EBU6018 Advanced Transform Methods revision questions.

1. List the various forms of the Fourier Transform.

i) Fourier Series:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_k) \qquad -\infty < t < \infty$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n.\ e^{jn\omega t}$$

ii) Continuous Fourier Transform:

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)dt$$

iii) Discrete Time Fourier Transform (DTFT):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\,\omega\, n}$$

iv) Discrete Fourier Transform (DFT):

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} = X(\omega)\Big|_{\omega=\frac{2\pi}{N}k} \qquad \text{for } k = 0,\ 1,\ \ldots,\ N\text{-}1$$

2. What is the Fourier Transform used for?

The Fourier Transform is used to transform from the time domain to the frequency domain. The output of the FT is the spectrum of the input signal, that is, the range of frequencies the signal contains.

3. What is the principal limitation of the Fourier Transform?

The principal limitation is that if the input signal is non-stationary then the distribution of frequencies with time is lost in the transform.

4. What are the basis functions of the Fourier transform?

   The basis functions are sine and cosine, which are orthogonal and so can be used to invert the transform.

5. What is meant by the term “Windowed Transform”?

   A windowed transform is a transform that uses a short duration function to isolate short segments of the input signal to allow the frequency content to be obtained for that short segment. The short duration function is then moved to allow the frequency content of a different segment to be obtained. The resulting output is therefore a function of time and frequency.

6. What advantage is gained by using a windowed transform?

   Windowed transforms are useful for non-stationary signals. The output not only gives the spectrum of the input signal but also the distribution of the frequencies with time.

   Note that the STFT and Wavelet Transform are both examples of windowed transforms.

7. What is the implication of the Uncertainty Principle

   a) in general?

   The UP is a “universal” rule that states that when you try to measure something then you change the value you are trying to measure. So it is not possible to measure anything accurately. The science of measurement is to make the inaccuracy as small as possible.

   b) specifically related to windowed transforms?

   Windowed transforms are used to find the frequency content of signals during short segments of the signal. So the distribution of frequencies in the signal are found as a function of time and frequency. However, to find the time that a frequency is present in the signal, the segment must be very short so that means we cannot accurately measure the frequency. If we want to measure the frequency accurately then we need a longer segment, so accurate time information is not possible.

If the length of the segment is Δ_t and the uncertainty of the value of the frequency is Δ_f then the UP states that $\Delta_t.\Delta_f = k$. That is, they are inversely proportional; if we want to know one more accurately then the other will be less accurate.

8. Compare the mathematical expressions for the Fourier Transform and the Short Time Fourier Transform.

Fourier Transform:

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)dt$$

Short Time Fourier Transform:

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^{*}(\tau - t)e^{-j\omega t}d\tau$$

In both of them, the input signal is convolved with the basis function. In the FT the basis function is the complex exponential. In the STFT the basis function is the product of the complex exponential and the window function.

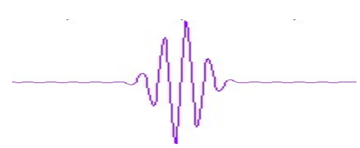
9. What is a Wavelet?

A wavelet is a short duration oscillatory signal, for example:

For a function to be a useful wavelet, it must satisfy the Admissibility Condition:

$$C_{\Psi} = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$

That is, the function must have zero mean, be of short duration and have the energy concentrated near the centre. The Haar function is the simplest wavelet, it is short duration, has zero mean but the energy is not well concentrated near the centre.



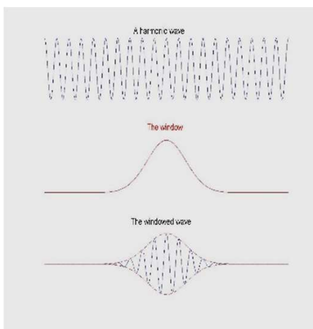
The wavelet function can be scaled

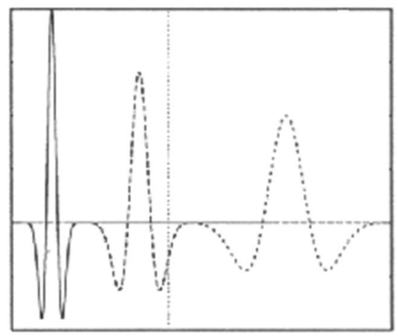
and translated.

10. What advantage does a Wavelet Transform have over a Short Time Fourier Transform?

The STFT basis function is the window multiplied by the complex exponential. If the window width is changed, the frequency of the complex exponential stays the same.

The WT basis function is a single function, if it is scaled then the frequency changes. For example, scaling to compress the wavelet by a power of 2 also doubles the frequency.



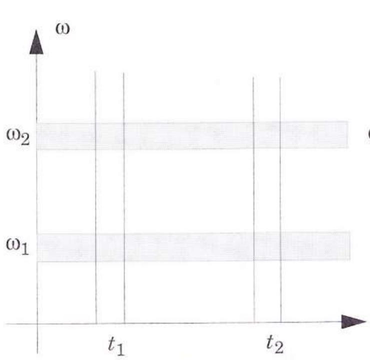


$$STFT(b,\omega) = \left\langle s(t), \gamma(t-b)e^{j\omega t} \right\rangle$$

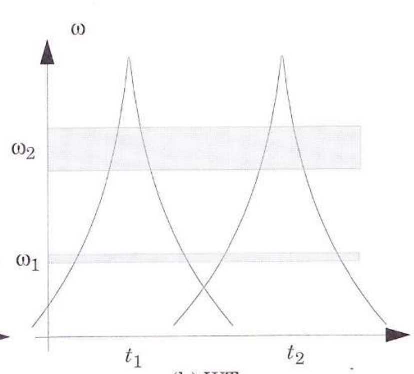
$$CWT(b,a) = \left\langle s(t), \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \right\rangle$$

In the STFT, a small window cannot see low frequencies, and a large window means that brief changes in frequency will be lost.

