Chap 13. Nielsen-Thurston Classification.

<u>Thm</u> (Thurston) Every $f \in \mathrm{Mod}(S)$ has a representative φ s.t.

① periodic: $\varphi^n = 1$

② reducible: $\varphi(M) = M$ some multicurve M

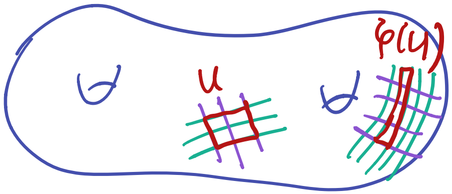
③ pseudo-Anosov:

$\exists$ transv. meas. fol's $\mathcal{F}_u, \mathcal{F}_s$ & $\lambda > 1$ s.t. "stretch factor"

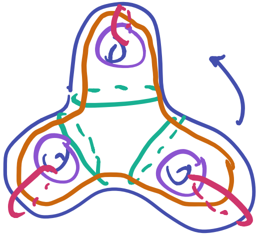
$$\varphi \cdot \mathcal{F}_u = \lambda \mathcal{F}_u$$

$$\varphi \cdot \mathcal{F}_s = \frac{1}{\lambda} \mathcal{F}_s$$

Moreover ③ is exclusive from ① & ②.



Examples



per.

& red.

CRS = $\emptyset$

per.

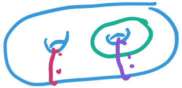
(not. red)

b/c quotient orbifold is

T_c

red

not per.



CRS = c

 $\circlearrowleft \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

pA

foliations $\longleftrightarrow$ eigenvectors

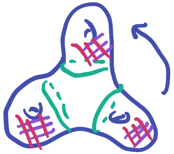
stretch factor $\longleftrightarrow$ eigenvalues.

λ

Birman-Lubotzky-McCarthy:

Canonical reduction system = intersection of max reduction systems.

Restatement of NTC: Every mapping class reduces to per & pA pieces.



a typical mapping class.

Jordan form

Torus case

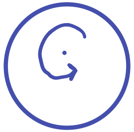
$f \in \mathrm{Mod}(T^2) \leftrightarrow A \in SL_2\mathbb{Z}$

$\leftrightarrow \tau \in \mathrm{Isom}^+(\mathbb{H}^2)$

Case 1 2 complex eigenvals

per. $\leftrightarrow$ τ rotation.

prop disc. $\Rightarrow |\tau| < \infty$.



Case 2. 1 real eigenval

red

product of eigenvals = det = 1.

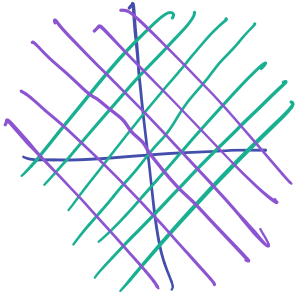
$\lambda = \pm 1$

$\Rightarrow$ eigenvector rational.

$\rightsquigarrow$ fixed curve

Case 3 2 real eigenvals. $\lambda, \frac{1}{\lambda}$

pA $\Rightarrow$ f (pseudo-) Anosov.



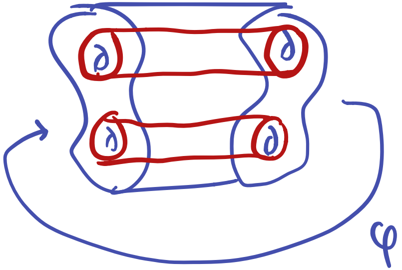
$X^2 + X + 1 = 0$

$(X-1)(X^2+X+1) = 0$

$X^3 = 1.$

3-manifolds

$f \in \mathrm{Mod}(S) \rightsquigarrow M_f$ = mapping torus



$[\varphi] = f$.

Thm (Thurston) $f \in \mathrm{Mod}(S)$ $\chi(S) < 0$.

- f per $\Longleftrightarrow$ M_f admits metric locally isometric to $\mathbb{H}^2 \times \mathbb{R}$.
- f red $\Longleftrightarrow$ M contains incompressible torus
  ↳ π_1-inj.
- f pA $\Longleftrightarrow$ M admits hyperbolic metric.
  $\Longrightarrow$ hard.

