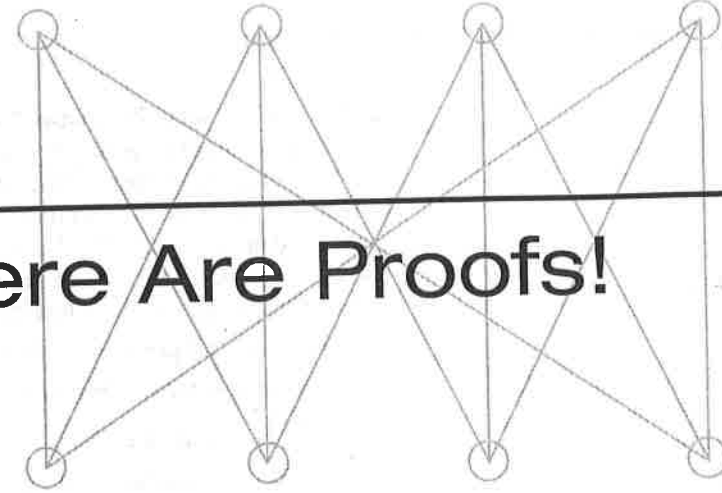
0

Yes, There Are Proofs!



"How many dots are there on a pair of dice?" The question once popped out of the box in a game of trivia in which one of us was a player. A long pause and much consternation followed the question. After the correct answer was finally given, the author (a bit smugly) pointed out that the answer "of course" was 6×7, twice the sum of the integers from 1 to 6. "This is because," he declared, "the sum of the integers from 1 to n is $\frac{1}{2}n(n+1)$, so twice this sum is $n(n+1)$ and, in this case, $n = 6$."

"What?" asked one of the players.

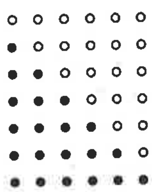


Figure 0.1

At this point, the game was delayed for a considerable period while the author found pencil and paper and made a picture like that in Fig. 0.1. If we imagine the dots on one die to be solid and on the other hollow, then the sum of the dots on two dice is just the number of dots in this picture. There are seven rows of six dots each—42 dots in all. What is more, a similar picture could be drawn for seven-sided dice showing that

$$2(1+2+3+4+5+6+7) = 7 \times 8 = 56$$

and, generally,

$$(*) \qquad 2(1+2+3+\cdots+n) = n \times (n+1).$$

Sadly, that last paragraph is fictitious. Everybody was interested in, and most experimented with, the general observation in equation $(*)$, but nobody (except an author) cared why. Everybody wanted to resume the game!

Pause 1 What is the sum $1+2+3+\cdots+100$ of the integers from 1 to 100? ▮

"Are there proofs?" This is one of the first questions students ask when they enter a course in analysis or algebra. Young children continually ask "why" but, for whatever reason, as they grow older, most people only want the facts. We take the view that intellectual curiosity is a hallmark of advanced learning and that the ability to reason logically is an increasingly sought after commodity in the world today. Since sound logical arguments are the essence of mathematics, the subject provides a marvelous training ground for the mind. The expectation that a math course will sharpen the powers of reason and provide instruction in clear thinking surely accounts for the prominence of mathematics in so many university programs today. So yes, **proofs—reasons** or **convincing arguments** probably sound less intimidating—will form an integral part of the discussions in this book.

In a scientific context, the term *statement* means an ordinary English statement of fact (subject, verb, and predicate in that order) that can be assigned a *truth value*; that is, it can be classified as being either true or false. We occasionally say *mathematical* statement to emphasize that a statement must have this characteristic property of being true or false. The following are all mathematical statements:

There are 168 primes less than 1000.

Seventeen is an even number.

$\sqrt{3}^{\sqrt{3}}$ is a rational number.

Zero is not negative.

Each statement is certainly either true or false. (It is not necessary to know which.) On the other hand, the following are not mathematical statements:

What are irrational numbers?

Suppose every positive integer is the sum of three squares.

The first is a question and the second a conditional; it would not make sense to classify either as true or false.

0.1 Compound Statements

"And" and "Or"

A *compound statement* is a statement formed from two other statements in one of several ways, for example, by linking them with "and" or "or." Consider

$9 = 3^2$ and $3.14 < \pi$.

This is a compound statement formed from the simpler statements "$9 = 3^2$" and "$3.14 < \pi$." How does the truth of an "and" compound statement depend on the truth of its parts? The rule is

> "p and q" is true if both p and q are true; it is false if either p is false or q is false.

Thus, "$-2^2 = -4$ and $5 < 100$" is true, while "$2^2 + 3^2 = 4^2$ and $3.14 < \pi$" is false.

In the context of mathematics, just as in everyday English, one can build a compound statement by inserting the word "or" between two other statements. In everyday English, "or" can be a bit problematic because sometimes it is used in an inclusive sense, sometimes in an exclusive sense, and sometimes ambiguously, leaving the listener unsure about just what was intended. We illustrate with three sentences.

