MATH 4803-MAR

Intro. Geometric Group Theory

Spring 2021 GATECH

Basic Info

Canvas
- → Piazza → 1^{st} HW
- → Grades
- → Gradescope
- → Web page
  - ↓
  - Calendar → HW, → Notes
  - Resources/Refs
  - Syllabus, Final Project

Assessments

10% Participation. (Piazza)
30% HW
30% Midterm
30% Final project

What is GGT?

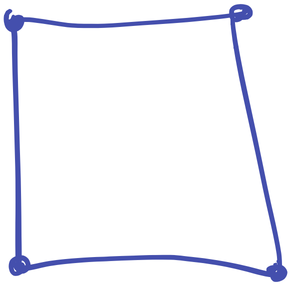
Groups are collections of symmetries of geometric objects

Use the geometry to learn about the algebraic properties of the group.

A group:

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

It is D_4



A group:

$SL_2(\mathbb{Z}) = \left\{ \begin{array}{l} 2\times 2 \text{ integer} \\ \text{matrices of} \\ \det = 1 \end{array} \right\}$

examples: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

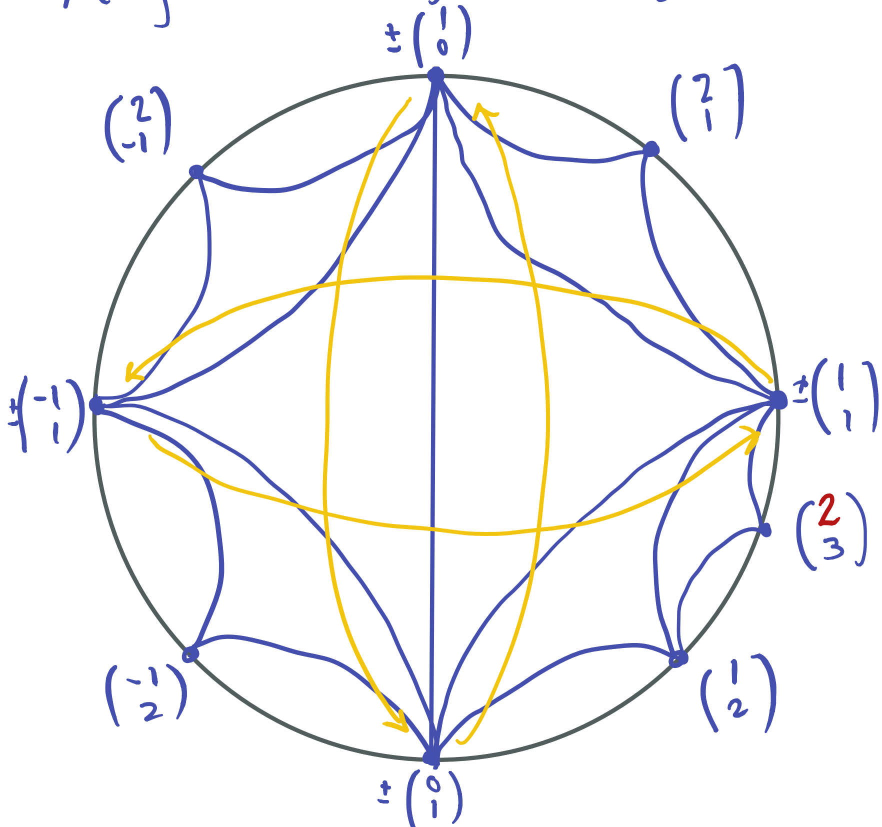
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Lin alg: eigenvals, eigenvectors, ...
lin transf. of $\mathbb{R}^2$

Group theory:

generators?
relations?
torsion?
subgroups?
quotients?

A geometric object: Farey graph



Not-obvious but true:

not a multiple

- all primitive integer vectors are vertices.
- each $A \in SL_2(\mathbb{Z})$ gives a symmetry of the graph.

Check: if v, w connected by edge then Av, Aw connected by edge

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ takes $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Rotation by π.

To do the check, prove:

$\begin{pmatrix} p \\ q \end{pmatrix}$ & $\begin{pmatrix} r \\ s \end{pmatrix}$ are connected by an edge

$$\Longleftrightarrow \det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1.$$

Check: if v, w connected by edge
then Av, Aw connected by edge

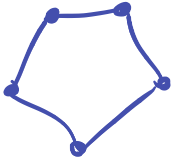
$$A\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \left(A\begin{pmatrix} p \\ q \end{pmatrix} \ A\begin{pmatrix} r \\ s \end{pmatrix} \right)$$

If $\begin{pmatrix} p \\ q \end{pmatrix}$ — $\begin{pmatrix} r \\ s \end{pmatrix}$ then $\det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1$
then $\det A\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1$ then $A\begin{pmatrix} p \\ q \end{pmatrix}$ — $A\begin{pmatrix} r \\ s \end{pmatrix}$

Overview of Course

Chap 1. Cayley graph

$G \longleftrightarrow$ graph



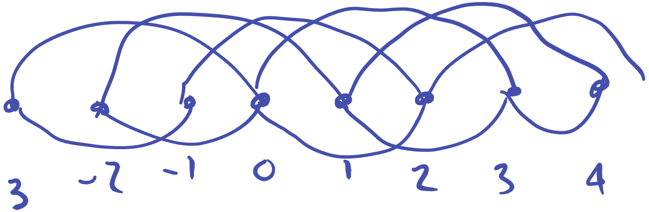
$\mathbb{Z}/5\mathbb{Z}$

0 ends

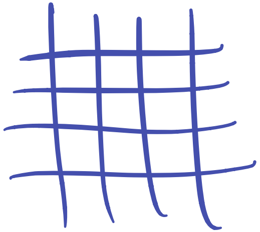
2 ends

$\mathbb{Z}$

2 ends.

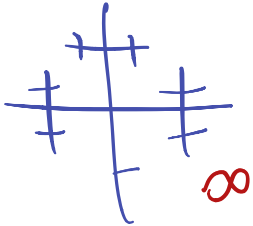


$\mathbb{Z}$ with gen set $\{3,2\}$



$\mathbb{Z}^2$

1 end

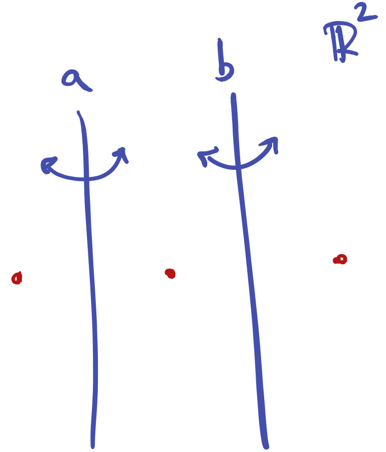


∞ ends

Chap 2 Coxeter gps

= groups gen by reflections

example



ab

Chap 3 Groups acting on trees

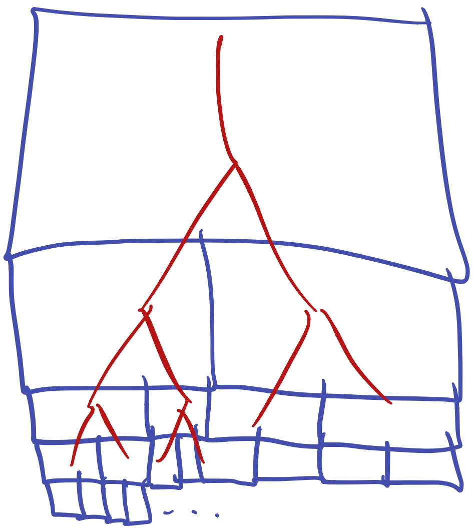
Free gps

F_n = gp with n gens & no relations

- Groups acting (freely) on trees $\longleftrightarrow$ free gps
- $F_3 \leq F_2$
- $F_\infty \leq F_2$

Chap 4 Baumslag-Solitar gps

see pic on web site.



hyperbolic plane. "treelike"

Chap 5 Word problem

Given a product of gens, is it id in gp.

Chap 8 Lamplighter gp

Chap 10 Thompson's gp.

Chap 11 Large scale properties

Thm. Every gp has 0, 1, 2, or ∞ many ends.

Announcements Jan 19

- Cameras on in class
- 1st HW assigned Thu, due Tue 3:30. Gradescope
- Lecture notes/HW posted on web site.
- Groups/topics due Feb 5
- Office hours Tue 11-12, Fri 2-3, appt

Q. How many symmetries does a cube have?
tetrahedron, icosahedron, ...?

Groups

G set

$G \times G \longrightarrow G$ mult.

w/ id

inv.

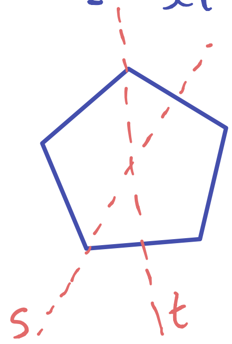
assoc.

example: symmetries of... anything.

Examples of finite groups

① Dihedral group D_n

= set of symmetries of n-gon



s, t are generators.

Since st is a rotation

relations:

$$s^2 = t^2 = id.$$

$$(st)^n = id.$$

This is a presentation for D_n:

generators

relators

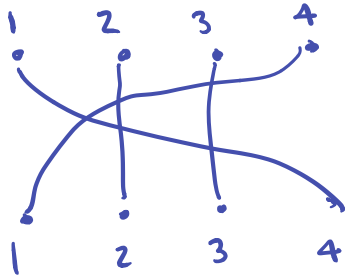
$$s = s\cancel{(st)^n}$$

$$\langle s, t \mid s^2 = t^2 = (st)^n = id \rangle$$

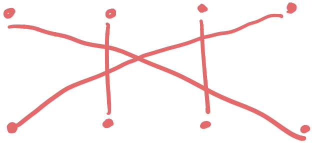
② Symmetric group.

S_n = set of permutations of $\{1,\dots,n\}$.

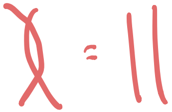
generators : $(i \;\; i+1) = \tau_i$



$(3\;4)(2\;3)(1\;2)(2\;3)(3\;4)$



Generators : $\tau_1, \dots, \tau_{n-1}$



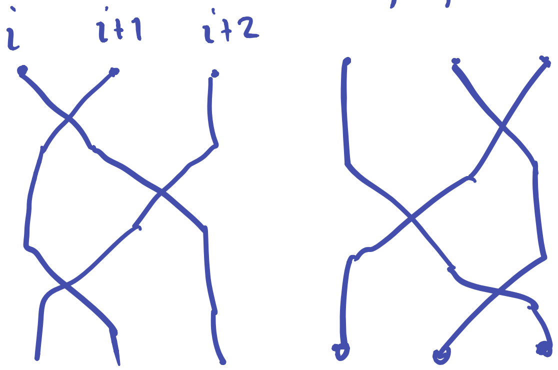
Relations : $\tau_i^2 = \mathrm{id}$.

$\tau_i \tau_j = \tau_j \tau_i \quad |i-j|>1$

$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$

$i = 1, \dots, n-2$

These give a presentation for S_n !

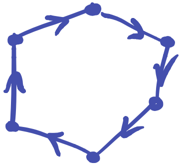


$\tau_i \tau_{i+1} \tau_i \qquad \tau_{i+1} \tau_i \tau_{i+1}$

③ Finite cyclic gps

$\mathbb{Z}/n\mathbb{Z}$

What is it the symmetries of?



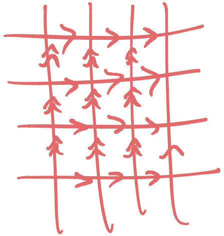
n-gon.

Presentation?

$$\langle a \mid a^n = id \rangle$$

④ Trivial group

$\langle \mid \rangle$ or $\langle a \mid a \rangle$



Examples of Infinite groups

① $\mathbb{Z}$

$\langle a \mid \rangle$

$a \quad a^{-1}$

$a^2 \quad a^{-2}$

$a^3 \quad a^{-3}$

② $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(a,b) : a,b \in \mathbb{Z}\}$

$\langle a, b \mid ab = ba \rangle$ $\quad a = (1,0)$, $b = (0,1)$

$aba = a^2 b = (2,1)$

③ $SL_n\mathbb{Z} = \{n \times n$ integer matrices with $\det = 1\}$

What is this the symmetries of?

Presentation?

Harder!

④ Free groups

no relations.

$F_2 = \{$ freely reduced finite words in $a, b\}$

word: finite sequence of a, b, a^{-1}, b^{-1}

freely reduced: no aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$

Multiplication: concatenate, then freely reduce

e.g. $aba^{-1} \cdot ab = abb$

Check this is a group.

id = empty word.

inverse = reverse & invert letters

e.g. $(abab)^{-1} = b^{-1}a^{-1}b^{-1}a^{-1}$

assoc. ✓

An issue: different reductions lead to same reduced word.

e.g. $aa^{-1}bb^{-1}$

or $b^{-1}aa^{-1}bb^{-1}$

Presentation: $\langle a, b \mid \ \rangle$

So...

$\mathbb{Z} \cong F_1$

& F_0 = trivial group.

Later in the class:

In $SL_2(\mathbb{Z})$:

$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

generate a free group.

so: $a^5 b^7 a^{-1} b^{-4} a \neq id$.

Next time: Free gps are important because every countable group is a quotient of a free group.

Announcements Jan 21

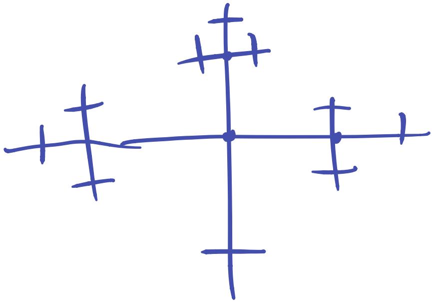
- Please turn cameras on
- HW1 due Tue 3:30 (I need to set up Gradescope)
- HW1 Lecture notes posted on web site. Need to add a reading prompt
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, by appt.

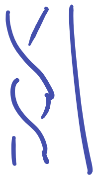
Examples of groups

D_n, S_n, $\mathbb{Z}/n$

$\mathbb{Z}^n$, $SL_n\mathbb{Z}$, F_n

What is F_n the symmetries of F_2?

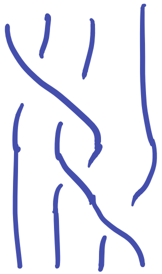




a^2

Braid groups B_n ← # of strands

elements:

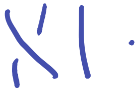


up to isotopy.

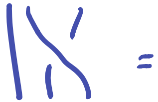
in B_4

multipl: concatenation

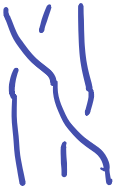
B_3



a



b



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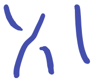


a^{-1}

Internal presentations

$\langle S \mid R \rangle$ is an internal presentation of G

if ① S is a generating set for G

② If two words in $S \cup S^{-1}$ are equal in G, they differ by a finite seq. of elements of $R \cup \{ss^{-1} : s \in S \cup S^{-1}\}$

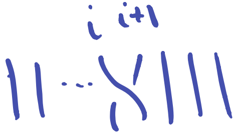
(replacing one side of an equality with another)

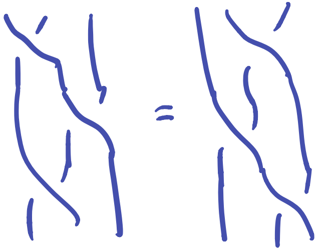
Fact. Every group has one: $S = G$
R = every possible equality.

__Example__ $B_n = \langle \sigma_1, \dots, \sigma_{n-1} : \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$

σ_i





Homomorphisms

$f: G \to H$

$f(ab) = f(a)f(b)$

Injective homomorphisms

"putting one group into another as a subgp"

- $\mathbb{Z}/n \to D_n$ rotations.
- $\mathbb{Z}/2 \to D_n$ reflection.
- $\mathbb{Z} \to F_2$

  $1 \mapsto a$

Which groups have inj homoms to B_n?

$\mathbb{Z}$ $\quad 1 \mapsto \sigma_1$ ✓

$\mathbb{Z}^2$ $\quad (1,0) \mapsto \sigma_1$, $(0,1) \mapsto \sigma_3$ $\quad$ assuming $n \geq 4$

$\mathbb{Z}/2$ $\quad$ Does B_n have an elt of order 2? No.

F_2 $\quad a \mapsto \sigma_1^2$, $b \mapsto \sigma_2^2$

Non-injective homoms

"forgetting (wisely)"

$\mathbb{Z} \to \mathbb{Z}/2$ even/odd

$\mathbb{Z} \to \mathbb{Z}/10$ 1's digit

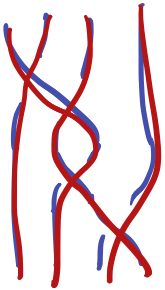
$D_n \to \mathbb{Z}/2$ flip?

$F_2 \to \mathbb{Z}^2$ exponent sum on a & b

$a \mapsto (1,0)$

$b \mapsto (0,1)$

$B_n \to S_n$



Normal subgps $N \leq G$

$$gNg^{-1} = N \quad \forall g \in G$$

$$\left\{\begin{array}{c}\text{kernels}\\ \text{of } G \to \square\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{normal subgps}\\ \text{of } G\end{array}\right\}$$

First Isomorphism Thm

If $f: G \to H$ surj homom. with kernel K

Then $H \cong G/K$.

First Isomorphism Thm

If $f: G \to H$ surj homom. with kernel K

Then $H \cong G/K$.

Surj. homoms

$\mathbb{Z} \to \mathbb{Z}/2$

$\mathbb{Z} \to \mathbb{Z}/10$

$D_n \to \mathbb{Z}/2$

$F_2 \to \mathbb{Z}^2$

$a \mapsto (1,0)$

$b \mapsto (0,1)$

$B_n \to S_n$

$\Longrightarrow$ 1st Isom. Thm

$\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2$

$D_n/\text{rotations} \cong \mathbb{Z}/2$

$D_n/\mathbb{Z}/n \cong \mathbb{Z}/2$

$F_2/F_2' = \mathbb{Z}^2$

commutator subgp.

$B_n/PB_n \cong S_n$

External presentation

$\langle S \mid R \rangle$

S = set

elts of R: equalities between words in $S \cup S^{-1}$ and id.

Not $ab = ba$, but $aba^{-1}b^{-1} = \mathrm{id}$

We obtain a group: $\underbrace{F(S)}_{\text{free gp on } S} \big/ \underbrace{\langle\langle R \rangle\rangle}_{\text{normal closure of } R}$

normal closure of R

= smallest normal subgp of $F(S)$ containing R.

= subgp of $F(S)$ gen. by elts of R & their conjugates.

HW. Internal & External presentations are equivalent.

Consequence. Every gp is a quotient of a free group.

$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} = \mathrm{id} \rangle = F^2 / \langle\langle \ \rangle\rangle$

ANNOUNCEMENTS JAN 26

$$\langle a, b, c, d \mid a b^2 c^3 a^{-1} d = 1 \rangle$$

- Cameras on for class
- HW1 due Thu 3:30
- Groups/topics due Feb 5
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How many symmetries does a cube have?

Symmetries

X = math object

$\mathrm{Sym}(X) = \{\text{symmetries of } X\}$

group under composition.

examples

X	$\mathrm{Sym}(X)$
regular n-gon	D_n
$\{1,\dots,n\}$	S_n
n-dim vector space over F	$GL_n F$
$\mathbb{R}^2$ as a vect sp	$GL_2\mathbb{R}$
Euclidean 2-space	$\mathrm{Aff}(\mathbb{R}^2)$ = line-preserving
topological space	$\mathrm{Homeo}(X)$
group G	$\mathrm{Aut}(G)$

Actions

An action of a group G on a math object X is a homomorphism

$$G \to \mathrm{Sym}(X)$$

G does something to X

Write: $G \circlearrowright X$

acts on

or a map

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

with

$$e \cdot x = x \quad \forall\, x \in X$$
$$g \cdot (h \cdot x) = (gh) \cdot x \quad \forall\, g, h \in G,\ x \in X$$

and the restriction

$$g \times X \to X$$

is in $\mathrm{Sym}(X)$ $\forall\, g \in G$

Examples

$D_n \circlearrowright$ n-gon (filled in or not)

$\mathbb{R}^2$

{vertices of n-gon}

{diagonals of n-gon}

$SL_2\mathbb{Z} \circlearrowright \mathbb{R}^2$ as a vector space

$\circlearrowright$ {vectors in $\mathbb{R}^2$}

$\circlearrowright$ {primitive vectors...}

($\circlearrowright$ Farey graph.)

Two vocab words:

1. If have $G \curvearrowright X$ say G is <u>represented</u> by symmetries of X.
2. An action is <u>faithful</u> if $G \to \mathrm{Sym}(X)$ is injective.

<u>Cayley's Thm</u>

Every group can be represented faithfully as a group of permutations

Rephrase: there is $G \hookrightarrow \mathrm{Sym}(X)$ X: a set.

$g \neq id \longmapsto$ not id

Pf. Take $X = G$ as a set. $\rightsquigarrow$ $\mathrm{Sym}(X)$ is a permutation group.

Given $g \in G$ need a permutation of $X = G$.

or $G \times G \to G$

$(g, h) \mapsto gh$

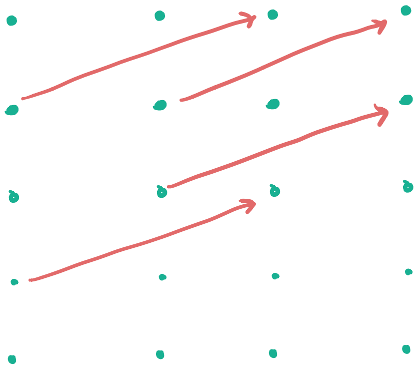
Need to check: defn of action. ✓

faithful

$\square$

Examples

$G = \mathbb{Z}^2$ $X = \mathbb{Z}^2$



The action of $(2,1)$ on $\mathbb{Z}^2$

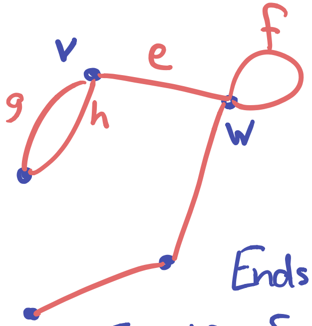
Graphs

A *graph* Γ is a set $V(\Gamma)$, a set $E(\Gamma)$,

vertices edges

and a function

$$\text{Ends}: E(\Gamma) \longrightarrow \{\{u,v\} : u,v \in V(\Gamma)\}$$

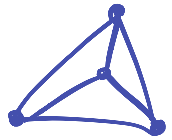


$\text{Ends}(f) = \{w\}$

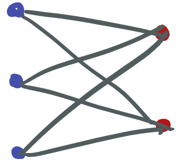
$\text{Ends}(e) = \{v,w\}$

Examples

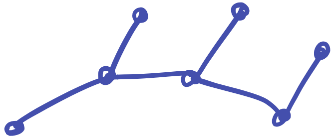
K_n = complete graph on n vertices



$K_{m,n}$ = complete bipartite graph ...



Tree = connected graph with no cycles.



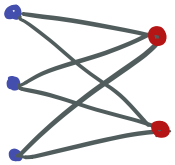
Symmetries of graphs

A **symmetry** of a graph Γ is a pair of bijections

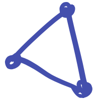
$$\alpha: V(\Gamma) \to V(\Gamma)$$
$$\beta: E(\Gamma) \to E(\Gamma)$$

preserving Ends function:

$$\mathrm{Ends}(\beta(e)) = \alpha\,\mathrm{Ends}(e)$$



Examples



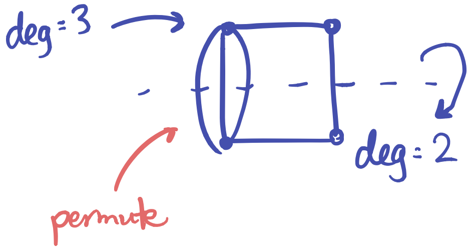
$$\mathrm{Sym}(K_n) \cong \mathrm{Sym}_n \cong \mathrm{Sym}(V(K_n))$$

$$\mathrm{Sym}(K_{m,n}) \cong \mathrm{Sym}_m \times \mathrm{Sym}_n$$

unless $m=n$.

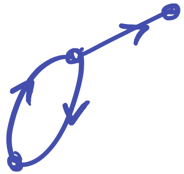
How many symmetries? 12

$\mathrm{Sym}_3 \times \mathbb{Z}/2$



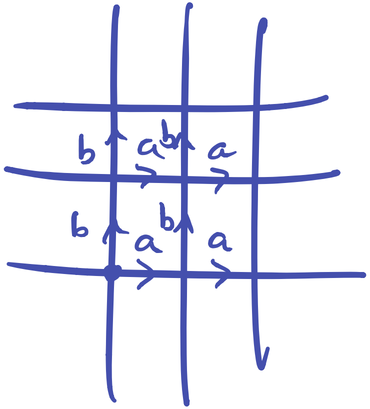
Graphs can have "decorations":

- directed edges

  "directed graph"

- labeled edges



$\mathrm{Sym}^+(\Gamma) = \{$ symmetries of Γ preserving decorations $\}$

Cayley's better theorm: Every group is faithfully rep. as symmetries of a graph. (next time)

Cayley graphs

G = group

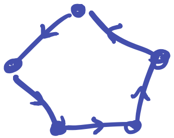
S = gen set.

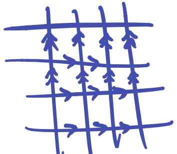
$\rightsquigarrow$ Cayley graph for G with respect to S

has vertices: G

edges:  $g \in G$, $s \in S$

Examples ① $G = \mathbb{Z}/n$ $S = \{1\}$ ② $G = \mathbb{Z}^2$ $S = \{(1,0), (0,1)\}$





Announcements Jan 28

RECORD

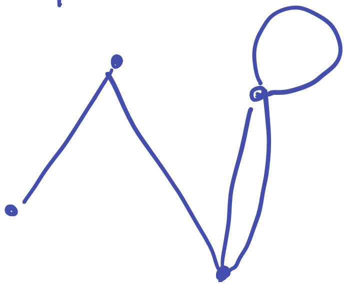
- Cameras on
- HW 2 due next week Thu 3:30
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, appt
- Way-too-early course evals

Goals today:

- Given a graph, what are its symmetries?
- Given a group, what graph(s) does it act on?

Why? e.g. <u>Thm.</u> If $G \circlearrowright$ tree freely then G is free.

Last time: ① Graphs



exactly 2 symmetries.

Symmetries: permutation of $V(\Gamma)$, $E(\Gamma)$ respecting Ends.

② Actions

$G \circlearrowright X$ or $G \times X \longrightarrow X$

$(g, x) \longmapsto g \cdot x$

means: $G \longrightarrow Sym(X)$

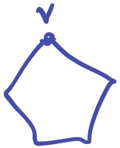
1.4 Orbits & stabilizers

Say $G \circlearrowright X$

$\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}$

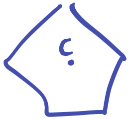
this is a subgroup

e.g. • $D_n \circlearrowright$ n-gon.



$\mathrm{Stab}(v) \cong \mathbb{Z}/2$

• $D_n \circlearrowright$ n-gon

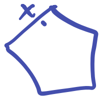


$\mathrm{Stab}(c) = D_n$

The action of G is free if $\mathrm{Stab}(x) = \{e\}$ $\forall x$

$\mathrm{Orb}(x) = \{g \cdot x : g \in G\}$

eg. $D_n \circlearrowright$ n-gon.



$|\mathrm{Orb}(x)| = 2n$

$|\mathrm{Orb}(v)| = n$

$|\mathrm{Orb}(c)| = 1$

$\mathrm{Stab}(x) = \{ g \in G : g \cdot x = x \}$

$\mathrm{Orb}(x) = \{ g \cdot x : g \in G \}$

__Thm.__ There is a bijection:

$\mathrm{Orb}(x) \longleftrightarrow$ left cosets of $\mathrm{Stab}(x)$

given by $\quad g \cdot x \longleftrightarrow g\,\mathrm{Stab}(x)$.

__Pf.__ Subtlety: well-definedness.

But $g \cdot x = h \cdot x \iff h^{-1} g \cdot x = x$

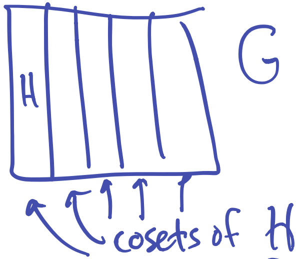
$\iff h^{-1} g \in \mathrm{Stab}(x) \iff g\,\mathrm{Stab}(x) = h\,\mathrm{Stab}(x)$ $\square$

__Cor__ (Orbit-Stab Thm)

If $|G| < \infty$ & $G \curvearrowright X$ then

$$|G| = |\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)|$$

__Pf.__ Lagrange's thm



__Example.__ How many symmetries does a cube have?

$|G| = 3 \cdot 8 = 24$ rotations

or $6 \cdot 8 = 48$ all symmetries

__Cor__^{2} If $\mathrm{Stab}(x) = \{e\}$

then: $G \leftrightarrow \mathrm{Orb}(x)$

1.5 Cayley graphs

$G \qquad S = \text{gen set}.$

$\rightsquigarrow \Gamma_{G,S}$

$V(\Gamma_{G,S}) = G$

$E(\Gamma_{G,S})$:

$$g \longrightarrow gs$$

$\forall\, g \in G$
$\quad s \in S$

<u>Fact</u>. $\Gamma_{G,S}$ is connected. (as undirected graph)

Why? Enough to show all vertices are connected by a path to e.
Given a vertex g, write as a product $g = s_1 \cdots s_n \quad s_i \in S$.

<u>Fact</u>. $G \circlearrowright \Gamma_{G,S}$ as a labeled, directed graph

gp elt   vertex   vertex

On vertices: $g \cdot h = gh$

This extends to edges:

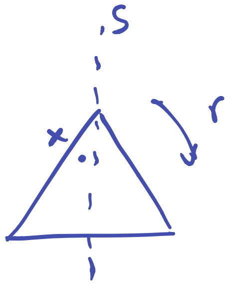
$$\begin{tikzcd} hs & & ghs \\ h \arrow[u, "s"] \arrow[rr, phantom, "\xrightarrow{\;g\;}"] & & gh \arrow[u, "s"] \end{tikzcd}$$

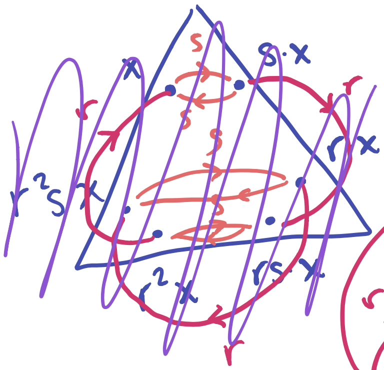
<u>Fact</u>. The action is faithful.

Examples

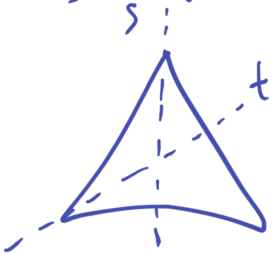
① $D_3 = Sym_3$

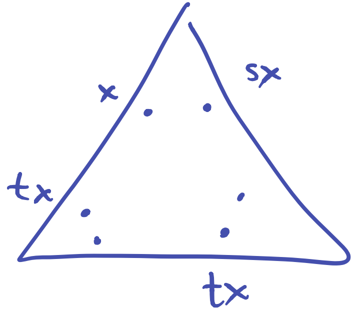
$S = \{r, s\}$





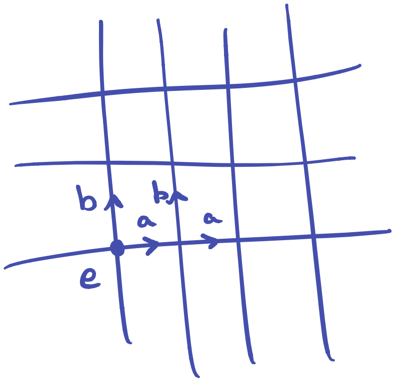
$D_3 = Sym_3$ $S = \{s, t\}$





exercise...

