EBU6018
Advanced Transform Methods

Multiresolution Analysis (MRA)

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Tackling the Redundancy Problem

- Because the CWT(a,b) is a continuous transform, it encodes more than enough information to reconstruct the signal. A sampled version is sufficient.
- a is scaling factor. b is translation term.
- Let us sample in a dyadic (power of 2) grid:

$$a = 2^{-m} \qquad b = n2^{-m}$$

$$CWT(2^{-m}, n2^{-m}) = \frac{1}{\sqrt{\left|2^{-m}\right|}} \int_{-\infty}^{\infty} s(t)\psi^*\left(\frac{t - n2^{-m}}{2^{-m}}\right) dt$$

$$d_{m,n} = CWT(2^{-m}, n2^{-m}) = \int_{-\infty}^{\infty} s(t)\psi^*{}_{m,n} dt$$

where $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$

Discrete (sampled) Transform

If the set of functions $\psi_{m,n}$ forms a frame, then we can recover s(t)

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_{m,n} \hat{\psi}_{m,n}(t)$$

where $\hat{\psi}_{m,n}(t)$ forms the dual frame

For the orthonormal frame, we have $\hat{\psi}_{m,n}(t) = \psi_{m,n}(t)$

So we get the *wavelet series* (transform and inverse):

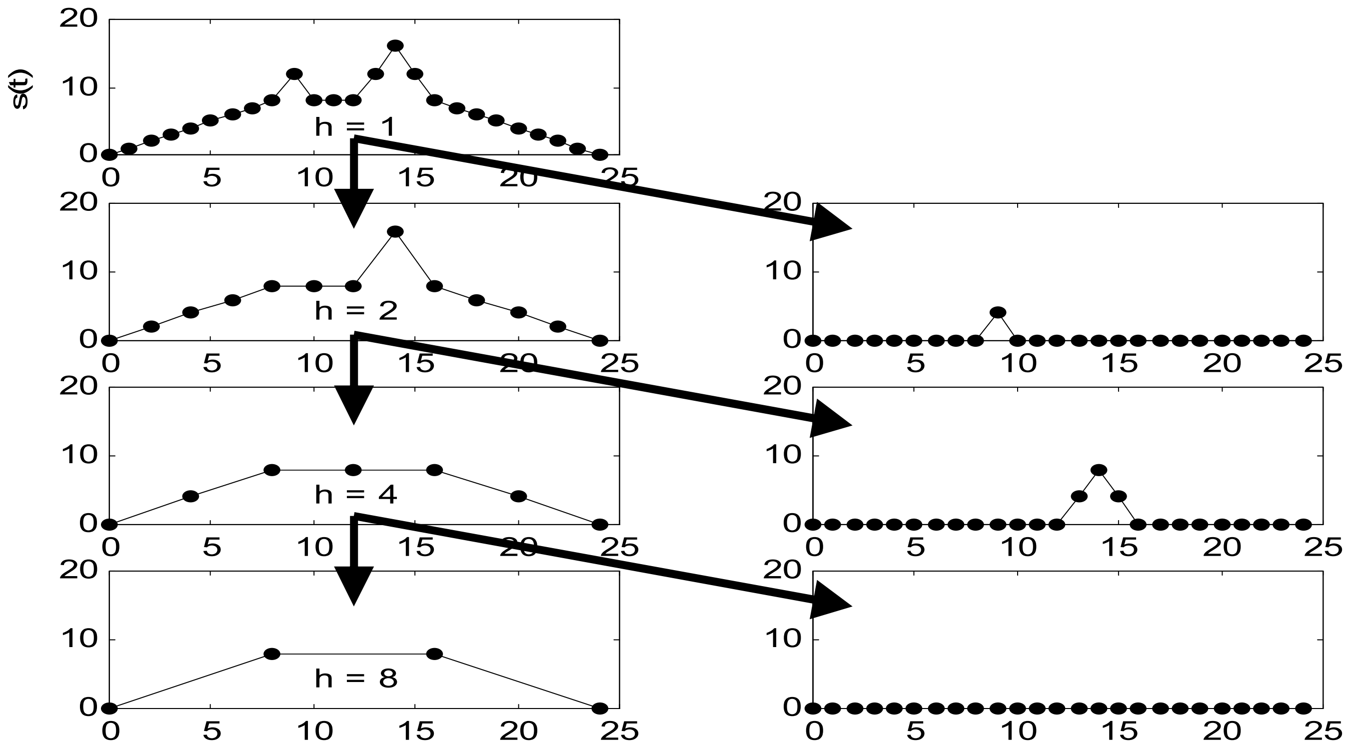
$$d_{m,n} = \int_{-\infty}^{\infty} s(t) \psi^*{}_{m,n} dt$$

