KIRBY

COMBINATORICS OF TRAIN TRACKS

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Introduction

The study of surfaces in topology, geometry and dynamical systems has undergone a remarkable revival in recent years due to the insight of William Thurston. Part of the purpose of this text is the basic theory of one aspect of his contributions, the theory of measured train tracks. These objects arise naturally when one looks for a "completion" of the set of curves in a surface, where there is already a rich geometric and combinatorial structure. The collection of all measured train tracks in a surface (up to a natural equivalence relation) forms a space $\mathcal{MT}$ which is homeomorphic to Euclidean space. It has a natural projective version, $\mathcal{PT}$, homeomorphic to a sphere which Thurston has interpreted as the boundary of a compactification of Teichmüller space. A homeomorphism of the surface acts on this sphere, and the dynamics of the action on the surface are often captured by the dynamics of the action on this sphere. There are many other questions in Mathematics of which train tracks allow a coherent treatment, for example, they may also be used to describe the degeneration of Kleinian groups.

The theory which we develop admits contrasting geometric and combinatorial viewpoints. While Thurston's original approach was comparatively geometric in nature, we shall develop the basic theory in Section 1 from an essentially combinatorial point of view. We will give a few new proofs and constructions and supply the details for various theorems due to Thurston. In Sections 2-5, we describe our work, though at several points in the exposition we derive some of the deeper facets of Thurston's theory.

The reader will find that we concentrate on the static aspects of the theory in the sense that we pursue structural properties of $\mathcal{PL}$. This means that we only rarely consider the action of the mapping class group on the boundary of Teichmuller space. There is much to be said, for example, about the equivalence relation on train tracks itself, for the underlying combinatorics of degenerate conformal structures on surfaces is both complicated and beautiful.

In Section 2, we describe this equivalence relation and derive some useful technical facts. We moreover define canonical models, called <u>standard tracks</u>, of equivalence classes. These standard tracks play a central role in our treatment. In Section 3, each measured track is shown to be equivalent to a <u>unique</u> measured standard track. This is done by introducing a second equivalence relation on measured train tracks, which, it turns out, gives the same equivalence classes as our first relation. This second equivalence relation leads us to a natural pairing on MT which generalizes intersections of simple closed curves on our surface. This pairing is jointly continuous, symmetric, non-degenerate and locally bilinear.

The collection of projective measures on any train track gives a cell in PT. Using the standard tracks we then give an explicit tiling of PT, called a <u>paving</u> to emphasize that cells are irregular shapes and sizes. The paving of PT is used in Section 4 to give a new proof of Thurston's result that the space PT is a sphere. This is proved by explicitly considering the cell complex described by our paving. (We do not show that this sphere compactifies Teichmuller space, since there are nice treatments of this in print.)

Section 5 contains by far the deepest of the new results presented here. Here we describe a means of refining a paving. A family $\{P_i\}_1^\infty$ of pavings is constructed whose elements are cofinal among all the cells in PT given by train tracks on our surface. This means that for any train track, the cell it defines is the union of cells given by tracks in P_n for n large enough. The value of cofinality is that the combinatorics of degenerate conformal structures are completely described by our family. As an application, we derive a faithful representation of the mapping class group as a group of affine interval exchange transformations.

The Addendum begins with the description of a local pairing on $\mathcal{MT}$ which is part of the general theory developed by Thurston. This pairing has all the properties listed above for the first pairing except that it is skew-symmetric rather than symmetric. In the latter part of the section we compute this pairing relative to our standard tracks.

We will assume throughout a familiarity with basic surface topology and with the rudiments of hyperbolic geometry. A good treatment of the needed geometric results may be found in the delightful paper of Peter Scott [S]. Familiarity with the earlier aspects of Thurston's work, especially the topic of measured foliations, while not necessary, will help to motivate some of the work herein. The book [FLP] covers these topics completely.

Our primary sources have been Thurston's Princeton Lecture Notes [T] and the Boulder, Colorado, seminar notes [TG]. Some relevant details may also be found in [K] and in [Ca]; the latter highlights the material here by its emphasis on the action of the mapping class group on $\mathcal{PT}$. The reader will also find [Ke1] and [Ke2] helpful for an understanding of the space $\mathcal{PT}$ as the boundary of Teichmuller space.

The authors are indebted to several people. We wish to thank William Thurston for his help and his incredible insight. His creativity has reinvigorated and united many fields of mathematics. We are grateful to John Morgan for his lectures at Columbia University on the Thurston theory. We also thank William Goldman for his notes from the Boulder seminars [TG] and for many useful comments, Bill Floyd for helpful discussions with the second author on the material in the Addendum, and Nat Kuhn for allowing us to use an excerpt from [Ku] as Section 1.4.

Section 1: The Basic Theory

We will begin with the basic definitions. The material of this section is mostly due to Thurston. Our treatment, however, involves some new proofs of facts from [T] and [TG].

(1) Train tracks and train tracks with stops

Let $F^s_{g,r}$ be an oriented surface of genus g with r boundary components and s punctures. When the topological type of the surface is fixed or is not important, we may simply call it F. Sometimes we think of the punctures as distinguished points of F; call their union Δ, and write $F_0 = F - \partial F - \Delta$. Let Σ be Δ together with one point in each component of ∂F. A train track τ on F is a finite collection of simple closed curves and one-dimensional cell-complexes, each made up of vertices (also called switches) and edges (also called branches) disjointly embedded in F_0. Edges are open 1-cells by convention so that τ is the disjoint union of its edges and vertices; circle components of τ are considered edges. Moreover, a train track avoids a neighborhood of $\partial F \cup \Delta$. We furthermore require that the following conditions hold.

(1) τ is C^1 away from its vertices.

(2) For each vertex v of τ, the tangent lines to τ at v defined using one-sided limits along edges incident to v agree.

(3) There are no "dead-ends" along τ; that is, for any vertex v of τ, there is an embedding $f:(0,1) \to \tau$ with $f(1/2) = v$ which is C^1 as a map into F.

(4) Each vertex of τ is at least tri-valent.

(5) If S is a component of the surface obtained by splitting F along τ, let $D_0(S)$ denote the double of S along the edges of S (edges are open so $D_0(S)$ is usually non-compact). Form $D(S)$ by adding to $D_0(S)$

the switches through which there are smooth arcs in the frontier of S. We require that $\chi(D(S)) < 0$.

Examples of train tracks may be found in Figure 1.1.

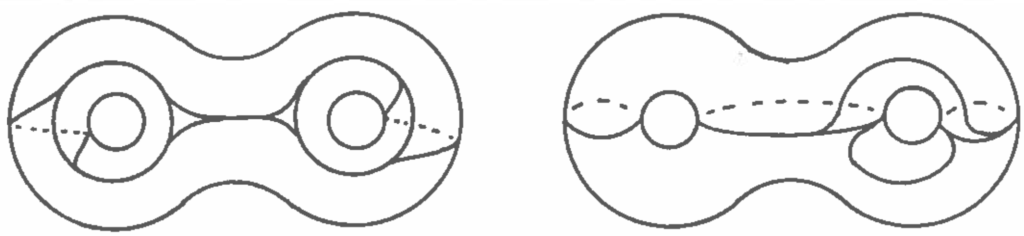


Figure 1.1

We define the notion of an <u>n-gon</u> D in F as follows. D is a smooth open two-disc in F_0 with n discontinuities in the tangent lines of FrD (Fr denotes the frontier). Condition (5) is equivalent to the requirement that no component of $F_0 - \tau$ is a null-gon, mono-gon, bi-gon, punctured null-gon or smooth annulus.

A cell-complex or manifold is <u>properly embedded</u> if it is embedded and intersects $\partial F \cup \Delta$ in Σ, if at all. Occasionally, we will need the more general concept of a train track with stops; this is a finite collection τ of properly embedded one-dimensional manifolds and cell-complexes. We require that conditions (1), (2) and (5) above hold as well as the following.

(3') There are no dead-ends in F_0.

(4') Each vertex of τ in F_0 is at least tri-valent.

We will assume unless otherwise stated that our train tracks with stops have at most one branch incident on a vertex in Σ. Vertices in Σ will be called <u>stops</u> of τ. An example of a track with stops is given in Figure 1.2.

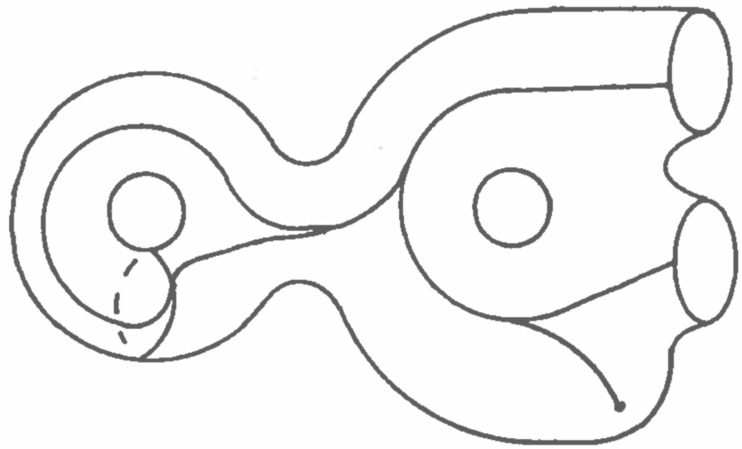


Figure 1.2

Another point of view we use for train tracks with stops is as follows. Let $F_1 = F - \Sigma$. Choose a complete hyperbolic metric of finite area on F_1 such that $\partial F - \Sigma$ is geodesic; each point of Σ then becomes a cusp. In this setting train tracks with stops are required to be imbedded in F_0 and are allowed to have non-compact edges that tend to points in Σ.

If σ and τ are train tracks (with or without stops) in F and σ is contained in τ as a point set, we say σ is a sub-track of τ and write $\sigma \subset \tau$. In this case, σ may be obtained from τ be deleting certain edges and then amalgamating any two edges which meet at a resulting bivalent vertex.

(2) Curves, arcs, and the Dehn-Thurston theorem

A primary motivation for studying train tracks is that large numbers of simple closed curves may be described using a single track. The isotopy class of a family C of disjoint, non-boundary parallel simple closed curves none of which is homotopic to a point or into a puncture is called a multiple curve.

The multiple curve C is carried by τ if there is a C^1 map $\phi: F \to F$ (called the supporting map) such that

(1) $\phi(C) \subset \tau$,

(2) ϕ is homotopic to the identity, and

(3) $d\phi_p|$ (tangent line to C) in non-zero for every $p \in C$.

One thinks of ϕ as squashing together strands of C which are nearly parallel. We also say that the track σ is carried by the track τ (denoted $\sigma < \tau$) using the same definition; condition (3) is legitimate by condition (2) in the definition of a train track. Note that $<$ is transitive.

The curves carried by τ may be described by labeling the branches of τ with integers; for each edge e, pick $x \in e$ and write $\mu_C^{(e)}$ for the number of points in ϕx^{-1}. The definition is independent of x. These integers satisfy a condition defined as follows. For each switch v of τ, fix a direction in $T_v\tau$ (the tangent line to τ at v). An edge incident to v may then be called incoming if it agrees with the direction, outgoing if not. The integers $\{\mu_C(e_i)\}$ with $e_1,\ldots,e_s$ incoming at v and $e_{s+1},\ldots,e_{s+t}$ outgoing, satisfy the switch condition:

$$\mu_C(e_1) + \ldots + \mu_C(e_s) = \mu_C(e_{s+1}) + \ldots + \mu_C(e_{s+t})$$

Conversely, any assignment of non-negative integers to branches of τ satisfying the switch condition defines a multiple curve.

Suppose next that W is a one-manifold properly embedded in F. If (1) no closed component of W is null homotopic or homotopic into a puncture of F, and

(2) the closure of no component of $W \cap F_0$ is homotopic into F or a puncture of F,

then the isotopy class (rel $\sum$) of W will be called a <u>multiple arc</u>. Train tracks with stops carry multiple arcs.

In a 1922 Breslau lecture, Dehn [D] described a one-to-one correspondence between the collection of multiple arcs in an oriented surface $F^s_{g,r}$ of negative Euler characteristic and a subset of $\mathbb{R}^M, M = 6g - 6 + 4r + 3s$. In 1976, Thurston independently rediscovered Dehn's result and extended it to a parametrization of (Whitehead equivalence classes of) measured foliations in F. (See [FLP] for the definition of Whitehead equivalence.) We will call this the <u>Dehn-Thurston parametrization</u> and will presently describe Dehn's result for multiple arcs. [FLP] contains a proof of the general Dehn-Thurston Theorem, and [P] contains a proof of the theorem we use below. We will derive a parametrization for (equivalence classes of) measured train tracks in Section 3 assuming only Dehn's original result; in fact, the arguments in Section 2 and 3 together with some simple results of Dehn give a self-contained proof of the full Dehn-Thurston Theorem.

A <u>pair of pants</u> is any surface homeomorphic to a closed disc minus two disjoint interior open discs. Choose an oriented standard pair of pants P with the boundary components labelled ∂_i and arcs $w_i \subset \partial_i, i = 1,2,3$, called <u>windows</u>, as in Figure 1.3(a).

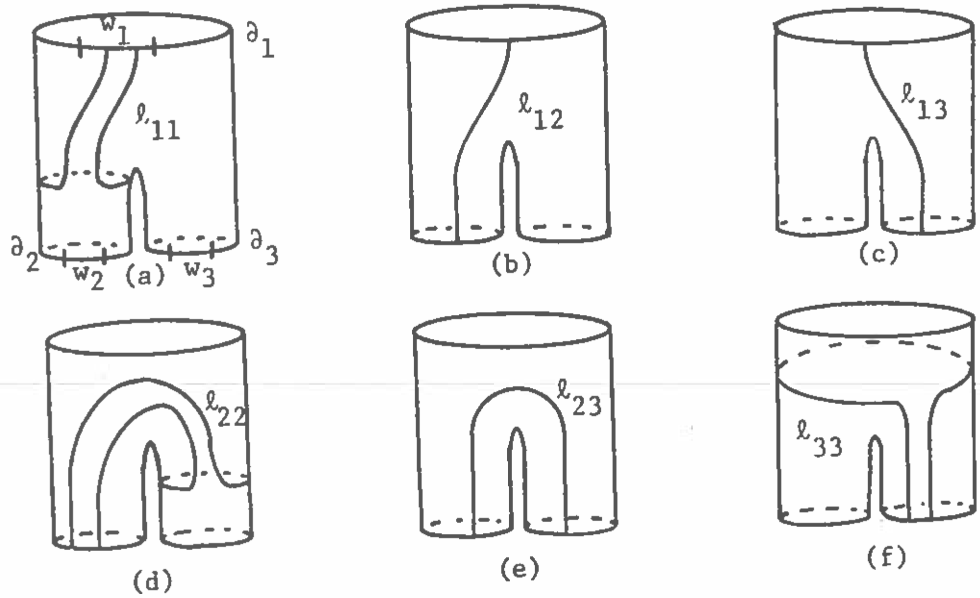


Figure 1.3

Let D be a one-manifold properly embedded in P with ∂D contained in the windows and so that no component of D is boundary parallel. A basic fact due to Dehn is that D is isotopic (keeping ∂D in ∂P but not necessarily fixed) to a unique disjoint union of certain of the arcs illustrated and labelled in Figure 1.3. The different possibilities are uniquely determined by the number m_i of times that D intersects $\partial_i, i = 1,2,3$, subject only to the restriction that $m_1 + m_2 + m_3$ is even. Moreover, D is isotopic fixing ∂P-(windows) pointwise to a one-manifold that consists of a disjoint union of arcs obtained by Dehn twisting those in Figure 1.3 along the $\partial_i, i = 1,2,3$. For each triple (m_1,m_2,m_3) with $m_1 + m_2 + m_3$ even, we choose a representative with no twisting for the isotopy class of the corresponding one-submanifold of P.

The simple structure of multiple arcs in P suggests that we decompose F into pairs of pants. A <u>pants decomposition</u> $\{K_i\}$ of F is an embedded closed one-submanifold in F so that each component Γ of $F\backslash \cup\{K_i\}$ is the interior of a pair of pants or a punctured null-gon. We do not require the closure of Γ in F to be an embedded pair of pants. Some examples of pants decompositions are given in Figure 1.4. Note that $\partial F \subset \cup\{K_i\}$, and for each puncture p of F, there is a uniqure "pants curve" K_i homotopic into p.