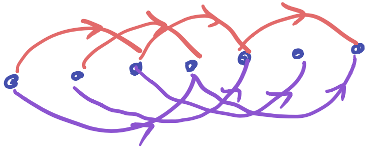
$\mathbb{Z}^2$



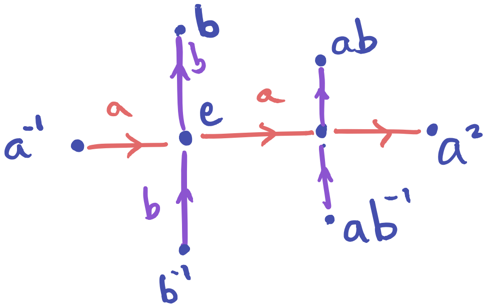
$S = \{a, b\}$, $a = (1,0)$, $b = (0,1)$

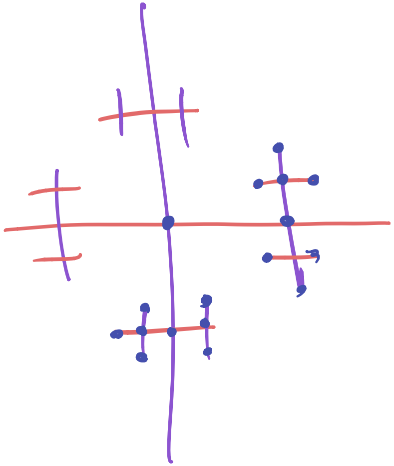
$\mathbb{Z}$ $\quad S = \{1\}$

or $\mathbb{Z}$ $\quad S = \{2,3\}$



F_2 $\quad S = \{a,b\}$





Cycles in $\Gamma_{G,S}$

$\longleftrightarrow$ relations.

ANNOUNCEMENTS FEB 2

- Cameras on
- HW 2 due Thu 3:30
- Groups/topics due Feb 5
- OH Fri 2-3, appt
- Way-too-early course feedback Canvas → Quizzes

Cayley graphs

G = group

S = genset

$\Gamma_{G,S}$ graph

vertices: G

edges: $g \xrightarrow{s} gs$

$\forall\ g \in G$
$s \in S$.

Last time: $G \curvearrowright \Gamma_{G,S}$

$$G \times V(\Gamma_{G,S}) \longrightarrow V(\Gamma_{G,S})$$

$$g \cdot h = gh.$$

gp elt vertex

This rule also tells you where edges go.

$$h \xrightarrow{s} hs \quad \xrightarrow{g} \quad gh \xrightarrow{s} ghs$$

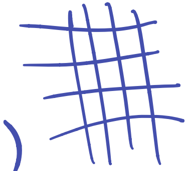
preserving labels/arrows.

Also: $G \hookrightarrow \mathrm{Sym}^+(\Gamma_{G,S})$

Thm. The natural map

$$\Phi : G \longrightarrow \mathrm{Sym}^+(\Gamma_{G,S})$$

defined above is an isomorphism.



Pf. Remains to show surjectivity.

Let $\overset{\alpha}{\cancel{\Phi}} \in \mathrm{Sym}^+(\Gamma_{G,S})$

Need: $\overset{\alpha}{\cancel{\Phi}} = \Phi(g)$ some $g \in G$.

Which g? Take "$g = \underset{\alpha}{\cancel{\Phi}}(e)$"

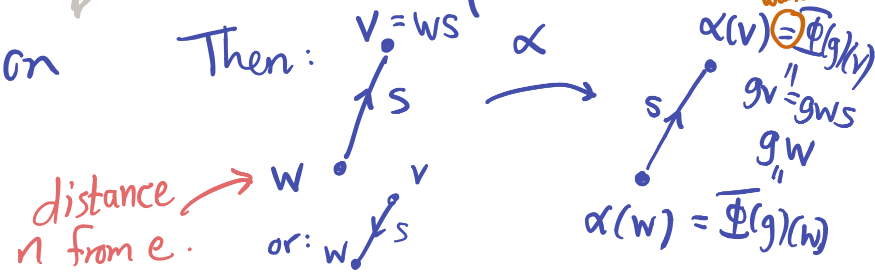
So $\overset{\alpha}{\cancel{\Phi}}$ & $\Phi(g)$ agree on the vertex e.

Induct on distance from e: we'll show $\Phi(g)$ & α agree on all vertices of distance n from e.

Base case: distance 0

Inductive step: Assume $\Phi(g)$, α agree on vertices of distance n from e. Say v has distance $n+1$

Then:



From Meier:

$G \circlearrowright X$ = top space.

(example: $\mathbb{Z}^2 \circlearrowright \mathbb{R}^2$)

A fundamental domain for the action is a subset $\mathcal{F} \subseteq X$ s.t.

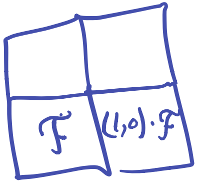
① $\mathcal{F}$ closed

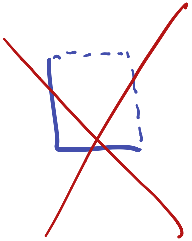
② $\bigcup_{g \in G} g \cdot \mathcal{F} = X$

③ connected

and no proper subset of $\mathcal{F}$ satisfies ① & ② & ③.

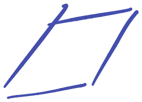
In the example can take $\mathcal{F}$ = unit square

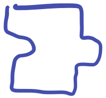




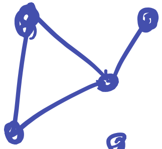
Issue #1. We don't know what closed subsets of a graph are.

Lots of fundamental domains:



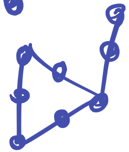


For us: $G \curvearrowright \Gamma$ = graph.



Γ

Γ' = (barycentric) subdivision of Γ (subdivide all edges of Γ).



Γ'

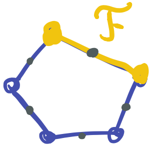
A fundamental domain for $G \curvearrowright \Gamma$ is a subgraph $\mathcal{F} \subseteq \Gamma'$ s.t.

1. $\mathcal{F}$ connected.
2. $\bigcup_{g \in G} g \cdot \mathcal{F} = \Gamma'$

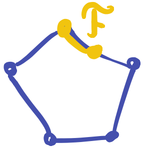
& $\mathcal{F}$ minimal with respect to ① & ②.

Examples

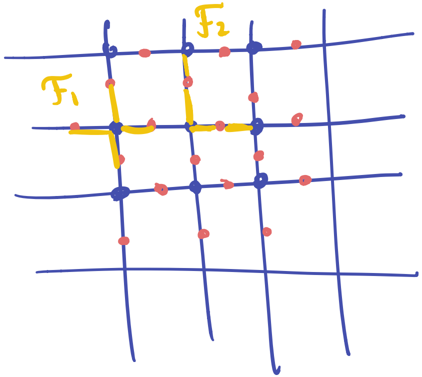
① $\mathbb{Z}/5 \curvearrowright$



② $D_5 \curvearrowright$



③ $\mathbb{Z}^2 \curvearrowright$



Thm. Say $G \curvearrowright \Gamma$ ← connected

& $F \subseteq \Gamma'$ connected subgraph.

& $\bigcup_{g \in G} g \cdot F = \Gamma \, (= \Gamma')$

(e.g. F = fund. domain)

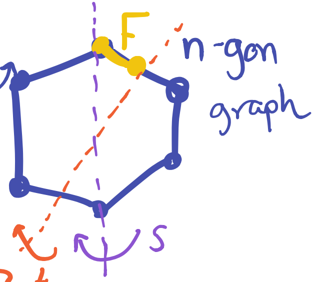
Let $S = \{ g \in G : g \cdot F \cap F \neq \emptyset \}$

Then S generates G.

The smaller F is, the smaller S is.

That's why we care about fundamental domains.

Example. $D_n \curvearrowright$



$S = \{ s, t \}$

Q. Is G acts faithfully and F is a fund domain, is S minimal? (Tolson).

Example next time: $\mathrm{Sym}_n \curvearrowright K_n$.

Thm. Say $G \curvearrowright \Gamma$ ← connected

& $F \subseteq \Gamma$, connected subgraph.

& $\bigcup_{g\in G} g\cdot F = \Gamma\,(=\Gamma')$

(e.g. F = fund. domain)

Let $S = \{g\in G : g\cdot F\cap F \neq \emptyset\}$

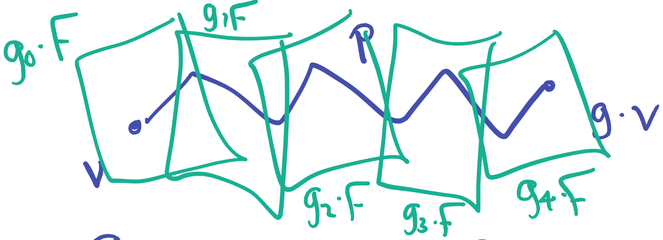
Then S generates G.

Proof. Let $g\in G$

Choose a vertex v in F.

Find a path p from v to $g\cdot v$

(Γ connected)



Choose $g_0,\dots,g_n\in G$

s.t. $g_0 = e$, $g_n = g$

$\bigcup_{i=1}^{n} g_i\cdot F$ contains p.

& $g_i\cdot F\cap g_{i+1}\cdot F\neq\emptyset$ $\forall i$.

Now: $g_0 F\cap g_1\cdot F\neq\emptyset \Longrightarrow g_1\in S$

$g_1 F\cap g_2 F\neq\emptyset$

$\Longrightarrow F\cap g_1^{-1}g_2\cdot F\neq\emptyset \Longrightarrow g_1^{-1}g_2\in S$

$\Longrightarrow g_2$ is a product of two elts of S.....

Announcements Feb 4

- Cameras on
- Grade/topic due Fri Gradescope
- HW 3 due Feb 11 3:30
- Abstracts due Feb 26
- Office hours Fri 2-3, Tue 11-12, appt.

Today: Generators from group actions. $SL_2\mathbb{Z}$

Fundamental domains ~ existence, ...

Thm $G \curvearrowright \Gamma$ connected

$F \subseteq \Gamma'$ subgraph.

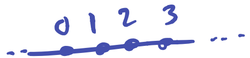
$$\bigcup_{g \in G} g \cdot F = \Gamma'$$

Then

$$S = \{ g \in G : g \cdot F \cap F \neq \emptyset \}$$

generates G.

---

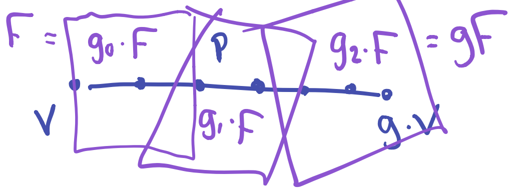
example: $\mathbb{Z} \curvearrowright$ 

$F =$  $\quad S = \{\pm 1, 0\}$

Pf. Let $g \in G$. Pick v = vertex of F.

Choose a path p from v to $g \cdot v$

(Γ connected)



Choose $g_0 \cdot F, \dots, g_n \cdot F$

s.t. $g_0 = e$, $g_n = g$, $p \subseteq \bigcup_{i=1}^{n} g_i \cdot F$

$g_i \cdot F \cap g_{i+1} \cdot F \neq \emptyset$.

Show by induction: g_i is a prod. of elts of $S^{\pm 1}$

$i = 0$ ✓

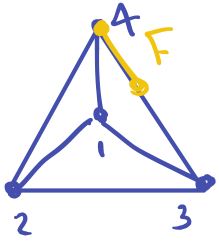
Assume true for i. WTS for $i+1$.

$g_{i+1} \cdot F \cap g_i \cdot F \neq \emptyset$

$\Longrightarrow g_i^{-1} \cdot g_{i+1} \cdot F \cap F \neq \emptyset$

$\Longrightarrow g_i^{-1} \cdot g_{i+1} = s \in S \Longrightarrow g_{i+1} = g_i s$ $\square$

Example 1. $S_n \circlearrowright K_n$



$S_n = \mathrm{Sym}_n$

F = half-edge from n to $n-1$.

S contains: $\mathrm{Stab}(n) = S_{n-1}$

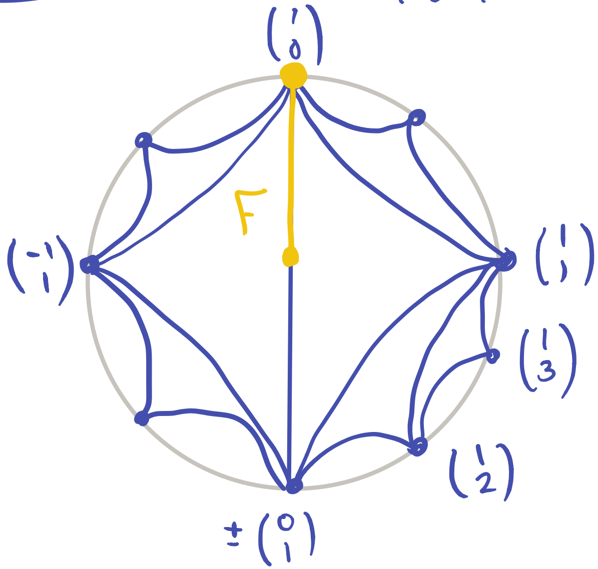
$(n{-}1 \; n)$ · ~~any elt of S_{n-2}~~

Can simplify: S_{n-1} $\leftarrow$ by induction: gen by adjacent transpositions.

$(n{-}1 \; n)$

$\Rightarrow$ S_n gen by adjacent transp.

Example 2. $SL_2\mathbb{Z} \circlearrowleft$ Farey graph.



Vertices: $\{\text{primitive } \mathbb{Z} \text{ vectors}\}/\pm$

edges: $\binom{p}{q} \text{———} \binom{r}{s} \iff \det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1.$

$\iff$ integer bases for $\mathbb{Z}^2$

Note: $\exists\, A \in SL_2\mathbb{Z}$ s.t. $A \cdot \binom{1}{0} = \binom{0}{1}$

$$A = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix}$$

must be -1

anything.

Better: $\exists\, A \in SL_2\mathbb{Z}$ s.t. $A \cdot \binom{1}{0} = \binom{p}{q}$

$$A = \begin{pmatrix} p & * \\ q & * \end{pmatrix}$$

Bezout

$* p + * q = 1$

$\Longrightarrow$ F need only 1 vertex of Γ....

Note: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ flips vertical edge.

in S!

Also need: $\mathrm{Stab}\binom{1}{0}$.

What about $\mathrm{Stab}\begin{pmatrix}1\\0\end{pmatrix}$?

$$\cancel{\begin{pmatrix}1 & p\\0 & q\end{pmatrix}} \quad \cancel{\begin{pmatrix}1 & n\\0 & 1\end{pmatrix}}$$

$$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & n\\0 & 1\end{pmatrix}$$

only need : $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$

and : $\cancel{\begin{pmatrix}-1 & -1\\0 & -1\end{pmatrix}}$ $-I$

(because first col is really $\begin{pmatrix}\pm 1\\0\end{pmatrix}$)

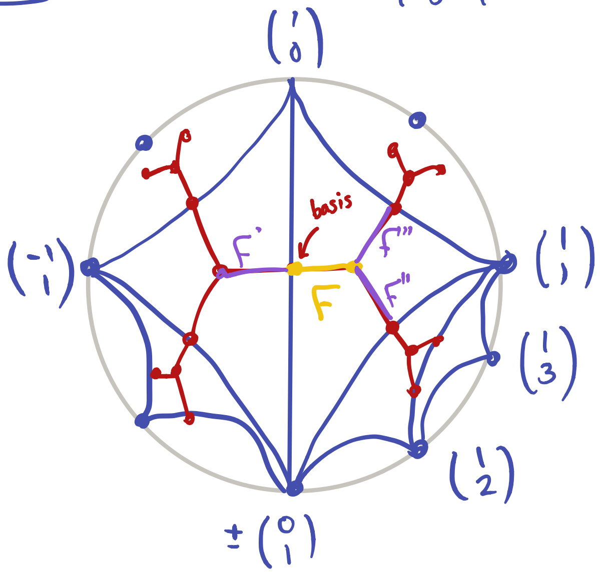
Finally : $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$

$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$

generates $SL_2\mathbb{Z}$.

Example ~~2~~ 3. $SL_2\mathbb{Z} \circlearrowright$ Farey ~~graph~~ tree.



$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ takes F to F'

$\begin{pmatrix}0&1\\-1&1\end{pmatrix}$ takes F to F'' & F'''

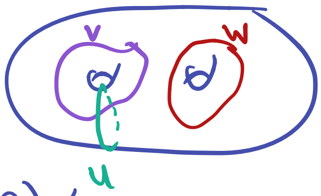
$\cancel{\begin{pmatrix}-1&0\\0&-1\end{pmatrix}}$ fixes F (and the whole tree)

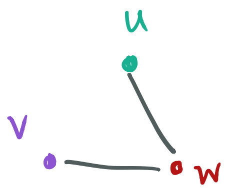
These gens have finite order: 4, 6, ~~2~~.

What is F?

gen set?

A more far out example.

Take S = surface 



$$MCG(S) = \mathrm{Homeo}(S) / \text{homotopy}.$$

We find generators using curve graph

vertices: simple closed curves in S / homotopy

edges: disjointness

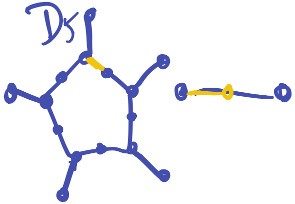
ANNOUNCEMENTS FEB 9

- Cameras on
- Abstract Feb 26 (consult w/ me)
- HW 3 due Thu 3:30
- OH Fri 2-3, appt

Today: Fundamental domains

D_∞.

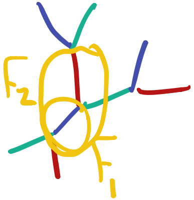
Fundamental domains



Have $G \circlearrowright \Gamma$

F = minimal, connected subgraph of Γ' so

$$\bigcup g \cdot F = \Gamma'.$$



Lemma. If $G \circlearrowright \Gamma$ = connected graph then the action has a fund. dom. F.

Pf. First assume $G \circlearrowright \Gamma'$ has finitely many orbits of edges. ⊛

Color each edge according to orbit.

Example: $D_5 \circlearrowright$ 10-gon has 1 orbit of edges

$\mathbb{Z}/5 \circlearrowright$ 10-gon has 2 orbits. of edges

Build F inductively.

Choose any edge. Call it F.

Find a new color edge (not in F) adjacent to F, and add it to F.

This stops by ⊛

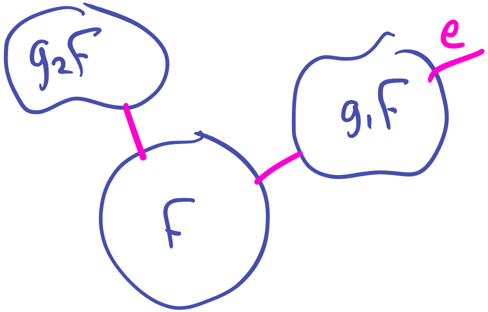
Need to show

$$\bigcup_{g\in G} g\cdot F = \Gamma'$$

Suppose not. There is an edge e not in $\bigcup g\cdot F$ and adjacent to it.

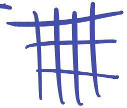
Say e adjacent to gF

Then $\underline{g^{-1}e}$ is adjacent to F.

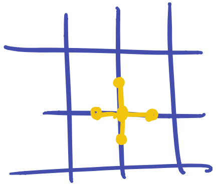


<u>example</u>

$\mathbb{Z}^2 \circlearrowright$



$F =$



This is a contradiction. We should have added $g^{-1}e$ already $\square$

We proved: F is a union of edges from different G-orbits.

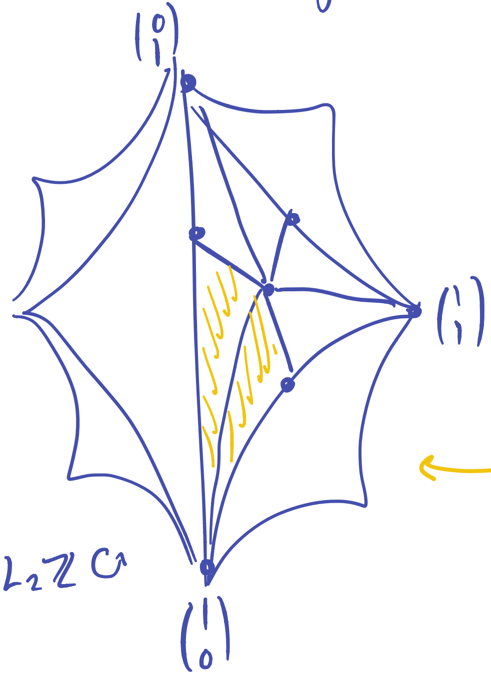
So: any two translates of F can only meet at vertices.

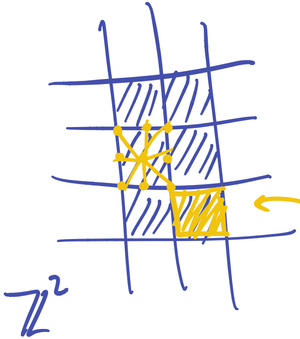
So: the $\{g\cdot F\}$ "tile" Γ'.

Aside: There is a higher dimensional version.

2D cell complex.

$SL_2\mathbb{Z} \circlearrowright$





Fundamental domain

Thm. Say $G \curvearrowright \Gamma$ = conn. graph

$H \le G$

$F_G \subseteq \Gamma'$ fund. dom for G

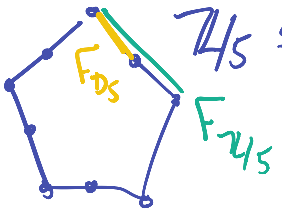
$F_H \subseteq \Gamma'$ fund dom for H

If $F_H = g_1 F_G \cup \cdots \cup g_n F_G$

then $[G:H] = n$.

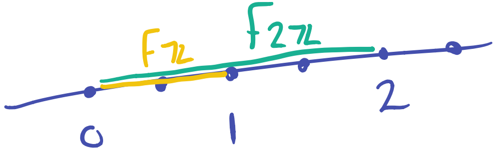
Examples

① $\mathbb{Z}/5 \le D_5$



② $n\mathbb{Z} \le \mathbb{Z}$

$n = 2$



Thm. Say $G \curvearrowright \Gamma$ = conn. graph

$H \le G$

$F_G \subseteq \Gamma'$ fund. dom for G

$F_H \subseteq \Gamma'$ fund dom for H

If $F_H = g_1 F_G \cup \cdots \cup g_n F_G$

then $[G:H] = n$.

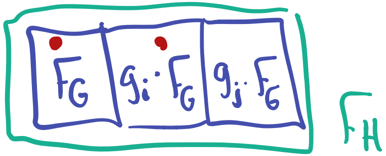
Pf. Define

$$\{g_i \cdot F_G\}_{i=1}^{n} \longrightarrow \{G/H\}$$

$$g_i \cdot F_G \longmapsto g_i H$$

Want: bijection

A picture:



Injectivity Suppose $g_i H = g_j H$

$\Longrightarrow g_i^{-1} g_j \in H$

$\Longrightarrow g_i^{-1} g_j$ does not identify two pts in interior of F_H

$\Longrightarrow g_i \cdot F_G = g_j \cdot F_G$

otherwise $g_i^{-1} g_j$ takes $g_j^{-1} \cdot F_G$ to $g_i^{-1} \cdot F_G$.

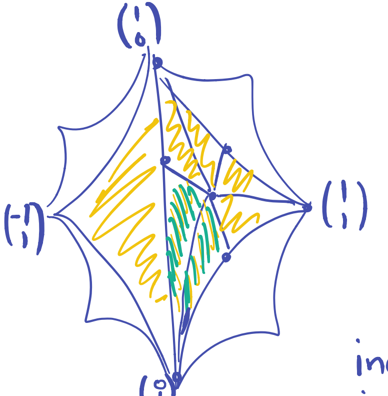
FINISH

An application

$SL_2(\mathbb{Z})[m]$ = level m congruence subgp of $SL_2\mathbb{Z}$

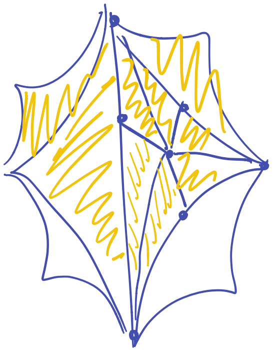
$= \{A \in SL_2(\mathbb{Z}) : A \equiv I \mod m\}.$

In $SL_2(\mathbb{Z})[2]$: $\pm I$, $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

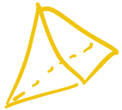


$SL_2(\mathbb{Z})[2]$

index 6 in $SL_2(\mathbb{Z})$



$SL_2(\mathbb{Z})[3]$

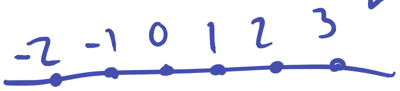


index 12 in $SL_2\mathbb{Z}$

Chap 2. Groups gen. by reflections

(Coxeter groups)

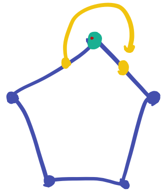
∞-gon.

Infinite Dihedral Gp $=$ Sym() $= D_\infty$

Sample elts: translate by n

reflect in vertex

reflect about middle of edge.



It is gen by reflections about 0, $1/2$.

More next time!

Announcements Feb 11

- Cameras on
- Abstracts Feb 26 : consult with me
- HW4 due Thu 3:30
- Office hours Fri 2-3, Tue 11-12, appt

Today: D_∞
Triangle gps
Coxeter groups.

From last time:

<u>Thm</u>. If $G \curvearrowright \Gamma$

fund dom F. and $g \cdot F = F$ $\Rightarrow g = id$.

$H \le G$

fund dom F_H

<u>and</u>. $F_H = g_1 \cdot F \cup \dots \cup g_n \cdot F$

then $[G:H] = n$.

index 2

e.g. $2\mathbb{Z} \le \mathbb{Z} \curvearrowright$ 

$2\mathbb{Z} \times 1 \le \mathbb{Z} \times \mathbb{Z}/2$ index 4

Noah's question:

Take $G \curvearrowright \Gamma$. F

index n $H \le G$ F_H

Now: K other gp.

$G \times K \curvearrowright \Gamma$

$H \times 1 \le G \times K$ index bigger.

Same fund domains as before?

If yes: seems like contradiction.

Fix

Infinite Dihedral Group

$\Gamma =$ 

$D_\infty = \mathrm{Sym}(\Gamma)$

Last time: gen. by

$a =$ refl. about 0

$b =$ refl. about $1/2$.

Presentation?

To start: $a^2 = b^2 = id$

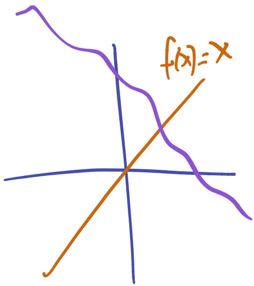
What else?

Typical elt of gp:

~~$aba^{-1}b^7a^4bab^{-1}$~~

~~really, this is:~~

~~$ababebab \rightsquigarrow a$~~



alternating word in a, b.

So all elts are:

reflections by IVT

about what?

translation by n $\leftarrow (ab)^n$ $\qquad (ab)^n a$

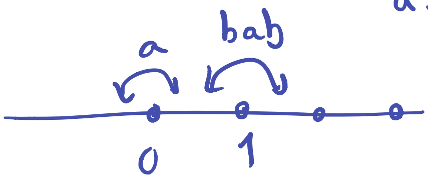
$(ba)^n$ $\qquad (ba)^n b$ $\qquad n \geq 0.$

translation by $-n$.

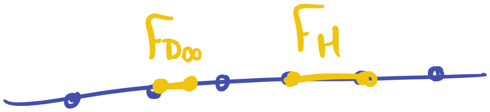
Presentation: $D_\infty \cong \langle a, b \mid a^2 = b^2 = id \rangle$

A subgp of D_∞

$H = \langle a, bab \rangle$ = subgp gen by a, bab in D_∞.



H is isomorphic to D_∞



$[D_\infty : H] = 2$.

By the way:

H = kernel of

$$D_\infty \to \mathbb{Z}/2$$

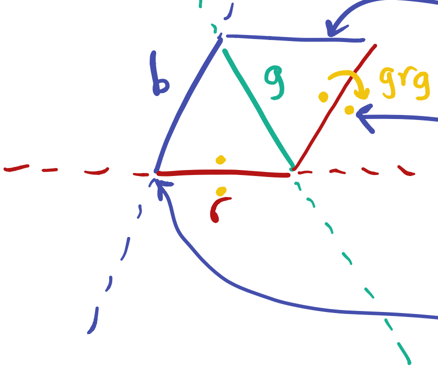
"count # of b's mod 2"

An explicit $H \to D_\infty$

$$a \longmapsto a$$
$$bab \longmapsto b$$

Triangle groups

W_{333} = gp gen. by reflections in



What are g, grg, gbg? mult. choice.

rb? no choice.

grg = reflection about image of r under g.

gbg

rb = rotation by $2\pi/3$

Goals: Fund. domain.

Presentation

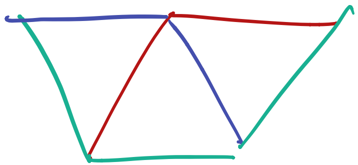
Some relations: $r^2 = b^2 = g^2 = id$

$(rb)^3 = (rg)^3 = (gb)^3 = id.$

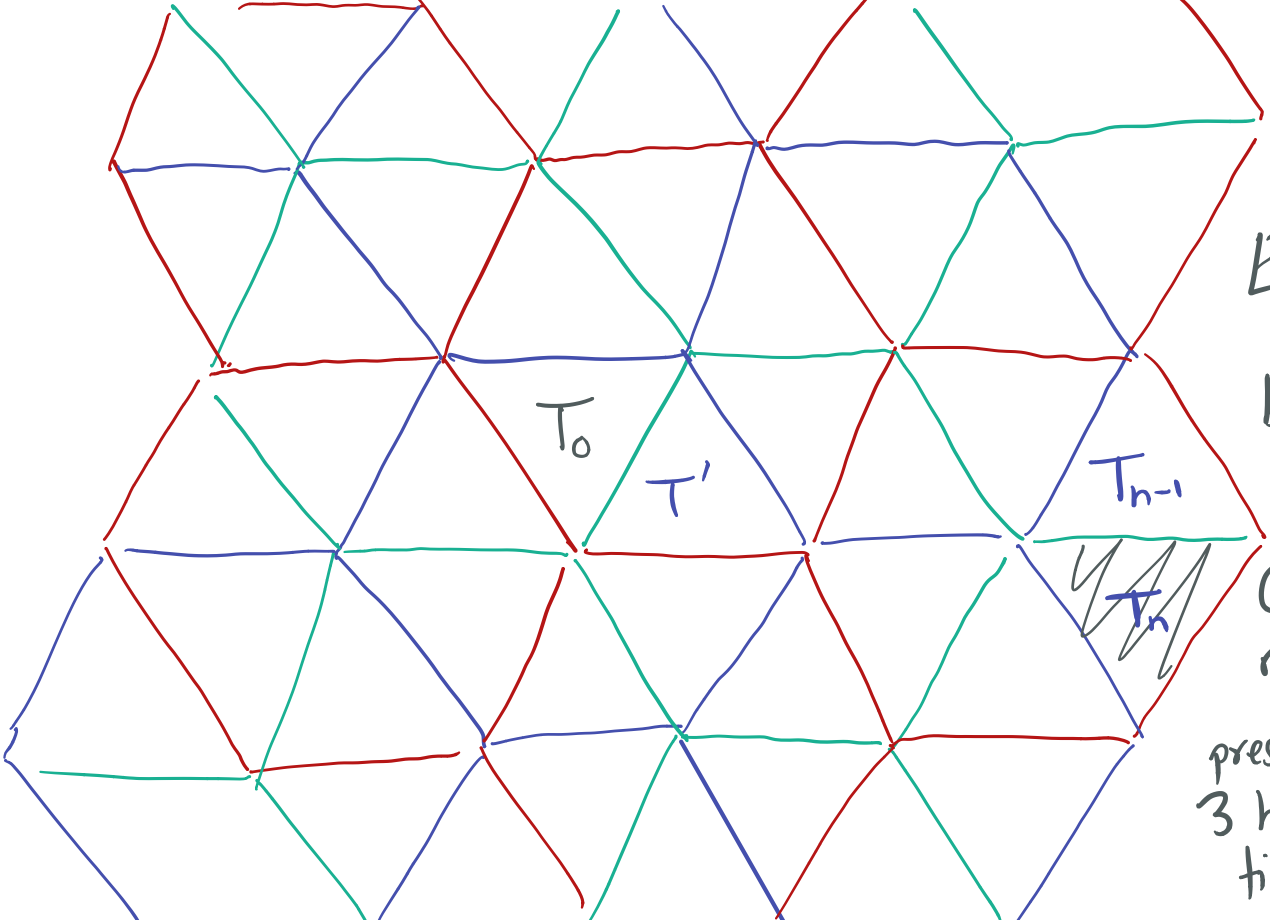
Guess for fund domain: original triangle.

