§ 5.2 *Magnetic Vector Potential*

In § 5.1 we got $\vec{B} = \nabla \times \left[\frac{\mu_0}{4\pi} \oint_C \frac{I d\vec{l}\,'}{R} \right] = \nabla \times \vec{A}$

Magnetic Vector Potential

Unit: *Wb*/*m* (Weber/m)

We call it M-vector potential, but not M potential directly. The reason is that there still exists ***M-scalar potential***.

According to …?, we know that ***the divergence*** has to be known to determine $\vec{A}$

$$\nabla \times \vec{A} = \vec{B}$$ By definition

$$\nabla \bullet \vec{B} = 0 \rightarrow \nabla \bullet \vec{A} = ?$$

Coulomb's Gauge 库仑规范

Let’s go on.

—— Vector Poisson Equation

$$\begin{cases} \nabla \bullet \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{J} \end{cases} \longrightarrow \nabla \times \left(\nabla \times \vec{A} \right) = \mu_0 \vec{J}$$

Using the formula of ***Vector Product***

$$\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2 \boldsymbol{a}$$

$$\nabla \times \left(\nabla \times \vec{A} \right) = \mu_0 \vec{J}$$

$$\nabla \times \left(\nabla \times \vec{A} \right) = \nabla(\nabla \bullet \vec{A}) - \nabla^2 \vec{A}$$

$$\therefore \nabla^2 \vec{A} = \nabla(\nabla \bullet \vec{A}) - \nabla \times \left(\nabla \times \vec{A} \right)$$

$$\nabla \bullet \vec{A} = 0 \qquad \therefore \nabla^2 \vec{A} = -\nabla \times \left(\nabla \times \vec{A} \right) = -\mu_0 \vec{J}$$

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$ —— ***Vector Poisson Equation***

Discussions $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

A vector Poisson equation may be decomposed into several scalar Poisson equations.

$$\begin{cases} \nabla^2 A_x = -\mu_0 J_x \\ \nabla^2 A_y = -\mu_0 J_y \\ \nabla^2 A_z = -\mu_0 J_z \end{cases}$$

$$\nabla^2 ? = -**$$

Scalar Poisson Equation $\nabla^2 \psi = -\dfrac{\rho}{\varepsilon_0}$

$$\nabla^2\psi = -\frac{\rho}{\varepsilon_0} \Rightarrow \psi = \frac{1}{4\pi\varepsilon_0}\int_{V_{\text{source}}}\left(\frac{\rho}{R_{\text{source-spot}}}\right)dV$$

Similarly, we get

$$A_* = \frac{\mu_0}{4\pi}\int_{V_{\text{source}}}\left(\frac{J_*}{R_{\text{source}-spot}}\right)dV$$

Compose: $$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\int_{V_{\text{source}}}\left(\frac{\vec{J}}{R_{\text{source-spot}}}\right)dV$$

$$\vec{B} = \nabla\times\vec{A} \qquad \vec{E} = -\nabla\psi$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{V_{\text{source}}} \frac{\vec{J}_V dV}{R_{\text{source-spot}}}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint_{C_{\text{source}}} \frac{d\vec{l}}{R_{\text{source-spot}}}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{S_{\text{source}}} \frac{\vec{J}_S dS}{R_{\text{source-spot}}}$$

$$d\vec{A} = \frac{\mu_0 I d\vec{l}}{4\pi R_{\text{source-spot}}}$$

Reasons to introduce this vector:

Its direction?

To make the algebra simpler:

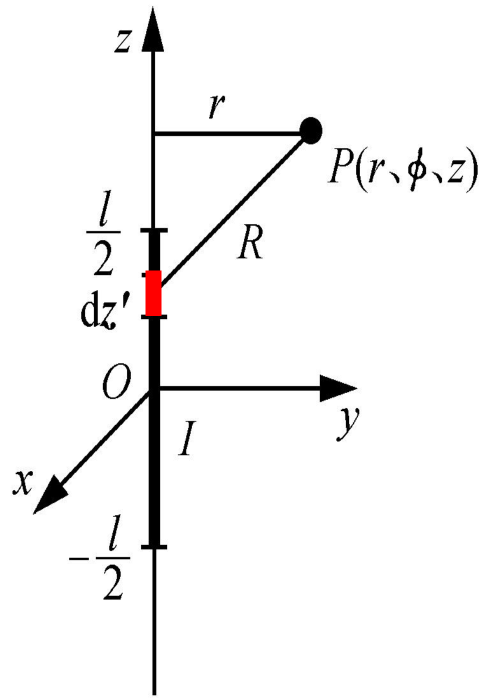
- **Coincide with the current in direction**
- **Be linear to the current element sometimes**
- **二阶偏微分方程常可分解成标量泊松方程形式**

