EBU6018
Fourier Transform

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Fourier Transform

- The Fourier Series can only be applied to periodic signals, giving an infinite number of discrete frequencies. However, periodic signals are non-informational.

- Non-periodic signals (signals containing information) cannot be analysed using the Fourier Series, the Fourier Transform (FT) is required.

- This gives us the bandwidth of a signal as the sum of an continuous infinity of sinusoids.

Fourier Transforms

The Fourier Transform (FT) is defined as:

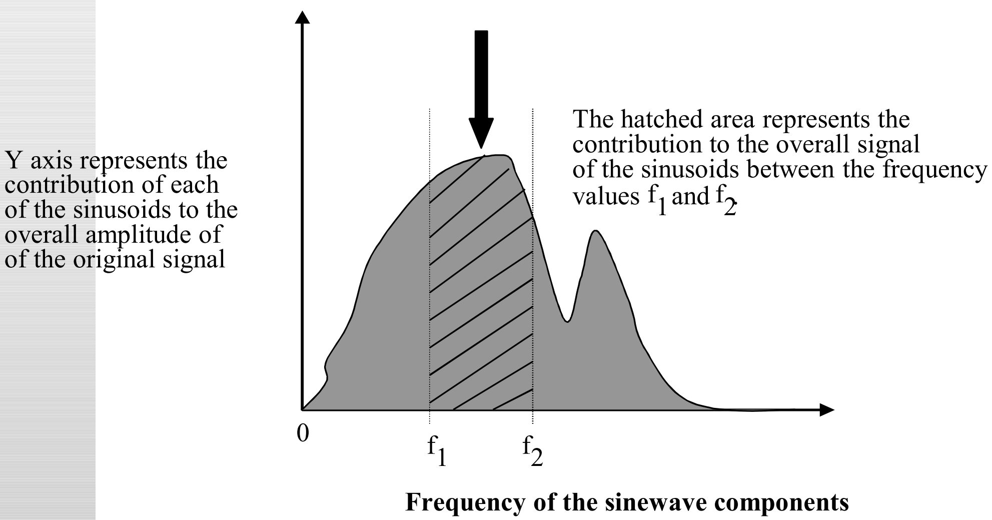
$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)\,dt$$

X(f) in LHS is values in the amplitude spectral density (ASD).

x(t) is the signal

The Fourier transform will always be denoted by an uppercase letter or symbol, whereas signals will usually be denoted by lowercase letters or symbols.

A frequency domain diagram showing spectral density



R&S spectrum analyzer (R&S FSP40) F, at QMUL

Fourier Transforms

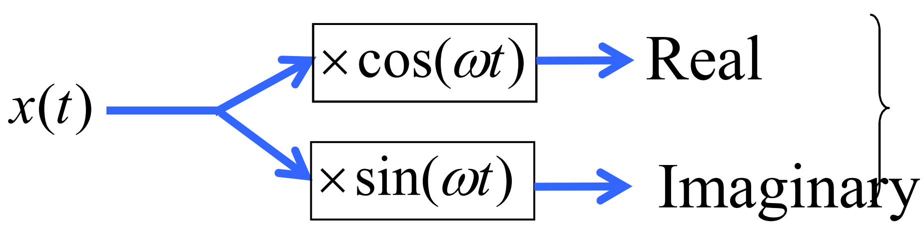
The Fourier Transform (FT) is defined as:

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)\,dt$$

By Euler's formula

$$= \int_{t=-\infty}^{t=\infty} \left(\cos(\omega t) - j\sin(\omega t)\right) \cdot x(t)\,dt$$

Implement the FT:



$$X(f) = \mathrm{Re} + j\,\mathrm{Im}$$

The Conditions for an FT

- **A signal is said to have a Fourier transform in the ordinary sense if the integral in the following equation converges (i.e. exists) .**

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)\,dt$$

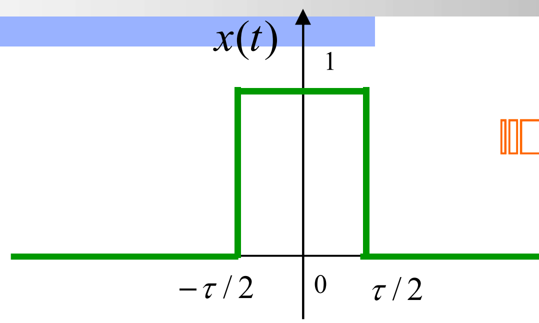
x(t) is "well behaved" if:

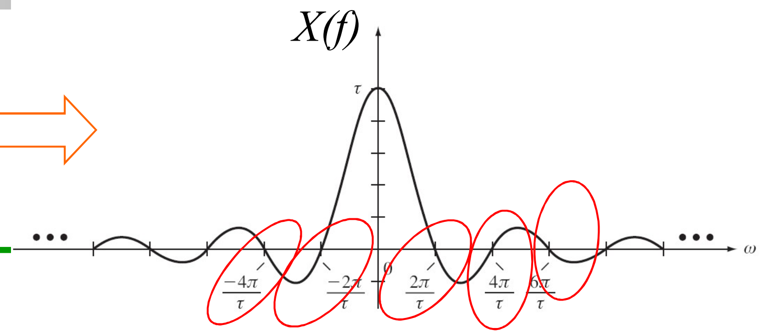
1. the signal x(t) has a finite number of discontinuities, maxima, and minima within any finite interval of time.
2. if x(t) is absolutely integrable $\int_{t=-\infty}^{t=\infty} |x(t)|\,dt \ < \ \infty$

However some common signals are not absolutely integrable.

