Chapter 1. Vector Analysis

Contents（相应小节和教材不一一对应）

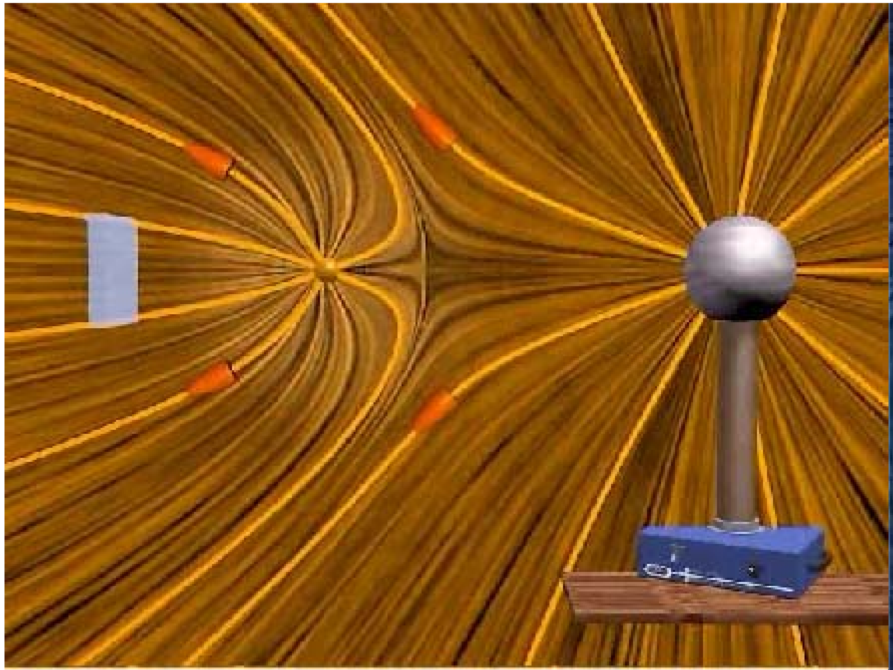
Aim of Vector Analysis

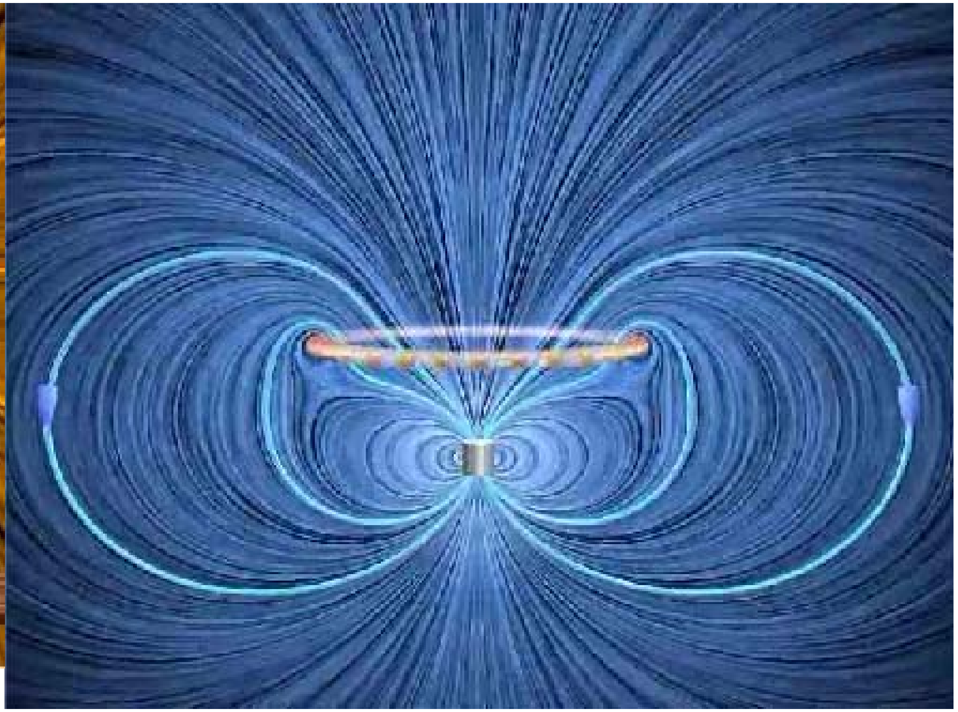
- It is the language used in the study of EMF.
- The widespread of vectors in EMF theory is due part to the fact that they provide compact mathematical representations of complicated phenomena and allow for easy visualization and manipulation.

1.1 Scalars, Vectors & Fields

- Scalar: A quantity completely specified by its **magnitude** ,without **direction**.
- Vector: A quantity, completely specified by both its **magnitude** and **direction**.
  - (how do we specify the direction of a vector? 3-D space, 3 numbers)
- Examples
  - Scalar: mass, length, voltage, temperature, “lifeblood” ….
  - Vectors: velocity, force, field intensity…

Scalars, Vectors & Fields





Concept of Field

- **Field:** **is a spatial distribution of a quantity, which may or may not be a function of time.**
- A region of space characterized by a physical property, having a uniquely determinable value at each point on given moment in the region.
  - 在指定的时刻，空间每一点如果可以用一个量唯一地描述，则该量函数定出了场.

$$Q(\vec{r},t)\Big|_{\substack{\vec{r}=\vec{r}_0 \\ t=t_0}} = Q_0$$

  - This physical **property of field** here may be either **scalar** **or** **vector**, such as gravitational or electromagnetic force or fluid pressure.
  - **In 3-D space, a vector relation is 3 scalar relations**

Characteristics of a Field

1. Existing in a **region of space**
2. Can be described in mathematical form
3. Relevant physical property are **continuously distributed** except for several points or surfaces

- **Steady (static) Fields**
  - Relevant physical property does not vary with time

$$\frac{\partial}{\partial t} = 0$$

- **Dynamic (time varying) Fields**
  - Field varies with time

$$\frac{\partial}{\partial t} \neq 0$$

A Contrast of *circuit vs. field*

Circuit----In "Electronic Systems"

we use *centralized parameters*.

i.e. macroscopical or average parameters

such as current, voltage, resistance

To solve the problem, we depend on **differential equations and scalar equations. (随时间的变化关系，没有空间)**

Field----In "EM Theory"

$$Q(\vec{r},t)\Big|_{\substack{\vec{r}=\vec{r}_0 \\ t=t_0}} = Q_0$$

we use **distributed parameters.**

i.e. microcosmic or specific parameters

such as E intensity, M intensity, potential, Poynting vector

To solve the problem, we depend on **partial differential equations and vector algebra.**

