Torelli Groups

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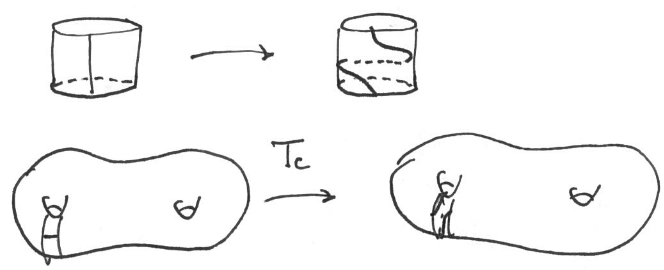
Background

Mapping class group:

S = surface.

$\mathrm{Mod}(S) = \pi_0 \, \mathrm{Homeo}^+(S, \partial S)$

Dehn twists:



<u>Thm</u> (Dehn 20's) For $g \geq 0$ $\mathrm{Mod}(S_g)$ is finitely generated by Dehn twists.

Symplectic rep $\quad \psi : \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z})$ $\quad$ action on $H_1(S_g; \mathbb{Z})$

<u>Thm</u>. For $g \geq 0$ ψ is surjective.

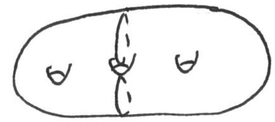
Torelli group: $\mathcal{I}(S_g) = \ker \psi$

e.g. $T_c \in \mathcal{I}(S_g)$ for any sep curve c.

Why study Torelli?
1. It is the non-linear part of $\mathrm{Mod}(S_g)$
2. It is π_1 (Torelli space)
   space of Riem. surf's w/ H_1-basis.
3. Every $\mathbb{Z}HS^3$ obtained from S^3 by cutting along S_g, regluing by $\mathcal{I}(S_g)$

I. Generation

<u>Thm</u> (Johnson '83). For $g \geq 3$ $\mathcal{I}(S_g)$ is finitely gen by bounding pair maps



$T_c T_d^{-1}$

Putman '12: # gens cubic in g.

Thm (Mess '86) $\mathcal{I}(S_2) \cong F_\infty$.

II. Johnson Homomorphism

$$\tau : \mathcal{I}(S_g^1) \longrightarrow \Lambda^3 H \qquad H = H_1(S_g^1; \mathbb{Z}).$$

three defns: algebra, alg. top., 3-manifolds
application: $\mathcal{I}(S_g)$ is distorted in $\mathrm{Mod}(S_g)$ (Broaddus-Farb-Putman)

$K(S_g) = \ker \tau$ "Johnson kernel"

<u>Thm</u> (Johnson '83) $K(S_g) = \langle T_c : c \text{ sep.} \rangle$

<u>Thm</u> (Ershov-He '17) $K(S_g)$ is finitely gen.

Johnson filtration: $\underset{\mathcal{I}(S_g)}{\overset{\shortparallel}{N_0(S_g)}} \trianglerighteq \underset{K(S_g)}{\overset{\shortparallel}{N_1(S_g)}} \trianglerighteq \cdots$

III. The abelianization

Birman-Craggs-Johnson homomorphisms

$$\mathcal{I}(S_g) \longrightarrow \mathbb{Z}/2$$

defined using Rochin invariant for 3-manifolds.

There are $\sum_{k=0}^{3} \binom{2g}{k}$ of these.

__Thm__ (Johnson '83) The abelianization of $\mathcal{I}(S_g)$ is given by

$$\tau \oplus \text{BCJs}$$

Also: Thm (Pitsch '08) Every $\mathbb{Z}HS^3$ obtained via $N_3(S_g)$.

IV. Higher finiteness properties

__Thm__ (Johnson-Millson-Mess '83) $H_3(\mathcal{I}(S_3);\mathbb{Z})$ is ∞-gen.

__Thm__ (Bestvina-Bux-M '08) $H_{3g-5}(\mathcal{I}(S_g);\mathbb{Z})$ is ∞-gen

Big Q. Is $H_2(\mathcal{I}(S_g);\mathbb{Z})$ finitely gen? Other H_k?

V. Representation stability

Johnson: parametrized Abel-Jacobi maps

$$\tau_i : H_i(\mathcal{I}_g^1; \mathbb{Q}) \longrightarrow \Lambda^{i+2} H \qquad 0 \le i \le 2g-2$$

<u>Thm</u> (Church-Farb '11)
- τ_i not injective $i>1$
- τ_2 surjective
- τ_i nonzero $1 \le i \le g$ $(\Rightarrow H_i \ne 0)$

$\rightsquigarrow$ invented/conjectured rep. stability

<u>Thm</u> (Boldsen-Dollerup '17) $H_2(\mathcal{I}(S_g); \mathbb{Z})$ finitely generated as an Sp-module.

<u>Also</u> Thm (Church-Putman '15) Fix k. Each $N_k(S_g)$ is generated by elements of small support, indep. of g.

GENERATING MCG

Alexander Trick Prop. $\mathrm{Mod}(D^2) = 1$.

Pf. For $\varphi \in \mathrm{Homeo}^+(D^2, \partial D^2)$ consider

$$\Phi(x,t) = \begin{cases} (1-t)\,\varphi\left(\frac{x}{1-t}\right) & 0 \le |x| \le 1-t \\ x & \text{o.wise} \end{cases}$$

$g=0$ Lemma $\mathrm{Mod}(\mathbb{R}^2) = 1$

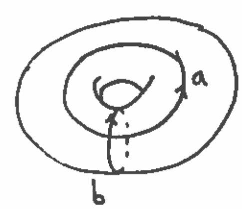
Pf. Straight line homotopy

$\Rightarrow$ Prop $\mathrm{Mod}(S^2) = 1$.

$g=1$ Thm. The map $\psi: \mathrm{Mod}(T^2) \to SL_2\mathbb{Z}$ (action on H_1) is an $\cong$

Pf. Injectivity: $K(G,1)$ theory (note $H_1 \cong \pi_1$)

Surjectivity: $T_a \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $T_b \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$



In particular: $\mathrm{Mod}(T^2)$ is gen. by Dehn twists.

$g>1$ Much more complicated!

Two ingredients: Complex of curves

Birman exact sequence.

Complex of Curves

$C(S_g)$ vertices: homotopy classes of scc in S_g
edges: disjoint reps

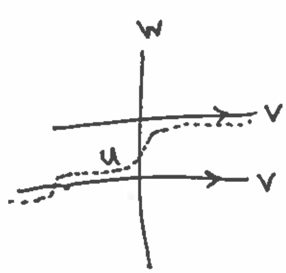
<u>Thm</u> (Lickorish '64) For $g \geq 2$ $C(S_g)$ is connected.

<u>Pf.</u> v, w vertices
To show v, w lie in same component.
Induct on $i(v,w)$

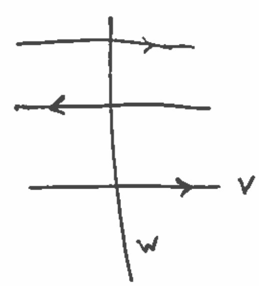
Base cases: $i(v,w) \leq 2$. follow from

<u>Lemma.</u> v, w fill $S_g \Rightarrow i(v,w) \geq 2g-1$
<u>Pf.</u> $v, w \rightsquigarrow$ cell decomp of S_g
$$2-2g = i(v,w) - 2i(v,w) + F \geq -i(v,w) + 1.$$

Now assume $i(v,w) \geq 3$. Must see:



<u>or</u>



In first case $i(u,v) = 1$ $i(u,w) < i(v,w)$ ▨

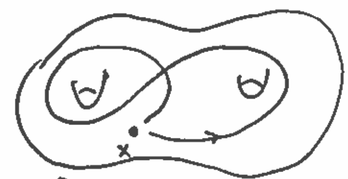
<u>Cor.</u> Complex of nonsep curves $N(S_g)$ is conn. $g \geq 2$

<u>Cor.</u> $\hat{N}(S_g)$ is conn. $g \geq 1$ vertices: nonsep curves
edges: $i = 1$.

Birman Exact Sequence

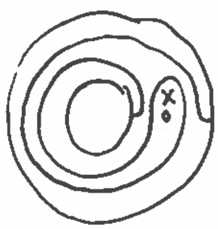
$\mathrm{Mod}(S,x) = \pi_0 \mathrm{Homeo}^+(S,x)$.

<u>Push map</u> $\quad \mathrm{Push} : \pi_1(S,x) \longrightarrow \mathrm{Mod}(S,x)$



example: α = simple loop

$\mathrm{Push}(\alpha) = T_c T_d^{-1} \quad c,d$ = left, right pushoffs



<u>Forgetful map</u> $\quad \mathrm{Forget} : \mathrm{Mod}(S,x) \longrightarrow \mathrm{Mod}(S)$

note $\mathrm{Im}\,\mathrm{Push} \subseteq \ker \mathrm{Forget}$.

<u>Thm</u>. For $\chi(S) < 0$: $\quad 1 \longrightarrow \pi_1(S,x) \longrightarrow \mathrm{Mod}(S,x) \longrightarrow \mathrm{Mod}(S) \longrightarrow 1$

is exact.

(For $\chi(S) \geq 0$, lose injectivity.)

<u>Cor</u>. $\mathrm{Mod}(S_{g,n})$ is fin. gen. by Dehn twists $g = 0,1$.

<u>Pf of Thm</u>. Long exact seq. for fiber bundle

$$\begin{array}{ccc} \mathrm{Homeo}(S,x) & \longrightarrow & \mathrm{Homeo}(S) \\ & & \downarrow \\ & & S \end{array}$$

plus: $\pi_{0,1} \mathrm{Homeo}(S) = 1$.

Groups Acting on Connected Complexes

Lemma. $G \circlearrowright X$ = connected graph transitive on vertices and ordered pairs of vertices

Say $v \bullet\!\!-\!\!\bullet\, w$ & $h \cdot w = v$

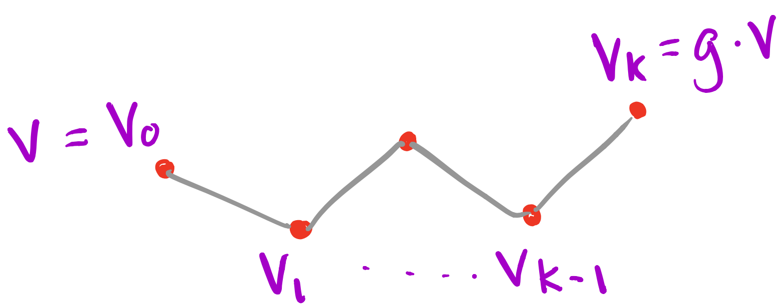
Then: $G = \langle \mathrm{Stab}_G(v), h \rangle$

Pf. Let $H = \langle \mathrm{Stab}_G(v), h \rangle$

$g \in G$.

Want to show $g \in H$.

Consider a path



Choose g_i s.t. $v_i = g_i \cdot v$

& $g_0 = id$, $g_k = g$.

Inductive hyp: $g_i \in H$.

Base case automatic. Assume $g_i \in H$.

Consider
$$\begin{tikzcd} v_i = g_i \cdot v \arrow[r, dash] & v_{i+1} = g_{i+1} \cdot v \end{tikzcd}$$

Apply g_i^{-1}:
$$\begin{tikzcd} v \arrow[r, dash] & g_i^{-1} g_{i+1} \cdot v \end{tikzcd}$$

Since G acts trans. on pairs of vertices...

Apply some r:
$$\begin{tikzcd} v \arrow[r, dash] & w = r g_i^{-1} g_{i+1} \cdot v \end{tikzcd}$$

Note: $r \in \mathrm{Stab}_G(v)$

$\rightsquigarrow \quad h r g_i^{-1} g_{i+1} \in \mathrm{Stab}_G(v) \subseteq H$

But $h, r, g_i^{-1} \in H \implies g_{i+1} \in H$. $\square$

PROOF OF FINITE GENERATION

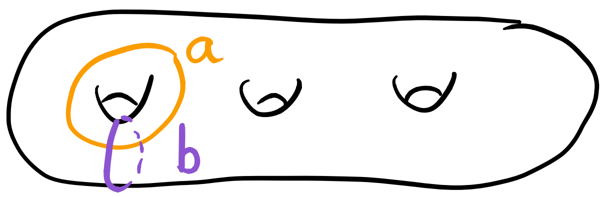
Theorem. For $g \geq 0$ $\mathrm{Mod}(S_g)$ is finitely gen. by Dehn twists.

Proof. Induct on g. Base cases $g=0,1$ ✓

Let $g \geq 2$.

$\mathrm{Mod}(S_g) \circlearrowright \hat{N}(S_g)$ satisfying Lemma.

Let



Check: $T_a T_b T_a(b) = a$

Lemma $\Rightarrow$ $\mathrm{Mod}(S_g) = \langle \mathrm{Stab}(a), T_a, T_b \rangle$

To show $\mathrm{Stab}(a)$ fin. gen. by Dehn twists.

$\mathrm{Stab}(a)/\langle T_a \rangle \cong \mathrm{Mod}(S_{g-1,2})$

cut along a.

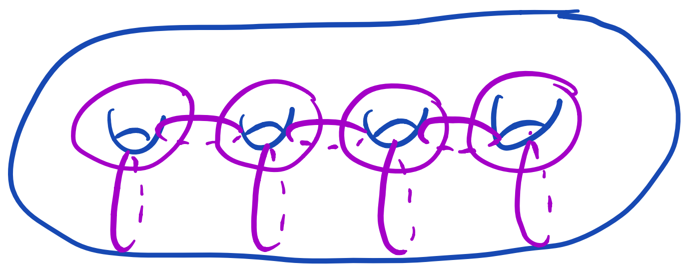
By induction $\mathrm{Mod}(S_{g-1})$ fin. gen. by Dehn twists.

$\Longrightarrow \mathrm{Mod}(S_{g-1,1})$ fin. gen. by Dehn twists

(Birman exact seq + usual gen. set for $\pi_1(S_g)$)

$\Longrightarrow \mathrm{Mod}(S_{g-1,2})$ fin. gen by DTs $\square$

Same proof:
- Finite gen. by DTs about nonsep curves (since π_1 gen by nonsep simple loops)
- Lickorish generators



Just check that each step works!

THE SYMPLECTIC REPRESENTATION OF MCG

The symplectic group

Consider $\mathbb{R}^{2g}$ with basis $(x_1, \ldots, x_g, y_g, \ldots, y_1)$ and standard symplectic form

$$\omega = \sum_{i=1}^{g} dx_i \wedge dy_i$$

Think of ω as a pairing on $\mathbb{R}^{2g}$ e.g.

$$\omega(x_1 + 2y_2,\ x_1 + y_1 + x_2) = 1 - 2 = -1$$

This is the unique nondegenerate, alternating bilinear form on $\mathbb{R}^{2g}$ up to change of basis.

Connection to surfaces:

$$(\mathbb{R}^{2g}, \omega) \cong (H_1(S_g;\mathbb{R}), \hat{i})$$

$Sp_{2g}(\mathbb{R})$ = subgp of $GL_{2g}\mathbb{R}$ preserving ω:

$$\omega(u,v) = \omega(Mu, Mv)$$

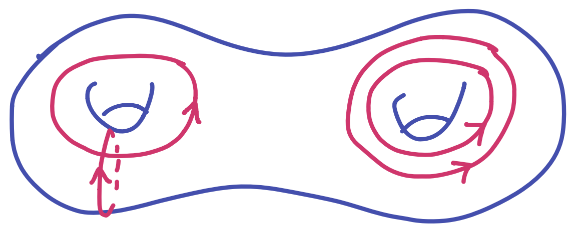
Similar with $\mathbb{Z}$.

Realizing H_1-classes by curves.

<u>Prop.</u> If $v \in H_1(S_g;\mathbb{Z})$ is primitive then $v = [c]$ where c is an oriented simple closed curve.

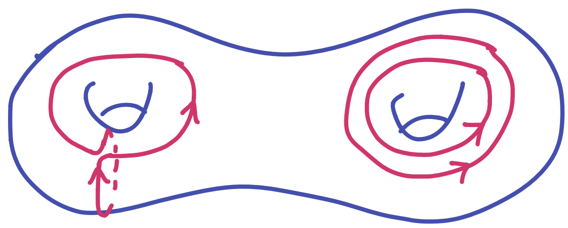
<u>Pf</u> (Meeks-Patrusky). Euclidean algorithm for scc's.

Step 1. Draw v naively:

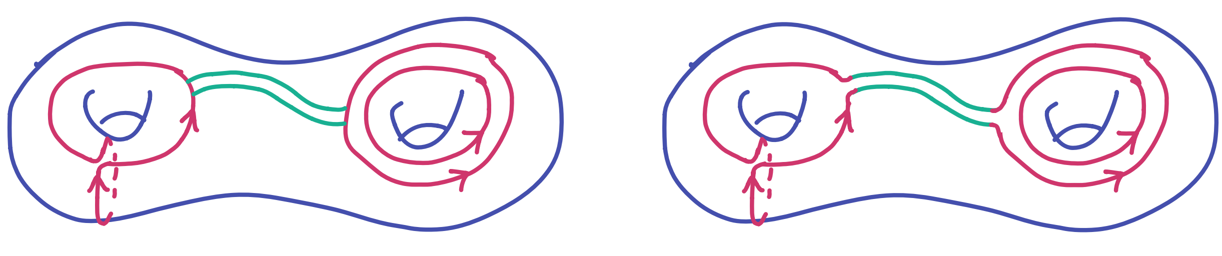


$$v = x_1 + y_1 + 2x_2$$

Step 2. Surger to remove crossings.



Step 3. Band surgeries to reduce the number of components



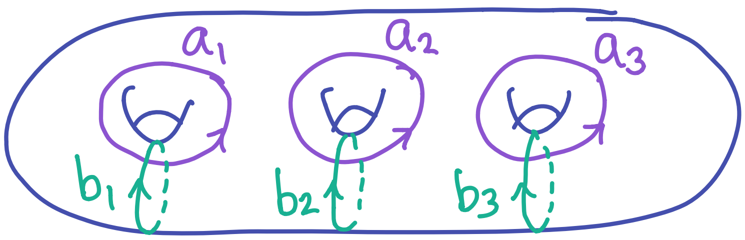
By Euclidean algorithm, this terminates in a connected curve! $\square$

Action of a Dehn Twist

Prop. Say a, b = oriented curves
Then $T_b^k([a]) = [a] + k\,\hat{i}(a,b)[b]$

N.B. Indep of orientation of b!.

A geometric symplectic basis:



Proof. Case 1. b separating

Choose geometric symplectic basis for $H_1(S_g)$ disjoint from b.

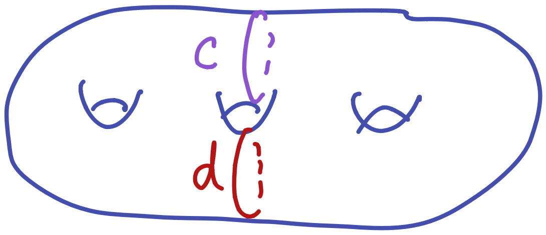
Case 2. b nonseparating.

Choose a geometric symplectic basis so b is one curve. Check for a = basis elt. Apply linearity of $\psi(T_b^k)$.

Some Elements of Torelli.

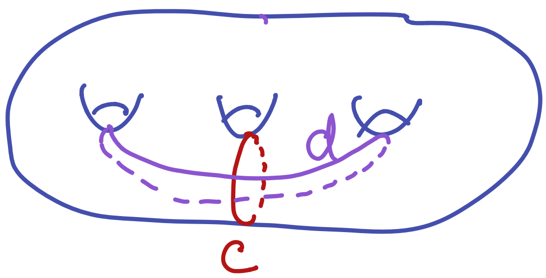
1. Dehn twists about sep curves

2. Bounding pair maps



$T_c T_d^{-1}$

3. Commutators of simply intersecting pairs



$[T_c, T_d]$

2 & 3 are special cases of:

$T_c T_d^{-1}$ where $[c] = [d]$

That all of these lie in Torelli follows immediately from the Prop.

4. $\pi_1(S_g) \trianglelefteq \mathrm{Mod}(S_{g,1})$

SURJECTIVITY OF THE Sp-REP

Thm $\Psi : \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z})$ is surjective.

$\boxed{\text{1st proof: transvections.}}$

A transvection in $Sp_{2g}(\mathbb{Z})$ is an elt whose 1-eigenspace is $(2g-1)$-dim.

$$\Updownarrow$$

$$\tau_v(u) = u + \omega(u,v)v \quad \text{(or a power)}$$

Fact. $Sp_{2g}(\mathbb{Z})$ is gen. by transvections.

Pf of Thm. Suffices to hit τ_v, v primitive.
Prop $\rightsquigarrow$ a s.t. $[a] = v$
$\Psi(T_a) = \tau_v$ $\square$

Want a proof that does not presuppose a genset for Sp.

2^{nd} proof: geometric symplectic bases

$Sp_{2g}(\mathbb{Z}) \longleftrightarrow$ symplectic bases for $\mathbb{Z}^{2g}$

$I \longleftrightarrow$ standard symplectic basis.

<u>Proof of Thm.</u> Given $A \in Sp_{2g}(\mathbb{Z})$, realize A as a <u>geometric</u> symplectic basis (supe up the proof of Prop above).

Realize I by standard geometric symplectic basis.

Apply <u>change of coordinates</u>: given two topologically equivalent configurations of curves, there is an element of $\mathrm{Mod}(S_g)$ taking one to the other. $\square$

Corollaries of the proof:

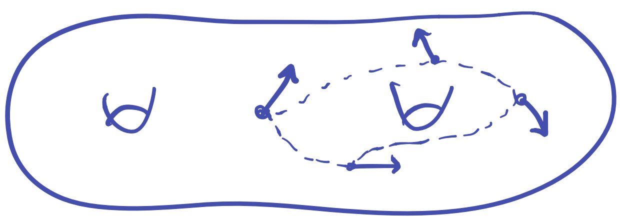
① $\mathcal{I}(S_g)$ acts transitively on s.c.c. reps of a homology class.

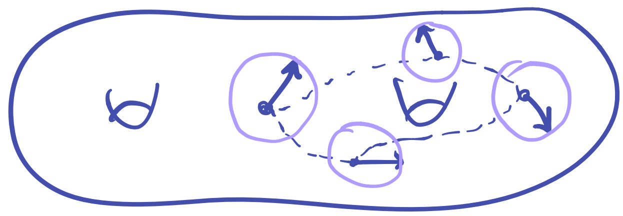
② $\mathcal{I}(S_g)$ acts trans. on sep curves/BPs inducing same splitting.

$\Longrightarrow$ conjugacy classification of such elts.

Unit Tangent Bundle Subgroups

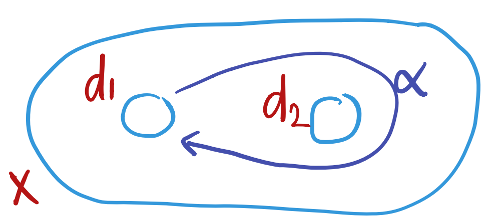
A loop in $\pi_1 UT(S_g) \rightsquigarrow$ isotopy of a disk in S_g





$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[d] & & \\
1 \arrow[r] \arrow[d] & \pi_1(UT(S_g)) \arrow[r] \arrow[d] & \mathrm{Mod}(S_g^1) \arrow[r] \arrow[d] & \mathrm{Mod}(S_g) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(S_g) \arrow[r] \arrow[d] & \mathrm{Mod}(S_{g,1}) \arrow[r] \arrow[d] & \mathrm{Mod}(S_g) \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}$$

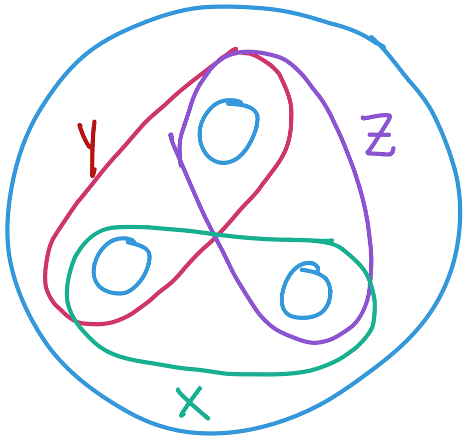
Example

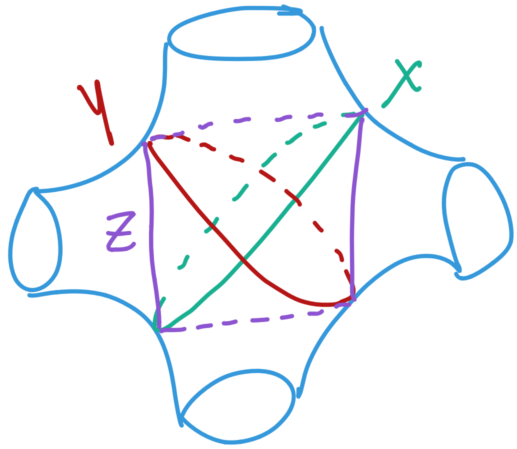


$\mathrm{Push}(\alpha) = T_x T_{d_1}^{-1} T_{d_2}^{-1}$

"do si do"

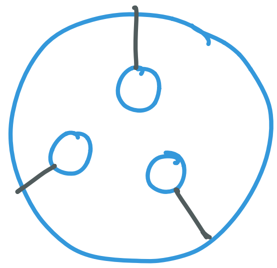
The Lantern Relation.





