1 First Try

Firm problem:

$$\max_{p_{jt}} E_0 \left[\sum_{t=0}^{\infty} M_t P_t \left(\frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \psi(p_{jt}, p_{jt-1}) \right) \right] \tag{1}$$

And we have the following:

$$y_{jt} = Z_t n_{jt} \tag{2}$$

$$y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t \tag{3}$$

$$\psi(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log \left(\frac{p_{jt}}{p_{jt-1}} \right) \right]^2 Y_t = \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log(p_{jt}) - \log(p_{jt-1}) \right]^2 Y_t \tag{4}$$

Which are, respectively, the production function, the demand for the variety j and the adjustment cost.

Since equations 2 and 3 are the same quantity, we can obtain the demand of labor:

$$Z_t n_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$n_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} \tag{5}$$

Plug equations 3 and 5 in the firm problem,equation 1, to obtain:

$$\max_{p_{jt}} E_0 \left[\sum_{t=0}^{\infty} M_t P_t \left(\frac{p_{jt}}{P_t} \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t - w_t \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] =$$

$$\max_{p_{jt}} E_0 \left[\sum_{t=0}^{\infty} M_t P_t \left(\left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}+1} Y_t - w_t \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] =$$

$$\max_{p_{jt}} E_0 \left[\sum_{t=0}^{\infty} M_t P_t \left(\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu-1-\mu}{\mu-1}} Y_t - w_t \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] =$$

$$\max_{p_{jt}} E_0 \left[\sum_{t=0}^{\infty} M_t P_t \left(\left(\frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}} Y_t - w_t \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] \tag{6}$$

The First Order Conditions from the problem above yield:

$$M_t P_t \left(-\frac{1}{\mu-1} \frac{1}{P_t} \left(\frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}-1} Y_t + \frac{\mu}{\mu-1} \frac{w_t}{P_t} \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}} \right) + M_{t+1} P_{t+1} \left(-\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}} \right) = \tag{7}$$

Lets calculate the $\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}}$ and $\frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}}$, using equation 3:

$$\frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}} = \frac{\mu}{\mu-1} \frac{2}{2\kappa} \log \left(\frac{p_{jt}}{p_{jt-1}} \right) \frac{Y_t}{p_{jt}} \tag{8a}$$

$$\frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt-1}} = -\frac{\mu}{\mu-1} \frac{2}{2\kappa} \log \left(\frac{p_{jt}}{p_{jt-1}} \right) \frac{Y_t}{p_{jt-1}} \tag{8b}$$

By forwarding every term in 8b forward one period of time, we obtain:

$$\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}} = -\frac{\mu}{\mu-1}\frac{2}{2\kappa}\log\left(\frac{p_{jt+1}}{p_{jt}}\right)\frac{Y_{t+1}}{p_{jt}} \tag{9}$$

Plug equations 8b and 9 in equation 7 to obtain:

$$M_t P_t \left(-\frac{1}{\mu-1}\frac{1}{P_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{1}{\mu-1}-1} Y_t + \frac{\mu}{\mu-1}\frac{w_t}{P_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}-1}\frac{Y_t}{Z_t} - \frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(\frac{p_{jt}}{p_{jt-1}}\right)\frac{Y_t}{p_{jt}}\right) + \\ + M_{t+1}P_{t+1}\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(\frac{p_{jt+1}}{p_{jt}}\right)\frac{Y_{t+1}}{p_{jt}} \tag{10}$$

Note that, in the first parenthesis of equation 10, everyone depends on Y_t, $\frac{1}{P_t}$ and $\frac{1}{\mu-1}$. Simplifying:

$$M_t \frac{P_t}{P_t}\frac{Y_t}{\mu-1}\left(-\left(\frac{p_{jt}}{P_t}\right)^{-\frac{1}{\mu-1}-1} + \mu\frac{w_t}{Z_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}-1} - \frac{\mu}{\kappa}\log\left(\frac{p_{jt}}{p_{jt-1}}\right)\frac{P_t}{p_{jt}}\right) + M_{t+1}P_{t+1}\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(\frac{p_{jt+1}}{p_{jt}}\right)\frac{Y_{t+1}}{p_{jt}} \tag{11}$$

Let $P_t = p_{jt} \forall t$, which is the symmetric solution:

$$M_t \frac{Y_t}{\mu-1}\left(-\left(\frac{P_t}{P_t}\right)^{-\frac{1}{\mu-1}-1} + \mu\frac{w_t}{Z_t}\left(\frac{P_t}{P_t}\right)^{-\frac{\mu}{\mu-1}-1} - \frac{\mu}{\kappa}\log\left(\frac{P_t}{P_{t-1}}\right)\frac{P_t}{P_t}\right) + M_{t+1}P_{t+1}\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(\frac{P_{t+1}}{P_t}\right)\frac{Y_{t+1}}{P_t} = \\ = M_t\frac{Y_t}{\mu-1}\left(-1 + \mu\frac{w_t}{Z_t} - \frac{\mu}{\kappa}\log\left(\frac{P_t}{P_{t-1}}\right)\right) + M_{t+1}\frac{P_{t+1}}{P_t}\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(\frac{P_{t+1}}{P_t}\right)Y_{t+1} \tag{12}$$

