A Numerical Analysis Project of Developing the Optimal Investment

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1 MOTIVATION

Motivation is the reason for people's actions, willingness and goals. Our motivation to work on this project came from the Markowitz Portfolio Theory. It tells us that when two investment opportunities have the same expected return, we desire more for the less risky one. When facing the opportunity to invest in many different assets, our goal is to choose an optimal portfolio that maximizes the investment return while minimizing risks. So our project aims to build a model of investments which attempts to capture the trade-offs and utilize this model to construct optimal portfolios for two assets.

2 PROBLEM STATEMENT

Our project considers two situations:
(i) The expected return of each asset is fixed.
(ii) The expected return of one asset depends on the proportion of wealth we invest in that asset.

For each situation, we assume the following two scenarios:
(a) Given a desired expected return μ_p, find α such that $\mu_p = w_1\mu_1 + w_2\mu_2$ where $w_1 = \alpha,\ w_2 = 1 - \alpha$.
(b) Find α such that the riskiness is minimized. Note that the riskiness is minimized when the expected variance σ_p^2 is minimized.

3 NOTATIONS and FORMULAS

1. Weights:

Let w be the investor's wealth. Let $w_1 = \alpha$ be the proportion of wealth invested in asset 1. Let $w_2 = 1 - \alpha$ be the proportion of wealth invested in asset 2. $0 \leq \alpha \leq 1$.

2. The Expected Returns:

Let μ_1 be the expected return of asset 1. Let μ_2 be the expected return of asset 2. Then the expected return of the portfolio can be expressed as $\mu_p = w_1\mu_1 + w_2\mu_2$.

3. The Expected Variances:

Let σ_1 be the riskiness of asset 1. Let σ_2 be the riskiness of asset 2. Let $\sigma_{1,2}$ denote the covariance of asset 1 and asset 2. Then the expected variance of the portfolio can be expressed as $\sigma_p^2 = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{1,2} + w_2^2\sigma_2^2$

4. The Fluctuations of the Future Value:

The fluctuations in the future value of the investment portfolio is modeled by the range $[V_1(t), V_2(t)]$ where $V_1(t) = we^{\mu_p t - \sigma_p \sqrt{t}}$ and $V_2(t) = we^{\mu_p t + \sigma_p \sqrt{t}}$.

4 METHODS

For scenario(a), we need to solve for α such that $w_1\mu_1 + w_2\mu_2 - \mu_p = 0$. For scenario(b), we want to solve for α such that σ_p^2 is minimized. Note that

$$\begin{aligned}\sigma_p^2 &= w_1^2\sigma_1^2 + 2w_1w_2\sigma_{1,2} + w_2^2\sigma_2^2 \\ &= \alpha^2\sigma_1^2 + 2\alpha(1-\alpha)\sigma_{1,2} + (1-\alpha)^2\sigma_2^2 \\ &= (\sigma_1^2 - 2\sigma_{1,2} + \sigma_2^2)\alpha^2 + (2\sigma_{1,2} - 2\sigma_2^2)\alpha + \sigma_2^2\end{aligned} \tag{1}$$

Since $\sigma_1^2 - 2\sigma_{1,2} + \sigma_2^2 > 0$, σ_p^2 is an upward parabola. Then σ_p^2 is minimized when its derivative is 0. Thus, we need to solve for α such that the derivative of σ_p^2 at α is zero. So problem for each scenario becomes a zero-finding problem. We use Bisection method to solve for an initial point after 3 iterations for α. Then we apply Newton's method to approximate α more precisely. Here we only use three iterations of Bisection method because the Bisection method is relatively slow to converge and a good intermediate approximation might be inadvertently discarded. So we just use Bisection method as a starter for the Newton's method, which is more efficient. And the money we are expected to receive at certain time $t \in [0, 1]$ is bounded by $V_1(t)$ and $V_2(t)$.