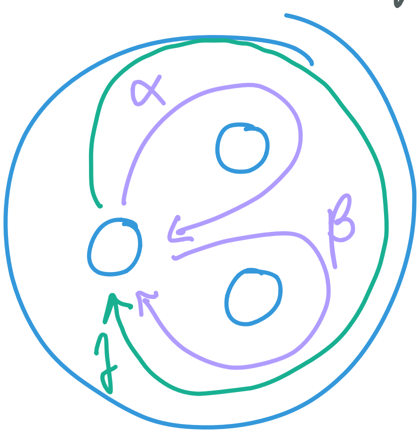
$$T_x T_y T_z = \prod_{i=1}^{4} T_{\partial_i}$$

Pf #1. Check action on



By symmetry, enough to check one (why?). Apply Alexander trick.

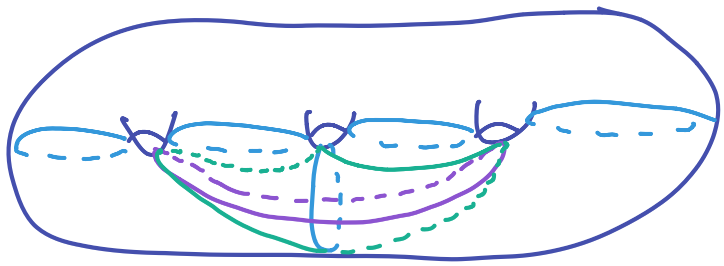
Pf #2. Consider push maps:



Note $\alpha\beta = \gamma$. Reinterpret as lantern relation.

Consequences of Lantern Relation.

1. Theorem $\mathrm{Mod}(S_g)^{ab} = 1 \qquad g \geq 3$



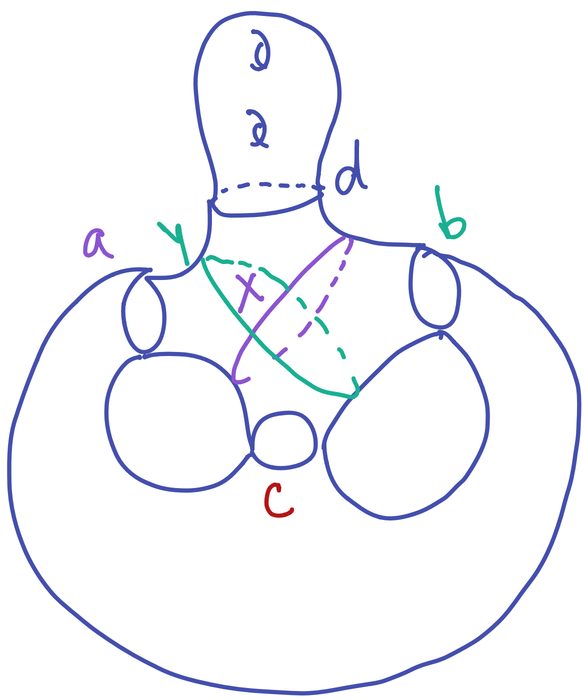
Pf. All 7 twists are conjugate

$\Rightarrow$ have same image t in $\mathrm{Mod}(S_g)^{ab}$

Lantern $\Rightarrow t^3 = t^4 \Rightarrow t = 1$.

2. Prop Dehn twists about sep curves are products of BP maps.

Pf.



Lantern relation $\Rightarrow$

$$T_d = (T_x T_a^{-1})(T_y T_b^{-1})(T_z T_c^{-1})$$

Torelli Groups Are Torsion Free

$\boxed{\text{Black Box}}$ If $f \in \mathrm{Mod}(S)$ has finite order, then $f = [\varphi]$ st φ = isometry

Theorem $\mathcal{I}(S_g)$ is torsion free

Pf. #1 Say $f \in \mathcal{I}(S_g)$ has finite order
Black box $\rightsquigarrow$ φ = isometry. $\Rightarrow$ iso. fix pts.
Lefschetz FPT:

$$\sum_{p=\text{fix pt}} \mathrm{ind}_p(\varphi) = \sum (-1)^i \,\mathrm{trace}\left(\varphi_* : H_i(S_g)^{\circlearrowleft}\right)$$

$$\#\text{ fixed pts} = 1 - 2g + 1 < 0 \qquad \square$$

Pf #2 Fact: $G \curvearrowright X \Rightarrow$

$$H_1(X/G;\mathbb{Q}) \cong H_1(X;\mathbb{Q})^G$$

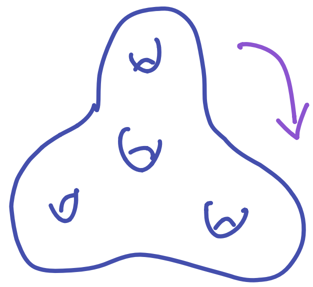
Assume now $g \geq 2$. As above $f \rightsquigarrow \varphi$
Take $G = \langle \varphi \rangle$. Have:

$$\mathrm{genus}(S_g/G) < g$$

$$\Rightarrow H_1(S_g/G) \not\cong H_1(S_g)$$

$$\Rightarrow G \not\subseteq \mathcal{I}(S_g). \qquad \square$$

Pf #3 Fact: f is either
- deck transf.
- rotation of a polygon

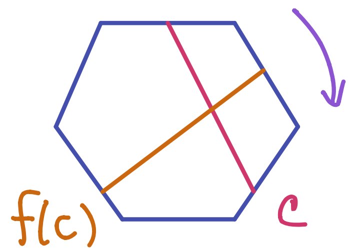


Deck transformations

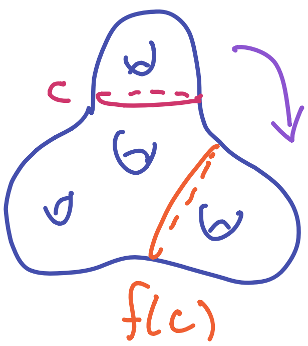
clearly not in $\mathcal{I}(S_g)$

Rotations of polygons

Choose a diag. that is a s.c.c. c in S_g



Note $i(c, f(c)) \leq 1$.



<u>Case 1</u>. c sep $\Rightarrow i(c, f(c)) = 0 \Rightarrow$ f not in Torelli ✓

<u>Case 2</u>. c nonsep, $i(c, f(c)) = 1$ ✓

<u>Case 3</u>. $|f| \geq 3 \Rightarrow c, f(c), f^2(c)$ cannot all be homologous (consider the labels they cut off) ✓

<u>Case 4</u>. $|f| = 2$, all diags are nonsep, all taken to homologous curves $\Rightarrow f$ = hyperelliptic involution ✓

Generating Torelli

Goal: $\mathcal{I}(S_g)$ is gen. by BP maps (and Dehn twists about sep curves)

Original proof: 1971 Birman gives presentation for $Sp_{2g}(\mathbb{Z})$
1978 Powell interprets relations
1980 Johnson, lantern relation

Want a proof analogous to $Mod(S_g)$ case.

Complex of homologous curves

Fix (primitive) $x \in H_1(S_g; \mathbb{Z})$

$\mathcal{C}_x(S_g)$ = subgraph of $\mathcal{C}(S_g)$ spanned by (unoriented) reps of x.

goal: Connected.

"borrowing complex"

Will use auxilliary complex $\mathcal{B}_x(S_g)$, the complex of cycles. Points of $\mathcal{B}_x(S_g)$ are simple, irredundant reps of x.

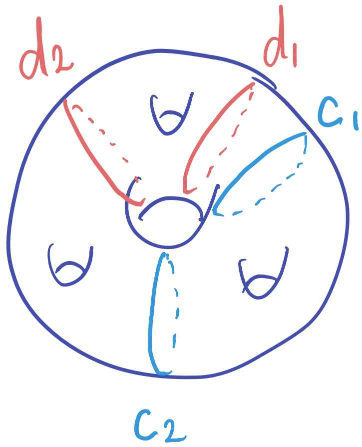
The Complex of Cycles

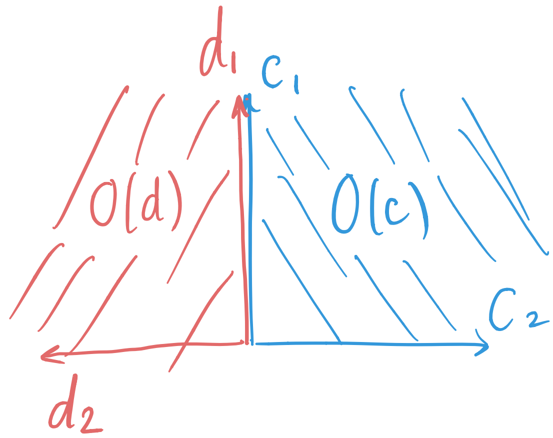
c = oriented multicurve, n components

$\rightsquigarrow \quad [0,\infty)^n \longrightarrow H_1(S_g;\mathbb{Z})$ orthant $O(c)$

$$A(S_g) = \coprod_c O(c) \;/\sim$$

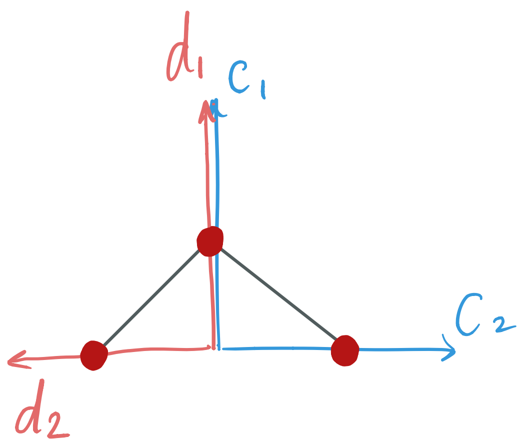
example.



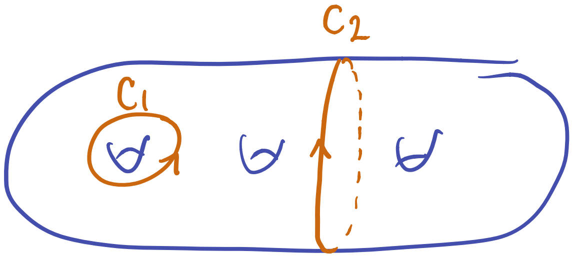


$$A_x(S_g) \subseteq A(S_g) \quad \text{reps of } x.$$

Say $x = [c_1]$

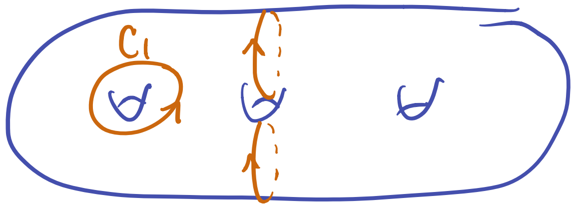


The cells of $A_x(S_g)$ are not necessarily compact:



If $[c_1] = x$ then $[c_1 + bc_2] = x \quad \forall\, b \in \mathbb{R}$

Or:



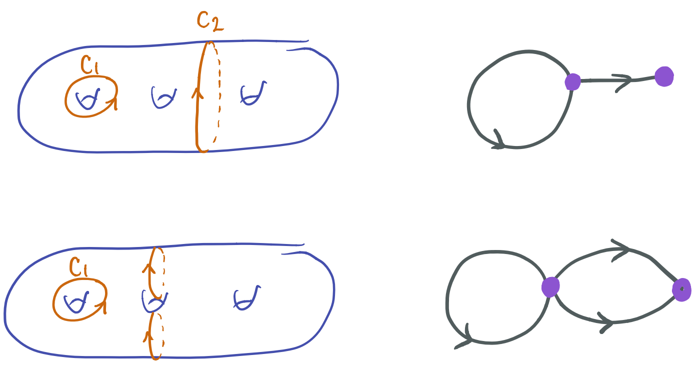
An oriented multicurve is <u>reduced</u> if

(1) the corresponding cell is compact

$\Longleftrightarrow$ (2) it has no homologically trivial subset

$\Longleftrightarrow$ (3) the dual directed graph is recurrent

Dual graphs:

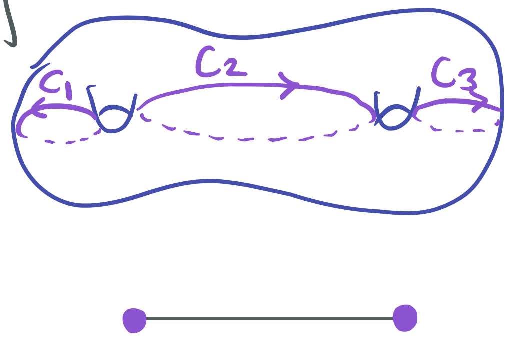


The complex of cycles $B_x(S_g)$ is the subcomplex of $A_x(S_g)$ whose cells correspond to reduced oriented multicurves.

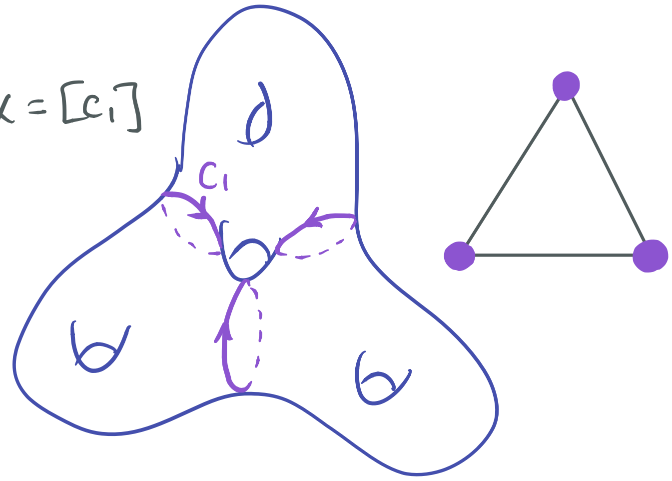
We'll show $B_x(S_g)$ is contractible.

Examples of cells

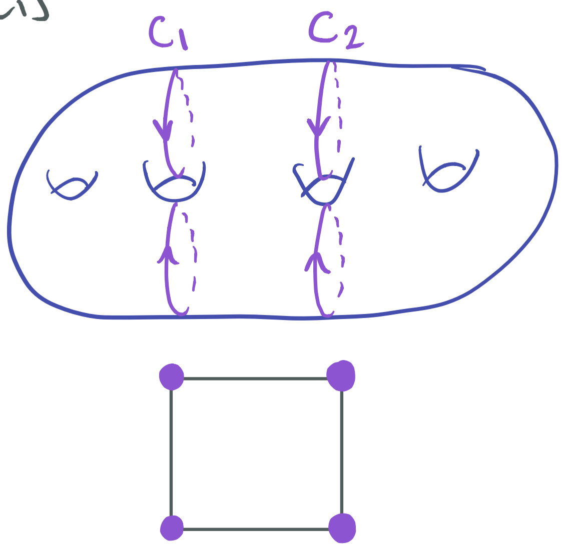
$X = [c_1]$

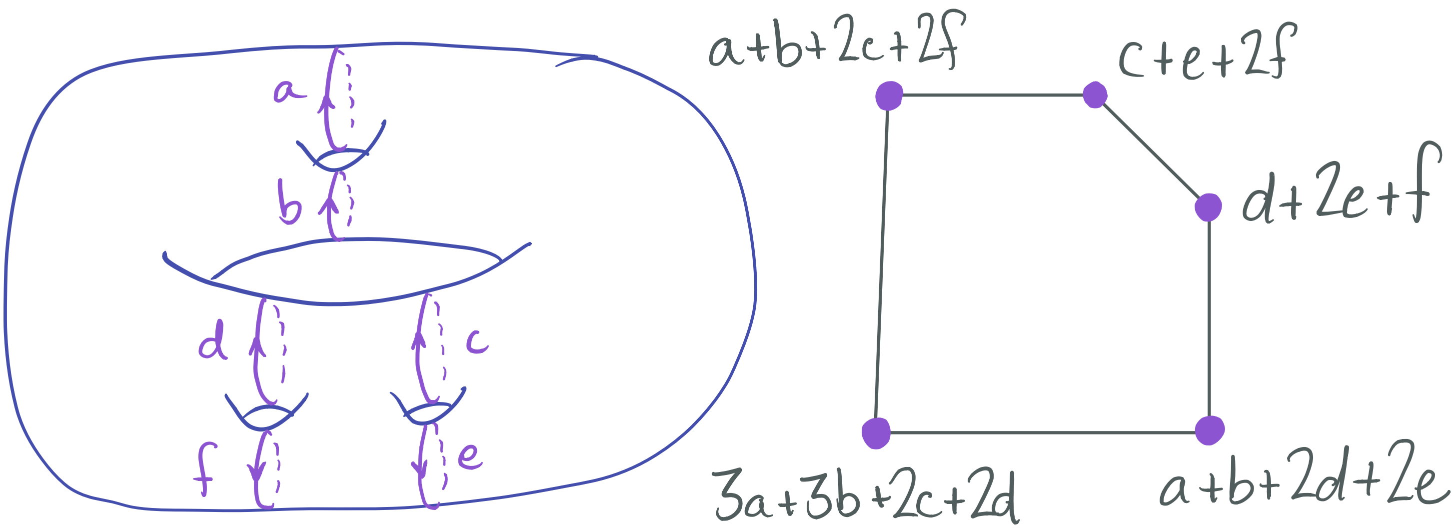


$X = [c_1]$



$X = [c_1] + [c_2]$





Q. Which polytopes arise?

Properties of Cells

Prop. The dim. of a cell = # compl. comp.'s -1.

Pf. Defn of homology.

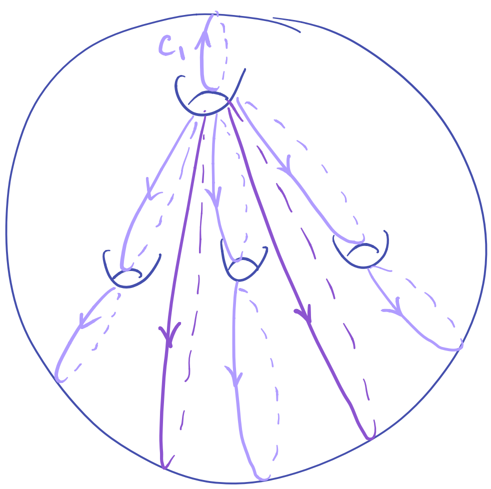
$\Longrightarrow$ vertices $\longleftrightarrow$ nonsep. multicurves.

Prop. Vertices of $B_x(S_g)$ are oriented multicurves with integral weights.

Pf. Given a vertex, consider a loop intersecting in one point.

Prop. $\operatorname{Dim} B_x(S_g) = 2g-3$.

Pf.



$x = [c_1]$.

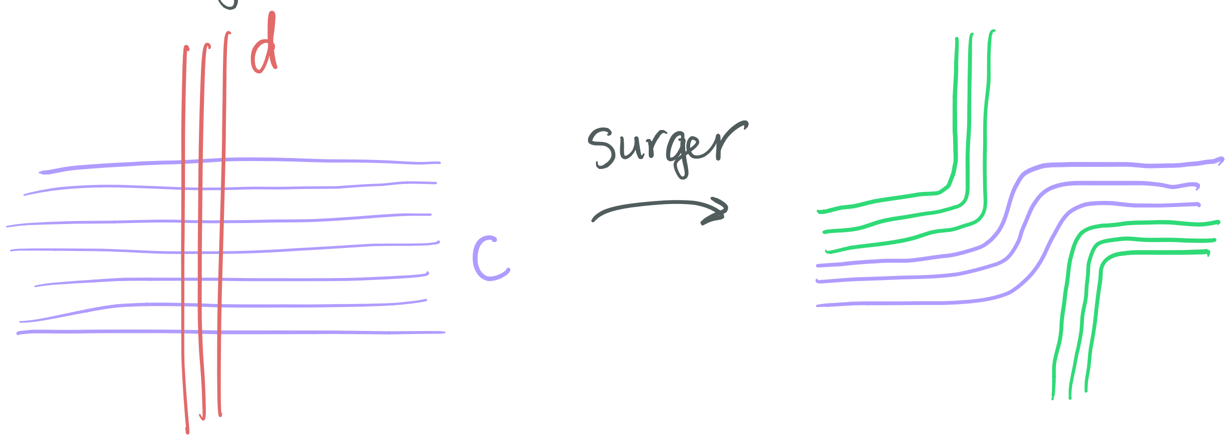
$\rightsquigarrow B_x(S_2)$ is a <u>graph</u>.

CONTRACTIBILITY

Theorem. $B_x(S_g)$ is contractible.

Surgery on 1-cycles

Say $c,d \in A_x(S_g)$. Thicken c,d according to weights and then:



If $[c]=[d]=x$, this procedure will result in a 1-cycle rep'ing x. Why?

$$H_1(S_g;\mathbb{Z}) \cong H^1(S_g;\mathbb{Z}) \cong \mathrm{Hom}(H_1(S_g;\mathbb{Z}),\mathbb{Z}) \longleftrightarrow [S_g, S^1]$$

The original c,d give maps $S_g \to S^1$ by integrating against width of annuli. The surgered picture corresponds to the map $S_g \to S^1$ obtained by integrating against both widths.

Prop. $A_x(S_g)$ is contractible

Pf. Fix some $c \in A_x(S_g)$. Consider:

$$F_t(d) = \text{Surger}\left(tc + (1-t)d\right) \qquad \square$$

Draining 1-cycles

Suppose $c \in A_x(S_g)$ is not reduced.
$\rightsquigarrow \{R_i\}$ subsurfaces with $\partial R_i \subseteq c$
$\text{Drain}_t(c) = c - t \sum \partial R_i$

Prop. $A_x(S_g)$ def. retracts to $B_x(S_g)$.
In partic. $B_x(S_g)$ is contractible.

Pf. Drain $\square$

In particular, $B_x(S_2)$ is a tree.

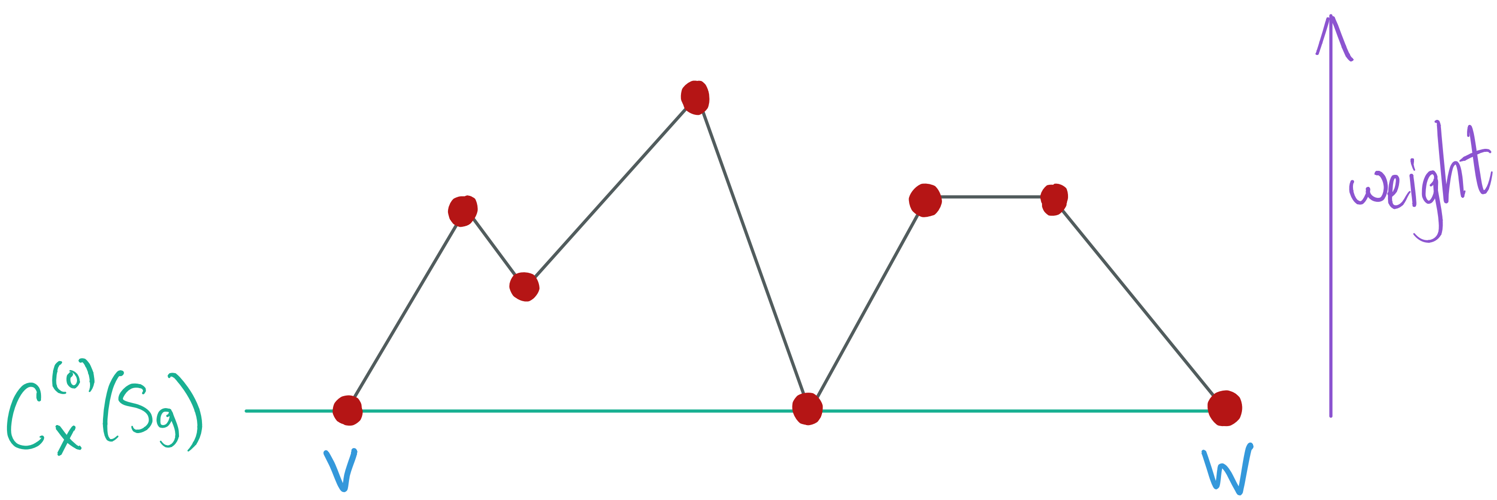
Connectivity of $C_x(S_g)$

Basic strategy

Define weight : $B_x(S_g) \longrightarrow \mathbb{Z}$
$\sum w_i c_i \longmapsto \sum w_i$

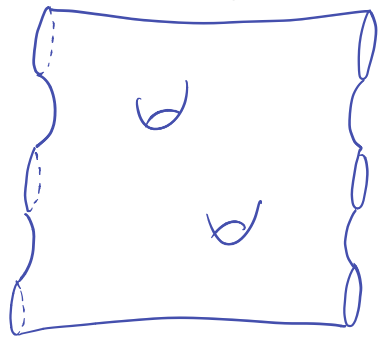
Note: $C_x(S_g) = \text{weight}^{-1}(1)$.

