Math 6421

Online or in person fine.

Some HW per week (drop 3) Gradescope.

Online office hours Fri 1:30-2:30 Teams

& by appt.

Overview of Course

What is alg. geom?

study of solns of polynomials

(using ring theory, etc.)

"linear alg, without the linear"

<u>much</u> harder.

Setup

k = field (usually can think $k = \mathbb{C}$)

$k[x_1, \dots, x_n]$ = ring of polynomials in $x_1, \dots, x_n$
with coeffs in k.

$\mathbb{A}^n = \mathbb{A}^n_k$ = affine n-space over k
$= \{(a_1, \dots, a_n) \in a_i \in k\}$

In bijection with k^n. In $\mathbb{A}^n$ no vect. sp structure,
so 0 not special etc.

For $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$:

$$Z(f_1, \ldots, f_r) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f_i(a_1, \ldots, a_n) = 0 \ \forall i\}$$

zero set or vanishing set

Some texts use V instead of Z

These are <u>affine</u> <u>algebraic</u> <u>varieties</u>.

<u>Special cases</u>

① $n = r = 1$

d = degree of poly.

Solving polynomials in 1 var.

k alg closed : exactly d solns (with mult)

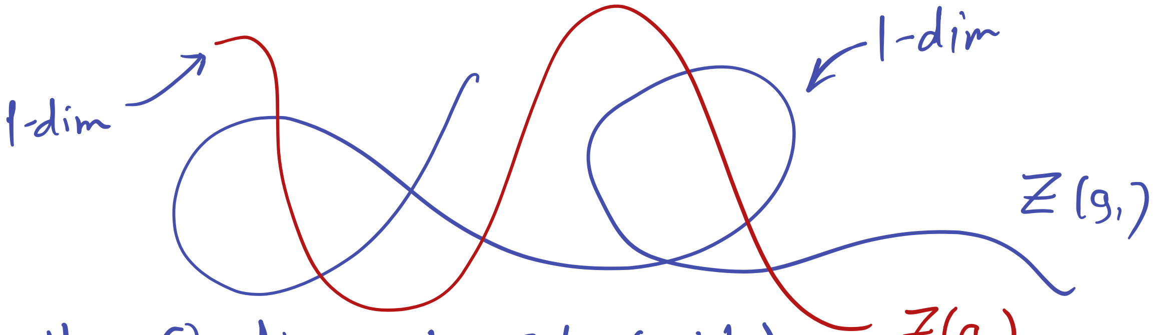
② Linear Algebra

Again, much harder when deg > 1 or # eqns > 1.

Example

$$g_1(x,y) = 3x^3 - 17xy^2 + 2xy + 4y^2 - 6$$
$$g_2(x,y) = x^5 - x^3y^2 + 3xy^2$$

We'll learn: $Z(g_i)$ is a "curve" in $\mathbb{C}^2$



Generically: 0-dim soln set (points)

Weak Bezout's Thm

If $f_1, f_2 \in \mathbb{C}[x,y]$, no common factors

$\deg f_i = d_i$

Then $|Z(f_1, f_2)| \le d_1 d_2$.

Chapter 1 The geometry/algebra dictionary

$\boxed{\text{Alg} \to \text{Geom}}$ Given $J \subseteq k[x_1,\dots,x_n]$ ideal

$\rightsquigarrow Z(J) = \{ a \in \mathbb{A}^n : f(a) = 0 \ \forall f \in J \}$

example $J = \langle f_1, \dots, f_r \rangle$ ideal generated by $f_1, \dots, f_r$

$= \{ g_1 f_1 + \dots + g_r f_r : g_i \in k[x_1, \dots, x_n] \}$

Then $Z(J) = Z(f_1, \dots, f_r)$ as above.

$\boxed{\text{Geom} \to \text{Alg}}$ Given $V \subseteq \mathbb{A}^n$

$\rightsquigarrow I(V) = \{ f \in k[x_1, \dots, x_n] : f(a) = 0 \ \forall a \in V \}$

this is an ideal.

We have:

$$\{\text{subsets of } \mathbb{A}^n\} \rightleftarrows \{\text{ideals in } k[x_1, \dots, x_n]\}$$

Neither is injective. Why?

$\leftarrow$ $\quad Z(x) = Z(x^2)$ $\quad$ more interesting direction

$\rightarrow$ $\quad$ all open sets in $\mathbb{C}$. $\longmapsto$ 0 ideal.

To fix latter, replace LHS with affine alg vars

For former, the example is the only issue. (taking powers).

The fix: For an ideal $J \subseteq R$, have

$$\text{rad}(J) = \{ r \in R : r^i \in J \text{ some } i \geq 1 \}$$

"radical"

Will use Hilbert's Nullstullensatz to show

$$\{\text{affine alg var's in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } k[x_1, \dots, x_n]\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

also:

$$\{\text{irreducible alg vars in } \mathbb{A}^n\} \longleftrightarrow \{\text{prime ideals} \dots\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\mathbb{A}^n = \{\text{pts in } \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals}\}$$

Chapter 2 Projective Varieties.

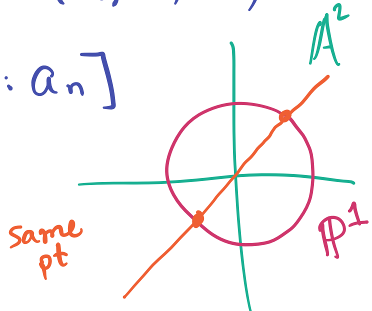
$$\mathbb{P}^n = \mathbb{P}^n_k = (k^{n+1} - 0)/\overset{k-0}{k^*}$$

where $v \sim w$ if $v = cw$, $c \in k^*$.

$= $ set of lines in k^{n+1}

Write equiv class of $(a_0, \dots, a_n)$ as $[a_0 : a_1 : \cdots : a_n]$

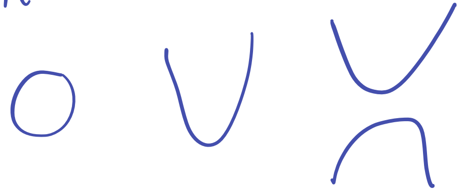
In $\mathbb{R}^2$:



We will study zero sets in $\mathbb{P}^n$ ....

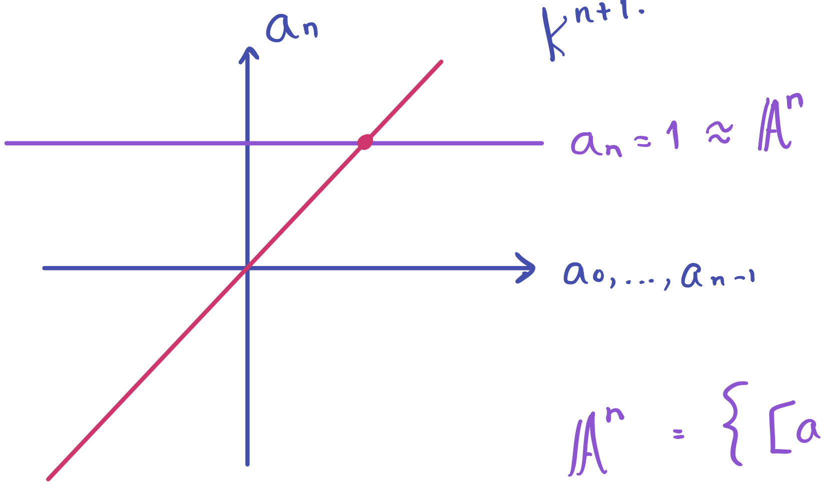
because: more symmetry.

In $\mathbb{A}^2$ have different conics



In $\mathbb{P}^2$ these are all same!

Can regard $\mathbb{P}^n$ as $\mathbb{A}^n \cup \mathbb{P}^{n-1}$.



$$\mathbb{A}^n = \{ [a_0 : \dots : a_n] : a_n \neq 0 \}$$

$$\mathbb{P}^{n-1} = \{ [a_0 : \dots : a_{n-1} : a_n] : a_n = 0 \}$$

Projective varieties

$f(a_0, \dots, a_n)$ is not well-defined on $\mathbb{P}^n$

e.g. $f(x,y) = x + y^2$

$f(-1,1) = 0$

$f(-2,2) = 2$

So can't say $f([-1:1]) = 0$.

But, if f is homogeneous (all terms have same degree d)

then $f(cv) = c^d f(v)$.

So $f(v) = 0 \iff f(cv) = 0$ ✓

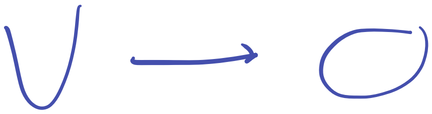
So: get zero sets in $\mathbb{P}^n$ for homog. poly's.

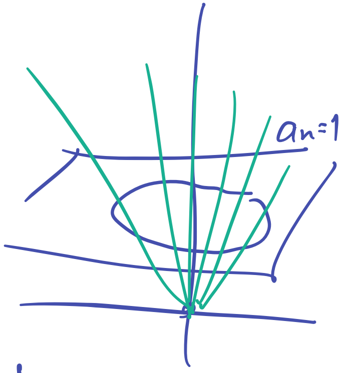
So for $f_1, \dots, f_r$ homogeneous.

$$Z(f_1, \dots, f_r) = \{a \in \mathbb{P}^n : f_i(a) = 0 \ \forall i\}$$

We'll see:

① Affine varieties have projective closures





② Cone on a proj. var. is an aff. variety.

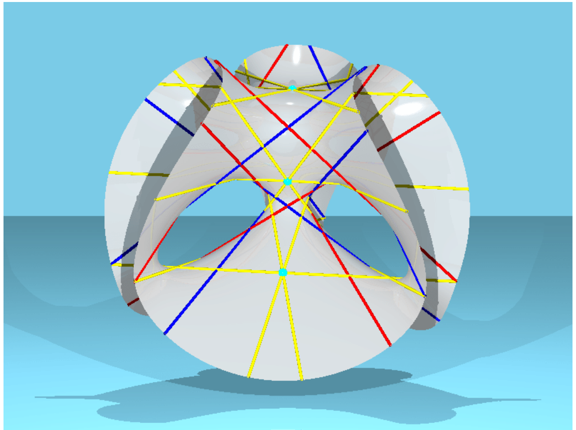
So, the theories are closely related.

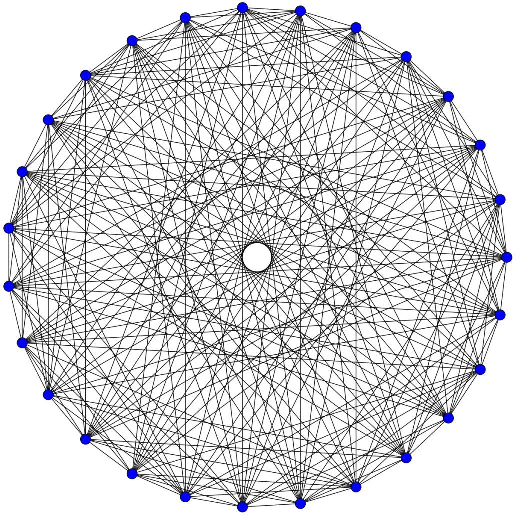
Next time: Better Bezout

$Z(f_1)$ curves in $\mathbb{P}^2$ of deg d_i

$Z(f_2)$ & f_1, f_2 no common factors.

Then $|Z(f_1) \cap Z(f_2)| = d_1 d_2$ (count with multiplicity).





Overview. From last time:

Chapter 1. The geometry/algebra dictionary

$f_1, \dots, f_r \in k[x_1, \dots, x_n]$

$Z(f_1, \dots, f_r) = \{a \in \mathbb{A}^n : f_i(a) = 0 \ \forall i\}$

"affine algebraic variety"

There is a bijection

$$\{\text{aav's in } \mathbb{A}^n\} \longleftrightarrow \left\{\begin{matrix}\text{rad ideals in} \\ k[x_1,\dots,x_n]\end{matrix}\right\}$$

$$V \longmapsto I(V)$$

$$Z(J) \longleftarrow J$$

Chapter 2. Projective varieties

$\mathbb{P}^n := (k^{n+1} - 0)/k^*$

$[(a_0, \dots, a_n)]$ written $[a_0 : \cdots : a_n]$

$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$

homogeneous poly's in $k[x_0, \dots, x_n]$

$\rightsquigarrow$ projective alg. var's

These are always compact and tend to have more symmetry/info (e.g. intersections at ∞).

Chapter 3 Classical constructions

① Segre embedding

$$\varphi_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

consequence: product of varieties is a variety

example: $g_1(x,y) = 3x^3 - 17xy^2$
$g_2(z,w) = z^5 - w^2 z^3$

Does $Z(g_1 g_2)$ work?

No. get $Z(g_1) \times \mathbb{P}^1$
$\mathbb{P}^1 \times Z(g_2)$
and more...

param space of all polys of $d=2$
$\{(a,b,c)\} = \mathbb{A}^3$

bad polys
$Z(b^2-4ac)$

② Veronese embedding

$$v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{d+n}{n}-1}$$

reduces the degree

For example: "Fermat cubic curve"

$$Z(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbb{P}^2$$

maps intersection of 9 quadrics in $\mathbb{P}^9$.

$\hookrightarrow$ Zero sets of quadratic.

Application: Complements of varieties are varieties

e.g. $\mathrm{Poly}_2(\mathbb{C}) = \{ax^2+bx+c : b^2-4ac \neq 0\}$
$GL_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc \neq 0\right\}$

③ Grassmannian

$Gr_{r,n} = \{ r\text{-dim planes thru } 0 \text{ in } k^n \}$

Note: $\mathbb{P}^n = G_{1,n+1}$

$Gr_{r,n}$ important in topology:
"classifying space for n-dim vector bundles"

We'll show this a proj. var

Plücker embedding:

$$Gr_{r,n} \longrightarrow \mathbb{P}(\Lambda^r k^n)$$

"parameter space of widgets is a widget"

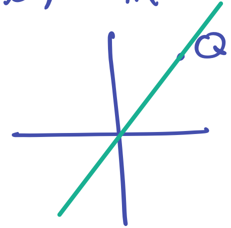
④ Blow up

(Fixing, not destroying)

The map $\mathbb{A}^2 \setminus 0 \longrightarrow \mathbb{P}^1$ does not extend to 0.

The blowup of $\mathbb{A}^2$ at 0:

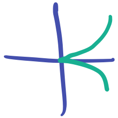
$$\{ (Q, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 : Q \in \ell \}$$

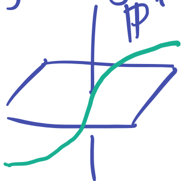


get a copy of $\mathbb{P}^1$ at 0.

Application: resolving singularities.

$x^3 = y^2$

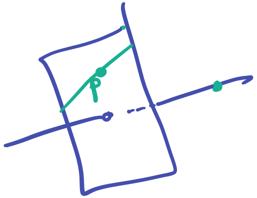




Chapter 4. Dimension, degree, smoothness

"expected properties"

Dim: $\dim_p V$ = (size of max chain of varieties at p) $- 1$



We'll show: behaves like dim in lin alg.

codim $V_i = c_i$ (V_i irred)

codim $V_1 \cap V_2 = c_1 + c_2$

generically.

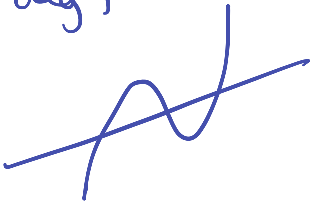
Degree: $V \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$ — k alg closed

$\dim V = k$.

$\deg V$ = generic/expected # intersections with $n-k$ plane

For $V = Z(f)$ "hypersurface"

$\deg V = \deg f$



Helps understand <u>number</u> of solns when $\dim = 0$.

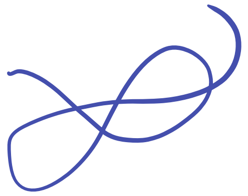
Smoothness: A $\mathbb{C}$-variety is smooth exactly when it is a manifold... can use manifold theory.

Chapter 5. Curves in the plane.

Setup $f \in \mathbb{C}[x_0, x_1, x_2]$ homog.

$C = Z(f) \subseteq \mathbb{P}^2$

Picture:



When $\deg f = 2$

C is a "conic"

Thm. (Five pts determine a conic)

Given $p_1, \dots, p_5 \in \mathbb{P}^2$ $\exists$ conic passing thru all p_i (generically unique)

Bézout's thm. $C_1 = Z(f_1)$ $\deg f_i = d_i$

$C_2 = Z(f_2)$

& f_1, f_2 no common factor

Then $|Z(f_1) \cap Z(f_2)| = d_1 d_2$

(count with mult.)

Thm. Given $\binom{d+1}{2}$ distinct pts in $\mathbb{P}^2$

$\exists$ deg d curve passing thru.

Cayley-Bacharach Thm

$C_1, C_2 \subseteq \mathbb{P}^2$ cubic curves

with $|C_1 \cap C_2| = 9$

If C_3 cubic curve passes thru 8 of the pts,

it passes thru the 9^{th}.

Smoothness for curves:

$$C = Z(f)$$

smooth a p if some

$\frac{df}{dx_i}(p)$ nonzero.

Thm. $f \in \mathbb{C}[x_0, x_1, x_2)$

irred, homog, deg d

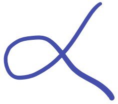
$\rightsquigarrow Z(f)$

Then # sing pts $\leq \binom{d-1}{2}$

Classification of cubic curves in $\mathbb{P}^2$

1 sing pt: equiv to

$Z(x^2 = y^3) \subseteq \mathbb{A}^2$



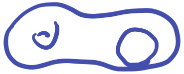
$Z(y^2 = x^2 + x^3)$
$\subseteq \mathbb{A}^2$

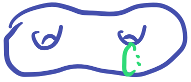
Smooth:

$Z(y^2 = 4x^3 - g_2 x - g_3)$

"Weierstrass curves"

Actual pic





Chapter 6. Special topics

Cayley-Salmon thm

Every smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ contains 27 lines.

e.g. $x^3+y^3+z^3+w^3$

find the lines!

Chapter 1. Affine alg. vars & the geometry/alg dictionary

Setup: $S \subseteq k[x_1,\dots,x_n]$

$$\rightsquigarrow Z(S) = \{a \in \mathbb{A}^n : f(a) = 0 \ \forall f \in S\}$$

"affine alg var"

Examples

① $\emptyset = Z(k[x_1,\dots,x_n])$
$= Z(1)$

Second = Makes sense since $(1) = k[x_1,\dots,x_n]$

② $\mathbb{A}^n = Z(0)$

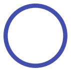
③ $(a_1,\dots,a_n) = Z(x_1-a_1,\dots,x_n-a_n)$

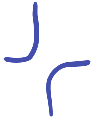
compare: lin alg.

④ (Hyper)planes

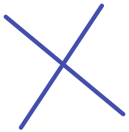
⑤ Conics $Z(f) \subseteq \mathbb{A}^2$ $\deg f = 2$.

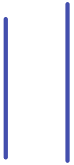
e.g. x^2-y^2-1 $\quad xy-1$ $\quad y-x^2$





$(x-y)(x+y)$ $\quad x^2-1$ $\quad x^2$

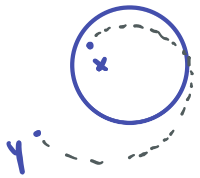




"double line"

Aside: Conics over $\mathbb{C}$ from a topological pt of view.

① $Z(xy-1)$ is connected over $\mathbb{C}$



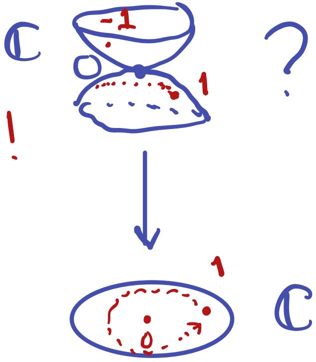
every pt connected to $(1,1)$

② $Z(x^2-y)$ over $\mathbb{C}$.

Have a map

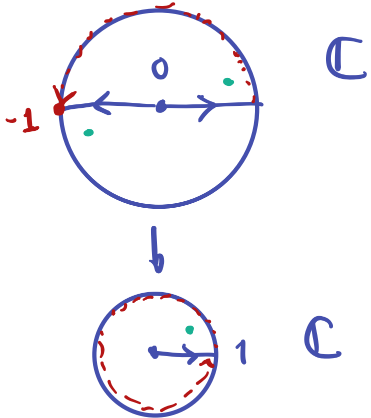
$$Z(x^2-y) \longrightarrow \mathbb{C}$$
$$(x,y) \longmapsto y$$

WRONG!



2 preimages (away from 0)

Right picture



"Riemann surface"

⑥ Algebraic groups

$SL_n k = Z(\det - 1) \subseteq \mathbb{A}^{n^2}$

$GL_n k$ complement of $Z(\det)$ by defn.

In general, complements of aar's are aav's (later)

To see $GL_n k$ as a variety:

$V = \{(x_{ij}, t) \in \mathbb{A}^{n^2+1} : \det(x_{ij})t - 1 = 0\}$

$\varphi : GL_n k \to V$

$A = (a_{ij}) \mapsto (a_{ij}, \frac{1}{\det A})$

is a bijection.

⑦ Twisted cubic

$C = \operatorname{Im} \varphi$ where

$\varphi : \mathbb{A}^1 \to \mathbb{A}^3$

$t \mapsto (t, t^2, t^3)$

As a variety

$C = Z(x^2 - y, x^3 - z)$

$= Z(x^2 - y, z - xy)$

intersection of two "quadrics"

C is also a determinantal var

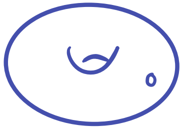
$C = \{(x,y,z) \in \mathbb{A}^3 : \operatorname{rank}\begin{pmatrix} 1 & x & y \\ x & y & z \end{pmatrix} < 2\}$

(Chris) Q. Is any int. of quadrics a det. var?

⑧ A family of (smooth) cubics

$$C_\lambda = Z(x(x-1)(x-\lambda) - y^2) \subseteq \mathbb{A}^2$$

$$\lambda \neq 0, 1 \qquad k = \mathbb{C}$$

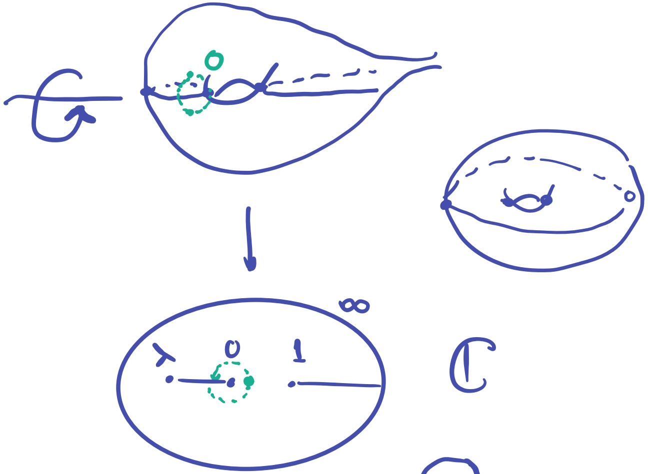
Claim: $C_\lambda \cong$ 

Like the $x - y^2$ example:

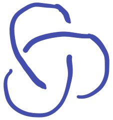
$$C_\lambda \longrightarrow \mathbb{A}^1$$
$$(x,y) \longrightarrow x$$

Other than $x = 0, 1, \lambda$

pts in $\mathbb{A}^1$ have two preims.



⑨ Trefoil



$$Z\left((x^2+y^2)^2 + 3x^2y - y^3\right)$$

intersect with $S^3 = \{(x,y) : |x^2| + |y^2| = 1\}$

"singularity theory"

exercise: Take complement of axes in $\mathbb{C}^2$ & intersect with S^3 ....

Nonexamples $k = \mathbb{C}$

① Fact. Every aff alg var in $\mathbb{A}^n$ is closed in Euclidean top

$\Rightarrow \{z : |z| < 1\}$ not an aav in $\mathbb{A}^n$

② Fact. The interior of any proper aav is $\emptyset$.

$\rightarrow \{z : |z| \leq 1\}$ not an aav.

③ Fact. Any proper variety in $\mathbb{A}^1$ is finite (by FTAlg)

$\Rightarrow \mathbb{Z} \subseteq \mathbb{C}$ is not a.a.v.

Basic Properties of aav's

① $\forall\ S \subseteq k[x_1, \dots, x_n]$ have

$$Z((S)) = Z(S)$$

(exercise)

② Intersections of aav's are aav

$$\bigcap_\alpha Z(I_\alpha) = Z\left(\bigcup I_\alpha\right)$$

③ Finite unions of aav's are aav's

$$V(I) \cup V(J) = V(\underbrace{IJ}_{\text{finite sums } \sum i_k j_k})$$

(exercise)

example $V(x) \cup V(y) = V(xy)$

Zariski Topology

A topology on a space X is a collection of sets, called closed sets such that

1. $\emptyset, X$ closed
2. Finite unions of closed sets are closed
3. Arbitray intersections of closed sets are closed.

Complements of closed sets called "open".

Def. Zariski topology on $\mathbb{A}^n$ has aav's as the closed sets.

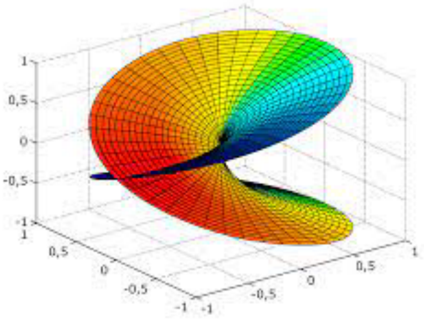
Basic properties $\Rightarrow$ this indeed is a topology.

The Zariski top. is strange:

1. All proper closed sets have $\emptyset$ interior.
2. Proper closed subsets of $\mathbb{A}^1$ are finite.
3. No two open sets are disjoint.
   $\Longrightarrow$ not Hausdorff.
4. compact $\not\Longrightarrow$ closed
   closed $\Longrightarrow$ compact

A concept: A set is Zariski dense iff every polynomial is det. by its values on that set.

e.g. $\mathbb{Z} \subseteq \mathbb{A}^1_{\mathbb{R}}$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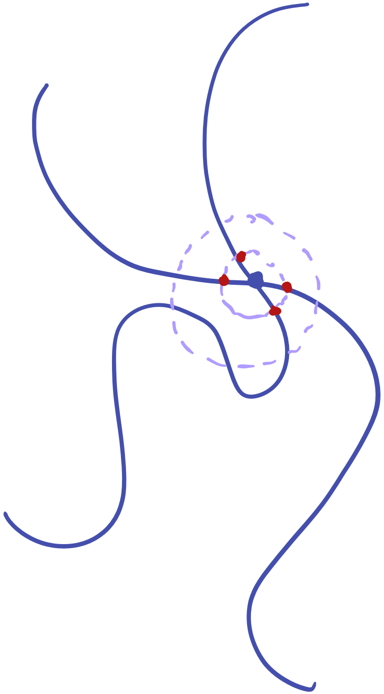
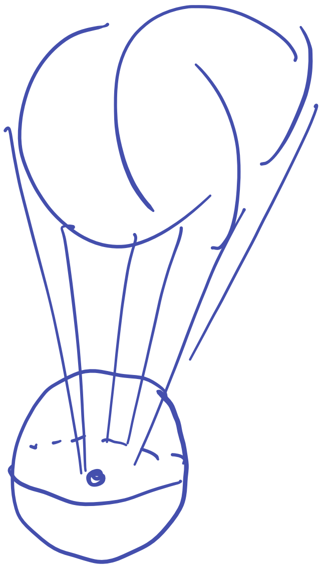
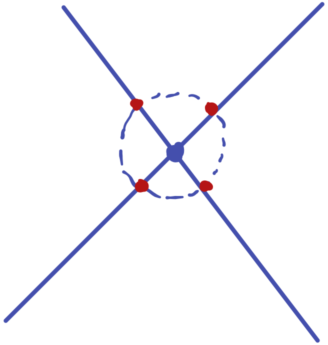






Hilbert Basis Thm

Thm. Every $Z(I)$ equals some $Z(f_1,\dots,f_r)$

i.e. every aav is the intersection of finitely many hypersurfaces

Lemma/Defn. R ring TFAE

① Every ideal in R is fin gen.

