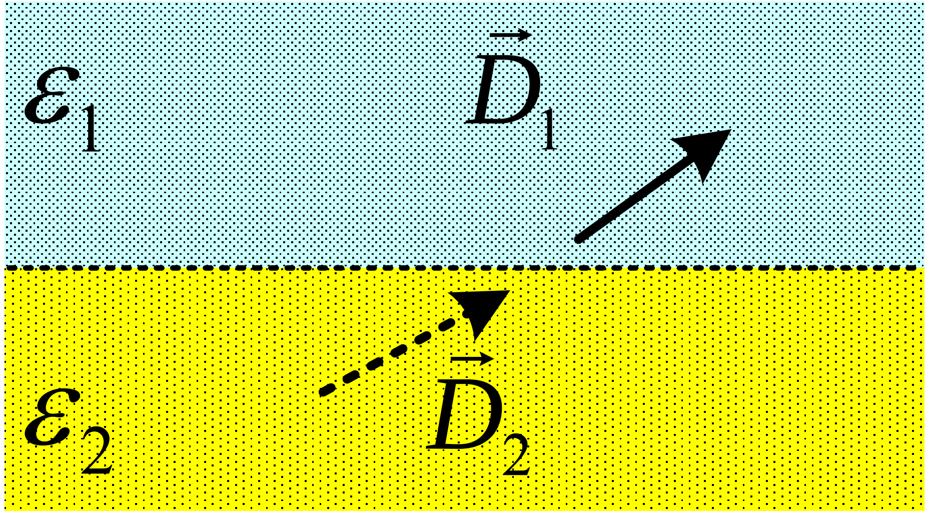
§ 2.9 Boundary Conditions for Electrostatics

- The Question
  - In general, E-intensity or potential may be obtained through two equations:
    - In source free region --- Laplace's Equation
    - Otherwise --- Poisson's Equation
  - **Boundary conditions are useful to**
    - determine the ***undecided constants*** in the solution;
    - turn the general solution into ***specific solution***.
  - Boundary Value Problem: solving partial differential equations under given boundary conditions.
    - Poisson's Equation subjected to boundary conditions
    - Laplace's Equation subjected to boundary conditions

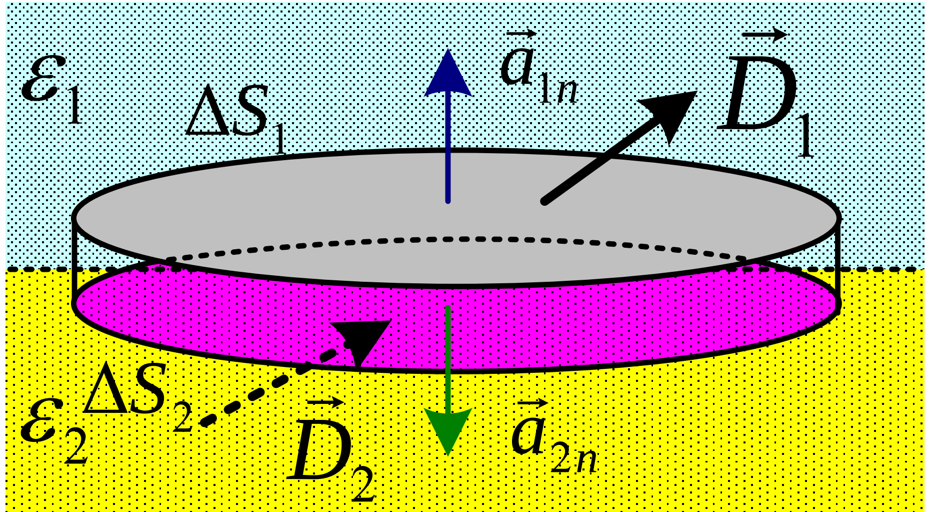
What is Boundary Conditions?

- **Equations governing the behavior of fields at the boundary (interface) between 2 mediums.**
- Categories of Boundary Conditions:
  - First: the field potential at boundary is given
    - *Dirichlet Problems* 狄理赫利问题
  - Second: the normal component of the derivative of the field potential at boundary is known
    - *Neumann Problems* 纽曼问题
  - Third: *Hybrid Problems* 混合问题

1. Boundary Conditions in Normal Direction



Construct an auxiliary closed surface of a very flat box.



