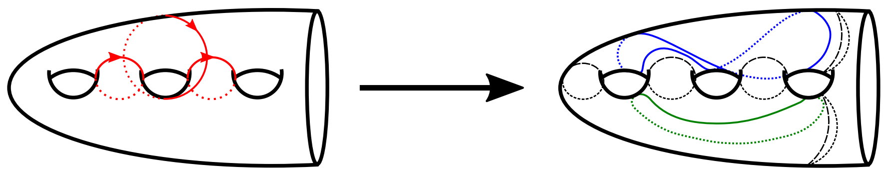
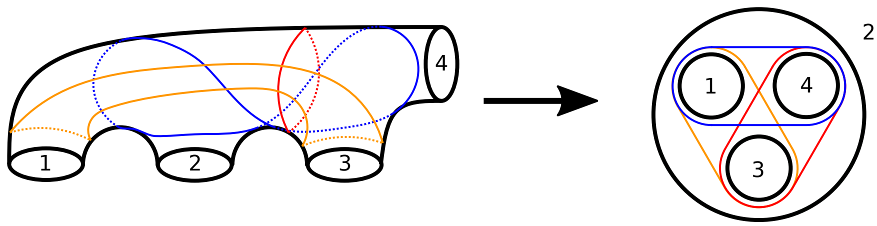
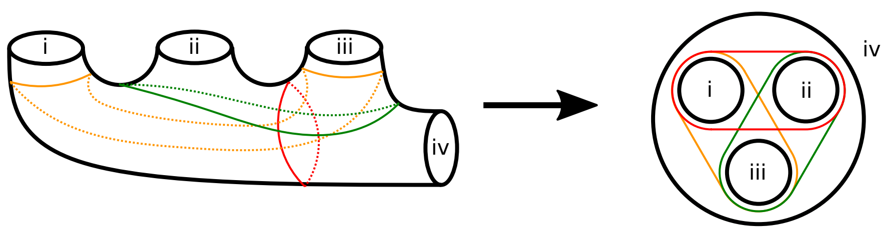
We will show that $T_b * [3\ 4\ 5\ 6]$ and $T_b^{-1} * [3\ 4\ 5\ 6]$, as described by Johnson, are themselves generated by Johnson maps. This is essentially Johnson's computation, merely clarified.

In the figure below, we have the chains obtained by applying T_b and T_b^{-1}, respectively, to the chain $[3\ 4\ 5\ 6]$. On the right, we have curves which bound a regular neighborhood of the chain on the left. $T_b * [3\ 4\ 5\ 6]$ and $T_b^{-1} * [3\ 4\ 5\ 6]$ is given by $T_B T_G^{-1}$, where B and G are the blue and green curves, respectively.



Using lantern relations on the indicated subsurfaces, we will realize $T_b * [3\ 4\ 5\ 6]$ as a product of simpler twists.





For the first subsurface, let ρ, ω be the red and orange curves, respectively. In the second subsurface, let ρ', ω' be the red and orange curves, respectively. By the lantern relations, we have $T_\rho T_\omega T_B = T_1 T_2 T_3 T_4$ and $T_G T_{\omega'} T_{\rho'} = T_i T_{ii} T_{iii} T_{iv}$, so $T_{\rho'}^{-1} T_{\omega'}^{-1} T_G^{-1} = T_{iv}^{-1} T_{iii}^{-1} T_{ii}^{-1} T_i^{-1}$. Note that T_B, T_ρ, T_ω commute with $T_G, T_{\rho'}, T_{\omega'}$ and $T_1, T_2, T_3, T_4, T_i, T_{ii}, T_{iii}, T_{iv}$ all commute by disjointness. Thus we have

$$(T_\rho T_\omega T_B)(T_{\rho'}^{-1} T_{\omega'}^{-1} T_G^{-1}) = (T_1 T_2 T_3 T_4)(T_{iv}^{-1} T_{iii}^{-1} T_{ii}^{-1} T_i^{-1}),$$

so

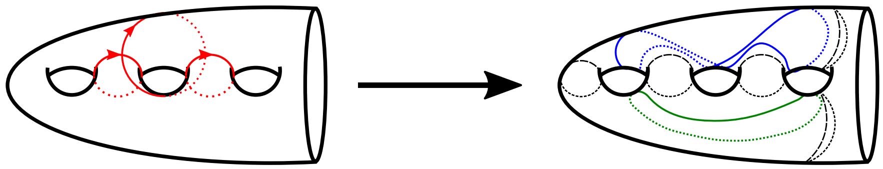
$$\begin{aligned}(T_\rho T_{\rho'}^{-1})(T_\omega T_{\omega'}^{-1})(T_B T_G^{-1}) &= (T_1 T_i^{-1})(T_2 T_{ii}^{-1})(T_3 T_{iii}^{-1})(T_4 T_{iv}^{-1}) \\ &= T_4 T_{iv}^{-1}.\end{aligned}$$

Next, note that ρ and ρ' bound a regular neighborhood of the chain giving $[1\ 2\ 3\ 4]$, and hence $T_\rho T_{\rho'}^{-1} = [1\ 2\ 3\ 4]$. Likewise, ω and ω' bound a regular neighborhood of the chain giving $[1\ 2\ 5\ 6]$, so $T_\omega T_{\omega'}^{-1} = [1\ 2\ 5\ 6]$. Finally, the boundary curves 4 and iv bound a regular neighborhood of the chain giving $[1\ 2\ 3\ 4\ 5\ 6]$, so $T_4 T_{iv}^{-1} = [1\ 2\ 3\ 4\ 5\ 6]$. Thus

$$[1\ 2\ 3\ 4]\ [1\ 2\ 5\ 6]\ T_b * [3\ 4\ 5\ 6] = [1\ 2\ 3\ 4\ 5\ 6].$$

Since $[1\ 2\ 3\ 4], [1\ 2\ 5\ 6]$, and $[1\ 2\ 3\ 4\ 5\ 6]$ are in Johnson's set of generators, it follows that $T_b * [3\ 4\ 5\ 6]$ is as well.

Finally, we can circumvent Johnson's argument for showing that $T_b^{-1} * [3\ 4\ 5\ 6]$ is in Johnson's set of generators. Indeed, the following figure depicts the curves B', G' with $T_b^{-1}[3\ 4\ 5\ 6] = T_{B'} T_{G'}^{-1}$:



Here, we note that these curves can be obtained by reflecting B and G across the vertical plane of this page. If we thus choose our auxiliary curves by reflecting $\rho, \rho', \omega, \omega'$ across the vertical plane, utilizing the given lantern relations, and reflecting back across the vertical plane, then our previous computations show that $T_b^{-1}[3\ 4\ 5\ 6]$ is indeed an element of Johnson's generating set.