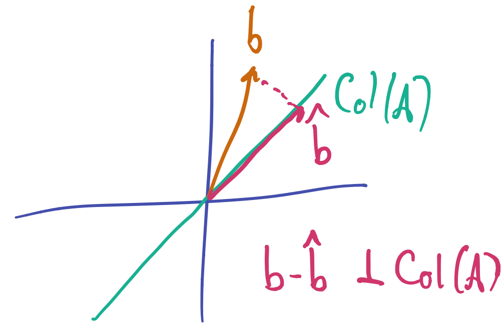
Chapter 6

Orthogonality

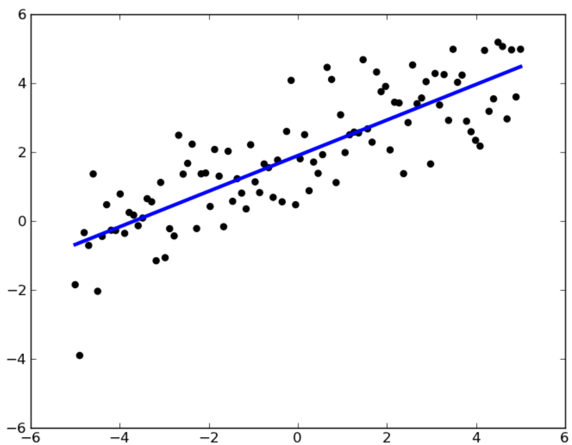
Where are we?

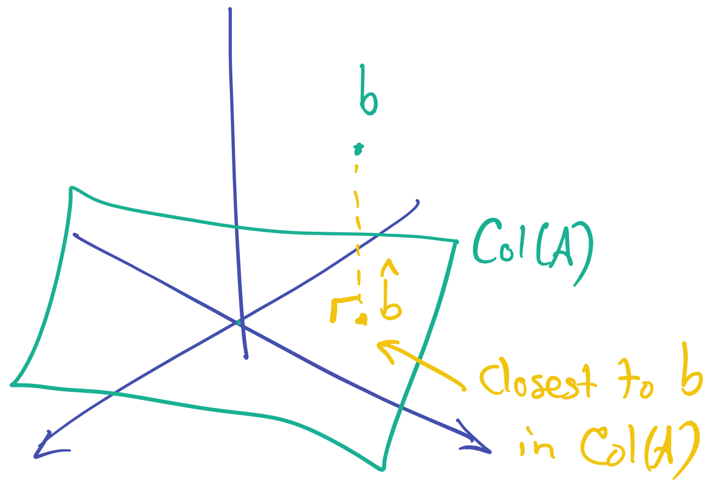


We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?





Solve $Ax = \hat{b}$ instead.

The answer relies on orthogonality.

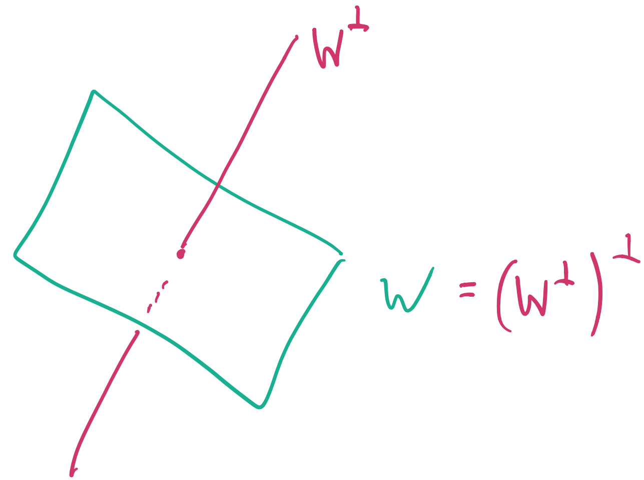
Section 6.2

Orthogonal complements

Orthogonal complements

W = subspace of $\mathbb{R}^n$ = plane thru 0.
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$? plane
What about the orthogonal complement of a plane in $\mathbb{R}^3$? line.



Demo

Demo

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$ (it's a null space!)
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$ (rank-nullity theorem!)
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.

For items 1 and 3, which linear transformation do we use?

