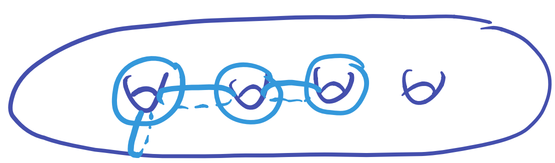
Johnson I

Thm $\mathcal{I}(S_g)$ is fin. gen. by Dehn twists for $g \geq 3$.

Basic strategy 1. List prospective generators $\{g_i\}$.
s.t. g_1 is a BP map of genus 1.
2. Show $\langle g_i \rangle \trianglelefteq \mathrm{Mod}(S_g)$.

This suffices since $\langle\langle g_i \rangle\rangle_{\mathrm{Mod}(S_g)} = \mathcal{I}(S_g)$.

Chains and BP maps.

A chain: 

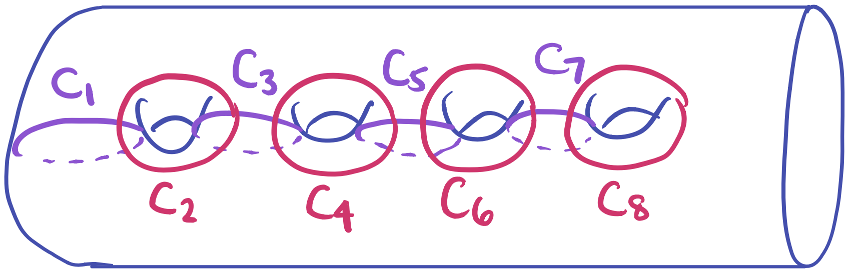
$\rightsquigarrow$ BP map.

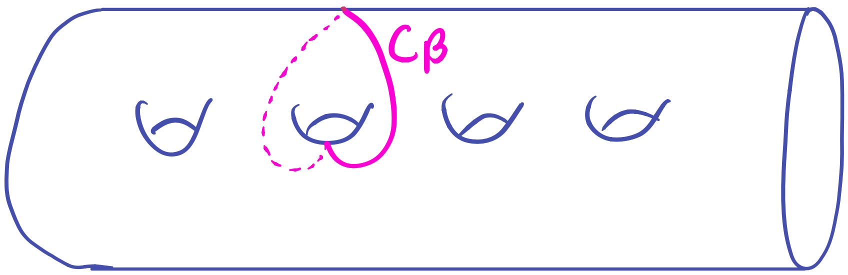
Given a chain, can resolve intersections to get another chain. Can also take subchains.

Given a chain $\mathrm{ch}(c_1, \ldots, c_n)$
$\mathrm{ch}(i_1 i_2 \cdots i_{k+1})$
denotes the chain you get by combining $c_{i_1}, \ldots, c_{i_2 - 1}$
$c_{i_2}, \ldots, c_{i_3 - 1}$ etc. dropping $c_{k+1}, \ldots, c_n$. "subchain"

Denote the BP-map $[i_1 i_2 \cdots i_{k+1}]$

Listing the Generators





Consider the chains:

$(c_1, \ldots, c_{2g})$ straight chain

$(c_\beta, c_5, \ldots, c_{2g})$ β-chain

Use same notation for subchains of β-chain:

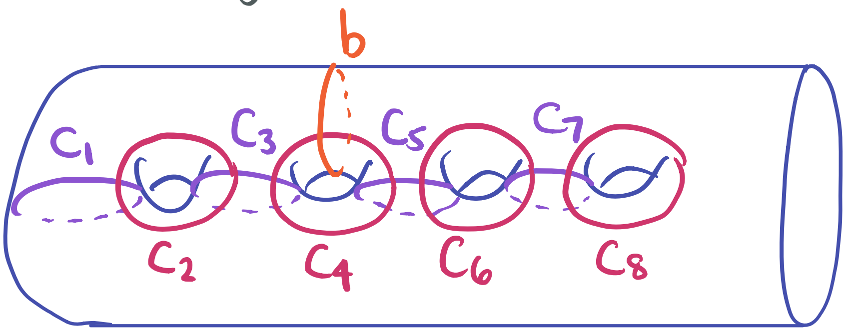
$(\beta i_1) = $ surger $c_\beta, c_5, \ldots, c_{i_1-1}$

Theorem. For $g \geq 3$ the odd subchain maps of straight chain & β-chain generate $\mathcal{I}(S_g^1)$

Since $\mathcal{I}(S_g^1) \twoheadrightarrow \mathcal{I}(S_g)$ this gives closed case as well.

