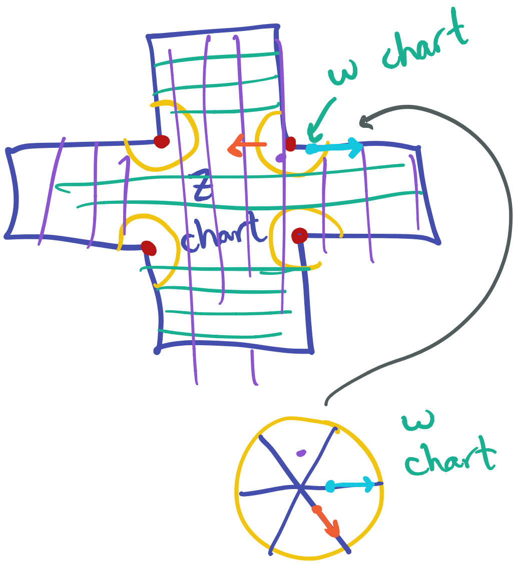


Z



$\eta \circ Z$

Example



In z-chart

$q = 1\, dz^2$ $\qquad \varphi_z(z) = 1$

In w-chart

$q = \varphi_w(z)\, dz^2 = 9z^4 dz^2$

Change of coords from w to z:

$z^3 + \text{const.}$

$$\varphi_z(z)\left(\frac{dz}{dw}\right)^2 = \varphi_w(z)$$

$$1 \cdot (3z^2)^2 = \varphi_w(z)$$

or

$q = \alpha\, dz^2$

$q = \alpha 9 z^4 dz^2$

$\rightsquigarrow$ foliations rotated by $\arg \alpha$.

Statement of Teich Thms

X, Y Riem surf's

A homeo $f: X \to Y$

is a Teich map if

$\exists$ qd's q_X initial differential

q_Y terminal.

& $K \in (0, \infty)$

s.t.

① $f(\text{zeros of } q_X)$

$= \text{zeros of } q_Y$

② At nonzero pts of q_X:

$$f(x+iy) = \sqrt{K}\, x + \frac{1}{\sqrt{K}}\, y$$

in natural coords

$\rightsquigarrow K_f = \max\{K, 1/K\}$

TET. X, Y Riem surfs

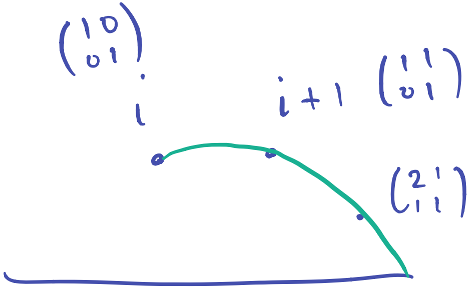
$f: X \to Y$ homeo

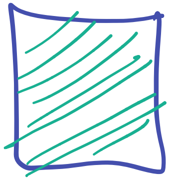
Then $\exists$ Teich map homotopic to f.

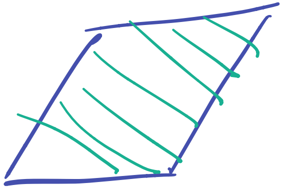
TUT. $h: X \to Y$ Teich map

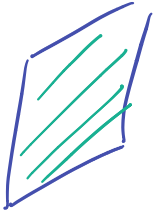
$f \sim h \Rightarrow K_f \geq K_h$

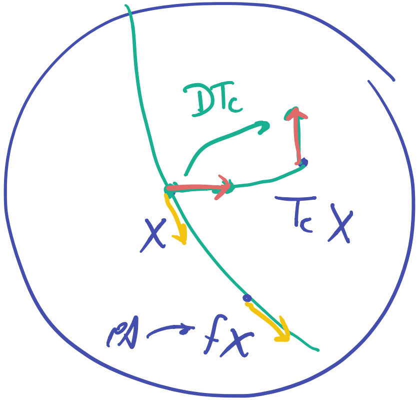
Equality $\Leftrightarrow f \circ h^{-1}$ conformal $\overset{g \geq 2}{\Longleftrightarrow} f = h$


$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
i
$i+1$
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$
$\begin{pmatrix}2&1\\1&1\end{pmatrix}$








DT_c
X
$T_c X$
fX