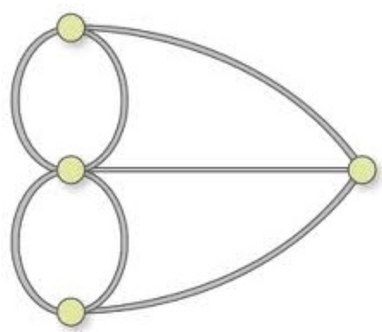
# The Königsberg Bridge Problem

The Bridges of Konigsberg

Is it possible to take a walk, cross each bridge exactly once, and return to where you started?

Or: Is the following pseudograph Eulerian?

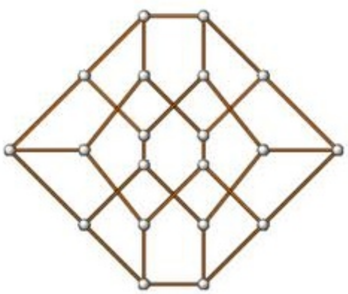


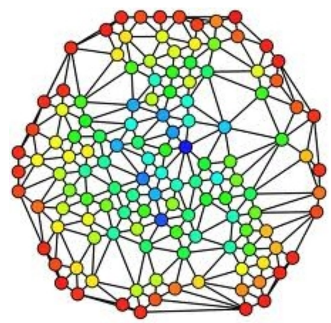
# Connectivity

We just argued that Eulerian graphs have no vertices of odd degree.

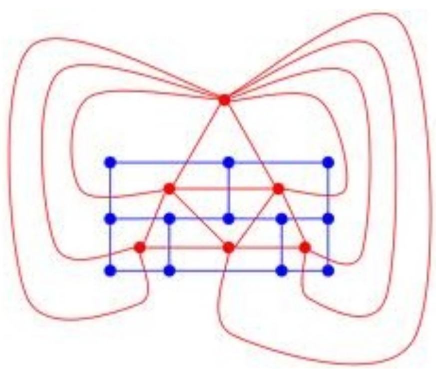
What else? Eulerian graphs must also be connected.

A pseudograph is connected if there is a walk between any two vertices.





connected



not connected

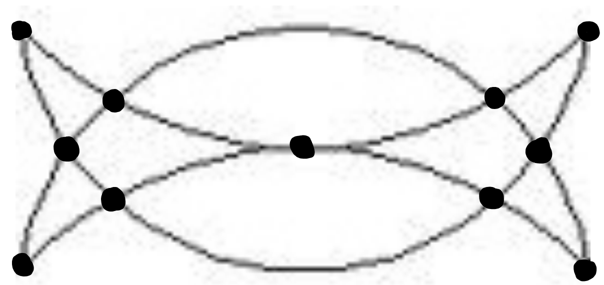
# Eulerian Pseudographs

THEOREM. A pseudograph is Eulerian if and only if it is connected and every vertex has even degree.

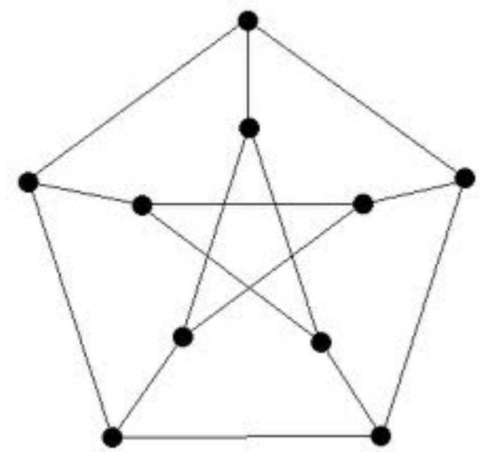
# Eulerian Pseudographs

For each pseudograph, find an Eulerian circuit if it exists.

