Proposal: Implied Ordinal Preferences

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Abstract

This paper investigates the the possibility of uncovering implied ordinal preferences when provided incomplete ordinal preferences for one side of a matching market. The work of Gale and Shapely's [8] provided the first algorithm for solving the stable marriage and attempted the more general college admissions problems (verified as different problems by Roth [13]). Much of the work in this field focuses on stability, which is not necessarily a requirement in systems where market participants are compelled [12]. In situations of compulsion, the preference u (un-assigned) is not an option. This arises in legally compulsory assignment situations such as jobs in the military or secondary school enrollment.

In these situation, complete ordinal preferences are not always provided. This can be due to a lack of time, lack of knowledge about preferences, excess of options, or some other system deficiency that does not encourage/enable participants to express preferences on all matching options.

This paper proposes a system where, given at least one preference for all participants on one side of the market, preferences may be supposed at a higher accuracy than random preference assumption.

The code to demonstrate the matching algorithms, optimization, and preference-based metrics can be found in Ian Shaw's Github Repository. [1]

[1] https://github.com/ieshaw/Imp_Ord_Pref

Contents

1 Formulation

Consider a worker i and an opportunity at a firm j. Suppose there are n firms, but worker i not expressed complete preferences, $n_i < n$. Suppose there are m workers in the compulsory market.

Consider the following terms

$$J = \text{set of opportunities at firms in the market}$$
$$|J| = n$$
$$P_i \in \mathbb{Z}^{+,n\times 1}$$
$$P_{ij} = \begin{cases} p & p \in \mathbb{Z}^+ > 0, \text{expressed ordinal preference of worker } i \text{ for job } j \\ 0 & \text{worker } i \text{ has not expressed an ordinal preference for job } j \end{cases}$$
$$P_i^C \in \mathbb{Z}^{+,n\times 1}$$
$$P_{ij}^C = \begin{cases} n - p & p \in \mathbb{Z}^+ > 0, \text{expressed ordinal preference of worker } i \text{ for job } j \\ 0 & \text{worker } i \text{ has not expressed an ordinal preference for job } j \end{cases}$$
$$J_i = \{j : P_{ij} \neq 0\}$$
$$|J_i| = n_i$$
$$J_i' = J \setminus J_i$$
$$|J_i'| = n - n_i$$

$S_{i,i'}$ = similarity of worker i and i'

Our proposed implied ordinal preference system P_i' is developed in the following manner. The metric r is the similarity of the worker in question with another worker multiplied by the complement ordinal ranking of that worker (so that metric increases as similar workers more prefer the positions in question), summed across all workers. The metrics are then sorted in descending order. Ties are broken randomly. Then the ordinal ranking of these metrics are considered the implied ordinal preference for the for previously unexpressed preferences.

$$r_{ij} = \sum_{i'}^{m-1} S_{i,i'} P_{i',j}^C$$
$$R_i = [r_{ij} : P_{ij} == 0]$$
$$R_i[k] \geq R_i[k+1]$$
$$P_i' \in \mathbb{Z}^{+,n\times 1}$$
$$P_{i,j}' = \begin{cases} p & p \in \mathbb{Z}^+ > 0, \text{ expressed ordinal preference of worker } i \text{ for job } j \\ n_i + k & R_i[k] == r_{ij}, \text{ implied ordinal preference of worker } i \text{ for job } j \end{cases}$$

1.1 Similarity Measures

We explore three types of similarity measure in our investigation: cosine similarity, normed Euclidean distance, and weighted Euclidean distances

Note that we use the complement of the ordinal preference, because having information up to the $n^{th} - 1$ preference is equivalent to having the n^{th} preference; we indicate having no preference information as a preference 0. Thus the most preferred choice is given a value of n.