Now, given $v, w \in C_x^{(0)}(S_g)$, we connect them in $B_x^{(1)}(S_g)$:



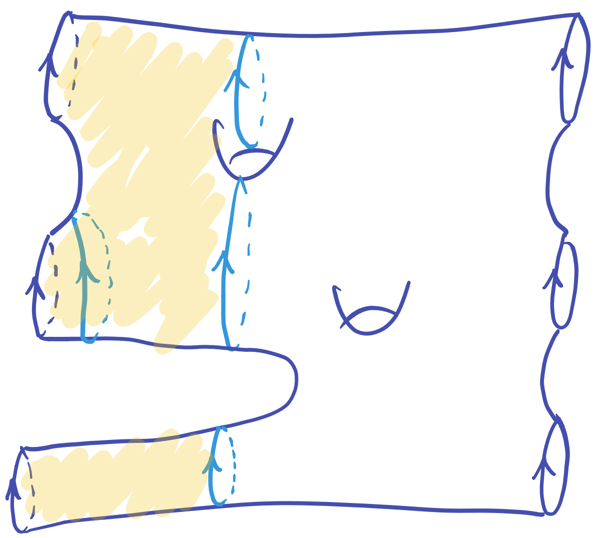
We then push the highest vertex down inductively until the path lies in $C_x(S_g)$.

Key idea: If we cut along a vertex of $B_x^{(0)}(S_g)$ we get

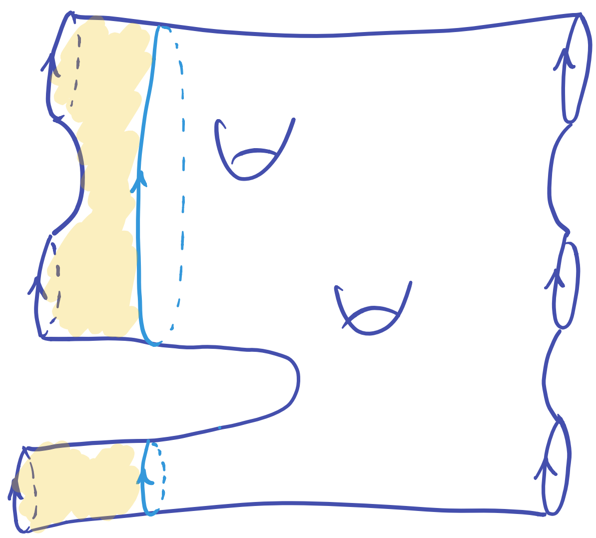


"cobordism"

What does an edge in $B_x(S_g)$ look like?



or

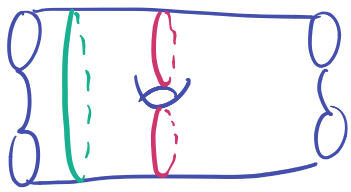


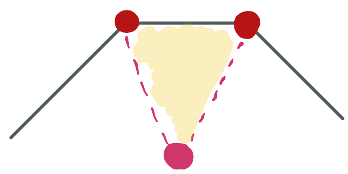
The region between transfers weight from one side to the other $\Longrightarrow$ the new vertex will have smaller weight iff there are fewer interior curves than boundary curves.

Call the edge on the right a <u>pants edge</u>.
This is the simplest way to reduce weight.

Proof that $Cx(S_g)$ is connected

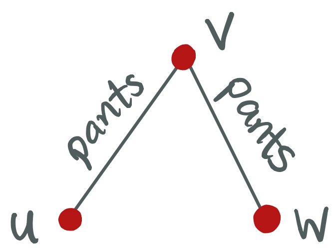
Step 1. Make maxima isolated, by making pants edges/triangles.



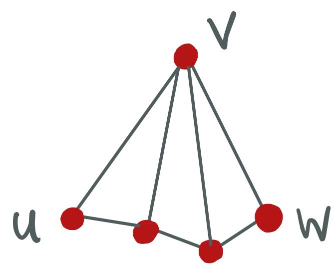


Step 2. Make highest edges into pants edges in same way

Step 3. Given



Connect uv to v by a seq of pants triangles



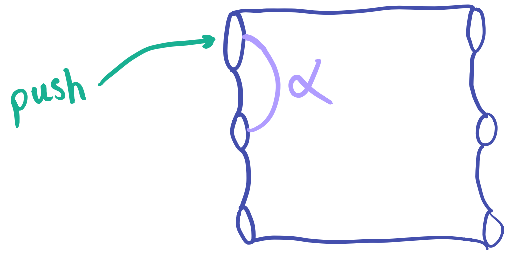
Can then push v down. Apply this process inductively.

To this end, consider the graph with

Vertices: pants edges emanating from v

$\longleftrightarrow$ certain arcs in $S \setminus v$

edges: disjoint arcs.



To show: connected

Notes.
- Every vertex is adjacent to one connecting first two components of $\partial(S\setminus v)$.
- Push maps (corresponding to 1^{st} ∂-comp act transitively on these.
- π_1(punctured sphere) has a simple gen set $\{x_i\}$

So: suffices to show that each $\mathrm{Push}(x_i)\cdot\alpha$ lies in same component as α.

Sample case: x_i lies on LHS of $S\setminus v$.
Then if β lies on RHS we have

$\square$

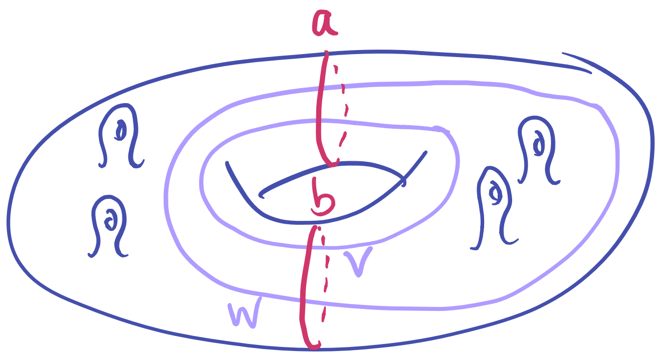
Proving Torelli is gen. by BP maps

Ingredient #1. $C_x(S_g)$ is connected ✓

Ingredient #2. Fact. Say $G \circlearrowright X$ = graph
$A \subseteq G$ s.t. $\forall$ edges $\overline{vw}$
$\exists$ $g \in A$ with $g \cdot v = w$.
Then $G = \langle A, \text{vertex stabs} \rangle$

Pf. Same as before

Ingredient #3. If vw is an edge of $C_x(S_g)$
$\exists$ BP map taking v to w.



$T_a T_b^{-1}(v) = w$

Thus it suffices to show $\mathrm{Stab}_{\mathcal{I}(S_g)}(v)$ is gen. by BP maps and Dehn twists about sep curves.

Two Birman Exact Sequences for Torelli

One Marked Point

$$1 \to \pi_1(S_g, p) \to \mathcal{I}(S_g, p) \to \mathcal{I}(S_g) \to 1$$

(restriction of usual BES)

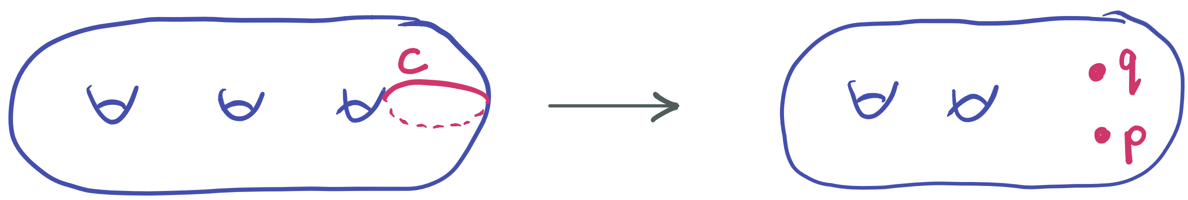
Two Marked Points

$$1 \to K \to \mathcal{I}(S_g, \{p,q\}) \to \mathcal{I}(S_g, p) \to 1$$

$K = [\pi, \pi] \qquad \pi = \pi_1(S_g \setminus p, q)$

will verify below!

What is defn of $\mathcal{I}(S_g, \{p,q\})$? It is the image of $\mathcal{I}(S_g, c)$ under the cutting map:



$\rightsquigarrow \mathcal{I}(S_g, \{p,q\}) = \ker\left(\mathrm{Mod}(S_g, \{p,q\}) \to \mathrm{Aut}\, H_1(S_g, \{p,q\})\right)$

Can define $\mathcal{I}(S_g, p)$ in same way, as further image under forgetting q. Get usual defn.

Stabilizers are gen. by BP maps...

First, $\mathrm{Stab}_{\mathcal{I}(S_g)}(c) \cong \mathcal{I}(S_{g-1}, \{p,q\})$ since:

$$1 \to \langle T_c \rangle \to \mathrm{Mod}(S_g, c) \to \mathrm{Mod}(S_g, \{p,q\}) \to 1$$

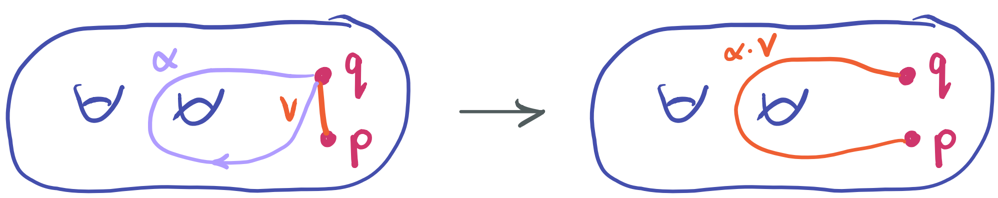
and $T_c \notin \mathcal{I}(S_g)$.

<u>Step 1.</u> K gen by BP maps & Dehn twists about seps.

$$\pi_1(S_g \setminus p, q) \circlearrowright H_1(S_g, \{p,q\}) \cong H_1(S_g) \oplus \mathbb{Z}$$

- trivial on first factor.
- action on 2nd factor is $\alpha \cdot v = [\alpha] + v$.

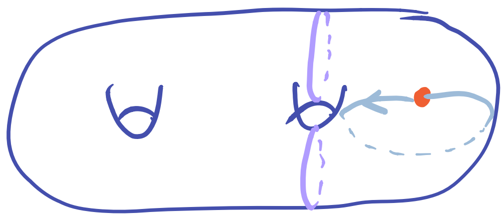
Check:



$\Rightarrow K = [\pi_1, \pi_1]$.

Now need: K gen. by simple (sep) loops.
Realize S_g as $4g$-gon with opp sides id'd.
and use the fact that commutator subgps are normally gen. by commutators of gens.

Step 2. $\pi_1(S_g,p) \subseteq \mathcal{I}(S_g,p)$ gen by BP maps.

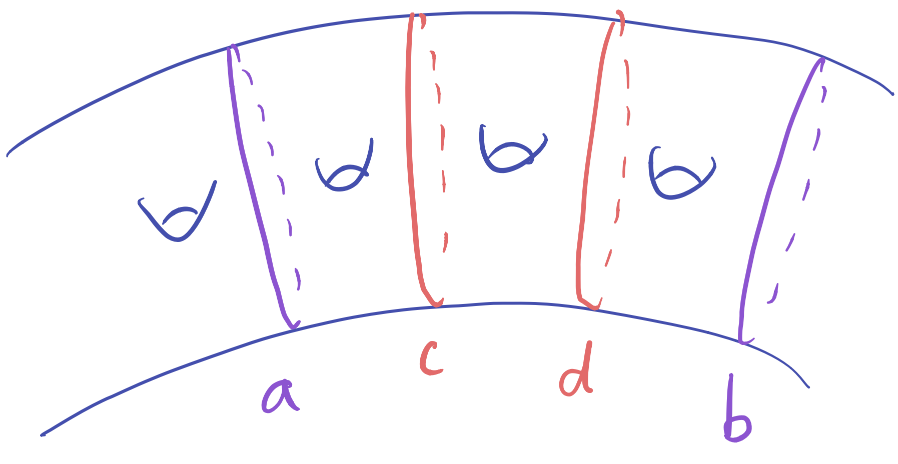
easy: 

$\square$

Even better:

Thm. $\mathcal{I}(S_g)$ is gen. by BP maps of genus <u>one</u>.

Pf.



$$T_a T_b^{-1} = (T_a T_c^{-1})(T_c T_d^{-1})(T_d T_b^{-1}).$$

Still need to address base case $g=2$!

Genus 2

$\mathcal{C}_x(S_2)$ is not connected ☹

Will instead use $\mathcal{I}(S_2) \circlearrowright \mathcal{B}_x(S_2)$.

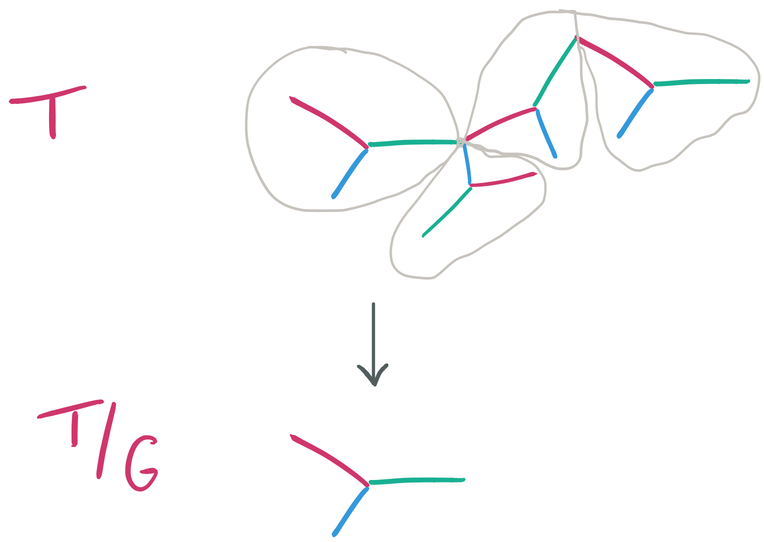
Already showed $\mathcal{B}_x(S_2)$ is a tree.

Will show $\mathcal{B}_x(S_2)/\mathcal{I}(S_2)$ is a tree.

Fact. If $G \circlearrowright T$ = tree and T/G = tree then G is (freely) gen. by vertex stabilizers.

Pf. Key point: $T/G \hookrightarrow T$

So T covered by translates of T/G



Fix X, a copy of T/G in T. ("tile")

Let $g \in G$. Suppose first that $g \cdot X \cap X \neq \emptyset$

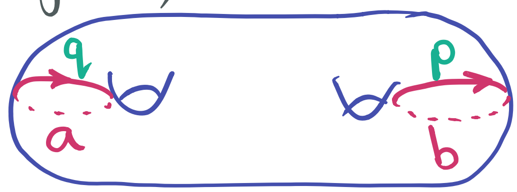
$\Rightarrow g \cdot X \cap X = \{v\} \Rightarrow g \in \mathrm{Stab}(v)$

(otherwise T/G would have a loop). Induct on tile distance. Free b/c no loops $\square$

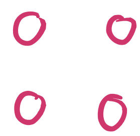
Remains: ① $Bx(S_2)/\mathcal{I}(S_2)$ = tree
② Stab's gen by Dehn twists (and free)

Proof of ① Weight fn. descends to quotient.
And $\exists!$ vertex of weight 1 (in quotient).

To show: if weight(v) > 1 $\exists!$ w with
weight(w) < weight(v) and 

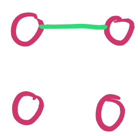
v looks like: 

$p, q \in \mathbb{Z}$

w given by nonsep curve c in middle.
$\longleftrightarrow$ curve in cut surface. 

c nonsep $\Longrightarrow$ c does not sep. left from right

weight(w) < weight(v) $\Longrightarrow$ c separates top from bottom.

Can think of c as an arc connecting top two circles. 

Make a graph: vertices: arcs as above
edges: disjointness.

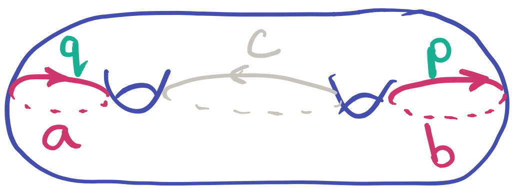
This connected (use usual trick with $PMod(S_{0,4})$ action). Adjacent vertices differ by sep twist.

Proof of ② Two kinds of vertices

Connected multicurves: Use Birman exact seq. as before (no change).

Note: $\pi_1(S_{1,1})$ is free.

Disconnected multicurves: By above argument can assume (up to Dehn twists about sep curves) that a stabilizer of a & b also fixes c.



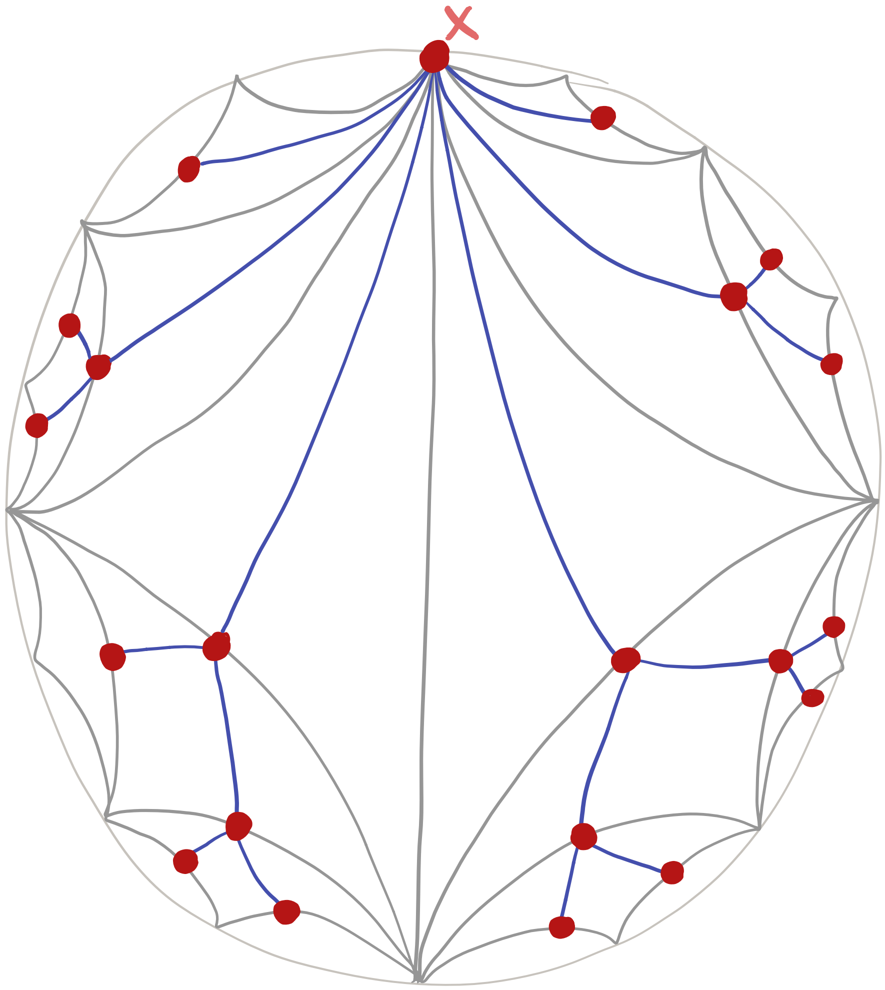
$\Rightarrow$ it is trival.

Note: $\mathrm{PMod}(S_{0,4})$ free.

We have shown $\mathcal{I}(S_2) = \langle T_c \mid c \text{ sep} \rangle$

With some more bookkeeping, we show it is freely gen. by Dehn twists, one for each symplectic splitting. $\square$

THE FAREY GRAPH AND $B_x(S_2)/\mathcal{I}(S_2)$.



one of these for each Lagrangian subspace containing x. All glued at x.

Vertices of $B_x(S_2)/\mathcal{I}(S_2)$ are minimal bases for Lagrangian subspaces of $H_1(S_2;\mathbb{Z})$ containing x.

(Minimal means that if the basis contains two elts, neither is x.)

Edges are for $\{a,b\} \longrightarrow \{a,a+b\}$
(If one of these is x, just drop it.)

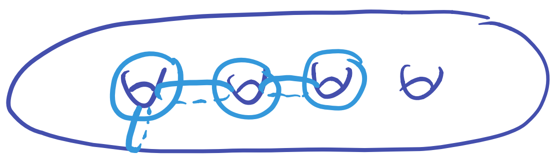
Johnson I

Thm $\mathcal{I}(S_g)$ is fin. gen. by Dehn twists for $g \geq 3$.

Basic strategy 1. List prospective generators $\{g_i\}$.
s.t. g_1 is a BP map of genus 1.
2. Show $\langle g_i \rangle \trianglelefteq \mathrm{Mod}(S_g)$.

This suffices since $\langle\langle g_i \rangle\rangle_{\mathrm{Mod}(S_g)} = \mathcal{I}(S_g)$.

Chains and BP maps.

A chain: 

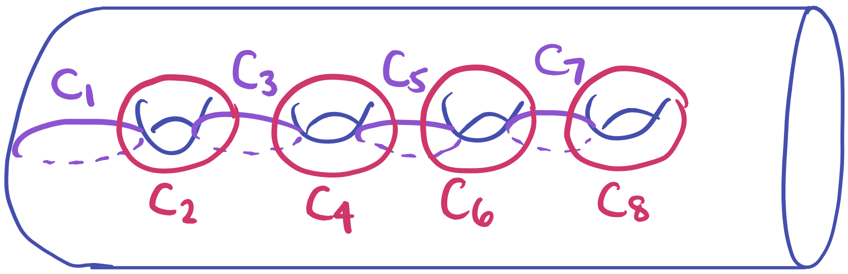
$\rightsquigarrow$ BP map.

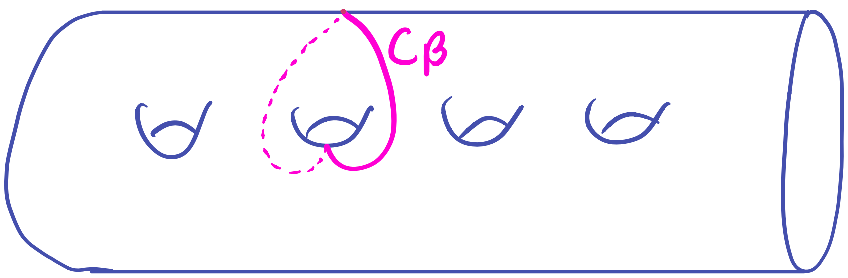
Given a chain, can resolve intersections to get another chain. Can also take subchains.

Given a chain $ch(c_1, \ldots, c_n)$
$ch(i_1 i_2 \cdots i_{k+1})$
denotes the chain you get by combining $c_{i_1}, \ldots, c_{i_2 - 1}$
$c_{i_2}, \ldots, c_{i_3 - 1}$ etc. dropping $c_{k+1}, \ldots, c_n$. "subchain"

Denote the BP-map $[i_1 i_2 \cdots i_{k+1}]$

Listing the Generators





Consider the chains:

$(c_1, \dots, c_{2g})$ straight chain

$(c_\beta, c_5, \dots, c_{2g})$ β-chain

Use same notation for subchains of β-chain:

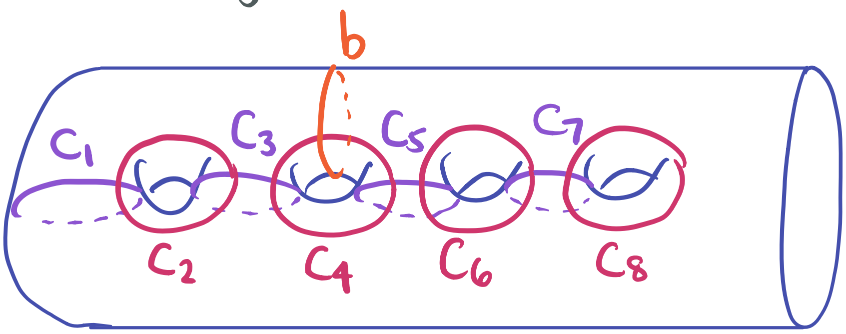
$(\beta i_1) =$ surger $c_\beta, c_5, \dots, c_{i_1-1}$

Theorem. For $g \geq 3$ the odd subchain maps of straight chain & β-chain generate $\mathcal{I}(S_g^1)$

Since $\mathcal{I}(S_g^1) \twoheadrightarrow \mathcal{I}(S_g)$ this gives closed case as well.

