Research Prospectus

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My research is on the group-theoretical, combinatorial, and dynamical aspects of surface homeomorphisms. For a closed surface S, its mapping class group $\mathrm{Mod}(S)$ is the group $\pi_0(\mathrm{Homeo}(S))$, that is, the group of homotopy classes of homeomorphisms of S. This group is central in mathematics because:

- it is π_1 of the moduli space of algebraic curves homeomorphic to S,
- it contains the gluing data required to build any 3-manifold (Heegaard),
- it contains the data needed to build any symplectic 4-man. (Donaldson),
- it classifies S-bundles over arbitrary spaces (via the monodromy), and
- it is isomorphic to the outer automorphism group of $\pi_1(S)$.

My work on mapping class groups draws techniques and ideas from—and answers questions raised by—the work of Artin, Dehn, Nielsen, Birman, Hain, McMullen, Thurston, and others, and makes use of Morse theoretic arguments à la Bestvina–Brady, combinatorial methods of Ivanov, and the tools of geometric group theory, algebraic topology, 3-manifold theory, and Teichmüller geometry. My research has had meaningful interactions with and external applications to algebraic geometry, 3-manifold theory, and complex dynamics; see §2, 4, and 5.

I will discuss five prongs of my research in separate sections: finiteness properties of the Torelli subgroup, representations of braid groups, normal subgroups of the mapping class group, dynamics of individual elements, and 1-dimensional complex dynamics.

1. Finiteness properties of Torelli groups

The mapping class group $\mathrm{Mod}(S_g)$ of a surface S_g of genus g has a symplectic representation. The kernel $\mathcal{I}(S_g)$ is called the Torelli group:

$$1 \to \mathcal{I}(S_g) \to \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1.$$

We think of the Torelli group $\mathcal{I}(S_g)$ as capturing the non-arithmetic (or, more mysterious) aspects of the mapping class group $\mathrm{Mod}(S_g)$.

A fundamental open problem is to determine if the Torelli group is finitely presentable when $g \geq 3$; in other words, does the Torelli group have a finite algebraic description?

A more general, and much-studied, problem is to determine which of the homology groups $H_k(\mathcal{I}(S_g))$ are finitely generated. Bestvina, Bux, and I determined that the cohomological dimension of $\mathcal{I}(S_g)$ is $3g-5$ (so all homology groups $H_k(\mathcal{I}(S_g))$ are all trivial for $k > 3g-5$), answering a 20-year-old question of Mess and Farb. We further showed that $H_{3g-5}(\mathcal{I}(S_g))$ is infinitely generated, sharpening a theorem of Akita, and answering a question of Mess from Kirby's problem list. Our work appeared in JAMS.

Theorem 1.1. *For $g \geq 2$, we have:*

- $\mathrm{cd}(\mathcal{I}(S_g)) = 3g-5$, *and*
- $H_{3g-5}(\mathcal{I}(S_g))$ *is infinitely generated.*

We in particular recover, and significantly extend, a celebrated theorem of Mess which says that $\mathcal{I}(S_2)$ is an infinitely generated free group.

We also studied the subgroup $\mathcal{K}(S_g)$ of $\mathcal{I}(S_g)$ that is generated by Dehn twists about separating curves. We proved that the cohomological dimension of $\mathcal{K}(S_g)$ is $2g-3$, also answering a question of Farb.

Theorem 1.2. *For $g \geq 2$, we have* $\mathrm{cd}(\mathcal{K}(S_g)) = 2g-3$.

The main contribution is a new combinatorial model for $\mathcal{I}(S_g)$ called the complex of minimizing cycles. This idea inspired papers by Hatcher and Irmer, and a paper of Hatcher and myself, where we give a simple proof of the theorem of Birman and Powell that the Torelli group is generated by bounding pair maps.

Analogous to $\mathrm{Mod}(S_g)$ is the group $\mathrm{Out}(F_n)$ (the group of outer automorphisms of a free group). We can think of $\mathrm{Out}(F_n)$ as the mapping class group of a graph. There is a corresponding Torelli group $\mathcal{I}(F_n)$, first studied by Nielsen and Magnus. Bestvina, Bux, and I showed the following (in Inventiones).

