# Section 2.7

## Bases

# Bases

$V$ = subspace of $\mathbb{R}^n$

A basis for $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \ldots, v_k\}$
2. $v_1, \ldots, v_k$ are linearly independent

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

Q. What is one basis for $\mathbb{R}^2$? $\mathbb{R}^n$? How many bases are there?

# Dimension

$V =$ subspace of $\mathbb{R}^n$

$\dim(V) =$ dimension of $V = k =$the number of vectors in the basis

(What is the problem with this definition of dimension?)

## Basis example

Find a basis for the $xy$-plane in $\mathbb{R}^3$? Find all bases for the $xy$-plane in $R^3$. (Remember: a basis is a set of vectors in the subspace that span the subspace and are linearly independent.)

## Bases for $\mathbb{R}^n$

Let us consider the special case where $V$ is equal to all of $\mathbb{R}^n$.

What are all bases for $V = \mathbb{R}^n$? Or, if we have a set of vectors $\{v_1, \ldots, v_k\}$, how do we check if they form a basis for $\mathbb{R}^n$? First, we make them the columns of a matrix....

- For the vectors to be linearly independent we need a pivot in every column.
- For the vectors to span $\mathbb{R}^n$ we need a pivot in every row.

Conclusion: $k = n$ and the matrix has $n$ pivots.

## The standard basis for $\mathbb{R}^n$

We have the standard basis vectors for $\mathbb{R}^n$:

$$
\begin{aligned}
e_1 &= (1, 0, 0, \ldots, 0) \\
e_2 &= (0, 1, 0, \ldots, 0) \\
&\vdots
\end{aligned}
$$

# Who cares about bases?

A basis $\{v_1, \ldots, v_k\}$ for a subspace $V$ of $\mathbb{R}^n$ is useful because:

Every vector $v$ in $V$ can be written in exactly one way:

$$v = c_1 v_1 + \cdots + c_k v_k$$

So a basis gives coordinates for $V$, like latitude and longitude. See Section 2.8.

# Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

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# Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

## Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\mathrm{Nul}(A)$
- the pivot columns of $A$ form a basis for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

What should you do if you are asked to find a basis for Span$\{v_1, \ldots, v_k\}$?

# Bases for planes

Find a basis for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

# Basis theorem

## Basis Theorem

If $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$, then

- any $k$ linearly independent vectors of $V$ form a basis for $V$
- any $k$ vectors that span $V$ form a basis for $V$

In other words if a set has two of these three properties, it is a basis:

spans $V$, linearly independent, $k$ vectors

We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

## Section 2.7 Summary

- A basis for a subspace $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{Span}\{v_1, \ldots, v_k\}$
  2. $v_1, \ldots, v_k$ are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\mathrm{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)
- Basis Theorem. Suppose $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$. Then
  - Any $k$ linearly independent vectors in $V$ form a basis for $V$.
  - Any $k$ vectors in $V$ that span $V$ form a basis.

## Typical exam questions

- Find a basis for the $yz$-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in $R^7$?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of $A$ is the number of pivots of $A$.
- True/false: If $b$ lies in the column space of $A$, and the columns of $A$ are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span $R^3$ must be linearly independent.