EBU6018
Advanced Transform Methods

Wigner-Ville Distribution

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Wigner-Ville Distribution

So far we have looked at transforms that compute the correlation between a signal and basis functions that are functions of time and frequency (or of scale and translation). The time-frequency resolution is determined by the basis functions.

These have been the STFT and the Wavelet Transform.

An alternative approach is to compute directly:

- Time-frequency energy density- signal’s energy density in both time and frequency
- (c.f. power spectrum: energy in frequency only)
- An example of this is the Wigner-Ville distribution

Looking back: Comparison of STFT and CWT

- Similarities:
  - signal is multiplied by a function, and the transform is computed separately for different segments of signals.
  - can be written in inner product form

$$STFT(b,\omega) = \left\langle s(t), \gamma(t-b)e^{j\omega t} \right\rangle$$

$$CWT(b,a) = \left\langle s(t), \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)\right\rangle$$

- Difference:
  - Fixed time duration and freq bandwidths of $\gamma(t)$
  - Variable time duration and bandwidth of $\psi(t)$

Looking forward: An "instantaneous power spectrum"

Recall the power spectrum:

$$P(\omega) = |S(\omega)|^2 = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

Power spectrum of a signal is the Fourier Transform of its autocorrelation function

where $R(\tau)$ is the autocorrelation function (acf)

$$R(\tau) = \int_{-\infty}^{\infty} s(t) s^*(t-\tau) dt = \int_{-\infty}^{\infty} s(t+\tau/2) s^*(t-\tau/2) dt$$

What happens if we use *instantaneous* autocorrelation:

$$R(t,\tau) = s(t+\tau/2) s^*(t-\tau/2)$$

instead of $R(\tau) = \int_{-\infty}^{\infty} R(t,\tau) dt$? We get

$$WVD_s(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2) s^*(t-\tau/2) e^{-j\omega\tau} d\tau$$

The WVD is the Fourier Transform of the instantaneous autocorrelation function

which is the *Wigner-Ville Distribution* (WVD).

Cross-WVD vs auto-WVD

Since we can define a cross - correlation,

we can also define a cross - Wigner - Ville distribution :

$$WVD_{s,g}(t,\omega) = \int_{-\infty}^{\infty} s(t+\tau/2)g^*(t-\tau/2)e^{-j\omega\tau}\,d\tau$$

Taking complex conjugates we find

$$WVD_{s,g}(t,\omega) = WVD^*_{g,s}(t,\omega)$$

So for the usual WVD ("auto - WVD") we have

$$WVD_s(t,\omega) = WVD_{s,s}(t,\omega) = WVD^*_s(t,\omega)$$

so the auto - WVD is always *real*.

Example: Gaussian function

Signal $s(t) = \sqrt[4]{\frac{\alpha}{\pi}} e^{-\alpha t^2/2}$ (normalized to unit energy)

For WVD, we get

$$WVDs(t,\omega) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{\alpha}{2}\left[\left(t+\frac{\tau}{2}\right)^2 + \left(t-\frac{\tau}{2}\right)^2\right]\right\} e^{-j\omega\tau} d\tau$$

$$= e^{-\alpha t^2} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{\alpha}{4}\tau^2\right\} e^{-j\omega\tau} d\tau$$