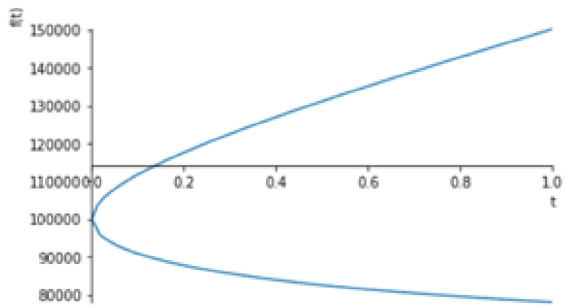
5 RESULTS

Situation(i)

For situation(i), we assume that an investor wishes to invest $w = \$100,000$ in the two assets. The weights are expressed as $w_1 = \alpha,\ w_2 = 1 - \alpha$. Let the assets have

$$\mu_1 = 0.03, \mu_2 = 0.09, \sigma_1 = 0.2, \sigma_2 = 0.4, \sigma_{1,2} = -0.02.$$

scenario(a): The investor wants to get a return $\mu_p = 0.08$.
By Bisection method and Newton's method, we found that the investor should invest approximately 0.16667 fraction of wealth in asset 1 and 0.83333 fraction of wealth in asset 2 to get an expected return $\mu_p = 0.08$. The variance (riskiness) σ_p^2 of the portfolio is approximately 0.10667. The fluctuations in the future value of the investment portfolio can be reflected from the following plot:

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v1t(1) = 78145.4281047113
v2t(1) = 150170.12504165436
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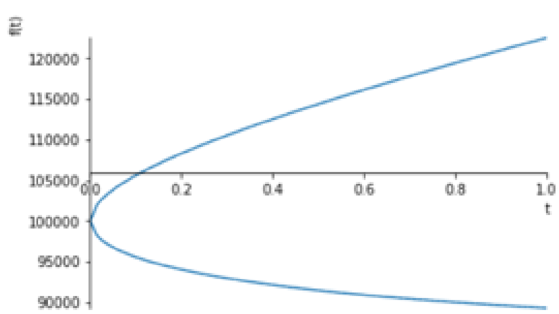
Here the horizontal axis represents the time $t \in [0, 1]$ and the vertical axis means the amount of money.
Putting the \$100,000 in a bank account paying a continuous compounding rate of 4%, the investor is going to receive $100,000 * (1 + \frac{0.04}{12})^{12} = \$104,074.15$ in one year.

Result: In this scenario, investing in the portfolio produces a riskiness of $\sigma_p^2 = 0.10667$, which is higher than the riskiness of asset 1 ($\sigma_1^2 = 0.2^2 = 0.04$). So I would recommend the investor to put money in a bank account since investing into the portfolio with expected return $\mu_p = 0.08$ is too risky.

scenario(b): The investor wants most to reduce the riskiness of the investment made in the two assets for any return.
By Bisection method and Newton's method, we found that the investor should invest approximately 0.75 fraction of wealth in asset 1 and 0.25 fraction of wealth in asset 2. This portfolio will give an expected return $\mu_p = 0.045$. The variance (riskiness) σ_p^2 of the portfolio is 0.025. The fluctuations in the future value of the investment portfolio can be reflected from the following plot:

