Relations b/w 2 Dehn twists

Prop. $i(a,b) = 1$ top. $\Longrightarrow$ alg.

$$T_a T_b T_a = T_b T_a T_b$$

"braid relation"

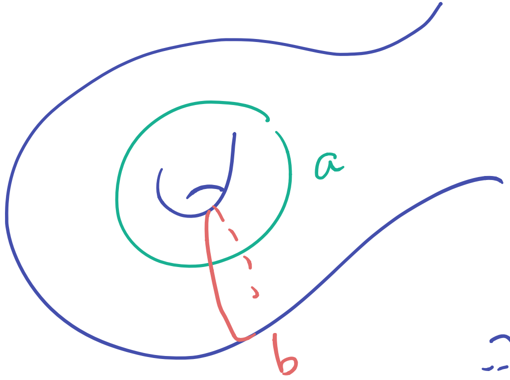
Pf. $(T_a T_b) T_a = T_b (T_a T_b)$

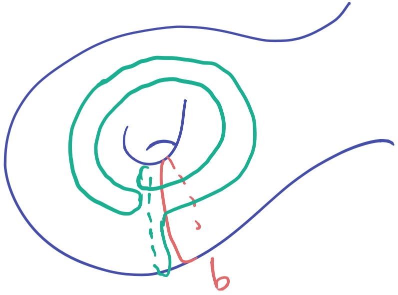
$\Longleftrightarrow \quad (T_a T_b) T_a (T_a T_b)^{-1} = T_b$

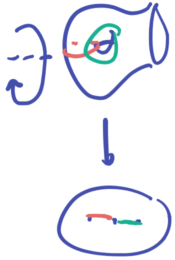
$\Longleftrightarrow \quad T_{T_a T_b(a)} = T_b$

$\Longleftrightarrow \quad T_a T_b(a) = b$

Change of coords







Converse!

Prop. $T_aT_bT_a = T_bT_aT_b$, $a \neq b$

$\Longrightarrow i(a,b) = 1.$

Pf. $T_aT_bT_a = T_bT_aT_b$

$\Longrightarrow T_aT_b(a) = b$

(as above).

So: $i(a,b) = i(a, T_aT_b(a))$

$= i(a, T_b(a))$

$= i(a,b)^2$

$\Longrightarrow i(a,b) = 0$ or $1 \ldots$ $\blacksquare$

Application

Given $\mathrm{Mod}(S_g) \to \mathrm{Mod}(S_g)$

If you can show

$T_a \longrightarrow T_{a'}$

Then curves $\longrightarrow$ curves

$a \longmapsto a'$

$i(a,b) = 1 \longmapsto i(a',b') = 1.$

Next: $\langle T_a, T_b \rangle$ $\forall a,b.$

Ping Pong Lemma

$G \circlearrowright X$ = set.

$g_1, g_2 \in G$

$X_1, X_2 \subseteq X$ nonempty disj.

$g_i^k(X_j) \subseteq X_i \quad i \neq j, \; k \neq 0.$

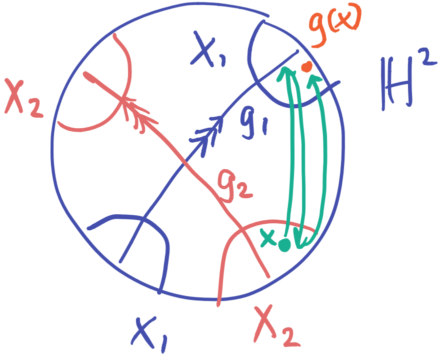
$\Longrightarrow \langle g_1, g_2 \rangle \cong F_2$

Pf. Let $g \in \langle g_1, g_2 \rangle$

WLOG (by conj)

$g = g_1^* g_2^* g_1^* \cdots g_2^* g_1^*$

Original source/application

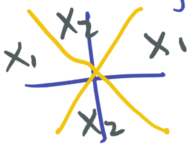


Second application:

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \cong F_2$$

$X_1 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2 : a > b \right\}$

X_2 similar.



Prop. $i(a,b) > 1$

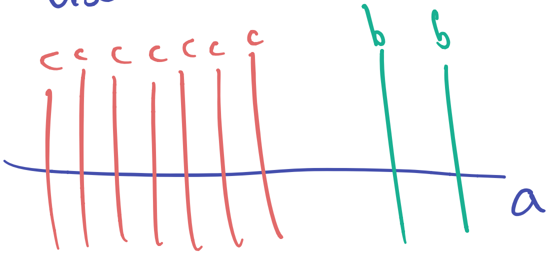
$\Rightarrow \langle T_a, T_b \rangle \cong F_2$.

