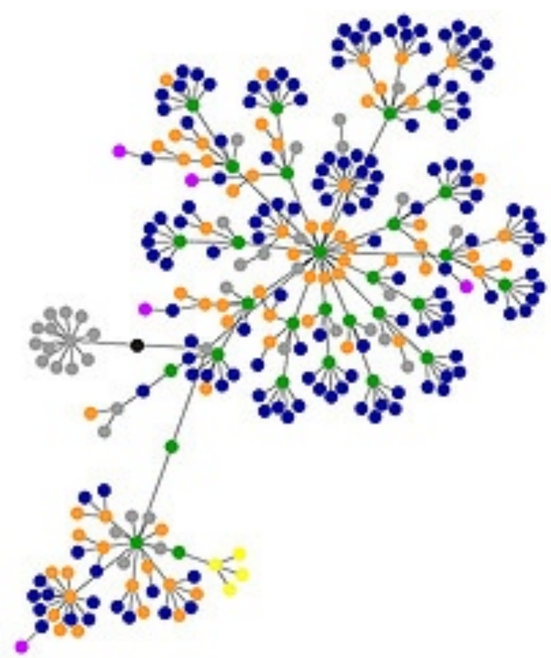
12 Trees

12.1 Trees and Their Properties

Trees

A circuit in a graph is a walk that starts and ends at the same point and traverses each edge at most once.

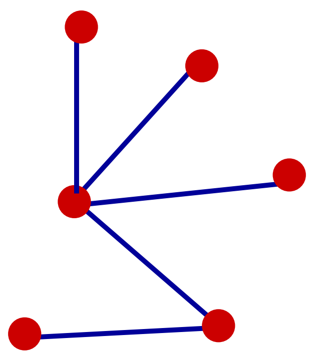
A tree is a connected graph with no circuits.



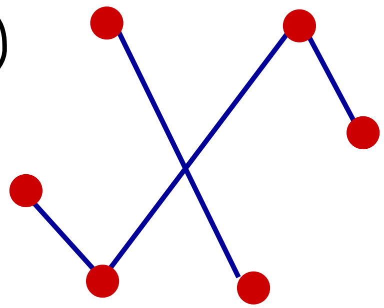
TREES

Which of the following graphs are trees?

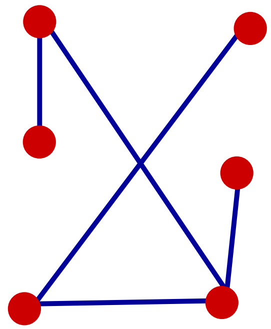
(i)



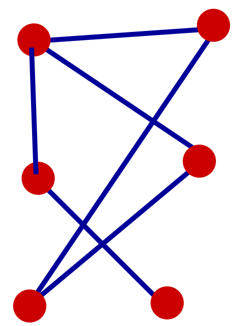
(ii)



(iii)



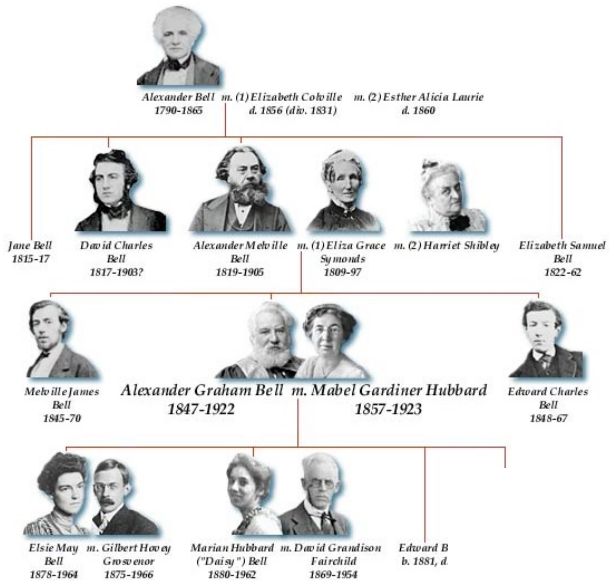
(iv)