Setup.

Humpries generators:



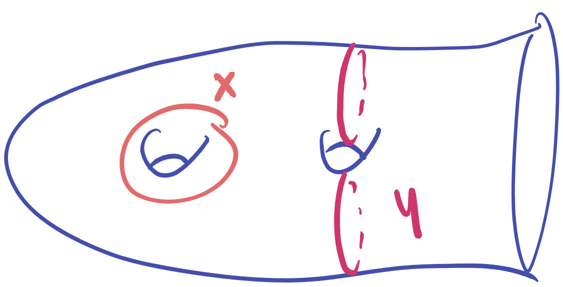
Let $J(S_g^1)$ & $J(S_g)$ denote groups gen by Johnson's generators.

As above, need to show

$$T_x * y = T_x y T_x^{-1} \in J(S_g)$$

$\forall$ $x \in$ Humphries set
$y \in$ Johnson set

In many cases $T_x * y$ equals y or is another Johnson gen:



Conjugating by a Positive Twist in the Chain

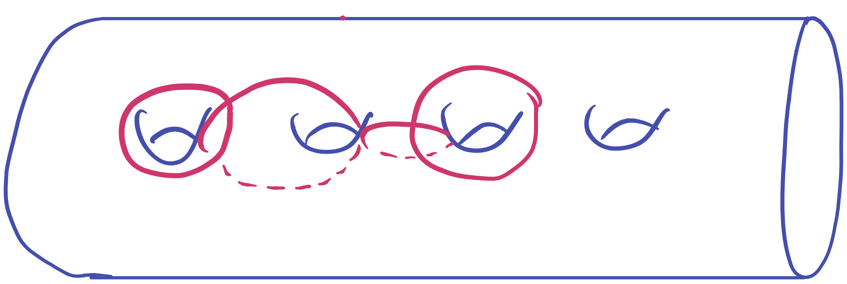
Prop

j in $\{i_1, \ldots\}$	$j+1$ in $\{i_1, \ldots\}$	$T_{c_j} * [i_1\ i_2 \cdots]$
✓	✓	$[i_1\ i_2 \cdots]$
✗	✗	$[i_1\ i_2 \cdots]$
i_m	✗	$[i_1 \cdots i_{m-1}\ i_m+1 \cdots]$
✗	i_m	$[i_1 \cdots i_{m-1}\ i_m-1 \cdots]$

The lemma completely characterizes commuting among straight Johnson & Humphries gens.

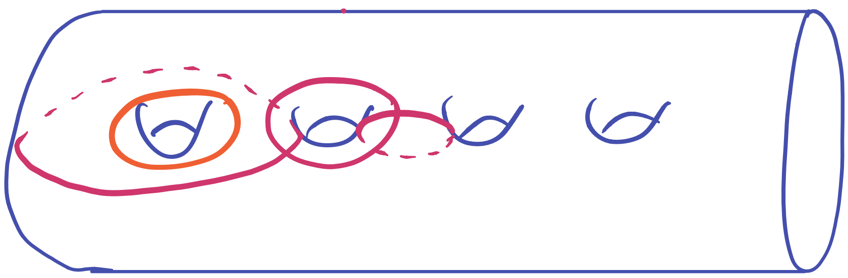
Pf (by examples)

Example $j=2$ $[2\ 3\ 5\ 6]$



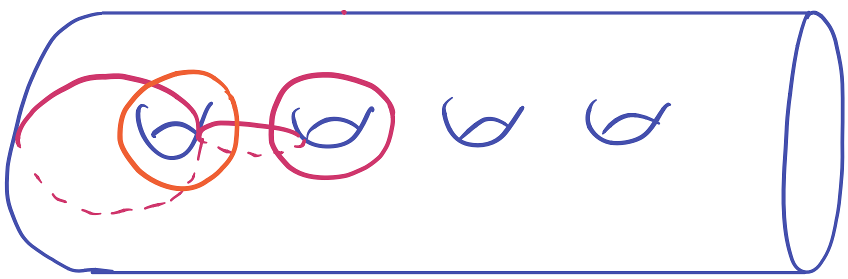
✓

Example $j=2$ $[1\ 4\ 5\ 6]$

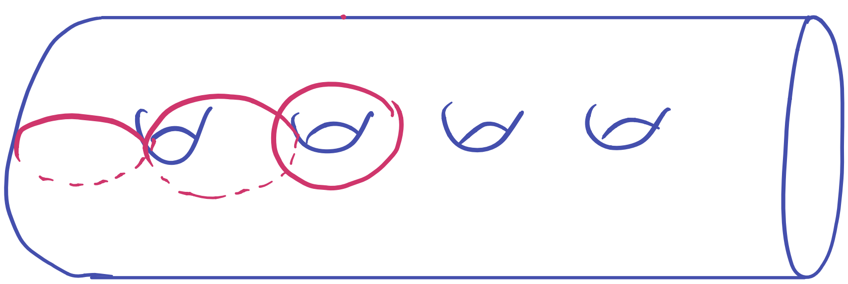


✓

Example $j=2$ $[1\ 3\ 4\ 5]$



$\downarrow$ $[1\ 2\ 4\ 5]$



✓

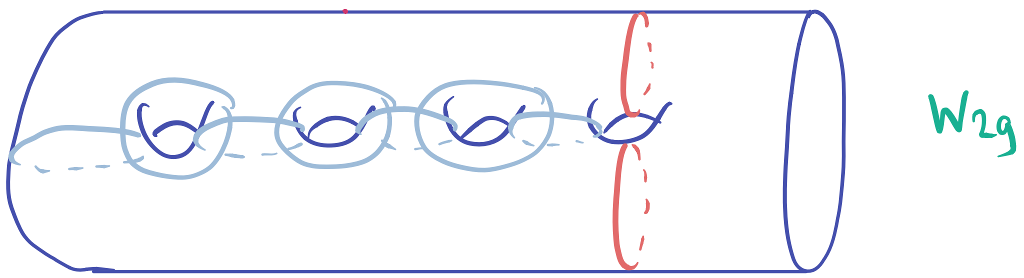
Third case similar.

Upshot of Lemma: If we conj a chain map by a <u>positive</u> twist about a curve in the chain we get a subchain map. What about negative twists?

Generating the Kernel of $\mathcal{I}(S_g^1) \to \mathcal{I}(S_g)$

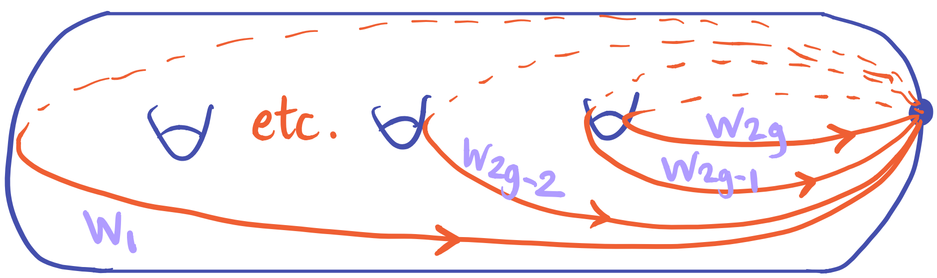
Define $W_i = [1\ 2\ 3 \cdots \hat{i} \cdots 2g+1]$ for $i \in \{1, \ldots, 2g+1\}$. "maximal odd subchain maps"

Each lies in $\pi_1 UT(S_g)$



Lemma. $\pi_1 UT(S_g)$ is gen by $W_1, \ldots, W_{2g}, T_b * W_1$

Pf. The corresponding push maps are



These clearly generate π_1. Need to get the fiber of $\pi_1 UT \to \pi_1$.

Claim. $T_b * W_1 = W_4 W_3^{-1} W_2 T_\partial$ $\square$

Cor. $\pi_1 UT(S_g^1) \leq \mathcal{J}(S_g^1)$

Pf. Use lantern to get $T_b * W_1$ (Cor 1 in paper).

Conjugating by a Negative Twist in the Chain

Prop. Fix a chain (c_i). The subgroup of $\mathcal{I}(S_g^1)$

$\langle$odd subchain maps of length $2k-1\rangle$

is normalized by $\langle T_{c_i}\rangle$.

Lemma. $\langle W_1, \dots, W_{2g+1}\rangle$ is normalized by $\langle T_{c_1}, \dots, T_{c_{2g}}\rangle$

Pf. Need to check $T_{c_i}^{\pm 1}(W_j) \in \langle w_1, \dots, w_{2g+1}\rangle$ in $\pi_1 UT(S_g^1)$.

Pf of Prop. Let $f = [i_1\ i_2 \cdots i_{2k}]$

Fix a c_j in original chain.

As above $T_{c_j} * f = f$ unless exactly one of $j, j+1$ is an $i_\square$.

Enlarge $(i_1, i_2, \dots, i_{2k})$ to include both j & $j+1$.

$\rightsquigarrow$ subchain of length $2k$ (c_i') w/ c_j basic

Lemma $\Longrightarrow$ $T_{c_j}^{\pm 1} * f$ is a product of maximal subchain maps of (c_i'). But these are all subchains of original chain $\square$

Two More Tools.

① Induction

If our Johnson & Humphries gens both lie in S'_{g-1}, can apply induction. Works because both sets of gens agree with $S'_{g-1} \hookrightarrow S'_g$ (this is one specific inclusion).

N.B. Base case is subgp of $\mathcal{I}(S'_2)$ gen. by BP maps i.e. $\pi_1 UT(S_2)$ i.e. $\langle W_1, \ldots, W_5 \rangle$.

② Controlled Change of Coords

Suppose $T_{c_i} \in$ Humphries
$f \in \mathrm{Mod}(S'_g)$ already known to normalize $\mathcal{J}(S'_g)$.
$h \in$ Johnson

Then checking $T_{c_i} * h \in \mathcal{J}(S'_g)$ is equiv. to checking $f * (T_{c_i} * h)$
$= T_{c_i} * (f * h) \in \mathcal{J}(S'_g)$

Proof of the Theorem

Need to conjugate all Johnson gens by $T_{c_i}^{\pm 1}, T_b^{\pm 1} \in$ Humphries.

<u>Step 1</u>. $T_{c_1}, T_{c_2}, T_{c_5}, \ldots, T_{c_{2g}}$ normalize $\mathcal{J}(S_g^1)$.

T_{c_1}, T_{c_2} in the straight chain, disjoint from β-chain.
$T_{c_5}, \ldots, T_{c_{2g}}$ in both chains.
Apply Prop.

<u>Step 2</u>. T_{c_3} normalizes $\mathcal{J}(S_g^1)$.

T_{c_3} normalizes straight chains by Prop.
By controlled change of coords, can restrict to consecutive β-chain maps.

$$[\beta \ 5 \ 6 \ \cdots].$$

If chain not maximal, use induction.
If it is, need a relation (Cor 2 in paper).
<u>But</u> this is one check.

<u>Step 3</u>. T_{c_4} normalizes $\mathcal{J}(S_g^1)$ similar to T_{c_3}

Step 4. T_b normalizes $\mathcal{J}(S_g^1)$.

β-chains: controlled change of coords reduces consecutive β-chains & induction reduces to

$T_b^{\pm 1} * [\beta \ 5 \ \cdots \ 2g+1]$ single check

Straight chains $[i_1 \cdots i_{2k}]$

Can assume $i_1 \leq 4$ $i_{2k} \geq 5$

(otherwise T_b commutes)

Change of coords reduces to: consecutive straight chain starting with 1, 2, 3, or 4. and going up to $2g$ or $2g+1$ (induction).

If it starts with 1 or 2 it is maximal, i.e. in $\pi_1 UT$.

Only two remaining cases:

$[4 \ 5 \ldots 2g+1]$ & $[3 \ 4 \ \cdots \ 2g]$.

$\square$

The Johnson Homomorphism.

Birman's question: Is $[\mathcal{I}(S_g) : \mathcal{K}(S_g)] < \infty$?

Will now define

$$\tau : \mathcal{I}(S_g^1) \to \Lambda^3 H$$

where $H = H_1(S_g^1; \mathbb{Z})$.

- $\Lambda^3 H \cong \mathbb{Z}^{\binom{2g}{3}}$ = free abelian group.
- $\mathcal{K}(S_g^1) \leq \ker \tau$

This answers Birman's question.

Later:
- τ captures all of $H^1(\mathcal{I}(S_g); \mathbb{Z})$.
- $\mathcal{K}(S_g) = \ker \tau$
- answer negatively a question of Chillingworth: is $\mathcal{K}(S_g)$ the subgroup of $\mathcal{I}(S_g)$ preserving all winding numbers of curves.
- All $\mathbb{Z}HS^3$ come from $\mathcal{K}(S_g)$
- $\mathcal{I}(S_g)$ exp. distorted in $\mathrm{Mod}(S_g)$
- and much more...

Tensor & Wedge Products

$$V \otimes W = \langle v \otimes w : v \in V, w \in W \rangle / \text{bilinearity}$$

basis: $e_i \otimes f_j \quad V = \langle e_i \rangle, W = \langle f_j \rangle$

$\rightsquigarrow (\dim V \cdot \dim W)$-dimensional.

$$\begin{aligned} V \wedge V &= (V \otimes V) / (v \otimes w = -w \otimes v) \\ &= (V \otimes V) / (v \otimes v). \\ &= \text{image of } V \otimes V \longrightarrow V \otimes V \\ &\qquad\qquad v \otimes w \longmapsto v \otimes w - w \otimes v \end{aligned}$$

basis: $e_i \wedge e_j \quad i \neq j \quad \rightsquigarrow \binom{n}{2}$-dimensional.

So $\Lambda^k V = \langle v_1 \wedge \cdots \wedge v_k \rangle / \text{bilinearity}$

and...

$$v_1 \wedge \cdots \wedge v_k = \text{sgn}(\sigma)\, v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}.$$

$\rightsquigarrow \binom{n}{k}$ dimensional.

$$\Lambda^k V = \text{image of } \otimes^k V \to \otimes^k V$$

$$v_1 \otimes \cdots \otimes v_k \longmapsto \sum_{\sigma \in \Sigma_k} \text{sgn}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

DEFINITION

Let $\Gamma = \pi_1(S_g^1) \cong F_{2g}$
$\Gamma' = [\Gamma, \Gamma]$

Consider:

$$1 \longrightarrow \frac{\Gamma'}{[\Gamma,\Gamma']} \longrightarrow \frac{\Gamma}{[\Gamma,\Gamma']} \longrightarrow \frac{\Gamma}{\Gamma'} \longrightarrow 1$$

or: $1 \longrightarrow N \longrightarrow E \longrightarrow H \longrightarrow 1$

The Johnson homomorphism is

$$\tau : \mathcal{I}(S_g^1) \longrightarrow \mathrm{Hom}(H, N)$$

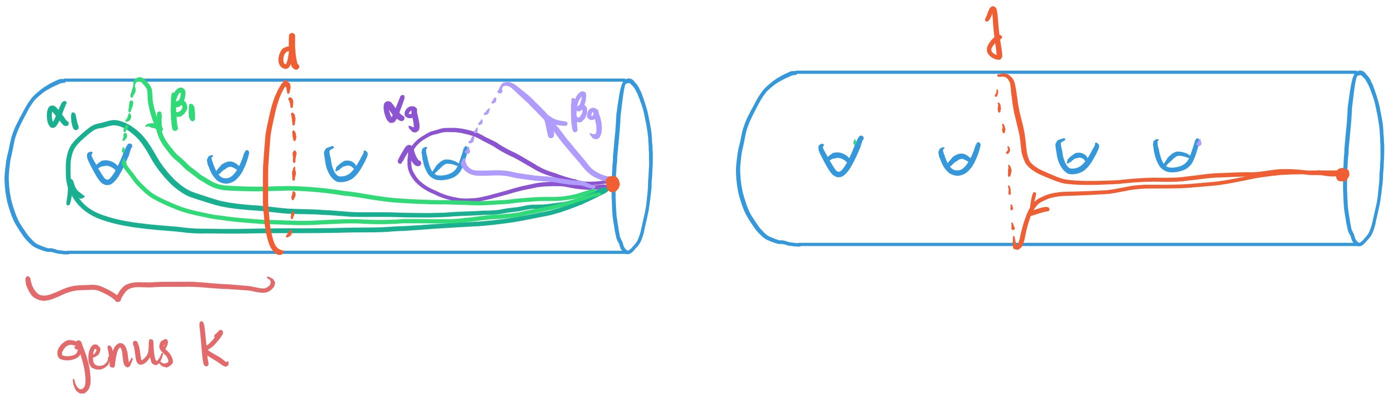
given by $\tau(f)(x) = f(e)e^{-1}$
where e is any lift of x to E.

What about $\Lambda^3 H$?

- $\Lambda^2 H \cong N$ via $a \wedge b \longmapsto [\tilde{a}, \tilde{b}]$
  where $\tilde{a}, \tilde{b}$ are lifts to E — see Putman's lecture notes
- $\mathrm{Hom}(H, \Lambda^2 H) \cong H^* \otimes \Lambda^2 H \cong H \otimes \Lambda^2 H$
- Will show $\mathrm{im}\,\tau$ is im $\Lambda^3 H \longrightarrow H \otimes \Lambda^2 H$
  $a \wedge b \wedge c \longmapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$

The Image of a Dehn Twist

Consider T_d



Action on Γ: $T_d(x) = x \quad x \in \{\alpha_{k+1}, \beta_{k+1}, \ldots \alpha_g, \beta_g\}$

$T_d(x) = \delta x \delta^{-1} \quad x \in \{\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k\}$.

So: $\tau(T_d)([x]) = 1$ or $[x, \delta]$ in N

but both equal 1 in N.

$\Longrightarrow \tau(T_d) = 0$.

This is just <u>one</u> Dehn twist, but:

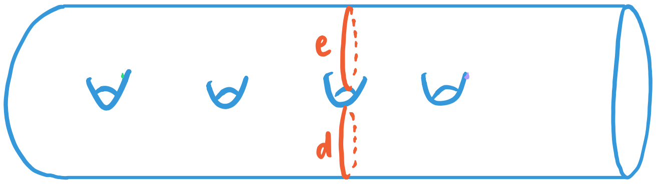
Naturality: If $h \in \mathrm{Mod}(S_g^1)$, $f \in \mathcal{I}(S_g^1)$

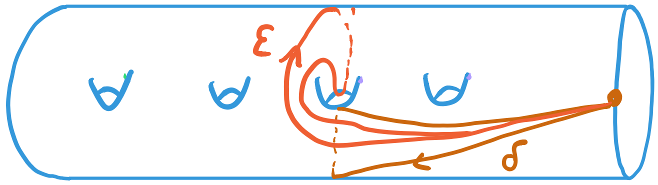
then $\tau(hfh^{-1}) = h\,\tau(f)$.

(straightforward)

Thus $K(S_g^1) \leq \ker \tau$.

The Image of a Bounding Pair Map





Let $f = T_d T_e^{-1}$

Compute: $f(\alpha_i) = \delta \alpha_i \delta^{-1} \quad i \leq k \qquad f(\alpha_i) = \alpha_i \quad i \geq k+1$

$f(\beta_i) = \delta \beta_i \delta^{-1} \quad i \leq k \qquad f(\beta_i) = \beta_i \quad i \geq k+1$

$f(\alpha_{k+1}) = \delta \varepsilon^{-1} \alpha_{k+1}$

Now write down all $f(x)x^{-1} \in N \cong \Lambda^2 H \qquad x \in \{\alpha_i, \beta_i\}$

$$f(\alpha_i)\alpha_i^{-1} = [\delta, \alpha_i] \longleftrightarrow [\beta_{k+1}] \wedge [\alpha_i]$$
$$f(\beta_i)\beta_i^{-1} = [\delta, \beta_i] \longleftrightarrow [\beta_{k+1}] \wedge [\beta_i]$$
$$f(\alpha_{k+1})\alpha_{k+1}^{-1} = \delta \varepsilon^{-1} \longleftrightarrow \sum [\alpha_i] \wedge [\beta_i]$$
$$f(x)x^{-1} = id \longleftrightarrow 0 \quad \text{otherwise.}$$

$$\Rightarrow \tau(f) = \sum_{i=1}^{k} \Big([\beta_i] \otimes \big([\beta_{k+1}] \wedge [\alpha_i]\big) - [\alpha_i] \otimes \big([\beta_{k+1}] \wedge [\beta_i]\big)\Big) + [\beta_{k+1}] \otimes \Big(\sum_{i=1}^{k} [\alpha_i \wedge \beta_i] \Big)$$

$$= \sum_{i=1}^{k} \big([\alpha_i] \wedge [\beta_i]\big) \wedge [\beta_{k+1}]$$

$\Rightarrow |\mathrm{Im}\,\tau| = \infty \Rightarrow$ Birman question.

The Image of Tau

Recall $\Lambda^3 H \hookrightarrow H \otimes \Lambda^2 H$
via $a \wedge b \wedge c \longmapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$

<u>Prop.</u> $\operatorname{im} \tau = \Lambda^3 H$.

By the naturality, $\operatorname{Im} \tau$ is an Sp-rep. And $\Lambda^3 H$ is one of two GL-irreps in $\Lambda^2 H \otimes H$.

<u>Pf. for $g=2$</u> Already have: $\boxed{x_1 \wedge y_1 \wedge y_2} \in \operatorname{Im} \tau$
(consider the "first" bounding pair).
Now use Sp to move this around.

Apply Sp-map $x_2 \longrightarrow y_2 \longrightarrow -x_2$
(other basis elements fixed).
to $\boxed{x_1 \wedge y_1 \wedge y_2}$ to get $\boxed{-x_1 \wedge y_1 \wedge x_2}$

Apply $x_1 \leftrightarrow x_2$, $y_1 \leftrightarrow y_2$ to get
$\boxed{x_2 \wedge y_2 \wedge y_1}$ $\boxed{-x_2 \wedge y_2 \wedge x_1}$ $\square$

Higher genus similar.

WHY WOULD JOHNSON THINK OF THIS?

$\mathcal{I}(S_g^1)$ is by definition the kernel of

$$\mathrm{Mod}(S_g^1) \longrightarrow \mathrm{Aut}(\Gamma/\Gamma'').$$

Then you notice that twists about sep curves conjugate by commutators and BP maps conjugate by non-commutators.

In other words the former acts trivially on $\Gamma/[\Gamma, \Gamma']$ and the latter does not.

The map τ measures this action.

The Johnson Homomorphism via Mapping Tori

$f \in \mathrm{Mod}(S_g^1) \rightsquigarrow$ mapping torus M_f.

$f \in \mathcal{I}(S_g^1) \Longrightarrow H^*(M_f) \cong H^*(S_g^1 \times S^1)$.

<u>but</u> ring structure (or, intersection theory) can be different.

Given f, want $\tau(f) \in \Lambda^3 H$.

By taking duals, this is the same as an alt. linear function that takes as input 3 elts of H and outputs a number

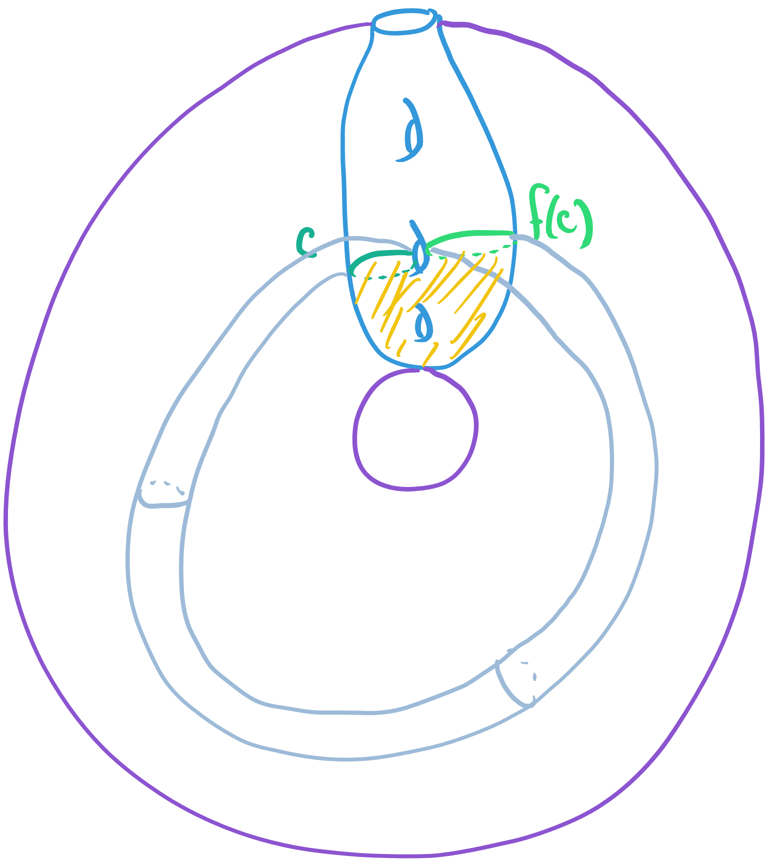
As above, there is an inclusion

$$H^* \cong H^1(S_g^1) \xrightarrow{i} H^1(M_f) \cong H_2(M_f, \partial)$$

The desired function is triple cup product or, dually, triple intersection.

