Announcements Mar 30

- Cameras on
- HW due Thu 3:30
- Office Hours Fri 2-3 appt
- Outline Apr 2 ~1 page, teams
- First draft Apr 9.
- Makeup points

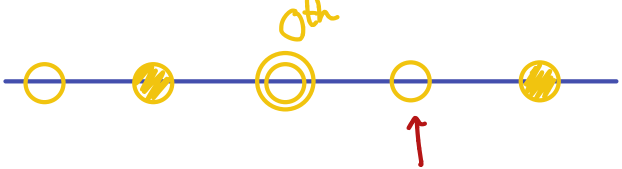
Today

Lamplighter groups

Diestel - Leader graphs

Lamplighter group (OHGGT)

Infinite street



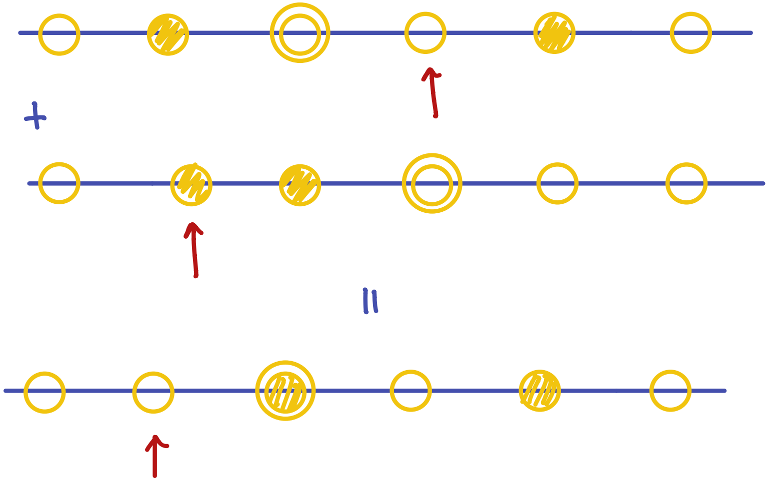
$L = \mathbb{Z} \times \left(\bigoplus_{\infty} \mathbb{Z}/2 \right)$ as a set.

$= \left\{ (k, \vec{x}) : k \in \mathbb{Z}, \ \vec{x} \in \bigoplus_{\infty} \mathbb{Z}/2 \right\}$

$=$ {configurations} or

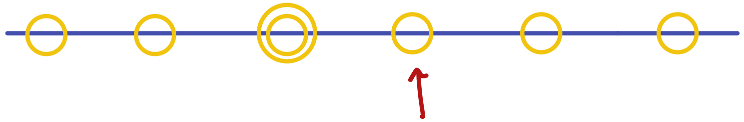
$=$ {actions}

Multiplication: stack and add

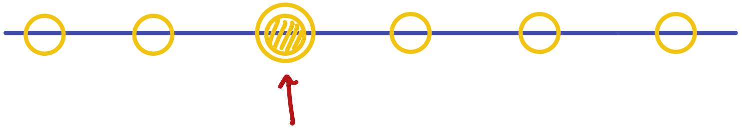


Generators

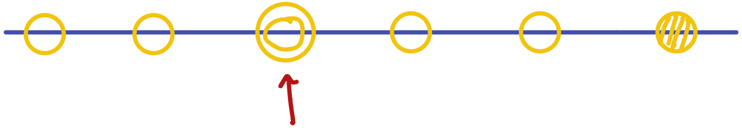
t:



a:



$t^3 a t^{-3}$



Presentation

$$L = \langle a, t \mid a^2 = id, \; (t^j a t^{-j})(t^k a t^{-k}) = (t^k a t^{-k})(t^j a t^{-j}) \rangle$$

A (faithful) representation ρ

First a notation for $\bigoplus_{\infty} \mathbb{Z}/2$.

$\mathbb{Z}/2[t,t^{-1}] = \{ \mathbb{Z}/2 \text{ poly's in } t, t^{-1} \}$

$$t^{-2} + 1 + t^5 \in \mathbb{Z}/2[t,t^{-1}]$$

$$\longleftrightarrow (\ldots 0, 1, 0, \underline{1}, 0, 0, 0, 0, 1, 0, \ldots)$$

0^{th} entry

$L = \{ (k, \vec{x}) \}$

$= \{ (k, P) : k \in \mathbb{Z}, P \in \mathbb{Z}/2[t,t^{-1}] \}$

$$\rho : L \longrightarrow GL_2(\mathbb{Z}/2[t,t^{-1}])$$

$$(k,P) \longmapsto \begin{pmatrix} t^k & P \\ 0 & 1 \end{pmatrix}$$

Thm. ρ is a faithful rep.