② R satisfies asc. chain cond: any $I_1 \subseteq I_2 \subseteq \dots$ eventually stationary.

Say R is Noetherian.

Fact. Fields are Noetherian.

Pf of Lemma.

① $\Rightarrow$ ② Let $I_1 \subseteq I_2 \subseteq \dots$

$\rightsquigarrow I = \bigcup I_i$ is an ideal.

I is f.g. by ①.

Some I_j contains all gens so $I_k = I$. $k \geq j$.

② $\Rightarrow$ ① If I not f.g. make $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ by adding on gen. at a time.

Prop. R Noetherian $\Rightarrow$ $R[x_1,\dots,x_n]$ Noeth.

In our case $R=k$, so HBT follows.

Pf. We'll do $R[x]$, rest is induction.

Say $I \subseteq R[x]$ not f.g.

Let f_0 = non-0 elt of I of min deg.

Given f_i: f_{i+1} = nonzero elt of $I \setminus \underbrace{(f_0,\dots,f_i)}_{J_i}$ of min deg.

Note $\deg f_i \leq \deg f_{i+1}$

Let a_i = lead coeff of f_i.

$$I_i = (a_0,\dots,a_i) \subset R.$$

R Noeth $\Rightarrow$ $I_0 \subseteq I_1 \subseteq \cdots$ eventually stat.

So $\exists\ m$ st $a_{m+1} \in (a_0,\dots,a_m)$

$$\Rightarrow a_{m+1} = \sum r_i a_i \quad r_i \in R$$

Let $f = f_{m+1} - \sum_{i=0}^{m} x^{\deg f_{m+1} - \deg f_i} r_i f_i$

This f cooked up so $\deg f < \deg f_{m+1}$

Thus $f \in J_m \Rightarrow f_{m+1} \in J_m$

contrad. $\square$

Hilbert's Nullstullensatz c.1900

Weak Nullst. k alg closed
Every max. ideal in $k[x_1,\dots,x_n]$ is of form $(x_1-a_1,\dots,x_n-a_n)$.

Strong Nullst. k alg closed
$I \subseteq k[x_1,\dots,x_n]$ ideal. Then

$$\mathbb{I}(Z(I)) = \sqrt{I}$$

i.e. $\left\{\begin{matrix}\text{aavs}\\ \text{in } \mathbb{A}^n\end{matrix}\right\} \xleftrightarrow{\text{bij}} \left\{\begin{matrix}\text{rad. ideals}\\ \text{in } k[x_1,\dots,x_n]\end{matrix}\right\}$

$$X \longmapsto \mathbb{I}(X)$$

$$Z(I) \longleftarrow\!\shortmid\ I$$

The WN implies other natural statements:

- Every proper ideal in $k[x_1,\dots,x_n]$ has a common zero.

  i.e. $I \subsetneq k[x_1,\dots,x_n] \Longrightarrow Z(I) \neq \emptyset$

- Converse: a family of polynomials with no common zeros generates whole $k[x_1,\dots,x_n]$.

Aside: SN is a generalization of Fund Thm Alg.

First, note

$(f) \in \mathbb{C}[z]$ radical

$\iff f$ has no rep. roots.

SN $\Rightarrow$ FTA because

$I(Z(f)) = \sqrt{(f)}$ implies

f has a root.

FTA $\Rightarrow$ SN because

f factors into linears

$\Rightarrow I(Z(f)) = \sqrt{(f)}$

Pf by example:

$$f(z) = (x-1)(x-3)^2$$

$$I(Z(f)) = I(\{1,3\}) = ((z-1)(z-3))$$

$$= \sqrt{(f)}.$$

Both WN & SN fail for k not alg. closed:

e.g. (x^2+1) radical in $\mathbb{R}[x]$

since $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$

But $\mathbb{I}(Z(x^2+1)) = \mathbb{I}(\emptyset) = \mathbb{R}[x]$.

Pf of WN $\Rightarrow$ SN

"Trick of Rabinowitz"

Say $g \in \mathbb{I}(Z(f_1,\dots,f_m))$

Want $g^{\text{some power}} \in (f_1,\dots,f_m)$.

The assumption $\Rightarrow$ a common zero of the f_i is a zero of g.

Thus $f_1,\dots,f_m, x_{n+1}g-1$ have no common zeros in $\mathbb{A}^{n+1}$

$$WN \Rightarrow (f_1,\dots,f_m, x_{n+1}g-1) = k[x_1,\dots,x_{n+1}]$$

$$\Rightarrow 1 = p_1f_1+\dots+p_mf_m + p_{m+1}(x_{n+1}g-1)$$

where $p_i \in k[x_1,\dots,x_{n+1}]$

Apply the map

$$k[x_1,\dots,x_{n+1}] \longrightarrow k(x_1,\dots,x_n)$$

$$x_i \longmapsto x_i$$

$$x_{n+1} \longmapsto 1/g$$

$$\rightsquigarrow 1 = p_1(x_1,\dots,x_n,\tfrac{1}{g})f_1+\dots+p_m(x_1,\dots,x_n,1/g)f_m \text{ in } k(x_1,\dots,x_n)$$

$$= \frac{\text{Something in } (f_1,\dots,f_m)}{g^{\text{power}}}$$

$\square$

Fact. Each $(x_1-a_1,\dots,x_n-a_n)$ is maximal.

Pf.

$$k[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n) \longrightarrow k$$

$$f \longmapsto f(a_1,\dots,a_n)$$

$$1 \longleftarrow\!\shortmid\; 1$$

This is $\cong$ so done.

Thm. k = field, K extension

If K is fin gen as a k-alg

then K is algebraic over k.

Pf of WN. Say $\mathfrak{m}$ = max ideal in

$R = k[x_1, \ldots, x_n]$

$\Rightarrow R/\mathfrak{m}$ is a field, fin gen as k-alg.

(since R is).

Have $k \cap \mathfrak{m} = \{0\}$. (else $\mathfrak{m} = R$)

$\rightarrow$ image $\bar{k}$ of k in $R/\mathfrak{m}$ is $\cong k$.

Thm $\rightarrow R/\mathfrak{m}$ alg. ext. of $\bar{k}$.

k alg closed $\Rightarrow R/\mathfrak{m} = \bar{k}$

Under $R \rightarrow R/\mathfrak{m}$

each $x_i \longmapsto \bar{a}_i \in \bar{k}$

Some $\bar{a}_i$ image of $a_i \in k$.

$$\Rightarrow \underbrace{\mathfrak{m} \supseteq (x_1 - a_1, \ldots, x_n - a_n)}_{= \mathfrak{m}'}$$

But $\mathfrak{m}'$ maximal

$\Rightarrow \mathfrak{m} = \mathfrak{m}'$ $\square$

Hilbert's N'satz

$$\{\text{aav's in } \mathbb{A}^n\} \longleftrightarrow \{\text{rad ideals } k[x_1,\dots,x_n]\}$$

$$V \longmapsto \mathbb{I}(V)$$

$$Z(V) \longleftarrow I$$

Nontrivial part: Z inj on rad ideals

Weak N'satz Max ideals in $k[x_1,\dots,x_n]$ are of form

$$(x_1-a_1,\dots,x_n-a_n)$$

Lemma. Assume ~~k alg closed~~ and uncountably infinite.

If $L \supseteq k$ field ext and L fin. gen. as k-algebra

Then L is algebraic over k.

$\exists\, u_1,\dots,u_r \in L$ so that each elt of L is a polynomial in u_i with coeffs in k.

each $u \in L$ is a root of poly in k:

$$k_n u^n + \dots + k_1 u + k_0 = 0.$$

Example $\mathbb{C}(x)$ not alg over $\mathbb{C}$, not fg alg

Lemma. Assume ~~k alg closed~~ and uncountably infinite.

If $L \supseteq k$ field ext and L fin. gen. as k-algebra

Then L is algebraic over k.

Pf. Suppose $u \in L$ not algebraic.

① The set $\left\{ \frac{1}{u-c} : c \in k \right\}$ uncountable & lin ind. over k.

Indeed, any lin. combo $\frac{b_1}{u-c_1} + \dots + \frac{b_q}{u-c_q}$ $b_i, c_i \in k$ gives u as a root of a poly (clear fractions)

② Let $u_1, \dots, u_r$ gens for L as k-alg.

$\rightsquigarrow \{u_1^{m_1} u_2^{m_2} \cdots u_r^{m_r}\}$ countable and is a k-basis for L.

This contradicts ①.

Lemma is true over arbitrary fields. Need

① Zariski's Lemma: L fin gen as k-alg $\iff$ L fin gen as k-module

② Noether normalization

~~Thm~~ Lemma. k = field, K extension

If K is fin gen as a k-alg

then K is algebraic over k.

Pf of WN. Say m = max ideal in

$R = k[x_1, \ldots, x_n]$

$\Rightarrow R/m$ is a field, fin gen as k-alg.

(since R is).

Have $k \cap m = \{0\}$. (else $m = R$)

$\rightarrow$ image $\bar{k}$ of k in R/m is $\cong k$.

~~Thm~~ Lemma $\Rightarrow R/m$ alg. ext. of $\bar{k}$.

k alg closed $\Rightarrow R/m = \bar{k}$

Under $R \rightarrow R/m$

each $x_i \longmapsto \bar{a}_i \in \bar{k}$

Some $\bar{a}_i$ image of $a_i \in k$.

$\Rightarrow m \supseteq (x_1 - a_1, \ldots, x_n - a_n)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{''}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad m'$

But m' maximal

$\Rightarrow m = m'$

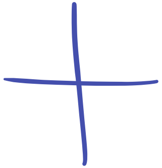
$\square$

Irreducibility

Basic example

① $Z(xy) \subseteq \mathbb{A}^2$

$\quad = Z(x) \cup Z(y)$



Say $Z(xy)$ reducible.

An nonempty aav is <u>reducible</u> if it is the union of two distinct, nonempty aav's.

The maximal* irred ~~closed subsets~~ sub aav's are the <u>irreducible components</u>.

* wrt inclusion

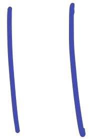
More examples

② $Z(x_1x_2, x_1x_3) \subseteq \mathbb{A}^3$

$\quad = Z(x_1) \cup Z(x_2, x_3)$

③ $Z(x^2-1) \subseteq \mathbb{A}^1$

$\quad = Z(x+1) \cup Z(x-1)$



not connected

④ A finite set in $\mathbb{A}^n$ is irred $\Longleftrightarrow$ ~~at most~~ one point

⑤ What about $\mathbb{A}^n$?

Prop. $X \subseteq \mathbb{A}^n$ aav.

X irred $\iff \mathbb{I}(X)$ prime.

Pf. ($\Leftarrow$) Say $\mathbb{I}(X)$ prime.

and $X = X_1 \cup X_2$

Then $\mathbb{I}(X) = \mathbb{I}(X_1) \cap \mathbb{I}(X_2)$

$\mathbb{I}(X)$ prime $\implies \mathbb{I}(X) = \mathbb{I}(X_1)$ WLOG ~~or $\mathbb{I}(X_2)$~~

(If $P = I \cap J$ then
$IJ \subseteq I \cap J = P \Rightarrow P = I$ or J).

(Prime $\implies$ radical) so SN $\implies$
$X = X_1$

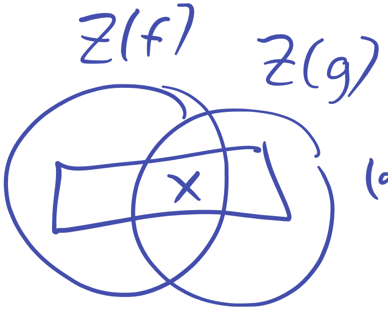
($\Rightarrow$) Say X irred & $fg \in \mathbb{I}(X)$

Then $X \subseteq Z(fg) = Z(f) \cup Z(g)$

$\implies X = (Z(f) \cap X) \cup (Z(g) \cap X)$

$\overset{\text{irred.}}{\implies} X = Z(f) \cap X$

$\implies X \subseteq Z(f) \implies f \in \mathbb{I}(X)$. $\square$



(a priori)

Consequences

① $\mathbb{A}^n$ irred since (0) prime

② $f \in k[x_1, \dots, x_n]$ irred

$\iff Z(f)$ irred.

[If $f = f_1 f_2$

$Z(f) = Z(f_1) \cup Z(f_2)$]

Dictionary

aav's	$\longleftrightarrow$	rad ideals
irred aav's	$\longleftrightarrow$	prime ideals
(in $\mathbb{A}^n$) pts	$\longleftrightarrow$	max ideals (in $k[x_1, \dots, x_n]$)

Decomposing into irreducibles

$k[x_1, \dots, x_n]$ Noetherian (Hilb. basis thm)

$\Rightarrow$ any desc. chain of aav's is... eventually stationary.

(Noetherian property for aav's)

Prop ① An aav can be written as a finite union of irred aav's

$X_1 \cup \dots \cup X_r$

② If $X_i \not\subset X_j \ \forall i \neq j$ the X_i unique.

In this case, X_i called the irred components of X.

Prop ① An aav can be written as a finite union of irred aav's

$$X_1 \cup \cdots \cup X_r$$

② If $X_i \not\subset X_j \ \forall i \neq j$ the X_i unique.

In this case, X_i called the <u>irred</u> components of X.

Pf of ① Let X be a minimal counterexample. If there is a counterex, a minimal one exists by Noetherian property.

under inclusion

Since it's a counterex, it's reducible:

$$X = X_1 \cup X_2$$

But X minimal $\Longrightarrow$ X_1, X_2 finite unions of irred's.

② Say

$$X = X_1 \cup \cdots \cup X_r$$
$$= X_1' \cup \cdots \cup X_s'$$

$$X_1 \subset \bigcup X_i'$$

In fact $X_1 \subset X_i'$ some i (otherwise X_1 reduces).

Next week: end of Chap 1

- Morphisms = polynomial maps

- Coordinate ring $k[V]$

$= \{\text{poly fns on } V\}$

$= k[x_1, \ldots, x_n] / \mathbb{I}(V)$

Morphisms

$X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aav's

$f: X \to Y$ is a <u>morphism</u>

if it's restriction of a polyn. map $\mathbb{A}^n \to \mathbb{A}^m$.

i.e. $\exists\ f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$

s.t. $f(x) = (f_1(x), \ldots, f_m(x))$

$\forall\ x \in X$.

A morphism is an isomorphism if it has an inverse morphism

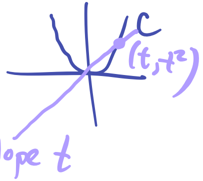
Examples

① Affine change of coords $\mathbb{A}^n \to \mathbb{A}^n$

linear map + translation.

This is $\cong \iff$ linear map is.

② $C = Z(y - x^2) \subseteq \mathbb{A}^2$



$f: \mathbb{A}^1 \to C$

$t \mapsto (t, t^2)$

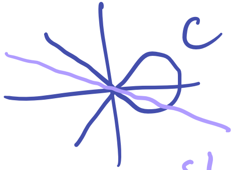
$f^{-1}: C \to \mathbb{A}^1$

$(x, y) \mapsto x$

In general, coord fns are morphisms

<u>isomorphism</u>

(3) $C = Z(x^3+y^2-x^2)$



$f: \mathbb{A}^1 \longrightarrow C$

$t \longmapsto (t^2-1,\ t(t^2-1))$

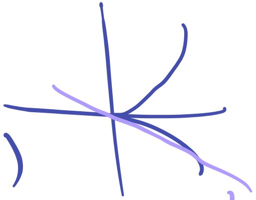
morphism, but not injective:

$f(1) = f(-1)$.

(4) $C = Z(y^2-x^3) \subseteq \mathbb{A}^2$

$f: \mathbb{A}^1 \longrightarrow C$

$t \longmapsto (t^2, t^3)$



bijective morphism,

but not $\cong$. Why? We need a new tool...

Facts about morphisms

(1) Morphisms are continuous wrt Zariski topology

$f: X \to Y$

$f^{-1}(Z(h_1, \dots, h_r))$

$= Z(h_1 \circ f, \dots, h_r \circ f)$

(2) Morphisms do not always map aav's to aav's

$Z(xy-1) \longrightarrow \mathbb{A}^1$

$(x,y) \longmapsto x$

Image is $\mathbb{A}^1 \setminus 0$

Coordinate Rings

X = aav

$\rightsquigarrow k[X] = \{ f|_X : f \in k[x_1, \dots, x_n] \}$

$= \{\text{poly fns on } X\}$

$=$ coord ring on X.

$k[X]$ is a ring, in fact a k-algebra. More:

$$k[X] = k[x_1, \dots, x_n] / \mathbb{I}(X)$$

So if $X = Z(xy-1)$

$$[1/x] \in k[X] \quad (!)$$

$$\parallel$$

$$[y]$$

First examples

① $k[\mathbb{A}^n] \cong k[x_1, \dots, x_n]$

② $k[p] \cong k$ (cf proof $(x_1 - a_1, \dots, x_n - a_n)$ maximal)

$$f \longmapsto f(p)$$

③ $k[X] \cong k^r$

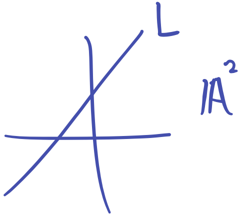
$X = p_1 \cup \dots \cup p_r$

$$f \longmapsto (f(p_1), \dots, f(p_r))$$

More examples

④ $L = Z(y - mx - b)$

$k[L] \cong k[x]$



First, any poly in x, y is equiv to a poly in x

$y \sim mx + b$

$$k[L] \longrightarrow k[x]$$

$$\boxed{xy + y \longmapsto x(mx+b) + mx + b}$$

$$[f(x,y)] \longrightarrow f(x, mx+b)$$

$$k[x] \longrightarrow k[L]$$

$$f(x) \longmapsto [f(x)]$$

These are inverses.

⑤ $C = Z(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$ cone

$k = \mathbb{C}$.

exercise: in $k[C]$

$$(x^3 + 2xy^2 - 2xz^2 + x) \sim (x - x^3)$$

Irreducibility & Coord rings

Prop. X irred $\iff k[X]$ integral domain

Pf. X irred $\iff \mathbb{I}(X)$ prime

$\iff k[x_1,\dots,x_n]/\mathbb{I}(X)$ integral domain

Fact. $k[X]$ gen. by coord fns,

$$X \to k$$
$$(a_1,\dots,a_n) \mapsto a_i$$

hence the name (?)

⑥ Twisted cubic

$$C = Z(y-x^2,\ z-x^3)$$

Will show C is irred.

Pf #1 $(y-x^2, z-x^3)$ prime.

Use $k[x,y,z] \cong (k[x,y])[z]$

$k[x,y] \cong (k[x])[y]$

Suppose $fg \in (y-x^2, z-x^3)$.

Division alg:

$f(x,y,z) = (z-x^3)f_1(x,y,z) +$ remainder (const in z)

$(z-x^3)f_1(x,y,z) + (y-x^2)f_2(x,y) + f_3(x)$

similar $g(x,y,z) = (z-x^3)g_1(x,y,z) - (y-x^2)g_2 + g_3(x)$

Since $fg \in (y-x^2, z-x^3) \Rightarrow f_3 = 0$ or $g_3 = 0$

Indeed if $f_3(x)$ & $g_3(x)$ both nonzero, can find a point in $\mathbb{A}^3$ where $y-x^2$, $z-x^3$ vanish but fg does not. $\blacksquare$

Alternately show:

$$k[x,y,z]\big/(y-x^2, z-x^3) \longrightarrow k[x]$$

Pf #2 $f: \mathbb{A}^1 \longrightarrow C$ is a surj. morphism...

$f \quad t \longmapsto (t, t^2, t^3)$

If $C = C_1 \cup C_2$ then $\mathbb{A}^1 = f^{-1}(C_1) \cup f^{-1}(C_2)$

$\implies \mathbb{A}^1$ irred CONTRAD.

Back to the dictionary

sub-aav's of $X \longleftrightarrow$ rad. ideals in $k[X]$

$Y \subseteq X \longmapsto k[Y] \subseteq k[X]$

irred $\longleftrightarrow$ prime

pts $\longleftrightarrow$ max ideals.

Pf. $3^{rd} \cong$ thm + prev. dictionary

Next time Every fin gen., reduced k-alg is some $k[X]$.

Last time

① Morphisms

$X \longrightarrow Y$

polynomial map

② Coordinate rings

$k[X]$: poly fns on X

$= \{ f|_X : f \in k[x_1,\dots,x_n] \}$

$= k[x_1,\dots,x_n] / \mathbb{I}(X)$

Wanted to show:

$\mathbb{A}^1 \longrightarrow Z(y^2 - x^3)$ not $\cong$.

Next: A morphism $X \to Y$ gives hom. $k[Y] \longrightarrow k[X]$

Pullbacks. $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aav's

$f : X \longrightarrow Y$ morphism.

$\rightsquigarrow f_* : k[Y] \longrightarrow k[X]$

$g + \mathbb{I}(Y) \longmapsto g \circ f + \mathbb{I}(X)$

or $[g] \longmapsto [g \circ f]$

Basic facts ① f_* is k-alg homom.

② $(fg)_* = g_* f_*$

③ f an $\cong \implies f_*$ an $\cong$

Basic facts ① f_* is k-alg homom.

② $(fg)_* = g_* f_*$

③ f an $\cong \Rightarrow f_*$ an $\cong$

[In other words, have contravariant functor

$$\text{aav's} \longrightarrow k\text{-algebras}$$
$$X \longmapsto k[X]$$
]

What is the image?

Examples

① $\mathbb{A}^1 \xrightarrow{\cong} Z(y-x^2) \subseteq \mathbb{A}^2$

$t \longmapsto (t, t^2)$

Pullback:

$f: \mathbb{C}[x,y]/(y-x^2) \longrightarrow \mathbb{C}[t]$

$g_1(x,y) = x \longmapsto t$

$g_2(x,y)\ y \longmapsto t^2$

This enough since x, y generate.

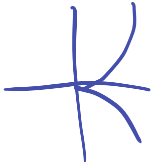
surjective ✓

injective ✓

So $\cong$

② $f: \mathbb{A}^1 \to Z(y^2 - x^3) \subseteq \mathbb{A}^2$

$t \longmapsto (t^2, t^3)$



Pullback:

$$\mathbb{C}[x,y]/(y^2-x^3) \longrightarrow \mathbb{C}[t]$$

$$x \longmapsto t^2$$

$$y \longmapsto t^3$$

Not surj: t not in image

so f is not an $\cong$.

Would be better:

$$\mathbb{C}[x,y]/(y^2-x^3) \not\cong \mathbb{C}[t]$$

Joshua's idea: compare transcendence degree over $\mathbb{C}[x]$.

(Can a transc. ext. of $\mathbb{C}[x]$ be $\cong$ to $\mathbb{C}[x]$?)

More refined: LHS free module over $\mathbb{C}[t]$ of rank 2

and RHS rank 1

③ $X = Z(xy-1)$

Will show $X \not\cong \mathbb{A}^1$

$k[\mathbb{A}^1] = k[x]$

$k[X] = k[x][x^{-1}]$

$= k[x, x^{-1}]$

$=$ Laurent polys

Want $k[x] \not\cong k[x, x^{-1}]$.

Suppose $\Phi : k[x, x^{-1}] \to k[x]$

$\Rightarrow \Phi(1) = 1$

$\Rightarrow \Phi(x)\Phi(x^{-1}) = 1$

$\Rightarrow \Phi(x), \Phi(x^{-1})$ units

$\Rightarrow \operatorname{Im}\Phi \subseteq \{\text{constant polys}\}$

$\blacksquare$

Next: Which alg's arise?

Defn. An alg is <u>reduced</u> if no nilpotent elts, i.e. no elts $r \neq 0$ with $r^k = 0$.

Thm

(1a) Every $k[X]$ is a fin gen. reduced k-alg. ✓

(1b) Every fin gen red. k-alg is a $k[X]$

(2a) $f: X \to Y$ morphism ✓
$\Longrightarrow f_*: k[Y] \to k[X]$ k-alg homom.

(2b) Every k-alg homom $R \to S$ of red fin gen k-alg is some f_* & f unique up to $\cong$.

So. The two categories are same (contravariant isomorphism):

aav's in $\mathbb{A}^n$ over all n $/\sim$ $\longleftrightarrow$ f.g. red k-algs. $/\sim$

Note. In 1950's Grothendieck removed 3 hypotheses: fin gen, red, alg closed
The corresp. geom objects are affine schemes

(1b) R = fg red k-alg

Choose a "presentation"

$$R \cong k[y_1, \dots, y_m] / J$$

$$\left(\begin{array}{l} y_i \text{ generators} \\ J \text{ relations.} \\ J = \ker k[y_1, \dots, y_m] \to R \end{array} \right)$$

R reduced $\Longrightarrow$ J radical.

Let $Y = Z(J) \subseteq \mathbb{A}^m$

SN $\Longrightarrow$ $k[Y] \cong R$. $\square$

(2b) See 2019 notes.

New varieties from old

① Products

Prop. The direct product of aav's is an aav.

② Complements of aav's

$V \subseteq \mathbb{A}^n$ aav.

$f \in k[V]$

$\rightsquigarrow V_f = V \setminus Z(f)$

$= \{p \in V : f(p) \neq 0\}$

Examples
- $GL_n k$
- $Poly_n = \{$polys of deg n with distinct roots$\}$

Prop. Any V_f is isomorphic to an affine variety with coord ring

$$k[V_f] \cong k[V][f^{-1}] = k[V]_f$$

"localization"

rat'l functions

$$\frac{\text{poly}}{f^k}$$

Prop. Any V_f is isomorphic to an affine variety with coord ring

$$k[V_f] \cong k[V][f^{-1}] = k[V]_f$$

Pf. Trick of Rabinowitz!

Let $J = \mathbb{I}(V) \subseteq k[x_1,\dots,x_n]$

$F \in k[x_1,\dots,x_n] \quad F \in [f]$

Set $J_F = (J, tF-1) \subseteq k[x_1,\dots,x_n,t]$

We'll show $V_f \cong W := Z(J_F) \subseteq \mathbb{A}^{n+1}$

$$W \rightleftarrows V_f$$

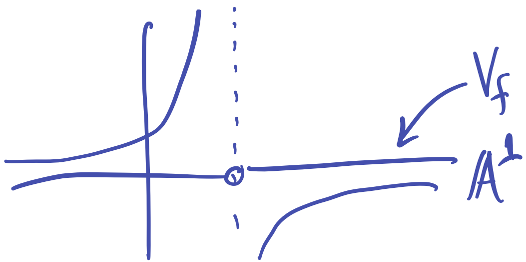
$$(x_1,\dots,x_n,y) \longmapsto (x_1,\dots,x_n)$$

$$\left(x_1,\dots,x_n,\frac{1}{F(x_1,\dots,x_n)}\right) \longleftarrow (x_1,\dots,x_n)$$

Check inverses, check second statement

Example $V = \mathbb{A}^1$

$f = x-1$



Chap 2 Projective varieties.

