RESEARCH STATEMENT

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My research interests are in nonlinear partial differential equations (PDEs), in particular those that model urban crime, biological aggregation, and chemotaxis. My main interest is studying the local and global well-posedness of these models. Indeed, one of the most fundamental questions that should be asked when studying PDEs is whether there is a unique solution, and if so whether the solution exists for all time (global solutions) or blows-up in finite time. Understanding these properties can help determine how robust a system is, as well as providing insight on the characteristics of the corresponding phenomena it is modeling.

Background

The use of mathematics to understand the physical world has been used for centuries, but the manner and degree to which it can used has drastically changed in recent years due to the invention of the computer and it's ability to perform incredibly complex and computation-intensive tasks. This is especially true in sociological and biological processes, where the ability to perform relatively inexpensive experiments using mathematical models has literally invented new fields of research and mechanisms for understanding complex systems. However, despite the rapid growth of computer capabilities, any viable mathematical model still requires significant simplification due to the degree of complexity of these processes, especially when human consciousness is involved. To this end, the objective of studying a mathematical model is not to recreate reality, but rather to shed some light onto the underlying dynamics of the process being modeled. Meanwhile, from a purely mathematical perspective, modeling these types of processes can lead to the construction of new and interesting PDE systems that demand the development of new mathematical theory. For instance, because these PDE models are not developed from conservation laws, as is true when modeling physical phenomena, classical mathematical theory and techniques do not generally apply.

As a graduate student, I have studied nonlinear PDE models for residential burglary "hotspots" (spatio-temporal areas of high density of crime) and aggregation equations, which model the competition between aggregation and diffusion. This work has been conducted under the guidance of Professor Andrea Bertozzi at UCLA. In the following sections I describe this work in more detail, and then conclude by discussing open problems that I would like to pursue in my postdoctoral research.

A PDE system for Urban Crime

Although crime is a ubiquitous feature of all societies, certain geographical locations have a higher propensity to crime than others. Indeed, residential burglary data exhibit areas of high crime density surrounded by areas of low crime density [30, 10]. There have been many studies indicating that the "repeat and near-repeat victimization effect," which states that crime in an area induces more crime in that and neighboring areas, leads to the residential burglary hotspots seen in the data ([36], [3, 20, 21, 22]). Short *et al.* develop in [33] an agent-based statistical crime model whose dynamics rely on this repeated victimization effect. The model assumes that criminal agents are moving across a bounded domain in $\mathbb{R}^2$ (in a manner to be described shortly) and committing burglaries probabilistically when encountering an opportunity. The probability of a house being burgled is determined by an attractiveness field A, where $A(x,t)$ is a scalar and measures how vulnerable a house is as a target. The motion of the criminal agents in this model is described by a Gaussian random walk with a bias

toward areas of high attractiveness values. More precisely, if $\rho(x, t)$ denotes the criminal density, then the continuum limit is the following:

$$\frac{\partial A}{\partial t} = \eta \Delta A - A + A\rho + A^o, \tag{1a}$$

$$\frac{\partial \rho}{\partial t} = \Delta \rho - 2\nabla \cdot [\rho \nabla \log A] + \overline{B} - A\rho. \tag{1b}$$

Above A^o is a constant representing the base attractiveness value, and $\overline{B}$ is a constant that controls the production rate of criminals. The expected number of burglaries occurring at (x, t) is given by the term $A(x, t)\rho(x, t)$, which is the *reaction* term in (1a). Notice that this term indicates that the attractiveness value increases with each burglary. A formal derivation of this model can be found in [33], where they use numerical and linear stability analysis to demonstrate that the model will display hotspots, if parameters that lead to an unstable steady-state solution are chosen. A more detailed weakly-nonlinear analysis also shows the existence of *sub*critical hotspots that bifurcate from the homogeneous steady-state. One interesting feature that distinguishes these two forms of hotspots is their reaction to *suppression*, i.e. the introduction of a term that reduces criminal density or attractiveness when these values become large (this term can be motivated by the introduction of law enforcement in areas of high-crime, see e.g. [14, 1]). In particular, while suppression can destroy *sub*critical hotspots, it tends to only displace *super*critical hotspots [34].

