Advanced Transform Methods

Time-frequency Analysis and the (Heisenberg) Uncertainty Principle

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Two Types of Signal

We saw in the introduction lecture that there are two types of signal:

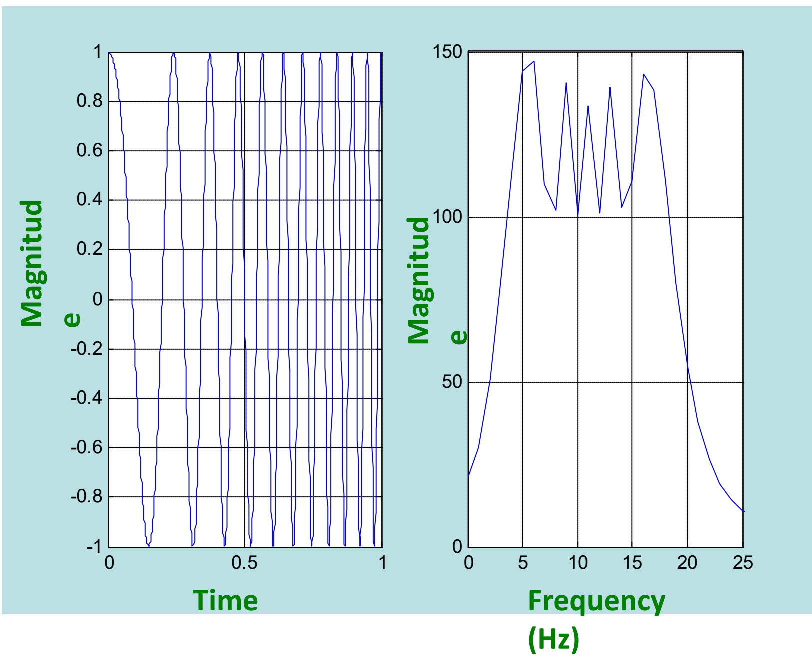
Stationary signals, in which the frequency content of the signal does not change with time, and

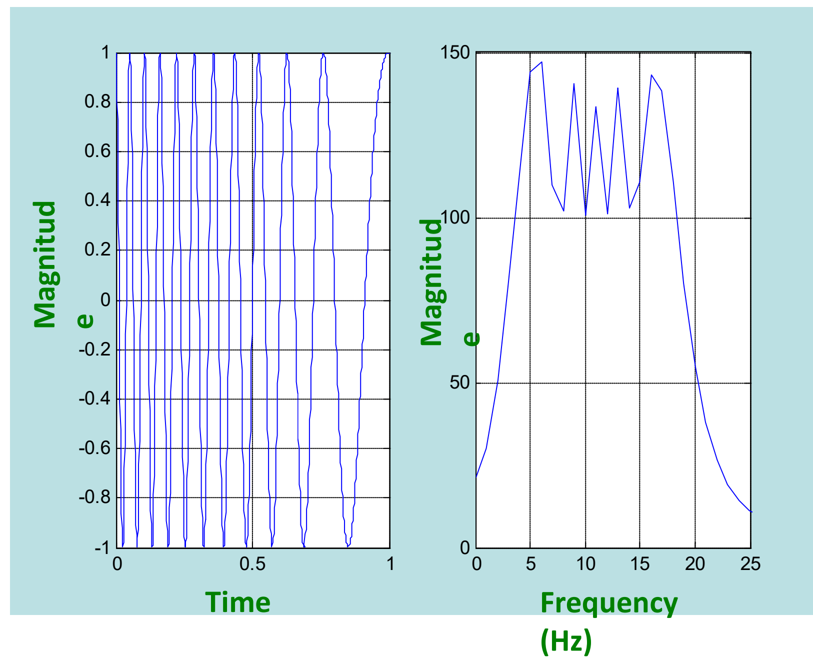
Non-Stationary signals, in which the frequency content changes with time.

Non-stationary signals contain useful information, and we need to find methods of analysing these signals, since the Fourier Transform itself omits the time information in the signal.

Example: non-stationary signals

Consider two linearly modulated sinusoids (chirps). The first with increasing frequency and the second with decreasing frequency.





In this case we have two nonstationary signals in time with identical FTs. Confusion arises and power spectrum is not very useful.

