EBU6018
Advanced Transform Methods

Eigenvalues/Eigenvectors

and the Karhunen-Loeve Transform (PCA)

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Eigenvalues

The eigenvalues, λ, of a square matrix A are the solutions of:

$|A - \lambda I| = 0$

As an example, consider the matrix A = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2 = 0$$

So $\lambda_1 = 1$ and $\lambda_2 = 3$

Eigenvectors

Eigenvectors v are the solutions of:

$(A - \lambda I)$v = 0

For λ_1= 1, $(A - I)v_{\lambda_1}$ = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so, v_1+v_2 = 0, $v_1 = -v_2$, v_{λ_1}=$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For λ_2= 3, $(A - I)v_{\lambda_2}$ = $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so, $-v_1$+v_2 = 0, $v_1 = v_2$, v_{λ_2}=$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Note that any scalar multiple of each eigenvector is OK. They can be normalised.

For a symmetric matrix, the eigenvalues are always real and the corresponding eigenvectors are always orthogonal.

Example 1

Evaluate the normalised eigenvectors of the covariance matrix of the following 2D dataset.

x	y
-3.01	-2.67
0.18	-2.12
-6.56	-4.24
1.60	1.92
2.78	-1.20
2.03	2.14

Example 1….solution

$$\bar{x} = -0.50$$

$$\bar{y} = -1.03$$

$$\text{VAR IN } x = \frac{\sum_{1}^{N} (x-\bar{x})^2}{N-1} = \underline{13.00}$$

$$\text{VAR IN } y = \frac{\sum_{1}^{N} (y-\bar{y})^2}{N-1} = \underline{6.60}$$

$$\text{COV}(x,y) = \frac{\sum_{1}^{N} (x-\bar{x})(y-\bar{y})}{N-1} = \underline{7.30}$$

$$\text{COV. MATRIX} = \begin{bmatrix} 13.00 & 7.30 \\ 7.30 & 6.60 \end{bmatrix}$$

Example 1….solution

FOR EIGENVALUES.

$$\left| \begin{bmatrix} 13.00 & 7.30 \\ 7.30 & 6.60 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (13.00 - \lambda) & 7.30 \\ 7.30 & (6.60 - \lambda) \end{vmatrix} = 0$$

$$\lambda^2 - 19.6\lambda + 32.51 = 0$$

$$\lambda_1 = 17.77, \quad \lambda_2 = 1.83$$

Example 1….solution

FOR EIGENVECTORS:

$$\lambda_1 = 17.77 \quad \begin{bmatrix} -4.77 & 7.30 \\ 7.30 & -11.17 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\phi_1 = \begin{bmatrix} 1.53 \\ 1.00 \end{bmatrix}$$

$$\lambda_2 = 1.83 \quad \begin{bmatrix} 11.17 & 7.30 \\ 7.30 & 4.77 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} -0.65 \\ 1.00 \end{bmatrix}$$

NORMALISE $\phi_1 = \dfrac{1}{\sqrt{3.34}} \begin{bmatrix} 1.53 \\ 1.00 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0.84 \\ 0.55 \end{bmatrix}}}$

NORMALISE $\phi_2 = \dfrac{1}{\sqrt{1.42}} \begin{bmatrix} -0.65 \\ 1.00 \end{bmatrix} = \underline{\begin{bmatrix} -0.55 \\ 0.84 \end{bmatrix}}$

Example 2

For the 2D data set given, determine:
i) Covariance matrix
ii) Eigenvalues
iii) Eigenvectors
iv) The KLT of the given 2D data set.

x	y
2.5	2.4
0.5	0.7
2.2	2.9
1.9	2.2
3.1	3.0
2.3	2.7
2.0	1.6
1.0	1.1
1.5	1.6
1.1	0.9

Example 2.....answers

Mean of x = 1.81
Mean of y = 1.91
Values of x with mean of x subtracted and values of y with the mean of y subtracted:

$$cov = \begin{bmatrix} 0.6166 & 0.6154 \\ 0.6154 & 0.7166 \end{bmatrix}$$

x	y
o.69	0.49
-1.31	-1.21
0.39	0.99
0.09	0.29
1.29	1.09
0.49	0.79
0.19	-0.31
-0.81	-0.81
-0.31	-0.31
-0.71	-1.01

Example 2.....answers

Eigenvalues = $\begin{bmatrix} 0.0491 \\ 1.2840 \end{bmatrix}$

Normalised eigenvectors =

$$\begin{bmatrix} -0.7352 & 0.6779 \\ 0.6779 & -0.7352 \end{bmatrix}$$

Multiplying the original data by this eigenvector matrix gives:

X'	Y'
-0.8280	-0.1751
1.7776	0.1429
-0.9921	0.3844
-0.2742	0.1304
-1.6758	-0.2095
-0.9129	0.1753
0.09911	-0.3498
1.1446	0.0464
0.4380	0.0178
1.2238	-0.1627