"To get into that college, you have to have a high school diploma or be over 25." (Both options are allowed.)

"That man is wanted dead or alive." (Here both options are quite impossible.)

"I am positive that either blue or white is in that team's logo." (Might there be both?)

Since mathematics does not tolerate ambiguities, we must decide precisely what "or" should mean. The decision is to make "or" inclusive: "or" always includes the possibility of both.

> "p or q" is true if p is true or q is true or both p and q are true; it is false only when both p and q are false.

Thus,

"$7 + 5 = 12$ or 571 is the 125th prime" and "25 is less than or equal to 25"

are both true sentences, while

"5 is an even number or $\sqrt{8} > 3$"

is false.

Implication

Many mathematical statements are *implications*; that is, statements of the form "p implies q," where p and q are statements called, respectively, the *hypothesis* and *conclusion*. The symbol $\rightarrow$ is read *implies*, so

Statement 1: "2 is an even integer $\rightarrow$ 4 is an even integer"

is read "2 is an even integer implies 4 is an even integer."

In Statement 1, "2 is an even number" is the hypothesis and "4 is an even number" is the conclusion.

Implications often appear without the explicit use of the word *implies*. To some ears, Statement 1 might sound better as

"If 2 is an even number, then 4 is an even number."

or

"4 is an even number only if 2 is an even number."

Whatever wording is used, common sense tells us that this implication is true.

Under what conditions will an implication be false? Suppose your parents tell you

Statement 2: If it is sunny tomorrow, you may go swimming.

If it is sunny, but you are not allowed to go swimming, then clearly your parents have said something that is false. However, if it rains tomorrow and you are not allowed to go swimming, it would be unreasonable to accuse them of breaking their word. This example illustrates an important principle.

> The implication "$p \rightarrow q$" is false only when the hypothesis p is true and the conclusion q is false. In all other situations, it is true.

In particular, Statement 1 is true since both the hypothesis "2 is an even number" and the conclusion "4 is an even number" are true. Note, however, that an implication is true whenever the hypothesis is false (no matter whether the conclusion is true or false). For example, if it were to rain tomorrow, the implication contained in Statement 2 is true because the hypothesis is false. For the same reason, each of the following implications is true.

If -1 is a positive number, then $2 + 2 = 5$.

If -1 is a positive number, then $2 + 2 = 4$.

Pause 2 Think about the implication, "If $4^2 = 16$, then $-1^2 = 1$." Is it true or false? ∎

The Converse of an Implication

The *converse* of the implication $p \to q$ is the implication $q \to p$. For example, the converse of Statement 1 is

"If 4 is an even number, then 2 is an even number."

Pause 3

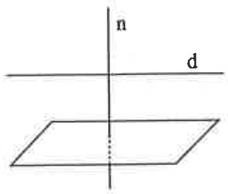
Write down the converse of the implication given in PAUSE 2. Is this true or false? ▮

Pause 4

A student was once asked to show that a certain line (in 3-space) was parallel to a certain plane.

Julie answered

If a line d is parallel to a plane and n is a line perpendicular to the plane, then d and n must be perpendicular,



and she proceeded to establish (correctly) that d was perpendicular to n. Was Julie's logic correct? Comment. ▮

Double Implication

Another compound statement that we will use is the double implication $p \leftrightarrow q$, read "*p if and only if q*." As the notation suggests, the statement "$p \leftrightarrow q$" is simply a convenient way to express

"$p \to q$" and "$p \leftarrow q$."

(We would be more likely to write "$q \to p$" than "$p \leftarrow q$.")

Putting together earlier observations, we conclude that

> The double implication "$p \leftrightarrow q$" is true if p and q have the same truth values; it is false if p and q have different truth values.

For example, the statement

"2 is an even number $\leftrightarrow$ 4 is an even number"

is true since both "2 is an even number" and "4 is an even number" are true. However,

"2 is an even number if and only if 5 is an even number"

is false because one side is true while the other is false.

Pause 5

Determine whether each of the following double implications is true or false.

(a) $4^2 = 16 \leftrightarrow -1^2 = -1$.
(b) $4^2 = 16$ if and only if $(-1)^2 = -1$.
(c) $4^2 = 15$ if and only if $-1^2 = -1$.
(d) $4^2 = 15 \leftrightarrow (-1)^2 = -1$. ▮

Negation

The *negation* of the statement p is the statement that asserts that p is not true. We denote the negation of p by "$\neg p$" and say "not p." The negation of "x equals 4" is the statement "x does not equal 4." In mathematical writing, a slash (/) through a symbol is used to express the negation of that symbol. So, for instance, $\neq$ means "not equal." Thus, the negation of "$x = 4$" is "$x \neq 4$." In succeeding chapters, we shall meet other symbols like $\in$, $\subseteq$, and $|$, each of which is negated with a slash, $\notin$, $\not\subseteq$, $\nmid$.

Some rules for forming negations are a bit complicated because it is not enough just to say "not p": We must also understand what is being said! To begin, we suggest that the negation of p be expressed as

"It is not the case that p."

