# Repetitions

**Question:** How many ways are there to put $r$ identical marbles into $n$ boxes, if you are allowed to put more than one marble per box?

First try 3 marbles into 10 boxes.

Case 1: All in same box $\binom{10}{1}$

Case 2: Two in one box, one in another $10 \cdot 9$

Case 3: All different boxes $\binom{10}{3} = 120$

Addition rule $\rightsquigarrow 120 + 90 + 10 = 220$.

What about 10 marbles in 3 boxes?

Lots of cases!

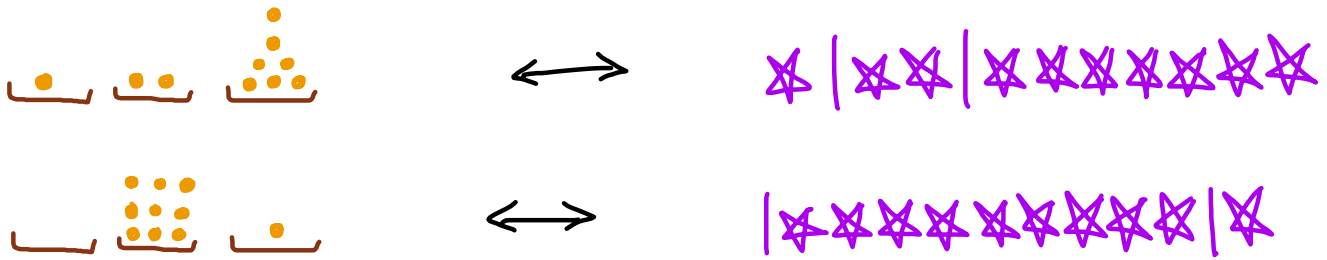
What to do?

# Stars and Bars

Can answer the last question by looking at it the right way:

The number of ways of putting 10 marbles into 3 boxes is the same as:

the number of binary strings with 10 zeros, 2 ones
(or 10 stars, 2 bars)



How many such strings are there?

$\binom{12}{2} = 66$ (choose which of the 12 spots will be stars.)

# REPETITIONS

QUESTION: How many ways are there to put $r$ identical marbles into $n$ boxes, if you are allowed to put more than one marble per box?

ANSWER: This is the same as the number of strings with $r$ stars and $n-1$ bars:

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

# Repetitions, Permutations, and Combinations

How many ways to put $r$ marbles in $n$ boxes if...

| | the marbles are indistinguishable | the marbles are distinguishable |
|---|---|---|
| at most one marble is allowed per box | $\binom{n}{r}$ | $P(n,r)$ |
| any number of marbles is allowed in a box | $\binom{n+r-1}{r}$ | $n^r$ |

# REPETITIONS

EXAMPLE: How many ways are there to choose 15 cans of soda from a cooler with (lots of) Coke, Dr. Pepper, Mtn Dew, RC cola, and Mr. Pibb?

FURTHER: What if I insist on at least 3 Cokes and exactly one Mr. Pibb?

# REPETITIONS

EXAMPLE. In how many ways can we choose 4 nonnegative integers $a$, $b$, $c$, and $d$ so that $a+b+c+d=100$?

What if $a, b, c$, and $d$ are natural numbers?

# REPETITIONS

EXAMPLE. How many ways are there to choose 4 integers $a, b, c,$ and $d$ so that:

$$a+b+c+d=15$$

$$a \geq -3,\ b \geq 0,\ c \geq -2,\ d \geq -1\ ?$$

# Generalized Permutations

Example. How many ways are there to arrange the letters of SYZYGY?

Example. What about MISSISSIPPI?

# Generalized Permutations

In general, say we have $n$ objects that fall into $k$ groups, with $n_i$ objects in the $i^{th}$ group. Two objects in the same group are indistinguishable, but objects in different groups are distinguishable. In how many ways can we order the objects?

$$P(n; n_1, \ldots, n_k) = n! / n_1!\, n_2! \cdots n_k!$$

This is also the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$

# Generalized Permutations

Example. Suppose there are 100 spots in the showroom of a car dealership. There are 15 (identical) sports cars, 25 compact cars, 30 station wagons, and 20 vans. In how many ways can the cars be parked?

# MORE PROBLEMS

1. How many numbers less than 1,000,000 have the sum of their digits equal to 19?

2. A shelf holds 12 books. How many ways to choose 5 books so no adjacent books are chosen?

3. You want to visit 5 towns twice each, but there is one town you don't want to visit twice in a row. How many different travel itineraries are there?