Section 3.5

Matrix Inverses

Section 3.5 Outline

- The definition of a matrix inverse
- How to find a matrix inverse
- Inverses for linear transformations

Inverses

To solve

$$Ax = b$$

we might want to "divide both sides by A".

We will make sense of this...

Inverses

$A = n \times n$ matrix.

A is invertible if there is a matrix B with

$$AB = BA = I_n$$

B is called the inverse of A and is written A^{-1}

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

The 2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of A.

Fact. If $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then A is not invertible.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

Solving Linear Systems via Inverses

Fact. If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

Solving Linear Systems via Inverses

What if we change b?

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 0 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

So finding the inverse is essentially the same as solving all $Ax = b$ equations at once (fixed A, varying b).

Some Facts

Say that A and B are invertible $n \times n$ matrices.

- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

What is $(ABC)^{-1}$?

A recipe for the inverse

Suppose $A = n \times n$ matrix.

- Row reduce $(A \,|\, I_n)$
- If reduction has form $(I_n \,|\, B)$ then A is invertible and $B = A^{-1}$.
- Otherwise, A is not invertible.

Example. Find $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

What if you try this on one of our 2×2 examples, such as $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$?

Why Does This Work?

First answer: we can think of the algorithm as simultaneously solving

$$Ax_1 = e_1$$
$$Ax_2 = e_2$$

and so on. But the columns of A^{-1} are $A^{-1}e_i$, which is x_i.

Matrix algebra with inverses

We saw that if $Ax = b$ and A is invertible then $x = A^{-1}b$.

We can also, for example, solve for the matrix X, assuming that

$$AX = C + DX$$

Assume that all matrices arising in the problem are $n \times n$ and invertible.

Invertible Functions

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a function $U : \mathbb{R}^n \to \mathbb{R}^n$, so

$$T \circ U = U \circ T = \text{identity}$$

That is,

$$T \circ U(v) = U \circ T(v) = v \text{ for all } v \in \mathbb{R}^n$$

Fact. Suppose $A = n \times n$ matrix and T is the matrix transformation. Then T is invertible *as a function* if and only if A is invertible. And in this case, the standard matrix for T^{-1} is A^{-1}.

Example. Counterclockwise rotation by $\pi/2$.

Which are invertible?

Poll

Which are invertible linear transformations of $\mathbb{R}^2$?

- reflection about the x-axis
- projection to the x-axis
- rotation by π
- reflection through the origin
- a shear
- dilation by 2

More rabbits

We can use our algorithm for finding inverses to check that

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1/8 & 0 & -3/2 \end{pmatrix}.$$

Recall that the first matrix tells us how our rabbit population changes from one year to the next.

If the rabbit population in a given year is $(60, 2, 3)$, what was the population in the previous year?

Summary of Section 3.5

- A is invertible if there is a matrix B (called the inverse) with

$$AB = BA = I_n$$

- For a 2×2 matrix A we have that A is invertible exactly when $\det(A) \neq 0$ and in this case

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \mid I_n)$.
- Invertible linear transformations correspond to invertible matrices.

Typical Exam Questions 3.5

- Find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

- Find a 2×2 matrix with no zeros that is equal to its own inverse.
- Solve for the matrix X. Assume that all matrices that arise are invertible:

$$AX(C + DX)^{-1} = B$$

- True/False. If A is invertible, then A^2 is invertible?
- Which linear transformation is the inverse of the clockwise rotation of $\mathbb{R}^2$ by $\pi/4$?
- True/False. The inverse of an invertible linear transformation must be invertible.
- Find a matrix with no zeros that is not invertible.
- Are there two different rabbit populations that will lead to the same population in the following year?