In work done in collaboration with Bertozzi [32], we study the well-posedness of classical solutions of (1). In particular, we prove local existence and uniqueness of classical solutions with H^4-initial data on the two-dimensional torus $\mathbb{T}^2$. Formally, letting

$$V^m = \{(u, v) \in H^m(\Omega) \times H^m(\Omega)\},$$

we proved the following theorem:

Theorem 1. *Given initial conditions $(A_0(x), \rho_0(x)) \in V^m$ for $m > 3$ such that $A_0(x) > A^o$, there exists a positive time $T > 0$ such that $A, \rho \in C([0, T]; C^2(\Omega)) \cap C^1([0, T]; C(\Omega))$ form a unique solution to (1) on the time interval $[0, T]$.*

Furthermore, we proved a continuation argument that provides a sufficient condition for global existence of the solution. More specifically, we prove that if $\|\nabla \rho\|_\infty$ remains bounded, then the solutions exists globally in time:

Theorem 2. *Given initial conditions $(A(x, 0), \rho(x, 0)) \in V^m$, $m \geq 4$ such that $A(x, 0) > A^o$ and 'no-flux' boundary conditions, there exist a maximal time of existence $0 < T_{max} \leq \infty$ and a unique solution $(A(x, t), \rho(x, t)) \in C([0, T_{max}); V^m) \cap C^1([0, T_{max}); V^{m-2})$ to the system (1). Furthermore, if T_{max} is finite, then $\lim_{t \to T_{max}} \|\nabla \rho\|_\infty = \infty$.*

Although Theorem 2 provides a condition for which the solution is guaranteed to exist globally, it remains an open question whether or not this condition is satisfied by (1). One complication in proving a bound for $\|\nabla \rho\|_\infty$ is that there does not exist a Lyapunov functional for (1).

Motivated by the relation between (1) and the Keller-Segel model for chemotaxis, we studied in [32] the following modified system on $\mathbb{R}^2$:

$$\epsilon \frac{\partial A}{\partial t} = \eta \Delta A - A + \beta \rho + A^o(x), \tag{2a}$$

$$\frac{\partial \rho}{\partial t} = \Delta \rho - 2\nabla \cdot (\rho \nabla A) + \overline{B}(x) - f(A)\rho. \tag{2b}$$

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Notice that because $A^o(x)$ and $\overline{B}(x)$ are now functions of the space variable, we must impose the constraint that they have sufficient decay as $|x| \to \infty$ in order for the system (2) to be well-posed. The system described by (2) makes three modifications to (1). First, the velocity field is now given simply by ∇A (as opposed to $\nabla \log A$). Second, the contribution to the rate of change in attractiveness value that comes from the density of criminals is now determined by a constant β as opposed to multiplying by the attractiveness (i.e. we replace $A\rho$ with $\beta\rho$). Finally, the criminal density decays with a rate of $f(A)$ as opposed to simply A, where f is some function that we assume has an upper and lower bound. We consider $L^1(\mathbb{R}^2)$ initial data with finite second moment. Note that the mass of solutions to (2) have an upper and lower bound, which we refer to as M_ρ^{max} and M_ρ^{min}. In particular, we consider the case when $\epsilon = 0$, which implies that

$$A(x) = \mathcal{B}_\eta * (\beta\rho + A^o), \tag{3}$$

where $\mathcal{B}_\eta$ is the Bessel potential. Defining $I_v = \int |x|^2 v \, dx$, we proved the following theorem in [32]:

Theorem 3. *Let $(A(x,0), \rho(x,0)) \in L^1(\mathbb{R}^2)$ be initial data such that $\left(\beta M_\rho^{min} - 4\pi\right) M_\rho^{min} > \pi I_{\overline{B}}$, and let ρ be the non-negative smooth solution to (2b). If A has reached a steady state and is defined by (3), then A, ρ are non-negative smooth solutions to (2) (for $\epsilon = 0$). Additionally, if the initial second moment is small enough. i.e. if:*

$$\int |x|^2 \rho \, dx \leq \frac{1}{K^2} \left[\left(\frac{\beta}{\pi} M_\rho^{min} - 4 \right) M_\rho^{min} - I_{\overline{B}} \right]^2, \tag{4}$$

then there exists a finite time singularity. Above, $K = \left[\frac{2\beta}{\pi} C \left(M_\rho^{max} \right)^{3/2} + a_1 \left(M_\rho^{max} \right)^{1/2} \right]$ (where C is a constant C and $a_1 = 4 \|\nabla \mathcal{B}_\eta(x)\|_1 |A^o(x)|_\infty$) and $I_{\overline{B}}(x) = \int \overline{B} dx$.

