Projective Varieties

$\mathbb{P}^n$ = compactification of $\mathbb{A}^n$ with one infinitely distant pt in each direction

$\rightsquigarrow$ compactifications of aff. alg. varieties.

Precisely: $\mathbb{P}^n = \{\text{1-dim subspaces of } k^{n+1}\}$

$$= \mathbb{C}^{n+1} \setminus \{0\} \big/ \sim$$

$x \sim y$ if $y = \lambda x$, $\lambda \in k$.

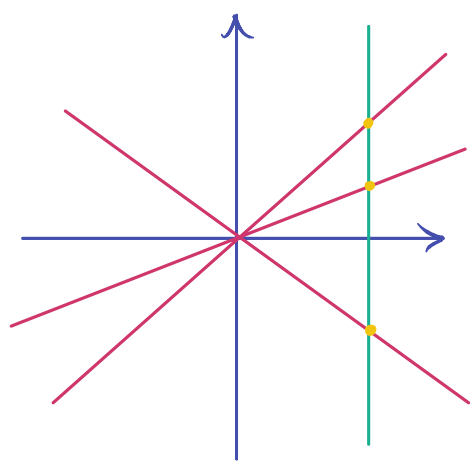
Write $[(x_0, \dots, x_n)]$ as $[x_0 : x_1 : \cdots : x_n]$

"homogeneous coords"

$\boxed{n=1}$ $\mathbb{P}^1 = \{[x_0 : x_1] : x_0 \text{ or } x_1 \text{ nonzero}\}$

$= \{[1 : x]\} \cup \{[0:1]\}$

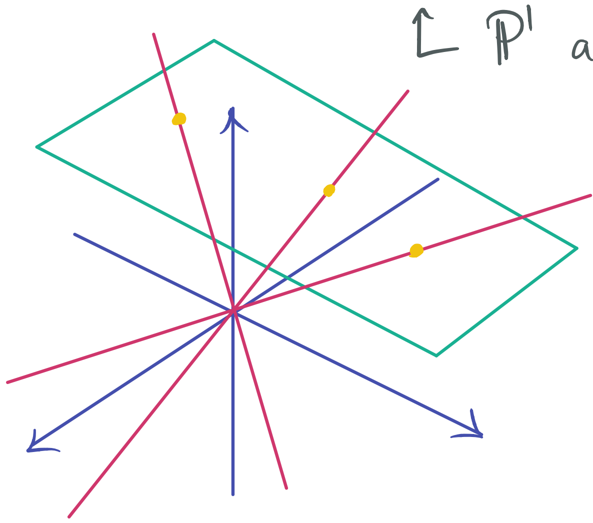
$= \mathbb{A}^1 \cup \infty$



The vertical line through 0 is the line at ∞.

$\boxed{n=2}$ $\mathbb{P}^2 = \{[1:x_1,x_2]\} \cup \{[0:x_1,x_2] : x_1 \text{ or } x_2 \neq 0\}$
$= \mathbb{A}^2 \cup \mathbb{P}^1$

$\mathbb{P}^1$ at infinity.



lines parallel to reference plane $x_0 = 1$.

More generally $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$.

But what is at ∞ is not canonical:

Let $U_j = \{[x_0 : \cdots : x_n]\}$ $x_j \neq 0$.
$\rightsquigarrow \mathbb{P}^n = U_j \cup \mathbb{P}^{n-1}$

The U_j form the standard affine cover of $\mathbb{P}^n$

For $k = \mathbb{C}$, the U_j make $\mathbb{P}^n$ into a complex manifold.

In the same way, it is an abstract algebraic variety (charts with polynomial transition maps).

There are no non-constant analytic functions on $\mathbb{P}^1$ = Riemann sphere. when $k = \mathbb{C}$.

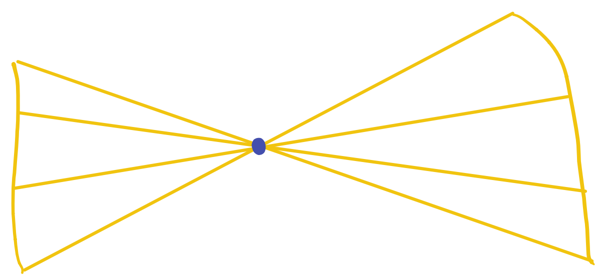
$\rightsquigarrow$ can't define a proj. variety as the zero locus of a collection of polynomials.

Say $f \in k[x_0, \ldots x_n]$ is homogeneous if all terms have the same degree.

Fact. The 0 set of a homogeneous poly. is a well defined subset of $\mathbb{P}^n$.

Pf. $f(\lambda x) = \lambda^d f(x)$.

i.e. 0-set is a union of lines through 0:



Def. A projective algebraic variety in $\mathbb{P}^n$ is the common zero set of a collection of homogeneous polynomials in $k[x_0, \ldots, x_n]$.

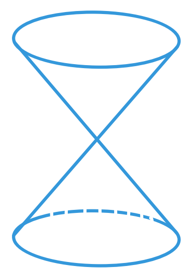
$X = Z(\{f_i\})$ as before.

Examples. ① $Z(0) = \mathbb{P}^n$

② $I = (x_0, \dots, x_n) = \{\text{polys with const term} = 0\}$
$\rightsquigarrow Z(I) = \emptyset$ "irrelevant ideal"

③ Finite sets of pts.
$[1 : \cdots : a_n] = Z(x_2 - a_2 x_0, \dots, x_n - a_n x_0)$

④ Hypersurfaces, e.g. conics:
$X = Z(f)$ $f = x^2 + y^2 - z^2$



There are 3 standard charts: U_x, U_y, U_z
$\rightsquigarrow$ hyperbola, hyperbola, circle.
(in general, intersecting with std. chart gives aff. alg. var.)

⑤ Grassmanians
$Gr_{r,n} = \{r\text{-dim vector subspaces of } k^n\}$
"Plücker embedding": (later).

⑥ X, Y proj. alg. vars $\Rightarrow X \times Y$ is one.
"Segre embedding" (later)

Grassmannians

Goal: Realize $G_{r,n}$ as a projective variety.

Wedge products

V = vector space

An r-wedge is a symbol

$$v_1 \wedge v_2 \wedge \cdots \wedge v_r \qquad v_i \in V$$

There is an equiv. relation given by

multilinearity: $v_1 \wedge \cdots \wedge (av_i + bv_i') \wedge \cdots \wedge v$

$$= av_1 \wedge \cdots \wedge v_r + bv_1 \wedge \cdots \wedge v_i' \wedge \cdots v_r$$

alternating: $v_1 \wedge \cdots \wedge v \wedge \cdots \wedge v \wedge \cdots \wedge v_r = 0$.

It follows that swapping two entries negates the r-wedge.

Compare with volumes, determinants.

$$\Lambda^r V = \{\text{finite sums of } r\text{-wedges}\}/\sim$$

If $B = (v_1, \ldots, v_n)$ is a basis for V, then $\{v_{i_1} \wedge \cdots \wedge v_{i_r} : i_1 < i_2 < \cdots < i_r\}$ is a basis for $\Lambda^r V \implies \dim \Lambda^r V = \binom{n}{r}$.

More About Wedge Products

Fact. If W is a subspace of V of dim d, $T \in \operatorname{Aut} W$, $\omega \in \Lambda^d W$, then $T(\omega) = (\det T)\,\omega$.

Pf. In ILA, we prove that det is the unique fn

$$\{\text{matrices}\} \longrightarrow \mathbb{R}$$

satisfying: ① $\det I = 1$

② multilinear in rows.

Note. There is a correspondence:

$$\Lambda^r V^* \longleftrightarrow \{\text{alt. multilinear maps } V \times \cdots \times V \longrightarrow k\}$$

$$v_1^* \wedge \cdots \wedge v_r^* \longmapsto v_1^* \otimes \cdots \otimes v_r^*$$

Geometric picture. We can think of ω = r-wedge as a choice of r-plane in V, equipped with a volume function (scalar mult.) of Euclidean volume. For an r-parallelpiped $P \subseteq k^n$, we can compute its volume by projecting to the r-plane. So ω is totally decomposable $\Longleftrightarrow$ it is given by the volume on a single plane.

To compute $(e_1 \wedge e_2 + e_3 \wedge e_4)(P)$ we must add the volumes of the projections to the e_1e_2-plane and the e_3e_4-plane.

To compute $((e_1 + e_2) \wedge e_3)(P)$ can either project to the $(e_1+e_2)e_3$-plane, or to both e_1e_3-plane and e_2e_3-plane and add together.

Plücker Embedding

$$F : Gr_{r,n} \longrightarrow \mathbb{P}(\Lambda^r V) \qquad V = k^n$$
$$\mathrm{Span}(v_1, \dots, v_r) \longmapsto v_1 \wedge \cdots \wedge v_r$$

Claim. F is well defined.

Pf. Say $(w_1, \dots, w_r)$ is another ordered basis & $w_i = \sum a_{ij} v_j$

Then $A = (a_{ij})$ is invertible

and $w_1 \wedge \cdots \wedge w_r = (\det A)\, v_1 \wedge \cdots \wedge v_r$

To show:
1. F is injective
2. $\mathrm{Im}(F)$ is closed.
3. $Gr_{r,n}$ locally can be given the structure of an affine variety & F restricts to an isomorphism between these local pieces of $Gr_{r,n}$ & Zariski open subsets of the image.

Def. Say $x \in \Lambda^r V$ is totally decomposable if it is an r-wedge.

Note: x is tot. dec. $\iff x \in \mathrm{Im}(F)$.

