Announcements Sep 8

- Please turn on your camera if you are able and comfortable doing so
- Current plan: Class on Monday in Howey L1/Teams (email forthcoming)
  - attendance optional
  - masks + distance from me expected
  - testing encouraged
- Quiz on 1.2 & 1.3 Friday. Open 6:30a–8p on Canvas/Assignments, 15 mins
- WeBWorK 1.2 & 1.3 due Thursday nite
- WeBWorK 2.1 & 2.2 due Tuesday nite
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Many, many TA office hours listed on Canvas
- Studio this Friday online; Studio for M02 will be recorded/streamed
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, old quizzes/exams
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Chapter 2

System of Linear Equations: Geometry

Where are we?

In Chapter 1 we learned to solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution. In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning. There are three main points:

Sec 2.3: $Ax = b$ is consistent $\Leftrightarrow$ b is in the span of the columns of A.

Sec 2.4: The solutions to $Ax = b$ are parallel to the solutions to $Ax = 0$.

Sec 2.9: The dim's of $\{b : Ax = b$ is consistent$\}$ and $\{$solutions to $Ax = b\}$ add up to the number of columns of A.

Section 2.1

Vectors

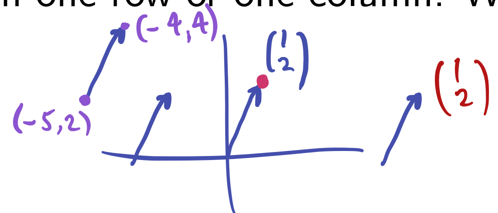
Outline

- Think of points in $\mathbb{R}^n$ as vectors.
- Learn how to add vectors and multiply them by a scalar
- Understand the geometry of adding vectors and multiplying them by a scalar
- Understand linear combinations algebraically and geometrically

Vectors

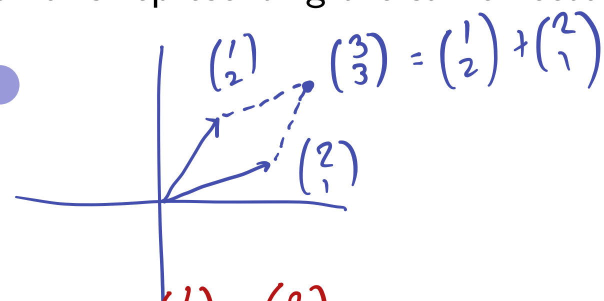
A vector is a matrix with one row or one column. We can think of a vector with n rows as:

- a point in $\mathbb{R}^n$
- an arrow in $\mathbb{R}^n$



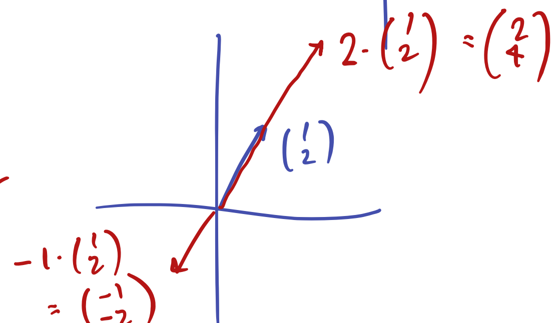
To go from an arrow to a point in $\mathbb{R}^n$, we subtract the tip of the arrow from the starting point. Note that there are many arrows representing the same vector.

Adding vectors / parallelogram rule ▸ Demo



Scaling vectors ▸ Demo

↳ multiply by real number



A scalar is just a real number. We use this term to indicate that we are scaling a vector by this number.

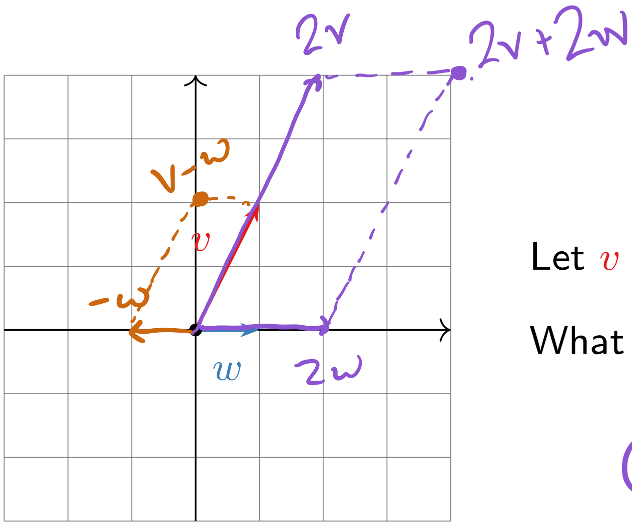
Linear Combinations

A linear combination of the vectors $v_1, \ldots, v_k$ is any vector

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

where $c_1, \ldots, c_k$ are real numbers.