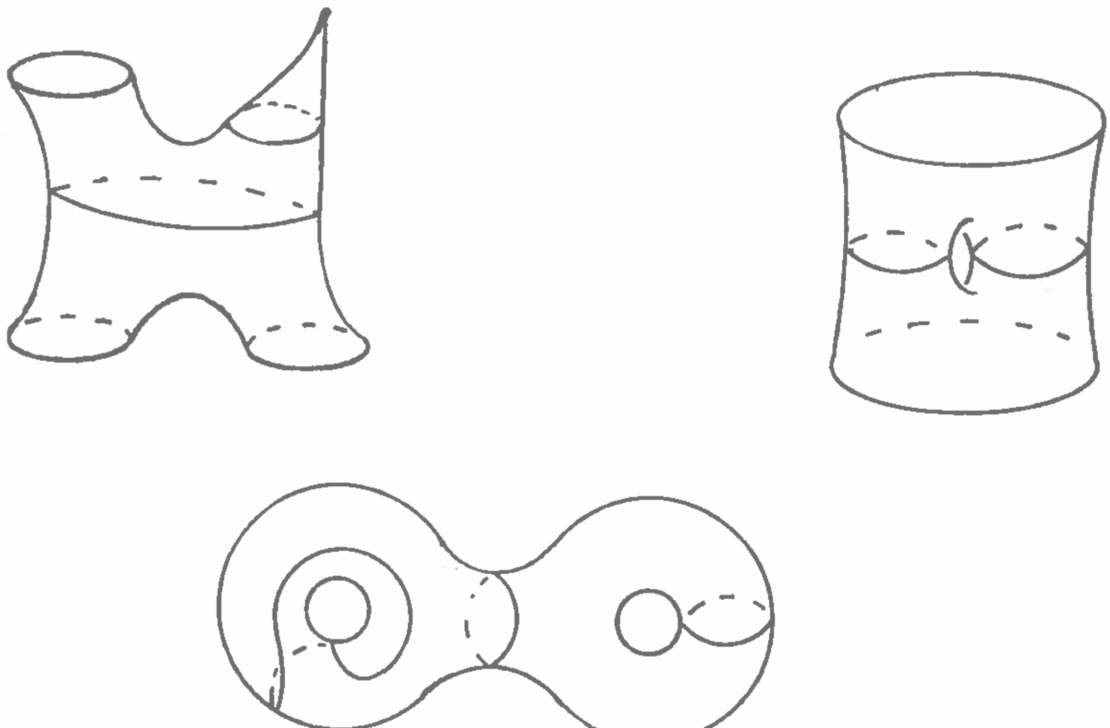


Figure 1.4

For each K_i not homotopic into a puncture of F, choose a small regular neighborhood A_i of K_i; if K_i is homotopic into a puncture p, let A_i be the union of a regular neighborhood of K_i and the component of $F \backslash \cup\{K_i\}$ containing p. Let $A = S^1 \times [-1,1]$ denote the standard oriented annulus and choose for each i an orientation-preserving "characteristic map" v_i that maps $S^1 \times 0$ to K_i as follows. If $K_i \subset F_0$ is not homotopic into a puncture, then v_i is a homeomorphism from A onto A_i; if K_i is a component of ∂F, then v_i is a homeomorphism of $S^1 \times [0,1] \subset A$ onto A_i; if K_i is homotopic into the puncture p, then v_i is a quotient map from

A to A_i collapsing $S^1 \times -1$ to p. In each $K_i \subset F_0$, choose an arc $u_i \subset K_i$ called a "window". Recall that we have chosen in each component $K_i \subset \partial F$ a point where multiple arcs may intersect (the points of $\Sigma - \Delta$). If $K_i \subset \partial F$, let the window u_i be a neighborhood of this point in K_i. If K_i is homotopic into a puncture of F, let u_i be an arbitrary arc in K_i. Finally, let G be the canonical projection of A onto the core $S^1 \times 0$.

Each component P_j of $F_0 \backslash \cup \{\mathrm{Int}\, A_i\}$ is a pair of pants embedded in F. Choose for every P_j an orientation-preserving homeomorphism f_j of P onto P_j carrying each component of $f_j^{-1} \circ v_i \circ G^{-1} \circ v_i^{-1}(u_i)$ to w_1, w_2 or w_3 in P whenever $A_i \cap P_j \neq \emptyset$. f_j is called the "characteristic map" of P_j.

Let C be a representative of a multiple arc $[C]$ in F. We define the (geometric) <u>intersection number</u> m_i of $[C]$ to be the minimum of $\mathrm{card}(C \cap D)$ taken over all curves $D \subset F_0$ isotopic to K_i. We may isotope C fixing ∂F pointwise so that C achieves the intersection number m_i with each component of ∂A_i, arranging that $C \cap \partial A_i \subset v_i \circ G^{-1} \circ v_i^{-1}(u_i)$. Finally, using the basic fact about one-submanifolds of P, we may isotope C so that $f_j^{-1}(C \cap P_j)$ is one of our models in P.

We define a <u>twisting number</u> t_i for each pants curve K_i not homotopic into punctures as follows:

-If $m_i = 0$, take t_i to be the number of components of $C \cap A_i$.

-If $m_i > 0$, then t_i is defined to be the intersection number of the isotopy class (fixing ∂A_i) of $v_i^{-1}(C \cap A_i)$ and $G^{-1}(v_i^{-1}(*))$, where $*$ is one of the boundary points of the window u_i. The sign of t_i is positive if some component of $v_i^{-1}(C \cap A_i)$ twists to the right in the oriented annulus A, and the sign of t_i is negative if some component of $v_i^{-1}(C \cap A_i)$ twists to the left in A.

We have indicated how to compute intersection numbers and twisting numbers of a multiple arc in F. The Dehn-Thurston Theorem asserts that the intersection numbers and twisting numbers uniquely determine this arc. (In Sections 2 and 3, we will prove a more general uniqueness result for measured train tracks.) For Euler characteristic reasons, there are $N = 3g-3+2r+2s$ curves in a pants decomposition of $F^s_{g,r}$ (s of them are homotopic into punctures).

Theorem 1.1 (Dehn-Thurston): *There is a parametrization of the collection of multiple arcs in $F^s_{g,r}$ by a subset of $(\mathbb{Z}^{+N}) \times \mathbb{Z}^{N-s}$. The point $(m_1,\ldots,m_N) \times (t_1,\ldots,t_{N-S})$ corresponds to a multiple arc if and only if the following conditions are satisfied.*

a) *If* $m_i = 0$, *then* $t_i \geq 0$, $i \leq N-S$.

b) *If* K_i, K_j, *and* K_k *bound an embedded pair of pants, then* $m_i + m_j + m_k$ *is even.*

c) *If* K_i *bounds an embedded torus-minus-a-disc, then* m_i *is even.*

As we have seen, to parametrize multiple arcs, we must decide upon a number of conventions. We define a basis A for multiple arcs on F to be a choice of pants decomposition and conventions as above (including the choice of arcs on P in Figure 1.3). In practice, to specify a basis A, we fix a choice of arcs in P, regard F as embedded in S^3, and draw pictures. v_i is chosen as the trivialization that extends across a disc in S^3 with boundary K_i. We draw and label K_i and v_i, and draw $f_j(\ell_{12})(f_j(\ell_{13})$, respectively), labelled 2 (3, respectively) in each P_j. For later use, we define a standard basis $A^s_{g,r}$ on $F^s_{g,r}$ in Figure 1.5 (using the arcs in Figure 1.3).

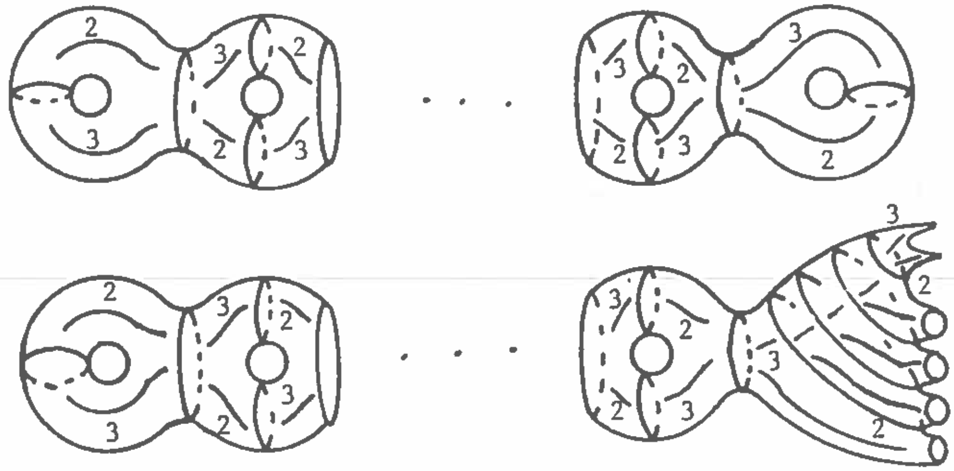


Figure 1.5

Example: Consider the pants decomposition on the surface F_2 indicated in Figure 1.6. We will draw a representative C of the multiple arc γ with Dehn-Thurston parameter values $(5,1,2)\times(1,-1,0) = (m_1,m_2,m_3)\times(t_1,t_2,t_3)$. There are five components of $C \cap A_1$ since $m_1 = 5$, and one of these twists to the right since $t_1 = +1$. Similarly, there is one component of $C \cap A_2$ since $m_2 = +1$, and it twists once to the left since $t_2 = -1$; there are two components of $C \cap A_3$ since $m_3 = 2$ and no twisting since $t_3 = 0$. Thus, we draw our representative of γ in each of the annuli A_i, $i = 1,2,3$, as in Figure 1.7a. We then connect up these arcs uniquely using the images under f_1 of arcs parallel to $\ell_{13},\ell_{23},\ell_{33}$ and under f_2 of arcs parallel to $\ell_{12},\ell_{23},\ell_{22}$ as shown in Figure 1.7b.

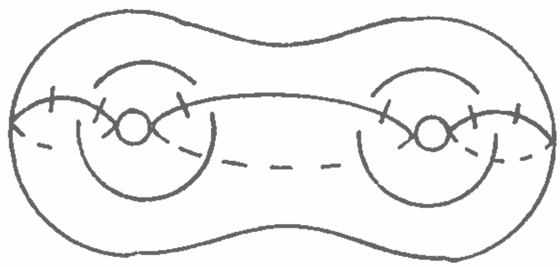
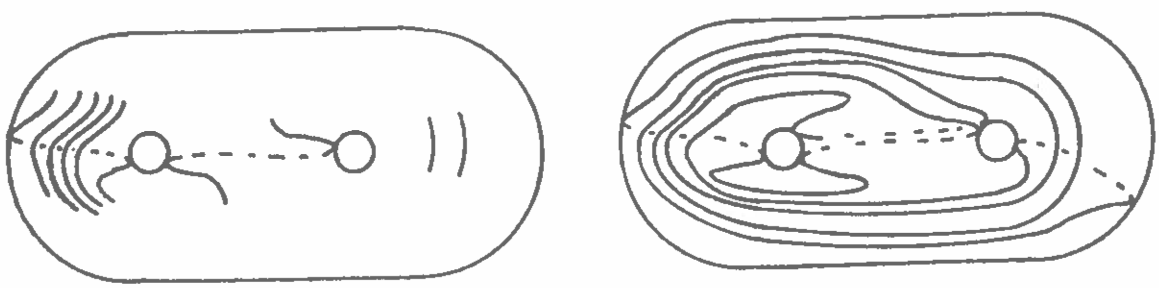


Figure 1.6



(a) (b)

Figure 1.7

As we have seen, to parametrize a multiple arc [C] in F, we isotope C so that each $C \cap A_i$ and $C \cap P_j$ is in canonical position. If K_i is a pants curve homotopic into a puncture of F and $m_i(C) \neq 0$, then we will homotope C so that the following conditions hold:

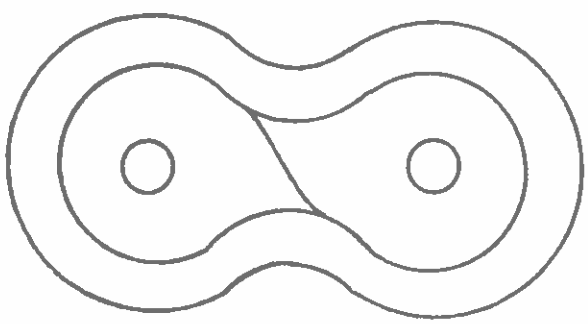
i) $v_i^{-1}C$ is $S^1 \times -1$ (collapsed by v_i to the puncture) together with a single arc running through A without twisting.

ii) $C \cap F \backslash A_i$ is a multiple arc with $t_i = 0$.

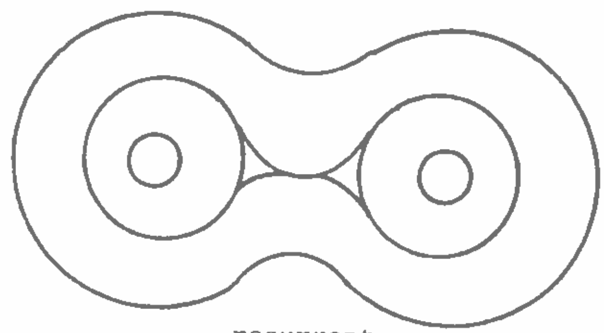
A representative such as C for $[C]$ is said to be in <u>good position</u> with respect to the basis $\mathcal{A}$.

3) Recurrence and Transverse recurrence

The train track τ is called recurrent if for each edge e of τ, there is a simple closed curve C_e carried by τ with supporting map ϕ and $e \subset \phi(C_e)$. It is not difficult to check that τ is recurrent if and only if τ carries a multiple curve C with $\mu_C(e) > 0$ simultaneously for each edge e. A similar definition of recurrence holds for a track with stops (C_e may be an arc with endjoints in $\sum$). Some recurrent and non-recurrent tracks are shown in Figure 1.8.



non-recurrent



recurrent

Figure 1.8

Define a train-path in τ to be any C^1 immersion $\rho:[n,m] \to F$, $n,m \in \mathbb{Z}\cup\{\pm\infty\}$, with image$(\rho) \subset \tau$ so that the restriction of ρ to each subinterval $(k, k+1)$ is a homeomorphism onto an edge of τ and $\rho(k)$ is a switch of τ for each $k \in [n,m]$. The integer $m-n$ is called the length of ρ. The idea behind the term recurrent is that every edge e of a recurrent track admits a train path which passes through e infinitely often (thus $m = \infty$, $n = -\infty$, or both), or passes through e into a stop at both ends. In fact, we may use this an an alternative definition of recurrence (see lemma 1.2).

If τ is an arbitrary train track, then there is a canonically defined maximal recurrent subtrack σ of τ. An edge e of τ lies in σ if and only if there is a curve or arc C in F carried by τ with $\mu_C(e) > 0$. For each such edge e_i of τ, we choose some curve or arc C_i with $\mu_{C_i}(e_i) > 0$ and let μ be the sum of the μ_{C_i}.

Let $C \subset F_0$ be a simple closed curve or a simple arc with endpoints in $\sum$ which is transverse to τ. We say that C hits τ efficiently if there is no bigon $B \subset F_0 - \tau - C$ whose frontier consists of two C^1 segments, one from C and one from τ. Dual to the notion of recurrence, τ is transversally recurrent if for each edge e of τ there is a non-trivial curve or arc C_e which hits τ efficiently so that C_e intersects e at least once. Two other equivalent definitions will be provided in theorem 1.3. An example of a track which is not transversally recurrent is given in Figure 1.9. We leave the verification of this as an exercise.

Assign weights to each edge. The triangle inequalities in T_1 and T_2 imply the edges labeled a,b,c and d must have 0 weight.

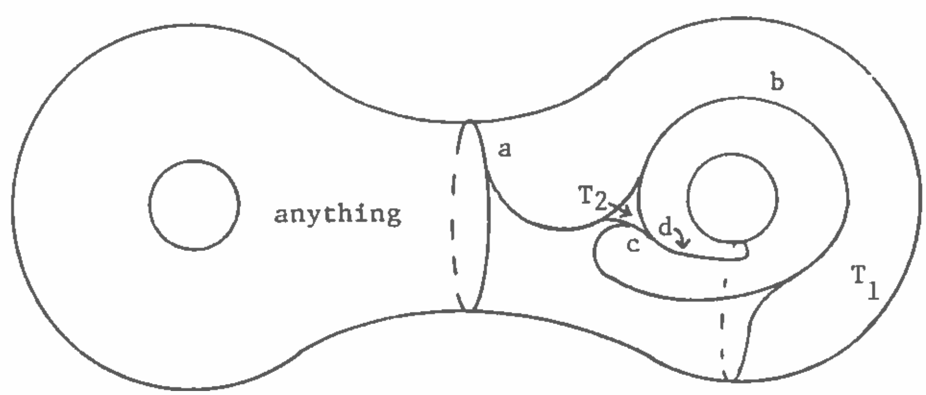


Figure 1.9

A track which is both recurrent and transversally recurrent is called bi-recurrent. There is an interesting duality between recurrence and transverse recurrence which we will explore in this section. We begin with the notion of measures.

A transverse measure μ on τ assigns to each edge e a non-negative real number $\mu(e)$, called the weight on e, and satisfies at each vertex v the switch condition

$$\mu(e_1) + \ldots + \mu(e_s) = \mu(e_{s+1}) + \ldots + \mu(e_{s+t}),$$

where $e_1, \ldots, e_s$ are the incoming edges at v and $e_{s+1}, \ldots, e_{s+t}$ are the outgoing ones. The pair (τ, μ) is called a measured train track; we will write $\mu > 0$ if $\mu(e) > 0$ for every edge e of τ.

When the weights on all edges are integral (rational) a measured train track describes a (weighted) multiple curve or arc as in subsection 1.2. When some weights are irrational the measured train track describes a measured geodesic lamination (to be defined in subsection (5)). Our basic approach however will be to work directly with the measured track whenever possible.