Then we should think for a minute or so about precisely what this means. For example, the negation of "25 is a perfect square" is the statement "It is not the case that 25 is a perfect square," which surely means "25 is not a perfect square." To obtain the negation of

"$n < 10$ or n is odd,"

we begin

"It is not the case that $n < 10$ or n is odd."

A little reflection suggests that this is the same as the more revealing "$n \geq 10$ and n is even."

The negation of an "or" statement is always an "and" statement and the negation of an "and" is always an "or." The precise rules for expressing the negation of compound statements formed with "and" and "or" are due to Augustus De Morgan (whose name we shall see again in Section 1.2).

> The negation of "p and q" is the assertion "$\neg p$ or $\neg q$."
>
> The negation of "p or q" is the assertion "$\neg p$ and $\neg q$."

For example, the negation of "$a^2 + b^2 = c^2$ and $a > 0$" is "Either $a^2 + b^2 \neq c^2$ or $a \leq 0$." The negation of "$x + y = 6$ or $2x + 3y < 7$" is "$x + y \neq 6$ and $2x + 3y \geq 7$."

Pause 6 What is the negation of the implication $p \to q$? ∎

The Contrapositive

The *contrapositive* of the implication "$p \to q$" is the implication "$(\neg q) \to (\neg p)$." For example, the contrapositive of

"If x is an even number, then $x^2 + 3x$ is an even number"

is

"If $x^2 + 3x$ is an odd number, then x is an odd number."

Pause 7 Write down the contrapositive of the implications in PAUSES 2 and 3. In each case, state whether the contrapositive is true or false. How do these truth values compare, respectively, with those of the implications in these Pauses? ∎

Quantifiers

The expressions *there exists* and *for all*, which quantify statements, figure prominently in mathematics. The universal quantifier *for all* (and equivalent expressions such as *for every*, *for any*, and *for each*) says, for example, that a statement is true *for all* integers or *for all* polynomials or *for all* elements of a certain type. The following statements illustrate its use. (Notice how it can be disguised; in particular, note that "for any" and "all" are synonymous with "for all.")

$x^2 + x + 1 > 0$ for all real numbers x.

All polynomials are continuous functions.

For any positive integer n, $2(1+2+3+\cdots+n) = n \times (n+1)$.

$(AB)C = A(BC)$ for all square matrices A, B, and C.

Pause 8

Rewrite "All positive real numbers have real square roots," making explicit use of a universal quantifier. ▮

The existential quantifier *there exists* stipulates the existence of a single element for which a statement is true. Here are some assertions in which it is employed.

There exists a smallest positive integer.

Two sets may have no element in common.

Some polynomials have no real zeros.

Again, we draw attention to the fact that the ideas discussed in this chapter can arise in subtle ways. Did you notice the implicit use of the existential quantifier in the second of the preceding statements?

"There exists a set A and a set B such that A and B have no element in common."

Pause 9

Rewrite "Some polynomials have no real zeros" making use of the existential quantifier. ▮

Here are some statements that employ both types of quantifiers.

There exists a matrix 0 with the property that $A+0 = 0+A$ for all matrices A.

For any real number x, there exists an integer n such that $n \leq x < n+1$.

Every positive integer is the product of primes.

Every nonempty set of positive integers has a smallest element.

To negate a statement that involves one or more quantifiers in a useful way can be difficult. It's usually helpful to begin with "It is not the case" and then to reflect on what you have written. Consider, for instance, the assertion

"For every real number x, x has a real square root."

A first stab at its negation gives

"It is not the case that every real number x has a real square root."

But what does this really mean? Surely,

"There exists a real number that does not have a real square root."

Notice that the negation of a statement involving the universal quantifier required a statement involving the existential quantifier. This is always the case.

> The negation of "For all something, p" is the statement "There exists something such that $\neg p$."
>
> The negation of "There exists something such that p" is the statement "For all something, $\neg p$."

For example, the negation of

"There exist a and b for which $ab \neq ba$"

is the statement

"For all a and b, $ab = ba$."

0.1.1 REMARK

The symbols $\forall$ and $\exists$ are commonly used for the quantifiers *for all* and *there exists*, respectively. For example, you might encounter the statement

$$\forall x, \exists n \text{ such that } n > x$$

or even, more simply,

$$\forall x, \exists n, n > x$$

in a book in real analysis. We won't use this notation in this book, but it is so common that you should know about it. ❖

What May I Assume?

In our experience, when asked to prove something, students often wonder just what they are allowed to assume. For the rest of this book, the answer is any fact, including the result of any exercise, stated **earlier** in the book. This chapter is somewhat special because we are talking *about* mathematics and endeavoring to use only *familiar* ideas to motivate our discussion. In addition to basic college algebra, here is a list of mathematical definitions and facts that the student is free to assume in this chapter's exercises.

