Announcements April 11

- WebWork 6.1 and 6.2 due Thursday
- Final Exam Wed May 4 8:00-10:50 (Sec H) and Mon May 2 2:50-5:40 (Sec J)
- Tell me now if you have a conflict (three exams in one day, Math 1553 in middle)
- Office Hours Tue 2-3 and Wed 2-3
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Shivang Tue 5-6, Baishen Wed 4-5, Matt Thu 3-4
- Math Lab, Clough 280
    - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
    - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
    - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Chapter 6

Orthogonality and Least Squares

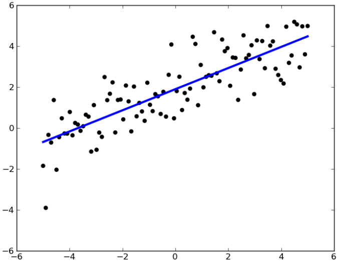
Section 6.1

Inner Product, Length, and Orthogonality

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Outline

- Dot products
- Dot products and orthogonality
- Orthogonal projection
- A formula for projection onto a line
- Orthogonal complements

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$\begin{aligned} u \cdot v &= \sum_{i=1}^{n} u_i v_i \\ &= u_1 v_1 + \cdots + u_n v_n \\ &= u^T v \end{aligned}$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

Dot product

Some properties of the dot product

- $u \cdot v =$
- $(u + v) \cdot w =$
- $(cu) \cdot v =$
- $u \cdot u$
- $u \cdot u = 0 \Leftrightarrow$

Dot product
and Length

Let v be a vector in $\mathbb{R}^n$

$$\begin{aligned}\|v\| &= \sqrt{v \cdot v} \\ &= \text{length (or norm) of } v\end{aligned}$$

Why?

Fact. $\|cv\| = c\|v\|$

v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.

Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$

Why?

Problem. Find a vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

Orthogonal Projections

Let W be a subspace of $\mathbb{R}^n$ and v a vector in $\mathbb{R}^n$.

$$\mathrm{proj}_W(v) = \text{orthogonal projection to } W \text{ of } v$$

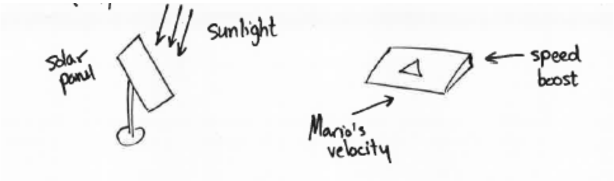
Say u and v are vectors in $\mathbb{R}^n$. Can project u to $\langle v \rangle = \mathrm{Span}\{v\}$.

Fact. $\mathrm{proj}_{\langle v \rangle}(u) = \frac{u \cdot v}{v \cdot v} v$

Why?

Orthogonal Projections

Many applications, including:



Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1, 1, -1)\}$. Find the equation of the plane $W^\perp$.

Problem. Let $W = \mathrm{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find the eqn of the line $W^\perp$.

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find the eqn of the line $W^\perp$.

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row} A)^\perp = \mathrm{Nul}\, A$$

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

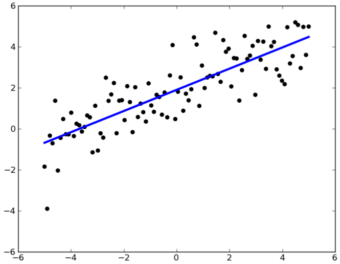
Section 6.2

Orthogonal Sets

Where are we?

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality. Last time we saw how to project onto a line. Now we will project onto higher-dimensional planes.

Outline

- Orthogonal bases
- A formula for projecting onto any subspace
- Breaking a vector into components

Orthogonal Sets

A set of vectors is orthogonal if each pair of vectors is orthogonal. It is orthonormal if in addition each vector is a unit vector.

Example.

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Fact. An orthogonal set of nonzero vectors is linearly independent.

Why?

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i w_i$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

In other words:

$$y = \sum_{i=1}^{k} \mathrm{proj}_{\langle u_i \rangle}(y)$$

Why?

What happens if y is not in W? The formula still works! But it gives the projection of y to W.

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i w_i$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Problem. Find the B-coordinates of $(6, 1)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}$$

Fact. Say that $\{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = \sum_{i=1}^{k} c_i w_i$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Problem. Find the B-coordinates of $(6, 1, -8)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$