Combinatorial Cubings, Cusps, and the Dodecahedral Knots

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Abstract. There are finitely many tessellations of 3-dimensional space-forms by regular Platonic solids. Explicit examples of constant curvature finite-volume 3-manifolds arising from these are well-known for all possibilities, except for the tessellation $\{5,\ 3,\ 6\}$. We introduce the dodecahedral knots D_f and D_s in S^3 to fill this gap. Techniques used illustrate the results on cusp structures and π_1-injective surfaces of alternating link complements obtained by Aitchison, Lumsden and Rubinstein **[ALR]**.

The Borromean rings and figure-eight knot arise from the tessellation of hyperbolic 3-space by regular ideal octahedra and tetrahedra respectively. We produce exactly four new links in S^3, corresponding to the tessellations $\{4,\ 3,\ 6\}$ and $\{5,\ 3,\ 6\}$ of $\mathbb{H}^3$, and united by a canonical construction from the Platonic solids.

The dodecahedral knot D_f is the third in an infinite sequence of fibred, alternating knots, the first member of which being the figure-eight. The complements of these new links contain π_1-injective surfaces, which remain π_1-injective after 'most' Dehn surgeries. The closed 3-manifolds obtained by such surgeries are determined by their fundamental groups, but are not known to be virtually Haken.

1. Introduction

Regular tessellations of space-forms by Platonic solids have played a significant rôle in the exploration and exposition of 3-dimensional geometries and topology. Table 1, derived from Coxeter **[Co1]**, **[Co2]**, gives all such tessellations, including those by solids with deleted vertices.

Remark 1.1. The tessellations $\{3,\ 3,\ 3\}$, $\{4,\ 3,\ 4\}$ and $\{5,\ 3,\ 5\}$ are self-dual. The links of vertices are respectively tetrahedra, octahedra and icosahedra, the Platonic solids with triangular faces. The corresponding edge degrees — 3, 4 and 5, the most famous Pythagorean triple — encapsulate the notions of positively-curved, flat and negatively-curved geometry.

The corresponding symmetry groups are well-understood in the spherical and flat spaceforms, as are the subgroups acting without fixed points. For the hyperbolic tessellations, finite-index torsion-free subgroups exist by Selberg's theorem, with corresponding quotient 3-manifolds having finite volume. Infinitely many such subgroups exist.

The dodecahedral tessellation $\{5,\ 3,\ 3\}$ gives rise to Poincaré's homology sphere $\mathcal{P}^3$, a manifold ubiquitous in geometric topology, associated with problems of smoothings and triangulations of manifolds. A beautiful description of $\mathcal{P}^3$ in terms of face identifications of a dodecahedron has been given by Seifert and Weber **[SW]**, where another such

Solid	Tessellations of spaceforms by Platonic solids			
	S^3	$\mathbb{R}^3$	$\mathbb{H}^3$, compact	$\mathbb{H}^3$, ideal
tetrahedra	$\{3,\ 3,\ 3\}$	none	none	$\{3,\ 3,\ 6\}$
	$\{3,\ 3,\ 4\}$			
	$\{3,\ 3,\ 5\}$			
icosahedra	none	none	$\{3,\ 5,\ 3\}$	none
octahedra	$\{3,\ 4,\ 3\}$	none	none	$\{3,\ 4,\ 4\}$
cubes	$\{4,\ 3,\ 3\}$	$\{4,\ 3,\ 4\}$	$\{4,\ 3,\ 5\}$	$\{4,\ 3,\ 6\}$
dodecahedra	$\{5,\ 3,\ 3\}$	none	$\{5,\ 3,\ 4\}$	$\{5,\ 3,\ 6\}$
			$\{5,\ 3,\ 5\}$	

Table 1

compact 3-manifold, the hyperbolic Seifert-Weber space, is also described. The latter manifold arises from the dodecahedral tessellation $\{5,\ 3,\ 5\}$. In both cases, opposite faces are identified in a natural fashion.

The cube is distinguished in that it tessellates all spaceforms. The cubical tessellation of $\mathbb{R}^3$ gives rise to the 3-dimensional torus, with flat geometry. Euclidean space does not admit regular tessellations other than by the cube. Nonetheless, the mysterious connections between the Platonic solids allows for an intriguing manifestation of dodecahedra even in this context. Thurston **[Th]** has shown how a dodecahedron can be flattened into a cube, and then allowed to 'tessellate' $\mathbb{R}^3$. Allowing orbifold structures, Thurston then shows how this tessellation induces a singular metric on S^3, with cone angle π concentrated along the Borromean rings. Using the universality of the Borromean rings in the construction of closed orientable 3-manifolds as branched covering spaces, Hilden, Lozano, Montesinos and Whitten **[H*]** demonstrate the significance of the dodecahedral tessellation $\{5,\ 3,\ 4\}$: its group of symmetries is rich enough to produce all closed 3-manifolds.

