Announcements Feb 22

- WebWork 2.3 and 2.5 due Thursday
- Homework 5 due in class Friday
- Quiz 5 on 2.3 and 2.5 in class Friday
- Midterm 2 in class Friday Mar 11 on Chapters 2 & 3
- Office Hours Tuesday and Wednesday 2-3, after class, and by appt in Skiles 244 or 236
- LA Office Hours: Scott Mon 12-1, Yashvi Mon 2-3, Baishen Wed 4-5, Matt Thu 3-4, Shivang Fri 10:30-11 + 12:30-1
- Math Lab, Clough 280
  - Regular hours: Mon/Wed 11-5 and Tue/Thu 11-5
  - Math 1553 hours: Mon-Thu 5-6 and Tue/Thu 11-12
  - LA hours: Matt Tue 11-12, Scott Tue 5-6, Baishen Thu 11-12, Yashvi/Shivang Thu 5-6

Section 2.5

Matrix Decompositions

Recall: If we want to solve $Ax = b$, we can:

- row reduce $(A|b)$, or
- find A^{-1}.

Today: the method of LU decomposition.

Computational complexity of row reduction: $n^4/3$
Computational complexity of LU decomposition: $4n^3/3$

LU Decomposition
Outline

- LU decompositions
- Using LU decompositions to solve $Ax = b$
- Finding LU decompositions: an example when A is square
- Finding LU decompositions: an example when A is not a square
- Application to electrical engineering (circuits)
- What do do when there are row swaps

LU Decomposition

An LU factorization of $A = m \times n$ is an expression

$$A = LU$$

where

- $L = m \times m$ unit lower triangular matrix
- $U = m \times n$ echelon form of A

Example.

$$\begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

To solve $Ax = b$, we write

$$\begin{aligned} Ax &= b \\ LUx &= b \end{aligned}$$

and

1. solve $Ly = b$, to obtain y,
2. solve $Ux = y$ to obtain x.

This approach uses only back substitution, *not* elimination.

LU Decomposition

Solving $Ax = b$

After writing $A = LU$, two steps:

1. solve $Ly = b$, to obtain y,
2. solve $Ux = y$ to obtain x.

Example.

$$\begin{pmatrix} 3 & 1 \\ -6 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

Solve $Ax = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$

Finding the LU Decomposition

We do row operations on A, using only row replacements, and doing them in the standard order. Then U is the reduced matrix and L records the negatives of the row operations.

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{pmatrix}$$

Finding the LU Decomposition

Why Does This Method Work?

Row operations are elementary matrices, so

$$
\begin{aligned}
E_4E_3E_2E_1A &= U \\
A &= (E_4E_3E_2E_1)^{-1}U \\
&= \left(E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}\right)U \\
&= LU
\end{aligned}
$$

Now use these facts

- each E_i is unit lower triangular, so each E_i^{-1} is as well
- the product of unit lower triangular matrices is lower triangular

Using LU to solve a linear system

We found:

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 24 & 1 & 8 \\ -12 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Use this to solve $Ax = \begin{pmatrix} 4 \\ 19 \\ -6 \end{pmatrix}$.

Finding the LU Decomposition

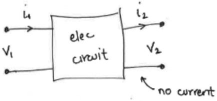
A non-square example

$$A = \begin{pmatrix} -2 & 1 & 3 \\ -4 & 4 & 1 \end{pmatrix}$$

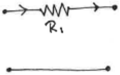
Solve $Ax = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Application to Electrical Engineering

In an electrical circuit, current i and voltage v often change by a linear transformation (by Ohm's law and Kirchoff's law).

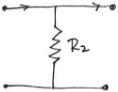


So $A\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ for some transfer matrix A.



series circuit

$$A = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$$

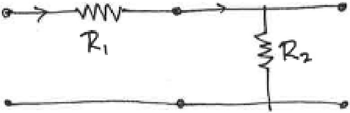


shunt circuit

$$A = \begin{pmatrix} 1 & 0 \\ -1/R_2 & 1 \end{pmatrix}$$

Application to Electrical Engineering

If we string these small circuits together we get a ladder circuit. The transfer matrix for the ladder circuit is the product of the matrices for the components. Why does this make sense?



The transfer matrix is:

$$\begin{pmatrix} 1 & 0 \\ -1/R_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -R_1 \\ -1/R_2 & 1 + R_1/R_2 \end{pmatrix}$$

Can you make a ladder circuit whose transfer matrix is

$$\begin{pmatrix} 1 & -8 \\ -0.5 & 5 \end{pmatrix}?$$

LU Decompostion

When there are row swaps

If row swaps are needed, we introduce a permutation matrix, P, so that

$$PA = LU$$

Example. $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$