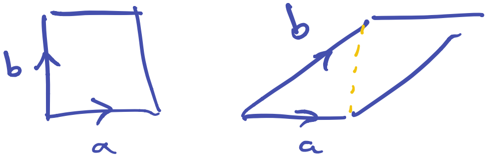
Chap 12. Moduli Space

$\mathcal{M}(S) = \{$ hyp / complex / alg / conformal structures on $S \} / \sim$



different in Teich

same in $\mathcal{M}$

$F \cdot X = X \quad \forall X$

$\ell_X(F^{-1}(c)) = \ell_{F \cdot X}(c) = \ell_X(c) \quad \forall c$

$\mathrm{Mod}(S) \circlearrowright \mathrm{Teich}(S)$

by pulling back metrics...

In terms of markings:

$$[\psi] \cdot (X, \varphi) = (X, \varphi \circ \psi^{-1})$$

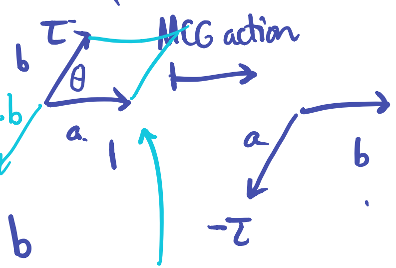
- Action is by isometries.
- $\mathrm{Stab}(X) = \mathrm{Isom}^+(X)$ finite
- Kernel is $\begin{cases} \mathbb{Z}/2 & g = 1, 2 \\ 1 & g \geq 3 \end{cases}$ $\leftarrow$ hyp. inv.

$$\mathcal{M}(S) = \mathrm{Teich}(S) / \mathrm{Mod}(S)$$

The torus

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:



as a mapping class: $a \to b$

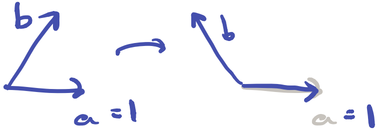
$b \to -a$

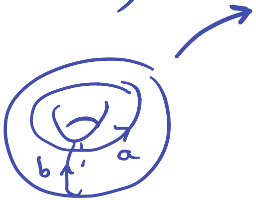
Prop. The action of $\mathrm{Mod}(T^2) = SL_2\mathbb{Z}$ on $\mathrm{Teich}(T^2) = \mathbb{H}^2$ is by Möbius transf. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \dfrac{az-b}{-cz+d}$

Pf. Check on generators.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:





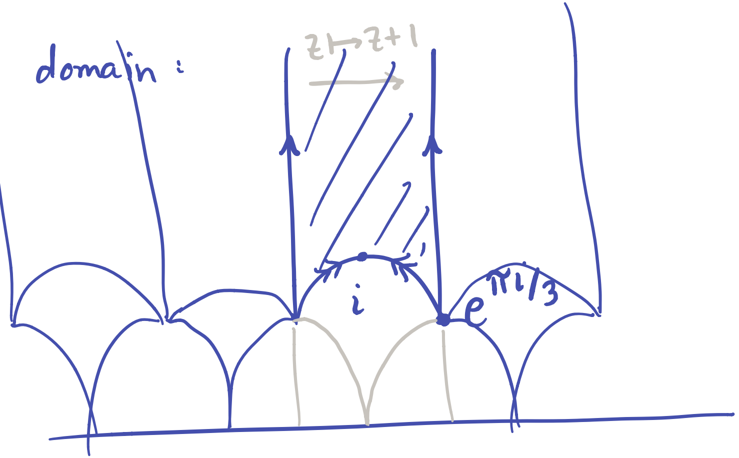
$z \longmapsto z-1$ ✓

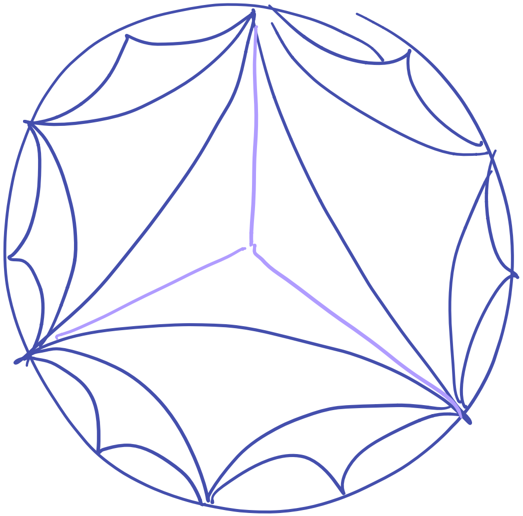
To put latter in std form, divide by $-\tau \rightsquigarrow -\dfrac{1}{\tau}$ in $\mathbb{H}^2$.