Orthog. proj to W

range: W
null space: $W^\perp$

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

Example

$W = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

$W = \mathrm{Row}(A)$

In other words:

Theorem. $A = m \times n$ matrix

$$\boxed{(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A}$$

or $(\mathrm{Col}\,A^T)^\perp = \mathrm{Nul}\,A$

or $(\mathrm{Col}\,A^T) = (\mathrm{Nul}\,A)^\perp$

$A = \begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}$

Geometry $\leftrightarrow$ Algebra

$W^\perp = (\mathrm{Row}\,A)^\perp = \mathrm{Nul}(A)$

Why?

(The row space of A is the span of the rows of A.)

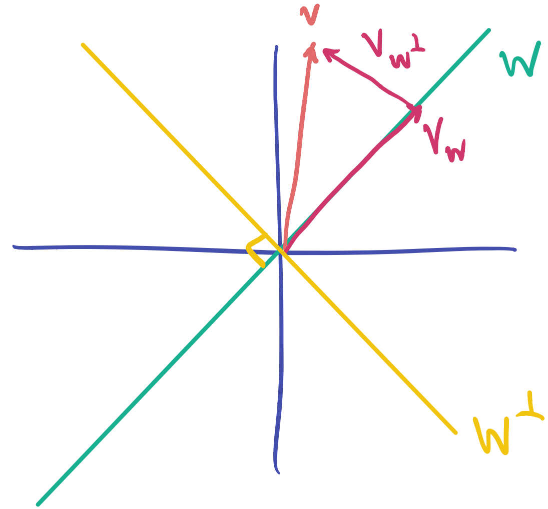
$\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Orthogonal decomposition

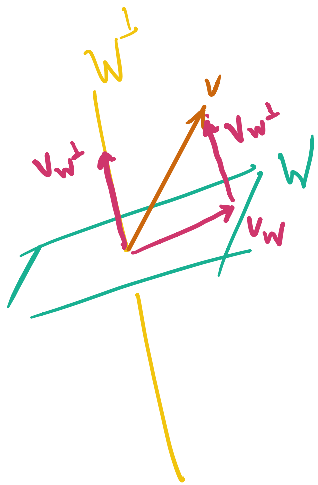
Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.



Why?



Demo

Demo

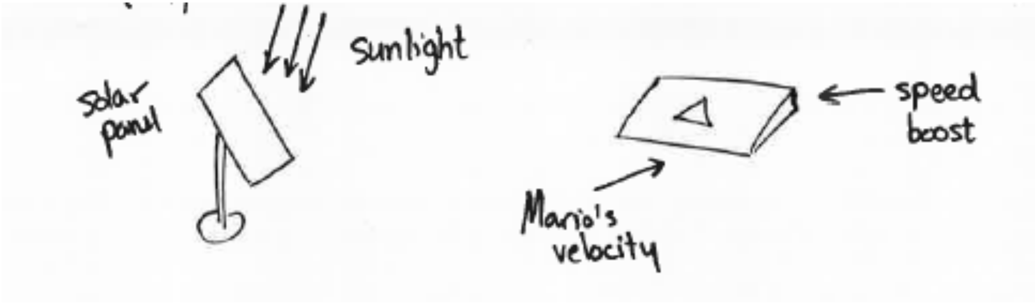
Next time: Find v_W and $v_{W^\perp}$.

V_W is... orthog. proj to W

Will give a formula.

Orthogonal Projections

Many applications, including:



Section 6.3

Orthogonal projection

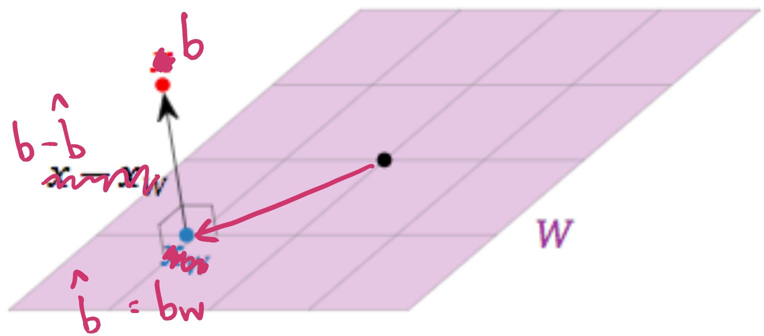
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections

Orthogonal Projections

Let b be a vector in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.

