Chap 8 DNB Thm.

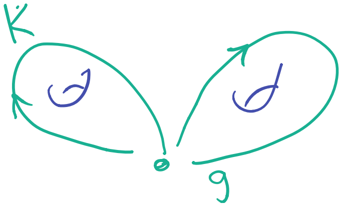
G = group

Inner autos:

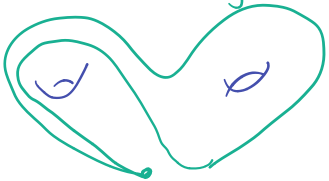
$$\Phi_k : G \to G$$
$$g \mapsto kgk^{-1}$$

Example: $G = \pi_1(S_g)$

Φ_k = push about k^{-1}.



kgk^{-1}



$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$$

Example $\Longrightarrow$

$$\sigma : \mathrm{Mod}^{\pm}(S_g) \longrightarrow \mathrm{Out}\,\pi_1(S_g)$$

topology algebra

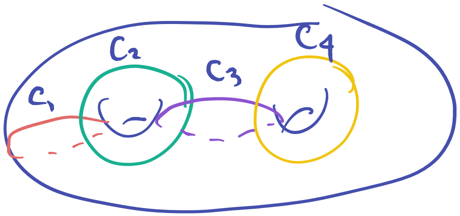
<u>Thm.</u> σ is $\cong$

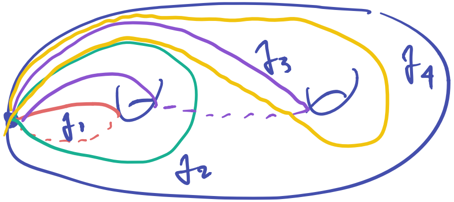
Injectivity: $K(G,1)$ theory

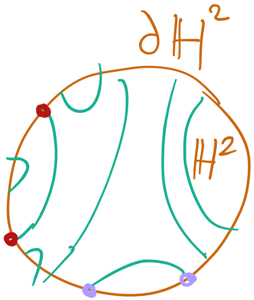
Surjectivity: $K(G,1)$ theory:

outer auto of $\pi_1 \longrightarrow$ homot. equiv.

Strategy







$c_i = [\gamma_i]$ conj class

Let $[\Phi] \in \mathrm{Out}\, \pi_1(S_g)$

① $\Phi(c_i)$ simple $\forall i$. $\Longleftrightarrow$ all pairs of lifts unlinked at $\partial \mathbb{H}^2$

② $i(\Phi(c_i), \Phi(c_{i+1})) = 1$ $\Longleftrightarrow$ a little more complicated.

③ $i(\Phi(c_i), \Phi(c_j)) = 0$ $\Longleftrightarrow$ all pairs of lifts unlinked at $\partial \mathbb{H}^2$

$|i-j| > 1$.

To show:

Φ preserves linking at $\partial \mathbb{H}^2$

Then Alex. method, change of coords...

Cayley graph

$G = \langle S \mid R \rangle$

gen set

vertices: G

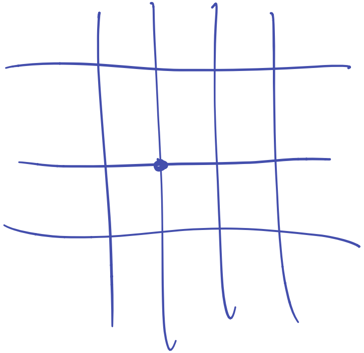
edges: $g \longrightarrow gs \quad s \in S$

Note $G \circlearrowright$ Cayley graph on left

path metric

$\rightsquigarrow$ metric on G.

Example: $\mathbb{Z}^2 = \langle a, b \mid [a,b] = 1 \rangle$

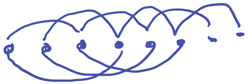


metric: taxicab.

$d(a^m b^n, id) = |m| + |n|$

Example $\mathbb{Z} = \langle 1 | \rangle$

$\mathbb{Z} = \langle 2, 3 | \cdots \rangle$



Quasi-isometries

X, Y metric spaces

$f : X \to Y$

Isometry: $d(f(x), f(y)) = d(x,y)$

Quasi-isometry: $\exists\, K, C, D$ s.t.

① $\frac{1}{K} d_X(x,y) - C \le d_Y(f(x), f(y)) \le K\, d_X(x,y) + C$

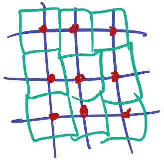
② D-nbd of $f(X)$ is Y

example $\mathbb{Z}^n \hookrightarrow \mathbb{E}^n$

$K = \sqrt{n}$

$C = 1$ (or 0).

