Section 7.4
Probability Theory

DEFINITIONS

An experiment is a procedure that yields one of a given set of outcomes.

The sample space of the experiment is the set of possible outcomes.

S = finite set


Blaise Pascal


Pierre Laplace

An event is a subset of the sample space:

$$A \subseteq S$$

The probability of an event A, assuming each outcome of the experiment is equally likely, is:

$$P(A) =$$

PROBABILITY FUNCTIONS

Say we do an experiment with outcomes $s_1, \ldots, s_n$. It might be that the s_i are not equally likely. For instance, consider an unfair die:

$$P(1) = 1/3$$
$$P(2) = P(3) = 1/12$$
$$P(4) = P(5) = P(6) = 1/6$$

What is the probability of rolling an even number?

Odd?

A 4, 5, or 6?

PROBABILITY FUNCTIONS

For an experiment with outcomes $S = \{s_1, \ldots, s_n\}$, a probability function is a function

$$P : S \to \mathbb{R}$$

with (i) $0 \leq P(s_i) \leq 1$ for all i. (the ≤ 1 is redundant)

(ii) $P(s_1) + \cdots + P(s_n) = 1$

If $A \subseteq S$ is an event, then

$$P(A) = \sum_{s_i \in A} P(s_i)$$

If each s_i is equally likely, then $P(s_i) = 1/|S|$

so $P(A) = \sum_{s_i \in A} \frac{1}{|S|} = |A|/|S|$, as before

Still true that:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) $P(A^c) = 1 - P(A)$

THE MONTY HALL PROBLEM

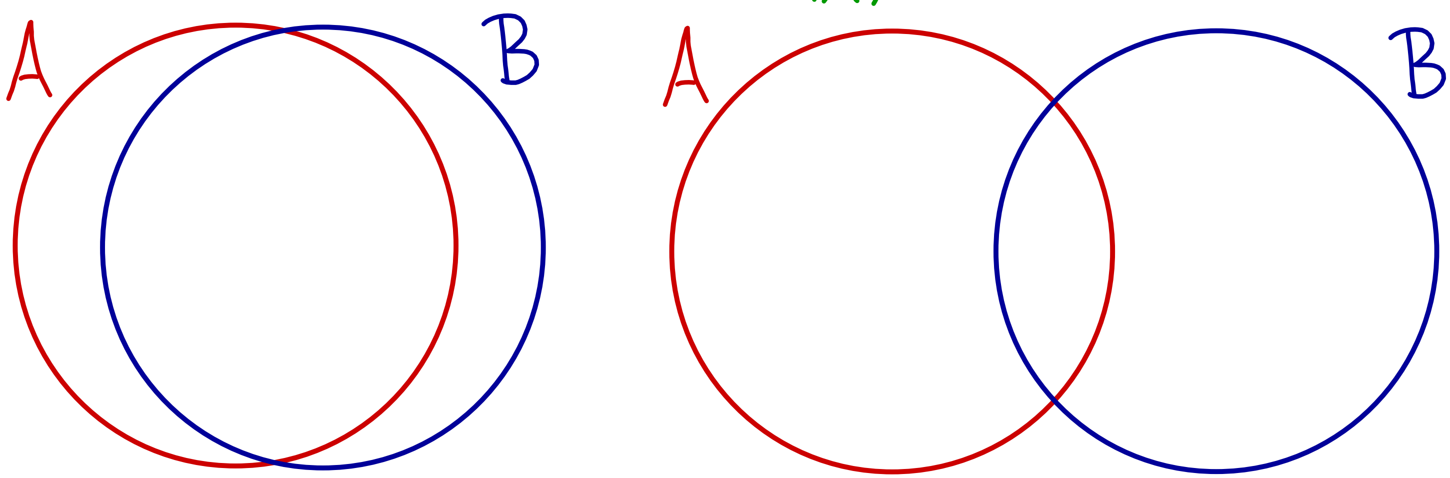
"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?"

THE MONTY HALL PROBLEM

CONDITIONAL PROBABILITY

Say A and B are events and $P(A) > 0$. The conditional probability of B given A is:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$



P(B|A) large

P(B|A) small

P(B) is same in both, but the knowledge of being in A makes a big difference.