Theorem 1.3. *For $n \geq 3$, we have:*

- $\mathrm{cd}(\mathcal{I}(F_n)) = 2n-4$, *and*
- $H_{2n-4}(\mathcal{I}(F_n))$ *is infinitely generated.*

This answers a question of Bridson–Vogtmann, and extends/sharpens a theorem of Smillie and Vogtmann. In the case $n = 3$, the last result generalizes the famous theorem of Krstić and McCool that $\mathcal{I}(F_3)$ is not finitely presented. We also give a geometric proof of the seminal theorem of Magnus that $\mathcal{I}(F_n)$ is finitely generated.

Bestvina, Bux, and I conjectured that our methods could be extended to show that $H_k(\mathcal{I}(S_g))$ is infinitely generated for $2g-3 \leq k \leq 3g-5$. This conjecture was recently confirmed by Gaifullin.

2. The Burau representation and the period map

The braid groups B_n are mapping class groups of punctured disks. What is more, there is a natural map $\mathrm{B}_{2g+1} \to \mathrm{Mod}(S_g)$. As a direct consequence, we obtain a symplectic representation $\mathrm{B}_{2g+1} \to \mathrm{Sp}_{2g}(\mathbb{Z})$. This representation is nothing other than the famous Burau representation of the braid group evaluated at $t = -1$. The kernel (modulo its center) is isomorphic to the hyperelliptic Torelli group $\mathcal{SI}(S_g)$ the centralizer in the Torelli group of a hyperelliptic involution.

Hain conjectured that $\mathcal{SI}(S_g)$ is generated by Dehn twists. At first Hain's conjecture seemed overly optimistic: one cannot expect that an infinite index subgroup of $\mathcal{I}(S_g)$ is generated by the generators of $\mathcal{I}(S_g)$ lying in that subgroup. Worse, there are other simple elements of $\mathcal{SI}(S_g)$, and at first it was not clear how to factor them into Dehn twists that lie in $\mathcal{SI}(S_g)$.

I started working on Hain's conjecture in 2005, first breaking the problem into three steps, and then completing the steps between 2005 and 2013 in two papers with Brendle and one paper with Brendle and Putman (Inventiones). In fact, we proved a theorem that is stronger than Hain conjectured.

Theorem 2.1. *$\mathcal{SI}(S_g)$ is generated by Dehn twists about separating curves that are preserved by the hyperelliptic involution and have genus at most two.*

As a consequence of Theorem 2.1, we obtain topological information about the branch locus of the period mapping from Torelli space to the Siegel upper half-plane. Let $\mathcal{H}_g^c$ denote the space obtained from this branch locus by adjoining curves of compact type.

Theorem 2.2. *For $g \geq 0$, the space $\mathcal{H}_g^c$ is simply connected.*

Brendle, Childers, and I proved two other results about $\mathcal{SI}(S_g)$.

Theorem 2.3. *For $g \geq 2$, we have:*

- $\mathrm{cd}(\mathcal{SI}(S_g)) = g - 1$, *and*
- $H_{g-1}(\mathcal{SI}(S_g))$ *is infinitely generated.*

In particular, $\mathcal{SI}(S_3)$ is not finitely presented. It it not known if $\mathcal{SI}(S_g)$ is finitely generated (or has finitely generated first homology) for $g \geq 3$.

There is a mod m version of $\mathcal{SI}(S_g)$: the level m subgroup $\mathrm{B}_n[m]$ of B_n is the kernel of the composition

$$\mathrm{B}_{2g+1} \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/m\mathbb{Z}).$$

Arnol'd proved that the group $\mathrm{B}_n[2]$ is exactly the pure braid group PB_n. Brendle and I prove the following (Crelle).

Theorem 2.4. *For any n, the group $\mathrm{B}_n[4]$ is equal to the subgroup generated by squares of Dehn twists, and is also equal to PB_n^2.*

One consequence is that $\mathrm{B}_3[4]$ is equal to the Brunnian subgroup of B_3; a priori these are totally different groups!

With Kordek, I recently proved that the first betti number of $\mathrm{B}_n[4]$ is

$$3\binom{n}{4} + 3\binom{n}{3} + \binom{n}{2}.$$

This should be compared with the first betti number of PB_n, which is $\binom{n}{2}$.

3. Normal subgroups of mapping class groups

Having already discussed the structure of two specific subgroups of the mapping class group, we now turn to the ambitious problem of classifying normal subgroups. A complete classification is almost certainly out of reach, but my work gives several broad characterizations.

