Section 1.2

Row reduction

Outline of Section 1.2

- Solve systems of linear equations via elimination
- Solve systems of linear equations via matrices and row reduction
- Learn about row echelon form and reduced row echelon form of a matrix
- Learn the algorithm for finding the (reduced) row echelon form of a matrix
- Determine from the row echelon form of a matrix if the corresponding system of linear equations is consistent or not.

Solving systems of linear equations by elimination

Example

Solve:

$$\begin{aligned} -y + 8z &= 10 \\ 5y + 10z &= 0 \end{aligned}$$

How many ways can you do it?

Example

Solve:

$$\begin{aligned} -x + y + 3z &= -2 \\ 2x - 3y + 2z &= 14 \\ 3x + 2y + z &= 6 \end{aligned}$$

Hint: Eliminate x!

Solving systems of linear equations with matrices

Example

Solve:

$$\begin{aligned} -y + 8z &= 10 \\ 5y + 10z &= 0 \end{aligned}$$

It is redundant to write x and y again and again, so we rewrite using (augmented) *matrices*. In other words, just keep track of the coefficients, drop the $+$ and $=$ signs. We put a vertical line where the equals sign is.

$$\left(\begin{array}{rr|r} -1 & 8 & 10 \\ 5 & 10 & 0 \end{array} \right) \rightsquigarrow$$

Example

Solve:

$$
\begin{aligned}
-x + y + 3z &= -2 \\
2x - 3y + 2z &= 14 \\
3x + 2y + z &= 6
\end{aligned}
$$

Again we rewrite using augmented matrices...

Row operations

Our manipulations of matrices are called row operations:

row swap, row scale, row replacement

If two matrices differ by a sequence of these three row operations, we say they are row equivalent.

Goal: Produce a system of equations like:

$$
\begin{aligned}
x \qquad\qquad &= 2 \\
y \qquad &= 1 \\
z &= 5
\end{aligned}
$$

What does this look like in matrix form?

Row operations

Why do row operations not change the solution?

Solve:

$$\begin{aligned} x + y &= 2 \\ -2x + y &= -1 \end{aligned}$$

System has one solution, $x = 1, y = 1$.

What happens to the two lines as you do row operations?

$$\left(\begin{array}{rr|r} 1 & 1 & 2 \\ -2 & 1 & -1 \end{array} \right) \rightsquigarrow$$

They pivot around the solution!

Row Reduction and Echelon Forms

Row echelon form

Remember our goal.

Goal: Produce a system of equations like

$$\begin{aligned} x \quad\quad\quad &= 2 \\ y \quad\quad &= 1 \\ z &= 5 \end{aligned}$$

Or at least...

Easier goal: Produce a system of equations like

$$\begin{aligned} x + 5y - 3z &= 2 \\ y + 7z &= 1 \\ z &= 5 \end{aligned}$$

Row Reduction and Echelon Forms

A matrix is in row echelon form if

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is easy to solve using back substitution.

The pivot positions are the leading entries in each row.

Reduced Row Echelon Form

A system is in reduced row echelon form if also:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

For example:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is even easier to solve.

Can every matrix be put in reduced row echelon form?

Reduced Row Echelon Form

Poll

Which are in reduced row echelon form?

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

REF:

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

RREF:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

Row Reduction

Theorem. Each matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm. That shows a matrix is equivalent to at least one matrix in reduced row echelon form.

Row Reduction Algorithm

To find row echelon form:

Step 1 Swap rows so a leftmost nonzero entry is in 1st row (if needed)

Step 2 Scale 1st row so that its leading entry is equal to 1

Step 3 Use row replacement so all entries below this 1 (or, pivot) are 0

Then cover the first row and repeat the three steps.

To then find reduced row echelon form:

- Use row replacement so that all entries above the pivots are 0.

Examples.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \left(\begin{array}{rrr|r} 0 & 7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right) \left(\begin{array}{rrr|r} 4 & -5 & 3 & 2 \\ 1 & -1 & -2 & -6 \\ 4 & -4 & -14 & 18 \end{array}\right)$$

Interactive Row Reducer

Solutions of Linear Systems

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

What are the solutions? Say the variables are x and y.

Solutions of Linear Systems: Consistency

Solve the linear system associated to:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Say the variables are x, y, and z.

A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

Example with a parameter

For which values of h does the following system have a solution?

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= h \end{aligned}$$

Solve this by row reduction and also solve it by thinking geometrically.

Summary of Section 1.2

- To solve a system of linear equations we can use the method of elimination.
- We can more easily do elimination with matrices. The allowable moves are row swaps, row scales, and row replacements. This is called row reduction.
- A matrix in row echelon form corresponds to a system of linear equations that we can easily solve by back substitution.
- A matrix in reduced row echelon form corresponds to a system of linear equations that we can easily solve just by looking.
- We have an algorithm for row reducing a matrix to row echelon form.
- The reduced row echelon form of a matrix is unique.
- Two matrices that differ by row operations are called row equivalent.
- A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.