

Chapter 4

Determinants

Where are we?

- We have studied the problem $Ax = b$
- We next want to study $Ax = \lambda x$
- At the end of the course we want to almost solve $Ax = b$

We need determinants for the second item.

Section 4.1

The definition of the determinant

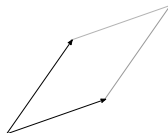
Outline of Sections 4.1 and 4.3

- Volume and invertibility
- A definition of determinant in terms of row operations
- Using the definition of determinant to compute the determinant
- Determinants of products: $\det(AB)$
- Determinants and linear transformations and volumes

Invertibility and volume

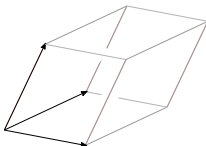
When is a 2×2 matrix invertible? \leftarrow Algebra

When the rows (or columns) don't lie on a line \Leftrightarrow the corresponding parallelogram has non-zero area. \leftarrow Geometry



When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane \Leftrightarrow the corresponding parallelepiped (3D parallelogram) has non-zero volume



Same for $n \times n$!

The definition of determinant

The **determinant** of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k , the determinant scales by k
4. $\det(I_n) = 1$

Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

The definition of determinant

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Problem. Just using these rules, compute the determinants:

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

A basic fact about determinants

Fact. If A has a zero row, then $\det(A) = 0$.

Fact. If A is a diagonal matrix then $\det(A)$ is the product of the diagonal entries.

Fact. If A is in row echelon form then $\det(A)$ is the product of the diagonal entries.

Why do these follow from the definition?

A first formula for the determinant

Fact. Suppose we row reduce A . Then

$$\det A = (-1)^{\# \text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$$

Use the fact to get a formula for the determinant of any 2×2 matrix.

Consequence of the above fact:

Fact. $\det A \neq 0 \Leftrightarrow A$ invertible

Computing determinants

...using the definition in terms of row operations

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} =$$

Computing determinants

...using the definition in terms of row operations

$$\det \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} =$$

A Mathematical Conundrum

We have this definition of a determinant, and it gives us a way to compute it.

But: we don't know that such a determinant function exists.

More specifically, we haven't ruled out the possibility that two different row reductions might give us two different answers for the determinant.

Don't worry! It is all okay.

We already gave the key idea: that determinant is just the volume of the corresponding parallelepiped. You can read the proof in the book if you want.

Fact 1. There is such a number \det and it is unique.

Properties of the determinant

Fact 1. There is such a number \det and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ **important!**

Fact 3. $\det A = (-1)^{\# \text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A) \det(B)$ **important!**

Fact 6. $\det(A^T) = \det(A)$ **ok, now we need to say what transpose is**

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A .

If you want the proofs, see the book. Actually Fact 1 is the hardest!

Powers

Fact 5. $\det(AB) = \det(A) \det(B)$

Use this fact to compute

$$\det \left(\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} \right)^5 \right)$$

What is $\det(A^{-1})$?

Poll

Suppose we know A^5 is invertible. Is A invertible?

1. yes
2. no
3. maybe

Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in \mathbb{R}^2 with vertices $(1, 2)$, $(4, 3)$, and $(2, 5)$?

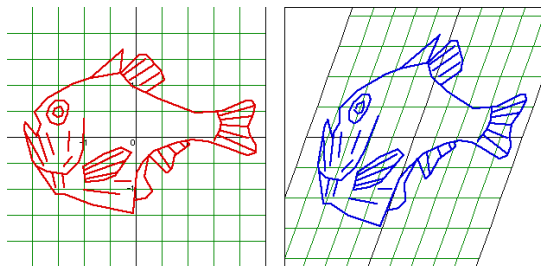
What is the area of the parallelogram in \mathbb{R}^2 with vertices $(1, 2)$, $(4, 3)$, $(2, 5)$, and $(5, 6)$?

Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of \mathbb{R}^n , then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

Summary of Sections 4.1 and 4.3

Say \det is a function $\det : \{\text{matrices}\} \rightarrow \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k , the determinant scales by k

Fact 1. There is such a function \det and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ **important!**

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

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Fact 8. If S is some subset of \mathbb{R}^n , then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?
- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T ?
- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T ?
- Find the volume of the triangular pyramid with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 0)$, and $(1, 2, 3)$.