Fourier Transform Limitation

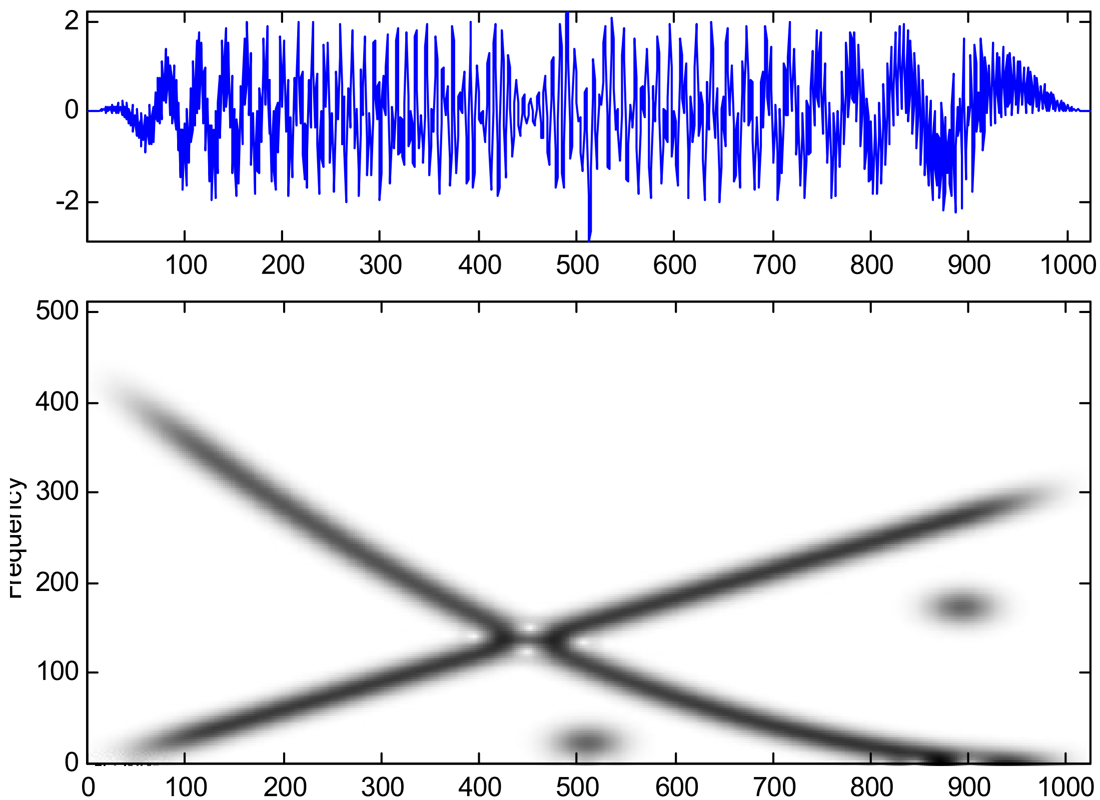
- We see that the Fourier Transforms of these two different signals are identical.
- The Fourier Transforms tell us the range of frequencies contained in the signals.
- But if we wanted to know how the frequencies changed with time then the Fourier Transform could not give that information.
- Time-frequency analysis studies both the time and frequency domains simultaneously.

Composite Signal Example

The following example (also from the introduction lecture) shows a signal of duration 1000mS, that appears to contain information. But what that information is cannot be easily seen.

A time-frequency analysis of that signal shows that there are 4 different pieces of information contained in it……..

Composite Signal Example



Composite Signal Example

- There is a chirp that decreases frequency non-linearly with time.
- A chirp that increases frequency linearly with time
- A short burst of about 20Hz at anout 500mS
- And a short burst of about 180Hz at about 900mS.
- NOTE that we have to say “about” because of the widths of the lines in the t-f plot

Time-Frequency Analysis

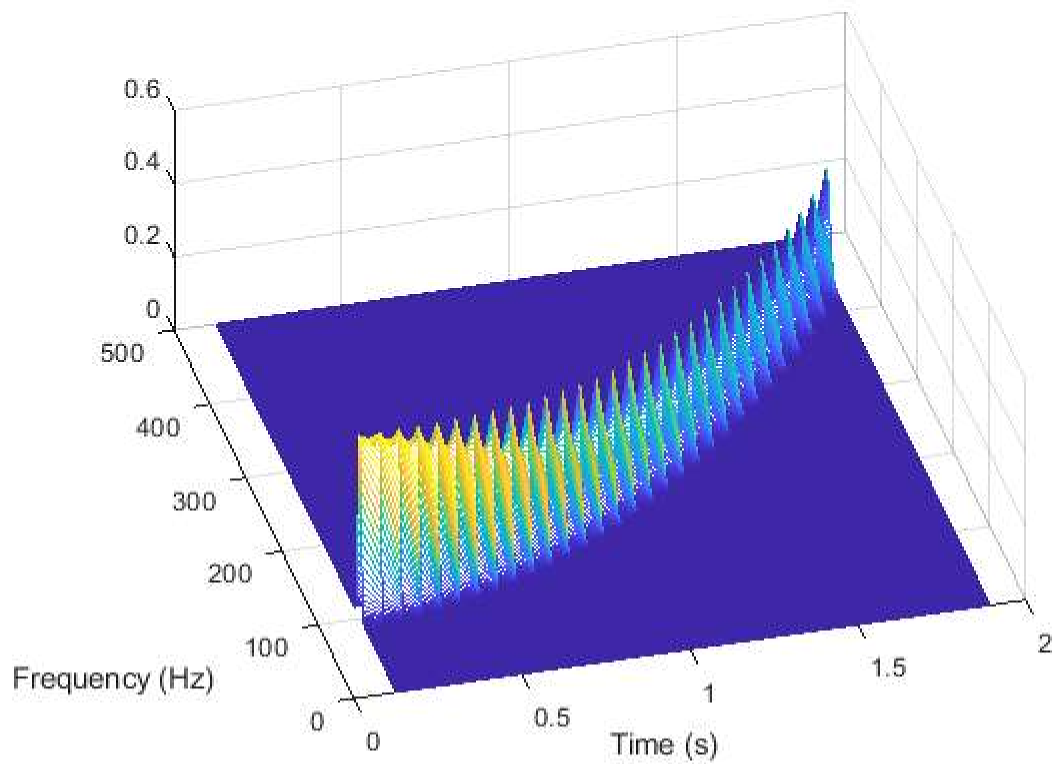
This example shows some of the applications of time-frequency analysis, including:

