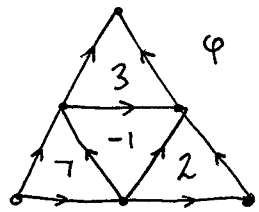
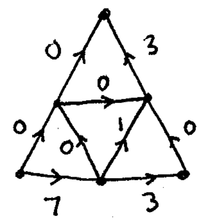
Examples of 2-cocycles

① $X = D^2$



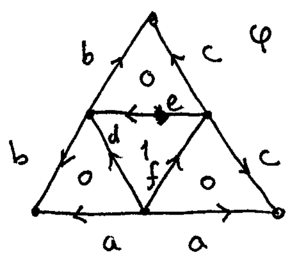
We know $H^2(D^2, \mathbb{Z}) = 0$
so $\varphi = \delta\psi$.
What is ψ?

Solution:



No obstructions.

② $X = S^2$



Want to show $[\varphi] \neq 0$ in $H^2(S^2, \mathbb{Z})$
i.e. no antiderivative ψ.

Any ψ with $\partial\psi = \varphi$ must satisfy:

writing a for $\psi(a)$

$$b + d = a$$
$$e + c = a$$
$$\left.\begin{aligned} b + f &= c \\ e + f &= d + 1 \end{aligned}\right\} \Rightarrow (b+d) - (e+c) = 1$$
$$\Rightarrow a - a = 1.$$

③ $X = T^3$, $G = \mathbb{Z}/2\mathbb{Z}$.

Realize T^3 as Δ-complex by subdividing cube into 6 tetrahedra, identifying opp faces of the cube. Let L = line segment in cube that is a loop in T^3, misses 1-skeletion. Declare $\varphi(T) = 1$ if $T \cap L \neq \phi$. Show $[\varphi] \neq 0$ in $H^2(T^3, \mathbb{Z}/2\mathbb{Z})$.

Product Structures

There are three natural products with homology & cohomology:

① Evaluation pairing:

$$H^k(X) \times H_k(X) \longrightarrow \mathbb{Z}$$

Can use this to show cocycles, or cycles, are nontrivial!

② Cup product:

$$H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X)$$
$$(\varphi, \psi) \longmapsto \varphi \cup \psi$$

$\rightsquigarrow$ $H^*(X)$ is a graded ring.

③ Cap product:

$$H^p(X) \times H_n(X) \longrightarrow H_{n-p}(X)$$
$$(\varphi, \alpha) \longmapsto \varphi \cap \alpha$$

Big Goal:

Poincaré Duality Theorem. Let M = compact, connected, oriented n-manifold. Then

$$H^p(M) \longrightarrow H_{n-p}(M)$$
$$\varphi \longmapsto \varphi \cap [M]$$

is an isomorphism.

We have already since examples of p-cocycles in n-manifolds of the form "intersect with this $(n-p)$-cycle". These are Poincaré duals.

Will see: under PD, cap product is intersection.

Cup Product

Want to define a product on $H_*(X)$.

There is a cross product $\quad H_i(X) \times H_j(Y) \longrightarrow H_{i+j}(X \times Y)$

$$(e_i, e_j) \longmapsto e_i \times e_j$$

Taking $X = Y$: $\quad H_i(X) \times H_j(X) \longrightarrow H_{i+j}(X \times X) \xrightarrow{?} H_{i+j}(X)$

Need a natural map $X \times X \to X$.

If X is a group, can multiply $\rightsquigarrow$ Pontryagin product.

Otherwise only natural map is projection $\rightsquigarrow$ stupid product.

For H^*, situation is better. Want

$$\cancel{H^i(X) \times H^j(X) \to H^{i+j}(X) \xrightarrow{?}}$$

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X \times X) \xrightarrow{?} H^{i+j}(X)$$

This requires a natural map $X \to X \times X$ $\rightsquigarrow$ diagonal!

This is the cup product.

We can also define cup product from scratch:

For $\varphi \in C^k(X, R)$, $\psi \in C^\ell(X, R)$ $\qquad R = $ ring.

the cup product $\varphi \cup \psi \in C^{k+\ell}(X, R)$ is

given by: $\quad (\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0, \ldots, v_k]}) \, \psi(\sigma|_{[v_k, \ldots, v_{k+\ell}]})$

for a simplex $\sigma : \Delta^{k+\ell} \to X$.