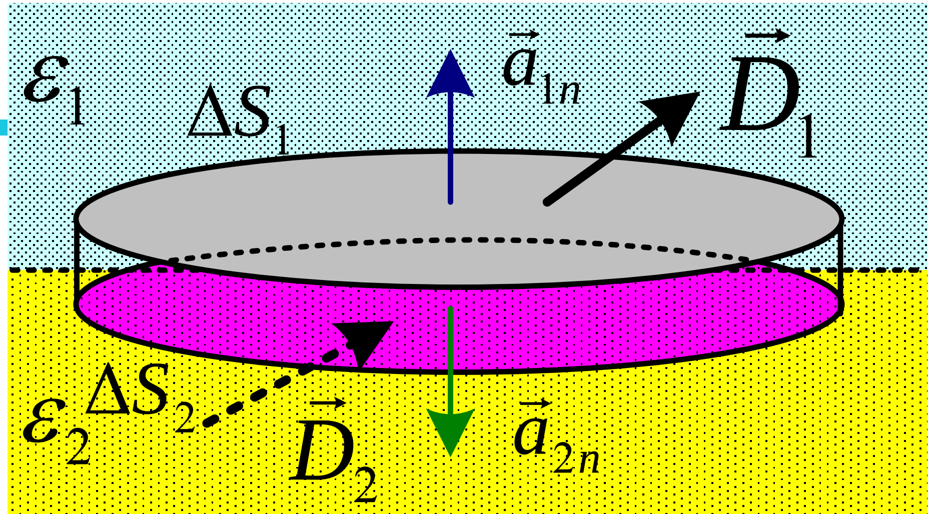
Applying Gauss's Law

$$\oint_S \vec{D} \bullet d\vec{S} = \rho_s \cdot \Delta S$$

$$\vec{D}_1 \cdot \Delta\vec{S}_1 + \vec{D}_2 \cdot \Delta\vec{S}_2 = D_{1n}\Delta S - D_{2n}\Delta S = \rho_S \Delta S$$

$$\boxed{\therefore D_{1n} - D_{2n} = \rho_s}$$

---surface density of free charges



$$D_{1n} - D_{2n} = \rho_s$$

$$\because \vec{E} = -\nabla \psi$$

$$\therefore \vec{D} \bullet \vec{a}_n = (\varepsilon \vec{E}) \bullet \vec{a}_n = \varepsilon(-\nabla \psi) \bullet \vec{a}_n = -\varepsilon \cdot \frac{\partial \psi}{\partial n}$$

$$\boxed{\therefore \varepsilon_2 \cdot \frac{\partial \psi_2}{\partial n} - \varepsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \rho_s}$$

Discussions

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

$$\boxed{\varepsilon_2 \cdot \frac{\partial \psi_2}{\partial n} - \varepsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \rho_s}$$

- **If there is no free charge at the boundary,**

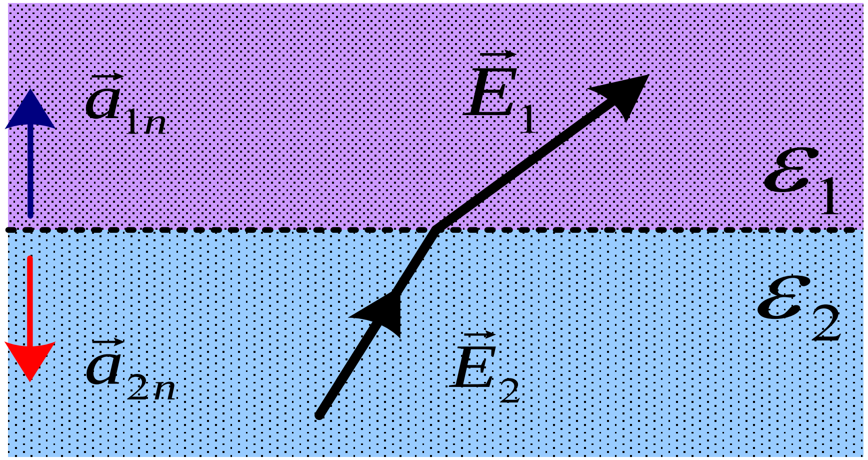
$$\boxed{D_{1n} = D_{2n}}$$

$$\boxed{\varepsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \varepsilon_2 \cdot \frac{\partial \psi_2}{\partial n}}$$