Periodic elements

"Easy Nielsen realization" (Fenchel)

Thm. $f \in \mathrm{Mod}(S)$ periodic $\Rightarrow f$ has a periodic rep:

$$\varphi^n = 1.$$

Pf. To show f has fixed pt in $\mathrm{Teich}(S)$.

Indeed: $f \cdot X = X$

$$\varphi^* X \sim X$$

Can change φ by isotopy so fixes X on nose.

Note $\langle f \rangle \cong \mathbb{Z}/m$

Fact. $\mathbb{Z}/m$ cannot act freely on a fin. dim contractible space.

(otherwise quotient is a fin. dim $K(\mathbb{Z}/m, 1)$ & $H^k(\mathbb{Z}/m) \neq 0$ arb. large k).

So f^j fixes $X \in \mathrm{Teich}(S)$ some j.

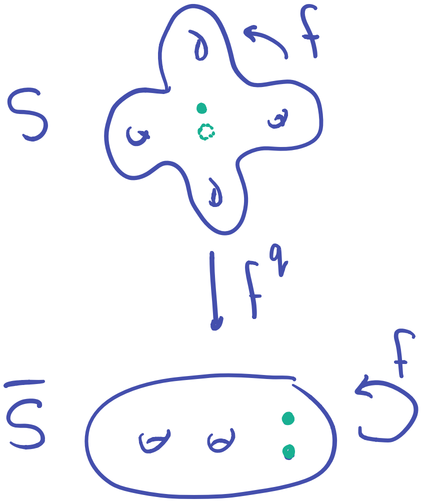
Special case. m prime.

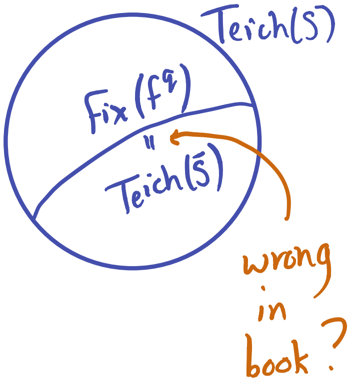
$f = (f^j)^k$ some k. $\Rightarrow f \cdot X = X$.

Assume now $m = pq$, p prime, q prime.

Note f^q has order p.

As above f^q fixes $X \in \mathrm{Teich}(S)$.





The map

$$\mathrm{Teich}(\bar{S}) \to \mathrm{Fix}(f^q)$$

is: lift complex structures.

Injectivity: Teich. U.T.*

$\bar{X} \neq \bar{Y} \in \mathrm{Teich}(\bar{S})$

$\rightsquigarrow$ Teich map of $\bar{S}$

$\rightsquigarrow$ Teich map of S between lifts X, Y.

$\Longrightarrow X \neq Y$.

Surjectivity: Special case.

$\square$

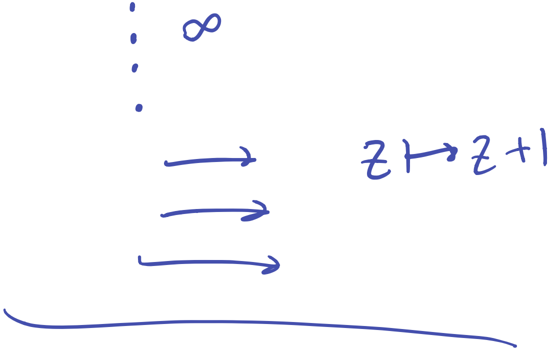
Outline of proof of NTC (Bers)

$f \in \mathrm{Mod}(S)$

$\tau(f) = \inf\{X \in \mathrm{Teich}(S) :$

$d(X, f \cdot X)\}$

"translation length"



To show:

$\tau(f) = 0$ & realized $\iff$ f periodic

$\tau(f)$ not realized $\iff$ f reducible

$\tau(f) > 0$ & realized $\iff$ f pA.

like torus case.