

# 1 DC Motor Circuit

As an electrical schematic a DC motor can be represented as:

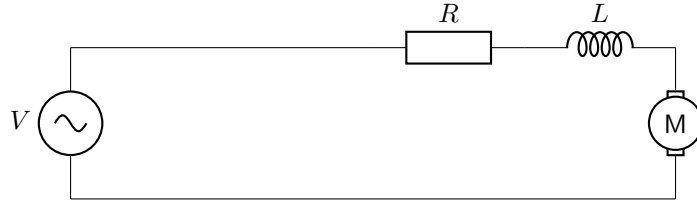


Figure 1: Electrical diagram of DC motor.

By introducing an armature constant of the motor  $K_t$ , the torque the motor produces,  $T$ , is related to the armature current of the motor by:

$$T = K_t i \quad (1)$$

Similarly the motor back emf,  $e$ , is related to the rotational speed,  $\dot{\theta}$  using the motor constant  $K_e$  by:

$$e = K_e \dot{\theta} \quad (2)$$

From Kirchhoff's law the sum of potential differences in a closed loop must be zero. This leads to:

$$iR + \frac{di}{dt}L = V - k_e \dot{\theta} \quad (3)$$

## 2 Inertia

The inertia of the motor armature must also be considered. From Newton's second law:

$$J\ddot{\theta} + B\dot{\theta} = T \quad (4)$$

Where  $J$  is the motor inertial constant, and  $B$  the motor damping constant.

## 3 State-space Equations

The following equations:

$$iR + \frac{di}{dt}L = V - k_e \dot{\theta} \quad (5a)$$

$$J\ddot{\theta} + B\dot{\theta} = T \quad (5b)$$

can be written in state-space form. Selecting  $\dot{\theta}$ ,  $\ddot{\theta}$  and  $i$  as the state variables:

$$\dot{\theta} = \dot{\theta} \quad (6)$$

For the inertia:

$$J\ddot{\theta} + B\dot{\theta} = T \quad (7a)$$

$$\ddot{\theta} + \frac{B}{J}\dot{\theta} = \frac{T}{J} \quad (7b)$$

$$\ddot{\theta} = \frac{T}{J} - \frac{B}{J}\dot{\theta} \quad (7c)$$

$$\ddot{\theta} = i\frac{K_t}{J} - \frac{B}{J}\dot{\theta} \quad (7d)$$

$$(7e)$$

For Kirchhoff's law:

$$iR + \frac{di}{dt}L = V - k_e\dot{\theta} \quad (8a)$$

$$i\frac{R}{L} + \frac{di}{dt} = \frac{V}{L} - \frac{k_e}{L}\dot{\theta} \quad (8b)$$

$$\frac{di}{dt} = \frac{V}{L} - \frac{k_e}{L}\dot{\theta} - i\frac{R}{L} \quad (8c)$$

$$(8d)$$

Using  $V$  as the input, and the motor position as the output:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{K_t}{J} \\ 0 & -\frac{k_e}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V \quad (9a)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} \quad (9b)$$

## 4 Current Sensing

As the torque is proportional to the current draw, the torque the motor produces can be found by measuring the current. Current feedback can be achieved by connecting a small resistor from the servo ground to the circuit ground and measuring the voltage drop across it. From Ohm's law  $V = IR$  so given the value of the resistor is known priori and  $V$  can be measured the current can be found.

This is shown in the diagram below:

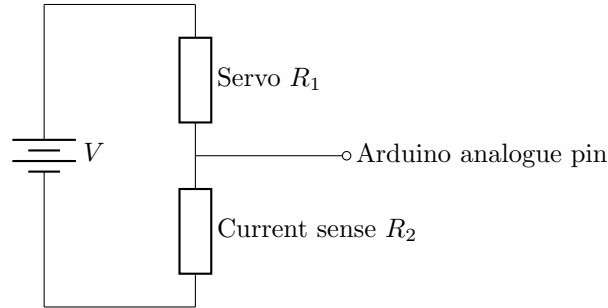


Figure 2: Electrical diagram of current sensing.

The voltage measurement can be done by the Arduino by tapping off of the centre of the potential divider ( $R_1$  and  $R_2$ ). The current sense resistor  $R_2$  should be small so as to not affect the operation of the servo. A value of  $1\Omega$  is sufficient for  $R_2$ .

## 4.1 Current smoothing

The problem with measuring the current from a servo motor is it is driven by a pulse width modulation (PWM) signal. For a torque controller to work it must receive a continuous value for the current. Smoothing of the current signal can be achieved with a resistor capacitor (RC) circuit:

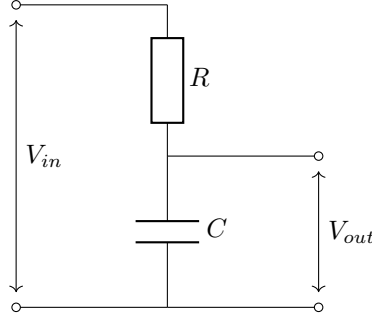


Figure 3: Electrical diagram of RC network.