Let $b(\tau)$ be the number of edges of τ. If we were to also allow negative measures on edges, the collection of edge weights which satisfy the switch conditions would be a sub-vector space $H(\tau) \subset \mathbb{R}^{b(\tau)}$. The collection of all possible transverse measures on τ is then $V(\tau) = H(\tau) \cap [\mathbb{R}^+ \cup \{0\}]^{b(\tau)}$, and the collection of all measures which are strictly positive is $V_0(\tau) = H(\tau) \cap [\mathbb{R}^+]^{b(\tau)}$. If τ is recurrent it supports a transverse measure $\mu > 0$. Conversely, if there is a measure $\mu > 0$ for τ, $V_0(\tau)$ is not empty and is therefore homeomorphic to an open cell of the same dimension as $H(\tau)$. Inside $V_0(\tau)$ we may then find a point with rational coordinates which approximates μ; multiplying all measures through to clear denominators gives a point in $V_0(\tau)$ with integer coordinates. These coordinates describe a multiple curve or arc C with $\mu_c > 0$ so τ is recurrent. Thus we have proven

Lemma 1.2: The track τ is recurrent if and only if it supports a transverse measure $\mu > 0$.

The dual notion is that of a tangential measure, which assigns to each edge e of τ a non-negative real number $\nu(e)$ subject to two conditions.

1) Let T be an n-gon in $F_0 - \tau$ with C^1 frontier edges $E_1,\ldots,E_n \subset \tau$, where

$$E_i = \bigcup_j \overline{e_{i,j}},$$

the $e_{i,j}$ branches of τ. Set $\nu(E_i) = \sum_j \nu(e_{i,j})$. For each i, we require

$$\nu(E_i) \leq \sum_{k\neq i} \nu(E_k).$$

2) Suppose A is an annulus in $F_0 - \tau$ with $F_r(A) = \gamma_1 \cup \gamma_2 \subset \tau$ where $\gamma_i = \bigcup_j \overline{e_{ij}}$, the $e_{i,j}$ branches of τ. Suppose further that γ_1 is smooth (hence γ_2 cannot be). With $\nu(\gamma_i) = \sum_j \nu(e_{i,j})$, we require

$$\nu(\gamma_1) \leq \nu(\gamma_2).$$

Other complimentary regions impose no conditions. As with transverse measures, write $\nu > 0$ if $\nu(e) > 0$ for every edge $e \subset \tau$. The analog of lemma 1.2 is the following.

Theorem 1.3: The following conditions on a train track τ are equivalent:

i) The track τ is transversally recurrent.

ii) The track τ admits a tangential measure $\nu > 0$.

iii) For every $\varepsilon > 0$ there is a complete hyperbolic metric on F_0 in which τ has geodesic curvature less than ε at each point.

The actual proof of theorem 1.3 occupies subsection 1.4. We give some preliminary material.

The collection of all tangential measures on τ has a structure similar to that for transverse measures. The first step in understanding this is the following lemma. Recall that $Fr(R)$ denotes the frontier of the region R in F.

Lemma 1.4: Suppose ν is an integral tangential measure on τ such that for each region R of $F - \tau$ which does not meet $\sum$ the sum of the measures $\nu(e)$ with $e \subset Fr(R)$ (counted twice if e occurs twice) is even. Then there is an imbedded multiple arc C meeting τ efficiently such that C intersects each edge e in exactly $\nu(e)$ points. If τ is a train track (without stops), then C is a multiple curve.

Proof: Add simple closed curves to τ to obtain a track whose complimentary regions are disks, annuli, and pairs of pants (possibly punctured). Set ν equal to zero on these new edges. The lemma reduces immediately to finding C in this case.

Pick $\nu(e)$ distinct points on each edge e of τ. We will construct C by connecting these points with arcs in each region of $F - \tau$. Let R be one such region. If R is a pair of pants with $Fr(R)$ missing $\sum$, complete as in the Dehn-Thurston parameterization. The arcs added in R meet $Fr(R)$ minimally so there are no bigons. Next, suppose R is an annulus with $Fr(R) = \gamma_1 \cup \gamma_2$, again with $\sum \cap Fr(R) = \emptyset$. Let E_i denote the sum of the numbers $\nu(e)$ with $e \subset \gamma_i$ (as usual if e occurs twice $\nu(e)$ is counted twice). Suppose $E_2 \geq E_1$. Select a vertex v of τ where γ_2 is not smooth and connect $\frac{1}{2}(E_2 - E_1)$ points to the left of v to the same number to the right (see Figure 1.10). The remaining E_1 points are then connected to the points on γ_2.

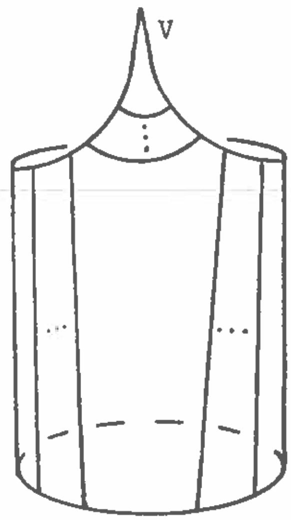


Figure 1.10

If R is an n-gon away from $\sum$ we proceed by induction on n. First let $n = 3$ and let E_1, E_2. and E_3 be the edge sums on $Fr(R)$. The equations

$$E_i = \sum_{j=1, j\neq i}^{3} n_{ij}$$

have unique integer solutions. Connect the $i^{\underline{th}}$ and $j^{\underline{th}}$ sides of R by n_{ij} arcs (Figure 1.11a) to fill in C. When $n > 3$ and the total edges sums are $E_1, \ldots, E_n$, select i so that $E_i + E_{i+1}$ is the minimum of $\{E_j + E_{j+1};\ 1 \leq j \leq n,$ indexed mod $n\}$. Cut R into a triangle and an $n-1$ gon, separating the $i^{\underline{th}}$ and $i+1^{\underline{st}}$ edges from the rest, and select on the new edge $E_i + E_{i+1}$ points (see Figure 1.11b). Completing in these two regions constructs C in R. If R is a punctured n-gon with $n = 1$ or 2, completion is easy.

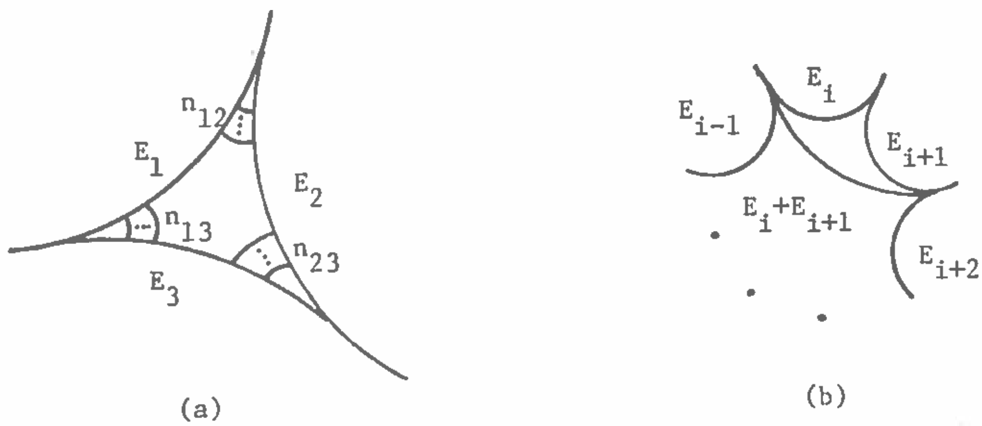
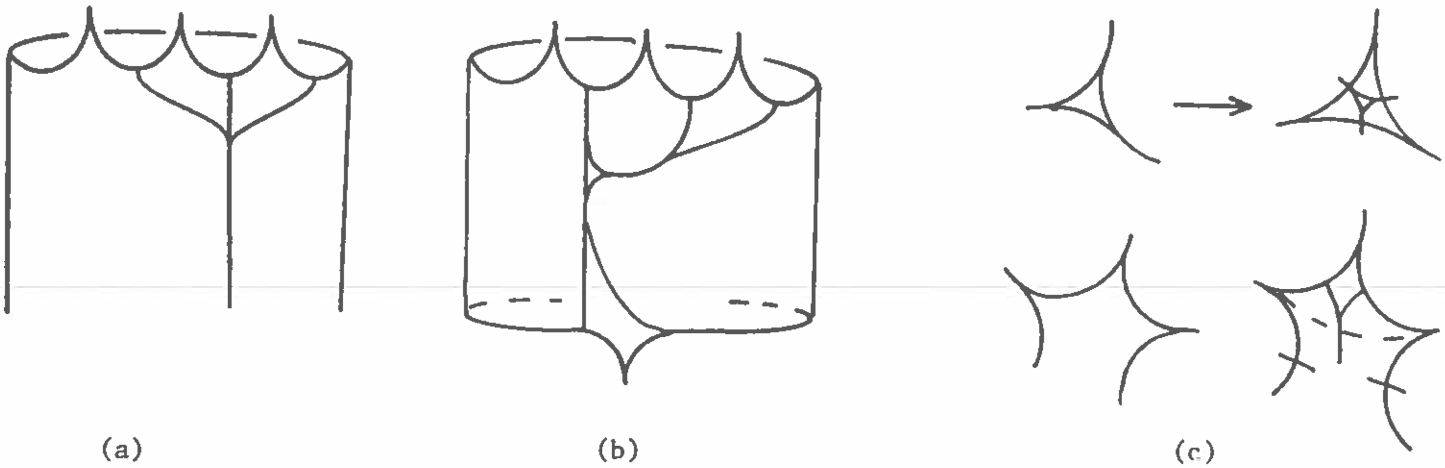


Figure 1.11

Finally, when $Fr(R)$ meets $\sum$, the completion process is analogous except that when the boundary sum is odd one point must be connected to $\sum$. □

A more general tangential measure will give a transverse measure on a <u>dual bi-gon track</u> defined as follows. Let τ be a track with tangential measure ν. Start with an edge $e^{\perp}$ transverse to each edge e of τ for which $\nu(e) > 0$. We follow through the proof of lemma 1.6, completing to a track $\tau^{\perp}$ in each region R of $F - \tau$. When R is a pair of pants with boundary components ∂_1, ∂_2 and ∂_3 and the edge sum at ∂_i equal to E_i, first connect the edges eminating from a single ∂_i as in Figure 1.12a. Then add one of the four tracks of Figure 2.7; use type (0) if E_1, E_2 and E_3 satisfy the three triangle inequalities, and type (i) if $E_i > E_j + E_k$, $\{i,j,k\} = \{1,2,3\}$. It is straightforward to solve the linear equations needed to find weights $\mu(e)$ for each edge e of $\tau^{\perp}$ satisfying the switch conditions so that for every $e \subset \tau$, $\mu(e^{\perp}) = \nu(e)$.



(a) (b) (c)

Figure 1.12

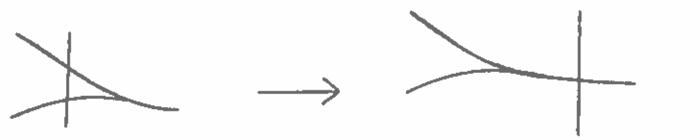
When R is an annulus, connect the edges again as in Figure 1.12 a, leaving one on each side of one vertex in γ_2. Then insert a triangle as in Figure 1.12b (assuming $E_2 \geq E_1$ with γ_2 at the top). Again the measure μ on $\tau^{\perp}$ is easy to find.

Finally, when R is an n-gon cut up into triangles as in the proof of 1.6, complete as in Figure 1.12c. Again the measure μ is easily defined. The process is analogous for regions whose frontier meets $\sum$.

The branched 1-submanifold $\tau^{\perp}$ is not a train track since there is one bigon in its compliment for each switch of τ. However, these are the only bad regions which arise and $\tau^{\perp}$ still carries the transverse measure μ with $\mu(e^{\perp}) = \nu(e)$ for every $e \subset \tau$. In the case where ν satisfies the conditions of Lemma 1.4, $(\tau^{\perp},\mu)$ determines the same curve C constructed before.

Notice that two measures on $\tau^{\perp}$ which give the same lamination differ only by moves which slide an edge across a bigon.

These moves induce an identification on tangential measures on τ by the relations



We see then that the collection of all tangential measures on τ is therefore the <u>quotient</u> of a Euclidean space by linear subspaces.

The second aspect of measured train tracks we will need to consider before proving theorem 1.3 is that of completing τ to a maximal track. Here a train track τ is called maximal if it is not a proper subtrack of any other track. Of course to any track which is not maximal we may simply add edges until the complimentary regions are trigons and punctured monogons. More interesting, however is

<u>Theorem 1.5</u>:(a) <u>If $g > 1$ or $r + s > 1$, every birecurrent track τ may be completed to a maximal birecurrent track (or to a maximal birecurrent track with stops)</u>.

(b) <u>If $g = r + s = 1$, every birecurrent track τ may be completed to a birecurrent track σ with one punctured bigon in $F - \sigma$ (or to a maximal birecurrent track with stops)</u>.

<u>Proof</u>: If $\sum$ is not empty, double F along $\partial F - \sum$. Replace τ by a track which includes $\partial F - \sum$ as an edge (Figure 1.13(a)). For each

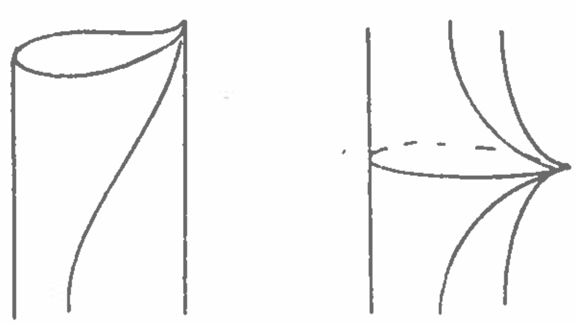


Figure 1.13 (a)

Figure 1.13 (b)

puncture, cut F along a horocycle to form F_0; double F_0 along these components, adding the horocycle as an edge and creating a switch on each side (Figure 1.13(b)) for each intersection with τ. This process allows us to reduce to the case where $\sum = \emptyset$.

An <u>orientation</u> for a track τ is a selection of an orientation for each edge of τ which is consistent along incident edges. Let τ be an <u>oriented</u>, recurrent, connected track and $e, e' \subset \tau$ any two edges. We claim that there is a train path $\rho: [0,n] \to \tau$ with $\rho([0,1]) = \bar{e}$, $\rho([n-1,n]) = \bar{e}'$, agreeing with the orientation on τ. To see this, fix e and let E denote the closure of the set of branches e' admitting such a train path. By maximality of E there is no branch $f \subset \tau - E$ incident on E which is oriented out of E (that is, there is no orientation preserving train path $\rho: [0,2] \to \tau$ with $\rho([0,1]) \subset E$, $\rho([1,2]) = f$). Suppose then that f is any edge in $\tau - E$ incident on E; f is oriented into E. By recurrence there is some simple closed curve C with $\mu_C(f) > 0$. Orient C compatibly with τ; to reach f, C must leave E along some edge, a contradiction. It follows that E is a full component of τ and since τ is connected $\tau = E$.

If, on the other hand, τ is <u>non-orientable</u>, recurrent and connected, <u>any</u> two oriented branches e, e' of τ may be connected by a train path as above which will be compatible with their orientations (including the case where $e = e'$ but opposite orientations). To prove this, simply pass to the orientable double cover $\hat{\tau}$ of τ (defined in the usual way). Choose an orientation for τ and select lifts $\hat{e}, \hat{e}'$ so that $\hat{e}$ projects onto e preserving orientation, as does $\hat{e}'$ onto e'. Recurrence of τ implies recurrence of $\hat{\tau}$ so we apply the previous argument to $\hat{\tau}$, $\hat{e}$, $\hat{e}'$ and push down.

Now we can build σ. First add circle components to τ until the complimentary regions are all disks, annuli and pairs of pants; each of the three types may be punctured but must have all its boundary components in τ. Any of these regions which is not a trigon or punctured monogon will be called <u>large</u>. We could now add an edge across any large region without violating the conditions on complimentary regions of train tracks, but there would be no guarantee that the result would be birecurrent if this is done carelessly.

Before giving the procedure to build σ, consider the problem of finding efficient curves. Let R be a large complimentary region for τ.

<u>Case 1</u>: If R is a disk, form τ' from τ by adding an edge e in R connecting two vertices of τ which are not consecutive on $Fr(R)$.

<u>Case 2</u>: If R is an annulus, form τ' by adding an edge e in R which connects the two components of $Fr(R)$ and is attached at one end to a vertex of τ which is a non-smooth point of $Fr(R)$.

<u>Case 3</u>: If R is a pair of pants, form τ' by adding an edge e in R connecting any two components of $Fr(R)$.

<u>Lemma 1.6</u>: <u>In cases 1,2 and 3, if τ is transversally recurrent, τ' will be as well.</u>

Proof: For case 1 choose edges e_1, e_2 of $Fr(R) \subset \tau$ on opposite sides of e in R. Let C_i be a curve which meets τ efficiently and crosses e_i. An isotopy of C_i (rel $F_0 - R$) arranges that C meets τ' efficiently. If C_1 or C_2 meets e, case 1 is complete, so suppose neither does. Let R_1, R_2 be the components of $R - e$ with $e_i \subset Fr(R_i)$. There are three possibilities to consider. Either C_1 meets R_2, C_2 meets R_1, or neither curve meets the opposite region. When C_1 meets R_2, we choose an imbedded arc α in R connecting two points of C_1, intersecting C_1 nowhere else, which crosses e once. Construct C by surgering C_1 along α; C meets e and is efficient. The second case is analogous. In the final situation, again choose $\alpha \subset R$ meeting e in one point, this time connecting C_1 to C_2, meeting $C_1 \cup C_2$ only at its end points. Form C' from $C_1 \cup C_2$ by surgering along α. This curve C^1 meets τ' and e correctly but unfortunately may not be imbedded.