To this end... take tiling of $\mathbb{E}^2$ by $\triangle$

& color the edges:



Critical point: this coloring is well-defined.



Each color is tiling by regular hexagons.

Check: r,g,b presene these 3 hexagonal tilings.

We just showed

Prop. The coloring is well defined.

Cor. If $g \in W_{333}$ & $g \cdot T_0 = T_0$ then $g = id$.

So the fund domain is at least as big as T_0.

To show T_0 is a fund domain, need that ~~W_{333} acts trans. on triangles.~~ Equivalently $W_{333} \cdot T_0 = \mathbb{H}^2$.

Prop. Let T be a triangle of the tesselation, and $T_0, T_1, \dots, T_n = T$ is a seq of triangles s.t. $T_i \cap T_{i+1}$ is an edge colored $c_i \in \{r, g, b\}$.

Then $c_1 \cdots c_n \cdot T_0 = T$.

Prop. Let T be a triangle of the tesselation.

and $T_0, T_1, \ldots, T_n = T$ is a seq of triangles s.t. $T_i \cap T_{i+1}$ is an edge colored $c_i \in \{r, g, b\}$.

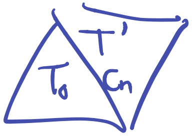
Then $c_1 \cdots c_n \cdot T_0 = T$.

Pf. Induct on n.

$n = 0$ ✓

Inductive hyp:

$c_1 \cdots c_{n-1} \cdot T_0 = T_{n-1}$

Define T' : 

Note $T' = c_n T_0$

Have $c_1 \cdots c_{n-1} T' = T_n$

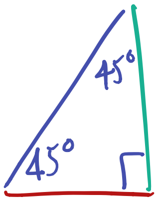
$c_1 \cdots c_{n-1} c_n T_0 = T_n$ $\square$

Coxeter groups: all generators have order 2.

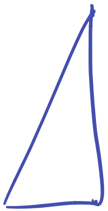
all other relations:

$$(ab)^n = \mathrm{id}.$$

e.g. D_n.



W_{244}



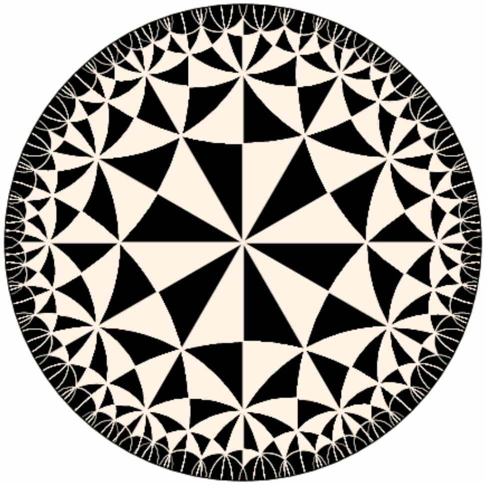
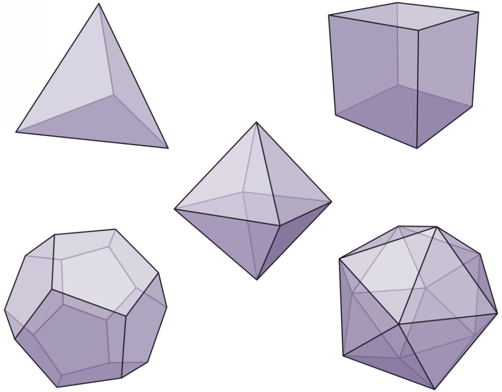


Figure 7 from Coxeter's address to the Royal Society of Canada



Announcements Feb 19

- Cameras on.
- HW 4 due Thu 3:30
- Abstracts Feb 26 : consult with me before Feb 26.
- Take home midterm Mar 4
- Fri office hours moved (requests?)

  Office hours Tue 11-12, appt.

Sample HW Solution

21. Let H be the subset of $\mathrm{Sym}_{\mathbb{Z}}$ so that $h \in H$ iff $\exists$ finite $C \subset \mathbb{Z}$, $k \in \mathbb{Z}$ s.t. $h(n) = n + k$ for $n \notin C$.

state problem.

(a) Show that H is a subgroup of $\mathrm{Sym}_{\mathbb{Z}}$.

white space!

__identity__: We see $e \in \mathrm{Sym}_{\mathbb{Z}}$ belongs to H by taking $C = \emptyset$, $k = 0$.

Scannable!

__inverses__: Let $h \in H$ with associated C, k. Let $C' = C + k$, $k' = -k$. Then h^{-1}, C', k' satisfy the required

systematic!

headings!

conditions. Indeed: $|C'| = |C| < \infty$ and

$$
\begin{aligned}
& n \notin C' \\
\Rightarrow\ & n-k \notin C \\
\Rightarrow\ & h(n-k) = n \\
\Rightarrow\ & h^{-1}(n) = n-k = n+k'.
\end{aligned}
$$

sequence of implications! (aligned)

details!

<u>Composition</u>. Let $h_1, h_2 \in H$ with associated C_1, k_1 & C_2, k_2.

Let $C' = (C_1 - k_2) \cup C_2$, $k' = k_1 + k_2$.

Then $h_1 h_2$, C', k' satisfy the required conditions since

parallel structure!

$$|C'| = |(C_1 - k_2) \cup C_2| \leq |C_1 - k_2| + |C_2| = |C_1| + |C_2| < \infty$$

and $\quad n \notin C'$

$$\Rightarrow n \notin C_2 \text{ and } n \notin C_1 - k_2$$

$$\Rightarrow n \notin C_2 \text{ and } n + k_2 \notin C_1$$

$$\Rightarrow h_1 h_2(n) = h_1(n + k_2) = n + k_1 + k_2 = n + k'$$

(b) Show that H is finitely generated.

Let $s = (0 \quad 1)$

$t = (\cdots -1 \quad 0 \quad 1 \quad 2 \cdots)$

We will show that s, t generate H.

Claim $(i \;\; i{+}1) = t^i s t^{i-1}$

Claim!

Pf of Claim.

Editing!

Let $h \in H$ with associated c, k.

Note $t^{-k}h$ has associated $c' = c - k$, $k' = 0$.

So $t^{-k}h$ can be regarded as an element of

$\mathrm{Sym}_{c'} \subseteq \mathrm{Sym}_{\mathbb{Z}} \ldots$

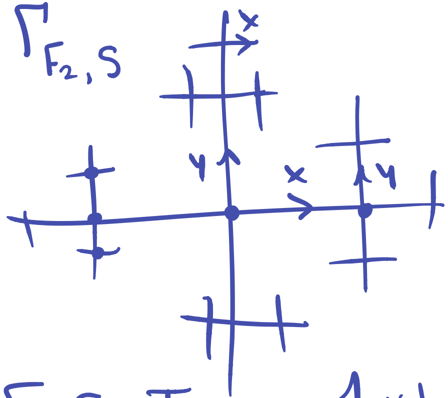
Chap 3 Groups acting on trees.

3.1 Free groups.

$F_2 = \langle x, y \mid \, \rangle$

$S = \{x, y\}$

$\Gamma_{F_2, S}$



So $F_2 \curvearrowright T_4$ = reg. 4 valent tree.

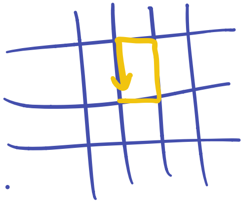
Why?

Non-backtracking
Paths in Cayley graph.

$\longleftrightarrow$ reduced words in x, y.

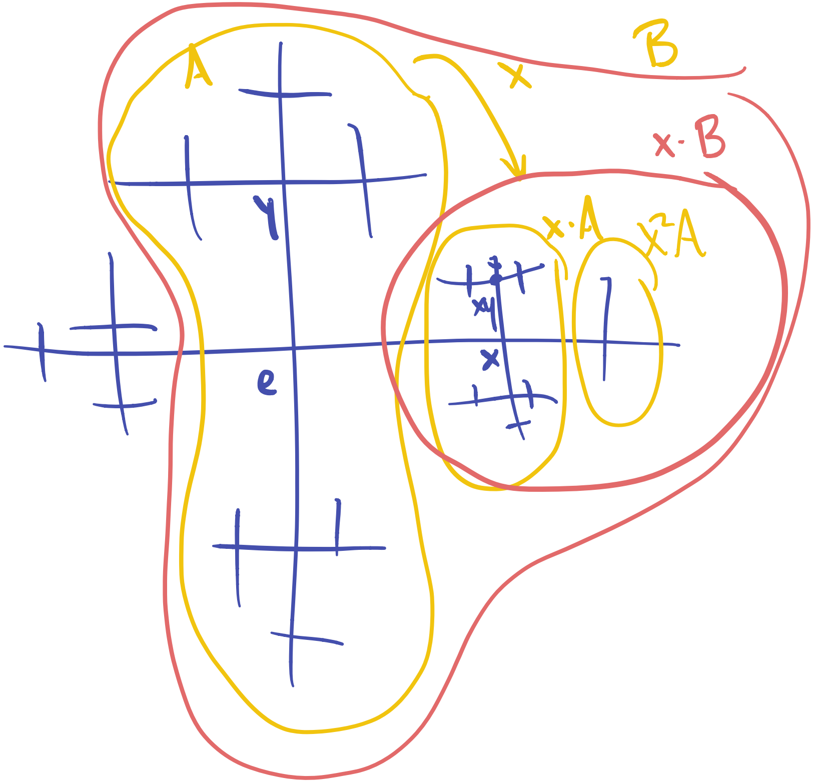
For free group with free gen set: no loops in Cayley graph.

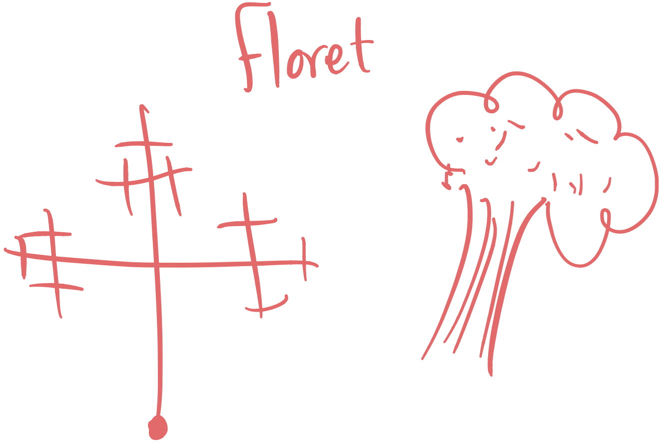
Or: relations among generators $\longleftrightarrow$ circuits in Cayley graph.



The action $F_2 \circlearrowright T_4$

What does x do?





<u>Goal</u> Let $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Then x, y generate a subgp of $SL_2\mathbb{Z}$, denoted $\langle x, y \rangle$

<u>Thm</u>. $\langle x, y \rangle \cong F_2$.

In other words, every nonempty freely reduced word in $x^{\pm 1}, y^{\pm 1}$ multiplies to a nontrivial matrix.

False if you replace the 2's with 1's.

Indeed...

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

$$= SL_2\mathbb{Z}.$$

proof: row reduction.

which is not free

because... torsion

<u>Exercise</u>. Free groups are torsion free.

Ping Pong Lemma

Say $G \curvearrowright X$ = set

$a, b \in G$

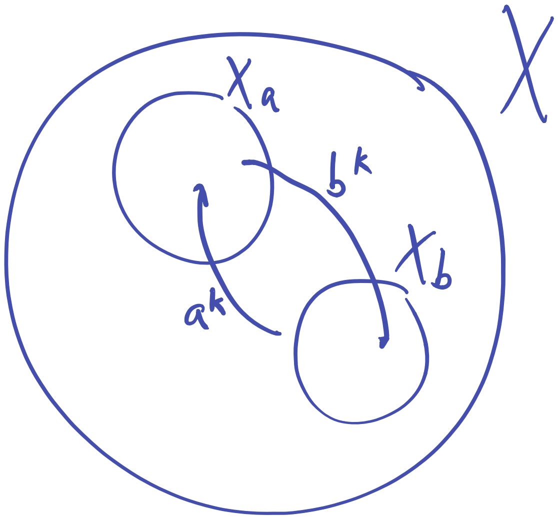
$X_a, X_b \subseteq X$

nonempty, disjoint

$a^k \cdot X_b \subseteq X_a \quad \forall k \neq 0$

$b^k \cdot X_a \subseteq X_b \quad \forall k \neq 0$

Then $\langle a, b \rangle \cong F_2$.



Pf by example

Q. Why is $abab^2a^3 \neq id$?

A. For any $x \in X_b$

$abab^2a^3 \cdot x$ in X_a hence $\neq x$.

Ping Pong Lemma

Say $G \curvearrowright X$ = set

$a, b \in G$

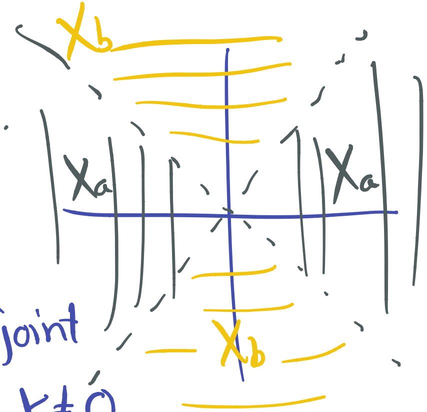
$X_a, X_b \subseteq X$

nonempty, disjoint

$a^k \cdot X_b \subseteq X_a \quad \forall k \neq 0$

$b^k \cdot X_a \subseteq X_b \quad \forall k \neq 0$

Then $\langle a, b \rangle \cong F_2$.



Application 1

$G = SL_2(\mathbb{Z})$

$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$X = \mathbb{Z}^2$

$X_a = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2 : |p| > |q| \right\}$

$X_b = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2 : |q| > |p| \right\}$

Check: If $\begin{pmatrix} p \\ q \end{pmatrix} \in X_b$, $k \neq 0$ then

$a^k \cdot \begin{pmatrix} p \\ q \end{pmatrix} \in X_a$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -99 \\ 101 \end{pmatrix} = \begin{pmatrix} 103 \\ 101 \end{pmatrix}$$

Check: If $\begin{pmatrix} p \\ q \end{pmatrix} \in X_b$, $k \neq 0$ then

$$a^k \cdot \begin{pmatrix} p \\ q \end{pmatrix} \in X_a$$

$$a^k \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} p \\ q \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$= \begin{pmatrix} p + 2kq \\ q \end{pmatrix}$$

But $|p + 2kq| \geq |2kq| - |p|$

$$= 2|k||q| - |p|$$

$$> 2|k||q| - |q|$$

$$> |q|$$

$\square$.

Application 2. $\mathrm{Homeo}(\mathbb{R})$

$\mathrm{Homeo}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} :$

f contin with

(contin) inverse $\}$

group op: $f \circ g$

Poll. If $f : \mathbb{R} \to \mathbb{R}$ continuous & bij

is f^{-1} automatically contin.

Yes

Invertible: horiz. & vert.

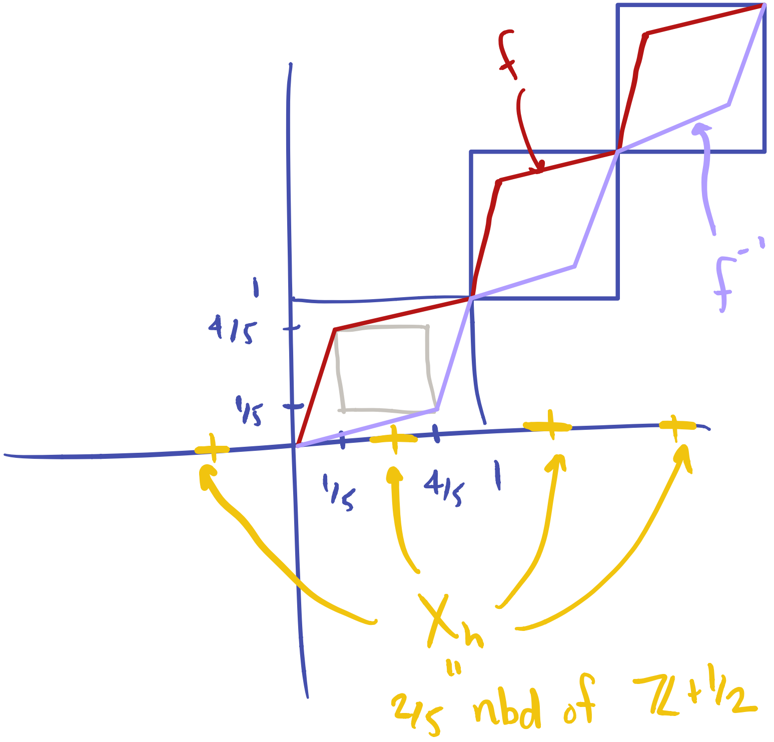
line test.

Inverse: flip over $y = x$.

Goal: $F_2 \leq \mathrm{Homeo}(\mathbb{R})$.

Let $f(x)$ be:

$X_f = \frac{2}{5}$ nbd of $\mathbb{Z}$



$g(x)$ is same,
shifted right by $\frac{1}{2}$
up by $\frac{1}{2}$

$$g(x) = f(x-\tfrac{1}{2})+\tfrac{1}{2}$$

<u>Prop.</u> $\langle f(x), g(x)\rangle \cong F_2$

Ping pong! $X = \mathbb{R}$

$$X_f = \bigcup_{n\in\mathbb{Z}} [n-\tfrac{1}{5}, n+\tfrac{1}{5}]$$

$$X_h = \bigcup_{n\in\mathbb{Z}} [n-\tfrac{1}{5}+\tfrac{1}{2}, n+\tfrac{1}{5}+\tfrac{1}{2}]$$

Next time: $F_3 \leq F_2$ (and $F_2 \leq F_3$)

index 2. ∞ index

& $F_\infty \leq F_2$

$\mathbb{Z} \leq \mathbb{Z}$

index n.

ANNOUNCEMENTS FEB 18

- Cameras on
- HW 5 due Thu
- Abstracts Feb 26 → consult with me ahead of time by meeting/chat/email
- Midterm Mar 4
- Fri office hour @ 10 (just tomorrow)
- Office hours Tue 11-12, appt.

Today: $F_3 \leq F_2$, $G \circlearrowright T$ freely $\Rightarrow$ G free

Ping pong lemma

$G \curvearrowright X$ = set

$a, b \in G$

$X_a, X_b \subseteq X$ disjoint, nonempty

$b^k(X_a) \subseteq X_b \quad \forall k \neq 0$

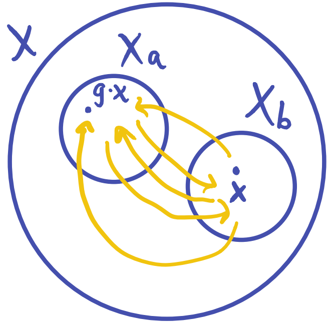
$a^k(X_b) \subseteq X_a \quad \forall k \neq 0$

Then $\langle a, b \rangle \cong F_2$.

Similar: $\langle a_1, \ldots, a_k \rangle \cong F_k$

Pf. If $g = a^* b^* a^* b^* a^*$ freely reduced word in a, b

then $g \neq id$



$g \cdot x \neq x$

$\Rightarrow g \neq id$.

Similar if g starts, ends in b.

If g starts with a, ends with b,

eg. $a^3 b^5 = g$

conjugate so starts, ends with a

$a a^3 b^5 a^{-1} \neq id \Rightarrow g \neq id$. $\square$

3.2 $F_3 \leq F_2$

$F_2 = \langle x, y \mid \ \rangle$

H = subset of F_2 consisting of reduced words of even length.

Let $a = x^2$, $b = xy$, $c = xy^{-1}$

Thm. ① $H \trianglelefteq F_2$ of index 2

② H is gen by a, b, c.

③ $H \cong F_3$

Pf. ① Consider

$$F_2 \longrightarrow \mathbb{Z}/2$$
$$g \longmapsto \text{mod } 2 \text{ word length}$$

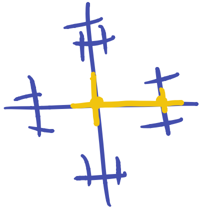
(or: $x \longmapsto 1$, $y \longmapsto 1$ this defines a homom)

② Write all words of length 2 in $\{x, y\}^{\pm 1}$ in terms of a, b, c.

e.g. $y^2 = c^{-1}b$ etc.

or use our thm...

1 edge + 6 half-edges



Let $a = x^2$, $b = xy$, $c = xy^{-1}$

Thm. ① $H \trianglelefteq F_2$ of index 2

② H is gen by a, b, c.

③ $H \cong F_3$

Pf. ③ Let $w = w_1 \cdots w_n$ freely red. word in a, b, c. Want: $w \neq id$.

Set $w_i = \alpha_i \beta_i$ $\alpha_i, \beta_i \in \{x, y\}^{\pm 1}$

We'll show: If w has a cancellation:

$$\cdots \beta_{i-1} \boxed{\alpha_i} \cancel{\beta_i} \cancel{\alpha_{i+1}} \boxed{\beta_{i+1}} \alpha_{i+2} \cdots$$

then no nearby cancellations:

$$\alpha_i \neq \beta_{i+1}^{-1}$$
$$\beta_{i+1} \neq \alpha_{i+2}^{-1}$$
$$\beta_{i-1} \neq \alpha_i^{-1}$$

In other words: α_i, β_{i+1} don't cancel.

Case by case check.

e.g.
	w_{i-1}	w_i	w_{i+1}	
	?	a^{-1}	b	
	(5 choices everything except a) ✓	$(x^{-1}\cancel{x^{-1}})$	$(\cancel{x}y)$ ✓	(5 choices)

$\square$

3.4 Free groups and actions on trees.

Say $G \circlearrowright \Gamma$ = graph.

The action is <u>free</u> if

$$g \cdot v = v \Rightarrow g = id.$$

$$\& \quad g \cdot e = e \Rightarrow g = id$$

$$\forall \, g \in G, \, v \in V(\Gamma), \, e \in E(\Gamma)$$

<u>example</u>. $F_2 \circlearrowright T_4$ free.

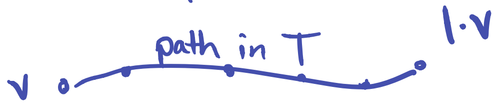
$\mathbb{Z}/n \circlearrowright$ n-cycle free

$D_n \circlearrowright$ n-cycle not free

$G \circlearrowright \Gamma_{G,S}$ free.

<u>Lemma</u>. Any action $\mathbb{Z}/2$ on a tree T is not free.

<u>Pf</u>. v = any vertex



Paths unique $\Rightarrow$ $\mathbb{Z}/2$ preserves the path.

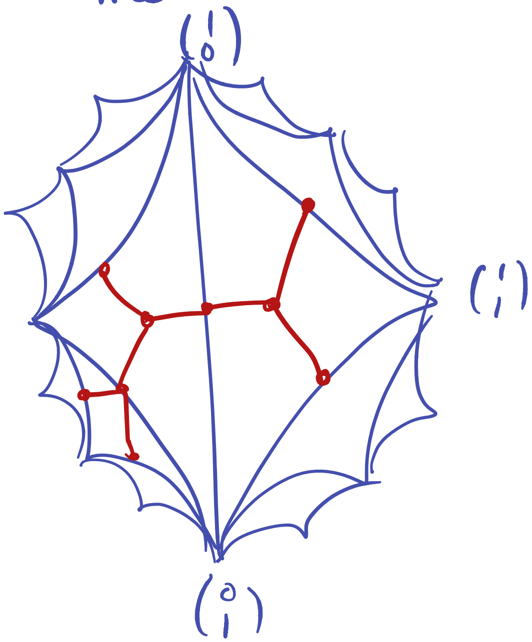
$\Rightarrow$ $\mathbb{Z}/2$ fixes the midpt of path

$\Rightarrow$ fixed edge or vertex. $\square$

Exercise: generalize to $\mathbb{Z}/m$.

<u>Cor</u>. If G has torsion (elt of finite order) then any $G \circlearrowright T$ not free.

Poll: Is $SL_2(\mathbb{Z})[2] \circlearrowright$ Farey tree free?



actually, $-I$ fixes the whole tree.

No. $-I$ has order 2...

Also can find a matrix that "rotates" any vertex

If an elt of $SL_2(\mathbb{Z})$ fixes an edge, it fixes both vertices:

$$\mathrm{Stab}(e) \leq \mathrm{Stab}(v)$$

Thm. If a group acts freely on a tree, it is free.

Cor. Subgroups of free gps a free (hard to prove directly).

Pf #1 of Thm Ping Pong

$G \curvearrowright T$ = tree freely.

F = fundamental dom.

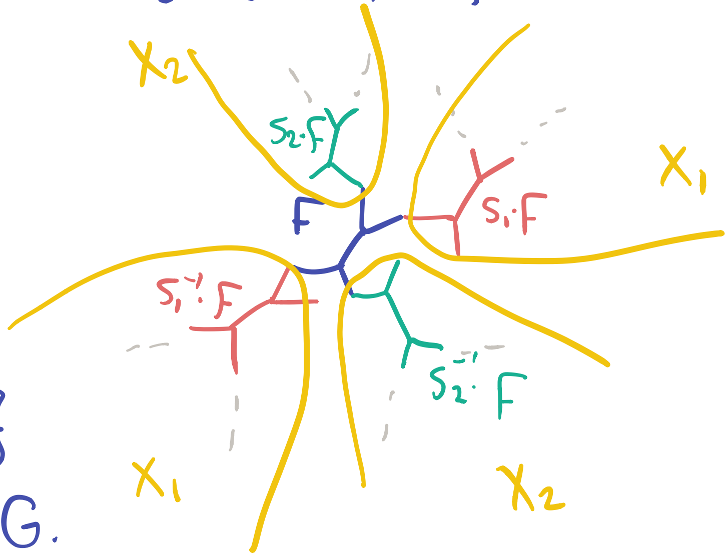
$S = \{g \in G : g \cdot F \cap F \neq \emptyset\}$

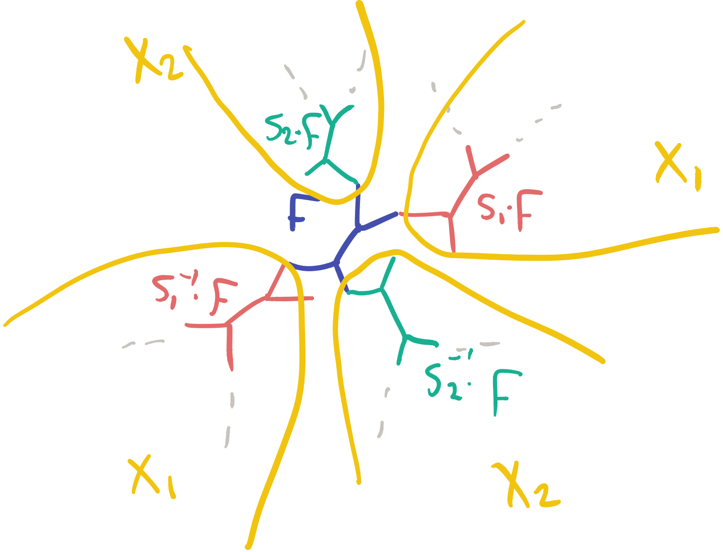
Earlier theorem: S generates G.

From S remove duplicates: x, x^{-1}.

To show: group gen. by S (i.e. G) is free.

$S = \{s_1, s_2, \dots\}$





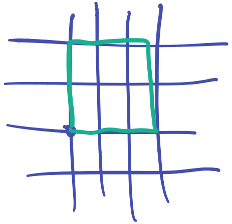
To check: $s_2 \cdot X_1 \subseteq X_2$.

$$s_2^{-1} F \xrightarrow[\text{to}]{\text{connected}} F \longrightarrow s_1 F \longrightarrow \overset{\text{rest of}}{X_1}$$

apply s_2:

$$F \longrightarrow s_2 F \longrightarrow (s_2 s_1 F) \longrightarrow s_2 \cdot X_1$$

ANNOUNCEMENTS FEB 23



- Cameras on
- HW5 due Thu
- Abstracts Fri Feb 26 Gradescope (Team submissions)
- Take home midterm March 4
- Office Hours moved to 1:00 Thu
- Regular office hours Tue 11, appt
- Ask for help on HW!

Today

- Ping pong lemma
- Free actions on trees $\leftrightarrow$ free groups
- Free actions on edges of trees $\leftrightarrow$ free products

Ping Pong Lemma II

Lemma 3.10

Have $G \circlearrowright X$ = set

$S \subseteq G$ ✓

$\forall s \in S \cup S^{-1}$: $X_s \subseteq X$ ✓

⓪ $p \in X \setminus \bigcup_s X_s$ ✓

and ① $s \cdot p \in X_s \quad \forall s \in S \cup S^{-1}$ ✓

② $s \cdot X_t \subseteq X_s \quad \forall t \neq s^{-1}$

↑ Meier says $\subsetneq$???

Then: $\langle S \rangle \cong F_S$

$\langle S \rangle$ means subgp gen by S.

F_S = free gp on S.

Distinctions from P.P.L. I :

① X_s's not disjoint

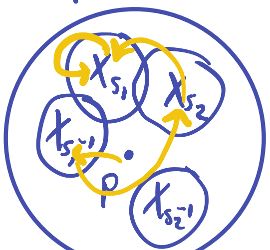
(replaced with existence of p)

② Only need $s^k \cdot X_t \subseteq X_s$ $k=1$.

(replaced with $s \cdot X_s \subseteq X_s$).

Pf. Look where p goes.

$s_1 \cdot s_1 \cdot s_2 \cdot s_1^{-1}$



3.4 Free gps & actions on trees

Thm. If a group acts freely on a tree, then it is free.

Pf #1 Say $G \curvearrowright T$ = tree

Let F = fund dom.

Call the $g \cdot F$'s tiles.

$S = \{ g \in G : g \cdot F \cap F \neq \emptyset \}$

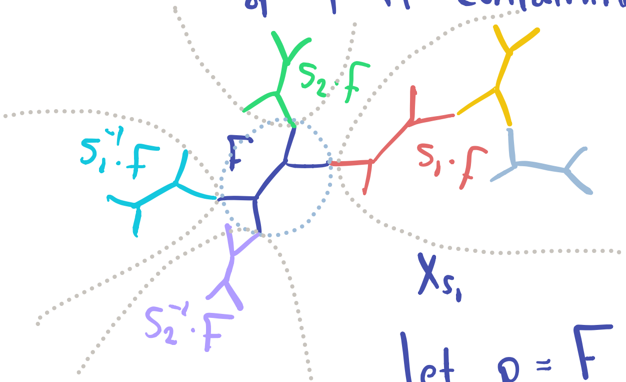
Previous thm $\Rightarrow$ S generates G.

To show: S generates a free gp.

Ping pong!

$X = \{\text{tiles}\}$

$X_s = \{$tiles that lie in component of $T \setminus F$ containing $s \cdot F\}$



Let $p = F$.

Freeness $\Rightarrow$ $s_i \cdot F \neq F \;\; \forall i$

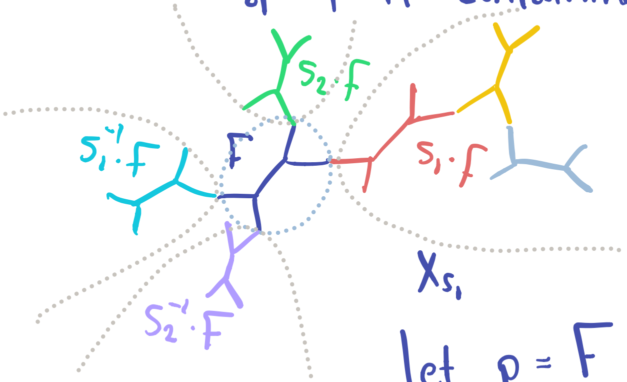
$\Rightarrow$ ⓪ in P.P.L. II.