$D = 1$



example $\mathbb{E}^n \to \mathbb{Z}^n$ "nearest pt"

Next $\pi_1(S_g) \to \mathbb{H}^2$

example $f : \mathbb{R} \to \mathbb{R}$

$f(x) = Kx$

or $f(x) = \begin{cases} Kx & x \text{ irrational} \\ Kx+1 & x \text{ rational}. \end{cases}$

Milnor–Švarc Lemma

X = proper, geod. metric space

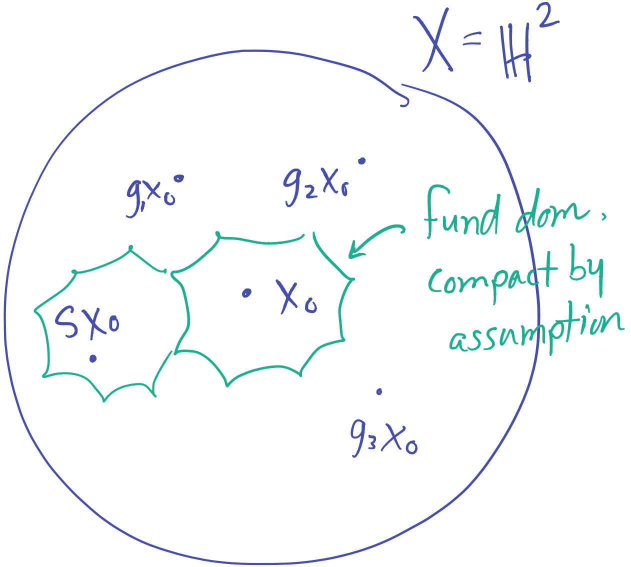
$G \circlearrowright X$ prop. disc. by isometries.

X/G compact

Then ① G is finitely generated

② G quasi-isom to X via any orbit map

$$g \longmapsto g \cdot x_0$$



$G = \pi_1(S_g)$

Gen set for $\pi_1(S_g)$:

elts that take fund dom to an adjacent one.

From Autos to QIs

$G = \text{gp} \qquad G = \langle S \rangle \quad |S| = \infty$

$\Phi \in \text{Aut}(G)$

$\rightsquigarrow$ quasi-isom of G.

$K = \max \{ \| \Phi^{\pm 1}(s) \| : s \in S \}$

$C = 0$

$D = 0$.

So:

$\Phi \in \text{Aut}\, \pi_1(S_g)$

$\downarrow$ ✓

quasi-isom of $\pi_1(S_g)$

$\downarrow$ ✓

quasi-isom of $\mathbb{H}^2$

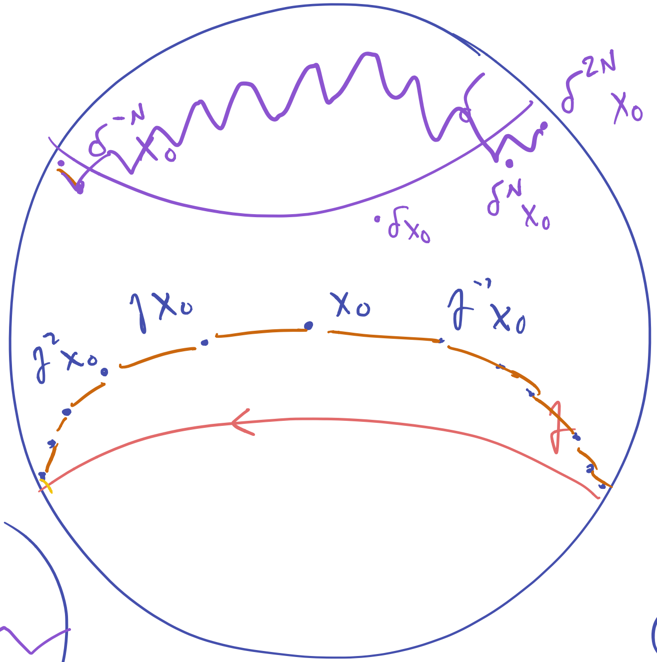
$\downarrow$ or

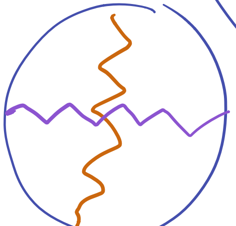
homeo of $\partial \mathbb{H}^2$

next

hence, linking preserved

Quasi-isometries of $\pi_1(S_g) = \mathbb{H}^2$ preserve linking.





Suppose $\gamma, \delta \in \pi_1(S_g)$ unlinked.

① Choose $N \gg 0$ large compared to QI consts.

$\rightsquigarrow$ orbit pts

$\gamma^i x_0$

$\delta^{Ni} x_0$ <u>far</u>

② Connect orbit pts by far paths P_γ, P_δ in $\pi_1(S_g)$

③ If $\Phi(\gamma), \Phi(\delta)$ linked, $\Phi(P_\gamma), \Phi(P_\delta)$ cross $\Longrightarrow$ CONT.

Gromov hyperbolic:

$\exists\, \delta$ s.t. for any triangle,

side 3 $\subseteq$ δ-nbd of side 1 $\cup$ side 2