Ch 6. Symplectic rep.

$\hat{i} : H_1(S_g;\mathbb{Z}) \times H_1(S_g;\mathbb{Z}) \longrightarrow \mathbb{Z}$

Can replace $\mathbb{Z}$ with $\mathbb{R}$

$\hat{i}$ is alternating, bilinear, nondegen.

$\forall \overset{\neq 0}{x}\ \exists y$ s.t. $\hat{i}(x,y) \neq 0$.

"symplectic"

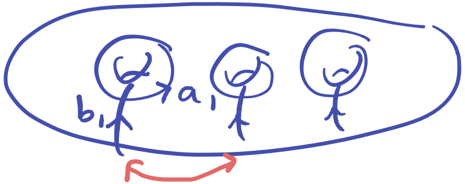
Symplectic basis for $H_1(S_g;\mathbb{Z})$:

x_i, y_i

$\hat{i}(x_i,y_i) = 1$ all other $\hat{i}$'s 0.

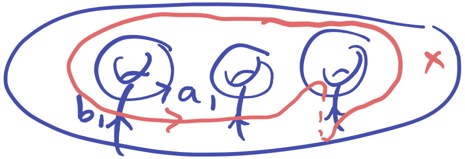
A geometric symplectic basis in S_g is a set of oriented curves $\{a_i, b_i\}$

s.t. $i(a_i,b_i) = 1$, all other i's 0



and $\{[a_i],[b_i]\}$ is a sympl. basis for $H_1(S_g;\mathbb{Z})$.

Aside: computing homology classes



$x = a_1 + a_2 + a_3 + b_3$

Since $\hat{i}(x, b_1) = 1$, the coeff on a_1 is 1.

i.e. $a_i^* = b_i$

Euclidean alg. for curves

Prop. A nonzero elt of $H_1(S_g;\mathbb{Z})$ is rep by a scc $\iff$ it is primitive

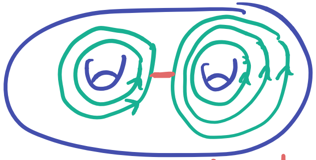
not a $\mathbb{Z}$ multiple

Pf. ($\Rightarrow$) Change of coords.

($\Leftarrow$) Example. $(2,0,3,0)$ in $H_1(S_2;\mathbb{Z})$

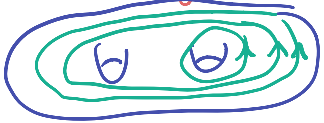
$$2x_1+3x_2$$

Start with:



Choose arc connect right-hand sides

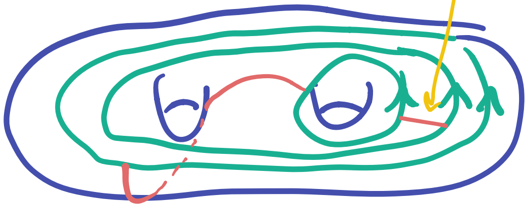
Surgery



Same H_1 class!

Repeat:

bad



\# curves in the two "bundles":

Step	1st bundle	2nd bundle
0	2	3
1	2	1
2	1	1
3	$\boxed{1}$	$\boxed{0}$

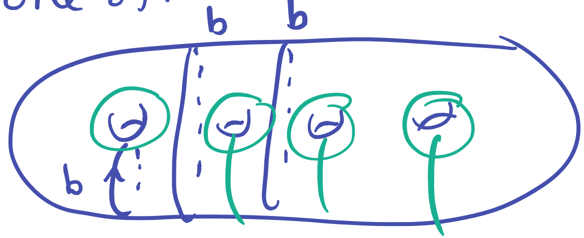
What we wanted!