Ivanov proved that $\mathrm{Aut}\,\mathrm{Mod}(S_g) \cong \mathrm{Mod}(S_g)$ when $g \geq 3$. He further proved that the abstract commensurator of $\mathrm{Mod}(S_g)$ (the group of isomorphisms between finite index subgroups of $\mathrm{Mod}(S_g)$) is again $\mathrm{Mod}(S_g)$. This work was sharpened by Farb, Ivanov, who showed that the automorphism group and abstract commensurator group of $\mathcal{I}(S_g)$ was again $\mathrm{Mod}(S_g)$. Let $\mathcal{K}(S_g)$ denote the normal subgroup $\mathcal{K}(S_g)$ of $\mathcal{I}(S_g)$ generated by Dehn twists about separating curves. Brendle and I improved on the work of Farb–Ivanov as follows.

Theorem 3.1. *For $g \geq 3$, we have*

$$\mathrm{Aut}(\mathcal{K}(S_g)) \cong \mathrm{Comm}(\mathcal{K}(S_g)) \cong \mathrm{Mod}(S_g).$$

Our theorem confirms a conjecture of Farb. It says that the very small subgroup $\mathcal{K}(S_g)$ remembers all of the algebraic structure of $\mathrm{Mod}(S_g)$. This theorem was generalized to open surfaces by Kida.

Our work was greatly generalized by Bridson–Pettet–Souto as follows. There is a filtration of $\mathrm{Mod}(S_g)$ called the Johnson filtration; the kth term $\mathcal{N}_k(S_g)$ is the kernel of the action on $\pi_1(S_g)$ modulo the kth term of its lower central series. The first three terms are $\mathrm{Mod}(S_g)$, $\mathcal{I}(S_g)$, and $\mathcal{K}(S_g)$.

In unpublished work, Bridson–Pettet–Souto showed that in each $\mathcal{N}_k(S_g)$ has automorphism group and abstract commensurator group $\mathrm{Mod}(S_g)$. In response to these works and others, Ivanov formulated the following.

Metaconjecture. *Every object naturally associated to a surface S and having a sufficiently rich structure has* $\mathrm{Mod}(S)$ *as its group of automorphisms.*

The "objects" of interest in the above papers are normal subgroups of $\mathrm{Mod}(S_g)$. With the metaconjecture in mind, Brendle and I proved a vast generalization of Bridson–Pettet–Souto, pointing to a much more general phenomenon. Say that a subsurface of S_g is *small* if it is contained in a subsurface of genus less than $g/3$ with connected boundary.

Theorem 3.2. *Let N be a normal subgroup of* $\mathrm{Mod}(S_g)$ *and assume that N has an element whose support is small. Then*

$$\mathrm{Aut}(N) \cong \mathrm{Comm}(N) \cong \mathrm{Mod}(S_g).$$

So the typical normal subgroup has automorphism group $\mathrm{Mod}(S_g)$. (Our hypothesis exactly rules out the pathological examples of Dahmani–Guirardel–Osin.) Our work validates the metaconjecture for normal subgroups of $\mathrm{Mod}(S_g)$.

With Clay and Mangahas, we construct many examples of free, normal subgroups of $\mathrm{Mod}(S_g)$. These are in the spirit of Dahmani–Guirardel–Osin but they have reducible elements (by Theorem 3.2, none has small support). We were thus led to the following conjectural dichotomy for normal subgroups of $\mathrm{Mod}(S_g)$:

- normal subgroups with automorphism group $\mathrm{Mod}(S_g)$
- normal right-angled Artin subgroups

To prove his original theorem about $\mathrm{Aut}(\mathrm{Mod}(S_g))$, Ivanov translated the problem into a combinatorial one. The complex of curves is a graph with vertices for isotopy classes of curves in S_g, and edges for disjointness. Ivanov proved that its automorphism group is $\mathrm{Mod}(S_g)$. Later works followed this pattern, and in each case the game was to show that the group of automorphisms of some particular curve complex was $\mathrm{Mod}(S_g)$. My thesis (Duke Math J.), for example, showed that the automorphism group of the pants complex is $\mathrm{Mod}(S_g)$.

In my work with Brendle, we give a very general condition for a curve complex to have automorphism group $\mathrm{Mod}(S_g)$. This is the first argument that applies to infinitely many (or even more than one) complex at once. Our work essentially resolves Ivanov's metaconjecture for simplicial complexes and normal subgroups.

4. Dynamical aspects of pseudo-Anosov mapping classes

A pseudo-Anosov element of the mapping class group is one that is locally modeled (away from a finite set) on the action of a hyperbolic element of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$. The typical mapping class is of this form.

A basic measure of complexity of a pseudo-Anosov map is its entropy. This number measures the amount of mixing being effected. A central question is: which real numbers are entropies of pseudo-Anosov maps?