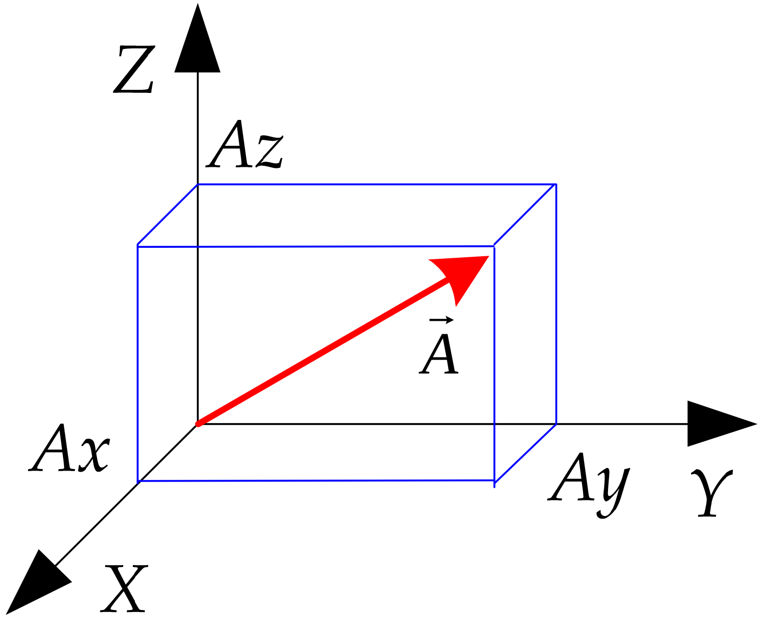
Vector (with magnitude and direction)

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

Unit Vector

Projection on X-axis

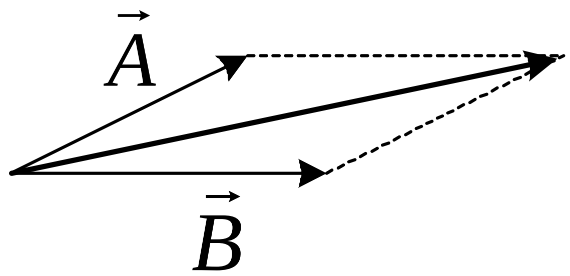
A_x is the component of $\vec{A}$ along x-axis, or the scalar projection of $\vec{A}$ on x-axis.



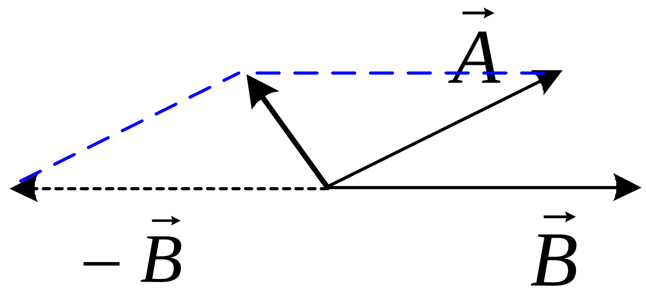
Vector Algebra

(1) The Sum $\boxed{\vec{A} \pm \vec{B}}$

$\vec{A}+\vec{B}=\vec{A}+\vec{B}$



$\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$



$$\vec{A}+\vec{B}=(A_x\vec{a}_x+A_y\vec{a}_y+A_z\vec{a}_z)+(B_x\vec{a}_x+B_y\vec{a}_y+B_z\vec{a}_z)=.....$$

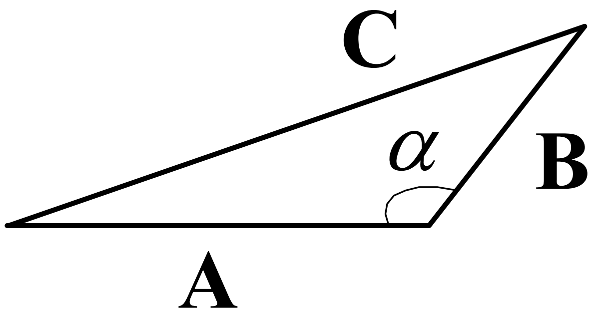
(2) The Products

Dot Products $\vec{A} \bullet \vec{B}$

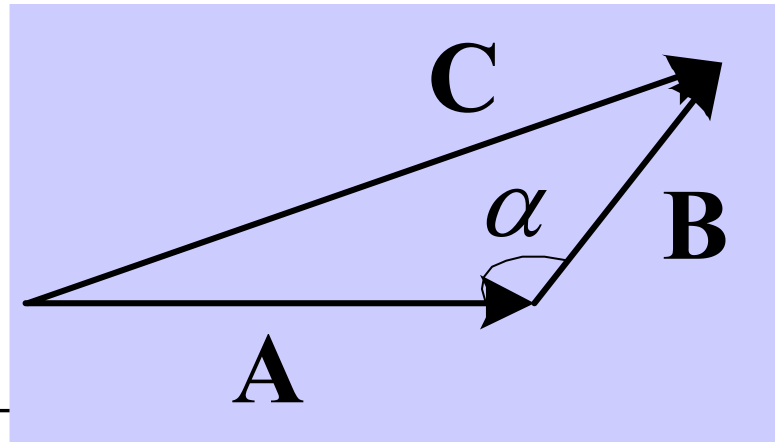
Cross Products $\vec{A} \times \vec{B}$

- **Dot Product, Scalar Product (点积,标量积)** **Is a scalar**
  - Quantity & Sign: $\vec{A} \bullet \vec{B} = A \cdot B \cdot \cos\theta_{AB}$
  - Modulus: $\left|\vec{A}\right| = \sqrt{\vec{A} \bullet \vec{A}}$
  - “正交”： $\vec{A} \bullet \vec{B} \equiv 0$
  - **Physical meaning: projection of one vector on another vector**

Exercise: Prove the following formula.



$$C = \sqrt{A^2 + B^2 - 2 \cdot A \cdot B \cdot \cos\alpha}$$



Thinking:

