SOLUTIONS

Module:	Advanced Transform Methods		
Module Code	EBU6018	Paper	B
Time allowed	2hrs	Filename	Solutions_1718_EBU6018_B
Rubric	ANSWER ALL FOUR QUESTIONS		
Examiners	Mr Andy Watson	Dr Song Yi-Zhe	

Question 1.

a) Suppose we have a set of M complex vectors $\psi_k = (\psi_k[0], \psi_k[1], \ldots, \psi_k[N-1])$, where $0 \le k \le M-1$ with pairwise inner products $\langle \psi_j, \psi_k \rangle \equiv \sum_{n=0}^{N-1} \psi_j[n]\psi_k^*[n]$.
Define what is meant for this set of vectors to be (i) orthogonal, and (ii) orthonormal.

[4 marks]

Answer:

i) Orthogonal means that different vectors must have zero inner product, i.e. $\langle \psi_j, \psi_k \rangle = 0$ for $j \ne k$. [2 marks]

ii) Orthonormal means that the vectors in the set must be orthogonal, and that each vector must have unit norm, i.e. $\|\psi_k\|^2 \equiv \langle \psi_k, \psi_k \rangle = 1$. Hence we must have $\langle \psi_j, \psi_k \rangle = \delta_{jk}$. [2 marks]

b) Consider the pair of vectors $\psi_0 = (1,1)$, $\psi_1 = \sqrt{\frac{1}{2}}(1,-1)$.
By calculating relevant inner products and norms, identify whether or not these vectors form an orthogonal or an orthonormal set.
Sketch these two vectors on a diagram to confirm your answer.

[8 marks]

Answer:

Calculating the inner product $\langle \psi_j, \psi_k \rangle$ we find:

$$\begin{aligned}\langle \psi_0, \psi_1 \rangle &= \sum_{n=0}^{1} \psi_0[n]\psi_1^*[n] \\ &= 1 \times \sqrt{\tfrac{1}{2}}1 + 1 \times \sqrt{\tfrac{1}{2}}(-1) \\ &= 0\end{aligned}$$

[1 mark]

Hence they are orthogonal. Now for the norms we find:

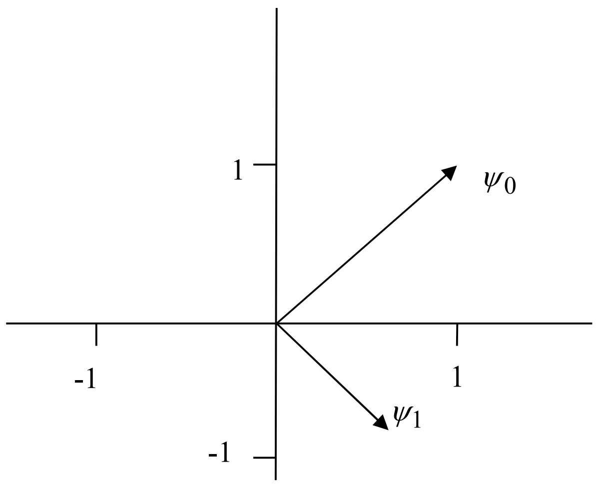
$$\begin{aligned}\|\psi_0\|^2 &= \langle \psi_0, \psi_0 \rangle \\ &= \sum_{n=0}^{1} \psi_0[n]\psi_0^*[n] \\ &= 1\times 1 + 1 \times 1 \\ &= 2\end{aligned}$$

So $\|\psi_0\| = \sqrt{2} \ne 1$. [2 marks]

$$\begin{aligned}\langle \psi_1, \psi_1 \rangle &= \sum_{n=0}^{1} \psi_1[n]\psi_1^*[n] \\ &= \left(\sqrt{\tfrac{1}{2}}\right)^2 (1)^2 + \left(\sqrt{\tfrac{1}{2}}(-1)\right)^2 \\ &= \tfrac{1}{2} + \tfrac{1}{2} = 1\end{aligned}$$

