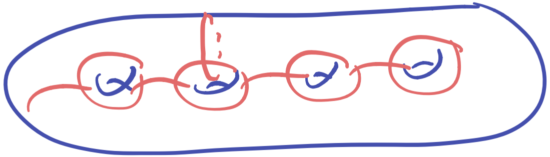
Chap 4. Generation.

fixing marked pts

Thm. $\mathrm{PMod}(S_{g,n})$ is finitely gen. by Dehn twists about nonsep curves.

Humphries:



$2g+1$

(minimal)

Application (later today?):

Every closed, orientable M^3 obtained from S^3 by Dehn surgery.

Application (next week?)

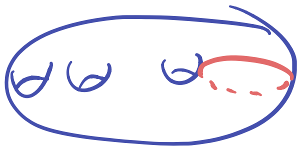
$$H_1(\mathrm{Mod}(S_g)) = 0$$

Thm. $PMod(S_{g,n})$ is finitely gen. by Dehn twists about nonsep curves.

Proof strategy

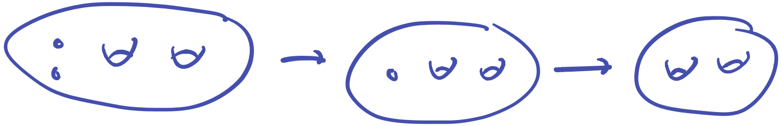
① Induction on genus:

$Mod(S_g)$ is gen. by stabilizers of nonsep curves



"complex of curves"

② Induction on punctures.



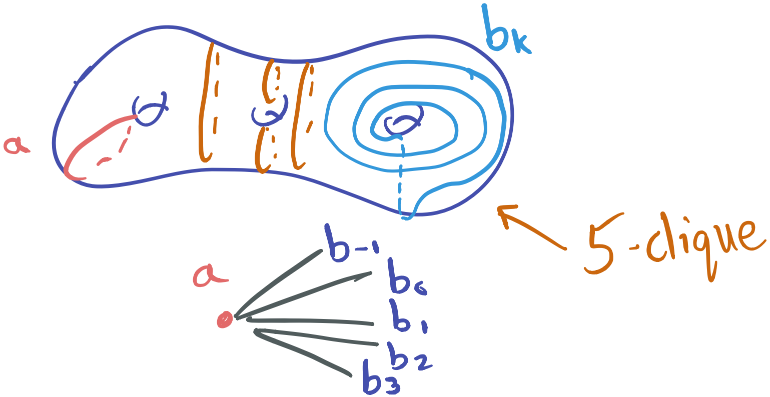
"Birman exact sequence"

Complex of curves (Harvey)

$C(S)$ has

vertices: isotopy classes of ess. s.c.c. in S

edges: disjointness.



Facts

① locally infinite

② connected (next!)

③ $\mathrm{Mod}^{\pm}(S) \xrightarrow{\cong} \mathrm{Aut}(C(S))$

(Ivanov)

applications...

$\mathrm{Aut}\,\mathrm{Mod}(S_g) \cong \mathrm{Mod}^{\pm}(S_g)$

$\mathrm{Isom}\,\mathrm{Teich}(S_g)$ "

④ $C(S)$ is hyperbolic

many applications...

& ∞-diameter.

exercise: find vertices of distance 3, 4, ...

Thm. $3g+n > 5$

$\mathcal{C}(S_{g,n})$ is connected.

Pf. Induct on $i(a,b)$.

(Say $n=0$)

Base cases:

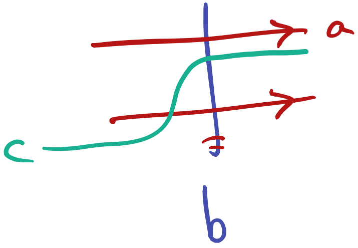
$i(a,b)=0$ ✓

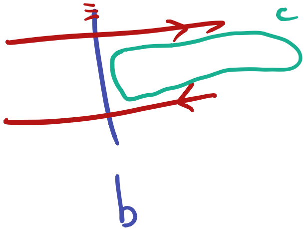
$i(a,b)=1$ ✓ change of coords.