$$= 2\exp\left\{-\left[\alpha t^2 + \frac{1}{\alpha}\omega^2\right]\right\}$$

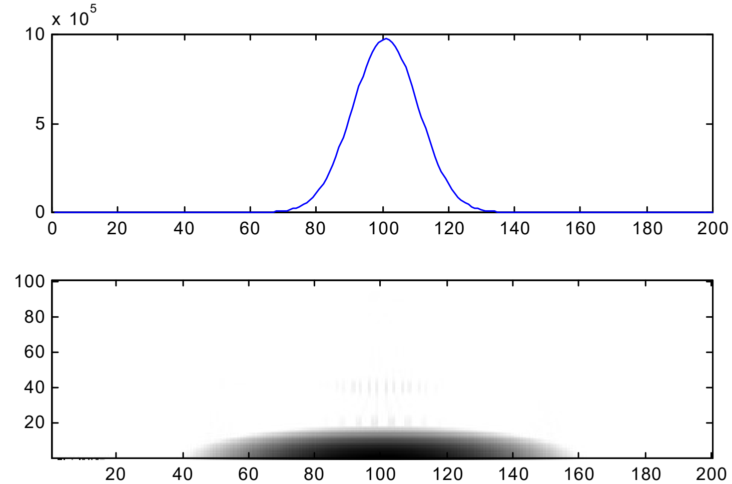
Gaussian in time, Gaussian in t-f

i.e. concentrated around (0,0).

α controls spread:

"time-width": $|t| < \sqrt{\frac{1}{\alpha}}$

"freq-width": $|\omega| < \sqrt{\alpha}$



Example 2: Gaussian chirplet

Signal: $$s(t) = \sqrt[4]{\frac{\alpha}{\pi}} \exp\left\{-\frac{\alpha}{2}t^2 + j\frac{\beta}{2}t^2\right\}$$