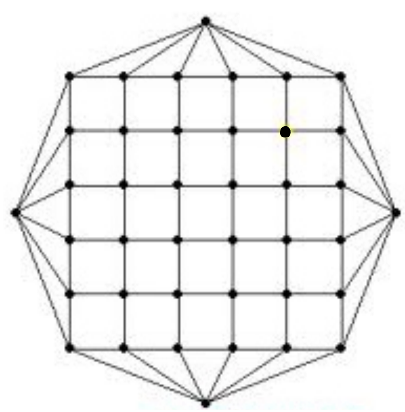
(i)



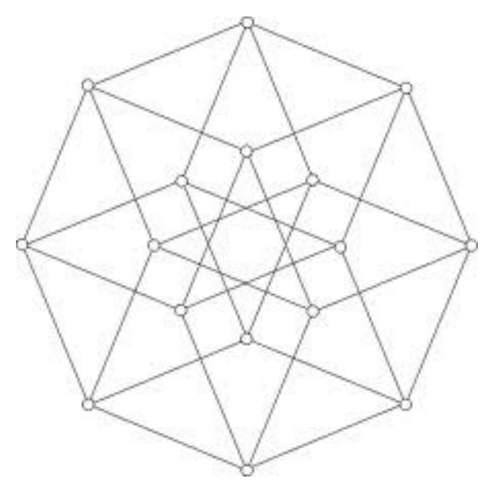
(ii)



(iii)

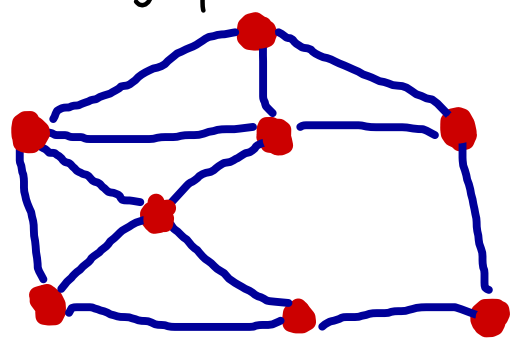


(iv)



# Hamiltonian Cycles

A Hamiltonian cycle in a pseudograph is a walk that visits each vertex exactly once:



Find one!

If a pseudograph has a Hamiltonian cycle, we say the pseudograph is Hamiltonian.

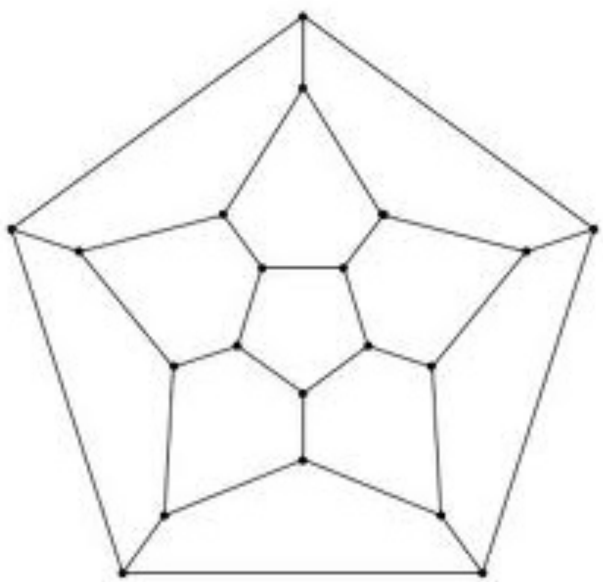
Euler: each edge once
Hamilton: each vertex once