① in PPL is by defn.

Remains to check ②.

$X = \{\text{tiles}\}$ tile in X_{s_2}

$X_s = \{$tiles that lie in component of $T \setminus F$ containing $s \cdot F\}$



Let $p = F$.

Freeness $\Rightarrow s_i \cdot F \neq F \quad \forall i$

$\Rightarrow$ ⓪ in P.P.L. II.

① in PPL is by defn.

Remains to check ②.

For ② we'll do: $s_1 \cdot X_{s_2} \subseteq X_{s_1}$

Consider the seq. of adjacent tiles:

$$s_1^{-1} \cdot F \longrightarrow F \longrightarrow s_2 F \longrightarrow \text{rest of } X_{s_2}$$

Apply s_1:

$$F \longrightarrow s_1 \cdot F \longrightarrow \underbrace{s_1 s_2 F \longrightarrow \text{rest of } s_1 \cdot X_{s_2}}_{s_1 \cdot X_{s_2} \subseteq X_{s_1}}$$

since T is a tree!

$\square$

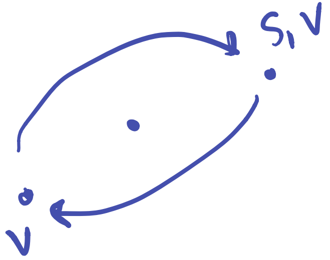
Tricky Special case: $s_1 \cdot X_{s_1} \subseteq X_{s_1}$

$$s_1^{-1}\cdot F \longrightarrow F \longrightarrow s_1 F \longrightarrow \text{rest of } X_{s_1}$$

Apply s_1:

$$F \longrightarrow s_1\cdot F \longrightarrow \underbrace{s_1 s_1 F \longrightarrow \text{rest of } s_1\cdot X_{s_1}}_{s_1\cdot X_{s_1} \subseteq X_{s_1}}$$

since T is a tree!



Fine if $s_1 s_1 F \neq F$

i.e. $s_1^2 = \mathrm{id}$.

Lemma from last time:

$\mathbb{Z}/2$ does not act freely on a tree.

$\Longrightarrow$ any gp with elt of order 2 does not act freely on a tree.

Is this page needed???

Cor. Subgroups of free gps are free.

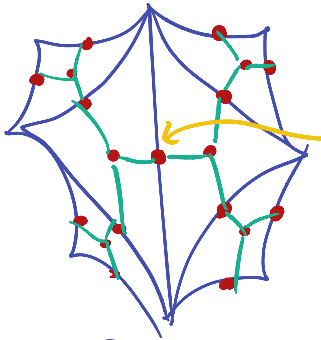
Pf. F = free gp

$F \curvearrowright T$ some tree (Cayley graph) freely.

Any subgp inherits a free action.

Apply the theorem. $\square$

Example $SL_2(\mathbb{Z})[m]$ is free $m \geq 3$.



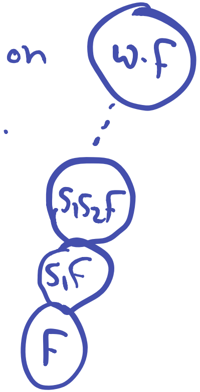
Check freeness.

Matrices fixing center vertex:

$\pm I \quad \pm\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

etc.

Thm. If a group acts freely on a tree, then it is free.



Pf #2 (ONGGT)

$G \curvearrowright T$ freely.

F = fund dom

$\rightsquigarrow S = \{s \in G : s\cdot F \cap F \neq \emptyset\}$.

Take a freely reduced word in S.

$w = s_1 \cdots s_k \qquad s_i \in S$.

Want: $w \neq \mathrm{id}$.

Will show: $w \cdot F \neq F$.

Will show

$F,\ s_1\cdot F,\ s_1s_2\cdot F, \ldots,\ s_1\cdots s_k\cdot F$

is a non-backtracking sequence of adjacent tiles.

Since T is a tree this implies

$s_1\cdots s_k\cdot F \neq F$.

Check $s_1\cdots s_i\cdot F$ adjacent, and not equal to $(s_1\cdots s_i)s_{i+1}\cdot F$

$s_{i+1}\cdot F$ adjacent to F (not equal F by freeness)

Apply $s_1\cdots s_i$ to both. $\square$

So: $\left\{\begin{matrix}\text{non backtracking}\\ \text{paths of tiles}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{freely red.}\\ \text{words in } S\end{matrix}\right\}$.

Ping Pong Lemma II

Lemma 3.10

Have $G \curvearrowright X$ = set ✓

$S \subseteq G$ ✓

$\forall s \in S \cup S^{-1}: X_s \subseteq X$ ✓

(0) $p \in X \setminus \bigcup_s X_s$ ✓

and (1) $s \cdot p \in X_s \quad \forall s \in S \cup S^{-1}$ ✓

(2) $s \cdot X_t \subseteq X_s \quad \forall t \neq s^{-1}$

↑ Meier says $\subsetneq$???

Then: $\langle S \rangle \cong F_S$

After class, we decided that we need to assume S has no elts of order 2.

Example: $\{\pm 1\} = \mathbb{Z}/2 \curvearrowright \{\pm 1\}$

$S = \{-1\}$ $p = +1$ $-1 \cdot p = -1$

$X_{-1} = \{-1\}$

(2) is vacuous here!

Pf of Thm is ok because we had a lemma about elts of order 2.

Noah suggested an alternate fix where we remove $t \neq s^{-1}$. Not sure if this version has any application

Announcements Feb 25

- Cameras on
- Abstracts Fri Feb 26 Gradescope (Team submissions)
- HW 6 due Thu
- Take home midterm March 4
- No office hour Fri this week.
- Regular office hours Tue 11, appt
- Ask for help on HW!

Today

- Ping pong lemma
- Free actions on trees $\leftrightarrow$ free groups
- Free actions on edges of trees $\leftrightarrow$ free products

Ping Pong Lemma II

Lemma 3.10

Have $G \curvearrowright X$ = set

$S \subseteq G$

$\forall s \in S \cup S^{-1} : X_s \subseteq X$

(0) $p \in X \setminus \bigcup_s X_s$

and (1) $s \cdot p \in X_s \quad \forall s \in S \cup S^{-1}$

(2) $s \cdot X_t \subseteq X_s \quad \forall t \neq s^{-1}$

↑ Meier says $\subsetneq$???

Then: $\langle S \rangle \cong F_S$

After class, we decided that we need to assume S has no elts of order 2.

Example: $\{\pm 1\} = \mathbb{Z}/2 \curvearrowright \{\pm 1\}$

$S = \{-1\} \quad p = +1 \quad -1 \cdot p = -1$

$X_{-1} = \{-1\}$

(2) is vacuous here!

Pf of Thm is ok because we had a lemma about elts of order 2.

Noah suggested an alternate fix where we remove $t \neq s^{-1}$. Not sure if this version has any application.

3.4 Free gps acting on trees

Thm. If a group acts freely on a tree, it is free.

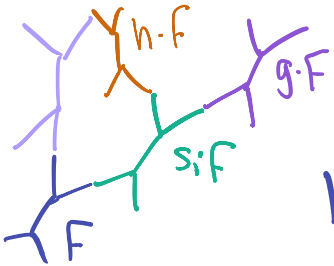
Pf #2 Let $G \circlearrowright T$ = tree freely.

F = fund. dom.

$\rightsquigarrow S = \{g : g\cdot F \cap F \neq \emptyset\}$ is a gen set $\Rightarrow \langle S\rangle = G$

Call the $g\cdot F$ tiles

The action preserves the tiling.



Want: no relations among S.

Let $\omega = s_1 \cdots s_k$ freely red. word in S/elt of G

Want $\omega \cdot F \neq F$ ($\Rightarrow \omega \neq \text{id}$)

treeness $\rightarrow$ True because the word ω gives... a non-backtracking path of tiles from F to $\omega\cdot F$:

$$F,\ s_1 F,\ s_1 s_2 \cdot F, \ldots, s_1 \cdots s_k \cdot F = \omega\cdot F$$

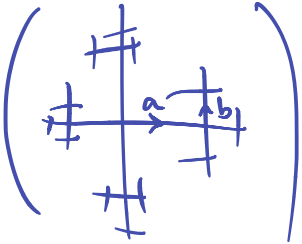
Indeed $s_1 \cdots s_{i+1}\cdot F$ adj to $s_1 \cdots s_i \cdot F$.

Freeness $\rightarrow$ Apply $s_1 \cdots s_i$ to F, $s_{i+1}\cdot F$ $\square$

Sudipta Kolay.

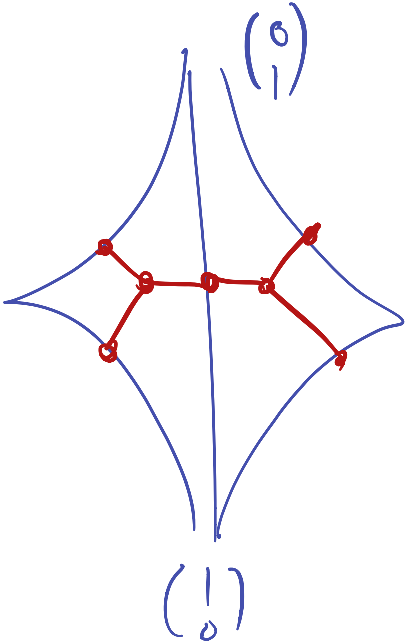
Examples

1. $\mathbb{Z} \circlearrowright$ 

2. Sym^+ 

3. $SL_2\mathbb{Z}[m] \circlearrowright$ Farey tree.

   freely $m \geq 3$.



One partic. vertex of Farey tree:

$\left\{\pm\binom{1}{0}, \pm\binom{0}{1}\right\}$ ← the elts of $SL_2\mathbb{Z}$ taking this vertex to itself: $\pm I, \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

of these, only I is $\equiv I \bmod m$

$m \geq 3$.

3.6 Free products of groups

A, B groups

a word in $A \amalg B$ is freely reduced if alternates between nontrivial elts of A & B, eg:

$$a_1 b_1 a_2 b_2 a_3$$

$$A * B = \{\text{freely red. words in } A \amalg B\}$$

group op: concat & reduce.

$$(a_1 b_1)(b_2 a_2) = a_1 (b_1 b_2) a_2$$

$$a_1 b_3 a_2$$

$$(b_3 = b_1 b_2 \text{ in } B)$$

$$\boxed{\begin{array}{l} \mathbb{Z} \cup \mathbb{Z} = \mathbb{Z} \\ \mathbb{Z} \amalg \mathbb{Z} : \text{2 copies of } \mathbb{Z}. \end{array}}$$

Prop. $A * B$ is well-defined: any word can be reduced to a unique freely red. word. Just like for F_n.

Examples

① $\mathbb{Z} * \mathbb{Z} \cong F_2$

$$(5\ 7\ {-3}\ 10)(-1\ 1)$$

$$= 5\ 7\ {-3}\ 9\ 1$$

$$(x^5 y^7 x^{-3} y^{10})(y^{-1} x)$$

$$= x^5 y^7 x^{-3} y^9 x$$

② $\mathbb{Z}/2 * \mathbb{Z}/2 \cong D_\infty$

alternating words in $1, 1$

" " " a, b

$\cong \langle a, b \mid a^2 = b^2 = id \rangle$

③ $\mathbb{Z}/2 * \mathbb{Z}/3 \cong ?$

$\cong \langle a, b \mid a^2 = b^3 = id \rangle$

Some true things

① $A, B \leq A * B$

② $A * B \twoheadrightarrow A$ (or B)

kernel : B.

③ $A * B \to A \times B$

kernel is free group (next time?)

e.g. $D_\infty \to \mathbb{Z}/2$

word length mod 2

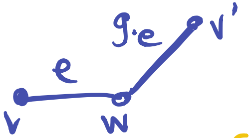
kernel : $\mathbb{Z}$

If $G \curvearrowright T$ denote stabilizer of v by G_v

Thm. If $G \curvearrowright T$ freely free & transitively on edges and fund dom $F = {}^{v}\bullet\!\overset{e}{\text{———}}\!\bullet^{w}$ and v, w in distinct G-orbits.
Then $G \cong G_v * G_w$.

Pf. Step 1. G is gen by G_v & G_w.

$$S = \{g : g \cdot F \cap F \neq \emptyset\}$$



$g \cdot e \in G_w$. since v, w in distinct orbits.

Step 2. Take a freely red word in $G_v \amalg G_w$

$$w = a_1 b_1 a_2 b_2 \cdots a_k b_k$$

To show $w \neq id$ or $w \cdot F \neq F$.

Like last time:

$F, a_1 F, a_1 b_1 F, \ldots$

is a nonbacktracking path.

e.g.

$\square$

tree   almost! $-I$ fixes everything.

Thm. If $G \curvearrowright T$ freely & transitively on edges

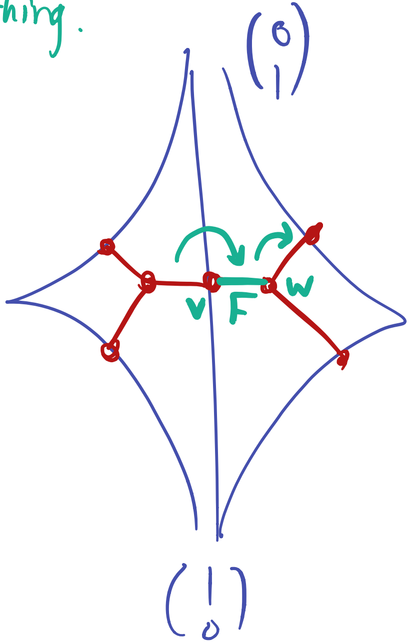
and fund dom $F = \overset{v}{\bullet}\!\!\overset{e}{\longrightarrow}\!\!\overset{w}{\bullet}$ ✓

and v, w in distinct G-orbits. ✓

Then $G \cong G_v * G_w$.

Application $SL_2\mathbb{Z} \curvearrowright$ Farey tree

$$PSL_2\mathbb{Z} = SL_2\mathbb{Z}/\pm I$$
$$\cong \mathbb{Z}/2 * \mathbb{Z}/3.$$



$\square$

Announcements Mar 2

- Cameras on
- HW 6 due Thu
- Midterm Mar 4-11
- Office hours Fri 2-3, Tue 11-12, appt.

Today: Free products & trees

Free products are virtually free.

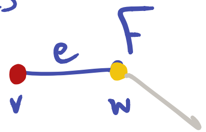
3.6 Free products

$A * B$

Thm. $G \curvearrowright T$ = tree.

freely, transitive on edges. redundant

2 orbits of vertices

fundamental domain 

Then $G \cong G_v * G_w$.

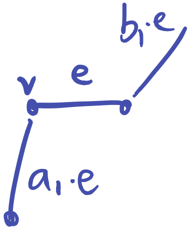
Pf. Step 1. $S = \{g \in G : g \cdot F \cap F \neq \emptyset\}$

$= G_v \cup G_w$

generates G

Step 2. Any freely red word $w = a_1 b_1 \cdots$

$a_i \in G_v \quad b_i \in G_w$



gives a nonbacktracking path from e and $w \cdot e$

the path is:

$e,\ a_1 e,\ a_1 b_1 e,\ \ldots$

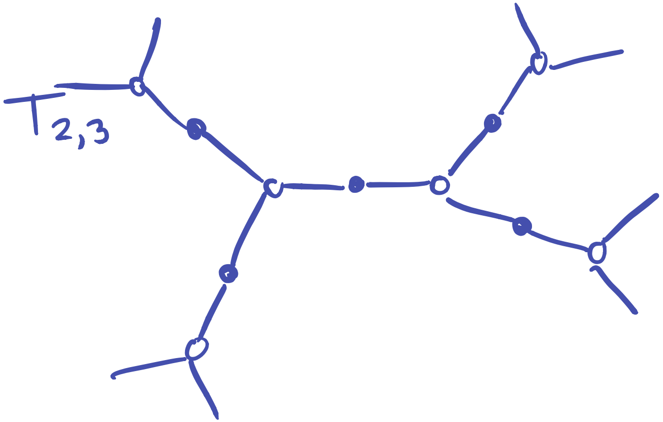
non back track $\Longrightarrow w \cdot e \neq e$

$\Longrightarrow w \neq \mathrm{id}$. $\square$

Application: $PSL_2\mathbb{Z} \cong \mathbb{Z}/2 * \mathbb{Z}/3$

3.8 A converse

Thm 3.28 Say $A * B$ is a free prod.
Then $\exists$ bipartite tree T and an action of $A*B$ satisfying the last theorem.
If $|A|, |B| < \infty$ then $T = T_{|A|,|B|}$



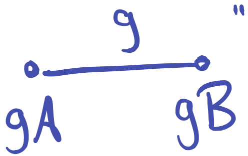
$T_{2,3}$

Pf. blue vertices : cosets of A

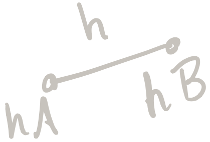
white vertices : cosets of B

in $A*B$.

edges:



"g edge"



$g \in A*B$

Action: left mult

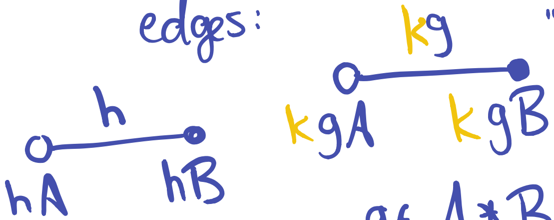
Q. When do g- & h-edges intersect?
A. $g^{-1}h \in A$ or B.

Pf. blue vertices : cosets of A

white vertices : cosets of B

in $A*B$.

edges:



"g edge"

$g \in A*B$

Check things:

① T is bipartite.

have A vertices

B vertices

because can't have $gA = hB$.

If $gA = hB$ then $h^{-1}gA = B$

but $id \in B \Rightarrow id \in h^{-1}gA$

$\Rightarrow h^{-1}gA = A$. But $A \neq B$.

② Action is free on edges. ✓

③ Two orbits of vertices ✓

(same as bipartiteness)

④ Transitively on edges ✓

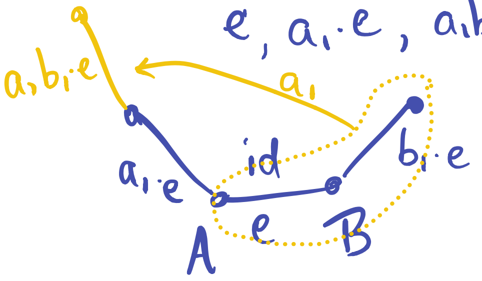
(5) T is connected $\quad g \in A * B$

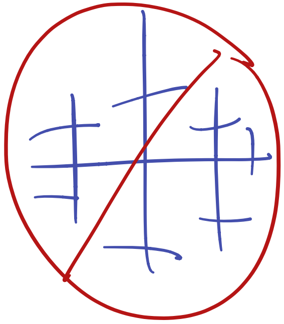
To connect id-edge e to g-edge:

write $g = a_1 b_1 \cdots$

the path of edges is

$e,\ a_1 \cdot e,\ a_1 b_1 \cdot e, \ldots$





(6) T is acyclic.

Nonbacktracking paths

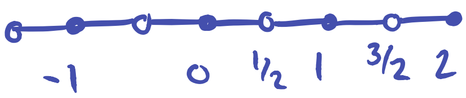
$\longleftrightarrow$ freely red. words

$\square$

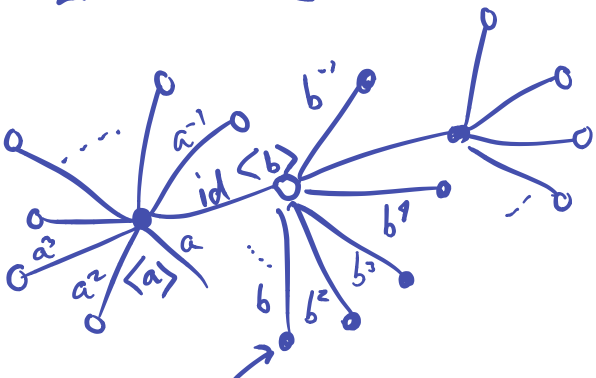
Examples

(1) $\mathbb{Z}/2 * \mathbb{Z}/2$

$\rightsquigarrow T_{2,2} =$



(2) $\mathbb{Z} * \mathbb{Z} \cong F_2 = \langle a, b \rangle$

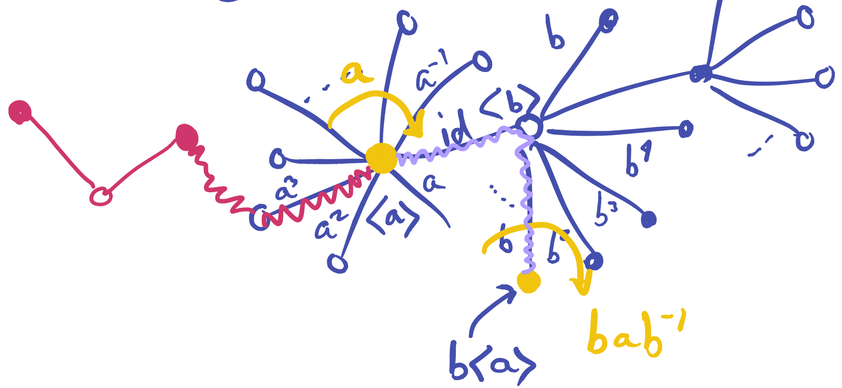


$T_{\infty,\infty}$ $\quad b\langle a \rangle$

The stabilizer of the vertex A is A

Prop. The stabilizer of gA is gAg^{-1}

$$\underline{gAg^{-1}}gA = gA$$



Poll. Consider $\langle a, bab^{-1}\rangle \leq F_2 \ldots$ $\langle a, c\rangle$

is it free?

Yes. Same as proof at start of class...

A reduced word in a, c gives a non-back path (2 edges for each "syllable").

3.8 Thm

Let A, B be finite groups then $A * B$ is virtually free (it has a free subgp of finite index).

Pf. We'll prove more: kernel K of

$$A * B \to A \times B$$

is free. Kernel has index $|A \times B| < \infty$.

Make the tree T for $A * B$ as above.

Check K acts freely.

Stabilizers of edges in $A * B$, hence K, are trivial.

Nontrivial Stabilizers of vertices in $A * B$ are of form gag^{-1} $a \in A$, $a \neq id$. which maps to $a \times id$ in $A \times B$

$\Rightarrow gag^{-1}$ not in K. $\square$

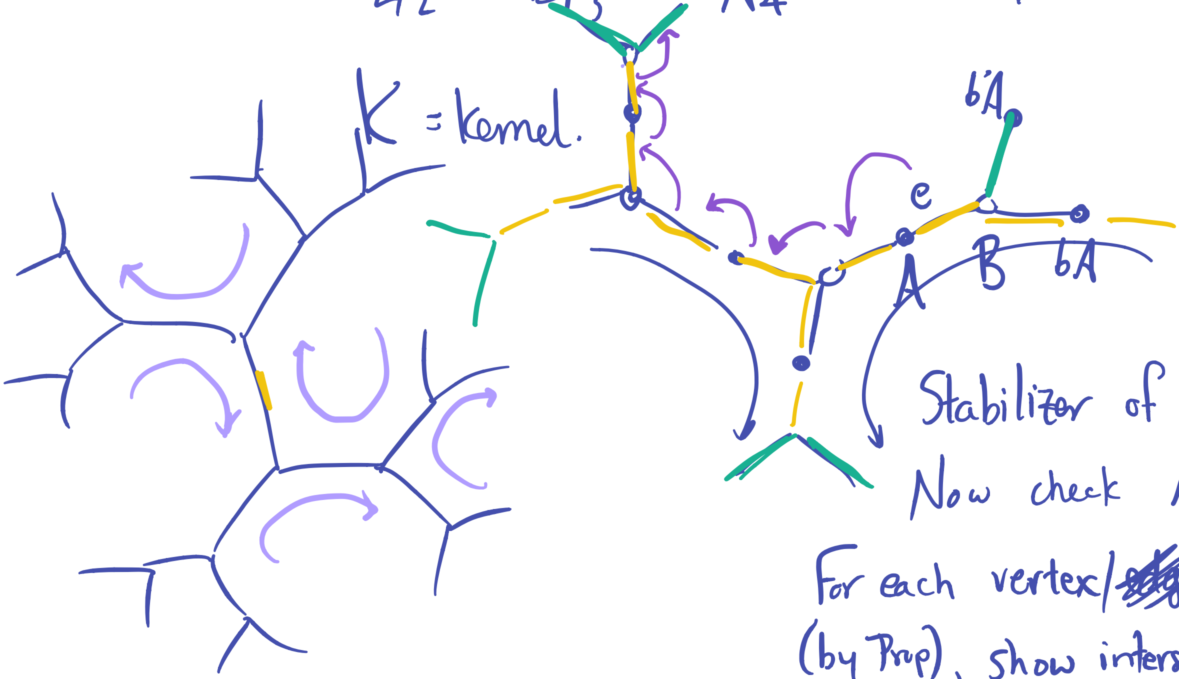
HW #20

A B

$\mathbb{Z}/2 * \mathbb{Z}/3 \curvearrowright T_{2,3}$

$\mathbb{Z}/2 * \mathbb{Z}/3 \to A_4$

K = kernel.

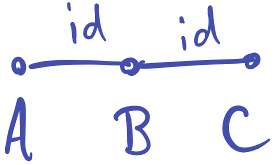


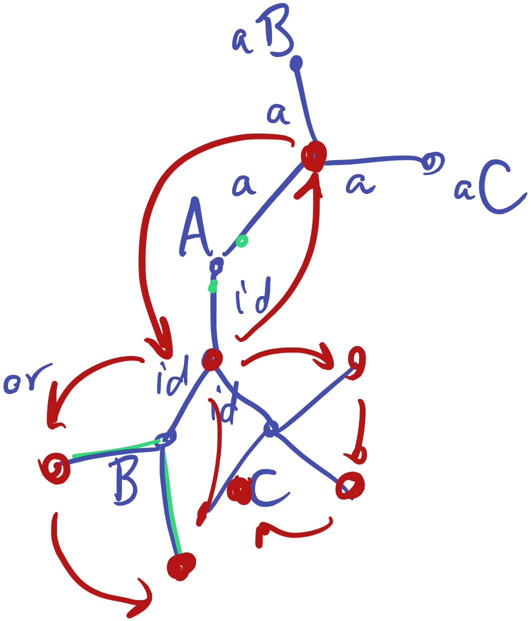
Stabilizer of A is A.

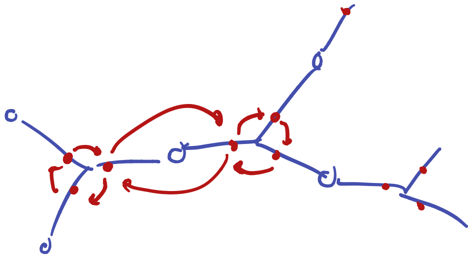
Now check $A \cap K = \{id\}$.

For each vertex/~~edge~~, find stabilizer (by Prop), show intersection with K is $\{1\}$.

Generalizing to $A * B * C$.

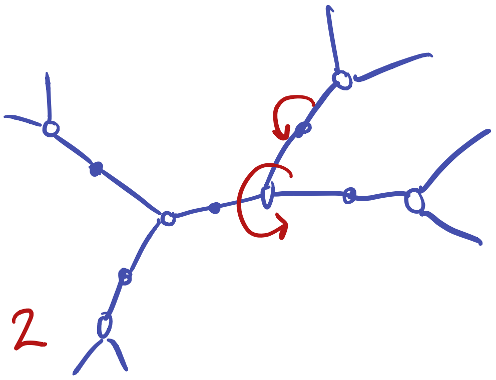






$\mathbb{Z}/2 * \mathbb{Z}/3$

Announcements Mar 9



- Cameras on
- Midterm due Thu 3/11 3:30
- First draft due ~~Mar 26~~ Apr 2
- Office Hours Wed 11-12, Thu 10-10:50, appt
  (no OH Fri this week)

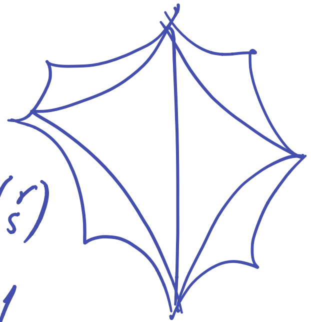
$\begin{pmatrix}1&2\\0&1\end{pmatrix} \in$
$SL_2(\mathbb{Z})[2]$
$= \{A \in SL_2\mathbb{Z} : A \equiv I \bmod 2\}$
$= \ker(SL_2\mathbb{Z} \to SL_2(\mathbb{Z}/2))$

Today: Word problem
Normal forms
BS(1,2)

$\pm\begin{pmatrix}p\\q\end{pmatrix}$ — $\pm\begin{pmatrix}r\\s\end{pmatrix}$

$\Updownarrow$

$\left|\det\begin{pmatrix}p&r\\q&s\end{pmatrix}\right| = 1.$



5 Word problem

Ball of radius n in $\Gamma_{G,S}$: union of paths from id of length $\leq n$

Given $G = \langle S \mid R \rangle$

$\{S \cup S^{-1}\}^* =$ words in $S \cup S^{-1}$

$\pi : \{S \cup S^{-1}\}^* \longrightarrow G$

Word Problem (Dehn): Determine if a given $w \in \{S \cup S^{-1}\}^*$ has $\pi(w) = \mathrm{id}$.

We say WP is solvable if there is an algorithm....

Equivalent to WP:

① Equality problem (does $\pi(w_1) = \pi(w_2)$)

(same as: $\pi(w_1 w_2^{-1}) = \mathrm{id}$?)

② determine which paths in Cayley graph are loops.

②' $\exists$ algorithm to draw ball of radius n in the Cayley graph.

First example: $G = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$

solution to WP: exponent sum.

Second example: $G = \langle a, b \mid \ \rangle = F_2$

Solution to WP: freely red.

$BS(m,n)$ harder... (later today)

A simple example of a group with unsolvable word problem

Donald J. Collins

Generators:

$$a, b, c, d, e, p, q, r, t, k.$$

Relations:

$$p^{10}a = ap,\ p^{10}b = bp,\ p^{10}c = cp,\ p^{10}d = dp,\ p^{10}e = ep,$$

$$qa = aq^{10},\ qb = bq^{10},\ qc = cq^{10},\ qd = dq^{10},\ qe = eq^{10},$$

$$ra = ar,\ rb = br,\ rc = cr,\ rd = dr,\ re = er,$$

$$pacqr = rpcaq, \qquad p^2adq^2r = rp^2daq^2,$$

$$p^3bcq^3r = rp^3cbq^3, \qquad p^4bdq^4r = rp^4dbq^4,$$

$$p^5ceq^5r = rp^5ecaq^5, \qquad p^6deq^6r = rp^6edbq^6,$$

$$p^7cdcq^7r = p^7cdceq^7,$$

$$p^8caaaq^8r = rp^8aaaq^8,$$

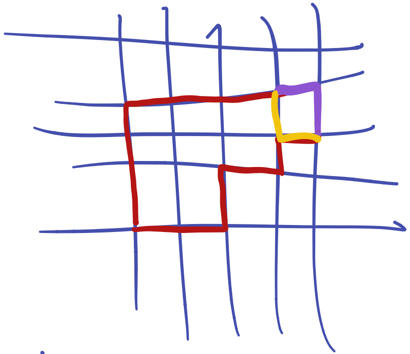
$$p^9daaaq^9r = rp^9aaaq^9,$$

$$pt = tp,\ qt = tq,$$

$$k(aaa)^{-1}t(aaa) = k(aaa)^{-1}t(aaa)$$

How can WP be hard?

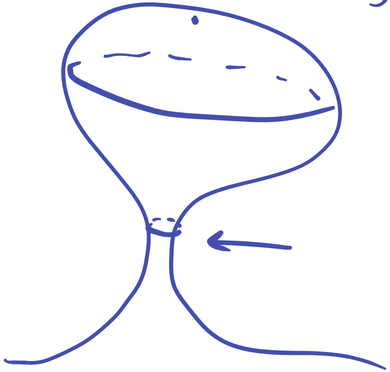
$\mathbb{Z}^2$ example



relations: pushing across squares.