Note. ① Every elt of $\Lambda^1 V$ is tot. dec.

② Every elt of $\Lambda^2 V$ is tot. dec. if $\dim V = 3$

③ If $v_1, \dots, v_4$ are lin ind, then $v_1 \wedge v_2 + v_3 \wedge v_4$ is not tot. dec.

exercise. hint: x tot dec $\Rightarrow x \wedge x = 0$.

Write the basis elts for $\Lambda^r V$ as

$$e_I = v_{i_1} \wedge \dots \wedge v_{i_r} \qquad I = \{i_1, \dots, i_r\}$$

Def. Any $x \in \Lambda^r V$ can be written uniquely as

$$x = \sum a_I e_I$$

The a_I are the Plücker coords on x.

(assoc. to the basis $(v_1, \dots, v_n)$ for V).

$\rightsquigarrow$ homog. coords in $\mathbb{P}(\Lambda^r V) = \mathbb{P}^{\binom{n}{r}-1}$

Example. Say $V = \mathrm{span}\{v_1, \dots, v_4\}$.

$\rightsquigarrow$ every $x \in \Lambda^2 V$ can be written uniquely as:

$$x = a_{12}(e_1 \wedge e_2) + \dots + a_{34}(e_3 \wedge e_4)$$

If x is tot. dec. then $x \wedge x = 0$. Compute

$$x \wedge x = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\, v_1 \wedge v_2 \wedge v_3 \wedge v_4$$

$\Rightarrow F(G_{2,4}) \subseteq \mathbb{P}^5$ satisfies a homog. eqn!

For injectivity, need to recover $\mathrm{Span}(v_1,\dots,v_r)$ from $x = v_1\wedge\cdots\wedge v_r$

Lemma. Given nonzero $x\in\Lambda^r V$, let

$$\varphi_x : V \longrightarrow \Lambda^{r+1}V$$
$$v \longmapsto v\wedge x$$

Then $\dim\ker\varphi_x \le r$ with $=$ iff x tot. dec.

If $x = v_1\wedge\cdots\wedge v_r$, $\ker\varphi_x = \mathrm{Span}(v_1,\dots,v_r)$.

Pf. Choose a basis $e_1,\dots,e_n$ for V s.t. $e_1,\dots,e_s$ is basis for $\ker\varphi_x$. $\quad s=\dim\ker\varphi_x$

Let $x = \sum a_I e_I$. Then:

$$\varphi_x(e_i) = e_i\wedge x = \sum a_I(e_i\wedge e_I) = \sum_{\substack{|I|=r\\ i\notin I}} \pm a_I e_{I\cup\{i\}}$$

So $\varphi_x(e_i)=0 \Rightarrow a_I=0$ for $i\notin I$, i.e. every nonzero term of x involves e_i

This is true for $1\le i\le s$, so $s\le r$. ✓

and $x = e_1\wedge\cdots\wedge e_s\wedge y$ for $y\in\Lambda^{r-s}V$

So $r=s \Rightarrow x$ tot. dec. ✓

other direction straightforward.

For the second statement: $\mathrm{Span}\{v_1,\dots,v_r\}\subseteq\ker\varphi_x$ & 1st statement implies these have the same dim. $\square$

Cor. F is injective.

Prop. $\mathrm{Im}(F)$ closed in $\mathbb{P}(\Lambda^k V)$.

In other words, showing here that $\mathrm{Im}(F)$ is a projective variety.

Pf. Lemma says $\dim \ker \varphi_x \leq r$, $=$ iff x tot dec.

$\implies \mathrm{rk}\, \varphi_x \geq n-r$, $=$ iff x tot dec.

Let $M_x =$ matrix for φ_x wrt basis $e_1, \dots, e_n$ for V & resulting basis e_I for Λ^{r+1}

If $x = \sum a_I e_I$, the nonzero entries of M_x are $\pm a_I$.

$\mathrm{rk}\, \varphi_x \leq n-r \iff$ all $(n-r+1) \times (n-r+1)$ minors vanish.

But minor condition is a collection of homog. polys of deg $n-r+1$ in the a_I. $\square$

Exercise. Find the polynomials for $G_{2,4}$. Show

$$F(G_{2,4}) = Z(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}).$$

Our next goal is to show that $\mathrm{Im}(F)$ is locally an affine alg. variety and to give explicit coordinates for this.

Local Coords on the Grassmannian

Let's consider a chart on $\mathrm{Im}(F)$ where some a_J is nonzero,

say $a_J = a_{1\cdots r} \neq 0$ (others differ by change of coords).

Let $B = r \times n$ matrix of rank r.

Image of $\mathrm{row}(B)$ under F is

$$(b_{11}e_1 + \cdots + b_{1n}e_n) \wedge \cdots \wedge (b_{r1}e_1 + \cdots + b_{rn}e_n) = \sum a_J e_J$$

Only the $b_{ij}e_j$ with $j \in J$ contribute to a_J, and further the a_J-coord is the $r \times r$ minor corresp. to first r rows & cols. (exercise)

So $a_J \neq 0$ means $r \times r$ matrix on the left is invertible.

$\rightsquigarrow$ can assume (by multiplying B on the left by the inverse)

$$B = \left(I_r \;\middle|\; \begin{matrix} b_{1,r+1} & \cdots & \cdots \\ \cdots & \cdots & b_{rn} \end{matrix} \right)$$

Any two distinct matrices of this form have different row sp's.

$\rightsquigarrow$ can think of the $b_{i,j}$ as local coords on $Gr_{r,n}$

i.e. bijection from $\mathbb{A}^{r(n-r)}$ to the $e_J \neq 0$ chart of $Gr_{r,n}$

Want to show this is an isomorphism of aff. alg. var's.

$\rightarrow$ The a_I are the $r \times r$ minors, which are poly's.

$\leftarrow$ Need to compute b_{ij}'s in terms of a_I's.

One example:

$$a_{23\cdots rj} = \begin{vmatrix} 0 & 0 & \cdots & 0 & b_{1j} \\ 1 & 0 & \cdots & 0 & \vdots \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & 1 & b_{rj} \\ 0 & \vdots & & & \end{vmatrix} = (-1)^{r+1} b_{1j}$$

$\square$

Homogeneous Ideals

Any $f \in k[x_1, \dots, x_n]$ is a sum of homogeneous pieces.

$$f = f^{(0)} + \dots + f^{(m)}$$

$$k[x_1, \dots, x_n] = \bigoplus_{d \geq 0} k[x_1, \dots, x_n]_{(d)}$$ "graded ring/k-alg"

Lemma. Let $I \subseteq k[x_1, \dots, x_n]$ ideal. TFAE:

1. I is gen. by homogeneous elts
2. $f \in I \Rightarrow f^{(d)} \in I \ \forall d$.

Such an I is called homogeneous.

Example. For any $N \geq 0$, $\bigoplus_{d \geq N} k[x_1, \dots, x_n]_{(d)}$

Pf. 2 $\Rightarrow$ 1 $\quad I = (f_1, \dots, f_r)$ by Hilbert basis thm.

Write $f_i = \sum f_i^{(d)} \rightsquigarrow I = (\{f_i^{(d)}\})$.

1 $\Rightarrow$ 2 $\quad I = (f_1, \dots, f_r)$ each f_i homog.

$f \in I \Rightarrow f = \sum a_i f_i \quad a_i \in k[x_0, \dots, x_n]$

$$\Rightarrow f^{(d)} = \sum a_i^{(d - \deg f_i)} f_i \in I \qquad \square$$

take the degree $d - \deg f_i$ part of a_i

Note. Not every elt of a homogeneous ideal is homogeneous. (add two elements of different degree).

Note. A projective alg. var can always be written as $Z(f_1, \dots, f_k)$ with $\deg f_i$ all the same, since

$$Z(f) = Z(x_0^k f, \dots, x_n^k f)$$

$\forall$ homog. f & $k \geq 0$.

Fact. ① The radical of a homogeneous ideal is homogeneous.
② An intersection, product, or sum of homogeneous ideals is homogeneous.

Also: The Zariski topology works for proj. alg. var's.

You check both of these.

Linear subspaces Let $L \subseteq \mathbb{A}^{n+1}$ linear subsp. of dim $k+1$

$\rightsquigarrow$ L given by $n-k$ linear eqns

$\rightsquigarrow$ linear subspace of dim k in $\mathbb{P}^n$

Will see: image of L isomorphic to $\mathbb{P}^k$.

example: $Z(x_2, x_3) = \{(a_0 : a_1 : 0 : 0) \mid a_0, a_1 \in k\} \subseteq \mathbb{P}^3$
$\cong \mathbb{P}^1$

Projective Nullstullensatz

Thm. Let k be alg. closed. $I \subseteq k[x_0,\dots,x_n]$ homog.

(i) $V(I) = \emptyset \iff (x_0,\dots,x_n) \subseteq \sqrt{I}$

(ii) $V(I) \neq \emptyset \implies I(V(I)) = \sqrt{I}$

So:

irrelevant ideal

$$\left\{\begin{array}{l}\text{projective alg.}\\ \text{subvars of } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{radical homog. ideals}\\ \text{of } k[x_0,\dots,x_n]\end{array}\right\} - \{(x_0,\dots,x_n)\}$$

$$X \longmapsto I(X) \text{ or } I_p(X)$$

p for projective

= ideal of homog. poly's vanishing on X.