SETUP.

Humpries generators:



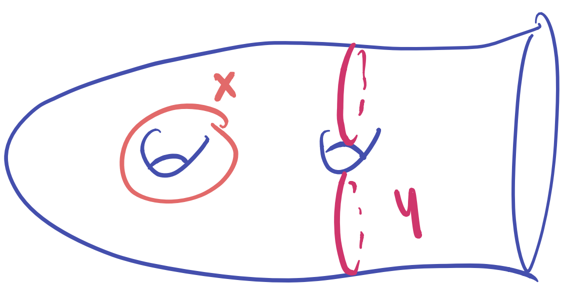
Let $J(S_g^1)$ & $J(S_g)$ denote groups gen by Johnson's generators.

As above, need to show

$$T_x * y = T_x y T_x^{-1} \in J(S_g)$$

$\forall$ $x \in$ Humphries set
$y \in$ Johnson set

In many cases $T_x * y$ equals y or is another Johnson gen:



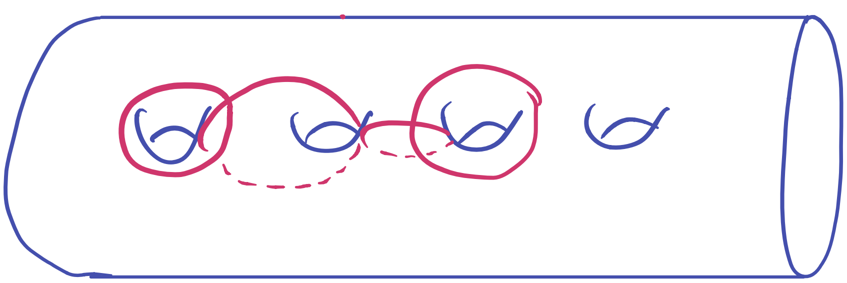
Lemma.

j in $\{i_1,\dots\}$	j+1 in $\{i_1,\dots\}$	$T_{c_j} * [i_1\ i_2 \cdots]$
✓	✓	$[i_1\ i_2 \cdots]$
✗	✗	$[i_1\ i_2 \cdots]$
i_m	✗	$[i_1 \cdots i_{m-1}\ i_m+1 \cdots]$
✗	i_m	$[i_1 \cdots i_{m-1}\ i_m-1 \cdots]$

The lemma completely characterizes commuting among straight Johnson & Humphries gens.

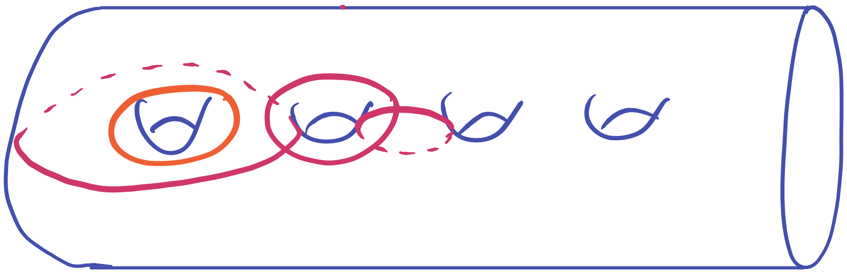
Pf (by examples)