Proj space

$\mathbb{P}^n$: compactification of $\mathbb{A}^n$
w/ one infinitely distant
pt in each direction.

$\rightsquigarrow$ compactification of aav's

Precisely:

$$\mathbb{P}^n = (k^{n+1} - 0)/k^*$$
$$= (k^{n+1} - 0) \,/\, \text{nonzero scaling}$$
$$= \text{space of lines thru } 0$$

So $x \sim y \iff x = \lambda y \quad \lambda \in k^*$

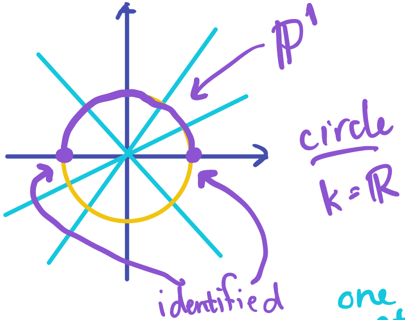
Write

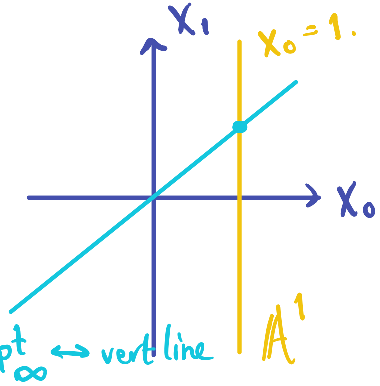
$[(x_0, \dots, x_n)]$ as $[x_0 : \dots : x_n]$

"homog. coords"

<u>$n=1$</u> pictures over $\mathbb{R}$

Two pics





algebraically:

$[x_0 : x_1]$

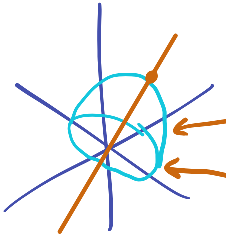
$$\mathbb{P}^1 = \{[1 : x_1]\} \cup \{[0 : 1]\}$$

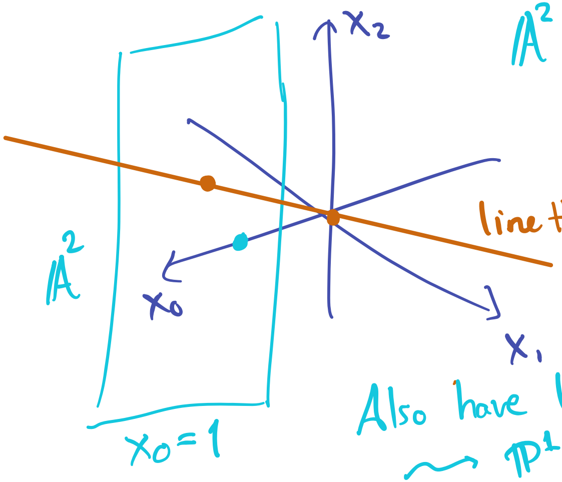
$$= \mathbb{A}^1 \cup \mathbb{A}^0 = pt$$

For $k = \mathbb{C}$, $\mathbb{P}^1_{\mathbb{C}}$ = Riemann sphere.

$$= \mathbb{C} \cup \{\infty\}$$

<u>$n=2$</u>





$\mathbb{A}^2 \cup \mathbb{P}^1$

Also have lines in x_1x_2-plane $\rightsquigarrow \mathbb{P}^1$

algebraically:

$$\mathbb{P}^2 = \underbrace{\{[1 : x_1 : x_2]\}}_{\mathbb{A}^2} \cup \underbrace{\{[0 : x_1 : x_2]}_{\mathbb{P}^1} \quad x_1, x_2 \text{ not both } 0\}$$

$$= \{[1 : x_1 : x_2]\} \cup \{[0 : 1 : x_2]\} \cup \{[0 : 0 : 1]\}$$

In general:

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$
$$= \mathbb{A}^n \cup \cdots \cup \mathbb{A}^0$$

This decomp. is not canonical.

Let $U_j = \{[x_0 : \cdots : x_n] : x_j \neq 0\}$

$$\rightsquigarrow \mathbb{P}^n = \underset{\mathbb{A}^n}{\underset{\shortparallel}{U_j}} \cup \underset{\mathbb{P}^{n-1}}{\underset{\shortparallel}{H_j}}$$

The U_j form the standard affine cover of $\mathbb{P}^n$.

For $k = \mathbb{C}$ the U_j give $\mathbb{P}^n$ structure of a $\mathbb{C}$ n-manifold.

Projective subspaces

Images in $\mathbb{P}^n$ of linear subspaces of k^{n+1}.

So a line in $\mathbb{P}^n$ is image of plane in k^{n+1}.

Through any two pts in $\mathbb{P}^n$ $\exists!$ line

Fact. Any two lines in $\mathbb{P}^2$ intersect.

Pf. Any two planes in k^3 intersect.

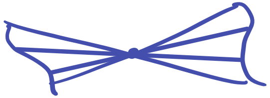
Projective varieties

$f \in k[x_0,\dots,x_n]$ is homog. if all terms have same degree.

Fact. The 0-set of f is well def. in $\mathbb{P}^n$.

Pf. $\lambda^d f(x) = f(\lambda x) = 0$
$\iff f(x) = 0$

Note: $Z(f)$ in $\mathbb{A}^{n+1}$ is a cone



A proj alg var in $\mathbb{P}^n$ is common 0-set of $f_1,\dots,f_r \in k[x_0,\dots,x_n]$ homog.

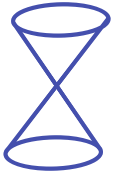
Examples

0. $Z(0) = \mathbb{P}^n$
   $Z(1) = \emptyset$.
1. $Z(x_0,\dots,x_n) = \emptyset$.
   $(x_0,\dots,x_n) = \{\text{polys w/ no const term}\}$
   "irrelevant ideal"
2. $Z(x_1 - a_1 x_0, \dots, x_n - a_n x_0) = [1 : a_1 : \cdots : a_n]$
3. $Z(x_0) =$ "hyperplane at ∞"
   $\cong \mathbb{P}^{n-1}$

④ Conics

$X = Z(f)$

e.g. $f = x^2 + y^2 - z^2$



$\exists$ std affine charts

$\rightsquigarrow$ circle, hyperbola,
hyperbola

⑤ Image of

$\varphi : \mathbb{P}^1 \to \mathbb{P}^3$

$\varphi([t_0 : t_1]) =$

$[t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3]$

This is a det. variety

$$\operatorname{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1$$

$\rightsquigarrow$ intersection of 3 quadrics.

" proj. rat'l normal curve of deg 3"

exercise: Im φ is the whole variety

Tayesh: 2nd, 3rd cols are multiples of 1st.

⑥ $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$$([x_0 : x_1], [y_0 : y_1]) \longmapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1]$$

$$\operatorname{Im} \varphi = Z(z_0 z_3 - z_1 z_2)$$

"quadric"

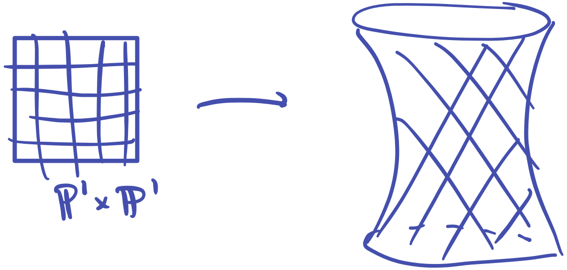


Image of "lines" on left are lines in quadric

e.g. $\varphi(\mathbb{P}^1 \times [1:0])$

$= Z(z_1, z_3)$.

Q. Do other lines in $\mathbb{P}^1 \times \mathbb{P}^1$ map to lines? (Tong)

Future

⑤ Grassmannians

$Gr_{r,n} = \{r\text{-dim planes in } k^n\}$

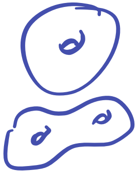
later!

⑥ Products of proj. alg. vars.

later!

⑦ Compact Riemann surfaces

later?



⑧ Moduli space

Homogenization

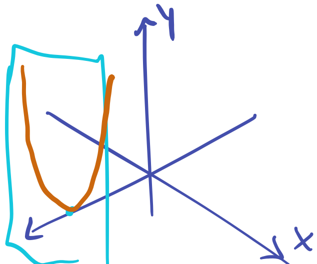
$f \in k[x_1, \ldots, x_n]$

$\rightsquigarrow h \in k[x_0, \ldots, x_n]$ homog.

Just add x_0 as needed.

Example $f(x,y) = y - x^2$

$\rightsquigarrow h(x,y,z) = yz - x^2$



So get the old parabola <u>plus</u> $[0:1:0]$

$\rightsquigarrow$ parabola + pt $\approx$ circle.

Example ② is also a homogenization.

Upshot: Any affine variety can be projectivized

$\rightsquigarrow$ compactness, more symmetry.

(Some of) HW assignment.

$$\begin{tikzcd} \underset{\text{roots}}{\mathbb{C}^n} \arrow[r, "\psi"] \arrow[d] & \underset{\text{coeff.}}{\mathbb{C}^n} \\ \mathbb{C}^n/\underbrace{\Sigma_n}_{\text{symm gp}} \arrow[ur, "\overline{\psi}"'] & \end{tikzcd}$$

map given by elem sym polys

$$(r_1, \ldots, r_n) \longmapsto \left(\sum_i r_i, \sum_{i \neq j} r_i r_j, \ldots, r_1 \cdots r_n\right)$$

$\overline{\psi}$ ← natural one.

$\cong$ of varieties.

Newton: these generate the invariants.

Surjective: FTA

Not injective: permuting roots.

HW #1. Show X/G is a.a.v. $G \circlearrowright X$ = a.a.v.

via $k[X/G] = k[X]^G$ "invariants"

#2. Show $\overline{\psi}$ is an $\cong$.

Projective closure

$X \subseteq \mathbb{A}^n$ aav.
$\subseteq \mathbb{P}^n$

The proj. closure

$\bar{X}$ is the closure of X in $\mathbb{P}^n$ in Zariski topology

Fact? Same as Eucl. closure.

Write $Z_a, \mathbb{I}_a$
$Z_p, \mathbb{I}_p$
to emphasize affine/proj

Closure: smallest closed set containing...

So proj closure: smallest proj. var containing....

(or largest homog. ideal...)

Easy: Euc. closure. $\subseteq$ proj. closure

So: If Euc closure is a pav, it is the proj clos.

Other dir of fact?

Fact. If $X = Z_a(I)$ then

$\bar{X} = Z_p(I_h)$

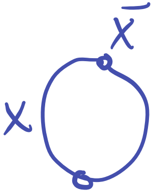
I_h = ideal gen by homog's of all elts of I.

Example.

$X_1 = Z(x_2 - x_1^2)$ $\quad$ $X_2 = Z(x_1 x_2 - 1)$

$$\left.\begin{aligned} \rightsquigarrow \bar{X}_1 &= Z(x_0 x_2 - x_1^2) \\ \bar{X}_2 &= Z(x_1 x_2 - x_0^2) \end{aligned}\right\} \text{same!}$$

"Extra" points: Take $x_0 = 0$.



In $\bar{X}_1$: $[0:0:1]$

In $\bar{X}_2$: $[0:0:1]$ & $[0:1:0]$

Not a coincidence: $\exists!$ conic in $\mathbb{P}^2$.

Why is $\bar{X}_i$ actually the proj. closure.

$\bar{X}_i$ is a proj var containg X_i

& $\bar{X}_i \setminus X_i$ finite.

more X_i dense in $\bar{X}_i$

or $\bar{X}_i$ = Euclidean closure of X_i.

note $[1:x:x^2] \longrightarrow [0:0:1]$

as $|x| \longrightarrow \infty$.

Fact. If $X = Z_a(f)$ then $\bar{X} = Z_p(f_h)$

homog.

But If $X = Z_a(f, g)$

$\bar{X}$ might not be $Z_p(f_h, g_h)$

Similar: $Z(x, y - x^2)$

example/exercise: $Z(y - x^2, z - xy)$

$\bar{X} \neq Z(wy - x^2, wz - xy) = \bar{X} \cup \{w = y = 0\}$

Homog. Ideals

Any $f \in k[x_1, \dots, x_n]$ is a sum of homog. terms

$$f = f^{(0)} + \dots + f^{(m)}$$

and

"graded ring"

$$k[x_1,\dots,x_n] = \bigoplus_{d \geq 0} \underbrace{k[x_1,\dots,x_n]_{(d)}}_{\text{homog deg } d \text{ polys (union } 0)}$$

Lemma. Let $I \subseteq k[x_1,\dots,x_n]$.

TFAE ① I gen by homog. elts

② $f \in I \Rightarrow f^{(d)} \in I \ \forall d$.

Such I called homog.

Pf ② $\Rightarrow$ ① $I = (f_1, \dots, f_r)$ (Hilbert BT)

Write $f_i = \sum f_i^{(d)} \rightsquigarrow I = (f_i^{(d)})$.

① $\Rightarrow$ ② $I = (f_1, \dots, f_r)$ each f_i homog.

($r < \infty$ since Noetherian)

$f \in I \Rightarrow f = \sum_i a_i f_i \quad a_i \in k[x_1,\dots,x_n]$

$$\Rightarrow f^{(d)} = \sum_i a_i^{(d-\deg f_i)} f_i \in I \quad \square$$

Note. Not all elts of homog ideals are homog.

Note. A par can always be written as $Z(f_1, \dots, f_k)$ with $\deg f_i$ all same. (mult. each f_i with non-max deg by power of x_0). FIX

Fact. ① I homog $\Rightarrow$ rad I homog.

② Intersection, sum, product of homog. ideals is homog.

③ I homog then:

I prime $\Leftrightarrow$ $\forall$ homog f, g have $(fg \in I \Leftrightarrow f$ or $g \in I)$

(If I homog, can test primeness only with homog elts)

Consequence: Zariski top. works for pav's.

Proj Nullstellensatz

Thm. k alg closed

$I \subseteq k[x_0,\dots,x_n]$ homog.

① $Z_p(I) = \emptyset \iff (x_0,\dots,x_n) \subseteq \operatorname{rad} I$

② $Z_p(I) \neq \emptyset \implies I_p(Z_p(I)) = \operatorname{rad} I$.

So:

$$\left\{ \begin{matrix} \text{pav's} \\ \text{in } \mathbb{P}^n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{rad. homog. ideals} \\ \text{in } k[x_0,\dots,x_n] \end{matrix} \right\} \Big\backslash \left\{ \begin{matrix} \text{irrelev.} \\ \text{ideal} \end{matrix} \right\}$$

Pf of Thm

① $Z_p(I) = \emptyset \iff Z_a(I) \subseteq \{0\}$.

$\iff \operatorname{rad} I = I_a Z_a(I)$

$\supseteq (x_0,\dots,x_n)$ (affine SN)

② Assume $Z_p(I) \neq \emptyset$.

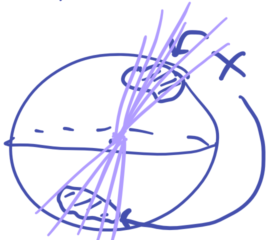
$f \in I_p(X) \iff f \in I_a C(X)$

$= I_a Z_a(I) = \operatorname{rad} I$

(affine SN)

$\square$

Pf uses cones:



$X \subseteq \mathbb{P}^n$

For $X \subseteq \mathbb{P}^n$

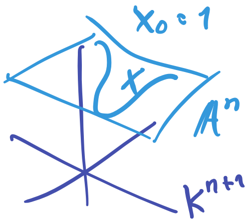
cone $C(X)$ is corresp union of lines in k^{n+1}.

Proj closure

Thm $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ aav

$I = \mathbb{I}_a(X)$

$\Longrightarrow \bar{X} = Z_p(I_h) \subseteq \mathbb{P}^n$



Pf. $\boxed{\subseteq}$ you

$\boxed{\supseteq}$ Say $G \in \mathbb{I}_p(\bar{X})$

$G \in k[x_0, \dots, x_n]$ homog.

$\Longrightarrow G = 0$ on $(\bar{X} \cap U_0) = X$

$x_0 \neq 0$.

$\Longrightarrow g = G|_{x_0 = 1}$ is 0 on X

$g \in k[x_1, \dots, x_n]$

$\Longrightarrow g \in \mathbb{I}_a(X) = I$

$\Longrightarrow g_h \in I_h$

$\Longrightarrow G = g_h x_0^t$ some t.

$$
\left[\begin{aligned}
G &= x_0^3 x_1 + x_0^2 x_1 x_2 + x_0^4 \\
g &= x_1 + x_1 x_2 + 1 \qquad x_0^2 \\
g_h &= x_0 x_1 + x_1 x_2 + x_0^2
\end{aligned}\right]
$$

$\Longrightarrow G \in I_h$ (since $g_h \in I_h$).

Thus $\mathbb{I}_p(\bar{X}) \subseteq I_h$ since $\bar{X}$ closed.

So $Z_p(I_h) \subseteq Z_p \mathbb{I}_p(\bar{X}) = \bar{X}$ $\square$

Example

$$X = Z(x, y - x^2) = \{0\} \leftrightarrow \begin{array}{c} z \;\; x \;\; y \\ [1:0:0] \end{array} \text{ in } \mathbb{P}^2$$

$$\rightsquigarrow \bar{X} = X \leftrightarrow U_z$$

$$\neq Z(x, yz - x^2) = \begin{array}{c} z \;\; x \;\; y \\ \{[1:0:0], \end{array}$$

$$[0:0:1]\}$$

at ∞.

Cor. $X = Z(f) \Longrightarrow \bar{X} = Z_p(f_h)$

Pf. $(f) = \{fg : g \in k[x_1, \dots, x_n]\}$ $\quad f_h$

$$\rightarrow \bar{X} \overset{\text{Thm}}{=} Z_p((fg)_h : g \in k[x_1, \dots, x_n])$$

$$= Z_p(f_h g_h : g \in k[x_1, \dots, x_n])$$

$$= Z_p(f_h) \qquad \square$$

Cor of Proj Null

$$\left\{ \begin{array}{l} \text{irred. proj vars} \\ Y \subseteq \mathbb{P}^n \\ Y \not\subseteq Z(x_0) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{irred affine} \\ \text{vars} \\ X \subseteq \mathbb{A}^n \end{array} \right\}$$

$$\bar{X} \longleftarrow X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$$

$$Y \longmapsto Y \cap U_0 \cong \mathbb{A}^n$$

Why you need irreducible: (Tuyesh)

$$x_0 x_2 - x_1^2$$

$$x_0^2 x_2 - x_1^2 x_0$$

Pf hint: polys $\longleftrightarrow$ polys.
(homog & dehomog).

Morphisms

Naive defn: polyn. maps.

Example $C = Z(xz - y^2)$

$$\varphi : \mathbb{P}^1 \longrightarrow C \subseteq \mathbb{P}^2$$
$$[s:t] \longmapsto [s^2 : st : t^2]$$

- φ is well def
- $\operatorname{im} \varphi = C$.

(This is a Veronese map)

In U_t chart, set $u = s/t$

$$u \longmapsto (u^2, u) \in U_z$$

In U_s: $v \longmapsto (v, v^2) \in U_x$.

These are affine morphisms.

Now for other direction...

$$\psi : C \longrightarrow \mathbb{P}^1$$
$$[x:y:z] \longmapsto \begin{cases} [x:y] & \text{on } U_x \\ [y:z] & \text{on } U_z \end{cases}$$

Defined on all of C: $x = z = 0 \Rightarrow y = 0$.

Well def. on C: $x, z \neq 0 \Rightarrow y \neq 0$ so

$$[x:y] = [yx : y^2] = [xy : xz] = [y:z]$$

On U_x, U_z: ψ is affine morphism.

but ψ is not globally polynomial.

No way to write ψ as $[f_1 : f_2]$

(exercise?)

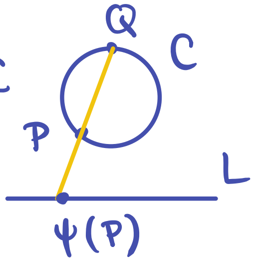
Aside: Stereographic proj.

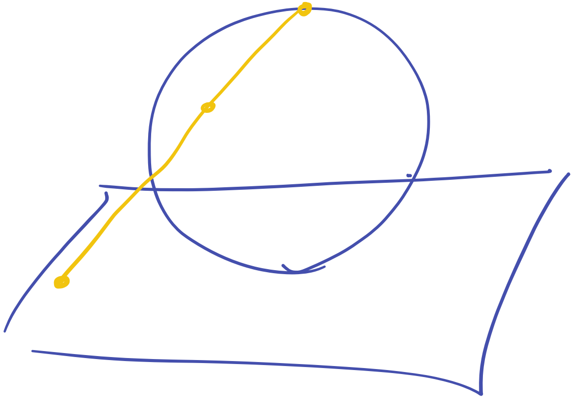
The map ψ can be defined as follows.

Let $Q = [1:0:0] \in C$ (pt at ∞)

$\begin{array}{ccc} x & y & z \end{array}$

$L = Z(x)$ line in $\mathbb{P}^2$





For $P = [a:b:c] \in C$, $P \neq Q$

The line PQ is $yc = zb$

and $PQ \cap L = \psi(P) = [0:b:c]$

We want (need?) this to a morphism, but not a poly.

From last time

Example $C = Z(xz - y^2)$

$\varphi : \mathbb{P}^1 \longrightarrow C \subseteq \mathbb{P}^2$

$[s:t] \longmapsto [s^2 : st : t^2]$

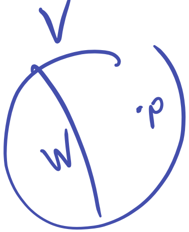
$\psi : C \longrightarrow \mathbb{P}^1$

$[x:y:z] \longmapsto \begin{cases} [x:y] & \text{on } U_x \\ [y:z] & \text{on } U_z \end{cases}$

Today: Morphisms, birational maps

Correspondence

PAV's $\longleftrightarrow$ fg extensions of k.



Morphisms of PAVs

$V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ pav's

$f : V \to W$ is a morphism if

$\forall\, p \in V$ $\exists$ homog polys

$f_0, \dots, f_m \in k[x_0, \dots, x_n]$ s.t. for

some ~~nonempty~~ open nbd of p

$f|_U$ agrees with

$U \longrightarrow \mathbb{P}^m$

$q \longmapsto [f_0(q) : \dots : f_m(q)]$

e.g. φ, ψ above.

Morphisms of PAVs

$V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ pav's

$f: V \to W$ is a <u>morphism</u> if

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$f|_U$ agrees with

$$U \longrightarrow \mathbb{P}^m$$
$$q \longmapsto [f_0(q) : \cdots : f_m(q)]$$

e.g. φ, ψ above.

Notes

1. Can also allow rat'l fns (clear denoms).
2. To have a well-def map, f_i must have same deg.
3. Also, f_i must not all vanish at a single pt.
4. Implicit: different fns agree on overlaps (since $\exists$ globally def f)

Isomorphism: if $\exists$ inverse morphism.

Examples

① φ, ψ above

$\mathbb{P}^1 \to C$ = parabola

are isomorphisms

Assume in U_x

e.g. $[s:t] \xmapsto{\varphi} [s^2 : st : t^2]$

$\xmapsto{\psi|_{U_x}} [s^2 : st] = [s:t]$

② Any homog rat'l fn $h : X \to k$

can be considered a morphism

$\varphi : X \longrightarrow \mathbb{P}^1$. If $h = f/g$ f, g homog same deg

$\varphi([x_0 : \dots : x_n]) = [f(x_0, \dots, x_n) : g(x_0, \dots, x_n)]$

③ Linear change of coords on $\mathbb{P}^n$.

Later: All isoms $\mathbb{P}^n \to \mathbb{P}^n$ are of this form.

Conseq 1. H = hyperplane in $\mathbb{P}^n$

$\Longrightarrow H \cong \mathbb{P}^{n-1}$

(using: restriction of $\cong$ is $\cong$)

Conseq 2. All conics in $\mathbb{P}^2$ are $\cong$. $(k = \mathbb{C})$

Conics $\longleftrightarrow$ symm bilin forms $\longleftrightarrow$ quad forms $\longleftrightarrow$ 3×3 matrices

$Z(x^2 + 4xy + 3y^2 + z^2)$ $(x\ y\ z)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

But: All symm. ~~inv~~ $\mathbb{C}$-matrices diagable...

Coord ring of PAVs

Can define:

$k[X] = k[x_0, \dots, x_n] / \mathbb{I}_p(X)$

Issue #1. The elts of $k[X]$ don't give well-def fns on X.

In Fact: Every ~~poly~~ rat fn on smooth X defined globally is const.

For $k = \mathbb{C}$ * this is Liouville's thm (bounded holom fns are const) plus fact that X compact.

* and X smooth.

Issue #2 Can have $X \cong Y$

$k[X] \not\cong k[Y]$

e.g. $\mathbb{P}^1 \cong X = Z_p(x) \subseteq \mathbb{P}^2 \rightsquigarrow k[X] \cong k[y,z]$ UFD

$Y = Z_p(x^2+y^2-z^2) \cong \mathbb{P}^1$

$\rightsquigarrow k[Y] \cong k[x,y,z]/(x^2+y^2-z^2)$

not UFD.

$z \cdot z = (x+iy)(x-iy)$

Fixes: ① $k(V)$

② rational maps

Rational maps

$X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$

A rational map

$$\varphi : X \dashrightarrow Y$$

is an eq class of expressions

$[f_0 : \cdots : f_m]$ s.t.

① $f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ homog. of same deg.

② $[f_0(p) : \cdots : f_m(p)] \neq [0 : \cdots : 0]$ some $p \in X$

③ $\forall\, p \in X$ if $[f_0(p) : \cdots : f_m(p)]$ is defined, it is in Y.

Two expressions are equiv. if they are equal where both defined.

example: φ, ψ from start of class:

$$[x:y:z] \longmapsto \begin{cases} [x:y] & \text{on } U_x \\ [y:z] & \text{on } U_z \end{cases}$$

Q. Why transitive?

Say φ is <u>regular</u> at p if $\varphi(p)$ defined for some expression representing φ.

So φ is <u>not</u> defined at non-reg. pts.

Rat. maps are like morphisms, but only def on open subset of X.

Example Cremona transformation.

$\varphi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$

$[x:y:z] \longmapsto [yz : xz : xy]$

Not def. at $[0:0:1]$ or any pt with two zeros.

In other words: $Z(x,y) \cup Z(y,z) \cup Z(x,z)$

Problem with rat'l maps:

can't nec. compose $f \circ g$

if $g(\operatorname{dom} g) \cap \operatorname{dom} f = \emptyset$.

Dominant maps

$\varphi : X \dashrightarrow Y$ is **dominant**

if $\varphi(\operatorname{dom} \varphi)$ nonempty open in Y

If φ dominant, can compose $\psi \circ \varphi$

Example Cremona map is dominant

and $\varphi \circ \varphi \approx \mathrm{id}$.

equal where both def.