- The product of nonzero real numbers is nonzero.
- The square of a nonzero real number is a positive real number.
- An even integer is one that is of the form $2k$ for some integer k; an odd integer is one that is of the form $2k + 1$ for some integer k.
- The product of two even integers is even; the product of two odd integers is odd; the product of an odd integer and an even integer is even.
- A real number is rational if it is a common fraction, that is, the quotient $\frac{m}{n}$ of integers m and n with $n \neq 0$.
- A real number is irrational if it is not rational. For example, π and $\sqrt[3]{5}$ are irrational numbers.
- An irrational number has a decimal expansion that neither repeats nor terminates.
- A prime is a positive integer $p > 1$ that is divisible evenly only by ± 1 and $\pm p$, for example, 2, 3 and 5.

Answers to Pauses

1. By equation (*), twice the sum of the integers from 1 to 100 is 100×101. So the sum itself is $50 \times 101 = 5050$.
2. This is false. The hypothesis is true, but the conclusion is false: $-1^2 = -1$, not 1.
3. The converse is "If $-1^2 = 1$, then $4^2 = 16$." This is true, because the hypothesis "$-1^2 = 1$" is false.
4. Julie's answer began with the **converse** of what she wanted to show. She intended to say

   If d is perpendicular to n, then the line is parallel to the plane.

   Then, having established that d was perpendicular to n, she would have the result.
5. **(a)** This is true because both statements are true.

   (b) This is false because the two statements have different truth values.

   (c) This is false because the two statements have different truth values.

   (d) This is true because both statements are false.
6. "Not $p \to q$" means $p \to q$ is false. This occurs precisely when p is true and q is false. So $\neg(p \to q)$ is "p and $\neg q$."

7. The contrapositive of the implication in PAUSE 2 is "If $-1^2 \neq 1$, then $4^2 \neq 16$." This is false because the hypothesis is true, but the conclusion is false.

   The contrapositive of the implication in PAUSE 3 is "If $4^2 \neq 16$, then $-1^2 \neq 1$." This is true because the hypothesis is false.

   These answers are the same as in Pauses 2 and 3. This is always the case, as we shall see in Section 0.2.

8. For all real numbers $x > 0$, x has a real square root.
9. There exists a polynomial with no real zeros.

True/False Questions

(Answers can be found in the back of the book.)

1. "p and q" is false if both p and q are false.
2. If "p and q" is false, then both p and q are false.
3. It is possible for both "p and q" and "p or q" to be false.
4. It is possible for both "p and q" and "p or q" to be true.
5. The implication "If $2^2 = 5$, then $3^2 = 9$" is true.
6. The negation of $a = b = 0$ is $a \neq b \neq 0$.
7. The converse of the implication in Question 5 is true.
8. The double implication "$2^2 = 5$ if and only if $3^2 = 9$" is true.
9. It is possible for both an implication and its converse to be true.
10. The statement "Some frogs have red toes" makes use of the universal quantifier "for all."
11. The negation of an existential quantifier is its own converse.

Exercises

The answers to exercises marked [BB] can be found in the Back of the Book.
You are urged to read the final paragraph of this section—What may I assume?—before attempting these exercises.

1. Classify each of the following statements as true, false, or not a valid mathematical statement.
   (a) [BB] An integer is a rational number.
   (b) [BB] Let x denote a real number.
   (c) [BB] The square of a real number is a positive number.
   (d) Where is Newfoundland?
   (e) The product of an integer and an even integer is an even integer.
   (f) Suppose this statement is false.
   (g) The product of five prime numbers is prime.
2. Classify each of the following statements as true, false, or not a valid mathematical statement. Explain your answers.
   (a) [BB] $4 = 2 + 2$ and $7 < \sqrt{50}$.
   (b) $4 \neq 2 + 2$ and $7 < \sqrt{50}$.
   (c) [BB] 5 is an even number and $16^{-1/4} = \frac{1}{2}$.
   (d) 5 is an even number or $16^{-1/4} = \frac{1}{2}$.
   (e) [BB] $9 = 3^2$ or $3.14 < \pi$.
   (f) $\sqrt{3}^{\sqrt{3}}$ is rational or $(-4)^2 = 16$.
   (g) [BB] $4 = 2 + 2 \rightarrow 7 < \sqrt{50}$.
   (h) 2 is an even integer $\rightarrow$ 6 is an odd integer.
   (i) [BB] Let n denote a positive integer.
   (j) $4 = 2 + 2 \leftrightarrow 7 < \sqrt{50}$.
   (k) $4 \neq 2 + 2 \rightarrow 7 < \sqrt{50}$.
   (l) $4 \neq 2 + 2 \leftrightarrow 7 < \sqrt{50}$.
   (m) $4 = 2 + 2 \rightarrow 7 > \sqrt{50}$.
   (n) [BB] The area of a circle of radius r is $2\pi r$ or its circumference is πr^2.
   (o) $2 + 3 = 5 \rightarrow 5 + 6 = 10$.
   (p) [BB] If a and b are integers with $a - b \geq 0$ and $b - a \geq 0$, then $a = b$.
   (q) If a and b are integers with $a - b > 0$ and $b - a > 0$, then $a = b$.
3. Rewrite each of the following statements so that it is clear that each is an implication.
   (a) [BB] The reciprocal of a positive number is positive.
   (b) The product of rational numbers is rational.

