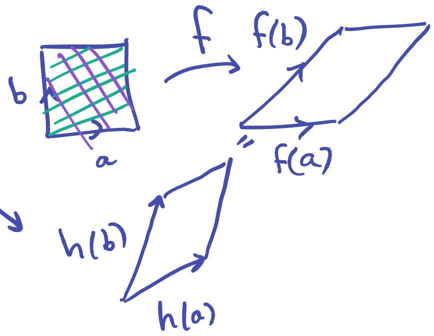
Teich Thms

TET. X, Y Riem surfs

$f : X \to Y$ homeo

$\exists$ Teich map $h \sim f$.

example.



TUT. $h : X \to Y$ Teich map

$f \sim h$

$\implies K_f \geq K_h$

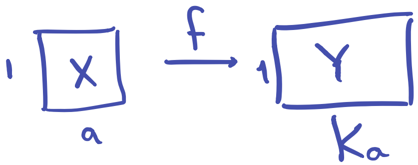
& equality $\iff f = h$ $(g \geq 2)$

Grötzch's Problem

The rectangle case of TUT

(The 1D version is MVT.
$K = |f'|$)

Thm. Given

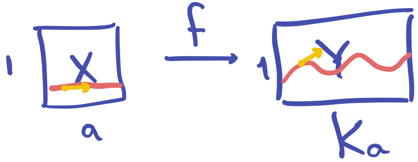


or. pres, side pres, almost smooth (smooth outside finite set)

Then $K_f \geq K$

& equality $\Longleftrightarrow$ f is the obvious map.

Thm. Given



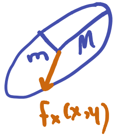
Then $K_f \geq K$

Pf. $K_f(x,y)$ = dil. at (x,y)

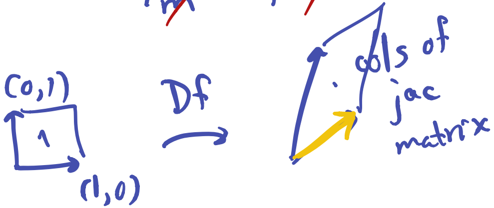
$j_f(x,y)$ = jacob. of f @ (x,y)

Claim 1. $|f_x(x,y)|^2 \leq K_f(x,y)\, j_f(x,y)$

Pf.



$M/\cancel{m} \cdot M\cancel{m}$



Uniqueness

For 1st $\leq$ to be $=$ need hor arcs $\mapsto$ hor arcs.

Symmetry: vertical arcs $\to$ vertical.

etc.

Claim 2. $\int_X |f_x(x,y)|\, dA \geq K\, \mathrm{Area}(X)$

Pf. Take $\underbrace{\int_0^a |f_x(x,y)|\, dx}_{\text{length}(f(\text{hor arc}))} \geq Ka$

y fixed

& integrate over y.

Now: $(K\,\mathrm{Area}(X))^2 \overset{②}{\leq} \left(\int_X |f_x(x,y)|\, dA\right)^2$

$\overset{①}{\leq} \left(\int_X \sqrt{K_f(x,y)}\sqrt{j_f(x,y)}\right)^2 dA$

$\overset{C\text{-}S}{\leq} \int_X K_f(x,y)\, dA \int_X j_f(x,y)\, dA$

$\leq K_f\, \mathrm{Area}(X)\, \mathrm{Area}(Y)$

$= K_f K\, \mathrm{Area}(X)^2$ $\square$

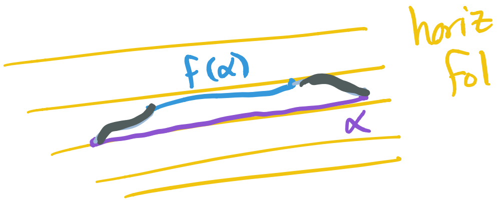
For TUT, need a version of Claim 2. But: leaves might not be closed...

Lemma. $q_Y \in QD(Y)$

$f : Y \longrightarrow Y$ $f \sim id$.

