Numerical Analysis of the Movement of a Damped, Driven Pendulum

Danming Wang, 5833587

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Group Members: Danming Wang (5833587), Xing Su (4502266), Xilin Huang (5562194), Zhixin Liu (7445224), Li Guo (7593601), Wenrui Wang(5813373)

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1 MOTIVATION

In real life, some objects oscillate in a similar way as how a damped and driven pendulum moves. Our project is inspired by these real-life situations, such as the oscillation driven by earthquake. We aim at building a model of a non-linear damped and driven pendulum to study its movement.

2 PROBLEM STATEMENT

Our project simulates a non-linear damped and driven pendulum. By applying different driven force, our pendulum's movement can change between chaotic and unchaotic.

We formulated our problem into an initial-value problem as follows:

$$\frac{d^2\theta}{dt^2} = -q\frac{d\theta}{dt} - \frac{g}{l}\sin\theta + b_0\cos\omega_D t = f(t,\theta),\ a \le t \le b,\ \theta(a) = \alpha_1, \theta'(a) = \alpha_2$$

Next, we can rewrite this second-order initial-value problem into a system of first-order initial-value problems by letting $u_1 = \theta$ and $u_2 = \frac{d\theta}{dt}$. Let $D = \{(t, u_1, u_2) | a \le t \le b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1, 2\}$. Then we get

$$f(t, u_1, u_2) = \begin{cases} u_1' & = u_2 = f_1(t, u_1, u_2) \\ u_2' & = -q \cdot u_2 - \frac{g}{l}\sin(u_1) + b_0 \cdot \cos(\omega_D t) = f_2(t, u_1, u_2) \end{cases}$$

with initial conditions $u_1(a) = \alpha_1$ and $u_2(a) = \alpha_2$. In this way, we can use the Runge-Kutta method for system of differential equations to solve for $u_1 = \theta$ and $u_2 = \theta' = \omega$.

3 NOTATIONS AND THEOREM

Notation:

1. θ: angular displacement
2. $\theta' = \omega$: angular velocity
3. g: the acceleration of gravity ($= 9.8m/s^2$)
4. l: length of the string
5. q: damping parameter
6. b_0: driving force parameter
7. ω_D: driving angular frequency

Mean Value Theorem: Suppose $f(x)$ is a function that satisfies both of the following:

1. $f(x)$ is continuous on the closed interval $[a, b]$.

2. $f(x)$ is differentiable on the open interval (a, b).

Then there is a number c such that $a < c < b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$.

4 ANALYSIS

4.1 Lipschitz Continuity

Proof. We can write $f(t, u_1, u_2)$ into matrix form:

$$\begin{aligned} f(t, u_1, u_2) &= \begin{bmatrix} f_1(t, u_1, u_2) \\ f_2(t, u_1, u_2) \end{bmatrix} \\ &= \begin{bmatrix} u_2 \\ -q \cdot u_2 - \frac{g}{l}\sin(u_1) + b_0 \cdot \cos(\omega_D t) \end{bmatrix} \end{aligned}$$

To prove that $f(t, u_1, u_2)$ is Lipschitz continuous on the set D, it suffices to prove that there exists $L > 0$ such that $\|f(t_1, u_1, u_2) - f(t_1, z_1, z_2)\|_2 \leq L\|(t_1, u_1, u_2) - (t_2, z_1, z_2)\|_2$ for any $(t_1, u_1, u_2), (t_2, z_1, z_2) \in D$.

Then it follows that

$$\|(t_1, u_1, u_2) - (t_2, z_1, z_2)\|_2 = \sqrt{(t_1 - t_2)^2 + (u_1 - z_1)^2 + (u_2 - z_2)^2}$$

