1 DC Motor Circuit

As an electrical schematic a DC motor can be represented as:

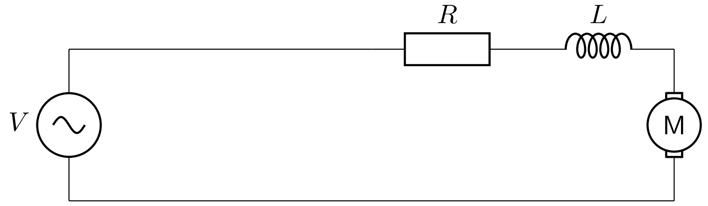

Figure 1: Electrical diagram of DC motor.

By introducing an armature constant of the motor K_t, the torque the motor produces, T, is related to the armature current of the motor by:

$$T = K_t i \tag{1}$$

Similarly the motor back emf, e, is related to the rotational speed, $\dot{\theta}$ using the motor constant K_e by:

$$e = K_e \dot{\theta} \tag{2}$$

From Kirchhoff's law the sum of potential differences in a closed loop must be zero. This leads to:

$$iR + \frac{di}{dt}L = V - k_e \dot{\theta} \tag{3}$$

2 Inertia

The inertia of the motor armature must also be considered. From Newton's second law:

$$J\ddot{\theta} + B\dot{\theta} = T \tag{4}$$

Where J is the motor inertial constant, and B the motor damping constant.

3 State-space Equations

The following equations:

$$iR + \frac{di}{dt}L = V - k_e \dot{\theta} \tag{5a}$$
$$J\ddot{\theta} + B\dot{\theta} = T \tag{5b}$$

can be written in state-space form. Selecting $\dot{\theta}$, $\ddot{\theta}$ and i as the state variables:

$$\dot{\theta} = \dot{\theta} \tag{6}$$

For the inertia:

$$J\ddot{\theta} + B\dot{\theta} = T \tag{7a}$$
$$\ddot{\theta} + \frac{B}{J}\dot{\theta} = \frac{T}{J} \tag{7b}$$
$$\ddot{\theta} = \frac{T}{J} - \frac{B}{J}\dot{\theta} \tag{7c}$$
$$\ddot{\theta} = i\frac{K_t}{J} - \frac{B}{J}\dot{\theta} \tag{7d}$$

(7e)

For Kirchhoff's law:

$$iR + \frac{di}{dt}L = V - k_e\dot{\theta} \tag{8a}$$
$$i\frac{R}{L} + \frac{di}{dt} = \frac{V}{L} - \frac{k_e}{L}\dot{\theta} \tag{8b}$$
$$\frac{di}{dt} = \frac{V}{L} - \frac{k_e}{L}\dot{\theta} - i\frac{R}{L} \tag{8c}$$
$$\tag{8d}$$

Using V as the input, and the motor position as the output:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{K_t}{J} \\ 0 & -\frac{k_e}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V \tag{9a}$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} \tag{9b}$$

4 Current Sensing

As the torque is proportional to the current draw, the torque the motor produces can be found by measuring the current. Current feedback can be achieved by connecting a small resistor from the servo ground to the circuit ground and measuring the voltage drop across it. From Ohm's law $V = IR$ so given the value of the resistor is known priori and V can be measured the current can be found.

This is shown in the diagram below:

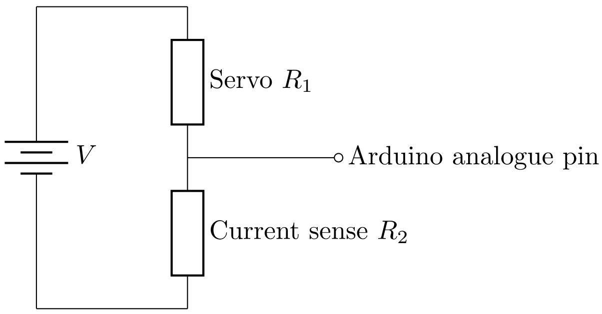

Figure 2: Electrical diagram of current sensing.

The voltage measurement can be done by the Arduino by tapping off of the centre of the potential divider (R_1 and R_2). The current sense resistor R_2 should be small so as to not affect the operation of the servo. A value of $1\,\Omega$ is sufficient for R_2.

4.1 Current smoothing

The problem with measuring the current from a servo motor is it is driven by a pulse width modulation (PWM) signal. For a torque controller to work it must receive a continuous value for the current. Smoothing of the current signal can be achieved with a resistor capacitor (RC) circuit:

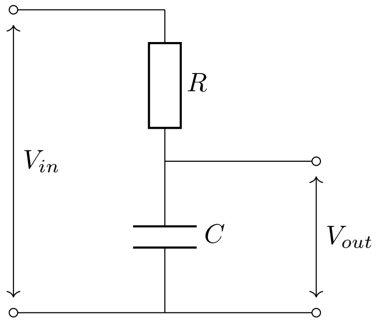

Figure 3: Electrical diagram of RC network.