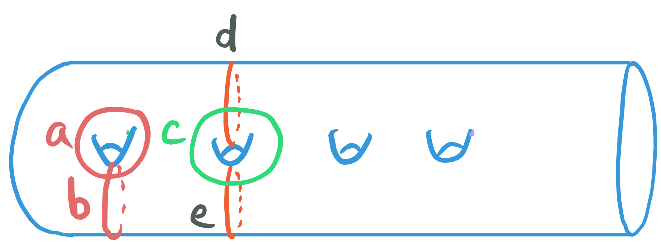
Let's check this does the right thing on BP maps.

First, let's give an explicit description of i:



If we "flow" c we get a tube in M_f from c to f(c) in one fiber. But c & f(c) are homologous so we can fill in the homology to get a closed surface, i.e. elt. of $H_2(M_f)$.

Now fix a BP map $T_d T_e^{-1}$



Only one curve from std geom. symplectic basis gets moved, namely c. The curves a & b give tori in M_f, intersecting at $(a \cap b) \times S^1$. The curve c gives a surface of genus 2 as in above picture, intersecting $(a \cap b) \times S^1$ in one point. This gives the term $[a] \wedge [b] \wedge [c]$, as desired. All other terms are zero.

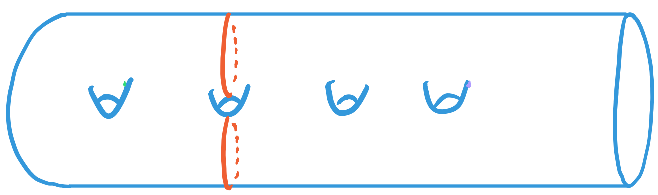
The Johnson Homomorphism via the Jacobian

Starting point: $\Lambda^3 H \cong H_3(T^{2g};\mathbb{Z})$.

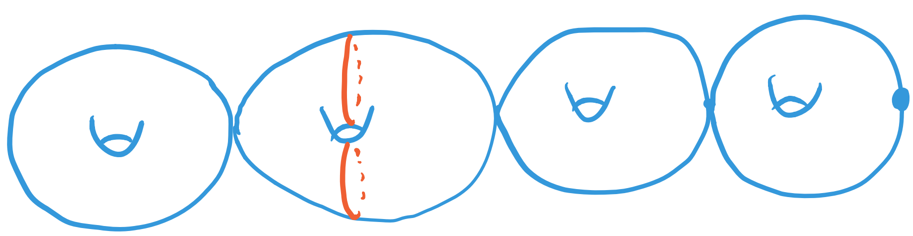
<u>and</u> we have a map $j: S_g^1 \to T^{2g}$ corresponding to $\pi_1(S_g^1) \to H_1(S_g^1)$.
By $K(G,1)$ theory, the map is unique up to homotopy.

Let $f \in \mathcal{I}(S_g^1)$. Then $j \circ f$ is homotopic to j <u>and</u> $\operatorname{im} j\circ f = \operatorname{im} j$.
The homotopy gives a mapping torus in T^{2g}, which represents an elt of $H^3(T^{2g})$!

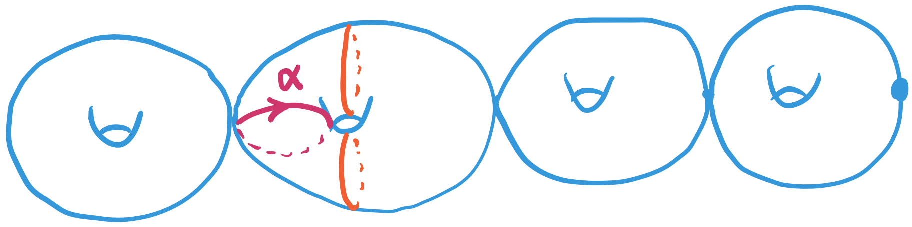
Again, lets check this does the right thing on BP maps. Consider:



Observe that j factors through:



Each torus maps to a coordinate 2-torus $x_i \wedge y_i$ in T^{2g}.



The BP map is obtained by pushing the leftmost torus along α.

What is happening in T^{2g}? The $x_1 \wedge y_1$ torus is moving in the y_2-direction, tracing out the 3-torus $x_1 \wedge y_1 \wedge y_2$.

But this is exactly the image of the BP map under τ!

<u>Another definition</u>: Moriyama (J. London Math Soc.) shows that $K(S_g^1)$ is the kernel of the action of $\mathrm{Mod}(S_g^1)$ on the compactly supp. cohomology group of the config. space of two points in S_g^1.

Can you see directly that this action is τ?

What about the Closed Case?

Since $\mathcal{I}(S_g) \cong \mathcal{I}(S_g^1)/\pi_1 UT(S_g)$
we have a map

$$\tau : \mathcal{I}(S_g) \longrightarrow \Lambda^3 H / \tau(\pi_1 UT(S_g))$$

Claim: $\tau(\pi_1 UT(S_g))$ = image of $H \longrightarrow \Lambda^3 H$
given by $x \longmapsto \omega \wedge x$
where $\omega = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$

Now, $\pi_1 UT(S_g)$ is gen. by BP maps of genus $g-1$:

$$\longmapsto (x_1 \wedge y_1 + \cdots + x_{g-1} \wedge y_{g-1}) \wedge y_g = \omega \wedge y_g$$

Now apply Sp-action. Note Sp fixes ω, and acts transitively on basis elements of H. That does it!

Put another way, the ambiguity is because a BP bounds <u>two</u> 2-chains in S_g, the difference being S_g.

The Chillingworth Class

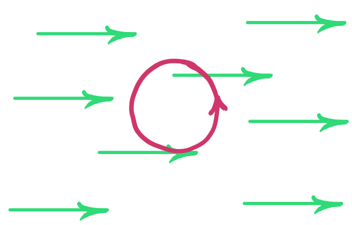
X = nonsingular vector field on S_g'.
$\omega_X(\gamma)$ = # times X rotates along γ.
well def on homotopy classes

X_1, X_2 two nonsingular vector fields

$\rightsquigarrow \omega_{X_2}(\gamma) - \omega_{X_1}(\gamma)$ is a cohomology

class, denoted $d(X_1, X_2)$.

A cohomology class is a function on cycles that is zero on boundaries $\Longleftrightarrow$ it is zero on small cycles (cocycle condition).
So $\omega_X(\gamma)$ is not a cocycle because it evaluates to ± 1 on small loops, but $d(X_1, X_2)$ is.



Let $f \in \mathcal{I}(S_g^1)$. Consider the function

$$e_{f,x}(\gamma) = \omega_x(f(\gamma)) - \omega_x(\gamma)$$

This is the change of winding number induced by f.

Two facts: ① $e_{f,x} = d(x, f^{-1}(x))$
② $e_{f,x}$ is indep. of x!

Proof of ①:
$$\begin{aligned} e_{f,x}(\gamma) &= \omega_x(f(\gamma)) - \omega_x(\gamma) \\ &= \omega_{f^{-1}(x)}(\gamma) - \omega_x(\gamma) \\ &= \langle d(f^{-1}(x), x), \gamma \rangle \end{aligned}$$

Proof of ②:
$$\begin{aligned} e_{f,x_2}(\gamma) - e_{f,x_1}(\gamma) &= \omega_{x_2}(f(\gamma)) - \omega_{x_2}(\gamma) - \omega_{x_1}(f(\gamma)) - \omega_{x_1}(\gamma) \\ &= \langle d(x_1, x_2), f(\gamma) \rangle - \langle d(x_1, x_2), \gamma \rangle \\ &= \langle d(x_1, x_2), f(\gamma) - \gamma \rangle. \end{aligned}$$
$\square$

So we may write e_f. This is an elt of $H^* \cong H$.

Fact. $e_{fh} = e_f + e_h$ for $f,h \in I(S_g^1)$.

i.e. $e : I(S_g^1) \longrightarrow H$ is a homomorphism.

Pf. $e_{fh}(\gamma) = \omega_x(fh(\gamma)) - \omega_x(\gamma)$
$$= \omega_x(f(h(\gamma))) - \omega_x(h(\gamma))$$
$$+ \omega_x(h(\gamma)) - \omega_x(\gamma)$$
$$= e_f(h(\gamma)) + e_h(\gamma)$$
$$= e_f(\gamma) + e_h(\gamma). \qquad \square$$

Let $t(f) = e_f^*$. This is the Chillingworth class.

The contraction $C : \Lambda^3 H \longrightarrow H$

$$x \wedge y \wedge z \longmapsto 2\left[\langle x,y\rangle z + \langle y,z\rangle x + \langle z,x\rangle y\right]$$

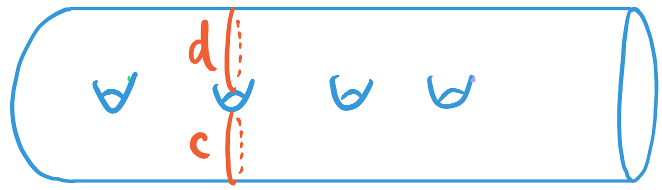
Theorem. $t(f) = C(\tau(f))$.

Need to check two things

① Naturality: $t(hfh^{-1}) = h\, t(f)$.

② t & $C \circ \tau$ agree on a BP map of genus 1.

Image of a BP map under t.

Consider: 

$$C(\tau(T_c T_d^{-1})) = C(x_1 \wedge y_1 \wedge y_2) = 2y_2$$

$T_c T_d^{-1}$ leaves all the a_i, b_i (std geom. symp. basis) fixed except a_2. It and its image a_2' bound a torus with two holes, call it N.

Fix a nonsingular vector field X on S_g, hence N. Cap N with disks, D & D', bounded by α_2, α_2'. Extend $X|_N$ to capped N minus 2 pts.

Fix nonsing. vector fields Y, Y' on D, D'.

Have $\omega_Y(a_2) = \omega_{Y'}(a_2') = +1$.

and $\omega_Y(a_2) - \omega_X(a_2) =$ index of singularity

$\omega_{Y'}(a_2') - \omega_X(a_2') =$ index of singularity.

But $\sum$ indices $= \chi(T^2) = 0$.

$\Rightarrow \omega_X(a_2') - \omega_X(a_2) = 2$ if you get the signs right. $\square$

Signed Stable Lengths – Irmer's Theorem

We may consider $C_x(S_g^1)$ as a directed graph: given two vertices of an edge, one lies to the left of the homology & one to the right.

$\rightsquigarrow$ signed distance d_s on vertices.

Signed stable length:

$$\phi_x(f) = \lim_{n\to\infty} \frac{d_s(v, f^n(v))}{n}$$

for $f \in \mathcal{I}(S_g^1)$. This is indep. of basept v.

$\rightsquigarrow$ $SSL : \mathcal{I}(S_g^1) \longrightarrow H$.

<u>Theorem</u>. $SSL = t/2$.

Birman-Craggs Homomorphisms

Will exhibit $\binom{2g}{0}+\binom{2g}{1}+\binom{2g}{2}+\binom{2g}{3}$ homomorphisms $\mathcal{I}(S_g^1) \to \mathbb{Z}/2$.

The last summand $\longleftrightarrow$ τ mod 2

Johnson: this is all of $\mathcal{I}(S_g^1)^{ab}$.

Fix $h: S_g \to S^3$ Heegaard embedding

Let $f \in \mathrm{Mod}(S_g)$

$M(h,f)$ obtained by regluing by f:

Say A, B are the handlebodies so $\partial A = \partial B = h(S_g)$.

Then $$M(h,f) = (A \amalg B) / x \sim hfh^{-1}(x)$$

$f \in \mathcal{I}(S_g) \Rightarrow M(h,f)$ a homology S^3.

$\rightsquigarrow$ Rochlin invariant* $\mu(h,f) \in \mathbb{Z}/2$

$\rho_h = \mu(h,\cdot)$ is a homom. $\mathcal{I}(S_g) \to \mathbb{Z}/2$.

Also: the homom. depends on h, but there are only finitely many $\longleftrightarrow$ self-linking forms.

Spin Structures.

A spin structure is a continuous choice of frame on the 1-skeleton that extends to the 2-skeleton

For an n-manifold M this gives a function
$$\pi_1 M \longrightarrow \pi_1 SO(n) \cong \mathbb{Z}/2.$$

$n>2.$

Spin structures on surfaces:

Rokhlin Invariant μ

$W = \mathbb{Z}HS^3$
$= \partial X$ where X = spin (or, parallelizable)

Such X always exists and signature $\sigma(X)$ is multiple of 8. Also $\sigma(X)/8$ mod 2 is indep. of choice of $X \rightsquigarrow \mu$

Note: $\mathbb{Z}HS^3$'s have unique spin structure.

Seifert Linking Forms

Fix $S_g \subseteq S^3$

$\rightsquigarrow$ bilinear Seifert linking form on $H_1(S_g;\mathbb{Z})$

$$L(x,y) = \text{linking}\ \#\ (x, y^+)$$

pushoff

Use $\mathbb{Z}/2$ coeffs and restrict to $y = x$

$\rightsquigarrow$ mod 2 self-linking form

= quadratic form on $H_1(S_g;\mathbb{Z}/2)$:

$$\omega(x+y) = \omega(x) + \omega(y) + \hat{\imath}(x,y)$$

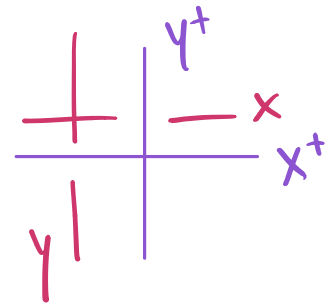
Johnson:

Quad. forms

$\updownarrow$

Spin struct's

Can see directly:



each intersection of x & y adds 2 crossings hence $+1$ to linking number.

Arf invariant: Fix a symplectic basis $\{x_i, y_i\}$

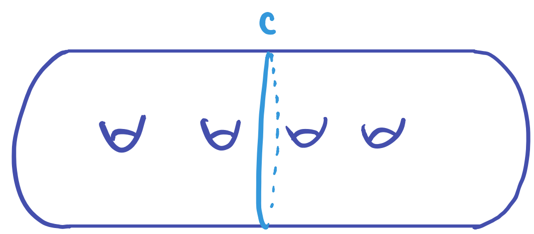
$$\alpha = \sum \omega(x_i)\,\omega(y_i)$$

For a knot K, the Arf invt is the Arf of the self linking # of any Seifert surface.

Computations

Fix $h: S_g \hookrightarrow S^3$
with Seifert linking form ω

Separating Twists



$M(h, T_c)$ obtained by Dehn surgery along c.
Gordon '75 & Gonzalez-Acuna '70:

$$\mu(M(h, T_c)) = \mathrm{Arf}(c)$$

But $h(S_g) \rightsquigarrow$ Seifert surface

$$\Rightarrow \rho_h(T_c) = \sum_{i=1}^{k} \omega(x_i)\omega(y_i)$$

BP maps.

Get a surgery description of the desired 4-manifold. End result:

$$\rho_h(T_c T_d^{-1}) = \begin{cases} 0 & \text{if } \omega(c) = 1. \\ \sum_{i=1}^{k} \omega(x_i)\omega(y_i) & \text{o.w.} \end{cases}$$

Upshot: ρ_h only depends on ω! Call it ρ_ω.

BCJ Maps Are Distinct

Lemma Fix ω = quad. form on H ← mod 2
Let $0 \neq a \in H$.
Then $\exists\ b \in H$ s.t. $\hat{i}(a,b) = 1$ & $\omega(b) = 1$.

Proof. Choose b_0 s.t. $\hat{i}(a,b) = 1$.
If $\omega(b_0) = 1$, done. Suppose not.
Extend a, b_0 to symplectic basis a, b_0, c, d.
Either $\omega(c)$, $\omega(d)$, or $\omega(c+d)$ must $= 1$.
Say $\omega(c) = 1$. Let $b = b_0 + c$. $\square$

Prop. If $\omega \neq \omega'$ then $\rho_\omega \neq \rho_{\omega'}$

Proof. Choose a s.t. $\omega(a) \neq \omega'(a)$.
Choose b as in Lemma.
Realize both by curves that intersect once.
$\rightsquigarrow$ sep curve c
Note $\rho_\omega(T_c) \neq \rho_{\omega'}(T_c)$. $\square$

Arf Zero

Prop. For a Heegaard embedding $S_g \subset S^3$
The Seifert linking form ω has Arf $= 0$.
Conversely, all such ω arise.

Pf. Forward direction. Waldhausen's theorem says all Heegaard surfaces are standard. The standard basis $\{x_i, y_i\}$ has
$\omega(x_i) = \omega(y_i) = 0 \implies \text{Arf} = 0$.

Reverse direction. Arf showed that all quadratic forms with same Arf invt differ by Sp. $\square$

Counting BCJ Maps

So we now want to count quadratic forms with Arf = 0. Counting all quadratic forms is easy:

Fix a basis $e_1, \dots, e_{2g}$.

$$\omega\left(\sum \alpha_i e_i\right) = \sum \alpha_i \omega(e_i) + \sum_{i<j} \alpha_i \alpha_j \hat{\imath}(e_i, e_j)$$

So...

A. 2^{2g} quad forms, determined by values on $\{e_i\}$.

Prop. There are $2^{g-1}(2^g+1)$ BCJ maps.

Proof. Induct on g. $\square$

These are not linearly indep!

Will show they form a $\binom{2g}{3} + \binom{2g}{2}$ dim space.

For open surfaces: $\binom{2g}{3} + \binom{2g}{2} + \underbrace{\binom{2g}{1} + \binom{2g}{0}}_{\text{disk pushing}}$.

The Space of BCJ maps

Let $H = H_1(S_g; \mathbb{Z}/2)$
Will define a $\mathbb{Z}/2$-algebra B based on H.
B is commutative w/ unit 1
has a generator $\bar{x}$ for each $x \in H$,
and relations:

① $\bar{x}^2 = x \quad \forall\ x \in H$

② $\overline{x+y} = \bar{x} + \bar{y} + \hat{\imath}(x,y) \quad \forall\ x,y \in H$

An element of B is a polynomial in the gens
$\rightsquigarrow$ degree
$B_K = \{ b \in B \mid \deg b \leq k \}$ vector space.

Goal: BCJ maps can be assembled to a surjective map $\sigma : \mathcal{I}(S_g^1) \longrightarrow B_3$ with naturality.

For $\mathcal{I}(S_g)$ we get a subspace of B_3, as some elts are 0
$\rightsquigarrow$ subspace of dim $\binom{2g}{3} + \binom{2g}{2}$

The Affine Space of Quadratic Forms

$\Omega = \{$quad. forms on $H\}$

<u>not</u> a vector space.

Still want to define linear functions on it.

For an abelian gp / vector space L,

a torsor / affine space over L

is a transitive free L-space, that is,

a set K with an action

$$+ : L \times K \longrightarrow K$$

with: $\ell_1 + (\ell_2 + k) = (\ell_1 + \ell_2) + k$

$\forall\, k_1, k_2\ \exists!\, \ell$ s.t. $\ell + k_1 = k_2$

So K is like L, but with no base pt.

e.g. $L = \mathbb{Z}/n$, $K =$ vertices of n-gon.

<u>Fact</u>. Ω is an affine space over H^1

In other words:

$$\omega_1, \omega_2 \in \Omega \Longrightarrow \omega_1 - \omega_2 \in H^1$$

$$\omega \in \Omega,\ \theta \in H^1 \Longrightarrow \omega + \theta \in \Omega$$

(exercise).

Polynomial Functions on Affine Spaces

V = vect. sp. over F
U = affine sp. over V
$f : U \to F$ is <u>linear</u> if $\exists$ linear $g : V \to F$
s.t. $f(v+u) = g(v) + f(u)$

Facts
- f linear, c const $\Rightarrow$ $f+c$ linear
- The linear fns on U form a vect sp. $L(U)$ over F.

Fact. $f \in L(U)$ determined by $g \in V^*$ & $f(u_0)$ some fixed $u_0 \in U$.
$\Longrightarrow \dim L(U) = \dim V + 1$.

A polynomial fn on U is a sum of products of linear ones.

Polynomial Functions on Ω

Consider $U = \Omega$, $V \in H'$

$x \in H \rightsquigarrow \bar{x} : \Omega \to \mathbb{Z}/2$

$\bar{x}(\omega) = \omega(x)$

Fact. $\bar{x}$ is linear.

$\{e_i\}$ = basis for H

$\Rightarrow \{\bar{e}_i\} \cup \{1\}$ = basis for $L(\Omega)$

Fact. $\overline{x+y} = \bar{x} + \bar{y} + \hat{\imath}(x,y)$

In $\mathbb{Z}/2$, $a^2 = a \rightsquigarrow$ above relation in B.

A Boolean polynomial is one made of square free monomials

The Arf invt is a quadratic Boolean poly:

$$\alpha = \sum_{i=1}^{g} \bar{x}_i \bar{y}_i$$

BCJ maps as Boolean Polys

Recall $\rho_\omega : \mathcal{I}(S_g) \longrightarrow \mathbb{Z}/2$

Dualize as $\sigma_f : \Omega \to \mathbb{Z}/2$

$\sigma_f(\omega) = \rho_\omega(f)$.

Let $\Psi = \{\omega \in \Omega : \alpha(\omega) = 0\}$

<u>Fact</u>. σ is a homom. from $\mathcal{I}(S_g)$ to vect. space of fns $\Psi \longrightarrow \mathbb{Z}/2$

We can rephrase our calculations:

$$\rho_\omega(T_c) = \sum \omega(x_i)\omega(y_i) \rightsquigarrow \sigma_{T_c}(\omega) = \sum \bar{x}_i(\omega)\bar{y}_i(\omega)$$

$$\rho_\omega(T_a T_b^{-1}) = \begin{cases} 0 & \omega(a) = 1 \\ \sum \omega(x_i)\omega(y_i) & \end{cases} = \left(\sum \omega(x_i)\omega(y_i)\right)\left(\omega(a)+1\right)$$

$$\rightsquigarrow \sigma_{T_a T_b^{-1}}(\omega) = \left(\sum \bar{x}_i(\omega)\bar{y}_i(\omega)\right)\left(\bar{a}+1\right)$$ cubic!

Use naturality to get all of B_3.

Remains to determine which elts are 0 on Ψ.

Answer: (linear functions) $\cdot\, \alpha$

Arf

BCJ for Open Surfaces

We can include S_g^1 into S_{g+1} and play the same game.

Everything works the same except now there are no cubic polys that are 0 on all of ψ.

$\rightsquigarrow$ image of BCJ is $\sum_{i=0}^{3} \binom{2g}{i}$ dim.

Johnson all gives a complete description of the relations among the ρ_ω.

Homology 3-Spheres and Torelli

Fix $S_g \subset S^3$ standard $\rightsquigarrow$ Heegaard splitting.
$f \in \mathrm{Mod}(S_g) \rightsquigarrow$ new manifold $M(f)$: change the gluing of the two handlebodies by f.

<u>Fact</u>. $M(f)$ is a homology $S^3 \iff f \in \mathcal{I}(S_g)$.

Since every closed, orientable 3-manifold is an $M(f)$ for some f (in some $\mathrm{Mod}(S_g)$), all homology 3-spheres arise this way.

Let $\mathcal{K}(S_g) = \ker \tau$

<u>Thm</u> (Morita). Every homology 3-sphere is $M(f)$ for some f in some $\mathcal{K}(S_g)$.