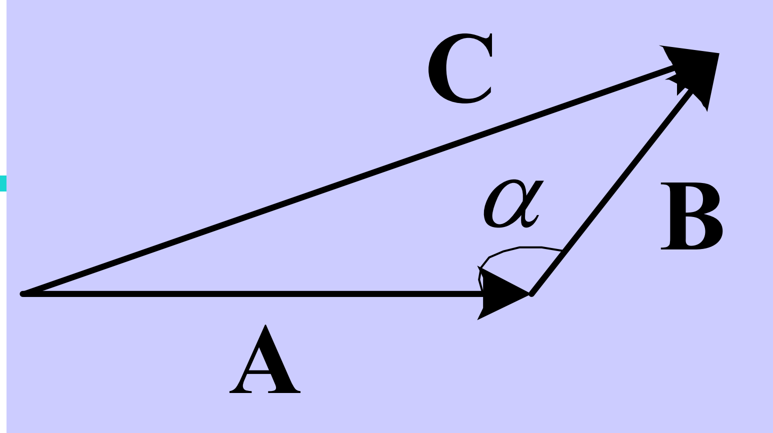
(1) C is actually the modulus of

$$C = \left|\vec{C}\right| = \sqrt{\vec{C} \bullet C}$$

(2) Vector $\vec{C}$ is actually the sum of $\vec{A}$ and $\vec{B}$.

$$\vec{C} = \vec{A} + \vec{B}$$

Solution:



$$C = \sqrt{A^2 + B^2 - 2 \cdot A \cdot B \cdot \cos\alpha}$$

$$C = \left|\vec{C}\right| = \sqrt{\vec{C} \bullet \vec{C}}$$

$$\vec{C} = \vec{A} + \vec{B}$$

$$\vec{C} \bullet \vec{C} = (\vec{A} + \vec{B}) \bullet (\vec{A} + \vec{B})$$

$$= \vec{A} \bullet \vec{A} + \vec{B} \bullet \vec{B} + 2 \cdot \vec{A} \bullet \vec{B}$$

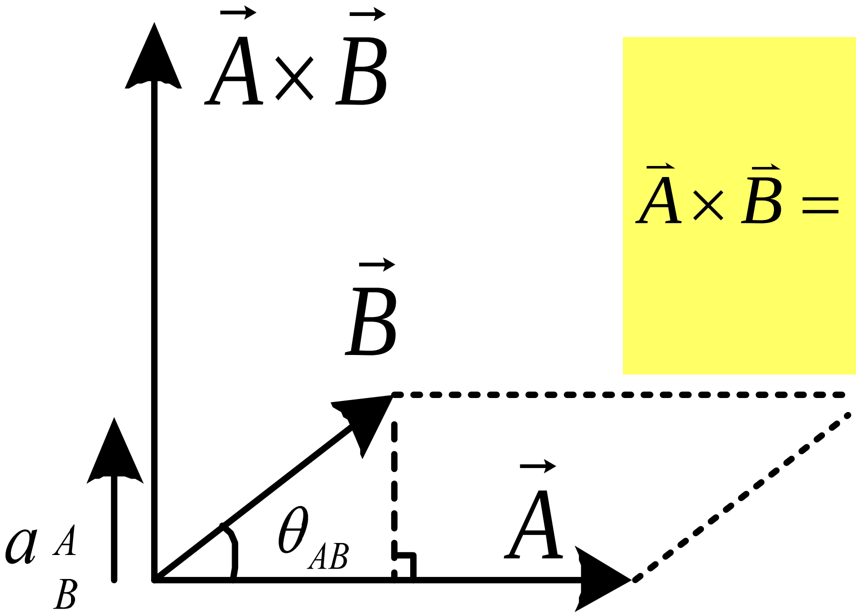
$$\vec{A} \bullet \vec{B} = A \cdot B \cdot \cos(\pi - \alpha)$$

? .

Cross Product, Vector Product (叉乘,矢量积)

$\vec{A}\times\vec{B}$ **Is a vector**

- Modulus: $\left|\vec{A}\times\vec{B}\right| = \left|A\cdot B\cdot\sin\theta_{AB}\right|$
- Direction: *right-hand rule*
- Physical meaning: Square of parallelogram



$$\vec{A}\times\vec{B} = \vec{a}_{AB}(A\cdot B\cdot\sin\theta_{AB}) = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

=?

四边形面积 $A\cdot(B\cdot\sin\theta_{AB})$

矢量叉乘的性质

1. $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

2. $\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

3. $\vec{A}\times(\vec{B}\times\vec{C})\neq(\vec{A}\times\vec{B})\times\vec{C}$

4. 标量三重积 *Scalar Triple Product*

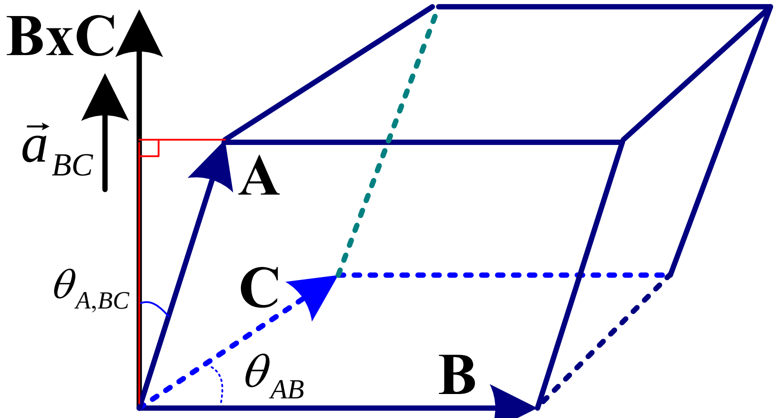
$$\vec{A}\bullet(\vec{B}\times\vec{C})=\vec{B}\bullet(\vec{C}\times\vec{A})=\vec{C}\bullet(\vec{A}\times\vec{B})$$

5. 矢量三重积 *Vector Triple Product*

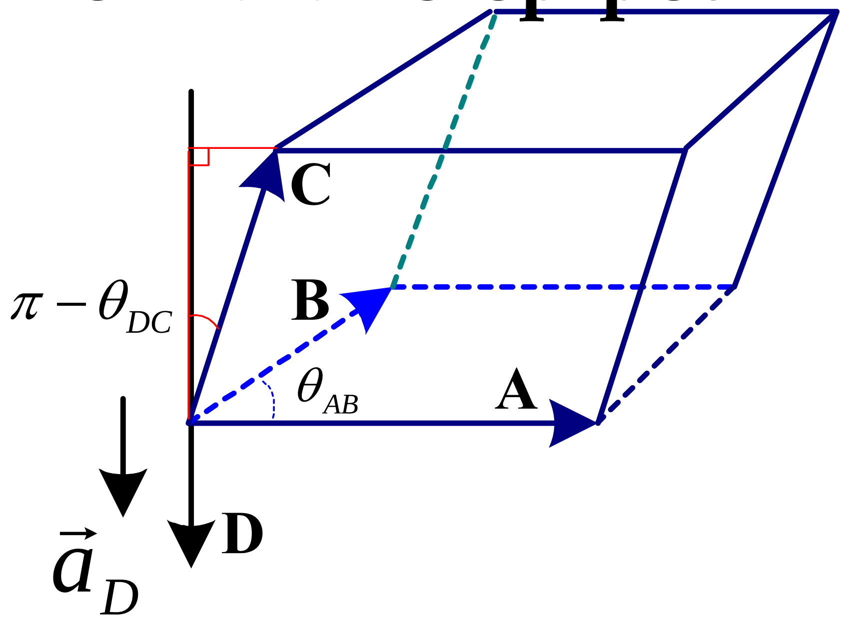
$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{C}\bullet\vec{A})-\vec{C}(\vec{A}\bullet\vec{B})$$

