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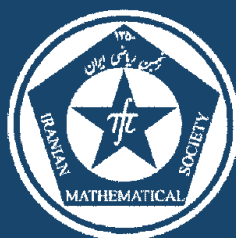
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## A SIMPLE PROOF OF ZARISKI'S LEMMA

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(Communicated by Rahim Zaare-Nahandi)

**ABSTRACT.** We give a simple proof for Zariski's Lemma.

**Keywords:** Zariski's Lemma.

**Keywords:** Primary: 12F05; Secondary: 13F10.

### 1. The result

Our aim in this very short note is to show that the proof of the following well-known fundamental lemma of Zariski follows from an argument similar to the proof of the fact that the field of rational numbers  $\mathbb{Q}$  is not a finitely generated  $\mathbb{Z}$ -algebra.

**Lemma 1.1** (Zariski's Lemma). *Let  $L$  be a field extension of a field  $K$ . Assume that for some  $\alpha_1, \dots, \alpha_n$  in  $L$ ,  $R = K[\alpha_1, \dots, \alpha_n]$  is a field. Then every  $\alpha_i$  is algebraic over  $K$ .*

In particular, if  $K$  is algebraically closed, then  $\alpha_i \in K$  for all  $i$ . This statement implies the so-called Hilbert's Weak Nullstellensatz, which states that when  $K$  is an algebraically closed field, every maximal ideal  $M$  of the polynomial ring  $K[x_1, \dots, x_n]$  is of the form  $M = (x_1 - a_1, \dots, x_n - a_n)$  with  $a_i \in K$  for all  $i$ .

Usually the proof of Zariski's Lemma depends on two technical lemmas due to Artin-Tate and Zariski, see [3, Proposition 3.2, and its subsequent comment]. Some textbooks on elementary algebraic geometry employ the Noether normalization lemma to prove Zariski's Lemma (see, e.g., [2, Theorem 1.15] and [5], (also see [1] and [4]).

Before giving the proof of the lemma, we recall the following two well-known facts.

**Fact 1.** If a field  $F$  is integral over a subdomain  $D$ , then  $D$  is a field.

**Fact 2.** If  $D$  is any principal ideal domain (or just a UFD) with infinitely many (non-associate) prime elements, then its field of fractions is not a finitely generated  $D$ -algebra.

*Proof of the Lemma.* We use induction on  $n$  for arbitrary fields  $K$  and  $L$ . For  $n = 1$  the assertion is clear. Let us assume that  $n > 1$  and the lemma is true for positive integers less than  $n$ . Now to show that it is true for  $n$ , one may assume that one of  $\alpha_i$ 's, say  $\alpha_1$ , is not algebraic over  $K$ . Since  $K[\alpha_1, \dots, \alpha_n] = K(\alpha_1)[\alpha_2, \dots, \alpha_n]$  is a field, by induction hypothesis, we infer that  $\alpha_2, \dots, \alpha_n$  are all algebraic over  $K(\alpha_1)$ . This implies that there are polynomials  $f_2(\alpha_1), \dots, f_n(\alpha_1) \in K[\alpha_1]$  such that all  $\alpha_i$ 's are integral over the domain  $A = K[\alpha_1][1/f_2(\alpha_1), \dots, 1/f_n(\alpha_1)]$ . Since  $R$  is integral over  $A$ , by Fact 1,  $A$  is a field. Consequently,  $A = K(\alpha_1)$ , which contradicts Fact 2.  $\square$

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### REFERENCES

- [1] E. Arrondo, Another elementary proof of the Nullstellensatz, *Amer. Math. Monthly* **113** (2006), no. 2, 169–171.
- [2] K. Hulek, Elementary Algebraic Geometry, Stud. Math. Libr. 20, Amer. Math. Soc. Providence, RI, 2003.
- [3] E. Kunz, Introduction to Commutative Algebra and Algebraic Geometry, Birkhäuser Boston, 1985.
- [4] J. McCabe, A Note on Zariski's Lemma, *Amer. Math. Monthly* **83** (1976), no. 7, 560–561.
- [5] M. Reid, Undergraduate Algebraic Geometry, London Math. Soc. Stud. Texts 12, Cambridge Univ. Press, Cambridge, 1988.

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