To prove this, the key is:

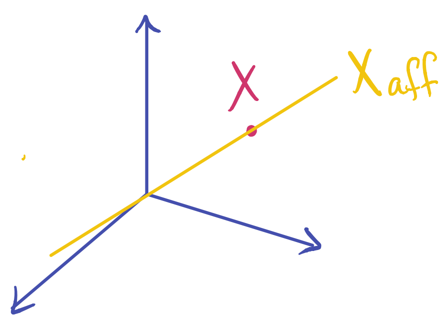
$$\{\text{proj. alg subvar's of } \mathbb{P}^n\} \longleftrightarrow \left\{\begin{array}{l}\text{nonempty closed cones in } k^{n+1}\\ \text{over affine alg. var's.}\end{array}\right\}$$

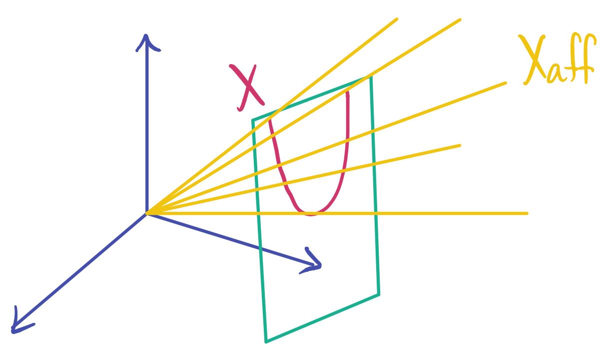
$$X \longrightarrow C(X)$$

cone

Example. $X = Z(x_1,\dots,x_n) = [1:0:\cdots:0] \in \mathbb{P}^n$

$\longleftrightarrow \{(\lambda,0,\dots,0) \in \mathbb{A}^{n+1} \mid \lambda \in k^*\} \cup \{\text{origin}\}$





Will use p/a for projective/affine,
$C(X)$ for the cone in $\mathbb{A}^{n+1}$ corresp. to $X \subseteq \mathbb{P}^n$

Note. I homog $\Rightarrow C(Z_p(I)) = Z_a(I)$

Note. f homog. $\quad f \in I_p(X) \Leftrightarrow f \in I_a(C(X))$

Thm. Let k be alg. closed. $I \subseteq k[x_0, \dots, x_n]$ homog.

(i) $V(I) = \emptyset \Leftrightarrow (x_0, \dots, x_n) \subseteq \sqrt{I}$

(ii) $V(I) \neq \emptyset \Rightarrow I(V(I)) = \sqrt{I}$

Pf. $Z_p(I) = \emptyset \Leftrightarrow Z_a(I) \subseteq \{0\}$

$\Leftrightarrow \sqrt{I} = I_a Z_a(I) \supseteq (x_0, \dots, x_n)$ (affine SN)

For $X = Z_p(I) \neq \emptyset$:

$f \in I_p(X) \Leftrightarrow f \in I_a(C(X)) = I_a Z_a(I) = \sqrt{I}$

by (affine) SN. $\square$

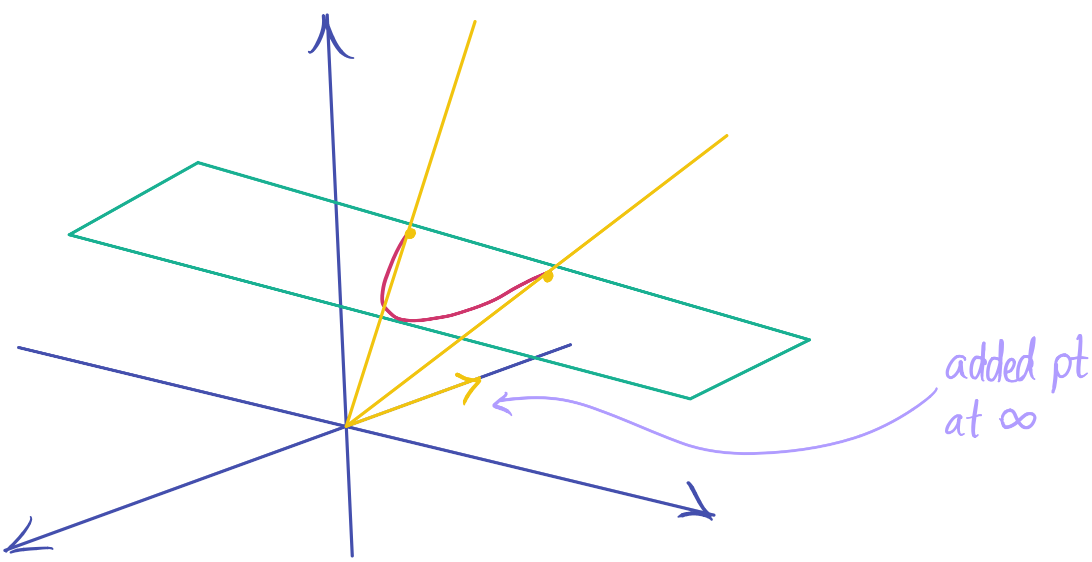
Projective Closure

X = affine alg. var in $\mathbb{A}^n$.
Have: $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$

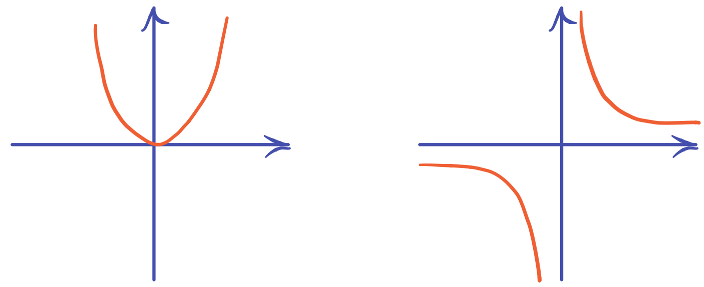
Def. The projective closure of X, denoted $\overline{X}$, is the closure of X in $\mathbb{P}^n$, where closure is wrt either the Euclidean or Zariski topology.

N.B. The projective closure depends on the embedding $\mathbb{A}^n \to \mathbb{P}^n$. We usually take the standard map $(a_1, \dots, a_n) \longmapsto [1 : a_1 : \cdots : a_n]$

Fact. If $X = Z(I)$ then $\overline{X} = Z_p(I_h)$ where I_h = ideal gen. by homogenized f, for $f \in I$.



Example. $X_1 = \{x_2 = x_1^2\}$ $\quad X_2 = \{x_1 x_2 = 1\}$



$\rightsquigarrow \bar{X}_1 = \{x_0 x_2 = x_1^2\}$ $\quad \bar{X}_2 = \{x_1 x_2 = x_0^2\}$

The projective closures are the same!
Note: $\bar{X}_1$ has one added point $[0:0:1]$ while $\bar{X}_2$ has two $[0:0:1]$, $[0:1:0]$.
Not a coincidence: there is only 1 projective conic in $\mathbb{P}^2$. (In $\mathbb{A}^2$ some conics meet ∞ in 1 pt, some in 2.)

What is $I(\bar{X})$? If $X = Z(f_i)$, is $I(\bar{X}) = Z(\bar{f}_i)$, where the $\bar{f}_i$ are the homogenizations of the f_i.

Yes for $x^2 - y$. Why?
No for $X = Z(y - x^2, z - xy) \subseteq \mathbb{A}^3$.

$= \{(t, t^2, t^3)\}$ "twisted cubic"

$\bar{X} \neq Z(wy - x^2, wz - xy) = \bar{X} \cup \{w = y = 0\}$

The extra line lies at ∞ in w-chart for $\mathbb{P}^3$.

Thm $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ aff. alg. var. $X = Z(I)$

Then $\overline{X} = Z(I_h) \subseteq \mathbb{P}^n$.

Pf. $\boxed{\overline{X} \subseteq Z(I_h)}$ Let $G \in Z(I_h)$. Want $g|_{\overline{X}} = 0$.

Setting $x_0 = 1 \rightsquigarrow G$ maps to $g \in I$.

$\Longrightarrow$ G vanishes on $\overline{X} \cap U_0 = X$

So X, hence $\overline{X}$, lies in $Z(I_h)$.

$\boxed{Z(I_h) \subseteq \overline{X}}$ Say G = homog. poly vanishing on $\overline{X}$

$\Rightarrow$ G vanishes on $\overline{X} \cap U_0 = X$

$\Rightarrow$ g vanishes on X (g defined as above).

$\Rightarrow g \in I$.

$\Rightarrow \overline{g} \in I_h$.

$\Rightarrow \overline{g} x_0^t = G$ some t.

$\Rightarrow G \in I_h$. $\square$

Also true that I_h is radical.

Morphisms of Projective Varieties

Start with an example:

$$V: \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$
$$[s:t] \longmapsto [s^2 : st : t^2]$$

Check this is well defined.

The image lies in $C = Z(xz - y^2)$, so

$$V: \mathbb{P}^1 \to C.$$

This is an example of a Veronese map.

"His work was severely criticised as unsound by Peano" (Wikipedia).

In the affine patch $U_t \cong \mathbb{A}^1 = \{[s:t] \mid t \neq 0\}$ we can denote s/t by u and write V as:

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^2 \cong U_z = \{[x:y:z] \mid z \neq 0\}$$
$$u \longmapsto (u^2, u) \longmapsto [u^2 : u : 1]$$

The image is $Z(x - y^2) \subseteq \mathbb{A}^2$.

Similarly, on U_s, V is given by

$$v \longmapsto (v, v^2) \longmapsto [v : v^2 : 1]$$

So on each std chart V restricts to a morphism of affine var's.