Given a word w with $\pi(w) = id$, can make it monotonically shorter using relations.

To have unsolvable WP must be that short words need <u>many</u> relations (which make the word much longer before getting shorter).



Dehn functions (OHGGT)

Word Problem for BS(1,2)

$BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$

Let $G = \{$ linear fns $g: \mathbb{R} \to \mathbb{R}$ of form $g(x) = 2^n x + \alpha$ with $\alpha \in \mathbb{Z}[1/2] \}$

Check: G is a group.

Have $f: BS(1,2) \longrightarrow G$

$$a \longmapsto g(x) = x+1$$
$$t \longmapsto g(x) = 2x.$$

Prop. f is an isomorphism.

Cor. f has solvable WP (evaluate $f(w)$).

Pf. Last time: well-def. $f(tat^{-1}) = f(a^2)$

Surj. $f(t^{-k}a^m t^k) = \left(g(x) = x + m/2^k\right)$

$f(a^n) = (g(x) = 2^n x)$

Inj. Say $f(w) = id$.

Key: exponent sum on t's is 0.

(take derivative, chain rule)

So: if there are t's there are t^{-1}'s.

Can conjugate so have

$$t \, \underline{a^k} \, t^{-1}$$

Replace with a^{2k}.

Eventually $a^n \Rightarrow n = 0$ $\square$

Example

$$\underbrace{t a^2 t^{-1}}_{a^4}\, a\, \underbrace{t^{-1} a^2 t}_{a}\, a\, \underbrace{t^{-1} a t}_{\text{uh oh!}}$$

$$a^4 a a a t^{-1} a t$$

conj by t

$$t a^7 t^{-1} a$$

$$a^{14} a$$

$$a^{15}$$

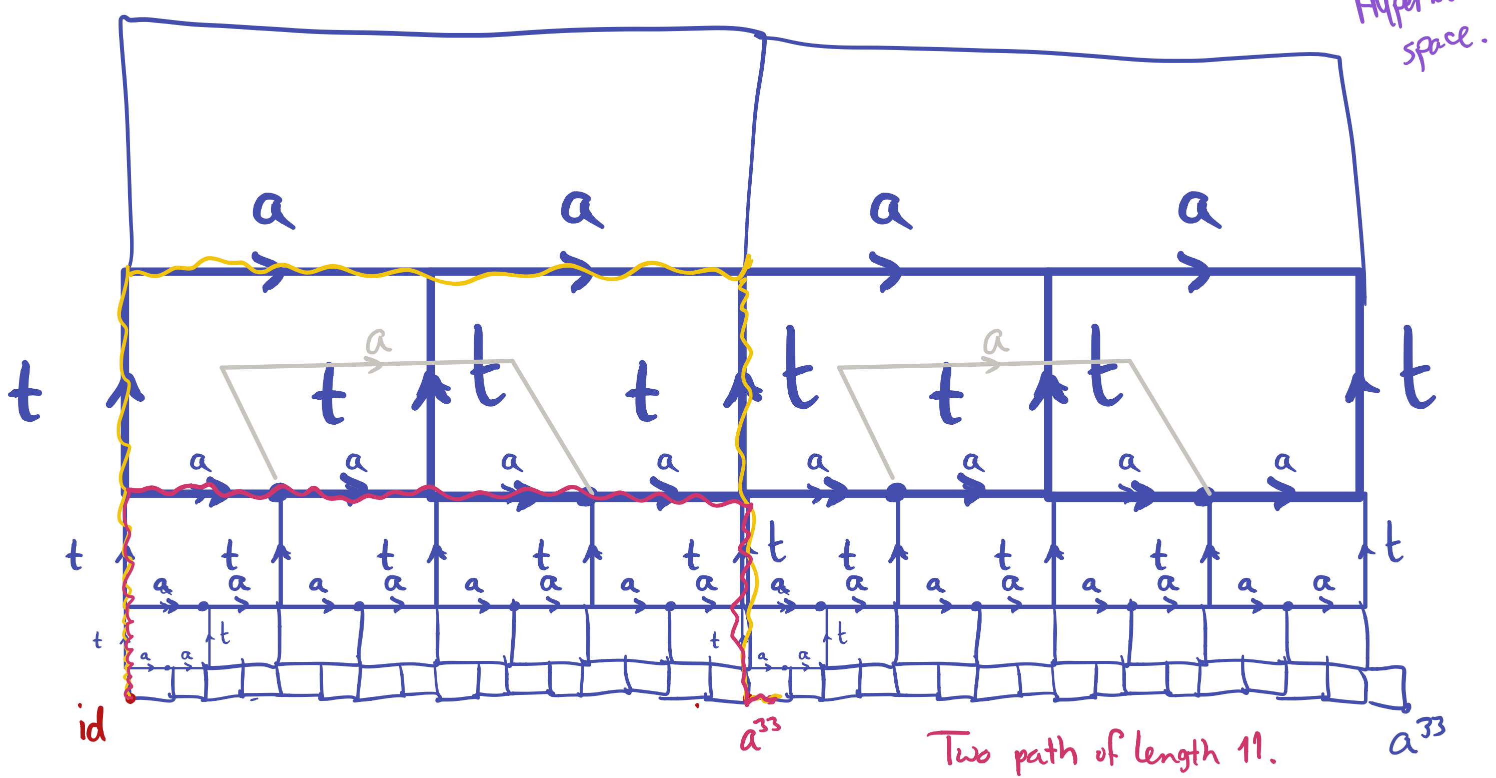
This shows:

If exp. sum on t is 0

then $w \overset{\text{conj}}{\sim} a^n$

Cayley graph for BS(1,2)

Poll: shortest path to a^{33}



ANNOUNCEMENTS MAR 11

- Cameras on
- Midterm due 11:59 pm
- No HW this week
- First draft due Apr 2
- Office hours by appt.

<u>Today</u>

Normal forms...

in $BS(1,2)$

in B_3

Hyperbolic plane?

Normal Forms

G = group

S = gen set

We have

$$\pi : \{\text{words in } S \cup S^{-1}\} \longrightarrow G$$

A <u>normal form</u> for G is an

$$\eta : G \longrightarrow \{\text{words in } S \cup S^{-1}\}$$

s.t. $\pi \circ \eta = \mathrm{id}$.

To tell if two elements are same, put them in normal form & compare.

This solves word problem.

We can also think of a normal form as a subset of $\{\text{words in } S \cup S^{-1}\}$, one word in $\pi^{-1}(g)$ for each $g \in G$.

<u>Examples</u>. ① $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

normal form: $\{a^m b^n : m, n \in \mathbb{Z}\}$

② F_2 normal form: freely red. words

Normal forms for $BS(1,2) = \langle a,t \mid tat^{-1} = a^2 \rangle$

We know

$$BS(1,2) \xrightarrow{\cong} \left\{ g(x) = 2^n x + m/2^k : \begin{array}{c} m,n,k \\ \in \mathbb{Z} \end{array} \right\}$$

$$a \longmapsto g(x) = x+1$$

$$t \longmapsto g(x) = 2x$$

We can check:

$$t^{-k} a^m t^k t^n \longmapsto g(x) = 2^n x + m/2^k.$$

Guess for normal form:

$$\{ t^{-k} a^m t^{k+n} \}$$

FAILS!

$$tat^{-1} = a^2$$

$k=-1$ $m=1$ $n=0$ $k=n=0$ $m=2$

The ambiguity is that $m/2^k$ might be reduced, i.e. m even.

The fix: write elements of $BS(1,2)$ as

$$g(x) = 2^n x + \frac{2m+1}{2^k}$$

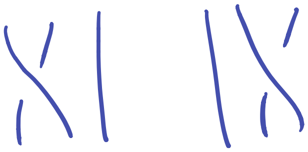
or $g(x) = 2^n x$

$\rightsquigarrow$ Normal form:

$$\{ t^{-k} a^{2m+1} t^{k+n} : \begin{array}{c} k,m,n \\ \in \mathbb{Z} \end{array} \} \cup \{ t^n : n \in \mathbb{Z} \}$$

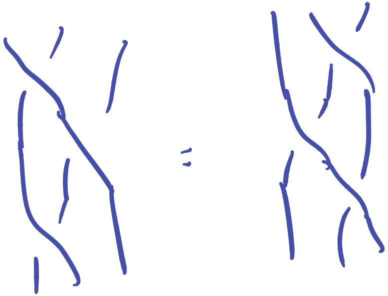
Normal form for B_3 (or B_n)

Generators:



σ_1 σ_2

Mutiplication is stacking.



$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 = \Delta$

Poll. Which are equiv to

$\underline{1\ 2\ 1}\ 1\ 2\ 2\ 2$?

$2\ 1\ \underline{2\ 1\ 2}\ 2\ 2$

$2\ 1\ 1\ \underline{2\ 1\ 2}\ 2$

$2\ 1\ 1\ 1\ \underline{2\ 1\ 2}$

$2\ 1\ 1\ 1\ 1\ 2\ 1$

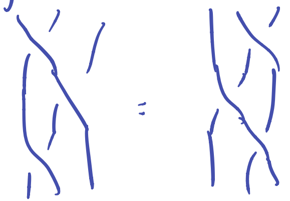
Garside Normal Form

Ingredient #1 :

$$B_3 \longrightarrow \mathbb{Z}$$
$$\sigma_1 \longmapsto 1$$
$$\sigma_2 \longmapsto 1$$

"signed word length"

Ingredient # 2 :



"half-twist"

$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 = \Delta$

Running example: $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}$

Step 1. Replace each σ_i^{-1} with $\Delta^{-1}\cdot$ pos. word.

Why can we do this?

$$\Delta^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$$ and $$\Delta^{-1} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$$

$$\Delta^{-1}\sigma_2\sigma_1 = \sigma_2^{-1} \qquad \Delta^{-1}\sigma_1\sigma_2 = \sigma_1^{-1}$$

example. $\sigma_1\sigma_2\sigma_1\underline{\sigma_2^{-1}} \rightsquigarrow \sigma_1\sigma_2\sigma_1\underline{\Delta^{-1}\sigma_2\sigma_1}$

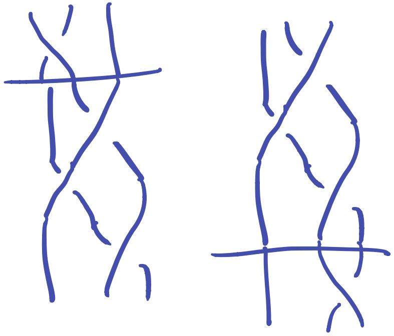
Step 2. Move all Δ^{-1} to the left.

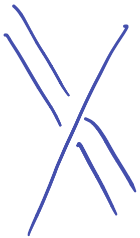
Why can we do this? $\sigma_i\Delta^{-1} = \Delta^{-1}\sigma_{n-i}$

example. $\rightsquigarrow \Delta^{-1}\sigma_2\sigma_1\sigma_2\sigma_2\sigma_1$

Check:

$$\sigma_i \Delta^{-1} = \Delta^{-1} \sigma_{n-i}$$





We now have $\Delta^i \cdot$ pos word $i \leq 0$.

<u>Step 3</u>. Find maximal i so our braid is

$\Delta^i \cdot$ pos word $i \leq 0$.

In our example:

$$\Delta^{-1} \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 = \cancel{\Delta^{-2} \overbrace{\sigma_2 \sigma_1 \sigma_2}^{\Delta} \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1}$$

How do we know our example is not

$\Delta^0 \cdot$ pos word?

It is!

$$\Delta^0 \sigma_2 \sigma_1$$

In general, use ingredient #1.

Alternate example: (not related to running example)

How do I know

$$\Delta^{-1}\sigma_1 \neq \Delta^0 \cdot \text{pos word?}$$

signed word length -2

signed word length ≥ 0.

Another example:

How do I know

$$\Delta^{-1}\sigma_1^4 \neq \Delta^0 \cdot \underbrace{\text{pos word}}_{\text{must have length } 1}?$$

signed word length 1

only 2 such words.

Step 4 Find all Δ^i · pos word equalling g.

Choose the smallest in lexicographic order.

example. $\Delta^0 \sigma_2 \sigma_1 \leftarrow$ normal form.

only candidates are ~~$\Delta^0 \sigma_1 \sigma_2$~~

~~$\Delta^0 \sigma_1^2$~~

~~$\Delta^0 \sigma_2^2$~~

Steps 3 & 4 use:

Thm. If two positive braids are equal, they differ by finitely many

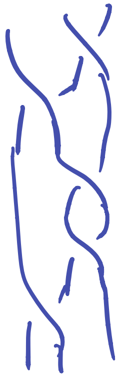
$$121 \longleftrightarrow 212$$

no inverses needed!

In fancy language:

The braid monoid B_n^+ embeds into B_n.

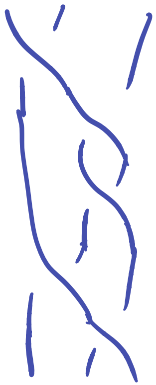
Prove this theorem?



21221

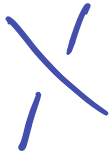
12121

11211

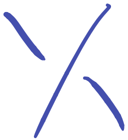


12212

positive crossing



neg. crossing



Announcements Mar 18

- Cameras on
- First draft due Apr 2
- HW due Thu 3:30
- Office Hours Fri 2-3, Tue 11-12, appt
- Talk to me about extra credit.

Question. What are all groups G of order $n!$ with $[G,G] = A_n$?

Hope: $G = S_n$.

Today

Burnside problem

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\rangle$$

"lantern relation"

Burnside Problem

A group is a <u>torsion group</u> if all elements have finite order.

Finite groups are all torsion gps

Easy to make infinite torsion groups:

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \cdots$$

$$\mathbb{Q}/\mathbb{Z}$$

These are not finitely generated (why?)

Q. (Burnside 1902) Is there a fin. gen. infinite torsion group?

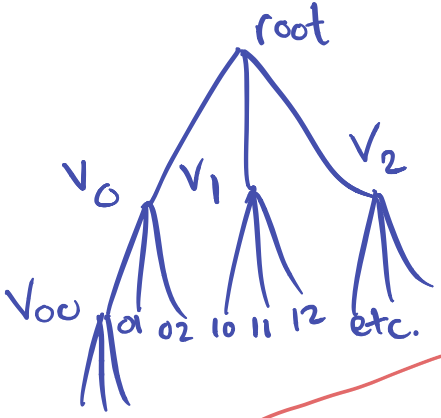
A. (Golod - Shafarevich '60s)

Yes.

We'll show an example from 80's by Gupta - Sidki using GGT.

Starting pt is...

T = rooted ternary tree



uncountable

$\mathrm{Sym}(T)$ = root-preserving symmetries.

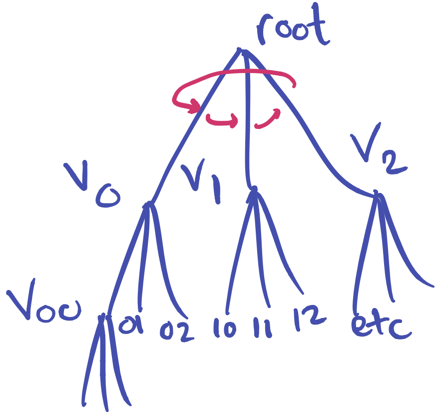
Important thing:

$$\mathrm{Sym}(T) \times \mathrm{Sym}(T) \times \mathrm{Sym}(T) \leq \mathrm{Sym}(T)$$

$\mathrm{Sym}(T)$ is self-similar.

Two elements of Sym(T)

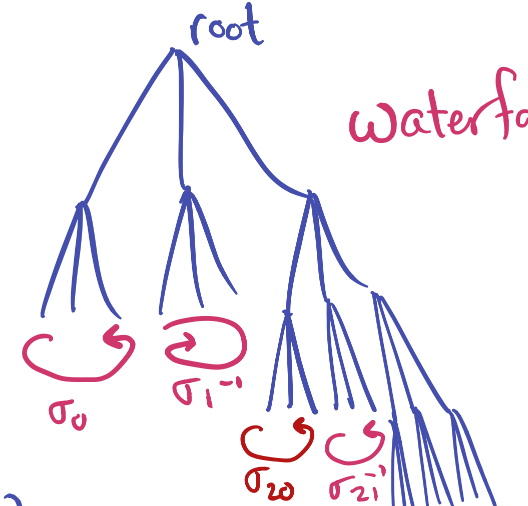
$\sigma : V_{n_1\cdots n_k} \to V_{(n_1+1)n_2\cdots n_k}$ mod 3



σ_L means do σ at V_L.

e.g. σ_0

$\omega = \sigma_0\sigma_1^{-1}\sigma_{20}\sigma_{21}^{-1}\sigma_{220}\sigma_{221}^{-1}\cdots$



ω_L means do ω at V_L.

Let $U = \langle \sigma, \omega \rangle \le \mathrm{Sym}(T)$

Let $U = \langle \sigma, \omega \rangle \leq \mathrm{Sym}(T)$

<u>Thm</u>. U is a fin. gen. ∞ torsion group.

⓪ fin gen ✓

① ∞

② torsion

Need a "normal form" for U.

Then will do ① & ②

A "normal form" for U

Lemma 1. Each elt of U can be expressed as

$$\sigma^k x_1 \cdots x_n$$

where $x_i \in \{\omega, \sigma\omega\sigma^{-1}, \sigma^2\omega\sigma^{-2}\}$

(kind of like B_n normal form).

Pf. Need a relation

$\sigma = \sigma^{-2} \Rightarrow \omega\sigma = \omega\sigma^{-2}$

$$\Rightarrow \boxed{\omega\sigma = \sigma(\sigma^2\omega\sigma^{-2})}$$

Use the relation to push σ's to the left.

example

$\omega\sigma\underline{\sigma\omega}$

$\sigma\underbrace{(\sigma^2\omega\sigma^{-2})}_{x_1}\ \sigma\underbrace{\omega}_{x_2}$

bad

$\sigma^2 \ \sigma^2(\sigma^2\omega\sigma^{-2})\sigma^{-2}\omega$

$\sigma^2 \underbrace{(\sigma\omega\sigma^{-1})}_{x_1}\underbrace{(\omega)}_{x_2}$ $\square$

① **Prop.** $|U| = \infty$.

We will find $K \lneqq U$

and $K \twoheadrightarrow U$.

The Prop follows.

Defining K

have $U \twoheadrightarrow \mathbb{Z}/3$

action on three edges from root. Works because ω & σ preserves the cyclic order.

K is the kernel.

(*) In terms of "normal form" these are the $\sigma^k x_1 \cdots x_n$ with $k=0$.

Let $H = \langle \omega, \sigma\omega\sigma^{-1}, \sigma^2\omega\sigma^{-2} \rangle$

Lemma. $K = H$.

Pf. Step 1. $H \leq K$ ✓

Step 2. $H \trianglelefteq U$.

finite check. conjugate each gen for H by gen for U, end up back in H

Step 3. $U/H \cong \mathbb{Z}/3$ by Lemma 1

Lemma $K \twoheadrightarrow U$

Pf. K maps to the copy of $\mathrm{Sym}(T)$ below vertex 0.

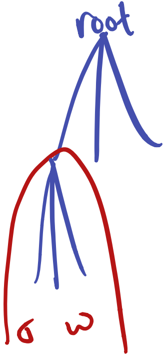
Want: Image of K contains

$$\sigma_0 \ \& \ \omega_0$$

Check on generators:

$$\omega \longmapsto \sigma_0$$
$$\sigma\omega\sigma^{-1} \longmapsto \omega_0$$
$$\sigma^2\omega\sigma^{-2} \longmapsto \sigma_0^{-1}$$

$\square$



② **Prop.** U is torsion: each elt has order a power of 3.

Pf. Induction on syllable length in normal form.

$$\sigma^k x_1 \cdots x_n$$

σ^k is a syllable, x_i is a syllable.

Idea. Given g, show g^3 is a product of 3 commuting elements of shorter syllable length

Proof by example

$g = \sigma\omega$ syll. length 2.

$\rightsquigarrow g^3 = \sigma\omega\sigma\omega\sigma\omega$

normal form has $k=0$:

$$\cancel{\sigma^3}(\sigma^{-2}\omega\sigma^2)(\sigma^{-1}\omega\sigma)\,\omega$$

lie in 3 diff. factors

$$\mathrm{Sym}(T) \times \mathrm{Sym}(T) \times \mathrm{Sym}(T)$$

all three pieces have syllable length 1. $\square$

Announcements Mar 23

- Cameras on
- HW due Thu 3:30
- Office Hours Fri 2-3, appt
  + makeup 10-11 Wed
- Progress report Apr 2 ~ 1 page
- First draft Apr 9
- Talk to me about makeup points!

Today

- Howson's thm
- Regular languages
- Automata

Howson's Thm

Thm 7.32 (1954)
If G, H f.g. subgps of F_n then $G \cap H$ is f.g.

A "counterexample" with F_n replaced by another group:

Take $F_2 \times \mathbb{Z}$ $\quad F_2 = \langle x, y \rangle \quad \mathbb{Z} = \langle z \rangle$

$G = F_2$ (first factor)

$H = \ker(F_2 \times \mathbb{Z} \to \mathbb{Z})$
all 3 gens $\mapsto 1$

To check: ① G f.g.
② H f.g.
③ $G \cap H$ not f.g.

② Claim: H is gen by $S = \{xz^{-1}, yz^{-1}\}, \cancel{xy^{-1}}$

Step 1. $\langle S \rangle$ normal.
To show: $gsg^{-1} \in \langle S \rangle$
$g = (\text{gen for } F_2 \times \mathbb{Z})^{\pm 1}$ $s \in S$.

example. $y(x^{-1}z)y^{-1} = (yz^{-1})(x^{-1}z)(y^{-1}z)$

Step 2. $\langle S \rangle \subseteq H$

Step 3. $(F_2 \times \mathbb{Z}) / \langle S \rangle \cong \mathbb{Z}$

We get $F_2 \times \mathbb{Z}$ subject to $x = z, y = z$

$F_2 \times \mathbb{Z}$ $\quad F_2 = \langle x, y \rangle$ $\quad \mathbb{Z} = \langle z \rangle$

$G = F_2$ (first factor)

$H = \ker(F_2 \times \mathbb{Z} \to \mathbb{Z})$

all 3 gens $\longmapsto 1$

Remains.

③ $G \cap H$ not fg.

$G \cap H$ is the subgp of F_2:

$$\ker F_2 \to \mathbb{Z}$$

$$x, y \longmapsto 1$$

(exponent sum 0).

Claim. $G \cap H$ is freely gen by

$$x^i y^{-i}$$

example. $x^5 y^{-5} x^3 y^{-3} y^2 x^{-2}$

[Very similar to HW problem:

$$\ker F_2 \to \mathbb{Z}^2$$

$$x \longmapsto (1,0)$$

$$y \longmapsto (0,1)$$

Freely gen by

$$\{x^i y^j x^{-i} y^{-j}\}$$]

$\square$

Hanna Neumann Conjecture (1957)

$$\mathrm{rk}(G \cap H) - 1 \leq (\mathrm{rk}(G) - 1)(\mathrm{rk}(H) - 1)$$

for $G, H \leq F_n$

Proved in 2011 by Friedman, Mineyev.

Our proof of Howson's thm

uses regular languages, automata.

Today: automaton version of Howson's thm. Thu: Howson's thm.

Languages

$S = \{x_1, \dots, x_n\}$ "alphabet"

$S^* = \{\text{words of finite length in } S\}$

Any subset $L \subseteq S^*$ is called a <u>language</u>

Examples

① $S = \{a, \dots, z\}$ $L = \{\text{words in OED}\}$

② $S = \{a\}$ $L = \{a^n : 3 \mid n\}$

③ $S = \{a, b, c\}$ $L = \{a^i b^j c^k : i > 0, j \geq 0, k \geq 0\}$

④ $S = \{\text{gen set for } G\}^{\pm 1}$ $L =$ words in S that equal id in G.

Automata (= simple computer)

S = alphabet (finite set)

An automaton M over S consists of a directed graph with decorations:

vertices: "states"

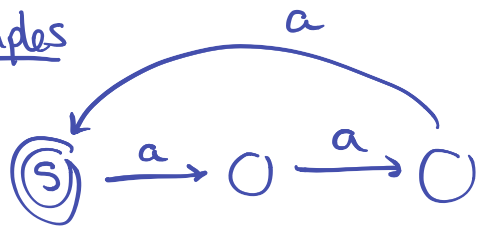
- some subset of vertices called start states $\textcircled{S}$
- some subset A of vertices called accept states $\textcircled{O}$
- edges labeled by elts of S.

If the graph is finite, M is a finite state automaton.

The language accepted by M is $\{w \in S^* : w$ given by a directed path in $M\}$

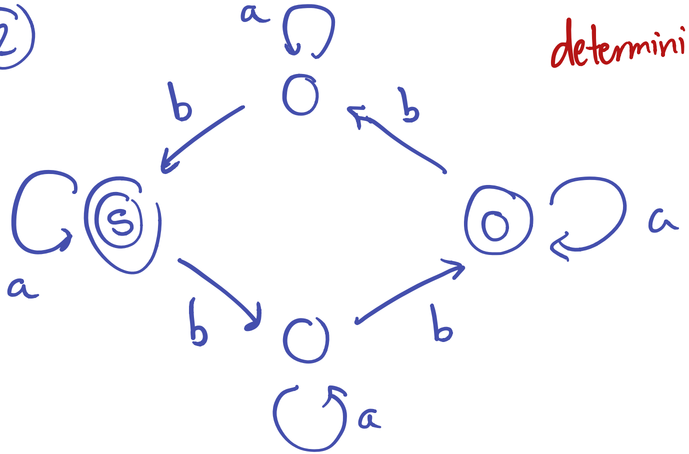
Examples

1.



$\rightsquigarrow L = \{a^i : 3 | i\}$

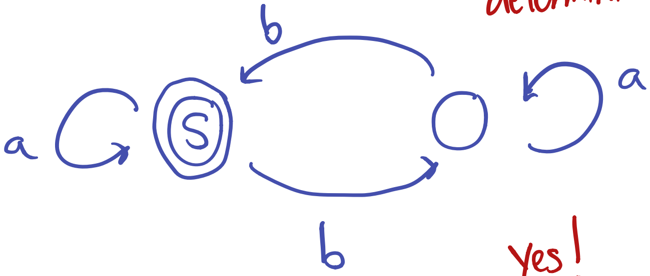
2



deterministic!

Poll: Is there a simpler automaton for same language?

deterministic!



yes!

$L = \{$ words with b-exponent even $\}$
sum

$a^3 b^5 a b$ ✓

Deterministic automata

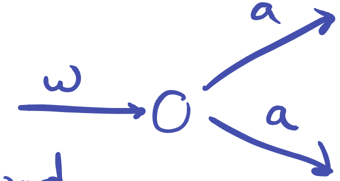
A <u>det. aut.</u> is a FSA with

- exactly one start state
- no two edges leaving same vertex have same label
- no edges with empty label (in Meier: empty = ε)

It is <u>complete</u> if each vertex has departing edges with all possible labels.

What's deterministic about it?

words ⟷ paths



The word wa corresponds to more than 1 path

To see if a word is in the accepted language, start at <u>the</u> start state, trace out the word/path, see if land at accept state.

A language is <u>regular</u> if accepted by a det. FSA.

Automaton version of Howson's Thm

Thm 7.11 Say $K, L \subseteq S^*$ are reg. languages. Then so are:

1. $S^* \setminus K$
2. $K \cup L$
3. ** $K \cap L$
4. $KL = \{w_K w_L : w_K \in K, w_L \in L\}$
5. $L^* = L \cup LL \cup LLL \cup \cdots$

** reg. lang. is automaton version of f.g.

Lemma 1. L accepted by a det. FSA (i.e. L is regular) $\Longrightarrow$ L accepted by a complete det FSA.

Pf. (exercise: add dead ends/fail states)

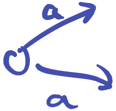
Lemma 2. L accepted by a non-det. FSA $\Longrightarrow$ L accepted by a det. FSA.

In other words: starting with a non-det FSA, Lemma 2 converts it to a det FSA, Lemma 1 converts to a complete det FSA.

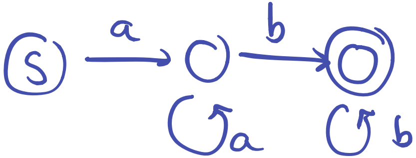
Lemma 2. L accepted by a non-det. FSA $\Rightarrow$ L accepted by a det. FSA.

Pf. Two steps:

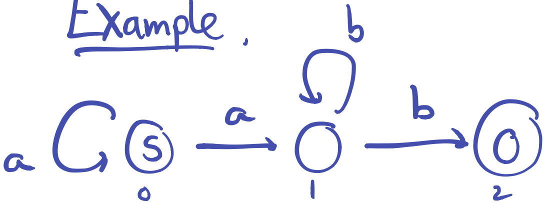
① Get rid of arrows with empty labels

② Get rid of 

③ Get rid of multiple start states.



Example.

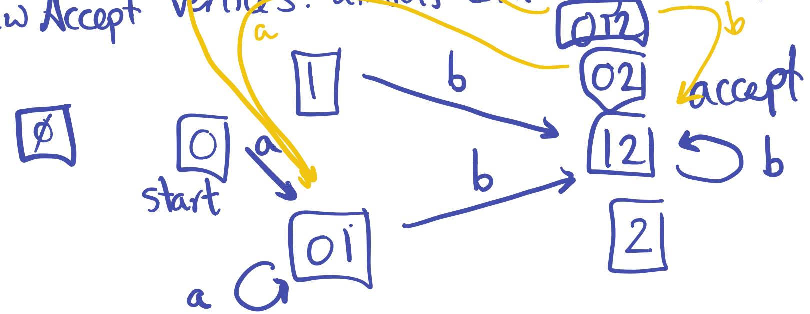


$L = \{a^i b^j : i,j>0\}$

New vertices: subsets of old vertices

New Start vertex: set of all old start vertices.

New Accept vertices: all sets containing an accept



We can now convert $\mathcal{M}_\varepsilon$ into a deterministic automaton, $\mathcal{D}$. The states of $\mathcal{D}$ consist of all the subsets of $V(\mathcal{M}_\varepsilon)$. The single start state of $\mathcal{D}$ is the subset of $V(\mathcal{M}_\varepsilon)$ consisting of all the start states of $\mathcal{M}_\varepsilon$. The accept states of $\mathcal{D}$ are the subsets of $V(\mathcal{M}_\varepsilon)$ that contain at least one accept state of $\mathcal{M}_\varepsilon$. In $\mathcal{D}$ there is an edge from U to U' labelled by x if, for each $v \in U$, there is an edge labelled x from v to some $v' \in U'$, and U' is entirely composed of such vertices. That is, there is an edge labelled x from U to the vertex corresponding to the set

$$U' = \{v' \in V(\mathcal{M}_\varepsilon) \mid v' \text{ is at the end of an edge labelled } x \text{ that begins at some } v \in U\}.$$

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ANNOUNCEMENTS MAR 25

- Cameras on
- HW due Thu 3:30
- Office Hours Fri 2-3, Tue 11-12
- Progress report Apr 2 ~ 1 page
- First draft Apr 9
- Talk to me about makeup points!

<u>Today</u>

- Howson's thm
- Regular languages
- Automata

Howson's Thm

Thm 7.32 (1954)

If G, H f.g. subgps of F_∞ then $G \cap H$ is f.g.

Original proof: algebraic topology

Languages

$S = \{x_1, \dots, x_n\}$ "alphabet"

$S^* = \{\text{words of finite length in } S\}$

Any subset $L \subseteq S^*$ is called a <u>language</u>

examples

① $L = \{a^i b^j : i, j > 0\} \subseteq \{a,b\}^*$

② Consider $H = \langle a^2, b \rangle \leq F_2$

L = reduced words in a, b, a^{-1}, b^{-1} corresponding to elts of H.

