

THE BINOMIAL THEOREM

THEOREM. For any x and y and any natural number n , we have:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \\ &= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n} x^0 y^n\end{aligned}$$

THE BINOMIAL THEOREM

PROBLEM. Expand $(2x^3 + y)^5$ and simplify.

PROBLEM. Expand $(x - \frac{1}{x})^6$ and simplify.

PROBLEM. Find the coefficient of x^{15} in $(x^2 - \frac{x}{3})^{11}$.

PASCAL'S TRIANGLE

THEOREM. The k^{th} entry in the n^{th} row of Pascal's triangle is $\binom{n}{k}$ for $n \geq 0$ and $0 \leq k \leq n$.

Note: The top row is considered to be row 0, and the leftmost entry is entry 0.

PROOF.

PASCAL'S TRIANGLE

What is the sum of the entries in the n^{th} row?

$$\begin{aligned} &1 = \\ &1 + 1 = \\ &1 + 2 + 1 = \\ &1 + 3 + 3 + 1 = \\ &1 + 4 + 6 + 4 + 1 = \\ &\vdots \\ &\cdot \end{aligned}$$

THE BINOMIAL THEOREM

$$(X+Y)^n = \sum_{k=0}^n \binom{n}{k} X^{n-k} Y^k$$

plug in...

to prove...

$X=1, Y=-1$	Inclusion-exclusion principle
$X=10, Y=1$	n^{th} row of P's $\Delta = 11^n$
$X=1, Y=1$	n^{th} row sum of P's $\Delta = 2^n$
$X=\sqrt{2}, Y=-1$	$\sqrt{2}$ is irrational

THE INCLUSION-EXCLUSION PRINCIPLE

THEOREM. $|A_1 \cup \dots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j|$
 $+ \dots + (-1)^{n+1} |A_1 \cap \dots \cap A_n|$

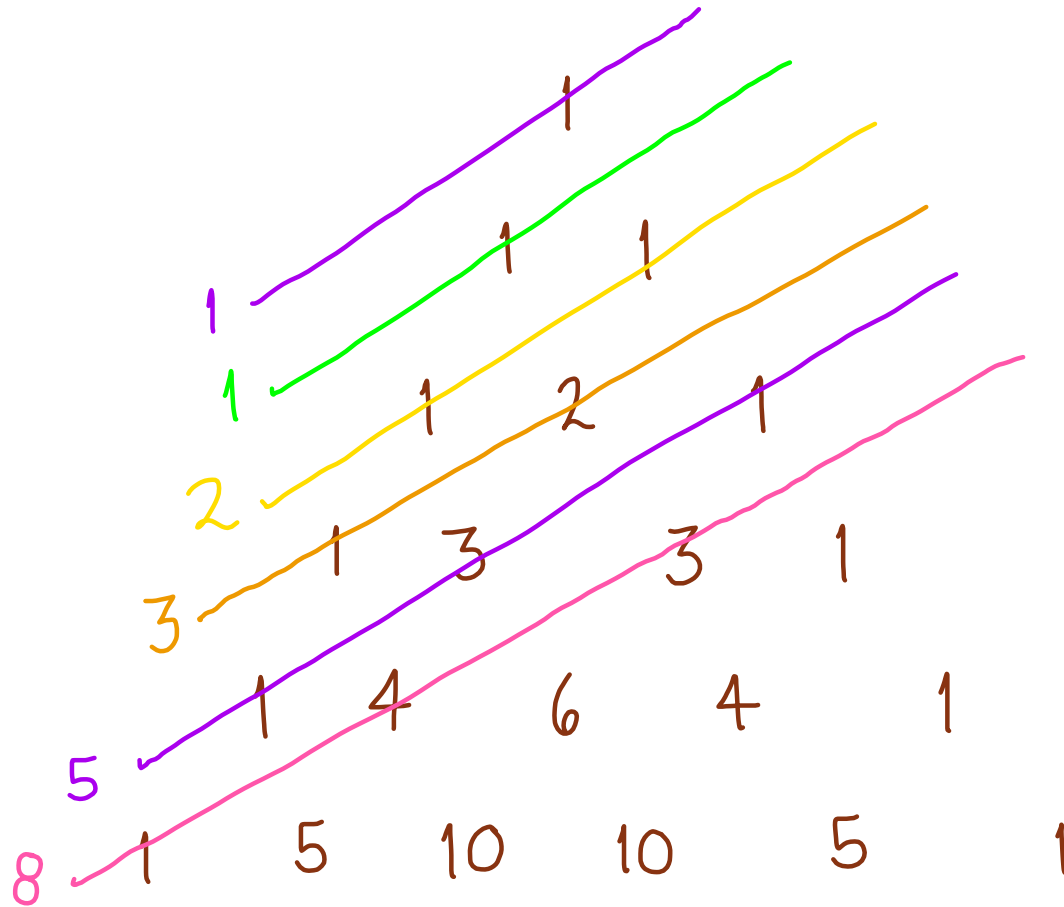
PROOF:

