SECTION 5.4

Solving Recurrence Relations– Generating Functions

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} - n/3$ (actually, this is first order)

Steps: 1

2

3

4

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} - n/3$, $a_0 = 1$

GENERATING FUNCTIONS

Sometimes counting problems, or recurrence relations can be solved using polynomials in a clever way.

Example: Find the number of solutions of

$$a+b+c=10$$

where a is allowed to be 2, 3, or 4
b is allowed to be 3, 4, or 5
c is allowed to be 1, 3, or 4

The answer is the coefficient of $x^{\boxed{}}$ in

$$(x^2+x^3+x^4)(x^3+x^4+x^5)(x+x^3+x^4)$$

e.g. $2+5+3 \leftrightarrow x^2x^5x^3$

This problem can be solved with a computer algebra system.

GENERATING FUNCTIONS

The *generating function* for the sequence

$$a_0, a_1, a_2, a_3, \ldots$$

is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

For example

$$a_n = 1 \longleftrightarrow 1, 1, 1, 1, \ldots \longleftrightarrow$$
$$a_n = n+1 \longleftrightarrow 1, 2, 3, 4, \ldots \longleftrightarrow$$
$$a_n = n \longleftrightarrow 0, 1, 2, 3, \ldots \longleftrightarrow$$

POWER SERIES

A generating function, as an object, is what is called a power series, that is, a formal sum ← Think "string"

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

These can be added, subtracted, and multiplied:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$f(x) + g(x) =$$
$$f(x)g(x) =$$

But we never plug in numbers for x, like with Taylor Series.

So generating functions should not be thought of as functions!

POWER SERIES

What about dividing?

1. Yes
2. No
3. Sometimes

POWER SERIES

What about dividing?

Amazingly, yes! as long as $a_0 \neq 0$.

$\frac{1}{f(x)}$ is the generating function so that $f(x) \cdot \frac{1}{f(x)} = 1$

Example: $f(x) = 1 + x + x^2 + \cdots$

What is a power series that, when multiplied by $f(x)$ gives 1?

a) $1+x$
b) $1-x$
c) $x-1$
d) x

POWER SERIES

What about dividing?

Amazingly, yes! as long as $a_0 \neq 0$.

$\frac{1}{f(x)}$ is the generating function so that $f(x) \cdot \frac{1}{f(x)} = 1$

Example: $f(x) = 1 + x + x^2 + \cdots$

What is a power series that, when multiplied by $f(x)$ gives 1?

$(1-x)f(x) = 1 + 0x + 0x^2 + \cdots = 1 \rightsquigarrow \frac{1}{f(x)} = 1-x$, or $f(x) = \frac{1}{1-x}$

We say $1/1-x$ is the generating function for $a_n = 1$.

EXAMPLES OF GENERATING FUNCTIONS

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \longleftrightarrow \quad a_n = 1$$

$$\frac{1}{1+x} = \quad \longleftrightarrow \quad a_n =$$

$$\frac{1}{1-ax} = \quad \longleftrightarrow \quad a_n =$$

$$\frac{1}{(1-x)^2} = \quad \longleftrightarrow \quad a_n =$$

What is the generating function for $a_n = n$?

$$a_n = n \longleftrightarrow$$

What about $a_n = -2n$?

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

The generating function for a_n is:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

Using $a_n = 2a_{n-1}$, and $a_0 = 1$, we can rewrite each term of $f(x)$:

Add up:

Solve for $f(x)$:

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS

Example: $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

PARTIAL FRACTIONS

Example: Rewrite $\frac{1-x}{1-5x+6x^2}$ as a sum of fractions where the denominator is linear.

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve $a_n = 5a_{n-1} - 6a_{n-2} \quad a_0 = 1, a_1 = 4$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve $a_n = a_{n-1} + a_{n-2}$ $\qquad a_0 = 0,\ a_1 = 1$

As above, get: $f(x) = \dfrac{x}{1-x-x^2}$

$a = \frac{1+\sqrt{5}}{2},\ b = \frac{1-\sqrt{5}}{2}$

Partial fractions: $1-x-x^2 = (1-ax)(1-bx)$

Note: $ab = -1,\ a+b = 1$

$a-b = \sqrt{5}$

$$f(x) = \frac{1/\sqrt{5}}{1-ax} - \frac{1/\sqrt{5}}{1-bx}$$

So $a_n = \dfrac{1}{\sqrt{5}}(a^n - b^n)$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: $a_n = 2a_{n-1} - n/3,\ a_0 = 1$

Example: $a_n = a_{n-1} + n^2 \quad a_0 = 0$ $\qquad a_n = 1^2 + \cdots + n^2$

REALLY, WHY GENERATING FUNCTIONS?

QUESTION. How many ways to write

$$a+b+c+d=6$$

where a is even, b is a multiple of 5, c is at most 4, and d is at most 1? (a,b,c,d nonneg integers)

e.g. making a fruit basket

What about $a+b+c+d=100$

or $a+b+c+d=n$?

REALLY, WHY GENERATING FUNCTIONS?

QUESTION. How many ways to write

$$a+b+c+d=n$$

where a is even, b is a multiple of 5, c is at most 4, and d is at most 1? (a,b,c,d nonne inte s)

$$A(x) = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$$

$$B(x) = 1 + x^5 + x^{10} + \cdots = \frac{1}{1-x^5}$$

$$C(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

$$D(x) = 1 + x$$

As before, the answer is obtained by multiplying polynomials

$$A(x)B(x)C(x)D(x) = \frac{1}{1-x^2}\,\frac{1}{1-x^5}\,\frac{1-x^5}{1-x}\cdot 1+x$$

$$= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Final answer: $n+1$ ways!