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Let $A, B \in M_{n\times n}(\mathbb{R})$. Then e^{tA} is the integral curve of $\tilde{A}$ starting at I

Thus $e^{t_0 A} e^{tB}$ is the integral curve of $\tilde{B}$ starting at $e^{t_0 A}$

Thus by spivak: $[A,B] = [\tilde{A},\tilde{B}]_e = \frac{1}{2}\lim_{h\to 0} \frac{c(h)-c(0)+c(-h)}{h^2}$ ⊛

where $c(t)$ is the "follow $\tilde{A}, \tilde{B}$ in square". By above we have that $c(t) = e^{tA}e^{tB}e^{-tA}e^{-tB}$. Thus we can plug this def'n into ⊛ and since we are dividing by h^2, omit terms of higher order

$$\frac{1}{2}\lim_{h\to 0}\frac{c(h)-c(0)+c(-h)}{h^2} \underset{\text{asymptotic}}{\cong} \Big[\left(1+hA+\tfrac{h^2A^2}{2}\right)\left(1+hB+\tfrac{h^2B^2}{2}\right)\left(1-hA+\tfrac{h^2A^2}{2}\right)\left(1-hB+\tfrac{h^2B^2}{2}\right) - 2I$$
$$+\left(1-hA+\tfrac{h^2A^2}{2}\right)\left(1-hB+\tfrac{h^2B^2}{2}\right)\left(1+hA+\tfrac{h^2A^2}{2}\right)\left(1+hB+\tfrac{h^2B^2}{2}\right)\Big] \Big/ h^2$$

on further note that $c(t)=c(-t)$ up to second order

again ignoring higher order terms → (also all first order cancels)

$$\cong \Big[I + h^2AB - h^2A^2 - h^2AB - h^2BA - h^2B^2 + 2\tfrac{h^2A^2}{2} + 2\tfrac{h^2B^2}{2} - 2I$$
$$+ I + h^2AB - h^2A^2 - h^2AB - h^2BA - h^2B^2 + 2\tfrac{h^2A^2}{2} + 2\tfrac{h^2B^2}{2}\Big] \Big/ h^2$$

$+A^2AB$ $\quad h^2AB$

lotsa cancel → $\cong (2AB - 2BA)\cdot h^2/h^2 = (2AB-2BA)$

So $[A,B] = \frac{1}{2}(2AB-2BA) = AB - BA$.