CONDITIONAL PROBABILITY EXAMPLE

A coin is flipped twice. The first flip is heads. What is the probability that both flips are heads?

Intuition:

Basic probability:

Conditional probability:

CONDITIONAL PROBABILITY EXAMPLE

I have two kids. One is a boy. What is the probability I have two boys?

CONDITIONAL PROBABILITY EXAMPLES

1. An urn has 10 white, 5 yellow, and 10 black marbles. A marble is chosen at random. We are told it is not black. What is the probability it is yellow?

2. We deal bridge hands at random to N, S, E, W. Together, N and S have 8 spades. What is the probability that E has 3 spades?

CONDITIONAL PROBABILITY EXAMPLE

Alice and Bob each roll a die. We are told that Alice rolled a higher number. What is the probability that Alice rolled a 3?

INDEPENDENCE

Events A and B are independent if

$$P(B|A) = P(B)$$

Since $P(B) = \frac{P(B \cap A)}{P(A)}$ we can say A and B are independent if:

$$P(A \cap B) = P(A)P(B)$$

Examples. 1. We roll two die. A = first comes up 2
B = second comes up 3

2. Two kids. B = 2 boys
A = at least one boy

INDEPENDENCE

Events A and B are independent if

$$P(B|A) = P(B)$$

Examples. 3. The Alice and Bob problem:

B = Alice rolled 3

A = Alice > Bob

4. Urn problem: 10 white, 5 yellow, 10 black.
   Are Y and B^c independent?

A CONDITIONAL PROBABILITY PROBLEM

We buy light bulbs from suppliers A and B.

30% of the bulbs come from A, 70% from B.

2% of the bulbs from A are defective

3% of the bulbs from B are defective.

What is the probability that a random bulb...

(i) is from A and defective?

(ii) is from B and not defective?

☆ (iii) is defective?

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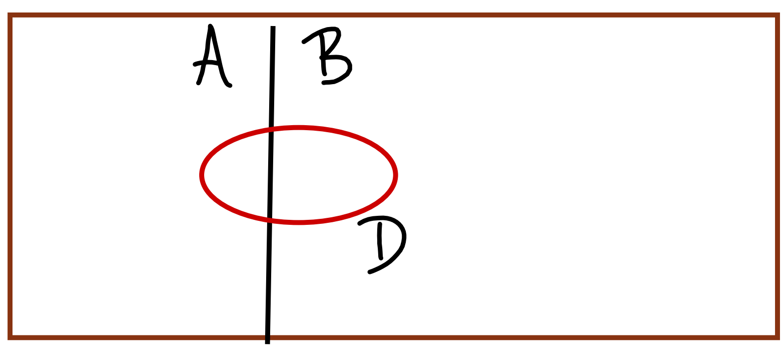


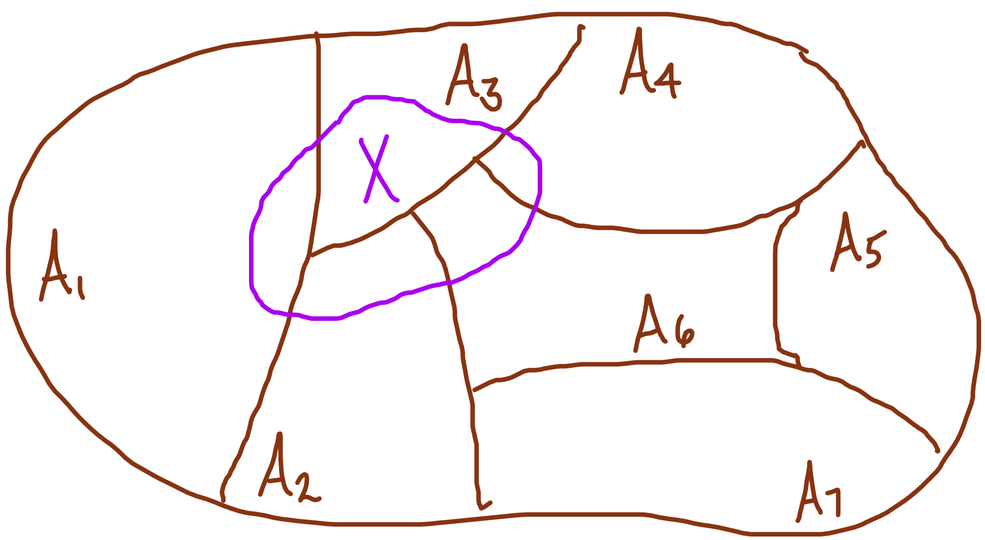
Reinterpret all questions in terms of areas.