For a fixed surface, the set of entropies of pseudo-Anosov maps is closed and discrete; in particular, there is a smallest one. For $H \leqslant \mathrm{Mod}(S_g)$, write $L(H)$ for the smallest entropy of a pseudo-Anosov element of H.

Penner showed that if we allow the genus of our surface to go to infinity we can find pseudo-Anosov maps with smaller and smaller entropies. Specifically:

$$L(\mathrm{Mod}(S_g)) \asymp \frac{1}{g}.$$

Farb, Leininger, and I set out to show that $L(\mathcal{I}(S_g))$ goes to zero at a slower rate than $L(\mathrm{Mod}(S_g))$. What we found is surprisingly stronger: the set of entropies in $\mathcal{I}(S_g)$ is bounded away from zero, independently of g (Amer. J. Math.).

Theorem 4.1. *For $g \geq 2$, we have $L(\mathcal{I}(S_g)) \asymp 1$.*

We generalize this in three directions. First, let $\mathrm{Mod}(S_g, k)$ be the subgroup of $\mathrm{Mod}(S_g)$ consisting of elements that fix a subspace of $H_1(S_g)$ of dimension at least k (the coefficients can be in any field). With Agol and Leininger, I proved the following, answering a question of Ellenberg (J. LMS).

Theorem 4.2. *For $g \geq 2$ and $0 \leq k \leq 2g$, we have*

$$L(\mathrm{Mod}(S_g), k) \asymp (k+1)/g.$$

This interpolates between Penner's result ($k = 0$) and Theorem 4.1 ($k = 2g$). With $\mathbb{Z}/2$-coefficients and $k = g$, we have $L(\mathrm{Mod}(S_g), k) = \mathrm{Mod}(S_g)[2]$.

The second direction concerns the Johnson filtration (see Section 3). We have the following theorem with Farb and Leininger.

Theorem 4.3. *Given $k \geq 1$, there exist constants m_k and M_k, with $m_k \to \infty$ as $k \to \infty$, so that*

$$m_k \leq L(\mathcal{N}_k(S_g)) \leq M_k$$

for all $g \geq 2$.

The point is that m_k and M_k are independent of g, in contrast to Penner's theorem above. We can paraphrase the theme as: *algebraic complexity implies dynamical complexity.*

The third direction generalizes from Torelli groups to arbitrary normal subgroups. With Lanier I recently proved the following.

Theorem 4.4. *If a pseudo-Anosov element of* $\mathrm{Mod}(S_g)$ *has entropy less than* $\log\sqrt{2}$ *then it normally generates* $\mathrm{Mod}(S_g)$*.*

In 1986, Long asked if there were any pseudo-Anosov normal generators for $\mathrm{Mod}(S_g)$. Our theorem gives a resounding 'yes.'

Another way to state Theorem 4.4 is: *if a pseudo-Anosov mapping class lies in any proper normal subgroup of* $\mathrm{Mod}(S_g)$*, its entropy is* $> \log\sqrt{2}$.

As such Theorem 4.4 generalizes my work with Agol, Farb, and Leininger to arbitrary normal subgroups. Lanier and I also give examples of pseudo-Anosov normal generators with arbitrarily large entropy and arbitrarily large translation distances on the complex of curves, disproving a conjecture of Ivanov.

(Lanier and I also proved that every periodic mapping class—besides the hyperelliptic involution—is a normal generator. This theorem has been applied by Mann–Wolff to show that surface group actions on the circle are rigid.)

Farb, Leininger, and I also proved a theorem that relates small entropies to 3-manifolds (in Advances). Fix some $L > 0$ and consider the set

$$\Psi(L) = \{\phi : S \to S \text{ pseudo-Anosov} \mid \text{entropy}(\phi) < L/|\chi(S)|\};$$

in this definition, S ranges over all surfaces. Penner's result implies that, for L large enough, this set of *small-entropy pseudo-Anosov maps* is infinite. Any $\phi \in \Psi(L)$ gives rise to a 3-manifold M_ϕ, its mapping torus. Denote by M_ϕ° the 3-manifold obtained from M_ϕ by deleting the orbit of each singular point of ϕ under the suspension flow.