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_{m,n} \psi_{m,n}(t)$$

So: wavelet series is not redundant (finite number of coefficients)

Multiresolution Analysis (MRA)

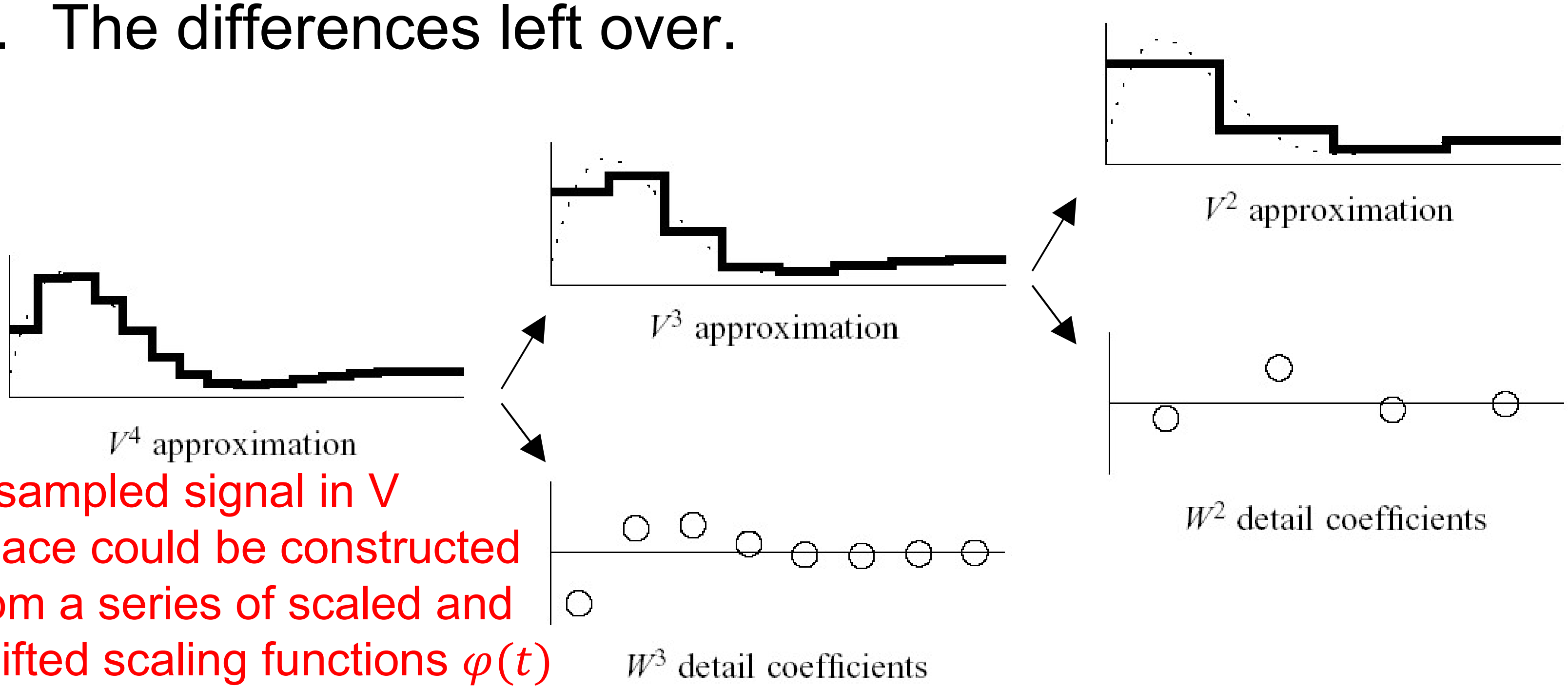
- Objective: To analyze a complicated function by dividing it into several simpler ones and studying them separately.