So $\|\psi_1\| = 1$. [2 marks]

Therefore only ONE of the vectors has unit norm, the other does not.
Hence this is an *orthogonal* set, but NOT *orthonormal*. [1 mark]



Sketch shows vectors at right angles, but ψ_0 longer than ψ_1. The vector ψ_1 has unit length, while ψ_0 has length $\sqrt{2}$. [2 marks]

c) Suppose a vector s can be represented by a set of vectors $\{\psi_k\}$ as $s = \sum_k c_k \psi_k$, where c_k are scalar coefficients and the vectors ψ_k are orthogonal. Show that, if ψ_k are also orthonormal, then $c_k = \langle s, \psi_k \rangle$. Also, calculate the equivalent expression for c_k if the vectors ψ_k are orthogonal but not orthonormal.

[7 marks]

Answer:

Expanding s in $\langle s, \psi_k \rangle$ we get

$$\begin{aligned}
\langle s, \psi_k \rangle &= \left\langle \left(\sum_j c_j \psi_j \right), \psi_k \right\rangle \\
&= \sum_j \langle c_j \psi_j, \psi_k \rangle \\
&= \sum_j c_j \langle \psi_j, \psi_k \rangle \\
&= \sum_j c_j \delta_{jk} \\
&= c_k
\end{aligned}$$

[2 marks]

For the equivalent form for orthogonal (not orthonormal) we get

$$\begin{aligned}
\langle s, \psi_k \rangle &= \left\langle \left(\sum_j c_j \psi_j \right), \psi_k \right\rangle \\
&= \sum_j \langle c_j \psi_j, \psi_k \rangle \\
&= \sum_j c_j \langle \psi_j, \psi_k \rangle \\
&= \sum_j c_j \left(\delta_{jk} \|\psi_j\|^2 \right) \\
&= c_k \|\psi_k\|^2
\end{aligned}$$

[4 marks]

and hence $c_k = \dfrac{1}{\|\psi_k\|^2} \langle s, \psi_k \rangle$. [1 mark]

d) Using the appropriate formula from part (c), derive expressions for the coefficients c_0 and c_1 for a vector $s = (s[0], s[1])$ represented by the pair of vectors ψ_0, ψ_1 in part (b) above.

[6 marks]

Answer

We have $\psi_0 = (1,1)$ and $\psi_1 = \sqrt{\tfrac{1}{2}}(1,-1)$, which are orthogonal but not orthonormal.

In fact we can use the formula $c_k = \dfrac{1}{\|\psi_k\|^2}\langle s, \psi_k \rangle$ for orthogonal and orthonormal, since the denominator will be 1 for orthonormal. [1 mark]

For c_0, first calculate $\langle s, \psi_0 \rangle = s[0]\psi_0[0] + s[1]\psi_0[1] = s[0]\times 1 + s[1]\times 1 = s[0] + s[1]$. [1 mark]

We also have $\|\psi_0\| = \sqrt{2} \neq 1$ from before, so for c_0 we have

$$c_0 = \frac{1}{\|\psi_0\|^2}\langle s, \psi_0 \rangle = \frac{1}{2}(s[0] + s[1])$$

[1 mark]

For c_1, we calculate $\langle s, \psi_1 \rangle = s[0]\psi_1[0] + s[1]\psi_1[1] = s[0]\times(1/\sqrt{2}) + s[1]\times(-1/\sqrt{2}) = (s[0] - s[1])/\sqrt{2}$.

[2 marks]

We also have $\|\psi_1\| = 1$ from before, so for c_1 we have

$$c_1 = \frac{1}{\|\psi_1\|^2}\langle s, \psi_1 \rangle = \frac{1}{\sqrt{2}}(s[0] - s[1])$$

[1 mark]

Question 2

(a) Sampling a signal in the time domain results in multiples of the baseband frequency spectrum separated by integer multiples of the sampling frequency.