It is possible, and quite easy, to alter C' into an imbedded, efficient curve meeting e. However, a more consistent approach is to proceed as follows. Associate to each edge e' of τ' the number $\nu(e')$ of times C^1 crosses e^1. These numbers determine an integral tangential measure on τ' and satisfy the parity condition of lemma 1.4. We then apply the lemma to find C. This finishes case 1.

For case 2, let γ be the core circle of R; γ meets e in one point. It is easy to see that γ hits τ' efficiently: any bigon for $\gamma \cup \tau'$ would lie in R and necessarily have an extra non-smooth point at a vertex of τ' where e meets τ. Suppose e' is another edge of τ and C' is a curve meeting it, efficient for τ. Put a tangential measure ν on τ by intersecting edges with C'. Let E_1, E_2 be the edge sums on $Fr(R)$ and set $\nu(e) = |E_1 - E_2|$. This gives a collection of integers satisfying the conditions of 1.4, the resulting curve is fitted to τ' and meets e'.

Case 3 is similar to case 2, and 1.6 is proven. □

Return now to the construction of σ in the proof of Theorem 1.5. We prove the theorem for the case of tracks without stops, the other case is similar. Form a finite, connected graph Γ as follows: Γ has a vertex for each component of τ and a vertex for each component of $F_0 - \tau$ which is an annulus or a pair of pants (either type might possibly be punctured). Attach edges between the vertex corresponding to a region and the vertices for its boundary components, one edge for each boundary component even if they are part of the same component of τ. Figure 1.14 gives an example. Vertices corresponding to regions are designated with an 0 and vertices for components of τ are solid. The sign + or − indicates whether or not the component is orientable. Fix an orientation on each orientable component of τ.

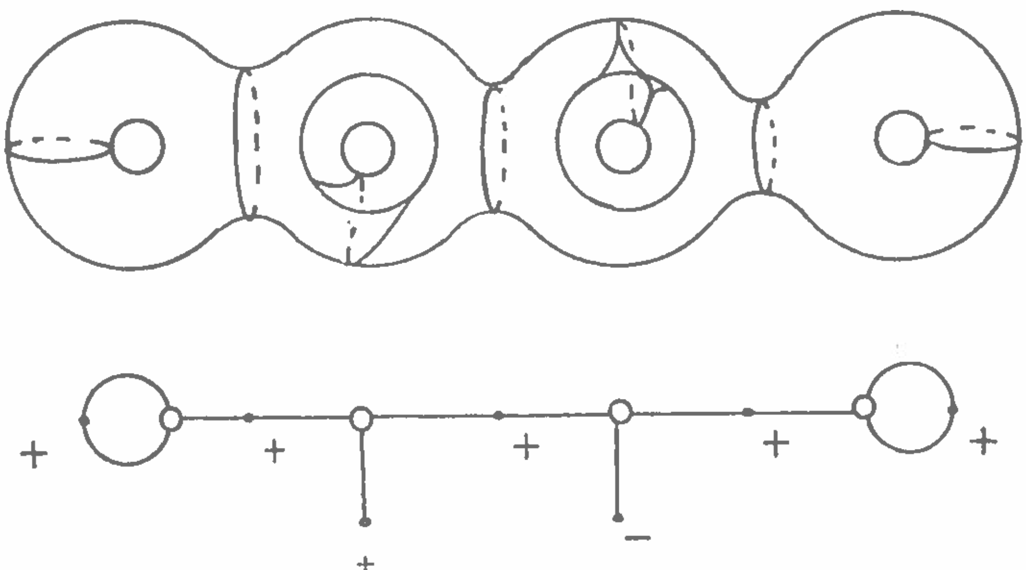


Figure 1.14

Consider first an oriented imbedded edge-loop γ in Γ involving the components $\tau_0, \tau_1, \ldots, \tau_n = \tau_0$ of τ in order. Let $R_1, \ldots, R_n$ be the intermediary regions. Add a branch e_i across each R_i connecting τ_{i-1} to τ_i as in 1.6; making sure that e_i agrees with the orientations of the oriented τ_i. Call the result τ'. There will then be a train path in τ' through each e_i so τ' is birecurrent. In this way we may reduce to the case where Γ is a tree.

Suppose Γ contains more than one vertex. A univalent vertex of Γ cannot represent a simple closed curve and must therefore have a switch. Furthermore, the region of $F - \tau$ it borders must be an annulus. Let γ be an oriented path from one end (a univalent vertex) of Γ to another. Attach branches across the corresponding regions of $F - \tau$ compatibly with orientations to construct τ'. By 1.6 τ' is transversely recurrent; unfortunately, it may not be recurrent. The annulus at each end, however, has become an n-gon with $n \geq 3$. If n is even so that this component is orientable, attach a branch across this n-gon which reverses this orientation. Do the same to the other end if it is also orientable. Our earlier argument on existence of train paths (the beginning of the proof of 1.5) guarantees that the new track is recurrent, and 1.6 says it is transversely recurrent. Finally we are reduced to the case where Γ is a single vertex, so that $F - \tau$ consists entirely of n-gons (possibly punctured).

If τ is non-orientable merely attach edges across the regions until the result is maximal. If τ is oriented we may still attach edges, preserving the orientation, until $F - \tau$ consists of 4-gons and punctured bi-gons. An easy computation shows that if K is the number of 4-gons and ℓ is the number of bigons, $k + \ell = -\chi(F_0)$. When F_0 is a punctured torus this is σ; if not, there must be at least two complimentary regions, R_1 and R_2. Form

τ' by adding edges (as in Figure 1.15) so that a train path into R_1 (following the orientation of τ) reverses direction and a train path into R_2 (travelling against the orientation) reverses direction. The track τ' is clearly recurrent and has acceptable complimentary regions. It is transversely recurrent by 1.6. Finally, τ' is non-orientable so the completion continues as before. This finishes the proof of theorem 1.4. □

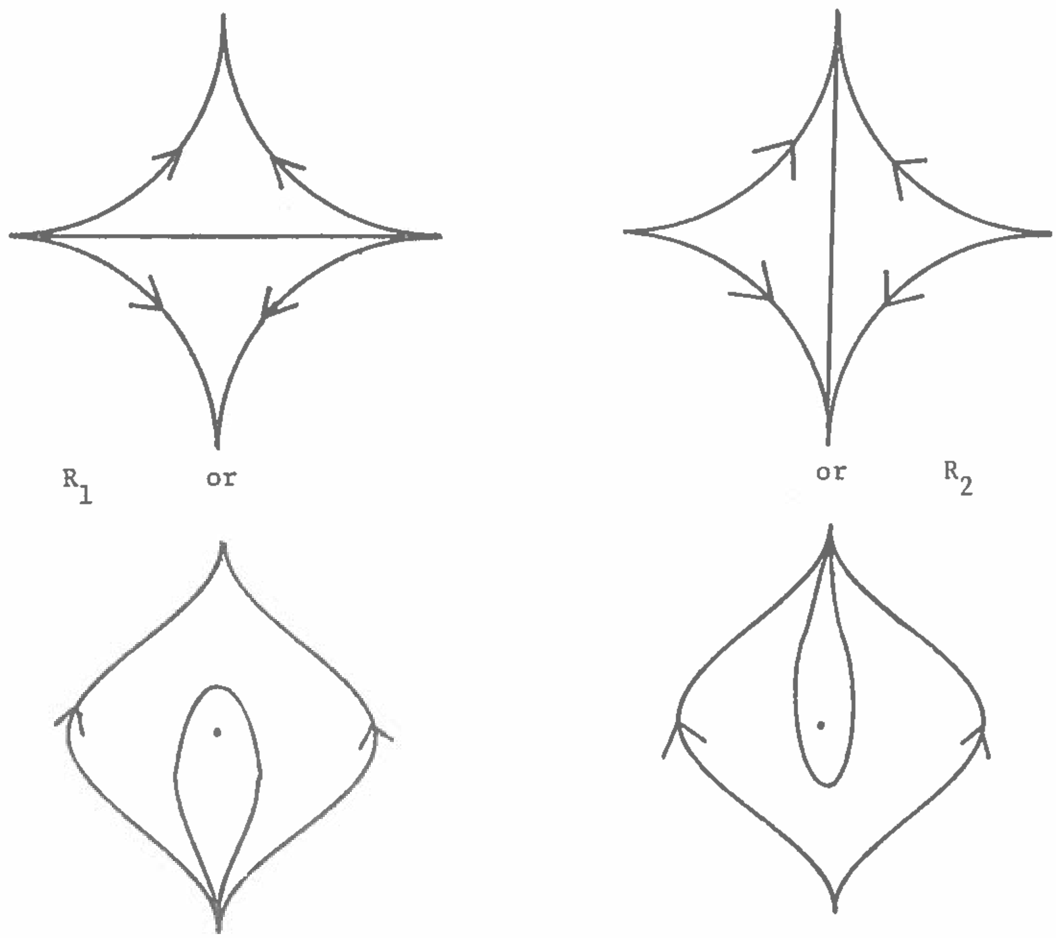


Figure 1.15

(4) Three Equivalent Definitions of Transverse Recurrence

This section is devoted to the proof of Theorem 1.3. We restate this for the convenience of the reader.

Theorem 1.3: The following conditions on a train track τ are equivalent:

(i) the track τ is transversely recurrent.

(ii) the track τ admits a tangential measure $\nu > 0$.

(iii) for every $\varepsilon > 0$ there is a complete hyperbolic metric on F in which τ has geodesic curvature less than ε at each point.

Proof: (i)$\Rightarrow$(ii): By the definition of transverse recurrence, for each edge of τ there is a simple closed curve C_e which hits τ efficiently and intersects the edge e.

Let A be some finite set of such curves, not necessarily disjoint, which hit every edge of τ. Let $\nu(e)$ be the total number of times, counted with multiplicity, that elements of A intersect e. Because elements of A hit every edge, $\nu(e) > 0$ for every e. ν is a tangential measure because given a complementary n-gon T with sides $E_1,\ldots,E_n$, a curve in A which enters T through E_1 cannot exit through E_1 be definition of efficiency (see figure 1.16).

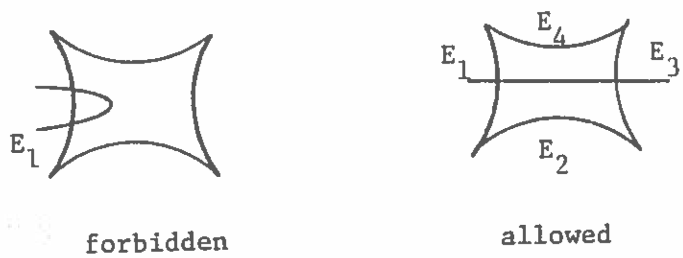


Figure 1.16

The curve must exit P through E_2, $E_3,\ldots,$ or E_n; hence it follows that

$$\nu(E_1) \le \nu(E_2) + \ldots + \nu(E_n)$$

and similarly for every other edge of P. The same argument applies when T is an annulus with one smooth boundary component. □

(i)⇒(iii): By Theorem 1.5 τ is contained in some maximal τ' which is transversely recurrent, and if τ' can be embedded with small geodesic curvature, certainly τ can. Thus without loss of generality we may assume τ is maximal.

<u>Theorem 1.7</u>: <u>Given positive constants L_0 and ε_0, there is a complete hyperbolic metric on F such that τ is piecewise geodesic and satisfies the following estimates</u>:

(a) <u>each edge of τ is a geodesic segment of length greater than L_0</u>.

(b) <u>at each switch the angle of deviation from a straight line is less than ε_0</u>.

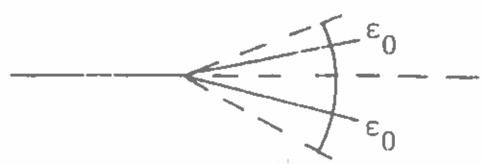


Figure 1.17

Controlling L_0 and ε_0 is a discrete analog of controlling the geodesic curvature, since geodesic curvature is a measure of angle of deviation per unit length. The following exercise completes the proof of (i) ⇒ (iii) given Theorem 1.7.

<u>Exercise</u>: Given ε, there exist L_0 sufficiently large and ε_0 sufficiently small that a piecewise geodesic τ satisfying (a) and (b) can be smoothed to have geodesic curvature less than ε. Hint: because of (a), there are disjoint embedded neighborhoods around each switch with segments of length $L_0/2$ entering and exiting. Such a thing can be smoothed explicitly in a variety of ways.

Proof of Theorem 1.7: For simplicity, assume that F is compact -- the punctured case will be covered later. The proof of (i)⇒(ii) above gives a tangential measure $\nu > 0$ satisfying the triangle inequality $\nu(A) \leq \nu(B) + \nu(C)$. In some cases this can be an equality as in figure 1.18.

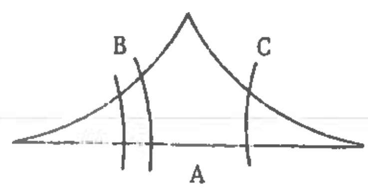


Figure 1.18

We can find a tangential measure satisfying the strict triangle inequality $\nu(A) < \nu(B) + \nu(C)$ if, given any pair of sides of a complementary triangle, we can find a simple closed curve which hits τ efficiently, enters through one side of the pair, and exits through the other. To find such a curve, note that a pair of sides must meet at a switch, so we can simply find an efficient path which crosses the edge on the other side of this switch and push it over as in figure 1.19.

Figure 1.19

Proceeding as before will now give a tangential measure satisfying the strict triangle inequality, and since all weights are integers we actually have:

$$\nu(A) \leq \nu(B) + \nu(C) - 1$$

The strategy now is to create a singular hyperbolic structure in which each edge e has length $L\nu(e)$. Given a complementary triangle with sides A, B, and C, the strict triangle inequality guarantees the existence of a hyperbolic triangle with sides $L\nu(A)$, $L\nu(B)$, $L\nu(C)$. Break up each side of the triangle into segments of length $L\nu(e)$ corresponding to edges e of τ

on the frontier of the triangle; then glue corresponding segments together to get a geometric structure on F. Each vertex of a triangle comes from the crotch of a switch. Number the switches $1, 2, \ldots, N$, and let θ_i denote the interior angle at the vertex corresponding to the i-th switch.

In the interior of a complementary triangle the geometric structure on F is obviously hyperbolic. At an interior point of an edge, the structure is hyperbolic because we are gluing together two half-planes (each with cone angle π) along geodesic boundaries, giving a hyperbolic surface (cone angle 2π). At the i-th switch we are gluing together two half-planes (each with cone angle π) and the interior of an angle θ_i for a total cone angle of $2\pi + \theta_i$. (This is often referred to as a hyperbolic structure with curvature $-\theta_i$ concentrated at the i-th switch.)

To correct this effect we "scallop" the edges of the complementary triangles: that is, we introduce breaks at each switch so that we can control the exterior angle, while leaving the edge lengths fixed (figure 1.20).

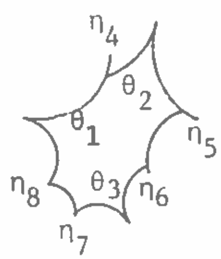


Figure 1.20

Let η_i denote the exterior angle of the region at the i-th switch. Actually, each of the two sides of a switch corresponds to an exterior angle of a scalloped point on a complementary region, but in out construction these angles will always be equal. Because the edges have constant length, we can still glue the scalloped regions together to obtain a singular hyperbolic structure with cone angle $2(\pi - \eta_i) + \theta_i = 2\pi - 2\eta_i + \theta_i$ at the i-th switch. Taking $\eta_i = 0$ simply gives us the unscalloped triangles and the old structure cone angle

$2\pi + \theta_i$ at the i-th switch; when $\eta_i = \theta_i/2$, the structure is non-singular. By changing η_i to $\theta_i/2$ we can attempt to construct a non-singular surface. However, we have also changed the vertex angles θ_i of the scalloped hyperbolic region. The bulk of the remainder of the proof is devoted to estimes showing that the changes of θ_i are small compared to the changes in the η_i. Then we can change η_i to half the <u>new</u> θ_i and use our estimates to show that repeating this process leads to convergence in the values of θ_i and η_i, giving a nonsingular hyperbolic structure.

To construct the scalloped regions, we may break them up into two sorts of pieces, triangles and polygonal slivers as in figure 1.21.

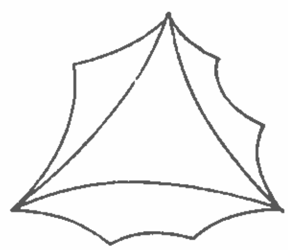


Figure 1.21

Lemma 1.9 below deals with the construction of the slivers and certain estimates which are important in the proof. It is proved by repeated application of Lemma 1.8 which covers triangular slivers. Lemma 1.10 takes care of the triangular pieces,and in Lemma 1.11 these results are combined into results about the scalloped regions.

The object of these estimates is to choose a value of L, the scale factor of the edge lengths, sufficiently large to guarantee the convergence discussed above. To this end, we use the following conventions: a statement like "for A sufficiently large" means for A greater than some a priori constant whose value is (implicitly) established in the course of the proof. Similarly, $x = O(y)$ means $|x| \leq K|y|$ where K is a constant which must be independent of the particular value of A, for example, and $x = o(y)$ means $|x| \leq K|y|$ where K tends to 0 as the bound for A (in this example) tends to infinity. Finally, $x \sim y$ means $x = y(1+o(1))$.

Lemma 1.8: Consider a triangular sliver as below with edges and angles labeled as in figure 1.22.