Row Sums IN PASCAL'S TRIANGLE

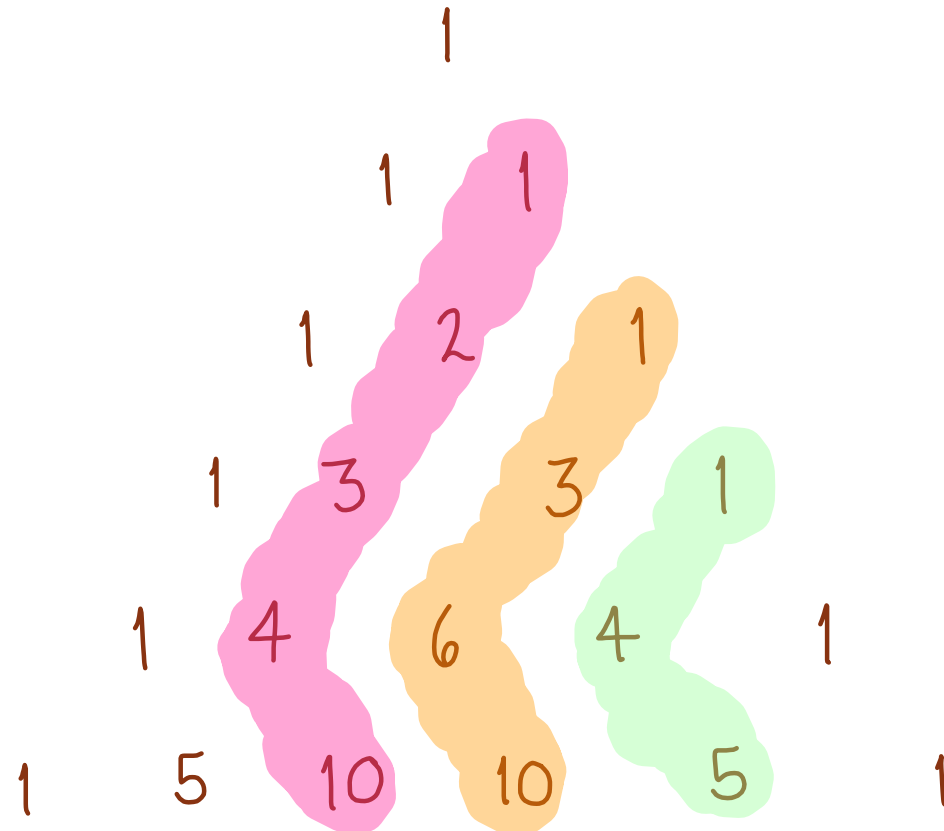
THEOREM. The sum of the entries in the n^{th} row of Pascal's triangle is 2^n .

PROOF.