Scale & then add the vectors v_i



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

1. $2v + 2w = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

2. $v - w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $\quad 1 \cdot v + (-1) \cdot w$

3. $2v - w = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ etc.

4. $\pi \cdot v + 0 \cdot w = \begin{pmatrix} \pi \\ 2\pi \end{pmatrix}$

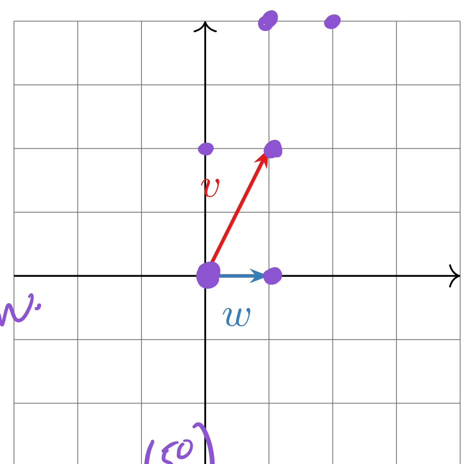
Poll

Is there a vector in $\mathbb{R}^2$ that is not a linear combination of v and w?

- yes
- $\boxed{\text{no}}$

Every pt in $\mathbb{R}^2$ is a lin combo of v, w.

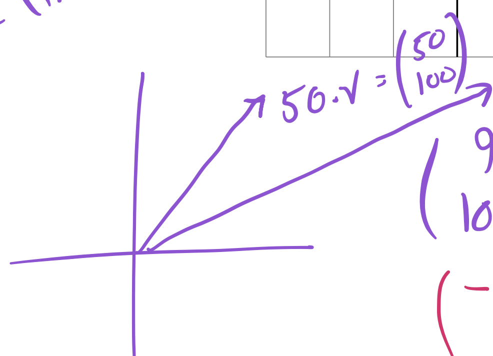
In language of Sec 2.2: the span of $\begin{pmatrix}1\\2\end{pmatrix}$ & $\begin{pmatrix}1\\0\end{pmatrix}$ is $\mathbb{R}^2$.



$v = \begin{pmatrix}1\\2\end{pmatrix}$

$w = \begin{pmatrix}1\\0\end{pmatrix}$

Secretly: solving a linear system.



$$\begin{pmatrix}98\\100\end{pmatrix} = \underline{50}\cdot\begin{pmatrix}1\\2\end{pmatrix} + \underline{48}\begin{pmatrix}1\\0\end{pmatrix}$$

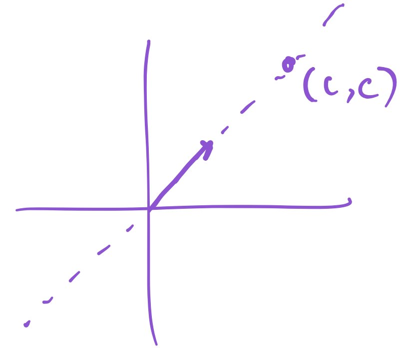
$$\begin{pmatrix}-1\\0\end{pmatrix} = \underline{0}\cdot\begin{pmatrix}1\\2\end{pmatrix} + \underline{-1}\begin{pmatrix}1\\0\end{pmatrix}$$

Linear Combinations

What are some linear combinations of $(1, 1)$?

line $y=x$

$c\cdot(1,1)=(c,c)$

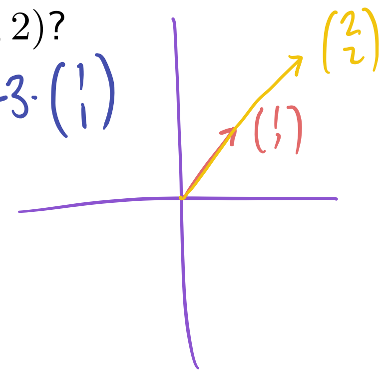


What are some linear combinations of $(1, 1)$ and $(2, 2)$?