Piecewise Approximation

Basic Concept: Decompose a fine-resolution signal into

1. A coarse-resolution version of the signal, and
2. The differences left over.



A sampled signal in V space could be constructed from a series of scaled and shifted scaling functions $\varphi(t)$

So: what about wavelets? Theory coming next...

Multiresolution Analysis

- If we remove the fine detail from the signal in space V^4
- and put that in space W^3 then the approximate signal left in V^3 and the detail in W^3 have no elements in common.
- So the signal in V space and the signal in W space at a given level of decomposition are orthogonal.
- Similarly for V^2 and W^2, etc.
- Starting at level 4 is arbitrary.

Multiresolution Analysis

With the Haar Function we saw that the scaling function can be constructed from a series of scaled and translated scaling functions. That is:

A *scaling function* $\phi(t)$ generates a nested subsequence of subspaces $\{V_j\}$,

$$\{0\} \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

and satisfies a *dilation* (refinement) equation

$$\phi(t) = \sum_k p_k \phi(at - k) \qquad \text{for integer } k$$

for some $a > 0$ and coefficients $\{p_k\}$

We consider $a = 2$ that corresponds to octave-scales, that is, dyadic.

Spaces of functions

The space V_0 is generated by $\{\phi(t-k)\}$ for integers k

In general, V_j is generated by $\{\phi(2^j t-k)\}$

Then: $$V_n = \bigoplus_{j=-\infty}^{n-1} W_j$$ (orthorgonal sum of subspaces).

$$V_{n+1} = \bigoplus_{j=-\infty}^{n} W_j = \bigoplus_{j=-\infty}^{n-1} W_j \oplus W_n = V_n \oplus W_n$$

Subspace W_n is the *orthogonal complementary subspace* of V_n in V_{n+1}.

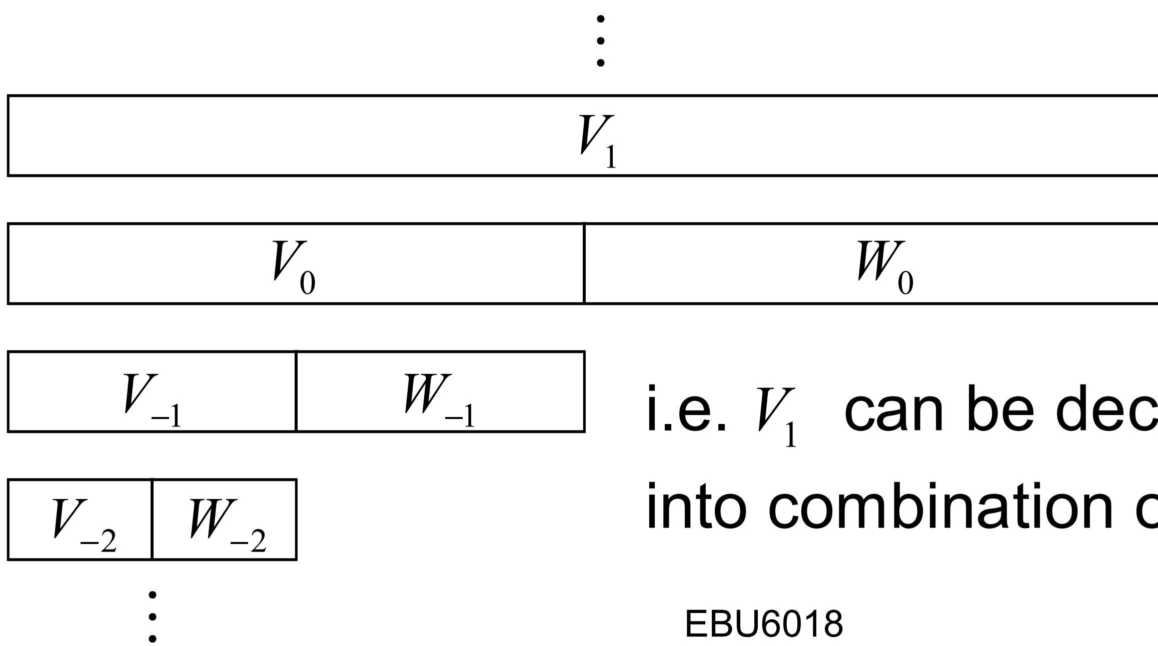
That is, we can reconstruct the original signal by adding together V and W at lower levels. (We could also provide compression my omitting the coefficients at low levels.)