Pf. Ping pong.

$X_1 = \{c : i(c,b) > i(c,a)\}$

X_2 similar.

Use our i-num. formulas.



In general: $j,k \neq 0$

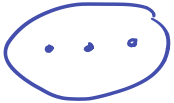
$\langle T_a^j, T_b^k \rangle \cong F_2$

unless

$i(a,b) = 1$ and

$\{j,k\}$ is $\{1,1\}$ $\{1,3\}$
$\{1,2\}$

$\begin{array}{cc} a & b \\ \shortparallel & \shortparallel \\ \sigma_1^2, & \sigma_2 \end{array}$



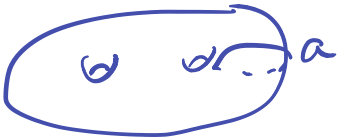
$112112 = 211211$

$abab = baba$

More Dehn twists: J. Mortada

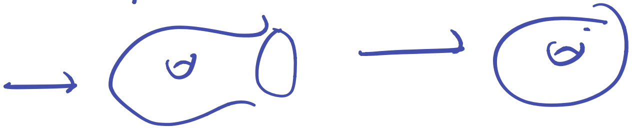
Cutting, capping, including

Later: want to prove things
by induction, hence understand
$\mathrm{Mod}(S_g, a)$



Cut along a:

Cap:

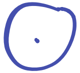


Including

Prop $S \subseteq S'$ no compl. disks

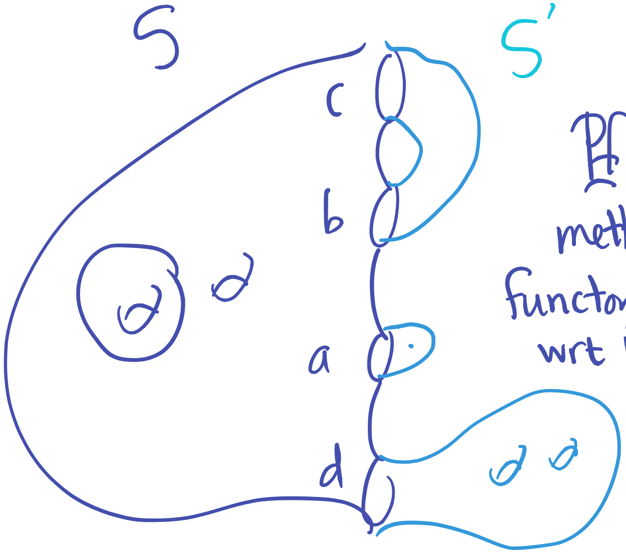
S closed in S'

$S \neq A$

a_i comp's of ∂S bounding 

$\{b_i, c_i\}$ bound 

$$\ker(\mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(S'))$$
$$= \langle T_{a_i}, T_{b_i} T_{c_i}^{-1} \rangle \cong \mathbb{Z}^k$$



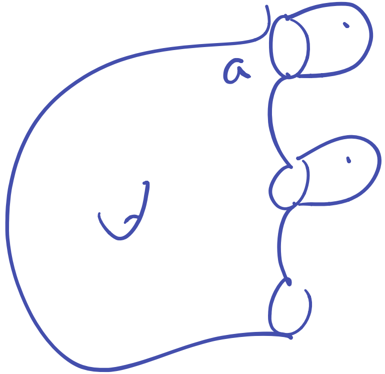
Pf. Alex. method is functorial wrt inclusion of surfaces.

Kernel:

$$\langle T_a, T_b T_c^{-1} \rangle \cong \mathbb{Z}^2$$

Capping.

Special case where $S \setminus S' = \odot$ $\qquad S \neq A$



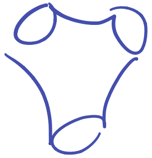
$\ker = \langle T_{a_i} \rangle \cong \mathbb{Z}^k$