Defn. $V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ proj. alg. var's.
$F: V \to W$ is a **morphism** if
$\forall\, p \in V$ $\exists$ homog. poly's $F_0, \dots, F_m \in k[x_0, \dots, x_n]$
s.t. for some nonempty open nbd U of p,
$F|_U : U \to W$ agrees with

$$\begin{aligned} U &\longrightarrow \mathbb{P}^m \\ q &\longmapsto [F_0(q) : \cdots : F_m(q)]. \end{aligned}$$

Implicit in the defn: the F_i all have same degree (otherwise they don't give a well-def. map).
Also: the F_i must not vanish simultaneously on U.

Example. $C = Z(zx - y^2) \subseteq \mathbb{P}^2$

$$C \longrightarrow \mathbb{P}^1$$

$$[x:y:z] \longmapsto \begin{cases} [x:y] & x \neq 0 \\ [y:z] & z \neq 0 \end{cases}$$

The map is defined on all pts of C since $x = z = 0 \Rightarrow y = 0$.
It is well defined: if $x, z \neq 0$, then $y \neq 0 \Rightarrow$

$$[x:y] = [yx : y^2] = [xy : xz] = [y:z]$$

So the map is a morphism, but not defined by any 1 polynomial.
In the U_x chart this map is $(u, u^2) \longmapsto u$.

This map is stereographic projection (see next page).

Defn. An isomorphism is an invertible morphism.

Example. Linear change of coords.

Example. The maps $C \longleftrightarrow \mathbb{P}^1$ defined above are inverses, hence isomorphisms.

Example. Let's show the above map $C \longrightarrow \mathbb{P}^1$ is an isomorphism.

The inverse map $f: \mathbb{P}^1 \longrightarrow C$ is

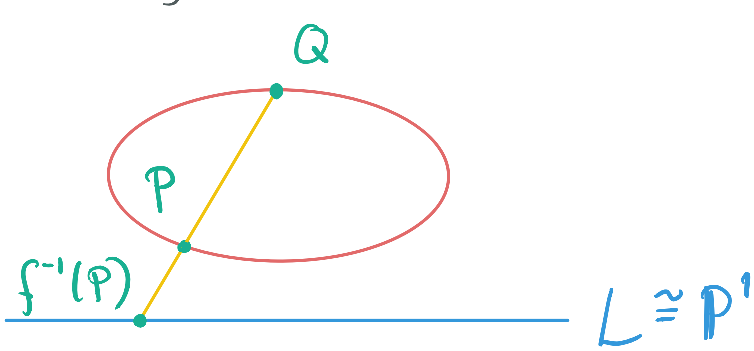
$$[s:t] \longmapsto [s^2 : st : t^2]$$

This is a morphism, inverse to the above map.

Geometric picture: Let $Q = [1:0:0] \in C$ & $L \subset \mathbb{P}^2$ is the line $\{x=0\}$

For $P = [a:b:c] \neq Q$ $\exists!$ line $\overline{PQ}$ through P & Q. It has eqn $yc = zb$. The intersection of $\overline{PQ}$ with L is $[0:b:c]$

Identifying L with $\mathbb{P}^1$, have $f^{-1}(P) = \overline{PQ} \cap L$



Segre Embedding

Goal: products of proj. alg. var's are proj. alg. var's.

First identify $\mathbb{P}^{(n+1)(m+1)-1}$ with the set of $(n+1)\times(m+1)$ matrices mod scalars. As in the Plücker situation, the set of matrices of rank at most 1 is a proj. alg. var: it is the zero set of the 2×2 minors, which are homogeneous.

Define $\varphi_{m,n}: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$

by $\varphi_{m,n}([x_0:\cdots:x_n],[y_0:\cdots:y_m]) = \begin{pmatrix} x_0y_0 & \cdots & x_0y_m \\ \vdots & & \\ x_ny_0 & \cdots & x_ny_m \end{pmatrix}$

Exercise. $\varphi_{m,n}$ maps $\mathbb{P}^n \times \{q\}$ and $\{p\}\times\mathbb{P}^m$ to linear subspaces. Similar if we consider linear subsp. $\times \{q\}$ and vice versa.

The map $\varphi_{m,n}$ is the Segre embedding and the image is the Segre variety.

Example. $\varphi_{1,1}: \mathbb{P}^1\times\mathbb{P}^1 \longrightarrow \mathbb{P}^3$

Image is $\left\{\begin{pmatrix} x & y \\ z & w\end{pmatrix} : xw-yz=0\right\}$

All 2×2 minors vanish, so the image of $\varphi_{m,n}$ lies in the corresponding variety of rank 1 matrices

Claim. $\varphi_{m,n}$ is injective.

Pf. Let $M = (m_{ij})$ be in the image, w/ $\varphi_{m,n}(a,b) = M$.

WLOG $a_0 = b_0 = m_{00} = 1$.

$$\Rightarrow b_j = c_{0j} \quad \forall \; 0 \le j \le m$$
$$a_i = c_{i0} \quad \forall \; 0 \le i \le n. \qquad \square$$

Claim. The image of $\varphi_{m,n}$ equals the variety $V_{m,n}$ of rank 1 matrices.

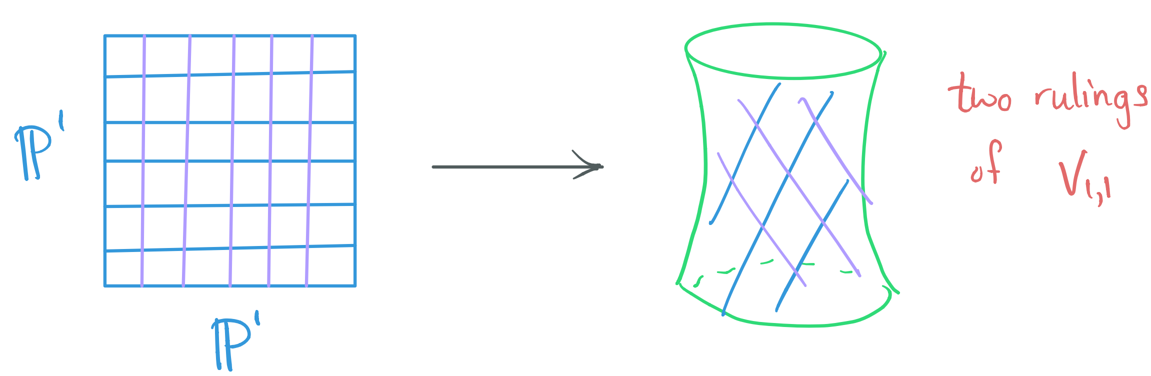
Pf. Say $M = (m_{ij})$ has rank 1. Scale so $m_{00} = 1$.

$\forall \; k,l \neq 0$ have $c_{kl} = c_{k0} c_{0l}$

Taking $a_0 = b_0 = 1$, $a_k = c_{k0}$, $b_l = c_{0l}$

get $\varphi_{m,n}(a,b) = M$. $\square$

Geometric picture for $\varphi_{1,1}$



$$[a:b]\times \mathbb{P}^1 \longmapsto \left\{ \begin{pmatrix} a & x \\ b & y \end{pmatrix} : \begin{array}{c} a,b \text{ fixed,} \\ ay - bx = 0 \end{array} \right\}$$

affine plane or projective line.

Algebraic Structure on $\mathbb{P}^n \times \mathbb{P}^m$

There are 2 options:

1. Use $\varphi_{m,n}$ to give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of a projective algebraic variety
2. Define an algebraic structure on $\mathbb{P}^n \times \mathbb{P}^m$ using sheaf theory, show that $\varphi_{m,n}: \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\cong} V_{m,n}$ (the maps in both directions are polynomial).

We can also accomplish 2 without sheaf theory. First, define the homogeneous coordinate ring of a projective variety $V \subseteq \mathbb{P}^n$ to be:

$$k[V] = k[x_0, \dots, x_n] / I_p(V)$$

This is the same as the affine coord. ring over the affine cone over V in $\mathbb{A}^{n+1}$

Then we define the alg. structure on $V \times W$ algebraically:

$$k[V \times W] = k[V] \otimes k[W]$$

Then a map $V \times W \longrightarrow U$ is a morphism if the induced map $k[U] \longrightarrow k[V \times W]$ is polynomial.

This is essentially what goes on in the sheaf theory defn.

Products of Projective Varieties

We can use the Segre map in order to give an algebraic structure to the product $V \times W$ where $V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ are algebraic varieties (could also use coord rings intead)

Say a polynomial in $x_0, \dots, x_n, y_0, \dots, y_m$ is bihomogeneous if it is homogeneous in each variable separately (i.e. treating y_i's as constants, get a homog. poly in x_i.)
Say a bihomog. poly. is balanced if the x- and y- degrees are equal.

Say $V = Z(f_1, \dots, f_k)$
$W = Z(g_1, \dots, g_l)$
Then $V \times W$ = Zero set of $\{f_i\} \cup \{g_j\}$
Can replace each f_i, g_j with a balanced polynomial since for a homog. poly f, have

$$Z(f) = Z(x_0 f, \dots, x_n f).$$

Once the f_i & g_j are balanced, we can substitute

$$z_{ij} \text{ for } x_i y_j$$

$\rightsquigarrow$ homog. poly in z_{ij}.

Then intersect with Segre variety.

Lines Intersecting 4 Skew Lines

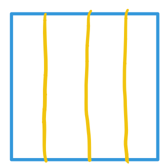
Thm Given a generic configuration of 4 skew lines $\ell_1, \dots, \ell_4$ in $\mathbb{P}^3$, exactly 2 lines intersecting all 4.