Similarly, orbifold structures on links in S^3 arise from the Seifert-Weber manifold in the guise of the Whitehead link, and from the tessellation of $\mathbb{H}^3$ by cubes with icosahedral vertex links via the 5_2-knot (**[Be]**, **[AR1]**). Both of these links are universal.

Other *closed* hyperbolic 3-manifolds arising from tessellations of $\mathbb{H}^3$ have been described in Best **[Be]**, and in Richardson and Rubinstein **[RR]**.

The tessellations of hyperbolic space by ideal Platonic solids are of equal interest. The most famous contemporary example is $\{3,\ 3,\ 6\}$, giving the figure-eight knot complement (again universal) as quotient (**[Th]**). An example of a link complement in S^3 whose complement is the quotient of $\{4,\ 3,\ 6\}$ is described in **[AR1]**. Thurston **[Th]** also shows that two octahedra of $\{3,\ 4,\ 4\}$ form the fundamental domain for a discrete subgroup of symmetries, with quotient again the complement of the Borromean rings in S^3.

The remaining tessellation $\{5,\ 3,\ 6\}$ of $\mathbb{H}^3$ by ideal dodecahedra has not been considered previously — no *explicit* link complement in any 3-manifold is known to have such a structure. We will construct two such examples in S^3, obtaining what we call the

dodecahedral knots D_f and D_s. Whether the tessellation $\{5,\ 3,\ 6\}$ leads to as rich a domain as the other dodecahedral tessellations remains to be seen.

Complements of alternating links

In each of the cases above, the resulting link in S^3 is alternating. Investigations of the hyperbolic structures of alternating link complements have been given by Lawson **[La]**, Menasco **[Me]**, Takahashi **[Ta]**, and more recently by Weeks **[We]**, seeking to generalize the beautiful constructions of Thurston **[Th]**. In each case, the aim has been to demonstrate the existence of a complete metric of constant curvature -1 on the complement, and to calculate various invariants from such a (unique) structure. This invariably necessitates determining a combinatorial description of the link complement as the union of two 'ideal' polyhedra, with face identifications, and then decomposing these polyhedra into ideal tetrahedra whose shapes and volumes can be calculated. At this stage of the procedure, there is no canonical way to proceed, and any structure hidden in the combinatorics at the polyhedral level is lost.

That some beautiful deeper combinatorial structure may have existed has been remarked in these papers, but neither revealed nor exploited explicitly.

Retrospectively, our starting point is two remarks of Thurston **[Th]**. The first is that the figure-eight knot can be arranged on the 1-skeleton of a tetrahedron, as a 'heuristic' that the complement admits a tetrahedral decomposition. In fact, there are two simple such arrangements, and we develop the second one. Thurston's second remark is that for the Borromean rings, face identifications have a beautiful naturality: "Faces are glued to their corresponding faces with 120° rotations, alternating in direction like gears" **[Th]**.

We describe how, with our arrangement of the figure-eight knot on the tetrahedron, these remarks are related, and generalize to face identifications of two identical polyhedra, producing all of the examples of alternating links considered. We illustrate with each of the ideal regular tessellations of $\mathbb{H}^3$, producing 4 new links in the process. Our favourites, arising from $\{5,\ 3,\ 6\}$, are a new fibred knot D_f, and a knot D_s possessing a high degree of symmetry. The existence and simplicity of this combinatorial structure of alternating link complements is described in detail in **[ALR]**. A more general context is described in **[AR2]**.

2. The general construction for 4-valent graphs

We recall the construction of **[ALR]**. Take an arbitrary finite connected planar graph Γ, all of whose vertices having degree 4. We also require that at any vertex, all regions meeting at the vertex are distinct. Two-colour the regions of the plane checker-board fashion using white and black, with the exterior white by convention. Assign signs '$+$' and '$-$' to the white and black regions respectively. Denote the resulting combinatorial polyhedron by Π_Γ^+.

