Torelli groups

$\psi : \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}\mathbb{Z}$

$\mathcal{I}(S_g) = \ker \psi.$

- $\mathcal{I}(S_g)$ hard/non-linear part of MCG.

- All $\mathbb{Z}HS^3$ are:

$$H_g \coprod_{\varphi} H_g$$

$\varphi \in \mathcal{I}(S_g)$

- $\mathcal{I}(S_g) = \pi_1(\text{Torelli space})$

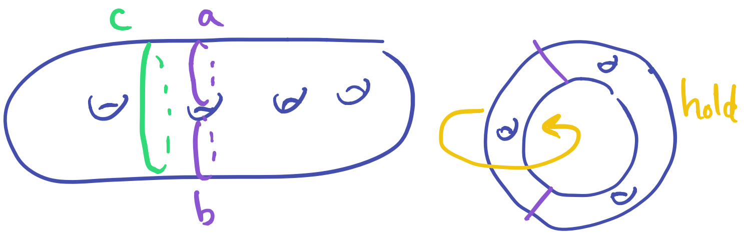
Space of Riem. surf's with homology framings

Examples of Elements

① T_c c sep.

② Bounding pair map

$T_a T_b^{-1}$ $[a]=[b]$
$i(a,b)=0.$



③ Fake bounding pair maps

$T_a T_b^{-1}$ $[a]=[b].$

④ $[T_a, T_b]$ $\hat{i}(a,b)=0.$

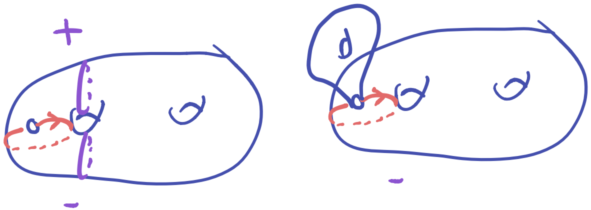
special case of 3

$T_a(T_b T_a^{-1} T_b^{-1}) = T_a T_{T_b(a)}^{-1}$

hom. to a.

⑤ Boundary ~~Point~~/handle pushes

Special case of 3



Generators

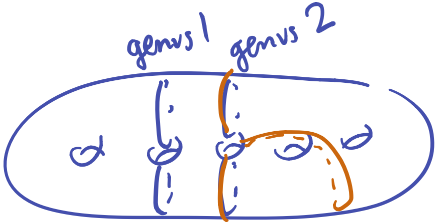
$$1 \to \mathcal{I}(S_g) \to \mathrm{Mod}(S_g) \xrightarrow{\psi} Sp_{2g}\mathbb{Z} \to 1$$

generators ← relators

Birman: presentation for Sp.

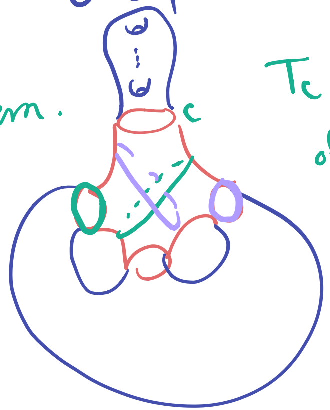
Powell ⇝ BP's & Sep. twists

Birman: Do these generate?



Johnson: ① Sep twists not needed

Lantern.

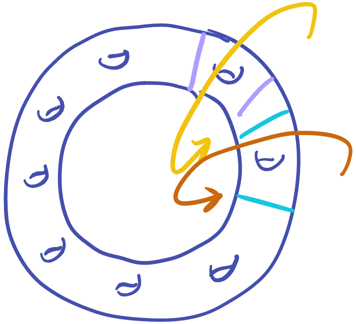


T_c = product of 3 BPs.

② Only BPs of genus 1 are needed.

So:

$\mathcal{I}(S_g)$ = normal closure in $\mathrm{Mod}(S_g)$ of a single BP of genus 1.



Johnson I: Finite generation.

Thm $g \geq 3$ $\mathcal{I}(S_g)$ f.g. by BPs of genus 1.

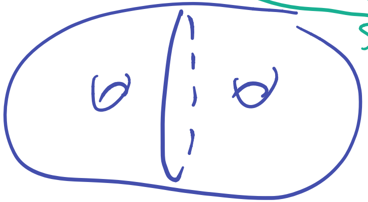
Pf idea. List $O(2^g)$ BPs $\{f_i\}$

Check $\langle f_i \rangle \trianglelefteq \mathrm{Mod}(S_g)$

$\Rightarrow \langle f_i \rangle = \mathcal{I}(S_g)$

$$\boxed{\begin{array}{c}\text{So:} \\ \mathcal{I}(S_g) = \text{normal closure in } \mathrm{Mod}(S_g) \\ \text{of a single BP of genus 1.}\end{array}}$$

$T_c T_d T_c^{-1} = T_{T_c(d)}$



Same H_1 splitting

Mess $\mathcal{I}(S_2) \cong F_\infty$

gen set $\longleftrightarrow$ H_1 splittings

Open Q. Explicit gen set.

Major Open Q. Is $\mathcal{I}(S_g)$ fin pres?

$H_2(G)$ ∞ gen $\Rightarrow$ G not f.p.

