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A SIMPLE PROOF OF ZARISKI'S LEMMA

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Abstract. We give a simple proof for Zariski's Lemma.
Keywords: Zariski's Lemma.
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1. The result

Our aim in this very short note is to show that the proof of the following well-known fundamental lemma of Zariski follows from an argument similar to the proof of the fact that the field of rational numbers $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-algebra.

Lemma 1.1 (Zariski's Lemma)**.** *Let L be a field extension of a field K. Assume that for some $\alpha_1, \ldots, \alpha_n$ in L, $R = K[\alpha_1, \ldots, \alpha_n]$ is a field. Then every α_i is algebraic over K.*

In particular, if K is algebraically closed, then $\alpha_i \in K$ for all i. This statement implies the so-called Hilbert's Weak Nullstellensatz, which states that when K is an algebraically closed field, every maximal ideal M of the polynomial ring $K[x_1, \ldots, x_n]$ is of the form $M = (x_1 - a_1, \ldots, x_n - a_n)$ with $a_i \in K$ for all i.

Usually the proof of Zariski's Lemma depends on two technical lemmas due to Artin-Tate and Zariski, see [3, Proposition 3.2, and its subsequent comment]. Some textbooks on elementary agebraic geometry employ the Noether normalization lemma to prove Zariski's Lemma (see, e.g., [2, Theorem 1.15] and [5], (also see [1] and [4]).

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Before giving the proof of the lemma, we recall the following two well-known facts.

Fact 1. If a field F is integral over a subdomain D, then D is a field.

Fact 2. If D is any principal ideal domain (or just a UFD) with infinitely many (non-associate) prime elements, then its field of fractions is not a finitely generated D-algebra.

Proof of the Lemma. We use induction on n for arbitrary fields K and L. For $n = 1$ the assertion is clear. Let us assume that $n > 1$ and the lemma is true for positive integers less than n. Now to show that it is true for n, one may assume that one of α_i's, say α_1, is not algebraic over K. Since $K[\alpha_1, \dots, \alpha_n] = K(\alpha_1)[\alpha_2, \dots, \alpha_n]$ is a field, by induction hypothesis, we infer that $\alpha_2, \dots, \alpha_n$ are all algebraic over $K(\alpha_1)$. This implies that there are polynomials $f_2(\alpha_1), \dots, f_n(\alpha_1) \in K[\alpha_1]$ such that all α_i's are integral over the domain $A = K[\alpha_1][1/f_2(\alpha_1), \dots, 1/f_n(\alpha_1)]$. Since R is integral over A, by Fact 1, A is a field. Consequently, $A = K(\alpha_1)$, which contradicts Fact 2.

$\square$

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