- Feature extraction (e.g. the short bursts of signal)…..looking for particular events and at what times they happen, e.g. electrocardiograms (ecgs), hydrocarbon exploration, etc.
- Trend analysis (e.g. the change of information with time), e.g. financial information, environmental data (weather, etc), etc.

Spectrogram

- The time-frequency plot is a plot in 2D of the energy in the signal at localities in time and frequency
- The more dense the lines then the greater the energy at that location.
- A 3D plot would give the information more clearly…….

3D t-f plot



https://uk.mathworks.com/help/signal/ref/pspectrum.html

Window Transform

- Putting a window over the signal and calculating the FT of the signal during the window allows us to find the frequencies present during the window. If we then move the window we find the frequencies present at a different time.
- This is the principle of the STFT (next lecture).

Window Transform

- We can see that if the window is very wide, then we cannot localise the frequency content very accurately in time.
- However we will be able to know the frequencies present during the window quite accurately because we will have many cycles of them to be able to calculate the frequencies.

Window Transform

- If the window is very narrow then the localisation in time is more accurate
- However the knowledge of the frequencies present will not be accurate because we will “see” only part of the cycle of the waveform and will not be able to know the frequency accurately.
- If I look out my window and see the road outside I cannot know how long the road is because the window is too narrow.

Uncertainty Principle

- Obviously the window cannot be zero width (to get accurate time information)
- Nor can it be the complete width of the signal (then we would not need the window)
- So there are limits to the accuracy of the time information and of the frequency information
- And there is a trade-off between them.
- This trade-off can be formulated by the Uncertainty Principle.

Uncertainty Principle

- The Uncertainty Principle states:
- The product of the uncertainty in time and the uncertainty in frequency is a constant.
- So the are inversely proportional.

$$\Delta_t \Delta_\omega \geq 1/2$$

Heisenberg Uncertainty Principle

$$\Delta_t \Delta_\omega \geq 1/2$$

This is the answer.
The following derivation is NON_EXAMINABLE and is included only for information…..
…..we can jump to slide 32.

Time Convolution

The Fourier transform of the convolution is the product of the transforms.

$$\int_{-\infty}^{\infty} \left[s_1(t) * s_2(t) \right] e^{-j\omega t} dt = S_1(\omega) S_2(\omega)$$

So the convolution of two signals is the inverse Fourier transform of the product of the transforms.

$$\int_{-\infty}^{\infty} s_1(\tau) s_2(t-\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_1(\omega) S_2(\omega) e^{j\omega t} d\omega$$

Frequency Convolution

To find the inverse Fourier transform of the convolution $S_1(\omega) * S_2(\omega) \equiv \mathrm{S}(\omega)$

$$\mathrm{s}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_1(\omega) * S_2(\omega)\right]e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}\left[\int_{-\infty}^{\infty}S_1(\alpha)S_2(\omega-\alpha)d\alpha\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j(x+\alpha)t}S_1(\alpha)S_2(x)d\alpha dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}S_1(\alpha)e^{j\alpha t}d\alpha\int_{-\infty}^{\infty}e^{jxt}S_2(x)dx$$

$$= 2\pi s_1(t)s_2(t)$$

Inverse Fourier Transform of the convolution of 2 Fourier transforms is 2π times the product of the signals.

