7.2 DIAGONALIZATION

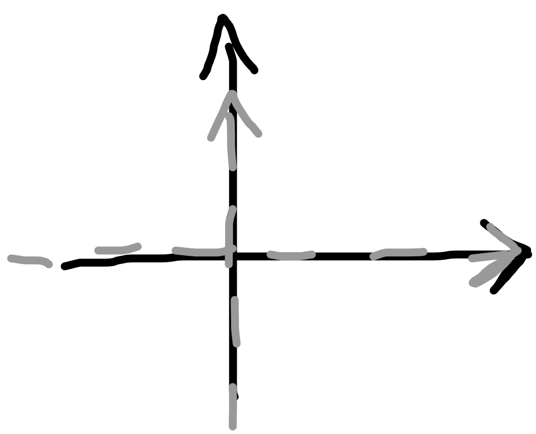
DIAGONALIZING MATRICES

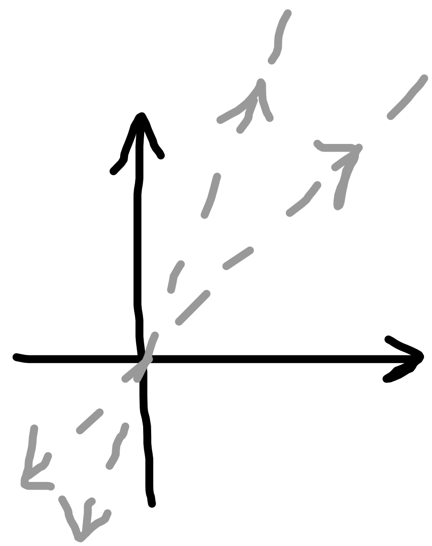
What does $\begin{pmatrix} 1 & 2 \\ -1 & -4 \end{pmatrix}$ do to $\mathbb{R}^2$?

We find eigenvectors:

eigenvalues:

It is similar to

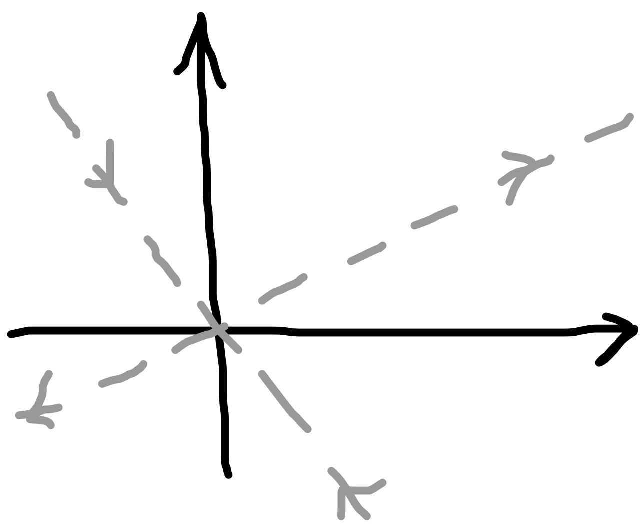




Diagonalizing Matrices

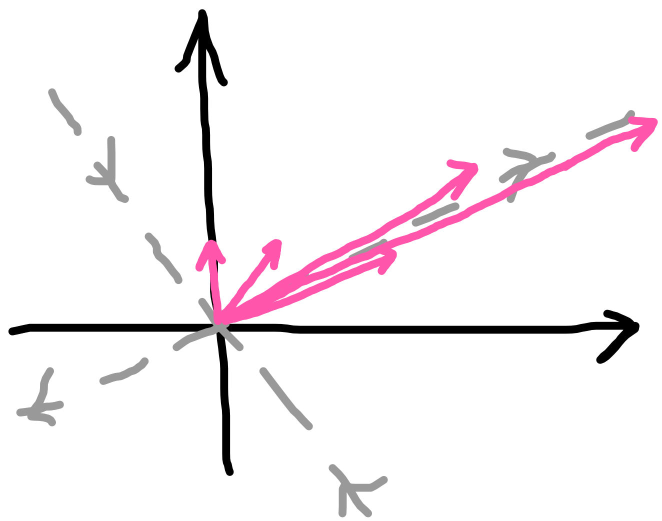
What about $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$?

similar to



Diagonalizing Matrices

What about powers of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$?