Essentially, Theorem 3 states that solutions to (2) with initial data that has small enough second moment and large enough mass will blow-up in finite time. Inequality (4) gives the an upper bound for the second moment in the initial crime density. It is interesting to compare our results to those obtained for the Keller-Segel model [12]. In particular, Calvez demonstrates a similar phenomenon regarding a critical mass value for the fully-parabolic Keller-Segel model on $\mathbb{R}^2$, where solutions for mass less than the critical value are global, while those with mass larger than the critical value blow-up in finite time.

One advantage of the system defined by (2) over (1) is that its solutions satisfy an energy functional that remains bounded for all time:

$$\mathcal{F}(t) = \int u \log(u) dx - \int uv \, dx + \frac{1}{2} \int |\nabla v|^2 \, dx + \int \alpha v^2 dx.$$

We prove global existence of solutions to (2) for small initial mass in a paper in preparation [31]. Additional estimates, such as a global bound on the L^∞-norm of the criminal density. We also study in [31] the system described by (1) on bounded domains in $\mathbb{R}^1$, and prove global existence by demonstrating a global bound on $\|\nabla\rho\|_\infty$ using the Moser-Alikakos iteration [2].

Aggregation Equation

Partial differential equations have also proved useful for studying aggregation in biological systems, a phenomenon which has been observed in numerous species, *e.g.* schools of birds and fish

[9, 35, 27, 17, 11]. In these settings, aggregation has typically been modeled by an advective term with a velocity field given by the gradient of a convolution the density with a potential, the latter of which is usually assumed to be non-increasing and radially symmetric. In addition to a term representing a tendency towards aggregation, PDEs modeling biological systems also typically involve a competing diffusion term representing dispersal, motivated by a species' desire to move away from an existing population.

As a graduate student, I have been interested in studying the well-posedness of an aggregation equation with general degenerate diffusion of the form:

$$\frac{\partial u}{\partial t} = \Delta A(u) - \nabla \cdot (u(\nabla \mathcal{K} * u)) \quad \text{in } [0, T) \times D, \tag{5}$$

where $A(u)$ represents diffusion, aggregation is controlled by convolving u with the kernel $\mathcal{K}$, and D is a bounded domain in $\mathbb{R}^2$. The system described by (5) has been studied in multiple contexts for various classes of $\mathcal{K}$ and diffusion A. Motivated by the belief that the system described by (5) can model chemotaxis (under appropriate choices for $\mathcal{K}$ and A), Patlak-Keller-Segel (PKS) and many subsequent researchers have studied this system to better understand the interplay between aggreagation and diffusion in this model [16, 29, 19, 23, 24], [8]. Classically, aggregation in the PKS system is modeled using the Newtonian or Bessel potential for $\mathcal{K}$. In a separate line of research, (5) has also been used to model over-crowding effects [35, 11], where they considered general diffusion A but restricted to smooth kernels $\mathcal{K}$.

While the above two research paths were motivated for studying separate applications, and the mathematical tools that were developed and employed in each case were distinct, an underlying feature of the system described by (5) that made it appropriate to model both phenomena was the presence of both aggregation and diffusion terms. Motivated by a desire to unite these separate research paths by providing a unified approach to solving (5), my collaborators and I considered in [5] a class of kernel $\mathcal{K}$ and aggregation A that simultaneously encompass the values used in [16, 29, 19, 23, 24] and [35, 11]. We were able to achieve our results by combining techniques that had been developed in the study of the aggregation equation with techniques that had been developed to study the PKS model.