0.2 Proofs in Mathematics

Many mathematical theorems are statements that a certain implication is true. A simple result about real numbers says that if x is between 0 and 1 then $x^2 < 1$. In other words, for any choice of a real number between 0 and 1, it is a fact that the square of the number will be less than 1. We are asserting that the implication

Statement 3: "$0 < x < 1 \rightarrow x^2 < 1$"

is true. In Section 0.1, the hypothesis and conclusion of an implication could be any two statements, even statements completely unrelated to each other. In the statement of a theorem or a step of a mathematical proof, however, the hypothesis and conclusion will be statements about the same class of objects, and the statement (or step) is the assertion that an implication is always true. The only way for an implication to be false is for the hypothesis to be true and the conclusion false. So the statement of a mathematical theorem or a step in a proof only requires proving that whenever the hypothesis is true the conclusion must also be true.

When the implication "$\mathcal{A} \rightarrow \mathcal{B}$" is the statement of a theorem, or one step in a proof, $\mathcal{A}$ is said to be a *sufficient* condition for $\mathcal{B}$ and $\mathcal{B}$ is said to be a *necessary* condition for $\mathcal{A}$. For instance, the implication in Statement 3 can be restated "$0 < x < 1$ is sufficient for $x^2 < 1$" or "$x^2 < 1$ is necessary for $0 < x < 1$."

Pause 10

Rewrite the statement "A matrix with determinant 1 is invertible" so that it becomes apparent that this is an implication. What is the hypothesis? What is the conclusion? This sentence asks which is a necessary condition for what? What is a sufficient condition for what? ∎

To prove that Statement 3 is true, it is **not** enough to take a single example, $x = \frac{1}{2}$ for instance, and claim that the implication is true because $(\frac{1}{2})^2 < 1$. It is **not** better to take ten, or even ten thousand, such examples and verify the implication in each special case. Instead, a **general argument** must be given that works for all x between 0 and 1. Here is such an argument.

> Assume that the hypothesis is true; that is, x is a real number with $0 < x < 1$. Since $x > 0$, it must be that $x \cdot x < 1 \cdot x$ because multiplying both sides of an inequality such as $x < 1$ by a positive number such as x preserves the inequality. Hence $x^2 < x$ and, since $x < 1$, $x^2 < 1$ as desired.

Now let us consider the converse of the implication in Statement 3:

Statement 4: "$x^2 < 1 \rightarrow 0 < x < 1$."

This is false. For example, when $x = -\frac{1}{2}$, the left-hand side is true, since $(-\frac{1}{2})^2 = \frac{1}{4} < 1$, while the right-hand side is false. So the implication fails when $x = -\frac{1}{2}$. It follows that this implication cannot be used as part of a mathematical proof. The number $x = -\frac{1}{2}$ is called a *counterexample* to Statement 4; that is, a specific example that proves that an implication is false.

There is a very important point to note here. To show that a theorem, or a step in a proof, is false, it is enough to find a single case where the implication does not hold. However, as we saw with Statement 3, to show that a theorem is true, we must give a proof that covers all possible cases.

Pause 11

State the contrapositive of "$0 < x < 1 \rightarrow x^2 < 1$." Is this true or false? ∎

Pause 12

Consider the statement "There exists a positive integer n such that $n^2 < n$." Would it make sense to attempt to prove this statement false with a counterexample? Explain.

Theorems in mathematics don't have to be about numbers. For example, a very famous theorem proved in 1976 asserts that if $\mathcal{G}$ is a planar graph then $\mathcal{G}$ can be colored with at most four colors. (The definitions and details are in Chapter 13.) This is an implication of the form "$\mathcal{A} \to \mathcal{B}$," where the hypothesis $\mathcal{A}$ is the statement that $\mathcal{G}$ is a planar graph and the conclusion $\mathcal{B}$ is the statement that $\mathcal{G}$ can be colored with at most four colors. The Four-Color Theorem states that this implication is true.

Pause 13 (For students who have studied linear algebra.) The statement given in PAUSE 10 is a theorem in linear algebra; that is, the implication is true. State the converse of this theorem. Is this also true?

A common expression in scientific writing is the phrase *if and only if* denoting both an implication and its converse. For example,

Statement 5: "$x^2 + y^2 = 0 \leftrightarrow (x = 0$ and $y = 0)$."

As we saw in Section 0.1, the statement "$\mathcal{A} \leftrightarrow \mathcal{B}$" is a convenient way to express the compound statement

"$\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{A}$."

The sentence "$\mathcal{A}$ is a *necessary and sufficient* condition for $\mathcal{B}$" is another way of saying "$\mathcal{A} \leftrightarrow \mathcal{B}$." The sentence "$\mathcal{A}$ and $\mathcal{B}$ are *necessary and sufficient conditions* for $\mathcal{C}$" is another way of saying

"($\mathcal{A}$ and $\mathcal{B}$) $\leftrightarrow \mathcal{C}$."