Example $j=2$ $\quad [2\ 3\ 5\ 6]$



✓

Example $j=2$ $[1\ 4\ 5\ 6]$



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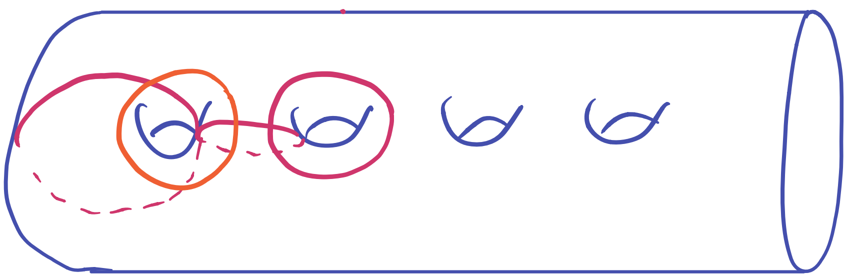
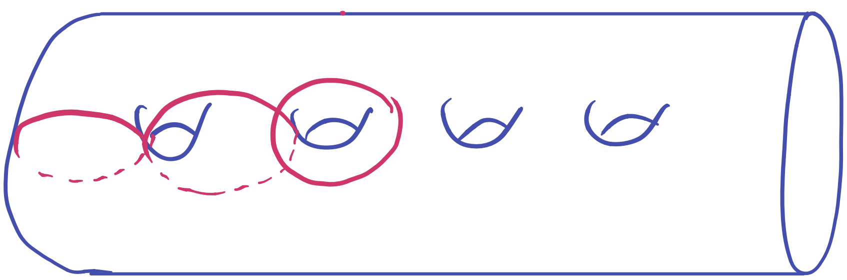
Example $j=2$ $[1\ 3\ 4\ 5]$



$\downarrow$

$[1\ 2\ 4\ 5]$



✓

Third case similar.

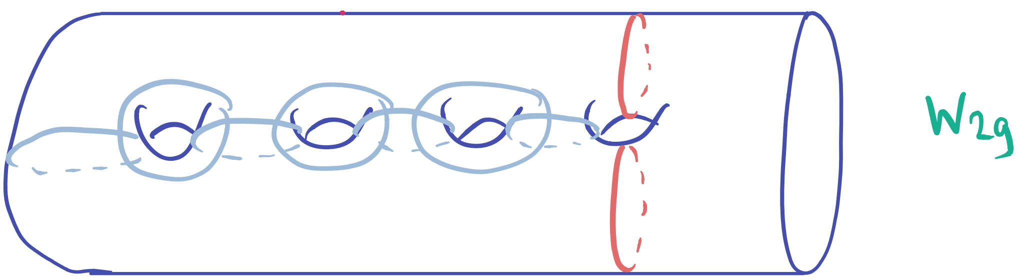
Generating the Kernel of $\mathcal{I}(S_g^1) \to \mathcal{I}(S_g)$

Define $W_i = [1\ 2\ 3 \cdots \hat{i} \cdots 2g+1]$ for $i \in \{1, \ldots, 2g+1\}$.

"maximal odd subchain maps"

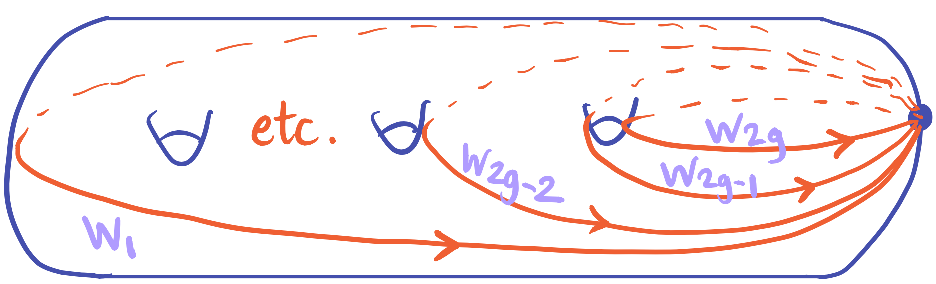
Fact. # of curves in a subchain is one less than # indices.

By change of coords, each lies in $\pi_1 UT(S_g)$



Lemma. $\pi_1 UT(S_g)$ is gen by $W_1, \ldots, W_{2g}, T_b * W_1$

Pf. The corresponding push maps are



tangentially framed.

These clearly generate π_1. Need to get the fiber of $\pi_1 UT \to \pi_1$.

Claim. $T_b * W_1 = W_4 W_3^{-1} W_2 T_\partial$ $\square$