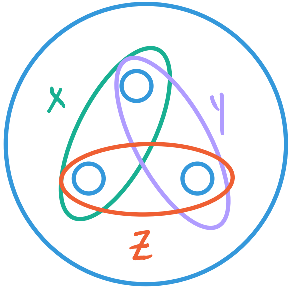
Chap 5. Presentations & H_1, H_2

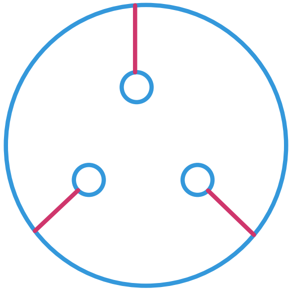
Lantern Relation $S_{0,4} \subseteq S$

$$T_x T_y T_z = \prod T_{\partial_i}$$

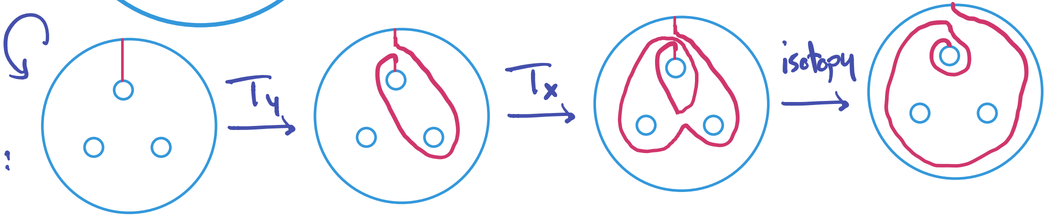


Pf #1 Alex Method:

Check relation on 3 arcs.

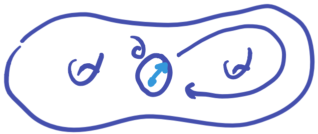


We'll do one arc:



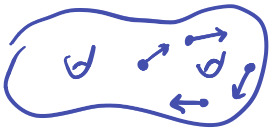
Pf #2 Boundary pushing $\chi(S) < 0$ $S^\circ = S \setminus$ open disk.

$$\text{Push} : \pi_1 UT(S) \longrightarrow \text{Mod}(S^\circ)$$



gen. of $\pi_1(\text{fiber}) \longmapsto T_\partial$

$\underline{\pi_1 UT(S)}$

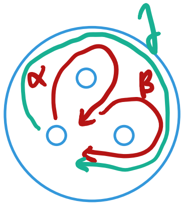


$$\begin{array}{c} S^1 \to UT(S) \\ \downarrow \\ S \end{array}$$

The lantern relation is:

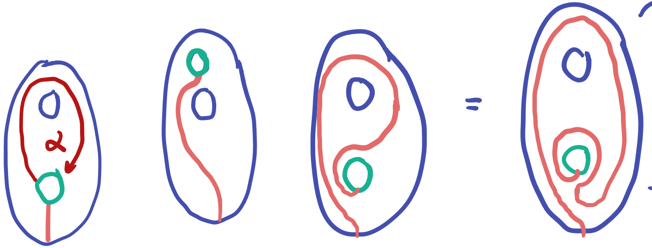
$$\text{Push}(\beta)\,\text{Push}(\alpha) = \text{Push}\,\gamma$$

push w/o rotating



$\beta\alpha = \gamma$ in $\pi_1(UT(S))$.

Why is this lantern relation?



$\text{Push}\,\alpha = T_x T_{\partial_1}^{-1} T_{\partial_2}^{-1}$

Thm $H_1(\mathrm{Mod}(S_g)) = 1$ $g \geq 3$.

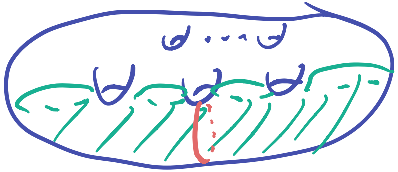
$H_1(G) \cong G^{\text{abel.}} = G/[G,G]$

So: no char. classes for S_g-bundles over S^1.

Pf. **Fact 1.** $\mathrm{Mod}(S_g)$ gen by T_c, c nonsep (Dehn-Lick.)