$15\cdot\begin{pmatrix}1\\1\end{pmatrix} + -9\cdot\begin{pmatrix}2\\2\end{pmatrix} = -3\cdot\begin{pmatrix}1\\1\end{pmatrix}$

$-18\cdot\begin{pmatrix}1\\1\end{pmatrix}$

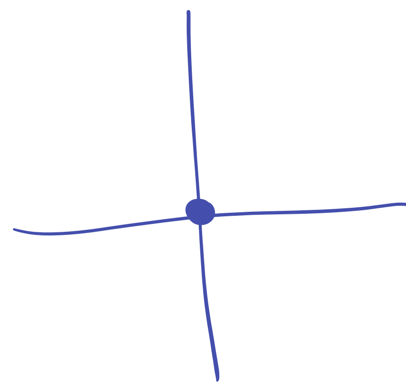
line $y=x$



What are some linear combinations of $(0, 0)$?

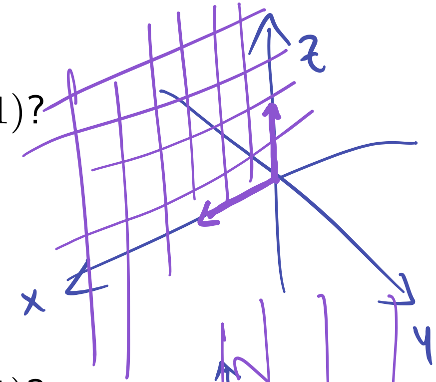
$c\cdot\begin{pmatrix}0\\0\end{pmatrix}$

just the origin.



Linear Combinations

What are all linear combinations of $(1, 0, 0)$ and $(0, 0, 1)$?

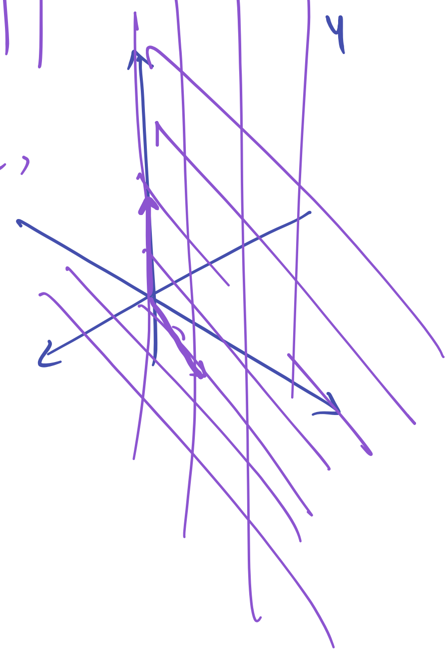


xz-plane

$$\begin{pmatrix} 77 \\ 0 \\ -19 \end{pmatrix} = 77 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-19) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

What are all linear combinations of $(1, 1, 0)$ and $(0, 0, 1)$?

Take plane from last example,
rotate by... 45°
about... z-axis



What are all linear combinations of $(3, 2, 4)$ and $(-4, 2, 1)$?

It's a plane
in $\mathbb{R}^3$

Summary of Section 2.1

- A vector is a point/arrow in $\mathbb{R}^n$
- We can add/scale vectors algebraically & geometrically (parallelogram rule)
- A linear combination of vectors $v_1, \ldots, v_k$ is a vector

$$c_1 v_1 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.

Typical exam questions

True/False: For any collection of vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, the zero vector in $\mathbb{R}^n$ is a linear combination of $v_1, \ldots, v_k$.

True/False: The vector $(1, 1)$ can be written as a linear combination of $(2, 2)$ and $(-2, -2)$ in infinitely many ways.

Describe geometrically the set of linear combinations of the vectors $(1, 0, 0)$ and $(1, 2, 3)$.

Suppose that v is a vector in $\mathbb{R}^n$, and consider the set of all linear combinations of v. What geometric shape is this?