For the WT, when the wavelet is scaled the number of oscillations is kept constant and the window width is changed. So the higher the frequency, the better the time resolution, and vice versa.



(a) STFT



(b) WT

Suppose we have a composite signal with a pulse at t_1 and another at t_2, and two frequencies ω_1 and ω_2. The resolution is constant for the STFT but varies for the WT.

Frequency resolution is better at low frequencies but time resolution is poorer, and vice-versa

REF: Qian 5.1 page 105.

11. What is a Spectrogram?

A Spectrogram is a plot of the energy in a signal as a function of time and frequency. It is the output of a STFT.

12. What is a Scalogram?

A Scalogram is a plot of the energy in a signal as a function of scale and translation. It is the output of a Wavelet Transform.

13. How can a wavelet transform be used for “feature extraction”?

Use a wavelet function similar in shape to the feature to be located, scale the wavelet function so that the duration is similar to the feature and then translate the scaled wavelet till it coincides with the feature.

14. How can a wavelet transform be used for “trend analysis”?

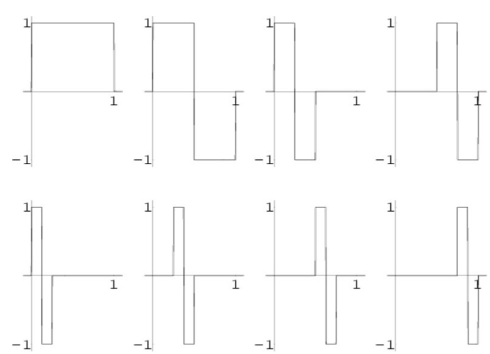
By removing the high frequency detail from the input data leaves the low frequency smoothed data.

15. Refer to a Haar Matrix to explain the answers to Q13 and Q14

Consider the 8x8 Haar Matrix:

Each row of the un-normalised matrix corresponds to one of the Haar Functions.

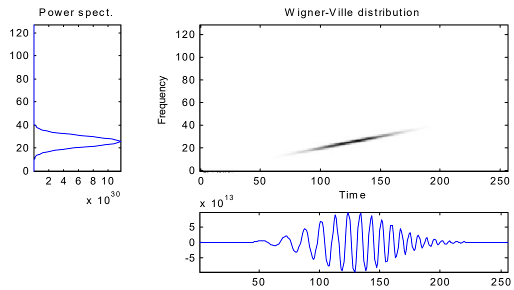
$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$



So multiplying the input data by the Transform Matrix will locate the location of a feature to be extracted.

16. Does the Uncertainty Principle hold for the Wigner-Ville Distribution?

The Uncertainty Principle always holds. The Wigner-Ville Distribution must satisfy the UP, but it is not a windowed transform so its resolution can be better.

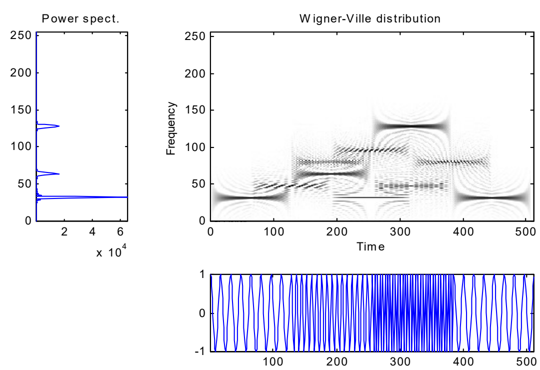


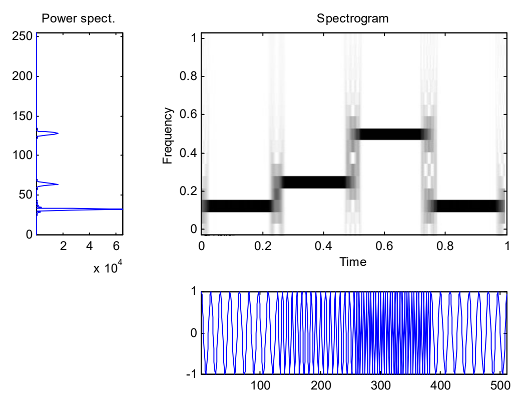
17. What is the main advantage of the Wigner-Ville distribution?

The main advantage is its improved resolution.

18. What is the main disadvantage of the Wigner-Ville Distribution?

The main disadvantage is the presence of cross-terms (noise) if the input is a composite signal.





19. Compare the DCT and the KLT.

The DCT and KLT are both used for compressing images.

The DCT is relatively simple to compute (purely real) but the KLT is computionally intensive.

For a given level of compression, the KLT gives the best quality of compressed image possible. However, the DCT is the next best.