Harer + Mumford **Fact 2.** Such T_c are conjugate (Change of coords)

Fact 3. $\exists$ lantern reln in S_g w/ all 7 curves nonsep.



(?!)

Given $\mathrm{Mod}(S_g) \longrightarrow A$

gens $T_{c_i} \longmapsto t$ by fact 2

by fact 1, Image is $\langle t \rangle$ (cyclic)

Fact 3: $t^3 = t^4 \Rightarrow t = 1$.

Presentations

We have them (see book)

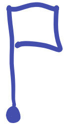
Next goal: proof of fin. presentability.

fin generation $\longleftrightarrow$ action on connected complex with finite quotient.

$H_2(G)$ $\longleftrightarrow$ fin presentability $\longleftrightarrow$ . . . . . simply connected . . .

(abelianized version)

Arc complex $\mathcal{A}(S)$



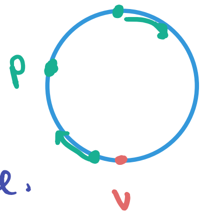
vertices: arcs / ~

edges: disjointness

k-simplices: (k+1) pairwise disjoint arcs.

(flag complex)

Thm. $\mathcal{A}(S_{g,n})$ contractible.

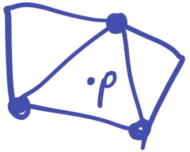


Pf (Hatcher) v = any vertex

Goal: homotope $\mathcal{A}(S_{g,n})$ into

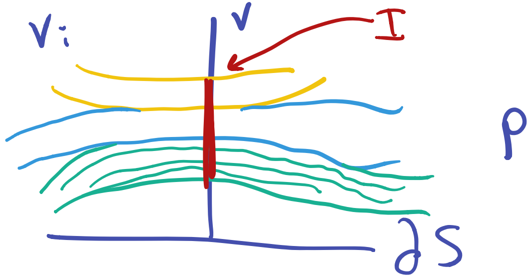
union of all simp. cont. v $\longrightarrow$ Star(v) $\simeq *$ so paths vary contin.

Let $p \in |\mathcal{A}(S_{g,n})|$

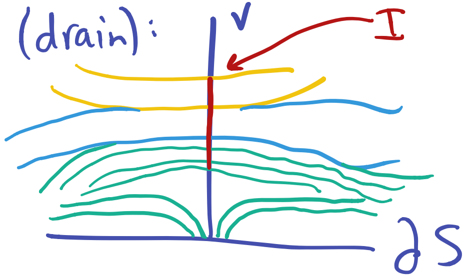


p = weighted sum of disjoint arcs

thicken arcs to bands, push together at v:



Homotopy (drain):



Prop. Say $G \curvearrowright X \simeq *$ w/o rotations

$\Longrightarrow \mathrm{Stab}(e) \subseteq \mathrm{Stab}(v)$

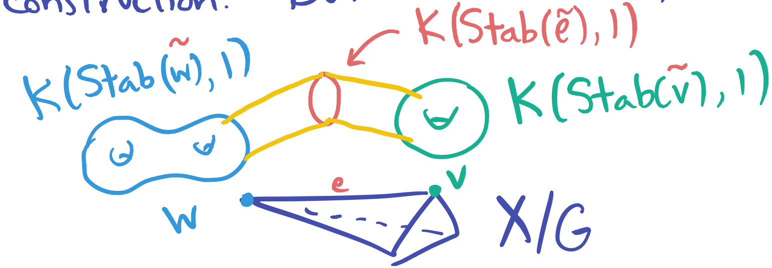
& ① X/G finite

② vertex stabs f.p.

③ edge stabs f.g.

Then G is f.p.

__Pf idea__ Borel construction: Build a $K(G,1)$ for G.



Thm. $\mathrm{Mod}(S_{g,n})$ fin pres.

Pf for $n>0$

Apply Prop.

Stab's are MCG's of simpler surfaces.

$\rightsquigarrow$ induction! $\square$

For $n=0$:

$$1 \to \pi_1(S_g) \to \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_g) \to 1$$

Quotient of a fp gp by a fg gp is f.p.

Also, alg. geom. proof: $\mathcal{M}_{g,n}$ is a quasi-proj variety.

Q. What presentation do you get from this proof?