Announcements Nov 22

next week.

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 5.6 & 6.1 due Tue @ midnight
- Office hrs: Tue 4-5 Teams
- Cumulative Final exam Tue Dec 14 6-8:50 pm on Teams.

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Chapter 6

Orthogonality

Where are we?

Spotify!

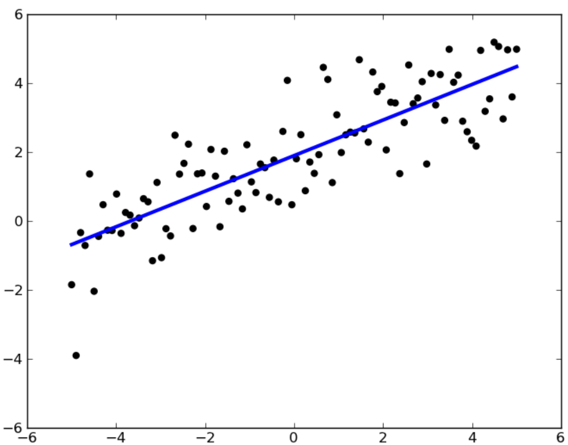
We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

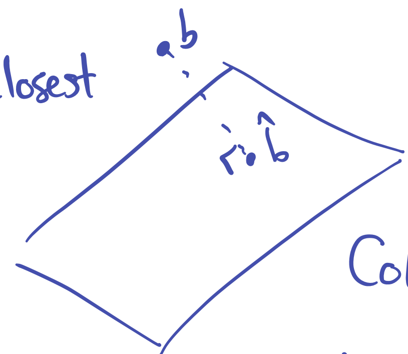
What if we can't solve $Ax = b$? How can we solve it as closely as possible?

$Ax = b$ has no soln

$\iff$ b not in Col(A)



Find $\hat{b}$ closest to b in Col(A)



Col(A)

Solve $Ax = \hat{b}$ instead.

The answer relies on orthogonality.

Section 6.1

Dot products and Orthogonality

Outline

- Dot products
- Length and distance
- Orthogonality

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$\begin{aligned} u \cdot v &= \sum_{i=1}^{n} u_i v_i \\ &= u_1 v_1 + \cdots + u_n v_n \\ &= u^T v \end{aligned}$$

Used when multiplying matrices.

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

$1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 5 + 10 + 18 = 33.$

Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \Leftrightarrow u = 0$

$(-1, -2, 3) \cdot (-1, -2, 3)$

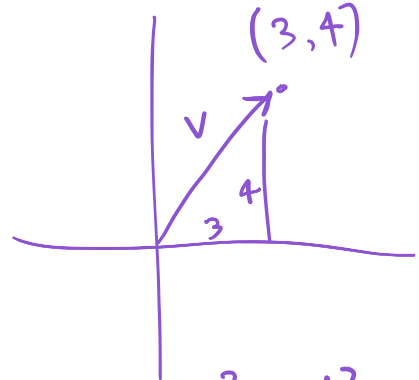
$= (-1)^2 + (-2)^2 + 3^2 \geq 0$

Length

Let v be a vector in $\mathbb{R}^n$

length

$$\|v\| = \sqrt{v \cdot v}$$
$$= \text{length of } v$$



Why? Pythagorean Theorem

$v \cdot v = 3^2 + 4^2 = 25$

$\sqrt{v \cdot v} = 5$

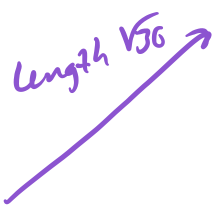
abs val length

Fact. $\|cv\| = |c|\|v\|$

v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$



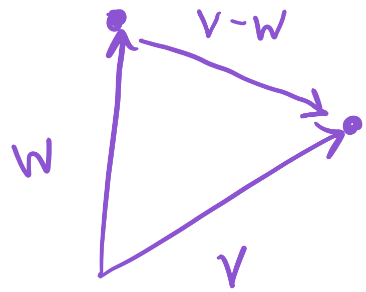
Scale so length is 1.

$$\|(1,2,3,4)\| = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{30}$$

$$\frac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Distance

The distance between v and w is the length of $v - w$ (or $w - v$!).



Note: $w - v = -(v - w)$

same length!