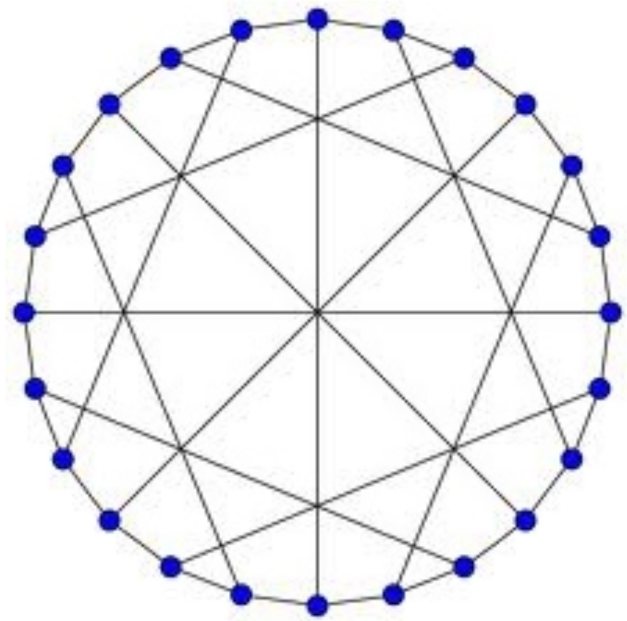
Sir William Rowan Hamilton

Note: A Hamiltonian cycle is isomorphic to an $n$-cycle.

# HAMILTONIAN CYCLES

Show that the following graphs are Hamiltonian.





In other words, find a Hamiltonian cycle in each.

# Hamiltonian Graphs

We saw that it is easy to tell if a graph is Eulerian or not.
To prove a graph is Hamiltonian, just find a Hamiltonian cycle.
But there is no easy method for showing a graph is not Hamiltonian.

You could check all paths of length $|V|$. Takes too long!

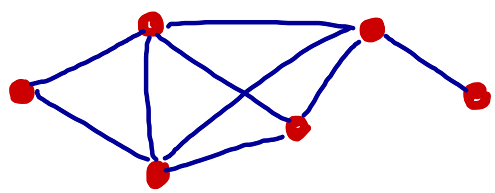
Better to use some basic facts:

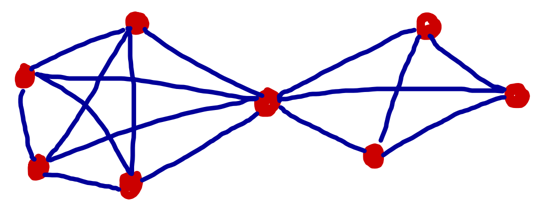
Let $\mathcal{H}$ be a Hamiltonian cycle in a pseudograph $G$

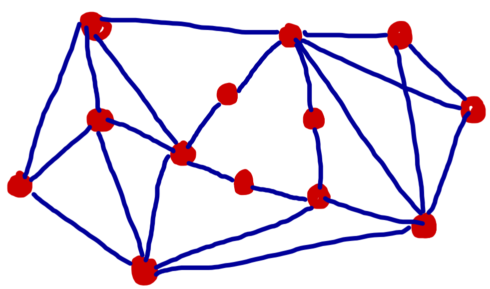
1. Every vertex of $G$ has exactly two edges of $\mathcal{H}$ passing through it.
2. The only cycle contained in $\mathcal{H}$ is $\mathcal{H}$.

# Hamiltonian Graphs

Prove that the following graphs are not Hamiltonian.