LAW OF TOTAL PROBABILITY

Say that events $A_1, \ldots, A_n$ form a partition of the sample space S, that is, the A_i are mutually exclusive ($A_i \cap A_j = \emptyset$ for $i \neq j$) and $A_1 \cup \cdots \cup A_n = S$.

Let $X \subseteq S$ be any event. Then

$$P(X) = P(A_1)P(X|A_1) + \cdots + P(A_n)P(X|A_n)$$



BAYES' FORMULA

How is $P(A|B)$ related to $P(B|A)$?

THEOREM: $$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

PROOF: $\blacksquare$

EXAMPLE. In the light bulb problem, say a randomly selected light bulb is defective. What is the probability it came from A?

BAYES' FORMULA

EXAMPLE. Coin A comes up heads 1/4 of the time.
Coin B comes up heads 3/4 of the time.
We choose a coin at random and flip it twice.
If we get two heads, what is the probability coin B was chosen?

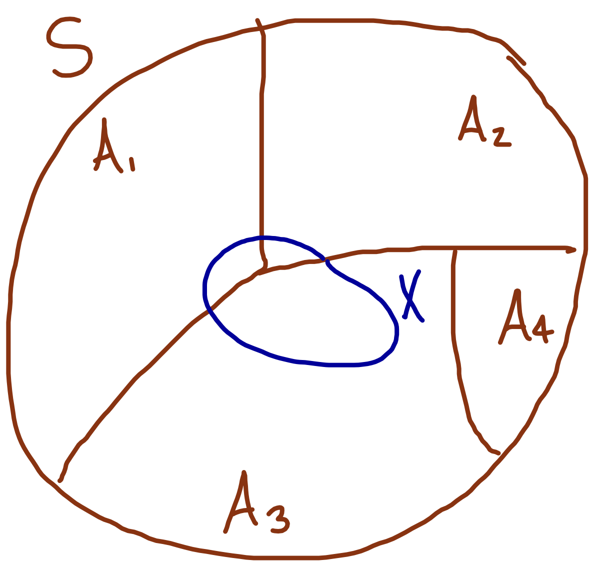
BAYES' FORMULA

Computing the denominator with the law of total probability

$A_1, \ldots, A_n$ pairwise mutually exclusive events with $A_1 \cup \cdots \cup A_n = S$ and $P(A_i) > 0$ for all i. Let X be an event with $P(X) > 0$. Then, for each j, we have:

$$P(A_j | X) = \frac{P(A_j) P(X | A_j)}{P(X)}$$

where $P(X) = P(A_1) P(X | A_1) + \cdots + P(A_n) P(X | A_n)$



$P(A_3 | X)$ big
$P(A_2 | X)$ small
$P(A_4 | X) = 0.$

EXAMPLE. Do a variant of the coin problem with 3 or more coins.

BAYES' FORMULA

PROBLEM. You have 3 cards. One is red on both sides, one is black on both sides, and one has a red side and a black side. You pick one card randomly and put it on the table. Its top side is red. What is the probability the other side is red?

BAYES' FORMULA

PROBLEM. There are 3 urns, A, B, and C that have 2, 4, and 8 red marbles and 8, 6, and 2 black marbles, respectively. A random card is picked from a deck. If the card is black we choose a marble from A, if it is a diamond we choose a marble from B, and otherwise choose a marble from C.

(a) What is the probability that a red marble gets drawn?

(b) If we know a red marble was drawn, what is the probability the card was hearts? diamonds?

Draw the picture!