Isolated Rectangular Pulse

$$x(t) = \begin{cases} 1, & -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2} \\ 0, & all\ other\ t \end{cases}$$





also denoted as $p_\tau(t)$

FT definition

even **odd**

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)dt = \int_{t=-\infty}^{t=\infty} \left(\cos(\omega t) - j\sin(\omega t)\right) \cdot x(t)dt$$

and x(t) is an even signal.

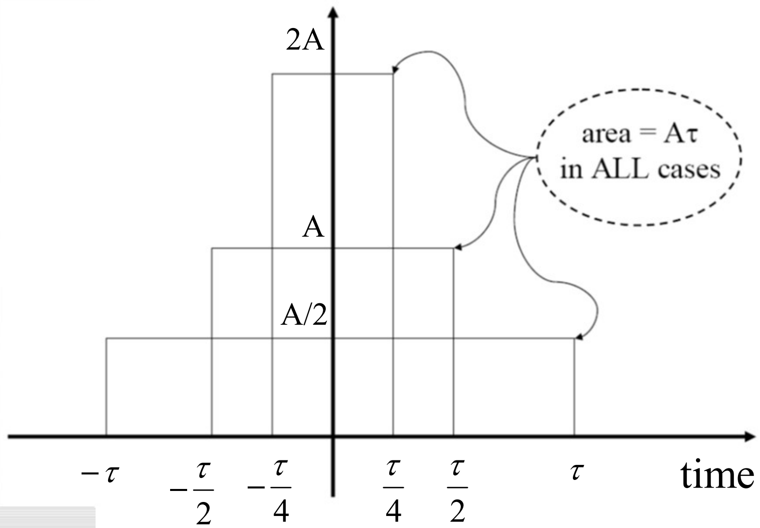
$$X(f) = 2\int_{0}^{\tau/2} (1)\cos(\omega t)dt = \frac{2}{\omega}\left[\sin(\omega t)\Big|_{t=0}^{t=\tau/2}\right] = \frac{2}{\omega}\sin\frac{\omega\tau}{2}$$

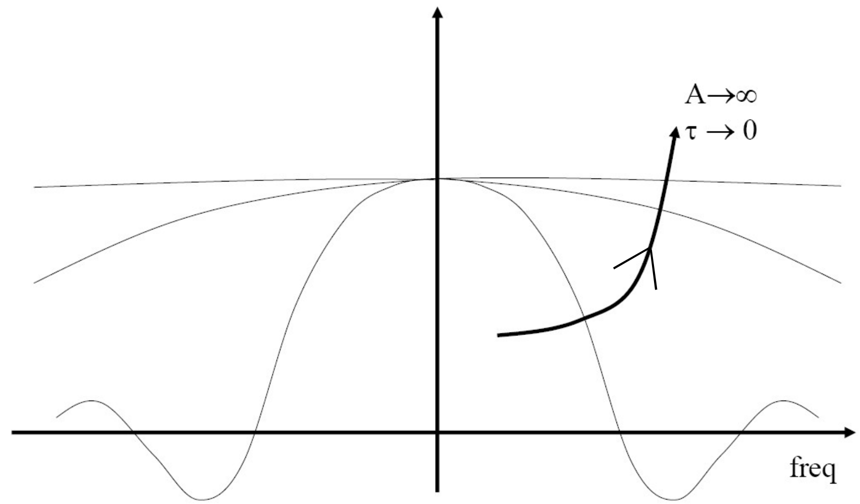
Let's recall the sinc function $\quad \mathrm{sinc}(a\omega) = \dfrac{\sin(a\pi\omega)}{a\pi\omega}$ $\quad$ Setting $\quad a = \dfrac{\tau}{2\pi}$

$$\mathrm{sinc}\left(\frac{\tau\omega}{2\pi}\right) = \frac{2}{\tau\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

Thus, $\quad X(f) = \tau\ \mathrm{sinc}\left(\dfrac{\tau\omega}{2\pi}\right)$

Isolated Rectangular Pulse





Inverse Fourier Transform

Given a signal x(t) with Fourier transform X(f), x(t) can be recomputed from X(f) by application of the inverse Fourier transform give by:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) \cdot e^{j\omega t} df$$

To denote the fact that X(f) is the Fourier transform of x(t), or that X(f) is the inverse Fourier transform of x(t), the transform pair notation:

$$x(t) \leftrightarrow X(f)$$

will sometimes be used.

One of most fundamental transform pairs in the Fourier Theory is the pair

$$p_\tau(t) \leftrightarrow \tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$$