This is a first theorem in enumerative geometry. Algebraic geometry was invented to solve problems like this.

Note: We won't actually need the alg. structure on $\mathbb{P}^1 \times \mathbb{P}^1$.

Outline.

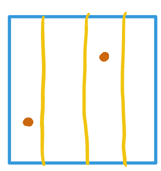
Step 1. Can assume ℓ_1, ℓ_2, ℓ_3 are my favorite triple of skew lines



$V_{1,1}$

Step 2. The set of lines intersecting (my favorite) ℓ_1, ℓ_2, ℓ_3 is the set of horizontal lines in the Segre variety $V_{1,1}$.

Step 3. Generically, ℓ_4 intersects $V_{1,1}$ in two points:



So the two corresponding horizontal lines are the only ones hitting $\ell_1, \dots, \ell_4$.

Lemma (Step 1) Let ℓ_1, ℓ_2, ℓ_3 & m_1, m_2, m_3 be two sequences of skew lines in $\mathbb{P}^3$. Then there is an element of $PGL_4(k)$ taking ℓ_i to m_i $\forall i$.

Pf. A line in $\mathbb{P}^3$ is a plane through 0 in k^4.

Take coords on k^4: x, y, z, w.

WLOG ℓ_1 given by $x = y = 0$

ℓ_2 given by $z = w = 0$.

ℓ_3 given by $x - z = y - w = 0$.

check these are skew!

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

We claim that there is an elt of PGL_4 taking m_1, m_2 to ℓ_1, ℓ_2. Indeed:

2 skew lines in $\mathbb{P}^3$ $\longleftrightarrow$ 2 planes in k^4 intersecting at 0.

there are many choices of basis $\longleftrightarrow$ (partitioned) basis for k^4

So just take one basis to another.

We claim that m_3 does not lie in the plane $x = 0$ or the plane $w = 0$. Indeed, if m_3 were in $x = 0$ it would intersect m_1 (in k^4 would have two 2-planes in a 3-dim subspace).

It follows that ℓ_3 & m_3 intersect $x = 0$ and $w = 0$ in one point each (a 3-plane and a 2-plane in k^4 intersect in a line if the 2-plane is not contained).

Denote these points of intersection for m_3 by:

$$[0:a:b:c] \quad \& \quad [d:e:f:0].$$

The line m_3 is determined by these points (a plane in k^4 is determined by 2 lines).

The corresponding points of intersection for ℓ_3 are

$$[0:1:0:1] \text{ and } [1:0:1:0]$$

since $\quad y=w \quad$ and $\quad x=z$.

A matrix in $PGL_4(k)$ fixing ℓ_1, ℓ_2 is of the form
$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ where each block is 2×2

We can choose A & D to make $b=e=0$. why?

Choose A s.t. $A\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}$, $A\begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$.

Similar for D.

Then the diagonal matrix $\begin{pmatrix} 1/d & & & 0 \\ & 1/a & & \\ & & 1/f & \\ 0 & & & 1/c \end{pmatrix}$

takes m_3 to ℓ_3. $\square$

Recall: A **quadric surface** is a proj. alg. var. in $\mathbb{P}^3$ cut out by a quadratic polynomial

Fact. If $V \subseteq \mathbb{P}^3$ is a quadric surface and $\ell \subseteq \mathbb{P}^3$ is a line then $\ell \cap V$ is either 1 pt, 2 pts, or V.
If $\ell \not\subseteq V$ then 2 pts of intersection is generic.

Pf. Think of ℓ as a plane thru O in k^4. It is described by 2 free variables. In other words, can write 2 of the var's, say z and w, in terms of the other 2, x & y.
So, to compute $\ell \cap V$, we can make this substitution into the defining poly. for V.

e.g. $xy + xz + yw \rightsquigarrow xy + x^2 + y^2$

If $\ell \subseteq V$, we get the zero polynomial.
Otherwise, we can dehomogenize, i.e. set $x=1$ or $y=1$.

$\rightsquigarrow$ quadratic poly in 1 var.
$\rightsquigarrow$ 1 or 2 solutions, latter being generic.

Lemma (Step 2). The family of lines $\mathcal{F}$ intersecting ℓ_1, ℓ_2, ℓ_3 (my fave 3 lines above) sweeps out the Segre variety $V_{1,1}$.

Pf. Certainly $V_{1,1} \subseteq \mathcal{F}$ (consider the horiz. lines in $\mathbb{P}^1 \times \mathbb{P}^1$).
Now suppose $\ell \subseteq \mathbb{P}^3$ intersects ℓ_1, ℓ_2, ℓ_3. Then ℓ meets $V_{1,1}$ in 3 pts. Fact $\Rightarrow \ell \subseteq V_{1,1}$. $\square$

Step 3 follows now, again using the Fact.
Genericity $\Rightarrow$ 2 pts in $V_{1,1}$ at different heights.

Veronese Maps

$$\left\{ \begin{array}{l} \text{degree } d \text{ homog. polynomials} \\ \quad \text{in } k[x_0, \dots, x_n] \end{array} \right\} = \begin{array}{l} k\text{-vector space on the} \\ \binom{d+n}{d} \text{ monomials of} \\ \text{degree } d. \end{array}$$

The d^{th} Veronese mapping of $\mathbb{P}^n$ is

$$V_d : \mathbb{P}^n \longrightarrow \mathbb{P}^m \qquad m = \binom{d+n}{d} - 1$$

$$[x_0 : \cdots : x_n] \longmapsto \underbrace{[x_0^d : x_0^{d-1} x_1 : \quad \cdots : x_n^d]}_{\text{all monomials of deg } d}$$

This is well def. b/c all the monomials have same degree and they do not vanish simultaneously.

$\boxed{n=1, d=2}$

$$V_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

$$[s:t] \longmapsto [s^2 : st : t^2]$$

Image is $V(xz - y^2)$ = conic in $\mathbb{P}^2$

We saw earlier this is an $\cong$ onto image.

$$[s^2 : st] \qquad [st : t^2]$$

$$[s:t] \qquad [s:t]$$

$\boxed{n=1,\ d=3}$ $\quad \nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3 \quad \longleftarrow$ use $[x:y:z:w]$ coords

$$[s:t] \longmapsto [s^3 : s^2t : st^2 : t^3]$$

Image is called the rational normal curve of deg 3.
It is the proj. closure of the twisted cubic.
It is:

$$Z(xw - yz,\ y^2 - xz,\ wy - z^2)$$

Prop. ν_d is an isomorphism onto its image.

Pf. Will describe the inverse.

Let W denote the image.

Denote the homog. coords on $\mathbb{P}^m$ by

$$z_I \qquad I = (i_0, \dots, i_n) \in \mathbb{N}^{n+1} \text{ with } |I| = \sum i_j = d.$$

At each pt of W, at least one of the x_i^d coords must be nonzero.

Let U_i be the pts with $x_i^d \neq 0$. $\rightsquigarrow$ cover of W.

Define: $U_i \longrightarrow \mathbb{P}^n$

$$z \longmapsto [z_{(1,0,\dots,d-1,\dots,0)} : z_{(0,1,0,\dots,d-1,\dots,0)} : \dots : z_{(0,\dots,d-1,\dots,0,1)}]$$

$=$ the coords of z indexed by $x_j x_i^{d-1}$.

This is well def because the maps agree on overlaps.

e.g. in $n=1, d=2$ case $U_s(z) = U_t(z) = [s^2 : st] = [st : t^2] = [s:t]$.

Check that the two maps are inverse morphisms $\square$

In general: $\mathrm{im}(V_d)$ is defined by

$$\{Z_I Z_J - Z_K Z_L : I, J, K, L \in \mathbb{N}^{n+1},\ I+J = K+L\}$$

$\boxed{n=1}$ $V_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^m$ $m = \binom{d+1}{d} - 1 = d-1.$

Image is the rational normal curve of deg d.

The defining eqns are 2×2 minors of

$$\begin{pmatrix} Z_{0,d} & Z_{1,d-1} & \cdots Z_{d-1,1} & Z_{d,0} \\ Z_{1,d-1} & Z_{2,d-2} & \cdots & Z_{d,0} & Z_{0,d} \end{pmatrix}$$

What is the point?

① Gives an interesting subvariety of $\mathbb{P}^N$

② Translates problems about deg d polys in $X_0, \dots, X_n$ into problems about linear polys in X_I

e.g. Say $X = Z(x^2 - 3yz) \subseteq \mathbb{P}^2$

$$V_2(X) = V_2(\{[x:y:z] : x^2 - 3yz = 0\})$$

$$= \{[x^2 : xy : xz : y^2 : yz : z^2] : x^2 - 3yz = 0\}$$

$$= V(X_0 - 3X_4) \subseteq \mathbb{P}^5.$$

Hypersurface Sections

F = nonzero homog. form of deg $m \geq 1$.

$\rightsquigarrow V(F) \subseteq \mathbb{P}^n$ hypersurface of deg. d.

For X = proj. alg. var, $V(F) \cap X$ is called a hypersurface section of X.

When $d=1$, we replace "surface" by "plane".

Thm $X \setminus (V(F) \cap X)$ is an affine variety.

Application. $\mathrm{Poly}_n = \{$poly's of deg n with nonzero discriminant$\}$ / scale

e.g. $\mathrm{Poly}_2 = \{ax^2+bx+c : b^2-4ac \neq 0\} \subseteq \mathbb{P}^2$

$= \{x^2+bx+c : b^2-4c \neq 0\} \subseteq \mathbb{A}^2$

Pf of Thm. $d=1$. In this case $V(F)$ is a hyperplane H.