Briefly explain how aliasing can be prevented.

[5 marks]

Answer:

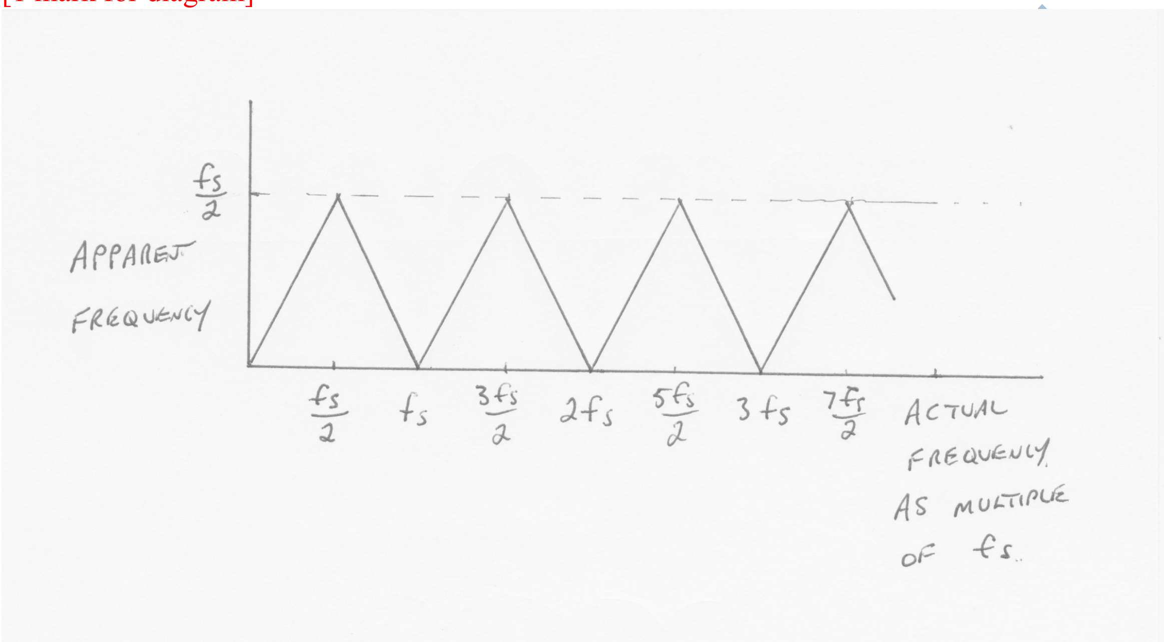
A bandlimiting filter must be used prior to sampling [1 mark]. This is a lowpass filter [1 mark]. The sampling frequency must be greater than twice the bandwidth of the signal to give a gap between spectra [1 mark].The filter cutoff frequency should be set at the upper limit of the baseband spectrum [1 mark]. The roll-off rate of the filter must be sufficiently great to provide adequate attenuation at the lower end on the first replicated spectrum[1 mark],

(b) With the aid of a diagram, demonstrate the effect of the folding of an aliased signal. For a sampling frequency of 4000 samples/second, what is the apparent frequency resulting from sampling signals of respectively 1000 Hz, 5000 Hz, 7500 Hz and 11,500 Hz?

[6 marks]

Answer:

(b) The term folding arises because of the mirror effect about multiples of $f_s/2$ (f_s is the sampling frequency) [1 mark].
[1 mark for diagram]



If we sample at 4000 samples/second:
1000Hz appears to be 1000Hz (as it should). [1 mark].
5000Hz appears to be 1000Hz [1 mark]
7500Hz appears to be 500Hz [1 mark]
11500Hz appears to be 500Hz [1 mark].

(c) The Short Time Fourier Transform is used to localise the frequency content of a signal in time in addition to the signal's frequency content.

i) List the steps involved in performing a STFT. **[6 marks]**

ii) Two factors influence the choice of window, namely shape and width. Briefly discuss these.