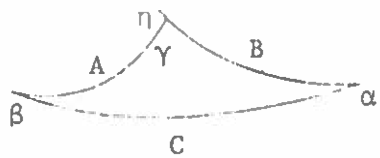


Figure 1.22

Suppose $0 < \eta < \pi/2$ and A and B sufficiently large. Then

(i) $A + B \geq C \geq A + B - 1$

(ii) Suppose A and η are varied by amounts ΔA, $\Delta\eta$, where ΔA is bounded and $0 < \eta + \Delta\eta < \pi/2$. If B is held constant, then $\Delta C = O(\Delta\eta) + O(\Delta A)$

(iii) $\alpha = O(e^{-B}\eta)$ $\qquad\qquad \beta = O(e^{-A}\eta)$

(iv) When A and η are varied as in (ii), $\Delta\alpha = O(e^{-B}\Delta\eta) + O(e^{-B}\Delta A)$, $\Delta\beta = O(e^{-A}\Delta\eta) + O(e^{-A}\Delta A)$.

Proof: The proof rests on a fundamental estimate (*), which will also be used in the proof of lemma 1.10, based on the hyperbolic law of cosines [T 2.6.9]:

$$\begin{aligned} \cosh C &= \cosh A \cosh B - \sinh A \sinh B \cos \gamma \\ &= \frac{1}{4} e^{A+B}[(1+e^{-2A})(1+e^{-2B}) - (1-e^{-2A})(1-e^{2B})\cos \gamma] \\ &= \frac{1}{4} e^{A+B} F(A,B,\gamma) \end{aligned}$$

Now $\cosh x + \sinh x = e^x$, and $\sinh x = \sqrt{\cosh^2 x - 1}$, so $\cosh^{-1} x = \log(x + \sqrt{x^2-1}) = \log(x(1 + \sqrt{1 - \frac{1}{x^2}}))$. Thus:

$$C = \log\left[\frac{1}{4} e^{A+B} F(A,B,\gamma)\left[1 + \sqrt{1 + \frac{e^{-2A-2B}}{(F/4)^2}}\right]\right]$$

$$= A + B + \log\frac{F}{2} + \log\frac{1 + \sqrt{1 + \frac{e^{-2A-2B}}{(F/4)^2}}}{2}$$

If A and B are sufficiently large, e^{-2A} and e^{-2B} are $o(1)$, so $F \sim 1 - \cos\gamma$. Using this and the fact that $\log(1 + o(1)) = o(1)$, we obtain the estimate

$$\log\frac{F}{2} = \log\left[\frac{1+\cos\gamma}{2}\right] + o(1)$$

Similarly, the second log term is

$$\log\left[\frac{1+\sqrt{1+o(1)}}{2}\right] = \frac{\log(1+1+o(1))}{2} = \log(1+o(1)) = o(1)$$

Hence we obtain the fundamental estimate:

$$C = A + B + \log\frac{1-\cos\gamma}{2} + o(1) \qquad (*)$$

(i) $\eta = \pi - \gamma$, so $\cos\gamma = -\cos\eta < 0$. $\frac{1-\cos\gamma}{2} \geq \frac{1}{2}$, so $\log\frac{1-\cos\gamma}{2} \geq -\log 2$, implying

$C > A + B - \log 2 + o(1) > A + B - 1.$

Remark: This implies for example that if A and B are sufficiently large, $\frac{\cosh A \cosh B}{\sinh C} \sim \frac{e^{A+B}/2}{e^{C}/2} = O(1)$. Estimates of this sort will be used without further comment.

(ii) We use Taylor's theorem to estimate C as a function of A and η. From the formula for F:

$$\frac{\partial F}{\partial A} = O(1) \qquad \frac{\partial F}{\partial \eta} = O(1)$$

$$\frac{\partial}{\partial A}\log\frac{F}{2} = \frac{2}{F}\frac{\partial F}{\partial A} = O(1)$$

$$\frac{\partial}{\partial A}\log\frac{F}{2} = \frac{2}{F}\frac{\partial F}{\partial A} = O(1)$$

since $F \geq 1$. Similarly, the partials of the second log term are $O(1)$. Thus $\frac{\partial C}{\partial A} = O(1)$, $\frac{\partial C}{\partial \eta} = O(1)$, and the result follows immediately from Taylor's theorem.

(iii) The hyperbolic law of sines [T 2.6.16] states:

$$\frac{\sin \alpha}{\sinh A} = \frac{\sin \gamma}{\sinh C} = \frac{\sin \eta}{\sinh C}, \text{ so}$$

$$\alpha = \sin^{-1}\left[\frac{\sinh A \sin \eta}{\sinh C}\right]$$

$\frac{\sinh A}{\sinh C} = O(e^{-B})$, so for B sufficiently large, $\alpha = O(e^{-B}\eta)$.

Similarly for β.

(iv) $\sin \alpha = \frac{\sinh A}{\sinh B} \sin \eta$.

Taking $\frac{\partial}{\partial \eta}$ and using (iii) to show that $\cos \alpha \sim 1$,

$$\begin{aligned} \cos \alpha \frac{\partial \alpha}{\partial \eta} &= \frac{\sinh A}{\sinh C} \cos \eta - \frac{\sinh A}{\sinh C} \sin \eta \cosh C \frac{\partial C}{\partial \eta} \\ &\sim e^{(A-C)} \cos \eta + e^{A-C} \sin \eta \, O(1) \\ &= O(e^{A-C}) \end{aligned}$$

To apply Taylor's theorem we must show that the partials of C satisfy a uniform bound over all permissible values of A and η in the neighborhood of the initial values A_0, η_0. By hypothesis, A varies in a bounded range, and by 1, C varies in a bounded range. Hence

$$\frac{\partial \alpha}{\partial \eta} = O(e^{A-C}) = O(e^{A_0 - C_0}) = O(e^{-B_0})$$

in the permitted range of A and η. By abuse of notation, we drop the subscripted zero on the initial values of the side lengths.

Similarly, taking $\frac{\partial}{\partial A}$

$$\frac{\partial\alpha}{\partial A} \sim e^{A-C} \sin \eta - e^{A-C} \sin \eta \frac{\partial C}{\partial A} = O(e^{A-C}) = O(e^{-B})$$

in the permitted range of A and η. The formulas for $\frac{\partial\beta}{\partial\eta}$ and $\frac{\partial\beta}{\partial A}$ are similar except that the first term in $\frac{\partial\beta}{\partial A}$ is missing. The result again follows from Taylor's theorem. □

<u>Lemma 1.9</u>: <u>Suppose angles</u> $0 \le \eta_i < \pi/4$ <u>are given</u>. <u>Then if</u> L <u>is sufficiently large, there is a polygonal sliver as in figure</u> 1.23 .

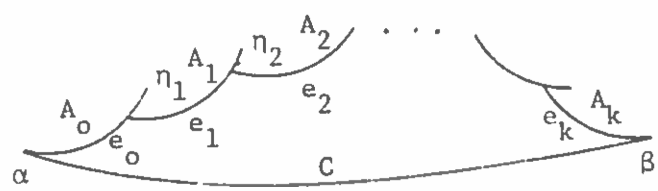


Figure 1.23

<u>with</u> $A_i = L\nu(e_i)$, $0 \le \eta_i < \pi/4$. C, α, <u>and</u> β <u>are continuous functions of the</u> η_i, <u>and</u>

(i) $\sum_0^k A_i \ge C \ge (\sum_0^k A_i) - k$

(ii) If the η_i are varied by $\Delta\eta_i$ such that $0 \le \eta_i + \Delta\eta_i < \pi/2$,

$$\Delta C = \sum_1^k O(\Delta\eta_i)$$

(iii) $\alpha,\beta = \sum_1^k O(e^{-L}\eta_i)$

(iv) $\Delta\alpha,\ \Delta\beta = \sum_1^k O(e^{-L}\Delta\eta_i)$

where the various constants may depend on k.

<u>Proof</u>: Induction on k. Because $\nu(e) \ge 1$ for all edges e, making L large will make all edges long. In the case $k = 1$, the existence of the triangle is

immediate and the estimates are simply those of Lemma 1.8. When $k > 1$, we break the sliver up into a shorter sliver and a triangle as in figure 1.24.

Figure 1.24

The exterior angle η of the triangle is simply $\eta_k + \beta_0$. If L is sufficiently large, induction hypothesis (ii) implies that $\beta_0 < \pi/4$, so that $\eta < \pi/2$ and Lemma 2 applies to the triangle. As to the estimates:

(i) This follows immediately from 1.i and induction hypothesis (i).

(ii) $\eta = \eta_k + \beta_0 = \sum_1^k O(\eta_i)$ be induction hypothesis (iii). Similarly,

$\Delta\eta = \Delta\eta_k + \Delta\beta_0 = \sum_1^k O(\Delta\eta_i)$ by (iv). By 1.ii, $\Delta C = O(\Delta\eta) + O(\Delta A)$,

and the result follows immediately from these formulas and induction hypothesis (ii).

(iii) By 1.iii, $\beta = O(e^{-L}\eta) = \sum_1^k O(e^{-L}\eta_i)$ by the formula for η. The result follows for α by symmetry.

(iv) By 1.iv, $\Delta\beta = O(e^{-L}\Delta\eta) + O(e^{-L}\Delta A)$. Using the formula for $\Delta\eta$ above and induction hypothesis (ii), the result follows. Again the result for α follows by symmetry. □

Lemma 1.10. Consider a hyperbolic triangle with labels as in figure 1.25:

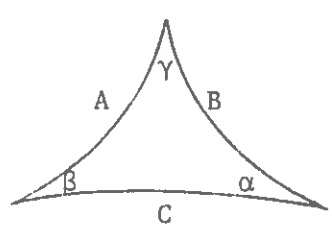


Figure 1.25

If A, B, and C are sufficiently long, and $A + B - C$ is sufficiently large, then:

(i) $\gamma \sim 2e^{(C-A-B)/2}$

(ii) If A, B, and C are varied by bounded amounts ΔA, ΔB, ΔC, then

$$\Delta\gamma \sim O(e^{(C-A-B)/2}\,\Delta C) + O(e^{(B-3A-C)/2}\,\Delta A) + O(e^{(A-3B-C)/2}\,\Delta B)$$

__Proof__: (i) The fundamental estimate (*) from Lemma 1 says:

$$C = A + B + \log \frac{1 - \cos\gamma}{2} + o(1)$$

$$= A + B + \log \frac{\sin^2\gamma}{2} = o(1), \text{ s0}$$

$$\sin \frac{\gamma}{2} \sim e^{(C-A-B)/2}$$

$$\frac{\gamma}{2} \sim e^{(C-A-B)/2}$$

(ii) The hyperbolic law of cosines states:

$$\cos\gamma = \frac{\cosh A \cosh B - \cosh C}{\sinh A \sinh B}, \text{ so}$$

$$-\sin\gamma \frac{\partial\gamma}{\partial C} = -\frac{\sinh C}{\sinh A \sinh B}$$

$$\frac{\partial\gamma}{\partial C} \sim \frac{e^{C-A-B}}{e^{(C-A-B)/2}} = e^{(C-A-B)/2}$$

$$-\sin\gamma \frac{\partial\gamma}{\partial A} = -\frac{1}{\sinh^2 A}\frac{\cosh B}{\sinh B}$$

$$\frac{\partial\gamma}{\partial A} \sim \frac{e^{-2A}}{e^{(C-A-B)/2}} = e^{(B-3A-C)/2}$$

and similarly for $\frac{\partial\gamma}{\partial B}$. Since A, B, and C vary by bounded amounts, $\frac{\partial\gamma}{\partial A} = O(e^{(C_0-A_0-B_0)/2})$ in the permitted range. Similarly for the other partials, and the result follows from Taylor's theorem. □

<u>Lemma 1.11</u>. <u>Suppose angles</u> $0 \leq \eta_i < \pi/4$ <u>are given</u>. <u>Then if</u> L <u>is sufficiently large</u>, <u>there is a scalloped region with these exterior angles</u> (<u>figure</u> 1.26),

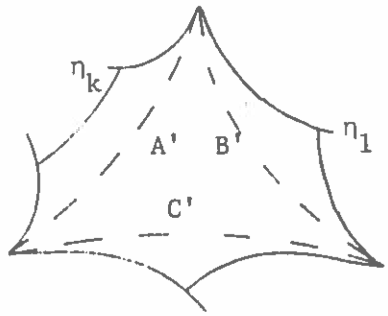


Figure 1.26

where the length of an edge e is $L\nu(e)$. α, β, and γ are continuous functions of the η_i,

(i) $\alpha = O(e^{-L/2})$

(ii) $\Delta\alpha = \sum O(e^{-L/2}\Delta\eta_i)$,

and identical estimates for β and γ obtain.

<u>Proof</u>: By Lemma 3, we can construct three slivers with long sides A', B', C'. From 2.i, $L\nu(C) \geq C'$, $A' > L\nu(A) - \text{const}$, $B' > L\nu(B) - \text{const}$, so

$$A' + B' - C' > L(\nu(A) + \nu(B) + \nu(C)) - \text{const} \geq L - \text{const}$$

and similarly for cyclic permutations of A', B', and C'. Thus if L is sufficiently large the triangle inequality holds, we can construct a hyperbolic triangle with these sides and angles α', β', and γ'. Furthermore, for L sufficiently large, Lemma 3 will apply to this triangle.

The angle α is simply α' plus the angles at the extreme vertices of two of the slivers. By 2.iii and 3.i, these are both $O(e^{-L/2})$, and the result follows.

Hence $\quad e^{(C'-A'-B')/2} = O(e^{-L/2})$. Similarly,

$e^{(B'-3A'-C')/2} \leq e^{(B'-A'-C')} = O(e^{-L/2})$, and

$e^{(A'-3B'-C')/2} = O(e^{-L/2})$. Thus

$$\Delta\gamma' = O(e^{-L/2}\ \Delta A') + O(e^{-L/2}\ \Delta B') + O(e^{-L/2}\ \Delta C')$$

$$= \sum O(e^{(-L/2)\Delta\eta_i}) \quad \text{by Lemma 2.}$$

On the other hand, $\Delta\gamma$ is the sum of $\Delta\gamma'$ and the changes in the extreme angles of the slivers, which are $\leq \sum O(e^{-L}\Delta\eta_i) = \sum O(e^{-L/2}\Delta\eta_i)$. Similarly for α and β. This completes the proof. $\square$

<u>Proof of Theorem 1.7</u> (conclusion): Choose $L \geq L_0$ so that Lemma 4 applies and gives:

(i) $\theta_i < \min(\varepsilon_0, \pi/8)$

(ii) $|\Delta\theta_i| \leq \sum |\Delta\eta_i|$, so in particular $|\Delta\theta_i| \leq |\Delta\eta_i|$.

For $j = 0,1,2,\ldots$ we construct a sequence of exterior angles at each switch by taking $\eta_{i0} = 0$, $\eta_{i1} = \theta_{i0}/2, \ldots\ \eta_{ij} = \theta_{i(j-1)}/2, \ldots$ where θ_{ij} is the interior angle of the scalloped region determined by the exterior angles η_{ij} for a fixed j. We will show that the η_{ij} converge to some η_i at least as fast as a geometric series. $\Delta\eta_{ij} = \eta_{i(j+1)} - \eta_{ij} = \dfrac{\theta_{ij} - \theta_{i(j-1)}}{2} = \dfrac{\Delta\theta_{i(j-1)}}{2}$. Using this and the fact that $|\Delta\theta_i| \leq |\Delta\eta_i|$, we see that $|\theta_{ij}| \leq \dfrac{|\theta_{i0}|}{2^j}$, so

$\theta_{ij} = \theta_{i0} + \sum_k \Delta\theta_{ik}$ converges to θ_i. Since $\eta_{ij} = \theta_{i(j-1)}/2$, η_{ij} converges to $\theta_i/2$. Since the θ_i are continuous functions of the η_i, Lemma 4 implies that there is a scalloped region with these particular values of η_i and θ_i, which can be glued together to give a nonsingular hyperbolic structure on F. The edges are then a piecewise geodesic embedding of τ, and because $L \geq L_0$ and $\theta_i < \varepsilon_0$, conclusions (a) and (b) are established.

<u>Exercise</u>: Complete the proof in the punctured case by providing an analysis parallel to Lemma 1.11 of a scalloped punctured monogon (figure 1.27).

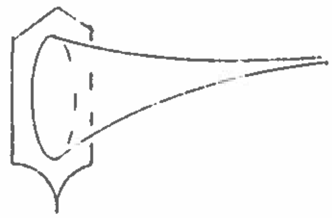


Figure 1.27

<u>Hint</u>: Construct such an object by dividing it into a sliver and a geodesic punctured monogon. A geodesic punctured monogon may be constructed by gluing together two congruent hyperbolic right triangles with one ideal vertex (figure 1.28).

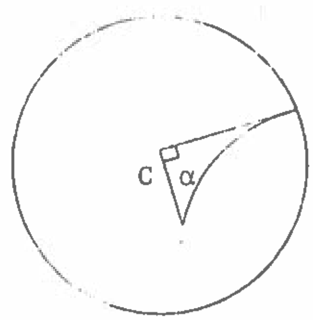


Figure 1.28

Use the trigonometric formula [T 2.16.12]:

$$\cosh C = \frac{1}{\sin \alpha} \qquad \square$$

(ii) $\Rightarrow$ (i): This follows from lemma 1.4.

(iii) $\Rightarrow$ (i): Once a metric is specified on F we may associate to each edge e of τ its length in this metric. What needs to be shown is that if ε is sufficiently small, these lengths satisfy the triangle inequality.