Multiresolution Analysis (cont)

Subspaces $\{V_j\}$ are nested

while subspaces $\{W_j\}$ are mutually orthogonal.

Consequently,
$$\begin{cases} V_j \cap V_l = V_j & l > j; \\ W_j \cap W_l = \{0\} & j \neq l; \\ V_j \cap W_l = \{0\} & j \leq l. \end{cases}$$



i.e. V_1 can be decomposed into combination of V_0 and W_0, etc.

Multiresolution Analysis (cont.)

If the Scaling function and the Wavelet function are orthogonal, as they are for the Haar Function, then we can say that because V and W are orthogonal:

Subspaces $\{W_j\}$ are generated by *wavelets* $\psi(t)$

And $\{V_j\}$ is generated by scaling functions $\phi(t)$

In other words, any $f_j(t) \in V_j$

can be written as $f_j(t) = \sum_k c_k^j \phi(2^j t - k),$

And……

Multiresolution Analysis (cont.)

….. any function $g_j(t) \in W_j$
can be written as $g_j(t) = \sum_k d_k^j \psi(2^j t - k),$
for some coefficients

$$\{c_k^j\}_{k \in z}, \ \{d_k^j\}_{k \in z}$$

Furthermore,

$$f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$$

$$f(t) \in V_j \Leftrightarrow f\left(t + \frac{1}{2^j}\right) \in V_j$$

Multiresolution Analysis (cont.)

Subspaces $\{V_j\}$ are called “approximation subspaces” and $\{W_j\}$ are the “wavelet subspaces”.

At any level j, V_j contains the smooth part and W_j contains the “details” of the original function.

$j \uparrow \Rightarrow f_j(t),$ a finer approximation of $f(t)$,

$j \downarrow \Rightarrow f_j(t),$ a coarser approximation of $f(t)$.

Two-Scale Relations

Since $\phi(t) \in V_0 \subset V_1,$ and $\psi(t) \in W_0 \subset V_1;$

we should be able to write $\phi(t)$ and $\psi(t)$ in terms of the bases that generate V_1 .

In other words, there exists two sequences $\{p_k\}, \{q_k\}$

such that

$$\phi(t) = \sum_k p_k \phi(2t - k),$$

$$\psi(t) = \sum_k q_k \phi(2t - k).$$

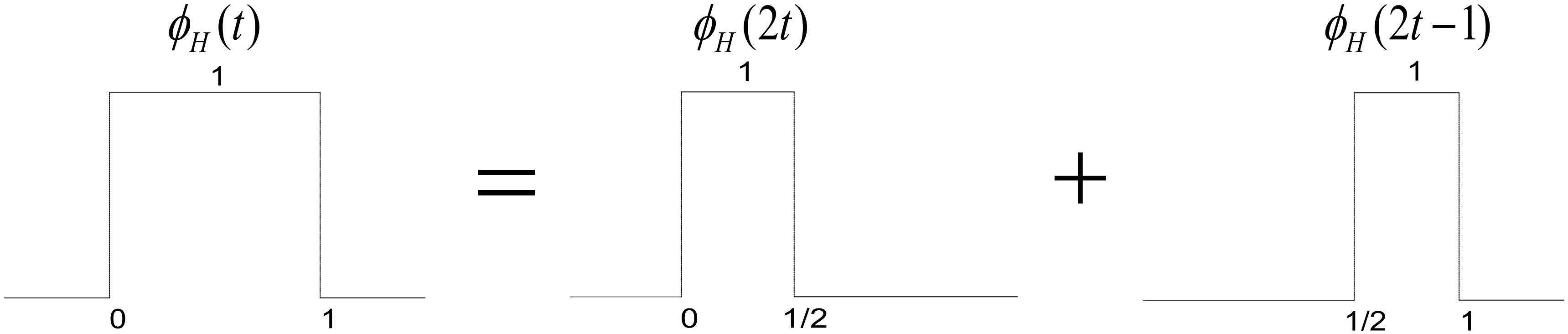
In general, for any j , the relationship between V_j, W_j with V_{j+1} is governed by