To show cup product induces a product on cohomology.

<u>Lemma</u> $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

<u>Pf</u> Say $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$, $\sigma : \Delta^{k+\ell+1} \to X$.

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|[v_0,\dots,\hat{v}_i,\dots,v_{k+1}])\,\psi(\sigma|[v_{k+1},\dots,v_{k+\ell+1}])$$

$$(-1)^k (\varphi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|[v_0,\dots,v_k])\,\psi(\sigma|[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}])$$

Last term of first sum cancels first sum of second.
Rest is $\delta(\varphi \cup \psi)(\sigma) = (\varphi \cup \psi)(\partial\sigma)$. ▨

Since $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi \pm \varphi \cup \delta\psi$

$\rightsquigarrow$ product of cocycles is a cocycle.

Also, the product of a cocycle and a coboundary is a coboundary:

$$\psi = \delta\theta, \quad \delta\varphi = 0 \implies \delta(\varphi \cup \theta) = \delta\varphi \cup \theta \pm \varphi \cup \delta\theta = \pm\, \varphi \cup \psi.$$

We thus have an induced cup product

$$H^k(X,R) \times H^\ell(X,R) \xrightarrow{\cup} H^{k+\ell}(X,R)$$

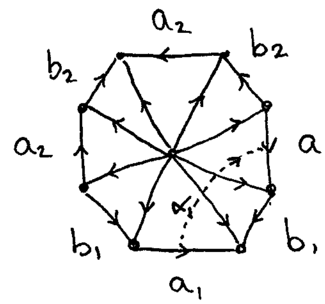
It is associative and distributive, since it is on cochain level.
If R has 1 then $H^*(X,R)$ has identity, namely:

$1 \in H^0(X,R)$ taking value $1 \in R$ on each 0-simplex.

Note: The canonical isomorphism between simplicial/singular H^* preserves $\cup$, so can switch back & forth.

EXAMPLE: SURFACES

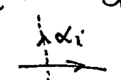
$X = M_g$. Will show $\cup : H^1(M_g,\mathbb{Z}) \times H^1(M_g,\mathbb{Z}) \longrightarrow H^2(M_g,\mathbb{Z}) = \mathbb{Z}$
is algebraic intersection.

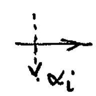


a_i, b_i form a basis for $H_1(M_g,\mathbb{Z})$.
UCT $\Rightarrow$ $H^1(M_g) \cong \mathrm{Hom}(H_1(M_g),\mathbb{Z})$
Basis for $H_1 \rightsquigarrow$ dual basis for H^1

$$a_i \rightsquigarrow \varphi_i \quad \begin{array}{l} a_1 \mapsto 1 \\ \text{others} \mapsto 0. \end{array}$$

Can represent φ_i, ψ_i by simplicial cocycle $\rightsquigarrow$ dotted arc. α_i, β_i.

α_i evaluates to 1 on an edge like 

-1 on an edge like 

Compute $\varphi_1 \cup \psi_1$ from definition.
Takes value 0 on all cells but SE,
where it takes value 1.

We know $H_2(M_g) = \mathbb{Z} = \langle [M_g] \rangle$ ← fundamental class
UCT $\Longrightarrow H^2(M_g,\mathbb{Z}) \cong \mathrm{Hom}(H_2(M_g),\mathbb{Z})$.
So which elt of $H^2(M_g,\mathbb{Z})$ is $\varphi_1 \cup \psi_1$?
We check $(\varphi_1 \cup \psi_1)([M_g]) = 1$
This tells us both that (i) $[M_g]$ generates $H_2(M_g)$
(ii) $\varphi_1 \cup \psi_1$ is dual to $[M_g]$,
hence a gen. for $H^2(M_g,\mathbb{Z})$.

In general, identifying $H^2(M_g,\mathbb{Z})$ with $\mathbb{Z}$:

$$\cup = \hat{\imath}$$
↖ algebraic intersection.

Suffices to check on generators.