

# 1 First Try

Firm problem:

$$\max_{p_{jt}} E_0 \left[ \sum_{t=0}^{\infty} M_t P_t \left( \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \psi(p_{jt}, p_{jt-1}) \right) \right] \quad (1)$$

And we have the following:

$$y_{jt} = Z_t n_{jt} \quad (2)$$

$$y_{jt} = \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t \quad (3)$$

$$\psi(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[ \log \left( \frac{p_{jt}}{p_{jt-1}} \right) \right]^2 Y_t = \frac{\mu}{\mu-1} \frac{1}{2\kappa} [\log(p_{jt}) - \log(p_{jt-1})]^2 Y_t \quad (4)$$

Which are, respectively, the production function, the demand for the variety  $j$  and the adjustment cost.

Since equations 2 and 3 are the same quantity, we can obtain the demand of labor:

$$\begin{aligned} Z_t n_{jt} &= \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t \\ n_{jt} &= \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} \end{aligned} \quad (5)$$

Plug equations 3 and 5 in the firm problem, equation 1, to obtain:

$$\begin{aligned} \max_{p_{jt}} E_0 \left[ \sum_{t=0}^{\infty} M_t P_t \left( \frac{p_{jt}}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} Y_t - w_t \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] &= \\ \max_{p_{jt}} E_0 \left[ \sum_{t=0}^{\infty} M_t P_t \left( \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}+1} Y_t - w_t \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] &= \\ \max_{p_{jt}} E_0 \left[ \sum_{t=0}^{\infty} M_t P_t \left( \left( \frac{p_{jt}}{P_t} \right)^{\frac{\mu-1-\mu}{\mu-1}} Y_t - w_t \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] &= \\ \max_{p_{jt}} E_0 \left[ \sum_{t=0}^{\infty} M_t P_t \left( \left( \frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}} Y_t - w_t \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] &= \end{aligned} \quad (6)$$

The First Order Conditions from the problem above yield:

$$M_t P_t \left( -\frac{1}{\mu-1} \frac{1}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}-1} Y_t + \frac{\mu}{\mu-1} \frac{w_t}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}} \right) + M_{t+1} P_{t+1} \left( -\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}} \right) = 0 \quad (7)$$

Lets calculate the  $\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}}$  and  $\frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}}$ , using equation 3:

$$\frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}} = \frac{\mu}{\mu-1} \frac{2}{2\kappa} \log \left( \frac{p_{jt}}{p_{jt-1}} \right) \frac{Y_t}{p_{jt}} \quad (8a)$$

$$\frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt-1}} = -\frac{\mu}{\mu-1} \frac{2}{2\kappa} \log \left( \frac{p_{jt}}{p_{jt-1}} \right) \frac{Y_t}{p_{jt-1}} \quad (8b)$$

By forwarding every term in 8b forward one period of time, we obtain:

$$\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}} = -\frac{\mu}{\mu-1} \frac{2}{2\kappa} \log\left(\frac{p_{jt+1}}{p_{jt}}\right) \frac{Y_{t+1}}{p_{jt}} \quad (9)$$

Plug equations 8b and 9 in equation 7 to obtain:

$$\begin{aligned} M_t P_t \left( -\frac{1}{\mu-1} \frac{1}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}-1} Y_t + \frac{\mu}{\mu-1} \frac{w_t}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\mu}{\mu-1} \frac{1}{\kappa} \log\left(\frac{p_{jt}}{p_{jt-1}}\right) \frac{Y_t}{p_{jt}} \right) + \\ + M_{t+1} P_{t+1} \frac{\mu}{\mu-1} \frac{1}{\kappa} \log\left(\frac{p_{jt+1}}{p_{jt}}\right) \frac{Y_{t+1}}{p_{jt}} \end{aligned} \quad (10)$$