$$\phi(2^j t) = \sum_k p_k \phi(2^{j+1} t - k),$$

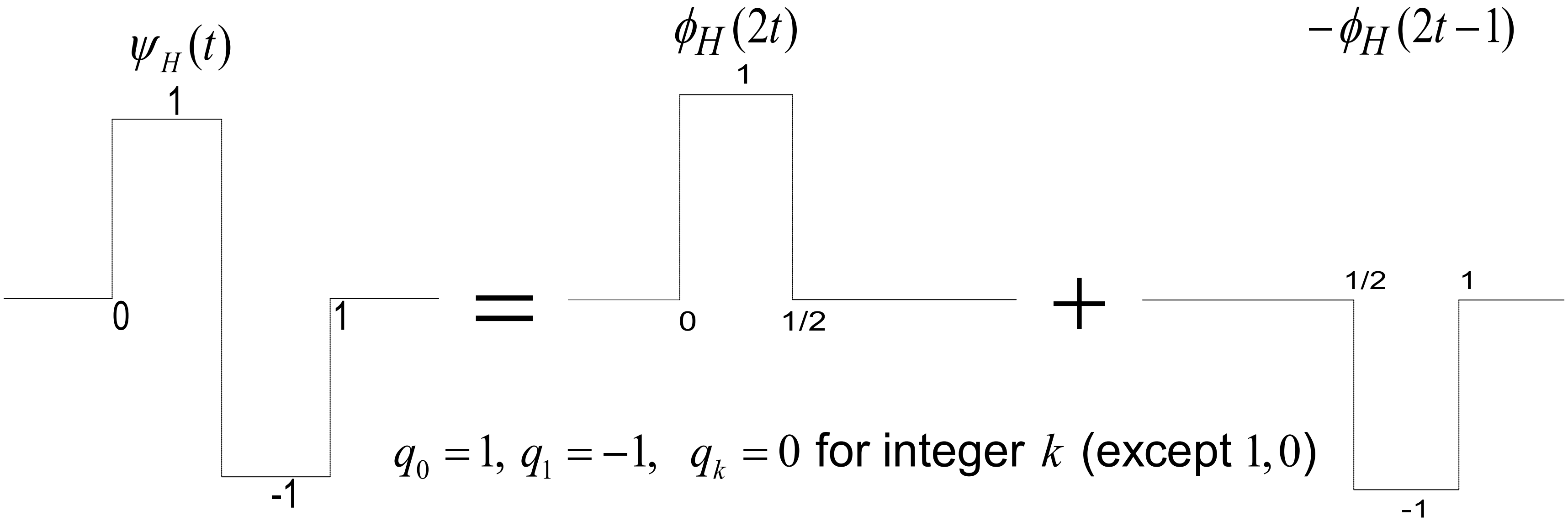
$$\psi(2^j t) = \sum_k q_k \phi(2^{j+1} t - k).$$

Two-Scale Relations (cont.)

Example: (H: Haar)



$p_0 = p_1 = 1, \quad p_k = 0$ for integer k (except $1, 0$)



$q_0 = 1, \; q_1 = -1, \quad q_k = 0$ for integer k (except $1, 0$)

Decomposition Relation

Since $V_1 = V_0 + W_0$ and $\phi(2t),\ \phi(2t-1) \in V_1$, there exist two pairs of sequences