Assume $i(a,b) \geq 2$.

Orient a.

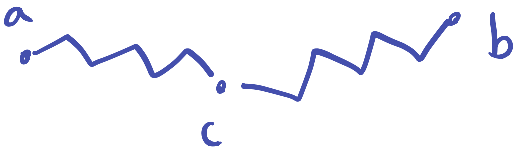
Two pictures:





Check: ① c essential

② $i(a,c),\ i(b,c) < i(a,b)$



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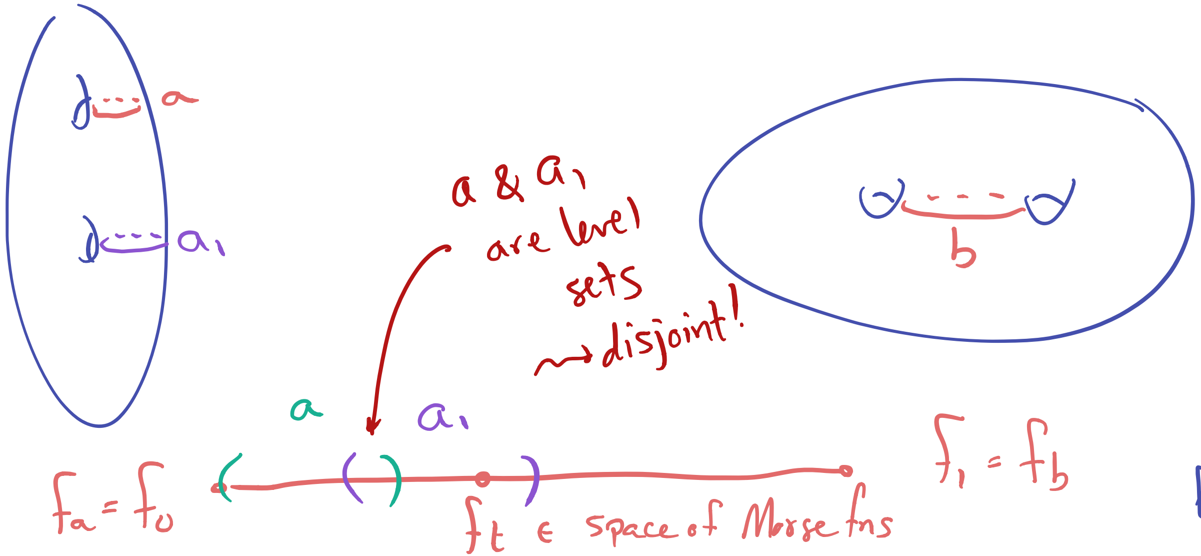
Cerf theory proof (Ivanov)

Given a, b. Choose Morse fns f_a, f_b s.t. a, b level sets on S.



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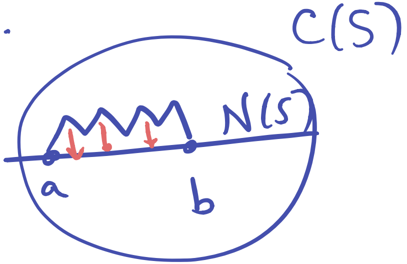
Complex of Nonsep curves

$N(S)$ = subcomplex of $C(S)$ spanned by nonseps.

Thm $N(S_g)$ connected $g > 1$.

Note. $N(S_{1,n})$ not connected!

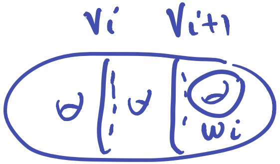
Pf of Thm.

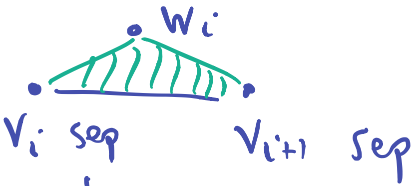


$a, b \in N(S)$

Connect by path v_i in $C(S)$.

Can assume no consec. v_i are sep.





If we have

either: sep is not needed.
or can replace with a nonsep.

