

THE MONTY HALL PROBLEM



"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?"



CONDITIONAL PROBABILITY

A coin is flipped twice. The first flip is heads. What is the probability that both flips are heads?

Intuition:

Basic probability:

Conditional probability:

CONDITIONAL PROBABILITY

I have two kids. One is a boy. What is the probability I have two boys?

CONDITIONAL PROBABILITY

1. An urn has 10 white, 5 yellow, and 10 black marbles. A marble is chosen at random. We are told it is not black. What is the probability it is yellow?
2. We deal bridge hands at random to N, S, E, W. Together, N and S have 8 spades. What is the probability that E has 3 spades?

CONDITIONAL PROBABILITY

Alice and Bob each roll a die. We are told that Alice rolled a higher number. What is the probability that Alice rolled a 3?

INDEPENDENCE

Events A and B are independent if
 $P(B|A) = P(B)$

Since $P(B|A) = \frac{P(B \cap A)}{P(A)}$ we can say A and B are independent if:
 $P(A \cap B) = P(A)P(B)$

Examples. 1. We roll two die. A = first comes up 2
 B = second comes up 3

2. Two kids. B = 2 boys
 A = at least one boy

INDEPENDENCE

Events A and B are independent if
 $P(B|A) = P(B)$

Examples. 3. The Alice and Bob problem:
 $B = \text{Alice rolled } 3$
 $A = \text{Alice} > \text{Bob}$

4. Urn problem: 10 white, 5 yellow, 10 black.
Are Y and B^c independent?

A CONDITIONAL PROBABILITY PROBLEM

We buy light bulbs from suppliers A and B.

30% of the bulbs come from A, 70% from B.

2% of the bulbs from A are defective

3% of the bulbs from B are defective.

What is the probability that a random bulb...

(i) is from A and defective?

(ii) is from B and not defective?

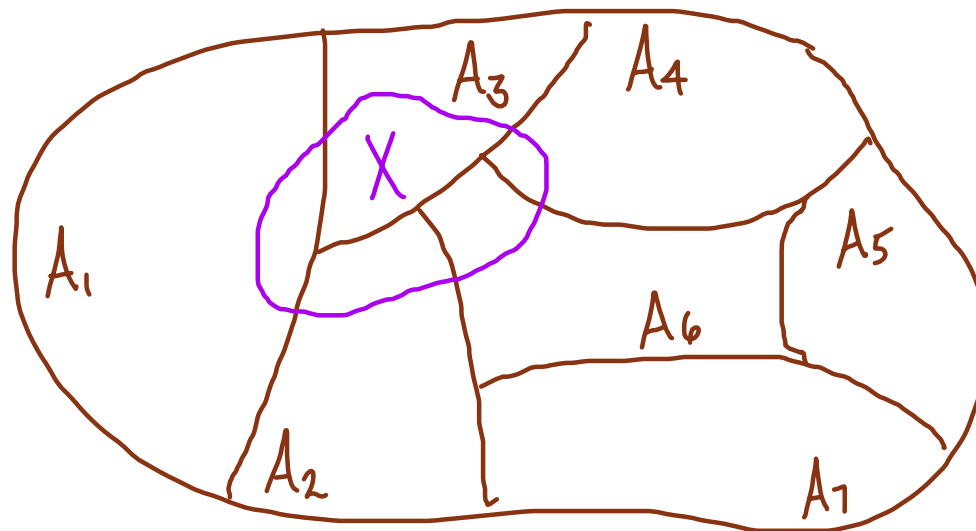
★ (iii) is defective?

LAW OF TOTAL PROBABILITY

Say that events A_1, \dots, A_n form a *partition* of the sample space S , that is, the A_i are mutually exclusive ($A_i \cap A_j = \emptyset$ for $i \neq j$) and $A_1 \cup \dots \cup A_n = S$.

Let $X \subseteq S$ be any event. Then

$$P(X) = P(A_1)P(X|A_1) + \dots + P(A_n)P(X|A_n)$$



BAYES' FORMULA

How is $P(A|B)$ related to $P(B|A)$?

THEOREM:
$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

PROOF:



EXAMPLE. In the light bulb problem, say a randomly selected light bulb is defective. What is the probability it came from A?

BAYES' FORMULA

EXAMPLE. Coin A comes up heads $1/4$ of the time.
Coin B comes up heads $3/4$ of the time.
We choose a coin at random and flip it twice.
If we get two heads, what is the probability coin B was chosen?

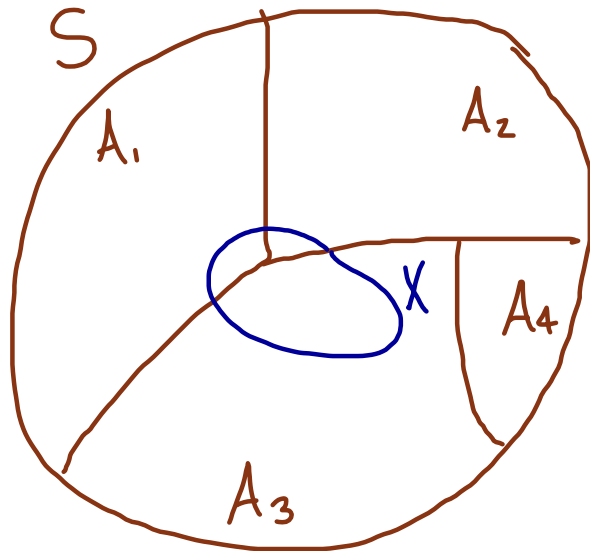
BAYES' FORMULA

Computing the denominator with the law of total probability

A_1, \dots, A_n pairwise mutually exclusive events with $A_1 \cup \dots \cup A_n = S$ and $P(A_i) > 0$ for all i . Let X be an event with $P(X) > 0$. Then, for each j , we have:

$$P(A_j | X) = \frac{P(A_j)P(X|A_j)}{P(X)}$$

where $P(X) = P(A_1)P(X|A_1) + \dots + P(A_n)P(X|A_n)$



$P(A_3|X)$ big
 $P(A_2|X)$ small
 $P(A_4|X) = 0$.

EXAMPLE. Do a variant of the coin problem with 3 or more coins

BAYES' FORMULA

PROBLEM. You have 3 cards. One is red on both sides, one is black on both sides, and one has a red side and a black side. You pick one card randomly and put it on the table. Its top side is red. What is the probability the other side is red?

BAYES' FORMULA

- PROBLEM.** There are 3 urns, A, B, and C that have 2, 4, and 8 red marbles and 8, 6, and 2 black marbles, respectively. A random card is picked from a deck. If the card is black we choose a marble from A, if it is a diamond we choose a marble from B, and otherwise choose a marble from C.
- (a) What is the probability that a red marble gets drawn?
 - (b) If we know a red marble was drawn, what is the probability the card was hearts? diamonds?

Draw the picture!