Now take an identical copy of Π_Γ^+, reverse all colours and signs, and denote the resulting polyhedron by Π_Γ^-. Each face ϕ_i of $\Pi_\Gamma^\pm$ is a combinatorial n_i-gon, with sign allocation σ_i, and we identify ϕ_i with the corresponding face ϕ_i' of $\Pi_\Gamma^\mp$ by a rotation of $\sigma_i.2\pi/n_i$, with a '+' sign denoting clockwise.

Denote the resulting topological space by $\overline{M}_\Gamma$, and let M_Γ denote $\overline{M}_\Gamma$ with deleted vertices. Finally, let $\mathcal{L}_\Gamma$ denote the alternating link in S^3 canonically associated to Γ, as in Figure 1. Observe that, viewed from the center of any region, crossings are of the sign assigned to that region.

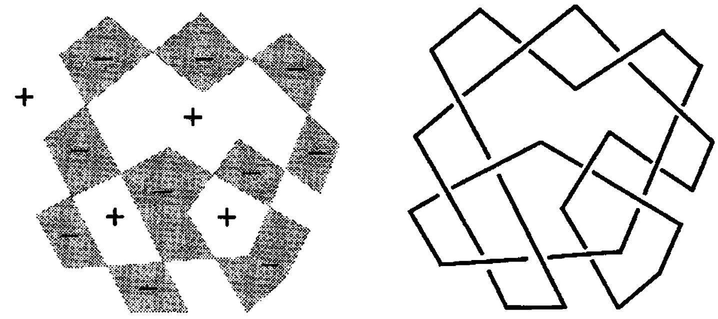

Figure 1

One of the results of **[ALR]** is

Theorem 2.1. *M_Γ is canonically homeomorphic to $S^3 - \mathcal{L}_\Gamma$. Each edge of M_Γ is of degree 4.*

3. The six examples arising from ideal tessellations

In each case, we describe an alternating link, and face identifications of the corresponding pair of identical polyhedra. That the link complement has a complete metric of constant curvature -1 follows immediately on declaring each polyhedron to be ideal and regular in hyperbolic space.

Example 1. The Borromean rings. Applying this construction to the graph $\Gamma_{\{3,4\}}$ underlying the octahedron, we recover Thurston's description of the complement of the Borromean rings of Figure 2. The universal cover is the tessellation $\{3,4,4\}$ of $\mathbb{H}^3$.

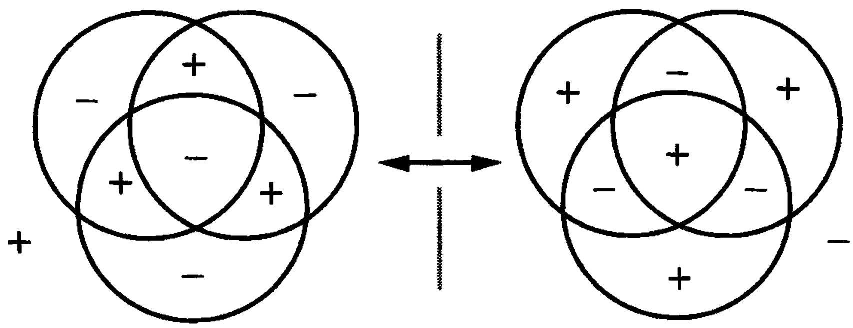

Figure 2

Example 2. The figure-eight knot. Take a tetrahedron, corresponding to the graph $\Gamma_{\{3,3\}}$ and 2-colour its faces black and white in the unique (up to symmetry) way so that no vertex is surrounded by regions all of the same colour. Assign the sign '+' to

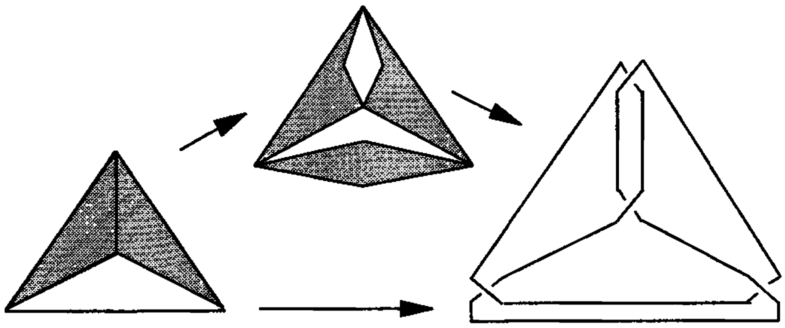

Figure 3

the white regions, ‘ $-$ ’ to the black. Now split each edge separating regions of the same colour to obtain a 4-valent graph 2-coloured as above. (Figure 3.)

