Scores: 1 2 3 4 5 6 7 8 9 10 Name Prof. M

Section K___

Mathematics 2602

Midterm 2

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1. Compute the following.

$100000001 \times 10000000000000000001 \mod 5$

$\equiv 1 \times 1 \mod 5$

$\equiv 1 \mod 5$

$17000000000001^{57} \mod 17$

$\equiv 1^{57} \mod 17$

$\equiv 1 \mod 17$

$15^{16} \mod 17$

$\equiv 1 \mod 17$ (Fermat)

2. Solve for n:

$$3n \equiv 4 \mod 7$$

$$5 \cdot 3n \equiv 5 \cdot 4 \mod 7$$

$$n \equiv 6 \mod 7$$

Find the smallest natural number n so that:

$$n \equiv 1 \mod 5$$
$$n \equiv 3 \mod 11$$

36

What is the second smallest n in the previous problem?

$36 + 5 \cdot 11$ (Chinese Remainder)

3. Suppose that $a_0 = 3$ and $a_n = a_{n-1} + 7$.

What is a_{10}?

$$a_{10} = 3 + 10 \cdot 7 = 73$$

What is a_n?

$$a_n = 3 + 7n$$

Compute the following:

$$\sum_{i=3}^{42} a_i$$

Do not simplify your answer.

$$S = \quad 3 \quad 10 \quad 17 \quad \ldots \quad 294$$

$$S = \quad 294 \quad 287 \quad 280 \quad \ldots \quad 3$$

$$2S = 297 \cdot 40$$

$$S = 297 \cdot 20$$

4. Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 16$.

$$x^2 - 8x + 16 = 0$$

$$\rightsquigarrow (x-4)^2 = 0$$

$$x = 4, 4$$

$$a_n = c4^n + dn4^n$$

$$1 = a_0 = c$$

$$16 = a_1 = 4 + d \cdot 4 \rightsquigarrow d = 3$$

$$a_n = 4^n + 3n4^n$$

5. Solve the recurrence relation

$$a_n = 5a_{n-1} + 6^n$$

with initial condition $a_0 = 7$.

$$q_n = c5^n$$

$$p_n = a6^n$$

$$a6^n = 5a6^{n-1} + 6^n$$

$$6a = 5a + 6$$

$$a = 6$$

$$\rightsquigarrow p_n = 6^{n+1}$$

$$a_n = c5^n + 6^{n+1}$$

$$a_0 = 7 = c + 6$$

$$\rightsquigarrow c = 1$$

$$a_n = 5^n + 6^{n+1}$$

6. For each of the following functions $f(n)$, find a function $g(n)$ below so that $f(n) \asymp g(n)$.

$$f(n) = \frac{n^7}{n^7 - n^5 + n^3 - n + 1}$$

E

$f(n) = 2n^3 + n\log(n^3)$

H

$f(n) = 2^n + n!$

A

$f(n) = 3^n + 5^n$

J

A.	$g(n) = n!$	F.	$g(n) = 3^n$
B.	$g(n) = \log(n)$	G.	$g(n) = 1$
C.	$g(n) = 2^n$	H.	$g(n) = n^3$
D.	$g(n) = n^7$	I.	$g(n) = n\log(n)$
E.	$g(n) = n$	J.	$g(n) = 5^n$

8. Recall Horner's algorithm for evaluating a poylnomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

at a real number $x = a$:

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Horner's algorithm (input: polynomial p(n),  real number a)

b=0;
for (i=0 to n) {
   b = a_{n-i} + b*a;
}
return b;
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For the polynomial $p(x) = 3x^2 - 7x - 9$ and the value $a = 4$, what values of b do we obtain?

i=0: 3

i=1: $-7 + 4 \cdot 3 = 5$

i=2: $-9 + 4 \cdot 5 = 11$

How many multipliations *and* additions are required to run Horner's algorithm on a polynomial of degree n?

$2(n+1)$

What is the smallest k so that Horner's algorithm is $\mathcal{O}(n^k)$?

$k = 1$

9. Use mathematical induction to prove that

$$1 + 3 + \cdots + (2n-1) = n^2.$$

Base case: $n = 1$

$1 = 1^2$ ✓

Ind. hyp: $1 + 3 + \cdots + (2k-1) = k^2$

Ind step: $1 + 3 + \cdots + (2(k+1)-1)$

$= [1 + 3 + \cdots + (2k-1)] + (2k+1)$

$= k^2 + 2k + 1$

$= (k+1)^2$ ✓

10. Use mathematical induction to prove that $5^n - 3^n$ is even.

Base case $n = 0$:

$$5^0 - 3^0 = 0 \checkmark$$

Ind. hyp. $5^k - 3^k$ even.

Ind. step. $5^{k+1} - 3^{k+1}$

$$= 5 \cdot 5^k - 3 \cdot 3^k$$

$$= 5 \cdot 5^k - 5 \cdot 3^k + 2 \cdot 3^k$$

$$= 5(5^k - 3^k) + 2 \cdot 3^k$$

$$= \text{even} + \text{even} = \text{even} \checkmark$$