$v =$ $w =$

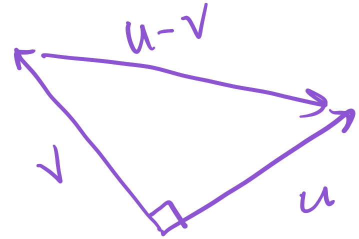
Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

$$v - w = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$$

$$\|v - w\| = \sqrt{0 + 3^2 + 4^2} = \sqrt{25} = 5$$

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$



Why? Pythagorean theorem again!

$$u \perp v \overset{\text{Pythag.}}{\Leftrightarrow} \|u\|^2 + \|v\|^2 = \|u - v\|^2 \qquad (u-v)\cdot(u-v)$$

$$\Leftrightarrow \cancel{u \cdot u} + \cancel{v \cdot v} = \cancel{u \cdot u} - 2u \cdot v + \cancel{v \cdot v}$$

$$\Leftrightarrow u \cdot v = 0$$

Problem. Find a nonzero vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

$$(1,2,3)\cdot(-1,-1,1) = 0$$

$$-1 - 2 + 3$$

Summary of Section 6.1

- $u \cdot v = \sum u_i v_i$
- $u \cdot u = \|u\|^2$ (length of u squared)
- The unit vector in the direction of v is $v/\|v\|$.
- The distance from u to v is $\|u - v\|$
- $u \cdot v = 0 \Leftrightarrow u \perp v$

Outline of Section 6.2

- Orthogonal complements
- Computing orthogonal complements

Orthogonal complements

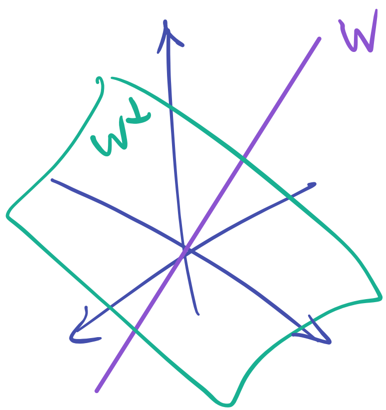
W = subspace of $\mathbb{R}^n$

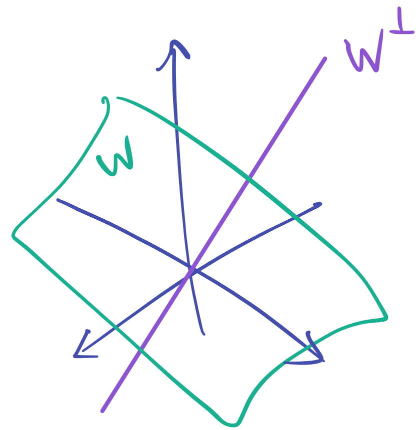
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

W perp

subspace

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?
What about the orthogonal complement of a plane in $\mathbb{R}^3$?





Demo Demo

Stefano:

$W = \mathbb{R}^n$ $\quad$ $W = \{0\}$

$W^\perp = \{0\}$ $\quad$ $W^\perp = \mathbb{R}^n$

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$ (it's a null space!)
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$ (rank-nullity theorem!)
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$

   k lin. eqns in n vars
   $x \cdot w_i = 0$.
5. The intersection of W and $W^\perp$ is $\{0\}$.

For items 1 and 3, which linear transformation do we use?

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1)\}$. Find the equation of the plane $W^\perp$. line.

Which $x = (x_1, x_2, x_3)$ are perpend. to $(1,1,-1)$?

$\mathrm{Nul}\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$

$$(x_1, x_2, x_3) \cdot (1,1,-1) = 0$$

$$x_1 + x_2 - x_3 = 0.$$

$W^\perp$ is the set of solns.

plane

Find a basis for $W^\perp$.

Vect param form.

1 pivot ✓

Orthogonal complements

Finding them

plane.

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find a system of equations describing the line $W^\perp$.

$$(x,y,z)\cdot(1,1,-1) = 0$$
$$(x,y,z)\cdot(-1,2,1) = 0$$

$$x + y - z = 0$$
$$-x + 2y + z = 0.$$

$$\mathrm{Nul}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

Find a basis for $W^\perp$.

VPF. 1 basis vector.

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^{\perp}$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

(VecParamForm)

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

See last 2 examples.

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

$W = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix}\right\}$

In other words:

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row}A)^\perp = \mathrm{Nul}\,A$$

e.g. $W = \mathrm{Row}\begin{pmatrix}1 & 1 & -1\\ -1 & 2 & 1\end{pmatrix}$

$W^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & -1\\ -1 & 2 & 1\end{pmatrix}$

$\begin{pmatrix}1 & 1 & -1\\ -1 & 2 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Geometry $\leftrightarrow$ Algebra

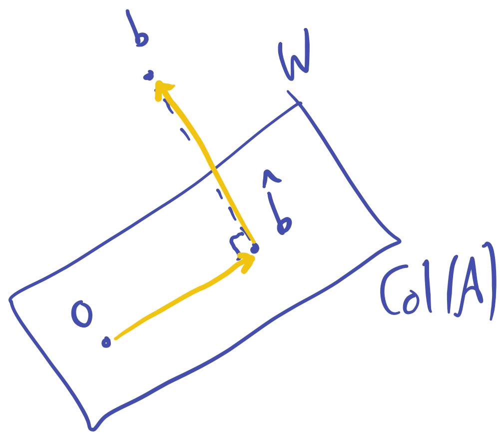
(The row space of A is the span of the rows of A.)