Carrying out face identifications yields the figure-eight knot complement. The two resulting ‘bigons’ can be squeezed back to a single edge to recapture the face identifications of tetrahedra as in Thurston’s description. Note that in removing a bigon, two edges are identified in each polyhedron $\Pi^{\pm}_{\Gamma}$, from different equivalence classes. Every edge in the quotient is thus of degree 6. The universal cover corresponding to this combinatorial structure is geometrically the tessellation $\{3,3,6\}$ of $\mathbb{H}^3$, giving rise to the complete structure on the knot complement.

Examples 3, 4. Two cubical links. There are *two* ways to 2-colour the regions of the graph $\Gamma_{\{4,3\}}$. These are depicted in Figure 4.

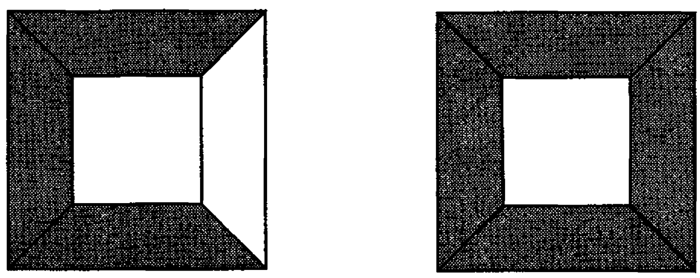

Figure 4

Proceed exactly as in the last example, observing that the introduction and deletion of bigons is unnecessary provided the link associated with such a 2-coloured trivalent graph is interpreted according to Figure 5.

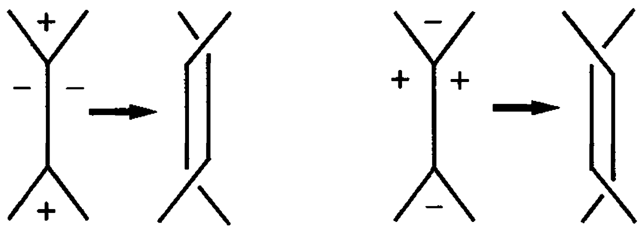

Figure 5

These two links obtained from the cube arise from the tessellation $\{4,3,6\}$ of $\mathbb{H}^3$, and are the links 8^3_4 and 8^4_1 in Rolfsen’s book, depicted in Figure 6.

In **[ALR]**, 4-valent graphs admitting a collapse to a 2-coloured 3-valent graph without bigons are called ‘balanced’: the construction applied here works for all such graphs.

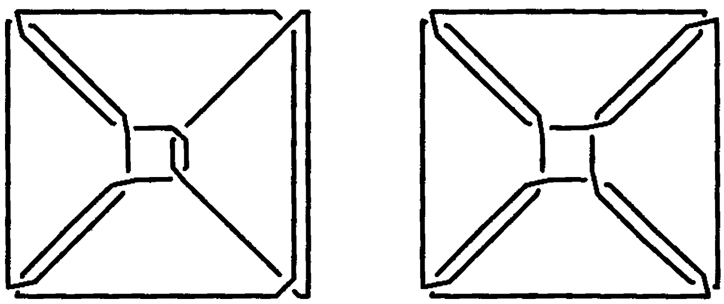

Figure 6

Remark 3.1. These two are the *only* links in Rolfsen's tables which have balanced bigons, in the sense of **[ALR]**, and no triangular regions.

There is another 3-component link also corresponding to the tessellation $\{4,3,6\}$ of $\mathbb{H}^3$, described in **[AR1]**. This does not obviously arise as part of our general construction.

Examples 5, 6. The two dodecahedral knots. The dodecahedron may be depicted combinatorially as in Figure 7.

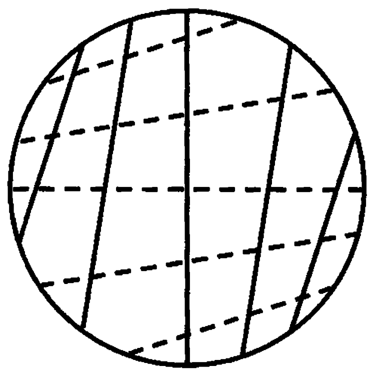

Figure 7

Up to symmetry and colour interchange, there are two allowable 2-colourings. These are depicted in Figure 8, with corresponding knots in Figure 9 denoted D_s and D_f arising from the tessellation $\{5,3,6\}$ of $\mathbb{H}^3$. The knot D_s has considerable symmetry, whereas D_f turns out to be fibred.