$H \cong \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$. $\mathbb{P}^n \setminus H \cong \mathbb{A}^n$.

General case. Apply ν_d:

$X \cong \nu_d(X) \subseteq \mathbb{P}^m$ proj. subvariety.

$\nu_d(V(F)) = Z(\text{linear}) \subseteq \mathbb{P}^m$.

Apply $d=1$ case.

Automorphisms of Projective Space

k alg closed

An element A of $GL_{n+1}(k)$ gives an automorphism of $\mathbb{P}^n$:

$$[v] \longmapsto [Av] \qquad v \in k^{n+1}$$

The matrices $k^* I$ induce the trivial automorphism

Thus, there is a map

$$\Phi : PGL_{n+1}(k) \longrightarrow \operatorname{Aut} \mathbb{P}^n$$

$GL_{n+1}(k) / k^* I$

We have that $PGL_{n+1}(k)$ is the complement in $\mathbb{P}^{(n+1)^2-1}$ of the hypersurface $\det = 0$.

By the last theorem, $PGL_{n+1}(k)$ is an affine variety.

Thm. The natural map Φ is an isomorphism.

Note. For $n=1$, $k=\mathbb{C}$ this is the fact that $PGL_2\mathbb{C}$ is the group of automorphisms of $\hat{\mathbb{C}}$.

Indeed: Möbius transf's are algebraic and algebraic maps are conformal, hence Möbius transf.

Pf for $n=1$ Let $\alpha \in \mathrm{Aut}\,\mathbb{P}^1$

Say $\alpha([0:1]) = [1:a]$

Let $A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$

Then $\beta = A \circ \alpha$ fixes $\infty = [0:1]$

$\rightsquigarrow$ β restricts to a morphism

$$\bar{\beta} : U_x \longrightarrow U_x$$

This $\bar{\beta}$ is a polynomial

It must be linear since it is an auto

(since k alg closed, if $\bar{\beta}$ were not linear, it would not be injective)

$\Longrightarrow$ $\bar{\beta}(y) = py + q \quad p \neq 0$

$\Longrightarrow$ $\beta(x,y) = (x, py + qx)$ (homogenize)

$\longleftrightarrow$ $B = \begin{pmatrix} 1 & q \\ 0 & p \end{pmatrix}$ $\det B = p \neq 0$.

$\Longrightarrow$ $\alpha(z) = A^{1} \circ \beta \in PGL_2 k$ $\square$

Pf. Φ is obviously injective. To show surjective.
Want to show any $\alpha \in \operatorname{Aut} \mathbb{P}^n$ preserves planes.
$\rightsquigarrow$ need an algebraic characterization.
Will show: a hypersurface H is a hyperplane iff:
① $\dim H = n-1$
② $\# H \cap L = 1 \ \forall$ lines $L \not\subseteq H$.

Let $H = Z(f)$ be a hypersurface
Let L be a line
Can apply an elt of $PGL_{n+1}(k)$ so that
$L = \{ [s:t:0:0] \mid s,t \in k \}$
Consider the chart $x_0 \neq 0$
$\rightsquigarrow L = \{ [s:1:0:0] \}$
Typically, plugging L into f gives a polynomial in s, same deg as f.
$\Longrightarrow L \cap H$ typically has $\deg f$ points.
The desired characterization follows.

wait! why are lines preserved?

An automorphism of $\mathbb{P}^n$ preserves ① & ②, hence it preserves hyperplanes.

Thus, the polynomials describing α are all linear, whence the theorem. $\square$

The real proof. For a proj. alg. var X,
$\mathrm{Pic}(X) = \{\text{line bundles over } X\}/\sim$

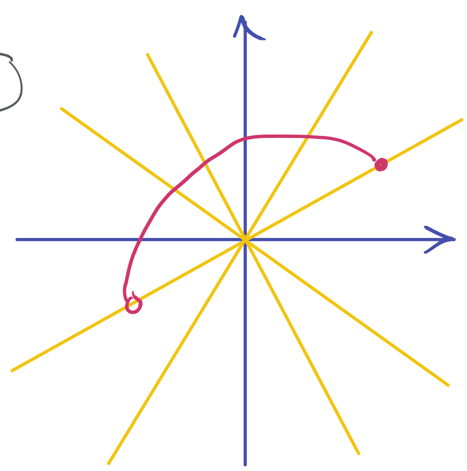
This is a group under $\otimes$ since $k \otimes k \cong k$.
It is a fact that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$
Elements are denoted $\mathcal{O}(n)$
$\mathcal{O}(0)$ = trivial bundle $\mathbb{P}^n \times k$
$\mathcal{O}(-1)$ = canonical bundle $\{(\ell, v) : \ell \in \mathbb{P}^n, v \in \ell\}$
$\mathcal{O}(1) = \mathcal{O}(-1)^{-1} = \mathcal{O}(-1)^*$. $\longleftarrow$ inverses in Pic are duals.

There are now several steps.

1. $\mathcal{O}(-1)$ has no global nonzero sections.
2. Global sections of $\mathcal{O}(1) \longleftrightarrow$ linear poly's on $\mathbb{P}^n$
3. $\mathrm{Aut}\,\mathbb{P}^n \longrightarrow \mathrm{Aut}\,\mathrm{Pic}(\mathbb{P}^n)$ (pullback).
   Moreover sections pull back to sections

1, 2, 3 $\Longrightarrow$ $\mathrm{Aut}\,\mathbb{P}^n \longmapsto 1 \leq \mathrm{Aut}\,\mathrm{Pic}\,\mathbb{P}^n$
and $\mathrm{Aut}\,\mathbb{P}^n$ preserves hyperplanes.

Idea for 1



Idea for 2 $\mathcal{O}(1)$ is dual to $\mathcal{O}(-1)$. So an elt of $\mathcal{O}(1)$ is a morphism

$$\mathcal{O}(-1) \longrightarrow \mathcal{O}(0) = \mathbb{P}^n \times k$$

that covers id map $\mathbb{P}^n \to \mathbb{P}^n$ and restricts to linear map on each fiber.

Let's remove 0-sections:

$$\begin{array}{ccc} \mathcal{O}(-1)\setminus 0 & \longrightarrow & \mathcal{O}(0)\setminus 0 \\ k^{n+1}\setminus\{0\} & \xrightarrow{(f_1, f_2)} & \mathbb{P}^n \times \{k \setminus 0\} \end{array}$$

The map f_1 must be the std projection.

& f_2 must be linear.

So an elt of $\mathcal{O}(1)$ is a linear map on each fiber.

A section is a polynomial map $\mathbb{P}^n \longrightarrow \mathcal{O}(1)$...

Conics in $\mathbb{P}^2$

A conic in $\mathbb{P}^2$ is $C = Z(f)$, $\deg f = 2$.
In U_z chart $(z=1)$ get conic in $\mathbb{A}^2$.

example. $\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}$ ← $z=0$, S^1 at ∞

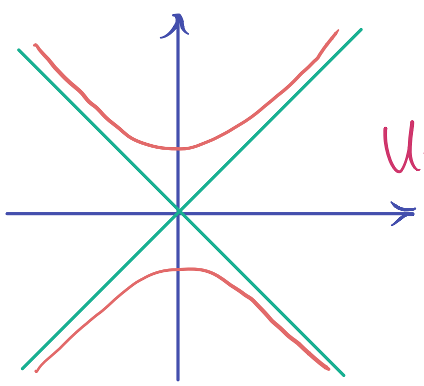
Lines in $\mathbb{P}^2 \longleftrightarrow L = Z(ux+vy+wz)$

L passes thru $\mathbb{P}^1$ at ∞

$\rightsquigarrow$ pt at ∞ is $u/v \in S^1$.

So L, L' meet at $\infty \iff u'/v' = u/v$

example (hyperbolas are ellipses).



$H = Z(x^2/a^2 - y^2/b^2 - z^2) \subseteq \mathbb{P}^2_{\mathbb{R}}$

In $U_z = \mathbb{P}^2$ chart: $x^2/a^2 - y^2/b^2 = 1$

$H \cap \mathbb{P}^1_{\{z=0\}} = [a : \pm b : 0]$

In U_x chart: $1/a^2 + y^2/b^2 = z^2$

$1 = a^2 z^2 + (a^2/b^2) y^2$ ellipse!

& $H \cap \mathbb{P}_{\{x=0\}}$: $z^2 = -y^2/b^2 \rightsquigarrow \phi$.

Say two conics are projectively equivalent if they differ by $PGL_3 k$. "linear change of coords"

Thm. Any conic in $\mathbb{P}^2_{\mathbb{C}}$ is projectively equivalent to exactly one of:

- $Z(x^2+y^2+z^2)$ ellipse/non-degen.
- $Z(x^2+y^2)$ 2 lines
- $Z(x^2)$ double line

Thm. Let $p_1, \dots, p_5 \in \mathbb{P}^2$.

- There exists a conic containing $p_1, \dots, p_5$
- The conic is unique if no 4 are collinear.
- It is nondegenerate if no 3 are collinear.

Pf. A conic is 0-set of

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

Not all of a,b,c,d,e,f are 0

$\rightsquigarrow [a : \cdots : f] \in \mathbb{P}^5$

$\rightsquigarrow \mathbb{P}^5 \longleftrightarrow$ Conics in $\mathbb{P}^2$

Conics passing thru $[\alpha : \beta : \gamma] \in \mathbb{P}^2$ form a hyperplane in $\mathbb{P}^5$ (substitute for x, y, z).