The orthogonal projection of b onto W the vector obtained by drawing a line segment from b to W that is perpendicular to W.



Fact. The following three things are all the same:

- The orthogonal projection of b onto W
- The vector b_W (the W-part of b) algebra!
- The closest vector in W to b geometry!

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

T means transpose:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Step 1. Find A^TA & A^Tb

Step 2. Solve $\underbrace{(A^TA)}_{\text{matrix}}x = \underbrace{A^Tb}_{\text{vector}}$

(This is an $Ax=b$ problem)

Step 3. $A \cdot (\text{any solution}) = b_W$

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Why? Choose $\widehat{x}$ so that $A\widehat{x} = b_W$. We know $b - b_W = b - A\widehat{x}$ is in $\boxed{W^\perp = \mathrm{Nul}(A^T)}$ and so

$$0 = A^T(b - A\widehat{x}) = A^T b - A^T A\widehat{x}$$

$$\rightsquigarrow A^T A\widehat{x} = A^T b$$

$$\mathrm{Nul}\,A = (\mathrm{Row}\,A)^\perp$$

$$\mathrm{Nul}\,A^T = (\mathrm{Row}\,A^T)^\perp = (\mathrm{Col}\,A)^\perp$$

Orthogonal Projections

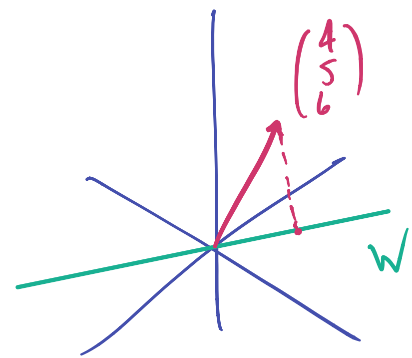
Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

What does the theorem give when $W = \mathrm{Span}\{u\}$ is a line?

$W = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$ $\quad b = \begin{pmatrix}4\\5\\6\end{pmatrix}$ $\quad A = \begin{pmatrix}1\\2\\3\end{pmatrix} = u$



Step 1. $A^T A = u \cdot u = \|u\|^2$

$A^T b = u \cdot b$

Step 2. Solve $A^T A x = A^T b$

$\|u\|^2 \cdot x = u \cdot b$

$x = \dfrac{u \cdot b}{\|u\|^2}$

Step 3. Multiply

$\dfrac{u \cdot b}{\|u\|^2} u$

scalar mult.

Orthogonal Projection onto a line

Special case. Let $L = \mathrm{Span}\{u\}$. For any vector b in $\mathbb{R}^n$ we have:

$$b_L = \frac{u \cdot b}{u \cdot u} u$$

$$b = b_L + b_{L^\perp}$$

Find b_L and $b_{L^\perp}$ if $b = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

$$\frac{u \cdot b}{u \cdot u} u = \frac{-2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix} = b_L$$

$$b_{L^\perp} = b - b_L = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Example. Find b_W if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T b$, then solve for x, then compute Ax.

1 2 3

Question. How far is b from W?

Orthogonal Projections

$\|v\| = \sqrt{v \cdot v}$

Example. Find b_W if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T b$, then solve for x, then compute Ax.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Step 1. $A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$A^T b = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$

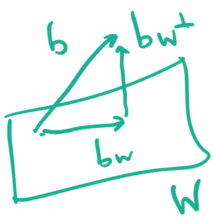
Step 2. Solve $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 2 & 11 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & 1 & 10 \end{array}\right) \rightsquigarrow \begin{pmatrix} 1 & 2 & 11 \\ 0 & -3 & -12 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 11 \\ 0 & 1 & 4 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right) \rightsquigarrow x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Step 3. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$

proj of b to W $= b_W$

Question. How far is b from W?



$$\|b_{W^\perp}\| = \|b - b_W\| = \left\| \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{3}$$

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

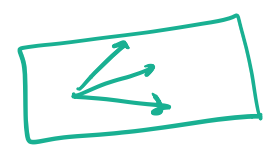
$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Special case. If the columns of A are independent then $A^T A$ is invertible, and so

$$b_W = A(A^T A)^{-1} A^T b.$$

Why? The x we find tells us which linear combination of the columns of A gives us b_W. If the columns of A are independent, there's only one linear combination.