[8 marks]

Answer:

i) The steps are:
1. Choose a window of finite length
2. Place the window on top of the signal at t=0

3. Truncate the signal using this window
4. Truncate the signal using this window
5. Incrementally shift the window to the right
6. Repeat from step 3 [6 marks: 1 for each]

ii)
Shape: If the windows are rectangular, then there is a problem avoiding overlap at the edges or allowing gaps when translating the window.[1 mark]

Discontinuities are also a problem [1 mark]

So windows that taper towards the edges are preferred such as Gaussian, Hamming, Elliptic, etc. [1 mark].These also help when synthesising (inverting) the transform [1 mark].

Width: Windows should be narrow enough to make sure that the portion of the signal within the window is stationary [1 mark] but this will not allow good localisation in the frequency domain [1 mark]. The window width is a trade-off between good time localisation and good frequency localisation. [1 mark]. The Uncertainty Principle limits the time/frequency localisation.[1 mark].

Question 3

(a) Multiresolution Analysis (MRA) is used to separate data into course and fine detail.

Apply the transform defined by

$$x_{n-1},i = (x_{n,2i}+x_{n,2i+1})/2$$
$$d_{n-1},i = (x_{n,2i}-x_{n,2i+1})/2$$

to the sequence

$$[x_{n,i}] = [6, 8, 3, 11, 9, 5, 7, 2]$$

Where i = 0…..7, is the index position in the sequence, and
n is the level. The next level is n-1.
At each level, calculate the sequencies for x_{n-1},i and d_{n-1},i
Continue till no further levels are possible.

i) State the significance of the first element in the final level.

ii) Has any information been lost in the process?

iii) Comment on how this process could be used to compress the data.

[12 marks]

Answer:

Applying the transform:

n=3 [6, 8, 3, 11, 9, 5, 7, 2]
n=2 [7, 7, 7, 4.5, -1, -4, 2, 2.5]
n=1 [7, 5.75, 0, 1.25, -1, -4, 2, 2.5]
n=0 [6.375, 0.625, 0, 1.25, -1, -4, 2, 2.5] [6 marks: 2 for each row]

i) The first element is the average of all the elements in the original sequence [1 mark].
ii) No information has been lost [1 mark]
iii) Because most of the values in the final level are small, potentially fewer bits would be required to store it [1 mark]. Where there are zeroes, they do not need to be stored, although the positions of the other values would need to be stored [1 mark]. Small values could be replaced by zeroes without significant loss of detail [1 mark], this can be done by applying a threshold value, the bigger the threshold the greater the loss of detail [1 mark].

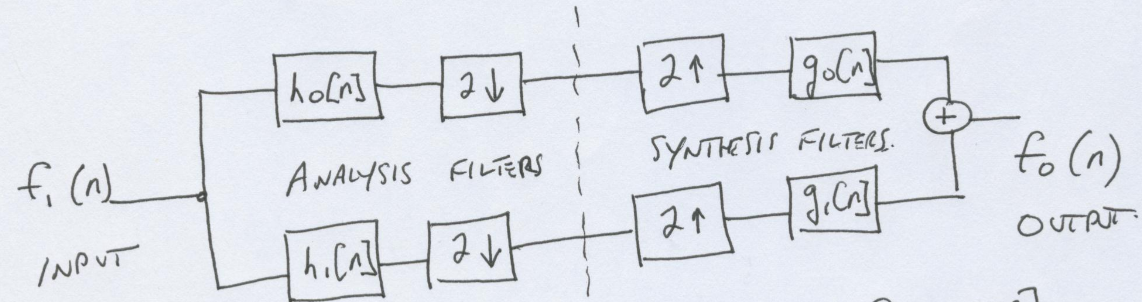
(b) With reference to a two-band subband coding and decoding system, explain the relationship between the analysis and synthesis filters.

