EBU6018
Advanced Transform Methods

Haar Transform

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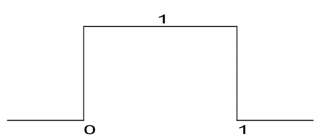
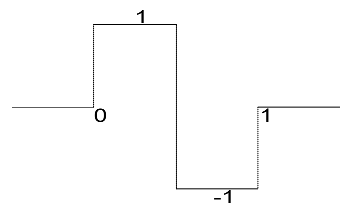
Haar Functions

The Haar Scaling Function and the Haar Wavelet Functions:





$\varphi_{00} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $\qquad$ $\psi_{00} = \begin{bmatrix} 1 & -1 \end{bmatrix}$

These are orthogonal

$$< \varphi_{00}, \psi_{00} > = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Wavelet Transforms from Filter Banks

- We saw in “Wavelet Transforms from Filter Banks” that we can perform wavelet transforms by using low and high pass filters to calculate the coefficients of the transform.
- The low and high pass filters are derived from the wavelet function we want to use.
- The low pass filter is used to calculate the coefficients of the approximate output sequence and the high pass filter to calculate the coefficients of the fine details.
- We used Haar functions as an example.
- However we know that we can obtain the output of a transform by using a transform matrix, for example the DCT.

Haar Matrix

These two functions can be written in matrix form:

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The normalised Haar Matrix is:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

4x4 Haar Matrix

The 4x4 Haar Matrix combines two stages of a Haar Wavelet transform:

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Normalised this is:

$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

Example 1

Apply the Haar Transform to the 4-point input sequence:

S[n] = [2, 5 -3, 7]

Show that this gives the same answer as the calculation using filter banks.

Example 1…Solution

8x8 Haar Matrix

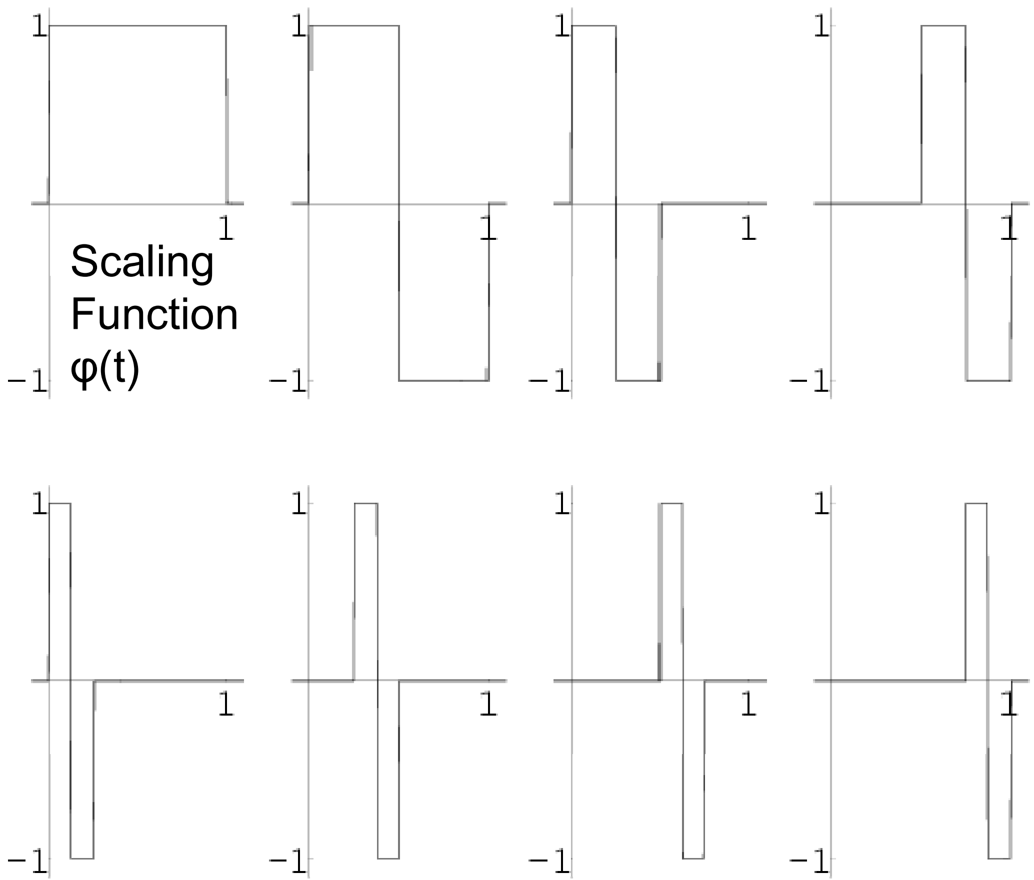
The un-normalised 8x8 Haar Matrix can be used to show how a Haar Matrix is derived:

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{c} \varphi_0(\mathrm{t}) \\ \psi_0(\mathrm{t}) \\ \psi_{1,0}(\mathrm{t}) \\ \psi_{1,1}(\mathrm{t}) \\ \psi_{2,0}(\mathrm{t}) \\ \psi_{2,1}(\mathrm{t}) \\ \psi_{2,2}(\mathrm{t}) \\ \psi_{2,3}(\mathrm{t}) \end{array}$$

The first row gives the average value of the input sequence. Then subsequent rows correspond to increasing frequencies (similar to DCT)

The matrix would need to be normalised before it could be applied directly to a transform.

Haar Functions



Wavelet Function:

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_{jk}(x) \equiv \psi(2^j x - k),$$

$\phi_{00}=\phi\,(x)$

$\psi_{00}=\psi\,(x)$

$\psi_{10}=\psi\,(2x)$

$\psi_{11}=\psi\,(2x\text{-}1)$

$\psi_{20}=\psi\,(4x)$

$\psi_{21}=\psi\,(4x\text{-}1)$

$\psi_{22}=\psi(4x\text{-}2)$

$\psi_{23}=\psi(4x\text{-}3)$

Inverse Haar Transform

The Haar Matrix is real and orthonormal.

The inverse Haar Transform can be derived from:

$H = H^*,\quad H^{-1} = H^T,\quad HH^T = I$ (the identity matrix)

So for the 4x4 Haar Matrix, the inverse transform is carried out using:

$$H_4^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

Example 2

For the 4x4 Haar matrix show that

$$HH^T = I$$

Example 3

Perform a Haar Transform on the 4-point input sequence :

S[n] = [1, 2, 3, 4]

Reconstruct the input sequence using the inverse Haar transform.

Example 3….Solution

Summary

- We have seen that a Haar Matrix can be constructed to perform Haar Transforms directly.
- The Haar Transform is fast because the matrix contains many zero terms.
- It can be used to identify frequency components in the signal to be analysed.
- It can be used for compression by reducing or eliminating the coefficients corresponding to high frequencies in the signal and then inverting the transform.

Reference

- Copyright © 1993, 1994, 1995, 1996, Nikos Drakos, Computer Based Learning Unit, University of Leeds.
  Copyright © 1997, 1998, 1999, Ross Moore, Mathematics Department, Macquarie University, Sydney.