(Note that n is used to index discrete steps in time and k is used to denote discrete iterations within a time step).

1 Backwards Euler

The Backwards Euler method is:

$$x_{n+1} = x_n + \Delta t \left[g\left(x_{n+1}\right)\right] \tag{1}$$

Or for a system of ODEs:

$$X_{n+1} = X_n + \Delta t \left[G\left(X_{n+1}\right)\right] \tag{2}$$

2 Newton's Method

Newton's method is given as:

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)} \tag{3}$$

For a system of equations Newton's method becomes

$$X^{k+1} = X^k - J^{-1}(X^k)F(X^k) \tag{4}$$

Where J is the Jacobian matrix:

$$J(X^k) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{5}$$

The product $J^{-1}(X^k)F(X^k)$ may be found by solving

$$J(X^k)S^k = -F(X^k) \tag{6}$$

For S^k. Then the Newton iteration is:

$$X^{k+1} = X^k + S^k \tag{7}$$

As the partial derivatives in the Jacobian may not be able to be determined analytically a finite difference method can be utilised

$$\frac{\partial f_n}{\partial x_n} \approx \frac{f(x+h) - f(x-h)}{2h} \tag{8}$$

3 Combining the Backwards Euler Method and Newton's Method

Rearranging the backwards Euler equation:

$$X_{n+1} = X_n + \Delta t \left[G\left(X_{n+1}\right)\right] \tag{9a}$$
$$F(X_{n+1}) = X_{n+1} - X_n - \Delta t \left[G\left(X_{n+1}\right)\right] = 0 \tag{9b}$$

When solving using Newton's method the X_n term in the backwards Euler method at the start of the iteration process and will be referred to as X_0.

Substituting into Newton's method:

$$X^{k+1} = X^k - J^{-1}(X^k)F(X^k) \tag{10a}$$
$$X^{k+1} = X^k - J^{-1}(X^k)\left(X^k - X_0 - \Delta t \left[G(X^k)\right]\right) \tag{10b}$$

The start index is denoted by $k = 0$ and the starting term X^0 may be found by using the forward Euler method

$$X^0 = X_{n-1} + \Delta t \left[G(X_{n-1}) \right] \tag{11a}$$

with X_{n-1} the result of the previous step.

4 Solution with Aircraft State-Space Equations

In matrix form the aircraft state-space equations are given as:

$$\dot{X} = AX + BU \tag{12}$$

The backwards Euler method is then:

$$F(X_{n+1}) = X_{n+1} - X_n - \Delta t \left[G\left(X_{n+1} \right) \right] = 0 \tag{13a}$$

$$F(X_{n+1}) = X_{n+1} - X_n - \Delta t \left[AX_{n+1} + BU \right] = 0 \tag{13b}$$

So that Newton's method is:

$$X^{k+1} = X^k - J^{-1}(X^k) \left(X^k - X_0 - \Delta t \left[AX^k + BU \right] \right) \tag{14}$$