$[x:y:z] \overset{\varphi}{\longmapsto} [yz : xz : xy]$

$\overset{\varphi}{\longmapsto} [x^2yz : xy^2z : xyz^2]$

$= [x:y:z]$ if $x,y,z \neq 0$.

Field of rat'l fns

$$k(X) = \left\{ f/g \;:\; \begin{array}{l} f,g \in k[x_0,\dots,x_n] \\ \text{homog of same deg} \\ g \notin \mathbb{I}_p(X) \end{array} \right\} \Big/ \sim$$

$f_1/g_1 \sim f_2/g_2$ if $f_1 g_2 - g_1 f_2 \in \mathbb{I}_p(X)$

$\rightsquigarrow$ Well-def fns on X (open subset of X) $g \neq 0$.

Thm. ① A rat'l map $\varphi: X \dashrightarrow Y$ is birational (has rat'l inverse) $\iff$ φ dominant & φ^* is $\cong$

② X, Y birat. equiv $\iff$ $k(X) \cong k(Y)$

So: equiv of categories

$$\left\{ \text{irred. } \overbrace{\text{quasi-proj}}^{\text{open subset of var}} \text{ vars} \right\} \longleftrightarrow \left\{ \text{f.g. field exts of } k \right\}$$

w/ birat. maps $\qquad$ w/ k-homoms.

From last time ...

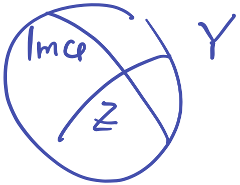
A pos. criterion for dominance

<u>Lemma</u>. $\varphi : X \dashrightarrow Y$ rat'l map b/w proj vars. Y irred. If $\exists$ $Z \subseteq Y$ par s.t. $\operatorname{im}\varphi$ contains $Y \setminus Z$ then φ is dom.

Defn of dominant: $\operatorname{im}\varphi$ not contained in proper subvar (assuming Y irred). $\Longleftrightarrow$ dense. in Z subsp. top. $\overset{??}{\Longleftrightarrow}$ image open (from last time)

<u>Pf of Lemma</u>. Follow your nose.

Contradict irreducibility.



Chap 3 Classical constructions

(Veronese, Segre, Grassmannian).

Veronese Maps

$k[x_0, \dots, x_n]_{(d)} = \left\{ \begin{array}{l} \text{deg } d \text{ homog} \\ \text{polys in } x_0, \dots, x_n \end{array} \right\}$

$\cong$ k-vect sp on the $\underline{\binom{d+n}{n}}$

monomials of deg d.

$$\underbrace{d}_{\#\text{balls}} + \underbrace{n}_{\#\text{vars} - 1} \text{ "slots"}$$

• • • | • • • • | •

choose n slots to put "bars"

$x_0^3 x_1^4 x_2^1$

The d-th Veronese map is

$$v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^m \quad m = \binom{d+n}{d} - 1$$

$$[x_0 : \dots : x_n] \longmapsto [x_0^d : \dots\dots]$$

all deg d monomials in $x_0, \dots, x_n$.

- v_d is well def ✓ (all deg d, don't all vanish)
- v_d is injective.
  look at $x_0^{d-1} x_i$ coords. where $x_0 \neq 0$.
  $[x_0^d : x_0^{d-1} x_1 : x_0^{d-1} x_2 : \dots$
  $\sim [x_0 : x_1 : x_2 \dots$

Examples

① $n=1, d=2$

$$\nu_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

$$[s:t] \longmapsto [s^2 : st : t^2]$$

$\mathbb{V}_{1,2} = \operatorname{im} \nu_2 = Z(xz - y^2)$

& ν_2 is $\cong$ onto image.

② $n=1, d=3$

$$\nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^{\binom{4}{1}-1} = \mathbb{P}^3$$

$$[s:t] \longmapsto [s^3 : s^2t : st^2 : t^3]$$

$\mathbb{V}_{1,3} = \operatorname{im} \nu_3$ = rat'l normal curve of deg 3

= proj clos. of twisted cubic:

$W = Z(xw - yz, y^2 - xz, wy - z^2)$

Easy: $\operatorname{im} \nu_3 \subseteq W$

Hard: $W \subseteq \operatorname{im} \nu_3$ (Arrondo)

Chris: maybe direct proof? Proj vars.

③ $n=1, d$

$\operatorname{im} \nu_d$ = rat'l norm. curve of deg d

= vanish. set of 2x2 dets

$$\begin{pmatrix} z_{0,d} & z_{1,d-1} & \cdots & z_{d-1,1} \\ z_{1,d-1} & z_{2,d-2} & \cdots & z_{d,0} \end{pmatrix}$$

$$z_{ij} \leftrightarrow s^i t^j \quad i+j=d.$$

Check: $z_{i,j} \cdot z_{k,\ell} = z_{i+k, j+\ell}$

④ Veronese surface

$$\nu_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$$

$$[s:t:u] \longmapsto [s^2 : t^2 : u^2 : st : su : tu]$$

$\operatorname{Im} \nu_2$ is van set for 2×2 minors

of $\begin{pmatrix} z_0 & z_3 & z_4 \\ z_3 & z_1 & z_5 \\ z_4 & z_5 & z_2 \end{pmatrix}$ (rank 1 condition)

For general deg 2:

$\operatorname{Im} \nu_2$ = van set for 2×2 minors

of $\underbrace{(z_{i,j})}_{\text{symmetric}}$ $\underbrace{z_{i,j}}_{\cancel{i \le j}} \longleftrightarrow x_{i-1} x_{j-1}$

Image of Veronese

$s^2 : t^2 : st \qquad s^2 : st \qquad t^2 : st$

Let $W = \operatorname{im} \nu_d \ (= V_{n,d})$ $\qquad I = (i_0, \ldots, i_n)$

Let $x^I \leftrightarrow x_0^{i_0} \cdots x_n^{i_n}$ $\qquad \sum i_j = d$

Prop. W is vanish. set of

$$\{ x^I x^J - x^K x^L : I + J = K + L \}$$

Q. Can this be written in terms of determinants

A. Yes?

$n+1$ rows
$d+1$ cols

check!

<u>Example of proof</u> $n=1$, $d=2$

$U_s(x) = [s^2 : st]$

$U_t(x) = [st : t^2]$

both equal $[s:t]$ on overlap.

Q. Proof of Prop?

Prop. $\nu_d : \mathbb{P}^n \to W$ is $\cong$ onto image.

Pf. Construct inverse.

On each pt of W at least one x_i^d is nonzero.

$\rightsquigarrow U_{x_i}$ cover W

Define $U_{x_i} \longrightarrow \mathbb{P}^n$

$x \longmapsto$ "$x_j x_i^{d-1}$ coords" (put in correct order)

as in proof of injectivity.

These agree on overlaps, give inverse to ν_d $\square$

A possible hint for proving the Prop:

$$\Theta : k[x^I] \longrightarrow k[x_0, \dots, x_n]$$

Show $\ker \Theta$ gen by the $x^I x^J - x^K x^L$.

Hypersurface Sections

f = nonzero homog poly of deg $d \geq 1$.

$\rightsquigarrow Z(f)$ = hypersurf of deg d.

For X = pav, $Z(f) \cap X$ called a hypersurf. section.

Thm. $X \setminus (Z(f) \cap X)$ is $\cong$ an affine alg var. (if not $\emptyset$)

Application.

$\text{Poly}_n/\sim$ = $\{$polys of deg n with nonzero discriminant$\}/\text{scale}$.

$\hookrightarrow$ homog: $\prod(x_i - x_j)$

is affine.

Pf for $d=1$ $Z(f)$ = hyperplane, WLOG $x_0 = 0$.

Pf of general d Apply V_d

hypersurf $Z(f) \rightsquigarrow$ hyperplane.

apply $d=1$ case. use fact that V_d(proj variety) = isomorphic proj var (next page).

example. $f = x^2 - 3yz \subseteq \mathbb{P}^2$

This $(x^2) - 3(yz)$ in Veronese coords $\rightsquigarrow$ linear!

Images of varieties

Prop. $X \subseteq \mathbb{P}^n$ pav

$\Rightarrow$ $\nu_d(X)$ is pav <u>any</u> d.

<u>Pf by example</u> (Harris)

$X = Z(x_0^3 + x_1^3 + x_2^3)$ hyperplane under ν_3

multiply by all x_i to

get 3 polys of deg 2×2 mult of the d you chose.

$$X = Z(x_0^4 + x_0 x_1^3 + x_0 x_2^3, \; x_1 x_0^3 + x_1^4 + x_1 x_2^3, \; x_2 x_0^3 + x_2 x_1^3 + x_2^4)$$

hyperplane ν_4

quadrics under ν_2

Apply $\nu_2 \rightsquigarrow$ 3 quadratics.

But $\operatorname{Im} \nu_2$ is van set of 6 quadratics.

So im X is van set of 9 quadratics.

Q. Suppose $Z \subseteq \mathbb{P}^n$ dense

& Z = image of a morphism.

on q.p. var

Then Z open?

Optional homework:

1. above
2. Image of ν_d =

$Z(x^I x^J - x^K x^L)$

Segre Map

Goal: products of pav's are pav's.

easy for affine space since $\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$

Note $\mathbb{P}^m \times \mathbb{P}^n$ not even homeo to $\mathbb{P}^{m+n}$

Identify $\mathbb{P}^{(m+1)(n+1)-1}$ with

$$M_{m+1,n+1}(k) / \text{scalar}.$$

Define $\varphi_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \longmapsto$$

$$\begin{pmatrix} x_0y_0 & \cdots & x_0y_n \\ \vdots & \ddots & \vdots \\ x_my_0 & \cdots & x_my_n \end{pmatrix} = \begin{pmatrix} x_0 \\ \vdots \\ x_m \end{pmatrix} (y_1 \cdots y_n)$$

"outer product"

$\operatorname{Im} \varphi_{m,n}$ = Segre variety.

Use homog coords

$$z_{ij} \leftrightarrow x_i y_j$$

Example $\varphi_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$$([x_0 : x_1], [y_0 : y_1]) \longmapsto \left[\begin{pmatrix} x_0y_0 & x_0y_1 \\ x_1y_0 & x_1y_1 \end{pmatrix}\right]$$

Note: $\det = 0 \implies \mathrm{rk} \leq 1.$

Also $\mathrm{rk} \neq 0 \implies \mathrm{rk} = 1$

Thus $\varphi_{1,1}$ well def &

$$\mathrm{Im}\,\varphi_{1,1} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \ : \ \det = 0\right\}$$

(lin alg: all rank 1 matrices are outer products)

Claim: $\varphi_{1,1}(\mathbb{P}^1 \times pt)$ is linear

($\leftrightarrow$ plane in k^4)

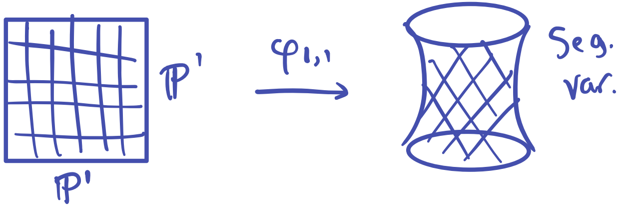
Pf. $pt = [1 : b]$

$$\varphi_{1,1}([x_0 : x_1], [1 : b])$$

$$= \left[\begin{pmatrix} x_0 & bx_0 \\ x_1 & bx_1 \end{pmatrix}\right]$$

$$z_{01} = b\,z_{00}$$
$$z_{11} = b\,z_{10}$$

intersection of 2 3-planes in k^4.



Prop. $\varphi_{m,n}$ injective.

Pf. Let $M = (m_{ij}) = \varphi_{m,n}(a,b)$

WLOG $a_0 = b_0 = 1 \Rightarrow m_{00} = 1$

Recover a, b from first col, row resp. $\square$

Prop. $\operatorname{Im} \varphi_{m,n} = \{\text{rank } 1 \text{ matrices}\}/\text{scale}$.

Pf. Use above lin alg fact or:

Say rk of $M = (m_{ij}) = 1$

Scale so $m_{00} = 1$

$\forall\, k, \ell \neq 0 \quad m_{k\ell} = m_{k0} m_{0\ell}$ (i^{th} col is $m_{0i} \cdot 1^{\text{st}}$)

Take a, b to be first col/row.

Algebraic structure on $\mathbb{P}^m \times \mathbb{P}^n$

$\varphi_{m,n}$ gives $\mathbb{P}^m \times \mathbb{P}^n$ an alg. structure:

- Varieties in $\mathbb{P}^m \times \mathbb{P}^n$ are intersections of vars in $\mathbb{P}^N$ with $\operatorname{im} \varphi_{m,n}$ (subspace Zar. topology)
- poly fns on $\mathbb{P}^m \times \mathbb{P}^n$ are poly fns on $\operatorname{im} \varphi_{m,n}$

Prop. Under this defn, subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ are zero sets of bihomog. polys*.

* if x_i, y_i are coords on $\mathbb{P}^m$, $\mathbb{P}^n$, each monomial has fixed deg in x_i & fixed deg in y_i. If the deg's are same, say the bihomog. poly is balanced.

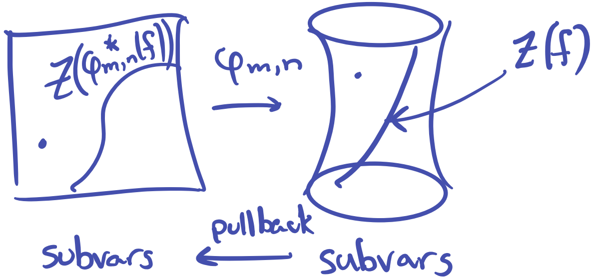
Pf. Given subvar of Segre var:

$$Z(f_1, \dots, f_r)$$

Each f_i pulls back to balanced poly in x, y. If $\deg f_i = d_i$, pullback has bi-degree (d_i, d_i)

e.g. $\varphi_{m,n}^*(z_{00}^2 - z_{01} z_{02})$

$$= (x_0 y_0)^2 - (x_0 y_1)(x_0 y_2)$$



Other direction: Given

$f_1, \dots, f_r$ bihomog. in x_i, y_i

can make each balanced

w/o changing zero set (cf last lecture):

replace f_i with

$\{y_0 f_i, \dots, y_n f_i\}$ $\square$

Notice: There are many more varieties in $\mathbb{P}^m \times \mathbb{P}^n$ than just products of varieties:

product of vars $\longleftrightarrow$ polys factorable as (poly in x)$\cdot$(poly in y).

- Another way to define products of proj vars:

$$k[X \times Y] = k[X] \otimes k[Y] \ ?$$

(~~Maybe~~ Probably with $k(X)$?)

- $X \times Y$ is a categorical product (satisfies univ property).

Given $\iota_X : Z \to X$

$\iota_Y : Z \to Y$

$\exists$ $\iota : Z \to X \times Y$

s.t. $\pi_X \circ \iota = \iota_X$ same for Y.

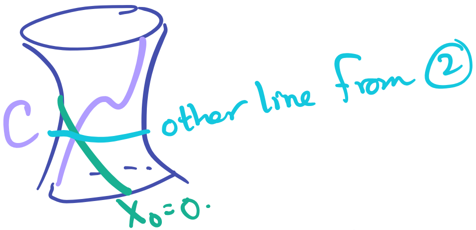
Example. Twisted cubic.

C = image of

$$[s:t] \longmapsto [s^3 : s^2t : st^2 : t^3]$$

Observe $C \subseteq \mathrm{Segre}_{1,1} \subseteq \mathbb{P}^3$.

$$\det\begin{pmatrix} s^3 & s^2t \\ st^2 & t^3 \end{pmatrix} = 0.$$



$\rightsquigarrow C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

Besides the eqn defining $\mathrm{Segre}_{1,1}$ there are 2 polys defining C in $\mathbb{P}^3$

1. $Z_{00}Z_{10} - Z_{01}^2$
2. $Z_{01}Z_{11} - Z_{10}^2$

1 Pulls back to C union a line:

$$x_0y_0x_1y_0 - (x_0y_1)^2$$

$$= x_0(\underbrace{y_0^2x_1 - x_0y_1^2}_{f}) \longleftrightarrow \begin{array}{l}\text{line } x_0 = 0 \\ \cup Z(f)\end{array}$$

Check: $\varphi_{1,1}$ maps $Z(f)$ bij to C.

Coord-free descriptions of V_d & $\varphi_{m,n}$

$\exists$ natural map

$$k^{n+1} \to \mathrm{Sym}^d(k^{n+1})$$
$$v \mapsto v^d$$

projectivizing gives V_d

e.g. $V_1 : \mathbb{P}^1 \to \mathbb{P}^2$

$$k^2 \longrightarrow \mathrm{Sym}^2 k^2$$
$$e_1, e_2 \qquad\qquad e_1^2 \quad e_1e_2 \quad e_2^2$$
$$(xe_1+ye_2) \longmapsto (xe_1+ye_2)^2 = x^2(e_1^2) + xy(e_1e_2) + y^2(e_2^2)$$

Similarly $\varphi_{m,n}$ comes from

$$k^{m+1} \times k^{n+1} \longrightarrow (k^{m+1}) \otimes (k^{n+1})$$

$$\mathrm{Sym}^d(V) = V^{\otimes d} / \text{rearranging terms.}$$

Grassmannian

$V = k^n$

$Gr_{r,n} = G_r(V)$

$= \{r\text{-dim subsp's of } V\}$

e.g. $G_{1,n} = \mathbb{P}^{n-1}$

Today: $Gr_{r,n}$ is a proj av.

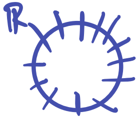
So: The "moduli/parameter space of r-dim lin. varieties is a variety"

Topology aside

B = space.

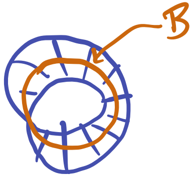
An r-plane bundle over B is a (bigger) space so "over" each $b \in B$, have r-plane.

examples. ① $B = S^1$ $r = 1$



$S^1 \times \mathbb{R}$

open annulus.

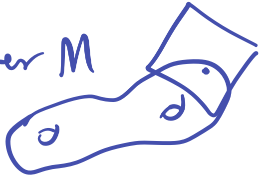


open Mobius band.

② M = smooth n-manifold

TM = n-plane bundle over M

M



Amazing fact:

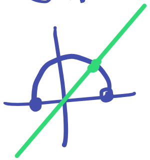
$$\left\{ \begin{matrix} r\text{-bundles} \\ \text{over } B \end{matrix} \right\}_{/\sim} \longleftrightarrow \left\{ B \longrightarrow G_{r,\infty} \right\}_{/\sim}$$

and given $B \to G_{r,n}$ can pull back the bundle over $G_{r,n}$.

Why? $G_{r,n}$ (and $G_{r,\infty}$) have canonical r-plane bundle E over them.

$$E \subseteq G_{r,n} \times k^n$$
$$\shortparallel$$
$$\{(W,v) : v \in W\}$$

example. $G_{1,2}$ $k=\mathbb{R}$.

$G_{1,2}$  $\longleftarrow$ O

Back to the goal: $Gr_{r,n}$ is par.

<u>Direct approach</u>

We define

$$Gr_{r,n} \longrightarrow \mathbb{P}^{\binom{n}{r}-1}$$

Given $W \in Gr_{r,n}$

$\rightsquigarrow$ basis $v_1, \dots, v_r$

$\rightsquigarrow$ $r \times n$ matrix

$\rightsquigarrow \left(\binom{n}{r} \text{ minors}\right) \in K^{\binom{n}{r}}$

Different bases give $r \times n$ matrices that differ by mult on left by invertible $r \times r$ matrix A.

This changes all minors by $\det A$.

$\rightsquigarrow$ well def pt in $\mathbb{P}^{\binom{n}{r}-1}$.

Need to show:
- injective
- image is variety.

For latter, show the image satisfies Plücker relations:

Denote by $M_{i_1 \dots i_r}$ the minor...

Given $i_1 < \dots < i_{r-1}$
$j_1 < \dots < j_{r+1}$

$$0 = \sum_{l=1}^{r+1} (-1)^l M_{i_1 \dots i_{r-1} j_l} M_{j_1 \dots \hat{j_l} \dots j_{r+1}}$$

$\rightsquigarrow$ many quadrics

Examples

- $W \in G_{1,3}$ $W = \mathrm{Span}\left\{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}\right\}$

$\rightsquigarrow (a_0 \ a_1 \ a_2)$

minors: a_0, a_1, a_2.

- $W \in G_{2,3}$ $W = \mathrm{Span}\{a, b\}$

$$\rightsquigarrow \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

minors $\longleftrightarrow$ cross product.

Plucker: $(-1) M_{01} M_{02}$

$i_1 = 0$
$j_1 j_2 j_3 = 012$ $+ M_{02} M_{01}$ $\ddot{\frown}$

Can see injectivity in both cases. & surjectivity to $\mathbb{P}^2$

Observation (lon): $G_{1,n} \cong G_{n-1,n}$

$G_{r,n} \cong G_{n-r,n}$

First nontrivial Plucker relation: $G_{2,4}$

$$M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23}$$

single defining poly.

Second approach: Wedge products

V = vect sp. over k

tensor product.

$$V^{\otimes r} = V \times \cdots \times V \,/\, \text{multilinearity}.$$

$$= \{\text{finite sums of } v_1 \otimes \cdots \otimes v_r\}$$

subject to

$$(av_1 + a'v_1') \otimes v_2 \otimes v_3$$

$$= a\, v_1 \otimes v_2 \otimes v_3 + a'\, v_1' \otimes v_2 \otimes v_3$$

Why? $\{\text{multilinear maps } V^r \longrightarrow W\}$

$$\longleftrightarrow \{\text{linear maps } V^{\otimes r} \longrightarrow W\}$$

Next...

$$\Lambda^r V = V^{\otimes r} / \text{alternating}.$$

$$= \{\text{finite sums } v_1 \wedge \cdots \wedge v_r\}$$

subject to multilinearity as above and: swapping two entries gives -1

So: $v_1 \wedge v_2 \wedge v_3 = -v_2 \wedge v_1 \wedge v_3$

and $v_1 \wedge v_1 \wedge v_2 = -v_1 \wedge v_1 \wedge v_2$

$$\Longrightarrow v_1 \wedge v_1 \wedge v_2 = 0$$

$(\text{char } k \neq 2)$

Why?

① $\{$alt. multilin. maps $V^r \to W\}$

$\longleftrightarrow \{$lin maps $\Lambda^r V \to W\}$

② $\Lambda^n K^n \cong K \Longrightarrow$ determinants exist and are unique.

③ Area functions in K^n

$$(e_1+e_2)\wedge e_3 = e_1\wedge e_3 + e_2\wedge e_3$$

area of proj to $(e_1+e_2)e_3$ plane = area of proj to e_1e_3 plane + area of proj to e_2e_3 plane

where e_1+e_2 declared to have length 1

Facts ① If $v_1,\dots,v_n$ basis for V then $\{v_{i_1}\wedge\cdots\wedge v_{i_r} : i_1<\cdots<i_r\}$ is a basis for $\Lambda^r V$

$\Longrightarrow \dim \Lambda^r V = \binom{n}{r}$

② $W \leq V$ subsp of dim r

$T \in \mathrm{Aut}(W)$

$\omega \in \Lambda^r W$

$\Longrightarrow T(\omega) = (\det T)\,\omega$

Plücker embedding

$$F : Gr_{r,n} \longrightarrow \mathbb{P}(\Lambda^r V) = \mathbb{P}^{\binom{n}{r}-1}$$

Fact 1

$$\mathrm{Span}\{v_1, \dots, v_r\} \longmapsto [v_1 \wedge \dots \wedge v_r]$$

Well def by fact ②

e.g. $v_1 \wedge v_2 = (v_1 + v_2) \wedge v_2$

$= v_1 \wedge v_2 + \cancel{v_2 \wedge v_2}$

$5 v_1 \wedge v_2 \sim v_1 \wedge v_2$

Todo: · F inj

· $\mathrm{Im}\, F$ is proj. var.

Will do at same time.

Defn. $x \in \Lambda^r V$ is totally decomposable if it's an r-wedge (not a sum)

Note: $\mathrm{Im}\, F = \{\text{totally dec}\}$

$e_1 \wedge e_2 + e_3 \wedge e_4$ is the simplest example of not-(tot. dec)

Lemma. Given nonzero $x \in \Lambda^r V$

Let $\varphi_x : V \longrightarrow \Lambda^{r+1} V$

$v \longmapsto v \wedge x$

① $\dim \ker \varphi_x \leq r$, with $=$ iff x tot. dec.

② If $x = v_1 \wedge \dots \wedge v_r$ then $\ker \varphi_x = \mathrm{Span}\{v_1, \dots, v_r\}$

Lemma. Given nonzero $x \in \Lambda^r V$

Let $\varphi_x : V \to \Lambda^{r+1} V$

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① $\dim \ker \varphi_x \leq r$, with $=$ iff x tot. dec.

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---

② $\Rightarrow F$ inj.

second half of ① $\Rightarrow \operatorname{im} F$ is a variety because:

$x \in \operatorname{im} F \iff x$ tot dec.

$\overset{①}{\iff}$ nullity $\varphi_x \geq r$.

$\iff \operatorname{rank} \varphi_x \leq n-r$

$\iff$ all $n-r+1$ minors vanish.

$Gr_{r,n} \to \Lambda^r V$

$\operatorname{Span}\{v_1, \ldots, v_r\} \to v_1 \wedge \cdots \wedge v_r$

$\mathbb{P}(\Lambda^r V) \to \mathbb{P}(\operatorname{Hom}_k(V, \Lambda^{r+1} V))$

$x \mapsto \varphi_x$

inj & linear, can apply $\mathbb{P}$

rank $\leq n-r$ defines closed subset of RHS

$\rightsquigarrow$ closed subset of RHS

$\overset{?}{\rightsquigarrow}$ closed subset of $\mathbb{P}(\Lambda^r V)$ (preim. of closed is closed).

Grassmannians

$Gr_{r,n} = \{r\text{-planes in } V = k^n\}$

Goal: this is proj. alg var.

Plücker embedding

$$F : Gr_{r,n} \longrightarrow \mathbb{P}(\Lambda^r V)$$

$$\mathrm{Span}\{v_1, \dots, v_r\} \longrightarrow [v_1 \wedge \cdots \wedge v_r]$$

To show: ① F inj.

② $\mathrm{Im}\, F$ closed.

Note: $\mathrm{Im}\, F = \{\text{tot. dec. elts}\}$

Tool: Wedging map

$x \in \Lambda^r V$

$$\rightsquigarrow \varphi_x : V \longrightarrow \Lambda^{r+1} V$$

$$v \longmapsto v \wedge x$$

Have $\varphi_x \in \mathrm{Hom}_k(V, \Lambda^{r+1} V)$

__Lemma__. $x \in \Lambda^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

Im F closed $\Longleftarrow$ ② equality $\Longleftrightarrow$ x tot. dec.