$$\vec{A}\bullet(\vec{B}\times\vec{C})=\vec{B}\bullet(\vec{C}\times\vec{A})=\vec{C}\bullet(\vec{A}\times\vec{B})$$

"Bulk of Parallelepiped"



Bottom Area= $B\cdot(C\cdot\sin\theta_{BC})$
Height= $A\cdot\cos\,\theta_{A,BC}$



- 标量三重积
  - 记忆1：“循环互换规律”
  - 记忆2：“平行六面体体积”

```
 ___     ___     ___     ___
('v')   ('v')   ('v')   ('v')
((  ))  ((  ))  ((  ))  ((  ))
-/-"--"----/-"--"----/-"--"----/-"--"--
```

1.2 Coordinates

What is Coordinates?

- Cartesian Coordinates $(x, y, z; \vec{a}_x, \vec{a}_y, \vec{a}_z)$
  Rectangular Coordinates
- Cylindrical Coordinates $(r, \varphi, z; \vec{a}_r, \vec{a}_\varphi, \vec{a}_z)$
- Spherical Coordinates $(r, \theta, \phi; \vec{a}_r, \vec{a}_\theta, \vec{a}_\phi)$

Cartesian Coordinates

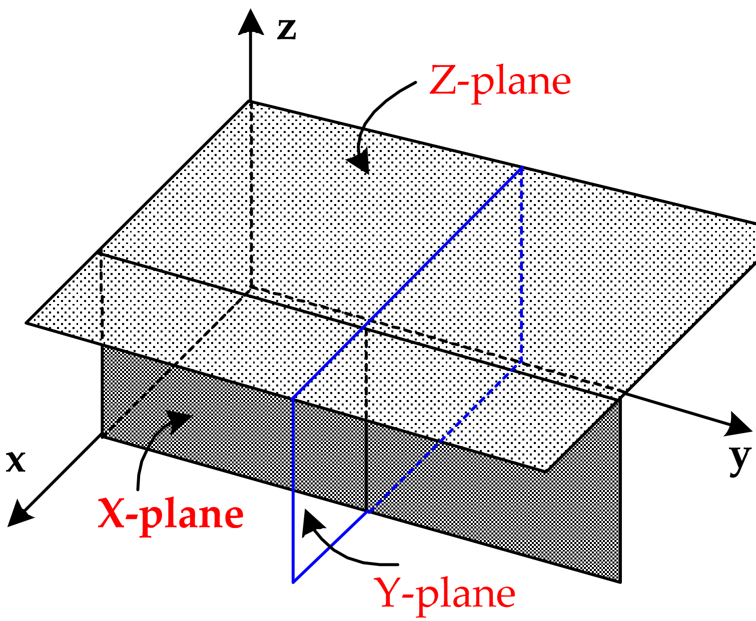
Vector expression: $\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$

Sum: $\vec{A} + \vec{B} = ?\,\vec{a}_x + ?\,\vec{a}_y + ?\,\vec{a}_z$

Dot product: $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A} = ? = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$

Cross product:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

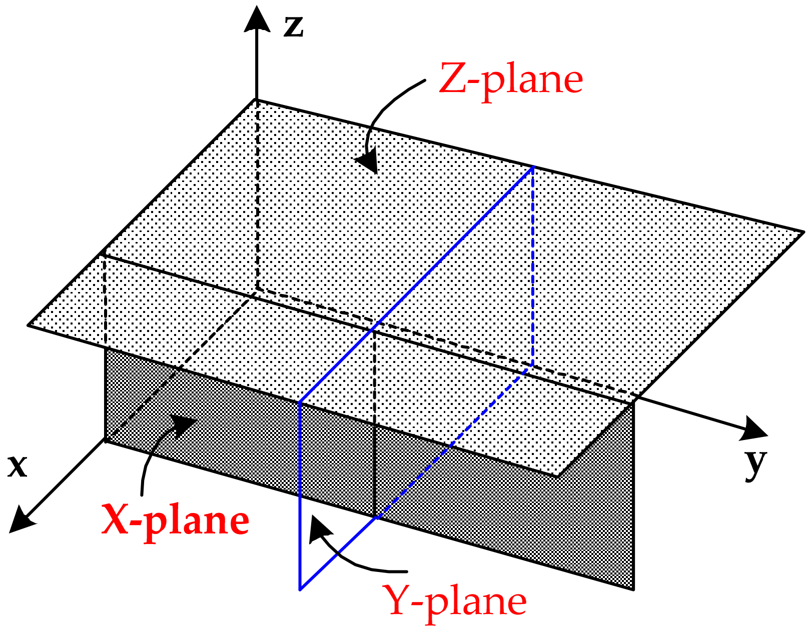


Differential Elements in Cartesian Coordinates

Differential Length *(vector)* $d\vec{l} = \vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz$

Differential Surface *(vector)* $d\vec{S}$

$$\begin{cases} d\vec{S}_x = \vec{a}_x dydz \\ d\vec{S}_y = \vec{a}_y dzdx \\ d\vec{S}_z = \vec{a}_z dxdy \end{cases}$$