Pf. inj: clear...

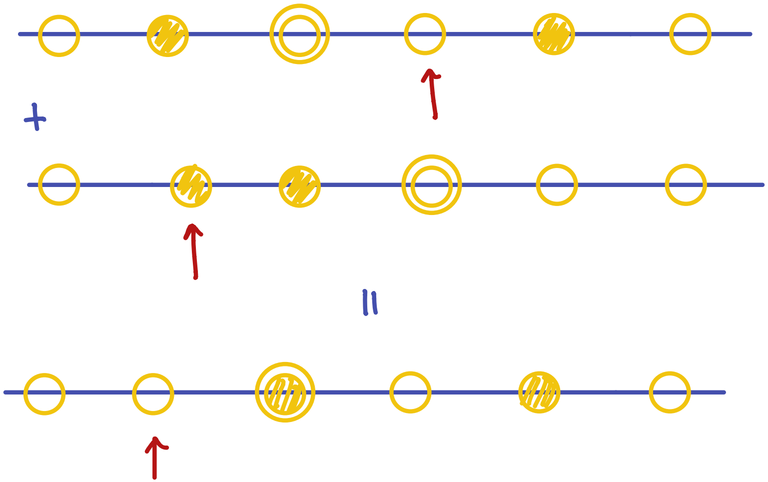
homom: Check relations

$$a^2 \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = I$$

other reln:

shears commute. $\square$

Example



$$\begin{pmatrix} t & t^{-1}+t^{2} \\ 0 & 1 \end{pmatrix}$$

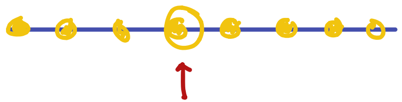
$$\begin{pmatrix} t^{-2} & t^{-1}+t^{-2} \\ 0 & 1 \end{pmatrix}$$

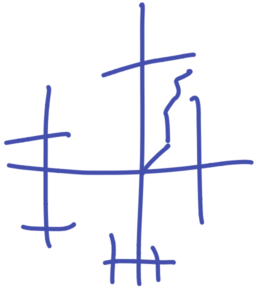
$$\begin{pmatrix} t & t^{-1}+t^{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-2} & t^{-1}+t^{-2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & 1 \cancel{+2t^{-1}} + t^{2} \\ 0 & 1 \end{pmatrix}$$

See OHGGT for a discussion of

- word length (1D - traveling salesman problemish) EASY!

- dead ends





- generalize

① L_n : lamps have $\mathbb{Z}/n$ states

② Wreath products

$L = \mathbb{Z}/2 \wr \mathbb{Z}$

$G \wr H$ is the lamplighter gp with
"map" H (like $\mathbb{Z}$ in L)
"lamp states" G (like $\mathbb{Z}/2$ in L)

$\mathbb{Z}/2 \wr \mathbb{Z}^2$ 2D - trav. sal. prob. HARD!

A little more:

$$G \wr H = \{(k, \vec{x}) : k \in H,\ \vec{x} : H \to G\}$$

$$H \ltimes \left(\bigoplus_{H} G\right)$$

1^{st} factor permutes coords of 2^{nd}.

example

$$\mathbb{Z}/2 \wr \mathbb{Z} = \{(k, \vec{x}) : k \in \mathbb{Z},\ \vec{x} : \mathbb{Z} \to \mathbb{Z}/2\}$$

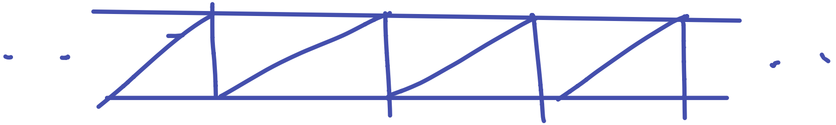
$$\mathbb{Z} \ltimes \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\right)$$

first factor permutes coords of 2^{nd}

Next: Cayley graph for L

<u>Diestel-Leader Graphs</u>

An old question: is every graph quasi-isometric to a Cayley graph?