$$(\{a_{2k}\},\ \{b_{2k}\})$$

such that

$$\phi(2t) = \sum_k \{a_{2k}\phi(t-k) + b_{2k}\psi(t-k)\};$$

$$\phi(2t-1) = \sum_k \{a_{2k-1}\phi(t-k) + b_{2k-1}\psi(t-k)\}.$$

Combining these two relations, we have

$$\phi(2t-l) = \sum_k \{a_{2k-l}\phi(t-k) + b_{2k-l}\psi(t-k)\}.$$

for all integer l

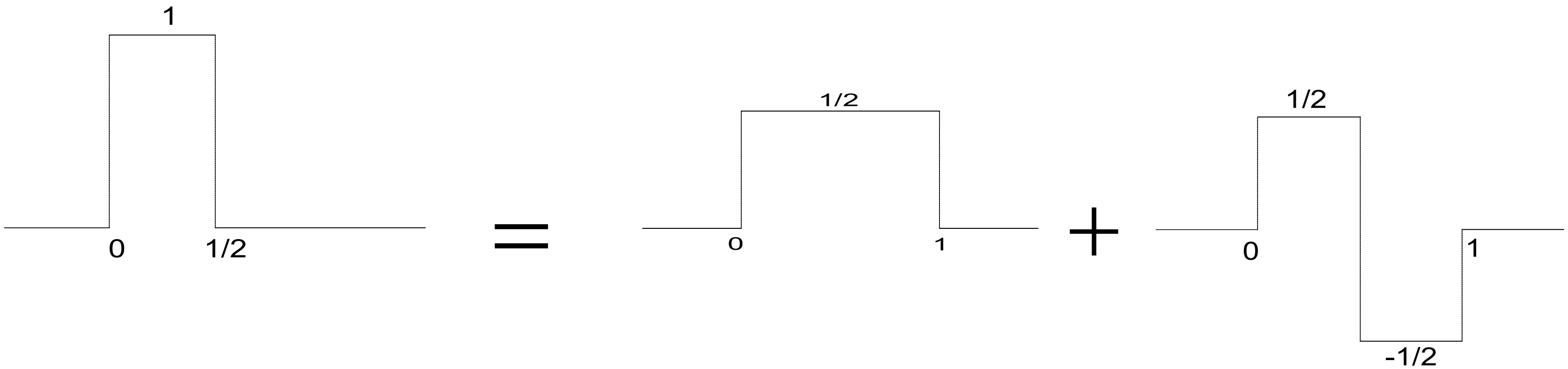
In general, we have

$$\phi(2^{j+1}t-l) = \sum_k \{a_{2k-l}\phi(2^j t-k) + b_{2k-l}\psi(2^j t-k)\}.$$

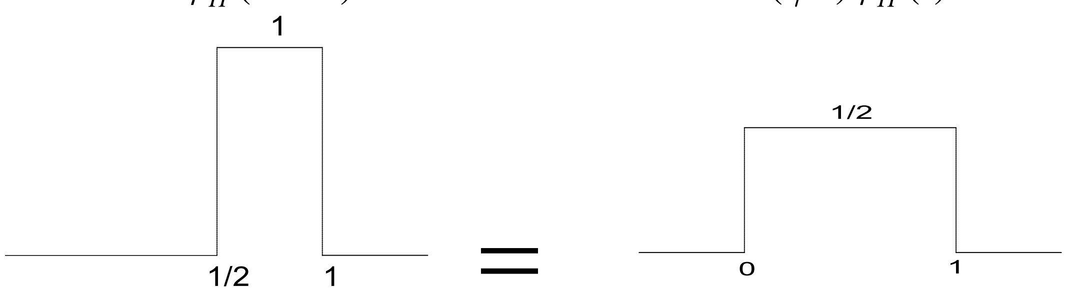
Decomposition Relation (cont.)

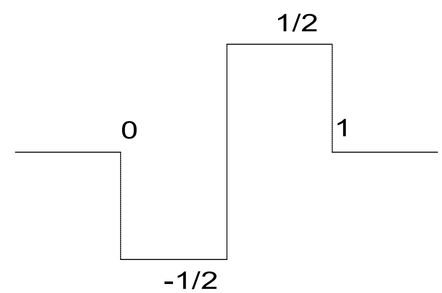
Example: (H: Haar)

$\phi_H(2t) \qquad = \qquad (1/2)\phi_H(t) \qquad + \qquad (1/2)\psi_H(t)$



$\phi_H(2t-1) \qquad = \qquad (1/2)\phi_H(t) \qquad + \qquad (-1/2)\psi_H(t)$





$$a_0 = 1/2,\ \ a_{-1} = 1/2,\ \ a_k = 0 \text{ for } k \in Z \setminus \{0,-1\}$$

$$b_0 = 1/2,\ b_{-1} = -1/2,\ \ b_k = 0 \text{ for } k \in Z \setminus \{0,-1\}$$

Summary

We have seen that a series of sampled values of a continuous signal can be divided into two sequences:

- One which is approximations
- The other fine detail
- These two sequences are orthogonal
- The approximations can be derived as a series of scaled and shifted scaling functions, each with a coefficient (same as a linear piecewise approximation)
- The fine detail can be derived as a series of scaled and shifted wavelet functions, each with a coefficient
- These two sequences can be recombined to form the original sequence