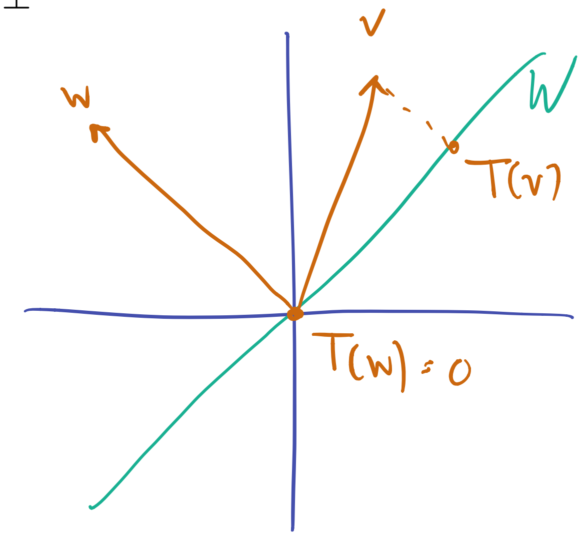


Projections as linear transformations

~~Skipping this slide this semester!~~

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Then

- T is a linear transformation
- $T(b) = b$ if and only if b is in W
- $T(b) = 0$ if and only if b is in $W^\perp$
- $T \circ T = T$
- The range of T is W



Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^T A)^{-1} A^T.$$

Why? Two slides ago we said

$$A(A^TA)^{-1}A^T b = b_W$$

Example. Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Properties of projection matrices

~~Skipping this semester!~~

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Let P be the standard matrix for T. Then

- The 1–eigenspace of P is W (unless $W = 0$)
- The 0–eigenspace of P is $W^\perp$ (unless $W = \mathbb{R}^n$)
- $P^2 = A$
- $\mathrm{Col}(P) = W$
- $\mathrm{Nul}(P) = W^\perp$
- A is diagonalizable; its diagonal matrix has m 1's & $n - m$ 0's where $m = \dim W$

You can check these properties for the matrix in the last example. It would be very hard to prove these facts without any theory. But they are all easy once you know about linear transformations!

Summary of Section 6.3

- The orthogonal projection of b onto W is b_W
- b_W is the closest point in W to b.
- The distance from b to W is $\|b_{W^\perp}\|$.
- Theorem. Let $W = \mathrm{Col}(A)$. For any b, the equation $A^TAx = A^Tb$ is consistent and b_W is equal to Ax where x is any solution.
- Special case. If $L = \mathrm{Span}\{u\}$ then $b_L = \frac{u \cdot b}{u \cdot u} u$
- Special case. If the columns of A are independent then A^TA is invertible, and so $b_W = A(A^TA)^{-1}A^Tb$
- When the columns of A are independent, the standard matrix for orthogonal projection to $\mathrm{Col}(A)$ is $A(A^TA)^{-1}A^T$
- Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$. Then
  - T is a linear transformation
  - etc.
- If P is the standard matrix then
  - The 1–eigenspace of P is W (unless $W = 0$)
  - etc.

Typical Exam Questions 6.3

- True/false. The solution to $A^TAx = A^Tb$ is the point in $\mathrm{Col}(A)$ that is closest to b.
- True/false. If v and w are both solutions to $A^TAx = A^Tb$ then $v - w$ is in the null space of A.
- Find b_L and $b_{L^\perp}$ if $b = (1, 2, 3)$ and L is the span of $(1, 2, 1)$.
- Find b_W if $b = (1, 2, 3)$ and W is the span of $(1, 2, 1)$ and $(1, 0, 1)$. Find the distance from b to W.
- Find the matrix A for orthogonal projection to the span of $(1, 2, 1)$ and $(1, 0, 1)$. What are the eigenvalues of A? What is A^{100}?