Nielson-Thurston Classification

Thm. Every $f \in \mathrm{Mod}(S)$ has a rep φ s.t.

① periodic: $\varphi^n = 1$

② reducible: $\varphi(M) = M$

③ pseudo-Anosov:

$$\varphi \cdot \mathcal{F}_u = \lambda \mathcal{F}_u$$
$$\varphi \cdot \mathcal{F}_s = \frac{1}{\lambda} \mathcal{F}_s$$

③ is exclusive from ① & ②

Exclusivity: we show in Chap 14

for any curve γ

$$\ell_X(f^n(\gamma)) \to \infty$$
$$\text{\rotatebox{90}{$\cong$}}$$
$$\lambda^n \ell(\gamma)$$

Proof. $\tau(f) = \inf_X d_{Teich}(X, f \cdot X)$

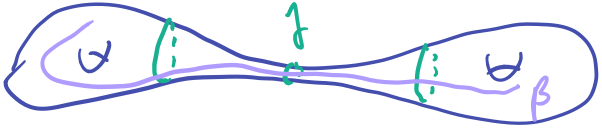
"translation length"

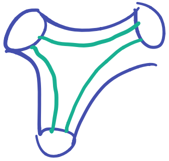
elliptic: $\tau(f) = 0$, realized $\Rightarrow$ f periodic ✓

parabolic: $\tau(f)$ not realized $\Rightarrow$ f reducible

loxodromic: $\tau(F) > 0$, realized $\Rightarrow$ f pA

Collar Lemma





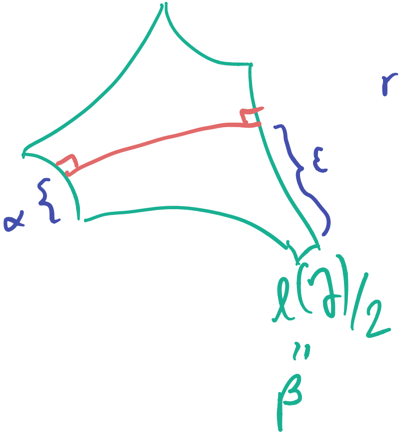
Prop. γ = scc on hyp $X \Longrightarrow$
r-nbd of γ is an embedded annulus
where $r = \sinh^{-1}\left(\dfrac{1}{\sinh \frac{1}{2}\ell(\gamma)}\right)$

Note $r \longrightarrow \infty$ as $\ell(\gamma) \longrightarrow 0$.

Cor. X = hyp surf
$\exists\ \delta$ s.t.
$\ell(\beta), \ell(\gamma) < \delta$
$\Longrightarrow i(\beta,\gamma) = 0$.

δ = universal $\forall X$ in all Teich(S).

Pf. Choose pants decomp $\{\gamma_1, \dots\}$
$\rightsquigarrow$ hyp pants $\rightsquigarrow$ right angled hexagons
$\rightsquigarrow$ right angled pentagons



rt ang pent formula:

$$\sinh \varepsilon \sinh \beta = \cosh \alpha$$
$$\geq 1$$

$\square$

Parabolic $\Longrightarrow$ Reducible

Assume $\tau(f)$ not realized.

Choose X_i s.t.

$$d(X_i, f\cdot X_i) \longrightarrow \tau(f)$$

Step 1. $\ell(X_i) \longrightarrow 0$.

Pf is essentially prop disc.

(next)

Step 2. Find reduction curves.

Wolpert Lemma: $d(X,Y) \le \tau(f)+1$

$$\Longrightarrow \ell_X(c) \le K\,\ell_Y(c)$$

some fixed K.

Choose $X = X_N$ s.t.

① $d(X, f\cdot X) \le \tau(f)+1$

② $\ell(X) < \left(\frac{1}{K}\right)^{3g-3}\delta$ (Step 1)

#curves in a pants decomp.

from Wolpert

collar lemma const.

Choose c s.t. $\ell_X(c) = \ell(X)$

Will show $c, f^{-1}(c), f^{-2}(c), \dots, f^{-(3g-3)}(c)$ is a reduction system.