→

The convolution of 2 Fourier transforms is 2π times the Fourier transform of the product of the signals

Parseval's Formula

From the time convolution formula,

$$\int_{-\infty}^{\infty} s_1(\tau)s_2(t-\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_1(\omega)S_2(\omega)e^{j\omega t}d\omega$$

So

$$\int_{-\infty}^{\infty} s_1(t)s_2(-t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_1(\omega)S_2(\omega)d\omega$$

If $s_2^*(t) = f(-t)$ then the Hermitian property, $s^*(t) \leftrightarrow S^*(-\omega)$ tells us

$$S_2^*(-\omega) = F(-\omega)$$

$$S_2(\omega) = F^*(\omega)$$

So

$$\int_{-\infty}^{\infty} s_1(t)f^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_1(\omega)F^*(\omega)d\omega$$

Energy

Parseval's formula states,

$$\int_{-\infty}^{\infty} s(t) f^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) F^*(\omega) d\omega$$

If *s=f*,

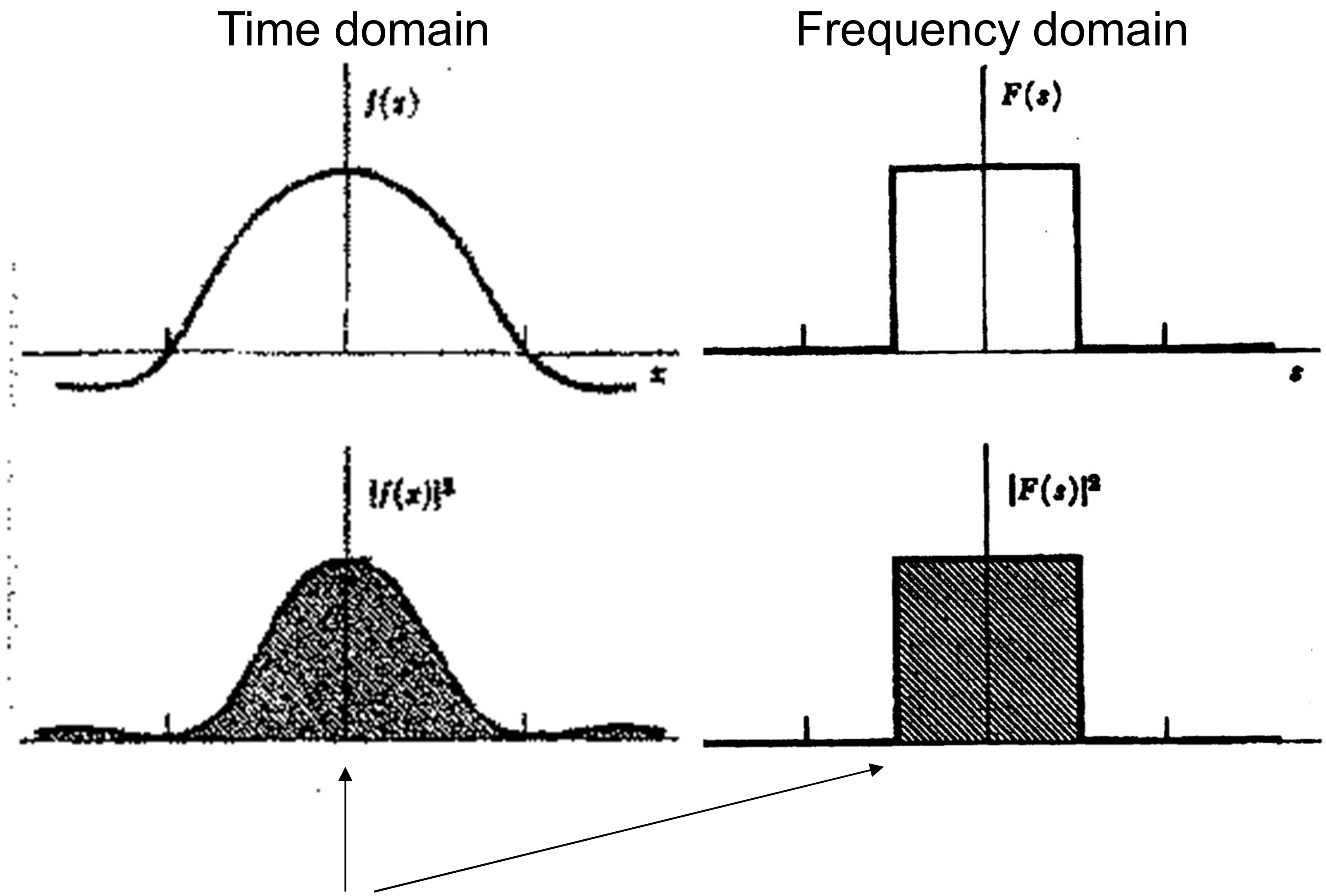
$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega$$

We can define energy in both time and frequency domains

$$E = \|s(t)\|^2 = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega$$

Energy is conserved

Parseval's Theorem in action



The two shaded areas (i.e., measures of the signal energy) are the same.