Defining $1+\pi_t = \frac{P_t}{P_{t-1}}$, equation 12 becomes:

$$M_t\frac{Y_t}{\mu-1}\left(-1 + \mu\frac{w_t}{Z_t} - \frac{\mu}{\kappa}\log(1+\pi_t)\right) + M_{t+1}(1+\pi_{t+1})\frac{\mu}{\mu-1}\frac{1}{\kappa}\log(1+\pi_{t+1})Y_{t+1} \tag{13}$$

Recall that 13 actually represents a first order condition, therefore:

$$M_t\frac{Y_t}{\mu-1}\left(-1 + \mu\frac{w_t}{Z_t} - \frac{\mu}{\kappa}\log(1+\pi_t)\right) + M_{t+1}(1+\pi_{t+1})\frac{\mu}{\mu-1}\frac{1}{\kappa}\log(1+\pi_{t+1})Y_{t+1} = 0 \tag{14}$$

Rearranging, we obtain:

$$\begin{aligned}
M_t Y_t\left(-1 + \mu\frac{w_t}{Z_t} - \frac{\mu}{\kappa}\log(1+\pi_t)\right) + M_{t+1}(1+\pi_{t+1})\mu\frac{1}{\kappa}\log(1+\pi_{t+1})Y_{t+1} = 0 \\
M_t Y_t\left(\mu\frac{w_t}{Z_t} - 1\right) + M_{t+1}(1+\pi_{t+1})\frac{\mu}{\kappa}\log(1+\pi_{t+1})Y_{t+1} = \frac{\mu}{\kappa}M_t Y_t\log(1+\pi_t) \\
\frac{\kappa}{\mu}\mu\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + (1+\pi_{t+1})\frac{M_{t+1}}{M_t}\frac{Y_{t+1}}{Y_t}\frac{\mu}{\kappa}\frac{\kappa}{\mu}\log(1+\pi_{t+1}) = \log(1+\pi_t) \\
\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + (1+\pi_{t+1})\frac{M_{t+1}}{M_t}\frac{Y_{t+1}}{Y_t}\log(1+\pi_{t+1}) = \log(1+\pi_t)
\end{aligned} \tag{15}$$

2 Second try

Firm problem, dropping P_t:

$$\max_{p_{jt}} E_0 \left[\sum_{t=0}^{\infty} M_t \left(\frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \psi(p_{jt}, p_{jt-1}) \right) \right] \tag{16}$$

Rearranging using 2 and 3:

$$\max_{p_{jt}} E_0 \left[\sum_{t=0}^{\infty} M_t \left(\left(\frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}} Y_t - w_t \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] \tag{17}$$

First order conditions with respect to p_{jt}:

$$M_t \left(-\frac{1}{\mu-1} \frac{Y_t}{P_t} \left(\frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}-1} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}} \right) + M_{t+1} \left(-\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}} \right) = \tag{18}$$

$$M_t \left(-\frac{1}{\mu-1} \frac{Y_t}{P_t} \left(\frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}-1} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left(\frac{p_{jt}}{p_{jt-1}} \right) \frac{Y_t}{p_{jt}} \right) +$$
$$+ M_{t+1} \left(\frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left(\frac{p_{jt+1}}{p_{jt}} \right) \frac{Y_{t+1}}{p_{jt}} \right) \tag{19}$$

Imposing $p_{jt} = P_t \ \forall t$, we obtain:

$$M_t \left(-\frac{1}{\mu-1} \frac{Y_t}{P_t} \left(\frac{P_t}{P_t} \right)^{-\frac{1}{\mu-1}-1} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \left(\frac{P_t}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left(\frac{P_t}{P_{t-1}} \right) \frac{Y_t}{P_t} \right) +$$
$$+ M_{t+1} \left(\frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left(\frac{P_{t+1}}{P_t} \right) \frac{Y_{t+1}}{P_t} \right) =$$
$$M_t \left(-\frac{1}{\mu-1} \frac{Y_t}{P_t} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \frac{Y_t}{Z_t} - \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left(\underbrace{\frac{P_t}{P_{t-1}}}_{1+\pi_t} \right) \frac{Y_t}{P_t} \right) + M_{t+1} \left(\frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left(\underbrace{\frac{P_{t+1}}{P_t}}_{1+\pi_{t+1}} \right) \frac{Y_t}{P_t} \right) =$$
$$\frac{M_t}{P_t} \frac{\mu}{\mu-1} Y_t \left(-\frac{1}{\mu} + \frac{w_t}{Z_t} - \frac{1}{\kappa} \log(1+\pi_t) \right) + M_{t+1} \left(\frac{\mu}{\mu-1} \frac{1}{\kappa} \log(1+\pi_{t+1}) \frac{Y_{t+1}}{P_t} \right) \tag{20}$$

This is a first order condition, so:

$$\frac{M_t}{P_t} \frac{\mu}{\mu-1} Y_t \left(-\frac{1}{\mu} + \frac{w_t}{Z_t} - \frac{1}{\kappa} \log(1+\pi_t) \right) + \frac{M_{t+1}}{P_t} \frac{\mu}{\mu-1} \left(\frac{1}{\kappa} \log(1+\pi_{t+1}) Y_{t+1} \right) = 0$$
$$M_t Y_t \frac{1}{\kappa} \log(1+\pi_t) = M_t Y_t \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + M_{t+1} Y_{t+1} \frac{1}{\kappa} \log(1+\pi_{t+1}) \tag{21}$$
$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{M_{t+1}}{M_t} \frac{Y_{t+1}}{Y_t} \log(1+\pi_{t+1}) \tag{22}$$