and

$$\begin{aligned} &\left\| \begin{bmatrix} f_1(t_1, u_1, u_2) \\ f_2(t_1, u_1, u_2) \end{bmatrix} - \begin{bmatrix} f_1(t_2, z_1, z_2) \\ f_2(t_2, z_1, z_2) \end{bmatrix} \right\|_2 \\ &= \left\| \begin{bmatrix} u_2 \\ -q \cdot u_2 - \frac{g}{l}\sin(u_1) + b_0 \cdot \cos(\omega_D t_1) \end{bmatrix} - \begin{bmatrix} z_2 \\ -q \cdot z_2 - \frac{g}{l}\sin(z_1) + b_0 \cdot \cos(\omega_D t_2) \end{bmatrix} \right\|_2 \\ &= \left\| \begin{bmatrix} u_2 - z_2 \\ -q \cdot u_2 - \frac{g}{l}\sin(u_1) + b_0 \cdot \cos(\omega_D t_1) - (-q \cdot z_2 - \frac{g}{l}\sin(z_1) + b_0 \cdot \cos(\omega_D t_2)) \end{bmatrix} \right\|_2 \\ &= \left\| \begin{bmatrix} u_2 - z_2 \\ q \cdot (z_2 - u_2) + \frac{g}{l}(\sin(z_1) - \sin(u_1)) + b_0 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2)) \end{bmatrix} \right\|_2 \\ &= \sqrt{(u_2 - z_2)^2 + \left[q \cdot (z_2 - u_2) + \frac{g}{l}(\sin(z_1) - \sin(u_1)) + b_0 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2))\right]^2} \end{aligned}$$

Notice that $(u_2 - z_2)^2 \leq (u_1 - z_1)^2 + (u_2 - z_2)^2 + (t_1 - t_2)^2$ since $(u_1 - z_1)^2 \geq 0$ and $(t_1 - t_2)^2 \geq 0$.

Next, we can calculate that

$$
\begin{aligned}
&\frac{\left[q\cdot(z_2-u_2)+\frac{g}{l}(\sin(z_1)-\sin(u_1))\right]^2}{(t_1-t_2)^2+(u_1-z_1)^2+(u_2-z_2)^2}\\
\leq&\frac{\left[q\cdot(z_2-u_2)+\frac{g}{l}(\sin(z_1)-\sin(u_1))\right]^2}{(u_1-z_1)^2+(u_2-z_2)^2}\\
=&\frac{q^2\cdot(z_2-u_2)^2+\frac{g^2}{l^2}(\sin(z_1)-\sin(u_1))^2+2q\frac{g}{l}(z_2-u_2)(\sin(z_1)-\sin(u_1))}{(u_1-z_1)^2+(u_2-z_2)^2}\\
=&\frac{q^2\cdot(z_2-u_2)^2}{(u_1-z_1)^2+(u_2-z_2)^2}+\frac{\frac{g^2}{l^2}(\sin(z_1)-\sin(u_1))^2}{(u_1-z_1)^2+(u_2-z_2)^2}\\
&\qquad+\frac{2q\frac{g}{l}(z_2-u_2)}{\sqrt{(u_1-z_1)^2+(u_2-z_2)^2}}\cdot\frac{(\sin(z_1)-\sin(u_1))}{\sqrt{(u_1-z_1)^2+(u_2-z_2)^2}}\\
\leq&\frac{q^2\cdot(z_2-u_2)^2}{(u_2-z_2)^2}+\frac{\frac{g^2}{l^2}(\sin(z_1)-\sin(u_1))^2}{(u_1-z_1)^2}+\frac{2q\frac{g}{l}(z_2-u_2)}{\sqrt{(u_2-z_2)^2}}\cdot\frac{(\sin(z_1)-\sin(u_1))}{\sqrt{(u_1-z_1)^2}}\\
=&q^2+\frac{g^2}{l^2}(\sin'(w))^2+\frac{2q\frac{g}{l}(z_2-u_2)}{|u_2-z_2|}\cdot\frac{(\sin(z_1)-\sin(u_1))}{|u_1-z_1|}\\
&\qquad\text{where } w \text{ is between } u_1 \text{ and } z_1 \text{ by Mean Value Theorem}\\
\leq&q^2+\frac{g^2}{l^2}(\cos(w))^2+\frac{2q\frac{g}{l}|z_2-u_2|}{|u_2-z_2|}\cdot\left|\frac{\sin(z_1)-\sin(u_1)}{u_1-z_1}\right|\\
\leq&q^2+\frac{g^2}{l^2}+2q\frac{g}{l}\cdot|\sin'(w)| \text{ where } w \text{ is between } u_1 \text{ and } z_1 \text{ by Mean Value Theorem}\\
=&q^2+\frac{g^2}{l^2}+2q\frac{g}{l}\cdot|\cos(w)|\\
\leq&q^2+\frac{g^2}{l^2}+2q\frac{g}{l},\ \text{since } |\cos(w)|\leq 1
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{2\left[q\cdot(z_2-u_2)+\frac{g}{l}(\sin(z_1)-\sin(u_1))\right]b_0\cdot(\cos(\omega_D t_1)-\cos(\omega_D t_2))}{(u_1-z_1)^2+(u_2-z_2)^2+(t_1-t_2)^2}\\
\leq&\left(\frac{2g(z_2-u_2)}{\sqrt{(u_2-z_2)^2}}+\frac{2q(\sin(z_1)-\sin(u_1))}{l\sqrt{(z_1-u_1)^2}}\right)\cdot\frac{b_0(cos(\omega_D t_1)-cos(\omega_D t_2))}{\sqrt{(t_1-t_2)^2}}\\
\leq&\left(2q+\frac{2g}{l}\cdot\frac{\sin(z_1)-sin(u_1)}{|z_1-u_1|}\right)\cdot\frac{b_0\cdot\omega_D(\cos(\omega_D t_1)-cos(\omega_D t_2))}{(t_1-t_2)\cdot\omega_D}\\
=&\left(2q+\frac{2g}{l}\cdot\sin'(s)\right)\cdot b_0\omega_D\cdot\cos'(v),\\
&\text{where s is between } z_1 \text{ and } u_1, \text{ and } v \text{ is between } \omega_D t_1 \text{ and } \omega_D t_2 \text{ by Mean Value Theorem}\\
\leq&\left(2q+\frac{2g}{l}\right)\cdot b_0\omega_D,\ \text{since } \cos'(v)=-\sin(v)\leq 1
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{b_0^2 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2))^2}{(u_1 - z_1)^2 + (u_2 - z_2)^2 + (t_1 - t_2)^2} \\
\leq &\frac{b_0^2 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2))^2}{\omega_D^2 (t_1 - t_2)^2} \cdot \omega_D^2 \\
\leq &b_0^2 \cdot (\cos'(T))^2 \cdot \omega_D^2 \text{ where } T \text{ is between } \omega_D t_1 \text{ and } \omega_D t_2 \text{ by Mean Value Theorem} \\
\leq &b_0^2 \omega_D^2, \text{ since } \cos'(T) = -\sin(T) \leq 1
\end{aligned}$$