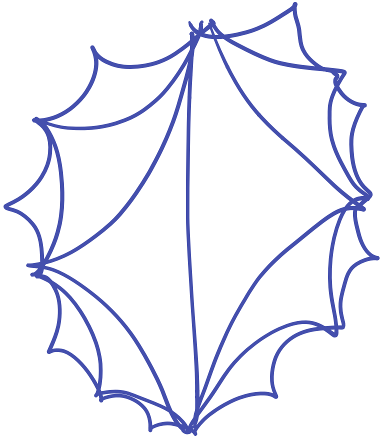
Modified complex $\hat{N}(S)$

Same vertices as $N(S)$

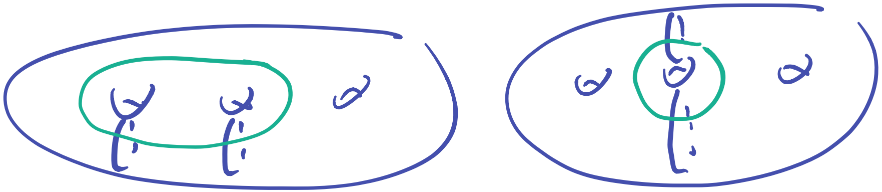
edges: $i(a,b) = 1$.

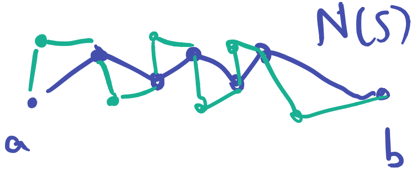
Thm. $\hat{N}(S_g)$ connected $g \geq 1$.

g = 1



Pf of Thm





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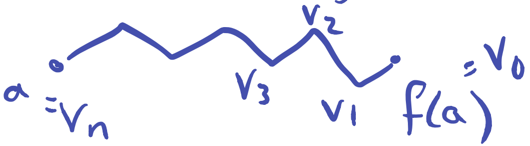
Prop. $\mathrm{Mod}(S)$ is gen. by stabilizers of (oriented) nonsep. s.c.c.

(Induction on genus).

Pf. Let $f \in \mathrm{Mod}(S_g)$

a = nonsep curve.

$\hat{N}(S_g)$



For each i:

$$T_{v_i} T_{v_{i+1}} (v_i) = v_{i+1} \quad \text{(braid reln)}$$

$$\text{So } \left(\prod T_{v_{i_j}}\right) f = \bar{f} \in \mathrm{Stab}(a)$$

all twists stabilize some nonsep curve.

$$\Rightarrow f \in \langle \text{stabilizers of nonsep curves} \rangle$$

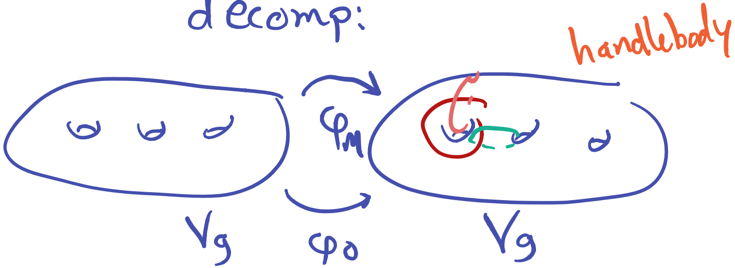
$$\in \langle \text{Dehn twists about nonseps, } \mathrm{Stab}(a) \rangle$$

Thm (Waldhausen) M^3 = closed, oriented 3-man

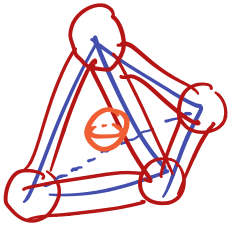
Then M^3 obtained from S^3 by $\underbrace{\text{Dehn surgery}}$

↳ remove disjoint collection of solid tori, reglue.

Pf. **Step 1**. M^3 has a Heegaard decomp:



Why? Triangulate M^3.
Thicken 1-skeleton.
That's one V_g.
The complement is other.