Johnson Homomorphism

$\tau : \mathcal{I}(S_g) \longrightarrow \Lambda^3 H$

$H = H_1(S_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$

Issue: $\mathcal{I}(S_g)$ acts triv. on

$H = \pi/[\pi,\pi]$ $\pi = \pi_1(S_g)$

Remedy: Look at action of $\mathcal{I}(S_g)$ on

$\pi/[\pi,[\pi,\pi]]$ 2-step nilpotent "like abelian"

$\Lambda^3 G = \{\text{formal sums of } g_1 \wedge g_2 \wedge g_3\}/\sim$

G abel. $a \wedge b \wedge c = -b \wedge a \wedge c \Rightarrow a \wedge a \wedge b = 0.$

$(a+a') \wedge b \wedge c = a \wedge b \wedge c + a' \wedge b \wedge c.$

e.g. $H^k(T^n) = \Lambda^k \mathbb{Z}^n$

Lower central series of G

$G_1 = G$

$G_2 = [G,G]$

$G_3 = [G,[G,G]]$

$G_4 = [G,[G,[G,G]]]$

Probe G by understanding G/G_k.

Johnson Homomorphism

$\tau : \mathcal{I}(S_g) \twoheadrightarrow \Lambda^3 H$

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Issue: $\mathcal{I}(S_g)$ acts triv. on

$H = \pi/[\pi,\pi] \quad \pi = \pi_1(S_g)$

Remedy: Look at action of $\mathcal{I}(S_g)$ on

$\pi/[\pi,[\pi,\pi]]$ 2-step nilpotent "like abelian"

Consider

$$1 \to \overset{N}{[\pi,\pi]/[\pi,[\pi,\pi]]} \to \overset{\overset{f(\tilde{x})}{\tilde{x}}}{\overset{E}{\pi/[\pi,[\pi,\pi]]}} \to \overset{x}{\overset{H}{\pi/[\pi,\pi]}} \to 1$$

Construct $\tau : \mathcal{I}(S_g) \to \underbrace{\mathrm{Hom}(H,N)}_{\text{abelian.}}$

Given $f \in \mathcal{I}(S_g)$

$x \in H$

need $\tau(f)(x) \in N$.

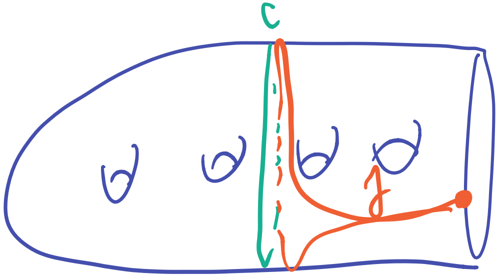
Lift x to $\tilde{x} \in E$.

$\rightsquigarrow f(\tilde{x})\,\tilde{x}^{-1} \in N$.

Image $\cong \Lambda^3 H$.

Computations

$$\boxed{\tau(T_c)=0} \quad c \text{ sep.}$$

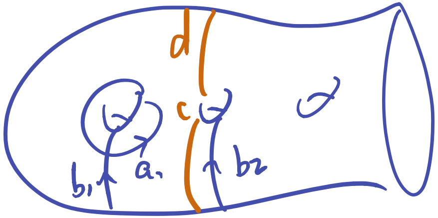


$T_c \longleftrightarrow$ conj. by $\gamma \in [\pi,\pi]$

$$f(\tilde{x})\ \tilde{x}^{-1}$$

$$\gamma\tilde{x}\gamma^{-1}\tilde{x}^{-1} = [y,z]\,\tilde{x}\,[y,z]^{-1}\tilde{x}^{-1} \in [\pi,[\pi,\pi]].$$

$$\tau(T_c T_d^{-1}) = a_1\wedge b_1\wedge b_2 \neq 0.$$



$\Longrightarrow$ $\mathcal{I}(S_g)$ not gen by sep twists.

Topological interpretation #1

$\alpha : \pi_1(S_g) \to \mathbb{Z}^{2g}$ abelianization.

$\rightsquigarrow A : (S_g, *) \to (T^{2g}, 0)$

Consider $A \circ \psi$ $[\psi] \in \mathcal{I}(S_g)$.

Since $[\psi] \in \mathcal{I}(S_g)$,

$$A \sim A \circ \psi.$$

The homotopy is a 3-man.

in $T^{2g} \rightsquigarrow \Lambda^3 H$.

$\boxed{\text{An elt of } \Lambda^3 H \text{ is a sum of 3-}\text{~~tori~~ mnflds in } T^{2g}}$

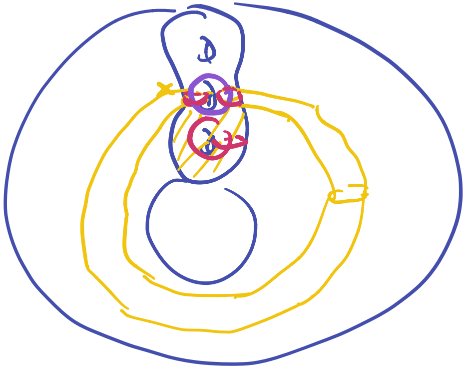
Top. interp #2

Given $f \in \mathcal{I}(S_g)$ need elt of $\Lambda^3 H$ or $(\Lambda^3 H)^* = \{\Lambda^3 H \to \mathbb{Z}\}$

$5\, \underline{x \wedge y \wedge z} + 7\, \underline{a \wedge b \wedge c}$

Given $f \in \mathcal{I}(S_g)$, $x \wedge y \wedge z \in \Lambda^3 H$ need a number:

Construct mapping torus M_f



$x \rightsquigarrow$ surface Σ_x in M_f

The desired number is

$$\hat{\iota}(\Sigma_x, \Sigma_y, \Sigma_z)$$