Section 2.2

Vector Equations and Spans

Outline of Section 2.2

- Learn the equivalences:

$$\text{vector equations} \leftrightarrow \text{augmented matrices} \leftrightarrow \text{linear systems}$$

- Learn the definition of span
- Learn the relationship between spans and consistency

Linear Combinations

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of in the span of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$? Super important!

yes $\Longleftrightarrow$ system below is consistent.

Write down an equation in order to solve this problem. This is called a vector equation.

$$x \cdot \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \cdot \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Notice that the vector equation can be rewritten as a system of linear equations. Solve it!

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned} \quad \longleftrightarrow \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

Now answer by row reduction. Is there a pivot on RHS? You check!!

Linear combinations, vector equations, and linear systems

Repeat of last slide in general language.

In general, asking:

Is b a linear combination of $v_1, \ldots, v_k$?

is the same as asking if the vector equation

$$x_1 v_1 + \cdots + x_k v_k = b$$

is consistent, which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

Span

$$\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} = \mathbb{R}^3$$

$$\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}9\\8\\1\end{pmatrix}\right\} = \mathbb{R}^3$$

Essential vocabulary word!

$$\begin{aligned}\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1v_1 + x_2v_2 + \cdots + x_kv_k \mid x_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)}\\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k\\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of two vectors in $\mathbb{R}^2$? $\boxed{1}$ $(0,0)$ (both vectors are zero)

or $\boxed{2}$ line if one is a multiple of other

or $\boxed{3}$ all of $\mathbb{R}^2$

Demo

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

$(0,0,0)$, line, plane, all of $\mathbb{R}^3$

→ all three vectors are multiples of each other

→ 3rd vector is in plane spanned by first 2

e.g. $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}9\\8\\0\end{pmatrix}$

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Span

Essential vocabulary word!

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 0 & 0 & 0 \\ 0 & 5 & -45 \end{array}\right)$$

CONSISTENT!

$$\begin{aligned} \text{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1v_1 + x_2v_2 + \cdots x_kv_k \mid x_i \text{ in } \mathbb{R}\} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k. \end{aligned}$$

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ in Span of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

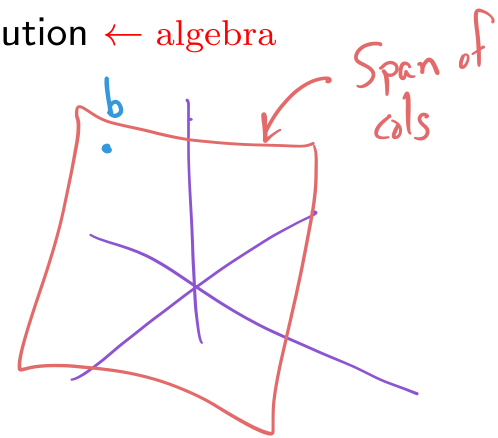
Four ways of saying the same thing:

beginning of 2.2

- b is in Span$\{v_1, v_2, \ldots, v_k\}$ $\leftarrow$ geometry
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $x_1v_1 + \cdots + x_kv_k = b$ has a solution $\leftarrow$ algebra
- the system of linear equations corresponding to

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

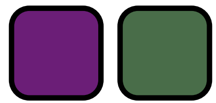


Demo Demo

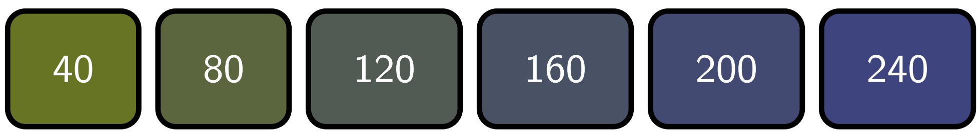
Application: Additive Color Theory

Consider now the two colors

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$$



For which h is $(116, 130, h)$ in the span of those two colors?



Summary of Section 2.2

- vector equations $\leftrightarrow$ augmented matrices $\leftrightarrow$ linear systems
- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

Typical exam questions

Is $\begin{pmatrix} 8 \\ 16 \\ 1 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Write down the vector equation for the previous problem.

True/False: The vector equation $x_1 v_1 + \cdots + x_k v_k = 0$ is always consistent.

True/False: It is possible for the span of 3 vectors in $\mathbb{R}^3$ to be a line.

True/False: the plane $z = 1$ in $\mathbb{R}^3$ is a span.