Mean Time and Time Duration

$$E = \int_{-\infty}^{\infty} \left|s(t)\right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|S(\omega)\right|^2 d\omega$$

- The signal's gravitational centres in the time domain.
  - Mean Time $$\langle t \rangle = \frac{1}{E} \int_{-\infty}^{\infty} t \left|s(t)\right|^2 dt$$
- The signal's energy spread in the time domain
  - Time Duration Δ_t $$\Delta_t^2 = \frac{1}{E} \int_{-\infty}^{\infty} \left(t - \langle t \rangle\right)^2 \left|s(t)\right|^2 dt$$

Mean Frequency and Frequency Bandwidth

$$E = \int_{-\infty}^{\infty} \left|s(t)\right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|S(\omega)\right|^2 d\omega$$

- The signal's gravitational centre in the frequency domain.
  - Mean Frequency $\langle \omega \rangle = \frac{1}{2\pi E} \int_{-\infty}^{\infty} \omega \left|S(\omega)\right|^2 d\omega$
- The signal's energy spread in the frequency domain.
  - Frequency Bandwidth Δ_ω

$$\Delta_\omega^2 = \frac{1}{2\pi E} \int_{-\infty}^{\infty} \left(\omega - \langle \omega \rangle\right)^2 \left|S(\omega)\right|^2 d\omega$$

Duration and Bandwidth

– Time Duration

$$\Delta_t{}^2 = \frac{1}{E}\int_{-\infty}^{\infty}(t^2 - 2t\langle t\rangle + \langle t\rangle^2)\left|s(t)\right|^2 dt$$

$$= \frac{1}{E}\int_{-\infty}^{\infty} t^2\left|s(t)\right|^2 dt - \frac{2\langle t\rangle}{E}\int_{-\infty}^{\infty}\left|s(t)\right|^2 t dt + \frac{\langle t\rangle^2}{E}\int_{-\infty}^{\infty}\left|s(t)\right|^2 dt$$

$$= \frac{1}{E}\int_{-\infty}^{\infty} t^2\left|s(t)\right|^2 dt - 2\langle t\rangle\langle t\rangle + \langle t\rangle^2 = \frac{1}{E}\int_{-\infty}^{\infty} t^2\left|s(t)\right|^2 dt - \langle t\rangle^2$$

– Frequency Bandwidth

$$\Delta_\omega{}^2 = \frac{1}{2\pi E}\int_{-\infty}^{\infty}(\omega^2 - 2\omega\langle\omega\rangle + \langle\omega\rangle^2)\left|S(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi E}\int_{-\infty}^{\infty}\omega^2\left|S(\omega)\right|^2 d\omega - \frac{2\langle\omega\rangle}{2\pi E}\int_{-\infty}^{\infty}\omega\left|S(\omega)\right|^2 d\omega + \frac{\langle\omega\rangle^2}{2\pi E}\int_{-\infty}^{\infty}\left|S(\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi E}\int_{-\infty}^{\infty}\omega^2\left|S(\omega)\right|^2 d\omega - 2\langle\omega\rangle^2 + \frac{\langle\omega\rangle^2}{E}E = \frac{1}{2\pi E}\int_{-\infty}^{\infty}\omega^2\left|S(\omega)\right|^2 d\omega - \langle\omega\rangle^2$$

Normalised in Time and Frequency

- Given a signal $s(t)$, can one find a signal with the same energy, bandwidth and duration but normalised such that the mean frequency and the mean time of the signal are both set to 0?
- Yes:

$$r(t) = e^{-jt\langle \omega \rangle} s(t + \langle t \rangle)$$

- That is, a shift left in time and a shift left in frequency. The shift in frequency is obtained by modulation in time.

Uncertainty Principle

To show :

if $\sqrt{t}\,s(t) \to 0$ as $|t| \to \infty$ (i.e. $s(t)$ decays fast enough)

and the signal has unit energy :

$$E = \int_{-\infty}^{\infty} |s(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega = 1$$

then

$$\Delta_t \Delta_\omega \geq \frac{1}{2}$$

where equality holds when $s(t)$ is a Gaussian, $s(t) = Ae^{-\alpha t^2}$

We will assume that $\langle t \rangle = 0$ and $\langle \omega \rangle = 0$. (Normalised)

{ NB Will do for simple case of real $s(t) = s^*(t)$ }

Squared time width is given by: $\Delta_t^2 = \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt$

and sq. frequency width by: $\Delta_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 d\omega$

with their product as:

$$\Delta_t^2 \Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 d\omega$$

From the time derivative property, we know

if $h(t) = \frac{d}{dt} s(t)$ then $H(\omega) = j\omega S(\omega)$

So using this in the frequency width we get

$$\Delta_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |j\omega S(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

