Planar graphs

Dan Margalit

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The complete graph on 5 vertices, denoted K_5, and the complete bipartite graph on two sets of 3 vertices, denoted $K_{3,3}$ both have the property that they cannot be embedded in the plane. These "forbidden graphs" are shown in Figure 1. We have the following famous theorem.

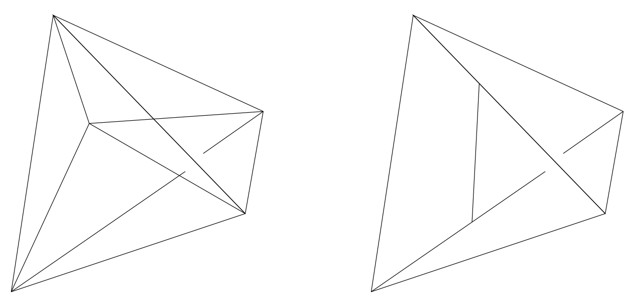

Figure 1: Forbidden graphs.

Theorem 1. *A finite graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.*

This theorem was proven in 1930 by Kuratowski and (independently) Frink–Smith. Kuratowski published it first, and henceforth the theorem has had his name attached to it.

It is not too difficult to show that the forbidden graphs do not embed in the plane, and it follows that any graph containing one of these does not embed in the plane. The real theorem is the other direction. Kuratowski's proof is purely topological, written in the language of Peano continua (a nonplanar Peano continuum is called "topologically gauche"). We present here a combinatorial proof due to Dirac–Schuster in 1954, as explained in Frank Harary's book "Graph theory."

First a lemma about blocks. A *cut point* of a graph is a vertex with the property that its complement is disconnected. A *block* is a connected graph without cut vertices. Any graph has a natural decomposition into blocks, which are glued along cut vertices.

Lemma 1. *If a block has at least 3 vertices, then there is a* cycle *(embedded circle) passing through any given pair of points.*

Proof. Let u be any vertex. Let $\mathcal{U}$ be the set of vertices which have a cycle passing through u and itself. Since there are at least 3 vertices and no cut points, $\mathcal{U}$ is nonempty. Suppose there is a vertex w which is not in $\mathcal{U}$ but is connected by an edge e to a vertex v in $\mathcal{U}$. Since v is not a cut vertex, there is another path from w to any cycle passing through u and v. But then we see a cycle through u and w. □

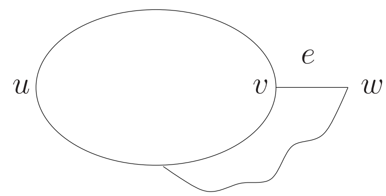

Figure 2: The picture for the lemma.

We are now ready for the proof of the theorem.

Proof. Suppose that the theorem is false. Let G be a counterexample with the smallest number of edges. That is, G is a nonplanar graph which contains no forbidden subgraph, and any graph with fewer edges is not a counterexample, and hence is planar.

Note that G is a block, for otherwise one of its subblocks would be a counterexample with fewer edges.

Let e be an arbitrary edge of G, with vertices u and v. We consider the graph $G-e$, obtained by removing the interior of e. By the minimality of G, we have that $G-e$ is planar. For the remainder of the proof, fix an embedding of $G-e$ in the plane.

We claim that $G-e$ is a block. Suppose not. If u and v are in the same block, then G is not minimal. Say that u and v are in different blocks, separated by a vertex w. Cut $G-e$ along w to obtain two graphs, each with a copy of w. If in the first graph, u is not connected to (its copy of) w, add that edge. Likewise for the other graph. Neither of these two graphs contains a forbidden graph, for otherwise G would. Since they each have fewer edges than G, they are planar. Embed each in the plane so the new edge is exterior (exercise). But now we can identify the two copies of w and draw an edge in the exterior connecting u to v—so G is planar.

Now, by the lemma, there is a cycle in $G-e$ passing through u and v. Choose $\mathcal{Z}$ to be such a cycle with the maximum number of interior regions. Cutting along $\mathcal{Z}$, the (closures of the) connected components are classified as *interior pieces* and *exterior pieces*; see Figure 3.

There must be at least one interior piece whose intersection with $\mathcal{Z}$ separates u from v along $\mathcal{Z}$. If not, then we could draw e inside $\mathcal{Z}$ and we would see G is planar.