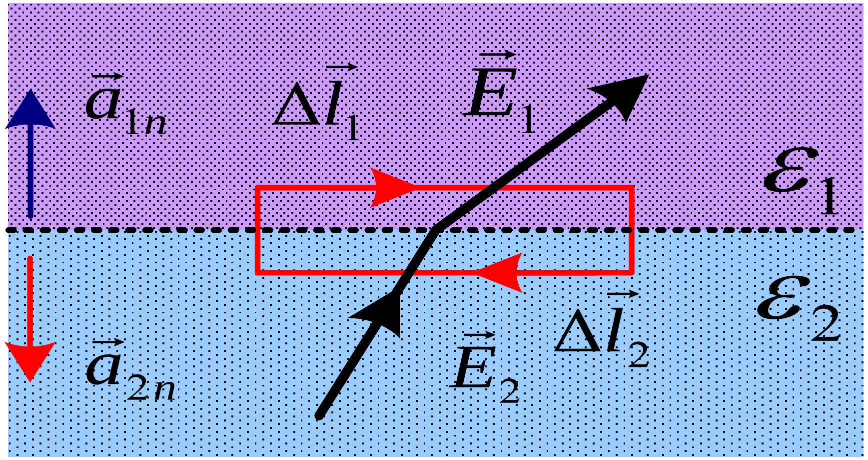
- **If the boundary is the interface of a conductor** $\boxed{D_{2n} = 0}$

$$\boxed{D_{1n} = \rho_s}$$

2. Boundary Conditions in Tangential Direction



Make an auxiliary closed path in rectangular shape, $\Delta h \rightarrow 0$



Applying the conservative law

$$\oint_c \vec{E} \bullet d\vec{l} = 0$$

$$\therefore \vec{E}_1 \cdot \Delta\vec{l}_1 + \vec{E}_2 \cdot \Delta\vec{l}_2 = E_{1t} \cdot \Delta l - E_{2t} \cdot \Delta l = 0$$

$$\boxed{\therefore E_{1t} = E_{2t}}$$

2. Boundary Conditions in Tangential Direction

- The tangential components of E-intensity are **always continuous** across the boundary.

$$E_{1t} = E_{2t}$$

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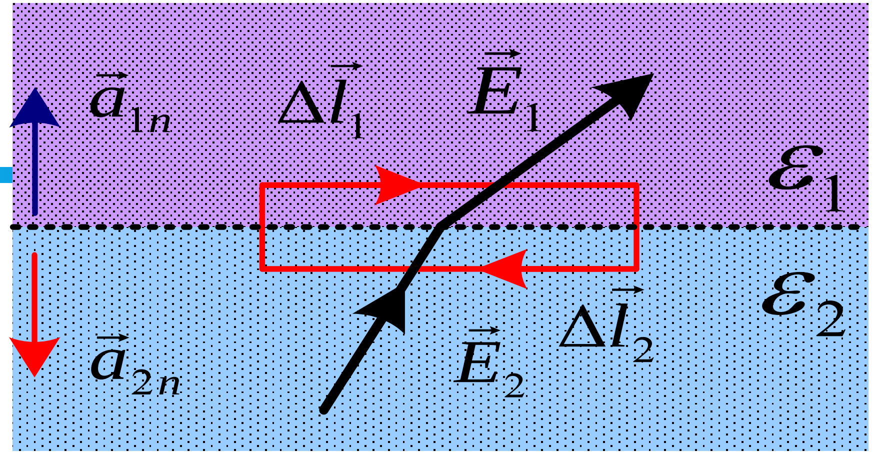


$$\Delta\psi = \psi_1 - \psi_2 = \int_a^{a+\Delta h} \vec{E} \bullet d\vec{l} \to 0 \qquad \Delta h \to 0$$

$$\boxed{\therefore \psi_1 = \psi_2}$$

- **E-potential is continuous across the boundary;**
- **In other word, E-potential is an integral function and all integral functions are continuous within their domains of definition.**