Note that, in the first parenthesis of equation 10, everyone depends on  $Y_t$ ,  $\frac{1}{P_t}$  and  $\frac{1}{\mu-1}$ . Simplifying:

$$M_t \frac{P_t}{P_t} \frac{Y_t}{\mu-1} \left( -\left(\frac{p_{jt}}{P_t}\right)^{-\frac{1}{\mu-1}-1} + \mu \frac{w_t}{Z_t} \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}-1} - \frac{\mu}{\kappa} \log\left(\frac{p_{jt}}{p_{jt-1}}\right) \frac{P_t}{p_{jt}} \right) + M_{t+1} P_{t+1} \frac{\mu}{\mu-1} \frac{1}{\kappa} \log\left(\frac{p_{jt+1}}{p_{jt}}\right) \frac{Y_{t+1}}{p_{jt}} \quad (11)$$

Let  $P_t = p_{jt} \forall t$ , which is the symmetric solution:

$$\begin{aligned} M_t \frac{Y_t}{\mu-1} \left( -\left(\frac{P_t}{P_t}\right)^{-\frac{1}{\mu-1}-1} + \mu \frac{w_t}{Z_t} \left(\frac{P_t}{P_t}\right)^{-\frac{\mu}{\mu-1}-1} - \frac{\mu}{\kappa} \log\left(\frac{P_t}{P_{t-1}}\right) \frac{P_t}{P_t} \right) + M_{t+1} P_{t+1} \frac{\mu}{\mu-1} \frac{1}{\kappa} \log\left(\frac{P_{t+1}}{P_t}\right) \frac{Y_{t+1}}{P_t} = \\ = M_t \frac{Y_t}{\mu-1} \left( -1 + \mu \frac{w_t}{Z_t} - \frac{\mu}{\kappa} \log\left(\frac{P_t}{P_{t-1}}\right) \right) + M_{t+1} \frac{P_{t+1}}{P_t} \frac{\mu}{\mu-1} \frac{1}{\kappa} \log\left(\frac{P_{t+1}}{P_t}\right) Y_{t+1} \end{aligned} \quad (12)$$

Defining  $1 + \pi_t = \frac{P_t}{P_{t-1}}$ , equation 12 becomes:

$$M_t \frac{Y_t}{\mu-1} \left( -1 + \mu \frac{w_t}{Z_t} - \frac{\mu}{\kappa} \log(1 + \pi_t) \right) + M_{t+1} (1 + \pi_{t+1}) \frac{\mu}{\mu-1} \frac{1}{\kappa} \log(1 + \pi_{t+1}) Y_{t+1} \quad (13)$$

Recall that 13 actually represents a first order condition, therefore:

$$M_t \frac{Y_t}{\mu-1} \left( -1 + \mu \frac{w_t}{Z_t} - \frac{\mu}{\kappa} \log(1 + \pi_t) \right) + M_{t+1} (1 + \pi_{t+1}) \frac{\mu}{\mu-1} \frac{1}{\kappa} \log(1 + \pi_{t+1}) Y_{t+1} = 0 \quad (14)$$

Rearranging, we obtain:

$$\begin{aligned} M_t Y_t \left( -1 + \mu \frac{w_t}{Z_t} - \frac{\mu}{\kappa} \log(1 + \pi_t) \right) + M_{t+1} (1 + \pi_{t+1}) \mu \frac{1}{\kappa} \log(1 + \pi_{t+1}) Y_{t+1} = 0 \\ M_t Y_t \left( \mu \frac{w_t}{Z_t} - 1 \right) + M_{t+1} (1 + \pi_{t+1}) \frac{\mu}{\kappa} \log(1 + \pi_{t+1}) Y_{t+1} = \frac{\mu}{\kappa} M_t Y_t \log(1 + \pi_t) \\ \frac{\kappa}{\mu} \mu \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + (1 + \pi_{t+1}) \frac{M_{t+1}}{M_t} \frac{Y_{t+1}}{Y_t} \frac{\mu}{\kappa} \frac{\kappa}{\mu} \log(1 + \pi_{t+1}) = \log(1 + \pi_t) \\ \kappa \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + (1 + \pi_{t+1}) \frac{M_{t+1}}{M_t} \frac{Y_{t+1}}{Y_t} \log(1 + \pi_{t+1}) = \log(1 + \pi_t) \end{aligned} \quad (15)$$