$P \hookrightarrow S_{0,3} =$

$\to \mathrm{Mod}(P) \longrightarrow \mathrm{PMod}(S_{0,3}) = 1$

$\ker \ \mathbb{Z}^3$

$P =$



Applications

$\mathrm{Mod}(P) \cong \mathbb{Z}^3$

$\mathrm{Mod}(S_1^1) \cong \langle a, b \mid aba = bab \rangle$

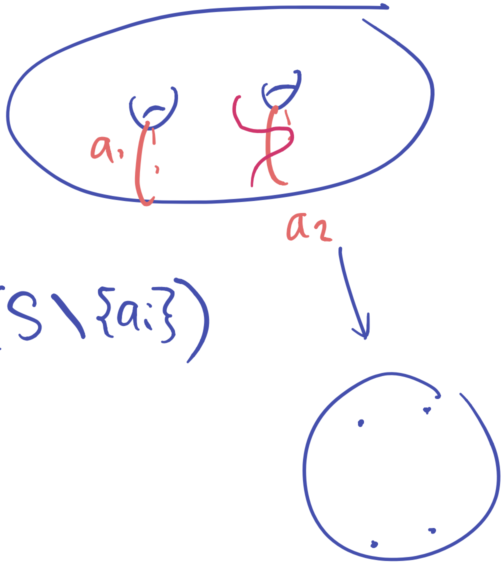
$\cong \pi_1(S^3 \setminus \ )$ $\quad \cong B_3$

$\cong \widetilde{SL_2 \mathbb{Z}}$

Cutting

$S = S_{g,n}$

$a_1, \ldots, a_k$ distinct, disjoint



There is a well-def map

Stab of $\{a_i\}$

$$\mathrm{Mod}(S, \{a_i\}) \longrightarrow \mathrm{Mod}(S \setminus \{a_i\})$$

with kernel $\langle T_{a_i} \rangle$

roughly

Pf. Apply inclusion homom to

$$S - \mathrm{Nbd}(\cup a_i) \hookrightarrow S$$

Q. Given $a_1, \ldots, a_k$

When is $\langle T_{a_1}^{e_1}, \ldots, T_{a_k}^{e_k} \rangle$ free?

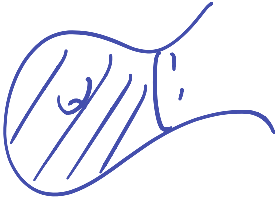
(Hamidi-Tehrani)

When is it a RAAG? (Runnels + refs)

Q (Afton) For which $G \leq MCG$

$\exists\, c, k$ s.t. $\langle G, T_c^k \rangle \cong G * \mathbb{Z}$

Q. When is it $= MCG \ldots$

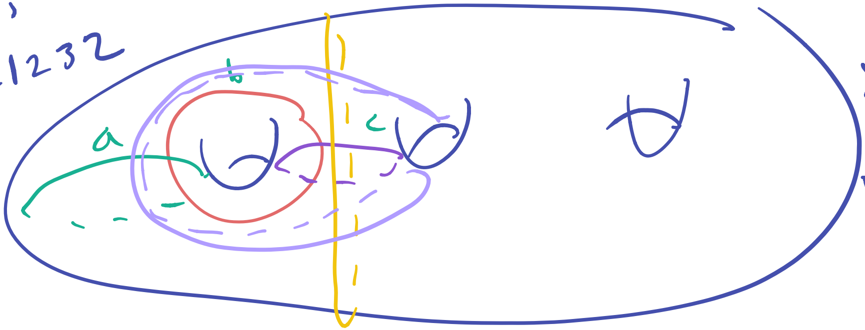
What replacing Dehn twists with  ?

Relations?

Or replace Dehn twists with "curve pushing maps" (Hadari

$\langle \sigma_1\sigma_2, \sigma_3\sigma_2 \rangle \subseteq B_4$

$1\,2\,3\,2\,\underline{1\,2} = 3\,2\,1\,2\,3\,2$

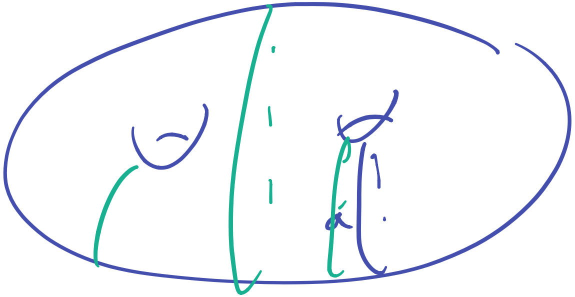


$x = T_a T_b$

$y = T_c T_b$

$B_n' \longrightarrow B_{2n}$

CRS



$CRS(T_a) = a$

$CRS(T_M) = M$

$CRS(fgf^{-1}) = f\,CRS(g)$

$CRS(f^n) = CRS(f)$

$f \leftrightarrow g \Rightarrow i(CRS(f), CRS(g)) = 0.$

$B_n \longrightarrow B_n$