

Backwards Euler method:

$$x_{n+1} = x_n + \Delta t [f(x_{n+1})] \quad (1a)$$

$$x_{n+1} = x_n + \Delta t [Ax_{n+1} + Bu] \quad (1b)$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \Delta t \left[\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right] \quad (1c)$$

Expanding this into two equations:

$$x_{a(n+1)} = x_{a(n)} + \Delta t [(a_1 x_{a(n+1)} + a_2 x_{b(n+1)}) + (b_1 u_1 + b_2 u_2)] \quad (2a)$$

$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (2b)$$

$$x_{b(n+1)} = x_{b(n)} + \Delta t [(a_3 x_{a(n+1)} + a_4 x_{b(n+1)}) + (b_1 u_1 + b_2 u_2)] \quad (3a)$$

$$x_{b(n+1)} = x_{b(n)} + a_3 x_{a(n+1)} \Delta t + a_4 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (3b)$$

Rearranging the $x_{b(n+1)}$ equation for $x_{b(n+1)}$:

$$x_{b(n+1)} = x_{b(n)} + a_3 x_{a(n+1)} \Delta t + a_4 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (4a)$$

$$x_{b(n+1)} [1 - a_4 \Delta t] = x_{b(n)} + a_3 x_{a(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (4b)$$

$$x_{b(n+1)} = \frac{x_{b(n)} + a_3 x_{a(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_4 \Delta t} \quad (4c)$$

Substituting this result into the $x_{a(n+1)}$ equation:

$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (5a)$$

$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + \quad (5b)$$

$$\frac{a_2 x_{b(n)} \Delta t + a_2 a_3 x_{a(n+1)} \Delta t \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t$$

$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + \quad (5c)$$

$$\frac{a_2 a_3 x_{a(n+1)} \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t$$

$$x_{a(n+1)} \left[1 - a_1 \Delta t - \frac{a_2 a_3 \Delta t \Delta t}{1 - a_4 \Delta t} \right] = x_{a(n)} + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t \quad (5d)$$

$$x_{a(n+1)} = \frac{x_{a(n)} + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_1 \Delta t - \frac{a_2 a_3 \Delta t \Delta t}{1 - a_4 \Delta t}} \quad (5e)$$

$$x_{a(n+1)} = \frac{x_{a(n)} + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{(1 - a_1 \Delta t) - (1 - a_1 \Delta t) a_4 \Delta t - a_2 a_3 \Delta t \Delta t} \quad (5f)$$

$$x_{a(n+1)} = \frac{x_{a(n)} + \frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_4 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_1 \Delta t - a_4 \Delta t + a_1 a_4 \Delta t \Delta t - a_2 a_3 \Delta t \Delta t} \quad (5g)$$

$$x_{a(n+1)} = \frac{[x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (1 - a_4 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} + \quad (5h)$$

$$\frac{a_2 x_{b(n)} \Delta t + a_2 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} + \frac{[x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (1 - a_4 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} + \frac{[x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (a_2 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} \quad (5i)$$

Rearranging the $x_{a(n+1)}$ equation for $x_{a(n+1)}$:

$$x_{a(n+1)} = x_{a(n)} + a_1 x_{a(n+1)} \Delta t + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (6a)$$

$$x_{a(n+1)} [1 - a_1 \Delta t] = x_{a(n)} + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (6b)$$

$$x_{a(n+1)} = \frac{x_{a(n)} + a_2 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_1 \Delta t} \quad (6c)$$

Substituting this result into the $x_{b(n+1)}$ equation:

$$x_{b(n+1)} = x_{b(n)} + a_3 x_{a(n+1)} \Delta t + a_4 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t \quad (7a)$$

$$x_{b(n+1)} = x_{b(n)} + \quad (7b)$$

$$\frac{a_3 x_{a(n)} \Delta t + a_2 a_3 x_{b(n+1)} \Delta t \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_1 \Delta t} +$$

$$a_4 x_{b(n+1)} \Delta t + (b_1 u_1 + b_2 u_2) \Delta t$$

$$x_{b(n+1)} = x_{b(n)} + \frac{a_3 x_{a(n)} \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_1 \Delta t} + a_4 x_{b(n+1)} \Delta t + \quad (7c)$$

$$\frac{a_2 a_3 x_{b(n+1)} \Delta t \Delta t}{1 - a_1 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t$$

$$x_{b(n+1)} \left[1 - a_4 \Delta t - \frac{a_2 a_3 \Delta t \Delta t}{1 - a_1 \Delta t} \right] = x_{b(n)} + \frac{a_3 x_{a(n)} \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_1 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t \quad (7d)$$