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v1t(1) = 89304.89512234372
v2t(1) = 122521.19911302069
```

Here the horizontal axis represents the time $t \in [0, 1]$ and the vertical axis means the amount of money.

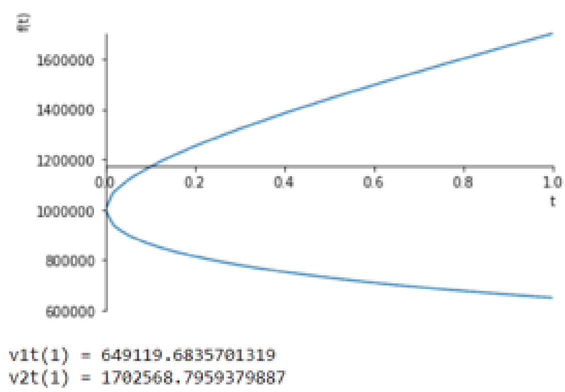
Result: In this scenario, investing in the portfolio produces a riskiness of $\sigma_p^2 = 0.025$, which is lower than the riskiness of asset 1 ($\sigma_1^2 = 0.2^2 = 0.04$) and the riskiness of asset 2 ($\sigma_2^2 = 0.16$). So I would recommend the investor to invest in this optimal portfolio since it is the least risky investment which will return much more money after one year than putting in a bank account.

Situation(ii): Consider certain assets which has an economy of scale, so that the expected return may increase as more resources are invested in that asset.

For situation(ii), we assume that an investor wishes to invest \$1,000,000 in the two assets. The weights are expressed as $w_1 = \alpha,\ w_2 = 1 - \alpha$. Let the assets have

$$\mu_1(w_1) = 0.0005e^{3w_1}, \mu_2(w_2) = 0.07, \sigma_1^2(w_1) = e^{-3w_1}, \sigma_2^2(w_2) = 0.4, \sigma_{1,2} = -0.01.$$

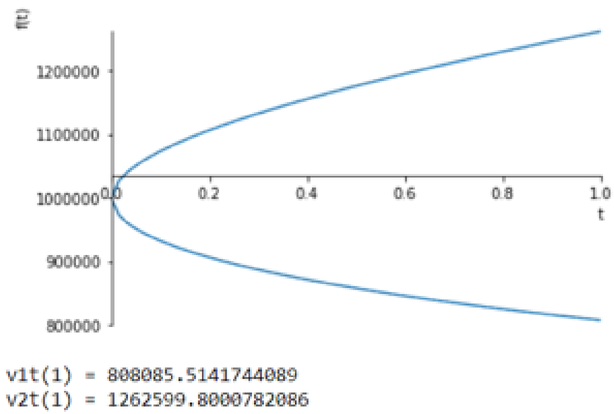
scenario(a): The investor wants to get a return $\mu_p = 0.05$.
By Bisection method and Newton's method, we found that the investor should invest approximately 0.29068 fraction of wealth in asset 1 and 0.70932 fraction of wealth in asset 2 to get an expected return $\mu_p = 0.05$. The variance (riskiness) σ_p^2 of the portfolio is approximately 0.23246. The fluctuations in the future value of the investment portfolio can be reflected from the following plot:



Here the horizontal axis represents the time $t \in [0, 1]$ and the vertical axis means the amount of money.
Putting the \$1,000,000 in a bank account paying a continuous compounding rate of 4%, the investor is going to receive $1,000,000 * (1 + \frac{0.04}{12})^{12} = \$1,040,741.54$ in one year.

Result: For situation(ii), the investor has much more wealth than in situation(i). In this scenario, investing in the portfolio produces a riskiness of $\sigma_p^2 = 0.23246$, which is lower than the riskiness of asset 1 ($\sigma_1^2 = e^{-3*0.2907} = 0.41810$). However, from the plot of the fluctuations in the future value of the investment portfolio, we can see that the future value in 1 year can be as low as \$649119.68 and can reach as high as \$1702568.80. If the investor is risk-averse, I would recommend the investor to put money in a bank account since investing into the portfolio with expected return $\mu_p = 0.08$ is too risky. Otherwise, the investor can choose to invest in this portfolio which may return a lot more money in one year than putting in a bank account.

scenario(b): The investor wants most to reduce the riskiness of the investment made in the two assets for any return.
By Bisection method and Newton's method, we found that the investor should invest all of his wealth in asset 1. This portfolio will produce an expected return $\mu_p = 0.01004$. The variance (riskiness) σ_p^2 of the portfolio is 0.049787. The fluctuations in the future value of the investment portfolio can be reflected from the following plot:



Here the horizontal axis represents the time $t \in [0, 1]$ and the vertical axis means the amount of money.

Result: In this scenario, investing in the portfolio produces a riskiness of $\sigma_p^2 = 0.049787$, which is lower than the riskiness of asset 2 ($\sigma_2^2 = 0.07$). However, the expected return $\mu_p = 0.01004$ is quite low. So even though investing in the portfolio has the potential of return $1,262,599.80 in one year, I would still recommend the investor to put money in a bank account rather than investing in this portfolio since putting in a bank account is risk-free and has a higher return than investing.