F inj $\Longleftarrow$ ③ If $x = a \cdot v_1 \wedge \cdots \wedge v_r$ tot dec

$\ker \varphi_x = \mathrm{Span}\{v_1, \dots, v_r\}$

__Lemma__. $x \in \Lambda^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

Given ①

② equality $\iff$ x tot. dec. $\iff$ $\mathrm{rk}\, \varphi_x \leq n-r$

③ If $x = a \cdot v_1 \wedge \cdots \wedge v_r$ tot dec

$\ker \varphi_x = \mathrm{Span}\{v_1, \dots, v_r\}$

Proof that ② $\Longrightarrow$ $\mathrm{Im}\, F$ closed:

Have $\Lambda^r V \longrightarrow \mathrm{Hom}_k(V, \Lambda^{r+1} V)$

$x \longmapsto \varphi_x$

injective & linear (check).

$\hookrightarrow$ uses $r < n$.

So can apply $\mathbb{P}$ ...

$$\begin{tikzcd} \mathbb{P}(\Lambda^r V) \arrow[r, "\text{linear}"] & \mathbb{P}(\mathrm{Hom}_k(V, \Lambda^{r+1} V)) \\ Gr_{r,n} \arrow[u, hook, "F"] \end{tikzcd}$$

$H: \mathbb{P}(\Lambda^r V) \to \mathbb{P}(\mathrm{Hom}_k(V, \Lambda^{r+1} V))$

by ②

Image of $Gr_{r,n}$ lies in set W of maps of rank $\leq n-r$. (alg cond)

$$Gr_{r,n} = Z(H^*(\overset{n-r+1}{\text{minor}} \text{ conditions}))$$

$$= H^{-1}(W \cap \mathrm{Im}\, H)$$

Lemma. $x \in \Lambda^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

② equality $\iff$ x tot. dec

③ If $x = a \cdot v_1 \wedge \cdots \wedge v_r$ tot dec

$\ker \varphi_x = \operatorname{Span}\{v_1, \ldots, v_r\}$

Pf. Choose basis $e_1, \ldots, e_n$ for V

$\rightsquigarrow$ basis e_I for $\Lambda^r V$

Assume $e_1, \ldots, e_s$ is basis for $\ker \varphi_x$

Pf of ① Want $s \leq r$

Say $x = \sum a_I e_I$

Fix some $i \in \{1, \ldots, s\}$

$\varphi_x(e_i) = 0 \iff a_I = 0$ when $i \notin I$

i.e. every non-0 term of x has an e_i.

Since true for $i \in \{1, \ldots, s\}$

every nonzero term uses $e_1, \ldots, e_s$

$\implies s \leq r$

Pf of ② Suppose $s = r$.

Then x is a mult. of $e_1 \wedge \cdots \wedge e_s$

other dir: $x = v_1 \wedge \cdots \wedge v_r$ apply ①

Pf of ③ $\operatorname{Span}\{v_1, \ldots, v_r\} \subseteq \ker \varphi_x$

but dim's same by ②. $\square$

Fact. $x \wedge x = 0 \iff x$ tot dec.

Local coords on Grassmannian

Consider chart on $\operatorname{Im} F$ where

$a_J \neq 0$. WLOG $a_J = a_{1\ldots r}$.
(others differ by permuting coords).

Let $B = r \times n$ matrix of rank r
($\operatorname{row} B$ = pt in $Gr_{r,n}$)

$F(\operatorname{row}(B))$ is

$(b_{11}e_1 + \cdots + b_{1n}e_n) \wedge \cdots \wedge (b_{r1}e_1 \cdots b_{rn}e_n)$

$= \sum a_J e_J$

Only the $b_{ij} e_j$ with $j \in J$ contribute to a_J

Further: a_J is the leftmost minor
$(J = 1 \ldots r)$

e.g. $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix}$ $\quad J = 1,2$

$\rightsquigarrow b_{11}b_{22} e_1 \wedge e_2 + b_{12}b_{21} e_2 \wedge e_1 + \cdots$

$= (b_{11}b_{22} - b_{12}b_{21}) e_1 \wedge e_2 + \cdots$

So $a_J \neq 0 \iff$ leftmost $r \times r$ matrix is invertible.

$\rightsquigarrow$ can mult. B on left to get

$$\left(I_r \middle| \begin{matrix} b_{1,r+1} & \cdots & b_{1,n} \\ & \ddots & \vdots \\ b_{r,r+1} & \cdots & b_{r,n} \end{matrix} \right) \overset{b_{ij}}{\longleftrightarrow} \text{copy of } \mathbb{A}^{r(n-r)}$$

It's a bijection since RREF unique.

It's also $\cong$ of aff. alg vars.

$\rightarrow$ The a_I are minors.

$\leftarrow$ Need to get b_{ij} as polys in a_I

One example

$$a_{23\ldots rj} = \begin{vmatrix} 0 & 0 & & 0 & b_{1j} \\ 1 & 0 & \cdots & \vdots & \vdots \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & b_{rj} \end{vmatrix} = (-1)^{r+1} b_{1j}$$

other cases similar.

The incidence correspondence

$$I = \{(W,v) : W \in Gr_{r,n},\ v \in \mathbb{P}(W)\} \subseteq Gr_{r,n} \times \mathbb{P}^{n-1}$$

Thm. I is a proj subvar of $Gr_{r,n} \times \mathbb{P}^{n-1}$

Applications

① $V \subseteq Gr_{r,n}$ proj sub var

$$\Longrightarrow \bigcup_{W \in V} W \subseteq \mathbb{P}^{n-1} \text{ is a subvar.}$$

Pf idea:

$$\begin{array}{ccc} & I & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ Gr_{r,n} & & \mathbb{P}^{n-1} \end{array}$$

$$\bigcup_{W \in V} W = \pi_2 \circ \pi_1^{-1}(V)$$

(2) $X \subseteq \mathbb{P}^n$ var.

$L_r(X)$ = locus of proj r-planes meeting X.

Prop. $L_r(X)$ is a proj subvar of $Gr_{r+1,n+1}$, hence $\rightsquigarrow$ var in $\mathbb{P}^n$ by prev appl.

Pf.
$$\begin{tikzcd} & I \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Gr_{r+1,n+1} & & \mathbb{P}^n \end{tikzcd}$$

$$L_r(X) = \pi_1 \circ \pi_2^{-1}(X)$$

Incidence Correspondence

$$I = I_{r,n} = \{(W,v) : v \in \mathbb{P}(W)\}$$
$$\subseteq Gr_{r,n} \times \mathbb{P}^{n-1}$$

$$\mathbb{P}(W) = \{W \setminus 0\}/\text{scale} \subseteq \mathbb{P}^{n-1}$$

Thm. $I_{r,n}$ is proj subvar of $Gr_{r,n} \times \mathbb{P}^{n-1}$

Fact 1. X, Y proj av's
$U \subseteq X$ open $\Rightarrow$ $U \times Y$ open in $X \times Y$

Pf. Suffices closed $\times Y$ is closed.
zero set of homog. polys in x_i

Can use same polys as fns on $X \times Y$.

Recall from Segre: Closed sets in $X \times Y$ are van sets of bihomog. poly's

(Topology)
Fact 2. $A \subseteq X$ closed
$\Leftrightarrow$ $\exists$ open cover $\{X_i\}$ of X s.t. $X_i \cap A$ closed in X_i (in subsp top)

Pf ($\Leftarrow$) $X_i \setminus A$ open in X_i hence X & $X \setminus A$ is union of these.

Fact 3 (LinAlg)

$$A = (I_r \mid B) \quad r \times n.$$

Then $v \in \text{Row}\, A \iff \forall i$

$$(i^{th} \text{ col of } A) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = v_i$$

Pf. $v \in \text{Row}\, A$

$\iff v \in \text{Col}\begin{pmatrix} I_r \\ B^T \end{pmatrix}$

$\iff \begin{pmatrix} I_r \\ B^T \end{pmatrix} x = v$ consistent

$\iff \begin{pmatrix} I_r \\ B^T \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = v \quad \square$

Fact 4 $f \in k[x_1, \ldots, x_n, y_0, \ldots, y_m]$ homog in y's then $Z(f)$ is closed in $\mathbb{A}^n \times \mathbb{P}^m$ in subsp top $\rightsquigarrow$ Pf: homog in x.

Pf of Thm. Cover $Gr_{r,n}$ by open sets $U_{a_{i_1 \cdots i_r}}$. By Facts 1+2 suffices to show

$$\left(U_{a_{i_1 \cdots i_r}} \times \mathbb{P}^{n-1}\right) \cap I_{r,n} \text{ closed in } U_{a_{i_1 \cdots i_r}} \times \mathbb{P}^{n-1}$$

We'll do $U_{a_{1 \cdots r}}$ i.e. subset of $Gr_{r,n}$ given by $\{A = (I_r \mid B)\}$

By Fact 3, the intersection given by

$$(i^{th} \text{ col of } B) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = v_{i+r}$$

This poly is homog in v_i $\overset{\text{Fact 4}}{\rightsquigarrow}$ closed subset of $U_{a_{1 \cdots r}} \times \mathbb{P}^{n-1}$

$\square$

More variety!

Four constructions

① Prop. $V \subseteq Gr_{r,n}$ subvar

$\Longrightarrow X = \bigcup_{W \in V} W$ subvar of $\mathbb{P}^{n-1}$

Pf.

$$\begin{tikzcd} & I_{r,n} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V \subseteq Gr_{r,n} & & \mathbb{P}^{n-1} \end{tikzcd}$$

$X = \pi_2 \circ \pi_1^{-1}(V)$

Need: • π_i are continuous

• π_i are closed.

② $X \subseteq \mathbb{P}^n$ var.

$L_r(X)$ = locus of proj r-planes meeting X

Prop. $L_r(X)$ subvar of $G_{r+1,n+1} =: \mathbb{G}_{r,n}$

Pf. $L_r(X) = \pi_1 \circ \pi_2^{-1}(X)$.

③ Joins

$X, Y \subseteq \mathbb{P}^n$ subvars

$J(X,Y) = \{$lines in $\mathbb{P}^n$ meeting both$\}$

Prop. $J(X,Y)$ subvar of $G_{2,n+1} = \mathbb{G}_{1,n}$

Pf. $J(X,Y) = L_1(X) \cap L_2(Y)$

④ Fano varieties

$F_r(X) = \{r\text{-planes contained in } X\}$
$\subseteq Gr_{r,n}$.

Projections are Morphisms

$X \subseteq \mathbb{P}^n,\ Y \subseteq \mathbb{P}^m$

Segre $\rightsquigarrow$ $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$

$$\pi_Y : X \times Y \longrightarrow Y$$
$$(x, y) \longmapsto y$$

Prop. π_Y is a morphism

Pf. Suffices to do

$$\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$$

Recall Segre:

$$(x, y) \longmapsto \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} (y_0 \cdots y_m)$$

$$= \begin{pmatrix} x_0 y_0 & \cdots & x_0 y_m \\ \vdots & & \vdots \\ x_n y_0 & \cdots & y_n y_m \end{pmatrix}$$

any nonzero row is the proj to Y.

(need to check agreement on overlap.
but all rows are multiples so ✓)

Prop. π_Y is closed.

Note: false in affine case

(project $xy=1$ to $\mathbb{A}^1$, get $\mathbb{A}^1 \setminus 0$)

Thm. $f: X \to Y$ morphism

$Z \subseteq X$ subvar. Then $f(Z) \subseteq Y$ subvar. (all proj)

Cor. X connected proj var

Then any (global) regular fn is const.

Pf. $X \xrightarrow{\text{reg fn}} \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ not surj

image is a subvar by Thm

$\Longrightarrow$ image is finite set of pts. Done by connected. $\square$

Tool: Graphs

$f: X \to Y$ morphism

$\rightsquigarrow \gamma_f : X \to X \times Y$

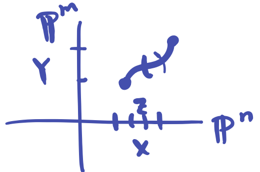
$x \mapsto (x, f(x))$

Image Γ_f is graph of f.

Lemma. Γ_f closed in $X \times Y$

& $\gamma_f : X \to \Gamma_f$ is $\cong$.

To prove Thm. Lemma allows us to assume f is $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$



Projections are closed

$X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$

$\rightsquigarrow X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$

$\pi_Y : X \times Y \to Y$

Prop. π_Y closed.

(false for affine! hyperbola!)

More generally...

Thm. Any $f : X \to Y$ morphism is closed

"compactness property"

Cor. Global reg fns const. if X conn.

Graphs $f : X \to Y$

$$\jmath_f : X \to X \times Y$$
$$x \mapsto (x, f(x))$$

$$\mathrm{Image}(\jmath_f) = \Gamma_f$$

$$\Gamma_f = \{(x,y) \in \mathbb{P}^n \times \mathbb{P}^m : f_i(x_0, \dots, x_n) = y_i \ \forall i\}$$

assuming $f = (f_0, \dots, f_m)$ on open set in X

Prop 1. • Γ_f closed in $X \times Y$

• $\jmath_F : X \to \Gamma_f$ is $\cong$.

Arrondo

Prop 1. • Γ_f closed in $X \times Y$

• $j_F : X \to \Gamma_f$ is $\cong$.

Pf. Morphism. At any $x \in X$ $\exists$ open U s.t. f given by $f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ same deg. On U, post-comp with Segre map gives

$$\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} (f_0 \cdots f_m) \rightsquigarrow x_i f_j$$

- this agrees on overlaps ✓
- same deg ✓
- image in Γ_f ✓
- $y_i f_j$ don't sim. vanish ✓

Closed. Let $(p,q) \notin \Gamma_f$ i.e. $f(p) \neq q$

Choose $U \subseteq \mathbb{P}^n$ nbd of p so f def on U by $f_0, \ldots, f_m$ of deg d.

Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ van set of 2×2 minors of $\begin{pmatrix} f_0 & \cdots & f_m \\ y_0 & \cdots & y_m \end{pmatrix}$ e.g. $f_0 y_1 = f_1 y_0$

- bihomog of deg $(d,1)$
- $(U \times \mathbb{P}^m) \cap Z^c$ open nbd of (p,q) in Γ_f^c

(Γ_f would be exactly Z. Problem is that f is only def. locally.)

Isomorphism. Inverse is projection $\square$

Prop. $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ closed.

"Main thm of elimination theory"

Lemma. $g_1, \dots, g_r \in \mathbb{P}(k[x_0, \dots, x_n])$ deg d

Regard $g_i \in \mathbb{P}^N$ (take coeffs)

Let $D \geq d$. Then

$\{(g_1, \dots, g_r) \in (\mathbb{P}^N)^r :$

$$k[x_0, \dots, x_n]_D \subseteq (g_1, \dots, g_r)_D$$

elts of deg D in the ideal

is ~~closed~~ open in $(\mathbb{P}^N)^r$

$$N = \binom{n+d}{d}$$

Gathmann

Pf of Lemma The condition

$$k[x_0, \dots, x_n]_D \subseteq (g_1, \dots, g_r)_D$$

equiv to

$$k[x_0, \dots, x_n]_D = (g_1, \dots, g_r)_D \quad (*)$$

Since $(g_1, \dots, g_r) = \{\sum h_i g_i : h_i \in k[x_0, \dots, x_n]\}$

$(*)$ equiv to:

$$F_D : (k[x_0, \dots, x_n]_{D-d})^r \to k[x_0, \dots, x_n]_D$$
$$(h_1, \dots, h_r) \mapsto \sum h_i g_i$$

being surjective, ie has rank $\dim k[x_0, \dots, x_n]_D = \binom{n+D}{D}$

$\iff$ one of the minors of F_D of that dim is not zero. $\square$

Prop. $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ closed.

Pf. Take coords $x_0, \dots, x_n$
$y_0, \dots, y_m$

Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$

Say $Z = Z(f_1, \dots, f_r)$

f_i of deg (d, d)

Let $a \in \mathbb{P}^m$

Let $g_i = f_i(\cdot, a)$

$\in k[x_0, \dots, x_n]$

Will show $a \notin \pi(Z)$ open condition.

$a \notin \pi(Z) \iff \nexists\, x \in \mathbb{P}^n$ s.t. $(x, a) \in Z$

$\iff Z_p(g_1, \dots, g_r) = \emptyset$.

$\overset{\text{WN}}{\iff} \sqrt{(g_1, \dots, g_r)} \supseteq (x_0, \dots, x_n)$

$\iff \exists\, d_i$ s.t. $x_i^{d_i} \in (g_1, \dots, g_r)\ \forall i$.

$\iff k[x_0, \dots, x_n]_D \subseteq (g_1, \dots, g_r)_D$ some D

take $D = \sum d_i$

open condition on coeffs of g_i

(lemma)

The coeffs of g_i are poly's in a, i.e. coords on $\mathbb{P}^m$

$\square$

Thm. Any $f: X \to Y$ morphism is closed

Pf. Say $Z \subseteq X$ closed.

$\gamma_f : X \xrightarrow{\cong} \Gamma_f$ (Prop 1)

$\Rightarrow \gamma_f(Z)$ closed in Γ_f, hence in $\mathbb{P}^n \times \mathbb{P}^m$

By Prop 2 $\pi(\gamma_f(Z)) = f(X)$ closed in $\mathbb{P}^m$

It is contained in Y, hence closed in Y $\square$

Chap 4 Dim, deg, smoothness.

V = vect sp.

$\dim V = \sup\{r : \exists$ strictly dec chain of lin subsp $V = V_0 \supset V_1 \supset \cdots \supset V_r\}$

X = top space

Applies to varieties.

Krull dimension is

$\dim X = \sup\{r : \exists$ strictly dec chain of closed irred sets $X = X_0 \supset \cdots \supset X_r\}$

Example. $\dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1$

Chap 4. Dim, deg, smoothness.

V = vect sp.

$\dim V = \sup \{ r : \exists$ strict dec chain of subsp. $V = V_0 \supset \cdots \supset V_r \}$

X = top sp. Krull dim

$\dim X = \sup \{ r : \exists$ strict dec chain of closed irred subsp $X = X_0 \supset \cdots \supset X_r (\neq \emptyset) \}$

we said $\emptyset$ not irred.

& $\dim \emptyset = \infty$.

X = variety

$\dim X$ = krull dim in Zar. top.

Example. $\dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1$

Facts ① If $X \neq \emptyset$, Hausdorff then $\dim X = 0$

(Hausdorff $\Rightarrow$ only irreds are pts).

② $\dim X = \sup \{ \dim X_i : X_i$ irred comp $\}$

③ $Y \subseteq X \Rightarrow \dim Y \leq \dim X$

& strict if no irred comp of (closure of) Y is irred comp of X.

④ X covered by U_i open $\Rightarrow \dim X = \sup \dim U_i$

Cor of ③ : X irred, $\dim X = 0$

$\implies X = pt.$

Want : $\dim \mathbb{A}^n = n.$

easy: $\geq n.$

Krull dim

A = ring

$\dim A = \sup\{r : \exists$ strict inc. $P_0 \subset \cdots \subset P_r$ of proper prime ideals $\}$

By our dictionary: $\dim X = \dim k[X].$

example

Prop. $\dim k[x_1, \dots, x_n] = n$

Cor. $\dim \mathbb{A}^n = n$

Cor. $\dim \mathbb{P}^n = n$ by ④

Example. $\dim Gr_{r,n} = r(n-r)$

$$\left(I \mid \quad \right)$$

$r \times (n-r)$

also using ④.

In $k[x,y]$:

$0 \subset (x) \subset (x,y) \; (\subset k[x,y])$

Prop. $\dim k[x_1, \ldots, x_n] = n$

Pf. Induct on n.

$n=0$ ✓

Gathmann
Comm Alg
class notes.
Ch 11

Inductive step

Say:

$0 = P_0 \subset P_1 \subset \cdots \subset P_m \subset k[x_1, \ldots, x_n]$

WLOG: $P_1 = (f)$ where f monic in x_n

← can asm P_1 principal since $k[x_1, \ldots, x_n]$ UFD.

In a non-UFD (prime) might not be prime.

Monic in x_n: leading term x_n^d

In quotient $k[x_1, \ldots, x_n]/P_1$

Show

shorter by 1 → $0 = \overline{P}_1 \subset \cdots \subset \overline{P}_m$ is str. inc. chain of prime ideals.

Now use:

$k[x_1, \ldots, x_{n-1}] \to k[x_1, \ldots, x_n]/P_1$

$x_i \longmapsto \overline{x}_i$

pull back $\overline{P}_i$. Get chain of strict inc* prime id's in $k[x_1, \ldots, x_{n-1}]$

Why is preim of $\overline{P}_2$ not 0?

$\square$

Example

$A = k[x,y]/(y^2 - x^3 + x)$

P = nonzero prime in A

$\varphi : k[x] \longrightarrow A$

$x \longmapsto \bar{x}$

Want $\varphi^{-1}(P) \neq 0$.

Subexample. Why is $\varphi^{-1}(y) \neq 0$?

$x - x^3 \longmapsto y^2 \in (y)$.

Next example

$A = k[x,y]/(\overset{f}{y^2 - x^3 + xy})$

Want $\varphi^{-1}(y) \neq 0$.

$f \in k[x][y]$ $\quad$ $\varphi(\text{const-in-}y \text{ term}) \in (y)$

$y^2 + (x)y - x^3$

$\varphi(x^3) \in (y)$

Where using monic??

Next goal

$X \subseteq \mathbb{P}^n$ variety

$\dim X =$ the d s.t.
$\exists$ finite map
$X \to \mathbb{P}^d$

Finite maps

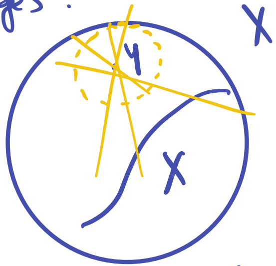
Defn 1. $f: X \to Y$ with dense image
and s.t $f_*: k[Y] \to k[X]$ finite,
meaning $k[X]$ f.g. module
over $\operatorname{im} f_*$

Defn 2. $f: X \to Y$ ~~dense image~~ surj.
& pt preimages are finite.

Why does every variety have a surj map to $\mathbb{P}^d$ with finite pt preimages?

$X \subsetneq \mathbb{P}^n$

Geom answer:



Stereographic proj $X \to \mathbb{P}^{n-1}$
with finite pt preims:
pt preims are $\mathbb{P}^1 \cap X$ = finite
Can iterate until get surj. map to $\mathbb{P}^d$

Noether normalization

Thm. A = fin gen k-alg

$\Rightarrow \exists\ y_1, \dots, y_d \in A$ alg. indep. s.t.

A is fg as $k[y_1, \dots, y_d]$ module.

On last slide $A = k[X]$.

(Can deduce Nullstellensatz from this.)

Think of y's as transcendental/indep and rest of A as dep. on those.

Example. $A = k[X_1, X_2]/(X_2^2 - X_1^3 + X_1)$

(as above)

$d = 1$, $y_1 = x_1$

x_2 satisfies $f \in k[x_1][Z]$

$$f(Z) = Z^2 - (x_1^3 - x_1)$$

$$\rightsquigarrow A = \{k[x_1] + x_2 k[x_1]\}$$

i.e. A gen by $x_2, 1$ as $k[x_1]$ module.

Notice f is monic in Z.
Can always do lin. change of coords to make it so.
The pf follows then as in example.

Pf of NN in special case:

A gen by one elt c.
(as k-mod)

If c transc. $\Rightarrow A = k[c]$ done.

If c alg $\Rightarrow f(c) = 0$ f monic deg d

$\Rightarrow A = k[z]/(f(z))$

& A gen as a module

by $1, c, \dots, c^{d-1}$

$\square$

Last time:

$$\dim X = \dim k[X]$$
$$= d \text{ s.t. } \exists \text{ finite } X \to \mathbb{P}^d.$$

Hulek Chap 3.

Today:

$$\dim X = \min_{p \in X} \dim T_p X$$
$$= \operatorname{tr\,deg}_k k(X)$$

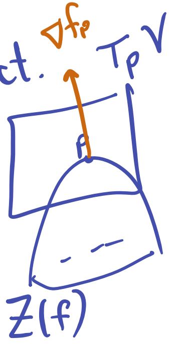
Tangent spaces

Note: f is well-def. object when X irred.

$X = Z(f) \subseteq \mathbb{A}^n$ irred. hypersurf.

$$\nabla f_p = \left(\frac{df}{dx_1}(p), \dots, \frac{df}{dx_n}(p) \right) \in k^n.$$

$\rightsquigarrow \nabla f_p \in (k^n)^*$ via dot product.



$$T_p X = p + \ker \nabla f_p$$

Write

$$f_p^{(1)} = \sum \left(\frac{df}{dx_i}(p) \right)(x_i - p_i)$$

"linear part of f at p"

$T_p X$ is the set of solns.

Examples

⓪ Hyperplane $H \subseteq \mathbb{A}^n$

$T_pH = H$ (exercise).

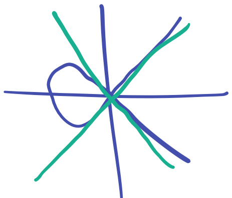
① Parabola $f(x,y) = y - x^2$

$\rightsquigarrow \nabla f = (-2x, 1)$

$\rightsquigarrow \nabla f_0 = (0,1)$

$\rightsquigarrow T_0X =$ x-axis.

② $X = Z(y^2 - x^2 - x^3)$

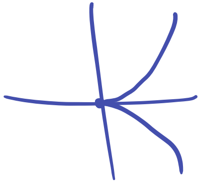


$\nabla f = (-2x - 3x^2, 2y)$

$\nabla f_0 = (0,0)$

$\rightsquigarrow T_0X = \mathbb{A}^2.$

③ $X = Z(y^2 - x^3)$



$\rightsquigarrow T_0X = \mathbb{A}^2$

④ $X = Z(x^m - y^m)$

Check: $T_{(a,b)}X$ is a line

if $\operatorname{char} k \nmid m$.

not irred — so we need more defns!

Projective varieties

To define T_pX, pass to affine chart, take tang. sp there, take proj. closure.

Tangent spaces & roots

Prop. $L \subseteq \mathbb{A}^n$ affine line.

$p \in L$

$X = Z(f) \subseteq \mathbb{A}^n$

Then $L \subseteq T_pX \iff$

$f|_L$ has a multiple root at p.

examples. ① $X = Z(y - x^2)$

$x^2 = 0$ ✓

② $X = Z(y^2 - x^3)$.

$L: y = tx$

$\rightsquigarrow (tx)^2 - x^3 = x^2(t^2 - x)$ ✓

mult. root at 0

Pf. Let $L(t) = (p_1 + b_1 t, \dots, p_n + b_n t)$

Let $g(t) = f|_L = f(p_1 + b_1 t, \dots, p_n + b_n t)$

Know $g(0) = f(p) = 0$.

Want $g'(0) = 0$. By chain rule:

$\frac{dg}{dt}(0) = 0 \iff \sum b_i \frac{df}{dx_i}(p) = 0$

$\iff L \subseteq T_pV$ $\square$

Tangent sp for general irred (not just hypersurf)

$X \subseteq \mathbb{A}^n \qquad X = Z(f_1, \dots, f_m)$

$$T_p X = \bigcap_{f \in I(X)} T_p Z(f)$$

exercise
$$= \bigcap_{i=1}^{m} T_p Z(f_i)$$

Smoothness

$X = Z(f) \subseteq \mathbb{A}^n$ irred hypersurf.