$\cong$ Q.I

DL(m,n) is a graph

D-L conjectured

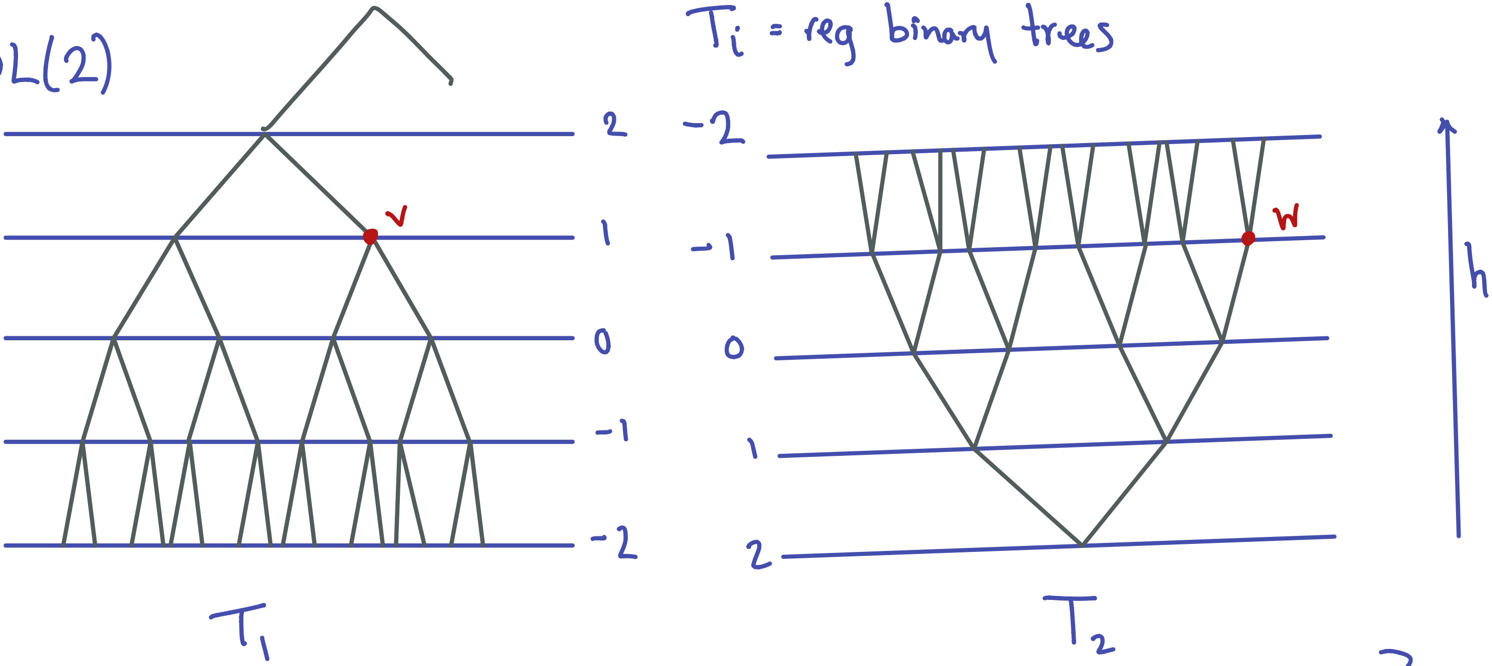
$m \neq n \Rightarrow$ DL(m,n) is not QI to a Cayley gr.

(proved by Eskin-Fisher-Whyte)

<u>But</u>. $DL(n) = DL(n,n)$ <u>is</u> the Cayley graph for L_n (lamp. gp w/ n states)

DL(2)