So we have $\Delta_\omega^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$

But from Parseval we have the conservation of energy :

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

so $\Delta_\omega^2 = \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} \left|\frac{d}{dt}s(t)\right|^2 dt$ since $h(t) = \frac{d}{dt}s(t)$

Therefore for the original product we get

$$\Delta_t^2 \Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt \int_{-\infty}^{\infty} \left|\frac{d}{dt}s(t)\right|^2 dt$$

So we have $$\Delta_t^2\Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt \int_{-\infty}^{\infty} \left|\frac{d}{dt}s(t)\right|^2 dt$$

The *Schwarz Inequality* states: $\|\psi_1\|^2\|\psi_2\|^2 \geq |\langle\psi_1,\psi_2\rangle|^2$ i.e.

$$\int_{-\infty}^{\infty} |f(t)|^2 dt \cdot \int_{-\infty}^{\infty} |g(t)|^2 dt \geq \left|\int_{-\infty}^{\infty} f(t)g(t)dt\right|^2$$

which we can use with $f(t) = ts(t)$ and $g(t) = \frac{d}{dt}s(t)$ to get

$$\Delta_t^2\Delta_\omega^2 = \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt \int_{-\infty}^{\infty} \left|\frac{d}{dt}s(t)\right|^2 dt \quad \geq \quad \left|\int_{-\infty}^{\infty} ts(t)\frac{d}{dt}s(t)dt\right|^2$$

{Remember using real $s(t)$ }

So far we have $\Delta_t^2 \Delta_\omega^2 \;\geq\; \left(\int_{-\infty}^{\infty} t s(t) \frac{d}{dt} s(t) dt \right)^2$

$\{s(t)$ etc are real$\}$

Now, try differentiating $s(t)^2$

$$\frac{d}{dt} s(t)^2 = 2 s(t) \frac{d}{dt} s(t)$$

so inserting this we get

$$\Delta_t^2 \Delta_\omega^2 \;\geq\; \left(\int_{-\infty}^{\infty} t s(t) \frac{d}{dt} s(t) dt \right)^2$$

$$= \left(\frac{1}{2} \int_{-\infty}^{\infty} t \frac{d}{dt} s(t)^2 \, dt \right)^2$$

So far we have $\Delta_t^2 \Delta_\omega^2 \ \geq \ \left(\frac{1}{2} \int_{-\infty}^{\infty} t \, \frac{d}{dt} s(t)^2 \, dt \right)^2$

Now remember *integration by parts* :

$$\int_a^b u dv = [uv]_a^b - \int_a^b v du$$

and use $u = t, \quad v = s^2(t), \quad du = dt, \quad dv = \frac{d}{dt} s^2(t) dt$

Then $\frac{1}{2} \int_{-\infty}^{\infty} t \frac{d}{dt} s(t)^2 \, dt = \frac{1}{2} [ts(t)^2]_{-\infty}^{\infty} - \frac{1}{2} \int_{-\infty}^{\infty} s(t)^2 \, dt$

But we specified that $\sqrt{t} s(t) \to 0$ as $|t| \to \infty$, so

$$[ts(t)^2]_{-\infty}^{\infty} = \infty s(\infty)^2 - (-\infty) s(-\infty)^2 = 0 - 0 = 0$$

So we now have

$$\Delta_t^2 \Delta_\omega^2 \geq \left(-\frac{1}{2} \int_{-\infty}^{\infty} s^2(t) dt \right)^2$$

So far we have

$$\Delta_t^2 \Delta_\omega^2 \geq \left(-\tfrac{1}{2} \int_{-\infty}^{\infty} s^2(t)\,dt \right)^2$$

But $\int_{-\infty}^{\infty} s^2(t)\,dt = 1$ since it is the energy of the signal,

so $\Delta_t^2 \Delta_\omega^2 \geq \left(-\tfrac{1}{2} \times 1 \right)^2$ i.e. $\underline{\underline{\Delta_t \Delta_\omega \geq \frac{1}{2}}}$

Which was what we wanted.