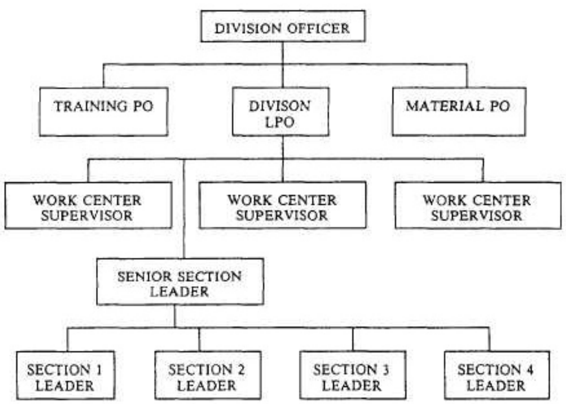


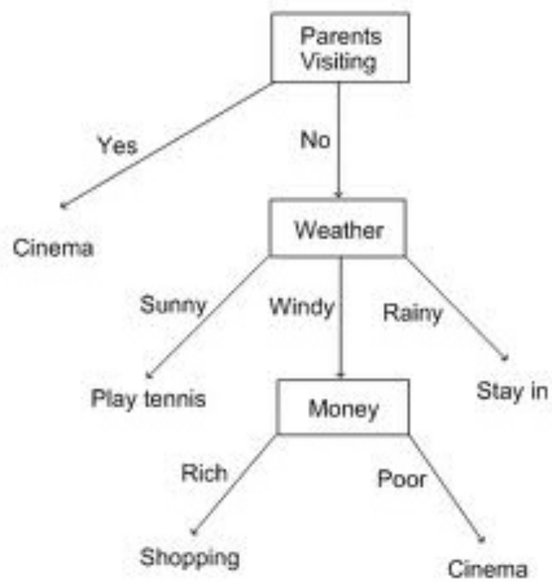
Trees

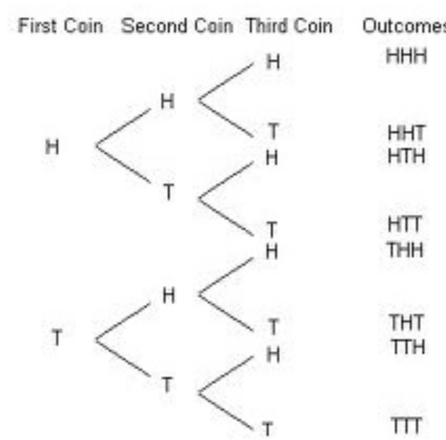
List all trees with 5 or fewer vertices up to isomorphism.

Applications of Trees









and many more...

Characterizing Trees

Theorem. Let G be a graph with n vertices. The following are equivalent:

(i) G is a tree (i.e. G is connected with no circuits)
(ii) G is connected and has no cycles.
(iii) G is connected and has $n-1$ edges.
(iv) Between any two vertices of G there is a unique walk that does not repeat any edges.

Also:

(v) G has $n-1$ edges and no cycles
(vi) G is connected, but removing any edge makes it disconnected.
(vii) G has no cycles, but adding any edge creates one.

etc...

Application To Chemistry

A hydrocarbon has the form C_nH_{2n+2}. Carbon has degree 4 and Hydrogen has degree 1.

So we get a graph with vertices and edges $\rightsquigarrow$!

Problem. Find all hydrocarbons for $n = 1, 2, 3, 4$.

CHARACTERIZING TREES

THEOREM. Let G be a graph with n vertices. The following are equivalent:

(i) G is a tree (i.e. G is connected with no circuits)
(ii) G is connected and has no cycles.
(iii) G is connected and has $n-1$ edges.
(iv) Between any two vertices of G there is a unique walk that does not repeat any edges.

SKETCH OF PROOF.
(i) $\Leftrightarrow$ (ii) Show, for any graph: cycle $\Leftrightarrow$ circuit
Idea: prune a circuit to get a cycle.
(i) $\Rightarrow$ (iii) Show that no circuits $\Rightarrow$ degree 1 vertex.
Then use induction on n.
(i) $\Rightarrow$ (iv) Given two different paths, construct a circuit.
(iv) $\Rightarrow$ (ii) In a cycle, can find two paths b/w same vertices.
(iii) $\Rightarrow$ (ii) Remove k edges to get a tree. Show $k=0$. ▨

12.2 Spanning Trees

Spanning Trees

A spanning tree for a graph G is a subgraph that is a tree and that contains every vertex.