In particular, since Dehn twists correspond to Dehn surgeries and $\mathcal{K}(S_g)$ is gen. by Dehn twists (later in the course), the following graph is connected: vertices - hom. 3-spheres
edges - Dehn surgery on a knot

Handlebody Groups

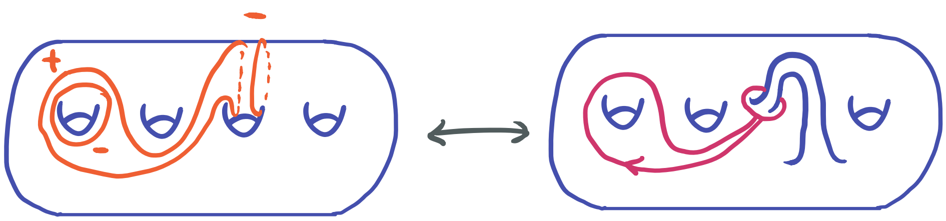
V_g = handlebody, $S_g = \partial V_g$

$H(S_g) = \{ f \in \mathrm{Mod}(S_g) : f \text{ extends over } V_g \}$

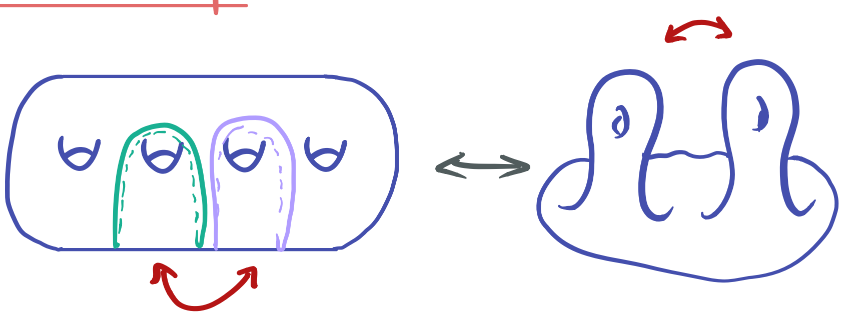
$\leq \mathrm{Mod}(S_g)$

Fact. If $c \subseteq S_g$ bounds a disk in V_g then $T_c \in H(S_g)$. (converse also true.)

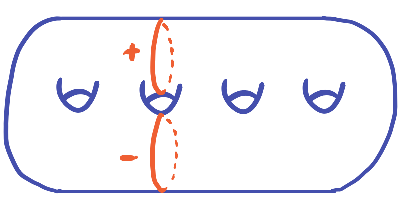
Dragging Feet



Handle Swaps



Bounding Pair Maps



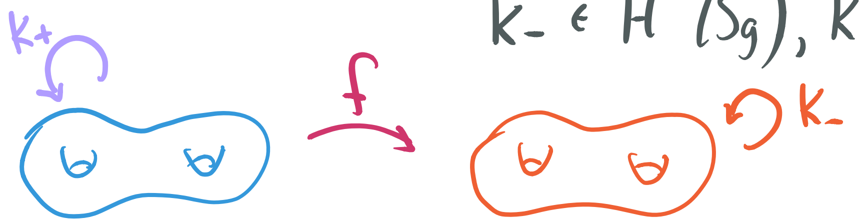
Handlebody Groups & Heegaard Splittings

Say $S_g \subseteq S^3$ Heegaard

$H^+(S_g)$, $H^-(S_g)$ the two handlebody groups.

<u>Fact</u>. $M(f) = M(h) \iff h = k_- f k_+$

$k_- \in H^-(S_g)$, $k_+ \in H^+(S_g)$



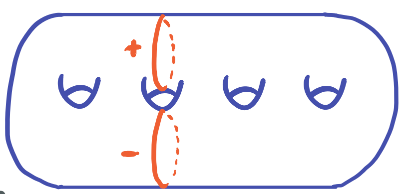
Handlebody Torelli Groups

V_g = handlebody

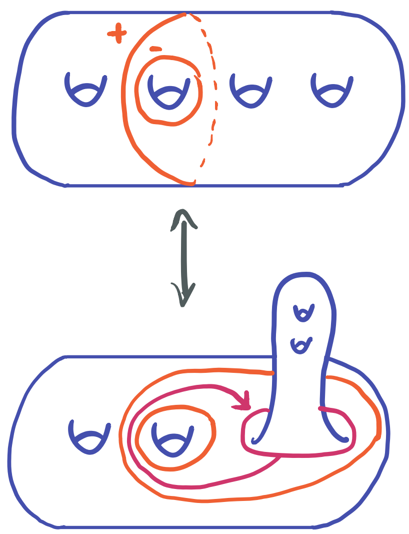
$S_g = \partial V_g$

$HI(S_g) = H(S_g) \cap I(S_g)$

Sample Elements



clearly in $H(S_g)$



Handlebody Torelli Under Johnson

Let $W_y \subseteq \Lambda^3 H$ subspace spanned by basis elts with a y.

Prop. $\tau(\mathcal{HI}(S_g)) = W_y$. (only need $\supseteq$).

Pf. $\boxed{\subseteq}$ $\mathcal{HI}(S_g)$ preserves $\langle\langle \beta_i \rangle\rangle$...

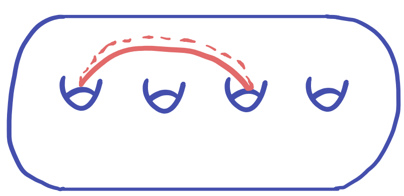
$\boxed{\supseteq}$ By naturality & existence of handle swaps in $\mathcal{H}(S_g)$, enough to exhibit:

$x_1 \wedge y_1 \wedge x_2$, $x_1 \wedge y_1 \wedge y_2$, $x_1 \wedge x_2 \wedge y_3$,
$x_1 \wedge y_2 \wedge y_3$, and $y_1 \wedge y_2 \wedge y_3$

The first 2 are given by the sample elts of $\mathcal{HI}(S_g)$.
The foot drag above acts on H by:

$$y_1 \longmapsto y_1 + y_3, \quad x_3 \longmapsto x_3 - x_1$$

Apply to the first 2 targets gives next 2.
Finally apply the twist:



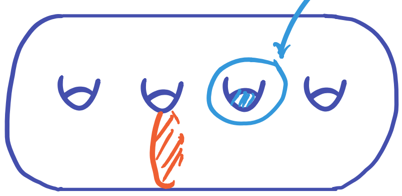
The action on H takes the second target to the sum of the second & fifth. $\square$

The Proof of Morita's Theorem

$S_g \subseteq S^3$ std/ Heegaard.

$H^+(S_g) = H(S_g)$
$H^-(S_g) = \iota H^+(S_g)\iota^{-1}$ where ι_* (action on H) swaps x's & y's.

bounds a disk on other.



bounds a disk on one side

But then $\tau(\langle H^+(S_g), H^-(S_g)\rangle) = W_x \cup W_y$
$= \Lambda^3 H.$

So can modify a given $M(f)$ so that $\tau(f') = 0$.

Johnson II

will write this for $\mathcal{I}(S_g')$

Let $K_g' = \ker \tau : \quad \mathcal{I}_g' \longrightarrow \Lambda^3 H$

$T_g' = \langle T_c : c \text{ sep} \rangle \leq \mathcal{I}(S_g')$

<u>Theorem</u>. $K_g' = T_g'$ & $K_g = T_g$

Strategy. We know $T_g' \leq K_g'$.
So τ factors:

$$\mathcal{I}_g' \to \mathcal{I}_g' / T_g' \twoheadrightarrow \Lambda^3 H \cong \mathcal{I}_g' / \ker \tau$$

Want: 2^{nd} map is $\cong$.
Will show: $\mathcal{I}_g' / T_g' \cong \mathbb{Z}^{\binom{2g}{3}}$

i.e. it ① has $\binom{2g}{3}$ gens & ② is abelian.

Consider the $\binom{2g}{3}$ straight chain maps

$[i \;\; i{+}1 \;\; j \;\; k]$

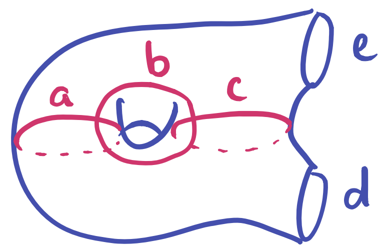
We will first observe that the τ-images span $\Lambda^3 H$, so they at least have a chance of generating $\mathcal{I}_g' / T_g'$.

Purported Generators for $\mathcal{I}_g' / \mathcal{T}_g'$

Prop. The τ-images of $[i\ \ i{+}1\ \ j\ \ k]$ span $\Lambda^3 H$.

Note: these are all BP maps of genus 1.

Observation. The τ-image of the BP map



$T_d T_e^{-1}$

is $a \wedge b \wedge c$ (ignoring signs)

Since: $a + c = d$ in $H \rightsquigarrow$

$$a \wedge b \wedge d = a \wedge b \wedge (a+c) = a \wedge b \wedge c$$

By the definition of the chain maps:

$$\tau([i\ \ i{+}1\ \ j\ \ k]) = c_i \wedge (c_{i+1} + \cdots + c_{j-1}) \wedge (c_j + \cdots + c_{k-1})$$

and these form a basis. Indeed, the $c_i \wedge c_j \wedge c_k$ do.

Here we get $c_i \wedge c_{i+1} \wedge c_{i+2}$ & $c_i \wedge c_{i+1} \wedge (c_{i+2} + c_{i+3})$

$= c_i \cancel{\wedge c_{i+1} \wedge c_{i+2}} + c_i \wedge c_{i+1} \wedge c_{i+3} \rightsquigarrow c_i \wedge c_{i+1} \wedge c_{i+3}$

& $c_i \wedge (c_{i+1} + c_{i+2}) \wedge c_{i+3} = c_i \cancel{\wedge c_{i+1} \wedge c_{i+3}} + c_i \wedge c_{i+2} \wedge c_{i+3}$

A Birman Exact Sequence for K_2'

Will use induction on g, with base case $g=2$.
We already know $\mathcal{I}_2 = \mathcal{T}_2$ and so $K_2 = \mathcal{T}_2$.
Next: $\mathcal{I}(S_2^1)$. Recall:

$$1 \to \pi_1 UT(S_2) \to \mathcal{I}_2' \to \mathcal{I}_2 \to 1$$

Need to determine $\pi_1 UT(S_2) \cap K_2'$.

First, we have:

$$\begin{tikzcd}
1 \arrow[r] & \langle T_\partial \rangle \arrow[r] & \mathrm{Mod}(S_g^n) \arrow[r] & \mathrm{Mod}(S_{g,1}^{n-1}) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u] & \pi_1 UT(S_g^{n-1}) \arrow[r] \arrow[u] & \pi_1(S_g^{n-1}) \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

We know that the $\mathbb{Z}$ is in K_2', so we can ignore it and deal with π_1.

Let $\alpha \in \pi_1(S_2^1)$.

did this when defining τ for closed S_g.

$$\tau(\mathrm{Push}(\alpha)) = [\alpha] \wedge \Theta \qquad \Theta = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$$

$$\Rightarrow \mathrm{Push}(\alpha) \in K_g \Leftrightarrow [\alpha] = 0 \Leftrightarrow \alpha \in \pi'.$$

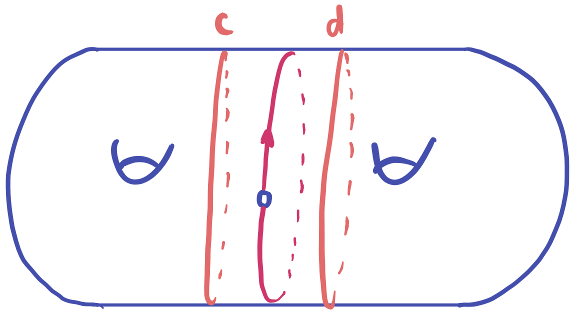
Intuitively: Push $\tilde{\alpha}$ conjugates by α. To be in K_g we must conjugate by a commutator.

Generating the Birman Kernel for K_2'

We just established

$$1 \to \pi_1 S_2' \to K_{2,1} \to K_2 \to 1.$$

We already saw $\pi_1(S_2)'$ is generated by simple separating loops. Now:



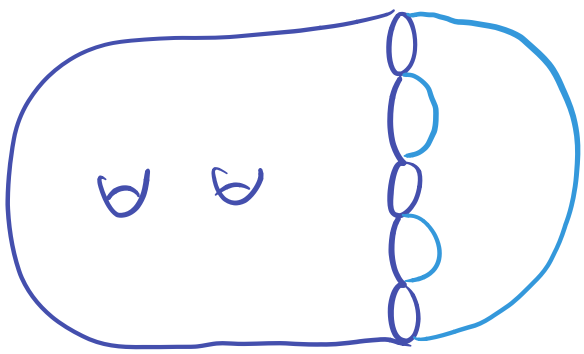
$T_c T_d^{-1}$

So $\pi_1(S_2)' \subseteq T_{2,1}$. It follows that $K_{2,1} \subseteq T_{2,1}$ hence $K_2' \subseteq T_2'$

Using the fact that each Dehn twist about a sep. curve in S_2 can be lifted to a Dehn twist about a sep curve in S_2'.

Two Boundary Components

In this paper, we define $\mathcal{I}_g^n$ as follows.
Embed S_g^n in S_{g+k} s.t. complement is connected.
Then $\mathcal{I}_g^n = \{ f \in \mathcal{I}_{g+k} : \text{supp } f \subseteq S_g^n \}$



This is not the subgroup acting trivially on homology
Eg a Dehn twist about a boundary curve.

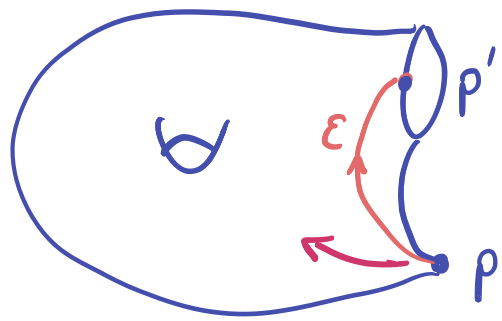
H is the subgroup acting trivially on $H_1(S_g^n, P)$
where P consists of one point in each component
of the boundary

Since the boundary twists are not in $\mathcal{I}_g^n$
we don't lose any info if we do point
pushing in $\mathcal{I}_{g,1}^{n-1}$ instead of $\mathcal{I}_g^n$

A Birman Exact Sequence for K_2^2

As above, suffices to consider

$K'_{2,1} \longrightarrow K'_2$



$P = \{p, p'\}$

An elt of $\mathrm{Mod}(S_{2,1})$ lies in $K_{2,1}$
$\iff$ it acts trivially on
$\pi / [\pi, [\pi, \pi]]$

More precisely if $f(x)x^{-1} \in [\pi, [\pi, \pi]]$ $\forall$
$x \in \pi_1(S'_2, p)$ or $x =$ arc from p to p'.

For $\alpha \in \pi_1(S'_2, p)$, action on π is conjugation for loops based at p & multiplication for arcs from p to p'.

So to act trivially on $\pi/[\pi, [\pi, \pi]]$
need $\alpha \in [\pi, [\pi, \pi]]$.

(So far this is necessary but not suff because of ∂ twisting)

Generating the Birman Kernel for K_2^2

Need to show we can realize each elt of $[\pi, [\pi,\pi]]$ as elt of K_2^2 and at the same time (a forteriori) as elt of T_2^2.

Want generators for $[\pi, [\pi,\pi]]$.
We already said the Mod-orbit of γ_1 generates $[\pi,\pi]$. So $[\pi, [\pi,\pi]]$ is gen. by conjugates of $[x, f(\gamma_1)]$ $\quad x \in \pi$
hence Mod-orbit of $[x, f(\gamma_1)]$

or $\quad [f^{-1}(x), \gamma_1]$

or $\quad [x, \gamma_1]$.

But γ_1 is in T_2^2. As the latter is normal, the commutator $[x,\gamma_1]$ is as well.

This sentence seems to eliminate much of Johnson's argument.

Summary:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S_2^1, p) & \longrightarrow & K_2^2 & \longrightarrow & K_2^1 & \longrightarrow & 1 \\ & & \cap\!\! \mid & & & & & & \\ & & T_2^2 & & & & & & \end{array}$$

Thus $K_2^2 = T_2^2$.

The Induction / Main Argument

As above we consider $\mathcal{I}_g' / \mathcal{T}_g'$

We want ① gen. by $2g$ choose 3 elts

② abelian

Begin with ①. We already listed our elts:

$[i \;\; i{+}1 \;\; j \;\; k] \qquad 1 \leq i < j < k \leq 2g{+}1.$

Let $\mathcal{J}_g' =$ subgp of $\mathrm{Mod}(S_g')/\mathcal{T}_g'$ gen. by these

As in Johnson I, suffices to show:

Prop. $\mathcal{J}_g'$ is normal in $\mathrm{Mod}(S_g')/\mathcal{T}_g'$ $\quad g \geq 2.$

Pf. Base case $g=2$. The $\binom{4}{3} = 4$ elts are

(images of)

$$W_5 = [1\;2\;3\;4]$$
$$W_4 = [1\;2\;3\;5]$$
$$W_3 = [1\;2\;4\;5]$$
$$W_1 = [2\;3\;4\;5]$$

From Johnson I, these & $W_2 = [1\;3\;4\;5]$ generate kernel of $\mathrm{Mod}(S_2') \longrightarrow \mathrm{Mod}(S_2)$.

Also from Johnson I:

$$W_5 W_4^{-1} W_3 W_2^{-1} W_1 = T_\partial \quad ✓$$

Now assume $g \geq 3$. Have a map:

$$\mathcal{J}'_{g-1} \longrightarrow \mathcal{J}'_g$$

This is injective since

Where is the injectivity used?

$$\begin{tikzcd}
\mathcal{J}'_{g-1} \arrow[r] \arrow[d, "\cong \text{ by induction}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{J}'_g \arrow[d] \\
\Lambda^3 H_{g-1} \arrow[r, hook] & \Lambda^3 H_g
\end{tikzcd}$$

So $\mathcal{J}'_k$ & $\mathcal{J}^2_k \hookrightarrow \mathcal{J}'_g$ $\quad 2 \leq k < g$

The proof has 5 parts:

① All straight 3-chain maps in $\mathcal{J}'_g$

② $\mathcal{J}'_g$ is normalized by T_{c_i}

③ $W_7 = [1\,2\,3\,4\,5\,6] \in \mathcal{J}^2_2 \subseteq \mathcal{J}'_g$

④ $W_1 = [2\,3\,4\,5\,6\,7]$, $W_2 = [1\,3\,4\,5\,6\,7]$, $\ldots$, $W_6 = [1\,2\,3\,4\,5\,7]$ & $T_b * W_1$, hence all 5-chain maps (= push maps) in $\mathcal{J}'_g$

⑤ T_b normalizes $\mathcal{J}'_g$

② & ⑤ imply the theorem.

① used for ②

③ used in ④ used in ⑤

Proof of ① Want all $[i\ j\ k\ l]$.
Induct on $j-i$. Base case $j-i=1$ vacuous.
The 5-chain $(i\ i{+}1\ j\ k\ l)$ gives an S_2^1.
$[i\ j\ k\ l]$ is a push map. But push maps are gen. by $[i\ i{+}1\ j\ k]$, $[i\ i{+}1\ j\ l]$, $[i\ i{+}1\ k\ l]$ & $[i{+}1\ j\ k\ l]$ & T_∂.
All these BPs have "shorter" first curve.

Proof of ② In Johnson I we showed the T_{c_i} normalize the group gen. by straight 3-chains.

Proof of ③ $W_7 = [1\ 2\ 3\ 4\ 5\ 6]$ lives on S_2^2
So it suffices to find a product of 3-chains in there whose product has same τ-image as W_7:
$[1\ 2\ 3\ 4][3\ 4\ 5\ 6][1\ 2\ 3\ 6][1\ 2\ 3\ 5]^{-1}$
$[1\ 4\ 5\ 6][2\ 4\ 5\ 6]^{-1}$.

Here is where we use $K_2^2 = T_2^{\,2}$

Proof of ④ Can hit W_7 with the T_{c_i} to get the other W_i (use part ②).
Another relation from Johnson I:

$$T_6 * W_1 = W_4 W_3^{-1} W_2 T_{\partial_3} \leftarrow \partial S_3^1$$

Proof of ⑤ As in Johnson I, reduce to $T_b * f$ where f is a consecutive 3-chain (controlled change of coords).
$\rightsquigarrow$ 3 nontrivial cases: [2 3 4 5], [3 4 5 6], & [4 5 6 7]

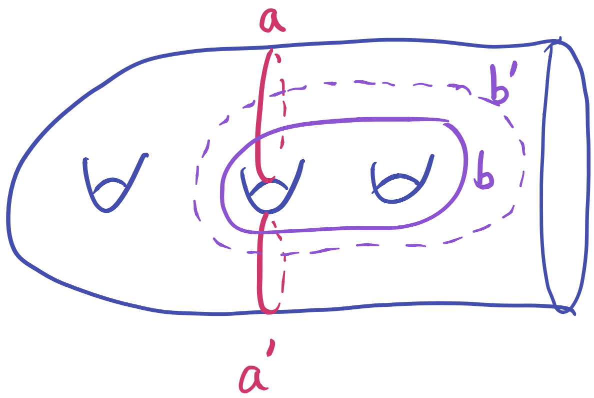
The first lies in S_2^1 ✓

The other required relations are in Johnson I.

Prop. $\mathcal{I}_g'/\mathcal{T}_g'$ is abelian

Proof. Want: all straight 3 chains centralize $\mathcal{I}_g'/\mathcal{T}_g'$.
But: $Z(\mathcal{I}_g'/\mathcal{T}_g')$ is characteristic in $\mathcal{I}_g'/\mathcal{T}_g'$, hence normal in $\mathrm{Mod}(S_g')/\mathcal{T}_g'$.
Thus: suffices to show [1 2 3 4] normalizes $\mathcal{I}_g'/\mathcal{T}_g'$.
i.e. $[[1\ 2\ 3\ 4], f]$ $\forall f \in$ gen set for $\mathcal{I}_g'/\mathcal{T}_g'$
Controlled change of coords $\rightsquigarrow$ 3 nontrivial cases:
[2 3 4 5], [3 4 5 6], [4 5 6 7]
First two are in S_2^1 & S_2^2, & image of a commutator is trivial under τ ✓

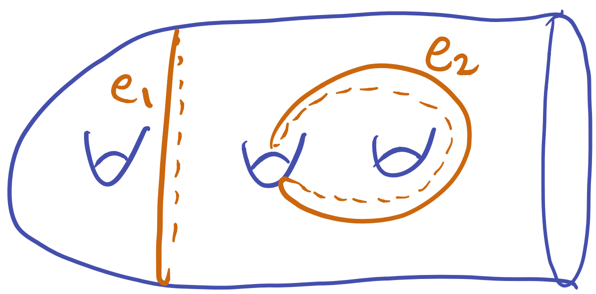
Last one is the following commutator:



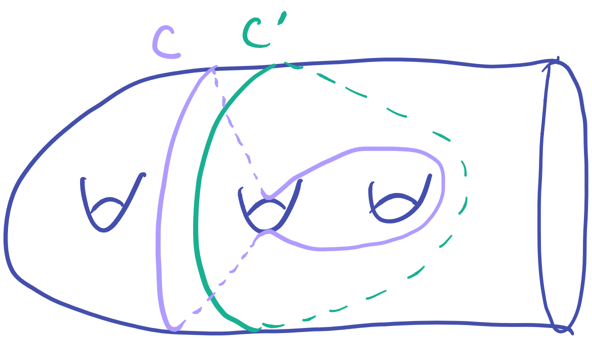
$$[T_a T_{a'}^{-1}, T_b T_{b'}^{-1}]$$

We use a lantern relation:

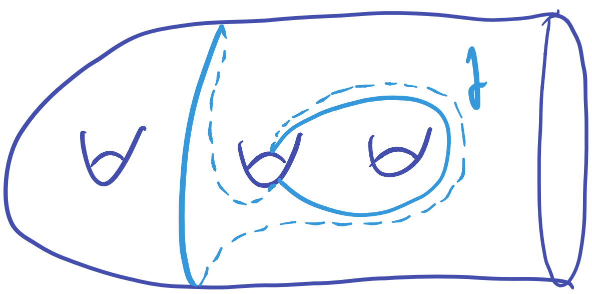
$$T_c T_{b'} T_d = T_b T_{c'} T_{e_1} T_{e_2}$$



b' is a band surgery of c' & e_1



$T_{a'} T_a^{-1} (\{b, b'\})$



Band surgery of e_1 & e_2

Rewrite as: $T_{c'}^{-1} T_b^{-1} T_c T_{b'} = T_d^{-1} T_{e_1} T_{e_2}$

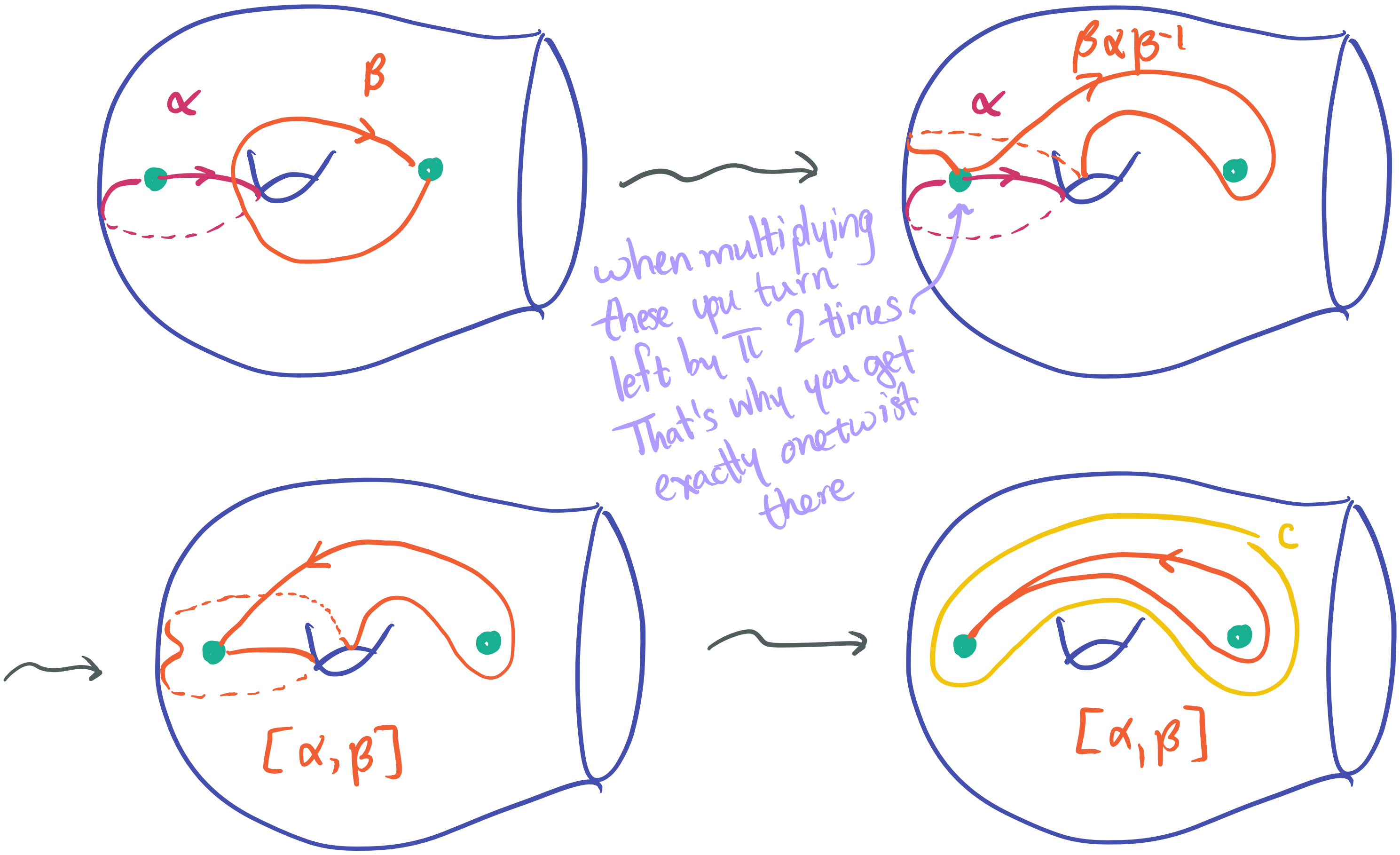
$$\begin{aligned} T_{c'}^{-1} T_b^{-1} T_c T_{b'} &= T_c T_{c'}^{-1} T_{b'} T_b^{-1} \\ &= [T_a^{-1} T_{a'}, T_b T_{b'}^{-1}] \end{aligned}$$

So the desired commutator is trivial in $\mathcal{I}(S_g^1)/K(S_g^1)$.