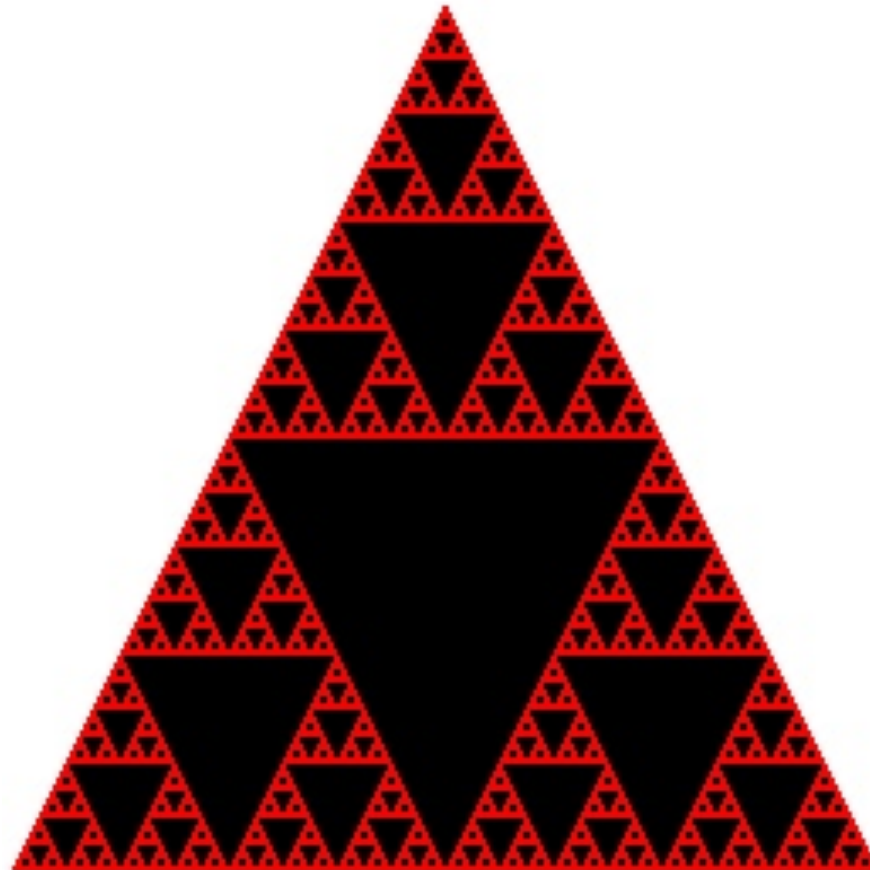
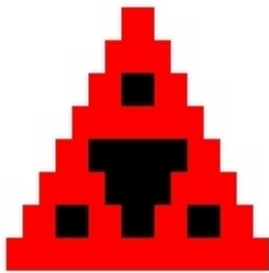
THE FIBONACCI NUMBERS IN PASCAL'S TRIANGLE



THE HOCKEY STICK THEOREM



Pascal's Triangle Mod 2



What about mod 3?

A Curious PROBABILITY

QUESTION. A professor hands back exams randomly. What is the probability that no student gets their own exam?

ANSWER. 5 students ~
 10 students ~
 100 students ~

DERANGEMENTS

A *derangement* of n objects that have some natural order is a rearrangement of the objects so that no object is in its correct position.

QUESTION. How many are there? Call the number D_n .

n	D_n	$P(D_n)$
1		
2		
3		
4		

What is the pattern?

A FORMULA FOR D_n

Let A_k be the permutations of n ordered objects with object k in the correct spot.

$$D_n = \left(\bigcup_{k=1}^n A_k \right)^c$$

$$D_4 =$$

$$D_4 =$$

THEOREM. $D_n =$

D_n AND e

THEOREM. $D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right)$

Recall: $e^x =$

$$\leadsto e =$$

$$e^{-1} =$$
$$\approx$$

So $D_n \approx$

$$\leadsto P(D_n) \approx$$

DERANGEMENTS

PROBLEM. Fifteen people check coats at a party and at the end they are handed back randomly. How likely is it that...

- (a) Tim gets his coat back?
- (b) Jeremy gets his coat back?
- (c) Jeremy and Tim get their coats back?
- (d) Jeremy and Tim get their coats back but no one else does?
- (e) The members of the Beatles get the right set of coats back (maybe not in the right order)?
- (f) Everyone gets their coat back?
- (g) Exactly one person gets their coat back?
- (h) Nobody gets their own coat back?
- (i) At least one person gets their coat back?