## 2 Second try

Firm problem, dropping  $P_t$ :

$$\max_{p_{jt}} E_0 \left[ \sum_{t=0}^{\infty} M_t \left( \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \psi(p_{jt}, p_{jt-1}) \right) \right] \quad (16)$$

Rearranging using 2 and 3:

$$\max_{p_{jt}} E_0 \left[ \sum_{t=0}^{\infty} M_t \left( \left( \frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}} Y_t - w_t \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{Z_t} - \psi(p_{jt}, p_{jt-1}) \right) \right] \quad (17)$$

First order conditions with respect to  $p_{jt}$ :

$$M_t \left( -\frac{1}{\mu-1} \frac{Y_t}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}-1} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\partial \psi(p_{jt}, p_{jt-1})}{\partial p_{jt}} \right) + M_{t+1} \left( -\frac{\partial \psi(p_{jt+1}, p_{jt})}{\partial p_{jt}} \right) = \quad (18)$$

$$M_t \left( -\frac{1}{\mu-1} \frac{Y_t}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{1}{\mu-1}-1} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \left( \frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left( \frac{p_{jt}}{p_{jt-1}} \right) \frac{Y_t}{p_{jt}} \right) + \quad (19)$$

$$+ M_{t+1} \left( \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left( \frac{p_{jt+1}}{p_{jt}} \right) \frac{Y_{t+1}}{p_{jt}} \right)$$

Imposing  $p_{jt} = P_t \forall t$ , we obtain:

$$M_t \left( -\frac{1}{\mu-1} \frac{Y_t}{P_t} \left( \frac{P_t}{P_t} \right)^{-\frac{1}{\mu-1}-1} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \left( \frac{P_t}{P_t} \right)^{-\frac{\mu}{\mu-1}-1} \frac{Y_t}{Z_t} - \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left( \frac{P_t}{P_{t-1}} \right) \frac{Y_t}{P_t} \right) + \quad (20)$$

$$+ M_{t+1} \left( \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left( \frac{P_{t+1}}{P_t} \right) \frac{Y_{t+1}}{P_t} \right) =$$

$$M_t \left( -\frac{1}{\mu-1} \frac{Y_t}{P_t} + \frac{\mu}{\mu-1} w_t \frac{1}{P_t} \frac{Y_t}{Z_t} - \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left( \underbrace{\frac{P_t}{P_{t-1}}}_{1+\pi_t} \right) \frac{Y_t}{P_t} \right) + M_{t+1} \left( \frac{\mu}{\mu-1} \frac{1}{\kappa} \log \left( \underbrace{\frac{P_{t+1}}{P_t}}_{1+\pi_{t+1}} \right) \frac{Y_{t+1}}{P_t} \right) =$$

$$\frac{M_t}{P_t} \frac{\mu}{\mu-1} Y_t \left( -\frac{1}{\mu} + \frac{w_t}{Z_t} - \frac{1}{\kappa} \log(1 + \pi_t) \right) + M_{t+1} \left( \frac{\mu}{\mu-1} \frac{1}{\kappa} \log(1 + \pi_{t+1}) \frac{Y_{t+1}}{P_t} \right)$$

This is a first order condition, so:

$$\frac{M_t}{P_t} \frac{\mu}{\mu-1} Y_t \left( -\frac{1}{\mu} + \frac{w_t}{Z_t} - \frac{1}{\kappa} \log(1 + \pi_t) \right) + \frac{M_{t+1}}{P_t} \frac{\mu}{\mu-1} \left( \frac{1}{\kappa} \log(1 + \pi_{t+1}) Y_{t+1} \right) = 0 \quad (21)$$

$$M_t Y_t \frac{1}{\kappa} \log(1 + \pi_t) = M_t Y_t \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + M_{t+1} Y_{t+1} \frac{1}{\kappa} \log(1 + \pi_{t+1})$$

$$\log(1 + \pi_t) = \kappa \left( \frac{w_t}{Z_t} - \frac{1}{\mu} \right) + \frac{M_{t+1}}{M_t} \frac{Y_{t+1}}{Y_t} \log(1 + \pi_{t+1}) \quad (22)$$