Power spectrum: $$|S(\omega)|^2 = \sqrt{\frac{4\pi(\alpha^2+\beta^2)}{\alpha}} \exp\left\{-\frac{\alpha}{\alpha^2+\beta^2}\omega^2\right\}$$

tells us *which* freqs $s(t)$ contains, not *when*. Compare:

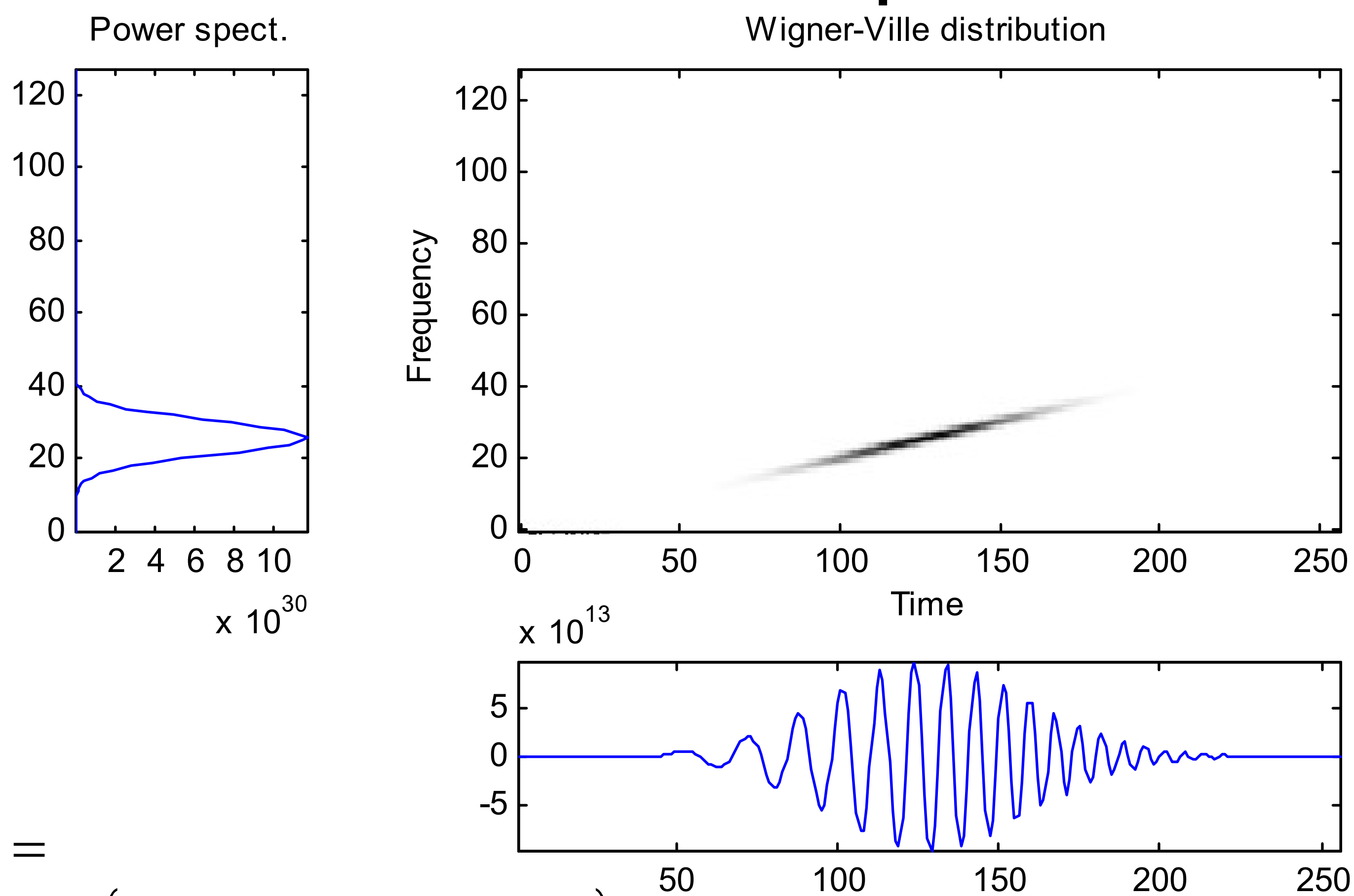
$$WVDs(t,\omega) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\left[\left(t+\frac{\tau}{2}\right)^2 + \left(t-\frac{\tau}{2}\right)^2\right] + \frac{j\beta}{2}\left[\left(t+\frac{\tau}{2}\right)^2 - \left(t-\frac{\tau}{2}\right)^2\right]} e^{-j\omega\tau} d\tau$$

