Covering Spaces.

In our proof of $\pi_1(S^1) \cong \mathbb{Z}$ we used $\mathbb{R} \to S^1$.
Can similarly show $\pi_1(T^2) \cong \mathbb{Z}^2$ using $\mathbb{R}^2 \to T^2$
or $\pi_1(S^1 \vee S^1) \cong F_2$ using $T_4 \to S^1 \vee S^1$.
In each case, $\pi_1(X)$ gives symmetries of the space lying above.

A <u>covering space</u> of X is an $\tilde{X}$ with
$$p: \tilde{X} \to X$$
satisfying: $\exists$ open cover $\{U_\alpha\}$ of X so that each $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to U_α.

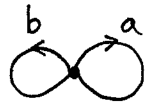
Examples. $\mathbb{R} \to S^1$ $\quad$ $\mathbb{R} \times I \to S^1 \times I$ $\quad$ $\mathbb{R}^2 \to T^2$ $\quad$ $S^2 \to \mathbb{RP}^2$
$S^1 \xrightarrow{\times n} S^1$ $\quad$ $\mathbb{R} \times I \to$ Möbius Strip $\quad$ $\mathbb{R}^2 \to$ Klein bottle

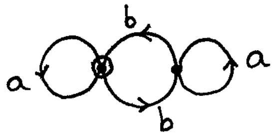
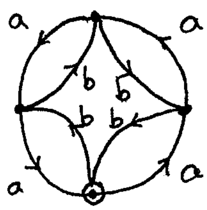
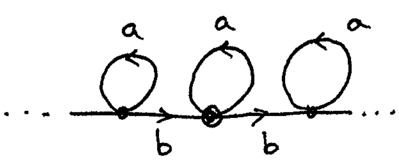
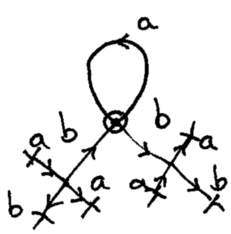
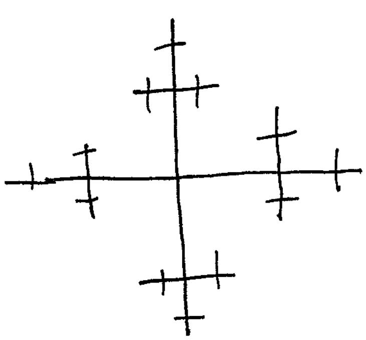
A <u>universal</u> covering space is a covering space that is simply connected.

We will see: ① $\pi_1(X) \leftrightarrow$ symmetries of univ. cover $\tilde{X}$
② Subgroups of $\pi_1(X) \leftrightarrow$ covers of X.

e.g. $X = S^1$.

① via path lifting, ② via path projecting

EXAMPLE $\quad X =$ 

$\tilde{X}$	$p_*(\pi_1(\tilde{X}))$
	$\langle a, b^2, bab^{-1} \rangle$
	$\langle a^2, b^2, ab \rangle$
	$\langle a^4, b^4, ab, ba, a^2b^2 \rangle$
	$\langle b^n a b^{-n} \rangle$
	$\langle a \rangle$
	1

FUNDAMENTAL THEOREM

$p : \tilde{X} \to X$ covering map
$G(\tilde{X})$ = deck transformation group
= p-equivariant symmetries of $\tilde{X}$

<u>Thm</u>: $1 \to \text{Normalizer}(p_*(\pi_1(\tilde{X}))) \to \pi_1(X) \to G(\tilde{X}) \to 1$

So: $p_* \pi_1(\tilde{X})$ normal $\Leftrightarrow$ $G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$

since change of basept for $\tilde{X}$
$\leftrightarrow$ conjugation of $p_* \pi_1(\tilde{X})$ in $\pi_1(X)$.

$p_*(\pi_1(\tilde{X})) = 1 \Leftrightarrow \tilde{X}$ = universal cover
$\Leftrightarrow G(\tilde{X}) \cong \pi_1(X)$

Converse: Any $\pi_1(X) \to G$ is realized by some $\tilde{X}$
Any $H \hookrightarrow \pi_1(X)$ is realized by some $\tilde{X}$.