$\subseteq \{a, b, a^{-1}, b^{-1}\}^*$

Language examples

① $L = \{a^i b^j : i,j > 0\} \subseteq \{a,b\}^*$

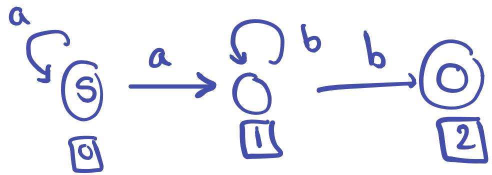
② Consider $H = \langle a^2, b \rangle \leq F_2$

L = reduced words in a, b, a^{-1}, b^{-1} corresponding to elts of H.

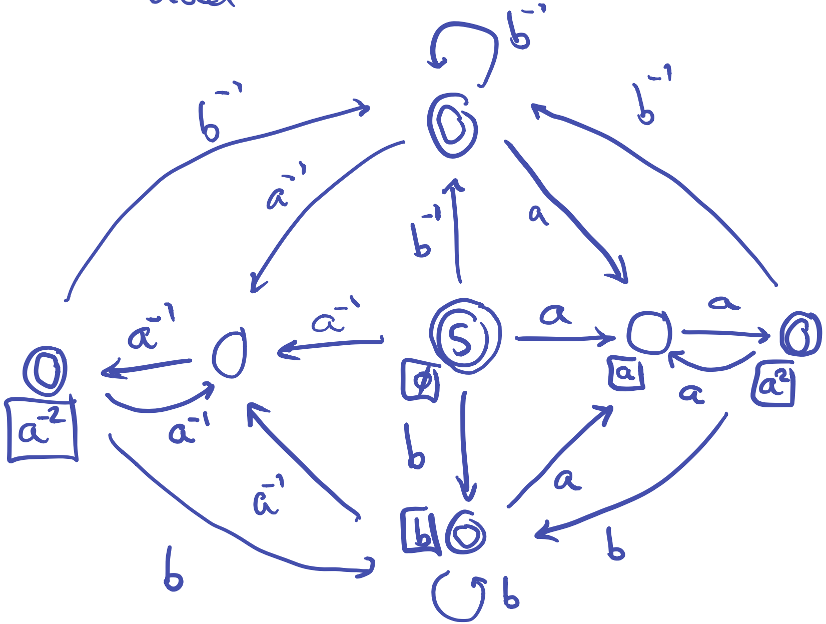
$\subseteq \{a, b, a^{-1}, b^{-1}\}^*$

Automaton examples

①



② Roughly states correspond to last letter used.



Deterministic FSA

FSA with

- one start state
- no edges w/ empty label
- ≤ 1 edge with a given letter starting from each vertex

$\rightsquigarrow$ regular languages

Complete: $= 1$ in 3rd bullet.

Tidying up a FSA

last time

Lemma 1. L accepted by det FSA

$\Rightarrow$ L accepted by complete det FSA

Lemma 2. L acc by non-det FSA

$\Rightarrow$ L acc by det FSA.

In other words: FSAs, det FSAs, compl. det FSAs all give same languages, i.e. regular lang's.

Lemma 2. L acc by non-det FSA

$\Longrightarrow$ L acc by det FSA.

Pf. Given FSA M, want to make it satisfy the 3 bullet pts without changing the accepted lang. We'll just do 3rd bullet:

- ≤ 1 edge with a given letter starting from each vertex

Let D be FSA with

Vertices $V(D) = P(V(M)) \setminus \emptyset$

Edges Let $U = \{v_1, \ldots, v_k\} \in V(D)$
$v_i \in V(M)$

3rd bullet

For each $a \in S$ (= alphabet)
Make an a-edge from U to

$$V = \bigcup_{i=1}^{k} \{v \in V(M) : \exists \text{ a-edge from } v_i \text{ to } v\}$$

1st bullet!

Start state $\{$start states in $M\}$

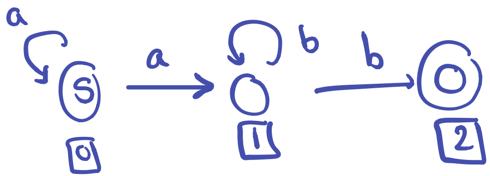
Accept states elts of $P(V(M))$ cont. accept st.

Key part of defn of D:

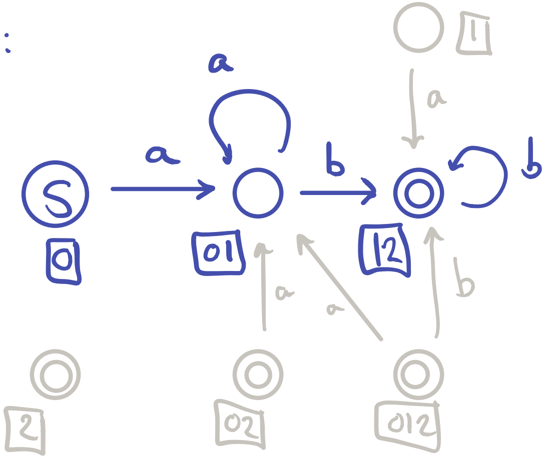
Make an a-edge from U to

$$V = \bigcup_{i=1}^{k} \{ v \in V(M) : \exists \text{ a-edge from } v_i \text{ to } v \}$$

M:



D:



Can cut the chaff

$\square$

Automaton version of Howson's Thm

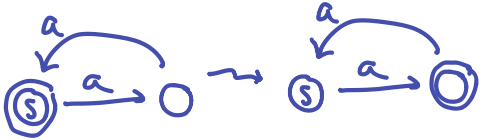
Thm 7.11 Say $K, L \subseteq S^*$ are reg. languages. Then so are:

1. $S^* \setminus K$
2. $K \cup L$
3. ** $K \cap L$
4. $KL = \{w_K w_L : w_K \in K, w_L \in L\}$
5. $L^* = L \cup LL \cup LLL \cup \cdots$

** reg. lang. is automaton version of f.g.

Pf. ① Toggle accept/non-accept states

example. $L = \{a^i : i \text{ even}\}$



② Say M_K, M_L FSA for K, L then $M_K \cup M_L$ is a FSA for $K \cup L$. Apply the 2 lemmas.

③ $K \cap L = S^* \setminus ((S \setminus K) \cup (S \setminus L))$

Apply ① & ②

Regular vs. finite gen.

Thm. S = fin. gen. set for G

Then $H \leq G$ is fin. gen

$\Longleftrightarrow$ H is image of reg. lang. $L \subseteq (S^{\pm 1})^*$

under $\pi : (S^{\pm})^* \longrightarrow H$

Idea of Pf: Generators for H

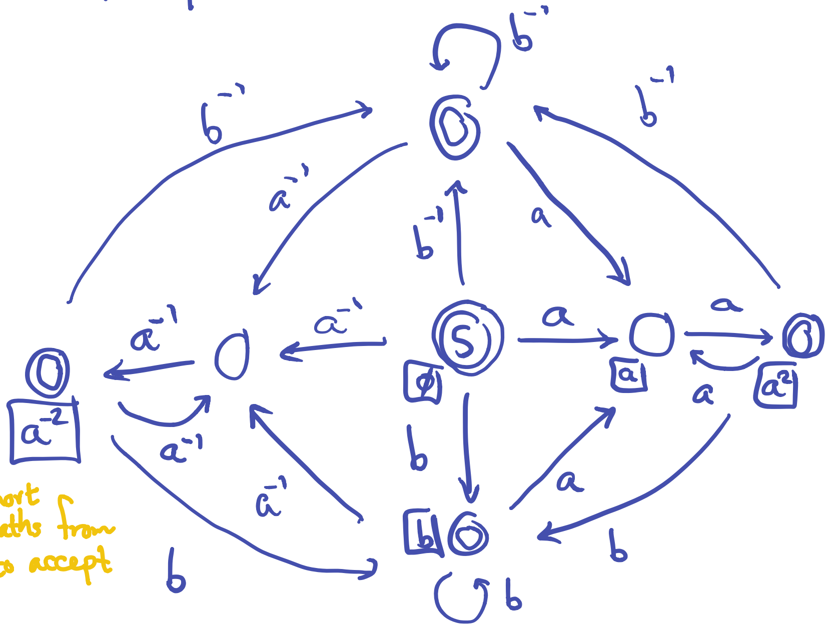
$\longleftrightarrow$ circuits in M

short
and paths from
start to accept

finitely many
since M is finite.

$\square$

Example. $H = \langle a^2, b \rangle$



Circuits: $a^2, a^{-2}, b, b^{-1}, ba^2, b^{-1}a^2, b^{-1}a^{-2}, ba^{-2}$

& b, a^2 generate H!

Freely reducing a language

Lemma 3. L = reg. lang over $S^{\pm 1}$

R = lang obtained from L by freely reducing.

Then R is regular.

Pf. Say L given by FSA M.

If we see
$$\begin{tikzcd} \circ \arrow[r, "s"] \arrow[rr, bend right] & \circ \arrow[r, "s^{-1}"] & \circ \end{tikzcd}$$

add empty edge

$\rightsquigarrow M'$

M' accepts all the words M did <u>plus</u> their freely reduced versions.

Let K = language of <u>all</u> freely reduced words in $S^{\pm 1}$

K is regular (exercise)

& $R = K \cap L(M')$

By Thm, R regular $\square$

Pf of Howson's thm

H, K fin gen. subgps of F_n

H, K are images of reg lang's

L_H, L_K by Thm.

By Lemma 3 we may assume

L_H, L_K consist of freely red. words, which are exactly elts of H, K (need a free gp for this!)

Other Thm $\Rightarrow$ $\underbrace{L_H \cap L_K}_{\text{elts of } H \cap K}$ regular.

Thm $\Rightarrow$ $H \cap K$ fin. gen.

$\square$

We can now convert $\mathcal{M}_\varepsilon$ into a deterministic automaton, $\mathcal{D}$. The states of $\mathcal{D}$ consist of all the subsets of $V(\mathcal{M}_\varepsilon)$. The single start state of $\mathcal{D}$ is the subset of $V(\mathcal{M}_\varepsilon)$ consisting of all the start states of $\mathcal{M}_\varepsilon$. The accept states of $\mathcal{D}$ are the subsets of $V(\mathcal{M}_\varepsilon)$ that contain at least one accept state of $\mathcal{M}_\varepsilon$. In $\mathcal{D}$ there is an edge from U to U' labelled by x if, for each $v \in U$, there is an edge labelled x from v to some $v' \in U'$, and U' is entirely composed of such vertices. That is, there is an edge labelled x from U to the vertex corresponding to the set

$$U' = \{v' \in V(\mathcal{M}_\varepsilon) \mid v' \text{ is at the end of an edge labelled } x \text{ that begins at some } v \in U\}.$$

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Announcements Mar 30

- Cameras on
- HW due Thu 3:30
- Office Hours Fri 2-3 appt
- Outline Apr 2 ~1 page, teams
- First draft Apr 9.
- Makeup points

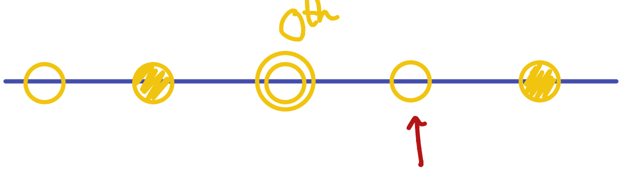
Today

Lamplighter groups

Diestel - Leader graphs

<u>Lamplighter group</u> (OHGGT)

Infinite street



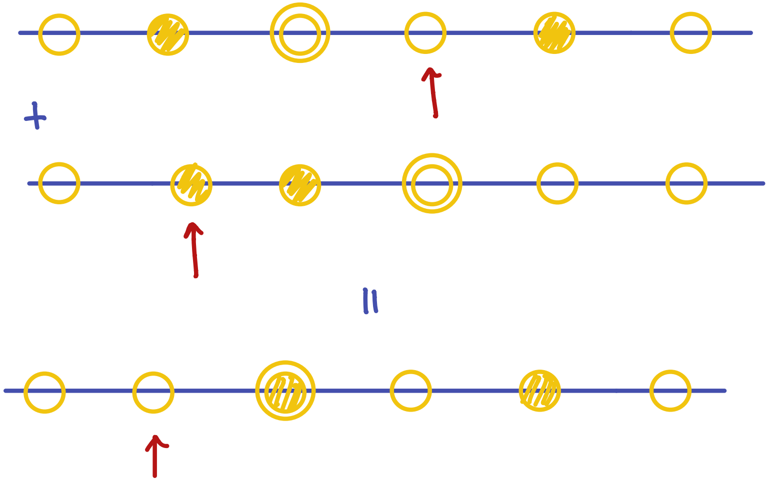
$$L = \mathbb{Z} \ltimes \left(\bigoplus_{\infty} \mathbb{Z}/2\right) \quad \text{as a } \underline{\text{set}}.$$

$$= \left\{ (k, \vec{x}) : k \in \mathbb{Z},\ \vec{x} \in \bigoplus_{\infty} \mathbb{Z}/2 \right\}$$

$= \{\text{configurations}\}$ or

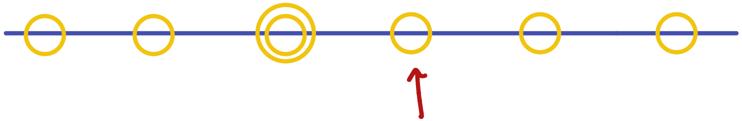
$= \{\text{actions}\}$

<u>Multiplication</u>: stack and add

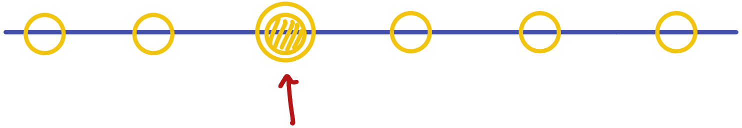


Generators

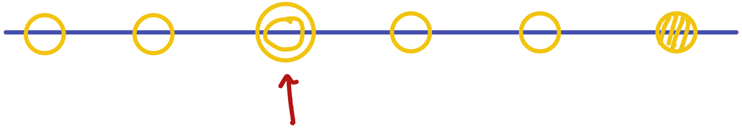
t:



a:



$t^3 a t^{-3}$



Presentation

$$L = \langle a, t \mid a^2 = \mathrm{id},\ (t^j a t^{-j})(t^k a t^{-k}) = (t^k a t^{-k})(t^j a t^{-j}) \rangle$$

A (faithful) representation ρ

First a notation for $\bigoplus_{\infty} \mathbb{Z}/2$.

$\mathbb{Z}/2[t,t^{-1}] = \{\mathbb{Z}/2 \text{ poly's in } t, t^{-1}\}$

$$t^{-2} + 1 + t^5 \in \mathbb{Z}/2[t,t^{-1}]$$

$$\longleftrightarrow (\ldots 0, 1, 0, \underline{1}, 0, 0, 0, 0, 1, 0, \ldots)$$

0^{th} entry

$L = \{(k, \vec{x})\}$

$= \{(k, P) : k \in \mathbb{Z}, P \in \mathbb{Z}/2[t,t^{-1}]\}$

$$\rho : L \longrightarrow GL_2(\mathbb{Z}/2[t,t^{-1}])$$

$$(k, P) \longmapsto \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$$

<u>Thm.</u> ρ is a faithful rep.

<u>Pf.</u> inj: clear...

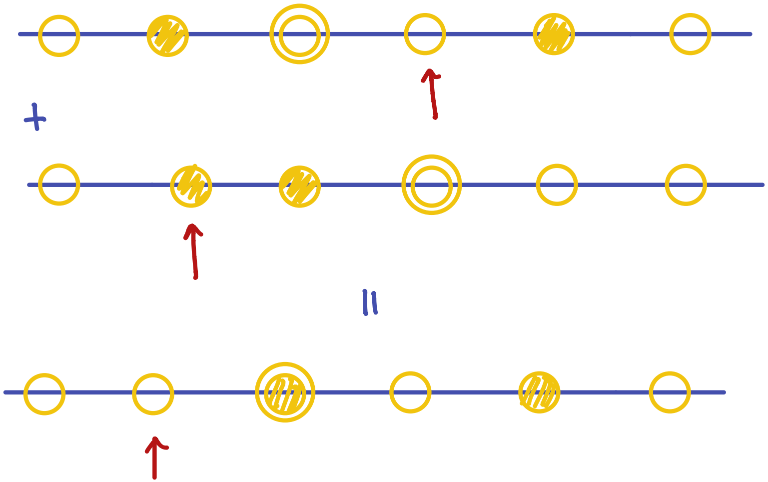
homom: Check relations

$$a^2 \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = I$$

other reln:

shears commute. $\square$

Example



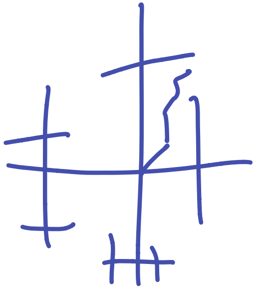
$$\begin{pmatrix} t & t^{-1}+t^{2} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} t^{-2} & t^{-1}+t^{-2} \\ 0 & 1 \end{pmatrix}$$

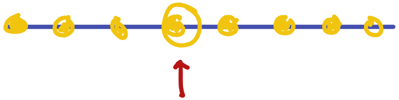
$$\begin{pmatrix} t & t^{-1}+t^{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-2} & t^{-1}+t^{-2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & 1 \cancel{+2t^{-1}} + t^{2} \\ 0 & 1 \end{pmatrix}$$

See OHGGT for a discussion of

- word length (1D - traveling salesman problemish) EASY!



- dead ends



- generalize

  ① L_n : lamps have $\mathbb{Z}/n$ states

  ② Wreath products

$L = \mathbb{Z}/2 \wr \mathbb{Z}$

$G \wr H$ is the lamplighter gp with

"map" H (like $\mathbb{Z}$ in L)

"lamp states" G (like $\mathbb{Z}/2$ in L)

$\mathbb{Z}/2 \wr \mathbb{Z}^2$ 2D - trav. sal. prob. HARD!

A little more:

$$G \wr H = \{(k, \vec{x}) : k \in H,\ \vec{x} : H \to G\}$$

$$H \ltimes \left(\bigoplus_H G \right)$$

1^{st} factor permutes coords of 2^{nd}.

example

$$\mathbb{Z}/2 \wr \mathbb{Z} = \{(k, \vec{x}) : k \in \mathbb{Z},\ \vec{x} : \mathbb{Z} \to \mathbb{Z}/2\}$$

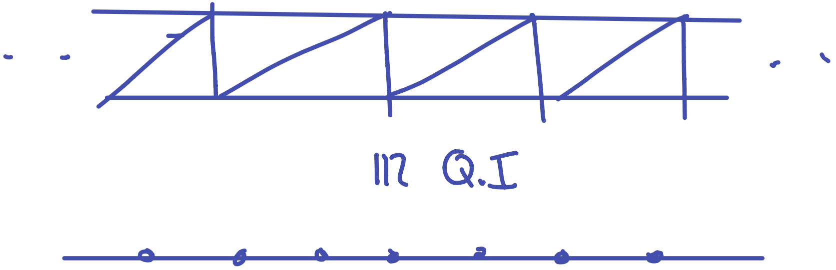
$$\mathbb{Z} \ltimes \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2 \right)$$

first factor permutes coords of 2^{nd}

Next: Cayley graph for L

Diestel-Leader Graphs

An old question: is every graph quasi-isometric to a Cayley graph?



$\mathrm{DL}(m,n)$ is a graph

D-L conjectured

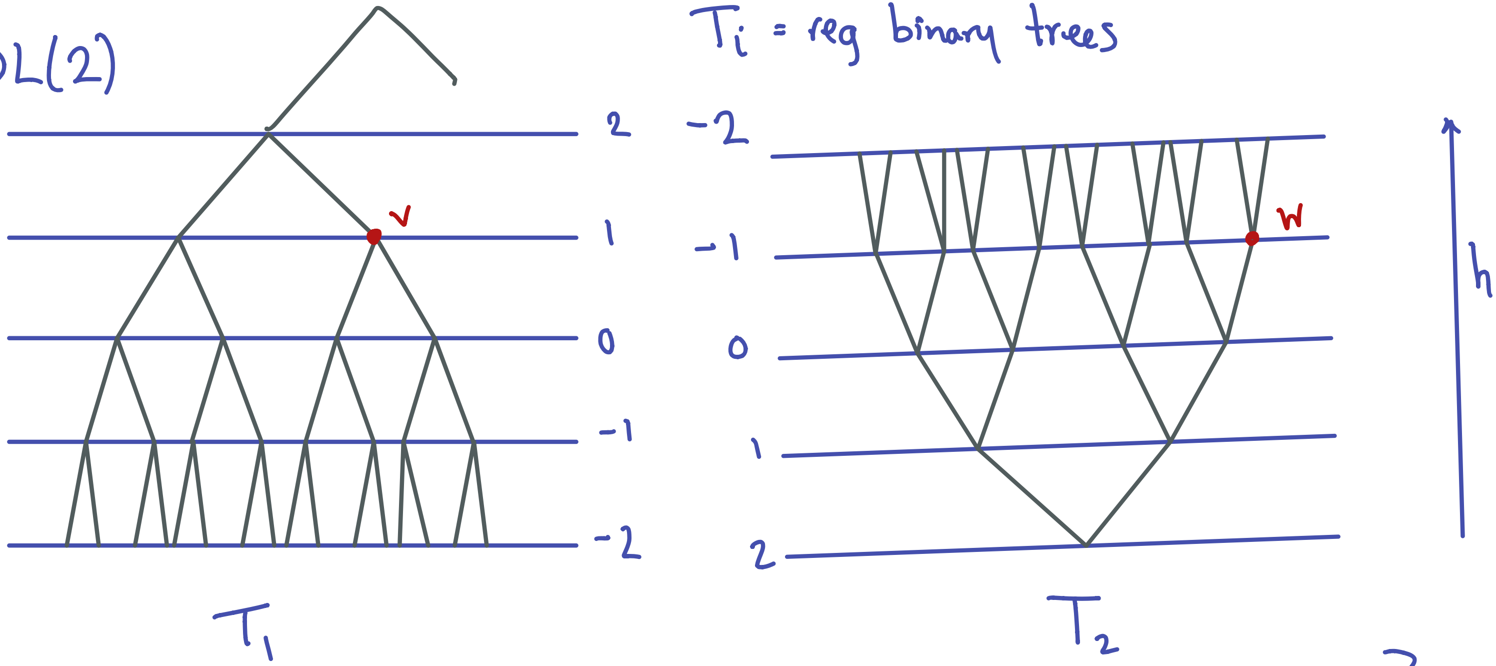
$m \neq n \Rightarrow \mathrm{DL}(m,n)$ is not QI to a Cayley gr.

(proved by Eskin-Fisher-Whyte)

But. $\mathrm{DL}(n) = \mathrm{DL}(n,n)$ is the Cayley graph for L_n (lamp. gp w/ n states)

DL(2)

T_i = reg binary trees



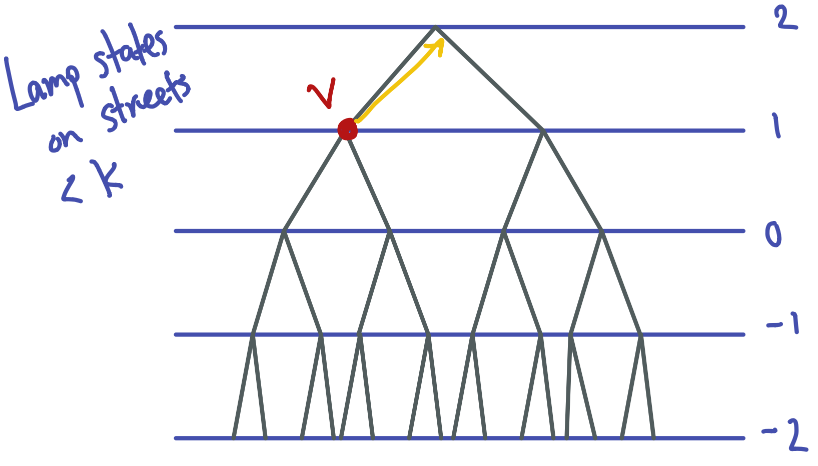
DL(2): vertices $\{(v,w) : v \in V(T_1),\ w \in V(T_2),\ h(v)+h(w)=0\}$

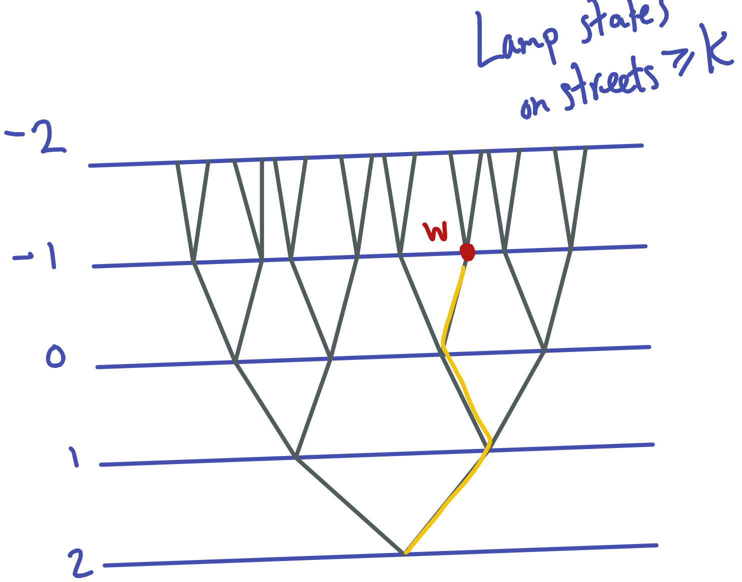
edges $(v,w) \text{---} (v',w')$ when $v \text{---} v'$ in T_1
$w \text{---} w'$ in T_2

Thm. $DL(2)$ = Cayley gr. for L wrt t, at

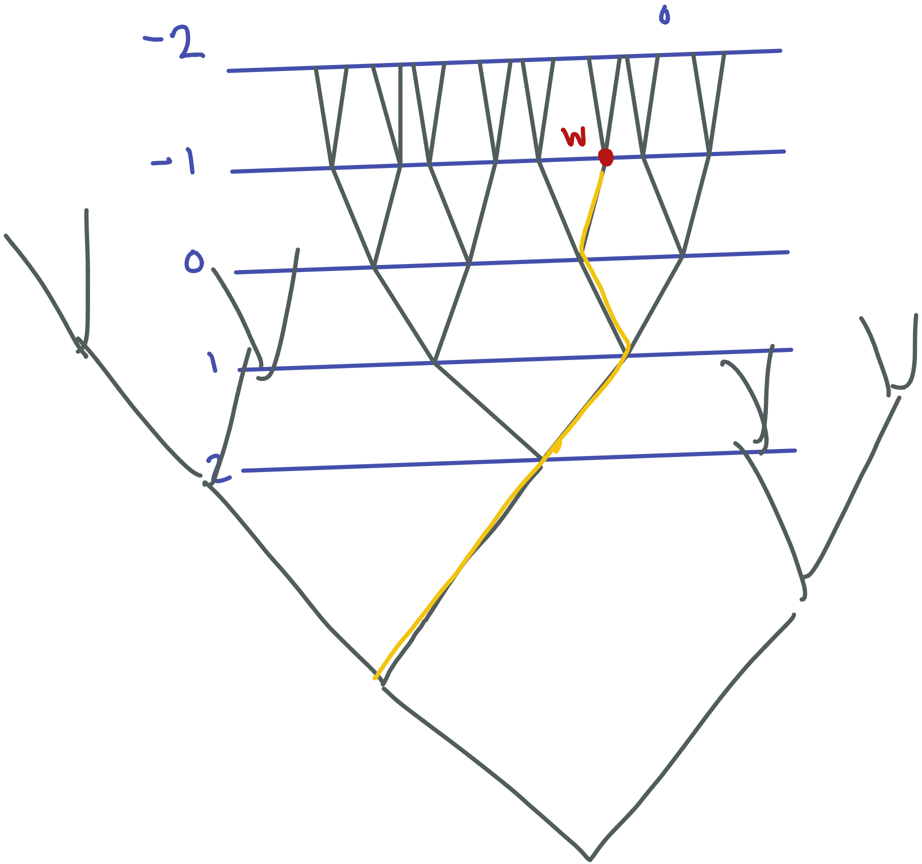
How to get a $(k, \vec{x})$ from a vertex of $DL(2)$?

k is easier: $h(v)$





Make up path from v, down path from w, concat $\rightsquigarrow$ ∞ string of L/R.



-2
-1
0
1
2
w
0

ANNOUNCEMENTS APR 1

- Cameras on
- HW due Thu 3:30
- OH Fri 2-3, Tue 11-12, appt
- Outline due Fri nite
- Makeup points

Today

Thompson's group.

Thompson's gp F

F = group of assoc. laws

or: How to get between all parenthesizations of an expression

x_0 : $a(bc) \to (ab)c$

x_1 : $a(b(cd)) \to a((bc)d)$

x_2 : $a(b(c(de))) \to a(b((cd)e))$

subscript $\leftrightarrow$ "depth"

Composition: do first, then the second.

Must allow for interpreting a, b, c as expressions themselves and allow expansions

$\dots a \dots \to \dots (a_1 a_2) \dots$

A, B, C expressions in a, b, c. $\longrightarrow$ $(A)(BC) = (AB)(C)$

example. $\underbrace{a}_{A}(\underbrace{b}_{B}(\underbrace{c(de)}_{C})) \xrightarrow{x_0} (ab)(c(de))$

A relation:

$$\begin{tikzcd}
a(b(c(de))) \arrow[r, "x_0"] \arrow[d, "x_2"'] & (ab)(c(de)) \arrow[d, "x_1"] \\
a(b((cd)e)) \arrow[r, "x_0"] & (ab)((cd)e)
\end{tikzcd}$$

So: $x_1 x_0 = x_0 x_2$ (right to left mult)

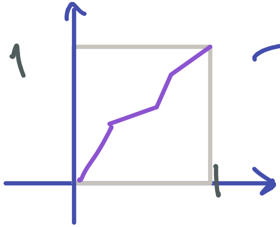
More generally:

$$x_n x_i = x_i x_{n+1} \quad i < n.$$

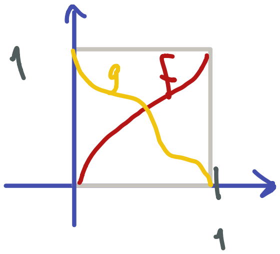
Q Is this really a group?

F via PL homeos

$F = \{$ orientation preserving,
piecewise linear
homeomorphisms
of $[0,1]$ with
dyadic
break points
with slopes powers of 2 $\}$
under fn composition.



Homeos A fn $f: [0,1] \to [0,1]$
is a homeo if it is contin
with contin. inverse.



f, g are
both homeos.

Orient. pres A homeo $f:[0,1]\to[0,1]$
is or. pres. if $f(0)=0$.

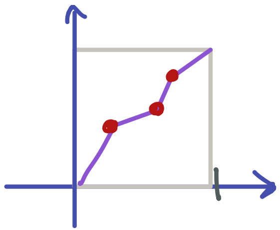
Piecewise liner What you think.
(finitely many line segments)

F via PL homeos

F = { orientation preserving,
piecewise linear
homeomorphisms
of $[0,1]$ with
dyadic
break points
with slopes powers of 2 }

under fn composition.

Break points

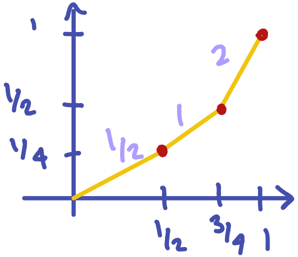


Dyadic $m/2^k$ $\quad k, m \in \mathbb{Z}$

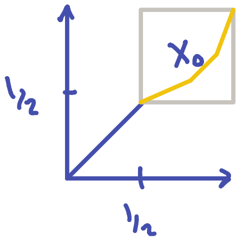
Why is this a group?

think about composition
or pres ✓
homeo ✓
break pts dyadic: exercise.
slopes powers of 2: chain rule

Examples

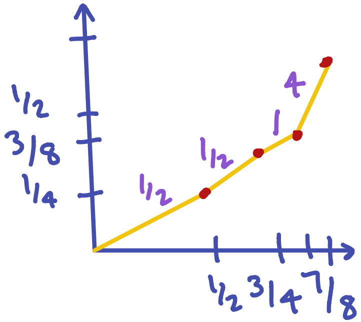


X_0



X_1

What is $X_1 X_0$?



Where is 15/16?

