MAYER-VIETORIS

Theorem ④ $A, B \subseteq X$ interiors cover X. There is long exact seq:

$$\cdots \to H_n(A\cap B) \to H_n(A)\oplus H_n(B) \to H_n(X) \to H_{n-1}(A\cap B) \to \cdots$$

$$x \mapsto x \oplus -x$$

$$x \oplus y \longmapsto x - y$$

$$x = x_A + x_B \mapsto \partial x_A$$

- Reduced version formally identical.
- Mayer-Vietoris is abelian version of Van Kampen: For $A\cap B$ path conn

$$MV \Longrightarrow H_1(X) = H_1(A)\oplus H_1(B) / H_1(A\cap B)$$

Examples ① $X = S^n$ $A, B =$ (neighborhoods of) hemispheres

$$\tilde{H}_i(A)\oplus \tilde{H}_i(B) = 0 \ \forall i.$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

② $X =$ Klein bottle $A, B =$ (nbhds of) Möbius bands

$A, B, A\cap B \simeq S^1 \rightsquigarrow$

$$0 \to H_2(X) \to H_1(A\cap B) \to H_1(A)\oplus H_1(B) \to H_1(K) \to 0$$

$$1 \mapsto 2 \oplus -2$$

$\Rightarrow H_2(K) = 0$

$$H_1(K) \cong H_1(A)\oplus H_1(B) / H_1(A\cap B) = (1,0)\oplus(1,1) / (-2,2)$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Excision

Theorem. Let $Z \subseteq A \subseteq X$ closure $Z \subseteq$ interior A
(3) Then $(X-\cancel{A}Z, A-Z) \hookrightarrow (X,A)$
induces an isomorphism on homology.

Equivalently: $A, B \subseteq X$, interiors cover X.
$(B, A\cap B) \hookrightarrow (X,A)$ induces $\cong$ on H_*
translation $B = X-Z$, $Z = X-B$.

Application: Invariance of Domain

Theorem: If nonempty open sets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ are homeomorphic, then $m=n$.

Proof: Let $x \in U$. $H_k(U, U-x) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - x)$ by Excision.
Long exact seq. for $(\mathbb{R}^m, \mathbb{R}^m - x)$:

$$\cdots \to H_k(\mathbb{R}^m) \to H_k(\mathbb{R}^m, \mathbb{R}^m - x) \to H_{k-1}(\mathbb{R}^m - x) \to H_{k-1}(\mathbb{R}^m) \to \cdots$$

$$\Rightarrow \quad H_k(\mathbb{R}^m, \mathbb{R}^m - x) \cong H_{k-1}(\mathbb{R}^m - x)$$

But $H_{k-1}(\mathbb{R}^m - x) \cong H_{k-1}(S^{m-1})$ since $\mathbb{R}^m - x$ def. ret to S^{m-1}

Thus:

$$H_k(U, U-x) = \begin{cases} \mathbb{Z} & k=m \\ 0 & \text{o.w.} \end{cases}$$

In other words, can detect m from homology groups. ∎

Excision also used to show $H_n(X,A) \cong \tilde{H}_n(X/A)$, so Theorem 2 implies Theorem 1. See Hatcher Prop 2.22

Remains to prove Excision and Mayer-Vietoris.

Idea: Subdivide.

Another homology: X = space

$\mathcal{U} = \{U_j\}$ collection of subspaces whose interiors cover X.

$C_n^{\mathcal{U}}(X)$ = chains $\sum n_i \sigma_i$ so each σ_i has image in some U_j

$\partial(C_n^{\mathcal{U}}(X)) \subseteq C_{n-1}^{\mathcal{U}}(X) \rightsquigarrow$ chain complex

$\rightsquigarrow H_n^{\mathcal{U}}(X)$

<u>Prop</u>: $H_n^{\mathcal{U}}(X) \cong H_n(X)$

Specifically, there is a subdivision operator $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ that is a chain homotopy inverse to $\iota : C_n^{\mathcal{U}}(X) \to C_n(X)$.

<u>Proof of Excision</u>. To show $H_n(B, A\cap B) \cong H_n(X, A)$.

Let $\mathcal{U} = \{A, B\}$

Note $C_n^{\mathcal{U}}(A)$ naturally identified with $C_n(A)$. by ρ and ι.

$$\Rightarrow \quad C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A) \longrightarrow C_n(X)/C_n(A)$$

induces isomorphism $H_n^{\mathcal{U}}(X,A) \cong H_n(X,A)$.

But:
$$C_n(B)/C_n(A\cap B) \longrightarrow C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$$

obviously an isomorphism: both are free on simplices lying in B but not A. So $H_n(B, A\cap B) \cong H_n^{\mathcal{U}}(X,A)$.

$\blacksquare$

Proof of Mayer-Vietoris. Recall $X = A \cup B$.
Let $\mathcal{U} = \{A, B\}$

There is a short exact seq. of chain complexes:

$$0 \longrightarrow C_n(A \cap B) \longrightarrow C_n(A) \oplus C_n(B) \longrightarrow C_n^{\mathcal{U}}(X) \longrightarrow 0$$

$$x \longmapsto x \oplus -x$$

$$x \oplus y \longmapsto x + y$$

$\rightsquigarrow$ long exact seq. in homology as before.
Substituting $H_n(X)$ for $H_n^{\mathcal{U}}(X)$ (Proposition)
$\rightsquigarrow$ Mayer-Vietoris sequence. $\blacksquare$

A description of $\partial : H_n(X) \longrightarrow H_{n-1}(A \cap B)$:

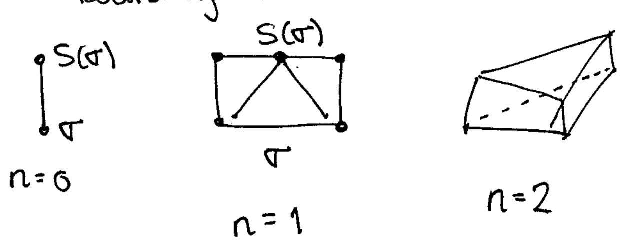
$\alpha \in H_n(X)$ rep. by cycle z

$z = x + y \quad x \in C_n(A),\ y \in C_n(B)$

$\partial x = -\partial y$ since $\partial z = 0$.

Set $\partial \alpha = \partial x$.

Proof of Prop. Let S = barycentric subdivision.
First show S is a chain homotopy equiv.
then take $\rho = S^N$.
Want $T : C_n(X) \longrightarrow C_{n+1}(X)$ s.t. $T\partial + \partial T = S - id$.
i.e. for any n-simplex σ want $(n+1)$-chain $T\sigma$ with boundary $S(\sigma) - \sigma - T\partial\sigma$



Do $n=1$ case on all 3 sides. Then join all simplices to barycenter on top.