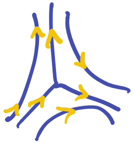
Pseudo-Anosov theory



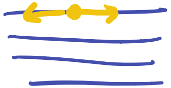
Part I. Stretch factors

Thm. $g \geq 2$, $f \in \mathrm{Mod}(S_g)$ pA. $\lambda(f)$ is alg int of deg $\leq 6g-6$.

Pf. Show $\lambda(f)$ is eigenval. of $\mathbb{Z}$ matrix of size $\leq 6g-6$.

Matrix comes from action on $H_1(S_g;\mathbb{Z})$ or subspace of $H_1(\tilde{S}_g;\mathbb{Z})$ $\quad \tilde{S}_g$ = branched double cover.

Pf. If $\mathcal{F}_u$ orientable then $(\mathcal{F}_u,\mu)$ is a 1-form ω on S_g.



$\cancel{|dy|}$ dy

$\varphi \cdot \mathcal{F}_u = \lambda \mathcal{F}_u \Rightarrow \omega$ is an eigenv. for $\psi(f)$.

If $\mathcal{F}_u$ not orientable, pass to orient. double cover. $\tilde{S}_g$

$\tilde{S}_g = \{(p,v) : p \in S_g,\ v \text{ points along } \mathcal{F}_u\}$

2-Fold cover, branched over odd sing.

$\tilde{S}_g$ has bounded genus, lift & apply prev case.

Q. Which alg. degrees occur for given S_g?

Strenner: exactly

$2, 4, 6, \ldots, 6g-6$

$3, 5, 7, \ldots, 3g-4$ or $3g-3$.

Q. What if you fix a subgp such as $\mathcal{I}(S_g)$.

<u>Fried's Conjecture</u>. $\lambda \in \mathbb{R}$ is a stretch factor $\Leftrightarrow$ all alg. conj's have abs val in $(1/\lambda, \lambda)$ except $\lambda, 1/\lambda$. (cf. Pankau, Kenyon)

<u>Spectrum of $\mathcal{M}(S)$</u> $\{\log \lambda(f) : f \in \mathrm{Mod}(S)\ pA\}$.

<u>Thm.</u> This is a closed, discrete subset of $\mathbb{R}$.

<u>Pf.</u> Set of alg. ints of deg $\leq N$ is discrete.

In particular, there is a smallest one.

Q. What is it? Only known $g=1,2$.

<u>Penner.</u> Smallest $\log \lambda(f)$ in $\mathrm{Mod}(S_g)$ $\asymp 1/g$.

Farb-<u>Leininger-M</u> Smallest $\log \lambda(f)$ in $\mathcal{I}(S_g) \asymp 1$

<u>Lanier-M</u> Any proper normal subgp

Thm. ρ = any Riem. metric on S.
α = any closed curve.

$$\lim_{n\to\infty} \sqrt[n]{\ell_\rho(f^n(\alpha))} = \lambda$$

i.e. $\ell_\rho(f^n(\alpha)) \sim \lambda^n$ geometry

Thm. a, b any s.c.c. in S

$$\lim_{n\to\infty} \sqrt[n]{i(f^n(a), b)} = \lambda$$

i.e. $i(f^n(a), b) \sim \lambda^n$ topology

Thm. $\alpha \in \pi_1(S)$

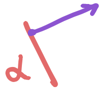
$$\lim_{n\to\infty} \sqrt[n]{|f^n(\alpha)|} = \lambda$$

word length

i.e. $|f^n(\alpha)| \sim \lambda^n$ group theory.

Thm. $\log \lambda$ = top. entropy of f. dynamics

Part II. Foliations



Poincaré recurrence for foliations

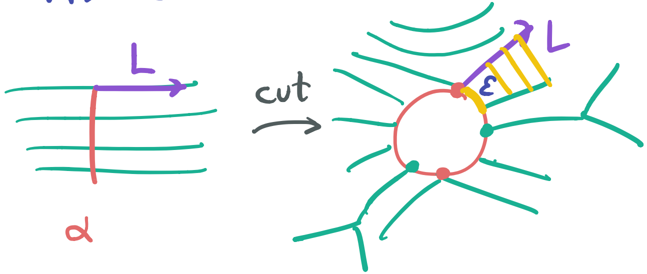
$(\mathcal{F}, \mu)$ meas. fol.