Example 1. current in finite thin wire

- Determine M-flux Density around.

$$\boxed{d\vec{A} = \frac{\mu_0 I d\vec{l}}{4\pi R_{\text{source-spot}}}}$$

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$



$$R = \sqrt{(z - z')^2 + r^2}$$

$$A_z = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{\mu_0 I \mathrm{d}z'}{4\pi \sqrt{(z - z')^2 + r^2}}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left[\frac{\sqrt{(l/2 - z)^2 + r^2} + (l/2 - z)}{\sqrt{(l/2 + z)^2 + r^2} - (l/2 + z)}\right]$$

$$A_z = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{\mu_0 I \mathrm{d}z'}{4\pi\sqrt{(z-z')^2 + r^2}}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left[\frac{\sqrt{(l/2-z)^2 + r^2} + (l/2-z)}{\sqrt{(l/2+z)^2 + r^2} - (l/2+z)}\right]$$

$$\vec{\boldsymbol{B}} = \nabla \times \vec{\boldsymbol{A}} = -\vec{\boldsymbol{e}}_\phi \frac{\partial A_z}{\partial r}$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{l/2-z}{\sqrt{(l/2-z)^2 + r^2}} + \frac{l/2+z}{\sqrt{(l/2+z)^2 + r^2}}\right]\vec{\boldsymbol{e}}_\phi$$

$$\vec{A}(r) = \vec{a}_z \frac{\mu_0 I}{4\pi} \ln\left[\frac{\sqrt{(l/2 - z)^2 + r^2} + (l/2 - z)}{\sqrt{(l/2 + z)^2 + r^2} - (l/2 + z)} \right]$$

If $l >> (r^2 + z^2)^{\frac{1}{2}}$

$$A_z \approx \frac{\mu_0 I}{4\pi} \ln\left(\frac{\sqrt{(l/2)^2 + r^2} + l/2}{\sqrt{(l/2)^2 + r^2} - l/2} \right)$$

$$\approx \frac{\mu_0 I}{4\pi} \ln\left(\frac{l}{r} \right)^2 = \frac{\mu_0 I}{2\pi} \ln\left(\frac{l}{r} \right)$$

$$A_z \approx \frac{\mu_0 I}{2\pi} \ln\left(\frac{l}{r}\right)$$

If the wire is infinite in length, $l \to \infty$

Assume a reference point *P*, and the distance from *P* to the axis is r_0.

$$A_{z0} \approx \frac{\mu_0 I}{2\pi} \ln\left(\frac{l}{r_0}\right)$$

$$\boxed{\vec{A} = \vec{a}_z \frac{\mu_0 I}{2\pi} (\ln\frac{l}{r} - \ln\frac{l}{r_0}) = \vec{a}_z \frac{\mu_0 I}{2\pi} \ln\frac{r_0}{r}}$$

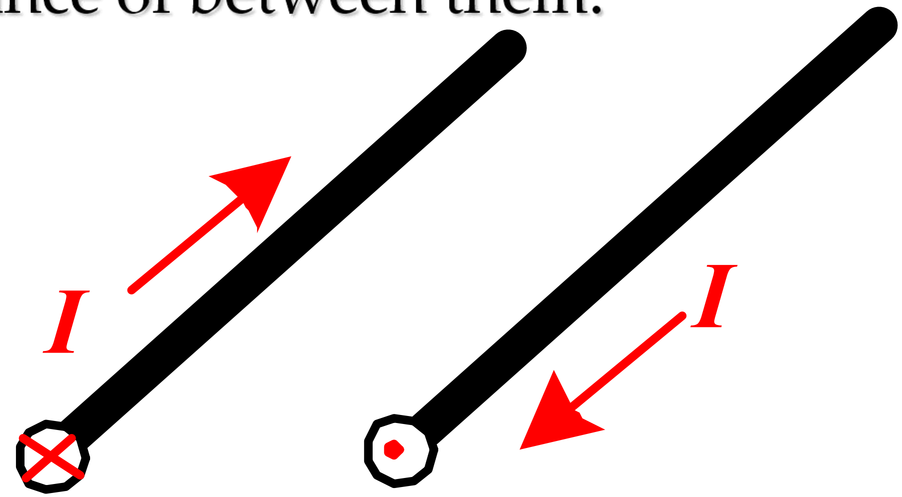
In practice r_0 is always assumed to be 1.

Example 2. parallel double lines

Parallel double lines with a distance of between them.