The cosine similarity of two workers i, i' when considering their preference vectors P, is

$$S_{i,i'} = \frac{P_i^C \bullet P_{i'}^C}{||P_i^C|| ||P_{i'}^C||}$$

Normed Euclidean distances is calculated as

$$S_{i,i'} = 1 - \frac{||P_i^C - P_{i'}^C||^2}{||P_i^C||||P_{i'}^C||}$$

The two measures above are chosen due to their established usage. Below we propose what we call weighted Euclidean distance, intended to punish dissimilar preferences more if highly preferred by one of th workers.

$$S_{i,i'} = 1 - \frac{1}{2n(n-1)^2}\sum_{k=1}^{n}(P_{i,k}^S + P_{j,k}^S)(P_{i,k}^S - P_{i',k}^S)^2$$

2 Analysis

2.1 Process

Given a set of preferences for one side of the market (does not need to be complete[complete means having preferences for all possibilities]), replace a certain proportion of known preferences with an "unknown" indicator (obviously have to work bottom up because if someone ranks their top 5 and their 7th, then they would know their 6th)[leaving the idea of knowing most and least preferred but not middle preferred for future work]. With each dropout propose two sets of preferences, one with random assignment up to complete preferences, and one using the implied ordinal preference system up to complete preference. Compare the Root-mean-squared error (RMSE) of the two proposed preference sets to the actual preference set.

2.2 Data Sources

Sources of Preference Data.
<u>Found</u>

1. Navy Medical Corps (Doctor) preferences for assignment. Seeking approval for use from Navy Medical Corps.

<u>Searching</u>

1. Air Force Officer Data. Reaching out to the Air Force Personnel Command.

2. New York School System. Maybe here? `https://data.nysed.gov/`.

3. Boston School System. This is the quote from the HKS report.

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 In response to an open records request,
the BPS provided the data from 2005 to
2010. http://www.pioneerinstitute.org/
blog/wp-content/uploads/2005-2010-
West-Zone-Sibling-Preference-Data.xlsx
```

2.3 Results

Here would be a plot of how for each data set, with increasing dropout rates the proposed implied similarity process has a smaller RMSE than random preference assignment, though they monotonically increase with increasing proportional dropout.

3 Literature Review

3.1 Completed Thus Far

The most important people in this field, and the winners of the 2012 Nobel Prize in Economics for their work in stable marriage matching, are Roth [2] and Shapely [3].

1962

The intent of this algorithm is to provide stable pairings between job owners and job seekers based on their ranked preferences. The algorithm's initial conception and definition of stability can be found in Gale and Shapely's 1962 publication in the January *The American Mathematical Monthly* [8]. The algorithm completes in polynomial time and was originally written for application in collage admissions.

1982

Roth explores the incentives of conveying true preferences and whether is it in everyone's best interest to do so. [12] In his work he specifically points to the applications to "civil servants with civil service positions".

1985

Roth explores the stable marriage problem specifically in the terms of 'firms and workers', also calling upon the lens of game theory. [14] He discussed an extension of the model from one assignment for each worker or firm, to multiple workers for each firm, to a situation where each firm can have multiple workers and each worker could have multiple firms. He also explored, under the constraint of stability, how in each model the optimal assignment set for one party (eg: firms) is the least optimal for the other (eg: workers). He elaborates that this final phenomenon creates difficulty in the institutional decision of how to formulate the matching algorithm.

1989

Irving explored indifference preferences and the follow on adoption for the Gale-Shapely algorithm. [9] This provides the theoretical framework allowing for indifference in our own formulation. Though much of his focus is on differing forms of stability (weak, strongly, and super) these lie outside of our investigation due to the Navy's authority to compel its members to placement.

1993

Roth, Rothblum, and Vande Vate explored the concept of partial matches, discovering in fact this forms a lattice of solutions as well. These fractional matches could represent lotteries or time splitting. [2]

3.2 Citations and Proposed Literature Review

Here we cite the literature thus far reviewed, and that which we intend to review. This list is not exhaustive as we know the review always grows throughout the investigation process. Any recommended literature will be enthusiastically added.

References

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[12] A. Roth. The economics of matching: Stability and incentives. *Mathematics of Operations Research*, 7(4), November 1982.

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[14] A. Roth. Conflict and coincidence of interest in job matching: Some new results and open questions. *Mathematics of Operations Research*, 10(3), August 1985.

[15] S. Mitchell S. Khuller and V. Vazirani. On-line algorithms for weighted bipartite matching and stable marriages. *Theoretical Computer Science*, 127:255–267, May Year.

4 Future Work

1. Adding in the owner preferences (show formulation)
2. Using regression or some sort of machine learning
3. Extending to non-compelling situations; such as there being an option for u (un-assignment).
4. Extend to wagering situations
5. Extend to situations where a market participant knows their top (say 1-5) preferences, their least preferred (say 15-20), but not necessarily their middle (6-14).