Thus,

$$\begin{aligned}
&(u_2 - z_2)^2 + \left[q \cdot (z_2 - u_2) + \frac{g}{l}(\sin(z_1) - \sin(u_1)) + b_0 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2))\right]^2 \\
= &(u_2 - z_2)^2 + \left[q \cdot (z_2 - u_2) + \frac{g}{l}(\sin(z_1) - \sin(u_1))\right]^2 + \\
&2\left[q \cdot (z_2 - u_2) + \frac{g}{l}(\sin(z_1) - \sin(u_1))\right] b_0 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2)) \\
&+ [b_0 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2))]^2 \\
\leq &\left(1 + q^2 + \frac{g^2}{l^2} + 2q\frac{g}{l} + \left(2q + \frac{2g}{l}\right) \cdot b_0 \omega_D + b_0^2 \omega_D^2\right) [(t_1 - t_2)^2 + (u_1 - z_1)^2 + (u_2 - z_2)^2]
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sqrt{(u_2 - z_2)^2 + \left[q \cdot (z_2 - u_2) + \frac{g}{l}(\sin(z_1) - \sin(u_1)) + b_0 \cdot (\cos(\omega_D t_1) - \cos(\omega_D t_2))\right]^2} \\
\leq &\sqrt{1 + q^2 + \frac{g^2}{l^2} + 2q\frac{g}{l} + \left(2q + \frac{2g}{l}\right) \cdot b_0 \omega_D + b_0^2 \omega_D^2} \cdot \sqrt{(t_1 - t_2)^2 + (u_1 - z_1)^2 + (u_2 - z_2)^2}
\end{aligned}$$

$\Rightarrow L = \sqrt{1 + q^2 + \frac{g^2}{l^2} + 2q\frac{g}{l} + \left(2q + \frac{2g}{l}\right) \cdot b_0 \omega_D + b_0^2 \omega_D^2} > 0$ □

4.2 Well Posed

Claim: $f(t, u_1, u_2)$ is continuous on D.