This relation will be used in Johnson III to show that the image of $K(S_g^1)$ is $\mathcal{I}(S_g^1)^{ab}$ is generated by images of genus 1 maps. In $\mathcal{I}(S_g^1)^{ab}$ the LHS is trivial and RHS gives $T_{e_1} T_{e_2} = T_d$ is $\mathcal{I}(S_g^1)^{ab}$.

AN INTUITIVE VERSION OF THE LAST RELATION

After crushing two handles:

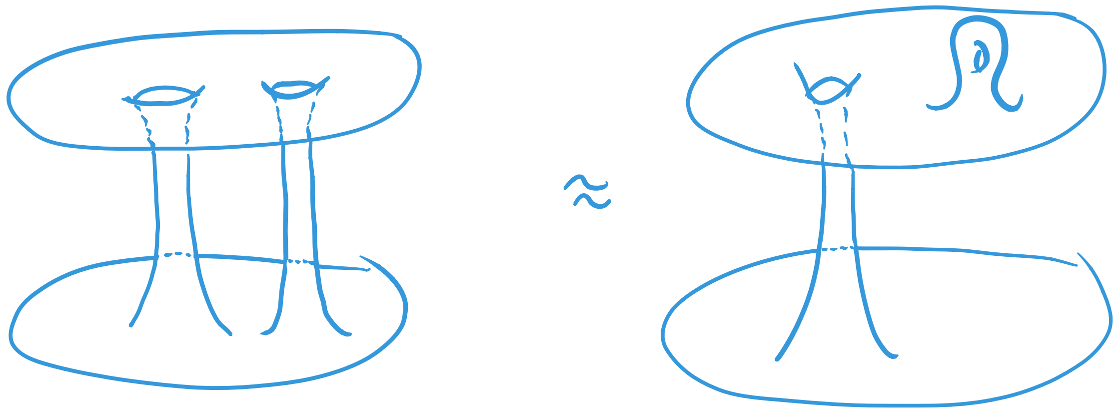


So back in S_3^1 the original commutator of BP maps is a multitwist about two genus one twists and a single genus 2 twist T_c & so the commutator is in $K(S_g^1)$.

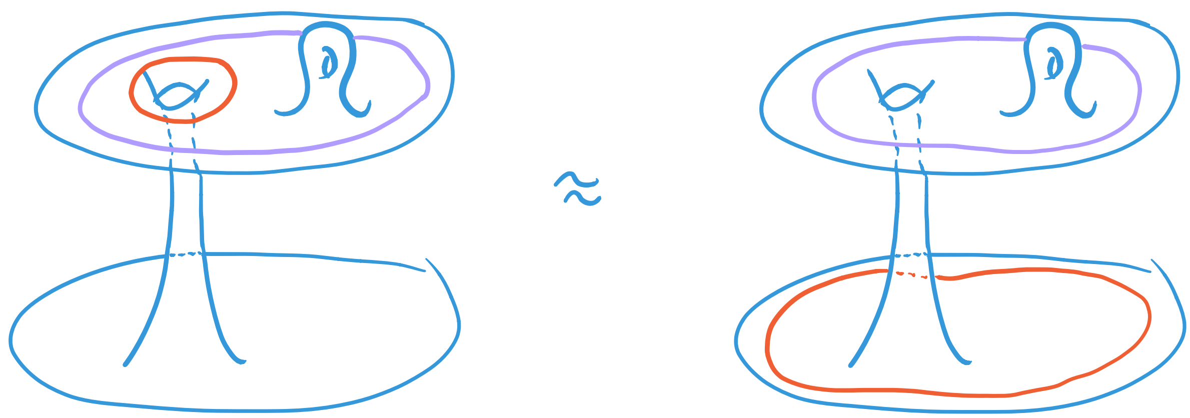
For Johnson III all we need is that in $\mathcal{I}(S_g^1)^{ab}$ T_c is a product of genus 1 twists.

A Finer Point from Previous Page

To get the first picture from Johnson's picture, need to believe:

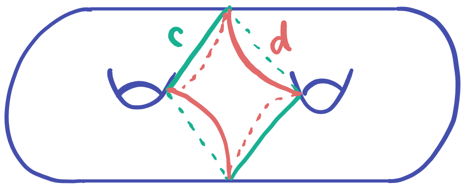


and



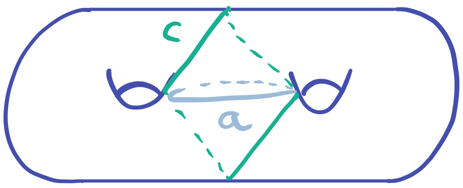
Aside: A Question of Johnson

Q. What is the normal closure in $\mathrm{Mod}(S_2)$?



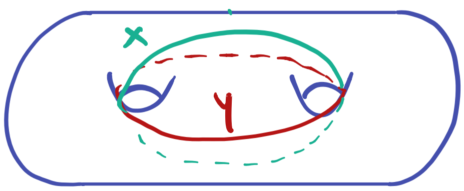
$T_c T_d^{-1}$

or:



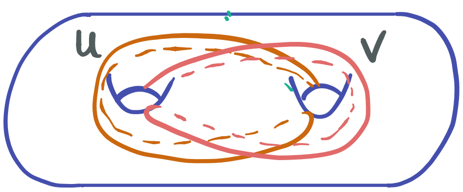
$[T_c, T_a]$

or:



$[T_x, T_y]$

Two different curves complete the lantern:



Lantern relation $\Rightarrow T_c T_d^{-1} = T_u T_v^{-1}$

$\Rightarrow T_c T_d^{-1} \in \operatorname{Ker} \mathcal{I}(S_2) \to \mathbb{Z}$

In fact it equals the kernel: By connectedness of the complex with vertices = sep curves & edges for $i=4$, all gens for $\mathcal{I}(S_2)$ have same image in $\mathcal{I}(S_2)/\langle\langle T_c T_d^{-1} \rangle\rangle$.

Johnson III: The Abelianization of Torelli

Main Goal: $\mathcal{I}(S_g^1)^{ab} \cong \mathbb{Z}^{\binom{2g}{3}} \oplus \mathbb{Z}/2^{\binom{2g}{2}+\binom{2g}{1}+\binom{2g}{0}}$

and: $\mathcal{I}(S_g)^{ab} \cong \mathbb{Z}^{\binom{2g}{3}-\binom{2g}{1}} \oplus \mathbb{Z}/2^{\binom{2g}{2}+\binom{2g}{1}}$

These are isomorphisms as abelian groups, but not as Sp-reps. To understand $\mathcal{I}(S_g^1)^{ab}$ as an Sp-rep, we need some setup.

Let $A_g^1 = \mathcal{I}(S_g^1)^{ab} = H_1(\mathcal{I}(S_g^1);\mathbb{Z})$
U_g^1 = mod 2 abelianization = $H_1(\mathcal{I}(S_g^1);\mathbb{Z}/2)$
T_g^1 = image of $K(S_g^1)$ in A_g^1
B_g^3 = boolean polys of deg 3 in H

Clearly $\mathrm{Mod}(S_g^1) \curvearrowright A_g^1$ & $\mathcal{I}(S_g^1)$ acts trivially
$\Longrightarrow A_g^1$ is an Sp-rep

The main work of the paper is:

Prop 1. $T_g^1 \cong \bigoplus^n \mathbb{Z}/2$

Prop 2. $U_g^1 \cong B_g^3$

Note: There is automatically a $\sigma: U_g^1 \twoheadrightarrow B_g^3$ by univ. property of U_g^1

Some Setup

Fact 1. The seq.

$$0 \to T_g' \longrightarrow A_g' \longrightarrow \Lambda^3 H \longrightarrow 0$$

is split exact.

Pf. We know

$$1 \longrightarrow K(S_g') \longrightarrow I(S_g') \xrightarrow{\tau} \Lambda^3 H \longrightarrow 0$$

is exact. Divide by $I(S_g')'$ to get the desired seq. It is split since $\Lambda^3 H$ is free. $\square$

Fact 2. $U_g' \cong A_g'/2A_g' \cong A_g' \otimes \mathbb{Z}/2$

Thus there are induced maps

$$T_g' \to U_g' \ , \qquad U_g' \xrightarrow{\tau \otimes \mathbb{Z}/2} \Lambda^3 H \bmod 2 \qquad \square$$

Fact 3. There is a split exact seq.

$$0 \to T_g' \longrightarrow U_g' \longrightarrow \Lambda^3 H \bmod 2$$

Pf. Tensor Fact 1 with $\mathbb{Z}/2$, use Fact 2, & Prop 1 above $\Rightarrow T_g' \otimes \mathbb{Z}/2 \cong T_g'$ $\square$

Note: Can regard T_g' as subgp of A_g' or U_g'

Statement of the Main Theorem

Theorem. The square

$$\begin{tikzcd}[column sep=small] & A_g^1 \arrow[dl, "\otimes \mathbb{Z}/2"'] \arrow[dr, "\tau"] & \\ B_g^3 \cong U_g^1 \arrow[dr, "\tau"'] & & \Lambda^3 H \arrow[dl, "\otimes \mathbb{Z}/2"] \\ & \Lambda^3 H \bmod 2 & \end{tikzcd}$$

is a pull back diagram.

Again, the point is that this diagram makes sense as an Sp-rep.

In general, the pullback of $\begin{tikzcd}[column sep=small] A \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$

is $D = \{(a,b) \in A \times B : p(a) = q(b)\} \leq A \times B$

This D is universal in that any other D' factors through D:

$$\begin{tikzcd}[column sep=small] & D' \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & D \arrow[dl] \arrow[dr] & \\ A \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

Example: Pullback bundles.

We think of D as combining the fibers of p & q together.

In the theorem, the two maps at the top are labeled. So the theorem means:

The map $\mathcal{A}_g' \longrightarrow \mathcal{U}_g' \oplus \Lambda^3 H$

$$x \longmapsto (x \otimes \mathbb{Z}/2, \tau(x))$$

is injective & image is

$$\{(u,\lambda) : \tau(u) = \lambda \otimes \mathbb{Z}/2\}$$

Pf (assuming Props 1 & 2)

Injectivity. Suppose $(x \otimes \mathbb{Z}/2, \tau(x)) = 0$

$\tau(x) = 0 \Longrightarrow x \in \mathcal{T}_g'$ by Fact 1.

Fact 3 $\Longrightarrow x = 0$.

Surjectivity. Let (u,λ) with $\tau(u) = \lambda \bmod 2$.

Fact 1 $\Rightarrow \tau$ surj $\rightsquigarrow f_1 \in \mathcal{A}$ s.t. $\tau(f_1) = \lambda$

$\Rightarrow \tau(f_1) = \tau(u) \bmod 2 \Rightarrow \tau(f_1 \otimes \mathbb{Z}/2 - u) = 0$

Fact 3 $\Longrightarrow f_1 \otimes \mathbb{Z}/2 - u = t \in \mathcal{T}_g'$

By Fact 1, may consider t as elt of $\mathcal{A}$.

Let $f = f_1 - t \rightsquigarrow$ image in $\mathcal{U}$ is

$f_1 \otimes \mathbb{Z}/2 - t = u$ & $\tau(f) = \tau(f_1) = \lambda$ $\square$

Conjugacy Relations

Working towards Prop 1. Need to know when Dehn twists & BP maps are conjugate in $\mathcal{I}(S_g')$.

Homology Chains. A chain in H or $H \bmod 2$ is a seq $(c_1, \ldots, c_n)$ s.t.

(a) $\hat{i}(c_i, c_{i+1}) = 1$ & $\hat{i}(c_i, c_j) = 0$ o.w.

(b) If n odd $c_1 + c_3 + \cdots + c_n$ primitive

Twists. For n even, get a well-def elt of $\mathcal{A}_g'$, call it $[c_1, \ldots, c_n]$.

Indeed: homology chains $\rightsquigarrow$ symplectic subspaces $\rightsquigarrow$ Dehn twists up to conj in $\mathcal{I}(S_g')$.

BP maps. For n odd, again get a well-def elt of $\mathcal{A}_g'$ called $[c_1, \ldots, c_n]$ or $[c_1, \ldots, c_{n-1} \mid c_1 + c_3 + \cdots + c_n]$

Indeed: homology chains $\rightsquigarrow$ geometric chains $\rightsquigarrow$ BP map, unique up to conj in $\mathcal{I}(S_g')$.

Note: $[c_1, c_2 \mid -c_3] = -[c_1, c_2 \mid c_3]$ (get inverse BP map).

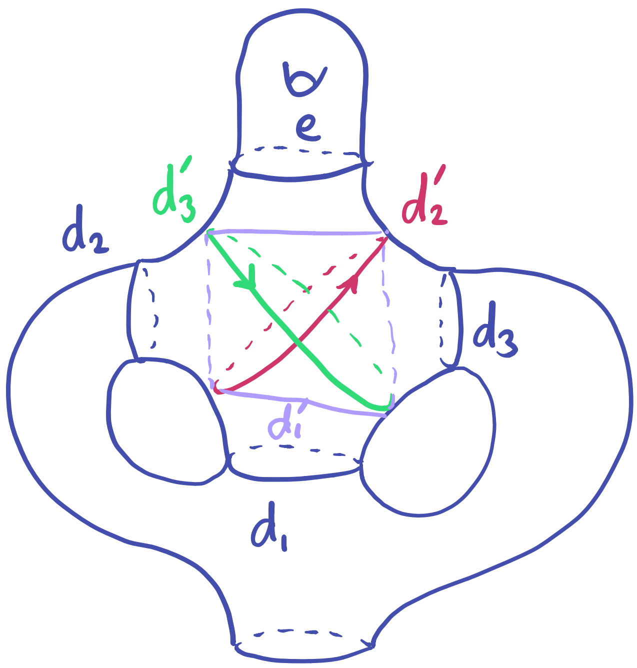
Naturality. For $h \in Sp$: $h * [c_1, \ldots, c_n] = [h(c_1), \ldots, h(c_n)]$

Dehn Twists are 2-Torsion

Lemma 1. If $\hat{i}(a,b)=1$ & $\langle d_1,d_2\rangle$ rationally closed in $\langle a,b\rangle^\perp$ & $\hat{i}(d_1,d_2)=0$ then in A_g'

$$[a,b\,|\,d_1+d_2]-[a,b\,|\,d_1]-[a,b\,|\,d_2]=[a,b]$$

Pf. Consider the lantern relation



$$T_e=(T_{d_3'}T_{d_3}^{-1})(T_{d_2'}T_{d_2}^{-1})(T_{d_1'}T_{d_1}^{-1})$$

Let $a,b=$ symp. basis for e-handle

In A:
$$T_{d_1'}T_{d_1}^{-1}=-[a,b\,|\,d_1]$$
$$T_{d_2'}T_{d_2}^{-1}=-[a,b\,|\,d_2]$$
$$T_{d_3'}T_{d_3}^{-1}=-[a,b\,|\,d_3]=[a,b\,|\,d_1+d_2]$$
$$T_e=[a,b]$$

$\square$

Lemma 2. If $\hat{i}(a,b)=1$ then $2[a,b]=0$ in $\mathcal{A}_g'$.

Pf. By change of coords, can replace d_i with $-d_i$ in Lemma 1. Thus plus $-[a,b|x]=[a,b|-x]$ gives
$$\begin{aligned}[a,b] &= -[a,b|d_1+d_2]+[a,b|d_1]+[a,b|d_2]\\ &= -[a,b]\end{aligned}$$ $\square$

Lemma 3. If a_1,b_1,a_2,b_2 is a symplectic subspace of H then $[a_1,b_1,a_2,b_2]=[a_1,b_1]+[a_2,b_2]$ in $\mathcal{A}_g'$.

Pf. See Johnson II, Cor to Thm 4B.

We have thus shown:

$\mathcal{T}_g'$ is a $\mathbb{Z}/2$ vector space, and it is generated by the 2-chain maps $[a,b]$

The Action of Level 2 on $\mathcal{U}$ is Trivial

$$\mathcal{M}_g^1[2] = \mathrm{Mod}(S_g^1)[2]$$
$$\mathrm{Sp}_g[2] = \mathrm{Sp}_{2g}(\mathbb{Z})[2]$$

Prop. $\mathcal{M}_g^1[2]$ acts trivially on $\mathcal{U}_g^1$.

If we believe that $\mathcal{U}_g^1 \cong B_g^3$ then this must be true!

Since $\mathcal{I}_g^1$ acts trivially on $\mathcal{U}$, the prop could be:

$\mathrm{Sp}_{2g}[2]$ acts trivially on $\mathcal{U}_g^1$

Prop implies $\mathcal{U}_g^1$ is an $\mathrm{Sp}_g(\mathbb{Z}/2)$-module.

$\rightsquigarrow$ projection $\tau : \mathcal{U}_g^1 \to B_g^3$ is $\mathrm{Sp}_g(\mathbb{Z}/2)$-module homomorphism.

Lemma. $\mathrm{Sp}_{2g}[2]$ is generated by squares of transvections

Cor. $\mathrm{Sp}_{2g}[2]$ is the normal closure in $\mathrm{Sp}_{2g}(\mathbb{Z})$ of any square transvection.

Pf of Prop. The kernel of $Sp_{2g}(\mathbb{Z}) \longrightarrow \operatorname{Aut} \mathcal{U}$ is normal in Sp. $\rightsquigarrow$ suffices to show a single square transv. is in kernel.

Suffices: $T_{c_1}^2$ acts trivially on images of straight & β-chain gens

β-chains: trivial since c_1 disjoint.

Straight chains: by controlled change of coords, reduce to consecutive chains starting w/ 2, namely $[2\ 3 \cdots 2k{+}1]$. Will write here for $k=1$, but proof works in general.

Johnson I: In $\mathcal{I}_g'$

$$T_{c_1} * [2\ 3][2\ 3]^{-1} = \left(T_{c_1}^{-1} * [2\ 3] \cdot [2\ 3]^{-1}\right)^{-1}$$

$\mathcal{U}_g'$ is abelian, so in $\mathcal{U}_g'$:

$$T_{c_1} * [2\ 3] = T_{c_1}^{-1} * [2\ 3] \quad \cancel{[2\ 3]^2}$$

$$T_{c_1}^2 * [2\ 3] = [2\ 3] \qquad \square$$

Cor. $[\mathcal{M}_g'[2], \mathcal{I}_g'] = (\mathcal{I}_g')^2$

Pf. $\boxed{\subseteq}$ Immediate from prop

$\boxed{\supseteq}$ Enough to show $\mathcal{I}_g'/[\mathcal{M}_g'[2], \mathcal{I}_g']$ is a $\mathbb{Z}/2$ vector space (since $\mathcal{U}_g'$ is universal).
It is abelian since $(\mathcal{I}_g')' \subseteq [\mathcal{M}_g'[2], \mathcal{I}_g']$.
It is gen. by images of BP maps.
So: suffices to show the square of any BP map is in $[\mathcal{M}_g'[2], \mathcal{I}_g']$.

In particular, we have: $\mathcal{U}_g' = \mathcal{I}_g' / [\mathcal{M}_g'[2], \mathcal{I}_g']$

follows from Prop $\longrightarrow$ and: $\mathcal{U}_g'$ is an $\mathrm{Sp}_g(\mathbb{Z}/2)$-module

Any chain $(c_1, \dots, c_n)$ in H mod 2 determines a unique element $[c_1 \cdots c_n]$ in $\mathcal{U}_g'$. This is in $\mathcal{I}_g'$ if n is even.

As usual, have naturality. So

$$\sigma : \mathcal{U}_g' \longrightarrow B_g^3 \text{ is an } \mathrm{Sp}_g(\mathbb{Z}/2)\text{-module hom.}$$

σ IS AN ISOMORPHISM FOR $g=3$.

It remains to show σ is an isomorphism.

Lemma. $\sigma : U_3' \longrightarrow B_3^3$ is an isomorphism.

Proof. Johnson I: I_3' is gen. by 42 elts
$\Rightarrow \dim U_3' \leq 42$
But σ is surjective &

$$\dim B_3^3 = \sum_{i=0}^{3} \binom{6}{i} = 20+15+6+1 = 42 \qquad \square$$

Strategy for general case: Use Sp-module structure to show $\ker \sigma$ is generated by elements of U_3', then apply the lemma.

Subalgebras of $\mathcal{B}_g^3$ from Subsurfaces.

Let X = symplectic subsp. of H mod 2.
$\rightsquigarrow \mathcal{B}_X^k$ = k-nomials in X.
$\rightsquigarrow \mathcal{B}_X^k \to \mathcal{B}_g^k$ injective.

Let $S_k^1 \subseteq S_g^1$ any subsurf of genus k
$\rightsquigarrow \mathcal{I}_k^1 \to \mathcal{I}_g^1$
$\rightsquigarrow \mathcal{U}_k^1 \to \mathcal{U}_g^1$

Lemma. (1) Image only depends on $H_1(S_k^1; \mathbb{Z}/2)$.
(2) For $k=3$, the map $\mathcal{U}_3^1 \to \mathcal{U}_g^1$ is injective.

Pf. (1) $\mathcal{M}_g^1[2]$ acts trivially on $\mathcal{U}_g^1$
(2) Commutativity of

$$\begin{tikzcd} \mathcal{U}_3^1 \arrow[r, "\cong"] \arrow[d] & \mathcal{B}_3^3 \arrow[d, hook] \\ \mathcal{U}_g^1 \arrow[r] & \mathcal{B}_g^3 \end{tikzcd}$$

$\square$

So for X as above, makes sense to define
$\mathcal{U}_X$ = image of $\mathcal{U}_k^1 \to \mathcal{U}_g^1$.