Differential Volume *(scalar)*

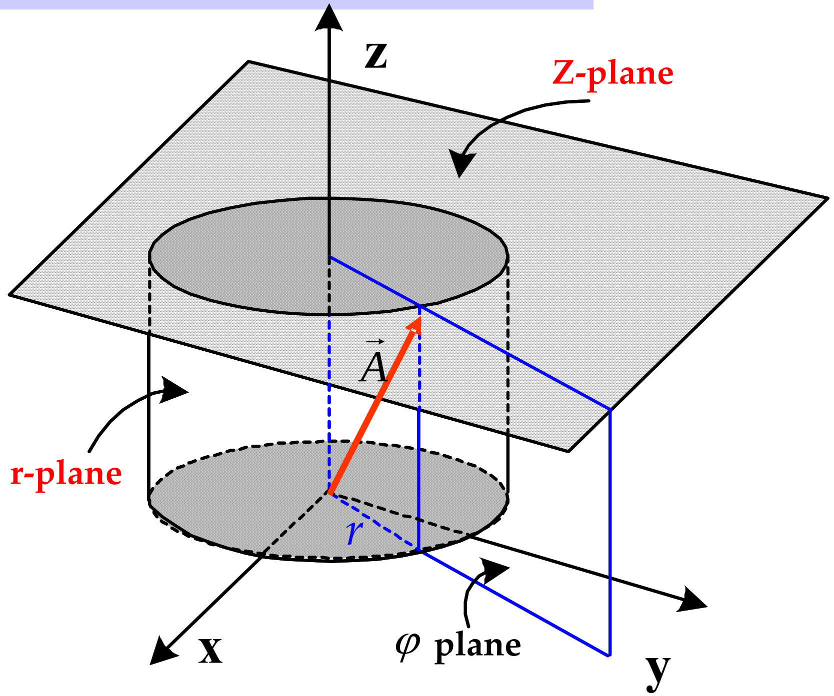
$$dV = dxdydz$$

Cylindrical Coordinates

Vector expression:

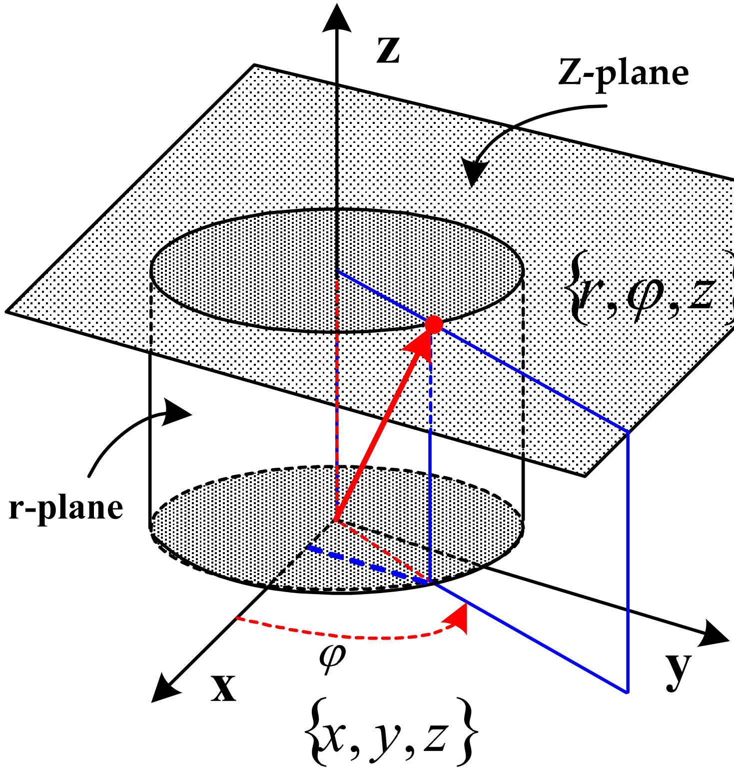
$$\vec{A} = A_r\vec{a}_r + A_\varphi\vec{a}_\varphi + A_z\vec{a}_z$$

Sum and product: ?

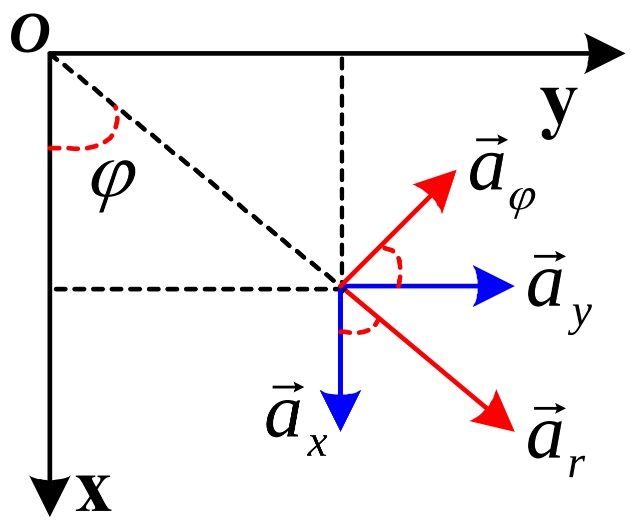


Refer to textbook **Bhag Singh Guru** pp.23-27

If angle is not constant, need convert to rectangular system firstly



Planform (俯视图)