$\exists\, M$ s.t. $\forall$ horiz. arcs α geodesic.

$$\ell_{q_Y}(f(\alpha)) \geq \ell_{q_Y}(\alpha) - M$$



Pf. $M = 2 \cdot$ max distance a pt moves under homotopy f to id.

α geodesic

$$\Longrightarrow \ell(f(\alpha)) + M \geq \ell(\alpha) \quad \square$$

Next: Analog of Claim 2 using this Lemma.

Prop. $h: X \to Y$ Teich map

init diff q_X term diff q_Y

hor stretch K, $f \sim h$ almost smooth

Then $\int_X |f_x| \, dA \geq K \, \mathrm{Area}(q_X)$

Pf. Define $\delta : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\delta(p, L) = \int_{-L}^{L} |f_x| \, dx$$

$$= \ell_{q_Y} f(\alpha_{p,L})$$

$\alpha_{p,L}$ = hor. arc length $2L$ thru p.

Also: $\ell_{q_Y}(h(\alpha_{p,L})) = 2KL$

Lemma $\Longrightarrow \ell_{q_Y}(f(\alpha_{p,L})) \geq 2KL - M$

some M.

So:

$$\int_X \delta(p,L) \, dA = \int_X \ell_{q_Y}(f(\alpha_{p,L})) \, dA$$

$$\geq \int_X (2KL - M) \, dA$$

$$= (2KL - M) \, \mathrm{Area}(X)$$

Fubini: $\int_X \delta(p,L) \, dA = \int_X \left(\int_{-L}^{L} |f_x| \, dx \right) dA$

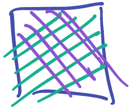
$$= 2L \int_X |f_x| \, dA$$

So: $\int_X |f_x| \, dA \geq \left(K - \frac{M}{2L}\right) \mathrm{Area}\, X$

$\forall L$. $\square$

Pf of TUT. Repeat Grötzch argument $\square$

Proof of TET



$X \in \mathrm{Teich}(S)$

$QD(X) = \mathbb{C}$-vector space

$\mathbb{R}$-dim $= 6g-6$ (Riemann-Roch)

Define $\|q\| = \int_X |\varphi| =$ area

$q = \varphi(z)\,dz^2$

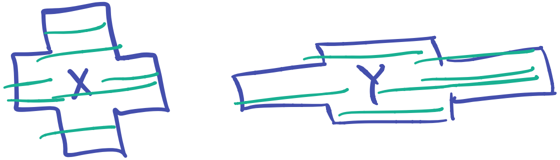
$QD_1(X) =$ open unit ball.

$\rightsquigarrow K = \dfrac{1+\|q\|}{1-\|q\|}$

$\rightsquigarrow Y \in \mathrm{Teich}(S)$

& Teich map $h : X \to Y$.

example.



$q = dz^2$

So $QD(X) \longleftrightarrow T_X\mathrm{Teich}(S)$

line in Teich(S) "exponential map"

$\rightsquigarrow \Omega : QD_1(X) \longrightarrow \mathrm{Teich}(S)$.

TET $\Longleftrightarrow$ Ω surjective.

Prop. Ω continuous

hard part!

Prop. Ω proper.

Also: Ω inj by TUT

& $\dim QD_1 = 6g-6$

Brouwer's Inv. of Domain:

Any proper, inj contin. map

$\mathbb{R}^n \to \mathbb{R}^n$

is a homeo.

Continuity uses Beltrami differentials

PDEs.

Teichmüller metric

$$d_{Teich}(X,Y) = \frac{1}{2}\log K$$

where K is dilatation of Teich map $h : X \to Y$.

Prop. d_{Teich} is a complete metric.

Prop. Teich lines above are geodesics in d_{Teich}. (TUT)

Prop. Teich(S) is a geodesic metric space (TET + prev. prop)

Prop. d_{Teich} for T^2 is hyp metric on $\mathbb{H}^2$ (up to multiple).