In particular: every 2-generator group is the symmetry group of some cover of $S^1 \vee S^1$

Lifting Properties

$p : \tilde{X} \to X$ covering space

A <u>lift</u> of $f : Y \to X$ is $\tilde{f} : Y \to \tilde{X}$ with $p\tilde{f} = f$.

Proposition 1 (Homotopy lifting property) Given a homotopy $f_t : Y \to X$ and $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, $\exists !\ \tilde{f}_t$ lifting f_t.

Proof: Same as S^1 case.

$Y =$ point $\rightsquigarrow$ path lifting property
$Y = I \rightsquigarrow$ homotopy lifting for paths

<u>Cor</u>: $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is injective.

Note: $p_*(\pi_1(\tilde{X}))$ is the subgroup of $\pi_1(X)$ consisting of loops that lift to loops.

Degree of a cover: $|p^{-1}(x)|$ is locally constant, hence constant

<u>Cor</u>: $X, \tilde{X}$ path connected.

$$\text{degree of } p = [\pi_1(X) : p_*(\pi_1(\tilde{X}))]$$

<u>Proof</u>: Let $H = p_* \pi_1(\tilde{X})$.

Define $\{\text{cosets of } H\} \longrightarrow p^{-1}(x_0)$

$$H[g] \longmapsto \tilde{g}(1).$$

Surjective: path proj. Injective: path lifting $\blacksquare$

Proposition 2 (Lifting existence criterion) Y = connected, locally path connected. We can lift $f : (Y, y_0) \to (X, x_0)$ to $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ iff

$$f_*(\pi_1(Y)) \leq p_* \pi_1(\tilde{X}).$$

<u>Proof</u>: $(\Rightarrow)$ $\tilde{f}$ exists $\Rightarrow f = p\tilde{f} \Rightarrow f_* = p_* \tilde{f}_*$

$$\Rightarrow \operatorname{Im} f_* \subseteq \operatorname{Im} p_*.$$

$(\Leftarrow)$ Suppose $\operatorname{Im} f_* \subseteq \operatorname{Im} p_*$. Want to <u>build</u> $\tilde{f}$.

Let $y \in Y$, γ a path from y_0 to y.
Prop 1 $\Rightarrow f\gamma$ has unique lift $\widetilde{f\gamma} : Y \to \tilde{X}$.
Define

$$\tilde{f}(y) = \widetilde{f\gamma}(1).$$

Why is $\tilde{f}$ well-defined?

Let γ' = another path from y_0 to y.
$\Rightarrow (f\gamma')(\overline{f\gamma})$ is a loop h_0 at x_0.
$\Rightarrow h_0 = f(\gamma\bar{\gamma}) \in f_*(\pi_1(Y))$
$\Rightarrow h_0 \in p_*(\pi_1(\tilde{X}))$ by assumption.
$\Rightarrow$ the lifted path $\tilde{h}_0$ is a <u>loop</u>.

Uniqueness of lifted paths $\Rightarrow \tilde{h}_0 = \widetilde{f\gamma}\,\widetilde{f\bar{\gamma}'}$
$\Rightarrow \widetilde{f\gamma}$, $\widetilde{f\gamma'}$ share common endpoint.

Exercise: $\tilde{f}$ continuous. $\blacksquare$

Proposition 3 (Uniqueness of lifts) Let $f: Y \to X$, Y connected. If lifts $\tilde{f}_1$, $\tilde{f}_2$ agree at one point, then they are equal.

<u>Proof</u>: Will show
$$A = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$$
is open and closed in Y.

Let $y \in Y$. Let U be open nbhd of Y as in definition of covering space.
Let $\tilde{U}_1, \tilde{U}_2$ be the components of $p^{-1}(U)$ containing $\tilde{f}_1(y)$, $\tilde{f}_2(y)$.
Continuity of $\tilde{f}_i \Rightarrow \exists$ nbhd N of y with
$$\tilde{f}_i(N) \subseteq \tilde{U}_i$$

- $\tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow \tilde{U}_1 \neq \tilde{U}_2 \Rightarrow \tilde{f}_1(N) \cap \tilde{f}_2(N) = \emptyset$
  $\Rightarrow A$ closed.
- $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{U}_1 = \tilde{U}_2 \Rightarrow \tilde{f}_1|_N = \tilde{f}_2|_N$
  Thus A open. ∎