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.



Why?

Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$\boxed{v = v_W + v_{W^\perp}}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^\perp$, so they must both be equal to 0.

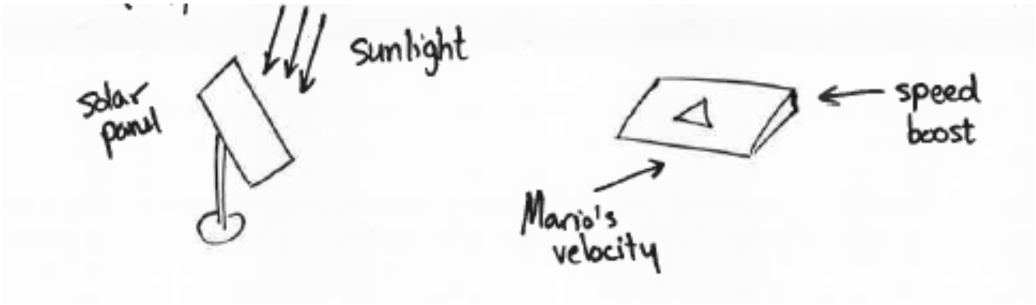
Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Summary of Section 6.2

- $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$
- Facts:
  1. $W^\perp$ is a subspace of $\mathbb{R}^n$
  2. $(W^\perp)^\perp = W$
  3. $\dim W + \dim W^\perp = n$
  4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
     $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
  5. The intersection of W and $W^\perp$ is $\{0\}$.
- To find $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find the null space.
- Every vector v can be written uniquely as $v = v_W + v_{W^\perp}$ with v_W in W and $v_{W^\perp}$ in $W^\perp$

Typical Exam Questions 6.2

- What is the dimension of $W^\perp$ if W is a line in $\mathbb{R}^{10}$?
- What is $W^\perp$ if W is the line $y = mx$ in $\mathbb{R}^2$?
- If W is the x-axis in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^\perp}$.
- If W is the line $y = x$ in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^\perp}$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^4$.
- What is the orthogonal complement of x_1x_2-plane in $\mathbb{R}^4$?

Section 6.3

Orthogonal projection

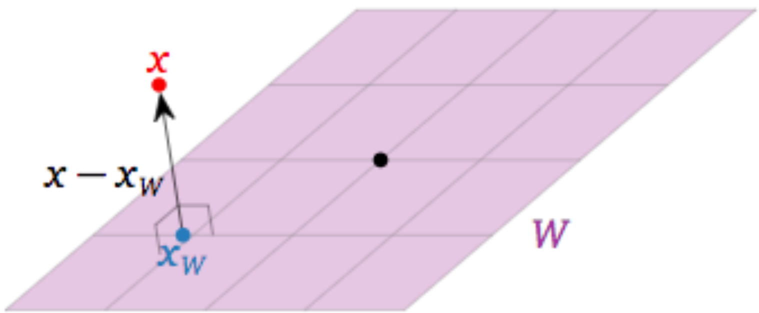
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections

Orthogonal Projections

Let b be a vector in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.

The orthogonal projection of b onto W the vector obtained by drawing a line segment from b to W that is perpendicular to W.



Fact. The following three things are all the same:

- The orthogonal projection of b onto W
- The vector b_W (the W-part of b)   algebra!
- The closest vector in W to b geometry!

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Why? Choose $\widehat{x}$ so that $A\widehat{x} = b_W$. We know $b - b_W = b - A\widehat{x}$ is in $W^\perp = \mathrm{Nul}(A^T)$ and so

$$0 = A^T(b - A\widehat{x}) = A^T b - A^T A\widehat{x}$$

$$\rightsquigarrow A^T A\widehat{x} = A^T b$$

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

What does the theorem give when $W = \mathrm{Span}\{u\}$ is a line?

Orthogonal Projection onto a line

Special case. Let $L = \mathrm{Span}\{u\}$. For any vector b in $\mathbb{R}^n$ we have:

$$b_L = \frac{u \cdot b}{u \cdot u} u$$

Find b_L and $b_{L^\perp}$ if $b = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.