In the case where τ is maximal and F has no punctures, each complementary region of τ is a triangle. Let T' be one such triangle. Lift T' to a triangle T in the hyperbolic plane. The problem now reduces to showing that if $T \subset \mathbb{H}^2$ has 0 angles and sides whose curvature is less than ε at each point, then the lengths of the sides of T satisfy the triangle inequality. This is an exercise in hyperbolic geometry. The details will be omitted. When τ is not maximal, or F has punctures, the regions to consider are n-gons, perhaps punctured, or annuli with one smooth boundary component. The arguments in these cases are similar. $\square$

(5) Laminations

A *lamination* L on F is a one-dimensional foliation of a closed subset of F_0. A *lamination with stops* is a closed subset L of $F_0 \cup \Sigma$ with $L \cap F_0$ a lamination. Actually, we will regard Σ as a set of deleted points in this subsection, so we will not distinguish between a lamination with stops and a lamination. A lamination is *carried* by a track τ in the same manner as a simple closed curve.

Suppose that $F\backslash\Sigma$ has a complete hyperbolic metric ρ of finite area with ∂F (minus Σ, as above) geodesic. (We will term such a metric a "hyperbolic structure.") Explicitly, we have chosen a (conjugacy class of) discrete faithful representation(s) of $\pi_1(F,*)$ onto $\Gamma \subset PSL_2\mathbb{R}$ so that H^2/Γ is a surface $F_* \subset F$ of finite area with the same cusps as F and each boundary component of F_* is a simple closed geodesic in F. Each component of $F-F_*$ must be an annulus-minus-a-point-on-the-boundary with hyperbolic structure. We call a component of $F-F_*$ a *spike*.

When the leaves of a lamination G are geodesic, G is termed a *geodesic lamination*. Clearly, no two leaves of G are parallel. Simple area and Gauss-Bonnet considerations show that $F-G$ has only finitely many components. Moreover, one can show that G has zero measure in F using the Poincare-Hopf Theorem.

Let M be the union over all cusps of F of the horocycles of length less than 2 together with the spikes of F. Another restriction on G is given by the following lemma.

Lemma 1.12: Suppose that ℓ is a leaf of a geodesic lamination G, and $\tilde{\ell}$ is a lift of ℓ to $\tilde{F}$. One of the following conditions holds.

(1) $\tilde{\ell}$ tends to a parabolic fixed point of Γ.

(2) $\tilde{\ell}$ tends to a stop in a spike.

(3) $\ell \cap M = \emptyset$.

Proof: In case $\tilde{\ell}$ interesects a small horocycle but does not satisfy (1), we may conjugate so that the parabolic transformation of the cusp is $z \mapsto z + 1$ in the upper half-space model. Since ℓ is geodesic, $\tilde{\ell}$ is a semi-circle perpendicular to the x-axis; since $\tilde{\ell}$ intersects a horocycle of length less than 2, $\tilde{\ell}$ has radius at least $\frac{1}{2}$. This is impossible since ℓ is simple.

In case $\tilde{\ell}$ intersects a spike but does not satisfy (2), there is a bi-gon in F with geodesic boundary, which is not possible. □

Let G be a geodesic lamination on F. Write A for the collection of all families of imbedded arcs in F with endpoints in $F-G$ which are transverse to G. A transverse measure on G is a function $\lambda: A \to \mathbb{R}^+ \cup \{0\}$ so that the following conditions hold.

(1) $\lambda(\alpha) = \lambda(\beta)$ whenever α is homotopic to β through families in A.

(2) $\lambda(\alpha) + \lambda(\beta) = \lambda(\gamma)$ whenever $\gamma = \alpha \cup \beta$ with $\alpha \cap \beta \subset \partial\alpha \cap \partial\beta$

(3) The support of λ is all of G; that is, if $\alpha \cap G \neq \emptyset$, then $\lambda(\alpha) > 0$.

We will usually suppress λ from the notation and call G a measured geodesic lamination, or simply a measured lamination.

The simplest example is a multiple curve where each component C_i is geodesic and has some weight $\mu_{C_i} \in \mathbb{R}^+$. The measure of any transverse arc α is then

$$\lambda(\alpha) = \sum_i \#(\alpha \cap C_i)\mu_{C_i}.$$

To require a geodesic lamination G to possess a transverse measure (of full support) further restricts G as in the following lemma.

Lemma 1.13: If x is a point in a leaf ℓ of a measured geodesic lamination (G,λ), then one and only one of the following possibilities holds.

(1) $x \in M$.

(2) x <u>is isolated and</u> ℓ <u>is either an isolated simple closed curve or an isolated arc connecting points in</u> $\sum$.

(3) x <u>is an accumulation point of</u> ℓ, <u>and</u> ℓ <u>is disjoint from</u> M.

<u>Proof</u>: If ℓ is a simple closed geodesic that is not isolated, then consider some lift $\tilde{\ell}$ of ℓ to the Poincaré disc; let $f \in \Gamma$ be a covering translation corresponding to $\tilde{\ell}$. We claim that there is a leaf k of G asymptotic to ℓ. If not, then there is a collection $\{k_i\}$ of leaves of G so that the endpoints at infinity of certain lifts $\{\tilde{k}_i\}$ of $\{k_i\}$ are arbitrarily near the fundamental points of f. Since f acts continuously on the circle at infinity we just have $\tilde{k}_i$ intersecting $f\tilde{k}_i$ for i sufficiently large, contradicting simplicity; our claim follows.

Let k be a leaf of G asymptotic to ℓ. Consider an arc α transverse to G intersecting both ℓ and k and with its endpoints in $F-G$. Since α contains the union of an infinite number of arcs, each meeting k once and all homotopic staying transverse to G, properties (1)-(3) show that $\lambda(\alpha) = \infty$, which is absurd. Thus, simple closed leaves of G are isolated and by property (3) have atomic transverse measure.

If ℓ is a leaf of G with both endpoints in $\sum$, then we may double $F-M$ and apply the above argument to conclude that ℓ is isolated and has atomic transverse measure. (Note that only part of $\partial(F-M)$ may be geodesic, but this does not affect the argument.)

Next, let $(D,\overline{G})$ denote the double of (F,G) along the spikes, so that D has only cusps and no spikes. Consider a collection of leaves of $\overline{G}$ all tending to a common cusp in D. If α is a small horosphere about this cusp and α_0 is the <u>finite</u> set $\alpha \cap$ (the set of leaves of $\overline{G}$ connecting cusps of D), then Poincare recurrence implies that $\lambda(\alpha_0) = \lambda(\alpha)$. It follows from property (3) that there are no leaves of $\overline{G}$ with only one end tending to a cusp; thus, no leaf of G may have exactly one endpoint in $\sum$.

Finally, suppose that ℓ is an infinite geodesic that has no endpoints in $\sum$. We may conjugate so that $\tilde{\ell}$ is the positive y-axis in the upper half-space model. We will show that there are covering translations f_n so that the fundamental points of f_n tend to 0 and ∞. The sequence $\{f_n\tilde{\ell}\}$ of lifts of ℓ give accumulating sequences as in (3).

To construct the sequence f_n of covering translations, we proceed as follows. Choose a point $*$ in ℓ, which will serve as base point, and parametrize ℓ by $t:\mathbb{R}\to F$ so that $t(0) = *$. Let $\{\alpha_n\}$ be a nested family of arcs transverse to ℓ of length $1/n$ with $*$ as midpoint. Let $r_n^+ > 0$ be the smallest parameter value so that $t(r_n^+) \cap \alpha_n \neq \emptyset$, and let $r_n^- < 0$ be the largest parameter value so that $t(r_n^-) \cap \alpha_n \neq \emptyset$. Define a loop γ_n in F based at $*$ by following the leaf ℓ from $*$ to $t(r_n^+)$, then traversing the part of α_n from $t(r_n^+)$ to $t(r_n^-)$, and finally following ℓ from $t(r_n^-)$ back to $*$. The loops γ_n are ε_n-quasi-geodesics where $\varepsilon_n \to 0$ as $n \to \infty$. Thus, γ_n is a bounded distance from a geodesic, and the bound tends to zero as $n \to \infty$. It follows that the endpoints at infinity of $\{\gamma_n\}$ tend to 0 and ∞, and hence the same is true of the fundamental points of the corresponding covering translations. The lemma follows. □

The previous lemma immediately gives the following two results.

<u>Proposition 1.14</u>: <u>If α is an arc transverse to a measured geodesic lamination G, then $\alpha \cap G$ is a disjoint union of a discrete set and a Cantor set (closed, perfect, no interior). Moreover, the isolated points of $\alpha \cap G$ are exactly the intersections of α with simple closed leaves and leaves connecting points in $\sum$.</u>

<u>Proposition 1.15</u>: <u>A measured geodesic lamination decomposes uniquely into a</u>

<u>finite number of minimal sets. Each minimal set is one of the following:</u>

(a) <u>a closed geodesic with atomic transverse measure</u>

(b) <u>an infinite geodesic connecting points of</u> $\sum$ <u>with atomic transverse measure</u>

(c) <u>a measured geodesic lamination</u> (G,λ) <u>where each leaf of</u> G <u>is a bi-infinite geodesic in</u> $F-M$ <u>which is dense in</u> G.

Suppose that (G,λ) is a measured geodesic lamination carried by a train track τ with supporting map ϕ. Define a measure μ on τ as follows. Let x be a point which is a regular value of ϕ in branch b_i of τ. $\phi^{-1}(x)$ is an arc α transverse to G, and we define $\mu_G(b_i) = \lambda(\phi^{-1}(\alpha))$. μ_G is independent of the regular point x in b_i, and satisfies the switch conditions by property (2) of transverse measures.

In the other direction, we show how a measured track (τ,μ) describes a measured goedesic lamination (G,λ) in F which is carried by τ.

<u>Construction 1.16</u>: Given a measured, transversally recurrent train track (τ,μ) (with stops) in F, we first describe how to construct a lamination L in F corresponding to (τ,μ); we must be careful to check that our collection of putative leaves foliates a closed subset of F. The second step is to straighten the lamination to a geodesic lamination G, and the final step is to a transverse measure to G.

Choose a hyperbolic structure on $F-\sum$. We denote the length of a path γ in F with respect to this metric by $\rho(\gamma)$. Note that edges of τ incident on stops will have infinite length.

We construct two foliations $\mathcal{F}$ and $\mathcal{F}^{\perp}$ (with singularities) on a neighborhood N of τ in F in the following manner. Fix some $\delta > 0$ with $2\delta < \min\limits_{b_i} \rho(b_i)$. For each branch b_i of τ, consider the rectangle

$R_i(\varepsilon)$ (with respect to ρ) of width $\varepsilon \cdot \mu(b_i)$, $\varepsilon > 0$, and length $\rho(b_i) - 2\delta$ (or infinite length if b_i is incident on stops). For ε sufficiently small, we may disjointly embed the $R_i(\varepsilon)$ in F so that the following conditions hold.

(1) The edges of $R_i(\varepsilon)$ are either parallel or perpendicular to b_i.

(2) b_i is equidistant from the two edges of $R_i(\varepsilon)$ parallel to b_i.

(3) The distance along b_i from a switch (not a stop) of b_i to a perpendicular edge of $R_i(\varepsilon)$ is exactly δ.

(4) Near a stop of τ, $R_i(\varepsilon)$ follows b_i into the cusp or spike.

Each rectangle $R_i(\varepsilon)$ is endowed with two natural foliations: $\mathcal{F}$ has leaves parallel to $b_i \cap R_i(\varepsilon)$, and $\mathcal{F}^{\perp}$ is perpendicular to $\mathcal{F}$. We extend these foliated rectangles to foliations with singularities on $N(\tau)$ in the obvious manner near switches as indicated in Figure 1.29; notice that the switch conditions guarantee consistency of these gluings. For some ε so that the $R_i(2\varepsilon)$ satisfy the conditions above, denote the foliated neighborhood of τ by $(N,\mathcal{F},\mathcal{F}^{\perp})$. It is clear that the isotopy class of $(N,\mathcal{F},\mathcal{F}^{\perp})$ is independent of the metric ρ and the scaling factor ε.

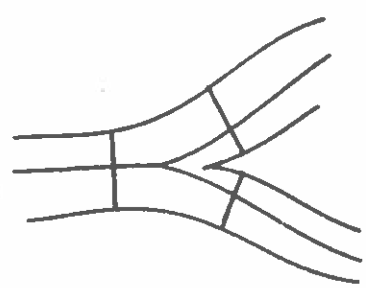


Figure 1.29

A leaf ℓ of F is said to be <u>singular</u> if it is not homeomorphic to a 1-dimensional manifold. For each i choose a leaf ξ_i of $F^\perp$ in the interior of R_i, and let θ_i denote the countable (perhaps finite) set $\xi_i \cap$ {singular leaves of F}. Enumerate the set θ_i as $\{x_j\}$. We construct an interval η_i together with a closed subset $K_i \subset \eta_i$ and a continuous map $\eta_i \xrightarrow{\psi_i} \xi_i$ so that the following conditions hold.

(1) Letting m denote Legesgue measure, $m(K_i) = m(\xi_i)$.

(2) $\psi_i|_{K_i}$ is one-to-one over points of $\xi_i - \theta_i$.

(3) $\psi_i|_{K_i}$ is two-to-one over points of θ_i.

The construction is as follows. Let m_i denote the measure on ξ_i given by adding to the Lebesgue measure, m, Dirac measure for each j of weight $m(\xi_i)2^{-j}$ at the point x_j. Let η_i be the interval of length $m(\xi_i) + \sum_j m(\xi_i)2^{-j}$ and set $t = 1 + \sum_j 2^{-j} \le 2$. The distribution of m_i takes ξ_i to η_i and is discontinuous exactly on θ_i. The inverse distribution $\psi_i\colon \eta_i \to \xi_i$, however, is continuous. Let $K_i \subset \eta_i$ be the complement of the interior of the intervals on which ψ_i is constant (mapping to θ_i). Note that $m(K_i) = m(\xi_i)$ and ψ_i satisfies conditions (2) and (3) above.

Now build a lamination L as follows. Foliate $K_i \times I \subset R_i(t\varepsilon)$ as before by lines parallel to b_i. If b_j and b_k are incident on b_i at a tri-valent switch, let

$$\psi\colon \eta_j \amalg \eta_k \to \xi_j \amalg \xi_k \to \xi_i$$

be the natural map. (The general case of an n-valent switch is similar.) Extend to form L by joining leaves $x \times I \subset R_i(t\varepsilon)$ and $y \times I \subset R_j(t\varepsilon) \amalg R_k(t\varepsilon)$ if $\psi^{-1} \circ \psi_i x = y$. This makes sense because ψ_i is two-to-one at x if and only if ψ is two-to-one at y, so there is a unique way to join leaves without intersections. Note that L is closed since cross-sections of L are closed by construction.

The next step is to replace L by a geodesic lamination G. The idea is that <u>because</u> τ <u>is transversally recurrent</u> each leaf ℓ of L will lift to $\tilde{\ell} \subset \mathbb{H}^2$ where $\tilde{\ell}$ has two well-defined limit (end) points in S^1_∞. We may then replace $\tilde{\ell}$ by the geodesic $\tilde{\ell}'$ these points determine. As the action of $\pi_1(F)$ on $\mathbb{H}^2$ is determined by its action on S^1_∞, $\tilde{\ell}'$ will descend to a simple geodesic $\tilde{\ell}'$ replacing ℓ.

Recall from lemma 1.4 that since τ is transversally recurrent there exists a multiple arc $C = \coprod C_i$ which meets τ efficiently and so that $C \cap e \neq \emptyset$ for every edge e of τ. Let F have a hyperbolic metric in which every C_i is a geodesic (see, e.g. [FLP]). Choose a leaf ℓ of L. Since τ has only a finite number of edges, there is some edge e of τ through which ℓ passes infinitely often (counting each recurrence of the same point separately if ℓ is closed). Let C_i be a component of C which meets e. Look at the collection $\tilde{C}^j_i$ of lifts of C_i and any lift $\tilde{\ell}$ of ℓ. Since $\tilde{C}^j_i$ and $\tilde{\ell}$ are geodesics, they meet in exactly one point. Each $\tilde{C}^j_i$ cuts $\mathbb{H}^2$ into two halves and its closure cuts S^1_∞ into two intervals. Once $\tilde{\ell}$ enters one half of $\mathbb{H}^2 - \tilde{C}^j_i$, it cannot escape. Since ℓ passes through e infinitely often, $\tilde{\ell}$ meets an infinite number of $\tilde{C}^j_i$. Each point of S^1_∞ which is a limit point of $\tilde{\ell}$ must therefore lie in the intersection of a sequence of intervals, and discreteness of the action of $\pi_1(F)$ on $\mathbb{H}^2$ tells us this intersection is a single point.

It now makes sense to replace each leaf of L by a geodesic to form G (note that many leaves of L may amalgamate to create one leaf for G).

Theorem 1.17: The collection G of geodesics given by construction 1.16 form a geodesic lamination.

Proof: The problem is to show G is closed, or equivalently, that the points $E(G)$ in $S^1_\infty \times S^1_\infty$ corresponding to pairs (x,y) where x and y are endpoints of some geodesic in $\widetilde{G}$ form a closed set. Our goal is to describe $E(G)$ as a nested intersection of closed sets. For this we return to τ and make use of C, $N(\tau)$ and F again.