For example, "a triangle has three equal angles" is a necessary and sufficient condition for "a triangle has three equal sides." We would be more likely to hear "In order for a triangle to have three equal angles, it is necessary and sufficient that it have three equal sides."

To prove that "$\mathcal{A} \leftrightarrow \mathcal{B}$" is true, we must prove separately that "$\mathcal{A} \to \mathcal{B}$" and "$\mathcal{B} \to \mathcal{A}$" are both true, using the ideas discussed earlier. In Statement 5, the implication "$(x = 0$ and $y = 0) \to x^2 + y^2 = 0$" is easy.

Pause 14 Prove that "$x^2 + y^2 = 0 \to (x = 0$ and $y = 0)$."

As another example, consider

Statement 6: "$0 < x < 1 \leftrightarrow x^2 < 1$."

Is this true? Well, we saw earlier that "$0 < x < 1 \to x^2 < 1$" is true, but we also noted that its converse, "$x^2 < 1 \to 0 < x < 1$," is false. It follows that Statement 6 is false.

Pause 15 Determine whether "$-1 < x < 1 \leftrightarrow x^2 < 1$" is true or false.

Sometimes, a theorem in mathematics asserts that three or more statements are *equivalent*, meaning that all possible implications between pairs of statements are true. Thus

"The following are equivalent:

1. $\mathcal{A}$
2. $\mathcal{B}$
3. $\mathcal{C}$"

means that each of the double implications $\mathcal{A} \leftrightarrow \mathcal{B}$, $\mathcal{B} \leftrightarrow \mathcal{C}$, $\mathcal{A} \leftrightarrow \mathcal{C}$ is true. Instead of proving the truth of the six implications here, it is more efficient just to establish the truth, say, of the sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{A}$$

of three implications. It should be clear that if these implications are all true then any implication involving two of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ is also true; for example, the truth of $\mathcal{B} \to \mathcal{A}$ would follow from the truth of $\mathcal{B} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{A}$.

Alternatively, to establish that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are equivalent, we could establish the truth of the sequence

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C} \to \mathcal{B}$$

of implications. Which of the two sequences a person chooses is a matter of preference, but is usually determined by what appears to be the easiest way to argue. Here is an example.

PROBLEM 1. Let x be a real number. Show that the following are equivalent.

(1) $x = \pm 1$.
(2) $x^2 = 1$.
(3) If a is any real number, then $ax = \pm a$.

Solution. To show that these statements are equivalent, it is sufficient to establish the truth of the sequence

$$(2) \to (1) \to (3) \to (2).$$

(2) $\to$ (1): The notation means "assume (2) and prove (1)." Since $x^2 = 1$, $0 = x^2 - 1 = (x+1)(x-1)$. Since the product of real numbers is zero if and only if one of the numbers is zero, either $x + 1 = 0$ or $x - 1 = 0$; hence $x = -1$ or $x = +1$, as required.

(1) $\to$ (3): The notation means "assume (1) and prove (3)." Thus either $x = +1$ or $x = -1$. Let a be a real number. If $x = +1$, then $ax = a \cdot 1 = a$, while if $x = -1$, then $ax = -a$. In every case, $ax = \pm a$ as required.

(3) $\to$ (2): We assume (3) and prove (2). We are given that $ax = \pm a$ for any real number a. With $a = 1$, we obtain $x = \pm 1$ and squaring gives $x^2 = 1$, as desired. ▲

In the index (see *equivalent*), you are directed to other places in this book where we establish the equivalence of a series of statements.

Direct Proof

Most theorems in mathematics are stated as implications: "$\mathcal{A} \to \mathcal{B}$." Sometimes, it is possible to prove such a statement *directly*; that is, by establishing the validity of a sequence of implications:

$$\mathcal{A} \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots \to \mathcal{B}.$$

PROBLEM 2. Prove that for all real numbers x, $x^2 - 4x + 17 \neq 0$.

Solution. We observe that $x^2 - 4x + 17 = (x-2)^2 + 13$ is the sum of 13 and a number, $(x-2)^2$, which is never negative. So $x^2 - 4x + 17 \geq 13$ for any x; in particular, $x^2 - 4x + 17 \neq 0$. ▲

PROBLEM 3. Suppose that x and y are real numbers such that $2x + y = 1$ and $x - y = -4$. Prove that $x = -1$ and $y = 3$.

Solution. $(2x + y = 1 \text{ and } x - y = -4) \to (2x + y) + (x - y) = 1 - 4$

$$\to 3x = -3 \to x = -1.$$

Also,

$$(x = -1 \text{ and } x - y = -4) \to (-1 - y = -4) \to (y = -1 + 4 = 3).$$ ▲

Many of the proofs in this book are direct. In the index (under *direct*), we guide you to several of these.