Break pts: Break pts of X_0

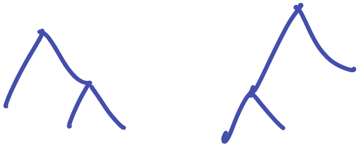
$\cup$ X_0^{-1}(Break pts of X_1)

$$\{1/2, 3/4\} \cup \underbrace{X_0^{-1}(\{1/2, 3/4, 7/8\})}_{\{3/4,\ 7/8,\ 15/16\}}$$

F via tree pairs

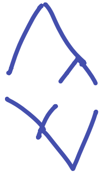
A tree pair is a pair of binary trees with same # of leaves

e.g.



S T

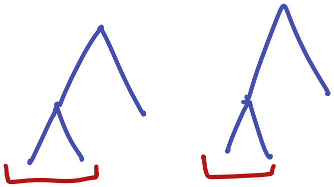
or



S

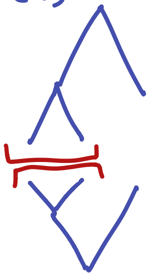
$\perp$

Reduced if no canceling carets:



S T

or



S

$\perp$

T reflected

$F = \{\text{reduced tree pairs}\}$

Mutiplication:

$$(S_2, T_2)\cdot(S_1, T_1)$$

is (S_2, T_1) if $T_2 = S_1$

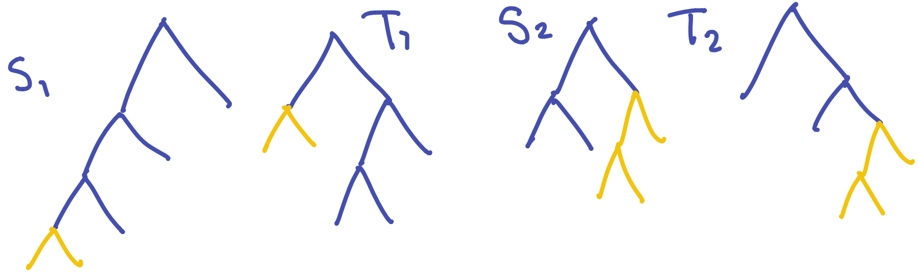
If $T_2 \neq S_1$, add carets until they are equal.

Mutiplication:

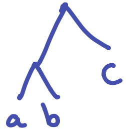
$$(S_2,\underbrace{T_2)\cdot(S_1}, T_1)$$

Is (S_2, T_1) if $T_2 = S_1$

If $T_2 \neq S_1$, add carets until they are equal.

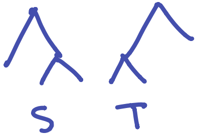


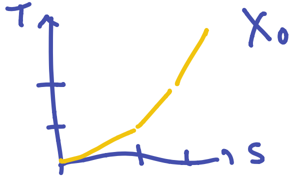
Carets $\longleftrightarrow$ parentheses.



$\longleftrightarrow$ (ab) c

$\longleftrightarrow$ dividing interval in half





Some Facts about F

① F is gen. by x_0, x_1

② F is finitely pres.

③ F contains

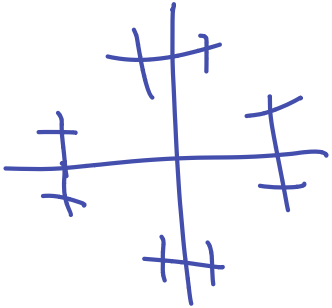
④ F

$\cancel{2}$ $\cancel{\alpha}$ ∞

Major open question

Q. Is F amenable?

A group is non-amenable if its Cayley graph admits a Ponzi scheme.



everyone passes \$1 to inward neighbor.

Yes $\Rightarrow$ F is a fin pres amen gp that is not elem. amenable.

No $\Rightarrow$ F is a fin pres non-amen gp with no free subgp.

ANNOUNCEMENTS APR 6

- Cameras on
- HW due Thu (your choice of 2)
- Draft due Fri
- Office Hours moved Wed 2-3, Tue 11-12, appt
- Makeup work

Today

Quasi-isometries

GEOMETRY VS. ALGEBRA

Thm. $G \underset{QI}{\cong} \mathbb{Z} \implies G$ has finite index subgp $H \cong \mathbb{Z}$.

geometry

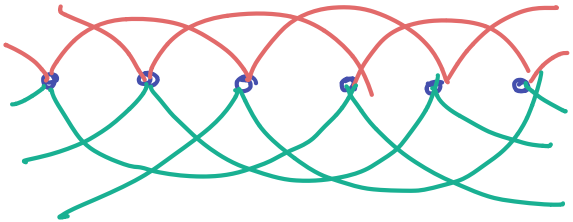
algebra

Two Cayley graphs for $\mathbb{Z}$

$S = \{1\}$

$d(-2,5) = 7$

$S = \{2,3\}$



$d(-2,5) = 3$

Looks like $\mathbb{R}$ from far away.

Will show: it is QI to $\mathbb{R}$.

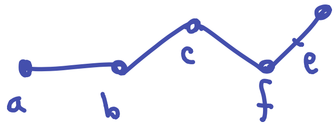
Metric space

(X, d_X) (Y, d_Y) metric spaces.

- $d(x,y) = 0 \iff x = y$
- $d(x,y) + d(y,z) \geq d(x,z)$
- $d(x,y) = d(y,x)$

meaning $d_X(x_1, x_2)$ is distance.

examples: graphs



$d(a,c) = 2$
$d(a,e) = 3\frac{1}{2}$

groups word metric/ distance in Cayley graph.

Isometries

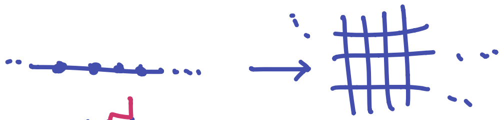
Equivalence of metric spaces

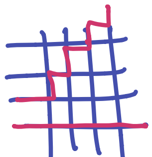
$f : X \longrightarrow Y$ is an isometric embedding if it doesn't change distances:

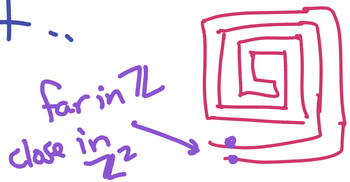
$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

$(\Longrightarrow \text{injective})$

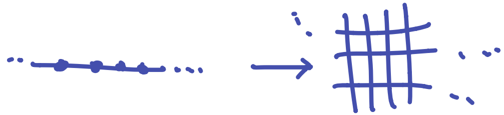
If f is also surjective, say f is an isometry.

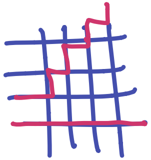
Examples. f: 

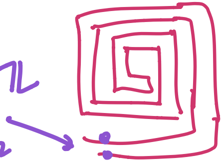
Two ~~An~~ isom. emb. 

Not: 

Examples

① $f:$ 

~~An~~ Two isom. emb. 

Not: far in $\mathbb{Z}$, close in $\mathbb{Z}^2$ 

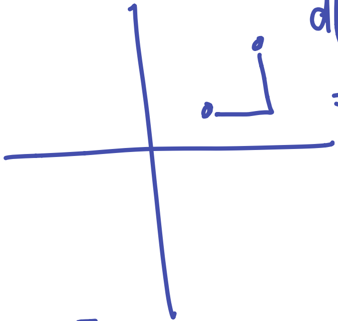
or any non-inj.

② $\mathbb{Z}^2 \xrightarrow{\text{inclusion}} \mathbb{R}^2$

Is this an isom. emb?

No with std metric on $\mathbb{R}^2$

Yes with "taxicab metric" on $\mathbb{R}^2$



$$d((x_1,y_1),(x_2,y_2)) = |x_1-x_2| + |y_1-y_2|$$

③ identity $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto n$, where first $\mathbb{Z}$ has $\{1\}$ metric, second $\mathbb{Z}$ has $\{2,3\}$ metric.

Not an isom. emb.

Bi-Lipschitz equivalence

$f : X \to Y$ is a bi-Lipschitz embedding if $\exists\ K \geq 1$ s.t.

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K\, d_X(x_1, x_2)$$

(K indep. of x_1, x_2).

If f also surj. then f is $\underbrace{\text{bi-Lip equiv.}}_{\text{equiv. rel.}}$

Thm. G = group

S, S' two finite gen sets

$\mathrm{id} : (G, d_S) \to (G, d_{S'})$

is a bi-Lip eq

Example

① $(\mathbb{Z}^2, \text{std}) \to (\mathbb{R}^2, \text{std})$ bi-Lip emb

$K = \sqrt{2}$

Thm. G = group

S, S' two finite gen sets

$\mathrm{id}: (G, d_S) \to (G, d_{S'})$

is a bi-Lip eq

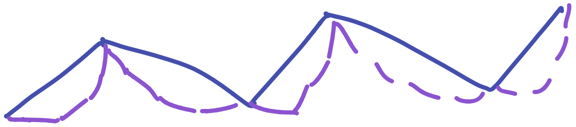
vertex set of Cayley graph.

If you include the edges, lose bi-Lip equiv.

Pf idea What is K?

$$K = \max \{ d_{S'}(\mathrm{id}, s) : s \in S \} \cup \{ d_S(\mathrm{id}, s) : s \in S' \}$$

Use triangle inequality.



Quasi-isometries

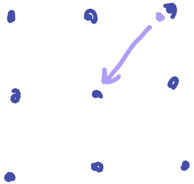
$f: X \to Y$ is a q.i. emb if $\exists\, K \geq 1, C \geq 0$ s.t.

$$\frac{1}{K} d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq K\, d(x_1, x_2) + C$$

If $\exists\, D \geq 0$ s.t. each pt of Y is within D of $f(X)$ then f is a quasi-isometry.

Examples. ① $(\mathbb{Z}^2, \text{std}) \longrightarrow (\mathbb{R}^2, \text{std})$

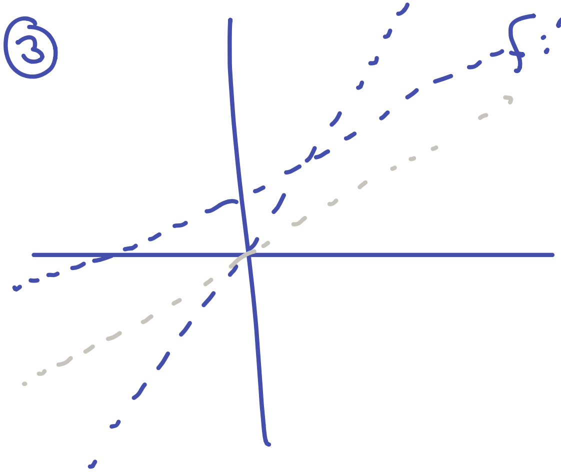
$K = \sqrt{2} \quad C = 0 \quad D = \sqrt{2}/2$



② $(\mathbb{R}^2, \text{std}) \longrightarrow (\mathbb{Z}^2, \text{std}) \qquad K = \sqrt{2} \quad D = 0.$

$(x, y) \longmapsto (\lfloor x \rfloor, \lfloor y \rfloor) \qquad C = \sqrt{2}$

3

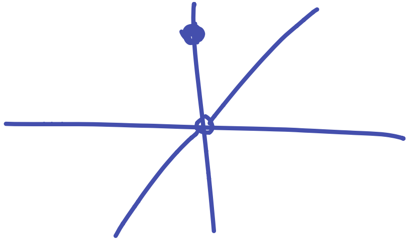


$f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases}5x & x\in\mathbb{Q}\\ 3x+1 & x\notin\mathbb{Q}\end{cases}$$

$K=5 \quad C=1 \quad D=1.$
(or 0)

4 $$f(x)=\begin{cases}5x & x\neq 0\\ 7 & x=0\end{cases}$$

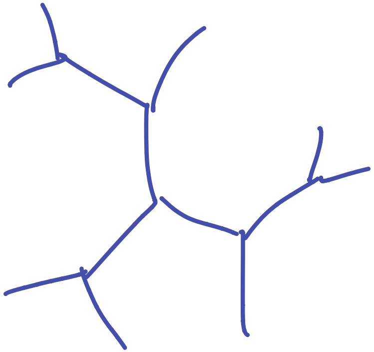


$K=5$

$C=7$

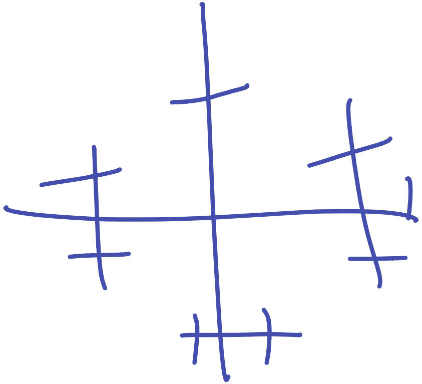
$D=1$

Poll.



T_3

QI $\simeq$



T_4

?

Yes. Ponzi scheme.

QI's are violent, but still have thm: $G \underset{QI}{\simeq} \mathbb{Z}$ then G has $\underset{\mathbb{Z}}{\underset{\cong}{H}} \overset{fi}{\leq} G$

Announcements Apr 8

- Cameras on
- HW due Thu – I forgot again!
- First draft due Fri – share on Teams & Reviews
- Office Hours Tue 11, appt. (more soon!)
- Makeup points

Today

- Milnor-Schwarz Lemma
- $G \underset{QI}{\simeq} \mathbb{Z} \Rightarrow G$ is virtually $\mathbb{Z}$

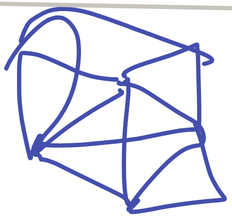
Quasi-isometries

(X, d_X), (Y, d_Y) metric spaces

$f : X \to Y$ is a quasi-isom. if

$$\tfrac{1}{K} d_X(x_1, x_2) - C \le d_Y(f(x_1), f(x_2)) \le K\, d_X(x_1, x_2) + C$$

and there is a D so all pts of Y are within distance D of $f(X)$

F = finite gp  $\underset{QI}{\simeq}$ $\Gamma = \bullet$ with $K = 1$, $C = \mathrm{diam}(F)$, $D = 0$

Milnor-Schwarz Lemma (Graph version)

(Fund Lemma of GGT)

<u>Thm</u>. $G \curvearrowright \Gamma$ = graph

geometric action: $\left\{ \begin{array}{l} \text{finite fund dom} \\ \quad (\text{or } \Gamma/G \text{ finite}) \\ \text{Action is prop. disc.} \end{array} \right.$

Then: G is fin. gen.

& $G \underset{QI}{\simeq} \Gamma$.

$G \curvearrowright \Gamma$ is properly discontinuous if $\forall$ $K \subseteq \Gamma$ finite subgraph

$\#\{g \in G : g \cdot K \cap K \neq \emptyset\} < \infty$.

P.D. and finite f.d. Needed for Thm because... might $G \curvearrowright \Gamma$ trivially.

Example. ⓪ F = finite gp $\Gamma = \bullet \Rightarrow F \underset{QI}{\simeq} \bullet$

with $K =$

$C =$

So: All finite gps are QI to trivial gp.

MILNOR - SCHWARZ LEMMA

(Fund Lemma of GGT)

<u>Thm</u>. $G \curvearrowright \Gamma$ = graph

finite fund dom

(or Γ/G finite)

Action is prop. disc.

Then: G is fin. gen.

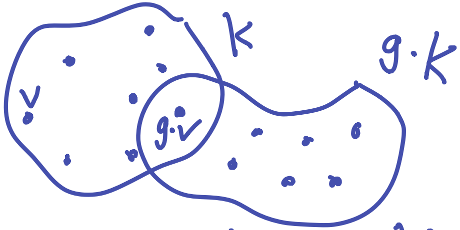
& $G \underset{QI}{\simeq} \Gamma$.

Examples. ① $\mathbb{Z} \curvearrowright$ 

② $SL_2(\mathbb{Z}) \curvearrowright$ Farey tree.

Prop. disc: a $\overset{\text{deg } 2}{\text{vertex}}$ of Γ

is a basis for $\mathbb{Z}^2$



means

g took a basis in K to another

basis in K.

$\Longrightarrow SL_2\mathbb{Z} \underset{QI}{\simeq} T_3 \underset{QI}{\simeq} T_4 \underset{QI}{\simeq} F_2 \underset{QI}{\simeq} F_k \quad k \geq 2$

Applications

① $H \leq G$ finite index.

$H \curvearrowright \Gamma$ = Cayley graph for G.

with finite fund. dom

& prop. disc (b/c G acts p.d.)

$\Longrightarrow H \underset{QI}{\simeq} G$.

② $N \trianglelefteq G$ N finite.

$G \curvearrowright \Gamma$ = Cayley graph for G/N.

with p.d. & finite fund dom.

b/c N is finite.

$\Longrightarrow G/N \underset{QI}{\simeq} G$

We say two groups differ by finite gps if can get from one to other by taking finite index subgps & quotients by finite groups.

[③ $(G,S) \underset{QI}{\simeq} (G,S')$]

Gromov's Program

Which fin. gen. gps are quasi-isometric?

We saw: G, H differ by finite groups $\Rightarrow G \underset{QI}{\simeq} H$

We say G is quasi-isometrically rigid if $G \underset{QI}{\simeq} H \Longrightarrow G, H$ differ by finite gps.

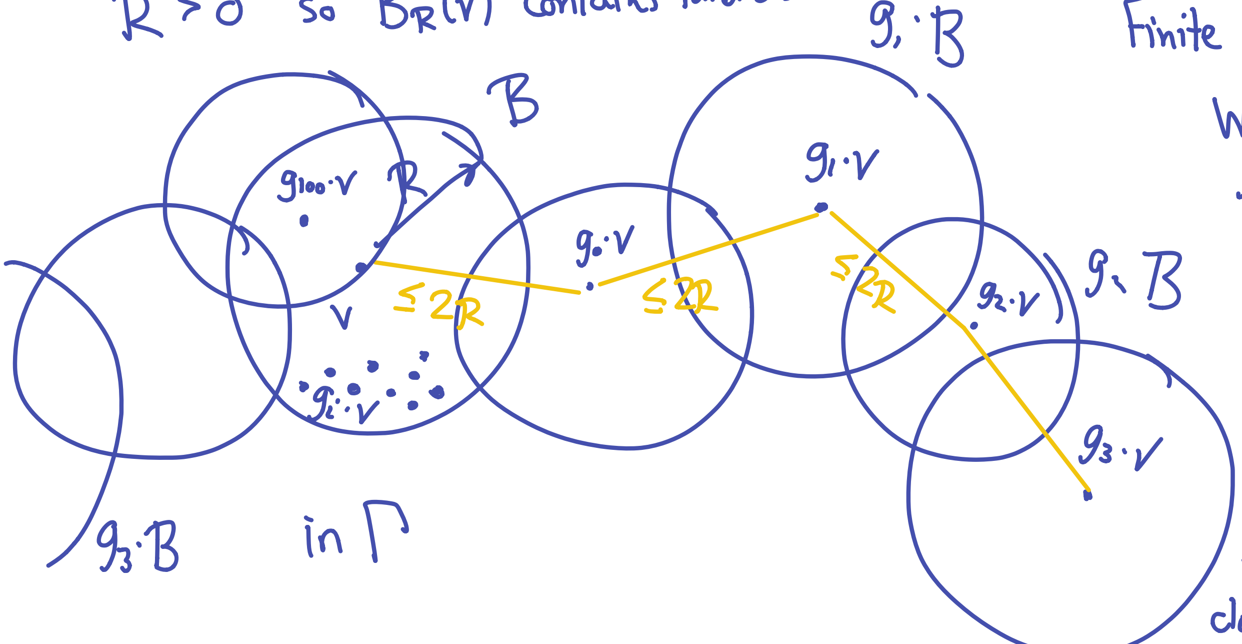
Examples

0. Trivial gp.
1. $\mathbb{Z}^n$ (we'll prove $n=1$ case soon)
2. Braid groups. & Mapping class groups.
3. Free groups.

Idea of Milnor-Schwarz

Choose vertex v.

$R > 0$ so $B_R(v)$ contains fund dom.



Get the finite gen set as usual

$S = \{ g : g \cdot B \cap B \neq \emptyset \}$ ✓

Finite by prop disc.

What are K, C, D?

Distances in Γ not much longer than in G by defn of R.

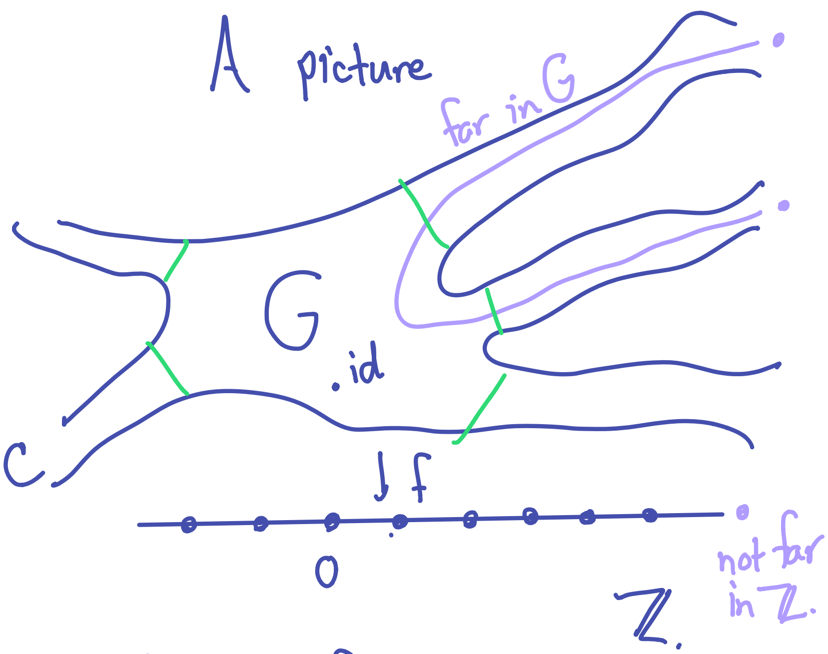
Opposite direction: If have g_i with $|g_i| \to \infty$ & $g_i \cdot v$ close to v, violate P.D.

Thm. G = fin. gen. gp.

$G \underset{QI}{\simeq} \mathbb{Z}$

$\Longrightarrow \exists\, H \underset{fi}{\leq} G,\ H \cong \mathbb{Z}$

A picture



Pf. Let $f: G \longrightarrow \mathbb{Z}$.

$$\frac{1}{K} d(x,y) - C \leq |f(x) - f(y)| \leq K\, d(x,y) + C$$

WLOG $f(\mathrm{id}) = 0$.

(change C if needed).

First Goal: 2 fingers, not 5

one left one right.

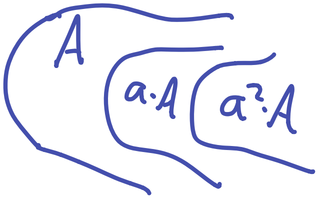
Step 1. G has ∞ order elt a.

Step 2. $|G/\langle a\rangle| < \infty$

For step 1, find $A \subseteq G$, $a \in G$

s.t. $a \cdot A \subsetneq A$

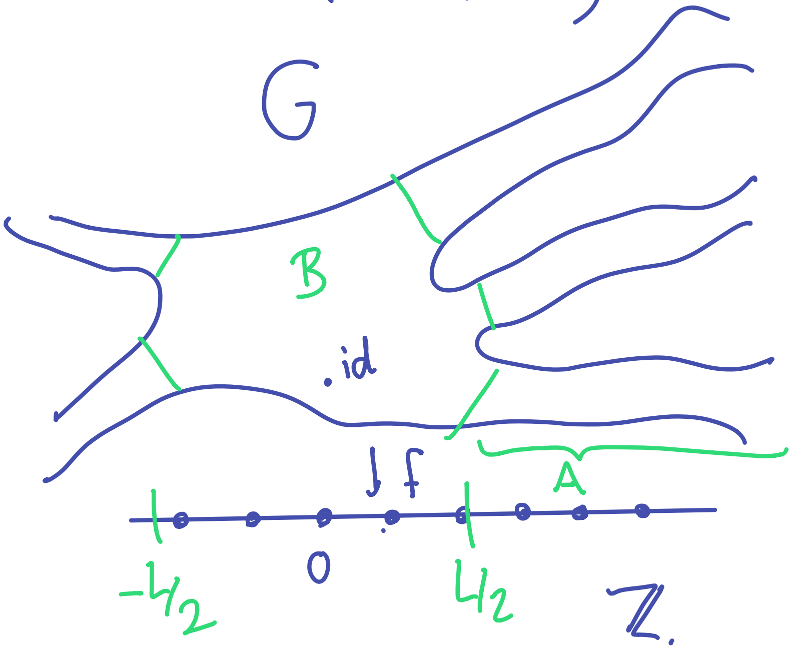
$\Longrightarrow |a| = \infty$.



How to find A?

Let $L \gg K, C$ $\quad (L = K + C)$

$B = f^{-1}([-L/2, L/2])$



Announcements Apr 13

- Cameras on
- Last HW due Thu
- Peer evaluations due Fri
- Presentations next week ~20
- Final draft due Apr 27 3:30.
- Makeup problems
- CIOS

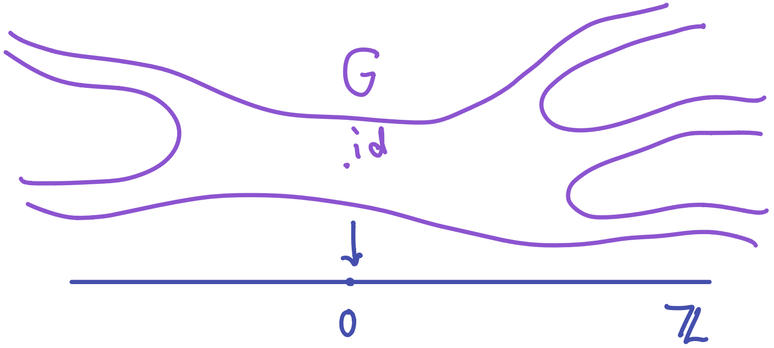
Today

- $G \underset{QI}{\cong} \mathbb{Z} \implies G \underset{v}{\cong} \mathbb{Z}$
- Ends of groups:
  Freudenthal-Hopf Thm

Thm. G = fin gen. gp

$$G \underset{QI}{\simeq} \mathbb{Z}$$

Then G has finite index subgp $H \cong \mathbb{Z}$.



Pf. Let $f: G \longrightarrow \mathbb{Z}$ qi.

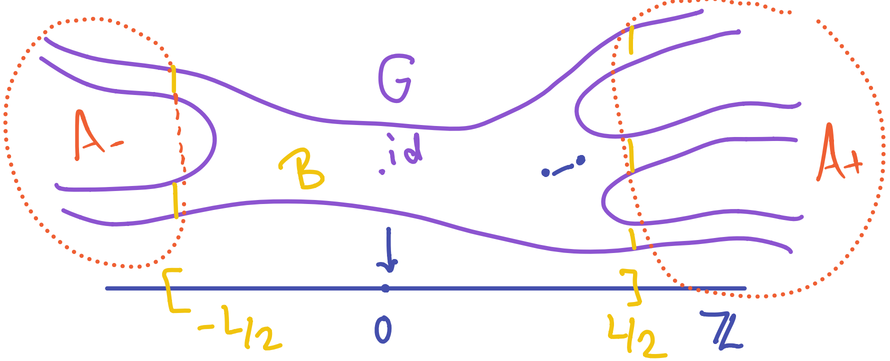
$$\tfrac{1}{K} d(x,y) - C \leq |f(x) - f(y)| \leq K d(x,y) + C$$

Also: D...

WLOG $f(\mathrm{id}) = 0$.

Step 1. G has ∞ order elt a

Step 2. $\langle a \rangle$ has finite index in G.



<u>Step 1.</u> G has ∞ order elt a

Suffices to find $A \subseteq G$, $a \in G$

$a \cdot A \subsetneq A$ (ping pong)

Let $L = K + C$ (e = edge in G $\Rightarrow$ $f(e)$ has length $\leq L$)

$B = f^{-1}([-L/2, L/2])$

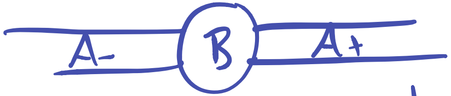
Note: if g, h connected in G, can't lie on opp. sides of B.

WLOG $G \setminus B$ has only unbounded pieces (if not, add any bounded pieces to B)

Let $A_+ = f^{-1}(L/2, \infty) \setminus B$

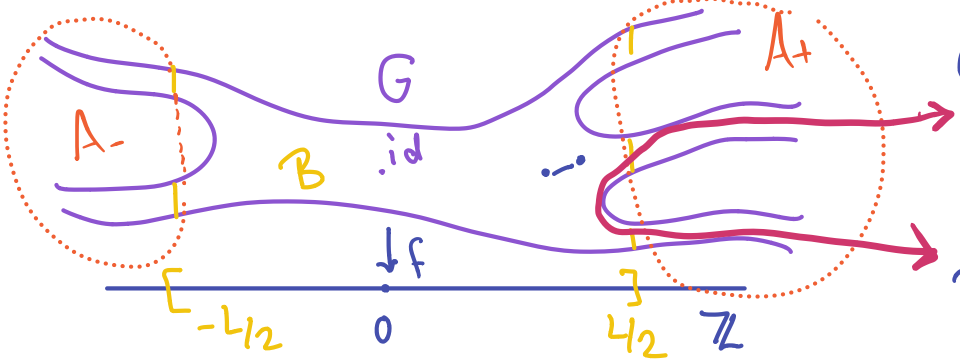
$A_- = f^{-1}(-\infty, -L/2) \setminus B$

Want this pic:



or: A_+, A_- each connected.

also, separate from eachother.



Let $L = K + C$ (e = edge in G $\Rightarrow$ $f(e)$ has length $\leq L$)

$$B = f^{-1}([-L/2, L/2])$$

Note: if g, h connected in G, can't lie on opp. sides of B.

Let $A_+ = f^{-1}(L/2, \infty) \setminus B$

$A_- = f^{-1}(-\infty, -L/2) \setminus B$

Claim 1. $G \setminus B$ has ≥ 2 pieces

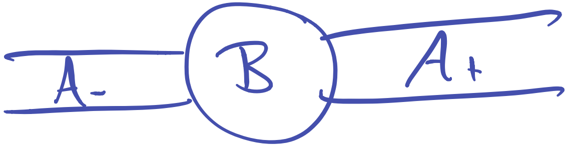
i.e. A_+, A_- not connected to each other.

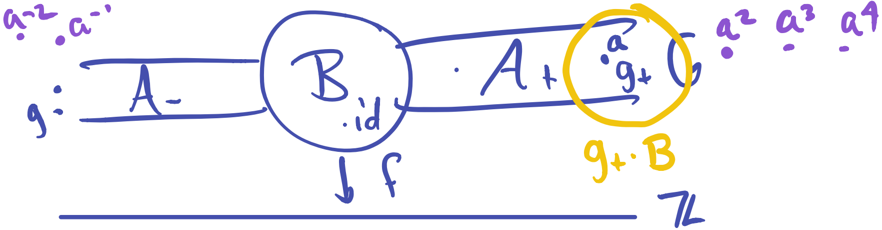
Pf. The above note.

Claim 2. $G \setminus B$ has ≤ 2 pieces.

Pf. Otherwise find arbitrarily far pt of G mapping to same pt of $\mathbb{Z}$. $\square$

Now Have:





Let $g, h \in G$ s.t. $g_+ \in A_+$, $g_- \in A_-$

$$d(id, g_\pm) > 2\,\mathrm{diam}(B)$$

Claim 3. For some $a \in \{g_+, g_-, g_+g_-\}$

$$a \cdot A_\pm \subsetneq A_\pm$$

Pf. Case 1. $g_+ \cdot A_+ \subseteq A_+$

Case 2. $g_- \cdot A_- \subseteq A_-$

Can we argue these cases are the same?

Case 3. Neither true. Think D_∞! $\square$

Step 2. $\langle a \rangle$ has finite index in G.

Claim 1. $d(id, a^n) \to \infty$ as $n \to \infty$

Claim 2. $\exists D$ s.t. D nbd of $\langle a \rangle$ in G is G.

Claim 3. $|G/\langle a \rangle| < \infty$.

Pf of Claim 1: By Step 1, a^n all distinct, but G locally finite (where we use G fin gen.).