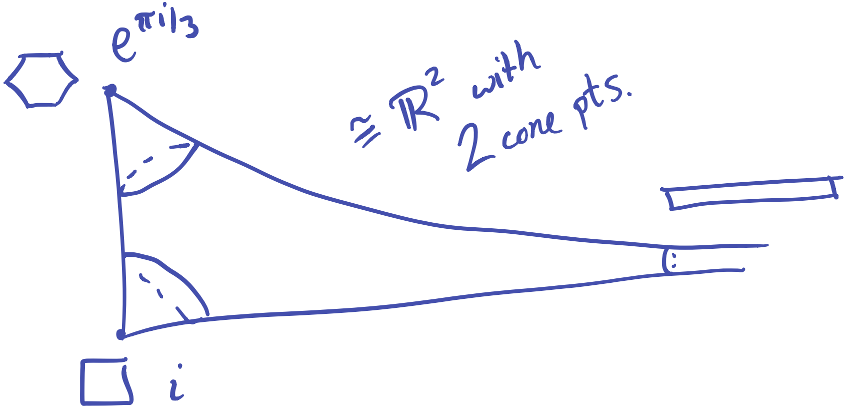
agrees with:

$$\frac{0z-1}{1z+0} = -\frac{1}{z}$$

Fund domain:







$\cong \mathbb{R}^2$ with 2 cone pts.

$PSL_2\mathbb{Z} \cong \mathbb{Z}/2 * \mathbb{Z}/3$

Proper Discontinuity

$G \circlearrowright X$ prop disc if

$\forall$ compact $K \subseteq X$

$\#\{g \in G : gK \cap K \neq \emptyset\} < \infty$.

Thm (Fricke) $\mathrm{Mod}(S_g) \circlearrowright \mathrm{Teich}(S_g)$

is prop. disc.

Thm + Teich metric $\Longrightarrow$ metric on $\mathcal{M}(S)$.

(inf of dist. b/w lifts).

Tool: Raw length spectrum.

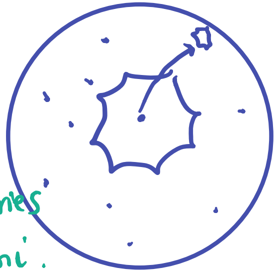
$$\mathrm{rls}(X) = \{\ell_X(c)\} \subseteq \mathbb{R}.$$

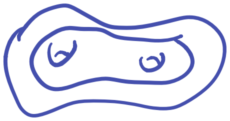
Lemma. $X \in \mathrm{Teich}(S)$. $\forall L$

Then $\#\{c : \ell_X(c) \leq L\} < \infty$

In partic, $\mathrm{rls}(X)$ closed, discrete in $\mathbb{R}$.

Pf. Prop. disc. of $\pi_1(S) \circlearrowright \mathbb{H}^2$.





$\square$

Birman-Series

cf. Mirzakhani.

Wolpert's Lemma

X_1, X_2 hyp. surfaces

$\varphi: X_1 \to X_2$ quasi-conf homeo $(K < \infty)$.

For all c:

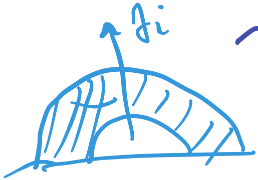
$$\frac{1}{K}\, \ell_{X_1}(c) \le \ell_{X_2}(\varphi(c)) \le K\, \ell_{X_1}(c)$$

"curves get stretched by at most K."

Pf. $\gamma_1, \gamma_2 \in \mathrm{Isom}^+(\mathbb{H}^2)$

$\updownarrow \quad \updownarrow$

$c \subseteq X_1 \quad \varphi(c) \subseteq X_2$

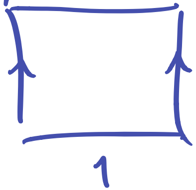


$\rightsquigarrow$ hyp annuli $\mathbb{H}^2 / \langle \gamma_i \rangle = A_i$

cover of X_i.

A_i is conformally equiv. to

a unique* std annulus



* Grötzch +

φ lifts to $\tilde{\varphi}: A_1 \to A_2$

(lifting criterion)

Same qc const.

Grotzch $\Rightarrow \frac{m_1}{K} \le m_2 \le K m_1$

$\Rightarrow$ lemma

Proof of PD

$d(X,Y) = \frac{1}{2}\log K$

$B \subseteq \mathrm{Teich}(S_g)$ compact.

$X \in B$ arbitrary.

$D = \operatorname{diam} B$.

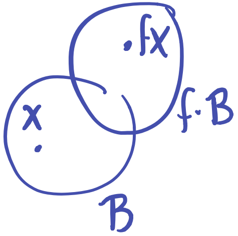
c_1, c_2 curves that fill S_g.

$L = \max\{\ell_X(c_1), \ell_X(c_2)\}$

Say $f \cdot B \cap B \neq \emptyset$.

(WTS finitely many such f)

$f \cdot B \cap B \neq \emptyset$



$\Rightarrow d(X, f \cdot X) \leq 2D$

Wolpert $\Rightarrow \ell_{f \cdot X}(c_i) \leq KL$

where $K = e^{4D}$

$\Rightarrow \ell_X(f^{-1}(c_i)) \leq KL$

Lemma $\Rightarrow$ finitely many choices for $f(c_1)$ & $f(c_2)$.

Alex method $\Rightarrow$ finitely many choices for f.

$\square$