L = ∞ half leaf.

α = arc transverse to $\mathcal{F}$

$\alpha \cap L \neq \emptyset \Rightarrow |\alpha \cap L| = \infty$.

Pf. WLOG L & α share endpt.



Choose small arc ε along new ∂.

Push along foliation.

$\rightsquigarrow$ sweep out rectangle.

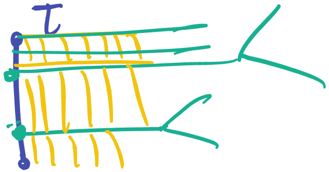
Can choose ε small enough so this rectangle never hits a singularity.

$\Rightarrow$ If L never hits ∂ again. can push forever. CONTRAD.

Really using: Can cover S by finitely many charts like

Cor. f pA $\Longrightarrow$ every leaf of $\mathcal{F}_u$ is dense.

Pf. τ = small arc transverse to $\mathcal{F}_u$



No closed leaves these swept out rectangles eventually return by Poinc. rec.

The union of these rectangles is the whole surface (otherwise the ∂ is a reducing curve).

Thm. $\mathcal{F}_u$ is uniquely ergodic i.e. μ is unique up to scale.

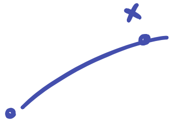
Part III. Dynamics.

Thm. φ pA $\Longrightarrow$ φ has dense orbit.

Pf. Claim. $U \neq \emptyset$, open, φ-invt $\Longrightarrow$ U dense.

Assume WLOG φ fixes sing's...

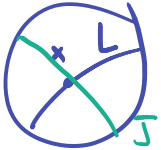
Choose:



L = leaf of $\mathcal{F}_s$

s = sing,

L dense $\Longrightarrow$



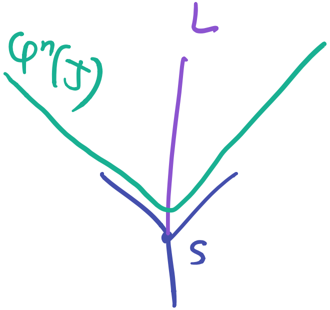
U

J in $\mathcal{F}_u$.

Apply powers of φ.

$x \longrightarrow s$

J gets longer



$\Longrightarrow \bigcup \varphi^n(J)$ dense.

$\Longrightarrow \underbrace{\bigcup \varphi^n(U)}_{=\,U.}$ dense

Now: Take $\{U_i\}$ countable basis for S.

Let $V_i = \bigcup_{n \in \mathbb{Z}} \varphi^n(U_i)$ satisfies claim, hence dense $\forall i$.

Baire category thm $\Longrightarrow \bigcap V_i$ dense

$\Longrightarrow \bigcap V_i \neq \emptyset$. Say $x \in \bigcap$

$\Longrightarrow \{f^i(x)\}$ intersects every U_i $\square$

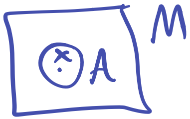
Thm. φ pA $\Rightarrow$ periodic pts dense.

Poincaré Recurrence. M = finite meas. sp.

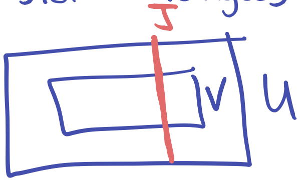
$T : M \to M$ meas. pres.

$A \subseteq M$ pos. meas.

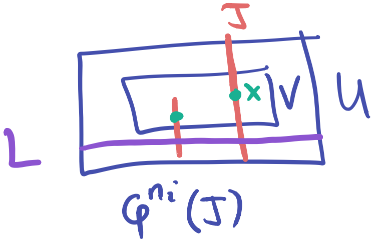
Then for a.e. $x \in A$ $\exists$ inc. seq n_i s.t. $T^{n_i}(x) \in A$.



Pf. Choose std rectangles



P.R. $\Rightarrow$ $\varphi^{n_i}(V) \cap V \neq \emptyset$.



continuous $\rightsquigarrow$ $J \to J$ (push right)

1D Brower $\Rightarrow$ fixed pt.

i.e. horiz leaf L mapping to itself.

Fund. thm of 1D dynamics:

Any map $f : [0,1] \to \mathbb{R}$ with $\operatorname{im} f \supseteq [0,1]$ has a fixed pt.

Apply to $L \cap U$. $\square$