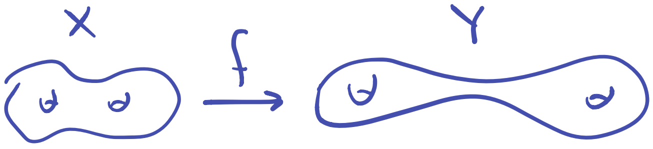
Chapter 11. Teich geom.

Basic question: $X, Y \in \text{Teich}(S)$



What is the best map?

Idea: Measure distortion

"dilatation"

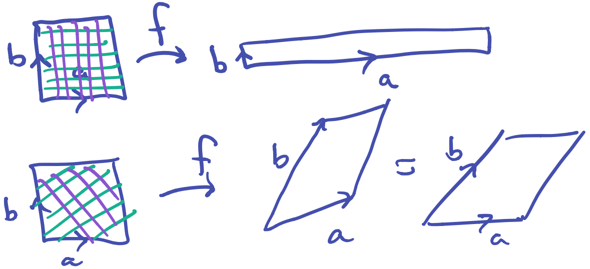
(at a pt)

$\rightsquigarrow$ metric on Teich(S)

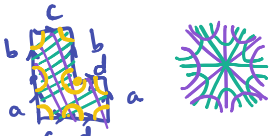
Take sup of dilatation over X

Take inf over f. Take log.

Teichmuller thm: existence & uniqueness of infimal f.



Higher genus:



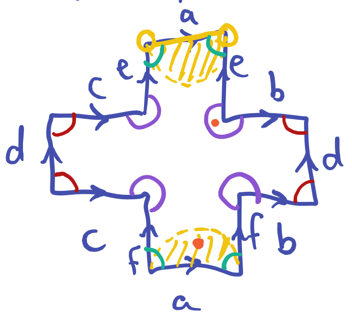
Complex structures

A complex structure on S consists of:

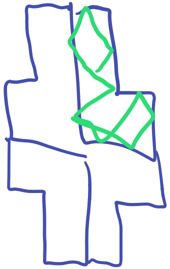
atlas of charts to $\mathbb{C}$ with holomorphic transition maps.

Riemann surface: S with complex structure.

Example of Riemann surface



$S = \maltese/\sim$



9 charts: "middle" identity map

6 edge charts: id on half-disk + diameter
translation on other half disk.

2 good corner charts: translation

1 bad corner chart:
apply $z^{1/3}$ + translation.

Complex str's vs Hyp strs. $\chi(S) < 0$.

$$\{\text{hyp strs on } S\} \longleftrightarrow \{\text{complex strs on } S\}$$

$\longrightarrow$ isometries of $\mathbb{H}^2$ are holomorphic. (Möbius tr)

+ Cartan-Hadamard: only simply conn. complete surface with $K=-1$ is $\mathbb{H}^2$.

$\longleftarrow$ uniformization thm: only simply conn Riem surf's are $\mathbb{H}^2$, $\mathbb{C}$, $\hat{\mathbb{C}}$.

Linear maps of $\mathbb{R}^2$ via $\mathbb{C}$-analysis

$1 \in \mathbb{C} \leftrightarrow (1,0) \in \mathbb{R}^2 \quad i \in \mathbb{C} \leftrightarrow (0,1) \in \mathbb{R}^2$

$U, V \subseteq \mathbb{C}$ open

$f : U \to V$ smooth

$$Df_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{R}$$

Can write as:

$$Df_p(z) = \alpha z + \beta \bar{z}$$

$$\alpha = \frac{(a+ic) - i(b+id)}{2}$$

$$\beta = \frac{(a+ic) + i(b+id)}{2}$$

Check $Df_p(1) = a + ic$

$Df_p(i) = b + id$

α, β called $f_z, f_{\bar{z}}$

$$Df_p(z) = f_z z + f_{\bar{z}} \bar{z}$$

Complex dilatation:

$$\mu_f = f_{\bar{z}} / f_z$$

$\mu_f = 0 \iff f$ holomorphic.

Dilatation of f

$$K_f(p) = \frac{|f_z(p)| + |f_{\bar z}(p)|}{|f_z(p)| - |f_{\bar z}(p)|} = \frac{1+|\mu_f(p)|}{1-|\mu_f(p)|} = d_{\mathbb{H}^2}(\mu_f(p), 0).$$

$= \text{eccentricity of } Df_p(S^1)$ $\qquad K_f = \sup_p K_f(p)$

to prove, write S^1 as $e^{i\theta}$, apply Df_p

$$\left| f_z(p)\, e^{i\theta} + f_{\bar z}(p)\, e^{-i\theta} \right|$$

$$1 - |\mu_f(p)| \le \cancel{|e^{i\theta}|}\,\cancel{|f_z(p)|}\, \left| 1 + \mu_f(p)\, e^{-2i\theta} \right| \le 1 + |\mu_f(p)|$$

Quasi-conformal maps

f is q.c. if $K_f < \infty$.

Holomorphic $\Longrightarrow$ 1-q.c.

Note: qc makes sense for Riem. surfaces since transitions maps are holomorphic.

$\boxed{\text{We only consider maps that are smooth outside a finite set.}}$

Fact. X, Y Riem surfs. The set of qc maps $X \to Y$ forms a group

Pf. $K_{f \circ g} \leq K_f K_g$

$K_{f^{-1}} = K_f$ $\square$

Teichmüller's extremal problem

Fix $f: X \to Y$ homeo.

Is this inf realized?

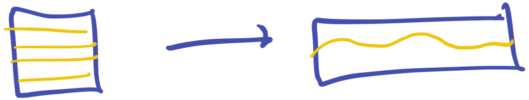
$$\inf\{K_h : h \sim f,\ h \text{ qc}\}$$

If so, what is min. map?

Teichmuller: existence & uniqueness.

$$\rightsquigarrow d_{Teich}(X,Y) = \frac{1}{2}\log K_h$$

Earlier, Grötzch did this for rectangles:

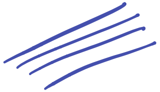


Extemal map is the obvious one & it is unique.

Measured foliations

Sing. foliation on S_g

locally:

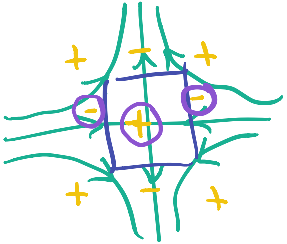


or

$k \geq 3$ prongs.

Prop. (Euler-Poincaré formula)

$$\chi(S) = \sum_{\text{sing}} \left(1 - \frac{k_i}{2}\right)$$



Special case: No singularities

$$\Longleftrightarrow \chi(S) = 0.$$

Pf (assuming foliation is orientable)

(W. Thurston)

Assume no singularities.

Triangulate