[13 marks]

Answer:

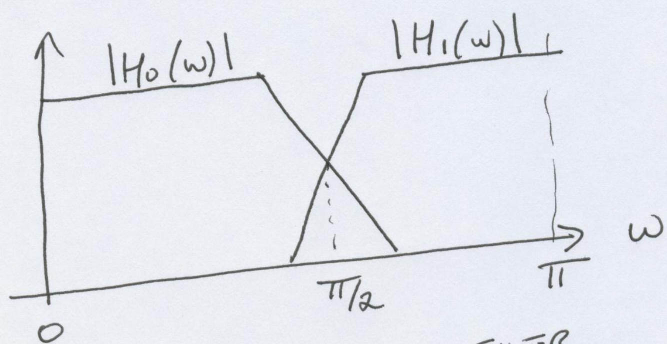
A block diagram of the two band system is shown below.

TWO-BAND SUBBAND CODING AND DECODING SYSTEM:



[2 MARKS]

ANALYSIS FILTER CHARACTERISTICS:



$h_0[n]$ IS LOWPASS FILTER
$h_1[n]$ IS HIGHPASS FILTER. [2 MARKS]

FOR PERFECT RECONSTRUCTION,

$$g_0(n) = (-1)^n h_1(n)$$
$$g_1(n) = (-1)^{n+1} h_0(n)$$
[2 MARKS]

OR
$$g_0(n) = (-1)^{n+1} h_1(n)$$
$$g_1(n) = (-1)^n h_0(n)$$
[2 MARKS]

ALSO, $h_i(n)$ AND $g_j(n)$ MUST BE BIORTHOGONAL.
[1 MARK]

The analysis filterbank is used to split the input sequence into two half-length sequences [1 mark]

Synthesis filterbanks combine the high and low pass subbands to produce the output [1 mark]

If the output and input are identical then we have used perfect reconstruction filters. [1 mark]

The synthesis filters are modulated versions of the analysis filters [1 mark].

Question 4

a) Explain what is meant by *multiresolution analysis*. Illustrate this concept with a diagram showing *piecewise approximation* of a signal.

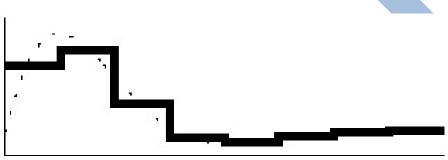
[4 marks]

Answer

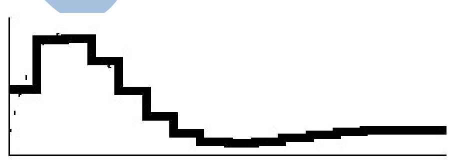
The basic concept is to decompose a fine-resolution signal into

A coarse-resolution version of the signal, [1 mark]

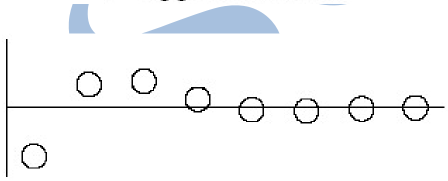
And the differences left over. [1 mark]



V^3 approximation



V^4 approximation



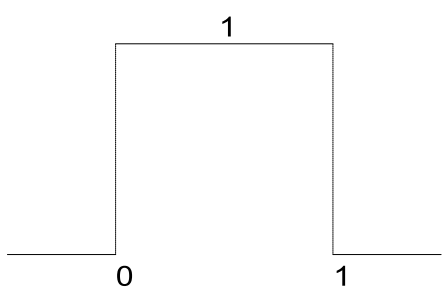
W^3 detail coefficients

The fine resolution signal on the right is decomposed into the coarse resolution signal and fine detail shown on the left. [2 marks: 1 for each decomposition]

b) Draw a diagram of the Haar scaling function $\phi(t)$ and wavelet function $\psi(t)$. Use diagrams to explain how $\phi(2t)$ and $\phi(2t-1)$ can be represented by combinations of $\phi(t)$ and $\psi(t)$, and how $\phi(t)$ and $\psi(t)$ can each be represented by combinations of $\phi(2t)$ and $\phi(2t-1)$.