$\Longrightarrow$ Set of conics passing through all p_i is $H_1 \cap H_2 \cap \cdots \cap H_5 \subseteq \mathbb{P}^5$

$\rightsquigarrow$ nonempty intersection (5 5-planes in k^6 thru 0 intersect in a line or more)

1 pt of intersection $\Longleftrightarrow \{H_i\}$ indep. $\square$

Intersection Theory

Q. When two varieties $X, Y \subseteq \mathbb{P}^n$ intersect in a finite set, what is the expected $\# X \cap Y$?

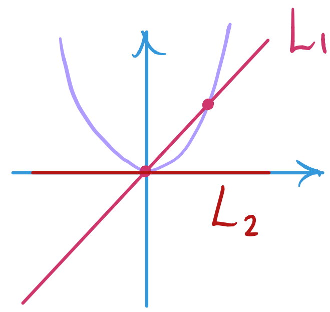
When studying $\operatorname{Aut} \mathbb{P}^n$ we saw for X a hypersurf. $Z(f)$ and Y a line L, have

$$|X \cap Y| = |Z(f|_L)|$$
$$\leq \deg f|_L$$
$$\leq \deg f$$

e.g. $X = Z(zy - x^2)$ $\deg = 2$

$L_1 = Z(y - x)$ $\deg = 1$

$L_2 = Z(y)$ $\deg = 1$



$|X \cap L_1| = 2$ $|X \cap L_2| = 1$ (order 2 intersection)

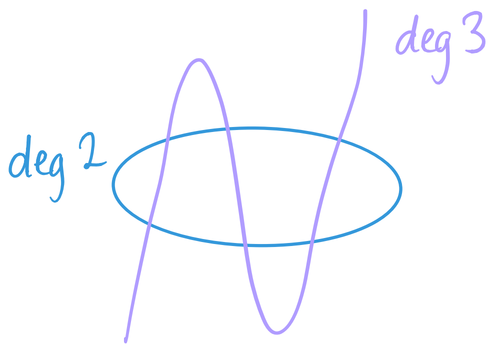
BEZOUT'S THEOREM

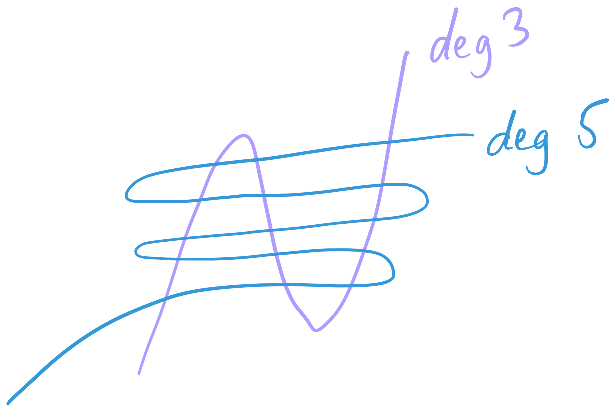
k alg closed.

$V = Z(f)$ hypersurface in $\mathbb{P}^n$.
Define $\deg V$ to be $\deg f$.

Thm. C, D curves in $\mathbb{P}^2$ of deg m, n.
If C, D have no irred comp's in common, then they intersect in mn points, counted with multiplicity.

Schematics:





Note Even without knowing what multiplicity means, the theorem tells us

$$1 \leq |C \cap D| \leq mn$$

Special cases

① C, D are both lines:
every pair of lines in $\mathbb{P}^2$ meets in a pt (possibly at ∞).

② $C = Z(f)$, $D = \text{line} = Z(\ell)$, $\ell = ax + by + cz$
Should get $\deg f$ intersections.
Use ℓ to eliminate z from f.
For typical affine chart, get a poly of $\deg = \deg f$ in one variable. Use: k alg closed.

example $\quad C = Z(yz - x^2) \quad D = Z(z - ax)$

$$C \cap D = (yz - x^2, z - ax)$$
$$= (axy - x^2)$$

Set $y = 1$: $ax - x^2 = 0 \rightsquigarrow x = 0, a$

$\rightsquigarrow$ solns $[0:1:0]$, $[a:1:a^2]$

Note: when $a = 0$ get one intersection of multiplicity 2.

special case Every conic meets the line at ∞ in 2 pts.
For instance, a circle is

$$(x-az)^2+(y-bz)^2=r^2z^2$$

& it always contains $[1:i:0]$ & $[1:-i:0]$.
So two circles intersect twice in $\mathbb{A}^2$ and twice in $\mathbb{P}^1_\infty$ (if concentric they meet with mult. 2 at both pts at ∞).
Similary, a hyperbola meets $\mathbb{P}^1_\infty$ at two points, corresponding to its asymptotes.

A parabola meets $\mathbb{P}^1_\infty$ at 1 pt, with multiplicity 2.

example of special case. $C=Z(x^2+y^2-z^2)$

$D=Z((x-z)^2+y^2-z^2)$

We have the 2 pts at $\mathbb{P}^1_\infty=\mathbb{P}^1_{z=0}$.
Setting $z=1$: $x^2+y^2=1$

$$(x-1)^2+y^2=1$$

$$\rightsquigarrow x^2-1=(x-1)^2-1$$

$$\rightsquigarrow 2x=1 \rightsquigarrow x=1/2$$

$$\rightsquigarrow [1/2 : \pm\sqrt{3}/2 : 1]$$

Resultants

Goal: find common solutions of two poly's.

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m$$
$$g(x) = b_0 + b_1 x + \cdots + b_n x^n$$

The resultant of f and g is the determinant of the $(m+n)\times(m+n)$ Sylvester matrix

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_m & & & \\ & a_0 & \cdots & & a_m & & \\ & & a_0 & \cdots & & a_m & \\ & & & \ddots & & & \\ & & & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & & & \\ & & & \ddots & & & \\ & & & & b_0 & b_1 \cdots & b_n \end{pmatrix}$$

Prop. $Z(f) \cap Z(g) \neq \emptyset \iff \mathrm{Res}(f,g) = 0$

The condition $Z(f) \cap Z(g) \neq \emptyset$ is same as the condition that f & g have no common factors.

Linear case $a_0 + a_1 x = 0$

$$b_0 + b_1 x = 0$$

Write as: $\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Nonzero soln $\Longrightarrow$ det $= 0$.

Quadratic case $a_0 + a_1 x + a_2 x^2 = 0$

$$b_0 + b_1 x + b_2 x^2 = 0$$

So far, 3 var's, 2 eqns. Also have:

$$a_0 x + a_1 x^2 + a_2 x^3 = 0$$

$$b_0 x + b_1 x^2 + b_2 x^3 = 0$$

$$\rightsquigarrow \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = 0.$$

Can see the rank of the matrix is at least 3 (look at first 3 rows).

Given a solution, scale so first entry is 1.

Want the solution to be of the form $(1, x, x^2, x^3)$

Since $(1, x^1, x^2)$ solves $a_0 + a_1 x + a_2 x^2$

& (x^1, x^2, x^3) solves $a_0 x^1 + a_1 x^2 + a_2 x^3$

& since solutions unique up to scale, it works!

(that is, each term is x times the previous).

An important point is that each term of the resultant is of the same "degree" (add the subscripts). In this case:

$$(a_0b_2 - a_2b_0)^2 = (a_0b_1 - a_1b_0)(a_1b_2 - a_2b_1)$$

In projective space

for this pt to not be in $Z(f)$, must be a term w/o x or y $\Rightarrow$ $z^{\deg f}$ term.

$C = Z(f)$, $D = Z(g)$

Assume WLOG $[0:0:1]$ is on neither curve

$$\rightsquigarrow \quad f(x,y,z) = z^m + a_1 z^{m-1} + \cdots + a_0$$

$$g(x,y,z) = z^n + b_1 z^{n-1} + \cdots + b_0$$

a_i = homog. poly in x,y of deg i

b_j = homog. poly in x,y of deg j.

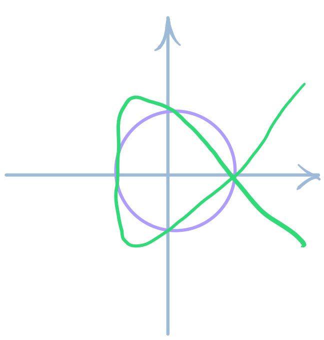
The resultant wrt z is a polynomial $R(x,y)$.

Prop. $R(x,y)$ is either 0 or homog of deg mn.

Example. $f(x,y,z) = x^2 + y^2 - z^2$

$g(x,y,z) = x^3 - x^2z - xz^2$

$\rightsquigarrow R(x,y) = -x^2y^4 \rightsquigarrow x = 0$ or $y = 0$.



$x = 0 \rightsquigarrow y^2 - z^2 = 0 \rightsquigarrow [0:1:1], [0:1:-1]$

$y = 0 \rightsquigarrow x^2 - z^2 = 0 \rightsquigarrow \underbrace{[1:0:1], [1:0:-1]}_{\text{mult. } 2}$

Pf. of Prop. To show $R(tx,ty) = t^{mn} R(x,y)$.

$$R(tx,ty) = \begin{pmatrix} 1 & ta_1 & t^2a_2 & \cdots & t^m a_m & & & \\ & t\cdot 1 & t^2 a_1 & \cdots & \cdots & t^{m+1} a_m & & \\ & & \ddots & & & & & \\ & & & t^{m-1}\cdot 1 & t^m a_1 & \cdots & & t^{2m-1} a_m \\ 1 & tb_1 & t^2 b_2 & \cdots & t^n b_n & & & \\ & t & t^2 b_1 & \ddots & & & & \\ & & & & 1 & tb_1 & \cdots & t^n b_n \end{pmatrix}$$

Multiply i^{th} row of a's by t^{i-1}, similar for b's.