The current flow is:

$$i(t) = \frac{V_{in}}{R} = \frac{V_R}{R} = C\frac{dV}{dt} \tag{10}$$

Also, for V_{out}:

$$\begin{aligned}
V_{out} &= V_C && (11a)\\
&= \frac{Q}{C} && (11b)\\
&= \frac{\int i(t)\ dt}{C} && (11c)\\
&= \frac{1}{C}\int i(t)\ dt && (11d)\\
&= \frac{1}{C}\int \frac{V_{in}}{R}\ dt && (11e)\\
&= \frac{1}{RC}\int V_{in}\ dt && (11f)\\
V_{out}(t) &= \frac{1}{RC}\int_0^t V_{in}\ dt && (11g)
\end{aligned}$$

A pulse wave is a simple case of what the RC network will need to be able to smooth and can be defined as:

$$f(x) = \begin{cases} 5 & \text{if } x < T/2 \\ 0 & \text{if } x > T/2 \end{cases} \tag{12}$$

To get the voltage across a capacitor charging start with Kirchhoff's voltage law:

$$V_{in} = V_R + V_C \tag{13}$$

For the current:

$$i_R = i_C = C\frac{dV_C}{dt} \tag{14}$$

So that:

$$V_R = i_R R = i_C R = C\frac{dV_C}{dt}R \tag{15}$$

$$V_{in} = C\frac{dV_C}{dt}R + V_C \tag{16a}$$
$$RC\frac{dV_C}{dt} = V_C - V_{in} \tag{16b}$$
$$\int \frac{1}{V_C - V_{in}} dV_C = \int \frac{1}{RC} dt \tag{16c}$$
$$-\ln(V_{in} - V_C) = \frac{t}{RC} + c \tag{16d}$$
$$V_{in} - V_C = Ae^{-\frac{t}{RC}} \tag{16e}$$
$$V_{C,charge}(t) = V_{in}\left[1 - e^{-\frac{t}{RC}}\right] \tag{16f}$$

(Assuming $V_C(t = 0) = 0 \rightarrow A = V_{in}$).

Similarly for discharging:

$$V_{in} - V_C = Ae^{-\frac{t}{RC}} \tag{17a}$$
$$0 - V_C = Ae^{-\frac{t}{RC}} \tag{17b}$$
$$V_{C,discharge}(t) = V_0\left[e^{-\frac{t}{RC}}\right] \tag{17c}$$

(Assuming $V_C(t = 0) = V_0 \rightarrow A = -V_0$).

For response analysis of the RC circuit it makes sense to evaluate the circuit as a transfer function.

$$V_{in}(t) = i(t)R + \frac{1}{C}\int i(t)\ dt \tag{18a}$$
$$\mathcal{L}(V_{in}(t)) = V_{in}(s) = I(s)R + \frac{1}{C}\frac{1}{s}I(s) \tag{18b}$$

$$V_{out}(t) = \frac{1}{C}\int i(t)\ dt \tag{19a}$$
$$\mathcal{L}(V_{in}(t)) = V_{in}(s) = \frac{1}{C}\frac{1}{s}I(s) \tag{19b}$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{Cs}I(s)}{I(s)R + \frac{1}{Cs}I(s)} \tag{20a}$$
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} \tag{20b}$$
$$G(s) = \frac{1}{RCs + 1} \tag{20c}$$

Similarly the pulse wave can be found as a transfer function:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\ dt && \text{(21a)}\\
&= \frac{1}{1-e^{-sT}} \int_0^{T/2} e^{-st} K\ dt && \text{(21b)}\\
&= \frac{K}{1-e^{-sT}} \int_0^{T/2} e^{-st}\ dt && \text{(21c)}\\
&= \frac{K}{1-e^{-sT}} \left[\frac{e^{-st}}{-s}\right]_0^{T/2} && \text{(21d)}\\
&= \frac{K}{(-s)(1-e^{-sT})} \left[e^{-st}\right]_0^{T/2} && \text{(21e)}\\
&= \frac{K}{(-s)(1-e^{-sT})} \left[e^{-Ts/2}-1\right] && \text{(21f)}\\
&= \frac{K(e^{-Ts/2}-1)}{(-s)(1-e^{-sT})} && \text{(21g)}\\
&= \frac{K(1-e^{-Ts/2})}{(s)(1-e^{-sT})} && \text{(21h)}\\
&= \frac{K}{s}\frac{1-e^{-Ts/2}}{1-e^{-sT}} && \text{(21i)}
\end{aligned}$$

Combining the transfer functions:

$$\begin{aligned}
&= \frac{Cs}{RCs+1}\frac{K}{s}\frac{1-e^{-Ts/2}}{1-e^{-Ts}} && \text{(22a)}\\
&= \frac{KC}{s}\frac{s(1-e^{-Ts/2})}{(RCs+1)(1-e^{-Ts})} && \text{(22b)}\\
&= \frac{KC(1-e^{-Ts/2})}{(RCs+1)(1-e^{-Ts})} && \text{(22c)}
\end{aligned}$$

Using the final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

$$\begin{aligned}
sF(s) &= s\frac{KC(1-e^{-Ts/2})}{(RCs+1)(1-e^{-Ts})} && \text{(23a)}\\
&= \frac{sKC - sKCe^{-Ts/2}}{(RCs+1)(1-e^{-Ts})} && \text{(23b)}\\
&= \frac{sKC - sKCe^{-Ts/2}}{RCs - RCse^{-Ts} + 1 - e^{-Ts}} && \text{(23c)}
\end{aligned}$$