Let $e_1,\ldots,e_k$ be the edges of τ. Select for each i a point p_i in $e_i \cap C$ with $p_i \in C_{j_i}$ and lifts $\widetilde{p}_i \in \widetilde{C}_{j_i}$. Let π_n be the collection of all oriented train paths α involving exactly n edges and meeting e_1 at least once. Each α lifts to one or more paths $\widetilde{\alpha}$ in $\widetilde{\tau}$ through $\widetilde{p}_1$ (one for each time α meets p_1). If the first and last edges of α are e_i and e_j, respectively, the ends of $\widetilde{\alpha}$ determine lifts $\widetilde{C}_i$ of C_i and $\widetilde{C}_j$ of C_j. These each divide $\mathbb{H}^2$ into two halves $H_i^\pm$, $H_j^\pm$ and S^1_∞ into two corresponding closed intervals $I_i^\pm$, $I_j^\pm$, where $\widetilde{\alpha}$ passes from H_i^- to H_i^+ and H_j^- to H_j^+. Define J_n to be the union over all $\widetilde{\alpha}$ of the products $I_i^- \times I_j^+ \subset S^1_\infty \times S^1_\infty$. Any infinite path along $\widetilde{\tau}$ containing $\widetilde{\alpha}$ will have its endpoint in J_n. Clearly $J_{n+1} \subset J_n$ for every n, and we let $J(e_1) = \bigcap_n J_n$. By requiring paths to meet e_i, we may similarly define $J(e_i)$. Finally, set $J_\tau = \bigcup_{e_i} J(e_i)$. Clearly J_τ is closed. Moreover the points of J_τ are exactly the endpoint pairs of the lifts of infinite train paths in τ. Recall that the Mobius band outside $\mathbb{H}^2 \cup S^1_\infty$ is the quotient of $S^1_\infty \times S^1_\infty$ -(diagonal) by the involution interchanging factors. The quotient map sends $J\tau$ to a set M_τ which is also closed.

As this result will be needed later we summarize with

<u>Lemma 1.18</u>: <u>For any transversally recurrent train track τ, the collection M_τ is closed in the Mobius band past infinity.</u> □

The set of endpoint pairs $E(G)$ is contained in J_τ, but it is much smaller. Again, using μ, build $N(\tau)$. The singular leaves of F may be used to describe certain restrictions on the train paths in τ. Let $S_1,\ldots,S_k$ be these leaves, indexed by the switches of τ. (Thus we may have $S_i = S_j$ for some $i \neq j$.) For each $n \geq 1$ let $S_i(n)$ denote the part of S_i which proceeds from the switch indexing S_i and follows exactly n branches of τ (Figure 1.30). Let $N_n(\tau)$ be $N(\tau) - \cup_i S_i(n)$. If $S_i = S_j$, both will be removed when n is large enough. The foliation F restricts to one on N_n.

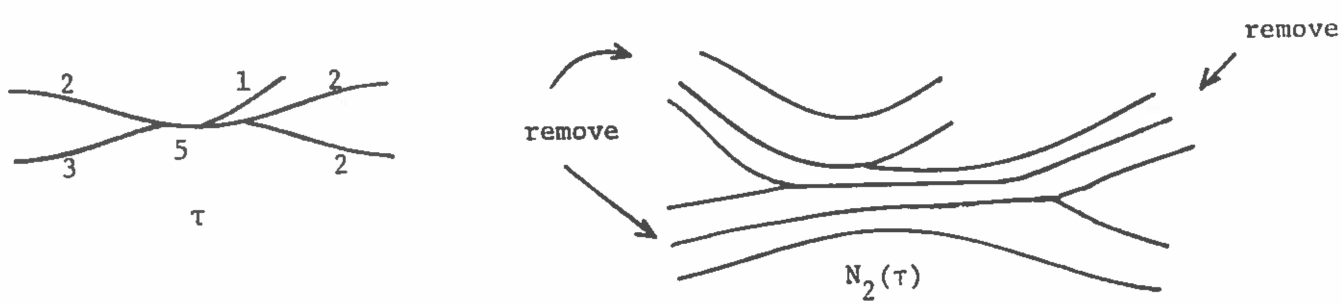


Figure 1.30

Consider now only those paths $\alpha \subset \tau$ for which there is a path $\beta \subset N_n$, β transverse to F, such that the natural map $\psi: N_n \to \tau$ sends β onto α. Use these as before to define a set J^n of endpoint pairs in $S^1_\infty \times S^1_\infty$ corresponding to lifts of infinite train paths in N_n. Again J^n is closed and we have $J^1 \supset J^2 \supset \ldots$. The intersection of these is $E(G)$ by construction and theorem 1.17 is proved. □

Remark: Without transverse recurrence there is no guarantee of a metric on F in which L may be straightened. We will see in section 2 however that arbitrary tracks give measured geodesic laminations because any track may be "refined" to one which is transversally recurrent.

Denote the map which supports G on τ by Ψ. If α is transverse to G, then we define the transverse measure $\lambda(\alpha)$ as follows. Let $\pi_i : R_i \to \xi_i$ denote projection. Thus, $\Psi(\alpha)$ is transverse to τ, and we define $\varepsilon\ \lambda(\alpha) = m(K_i \cap \Psi_i^{-1} \circ \pi_i \circ \Psi(\alpha))$.

We close this subsection by showing that every measured geodesic lamination arises from Construction 1.16 for some measured train track (τ,μ).

Proposition 1.19: Every geodesic lamination G is carried by a recurrent train track τ.

Remark: If λ is a measure on G, then τ inherits a measure μ from λ, as before. In Section 2, we will show how to derive a bi-recurrent track from (τ,μ) which carries G. Thus, every measured geodesic lamination is in fact carried by a bi-recurrent track.

Proof: Let $N_\varepsilon(G)$ be an ε-neighborhood of G. $\partial N_\varepsilon(G)$ inherits a natural simplicial structure, and there is a bound K (depending only on the topology of the surface) to the number of vertices in $\partial N_\varepsilon(G)$. Note that there is some $\delta > 0$ so that two points of G that are within $K\delta$ of each other have close tangent directions. Moreover, since G has zero area, for ε sufficiently small, $N_\varepsilon(G)$ contains no disc of diameter δ.

We foliate δ-neighborhoods of the vertices of $\partial N_\varepsilon(G)$ by arcs transverse to G and extend to a foliation of $N_\varepsilon(G)$, keeping the leaves transverse to G. Let τ be the "core" of $N_\varepsilon(G)$, and define a carrying map $\Psi : F \to F$ by collapsing the leaves of the foliated neighborhood of $N_\varepsilon(G)$ onto τ. □

Remark: This proof is borrowed from [Ca]. [T] proves this result by passing to the universal cover and flowing along horocyclic arcs to get a train track with small curvature. This approach requires showing that this flow is Lipshitz. Yet another proof can be given using the Dehn-Thurston Theorem and our standard models, which are introduced in the next section. See the remark before Lemma 2.7.

Proposition 1.20: If τ is birecurrent and μ_1,μ_2 are measures on τ, the geodesic laminations (G_1,λ_1) and (G_2,λ_2) constructed respectively from (τ,μ_1) and (τ,μ_2) coincide if and only if $\mu_1 = \mu_2$.

Proof: If α is an arc transverse to τ, $\lambda_i(\alpha) = \mu_i(\alpha)$ for $i = 1,2$ by construction. Since λ_i completely determines G_i we have only to show λ_i is uniquely defined by μ_i (on any arc in A_i). Fix i and let α be an arc in A_i. (Recall this means α is transverse to G_i and has its endpoints in $F-G_i$). By additivity, we may assue that α has one end outside $N(\tau)$, that is, if ψ is the map supporting G_i on τ, $\psi(\alpha)$ is an arc with one end on τ and one end outside $N(\tau)$. The endpoint p which maps to τ lies between two leaves of G_i. Since these are geodesics, they cannot remain parallel so there is a path in $F-G_i$ from p to some point q where $\Psi(q)$ is in $F-N(\tau)$. Slide α in this direction, using homotopy invariance of $\lambda_i(\alpha)$. Two problems may occur,

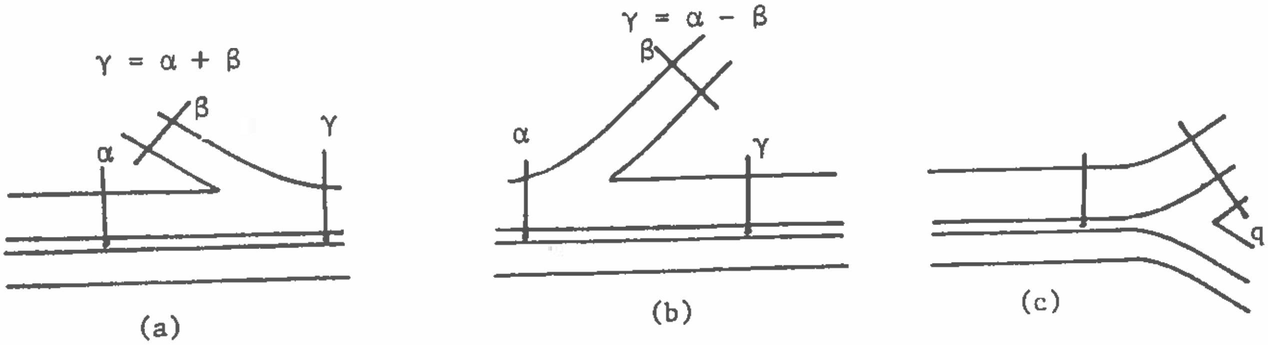


Figure 1.31

illustrated in Figure 1.31(a),(b). In each case, since the measure $\lambda_i(\beta)$ is known we reduce to computing $\lambda_i(\gamma)$ and continue the sliding towards q. Eventually, the needed arc emerges (see Figure 1.31(c)) and becomes one whose measure is known.

Now we know that when $\mu_1 = \mu_2$ every $\alpha \in A_1 \cap A_2$ satisfies $\lambda_1(\alpha) = \lambda_2(\alpha)$. But each G_i is closed and every arc α in A_i meets G_i in the disjoint union of a discrete set and a Cantor set; hence α is isotopic through arcs in A_i to $\alpha' \in A_1 \cap A_2$. This means the measures λ_1 and λ_2 have the same support, so $G_1 = G_2$.

For the converse we must show two distinct bi-infinte paths in τ cannot have lifts to $\mathbb{H}^2$ with the same endpoint pairs. Let $\tilde{\rho}_1, \tilde{\rho}_2$ be paths consisting of edges of $\tilde{\tau}$ with the same ends in S^1_∞. If there is an embedded n-gon in $\mathbb{H}^2$, $n \leq 2$, whose closure lies in $\mathbb{H}^2$ and whose frontier has an edge in each $\tilde{\rho}_i$, this n-gon gives rise to an embedded m-gon, $m \leq n$, in $\mathbb{H}^2 - \tilde{\tau}$ with frontier in $\tilde{\tau}$. This projects to an m-gon in $F - \tau$, contradicting the definition of a train track. If there is an embedded n-gon, $n \leq 2$, as above except that one of the vertices of its frontier lies in S^1_∞, we argue as follows. Assume first that $n = 2$; call the bigon B. Let s be the switch where the interior vertex of B lies, p the point on S^1_∞ at the other vertex of B, and $\tilde{e}_1, \tilde{e}_2$ the edges incident on s which lie in the sides of B. Index these edges so that when travelling away from s $\tilde{e}_1$ lies to the right of B and $\tilde{e}_2$ to the left. Let α_L be the oriented half-infinite path from s to p along $\tilde{\tau}$ which begins with e_1 but afterwards follows the branch to the left at each switch. Similarly, define α_R to be the oriented path beginning with e_2 but switching continually to the right (Figure 1.32). Clearly α_L and α_R converge to p and bound another bigon $B' \subset \mathbb{H}^2$. By construction, there

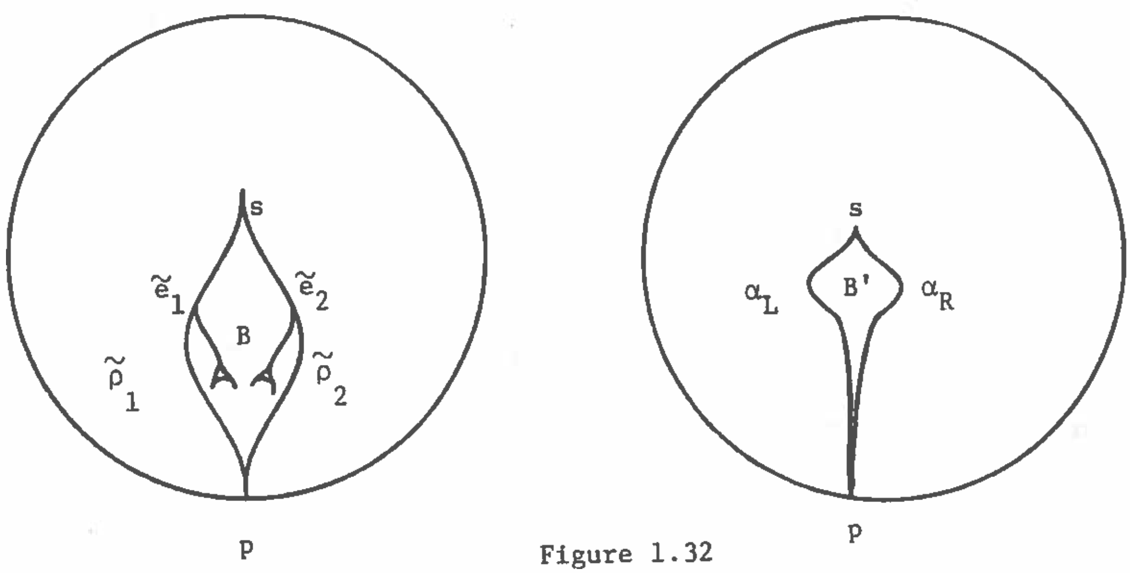


Figure 1.32

are no paths out of α_L or α_R into B' which agree with their orientation. Furthermore, any path into α_L from B' must either give rise to an embedded null-gon, monogon, or bigon with closure in $\mathbb{H}^2$ or a 3-gon and an embedded monogon with vertex in S^1_∞.

Suppose first that the bigon B' is embedded in $\mathbb{H}^2 - \tilde{\tau}$. Any covering translation T must either map B' off itself or preserve it setwise. However $T(B') = B'$ implies that $T(s) = s$, so T is the identity map. Therefore B' embeds in $F - \tau$. Once again this contradicts the definition of a track.

The next case to consider is when B' is a monogon with its vertex at $p \in S^1_\infty$. Area considerations then say that if B' does not imbed in F, p is a parabolic point and $Fr(B')$ is isotopic to a horocycle. In F, B' becomes a punctured null-gon which again is not legal.

The final situation is when $\tilde{\rho}_1$ and $\tilde{\rho}_2$ bound a bigon with both its vertices in S^1_∞. An argument similar to the one before allows us to reduce to an earlier case, or to an embedded version of this bigon in $\mathbb{H}^2$ which could only cover an annulus with two smooth components or a bi-infinite embedded bigon. The first case is not allowed, the second is not possible, proposition 1.20 is proven. □

(6) Measured Lamination Spaces

The measured lamination space ML is the collection of all measured goedesic laminations on F, where F has some fixed hyperbolic structure. The topology on ML is induced by the weak topology of measures supported on the open Möbius band beyond infinity. (Points on this Mobius band correspond uniquely to geodesics in $\mathbb{H}^2$, see for instance [C], and we induce a measure via λ.) ML_0 is the subspace of measured laminations with compact support; elements of ML_0 lie outside of a small neighborhood of each stop. Multiplying the measure by a positive constant gives an $\mathbb{R}^+$ action on both spaces. The projective lamination spaces are $PL = (ML - 0)/\mathbb{R}^+$ and $PL_0 = (ML_0 - 0)/\mathbb{R}^+$; where 0 is the empty lamination. None of these spaces depend up to homeomorphism on the hyperbolic structure given to F since they can be defined without the metric by using the circle at infinity.

We will also need to build the space of measured laminations with stops. Pick $\sum' \subset \sum$ consisting of s' of the points in Δ and r' of the points from ∂F. Define $ML_{r',s'}$ (with its projective version $PL_{r',s'}$) to be the measured geodesic laminations with stops in $\sum'$. These spaces are independent of the metric and the choice of the s' and r' points. Notice also that $PL_0 = PL_{0,0}$ and $ML_0 = ML_{0,0}$.

Perhaps a more agreeable interpretation is given by choosing a complete hyperbolic metric on $F - \sum$ so that each component of $\partial F - \sum$ is a geodesic. $ML_{r',s'}$ is then the space of measured geodesic lamination in $F - \sum$ which lie outside a neighborhood of the points of $\sum - \sum'$. Define the Teichmüller space $T^s_{g,r}$ to be the collection of all such complete hyperbolic metrics of finite area on $F - \sum$ with $\partial F - \sum$ geodesic, modulo the equivalence induced by homeomorphisms $f: F \to F$ with $f|\sum$ equal to the identity and f homotopic to the identity rel $\sum$. The space $PL_{r,s}$ may then be identified with the boundary of a compactification of $T^s_{g,r}$.

Addendum: The Piecewise Symplectic Structure of ML

In addition to the intersection pairing $\langle \cdot , \cdot \rangle : ML \times ML \to \mathbb{R}^+ \cup \{0\}$ discussed in Section 3, there is also a pairing $\{ \cdot , \cdot \}_\tau : V(\tau) \times V(\tau) \to \mathbb{R}$ defined for any track τ with stops. (We will identify $V(\tau)$ with the corresponding subset of ML when convenient.) This pairing will be seen to be a jointly continuous extension of homology intersection numbers of oriented curves and will be invariant under splitting and shifting of the track τ. It will follow that the pairing is well-defined on neighborhoods in ML independent of the track chosen to carry the neighborhood. Moreover, as with the pairing $\langle \cdot , \cdot \rangle$, our new pairing will be seen to be natural in the sense that it is invariant under the action of the mapping class group on ML.