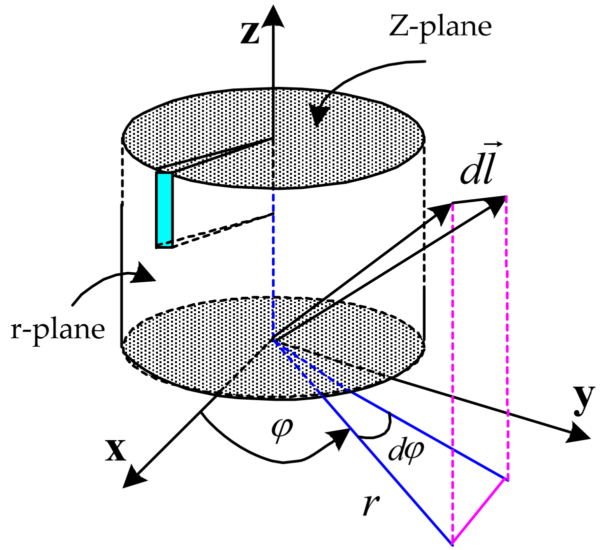


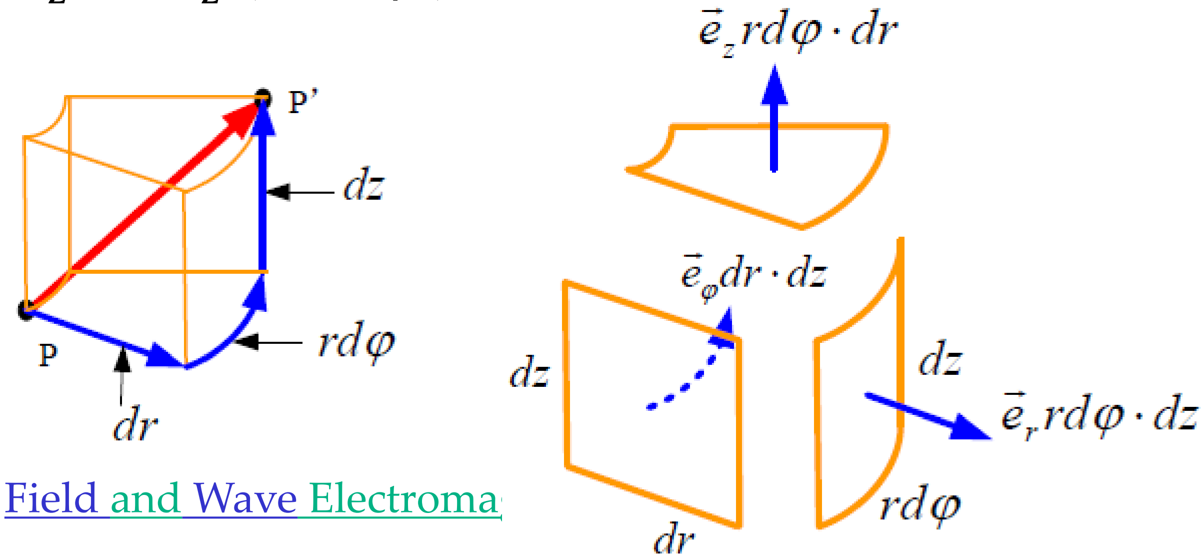
Differential Elements in Cylindrical Coordinates

Differential Length *(vector)* $d\vec{l} = \vec{a}_r dr + \vec{a}_\varphi (r \cdot d\varphi) + \vec{a}_z dz$

Differential Surface *(vector)* $d\vec{S}$

$$\begin{cases} d\vec{S}_r = \vec{a}_r (r \cdot d\varphi) dz \\ d\vec{S}_\varphi = \vec{a}_\varphi dr dz \\ d\vec{S}_z = \vec{a}_z (r \cdot d\varphi) dr \end{cases}$$

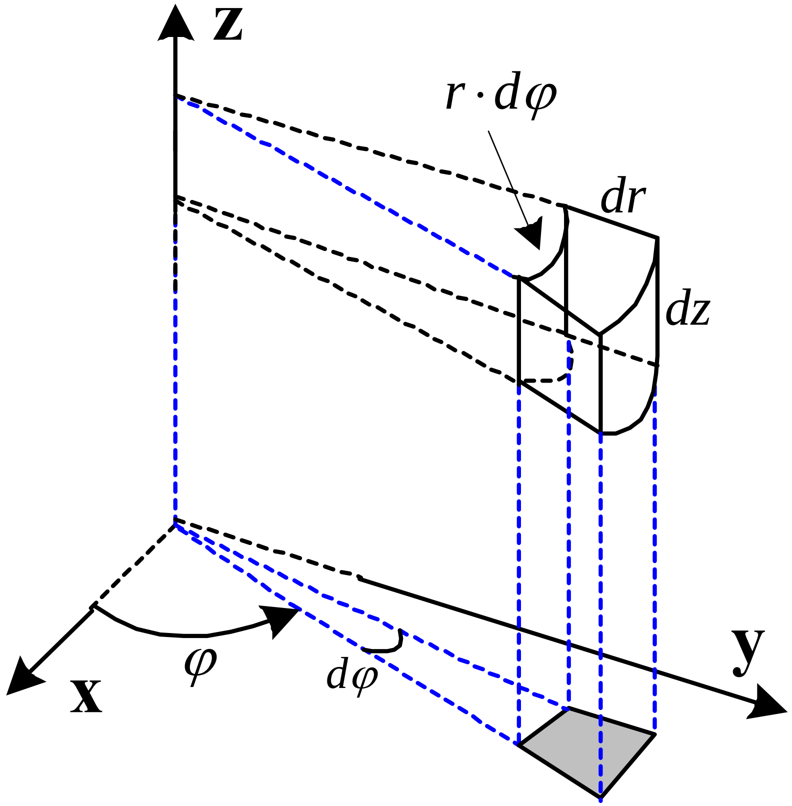




Differential Elements in Cylindrical Coordinates

Differential Volume *(scalar)*

$$dV = dr \cdot (r d\varphi) \cdot dz$$



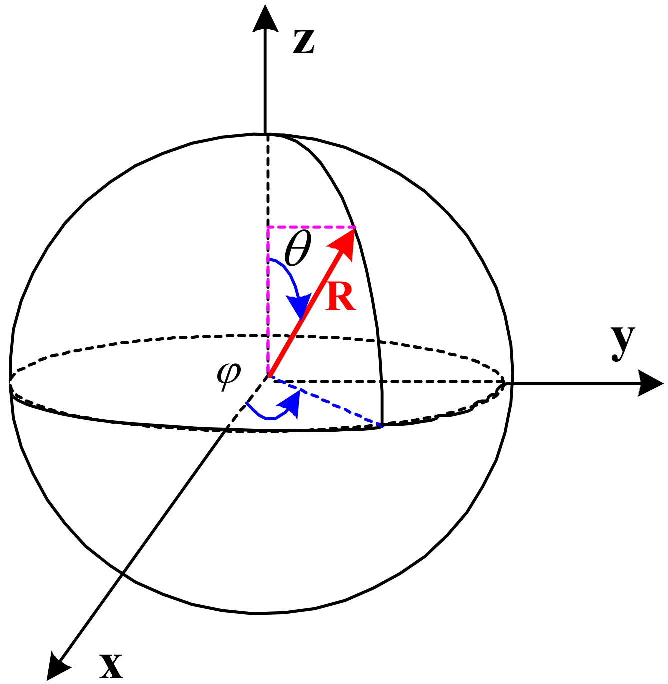
- Relationship between cylindrical and rectangular coordinates: Equation (2.36) in textbook page 25. （**Bhag Singh Guru**）

Spherical Coordinates

Vector expression: $\vec{A} = A_R \vec{a}_R + A_\theta \vec{a}_\theta + A_\varphi \vec{a}_\varphi$