$$x_{b(n+1)} = \frac{x_{b(n)} + \frac{a_3 x_{a(n)} \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_1 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_4 \Delta t - \frac{a_2 a_3 \Delta t \Delta t}{1 - a_1 \Delta t}} \quad (7e)$$

$$x_{b(n+1)} = \frac{x_{b(n)} + \frac{a_3 x_{a(n)} \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_1 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{(1 - a_1 \Delta t) - (1 - a_1 \Delta t) a_4 \Delta t - a_2 a_3 \Delta t \Delta t} \quad (7f)$$

$$x_{b(n+1)} = \frac{x_{b(n)} + \frac{a_3 x_{a(n)} \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_1 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{1 - a_1 \Delta t - a_4 \Delta t + a_1 a_4 \Delta t \Delta t - a_2 a_3 \Delta t \Delta t} \quad (7g)$$

$$x_{b(n+1)} = \frac{x_{b(n)} + \frac{a_3 x_{a(n)} \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{1 - a_1 \Delta t} + (b_1 u_1 + b_2 u_2) \Delta t}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} \quad (7h)$$

$$x_{b(n+1)} = \frac{[x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (1 - a_1 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} + \quad (7i)$$

$$\frac{a_3 x_{a(n)} \Delta t + a_3 (b_1 u_1 + b_2 u_2) \Delta t \Delta t}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)}$$

$$x_{b(n+1)} = \frac{[x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (1 - a_1 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} + \frac{[x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (a_3 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} \quad (7j)$$

Instead solving using the matrix equation:

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \Delta t \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (8a)$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} a_1 \Delta t & a_2 \Delta t \\ a_3 \Delta t & a_4 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} + \begin{bmatrix} b_1 \Delta t & b_2 \Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (8b)$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} - \begin{bmatrix} a_1 \Delta t & a_2 \Delta t \\ a_3 \Delta t & a_4 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1 \Delta t & b_2 \Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (8c)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} - \begin{bmatrix} a_1 \Delta t & a_2 \Delta t \\ a_3 \Delta t & a_4 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1 \Delta t & b_2 \Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (8d)$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_1 \Delta t & a_2 \Delta t \\ a_3 \Delta t & a_4 \Delta t \end{bmatrix} \right) \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1 \Delta t & b_2 \Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (8e)$$

$$\begin{bmatrix} 1 - a_1 \Delta t & -a_2 \Delta t \\ -a_3 \Delta t & 1 - a_4 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + \begin{bmatrix} b_1 \Delta t & b_2 \Delta t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (8f)$$

$$\begin{bmatrix} 1 - a_1 \Delta t & -a_2 \Delta t \\ -a_3 \Delta t & 1 - a_4 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + b_1 \Delta t u_1 + b_2 \Delta t u_2 \quad (8g)$$

$$\begin{bmatrix} 1 - a_1 \Delta t & -a_2 \Delta t \\ -a_3 \Delta t & 1 - a_4 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} \\ x_{b(n)} \end{bmatrix} + (b_1 u_1 + b_2 u_2) \Delta t \quad (8h)$$

$$\begin{bmatrix} 1 - a_1 \Delta t & -a_2 \Delta t \\ -a_3 \Delta t & 1 - a_4 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t \\ x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t \end{bmatrix} \quad (8i)$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \begin{bmatrix} 1 - a_1 \Delta t & -a_2 \Delta t \\ -a_3 \Delta t & 1 - a_4 \Delta t \end{bmatrix}^{-1} \begin{bmatrix} x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t \\ x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t \end{bmatrix} \quad (8j)$$

$$\begin{bmatrix} x_{a(n+1)} \\ x_{b(n+1)} \end{bmatrix} = \frac{1}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} \begin{bmatrix} 1 - a_4 \Delta t & a_2 \Delta t \\ a_3 \Delta t & 1 - a_1 \Delta t \end{bmatrix} \begin{bmatrix} x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t \\ x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t \end{bmatrix} \quad (8k)$$

$$\begin{bmatrix} x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t \\ x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t \end{bmatrix}$$

Therefore:

$$x_{a(n+1)} = \frac{[x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (1 - a_4 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} + \frac{[x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (a_2 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} \quad (9a)$$

$$x_{b(n+1)} = \frac{[x_{a(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (a_3 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} + \frac{[x_{b(n)} + (b_1 u_1 + b_2 u_2) \Delta t] (1 - a_1 \Delta t)}{(1 - a_1 \Delta t)(1 - a_4 \Delta t) - (-a_2 \Delta t)(-a_3 \Delta t)} \quad (10a)$$