$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

etc.

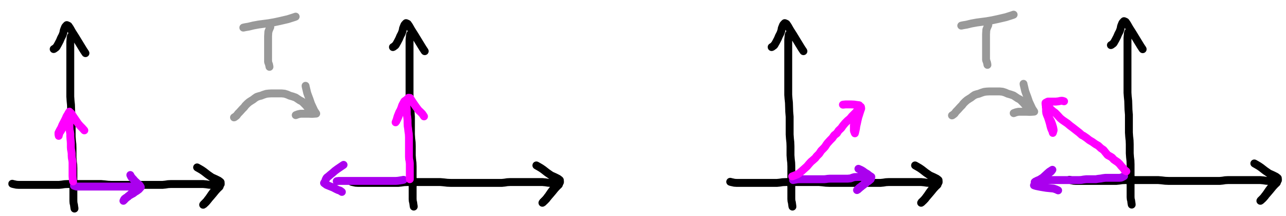
We conclude:

Similar Matrices

Two matrices A and B are *similar* if there is a matrix C so that

This means that A and B are essentially the same, just written with respect to different bases.

EXAMPLE. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the y-axis. We write T with respect to two different bases:



Similar Matrices

Show that the following matrices are similar:

1. $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$

Hint: Write the preferred basis for one in terms of the preferred basis for the other, as in the previous example.

Diagonalizable Matrices

A matrix is diagonalizable if it is similar to a diagonal matrix.

If a matrix A is diagonalizable, it is easy to compute powers of A:

$$A = CDC^{-1}$$

$$\longrightarrow A^k =$$

$$=$$

Computing D^k is a snap:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} =$$

So finding A^{1000} only requires matrix multiplications.

DIAGONALIZABLE MATRICES

1. Compute $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^5$.

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^5 =$$

2. We saw $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$. Use this to find an explicit formula for F_n. How does this relate to our old method?

Eigenvalues and Similarity

Theorem. Similar matrices have the same eigenvalues.

Proof. Say $B = CAC^{-1}$.

$$\det(B-\lambda I) = \det(CAC^{-1}-\lambda I) = \det(CAC^{-1}-\lambda CIC^{-1})$$
$$= \det(C(A-\lambda I)C^{-1}) = \det(C)\det(A-\lambda I)\det(C^{-1})$$
$$= \det(A-\lambda I).$$

Theorem. If a matrix A is similar to a diagonal matrix D, the eigenvalues of A are the same as the diagonal entries of D.

Diagonalizable?

Theorem. An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Theorem. A matrix is diagonalizable if and only if each eigenvalue of multiplicity k has k linearly independent eigenvectors.

Theorem. If an $n \times n$ matrix has n distinct eigenvalues, it is diagonalizable.

The number of linearly independent λ-eigenvectors for A equals the number of free parameters in the solution of $Av = \lambda v$.

DIAGONALIZABLE?

1. Is $\begin{pmatrix} 2 & -3/2 \\ 0 & 1/2 \end{pmatrix}$ diagonalizable?

2. Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?

3. Is $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ diagonalizable?

Diagonalization Recipe

Say A is diagonalizable, so $A = CDC^{-1}$. How to find C and D?

- Put the eigenvalues of A in some order: $\lambda_1, \ldots, \lambda_n$.
- Choose n linearly independent eigenvectors $v_1, \ldots, v_n$, where the eigenvalue for v_i is λ_i.

$$D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Then need to find C^{-1}.

Why this works:

Diagonalization Recipe

Diagonalize the following matrices:

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$$

Recall: To find C^{-1}, write

$$(C \mid I)$$

Row reduce:

$$(I \mid C^{-1})$$

Diagonalization

Are the following matrices diagonalizable? If so, diagonalize.

$$\begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$