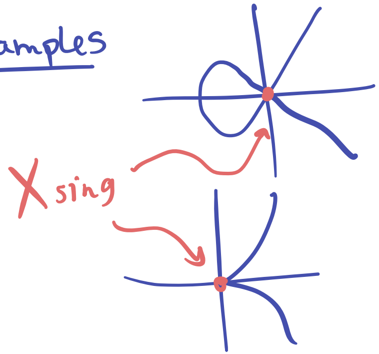
$p \in X$ smooth if $\nabla f_p \neq 0$. $\iff T_p X \cong \mathbb{A}^{n-1}$

singular o.w. $\iff T_p X = \mathbb{A}^n$

$\rightsquigarrow X_{smooth}$

$X_{sing} = X \setminus X_{smooth}$.

examples



Prop. $X = Z(f) \subsetneq \mathbb{A}^n$ irred.

$X_{smooth} \subseteq X$ open, dense

Pf (char $k = 0$)

To show: ① X_{sing} closed

② $X_{smooth} \neq \emptyset$.

① $X_{sing} = Z\left(f, \frac{df}{dx_1}, \ldots, \frac{df}{x_n}\right)$

② Assume $X_{sing} = X$.

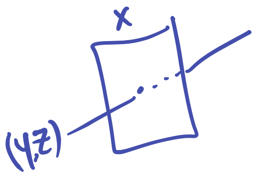
$\Longrightarrow \frac{df}{dx_i} \in (f) \quad \forall i$. since f irred.

Since f not const, this is a contrad. (look at degrees)

Note over $\mathbb{C}$, X smooth is a complex manifold (inverse fn thm).

The reducible case

If X has irred. comp's $\{X_i\}$ say p is smooth if it lies in exactly one X_i & is smooth as a pt in X_i.



$X = Z(xy, xz)$

O is smooth in both components.

but not smooth by our defn.

Back to dimension

Let's write

$$\dim X = \min_{p \in X} \dim T_p X$$

for X irred.

If X is red. with irred comp's X_i,

$$\dim X = \max \dim X_i .$$

Above examples ① - ④ have dim 1.

Prop. $X = Z(f) \subsetneq \mathbb{A}^n$ irred. hypersurf.

$\Longrightarrow \dim X = n-1$.

Prop. $X \subseteq \mathbb{A}^n$ irred.

$\exists$ open, dense $X_0 \subseteq X$ s.t.

$\dim T_p X = \dim X \quad \forall p \in X_0$.

Lemma. $X \subseteq \mathbb{A}^n$ irred. $\forall r \in \mathbb{N}$. The set

$S_r(X) = \{ p \in X : \dim T_p X \geq r \}$ is closed

Pf. Say $\mathbb{I}(X) = (f_1, \dots, f_m)$

$$T_p X = \bigcap Z\left((f_i)_p^{(1)} \right)$$

$$\Rightarrow \dim T_p X = n - \operatorname{rank}\left(\frac{df_i}{dx_j}(p) \right) \quad \square$$

Pf of Prop. Let $r = \dim X \rightsquigarrow S_r X = X$,

$S_{r+1} X \neq X$. $\square$

set of pts with wrong dim

Back to smoothness

X irred, maybe not hypersurf.

$X = Z(f_1, \dots, f_m)$

$p \in X$ is <u>smooth</u> if

$$\operatorname{rank}\left(\frac{df_i}{dx_j}\right) = m.$$

<u>Fact</u>.

p smooth $\iff \dim T_pX = \dim X$

Codim.

$X = Z(f_1, \dots, f_m)$ irred. $\subseteq \mathbb{A}^n$

$$\operatorname{codim} X = n - \dim X$$
$$= \operatorname{rank}\left(\frac{df_i}{dx_j}(p)\right) \quad p \text{ smooth.}$$

$\Longrightarrow \operatorname{codim} X \leq m.$

(also true for red.)

Alg char of dim

We'll show

$$\dim X = \operatorname{trdeg}_k k(X)$$

$$\overset{\text{algebra}}{=} \dim k[X]$$

Hypersurface case

$$k[X] = k[x_1, \ldots, x_n]/(f)$$

WLOG f uses x_1

$$k(X) = k(x_2, \ldots, x_n)[x_1]/(f)$$

which clearly(?) has transc. deg $n-1$.

Goal for today:

coord free description of tangent spaces.

We'll show

$$T_pX \cong \left(M/M^2\right)^* \cong \left(m/m^2\right)^*$$

$$M \subseteq k[x_1,\dots,x_n]$$

$$m \subseteq k(x_1,\dots,x_n)$$

max ideals at p:

$$M = (x_1-p_1,\dots,x_n-p_n)$$

fns that vanish at p

Cotangent spaces

$$T_p^*V = \text{dual of } T_pV$$
$$= \{\text{linear } T_pV \to k\}$$
$$= \text{"linear forms"}$$

Notation: $g \in k[x_1,\dots,x_n]$

d_pg = diff. of g at p.

e.g. $g(x,y) = x^2 + xy + x$

$$dg = (2x+y+1)\frac{d}{dx} + x\frac{d}{dy}$$

$$d_0g = \frac{d}{dx}. \in (k^n)^*$$

Prop. Let $X = Z(f_1, \dots, f_m) \subseteq \mathbb{A}^n$

$p \in X$ $\quad g \in k[X]$

$\rightsquigarrow d_p g$ is lin form on $T_p X$.

Justin R. Smith

Pf. To show well-def.

Say $G_1, G_2 \in k[x_1, \dots, x_n]$ rep. g.

$\Rightarrow G_1 - G_2 = \sum h_i f_i \quad h_i \in \mathbb{I}(X)$

$\Rightarrow d_p(G_1 - G_2) = \sum\left((d_p h_i) f_i(p) + h_i(p)(d_p f_i)\right)$

product rule.

0 by defn

$T_p V$ defined so that this form evals to 0 on it.

$= 0$ $\qquad \square$

Prop. Same X. Differentiation induces

surj $\quad M \longrightarrow T_p^* V$

with kernel M^2.

Pf. Setup. WLOG $p = 0$.

WLOG $T_p V = \langle x_1, \dots, x_r \rangle$

(change of coords)

Let $\tilde{M} = (x_1, \dots, x_n) \subseteq k[x_1, \dots, x_n]$

Its image in $k[X]$ is M.

Prop. Same X. Differentiation induces surj $M \longrightarrow T_p^* X$ with kernel M^2. So: $T_p^* X = M/M^2$.

Pf. Setup. WLOG $p = 0$.

WLOG $T_p X = \langle x_1, \dots, x_r \rangle$ (change of coords)

Let $\tilde{M} = (x_1, \dots, x_n)$

Its image in $k[x]$ is M.

Surjectivity: let $\ell = \sum c_i x_i^* \in T_p^* X$

Then $L = \sum c_i x_i$ has $dL = \ell$

Kernel: Say $g \in M$, $d_0 g \equiv 0 \in T_p^* X$ & g is image of $G \in \tilde{M}$

So $d_0 G \equiv 0$ on $T_0 X$ (first Prop)

Then $d_0 G = \sum \lambda_j (d_0 f_j)$ (by defn of $T_p X$)

Let $\bar{G} = G - \sum \lambda_j f_j$

Then $\bar{G}$ still maps to g in $k[X]$.

But $d_0 \bar{G} \equiv 0$ on $T_0 \mathbb{A}^n$

$\Longrightarrow$ const & lin. terms of $\bar{G}$ vanish.

$\Longrightarrow \bar{G} \in \tilde{M}^2 \Longrightarrow g \in M^2$ $\square$

Moving the $*$

R = ring, $M \subseteq R$ max ideal.

$\rightsquigarrow R \cdot M \subseteq M$, $R \cdot M^2 \subseteq M^2$

So M, M/M^2 modules over R

Also, mult by M on M/M^2 is 0 map.

So M/M^2 is R/M-module
$\hookleftarrow$ field!

i.e. M/M^2 is vect sp. over R/M

So $T_pV = (M/M^2)^*$ makes sense

$(M/M^2)^*$ is called Zariski tangent sp.

Differentials

Prop. $f: X \to Y$ morphism of aav's
$p \in X$

$\rightsquigarrow f_*: T_pX \to T_{f(p)}Y$

Pf. $f_*: k[Y] \to k[X]$

prein of M is N = max ideal for $f(p)$
= fns vanish @ $f(p)$

So $N/N^2 \longrightarrow M/M^2$ $\square$

Coord free descr. of differential

Prop. $X \subseteq \mathbb{A}^n$ irred.

$f \in k[X]$

Then $f - f(p) \in \mathfrak{m}$.

and $d_p f$ = image of $f - f(p)$

in $\mathfrak{m}/\mathfrak{m}^2 = T_p^* V$

Pf. Subtracting $f(p)$ kills const term.

Modding by $\mathfrak{m}^2$ kills quad & higher terms. $\square$

you!

Example $X = Z(x^3 - y^2) \subseteq \mathbb{A}^2$

At $p = (1,1)$ can see $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$:

$$\mathfrak{m} = (x-1, y-1)$$

$$\rightsquigarrow \mathfrak{m}^2 = (x^2 - 2x + 1, (x-1)(y-1), \overset{x^3}{\cancel{y^2}} - 2y + 1)$$

$$\rightsquigarrow y - 1 = (x^3+1)/2 - 1$$

$$= (x(2x-1)+1)/2 - 1$$

$$= (2x^2 - x + 1)/2 - 1$$

$$= (3x-1)/2 - 1$$

$$= \tfrac{3}{2}(x-1)$$

At $p = (0,0)$ can see $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$:

$$\mathfrak{m} = (x, y), \quad \mathfrak{m}^2 = (x^2, xy, y^2)$$

$$\rightsquigarrow \mathfrak{m}/\mathfrak{m}^2 = \{ax + by\}$$

Projective varieties

$\mathcal{O}_{X,p} = \{ f/g \in K(X) : g(p) \neq 0 \}$

$m \in f/g \in \mathcal{O}_{X,p}$ s.t. $f(p) = 0$

max ideal.

Lemma. X, M, m, p as above.

$M/M^2 \cong m/m^2$

Pf. WLOG $p = 0$.

Inclusion $M \hookrightarrow m$

Induces injection

$M/M^2 \hookrightarrow m/m^2$

Surj Let $f/g \in m/m^2$ so $g(0) \neq 0$

$\rightsquigarrow f/g(0) - f/g$

$= f\left(1/g(0) - 1/g \right) \in m^2$

So $f/g(0) = f/g$ in m/m^2

in $M \subseteq k[X]$

$\square$

Cor 1. $f : X \dashrightarrow Y$ rat.

$\rightsquigarrow f_* : T_p X \to T_{f(p)} Y$.

Cor 2. X, Y birat $\Rightarrow \dim X = \dim Y$

Back to dim

Thm. $X \subseteq \mathbb{A}^n$ irred

$$\dim X = \operatorname{tr deg}_k k(X).$$

Milne

Also: $\operatorname{trdeg}_k k(X) = \dim k[X]$.

Pf. If it's true for hypersurfaces, true for all X since every X is birat. equiv to hypersurf. (Noether norm).

Hulek

For hypersurfaces:

We proved $\dim = n-1$ so suff. to show $\operatorname{tr deg} = n-1$

$X = Z(f) \subseteq \mathbb{A}^n$ f irred.

$\rightsquigarrow k[X] = k[x_1, \dots, x_n]/(f)$

WLOG f uses x_1

$$k(X) = \underbrace{k(x_2, \dots, x_n)}_{\text{transc. basis.}}[x_1]/(f)$$

Blow Ups

or: Zooming in

Two problems

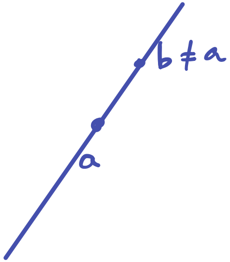
① Varieties have singularities

② Rational maps not def everywhere

$\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$

def. on $\mathbb{P}^n \setminus a$

No way to extend over a.

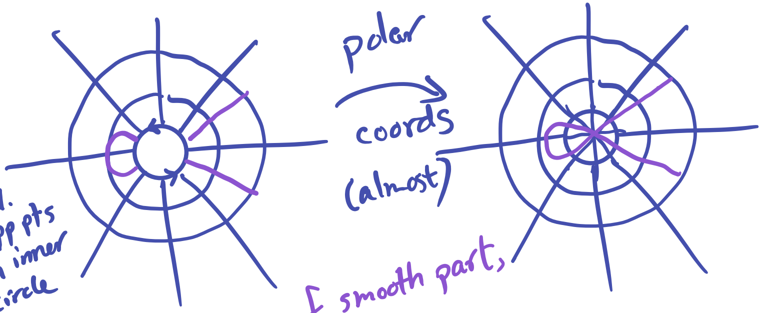


Blowup is a tool for fixing these.

Idea of blowup

Replace pt p with set of lines thru p

Picture over $\mathbb{R}$:



Id. opp pts on inner circle

Möbius band

take preim of smooth part, then take closure singularity gone!

The blowup of $\mathbb{A}^2$ at O

$$\pi : \mathbb{A}^n \setminus 0 \longrightarrow \mathbb{P}^{n-1}$$
$$(a_1, \dots, a_n) \longmapsto [a_1 : \cdots : a_n]$$

$\Gamma_\pi \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ graph.

$\widetilde{\mathbb{A}}^n = \overset{\text{Zar.}}{\text{closure}}$ of Γ_π in $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

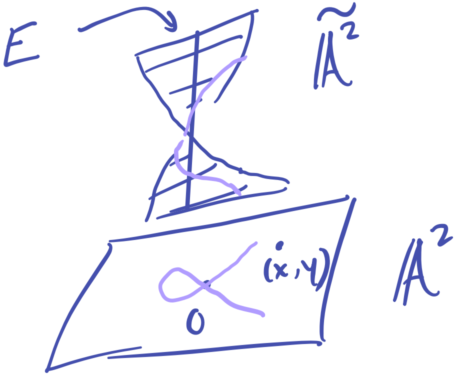
↳ blowup of $\mathbb{A}^n$ at O.

$n=2$ case

$\pi(x,y) = [x:y]$ (or x/y)

$\widetilde{\mathbb{A}}^2 = \{((x,y),[t_0:t_1]) : x t_1 = y t_0\}$

Check: this is the closure of Γ_π.



Projection to $\mathbb{A}^2$ induces

$$p : \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$$

and $p^{-1}(x,y) = \begin{cases} ((x,y),[x:y]) & (x,y) \neq O \\ \underbrace{(0,0) \times \mathbb{P}^1}_{E \,=\, \text{exceptional line/divisor}} & (x,y) = O. \end{cases}$

Fact. p induces $\widetilde{\mathbb{A}}^2 \setminus E \xrightarrow{\sim} \mathbb{A}^2 \setminus 0$

Affine cover of $\tilde{\mathbb{A}}^2$

$\mathbb{P}^1$ has std. aff. cover U_0, U_1

$\rightsquigarrow \tilde{\mathbb{A}}^2 = V_0 \cup V_1 \quad V_i \subseteq \mathbb{A}^2 \times \mathbb{A}^1$

where

$$V_0 = \{((x,y),[1:t_1]) : xt_1 = y\}$$

$$V_1 = \{((x,y),[t_0:1]) : x = yt_0\}$$

Note: $V_i \cong \mathbb{A}^2$

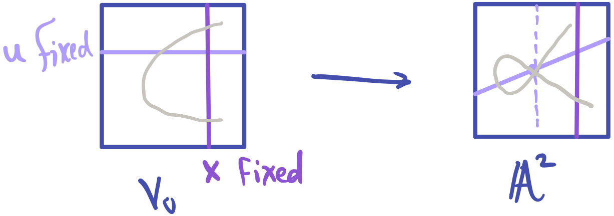
V_0 coords: $x, u = t_1$

V_1 coords: $y, v = t_0$

So $V_0 = \{((x,ux),[1:u])\} = \{(x,u)\}$

$V_1 = \{((vy,y),[v:1]\} = \{(y,v)\}$

Under $p : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$



Hor lines $\longrightarrow$ lines thru origin

(get all but vertical)

Vert lines $\longrightarrow$ vert lines.

Similar for V_1.

Resolving singularities

Say $X \subseteq \mathbb{A}^n$ sing. set S

A resolution is

$p: \tilde{X} \longrightarrow X$ s.t. $\tilde{X}$ nonsing

& restr. $\tilde{X} \setminus p^{-1}(S) \longrightarrow X \setminus S$

is an isomorphism.

Resolution for

curves: blow up pts

surfaces over $\mathbb{C}$: Jung, Walker
Zariski '35

3-folds char $= 0$: Zariski
Annals '44

3-folds char $\neq 0$: Abhiyankar (Z's student)

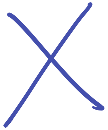
All varieties char 0: Hironaka ~'70

char $\neq 0$ open.

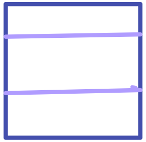
We'll look at curves :)

Example 1

$C = Z(x^2 - y^2)$



resolution:



in $\widetilde{\mathbb{A}}^2$

Higher dim version:

$X = Z(x^2 + y^2 - z^2)$

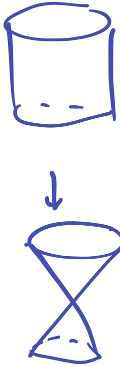
$\tilde{X} = Z(x^2 + y^2 - 1)$

$\tilde{X} \longrightarrow X$

$(x, y, z) \longmapsto (xz, yz, z)$

$xy\text{ plane} \longmapsto pt$

$\tilde{X}$



X

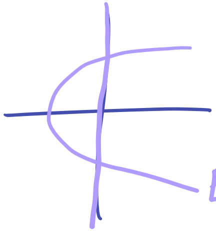
Example 2

$C = Z(y^2 - x^2 - x^3)$

$p^{-1}(C) = \{((x,y), [t_0 : t_1]) : y^2 = x^3 + x^2,\ t_0 y = t_1 x\}$

$p^{-1}(C) \cap V_0 = \{((x, xu), [1 : u]) : x^2(x + 1 - u^2) = 0\}$

$= \{(x, u) : x^2(x + 1 - u^2) = 0\} \subseteq \mathbb{A}^2$



E

$p^{-1}(C) = \text{parabola} \setminus pt$

closure $\tilde{C}$ is parabola.

Smooth!

Example 3

$C = Z(y^2 - x^3)$

$p^{-1}(C) \cap V_0 = \{(x, u) : \cancel{(xu)^2 = x^3}\}$

$x^2(x - u^2)$

$\rightsquigarrow$ parabola.

Aside: link of cusp is $(3,2)$-curve on T^2

(trefoil)

Blowing up higher-dim subvars

Algebra version: Harris

$Y \subseteq X \subseteq \mathbb{A}^n$ aav's

$Y = Z(f_0, \dots, f_m)$ $f_i \in k[X]$

Define:

$\varphi : X \dashrightarrow \mathbb{P}^m$

$x \longmapsto [f_0(x) : \dots : f_m(x)]$

regular on $X \setminus Y$

$\Gamma_\varphi \subseteq \mathbb{A}^n \times \mathbb{P}^m$ & $p : \Gamma_\varphi \to X$

closure is

$Bl_Y(X)$ blowup of X at Y.

$p^{-1}(Y)$ "exceptional divisor"

example $0 = Y \subseteq X = \mathbb{A}^2$

$Y = Z(x, y)$

$\varphi : \mathbb{A}^2 \to \mathbb{P}^1$

$(x, y) \longmapsto [x : y]$

Can do similar for proj var's

(use homog. polys).

Topological version:

Read in Harris.

Idea: replacing pts in Y with space of normal directions.

e.g. $Y = z$-axis in $\mathbb{A}^3$: pts in Y get replaced with $\mathbb{P}^1$

Thm X variety

$\varphi : X \dashrightarrow \mathbb{P}^n$ rat'l

Then $\exists$

$$\begin{tikzcd}
X_k \arrow[d] \arrow[dddr, "\text{morphism}"] & \\
\vdots \arrow[d, "\text{blowups}"'] & \\
X_1 \arrow[d] & \\
X = X_0 \arrow[r, dashed] & \mathbb{P}^n
\end{tikzcd}$$

So: a rat'l map is a reg map on some blowup.

HEISUKE HIRONAKA

"Resolution of singularities of an algebraic variety over a field of characteristic 0." Annals of Math.

Introduction

Let X be complex-(resp. real-)analytic space, i.e., an analytic **C**-(resp. **R**-)space in the sense defined in §1 of Chapter 0. *We ask if there exists a morphism of complex-(resp. real-)analytic spaces, say* $f: \tilde{X} \to X$, *such that*:

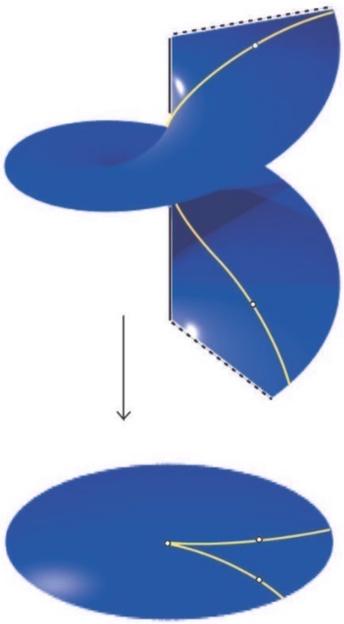
(1) $\tilde{X}$ *is a complex-(resp. real-)analytic manifold*, i.e., *a non-singular complex-(resp. real-)analytic space*,

(2) *if* V *is the open subspace of* X *which consists of the simple points of* X, *then* $f^{-1}(V)$ *is an open dense subspace of* $\tilde{X}$ *and* f *induces an isomorphism of complex-(resp. real-)analytic manifolds*: $f^{-1}(V) \xrightarrow{\approx} V$, *and*

(3) f *is proper*, i.e., *the preimage by* f *of any compact subset of* X *is compact in* $\tilde{X}$.

This is the problem which we call *the resolution of singularities in the category of complex-(resp. real-)analytic spaces*, or more specifically, the resolution of singularities of the given complex-(resp. real-)analytic space X. If X is a reduced *complex*-analytic space, then the open subspace V is dense in X and therefore the condition (2) implies that f is a *modification*. (The term '*reduced*' means that the structural sheaf of local rings has no nilpotent elements.) It should be noted, however, that V is not always dense if X is a reduced *real*-analytic space. So far as the resolution of singularities is concerned, we are particularly interested in the case of reduced complex-(resp. real-)analytic spaces. As for the general case in which X may not be reduced, we have a better formulation of the problem in terms of normal flatness. (See Definition 1, § 4, Ch. 0.)

The most significant result of this work is the solution of the above problem for the case in which X has *an algebraic structure*; that is to say, X is covered by a finite number of coordinate neighborhoods, each of



Degree

Harris.

$X = Z(f)$ hypersurf.

$\rightsquigarrow \deg X$ defined as

$\deg f$.

More generally:

$X \subseteq \mathbb{P}^n$ irred, k-dim

$\rightsquigarrow \deg X$ is

① deg of any hypersurf in $\mathbb{P}^{k+1}$ birat eq to X

② the deg of a cover $X \to \mathbb{P}^k$

③ # pts of int. of generic $(n-k-1)$-plane with X

If X is a complex manifold

$\rightsquigarrow [X] \in H_{2k}(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z}$

is the degree.

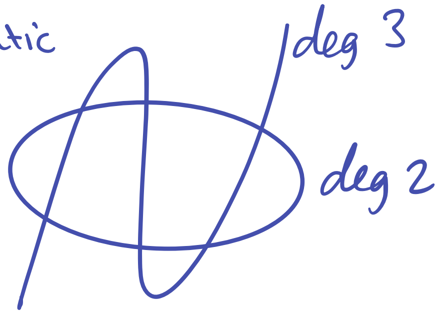
(also works for singular X).

Chapter 5 Curves or Bézout & applications.

k = alg closed.

B's thm. $C, D \subseteq \mathbb{P}^2$ curves of deg m, n. If C, D have no irred comp's in common then they intersect mn times with mult.

schematic



Special cases

① C, D lines $\rightsquigarrow$ 1 pt.

$\mathbb{P}^2$ exists so all lines intersect.

From this: all curves int. the right # of times.

(Like how solving $x^2+1=0$ allows to solve all polynomials)

② $C = Z(f)$ curve

$D = Z(ax+by+z)$ line.

Use $ax+by+z$ to elim. z from f

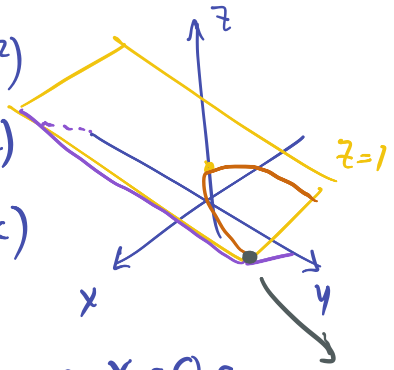
Passing to a generic affine chart, get a poly of $\deg = \deg f$ in one var.

Apply FTA.

you finish the proof of Bezout in this case.

<u>example</u>. $C = Z(yz - x^2)$

$D = Z(z - ax)$



$C \cap D = (yz - x^2, z - ax)$

$= (axy - x^2)$

Set $y=1$: $ax - x^2 = 0 \rightsquigarrow x = 0, a$

$\rightsquigarrow [0:1:0]$ & $[a:1:a^2]$

When $a = 0$ get one pt of mult 2.

Special case of ②: Every conic meets line at ∞ in 2 pts w/ mult.

(cont.)

Special case of ②: Every conic meets line at ∞ in 2 pts w/ mult.

e.g. circle $(x-az)^2+(y-bz)^2=r^2z^2$

always contains $[1:i:0]$ & $[1:-i:0]$

If C, D both circles, they meet at those 2 pts plus 2 more in $\mathbb{A}^2$

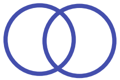
unless... concentric, in which case the 2 pts at ∞ have mult 2.

Similar: • hyperbola meets line at ∞ at the asymptotes

• parabola meets it at 1 pt with mult 2. (prev. ex. $a=0$)

example $C = Z(x^2+y^2-z^2)$

$D = Z((x-z)^2+y^2-z^2)$

circles 

you: find the 2 pts not at ∞.

Resultants

Goal: find common zeros of two polys

Say $f(x) = a_0 + \cdots + a_m x^m$

$g(x) = b_0 + \cdots + b_n x^n$

The resultant $\mathrm{Res}(f,g)$ is det of the $(m+n)\times(m+n)$ Sylvester matrix.

$$\begin{array}{c}\begin{pmatrix} a_0 & a_1 & \cdots & a_m & & \\ & a_0 & \cdots & \cdots & a_m & \\ & & \ddots & & & \ddots \\ & & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & & \\ & \ddots & & \ddots & & \\ & & b_0 & \cdots & b_n & \end{pmatrix}\end{array} \begin{array}{l} n \text{ times} \\ \\ m \text{ times} \end{array}$$

Prop. $\mathrm{Res}(f,g) = 0 \iff Z(f) \cap Z(g) \neq \emptyset$.

equiv: f,g no common factors.

Linear case

$$a_0 + a_1 x = 0$$
$$b_0 + b_1 x = 0.$$

$$\rightsquigarrow \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \quad \checkmark$$

Quadratic case

$$a_0 + a_1 x + a_2 x^2 = 0$$
$$b_0 + b_1 x + b_2 x^2 = 0$$

$$\rightsquigarrow \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix}$$

<u>Note</u>

$(1, x, x^2)$ solves 1^{st} eqn

$\Leftrightarrow (x, x^2, x^3)$ solves second.