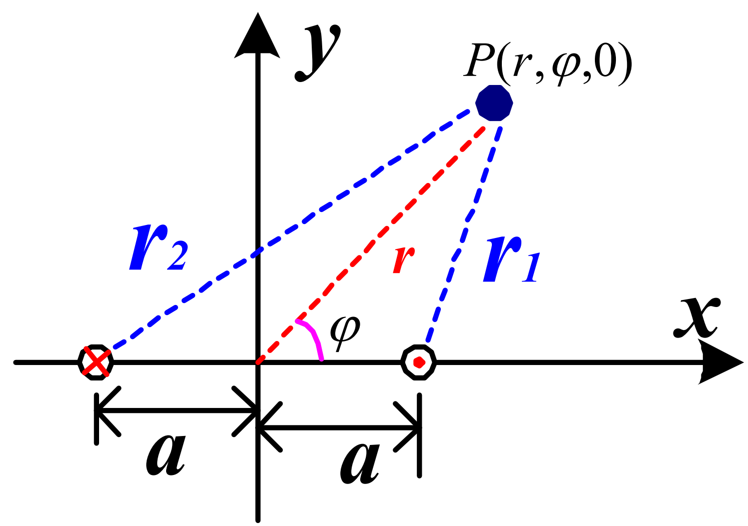
Please determine the M-flux density in space.

In this example we try to use M-vector potential around.



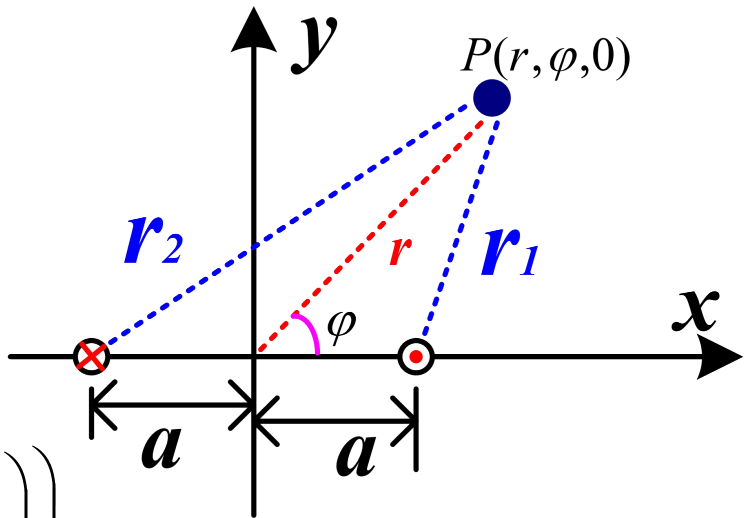
Analysis:

1. Direction of $\vec{A}$
2. Direction of $\vec{B}$
3. The coordinates?



From example 1, we know that around infinite thin wire …

$$\vec{A}_1 = \vec{a}_z I_1 \frac{\mu_0}{2\pi} \cdot \ln \frac{1}{r}$$



At a certain point in space *P*

$$\vec{A} = \vec{A}_1 + \vec{A}_2$$

$$= \left(+\vec{a}_z I\right)\left(\frac{\mu_0}{2\pi} \cdot \ln\left(\frac{1}{r_1}\right)\right) + \left(-\vec{a}_z I\right)\left(\frac{\mu_0}{2\pi} \cdot \ln\left(\frac{1}{r_2}\right)\right)$$

$$\therefore \vec{A} = \vec{a}_z\left(\frac{\mu_0 \cdot I}{2\pi} \cdot \ln\left(\frac{r_2}{r_1}\right)\right) = \ldots \ln\left(\frac{a^2 + r^2 + 2ar \cdot \cos\varphi}{a^2 + r^2 - 2ar \cdot \cos\varphi}\right)$$

$$\vec{B} = \nabla \times \vec{A} = ?$$

$$\nabla \times \left(u\vec{A}\right) = \nabla u \times \vec{A} + u\nabla \times \vec{A}$$

$$\vec{A} = \vec{a}_z \left(\frac{\mu_0 \cdot I}{2\pi} \cdot \ln \frac{a^2 + r^2 + 2ar \cdot \cos\varphi}{a^2 + r^2 - 2ar \cdot \cos\varphi} \right)$$

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left(A\vec{a}_z \right) = \nabla A \times \vec{a}_z + 0$$

$$= \cdots = \left(\vec{a}_r \frac{\partial A}{\partial r} + \vec{a}_\varphi \frac{\partial A}{r\partial \varphi} \right) \times \vec{a}_z = ?$$

Please finish this example after class.

Homework

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- **Exercises: 5.10,5.12,5.14,5.15**