Note: Only graphs have spanning trees.

If we have a subgraph H of a weighted graph G, the weight of H is

A minimal spanning tree for a weighted graph is

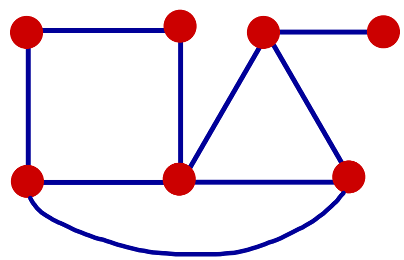
Application: Given a network of roads

Spanning Trees

How to find a spanning tree?

One answer: Delete all edges until there are no cycles.

Example. How many spanning trees can you find?

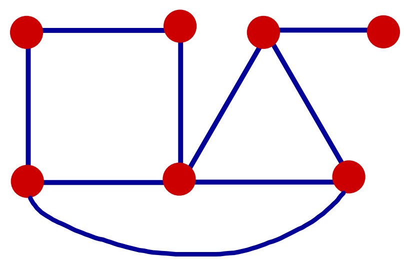


Question. How to find all spanning trees? How many are there?

Could hunt for cycles, delete edges. Inefficient!

Depth-First Search and Breadth-First Search

Depth-first: Start at some point in the graph.
Draw a long path, go as far as possible.
When you hit a wall (= degree 1 vertex),
or an edge that creates a cycle with your
path, back up one step and go in a new direction.



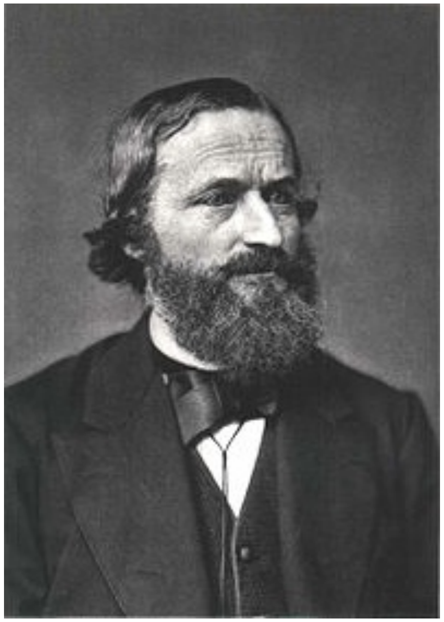
Breadth-first:

Kirchhoff's Theorem

Given a graph with vertices $v_1, \ldots, v_n$, make a matrix M with (i,i)-entry the degree of v_i and all other (i,j)-entries given by:
-1 if v_iv_j is an edge
0 otherwise

Theorem. Given a graph G, make the matrix M as above. Delete the i^{th} row and the j^{th} column to obtain a matrix M'. Then:

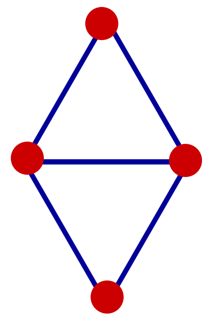
$$(-1)^{i+j}\det(M') = \#\text{ spanning trees for } G.$$



Gustav Kirchhoff

KIRCHHOFF'S THEOREM

EXAMPLE.



Find them all!

12.3 Minimal Spanning Tree Algorithms

Krushkal's Algorithm

Goal: Find a minimal spanning tree for a given graph.

Want something more efficient than enumerating all trees.

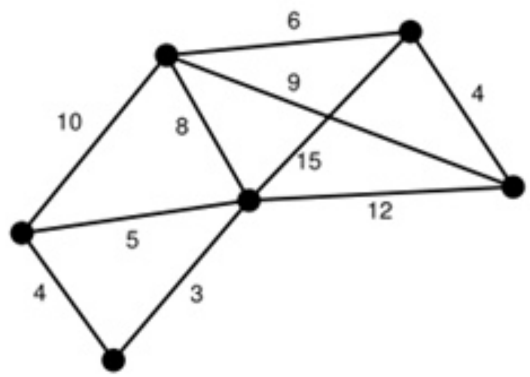
The Algorithm. Set $T = \emptyset$.
Consider all edges e so $T \cup \{e\}$ has no circuits.
Choose the edge e of smallest weight with this property.
Replace T with $T \cup \{e\}$.
Repeat until T is a spanning tree.