Proof. To prove $f(t, u_1, u_2)$ is continuous on D, it suffices to prove that $f_1(t, u_1, u_2)$ and $f_2(t, u_1, u_2)$ are continuous on D.
Since $f_1(t, u_1, u_2) = u_2$, for all $(t, u_1, u_2) \in D$, $f_1(t, u_1, u_2) = u_2$ is continuous.
Since $f_2(t, u_1, u_2) = -q \cdot u_2 - \frac{g}{l}\sin(u_1) + b_0 \cdot \cos(\omega_D t)$ for all $(t, u_1, u_2) \in D$ and u_2, $\sin(u_1)$, and $\cos(\omega_D t)$ are continuous, by Algebraic Continuity Theorem, $f_2(t, u_1, u_2)$ is continuous on D.
Therefore, $f(t, u_1, u_2)$ is continuous on D. □

Since $f(t, u_1, u_2)$ is continuous on D by claim and it also satisfies Lipschitz continuity on D in the variables u_1, u_2 by section 4.1, the initial value problem is well posed by Theorem 5.6 in textbook.

Thus, there exists an unique solution for $u_1 = \theta$ and $u_2 = \theta' = \omega$.

5 METHODS

To approximate the solution of the system of first-order initial-value problems, we are going to use the Runge-Kutta method for system of differential equations.

6 RESULTS

Fix $q = 0.5,\ g = 9.8,\ l = 9.8,\ \omega_D = \frac{2}{3},\ h = 0.1$. We will see how an increase in b_0, the driving force parameter, will change the oscillation from non-chaotic to chaotic.

Firstly, take $b_0 = 0.9$ and consider $0 \le t \le 100$, $\theta(0) = 0$, $\theta'(0) = 1$. By using the Runge-Kutta method, we get approximations of θ and θ' at different times, and we can plot our approximations with respect to t as follows:

Figure 1: Angular displacement of the pendulum at various times.

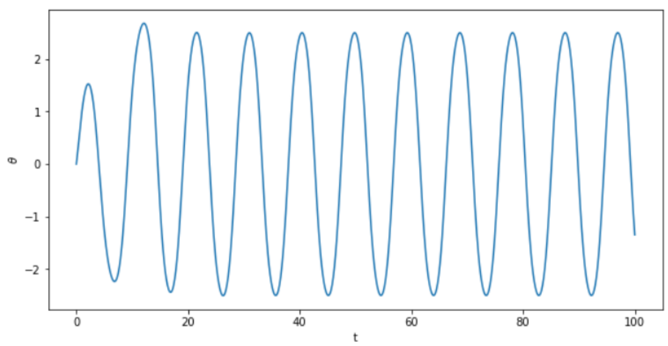
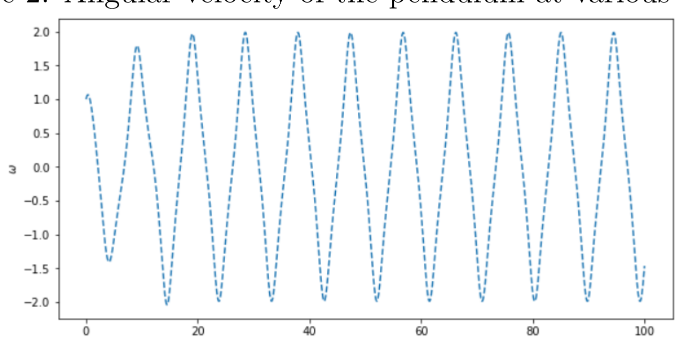
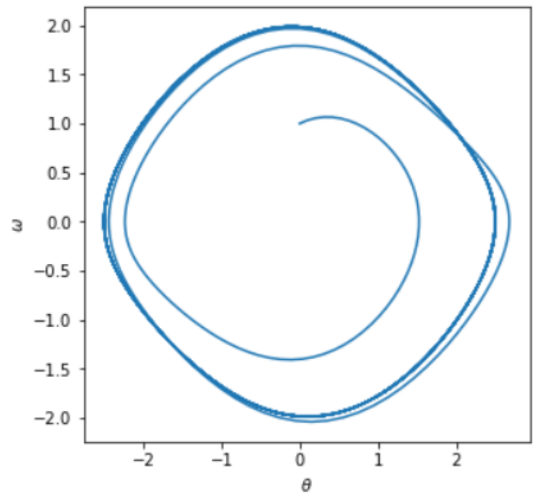


Figure 2: Angular velocity of the pendulum at various times.