Any exterior piece must intersect $\mathcal{Z}$ in exactly two points: once in each component of $\mathcal{Z}-\{u,v\}$. An exterior piece cannot intersect in one point, for that point would then be a

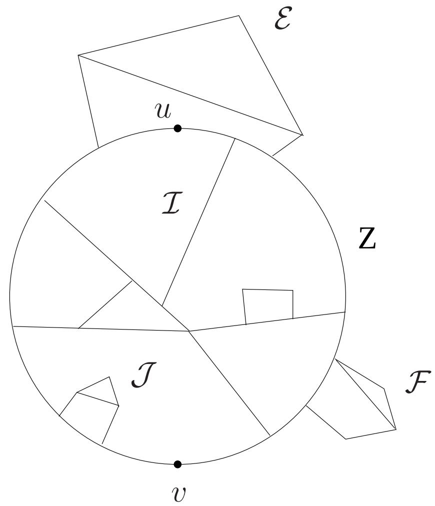

Figure 3: Interior and exterior pieces. The exterior piece $\mathcal{E}$ is allowed, the exterior piece $\mathcal{F}$ is not. The interior piece $\mathcal{I}$ separates u from v (along $\mathcal{Z}$) and also separates the points of $\mathcal{E}$ on $\mathcal{Z}$. The interior piece $\mathcal{J}$ does not separate either pair (hence it can be "flipped" outside $\mathcal{Z}$).

cut point. Also, an exterior piece cannot intersect the closed subpath of $\mathcal{Z}$ from u to v in more than one point, because that would violate the maximality condition on $\mathcal{Z}$.

Claim: There is an interior piece $\mathcal{I}$ with the property that its intersection with $\mathcal{Z}$ separates u from v along $\mathcal{Z}$ and also separates (along $\mathcal{Z}$) the two intersection points of some exterior piece $\mathcal{E}$ with $\mathcal{Z}$.

Let $\mathcal{J}$ be any interior piece which separates u from v, if it does not separate the two points of any exterior piece on $\mathcal{Z}$, then $\mathcal{J}$ can be "flipped" to the outside of $\mathcal{Z}$, preserving the planarity of $G - e$. But then we can flip all such interior pieces, violating the fact that any embedding of $G - e$ must have one interior piece separating u from v. The claim is proven.

We are now ready for the "end game". We start by drawing $\mathcal{Z}$, u, v, e (dashed), and a path of $\mathcal{E}$ (from the claim), as in Figure 4.

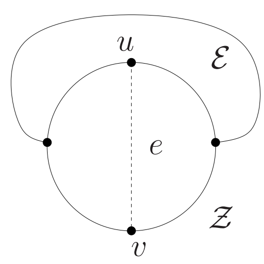

Figure 4: The setup for the end game.

We call u, v, and the two vertices of $\mathcal{E}$ on $\mathcal{Z}$ the four *distinguished points*. The four regions of $\mathcal{Z}$ demarcated by the distinguished points will be called *quadrants*.

There are now several cases to consider. If $\mathcal{I}$ has points in opposite quadrants, then we see a $K_{3,3}$ in G, as in the top left of Figure 5.

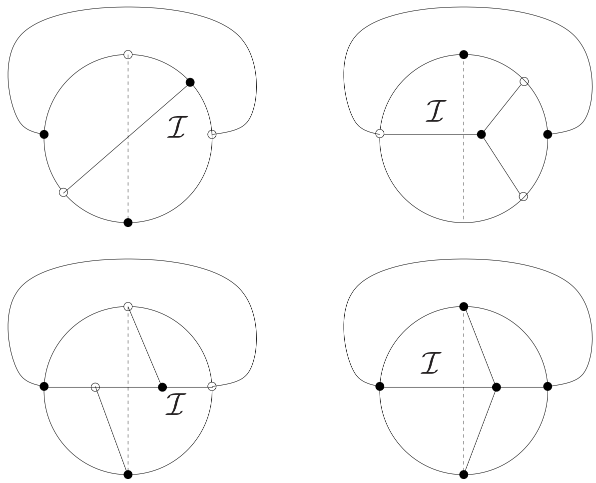

Figure 5: The cases for the end game.

If $\mathcal{I}$ has vertices in adjacent quadrants, then the only way to avoid being in the previous case is if $\mathcal{I}$ also has a vertex at one of the distinguished points ($\mathcal{I}$ has to separate both pairs of distinguished points). Again, we see a $K_{3,3}$ in G (top right of Figure 5).

The only way to avoid the the first two cases is if the intersection of $\mathcal{I}$ with $\mathcal{Z}$ is exactly the 4 distinguished points. There are two subcases. Either the two shortest paths connecting the pairs of distinguished points intersect in one point or more than a point. If they intersect in more than a point, we see a $K_{3,3}$ (bottom left of Figure 5). Otherwise, we see a K_5, as in the bottom right of Figure 5. This completes the proof. $\square$