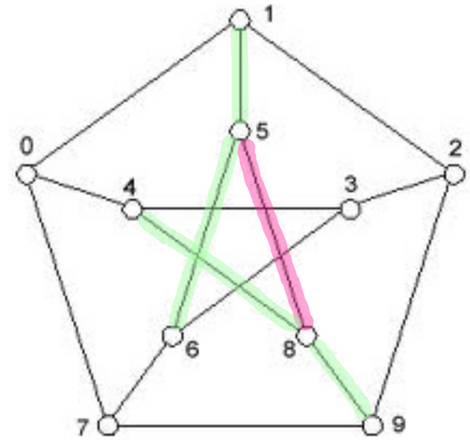






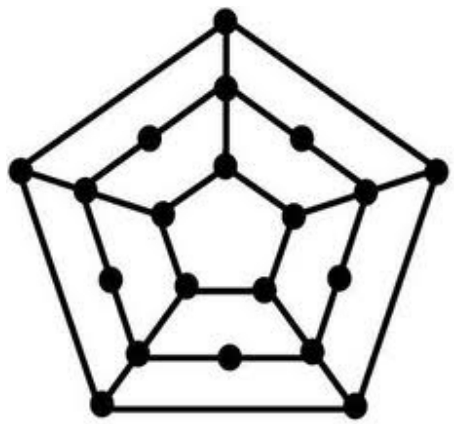
# The Petersen Graph

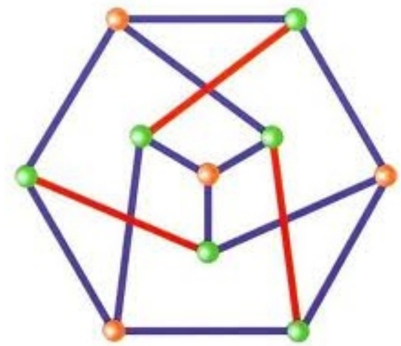
**Proposition.** The Petersen graph is not Hamiltonian.

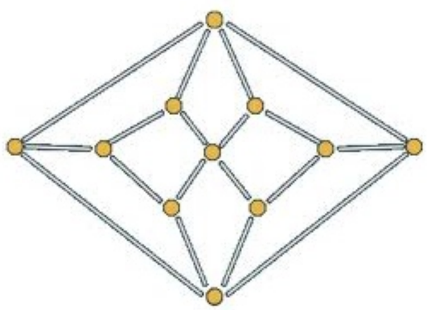


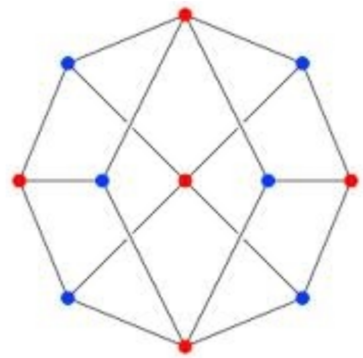
# Hamiltonian Graphs

Which of the following graphs are Hamiltonian?









# Hamiltonian Graphs

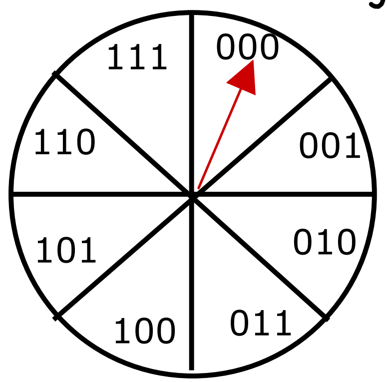
Which $K_n$ are Hamiltonian?

Which $K_{m,n}$ are Hamiltonian?

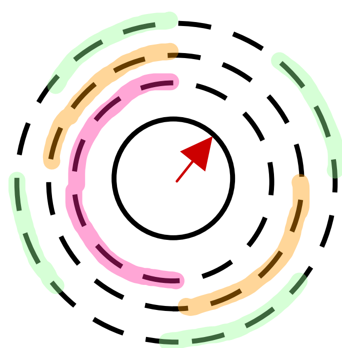
What about the knight graph on a chessboard?

# GRAY CODES

We can record the position of a rotating pointer with a bit string:




Can read the position of the arrow with 3 sets of contacts:

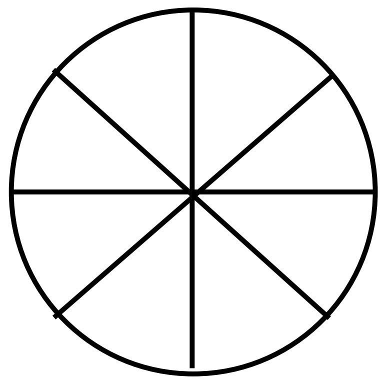


Problem: A small error could give 100 instead of 011

→ all 3 bits wrong!

# GRAY CODES

To fix this, want to number so that adjacent regions differ by one bit.



At first, not obvious how to do this.
But: such a numbering is just a Hamiltonian cycle in the n-cube.