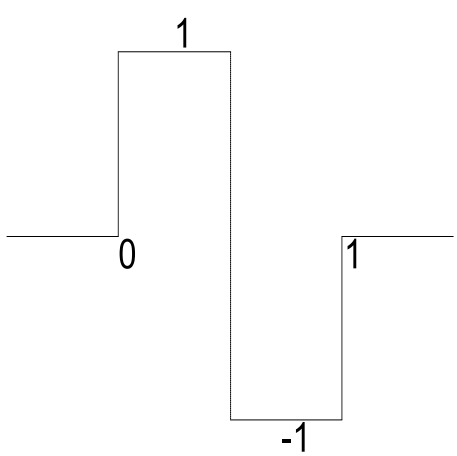
[6 marks]

Answer



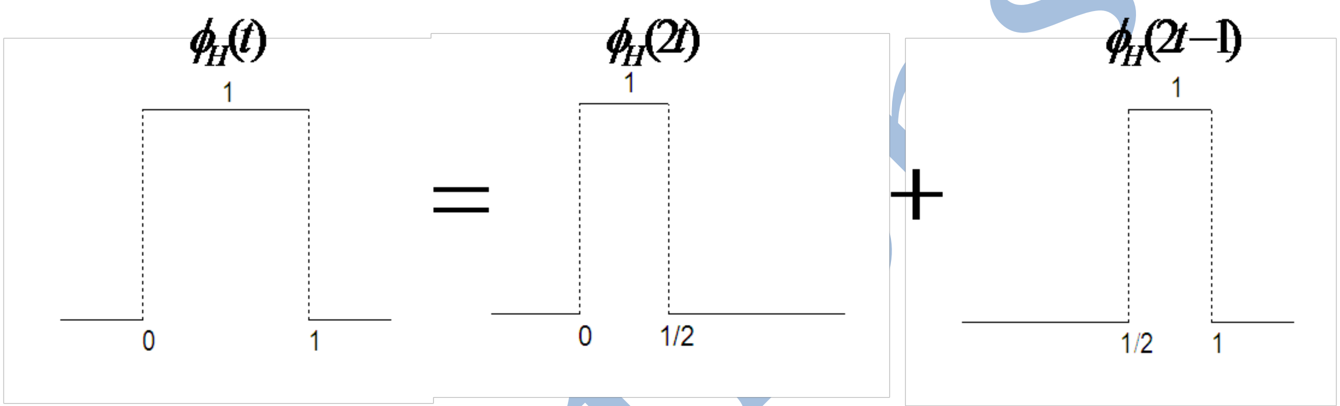
[1 mark]