Proof by Cases

Sometimes a direct argument is made simpler by breaking it into a number of cases, one of which must hold and each of which leads to the desired conclusion.

PROBLEM 4. Let n be an integer. Prove that $9n^2 + 3n - 2$ is even.

Solution. *Case 1.* n is even.

Recall that an integer is even if and only if twice another integer. So here we have $n = 2k$ for some integer k. Thus $9n^2 + 3n - 2 = 36k^2 + 6k - 2 = 2(18k^2 + 3k - 1)$, which is even.

Case 2. n is odd.

An integer is odd if and only if it has the form $2k + 1$ for some integer k. So here we write $n = 2k + 1$. Thus $9n^2 + 3n - 2 = 9(4k^2 + 4k + 1) + 3(2k + 1) - 2 = 36k^2 + 42k + 10 = 2(18k^2 + 21k + 5)$, which is even. ▲

In the index, we guide you to other places in this book where we give proofs by cases. (See *cases*.)

Prove the Contrapositive

A very important principle of logic, foreshadowed by PAUSE 7, is summarized in the next theorem.

0.2.1 THEOREM

"$\mathcal{A} \to \mathcal{B}$" is true if and only if its contrapositive "$\neg\mathcal{B} \to \neg\mathcal{A}$" is true.

Proof

"$\mathcal{A} \to \mathcal{B}$" is false if and only if $\mathcal{A}$ is true and $\mathcal{B}$ is false; that is, if and only if $\neg\mathcal{A}$ is false and $\neg\mathcal{B}$ is true; that is, if and only if "$\neg\mathcal{B} \to \neg\mathcal{A}$" is false. Thus the two statements "$\mathcal{A} \to \mathcal{B}$" and "$\neg\mathcal{B} \to \neg\mathcal{A}$" are false together (and hence true together); that is, they have the same truth values. The result is proved. ●

PROBLEM 5. If the average of four different integers is 10, prove that one of the integers is greater than 11.

Solution. Let $\mathcal{A}$ and $\mathcal{B}$ be the statements

$\mathcal{A}$: "The average of four integers, all different, is 10."

$\mathcal{B}$: "One of the four integers is greater than 11."

We are asked to prove the truth of "$\mathcal{A} \to \mathcal{B}$." Instead, we prove the truth of the contrapositive "$\neg\mathcal{B} \to \neg\mathcal{A}$", from which the result follows by Theorem 0.2.1.

Call the given integers a, b, c, d. If $\mathcal{B}$ is false, then each of these numbers is at most 11 and, since they are all different, the biggest value for $a + b + c + d$ is $11 + 10 + 9 + 8 = 38$. So the biggest possible average would be $\frac{38}{4}$, which is less than 10, so $\mathcal{A}$ is false. ▲

Proof by Contradiction

Sometimes a direct proof of a statement $\mathcal{A}$ seems hopeless: We simply do not know how to begin. In this case, we can sometimes make progress by assuming that the negation of $\mathcal{A}$ is true. If this assumption leads to a statement that is obviously false (an *absurdity*) or to a statement that contradicts something else, then we will have shown that $\neg\mathcal{A}$ is false. So, $\mathcal{A}$ must be true.

PROBLEM 6. Show that there is no largest integer.

Solution. Let $\mathcal{A}$ be the statement "There is no largest integer." If $\mathcal{A}$ is false, then there is a largest integer N. This is absurd, however, because $N+1$ is an integer larger than N. Thus $\neg\mathcal{A}$ is false, so $\mathcal{A}$ is true. ▲

Remember that *rational number* just means common fraction, the quotient $\frac{m}{n}$ of integers m and n with $n \neq 0$. A number that is not rational is called *irrational*.

PROBLEM 7. Suppose that a is a nonzero rational number and that b is an irrational number. Prove that ab is irrational.

Solution. We prove "$\mathcal{A}$: ab is irrational" by contradiction. Assume $\mathcal{A}$ is false. Then ab is rational, so $ab = \frac{m}{n}$ for integers m and n, $n \neq 0$. Now a is given to be rational, so $a = \frac{k}{\ell}$ for integers k and ℓ, $\ell \neq 0$ and $k \neq 0$ (because $a \neq 0$). Thus,

$$b = \frac{m}{na} = \frac{m\ell}{nk}$$

with $nk \neq 0$, so b is rational. This is not true. We have reached a contradiction and hence proved that $\mathcal{A}$ is true. ▲

Here is a well-known but nonetheless beautiful example of a proof by contradiction.

PROBLEM 8. Prove that $\sqrt{2}$ is an irrational number.

Solution. If the statement is false, then there exist integers m and n such that $\sqrt{2} = \frac{m}{n}$. If both m and n are even, we can cancel 2's in numerator and denominator until at least one of them is odd. Thus, *without loss of generality*, we may assume that not both m and n are even.

