Backwards Euler method:

$$x_{n+1} = x_n + \Delta t \left[f\left(x_{n+1}\right) \right] \tag{1a}$$
$$x_{n+1} = x_n + \Delta t \left[A x_{n+1} + B u \right] \tag{1b}$$
$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \Delta t \left[\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right] \tag{1c}$$

Expanding this into two equations:

$$x_{a(n+1)} = x_{a(n)} + \Delta t \left[\left(a_1 x_{a(n+1)} + a_2 x_{b(n+1)} \right) + (b_1 u_1 + b_2 u_2) \right] \tag{2a}$$
$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \tag{2b}$$

$$x_{b(n+1)} = x_{b(n)} + \Delta t \left[\left(a_3 x_{a(n+1)} + a_4 x_{b(n+1)} \right) + (b_1 u_1 + b_2 u_2) \right] \tag{3a}$$
$$x_{b(n+1)} = x_{b(n)} + a_3 x_{a(n+1)} \Delta t + a_4 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \tag{3b}$$

Rearranging the $x_{b(n+1)}$ equation for $x_{b(n+1)}$:

$$x_{b(n+1)} = x_{b(n)} + a_3 x_{a(n+1)} \Delta t + a_4 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \tag{4a}$$
$$x_{b(n+1)} \left[1 - a_4 \Delta t \right] = x_{b(n)} + a_3 x_{a(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \tag{4b}$$
$$x_{b(n+1)} = \frac{x_{b(n)} + a_3 x_{a(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_4 \Delta t} \tag{4c}$$

Substituting this result into the $x_{a(n+1)}$ equation:

$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \tag{5a}$$
$$\begin{aligned} x_{a(n+1)} = x_{a(n)} &+ a_1 x_{a(n+1)} \Delta t + \\ &\frac{a_2 x_{b(n)} \Delta t + a_2 a_3 x_{a(n+1)} \Delta t \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + \\ &(b_1 u_1 + b_2 u_2) \Delta t \end{aligned} \tag{5b}$$
$$\begin{aligned} x_{a(n+1)} = x_{a(n)} &+ a_1 x_{a(n+1)} \Delta t + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + \\ &\frac{a_2 a_3 x_{a(n+1)} \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t \end{aligned} \tag{5c}$$
$$x_{a(n+1)} \left[1 - a_1 \Delta t - \frac{a_2 a_3 \Delta t \Delta t}{1 - a_4 \Delta t} \right] = x_{a(n)} + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t \tag{5d}$$
$$x_{a(n+1)} = \frac{x_{a(n)} + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_1 \Delta t - \frac{a_2 a_3 \Delta t \Delta t}{1 - a_4 \Delta t}} \tag{5e}$$

Rearranging the $x_{a(n+1)}$ equation for $x_{a(n+1)}$:

$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \tag{6a}$$
$$x_{a(n+1)} \left[1 - a_1 \Delta t \right] = x_{a(n)} + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \tag{6b}$$
$$x_{a(n+1)} = \frac{x_{a(n)} + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_1 \Delta t} \tag{6c}$$

Substituting this result into the $x_{b(n+1)}$ equation:

$$x_{b(n+1)} = x_{b(n)} + a_3 x_{a(n+1)}\Delta t + a_4 x_{b(n+1)}\Delta t + (b_1u_1 + b_2u_2)\Delta t \tag{7a}$$

$$\begin{aligned} x_{b(n+1)} = x_{b(n)} + &\frac{a_3 x_{a(n)}\Delta t + a_2 a_3 x_{b(n+1)}\Delta t\Delta t + a_3(b_1u_1 + b_2u_2)\Delta t\Delta t}{1 - a_1\Delta t} + \\ &a_4 x_{b(n+1)}\Delta t + (b_1u_1 + b_2u_2)\Delta t \end{aligned} \tag{7b}$$