Can see rank ≥ 3 (look at 1^{st} 3 rows)

$\Rightarrow \dim \ker \leq 1$

So $\det = 0 \Rightarrow \dim \ker = 1$

Observe: Can artificially make 2 new eqns

$$a_0 x + a_1 x^2 + a_2 x^3$$
$$b_0 x + b_1 x^2 + b_2 x^3$$

Now we have 4 lin eqns in the "variables" x, x^2, x^3

Take a vector in null space of Sylv. with first entry 1

$$\begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = 0$$

Prop. $\mathrm{Res}(f,g)=0 \iff f,g$ have common root.

Pf. Say α = root of f & g.

$\rightsquigarrow \exists$ polys f_1, g_1 of deg $m-1, n-1$

s.t. $f(x) = (x-\alpha) f_1(x)$

$g(x) = (x-\alpha) g_1(x)$

$$\Rightarrow f(x) g_1(x) - g(x) f_1(x) = 0.$$

Both terms have deg $m+n-1$.

Equating coeffs to 0 gives

$m+n$ lin eqns in $m+n$ vars.

(coeffs of f_1, g_1)

The matrix is the Sylv. matrix.

The existence of a soln shows $\mathrm{Res}(f,g)=0$.

Conversely, say $\mathrm{Res}(f,g)=0$. As above get soln f_1, g_1 to

$$f(x) g_1(x) - g(x) f_1(x) = 0.$$

A root α of f must be a root of g or f_1. If g, done.

If f_1, cancel $(x-\alpha)$ from f & f_1 continue inductively.

$\square$

In proj space

$C = Z(f)$

$D = Z(g)$

Assume WLOG $[0:0:1]$ on neither.

($\exists$ purely z term)

$\rightsquigarrow f(x,y,z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$

$g(x,y,z) = z^n + \cdots + b_0$

a_i, b_i homog polys in x,y of deg $m-i, n-i$

$\rightsquigarrow R(x,y)$ resultant wrt z

poly in x,y.

Prop. $R(x,y)$ either $\equiv 0$ or deg mn.

Example. $f(x,y,z) = x^2+y^2-z^2$

$g(x,y,z) = x^3 - x^2z - xz^2$

$\rightsquigarrow R(x,y) = -x^2y^4 \rightsquigarrow x = 0$ or $y = 0$

2 roots w/ mult

4 roots w/ mult.

$x=0: \; y^2 - z^2 = 0 \rightsquigarrow [0:1:1], [0:1:-1]$

both have mult 2. $\longrightarrow$ $y=0: \; x^2 - z^2 = 0 \rightsquigarrow [1:0:1], [1:0:-1]$

Resultants

$f(x) = a_0 + a_1 x + \cdots + a_m x^m$

$g(x) = b_0 + \cdots + b_n x^n$

$\rightsquigarrow$ Sylvester matrix

$$\begin{array}{cl} \begin{pmatrix} a_0 & \cdots & a_m & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_n \end{pmatrix} & \begin{array}{l} n \text{ times} \\ \\ \\ m \text{ times} \\ \\ \\ \end{array} \end{array}$$

det. is $\mathrm{Res}(f,g)$.

Prop. $\mathrm{Res}(f,g) = 0 \iff$ common factor.

Lemma. f, g have common factor $\iff$

$\exists\, s,t$: $\deg s < \deg g$

$\deg t < \deg f$

$fs + gt = 0$.

Pf. ($\Rightarrow$) f, g have common factor

$\Rightarrow f = hf_1 \quad g = hg_1$

$\Rightarrow fg_1 - gf_1 = 0$.

($\Leftarrow$) $fs + gt = 0$. Assume no comm. fac.,

$\Rightarrow$ roots of f are roots of gt

$\Rightarrow$ roots of f are roots of t

but $\deg t < \deg f$. $\square$

Prop. $\operatorname{Res}(f,g) = 0 \iff$ common factor.

Lemma. f, g have common factor $\iff$

$\exists\ s, t:$ $\deg s < \deg g$

$\deg t < \deg f$

$fs + gt = 0.$

Pf of Prop. Want to know existence of s, t as in Lemma.

Let $P(x) = f(x)\, s(x) + g(x)\, t(x)$

$$s = \sum_{i<n} s_i x^i \quad t = \sum_{i<m} t_i x^i$$

$$\rightsquigarrow P(x) = (a_m s_{n-1} + b_n t_{m-1}) x^{m+n-1} + \cdots$$

Solving for $(s_0, \dots, s_{n-1}, t_0, \dots, t_{m-1})$

or:

$$(s_0, \dots, s_{n-1}, t_0, \dots, t_{m-1}) \cdot \begin{pmatrix} a_0 & \cdots & a_m & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_n \end{pmatrix} = 0$$

$\square$

Curves in $\mathbb{P}^2$ version

$C = Z(f)$, $D = Z(g)$

WLOG $[0:0:1] \notin C \cup D$.

$\Rightarrow$ f, g have Z-only term, which must be lead term.

$\rightsquigarrow$ $\left.\begin{array}{l} f = a_0 Z^m + a_1 Z^{m-1} + \dots \\ g = b_0 Z^n + b_1 Z^{n-1} + \dots \end{array}\right\}$ deg m, n as poly's in Z, coeffs in $k[x,y]$.

a_i, b_i homog. deg i in x, y.

$\rightsquigarrow$ $R(x,y) = \mathrm{Res}(f,g)$ poly in x, y.

Prop. $R(x,y)$ is either: $\equiv 0$ or homog. of deg mn.

Pf. To show $R(tx,ty) = t^{mn} R(x,y)$

$$R(tx,ty) = \begin{pmatrix} a_0 & ta_1 & t^2a_2 & \cdots & \\ & ta_0 & t^2a_1 & \cdots & \\ & & & & \\ b_0 & tb_1 & \cdots & & \\ & tb_0 & t^2b_1 & \cdots & \end{pmatrix}$$

Mult. rows by $t^0, t^1, \dots, t^{n-1}, t^0, \dots, t^{m-1}$

factor: $n(n-1)/2 + m(m-1)/2$

Divide cols by $t^0, \dots, t^{m+n-1}$ $\rightsquigarrow \mathrm{Res}(f,g)$

factor: $(m+n)(m+n-1)/2$

Difference is mn. $\square$

Bézout's Thm $C, D \subseteq \mathbb{P}^2$ curves of deg m, n w/ no common irred comp. Then they intersect mn times with mult.

Pf. **Setup**

① Suffices to consider C, D irred.

② $\dim C \cap D = 0$
$\Rightarrow |C \cap D| < \infty$.

③ WLOG change coords so $x \neq 0$ at all pts of $C \cap D$.

④ Say $C = Z(f)$, $D = Z(g)$
$\rightsquigarrow R(x,y)$

Step 1. $R(x,y)$ homog. of deg mn

If $R(x,y) \equiv 0$ then $\forall [a:b] \in \mathbb{P}^1$ f, g have common 0, violating ②.

Apply Prop.

Write $R(x,y) = x^{mn} R_*(y/x)$ where R_* is poly in $t = y/x$ of deg $\leq mn$.

Step 2. $\deg R_* = mn$.

$\deg R_* < mn \Leftrightarrow$ no y^{mn} term. $\Leftrightarrow$ all terms of R have $x \Leftrightarrow R(0,1) = 0$ violates ③.

Step 3 {Roots of R_*}

$\longleftrightarrow$ $C \cap D$ w/ mult.

($\rightarrow$) If α is a root of R_*

then $\alpha = a/b$ with $R(a,b) = 0$.

$\Rightarrow$ $f(a,b,z)$, $g(a,b,z)$ have

common root c $\rightsquigarrow$ pt of $C \cap D$ of form $[a:b:c]$.

($\leftarrow$) $[a:b:c] \in C \cap D$ $a \neq 0$

$\Rightarrow$ b/a root of R_*

Need Setup ⑤ No two pts $[a:b:c], [a:b:c'] \in C \cap D$. $\Longleftrightarrow$ no pts of $C \cap D$ lie on line $\parallel$ to z-axis.

No 2 pts of $C \cap D$ colin with $[0:0:1]$

$\square$ $\Updownarrow$

Define multiplicity now :)

~~# common roots c w/ mult. corresp. to given α.~~

~~or deg of c as root of $f-g$ @ (a,b).~~ more prec.

~~Claim. This equals deg. of α as root of R_*~~

Mult. defined as mult of root of R_*.

Right defn of mult

Fitchett.

Let $p \in C \cap D$

Assume p in std aff. chart $Z = 1$.

Define

$$i(C \cap D, p) = \dim_k \left(\frac{\mathcal{O}_p}{(f,g)_p} \right)$$

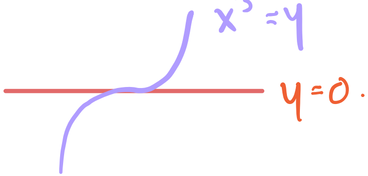
$\mathcal{O}_p$ = rat'l fns def. at p.

$(f,g)_p$ = ideal gen by f, g in $\mathcal{O}_p$.

"localization": allow denominators that don't vanish at p

i.e. denoms don't lie max ideal at p which is $(x_1 - p_1, \ldots, x_n - p_n)$.

Example 1. $k = \mathbb{C}$



$f(x,y) = y - x^3$

$g(x,y) = y$

exercise.

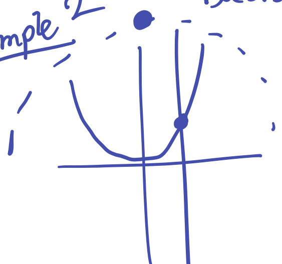
$$\frac{\mathcal{O}_p}{(f,g)_p} = \frac{\mathbb{C}[x,y]_{(x,y)}}{(y-x^3, y)_{(x,y)}} \cong \left(\mathbb{C}[x,y] \big/ (y-x^3, y) \right)_{(x,y)}$$

$$\cong \left(\mathbb{C}[x] \big/ (x^3) \right)_{(x)} \cong \mathbb{C}^3 \text{ as } \mathbb{C}\text{-v.s.}$$

basis $1, x, x^2$

exercise.

Example 2



Why are the multiplicities the same?

Write $I_p(C,D)$.

or $I_p(f,g)$

Fulton

Gims (?)

Axioms

① $I_p(f,g) = I_p(g,f)$

② $I_p(f,g) = \begin{cases} \infty & p \text{ in a common comp.} \\ 0 & p \notin C \cap D \\ \in \mathbb{N} & \text{otherwise.} \end{cases}$

③ C, D lines, $p \in C \cap D \Rightarrow I_p(f,g) = 1$

④ $I_p(f_1 f_2, g) = I_p(f_1, g) + I_p(f_2, g)$

⑤ $I_p(f,g) = I_p(f, g + fh)$ if $\deg h = \deg g - \deg f$.

<u>Thm</u>. An $I_p(f,g)$ satisfying the axioms exists and is unique.

Final HW

7 problems on web site

or project on Teams 1-2 "pages"

Learn something & tell us about it.

Ideas: Robotics

Splines

Image deblurring.

Divisors.

Riemann-Roch Thm

Gröbner bases

Fano varieties

Poncelet's porism

Hironaka's thm

Sheaves / Schemes.

or creative writing
artwork.

Computing multiplicities

$C = Z(x^2+y^2-z^2)$

$D = Z(x^2+y^2-2z^2)$

$C \cap D = [\pm i : 1 : 0]$

Via axioms

$$I_p(x^2+y^2-z^2,\ x^2+y^2-2z^2)$$

$$= I_p(x^2+y^2-z^2,\ z^2) \quad \text{“row op”}$$

$$= I_p(x^2+y^2,\ z^2)$$

$$= 2I_p(x^2+y^2,\ z)$$

$$= 2I_p(x+iy, z) + 2I_p(x-iy, z)$$

$$= 2+0 \quad \text{or} \quad 0+2 \qquad \text{“lines”}$$

dep. on p

Via resultant

$$R(x,y) = \det\begin{pmatrix} -1 & 0 & x^2+y^2 & 0 \\ 0 & -1 & 0 & x^2+y^2 \\ -2 & 0 & x^2+y^2 & 0 \\ 0 & -2 & 0 & x^2+y^2 \end{pmatrix}$$

$$= (x^2+y^2)^2$$

$$\rightsquigarrow R_*(t) = (1+t^2)^2$$

$$= (1+it)^2(1-it)^2$$

via Local rings

$f = Z(y-x^2)$ $g = Z(y)$

$$\mathbb{C}[x,y]_{(x,y)} \big/ (f,g)_{(x,y)}$$

$$\cong \left(\mathbb{C}[x,y]\big/(f,g)\right)_{(x,y)}$$

$$\stackrel{\text{as a set}}{=} \left\{ \frac{ax+b}{cx+d} : d \neq 0 \right\}$$

WTS : dim = 2

basis 1, x.

rationalize:

$$\frac{ax+b}{cx+d} \cdot \frac{-cx+d}{-cx+d}$$

$$= \frac{-acx^2 + (ad-bc)x + bd}{d^2 - c^2x^2}$$

$$= \left(\frac{ad-bc}{d^2}\right)x + \frac{b}{d}$$

exercise. Do example on last slide this way.

easier. Fitchett example $x^3 = y$, $y = 0$.

Easy conseq's of Bézout

1. If $|C \cap D| = mn$, all intersections are transverse (mult 1)
2. If $|C \cap D| > mn$, common irred. comp.
3. Any two proj. curves intersect.
4. $|C \cap L| = m$ with mult. (L line).

More conseq's

5. C irred has at most $\binom{d-1}{2}$ sing pts.

   Gathmann Prop 13.5

   Fulger Cor 8.14

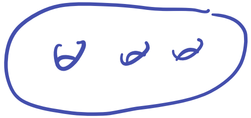
   $d = \deg C$

6. Degree genus formula.

   C smooth $\Rightarrow g = \binom{d-1}{2}$

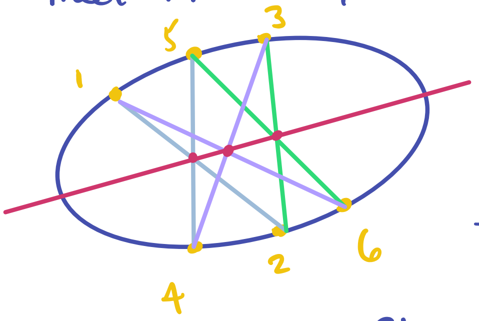
   Kerr Sec 14.3.

really

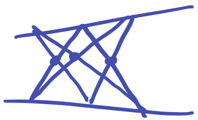


7. Pascal's mystic hexagon.

If a hexagon is inscribed in an irred conic then opp sides meet in colin pts.



Pappus



Stevens Prop 3.2.1

8. Cayley - Bacharach Thm

Kerr Thm 15.12

Cor 15.2.2

C, D cubic $|C \cap D| = 9$

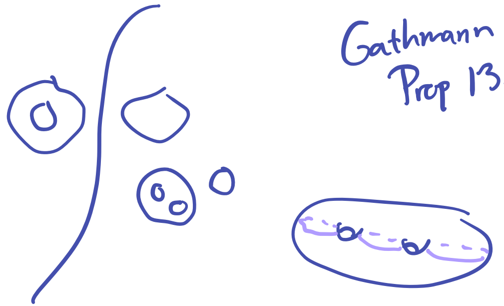
Any cubic passing thru 8 of the pts, passes thru the 9th

9. Space of smooth? curves of deg d in $\mathbb{P}^2$ is a complex proj space of dim $d(d+3)/2$

Moonen p.240

10. Harnack's thm

A smooth real proj. curve in $\mathbb{P}^2$ has at most $\binom{d-1}{2}+1$ loops



Gathmann Prop 13.11

(11) A nonsing cubic has 9 pts of inflection

Gims Cor 5.4

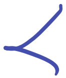
(12) Aut $\mathbb{P}^n \cong GL_{n+1} k$.

Gathmann Prop 13.4

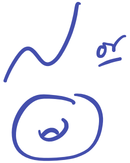
Classific. of conics: 

(13) Classific. of irred singular cubics

$y^2 - x^3 - x^2$ 

$y^2 - x^3$ 

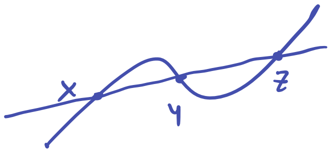
and smooth cubics:

$y^2 = 4x^3 - ax - b$ 

Hulek Prop 4.11

(14) Smooth cubics are groups!

Dolgachev p.52



Mystic Hexagon

Prop. Say C, D deg n
$|C \cap D| = n^2$
Assume mn of the int pts lie on irred curve E of deg m
Then the remaining $n(n-m)$ pts lie on a curve F of $\leq n-m$.

Other pf Shafarevich

Pf of Mystic Hex

Say vertices are $p_0, \dots, p_5$
Let L_i = line thru $p_i p_{i+1}$ (mod 5)
Let $L = L_0 L_2 L_4$ $L' = L_1 L_3 L_5$ cubics.
L & L' have no common factor.
Bézout $\Rightarrow$ $|L \cap L'| \leq 9$.
If < 9, nothing to do.
6 pts of $L \cap L'$ are $p_0, \dots, p_5$ which lie on conic.
The other 3 line on a line by the Prop. $\square$

Prop. Say C, D deg n
$|C \cap D| = n^2$
Assume mn of the int pts lie on irred curve E of deg m
Then the remaining $n(n-m)$ pts lie on a curve F of deg $\leq n-m$.

Pf. Say C, D, E given by f, g, h.
Let $[a:b:c] \in E \setminus (C \cap D)$
Let $F_0 = Z(p)$

$$p = g(a,b,c) f - f(a,b,c) g$$

deg $\leq n$.

Then $|E \cap F_0| \geq mn+1$ b/c it contains mn pts of $E \cap (C \cap D)$ and $[a:b:c]$.

Bézout $\Rightarrow$ F_0, E have common comp.

E irred, so it is a comp. of F_0

$$\Rightarrow p = \overset{E}{h} q \qquad \deg q \leq n-m$$

Let $F = Z(q)$

Each $[u:v:w]$ in $(C \cap D) \setminus E$ satisfies $f=0$, $g=0$ thus satisfies $p=0$

Also $h(u,v,w) \neq 0 \Rightarrow q(u,v,w) = 0$.

i.e. $[u:v:w] \in F$. $\square$

Curves thru given pts

Terry Tao (Husmöller)

Existence

① Thru 2 pts ∃ line.

$ax + by = c$

2 constraints (given pts)
3 unknowns a, b, c.

② Thru 5 pts ∃ quadric

(same lin. alg)

Bézout ⟹ if 3 are collinear

then quadric is reducible
(not a conic) ⇝ union of 2 lines.

③ Thru 9 pts ∃ cubic (same lin alg).

Uniqueness

Need some kind of general posn.

① always unique (if 2 pts distinct)

② if all 5 pts collinear, then can take that line & any other line.

BUT if no 3 collinear get uniqueness: can't be 2 distinct lines by hyp.
can't be two different quadrics by Bezout.

③ If 8 of 9 pts lie on conic C then many $C \cup L$ are cubics containing the 9 pts.

Even without this, uniqueness harder to come by.

Say $C_0 = Z(f_0)$
$C_\infty = Z(f_\infty)$ cubics.

& $|C_0 \cap C_\infty| = 9$.

Then $C_t = Z(f_0 + t f_\infty)$ $t \in k \cup \infty$ contains all 9 pts.

But these are only ones going through the 9 pts, or even 8 of them.

Cayley-Bacharach thm k alg closed.

If D is a cubic curve passing thru 8 pts of $C_0 \cap C_\infty$ then $D = C_t$ some t. In partic, D passes thru the 9^{th} pt.

Pf. Assume $\exists$ D passing thru 8 pts $a_1, \dots, a_8$ of $C_0 \cap C_\infty$.

Say $C_0 = Z(p_0)$ $C_\infty = Z(p_\infty)$

$D = Z(p)$ Want: $D = C_t$

Assume $D \neq C_t$ any t. (*)

Claim 1. No 4 of the a_i collin.

Pf. Bézout $\Rightarrow$ C_0, C_∞ would both contain this line.

$\Rightarrow |C_0 \cap C_\infty| = \infty > 9$.

Claim 2. No 7 of the a_i lie on quadric.

Pf. Same.

Claim 3. Any 5 of the a_i determine a unique quadric

Pf. If 5 pts lie on two quadrics E, F

Bezout $\Rightarrow$ $E \cap F$ contains line L

Claim 1 $\Rightarrow$ L contains at most 3 of the a_i.

The other ≥ 2 pts must lie on other comp of E (line) & other comp of F. (line)

Both are lines. Must be same line. So $E = F$.

Claim 4. No 3 of the a_i collin.

Pf. Say $a_1, a_2, a_3 \in L$ line

Claim 1 $\Rightarrow a_i \notin L$ $i > 3$.

$a_4, \dots, a_8$ lie on unique quadric E.
(Claim 3)

Let b be another pt on L
c be another pt not on E or L.

By lin alg $\exists$ cubic

$$q = x p + y p_0 + z p_\infty$$

vanishing at b, c. (3 vars
2 eqns)

By ⊛ $q \neq 0$.

Fact A linear interp
b/w C_0, C_∞, D
cont. $a_1, \dots, a_8$

Let $F = Z(q)$.

$F \cap L$ contains a_1, a_2, a_3, b

Bezout $\Rightarrow F = L \cup$ quadric

The quadric contains
$a_4, \dots, a_8$ (p, p_0, p_∞ all
vanish at
$a_1, \dots, a_8$)

By uniqueness of E:

$$F = L \cup E$$

but c not in E, L hence
not in F. contradiction

$\square$

Claim 5. No 6 of $a_1, \dots, a_8$ lie on a quadric.

Pf. Say $a_1, \dots, a_6$ lie on Q = quadric.

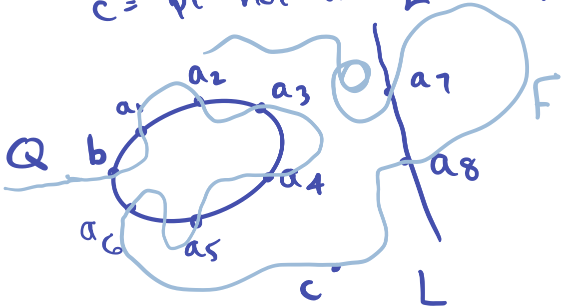
Claim 4 $\Rightarrow$ $Q \neq L_1 \cup L_2$

$\Rightarrow$ Q conic.

Let L = line thru a_7, a_8.

b = another pt on Q

c = pt not on L or Q



As before, have nonzero cubic

$$q = xp + yp_0 + zp_\infty$$

vanishes at b, c. Also at $a_1, \dots, a_8$

Let $F = Z(q)$. Note $b, c \in F$.

F contains $a_1, \dots, a_6, b \in Q$

$\Rightarrow F = Q \cup$ line

The line is L, but L hence F does not contain c.

Contrad.

Finishing...

Let L = line thru a_1, a_2

Q = quadric thru $a_3, \dots, a_7$

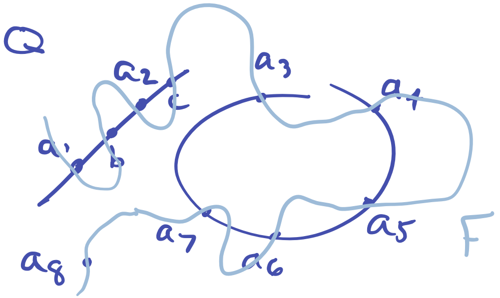
Claim 3 $\Rightarrow$ Q unique

Claim 4 $\Rightarrow$ Q conic (can't be 2 lines)

Claim 4 $\Rightarrow$ $a_8 \notin L$

Claim 5 $\Rightarrow$ $a_8 \notin Q$

Let $b, c \in L \setminus Q$



Again $\exists$ non-0 cubic

$q = xp + yp_0 + zp_\infty$

vanishing on b, c $\rightsquigarrow$ $F = Z(q)$

$F \cap L$ contains a_1, a_2, b, c

Bézout $\Rightarrow$ $F = L \cup$ quadric

The quadric contains $a_3, \dots, a_7$

So it is Q

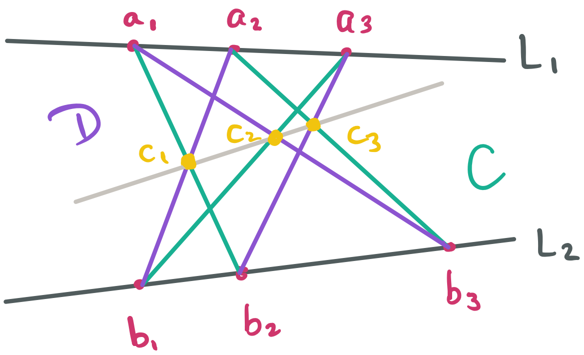
So $F = L \cup Q$

$\Rightarrow$ a_8 not in F.

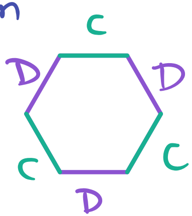
But F is a lin interp. of 3 cubics cont. $a_1, \dots, a_8$.

contradiction. $\square$

Proof of Pappus



C, D are cubics given by triples of lines in hexagon.



E given by L_1, L_2, line thru c_1, c_2

Cayley-Bacharach $\Rightarrow$ E contains c_3

(We assumed $c_1 \neq c_2$, o/w nothing to prove.)

Pascal's Mystic Hexagon similar

(note: the c_i can't all lie on the conic by Bezout.)

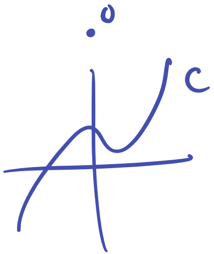
Smooth cubics are groups

$C = Z(y^2 - x^3 - ax - b) \subseteq \mathbb{P}^2$

smooth

$O = [0:1:0] \in C$

pt at ∞



For $c = [u:v:w]$

let $\bar{c} = [u:-v:w]$ refl. thru x-axis. in $\mathbb{A}^2$ plane

so $\bar{O} = O$.

Define $a+b = \bar{c}$

Thm. C is an abel. gp.

Pf. identity : O.

inverse : $c + \bar{c} = O$.

abelian : ✓

associativity: assume WLOG

$o, a, b, c, a+b, b+c, -(a+b),$
$-(b+c)$ all distinct from
each other and
$-((a+b)+c)$ & $-(a+(b+c))$

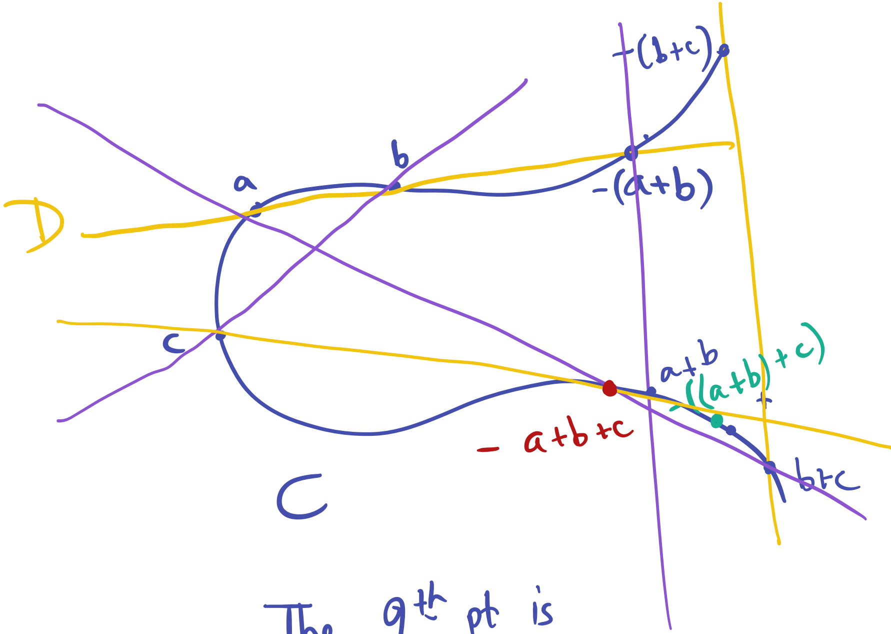
(uses smoothness)

Let $D = \overline{ab}, \overline{c(a+b)}, \overline{o(b+c)}$

$E = \overline{o\,a+b}\ \ \overline{bc}\ \ \overline{a(b+c)}$

C & D cubics meeting at 9 pts,
no common comp.