Maybe this page is not so essential for understanding

Carrying

X = $2k$-dim symp. subsp. of H mod 2

Say X carries $f \in \mathcal{U}_k'$ if $f \in \mathcal{U}_X$

i.e. $\exists\, S_k^1 \subseteq S_g^1$ s.t. $H_1(S_k^1; \mathbb{Z}/2) = X$

& $\tilde{f}$ with $\mathrm{supp}(\tilde{f}) = S_k^1$ s.t.

$\tilde{f} \longmapsto f$

Cor. Say $\dim X \le 6$. If $f, h \in \mathcal{U}_g'$ carried by X, then $f = g \iff \sigma(f) = \sigma(g)$ in B_g^3.

Pf. Extend X so $\dim X = 6$.

$\Rightarrow \sigma : \mathcal{U}_X \longrightarrow B_X^3$ is an isomorphism. $\square$

Note. $h \in Sp_g[2]$, $f \in \mathcal{U}_g'$, f carried by X

$\Rightarrow h * f$ carried by $h(X)$.

Maybe this page is not so essential for understanding

σ IS AN ISOMORPHISM

Basic outline. ① Show $\mathcal{T} = \mathcal{T}_g^1$ is ker of

$$\sigma : \mathcal{U}_g^1 \longrightarrow B_g^3 / B_g^2$$

Have gens for $\mathcal{T}_g^1$: 2-chain maps

② Find gens for S = Kernel of

$$\sigma : \mathcal{U}_g^1 \longrightarrow B_g^3 / B_g^1$$

$$= \ker \quad \sigma : \mathcal{T}_g^1 \longrightarrow B_g^3 / B_g^1$$

As a module over $Sp_g^1[2]$ it is gen. by

$$\partial(a,b,c) = [b,c] - [a+b,c] + [a,b+c] - [a,b]$$

for any 3-chain (a,b,c).

(Check this is really in the kernel!)

③ Find gens for R = kernel of

$$\sigma : \mathcal{U}_g^1 \longrightarrow B_g^3 / B_g^0$$

$$= \ker \quad \sigma : S \longrightarrow B_g^3 / B_g^0$$

If we rename $\partial(a,b,c)$ as $[a+c]$ (since, it turns out, the element only depends on this sum) then R is gen by $\partial(x,y) = [x+y] - [x] - [y]$ for any 2-chain (x,y).

④ $R = \mathbb{Z}/2$ & $\sigma : R \longrightarrow B_g^0$ is $\cong$. That's it!

STEP ①

Lemma. kernel of $\mathcal{U}_g^1 \longrightarrow B_g^3 / B_g^2 = \Lambda^3 H \bmod 2$ is $\mathcal{T}_g^1$

Proof. In his paper defining τ, Johnson shows:

$$\begin{tikzcd} A_g^1 \arrow[r, "\tau"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & \Lambda^3 H \arrow[d, "\otimes \mathbb{Z}/2"] \\ B_g^3 \arrow[r] & \Lambda^3 H \bmod 2 \end{tikzcd}$$

Just check on generators

Now tensor with $\mathbb{Z}/2$:

$$\begin{tikzcd} \mathcal{U}_g^1 \arrow[d, "\sigma"'] \arrow[dr, "\tau \otimes \mathbb{Z}/2"] \arrow[dr, phantom, "\circlearrowright", shift right=3] & \\ B_g^3 \arrow[r] & \Lambda^3 H \bmod 2 \cong B_g^3 / B_g^2 \end{tikzcd}$$

So $\tau \otimes \mathbb{Z}/2$ is same as $\mathcal{U}_g^1 \to B_g^3 \to B_g^3 / B_g^2$

By Fact 3, kernel is $\mathcal{T}_g^1$ $\square$

STEP ②

Let's first check $\partial(a,b,c)$ is in $\ker\left(U_g^1 \longrightarrow B_g^3/B_g^1\right)$

$$\partial(a,b,c) = [b,c] - [a+b,c] + [a,b+c] - [a,b]$$

Why the signs if T_g^1 is a $\mathbb{Z}/2$ vect sp?

$$\rightsquigarrow \bar{b}\bar{c} + \overline{a+b}\,\bar{c} + \bar{a}\,\overline{b+c} + \bar{a}\bar{b}$$

$$= \bar{b}\bar{c} + (\bar{a}+\bar{b}+1)\bar{c} + \bar{a}(\bar{b}+\bar{c}+1) + \bar{a}\bar{b}$$

$$= \cancel{\bar{b}\bar{c}} + \cancel{\bar{a}\bar{c}} + \cancel{\bar{b}\bar{c}} + \bar{c} + \cancel{\bar{a}\bar{b}} + \cancel{\bar{a}\bar{c}} + \bar{a} + \cancel{\bar{a}\bar{b}}$$

$$= \bar{a} + \bar{c} \qquad \underline{\text{linear}} \text{ in } B_g^1.$$

Since σ is $\cong$ for $g=3$: $\partial(a,b,c)$ only dep. on $a+c$ for $g=3$ (also true in general).

Lemma. S is gen (as a module) by any one $\partial(a,b,c)$.

Pf for $g=3$. In this case $\sigma : T_g^1 \xrightarrow{\cong} B_g^2$

and so $\sigma : S \xrightarrow{\cong} B_g^1$

So enough to show $\sigma(\partial(a,b,c))$ is a module gen for B_g^1. But a module gen. for B_g^1 is any non-0 vector. By above calculation $\sigma(\partial(a,b,c)) = \bar{a}+\bar{c}$. This is non-0 by defn of a chain. $\square$

Note: the $g=3$ case is a waste of time since we already know σ is $\cong$ here. But it gives the idea.

For $g>3$. Use naturality & move stuff around.

STEP (3)

As above $\partial(a,b,c) = [e]$ where $e = a+c$.

Have naturality: $h \in Sp_g[2] \Longrightarrow h * [e] = [h(e)]$.
(this actually requires e to live in Sp space of dim ≥ 4 for change of coords).

Check $\partial(x,y)$ in $\ker \mathcal{U}_g^1 \longrightarrow \mathcal{B}_g^3 / \mathcal{B}_g^0$

$\partial(x,y) = [x+y] - [x] - [y] = \hat{\imath}(x,y) \in \mathcal{B}_g^0$. ✓

Lemma. $\partial(x,y) = 0$ in $\mathcal{U}_g^1$ if $\hat{\imath}(x,y) = 0$
$\partial(x,y)$ indep of x,y if $\hat{\imath}(x,y) = 0$
In particular, $R \cong \mathbb{Z}/2$ & τ is $\cong$.

Pf. Make x,y sit in genus 3 subsurf. by change of coords.

This (more or less) proves the theorem. $\square$

Generating the Johnson Kernel

Thm (Ershov–He '17, Church–Ershov–Putman '17)
$K(S_g)$ is finitely generated for $g \geq 4$.

Notes. ① Not known for $g=3$.
② No explicit finite gen set is known.

History ① Biss–Farb '05 $K(S_g)$ is not f.g.
② Erratum '09
③ Dimça–Papadima '13: $K(S_g)^{ab}$ is f.g.
④ Morita–Sakasai–Suzuki '17: $K(S_g)^{ab}$ has $O(g^5)$ generators (after EH & CEP).

The CEP results are more general:

Main Theorem. For $k \geq 3$, $g \geq 2k+1$, $b \in \{0,1\}$ every subgp of $\mathcal{I}_g^b$ containing k^{th} term of LCS of $\mathcal{I}_g^b$ is fin. gen.

$k=2$ case: subgps containing $[\mathcal{I}_g^b, \mathcal{I}_g^b]$ also proved for $g=4$.

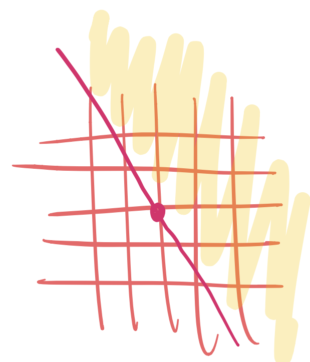
Special case: terms of Johnson filtration.

Bieri-Neumann-Strebel Invariants

G = group $\rightsquigarrow$ $G^* = \mathrm{Hom}(G, \mathbb{R})$

$\Sigma(G) = \{\rho \in G^* :$ the sugraph of the Cayley graph of G spanned by $g \in G$ with $\rho(G) \geq 0$ is connected$\}$

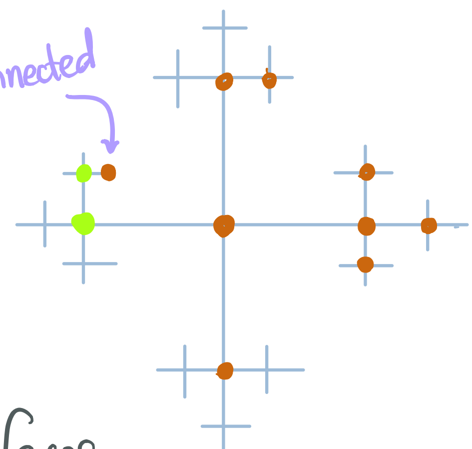
Examples. ① $\Sigma(\mathbb{Z}^n) = (\mathbb{Z}^n)^*$



② $\Sigma(F_2) = \{0\}$

e.g. consider $\rho : F_2 \longrightarrow \mathbb{R}$

$\rho(a) = 1,\ \rho(b) = 0$



③ M = hyperbolic 3-manifold.

$\Sigma(\pi_1(M))$ = cone on open fibered faces.

Theorem (BNS). Say G = fin gen group & $[G,G] \leq H \leq G$

Then: H f.g. $\iff \{\rho \in G^* : \rho|_H = 0\} \subseteq \Sigma(G)$

Prop. $\Sigma(G)$ is invariant under $\mathrm{Aut}(G)$.

Proof Outline

Will show $\Sigma(\mathcal{I}_g) = \mathcal{I}_g^*$ $\quad g \geq 4$.
Then apply above theorem.

Claim 1. $\exists$ finite gen set S for $\mathcal{I}_g$ consisting of genus 1 BPs s.t.

$$\{\rho \in \mathcal{I}_g^* : \rho(s) \neq 0 \ \forall s \in S\} \subseteq \Sigma(\mathcal{I}_g)$$

For next step note $\mathrm{Mod}(S_g) \circlearrowright \mathcal{I}_g$

$$\rightsquigarrow \mathrm{Mod}(S_g) \circlearrowright \mathcal{I}_g^*$$

$$\rightsquigarrow \mathrm{Mod}(S_g) \longrightarrow GL(\mathcal{I}_g^*)$$

Pull back the Zariski topology to $\mathrm{Mod}(S_g)$.

Claim 2. In this topology, $\mathrm{Mod}(S_g)$ is an irreducible space (i.e. it is not the union of proper closed subspaces.)

Fix $\rho \neq 0$ in $\mathcal{I}_g^*$. For each $s \in S$ let

$$Z_s = \{f \in \mathrm{Mod}(S_g) : f \cdot \rho(s) = 0\}$$

Claim 3. Z_s is a proper, closed subset of $\mathrm{Mod}(S_g)$.

Claims 2 & 3 imply:

$$\bigcup_{s \in S} Z_s \subsetneq \mathrm{Mod}(S_g)$$

Choose $f \in \mathrm{Mod}(S_g)$ in the complement.
That is: $f \cdot \rho(s) \neq 0 \quad \forall\, s \in S$.
Claim 1 $\Longrightarrow f \cdot \rho \in \Sigma(\mathcal{I}_g^*)$.

But $\Sigma(\mathcal{I}_g^*)$ is invariant under automorphisms of $\mathcal{I}_g$
$\Longrightarrow \rho \in \Sigma(\mathcal{I}_g^*)$. But ρ was arbitrary. $\square$

We know $\mathcal{I}_g$ is fin. gen.
So by BNS theorem, all subgroups of $\mathcal{I}_g$ containing the commutator subgroup, including $K(S_g)$ are finitely generated.

More About BNS

Prop. For G a group, $\Sigma(G)$ is indep. of gen set.

Pf. Say T, S are gen sets.
Fix $\rho \in G^*$. Say $\rho \in \Sigma(G,T)$. Want $\rho \in \Sigma(G,S)$.

Idea. Say $\rho(g), \rho(g') \geq 0$. Choose $s \in S$ s.t. $\rho(s) > 0$.
For big N, $\rho(s^N g), \rho(s^N g') \gg 0$. Connect using s^N-translate of T-path from g to g' with $\rho \geq 0$.

Claim 1. $\forall\, n \in \mathbb{Z}$ $\{g \in G : \rho(g) \geq n\}$ is T-connected.

Pf. Follows by translating $\{g : \rho(g) \geq 0\}$.

Claim 2. $\exists\, n$ s.t. for any $g \in G$ & $t \in T$ $\exists$ path in S-Cayley graph from g to gt s.t. if v is a vertex on this path then $\rho(v) > \max\{\rho(g) - n, \rho(gt) - n\}$

Pf. For each $t \in T$ choose word w_t in S rep'ing t. Choose n larger than $|\rho(w)|$ whenever w is a prefix or suffix of any w_t. Connect g to gt by $g \cdot w_t$.

Claim 3. $\exists\, s \in S \cup S^{-1}$ s.t. $\rho(s) > 0$.

Pf. $\rho \neq 0$.

Finish the proof as in the idea above.

As a cor, we obtain the Prop which says $\Sigma(G)$ is invariant under automorphisms. Indeed, say $\rho \in \Sigma(G)$, $\alpha \in \operatorname{Aut} G$. Then $\{g : \rho(g) \geq 0\}$ is connected wrt some gen set X. So $\{g : \rho(\alpha^{-1}(g)) \geq 0\}$ is connected wrt to $\alpha^{-1}(X)$. Want it to be connected wrt X. But this is just a change of gen set.

A Lemma about BNS

Lemma (Ershov-He) G = fin. gen. group, $\rho \in G^*$ nonzero.
Say $\exists\ x_1, \ldots, x_n \in G$ s.t.

① G is gen. by the x_i.
② $\rho(x_1) \neq 0$.
③ For $2 \leq i \leq n$ $\exists\ j < i$ s.t. $\rho(x_j) \neq 0$ and
$$[x_j, x_i] \in \langle x_1, \ldots, x_{i-1} \rangle$$
Then $\rho \in \Sigma(G)$.

Special case of ③: $x_i \leftrightarrow x_j$.
This goes back to Koban-McCammond-Meier (essentially).

Actions on $\mathbb{R}$-trees

$T = \mathbb{R}$-tree (a space w/ unique paths b/w pts)
$G \curvearrowright T$ by isometries.
$\rightsquigarrow \ell : G \to \mathbb{R}$ (translation) length fn.
The action is...

- *nontrivial* if no global fixed pts
- *exceptional* if no invariant lines
- *abelian* if $\exists\ \rho \in G^*$ s.t. $\ell = |\rho|$ (say action is associated to ρ).

Lemma. (Brown '87) Say $\rho \in G^*$. $\exists$ exceptional, nontrivial, abelian action of G on an $\mathbb{R}$-tree assoc. to ρ $\iff$ $\rho \notin \Sigma(G)$.

Say $G \circlearrowright T = \mathbb{R}$-tree, $g \in G$.

$\rightsquigarrow$ characteristic subtree T_g

g elliptic $\rightsquigarrow$ fixed pts

g hyperbolic $\rightsquigarrow$ axis

Facts ① $g \leftrightarrow h$ hyps $\Rightarrow T_g = T_h$

② $g \leftrightarrow h$, h hyp $\Rightarrow T_g \supseteq T_h$

Commuting graph $X \subseteq G \rightsquigarrow C(X)$ = graph with vertex set X and edges for commuting.

Domination $X, Y \subseteq G$. Say Y dominates X if every elt of X commutes with some elt of Y

Lemma (KMM) G = group, $\rho \in G^*$. If $\exists$ $X, Y \subseteq G$ s.t.

① $\rho(y) \neq 0 \ \forall y \in Y$

② $C(Y)$ connected

③ Y dominates X

④ X generates G

Then $\rho \in \Sigma(G)$.

Pf. Suppose $\exists$ abelian action of G on $\mathbb{R}$-tree T assoc. to ρ.

By ① each $y \in Y$ acts as hyperbolic.

By ② there is a common characteristic subtree T_Y.

By ③ $T_x \supseteq T_Y$ $\forall x \in X$.

By ④ T_Y invariant under G

$\Longrightarrow$ action is not exceptional. $\square$

Proof of Ershov-He is essentially same.

CLAIM 1

Claim 1. $\exists$ finite gen set S for I_g consisting of genus 1 BPs s.t.

$$\{\rho \in I_g^* : \rho(s) \neq 0 \ \forall s \in S\} \subseteq \Sigma(I_g)$$

Pf. By Johnson I there is a finite set X of genus 1 BP maps that generates I_g.

Make a graph Γ w/ vertices the genus 1 BPs in S_g and edges for disjointness. Putman trick $\Rightarrow$ Γ connected. Let S be a set of BP maps that contains X and corresponds to a connected subset of Γ.

Enumerate the elts of S as $s_1, \dots, s_n$ s.t. $\forall i \ \exists$ $j < i$ with $s_i \leftrightarrow s_j$ (enumerate by increasing distance from a basepoint in Γ).

Choose ρ with $\rho(s_i) \neq 0 \ \forall i$. Apply the Ershov-He lemma.

Claim 2

Claim 2. In this topology, $\mathrm{Mod}(S_g)$ is an irreducible space (i.e. it is not the union of proper closed subspaces.)

Facts about irred. spaces

① Y irred. top. space, $X \to Y$ set map $\Rightarrow$ pullback topology on X is irred.

② $Y \to Z$ cont., Y irred $\Rightarrow$ $\mathrm{im}(Y)$ irred.

③ $Z \subseteq W$ subsp. irred $\Leftrightarrow$ $\overline{Z}$ irred.

Pf. By ①, enough to show image of $\mathrm{Mod}(S_g^1)$ in $GL(\mathcal{I}_g^{1\,*})$ is irred.

Recall $(\mathcal{I}_g^1)^{ab} \otimes \mathbb{R} \cong (\mathcal{I}_g^1)^* \cong \Lambda^3 H$ natural.

Image of $\mathrm{Mod}(S_g^1)$ is image of $Sp_{2g}(\mathbb{Z})$.
under $\iota : GL_{2g}(\mathbb{R}) \to GL(\Lambda^3 H)$

Classical: Zariski closure of $Sp_{2g}(\mathbb{Z})$ in $GL_{2g}(\mathbb{R})$ is $Sp_{2g}(\mathbb{R})$, which is a connected alg. gp, hence irred.

So ③ $\Rightarrow$ $Sp_{2g}(\mathbb{Z})$ is irred.

The map ι is Zariski continuous.

So ② $\Rightarrow$ $\iota(Sp_{2g}(\mathbb{Z}))$ is irred. $\square$

Claim 3

Fix $\rho \neq 0$ in I_g^*, $s \in S \rightsquigarrow Z_s = \{f \in \mathrm{Mod}(S_g) : f \cdot \rho(s) = 0\}$

Claim 3. Z_s is a proper, closed subset of $\mathrm{Mod}(S_g)$.

Pf. For fixed s, the condition $\rho(s) = 0$ is Zariski closed
$\Rightarrow Z_s$ closed.
Suppose $Z_s = \mathrm{Mod}(S_g)$.
$\Rightarrow (f \cdot \rho)(s) = \rho(fsf^{-1}) = 0 \quad \forall f \in \mathrm{Mod}(S_g)$
$\Rightarrow \rho$ vanishes on all BP maps of genus 1
$\Rightarrow \rho = 0$. $\square$

BNS - THE EASY CASE

following Putman

Prop. G = f.g. gp
$\rho : G \to \mathbb{Z}$ surjective
$H = \ker \rho$
$\rho, -\rho \in \Sigma(G)$.
Then H f.g.

Pf. Choose $t \in G$ s.t. $\rho(t) = 1$.
G f.g. $\Rightarrow \exists$ finite $S \subseteq H$ s.t. $S \cup \{t\}$ gens. G.
$\rightsquigarrow$ S normally generates H.
$\rightsquigarrow$ H gen. by

$$\bigcup_{k=-\infty}^{\infty} t^k S t^{-k}$$

Claim. H gen. by $S_+ = \bigcup_{k=0}^{\infty} t^k S t^{-k}$

Pf. Since $\rho \in \Sigma(G)$, any $h \in H$ can be written as $t^{i_1} s_1 t^{i_2} s_2 \cdots t^{i_n} s_n$
where running totals $i_1 + \cdots + i_k$ are $\geqslant 0$
& $i_1 + \cdots + i_n = 0$. The claim follows.

Example: $h = t^2 s_1 t^{-1} s_2 t^3 s_3 t^{-4}$
$= (t^2 s_1 t^{-2})(t s_2 t^{-1})(t^4 s_3 t^{-4})$ //

Similarly, $-\rho \in \Sigma(G) \Rightarrow H$ gen. by

$$S_- = \bigcup_{k=-\infty}^{0} t^k S t^{-k}$$

So $\forall s \in S$, can write tst^{-1} as product of elts of S_-. $\Rightarrow \exists N \geq 0$ s.t. the gp. gen by

$$S_{-N,0} = \bigcup_{k=-N}^{0} t^k S t^{-k}$$

contains tSt^{-1}.

Let H' = gp. gen by $S_{-N,0}$.
WTS $H' = H$.

Claim. $tH't^{-1} \subseteq H'$

Pf. $tH't^{-1}$ gen by $\bigcup_{k=-N+1}^{1} t^k S t^{-k}$

All in H' by defn except the tSt^{-1}. But we actually chose N, hence H', so this is true.

Of course $H' \subseteq H$. Remains to show $H \subseteq H'$.
Applying claim iteratively $\Rightarrow t^k S t^{-k} \subseteq H' \; \forall k \geq 0$.
But as above these generate H. $\square$

Problems on Torelli Groups

Generation

- Find a conceptual proof $\mathcal{I}(S_3)$ is fin. gen.
- Is there a gen set for $\mathcal{I}(S_g)$ with $\binom{2g+1}{3}$ elts?
- Find an explicit gen set for $\mathcal{I}(S_2)$.
- Find an explicit finite gen set for $\mathcal{K}(S_g)$.
  Other terms of Johnson filtration, terms of LCS.
- Is $\mathcal{K}(S_3)$ finitely generated?
- Is $\mathcal{K}(S_g)$ generated by twist differences $T_c T_d^{-1}$?
  Other terms of Johnson filtration?
- Find explicit gen sets for terms of Johnson filtration.

Relations

- Is $\mathcal{I}(S_g)$ finitely presented?

  What about $\mathcal{K}(S_g)$, terms of Johnson filtration, etc.?

- Show that two elements of $\mathcal{I}(S_g)$ either commute or generate a free group.

  Or just do it for BP maps, or just for $\mathcal{K}(S_g)$.

Cohomology

- Compute $H_k(\mathcal{I}(S_g))$

  Is it f.g. for any $g \geq 3$ $\quad 2 \leq k < 2g-3$?

- What is cohomological dimension of terms of Johnson filtration?

- Show the largest free abelian subgp has rank $g-1$.

- Show $H_{2g-3}(\mathcal{K}(S_g))$ is ∞-gen.

Hyperelliptic Torelli

- Find a simpler proof that $\mathcal{SI}(S_g)$ is gen. by Dehn twists.

- Is $\mathcal{SI}(S_g)$ finitely generated? presented?

- Is $\mathcal{SI}(S_g)^{ab}$ finitely generated?

- Find natural gen sets for congruence subgroups $\mathrm{SMod}(S_g)[m]$. What are the abelianizations?

Stretch Factors

- Improve the gap between $\sqrt{2}$ & 62 for the smallest stretch factor in $\mathcal{I}(S_g)$.

- Is the smallest stretch factor in $\mathcal{I}(S_g)$ smaller than that in $\mathcal{K}(S_g)$? Further terms of Johnson filtration?

- Which algebraic degrees for stretch factors arise in $\mathcal{I}(S_g)$?

- What is the smallest stretch factor in $\mathrm{Mod}(S_g)[2]$?

Embeddings

- Show that any embedding $\mathcal{I}(S_g) \hookrightarrow \mathrm{Mod}(S_g)$ is standard.

- Show any non-abelian map $\mathcal{I}(S_g) \longrightarrow \mathrm{Mod}(S_k)$ is trivial if $g \neq k$.

- Similar for $\mathcal{K}(S_g)$, other terms of Johnson filtration.

Miscellanea

- Find a simple description of BCJ maps, using double covers.

- Find the image of the second Johnson homomorphism.

- Which subsets of $\mathcal{I}(S_g)$ give all $\mathbb{Z}HS^3$s?

- Determine if the even MMM classes vanish on $\mathcal{I}(S_g)$.