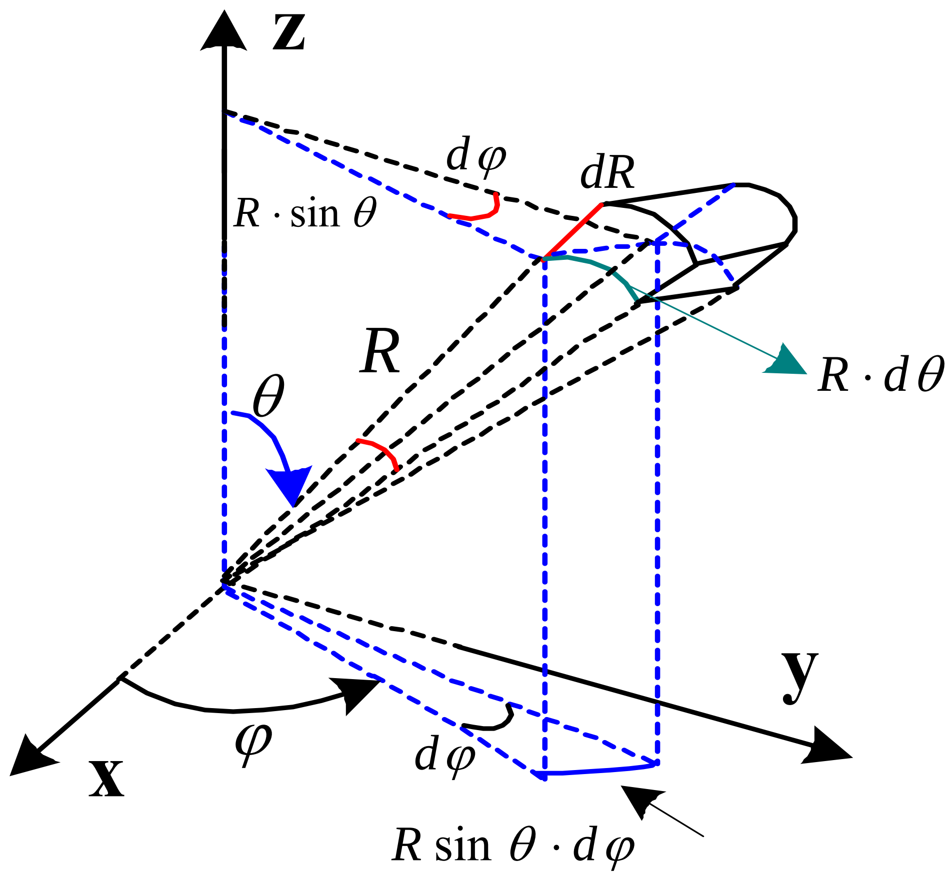
Sum and product: **?**

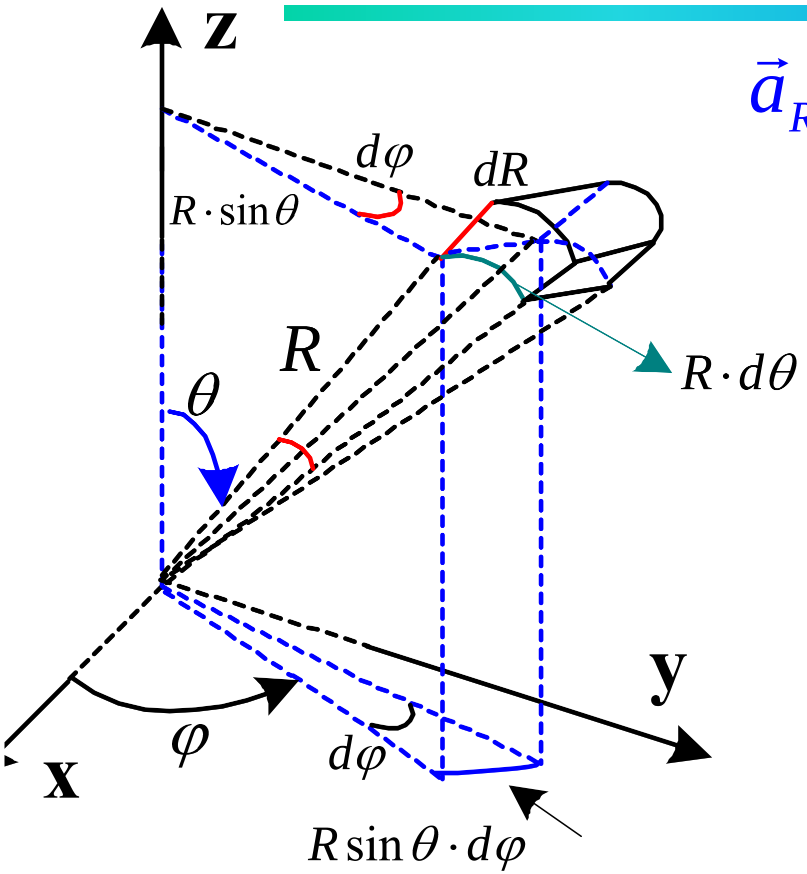
If angle is not constant, need convert to rectangular system firstly



Differential Length *(vector)*

$$d\vec{l} = \vec{a}_R dR + \vec{a}_\theta (R \cdot d\theta) + \vec{a}_\varphi (R \cdot \sin\theta \cdot d\varphi)$$





$$\vec{a}_R dR \quad \vec{a}_\theta (R\cdot d\theta) \quad \vec{a}_\varphi (R\cdot\sin\theta\cdot d\varphi)$$