$$\begin{aligned} x_{b(n+1)} = x_{b(n)} + &\frac{a_3 x_{a(n)}\Delta t + a_3(b_1u_1 + b_2u_2)\Delta t\Delta t}{1 - a_1\Delta t} + a_4 x_{b(n+1)}\Delta t + \\ &\frac{a_2 a_3 x_{b(n+1)}\Delta t\Delta t}{1 - a_1\Delta t} + (b_1u_1 + b_2u_2)\Delta t \end{aligned} \tag{7c}$$

$$x_{b(n+1)}\left[1 - a_4\Delta t - \frac{a_2 a_3\Delta t\Delta t}{1 - a_1\Delta t}\right] = x_{b(n)} + \frac{a_3 x_{a(n)}\Delta t + a_3(b_1u_1 + b_2u_2)\Delta t\Delta t}{1 - a_1\Delta t} + (b_1u_1 + b_2u_2)\Delta t \tag{7d}$$

$$x_{b(n+1)} = \frac{x_{b(n)} + \frac{a_3 x_{a(n)}\Delta t + a_3(b_1u_1 + b_2u_2)\Delta t\Delta t}{1 - a_1\Delta t} + (b_1u_1 + b_2u_2)\Delta t}{1 - a_4\Delta t - \frac{a_2 a_3\Delta t\Delta t}{1 - a_1\Delta t}} \tag{7e}$$

Instead solving using the matrix equation:

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \Delta t \left[\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right] \tag{8a}$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} a_1\Delta t & a_2\Delta t \\ a_3\Delta t & a_4\Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} + \begin{bmatrix} b_1\Delta t & b_2\Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8b}$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} - \begin{bmatrix} a_1\Delta t & a_2\Delta t \\ a_3\Delta t & a_4\Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1\Delta t & b_2\Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8c}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} - \begin{bmatrix} a_1\Delta t & a_2\Delta t \\ a_3\Delta t & a_4\Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1\Delta t & b_2\Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8d}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_1\Delta t & a_2\Delta t \\ a_3\Delta t & a_4\Delta t \end{bmatrix}\right) \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1\Delta t & b_2\Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8e}$$

$$\begin{bmatrix} 1 - a_1\Delta t & -a_2\Delta t \\ -a_3\Delta t & 1 - a_4\Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1\Delta t & b_2\Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8f}$$

$$\begin{bmatrix} 1 - a_1\Delta t & -a_2\Delta t \\ -a_3\Delta t & 1 - a_4\Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + b_1\Delta t u_1 + b_2\Delta t u_2 \tag{8g}$$

$$\begin{bmatrix} 1 - a_1\Delta t & -a_2\Delta t \\ -a_3\Delta t & 1 - a_4\Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + (b_1u_1 + b_2u_2)\Delta t \tag{8h}$$

$$\begin{bmatrix} 1 - a_1\Delta t & -a_2\Delta t \\ -a_3\Delta t & 1 - a_4\Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} + (b_1u_1 + b_2u_2)\Delta t \\ x_{b(n)} + (b_1u_1 + b_2u_2)\Delta t \end{bmatrix} \tag{8i}$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} 1 - a_1\Delta t & -a_2\Delta t \\ -a_3\Delta t & 1 - a_4\Delta t \end{bmatrix}^{-1} \begin{bmatrix} x_{a(n)} + (b_1u_1 + b_2u_2)\Delta t \\ x_{b(n)} + (b_1u_1 + b_2u_2)\Delta t \end{bmatrix} \tag{8j}$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \frac{1}{(1 - a_1\Delta t)(1 - a_4\Delta t) - (-a_2\Delta t)(-a_3\Delta t)} \begin{bmatrix} 1 - a_4\Delta t & a_2\Delta t \\ a_3\Delta t & 1 - a_1\Delta t \end{bmatrix} \tag{8k}$$

$$\begin{bmatrix} x_{a(n)} + (b_1u_1 + b_2u_2)\Delta t \\ x_{b(n)} + (b_1u_1 + b_2u_2)\Delta t \end{bmatrix} \tag{8l}$$