$\rightsquigarrow$ k^{th} col. has common factor of t^{k-1}.

Pull them out, get $R(x,y)$.

When we multiplied the rows, Scaled det by

$$t^{1+\cdots+n-1}\, t^{1+\cdots+m-1} = t^{(m(m-1)+n(n-1))/2}$$

When we pulled out the t's we divided det by

$$t^{1+\cdots+(m+n-1)} = t^{(m+n)(m+n-1)/2}$$

The difference in exponents is mn, as desired $\square$

$f(z) = s_0 z^m + \dots + s_m$ & $g(z) = t_0 z^n + \dots + t_n$

$s_0, t_0 \neq 0$

Prop. $R(f,g) = 0$ iff f & g have a common root.

Pf. Say α is a root of f & g

$\rightsquigarrow$ $\exists$ poly's f_1, g_1 of degree $m-1, n-1$ s.t.

$$f(x) = (x-\alpha) f_1(x) \quad g(x) = (x-\alpha) f_2(x)$$

$$\Longrightarrow \quad f(x) g_1(x) - g(x) f_1(x) = 0.$$

Equate the coeff's of $x^{m+n-1}, \dots, x, 1$ to 0 gives a system of $m+n$ linear eqns in $m+n$ unknowns (coeffs of f_1, g_1). The matrix is the Sylvester matrix for f, g. The existence of a solution shows $\mathrm{Res}(f,g) = 0$.

Conversely, say $\mathrm{Res}(f,g) = 0$

Since the matrix has $\det = 0$, as above there is a solution f_1, g_1 to $f(x) g_1(x) - g(x) f_1(x) = 0$. A root α of f must be a root of g or f_1. If g, done. If f_1, cancel $(x-\alpha)$ from f & f_1, continue inductively. Since $\deg f_1 < \deg f$, eventually get a root of f & g. $\square$

Pf of Bézout. Suffices to prove for C, D irred.

$C \cap D$ has dim $0 \Rightarrow |C \cap D| < \infty$.

Change coords so $x \neq 0$ $\forall$ pts of $C \cap D$.

Write $C = Z(f)$, $D = Z(g)$.

$$f(x,y,z) = s_0 z^m + \cdots + s_m$$

$$g(x,y,z) = t_0 z^n + \cdots + t_n$$

If $R(x,y) \equiv 0$ then $\forall$ $[a:b] \in \mathbb{P}^1$ have $f(a,b,z)$ & $g(a,b,z)$ have common zero, violating $|C \cap D| < \infty$.

Thus $\deg R(x,y) = mn$.

Write $R(x,y) = x^{mn} R_*(y/x)$ where $R_*(t)$ is a poly. of deg $\leq mn$ in $t = y/x$.

Same as: there is a y^{mn} term. If not, each term has an x.

Claim: $\deg R_* = mn$. If $\deg R_* = r < mn$, then $R(x,y) = x^{mn-r} P(x,y)$ where $P(x,y)$ is homog. poly of deg r not divis. by $x \Rightarrow R(0,1) = 0$ $\Rightarrow \exists$ $[0:1:c] \in C \cap D$, contradiction.

Let $\alpha_1, \ldots, \alpha_m$ be roots of R_*.

Each $\alpha_i = b_i/a_i$ with $R(a_i, b_i) = 0$.

$\rightarrow$ $f(a_i, b_i, z)$ & $g(a_i, b_i, z)$ have common root c_i $\forall i$.

$\Rightarrow$ $[a_i : b_i : c_i] \in C \cap D$ $\forall i$.

Conversely, $[a:b:c] \in C \cap D$ (so $a \neq 0$)

$\implies b/a$ is a root of R_*

So $C \cap D$ has exactly mn roots, provided we count multiplicity as roots of R_* $\square$

Of course we cheated at the end: we defined the multiplicity just to make the theorem work. To define multiplicity properly, need schemes. But we definitely proved Weak Bézout: at most mn pts of intersection. We'll see this already has deep implications.

Applications of Bézout's Theorem

Prop. Say C, D are projective curves of degree n in $\mathbb{P}^2$ with $|C \cap D| = n^2$.
Assume mn of the intersection pts lie on an irred. curve of degree m. Then the remaining $n(n-m)$ pts lie on a curve of deg at most $n-m$.

Pf. Say C, D, E defined by $f, g, h \in k[x,y,z]$.
Let $[a:b:c] \in E \setminus (C \cap D)$. Let $F = Z(p)$ where

$$p(x,y,z) = g(a,b,c) f(x,y,z) - f(a,b,c) g(x,y,z).$$

Then $|E \cap F| \geq mn+1$ since it contains

$$C \cap D \cap E \text{ and } [a:b:c].$$

Bézout $\Rightarrow$ E, F have a common component
E irred $\Rightarrow$ $p = hq$ some $q \in k[x,y,z]$, with $\deg q \leq n-m$.
Each $[u:v:w]$ in $(C \cap D) \setminus E$ satisfies $f=0$ & $g=0$, thus $p(u,v,w) = 0$ & $h(u,v,w) \neq 0$
so $q(u,v,w) = 0$, that is, $[u:v:w] \in F$ $\square$

Cor. (Pascal's Mystic Hexagon) Consider a hexagon inscribed in a conic. The three pairs of opposite sides meet in 3 collinear pts.

Pf. Say the vertices are $p_0, \dots, p_5$

Let L_i be the line $p_i p_{i+1}$.

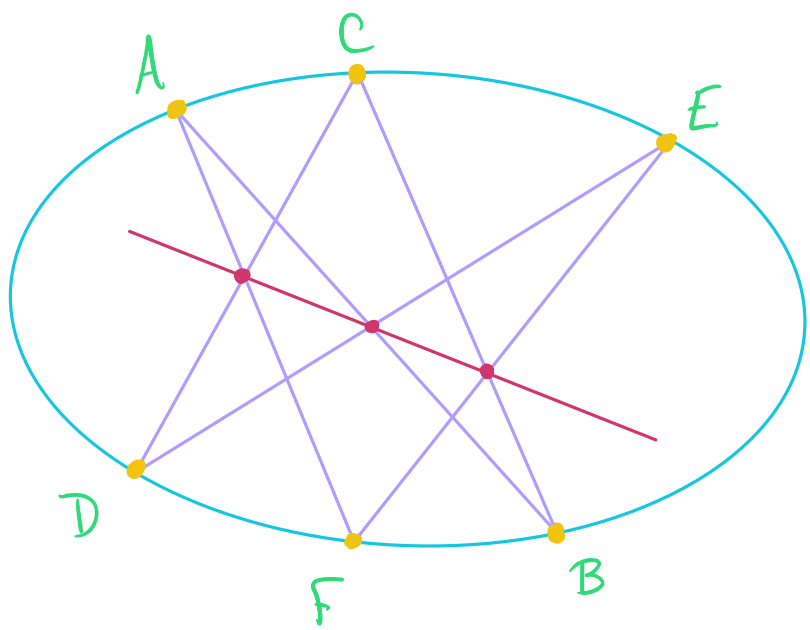
Let $L = L_0 L_2 L_4$ & $L' = L_1 L_3 L_5$ cubics.

L & L' have no common factor

Bézout $\Rightarrow$ they intersect at ≤ 9 distinct points.

6 are the vertices of the hexagon, which lie on the conic (deg 2).

The other 3 lie on a line by the prop. $\square$



See also Pappus' thm (the case of a degen. conic).
And Poncelet!

MULTIPLICITY

First version. Write $I_p(C,D)$ for multiplicity of p as a point of $C \cap D$.

Or $I_p(f,g)$ where $C = Z(f)$, $D = Z(g)$.

$\rightsquigarrow$ Axioms: ① $I_p(f,g) = I_p(g,f)$

② $$I_p(f,g) = \begin{cases} \infty & p \text{ in a common component} \\ 0 & p \notin C \cap D \\ \in \mathbb{N} & \text{otherwise} \end{cases}$$

③ C, D lines, $p \in C \cap D \Rightarrow I_p(f,g) = 1$.

④ $I_p(f_1 f_2, g) = I_p(f_1, g) + I_p(f_2, g)$

⑤ $I_p(f,g) = I_p(f, g+fh)$ if $\deg h = \deg g - \deg f$

Thm. $\exists!$ I_p defined for all proj. curves in $\mathbb{P}^2$.

If $p \in C \cap D$ and $p \notin$ common component then $I_p(C,D)$ given as follows: remove any common component and choose a coord system s.t. $[1:0:0]$ does not lie in:

Proof in Gim's notes on the web site.

- $C \cap D$
- any line containing distinct pts of $C \cap D$
- the tangent lines to C, D at a pt of $C \cap D$

(possible by Weak Bézout). For any such $p = [a:b:c]$ define $I_p(f,g)$ by exponent of $bz - cy$ in $R_{f,g}$.

Second version. $I_p(f,g) = \dim_k \dfrac{\mathcal{O}_p(\mathbb{A}^2)}{(f,g)}$

$\mathcal{O}_p(\mathbb{A}^2)$ = ring of regular fns at p
= rational fns where denom. does not vanish at p

Example. $f(x,y) = x$
$g(x,y) = y$
$p = (0,0)$
$\rightsquigarrow \mathcal{O}_p(\mathbb{A}^2)/(f,g)$ = const. fns.

Getting closer to sheaf theory...