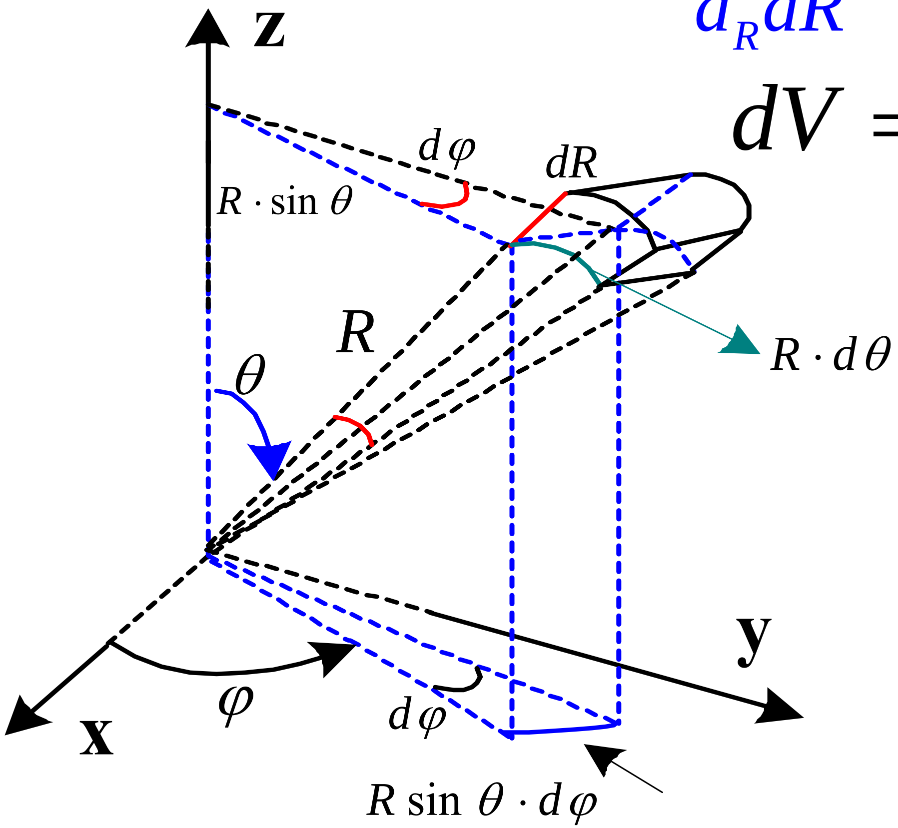
$$d\vec{S}$$

$$\begin{cases} d\vec{S}_R = \vec{a}_R (R\sin\theta\cdot d\varphi)\cdot(Rd\theta) \\ d\vec{S}_\theta = \vec{a}_\theta (R\sin\theta\cdot d\varphi)\cdot dR \\ d\vec{S}_\varphi = \vec{a}_\varphi (Rd\theta)\cdot dR \end{cases}$$

Differential Volume *(scalar)*

$$\vec{a}_R dR \quad \vec{a}_\theta (R \cdot d\theta) \quad \vec{a}_\varphi (R \cdot \sin\theta \cdot d\varphi)$$

$$dV = (Rd\theta) \cdot (R\sin\theta \cdot d\varphi) \cdot dR$$



Differential Length *(vector)*

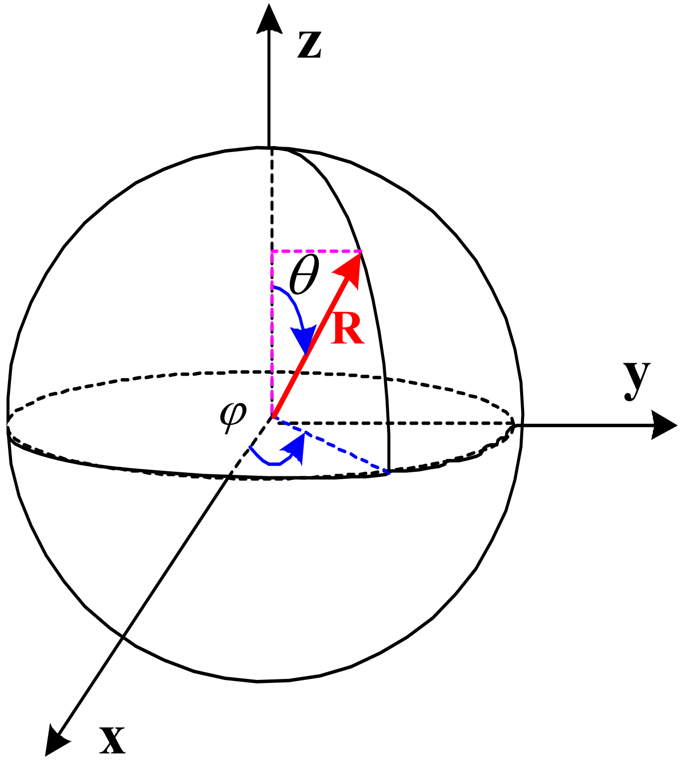
$$d\vec{l} = \vec{a}_R dR + \vec{a}_\theta (R \cdot d\theta) + \vec{a}_\varphi (R \cdot \sin\theta \cdot d\varphi)$$

Differential Surface *(vector)* $d\vec{S}$

$$\begin{cases} d\vec{S}_R = \vec{a}_R (R\sin\theta \cdot d\varphi) \cdot (Rd\theta) \\ d\vec{S}_\theta = \vec{a}_\theta (R\sin\theta \cdot d\varphi) \cdot dR \\ d\vec{S}_\varphi = \vec{a}_\varphi (Rd\theta) \cdot dR \end{cases}$$

Differential Volume *(scalar)*

$$dV = (Rd\theta) \cdot (R\sin\theta \cdot d\varphi) \cdot dR$$



- Relationship between spherical and rectangular coordinates: Equation (2.43b) in textbook page 30. **Bhag Singh Guru**

Line, surface and volume integrals

Line integral

$$\int_a^b f(x)dx \Longrightarrow \int_c \vec{F}\cdot d\vec{l}$$

Surface integral

$$\int_s f ds \Longrightarrow \int_s \vec{F}\cdot d\vec{s}$$

Volume integrals

$$\int_v f dv$$

A Summary of Vector Algebra

$\vec{A}+\vec{B}$ $\quad$ $\vec{A}\bullet\vec{B}$ $\quad$ $\vec{A}\times\vec{B}$

Scalar Product $\quad$ *Vector Product*

标量三重积： *Scalar Triple Product*

$$\boxed{\vec{A}\bullet(\vec{B}\times\vec{C})=\vec{B}\bullet(\vec{C}\times\vec{A})=\vec{C}\bullet(\vec{A}\times\vec{B})}$$

矢量三重积： *Vector Triple Product*

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{C}\bullet\vec{A})-\vec{C}(\vec{A}\bullet\vec{B})}$$

A Summary of Coordinates

$$\begin{cases} \text{Cartesian Coordinates} & (x, y, z; \vec{a}_x, \vec{a}_y, \vec{a}_z) \\ \text{Cylindrical Coordinates} & (r, \varphi, z; \vec{a}_r, \vec{a}_\varphi, \vec{a}_z) \\ \text{Spherical Coordinates} & (r, \theta, \phi; \vec{a}_r, \vec{a}_\theta, \vec{a}_\phi) \end{cases}$$

Differential Elements

$d\vec{l}$ $d\vec{S}$ dV

Refer to Table 2.1 in textbook pp.28-31
Bhag Singh Guru

请大家重视并掌握三个坐标系下的这三类微分元，它们是我们解决实际问题时的必备工具。

A Summary of field

Circuit----In "Electronic Systems"

we use *centralized parameters*.
i.e. macroscopical or average parameters
such as current, voltage, resistance

***To solve the problem, we depend on* differential equations and scalar equations.**

Field----In "EM Theory"

$$Q(\vec{r},t)\Big|_{\substack{\vec{r}=\vec{r}_0 \\ t=t_0}} = Q_0$$

we use distributed parameters.
i.e. microcosmic or specific parameters
such as E intensity, M intensity, potential, Poynting vector

***To solve the problem, we depend on* partial differential equations and vector algebra.**