Squaring both sides of $\sqrt{2} = \frac{m}{n}$, we get $m^2 = 2n^2$, so m^2 is even. Since the square of an odd integer is odd, $m = 2k$ must be even. This gives $4k^2 = 2n^2$, so $2k^2 = n^2$. As before, this implies that n is even, contradicting the fact that not both m and n are even. ▲

Answers to Pauses

10. "A matrix A has determinant $1 \to A$ is invertible."

    Hypothesis: A matrix A has determinant 1.
    Conclusion: A is invertible.

    The invertibility of a matrix is a necessary condition for its determinant being equal to 1; determinant 1 is a sufficient condition for the invertibility of a matrix.

11. The contrapositive is "$x^2 \geq 1 \to (x \leq 0$ or $x \geq 1)$." This is true, and here is a proof.

    Assume $x^2 \geq 1$. If $x \leq 0$, we have the desired result, so assume $x > 0$. In this case, if $x < 1$, we have seen that $x^2 < 1$, which is not true, so we must have $x \geq 1$, again as desired.

12. No it would not. To prove the statement false requires proving that $n^2 \geq n$ for **all** positive integers n. A single example is not going to make this point.

13. The converse is "An invertible matrix has determinant 1." This is false: for example, the matrix $A = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ is invertible (the inverse of A is A itself), but $\det A = -1$.
14. Assume that $x^2 + y^2 = 0$. Since the square of a real number cannot be negative and the square of a **nonzero** real number is positive, if either $x^2 \neq 0$ or $y^2 \neq 0$, the sum $x^2 + y^2$ would be positive, which is not true. This means $x^2 = 0$ and $y^2 = 0$, so $x = 0$ and $y = 0$, as desired.
15. This statement is true. To prove it, we must show that two implications are true.
($\longrightarrow$) First assume that $-1 < x < 1$. If $0 < x < 1$, then we saw in the text that $x^2 < 1$ while, if $x = 0$, clearly $x^2 = 0 < 1$. If $-1 < x < 0$, then $0 < -x < 1$ (multiplying an inequality by the negative number -1 reverses it) so, by the argument in the text $(-x)^2 < 1$, that is, $x^2 < 1$. In all cases, we have $x^2 < 1$. Thus $-1 < x < 1 \rightarrow x^2 < 1$ is true.
($\longleftarrow$) Next we prove that $x^2 < 1 \rightarrow -1 < x < 1$ is true. Assume $x^2 < 1$. If $x \geq 1$, then we would also have $x^2 = x \cdot x \geq x \cdot 1 = x \geq 1$, so $x^2 \geq 1$, which is not true. If we had $x \leq -1$, then $-x \geq 1$ and so $x^2 = (-x)^2 \geq 1$, which, again, is not true. We conclude that $-1 < x < 1$, as desired.

True/False Questions

(Answers can be found in the back of the book.)

1. If you want to prove a statement is true, it is enough to find 867 examples where it is true.
2. If you want to prove a statement is false, it is enough to find one example where it is false.
3. The sentence "$\mathcal{A}$ is a sufficient condition for $\mathcal{B}$" is another way of saying "$\mathcal{A} \rightarrow \mathcal{B}$."
4. If $\mathcal{A} \rightarrow \mathcal{B}$, $\mathcal{B} \rightarrow \mathcal{C}$, $\mathcal{C} \rightarrow \mathcal{D}$, and $\mathcal{C} \rightarrow \mathcal{A}$ are all true, then $\mathcal{D} \rightarrow \mathcal{B}$ must be true.
5. If $\mathcal{A} \rightarrow \mathcal{B}$, $\mathcal{B} \rightarrow \mathcal{C}$, $\mathcal{C} \rightarrow \mathcal{D}$, and $\mathcal{C} \rightarrow \mathcal{A}$ are all true, then $\mathcal{B} \leftrightarrow \mathcal{C}$ must be true.
6. The contrapositive of "$\mathcal{A} \rightarrow \mathcal{B}$" is "$\neg\mathcal{B} \rightarrow \mathcal{A}$."
7. "$\mathcal{A} \rightarrow \mathcal{B}$" is true if and only if its contrapositive is true.
8. π is a rational number.
9. 3.141 is a rational number.
10. If a and b are irrational numbers, then ab must be an irrational number.
11. The statement "Every real number is rational" can be proved false with a counterexample.
12. The statement "There exists an irrational number that is not the square root of an integer" can be proved false with a counterexample.

Exercises

The answers to exercises marked [BB] can be found in the Back of the Book.

You are urged to read the final paragraph of Section 0.1—What may I assume?—before attempting these exercises.

1. What is the hypothesis and what is the conclusion in each of the following implications?
   (a) [BB] The sum of two positive numbers is positive.
   (b) The square of the length of the hypotenuse of a right-angled triangle is the sum of the squares of the lengths of the other two sides.
   (c) [BB] All primes are even.
   (d) Every positive integer bigger than 1 is the product of prime numbers.
   (e) The chromatic number of a planar graph is 3.
2. [BB: (a), (c)] In each part of Exercise 1, what condition is necessary for what? What condition is sufficient for what?