Announcements Mar 28

- WebWork 5.2 and 5.3 due Thursday
- Quiz 8 on 5.2 and 5.3 on Friday
- Homework 7 due Friday April 8
- Midterm 3 in class Friday April 8 on Chapter 5
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 5.3

Diagonalization

Outline

- Taking powers of diagonal matrices is easy
- Taking powers of diagonalizable matrices is still easy
- Algebraic multiplicity vs geometric multiplicity vs diagonalizability
- Application: networks

Powers of diagonal matrices

We have see that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, A^k is easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10}$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose we need to compute

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^{10}$$

What would we do?

Earlier in the notes, we saw this matrix is similar to a diagonal one:

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \quad \text{“diagonalization”}$$

So...

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if it is similar to a diagonal matrix:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

How does this factorization of A help describe what A does to $\mathbb{R}^n$?

Diagonalization

Theorem. A is diagonalizable $\Leftrightarrow A$ has n linearly independent eigenvectors.

In this case

$$A = (v_1 \ v_2 \cdots v_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} (v_1 \ v_2 \cdots v_n)^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order).

Why?

Diagonalization

Fact. If A is diagonalizable, bases for the eigenspaces give a basis for $\mathbb{R}^n$.

Why?

Example

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Poll

Poll

Which are true?

1. if A is diagonalizable then A^2 is
2. if A is diagonalizable then A^{-1} is
3. if A^2 is diagonalizable then A is
4. if A is diagonalizable and B is similar to A then B is

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

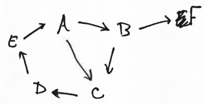
Then

1. $d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$

   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

Application: Social Networks

Consider the social network below.



- We want to find communities, say, a group of people so there is a direct path connecting any two.
- Make a matrix, M, whose ij-entry is the number of arrows from i to j.
- Then the ij entry of M^2 is the number of paths of length 2 to i to j. Why?
- Similar for M^3, etc.
- So the ij entry of $M + M^2 + \cdots M^k$ is the number of paths of length at most k. We look for positive minors.

The leading eigenvalue is a measure of how connected the network is.

Application: Business

Say your car rental company has 3 locations. Make a matrix M whose ij entry is the probability that a car at location i ends at location j. For example,

$$M = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Note the columns sum to 1. The eigenvector with eigenvalue 1 is the steady-state. Any other vector gets pulled to this state. Applying powers of M gives the state after some number of iterations.