Finally, Schwartz' inequality is exact (an equality) when the two functions ψ_1 and ψ_2 are colinear, $\psi_1 = c\,\psi_2$

With our functions $\psi_1 = ts(t)$ and $\psi_2 = \frac{d}{dt} s(t)$

this happens for Gaussian $s(t) = Ae^{-\alpha t^2}$,

$$\psi_2 = \tfrac{d}{dt} s(t) = -2\alpha t A e^{-\alpha t^2} = -2\alpha t s(t) = -2\alpha \psi_1$$

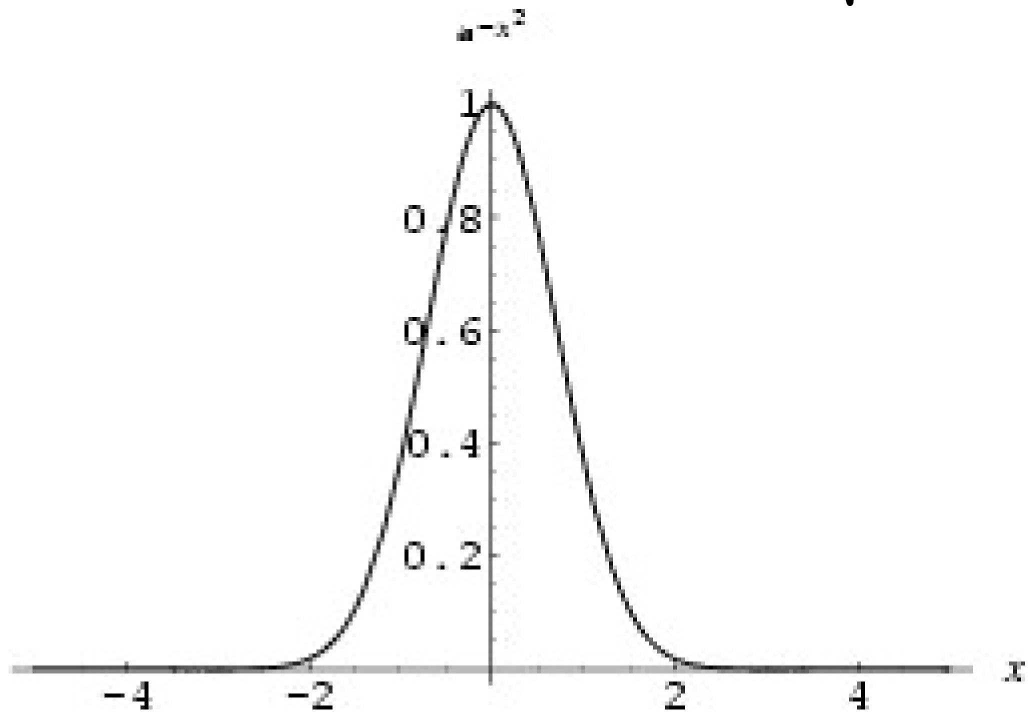
So $\Delta_t \Delta_\omega = \frac{1}{2}$ for a Gaussian

Uncertainty Principle

- We can easily find signals normalised in both time and frequency simultaneously, but it is not possible to *localise* a function in time and frequency simultaneously.
- Different definitions of the Fourier Transform yield different versions of the Uncertainty Principle.
- The Time-Bandwidth Product $\Delta_t \Delta_\omega$ is a measure of the pulse complexity
- A signal fixed in time has infinite bandwidth
- A signal fixed in frequency has infinite duration
- Discovered by Heisenberg and Applied to quantum mechanics

Special Properties of the Gaussian

- Simple form $$g(t) = e^{-at^2}$$
- General form $$g(t) = ae^{jbt}e^{-c(t-d)^2}$$
- Normalised form $$g(t) = \sqrt[4]{\frac{\alpha}{\pi}}\, e^{-\frac{\alpha}{2}t^2}$$



Uncertainty Principle and Gaussian functions

$$\frac{d}{dt}e^{-at^2} = -2ate^{-at^2}$$

$$E^2\Delta_t^2\Delta_\omega^2 = \left(\int_{-\infty}^{\infty} t^2\left|e^{-at^2}\right|^2 dt\right)\left(\int_{-\infty}^{\infty}\left|\frac{d}{dt}e^{-at^2}\right|^2 dt\right)$$

$$= \left(\int_{-\infty}^{\infty} t^2\left(e^{-at^2}\right)^2 dt\right)\left(\int_{-\infty}^{\infty}\left(-2ate^{-at^2}\right)^2 dt\right)$$

$$= \left(\int_{-\infty}^{\infty} -2at^2e^{-at^2}e^{-at^2} dt\right)\left(\int_{-\infty}^{\infty} -2at^2e^{-at^2}e^{-at^2} dt\right)$$

$$= \left(\int_{-\infty}^{\infty} te^{-at^2}\frac{d}{dt}e^{-at^2} dt\right)\left(\int_{-\infty}^{\infty} te^{-at^2}\frac{d}{dt}e^{-at^2} dt\right)$$

In fact, the Gaussian is the *only* function that gives equality in the uncertainty relationship.