$\phi_H(t)$



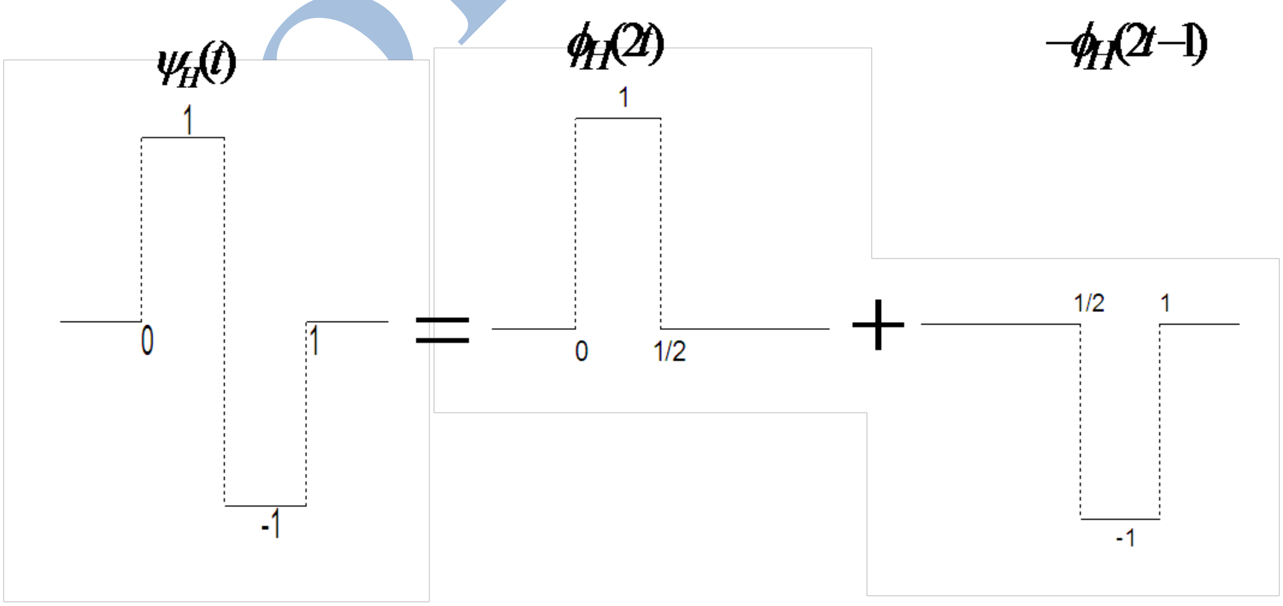
[1 mark]

$\psi_H(t)$



$\phi_H(t)$
1
0
1
=
$\phi_H(2t)$
1
0
1/2
+
$\phi_H(2t-1)$
1
1/2
1

[2 marks]



$\psi_H(t)$
1
0
1
-1
=
$\phi_H(2t)$
1
0
1/2
+
$-\phi_H(2t-1)$
1/2
1
-1

[2 marks]

c) Given the dilation equations $\phi(t/2) = 2\sum_i h_0[i]\phi(t-i)$ and $\psi(t/2) = 2\sum_i h_1[i]\phi(t-i)$, write down the filters $h_0[i]$ and $h_1[i]$ for the Haar basis.

[4 marks]

Answer

Haar low-pass filter $h_0[0] = h_0[1] = \frac{1}{2}$ and

high-pass filter $h_1[0] = \frac{1}{2}$, $h_1[1] = -\frac{1}{2}$. [4 marks: 2 for each]

d) i) Let $\phi_{m,n}(t) = 2^{m/2}\phi(2^m t - n)$, and let the scaling function coefficients be given by $c_{m,n} = \int_{-\infty}^{\infty} s(t)\phi^*{}_{m,n}(t)dt$ for integers n and m.

Show that: $c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$. **[9 marks]**

ii) State the corresponding equation for the fine detail coefficients,

$d_{m,n} = \int_{-\infty}^{\infty} s(t)\psi^*{}_{m,n}(t)dt$. **[2 marks]**

Answer

i)

Let us define

$c_{m,n} \equiv s[n] \equiv s(t)\big|_{t=2^{-m}n}$ for large m.

Using the dilation equation $\phi(t/2) = 2\sum_n h_0[n]\phi(t-n)$ we get

$$c_{m-1,n} = \int_{-\infty}^{\infty} s(t)\phi^*{}_{m-1,n}(t)dt = 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)\phi^*\left(\tfrac{2^m t - 2n}{2}\right)dt$$

$$= 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)2\sum_i h_0[i]\phi^*(2^m t - 2n - i)dt$$

$$= \sqrt{2}\sum_i h_0[i]\int_{-\infty}^{\infty} s(t)\phi^*{}_{m,2n+i}(t)dt = \sqrt{2}\sum_i h_0[i]c_{m,2n+i}$$

i.e. $$c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$$

[9 marks: 1 for definition, 1 for dilation equation, 1 for each of the steps in the derivation]

Also, $d_{m-1,n} = \sqrt{2}\sum_i h_1[i-2n]c_{m,i}$. [2 marks]