Though the pairing $\{ \quad , \quad \}_\tau$ can be defined for a train track τ with stops and the results of this addendum apply in this setting, we will restrict attention to track without stops; the technical details required for discussing tracks with stops obfuscate the relevant mathematics.

Suppose, then, that τ is a (generic) track (without stops) and that s is a switch of τ with incident edges a_s, b_s and c_s as indicated in Figure A.1. Suppose, moreover, that the surface is oriented as indicated with b_s to the left and c_s to the right as we transverse a_s towards s. If $\mu, \nu \in V(\tau)$, we define

$$\{\mu,\nu\}_\tau = \frac{1}{2} \sum_s \det \begin{pmatrix} \mu(c_s) & \mu(b_s) \\ \nu(c_s) & \nu(b_s) \end{pmatrix}.$$

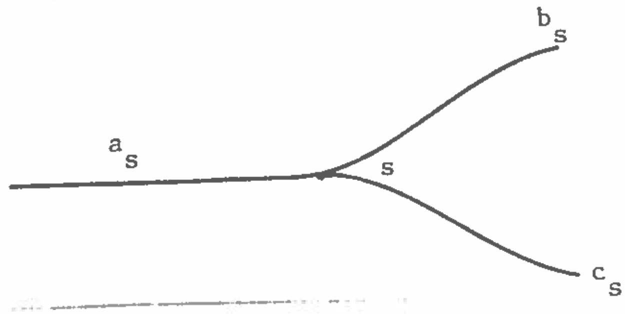


Figure A.1

Lemma A.1: a) $\{\cdot, \cdot\}_\tau$ is a skew-symmetric bi-linear pairing on $V(\tau)$.

b) If σ arises from τ by splitting, shifting or isotopy, then $\{\cdot, \cdot\}_\tau | V(\sigma) = \{\cdot, \cdot\}_\sigma$.

Proof: The assertion of part a) is obvious. Part b) follows by checking the various possibilities. □

We claim that the pairing is well-defined on neighborhoods in ML_0 in the following sense.

Lemma A.2: If G and H are measured geodesic laminations which are both carried by the tracks τ and τ', then $\{G,H\}_\tau = \{G,H\}_{\tau'}$.

Proof: We use the construction of Lemma 5.5 to build a branched one-submanifold σ of F so that $V(\sigma) = V(\tau) \cap V(\tau')$, $\sigma < \tau$ and $\sigma < \tau'$. There is a symplectic pairing $\{\cdot, \cdot\}_\sigma$ on $V(\sigma)$ defined as before. We collapse bi-gons and annuli complimentary to σ in F along the leaves of the foliated neighborhood $F^\perp$ of τ' to obtain a track σ' so that $\sigma' < \tau'$. Note that $V(\tau') \supset V(\sigma') \supset V(\sigma)$. It is straightforward to check that $\{\cdot, \cdot\}_{\tau'}|_{V(\sigma)} = \{\cdot, \cdot\}_\sigma$. By Theorem 2.11, τ' refined to σ', so $\{\cdot, \cdot\}_{\tau'}|_{V(\sigma')} = \{\cdot, \cdot\}_{\sigma'}$ by Lemma A.1 b). The result follows by symmetry. □

Together these lemmas imply that $\bigcup_\tau(\{\cdot, \cdot\}_\tau)$ defines a natural piecewise symplectic (bi-linear and skew-symmetric) pairing on ML_0. An equivalent point of view is that $\bigcup_\tau(\{\cdot, \cdot\}_\tau)$ defines a symplectic structure on the PL manifold ML_0.

There is an alternative definition of the pairing which we will give presently. Whereas the first definition is convenient computationally, the second definition will be useful theoretically. It will follow from our second definition, for instance, that $\{\cdot,\cdot\}_\tau$ is non-degenerate on $V(\tau)$, a fact which is difficult to prove directly from our first definition.

Recall that a track is orientable is there is an orientation of its branches that is consistent along incident edges and that a track is orientable if and only if each complimentary region has an even number of sides. Moreover, a measure on an oriented track describes a real homology cycle of the underlying surface F.

<u>Lemma A.3</u>: If τ is an orientable track, then $\{\cdot,\cdot\}_\tau : V(\tau) \times V(\tau) \to \mathbb{R}$ is the homology intersection number of the corresponding real cycles.

<u>Proof</u>: Choose an orientation on τ and a neighborhood N of τ as in Construction 1.16; it makes sense, then, to refer to the left and right sides of τ in N. Consider two isotopic copies τ_1 and τ_2 of τ in N, τ_1 pushed off to the left and τ_2 pushed off to the right. Near a switch s of τ, τ_1 and τ_2 are situated as in Figure A.2. We compute the homology intersection number of the cycles (τ,μ) and $(\tau_1,\nu) + (\tau_2,\nu)$ to be $\mu(c_s)\nu(b_s) - \mu(b_s)\nu(c_s)$, as desired. (Note that a different choice of orientation on τ does not affect our argument since homology intersection is bi-linear.) $\square$

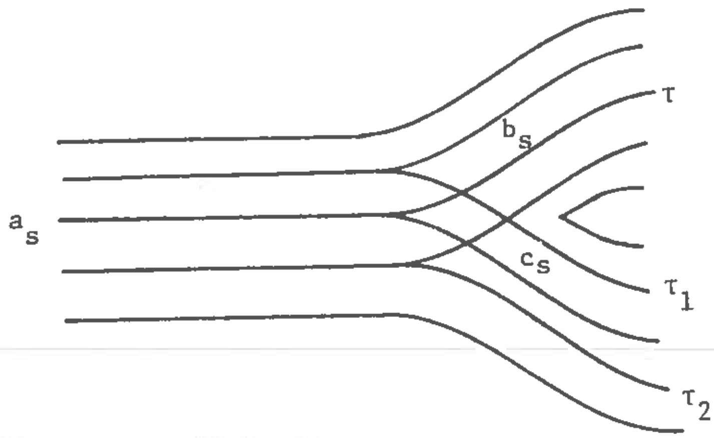


Figure A.2

If τ is a track in a surface F, we may construct an orientable track $\tilde{\tau}$ in a surface $\tilde{F}$ so that $\tilde{F} \xrightarrow{p} F$ is a two-fold branched cover with one branch point in each complimentary region of τ in F. $\tilde{\tau}$ is the full pre-image of τ under p; note that $\tilde{\tau}$ is topologically the (unbranched) orientation cover of τ in the sense of oriented graphs. An example of $(\tilde{F},\tilde{\tau})$ covering (F,τ) is given in Figure A.3.

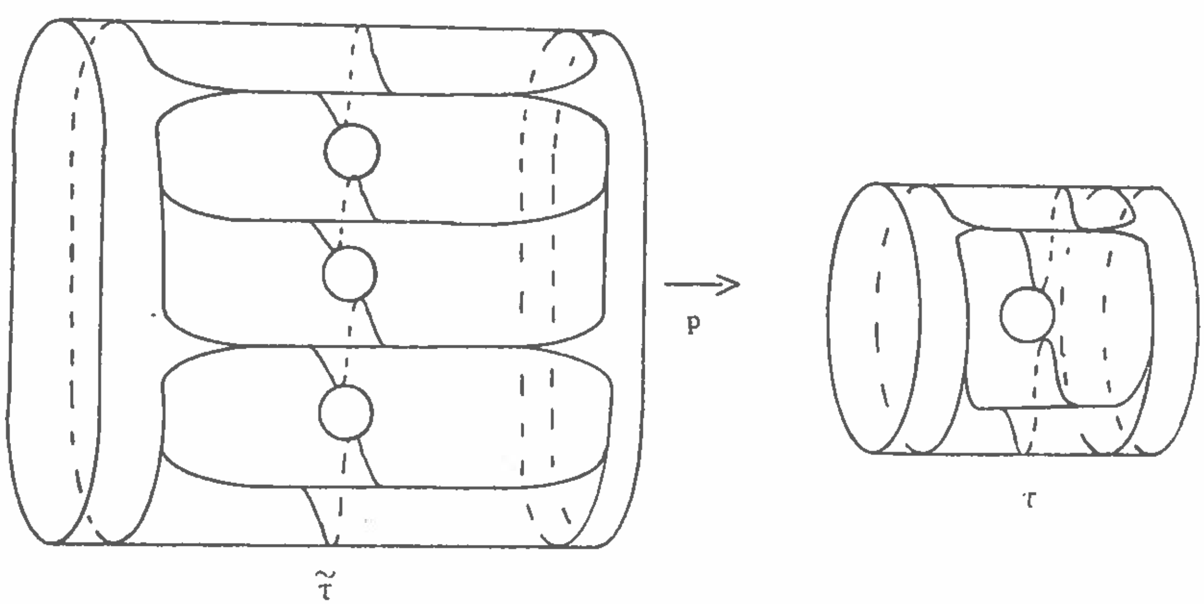


Figure A.3

Let $\iota:\widetilde{F} \to \widetilde{F}$ denote the covering involution of $p:\widetilde{F} \to F$. If ω is a choice of orientation on $\widetilde{\tau}$, then $\iota\omega = -\omega$. Moreover, if $\sigma \subset \tau$ is an orientable sub-track (for instance, if σ is an embedded canonical curve of τ), then $p^{-1}\sigma$ consists of two components $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$, and $\iota(\widetilde{\sigma}_1,\omega) = (\widetilde{\sigma}_2,-\omega)$. If $\sigma \subset \tau$ is non-orientable, then $\widetilde{\sigma} = p^{-1}\sigma$ is connected and $\iota(\widetilde{\sigma},\omega) = (\widetilde{\sigma},-\omega)$.

For convenience, we may complete τ to a maximal bi-recurrent track without stops (of the same name) by Theorem 1.4 so that each complimentary region of τ in F has area π. Each complimentary region of $\widetilde{\tau}$ is then a 6-gon or a punctured bigon. One computes that $\chi(\widetilde{F}) = 4\chi(F)$ using the Gaus-Bonnet and Hurwitz formulas. Moreover, each puncture or boundary component of $F = F^s_{g,r}$ has two distinct lifts to $\widetilde{F}$. It follows that

$$\dim_{\mathbb{R}} \widetilde{H}_1(F;\mathbb{R}) = \begin{cases} 8g - 6, & r + s = 0 \\ 8g - 7 + 4(r+s), & r + s \neq 0. \end{cases}$$

A standard spectral sequence argument (see, for instance, [B]) shows that the $+1$ eigen-space E^+ of the action ι_* of ι on $H_1(\widetilde{F};\mathbb{R})$ is isomorphic to $H_1(F;\mathbb{R})$, so

$$\dim_{\mathbb{R}} E^+ = \begin{cases} 2g, & r + s = 0 \\ 2g + (2r\ 2s-1), & r + s \neq 0. \end{cases}$$

This tells us that the -1 eigen-space E^- of ι_* in $H_1(\widetilde{F};\mathbb{R})$ has dimension $6g - 6 + 2(r+s)$.

Choose, once and for all, an orientation ω on $\widetilde{\tau}$. We define a map $\ell:V(\tau) \to H_1(\widetilde{F};\mathbb{R})$ as follows. If $\mu \in V(\tau)$, let $\ell(\mu)$ be the homology class of the real cycle in $\widetilde{F}$ obtained by taking the full pre-image of (τ,μ) under p, oriented accoring to ω.

<u>Lemma A.4</u>: ℓ is an isomorphism of $V(\tau)$ onto a cone C in E^-.

<u>Proof</u>: Let $H(\tau)$ denote the $\mathbb{R}$-vector space of measures allowed to be both positive and negative (satisfying the switch conditions) supported on τ. The natural <u>linear</u> extension $\ell : H(\tau) \to H_1(\tilde{F};\mathbb{R})$ clearly has image contained in E^-. Since $\dim_{\mathbb{R}} H(\tau) = 6g - 6 + 2(r+s) = \dim_{\mathbb{R}} E^-$, it suffices to show that ℓ is monic on $H(\tau)$.

If b_i is a branch of τ, then we choose a lift $\tilde{b}_i$ of b_i to $\tilde{\tau}$ and define $\bar{b}_i = \iota(\tilde{b}_i)$. Consider the CW- decomposition of $\tilde{F}$ with $\tilde{\tau}$ as 1-skeleton. The set $\{(\tilde{b}_i,\omega)\} \cup \{(\bar{b}_i,\omega)\}$ of oriented branches of τ forms a basis for the one-chains, the set of (oriented) simply-connected complimentary regions of $\tilde{\tau}$ in $\tilde{F}$ forms a basis for the two-chains, and ι acts cellularly.

Suppose that R is an (oriented) simply-connected complimentary region of $\tilde{\tau}$ in $\tilde{F}$, and let ∂ denote the boundary map of the CW- decomposition. If $\partial R = \sum u_i(\tilde{b}_i,\omega) + \sum v_i(\bar{b}_i,\omega)$, then one sees immediately that $u_i = -v_i$. Moreover, if $\mu \in V(\tau)$ and $\ell(\mu)$ is represented by the one-chain $\sum n_i(\tilde{b}_i,\omega) + \sum m_i(\bar{b}_i,\omega)$, then it follows from the definition of ℓ that $n_i = m_i$. Thus $\operatorname{Im} \ell \cap \operatorname{Im} \partial = \emptyset$, and the lemma follows. □

<u>Proposition A.5</u>: $\{\cdot\,,\cdot\}_\tau$ is non-degenerate on $V(\tau)$.

<u>Proof</u>: The previous two lemmas show that $\{\cdot\,,\cdot\}_\tau$ may be computed by passing to an oriented cover, computing homology intersection numbers, and dividing by 2 (from the two-fold cover, not the 1/2 in the definition of $\{\cdot\,,\cdot\}_\tau$). Since the homology intersection pairing on $\tilde{F}$ is natural with respect to the homeomorphism ι of $\tilde{F}$ and is, moreover, non-degenerate, its restriction of E^- is also non-degenerate. Finally, since the cone C has interior in E^-, $\{\cdot\,,\cdot\}_\tau$ is non-degenerate on $V(\tau)$. □

Our final goal is to compute the pairing $\{\cdot\,,\,\cdot\}_\tau$ where τ is one of our standard tracks in the paving P_1 of ML_0. For this purpose, we recall the parametrization of $ML(F^3_{0,0})$ described after Corollary 3.5.

If $\mu_1,\mu_2 \in V(\tau)$ where τ is a basic track (with stops) in $F^3_{0,0}$, let $\underline{x}$ be the vector $(\mu_{11}(\mu_1),\mu_{12}(\mu_1),\mu_{13}(\mu_1),\mu_{22}(\mu_1),\mu_{23}(\mu_1),\mu_{33}(\mu_1))$ of measures on the branches identified in Figure 2.7, and let $\underline{y}$ be the similar six-tuple corresponding to the measure μ_2 on τ. Using the first definition of the piecewise symplectic pairing (for tracks with stops), one computes that $\{\mu_1,\mu_2\}_\tau = \underline{xAy^t}$ <u>independent of the basic track</u> τ, where

$$\partial A = \begin{pmatrix} 0 & 0 & -2 & 4 & 0 & -4 \\ 0 & 0 & -1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 & -1 & 0 \\ -4 & -2 & 0 & 0 & 0 & 4 \\ 0 & -1 & 1 & 0 & 0 & 2 \\ 4 & 0 & 0 & -4 & -2 & 0 \end{pmatrix}$$

Similarly, suppose that σ is a connector (with stops) in $F^2_{0,0}$ and $\mu \in V(\tau)$. We associate to the measure μ two parameters $m(\mu)$ and $t(\mu)$ as follows: $m(\mu)$ is defined to be the intersection number as before, and $t(\mu)$ is simply the twisting number with one exception: if $m(\mu) = 0$ and the track σ is the connector of type -2, then we take $t(\mu)$ to be negative. (This is in contrast to our convention that a twisting number is non-negative if the corresponding intersection number vanishes.)

Suppose that $\mu_1,\mu_2 \in V(\sigma)$. One again computes directly that

$$\{\mu_1,\mu_2\}_\sigma = \left(m(\mu_1),t(\mu_1)\right)B\begin{pmatrix} m(\mu_2) \\ t(\mu_2) \end{pmatrix}.$$

<u>independent of the connector</u> σ, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Fix a standard track τ (without stops) in our paving P_1 of $ML_0(F)$. Let N denote the number of non boundary- or puncture-parallel pants curves and M the number of pairs of pants in our chosen basis for $A'(S)$. The parameters $\mu^u_{i,j}, 1 \le i \le j \le 3$, $u = 1,\ldots,M$ and (m_k, t_k), $k = 1,\ldots,N$ describe a PL monomorphism from a subset $\Lambda(\tau) \subset (\mathbb{R}^+)^{6M} \times (\mathbb{R}^+ \times \mathbb{R})^N$ onto $V(\tau) \subset ML_0$. The next result follows immediately from the computations above.

<u>Proposition A.6</u>: The pairing $\{\cdot\,,\,\cdot\}_\tau$ on $V(\tau)$ pulls back to a pairing on $\Lambda(\tau)$ given by the restriction of the direct sum of M copies of the matrix A and N copies of the matrix B <u>independent of the standard track</u> τ on F. $\square$

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