Theorem 4.5. *Fix* $L > 0$*. The set* $\{M_\phi^\circ \mid \phi \in \Psi(L)\}$ *is finite.*

In other words, the infinite set of small-entropy pseudo-Anosov maps is "generated" by (or, flow-equivalent to) a finite set of examples (after deleting the singular sets). This answers a question posed by McMullen. Our work inspired a paper by Agol, who (among other things) gave a new proof of our theorem and a paper by Algom-Kfir and Rafi, who gave a version of our theorem for $\mathrm{Out}(F_n)$, a paper by Kojima–McShane, who gave a simple proof of a weak version of our theorem, and a forthcoming paper by Leininger, Minsky, Souto, and Taylor.

We also made the following conjecture, which would give yet another point of view on the small-entropy pseudo-Anosov maps.

Symmetry Conjecture. *Any pseudo-Anosov map realizing the smallest entropy for a given surface can be decomposed as a homeomorphism supported on a subsurface of uniformly small complexity multiplied by a periodic homeomorphism.*

Such a theorem would show that Penner's examples are universal, in other words that all small entropy maps come from his construction.

In related work, Strenner, Yurttaş and I give a quadratic-time algorithm for determining if a mapping class is pseudo-Anosov and in that case producing the entropy and associated foliations. This improves on work of Bell–Webb, who showed that there is a polynomial time algorithm to determine if a mapping class is pseudo-Anosov.

5. Twisted rabbit problems in complex dynamics

A topological polynomial is a branched cover $\mathbb{C} \to \mathbb{C}$. We say that a topological polynomial is postcritically finite if the set of all forward orbits of critical points is finite. We consider topological polynomials up to conjugacy and homotopy.

One way to obtain a new postcritically finite topological polynomial from an old one is to compose the old one with a homeomorphism of $\mathbb{C}$ (or mapping class) that preserves the postcritical set pointwise. Then the question is: *is the new topological polynomial equivalent to an actual polynomial, and if so which one?*

This problem was originally posed by Hubbard for the special case of the so-called rabbit polynomial and where the mapping class was a power of a specific Dehn twist. This version is known as the twisted rabbit problem. The specificity of the question was meant to illustrate the difficulty of the general problem.

The twisted rabbit problem was solved in a breakthrough paper by Bartholdi–Nekrashevych in 2006. The main tool in their approach is an algebraic invariant called the iterated monodromy group. Their paper deals with the case of degree 2 with 3 postcritical points. Recently, Kelsey–Lodge extended their work to the case of degree 2 and 4 postcritical points. It seems daunting to deal with higher degree and/or more postcritical points using iterated monodromy groups.

In work with Belk, Lanier, and Winarski, we give a new approach to the twisted rabbit problem that is inspired by the modern theory of mapping class groups. In particular we give a simple-to-describe algorithm for solving the twisted rabbit problem, with any topological polynomial and mapping class as input. As an application, we are able to solve two natural analogs of the twisted rabbit problem, where the degree or the number of postcritical points is arbitrary.

The main new idea is to consider a combinatorial model X_n for Teichmüller space. The vertices are homotopy classes of trees in the complex plane with n marked points. Given a vertex of X_n, and a topological polynomial f, we can perform a pullback operation: take the preimage of the tree, and in that take the convex hull of the postcritical set. This operation gives a pullback map $p_f : X_n \to X_n$. If f is a polynomial then the Hubbard tree v_f for f is a fixed point for p_f. Interestingly, this pullback map is qualitatively different than Thurston's pullback map.

Theorem (in progress). *If f is a polynomial with n postcritical points, and v is a vertex of X_n then for k large enough $p_f^{\circ k}(v)$ lies in the 2-neighborhood of v_f.*

This theorem gives an iterative algorithm for finding the Hubbard tree, which completely determines the polynomial: iterate the pullback map, and at each step search the (finite) 2-neighborhood for the Hubbard tree.

In the obstructed (non-polynomial) case we expect to be able to find the obstructing curves by applying the pullback map to an augmented version of X_n.

I will end by describing some examples of twisted rabbit problems that we have solved with our techniques. We consider the n-eared rabbit polynomial f, which has $n+1$ postcritical points. And let c be any convex simple closed curve containing two of the postcritical points. Then the polynomial $T_c^n \circ f$ only depends on the 4-adic expansion of n (in a way that we completely understand), as in the Bartholdi–Nekrashevych case.

More generally, if we take c to surround k postcritical points, then the polynomial $T_c^n \circ f$ only depends on the 2^k-adic expansion of n (this is a new phenomenon). There are many other variations (in higher degree) we can understand. But again the main point here is the jump from 4 postcritical points to arbitrarily many, and degree 2 to arbitrary degree.