E passes thru 8 hence 9th by Cay-Ba



The 9th pt is
$-((a+b)+c)$

The line thru $a, b+c$ meets C
in $-(a+(b+c))$ & $-((a+b)+c)$
hence equal. $\square$

Tao says: Pascal is a degen. case of the assoc. law on cubic

& Pappus is a degen. case of Pascal

Mordell's Thm. $\mathbb{Q}$ pts on C form a fin. gen. abel. gp.

Goal: Classify cubic curves.

Hulek

i.e. $C = Z(f) \subseteq \mathbb{P}^2$

deg 3 (char $k = 0$)

proj. equiv: $GL_3 k$

4 cases: ① 3 lines

② conic + line

③ sing irred

④ smooth irred cubic.

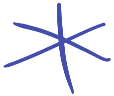
<u>Case 1</u> 3 lines.

lines in $\mathbb{P}^2 \longleftrightarrow$ pts in $\mathbb{P}^2$

via orthog. compl. in k^3

<u>Prop</u>. C = union of 3 lines

Then C is proj eq to exactly one of

① $Z(xyz)$ 

② $Z(xy(x+y))$ 

③ 

④ 

Pf. Translate to problem about pts in $\mathbb{P}^2$

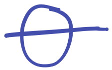
① $\Longleftrightarrow$ collinear, distinct...

Case 2: Conic + line

Prop. C = conic + line = $Q \cup L$

The C is proj eq to exactly one of

① $Z((xz-y^2)y)$

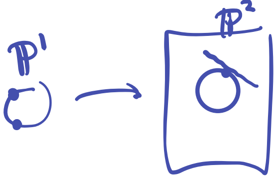


② $Z((xz-y^2)x)$

Pf. We already showed (using quad. forms) Q is proj equiv to $Z(x^2+y^2+z^2)$

$\sim Z(xz-y^2)$

Bézout $\Rightarrow$ 2 cases



① $|Q \cap L| = 2$

② $|Q \cap L| = 1$

Q is image of $\mathbb{P}^1 \to \mathbb{P}^2$

Up to linear change of coords in $\mathbb{P}^1$ can assume int. pts are

① $[1:0:0]$ & $[0:0:1]$

② $[0:0:1]$

Show $\exists$ linear change of coords on $\mathbb{P}^2$ realizing this reparameterization.

L is hence determined. $\square$

If the param of Q is

$$[t:u] \longmapsto [t^2 : tu : u^2]$$

If the re-parametrization in $\mathbb{P}^1$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

the reparam. in $\mathbb{P}^2$ is

$$\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

Case 3 Sing. irred. cubics.

Prop. C = sing. irred cubic.

Then C is proj equiv to exactly one of

① $Z(y^2z - x^3 - x^2z)$ 

② $Z(y^2z - x^3)$ 

Fact. $X = Z(f) \subseteq \mathbb{P}^2$

$p \in X \qquad L$ = line

Then $I_p(X, L) = \mathrm{mult}_p(f|_L)$

Pf. Change coords so $p = [0:0:1]$

& $L = Z(y)$

Let $\bar{f}(x) = f(x, 0, 1)$

$$I_p(X, L) = \dim \mathcal{O}_{\mathbb{P}^2, [0:0:1]} / (f, y)_{[0:0:1]}$$

$$= \dim \mathcal{O}_{\mathbb{A}^2, (0,0)} / (f, y)_{(0,0)}$$

$$= \dim \left(k[x,y]/(f,y) \right)_{(0,0)}$$

$$= \dim \left(k[x]/(\bar{f}) \right)_0 = \text{smallest degree of a term of } \bar{f},$$

$$= \mathrm{mult}_p(f|_L)$$

example.

$$f(x,y,1) = x^3y + x^2y^2 + x^2 + x^3$$

$$\bar{f}(x) = x^2$$

Cor 1. $X = Z(f) \subseteq \mathbb{P}^2$, $p \in X$

TFAE
1. $\mathrm{mult}_p(f|_L) > 1$
2. $L \subset T_pX$
3. $I_p(X, L) > 1$

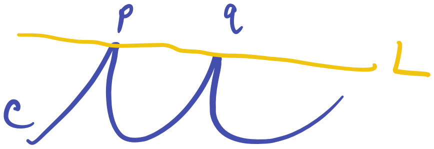
Pf. We already ① $\Leftrightarrow$ ②

Fact gives ① $\Leftrightarrow$ ③

Cor 2. $C \subseteq \mathbb{P}^2$ cubic curve.

Then C has at most 1 sing pt.

Pf. Suppose p, q singular, $p \neq q$.



Let $L = \overline{pq}$ (line)

$T_pC \cong T_qC \cong \mathbb{A}^2$

Cor 1 $\Rightarrow$ $I_p(C, L) \geq 2$

$I_q(C, L) \geq 2$.

Contradicts Bezout $\square$

Higher deg version essentially same:

$\leq \binom{d-1}{2}$ sing pts.

Gathmann Lemma 13.5

Pf of Case 3 Prop

Assume the sing. is at $[0:0:1]$

$\rightsquigarrow f = bx^3 + cx^2y + dxy^2 + ey^3 + q(x,y)$

$q(x,y)$ = quad form in x,y.

(Since $(0,0) \in C$, no const. term.
Since $(0,0)$ sing., no linear terms.)

Have $q(x,y) \not\equiv 0$ because then
f factors into product of 3 linears.

(divide by $y^3 \rightsquigarrow$
poly of deg 3 in x/y ...)

Can factor $q(x,y) = l_0(x,y)\, l_1(x,y)$

Case 1. l_0, l_1 not multiples

Case 2. $l_0 = c\, l_1$ (multiples).

Clever change of vars.

e.g. in Case 2, WLOG $l_0 = l_1 = y$

$\rightsquigarrow f = bx^3 + cx^2y + dxy^2 + ey^3 + y^2$

(linear) Change of vars: $x = x' - \frac{c}{3b} y$

gets rid of x^2y term

etc... $\square$

Case 4 Smooth irred cubics.

Prop. C smooth irr cubic

Then C is equiv to some Weierstrass curves.

$$C_{b,c} = Z(f_{b,c})$$

$$f_{b,c} = y^2 - 4x^3 + bx + c$$

$\heartsuit$

Flex pts & Hessians

$p \in C$ is a flex pt (or inflection pt)

if $I_p(C, T_pC) \geq 3$

If $C = Z(f) \subseteq \mathbb{P}^2$

$$\rightsquigarrow H_f = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0 \leq i,j \leq 2}$$

Have: H_f is $\equiv 0$ or homog of deg $3(d-2)$

$\rightsquigarrow$ Hessian curve $H \subseteq \mathbb{P}^2$

Prop. $H \cap C = \{\text{flex pts of } C\}$

Cor. C has a flex pt.

Discriminants

Define $\mathrm{Disc}(f_{b,c})$ to be $b^3 - 27c^2$

Fact. If α_i are roots of f.

$$\mathrm{Disc}(f_{b,c}) = a_n^{2n-2} \prod_{i \neq j} (\alpha_i - \alpha_j)$$

Define $\mathrm{Disc}(C_{b,c}) := \mathrm{Disc}(f_{b,c})/16$

Prop. $C_{b,c}$ smooth $\Leftrightarrow \mathrm{Disc}(C_{b,c}) \neq 0$.

Pf of Prop. Let $p =$ flex pt of C

WLOG $p = [0:0:1]$

& $T_pC = Z(x) = L$

$\rightsquigarrow f|_L$ has 0 of order 3.

at 0.

$\rightsquigarrow f = -y^3 + x(ax^2 + by^2 + cz^2 + dxy + exz + gyz)$

No quadratic terms (flex pt)

Plugging in $x = 0$ needs to give deg 3 in y.

p smooth $\Rightarrow c \neq 0$.

Clever change of coords $\square$

Smooth cubic curves

Hulek

Last time: every smooth irred cubic in $\mathbb{P}^2$ is proj. equiv to

$$C_{b,c} = Z(f_{b,c})$$

$$f_{b,c} = y^2 - 4x^3 + bx + c.$$

Also: $C_{b,c}$ smooth $\Longleftrightarrow$ $\mathrm{Disc}(f_{a,b}) \neq 0$

$$\mathrm{Disc}(f_{a,b}) = b^3 - 27c^2$$

Conseq. $\{\text{Smooth } C_{b,c}\}$ is $\cong$ a.a.v.

j-invt

$$j : \{C_{b,c}\}^{\text{smooth}} \longrightarrow \mathbb{C}$$

$$C_{b,c} \longmapsto \frac{b^3}{b^3 - 27c^2}$$

Equiv. reln on $\{C_{b,c}\}$: differ by proj aut fixing $[0:0:1]$. y coord ?

Prop. $C_{b,c} \sim C_{b',c'}$ (smooth) $\Longleftrightarrow$ same j

Lemma. Any proj aut. fixing $[0:0:1]$ is of form

$$x \longmapsto u^2 x$$
$$y \longmapsto u^3 y$$

Pf. lin alg...

Prop. $C_{b,c} \sim C_{b',c'} \iff$ same j
(smooth)

Pf. Special case $j=0$

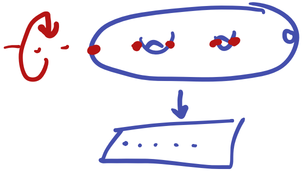
$\Rightarrow$ easy using lemma.

$\Leftarrow$ $j=0 \Rightarrow b=b'=0,\ c\neq 0.$

Choose u s.t. $c' = c/u^6$ $\square$

By Prop:

$$j : \{\text{smooth } C_{b,c}\}/\sim \ \longleftrightarrow\ \mathbb{C}.$$



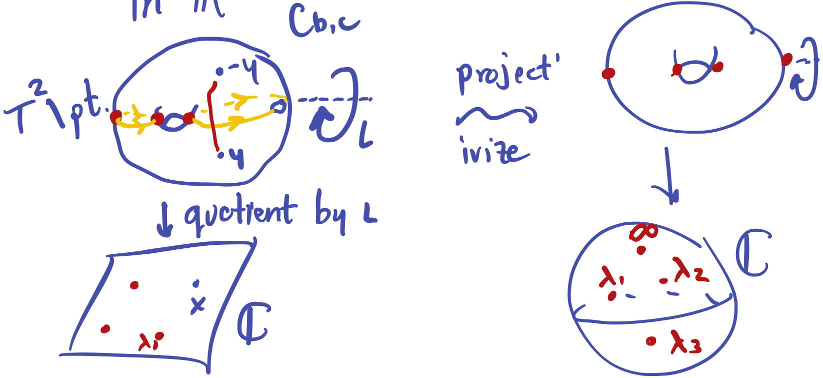
Another point of view $k=\mathbb{C}$.

Every $C_{b,c}$ is homeo to $\circledcirc = T^2$

$$y^2 = 4x^3 - bx - c = 4(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

$C_{b,c}$ has an involution $(y,x)\to(-y,x)$

In $\mathbb{A}^2$

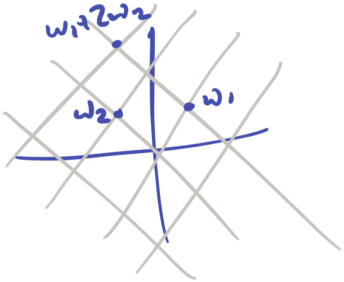


Upshot: $C_{b,c} \cong T^2$ as Riemann surf. (complex manifold)

Another way to make a torus

$\omega_1, \omega_2 \in \mathbb{C} \rightsquigarrow$

$$\Lambda = \{\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\}$$



$$E_\Lambda = \mathbb{C}/\Lambda \cong T^2$$ "elliptic curve"

equiv: biholomorphism.

Will show: $\{E_\Lambda\}/\sim \; \Leftrightarrow \; \left\{\begin{matrix}\text{smooth}\\ C_{b,c}\end{matrix}\right\}/\sim$
$\updownarrow \mathbb{C}$

Equivalence on $\{E_\Lambda\}$.

Given Λ, can rotate, ~~flip~~ negate, scale so

$\omega_1 = 1$

$\operatorname{Im} \omega_2 > 0$ $\quad (\omega_1 = 1, \; \omega_2 = \tau)$

$E_\Lambda \cong E_\tau$ $\quad \tau \in$ upper half plane

Moreover: $SL_2\mathbb{Z} \circlearrowright$ upper half-plane by Möbius transf.

Fact. $E_\tau \sim E_{\tau'} \Leftrightarrow \tau \sim \tau' \bmod SL_2\mathbb{Z}$.

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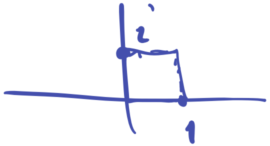
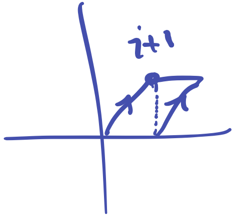
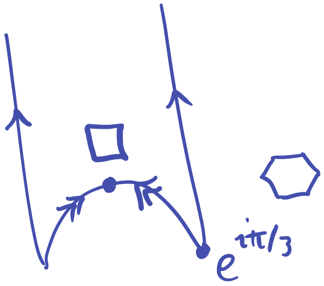
Example $\tau = i$.



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot i = \frac{1 \cdot i + 1}{0 \cdot i + 1} = i + 1 = \tau'$$

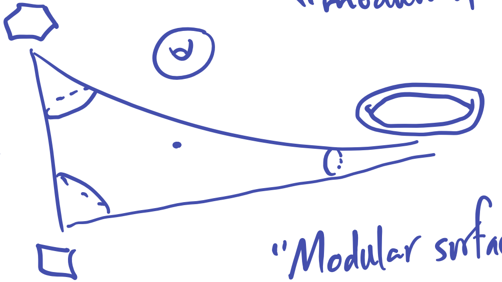


Fund dom for $SL_2\mathbb{Z} \curvearrowright \mathbb{H}$ (upper half plane)



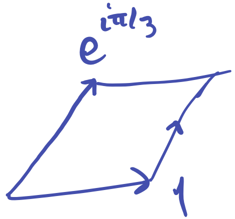
Quotient

$\{E_\Lambda\}/\sim$ =



Note: Mod. surf homeo to $\mathbb{C}$

Hexagonal torus



Cut & paste into a reg. hexagon

Now have:

$$\left\{\begin{matrix}\text{smooth}\\ C_{b,c}\end{matrix}\right\}/\sim \quad \& \quad \underset{\text{mod.surf}}{\{E_\Lambda\}/\sim}$$

both homeo to $\mathbb{C}$.

Want: Map between them.

<u>Weierstrass $\wp$ function</u>

Assume $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$

$$\wp(z) = \wp_\Lambda(z) =$$

$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Invariant under Λ, i.e.

it is a fn on E_Λ

Get a map:

$$\varphi : E_\Lambda \longrightarrow C_{b,c}$$

$$z \longmapsto [1 : \wp(z) : \underset{y}{\wp'(z)}]$$

where

$$b = 60 \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^4}$$

$$c = 140 \sum \frac{1}{\omega^6}$$

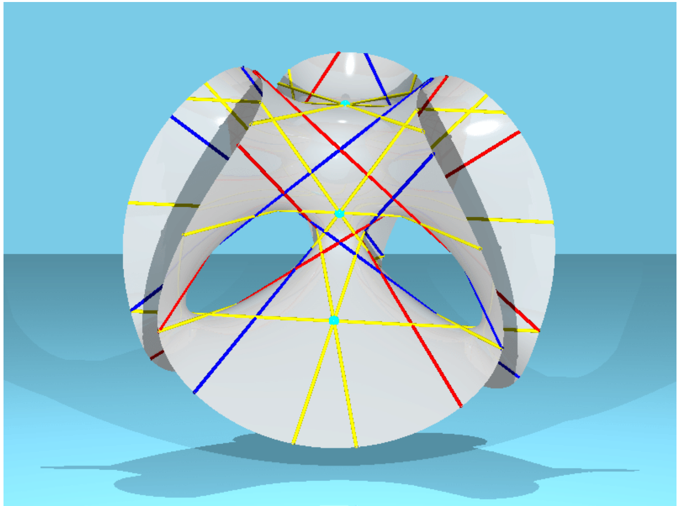
Works because

$$(\wp')^2 = 4\wp^3 - b\wp - c.$$

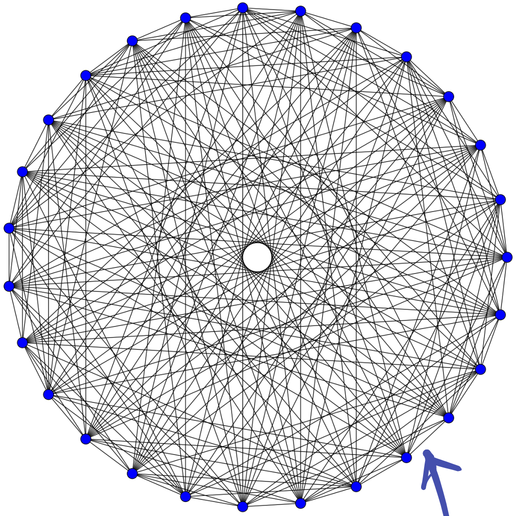
This the desired map $\underset{\text{mod. surf.}}{\{E_\Lambda\}/\sim} \to \left\{\begin{matrix}\text{Smooth}\\ C_{b,c}\end{matrix}\right\}$

Injectivity: J-invt.

Surj: J is holom. nonconst. map



Clebsch



Cayley - Salamon Thm: Every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.

and the (non)-intersection pattern given by

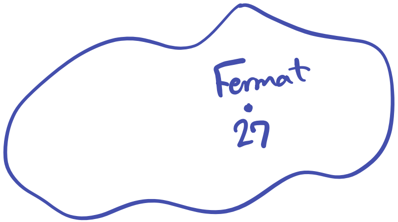
Basic strategy

① Show that

$Z(x^3+y^3+z^3+w^3)$ "Fermat curve"

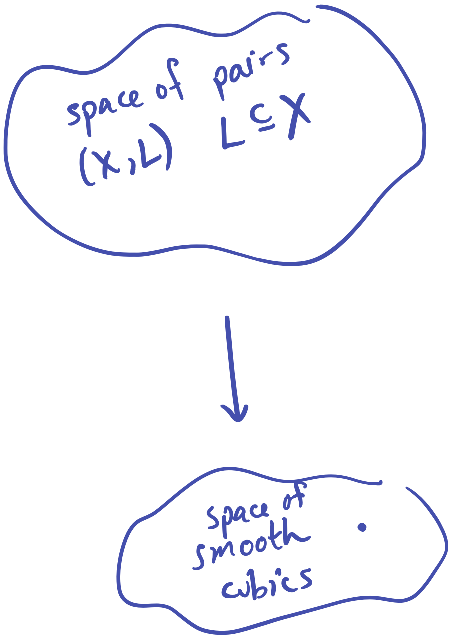
has exactly 27 lines.

② The # of lines is locally const. in moduli space of smooth cubic surfaces (which is connected)



smooth cubic surfaces

In alg. top. language.



deg 27 cov. space.

27 LINES

A cubic surf. is $S = Z(f) \subseteq \mathbb{P}^3$ where $\deg f = 3$.

<u>Cayley-Salamon Thm</u>

S smooth $\Longrightarrow$ S contains exactly 27 lines

Gathmann

<u>Strategy</u>. Show ① that <u>some</u> S has 27 lines and

② # lines is locally const. in space of smooth cubic surfaces.

The <u>some</u> S is Fermat cubic:

$$Z(x_0^3 + x_1^3 + x_2^3 + x_3^3)$$

Lemma. The Fermat cubic X has 27 lines (exactly)

$$X = Z(x_0^3 + x_1^3 + x_2^3 + x_3^3)$$

Pf. X invt under permutation of coords.

Up to such permut, any line is
$$x_0 = a_2 x_2 + a_3 x_3$$
$$x_1 = b_2 x_2 + b_3 x_3$$
(move the 2 pivots to left)

Such a line lies in $X \iff$

$$0 = (a_2 x_2 + a_3 x_3)^3 + (b_2 x_2 + b_3 x_3)^3 + x_2^3 + x_3^3$$

Compare coeffs of LHS=0 & RHS

$\rightsquigarrow$ $a_2^3 + b_2^3 = -1$ (1) x_2^3 term

$a_3^3 + b_3^3 = -1$ (2)

$a_2^2 a_3 = -b_2^2 b_3$ (3)

$a_2 a_3^2 = -b_2 b_3^2$ (4)

If a_2, b_2, a_3, b_3 all $\neq 0$ then $(3)^2/(4)$

$\rightsquigarrow a_2^3 = -b_2^3$ contradicting (1).

$\boxed{\text{So WLOG } a_2 = 0.}$

(1) $\Rightarrow b_2^3 = -1$

(3) $\Rightarrow b_3 = 0$

(2) $\Rightarrow a_3^3 = -1$

permute coords

other 18.

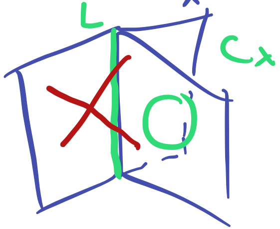
$\rightsquigarrow$ 9 lines (3 choices for each $\sqrt[3]{-1}$) $\square$

How are the lines related?

Intersection pattern: (complement of) Schlafli graph

Claim: Each of the 27 lines in a cubic surface intersects 10 of the others.

Idea: Given one line L, consider the family of planes $\{P_\lambda\}_\lambda \subseteq \mathbb{P}^3$ $\lambda \in \mathbb{P}^1$ containing L.



$P_\lambda \cap S$ = cubic curve X_λ

By our classification:

X_λ = 3 lines $L \cup L' \cup L''$ (C_λ)

or $L \cup C_\lambda$ conic

(need to rule out double lines).

C_λ being 2 lines or conic is a smoothness/Jacobian condition

$\rightsquigarrow$ deg 5 poly in λ. (discrim).

For each of the 5 roots, get 2 lines intersecting L.

Moduli space of smooth cubic surfaces

All cubic surfaces:

$$\mathbb{P}^{19} = \mathbb{P}^{\binom{3+3}{3} - 1}$$

3 balls in 4 boxes

Claim:

Smoothness for $Z(f)$

$\iff \operatorname{rk}(df/dx_i) \neq 0.$

$\left(\begin{array}{l} \iff \operatorname{rk} \geq 1 \\ \iff \text{tangent space} \leq 2. \end{array} \right)$

Lucky accident: The zeros of $\frac{df}{dx_i}$ are all on $Z(f)$

Why? Euler's identity

$$3f = \sum x_i \frac{df}{dx_i}$$

By claim: Moduli space of smooth cubic surfaces is complement of intersection of 4 hypersurfs in $\mathbb{P}^{19}$.

$\rightsquigarrow$ dense & open in $\mathbb{P}^{19}$.

$\implies$ moduli sp. is connected (codim reasons)

The Incidence Correspondence

U = mod. sp. of sm. cub. surf

$G_{1,3} = Gr_{2,4}$

$M = \{(X,L) : L \subseteq X\} \subseteq U \times Gr_{2,4}$

There is projection $\pi : M \to U$.

$(X,L) \to X$

Have:

$\#$ lines in X $= |\pi^{-1}(X)|$

WTS this # is indep of X.

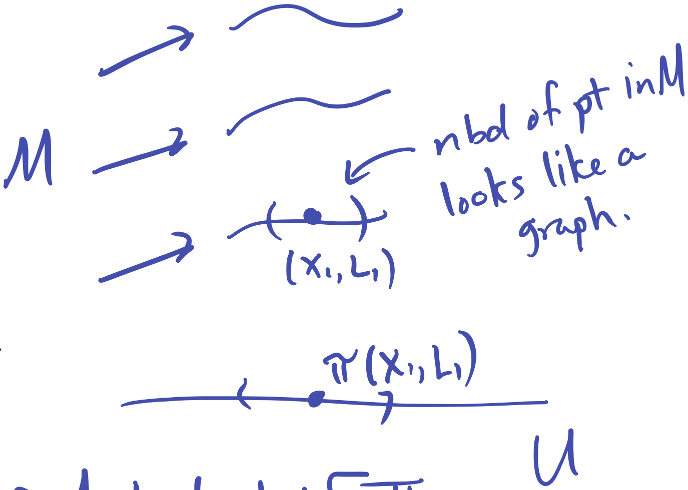
Two ways of rephrasing this:

① $M \to U$ is a cov. sp.

② $|\pi^{-1}(X)|$ is locally const.

Main idea M is locally* the graph of a continuously diff. fn $U \to Gr_{2,4}$.

* in classical topology.



Pf. Apply Implicit Fn Thm. $\square$

Blowing up $\mathbb{P}^2$

Thm. Every smooth cubic surface S is the blowup of $\mathbb{P}^2$ at 6 pts.

Cor. $S \overset{\text{homeo}}{\cong} \mathbb{C}P^2 \#_6 \overline{\mathbb{C}P^2}$

$\Rightarrow \pi_1(S) = 1$

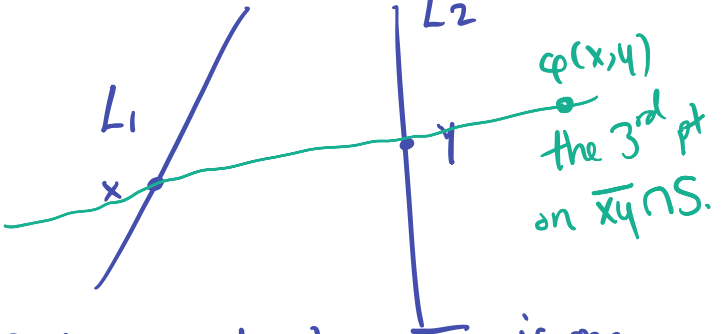
$\Rightarrow H_2(S) \cong \mathbb{Z}^7$

(intersection form type $(1,-6)$...)

Idea of Thm

Further analysis of above work
$\Rightarrow$ S has 2 disjoint lines L_1, L_2.
(we found the ones that intersect)

Define map $\varphi: L_1 \times L_2 \dashrightarrow S$



Works except when $\overline{xy}$ is one of the 27 lines.

Need to blow up $L_1 \times L_2$
in 5 pts to get well def map.

And: $L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

$\mathbb{P}^1 \times \mathbb{P}^1$

$\cong$ $\mathbb{P}^2$ blown up at 1 pt.

(stereographic proj.