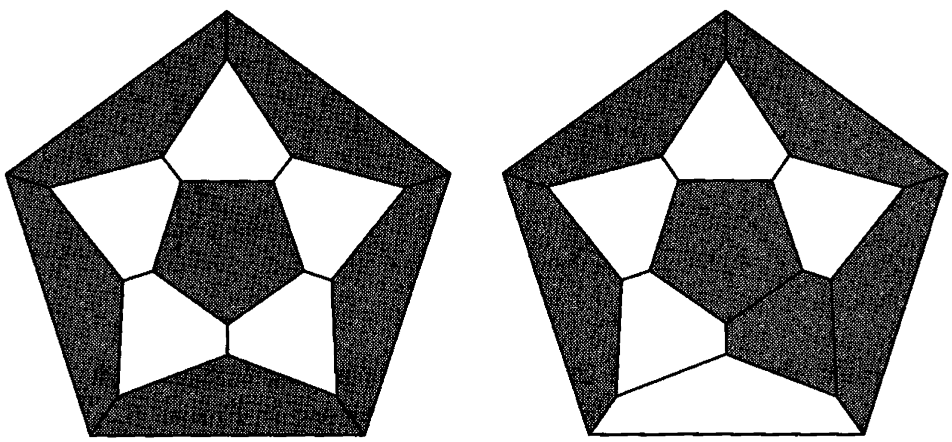

Figure 8

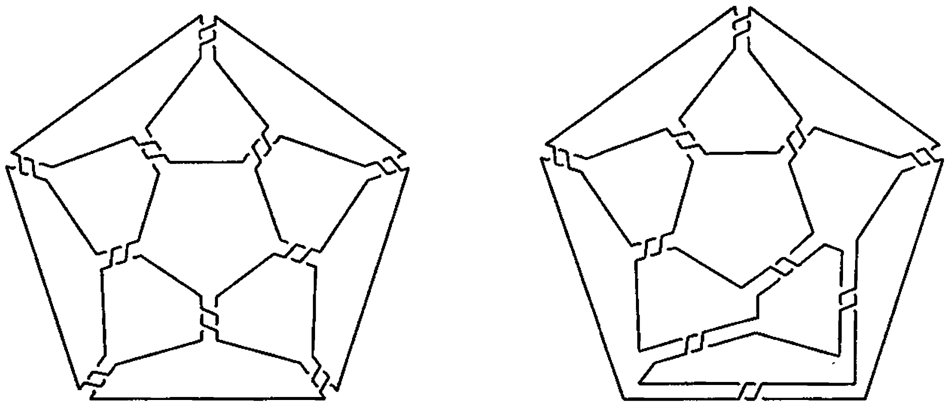

Figure 9

Added in Proof. Alan Reid and Walter Neumann have demonstrated some fascinating properties of these dodecahedral knots, in the context of their beautiful work on arithmetic structures [NR].

Hatcher has used similar ideas in [Ha], and it seems likely that Thurston is aware of the general construction, particularly since we have found the fibred dodecahedral knot in [Ri], referred to by Riley as 'Thurston's knot'. It is clear from the construction above that the complement of D_f admits an orientation reversing involution. The complements of both D_f and D_s contain totally geodesic immersed surfaces with respect to the complete metric of constant curvature.

4. Some fibred alternating knots from balanced links

We begin with a characterization of a class of colourable graphs.

Lemma 4.1. *Suppose Γ is a connected trivalent planar Hamiltonian graph. Then Γ can be 2-coloured with no vertex surrounded by regions of the same colour.*

Such a graph arises by drawing a circle as the equator of the sphere, and adding disjointly embedded arcs with endpoints on the equator. Colour one hemisphere white, the other black.

Remark 4.2. The 2-colourings of the cube and dodecahedron described above show that a graph with Hamiltonian circuit need not have a *unique* 2-colouring, and that the resulting alternating link may have more than one component.

A particularly nice class arises by taking the sequence of graphs Γ_t generalizing Figure 7: instead of 5 arcs in each hemisphere, take $2t - 1$ for any natural number t, with t arcs at the back meeting the equator in the left and right regions of the front. The top arc at the back meets the equator between the front t^{th}-and $(t+1)^{\text{st}}$-arcs numbered from the left. Observe that Γ_1 is a tetrahedron, whereas Γ_3 is the dodecahedron.