Wolpert ②

Have $\ell_X(f^{-i}(c)) = \ell_{f^i X}(c) \le K^i\,\ell(c) < \delta$

So the $c, f^{-1}(c), f^{-2}(c), \dots, f^{-(3g-3)}(c)$ are disjoint by collar lemma. Must repeat (only $3g-3$ disjoint curves)! $\square$

Loxodromic $\Rightarrow$ pA

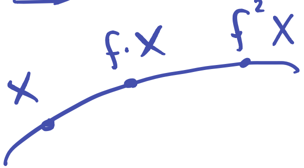
Choose X s.t.

$$d(X, f\cdot X) = \tau(f) > 0.$$

Let γ = Teich geod $X \longleftarrow f(X)$

Claim: $f\cdot\gamma = \gamma$

Claim $\Rightarrow$



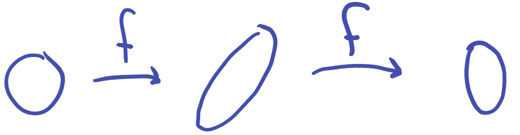
Violates

Let $h: X \to X$ Teich map in homotopy class of f.

Then h^2 is a Teich map in homotopy class of f^2.

We have: Initial & terminal qd's for h are equal.

If not: $d(X, f^2\cdot X) < 2\,d(X, f\cdot X)$

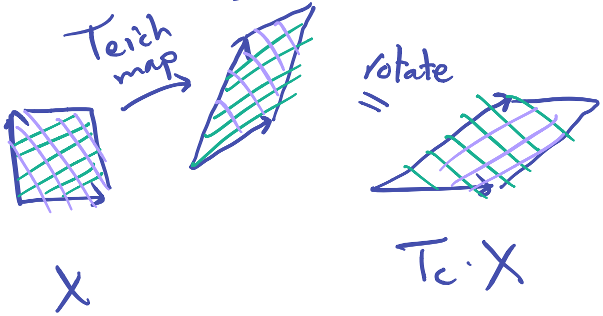


If init $\neq$ term

By the yellow box: h, hence f is pA with foliations from initial qd $\square$

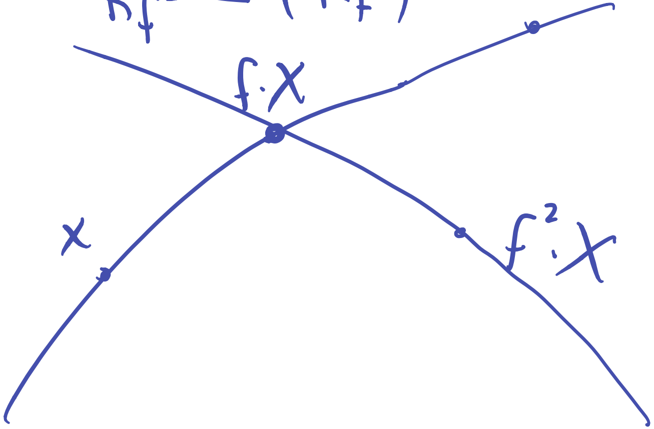
Picture for $T_c \in \mathrm{Mod}(T^2)$

"same" foliations in $\mathbb{R}^2$



Initial $\neq$ Terminal $\Longrightarrow$

$$K_{f^2} < (K_f)^2$$

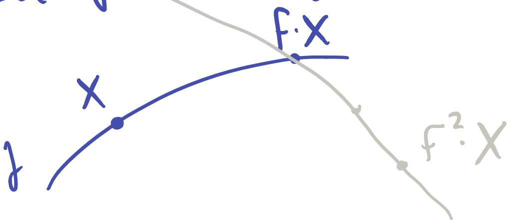


Loxodromic $\Rightarrow$ pA

Choose X s.t.

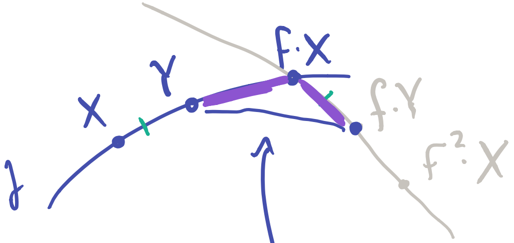
$$d(X, f\cdot X) = \tau(f) > 0.$$

Let γ = Teich geod $X \longrightarrow f(X)$



Claim: $f\cdot\gamma = \gamma$

Pf: Must rule out above picture.



$d(Y, f\cdot Y) < d(X, f\cdot X)$

violating $d(X, f\cdot X) = \tau(f)$

Indeed: purple path has length

$$d(X, f\cdot X).$$

Minimality of $X \Rightarrow f\cdot Y$

lies on γ.....

Some things about pA's

f pA with $\mathcal{F}_u, \mathcal{F}_s, \lambda$.

h commutes with f

$\Longrightarrow$ h preserves $\mathcal{F}_u, \mathcal{F}_s$

$\Longrightarrow$ h pA with same foliations $\Longrightarrow$ h is a power of a root of f.

or h periodic (if $\mathcal{F}_u, \mathcal{F}_s$ have symmetries)

$\Longrightarrow$ Centralizer of f is virtually cyclic.