Pf of Claim 2. Claim 1 $\Rightarrow$

$d(a^m, a^n) \to \infty$ $|m-n| \to \infty$

$\Rightarrow f(a^i) \to \infty$ $f(a^{-i}) \to -\infty$

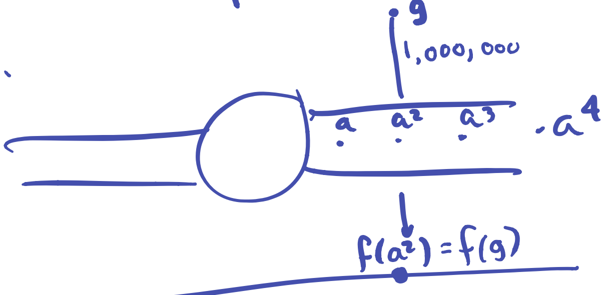
Claim 2. $\exists D$ s.t. D nbd of $\langle a\rangle$ in G is G.

Pf of Claim 2. Claim 1 $\Rightarrow$

$d(a^m, a^n) \to \infty \quad |m-n| \to \infty$

$\Rightarrow f(a^i) \to \infty \quad f(a^{-i}) \to -\infty$

If there were pts in G arbit. far from $\langle a\rangle$ then arb far pts in G would map to "same" pt in $\mathbb{Z}$.



Claim 3. $|G/\langle a\rangle| < \infty$.

Let Γ = Cayley graph for G

$\Gamma/\langle a\rangle$ has one vertex for all a^n

& locally finite,

& finite diam by Claim 2

$\Rightarrow \Gamma/\langle a\rangle$ finite

But vertices of $\Gamma/\langle a\rangle$ are the cosets of $\langle a\rangle$ in G. $\square$

Ends of Groups

Freudenthal-Hopf Thm

G = fin gen gp

$\Rightarrow$ G has $0, 1, 2$, or (∞ many) ends

Some defns:

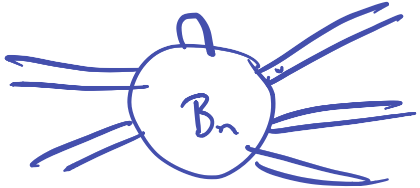
Γ = connected graph, locally finite.

v = base vertex.

B_n = ball of radius n around v.

$\|\Gamma \setminus B_n\|$ = # unbounded pieces of $\Gamma \setminus B_n$.

Γ

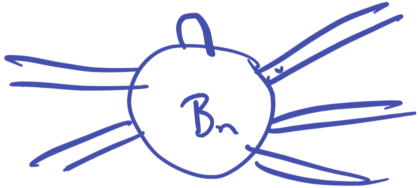


$\|\Gamma / B_n\| = 5$

$$e(\Gamma) = \lim_{n \to \infty} \|\Gamma \setminus B_n\|.$$

$\|\Gamma \setminus B_n\| = \#$ unbounded pieces of $\Gamma \setminus B_n$.

Γ

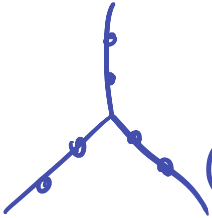


$\|\Gamma \setminus B_n\| = 5$

$$e(\Gamma) = \lim_{n\to\infty} \|\Gamma \setminus B_n\|.$$

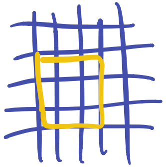
Examples ① Γ finite.

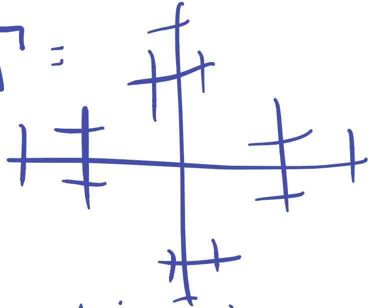
$\Rightarrow e(\Gamma) = 0$



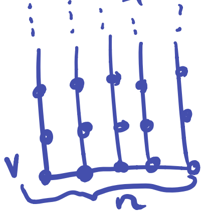
② $\Gamma =$ 

$e(\Gamma) = 2$

③ $\Gamma =$  $e(\Gamma) = 1$

④ $\Gamma =$ 

$e(\Gamma) = \infty$

⑤ $\Gamma =$ 

$e(\Gamma) = n$.

Announcements Apr 15

- Cameras on
- Peer evaluations due ~~Fri~~ Sun
- Presentations next week ~20 mins
- Final draft due Apr 27 3:30.
- Office Hours Fri - postponed.
- Makeup problems
- CIOS

Today

- Ends of groups:
  Freudenthal - Hopf Thm
- Summary

Ends of Groups

Freudenthal-Hopf Thm

G = fin gen gp

$\Rightarrow$ G has 0, 1, 2, or (∞ many) ends

Some defns:

Γ = connected graph, locally finite.

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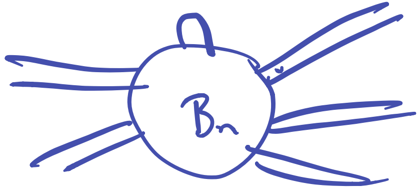
B_n = ball of radius n around v.

$\|\Gamma \setminus B_n\|$ = # unbounded pieces of $\Gamma \setminus B_n$.

Γ

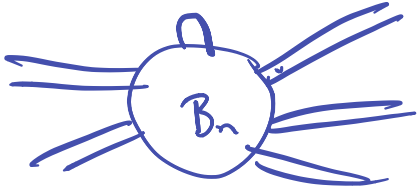


$\|\Gamma \setminus B_n\| = 5$

$$e(\Gamma) = \lim_{n\to\infty} \|\Gamma \setminus B_n\|.$$

$\|\Gamma \setminus B_n\|$ = # unbounded pieces of $\Gamma \setminus B_n$.

Γ



$\|\Gamma \setminus B_n\| = 5$

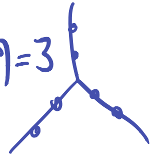
$$e(\Gamma) = \lim_{n\to\infty} \|\Gamma \setminus B_n\|.$$

Examples ① Γ finite.

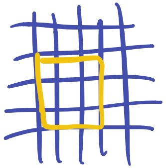
$\Longrightarrow e(\Gamma) = 0$

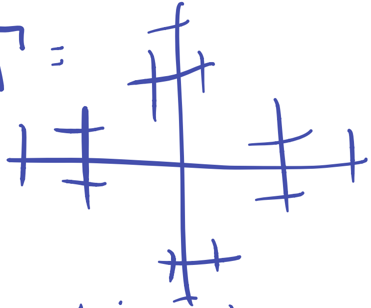
② $\Gamma =$ 

$e(\Gamma) = 2$

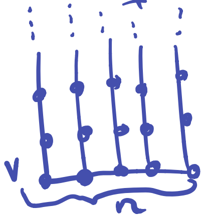
2.5 $\Gamma =$ 

$e(\Gamma) = 3$

③ $\Gamma =$  $e(\Gamma) = 1$

④ $\Gamma =$ 

$e(\Gamma) = \infty$

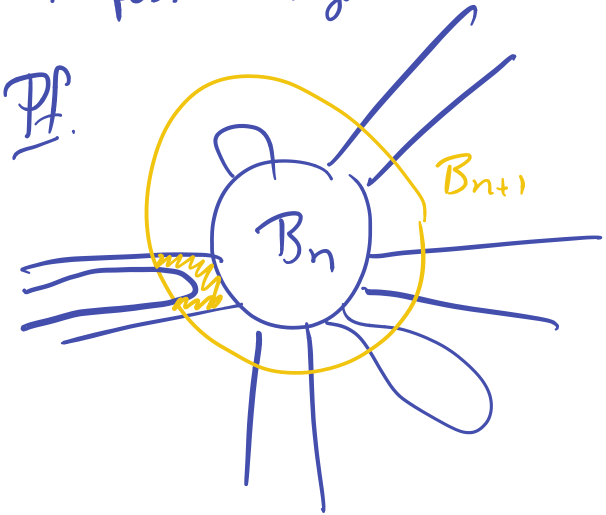
⑤ $\Gamma =$ 

$e(\Gamma) = n.$

$e(\Gamma)$ is well defined

Lemma. $\|\Gamma \setminus B_n\|$ is a non-decreasing seq. of pos. integers.

Pf.



When you take a unbdd subgraph and remove a bounded subgraph $(B_{n+1} \setminus B_n)$ it becomes ≥ 1 unbounded piece

(also, pieces can't merge when you remove stuff.)

Cor. $e(\Gamma)$ is well-def.

Next goal: $e(\Gamma)$ is a QI invariant

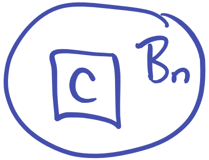
Alternate defn

$$e_c(\Gamma) = \sup\{\|\Gamma \setminus C\| : C \subseteq \Gamma \text{ finite}\}$$

Lemma. $e_c(\Gamma) = e(\Gamma)$.

Pf. $\geq$ sup over bigger set.

$\leq$ Any such C is contained in a B_n. Use argument from last slide.



$\|\Gamma \setminus B_n\| \geq \|\Gamma \setminus C\|$.

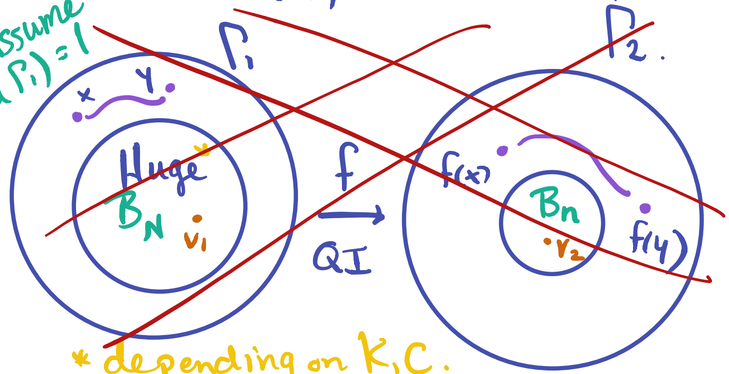
#Ends is a QI invariant

Prop. If $\Gamma_1 \underset{QI}{\simeq} \Gamma_2$ then

$$e(\Gamma_1) = e(\Gamma_2).$$

Pf. Let's convince ourselves that $e(\Gamma_1) = 1 \Leftrightarrow e(\Gamma_2) = 1$.

Assume $e(\Gamma_1) = 1$



* depending on K, C.

~~Want: x, y connected outside B_N.~~

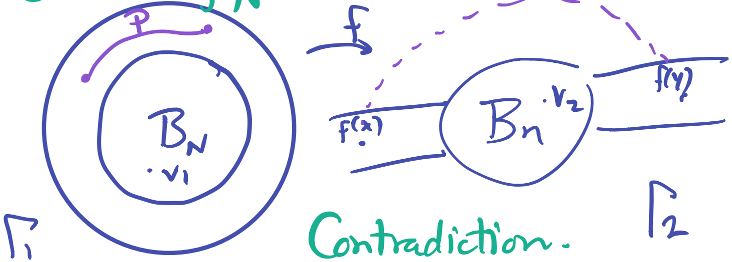
~~Note: f(x), f(y) outside B_n.~~

~~Thus they are connected by path P outside B_n.~~

Assume $e(\Gamma_1) = 1$

Assume B_n cuts Γ_2 in two unbdd pieces.

Choose big N.



Contradiction.

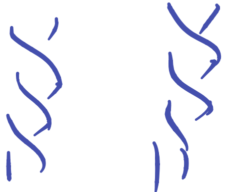
Poll

Q. How many ends does braid gp B_n have?

$B_1 \cong 1 \Rightarrow e(B_1) = 0$

$B_2 \cong \mathbb{Z} \Rightarrow e(B_2) = 2$

B_3 ???



A. $e(B_n) = 1 \quad n \geq 3.$

Step 1.

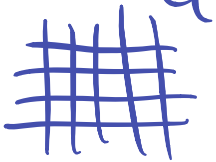
Pf. $e(B_n) = e(PB_n)$

since $[B_n : PB_n] = n! < \infty$

Step 2. $PB_n \cong PB_n/\mathbb{Z} \times \mathbb{Z}$

Fact. If G, H infinite,

$e(G \times H) = 1.$

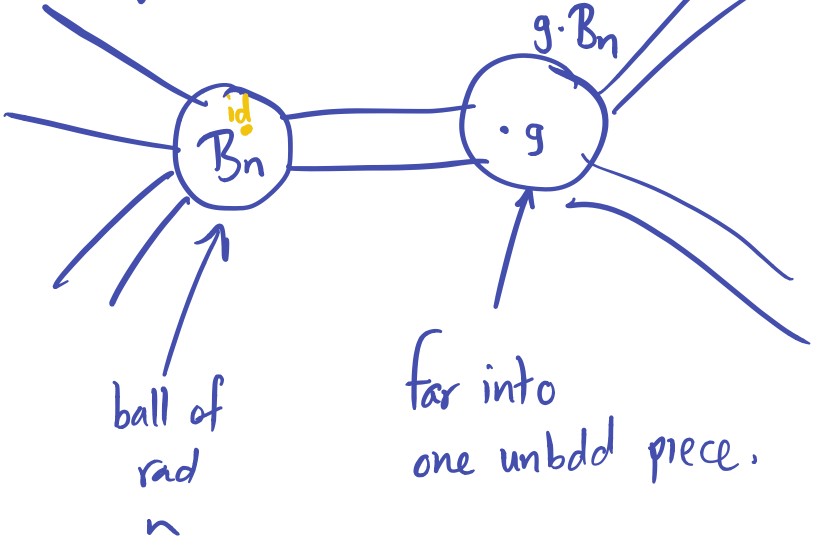


Freudenthal-Hopf Thm

G = fin gen.

Then $e(G) \in \{0, 1, 2, \infty\}$.

Assume 3 ends



$\Longrightarrow$ 4 ends etc...

Some Ends we know

$e(\mathbb{Z}) = 2$

$e(\text{finite gp}) = 0$

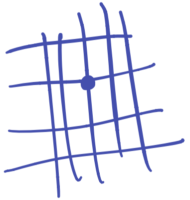
$e(\mathbb{Z}^n) = 1 \quad n \geq 2.$

$e(F_k) = \infty \quad k \geq 2.$

$e(B_n) = 1 \quad n \geq 3$

$e(SL_2\mathbb{Z}) = \infty$

$e(W_{333}) = 1$



Different # ends

$\Rightarrow$ not quasi-isometric.

Also sometimes gps with same # of ends are not QI.

example $\mathbb{Z}^m, \mathbb{Z}^n \quad m \neq n.$

(different growth rates)
$\sim r^n$

Who cares if groups are not QI?

· prop disc. ← doesn't lose info about G.
· finite fund dom. ← does lose info about Γ.

Milnor-Schwarz: If G acts geometrically on Γ then $G \underset{QI}{\simeq} \Gamma$.

<u>So</u>. If $G, H \underset{\text{geom.}}{\circlearrowright} \Gamma$ then $G \underset{QI}{\simeq} H$.

Infinitely many corollaries: B_n does not act geom. on $\mathbb{R}$, or tree.

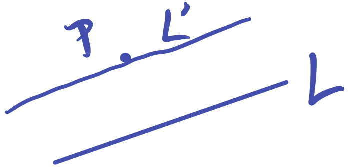
$SL_2\mathbb{Z}$ does not act geom. on $\mathbb{R}^2$.

Geometry, Topology, and Group Theory

Last time: We've learned so much!

Certain groups can/cannot act (geometrically) on the same graph/space.

Today: There's so much more to learn!



Hyperbolic Geometry

Euclid's Postulates 1-4 boring.

5 Given a point P not on line L

$\exists!$ line L' through P & not intersect L.

Lobachevsky / Poincaré : There is geometry without 5

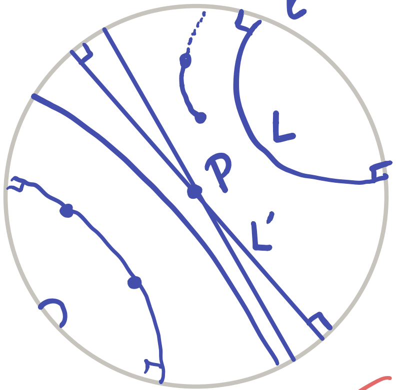
$\rightsquigarrow$ Hyperbolic plane.

Hyperbolic Plane $\mathbb{H}^2$

open disk.

Defn 1

Compare Farey Graph.

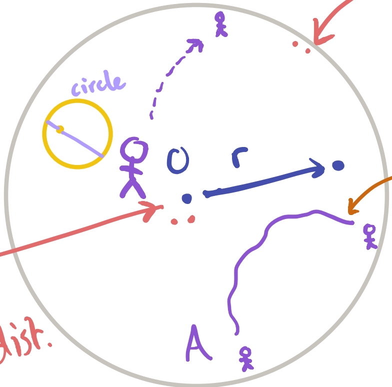


The straight lines are pieces of circles/lines $\perp$ to boundary. $\Longrightarrow$ metric is a multiple of one below

Riemannian geometry

really far apart.

Defn 2.



geodesic

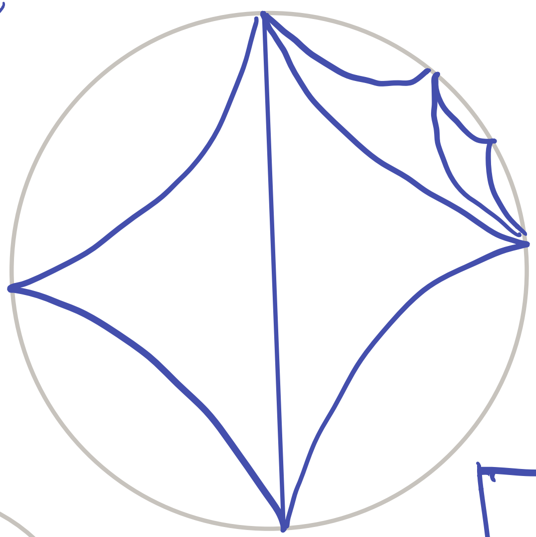
distances almost same as Eucl.dist.

compare BS(1,2)

Metric: $\dfrac{\text{Euclidean metric}}{(1-r^2)}$

$\Longrightarrow$ straight lines as above.

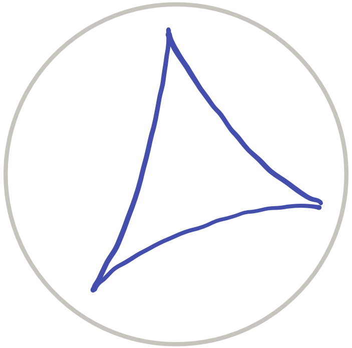
Farey graph



all triangles congruent in $\mathbb{H}^2$

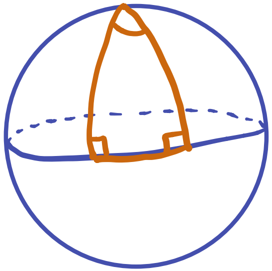
all have interior angles 0

(all triangles "skinny")



sum of interior angles $< \pi$.

Compare spherical geometry

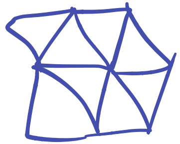


interior angles $> \pi$

Which groups act on $\mathbb{H}^2$?

For $\mathbb{E}^2$ have reflection groups, e.g. W_{333}

and $\mathbb{Z}^2$



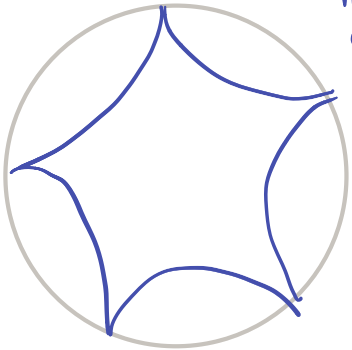
all of these coming from tilings

Let's look for tilings of $\mathbb{H}^2$.

Looking for tiles in $\mathbb{H}^2$

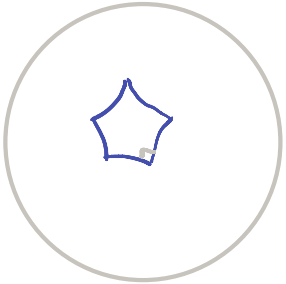
int. $\measuredangle$'s $\sim 3\pi/5$

small n-gons have nearly Euclidean interior angle sums $\pi(n-2)$



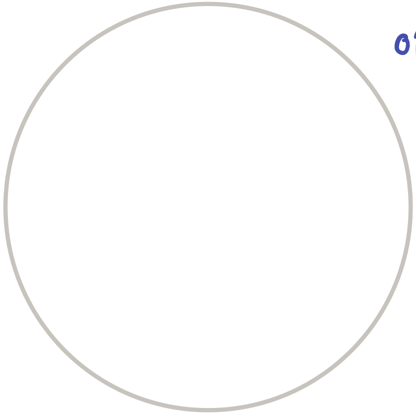
interior angles 0

IVT $\Rightarrow$ $\exists$ regular right angled pentagon!



Now tile!

Aside: Defn #3 of $\mathbb{H}^2$.

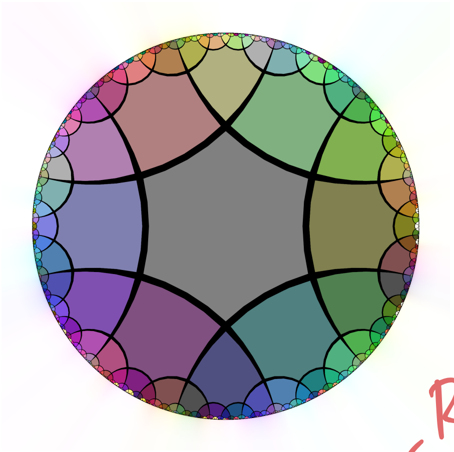


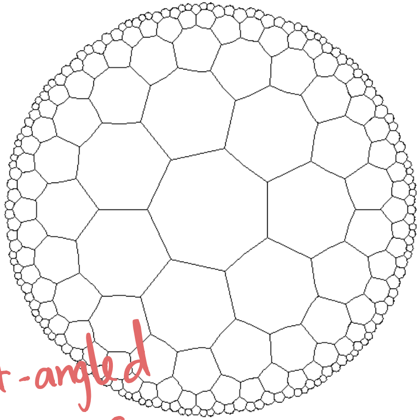
open unit disk

Isometries are:

$$\left\{ \begin{array}{c} \text{Möbius transformation} \\ \text{preserving open unit disk} \end{array} \right\}$$

$$\longleftrightarrow \left\{ f(z) = \frac{az+b}{cz+d} : \begin{array}{c} a,b,c,d \\ \in \mathbb{R} \end{array} \right\}$$





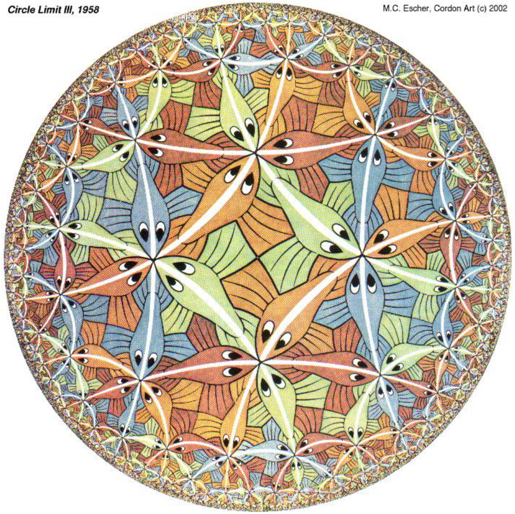
Right-angled Coxeter/reflection gps

reflection group

$$\langle x_1,\dots,x_5 : (x_1x_2)^2=(x_2x_3)^2=\dots=(x_5x_1)^2=\mathrm{id}\rangle$$

Now have <u>many</u> new gps, not QI to Euclidean gps
W_{333} etc.

Circle Limit III, 1958 M.C. Escher, Cordon Art (c) 2002

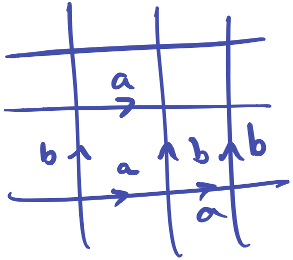


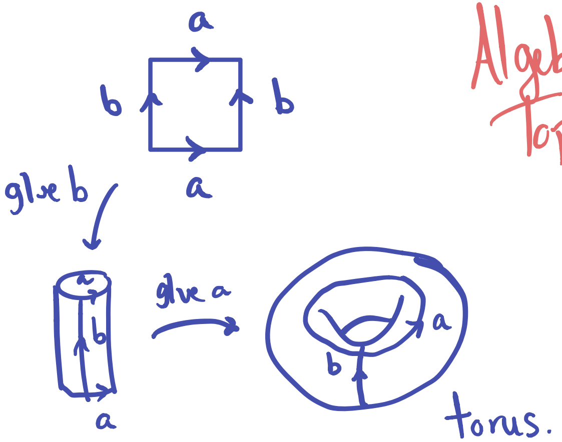
Connection to Topology

Algebraic Topology.

the loop around the square is the relation

$\mathbb{E}^2$



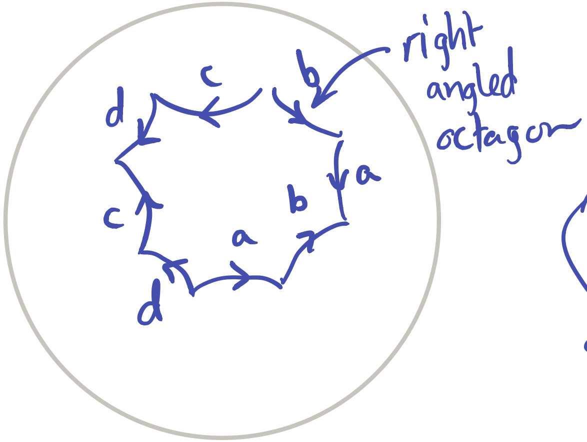


$$\langle a, b : aba^{-1}b^{-1} = id \rangle$$

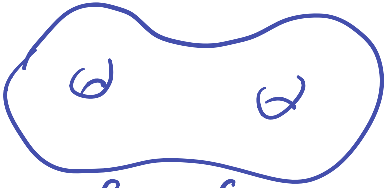
$$\cong \mathbb{Z}^2$$

QI to $\mathbb{E}^2$

$\mathbb{H}^2$



S_2

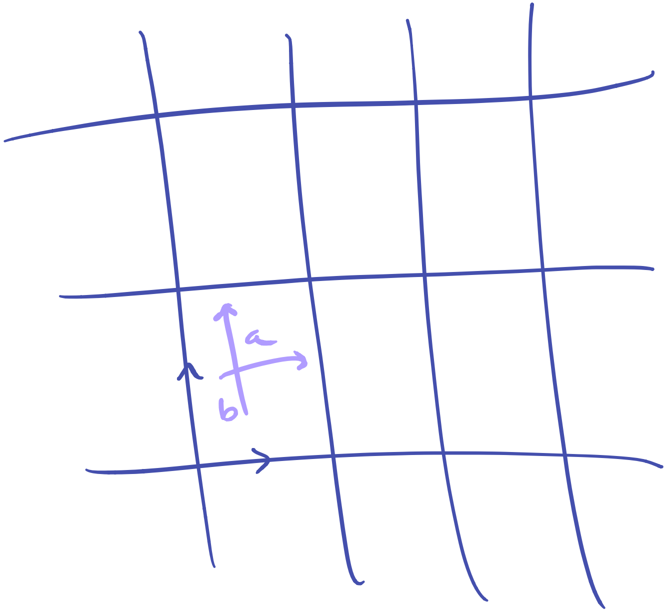


Surface of genus 2.

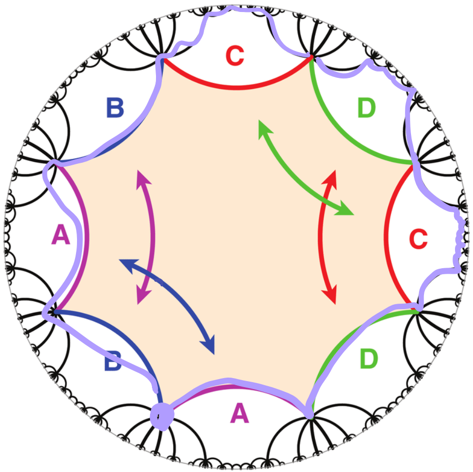
$$\langle a, b, c, d : aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

fundamental gp of S_2

QI to $\mathbb{H}^2$



$\mathbb{E}^2$

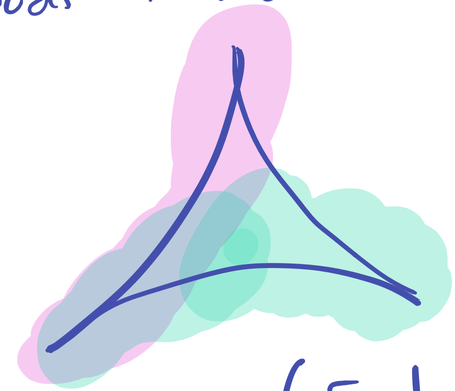


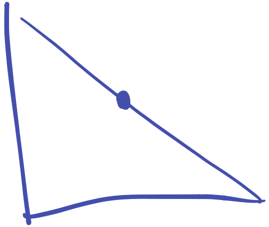
$\mathbb{H}^2$

Milnor-Schwarz:
fund. gp of $S_2 \underset{QI}{\simeq} \mathbb{H}^2$

Hyperbolic Groups à la Gromov

A space is δ-hyperbolic if for any triangle, the δ-neighborhoods of two sides together contain the 3rd side.





Facts.
- $\mathbb{H}^2$ is δ-hyperbolic ($\delta = \log 2$?)
- δ-hyp. is a QI invt $\Rightarrow$ fund gp of S_2 is δ-hyp.

Two Theorems of Gromov

Thm. Most groups are δ-hyperbolic.

Geometry

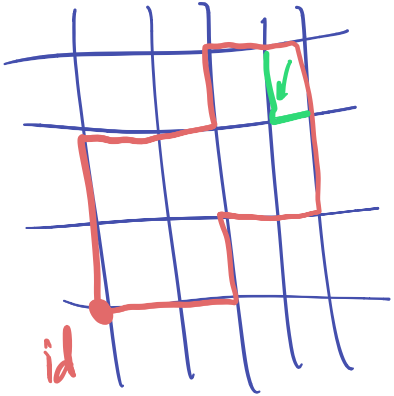
Thm. A group is δ-hyp $\Longleftrightarrow$ its word problem is solvable in linear time.

Group theory

Why does fund gp of S_2 have linear time soln to WP?

$\langle a,b,c,d :$
$aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

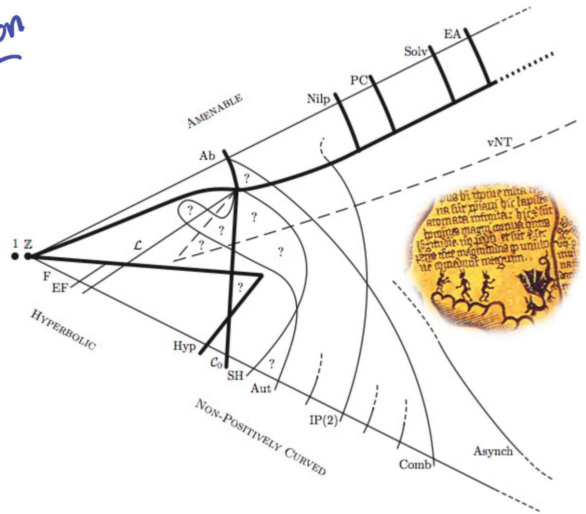
Any closed loop in Cayley graph must use ≥ 6 sides of a single octagon



So can replace word of length 6 with word of length 2

SHORTENING.

Bridson



Here there be dragons.

Key: Ab — abelian, Nilp — nilpotent, PC — polycyclic, Solv — solvable, EA — elementary amenable, F = free, EF — elementarily free, $\mathcal{L}$ — limit, Hyp — hyperbolic, $\mathcal{C}_0$ — CAT(0), SH — semi-hyperbolic, Aut — automatic, IP(2) — quadratic isoperimetric inequality, Comb — combable, Asynch — asynchronously combable, vNT — the von Neumann–Tits line. The question marks indicate regions for which it is unknown whether any groups are present.