T_i = reg binary trees



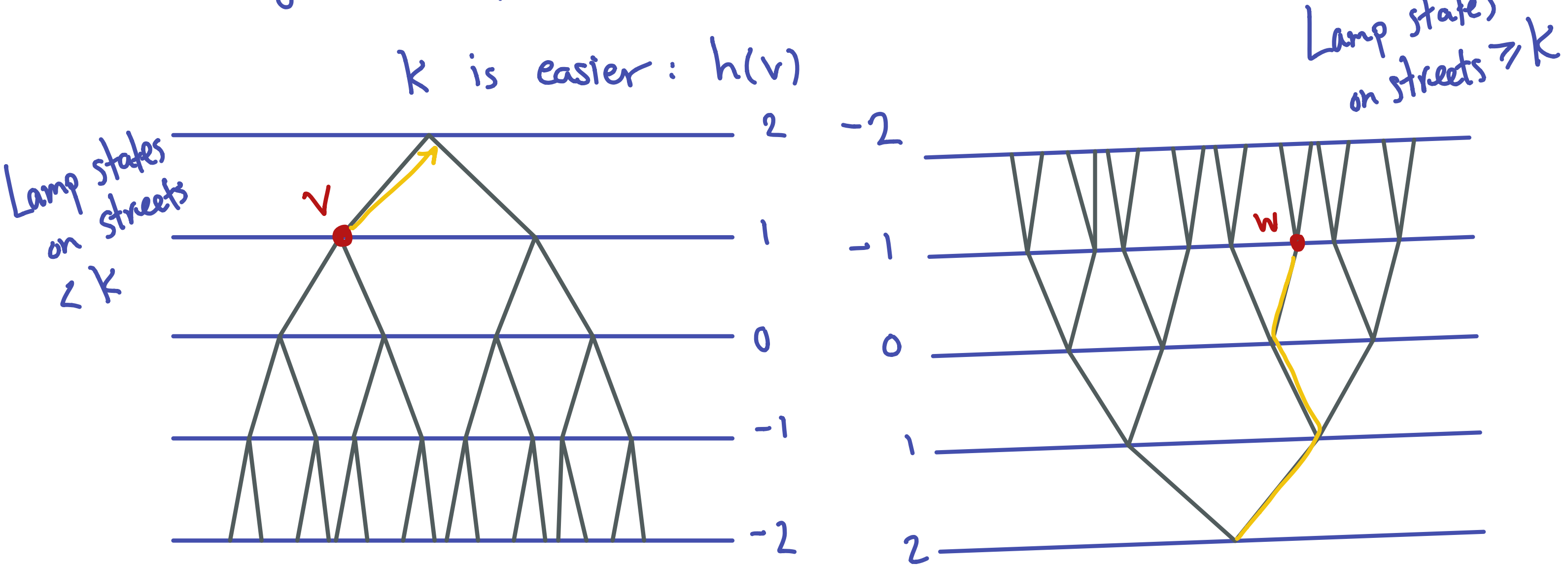
DL(2): vertices $\{(v,w) : v \in V(T_1),\ w \in V(T_2),\ h(v)+h(w)=0\}$

edges $(v,w) \text{---} (v',w')$ when $v \text{---} v'$ in T_1, $w \text{---} w'$ in T_2

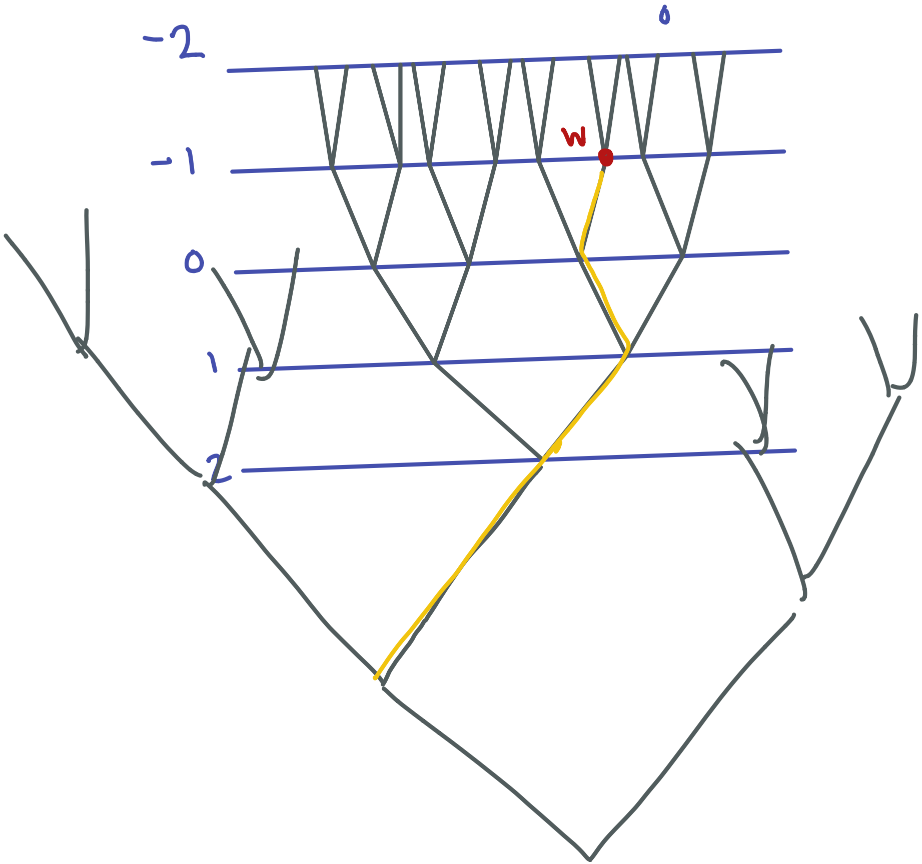
Thm. $DL(2)$ = Cayley gr. for L wrt t, at

How to get a $(k, \vec{x})$ from a vertex of $DL(2)$?

k is easier: $h(v)$



Make up path from v, down path from w, concat $\rightsquigarrow$ ∞ string of L/R.



-2
-1
0
1
2
w