A Gaussian transforms to a Gaussian

$$\int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \cos(\omega t) e^{-at^2} dt - j \int_{-\infty}^{\infty} \sin(\omega t) e^{-at^2} dt$$

$$= \int_{-\infty}^{\infty} \cos(\omega t) e^{-at^2} dt = \sqrt{\frac{\pi}{a}} e^{-\omega^2/4a}$$

The narrower a Gaussian is in one domain, the broader it is in the other domain.

Gaussian and Convolution- The Central Limit Theorem

The Central Limit Theorem says:

The convolution of the convolution of the convolution etc. of any signal approaches a Gaussian.

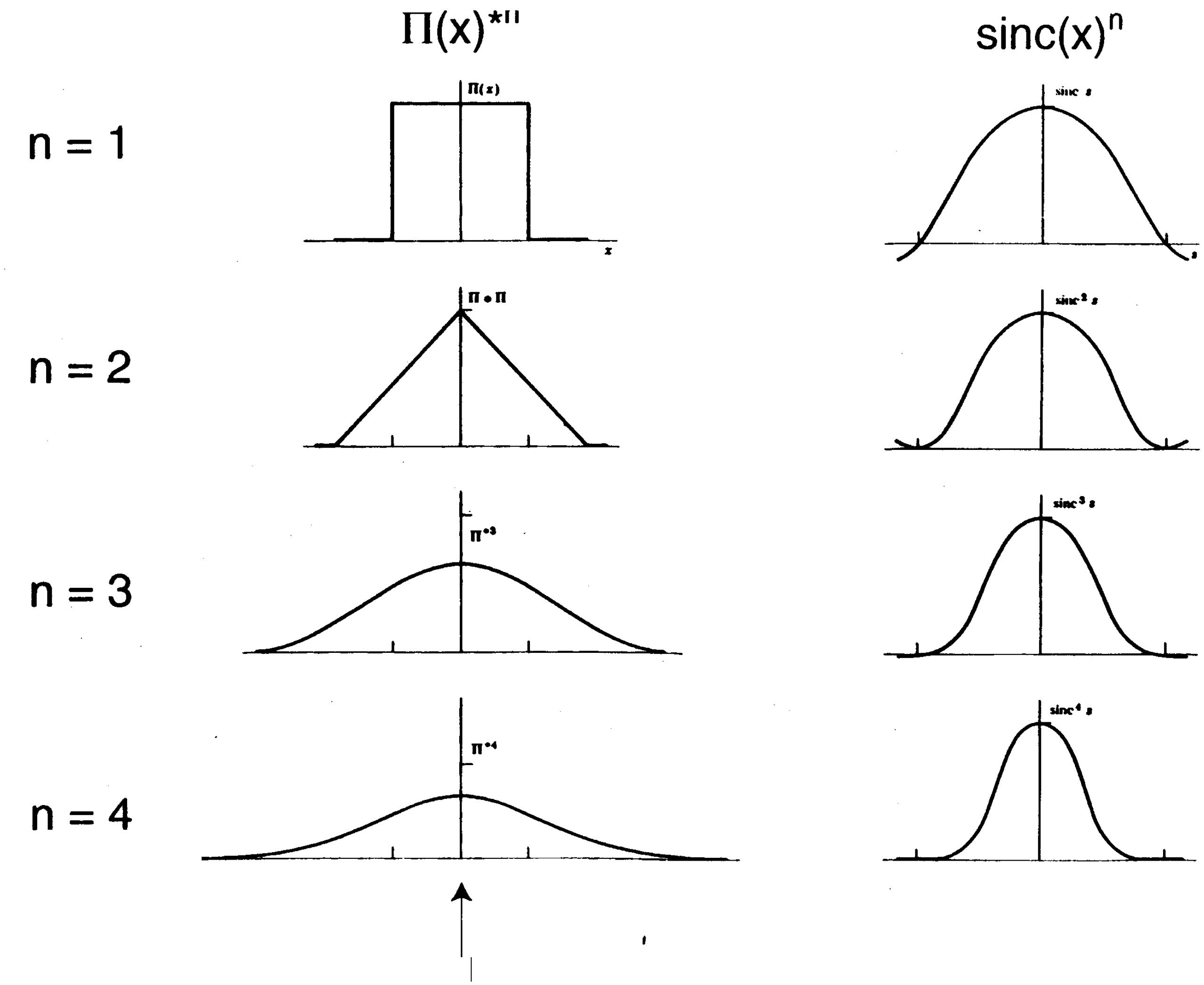
Mathematically,

$$f(x) * f(x) * f(x) * f(x) * \ldots * f(x) \rightarrow e^{(-x/a)2}$$

or: $$f(x)^{*n} \rightarrow exp[(-x/a)^2]$$

The Central Limit Theorem is why everything has a Gaussian distribution.

The Central Limit Theorem for a square function, $\Pi(x)$



Note that $P(x)^{*4}$ already looks like a Gaussian!