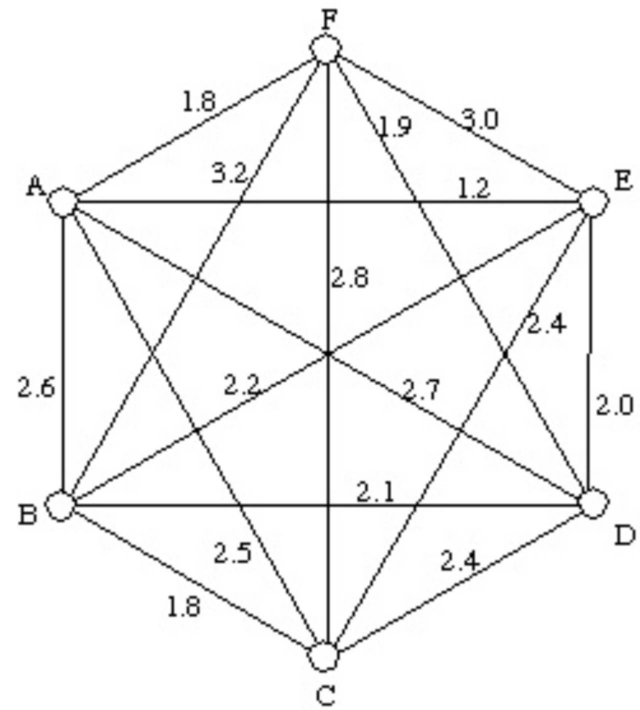
Note: The number of steps is

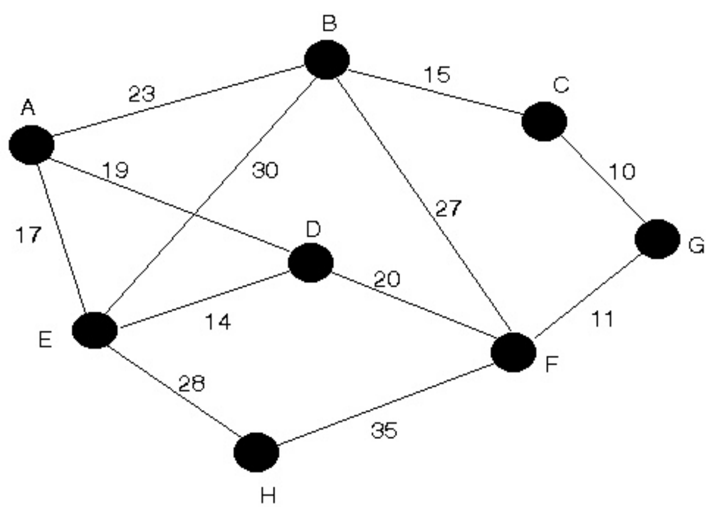
Krushkal's algorithm is an example of a "greedy algorithm"

Krushkal's Algorithm

Find minimal spanning trees for the following weighted graphs.







KRUSHKAL'S ALGORITHM

Why does the algorithm work?

Let $e_1, \ldots, e_{n-1}$ be the edges chosen by Krushkal's algorithm, in order.

Prove the following statement by induction:

$\{e_1, \ldots, e_k\}$ is contained in some minimal spanning tree.

Base case: $k=0$, i.e. $\emptyset$ contained in some minimal spanning tree. ✓

Suppose $\{e_1, \ldots, e_k\}$ contained in some minimal spanning tree T, but e_{k+1} is not in T. $\rightsquigarrow$ $T \cup e_{k+1}$ has a cycle.
There is an edge f contained in this cycle that is not equal to $e_1, \ldots, e_{k+1}$ (the e_i form a tree, so they form no cycles).
Now, f and e_{k+1} have same weight, otherwise weight of $T - f + e_{k+1}$ is less than weight of T. We see $T - f + e_{k+1}$ is the desired tree. ▨

Prim's Algorithm

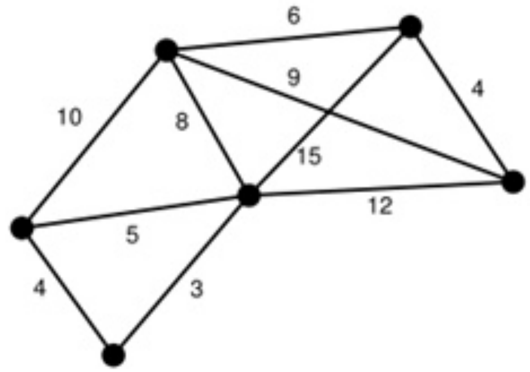
Idea: Grow a tree from a vertex.