The current flow is:

$$i(t) = \frac{V_{in}}{R} = \frac{V_R}{R} = C \frac{dV}{dt} \quad (10)$$

Also, for  $V_{out}$ :

$$V_{out} = V_C \quad (11a)$$

$$= \frac{Q}{C} \quad (11b)$$

$$= \frac{\int i(t) dt}{C} \quad (11c)$$

$$= \frac{1}{C} \int i(t) dt \quad (11d)$$

$$= \frac{1}{C} \int \frac{V_{in}}{R} dt \quad (11e)$$

$$= \frac{1}{RC} \int V_{in} dt \quad (11f)$$

$$V_{out}(t) = \frac{1}{RC} \int_0^t V_{in} dt \quad (11g)$$

A pulse wave is a simple case of what the RC network will need to be able to smooth and can be defined as:

$$f(x) = \begin{cases} 5 & \text{if } x < T/2 \\ 0 & \text{if } x > T/2 \end{cases} \quad (12)$$

To get the voltage across a capacitor charging start with Kirchhoff's voltage law:

$$V_{in} = V_R + V_C \quad (13)$$

For the current:

$$i_R = i_C = C \frac{dV_C}{dt} \quad (14)$$

So that:

$$V_R = i_R R = i_C R = C \frac{dV_C}{dt} R \quad (15)$$

$$V_{in} = C \frac{dV_C}{dt} R + V_C \quad (16a)$$

$$RC \frac{dV_C}{dt} = V_C - V_{in} \quad (16b)$$

$$\int \frac{1}{V_C - V_{in}} dV_C = \int \frac{1}{RC} dt \quad (16c)$$

$$-\ln(V_{in} - V_C) = \frac{t}{RC} + c \quad (16d)$$

$$V_{in} - V_C = Ae^{-\frac{t}{RC}} \quad (16e)$$

$$V_{C,charge}(t) = V_{in} \left[ 1 - e^{-\frac{t}{RC}} \right] \quad (16f)$$

(Assuming  $V_C(t=0) = 0 \rightarrow A = V_{in}$ ).

Similarly for discharging:

$$V_{in} - V_C = Ae^{-\frac{t}{RC}} \quad (17a)$$

$$0 - V_C = Ae^{-\frac{t}{RC}} \quad (17b)$$

$$V_{C,discharge}(t) = V_0 \left[ e^{-\frac{t}{RC}} \right] \quad (17c)$$

(Assuming  $V_C(t=0) = V_0 \rightarrow A = -V_0$ ).

For response analysis of the RC circuit it makes sense to evaluate the circuit as a transfer function.

$$V_{in}(t) = i(t)R + \frac{1}{C} \int i(t) dt \quad (18a)$$

$$\mathcal{L}(V_{in}(t)) = V_{in}(s) = I(s)R + \frac{1}{C} \frac{1}{s} I(s) \quad (18b)$$

$$\frac{I(s)}{V_{in}(s)} = \frac{1}{R + \frac{1}{Cs}} \quad (18c)$$

$$G(s) = \frac{Cs}{RCs + 1} \quad (18d)$$

Similarly the pulse wave can be found as a transfer function:

$$\mathcal{L} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \quad (19a)$$

$$= \frac{1}{1 - e^{-sT}} \int_0^{T/2} e^{-st} K dt \quad (19b)$$

$$= \frac{K}{1 - e^{-sT}} \int_0^{T/2} e^{-st} dt \quad (19c)$$

$$= \frac{K}{1 - e^{-sT}} \left[ \frac{e^{-st}}{-s} \right]_0^{T/2} \quad (19d)$$

$$= \frac{K}{(-s)(1 - e^{-sT})} [e^{-st}]_0^{T/2} \quad (19e)$$

$$= \frac{K}{(-s)(1 - e^{-sT})} [e^{-Ts/2} - 1] \quad (19f)$$

$$= \frac{K(e^{-Ts/2} - 1)}{(-s)(1 - e^{-sT})} \quad (19g)$$

$$= \frac{K(1 - e^{-Ts/2})}{(s)(1 - e^{-sT})} \quad (19h)$$

$$= \frac{K}{s} \frac{1 - e^{-Ts/2}}{1 - e^{-sT}} \quad (19i)$$

Combining the transfer functions:

$$= \frac{Cs}{RCs + 1} \frac{K}{s} \frac{1 - e^{-Ts/2}}{1 - e^{-Ts}} \quad (20a)$$

$$= \frac{KC}{s} \frac{s(1 - e^{-Ts/2})}{(RCs + 1)(1 - e^{-Ts})} \quad (20b)$$

$$= \frac{KC(1 - e^{-Ts/2})}{(RCs + 1)(1 - e^{-Ts})} \quad (20c)$$

Using the final value theorem:  $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$ .

$$sF(s) = s \frac{KC(1 - e^{-Ts/2})}{(RCs + 1)(1 - e^{-Ts})} \quad (21a)$$

$$= \frac{sKC - sKCe^{-Ts/2}}{(RCs + 1)(1 - e^{-Ts})} \quad (21b)$$

$$= \frac{sKC - sKCe^{-Ts/2}}{RCs - RCse^{-Ts} + 1 - e^{-Ts}} \quad (21c)$$