Our investigation of (5) focused on the balance between aggregation and diffusion: depending on the relative strength of these terms, the solutions exhibit different qualitative behaviors. This work was inspired by the work in [8], where they consider $A(u) = u^m$ and the Newtonian Potential and prove that *sub*critical solutions, which blow-up in finite time, emerge when the aggregation term dominates, and *super*critical solutions, which exist globally in time, emerge when the diffusion term dominates. Our goal in [5] was to generalize the notion of criticality in [8] and study the well-posedness of (5) for a more general choices of $\mathcal{K}$ and A. We first extended the existence theory of [7] to include singular kernels, including the Newtonian and Bessel potentials. This theory relies on *a priori* bounds for $\|u\|_\infty$. The regularity of the kernels considered in [7] lead to a global bound on this norm, which provides for global existence theory. The case we consider is we only obtain a local bound with the help of the Alikakos iteration [2]. As mentioned earlier the global existence theory is a generalization of the work done in [8].

We associate a *critical exponent*, $m^* = (p+1)/p$, to each kernel $\mathcal{K} \in L^{p,\infty}$ for some $d/(d-2) < p < \infty$. Correspondingly, we say that (5) is:

$$subcritical \text{ if:} \quad \liminf_{z \to \infty} \frac{A'(z)}{z^{m^\star - 1}} = \infty,$$

$$critical \text{ if:} \quad 0 < \liminf_{z \to \infty} \frac{A'(z)}{z^{m^\star - 1}} < \infty,$$

$$supercritical \text{ if:} \quad \liminf_{z \to \infty} \frac{A'(z)}{z^{m^\star - 1}} = 0.$$

We proved global existence for subcritical problems and finite time blow-up for a certain class of supercritical problems. In the critical case, similarly to the PKS system, there is a critical mass which separates global existence to finite time blow-up. In this work we determined the sharp critical value for a wide range of general diffusion and kernels. We extended the results of [5] to all of $\mathbb{R}^2$ in a paper not yet submitted for publication [4]. Furthermore, currently I am working, in collaboration with Jacob Bedrossian, on studying (5) with nonlinear velocity field of the form $\nabla F(\mathcal{K} * u)$.

Research Agenda

During my postdoctoral tenure, I intend to continue doing research in PDE models for sociological and biological phenomena, as I believe studying models of these types simultaneously impacts two fields of research. From a purely mathematical perspective, the generation of new theory and techniques is often necessary to understand these models; while from a sociological perspective, interesting and complex problems from the real-world can be solved mathematically with relatively minimal financial costs. Furthermore, these kinds of PDEs facilitate cross-disciplines collaborations, which I hope to continue to embrace as a postdoc. Below I discuss some specific avenues of research in which I believe there are interesting open problems.

NON-LOCAL CRIME MODELS: One of the key assumptions made in [33] is that criminal agents commit crimes in their own neighborhoods; this is reflected by the fact that the velocity field described in (1) is determined using only local information. I would like to study a model which does not rely on this assumption. Indeed, in practice criminal agents are very likely to be familiar with areas surrounding their neighborhoods, which in general may be a relatively large zone of the city. Hence, criminals will have nonlocal information about their environments, and it is highly probable that they will use this information to decide in which direction to travel [6, 28]. The effect of nonlocal factors can be modeled by changing the velocity field in the advective term of (1b) from $\nabla \log A$ to $\nabla \mathcal{K} * A$, with an appropriate choice of kernel $\mathcal{K}$.

CRIME MODELS WITH FRACTIONAL DIFFUSION: Studies in criminology have led to two fundamentally different types of criminology behavior: 'marauders' and 'commuters' [25]. The former commit crimes close to their home base, while the latter commute to commit their crimes. I am interested in studying a model which is based on 'commuter' criminal behavior. This can be modeled by assuming that the criminals are following a Lévy flight (using a 'heavy-tail' distribution for the distance traveled by the criminal agents) instead of a Brownian random walk (Gaussian distribution for the distance traveled by a criminal agent)[13, 26, 15, 18]. In terms of a PDE model, this corresponds to the generalization of the diffusion term in the criminal density to fractional diffusion $-(-\Delta^\gamma)\rho$ for $0 \leq \gamma < 1$. Note that the case when $\gamma = 1$ gives the original model (1). I would like to study global well-posedness of this system and to determine whether or not there exist emerging crime patterns.

Remark 1. The two proposed problems above are fundamentally different not only mathematically but also from the application point of view. In the first case (non-local crime models) I am roughly interested in modeling the use of nonlocal information in the decision of a criminal to move in a certain direction. In a sense, this system is modeling the 'commuter' criminal behavior.

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