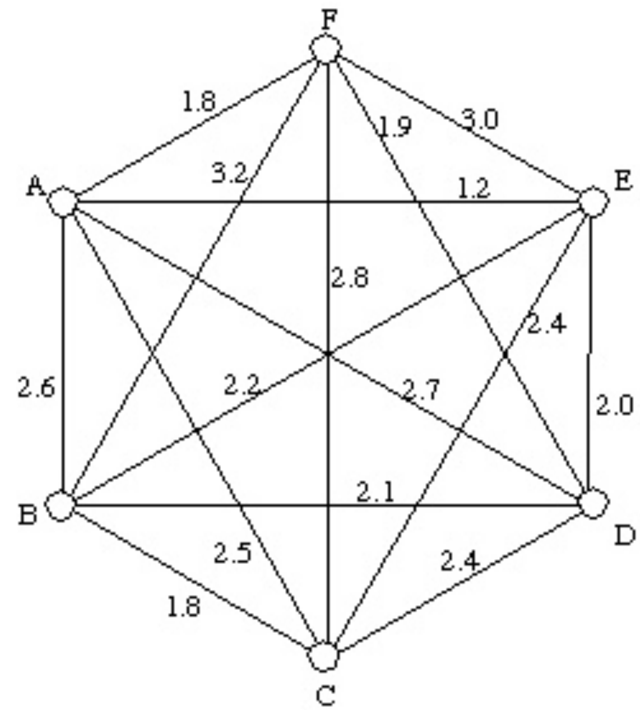
The algorithm. Set $T = v$ (any vertex)
Choose an edge e of minimal weight so $T \cup \{e\}$ is a tree
Replace T with $T \cup \{e\}$.
Repeat until T is a spanning tree.

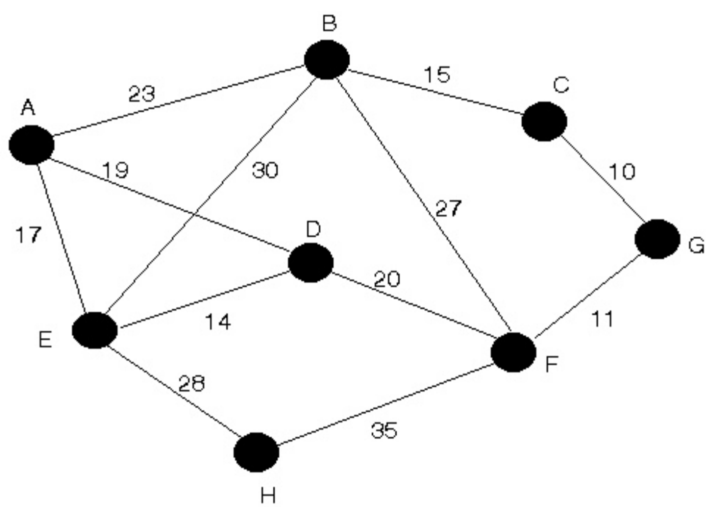
Note: We know $T \cup \{e\}$ is a tree if $T \cap e$ is a single vertex.

Prim's Algorithm

Find minimal spanning trees for the following weighted graphs.







Krushkal's Algorithm vs. Prim's Algorithm

What is the complexity?

size = # edges
cost = # comparisons

Krushkal: $\mathcal{O}(n\log n + n^2)$

Prim: $\mathcal{O}(n^2)$

Check these! Idea: order the remaining edges. Then, need to check which can be added to the current tree by comparing the endpoints of each edge with the vertices of the current tree.

The advantage over Krushkal's algorithm is that there are fewer edges to check at each step. In fact, Prim is $\mathcal{O}(n^2)$.