Proposition 4.3. *Each of the graphs Γ_t gives rise to an alternating fibred knot $K_t = \mathfrak{L}_{\Gamma_t}$. The knot K_1 is the figure-eight knot, and K_3 is the dodecahedral knot D_f.*

Proof. The resulting link is fibred since the construction yields a plumbing of Hopf bands onto two sides of a disc, along the arcs of the graph. We invoke the results of Murasugi **[Mu]** and Stallings **[St]**, who show such links are fibred.

That the resulting link has one component is a simple induction on t, adding additional Hopf bands on either side of the middle edge of each side of the disc. □

5. Dehn surgeries

Every non-trivial Dehn surgery on K_t is determined by prescribing a Dehn surgery coefficient $\rho = (p, q) \neq \infty$. Denote the resulting 3-manifold by $M_{t,\rho}$.

Theorem 5.1. *For each $\rho \neq \infty$ and $t > 2$, $M_{t,\rho}$ is irreducible, has universal cover homeomorphic to $\mathbb{R}^3$, and contains an immersed π_1-injective surface satisfying the 4-plane, 1-line condition. Hence $M_{t,\rho}$ has homotopy type determined by its fundamental group.*

Sketch of Proof. Each of the trivalent polyhedra $\Pi^{\pm}_{\Gamma_t}$ has $2(2t-2)$ pentagonal faces, four $(t+2)$-gons, $(12t-6)$ edges and $(8t-4)$ vertices. Each polyhedron can be decomposed into $(8t-4)$ cubes in the standard manner (see **[AR1]** for example). After face identifications, all edges of $S^3 - K_t$ have degree $(t+2)$, 5, or 6. The former two values occur along introduced edges joining the centers of the polyhedra through points at the center of faces.

Consider a cube in the ideal cubing $\{4, 3, 6\}$ of $\mathbb{H}^3$, and bisect it symmetrically into 8 isometric subcubes by planes orthogonal at the centre, and orthogonal to the edges. Endow each of the cubes of $\Pi^{\pm}_{\Gamma_t}$ with the geometry of one of these subcubes, with the distinguished vertex at a vertex of $\Pi^{\pm}_{\Gamma_t}$. The resulting singular metric is complete, and has negative curvature at every point. The structure of the cusps is depicted in Figure 10, where there are $(16t-8)$ equilateral triangles in the decomposition of the torus. Such pictures occured originally in **[Th]**. Generators for the homology of the peripheral torus of the knot have been labelled. These are sufficiently long for $t \geq 3$ that any non-trivial Dehn surgery, in the sense of Gromov-Thurston (**[AR]**, **[GT]**) always yields a closed Cartan-Hadamard manifold with negative curvature along the core of the sewn-in solid torus, and with the metric away from the cusp remaining unaltered.

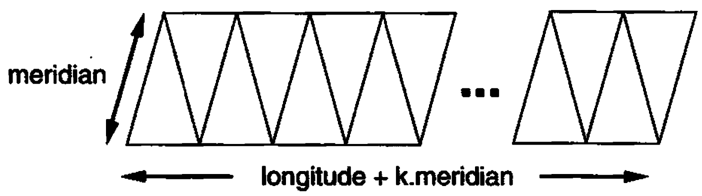

Figure 10

The immersed surface obtained by taking the union of squares bisecting each of the cubes of the decomposition of $S^3 - K_t$ is π_1-injective, being isotopic to a (singular) totally geodesic surface. Since this surface is in the 'thick' part of $S^3 - K_t$, it survives to produce an injective surface after surgery. This surface satisfies the conclusions of the theorem. For further details, see **[ALR]**, **[AR1]** and **[AR2]**. □

Remarks 5.2. The symmetric dodecahedral knot also belongs to an infinite family, obtained from the trivalent graphs $\mathcal{G}_k$, $k \geq 1$. These are obtained by drawing concentric $(k+1)$-gons in the plane, rotated relative to each other, and filling the annular region between them by $2k+2$ pentagons. The results on surgery also apply to this class, when $k > 2$. A similar argument applies to the cubical links described above.

The resulting closed 3-manifolds are not known to be virtually Haken.

Remark 5.3. The 14-sided polyhedron corresponding to $\mathcal{G}_5$ can be realized in hyperbolic space as the fundamental domain of the group action giving rise to Löbell's manifold **[Lö]**, the first closed hyperbolic 3-manifold to appear in the literature.

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