And the relation between $\omega(t)$ and $\theta(t)$ can be plotted as follows:

Figure 3: Relation between the angular velocity and angular displacement



From figure 1 and 2, we can see that both angular displacement and angular velocity change in a periodic way with respect to time. From figure 3, we can see that the orbit on the diagram will eventually enter the final state of travelling on a nearly elliptical cycle after undergoing a chaotic transient process.

Next, take $b_0 = 1.15$ and consider $0 \leq t \leq 1000$, $\theta(0) = 0$, $\theta'(0) = 1$. By using the Runge-Kutta method, we get approximations of θ and θ' at different times, and we can plot our approximations with respect to t as follows:

Figure 4: Angular displacement of the pendulum at various times.

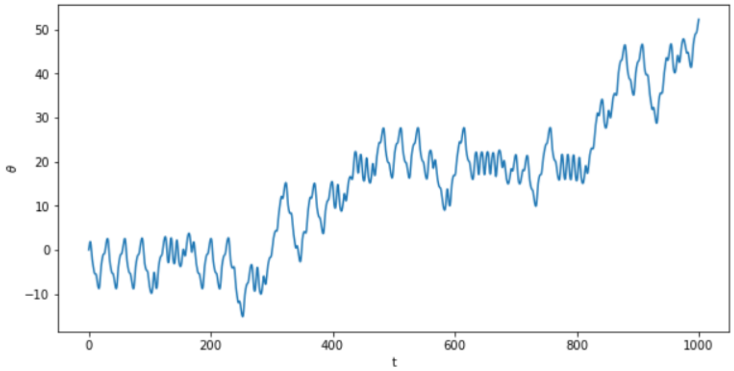
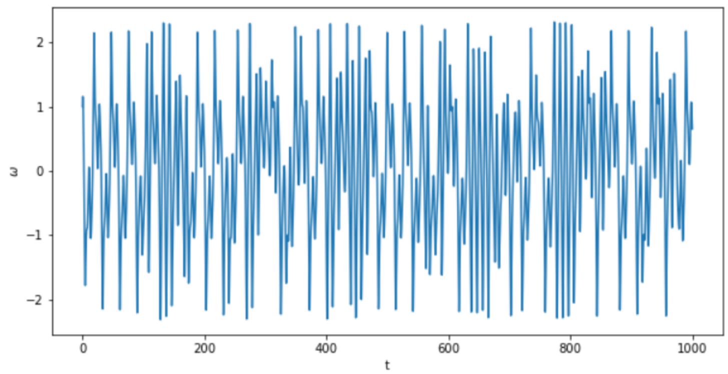
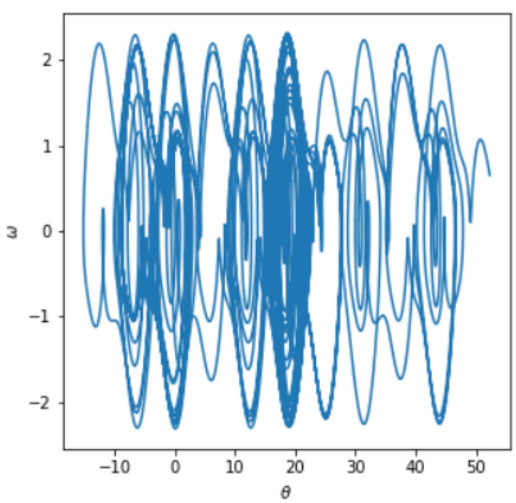


Figure 5: Angular velocity of the pendulum at various times.



And the relation between $\omega(t)$ and $\theta(t)$ can be plotted as follows:

Figure 6: Relation between the angular velocity and angular displacement



From figure 4 and 5, we can see that the angular displacement and angular velocity no longer change in a periodic way with respect to time. From figure 6, we can see that the orbit on the diagram is complicated and disordered. Thus, when taking $b_0 = 1.15$, the movement of this non-linear damped and driven pendulum becomes chaotic and unpredictable.

7 CONCLUSION

Given all the other variables are fixed. If we only increase the driving force, then after a certain point, the movement of a non-linear damped and driven pendulum will change from non-chaotic and predictable to chaotic and unpredictable. Our simulation may be used in the prediction of earthquakes.