The symplectic rep

$$\psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}\mathbb{Z}$$
$$\qquad\qquad\qquad\qquad \shortparallel$$
$$\qquad\qquad\qquad \mathrm{Aut}(H_1(S_g;\mathbb{Z}); \hat{\imath})$$

<u>Prop</u>. $\psi(T_b^k)[a] = [a] + k\,\hat{\imath}(a,b)[b]$

<u>Pf</u>. By change of coords, b is one of:



Choose a compatible geom. sympl. basis

Check the formula on the basis. $\square$

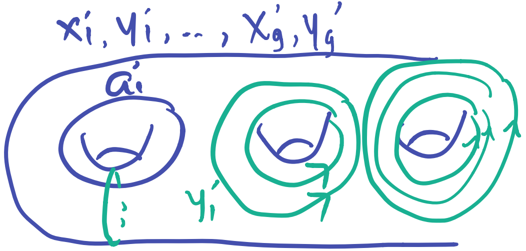
We see: ψ has kernel.

e.g. T_b, b sep.

$\mathrm{Ker}\,\psi$ called Torelli gp (Monday).

Surjectivity

$$\boxed{\psi : \mathrm{Mod}(S_g) \longrightarrow \mathrm{Sp}_{2g}\mathbb{Z}}$$



Thm. ψ is surjective.

Pf #1. Hit the "elementary matrices"

Pf #4. Hit the transvections: $\tau_v(w) = w + \hat{\imath}(v,w)\,v$

fixed set codim 1.

$$\mathrm{Sp}_{2g}\mathbb{Z} = \langle \tau_v : v \text{ prim} \rangle$$

Find T_c s.t. $\psi(T_c) = \tau_v$ using Eucl. alg.

Pf #3. Given $M \in \mathrm{Sp}_{2g}\mathbb{Z}$, $M(\text{std basis}) = \text{symplectic basis } B$.

Can soup up Euc. alg to get a geom. symp. basis $\tilde{B}$ representing B. By C of Coords $\exists\, f \in \mathrm{Mod}(S_g)$ $f\left(\substack{\text{std} \\ \text{geom} \\ \text{basis}}\right) = \tilde{B}$.

Residual finiteness

G is resid. fin if $\displaystyle\bigcap_{\substack{\Gamma \leq G \\ \text{f.i.}}} \Gamma = 1$

or. $\forall\, f \overset{\neq 1}{\in} G \;\; \exists$ finite F, $\rho: G \to F$

s.t. $\rho(f) \neq \mathrm{id}$.

Thm. $\mathrm{Mod}(S_g)$ is resid. finite.

Pf. $g = 0, 1$ easy.

$\psi(f) \neq \mathrm{id} \rightsquigarrow$ use rf'ness of $Sp_{2g}\mathbb{Z}$.

Remains to deal with $f \in$ Torelli $= \ker(\psi)$.

Fact: $\ker\psi$ is torsion free.

Assume now $|f| = \infty$. Want finite F, $\rho : \mathrm{Mod}(S_g) \to F$, $\rho(f) \neq 1$.

Choose a hyp. metric on S_g $\rightsquigarrow$ $\rho : \pi_1(S_g) \to PSL_2\mathbb{R} = \mathrm{Isom}^+ \mathbb{H}^2$

$\mathrm{Im}\,\rho \leq PSL_2 A$ $\quad A$ = fin gen subring of $\mathbb{R}$.

Such A is res. finite. (black box)

length of curves $\longleftrightarrow$ traces of elts of $PSL_2\mathbb{R}$

$|f| = \infty \implies \exists\, \gamma \in \pi_1(S_g)$ s.t. $\ell(\gamma) \neq \ell(f(\gamma)) \in A$

A res. fin. $\implies \exists$ finite quotient Q st $\ell(\gamma) \neq \ell(f(\gamma))$ in Q.

Let $H = \ker(\pi_1(S_g) \to PSL_2 A \to PSL_2 Q)$. $H \overset{f.i.}{\leq} \pi_1(S_g)$

Take. $F = \mathrm{Out}\left(\pi_1(S_g)/H\right)$. $\square$