Summary --- Boundary Conditions

1. In normal direction

$$\boxed{D_{1n} - D_{2n} = \sigma_{fc}}$$

$$\boxed{\varepsilon_2 \cdot \frac{\partial \psi_2}{\partial n} - \varepsilon_1 \cdot \frac{\partial \psi_1}{\partial n} = \sigma_{fc}}$$

2. In tangential direction

$$\boxed{E_{1t} = E_{2t}}$$

$$\boxed{\psi_1 = \psi_2}$$

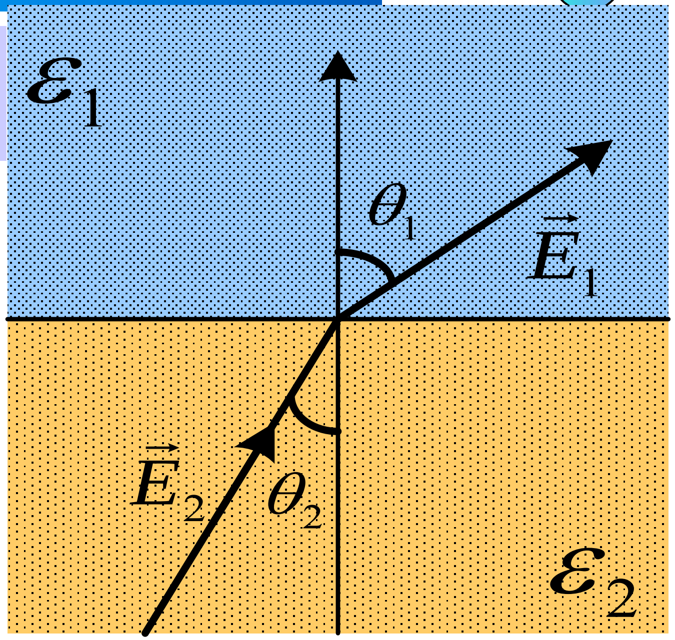
3. Applications of Boundary Conditions

$$\because D_{1n} = D_{2n} \Rightarrow \varepsilon_1 \cdot E_{1n} = \varepsilon_2 \cdot E_{2n}$$

$$E_{1t} = E_{2t}$$

$$\therefore \begin{cases} E_1 \cdot \sin\theta_1 = E_2 \cdot \sin\theta_2 \\ \varepsilon_1 \cdot E_1 \cdot \cos\theta_1 = \varepsilon_2 \cdot E_2 \cdot \cos\theta_2 \end{cases}$$

$$\boxed{\therefore \frac{tg\theta_1}{tg\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}}$$



Similar to the refraction of light

Exercise 3.27 in textbook p127

Example 2.

- A conductor ball. Radius a. E-potential U.
- Please determine the potential outside the ball.
- Analysis:
  - Any Symmetry? —— Yes, point symmetry.
  - How many approaches to determine E-potential?
    - Direct solution——via integral or sum
    - Via E-intensity
    - Via differential equations

Solution 1.

Because ??? we obtain $$\nabla^2\psi = 0$$

Because ??? we infer $$\psi = \psi(r)$$

Express Laplace's Equ. in spherical coordinates

$$\nabla^2\psi = \frac{1}{r^2}\frac{d}{dr}r^2\frac{d\psi}{dr} = 0$$

Through integral of above equ. $$\psi = -\frac{C_1}{r} + C_2$$

Via boundary conditions

$$\psi = \begin{cases} r > a & \frac{a}{r}\cdot U \\ r = a & U \\ r < a & U \end{cases}$$

$$\psi = -\frac{C_1}{r} + C_2$$

Determine the undecided constants via Boundary Conditions

If $r \to \infty$, we know $\psi = 0$, and then $C_2 = 0$

If $r = a$ and $\psi = U$, we infer $C_1 = -aU$

$$\psi = \begin{cases} r > a, & \frac{a}{r} \cdot U \\ r \le a, & U \end{cases}$$

- Please check that E-potential is continuous across the sphere of the conductor ball, i.e. $\psi_1 = \psi_2$

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Now, let's go on --->>>