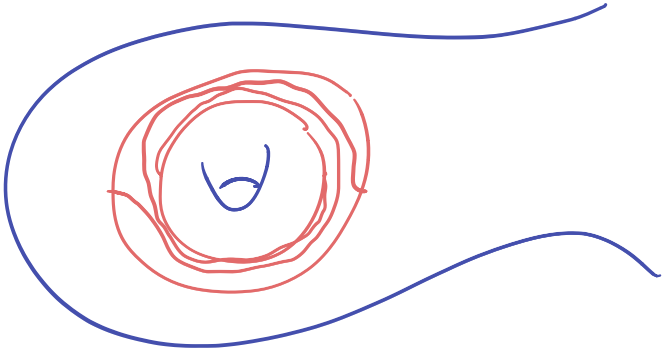


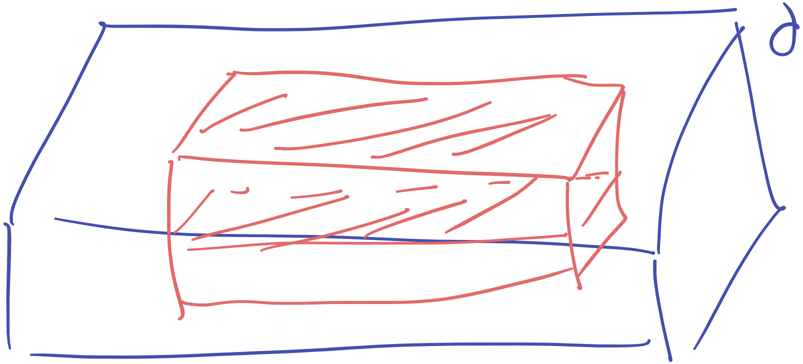
Step 2. Use fact that $\mathrm{Mod}(S_g)$ is gen by Dehn twists

M^3 has Heeg. decomp with φ_M
S^3 has - - - - with φ_0

$\varphi_M \varphi_0^{-1} \in \mathrm{Homeo}(S_g)$
$\shortparallel$
product of T_a

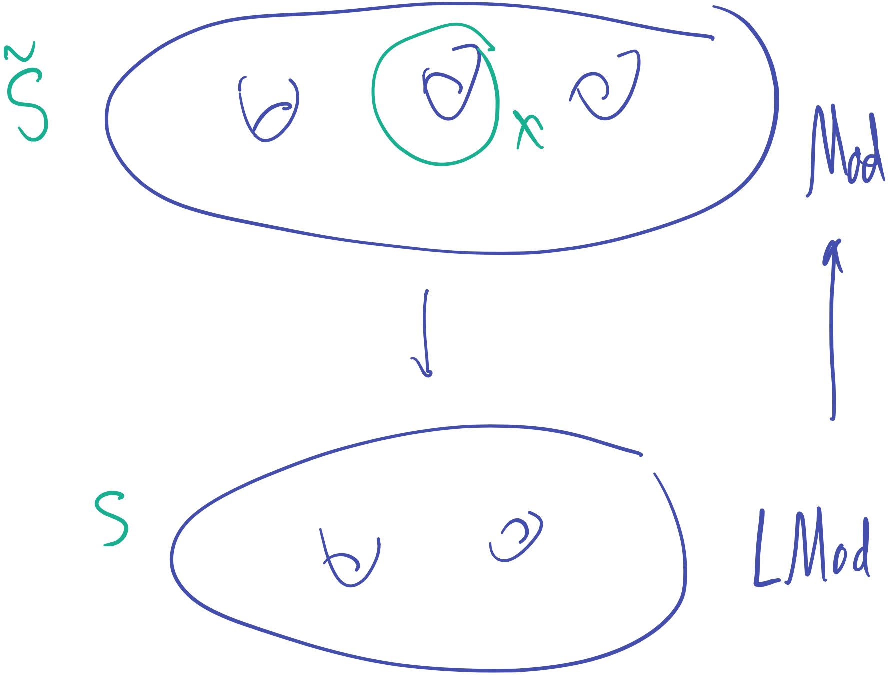
Putman: A note on...





$\partial V_g = S_g$

Rolfsen: Knots & Links.



Given $S, \tilde{S}$, $x \in H_1(\tilde{S})$ is $|\mathrm{LMod}(S) \cdot x| = \infty$?