$$= e^{-\alpha t^2} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{4}\tau^2} e^{-j(\omega-\beta t)\tau} d\tau$$

$$= 2e^{-\left[\alpha t^2 + \frac{1}{\alpha}(\omega-\beta t)^2\right]}$$

i.e. energy concentrated at $\omega = \beta t$, changing with time.

Illustration: chirplet



$$s(t) = \sqrt[4]{\frac{\alpha}{\pi}} \exp\left\{-\frac{\alpha}{2}(t-t_0)^2 + j\frac{\beta}{2}t^2\right\}$$

Time-limited or band-limited signals

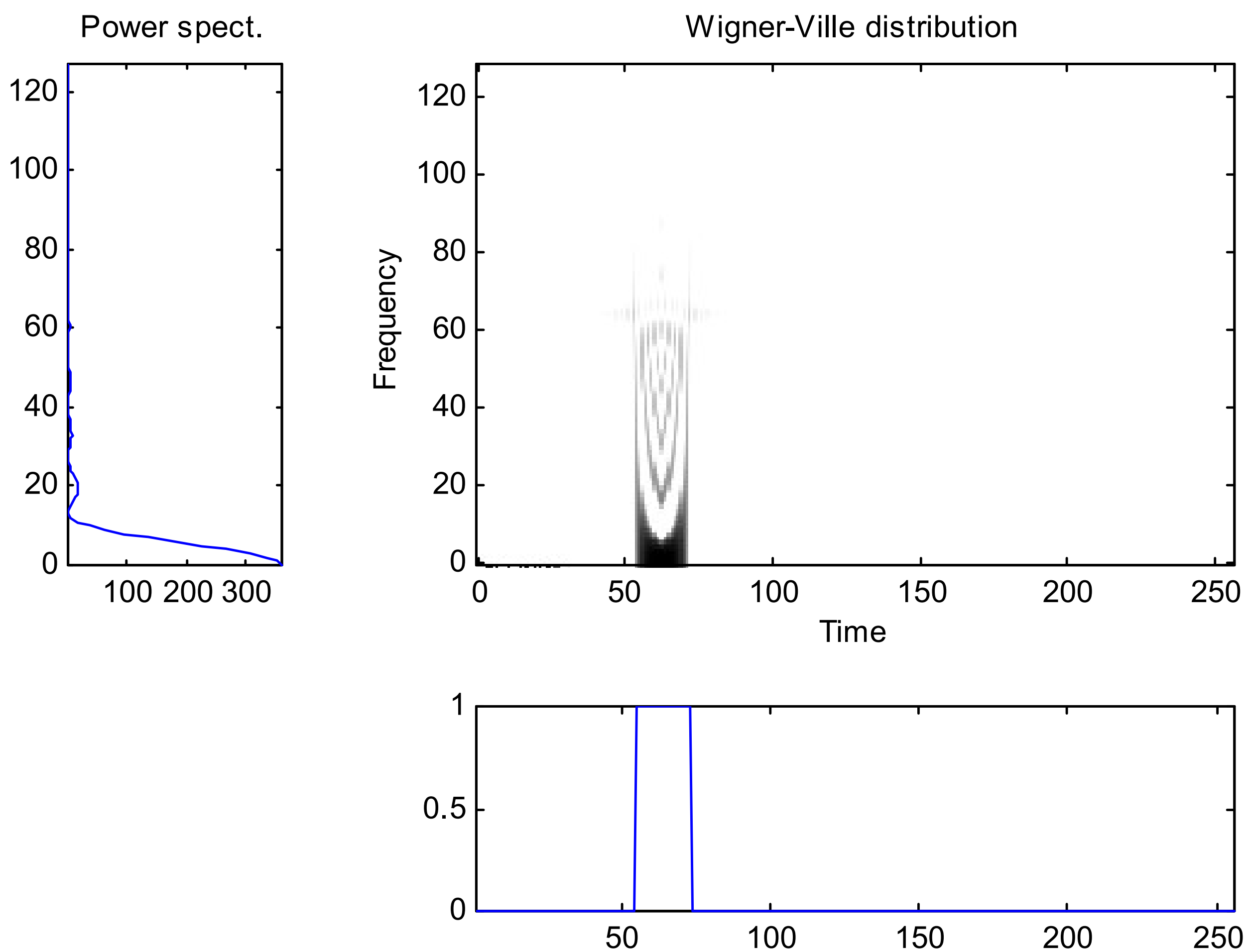
If $s(t)$ is time-limited, i.e. $s(t) = 0$ outside some interval $[t_{0,} t_1]$, then the WVD is also time-limited, i.e.

$$WVD_s(t, \omega) = 0 \text{ for } t \notin [t_0, t_1]$$

since no value for τ can make both $s(t + \tau / 2)$ and $s(t - \tau / 2)$ non-zero if t is outside this range.

A similar result is true for freq band-limited signals.

Example: WVD of a Time-limited signal



WVD Properties: Time Marginal Condition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} WVD_s(t,\omega)d\omega = \int_{-\infty}^{\infty} s\left(t+\tfrac{\tau}{2}\right)s*\left(t-\tfrac{\tau}{2}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\,d\omega d\tau$$

$$= \int_{-\infty}^{\infty} s\left(t+\tfrac{\tau}{2}\right)s*\left(t-\tfrac{\tau}{2}\right)\delta(\tau)d\tau$$

$$= |\, s(t)\,|^2$$

So intregral over frequency of WVD is the signal power density at time t

Compare similar result for probability densities :

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)dy$$

Frequency Marginal Condition

$$\int_{-\infty}^{\infty} WVD_s(t,\omega)dt = \int_{-\infty}^{\infty} s\left(t+\frac{\tau}{2}\right)s^*\left(t-\frac{\tau}{2}\right)\int_{-\infty}^{\infty} e^{-j\omega\tau}\,dt d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} \int_{-\infty}^{\infty} s(t)s^*(t-\tau)dt d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j\omega\tau} R(\tau)d\tau$$

$$= |S(\omega)|^2$$

So intregral over time of WVD is the power spectral density

We also have :

$$\frac{1}{2\pi}\int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} WVD_s(t,\omega)dt d\omega = \frac{1}{2\pi}\int_{-\infty}^{-\infty} |S(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$$

i.e. the WVD is unitary : the energy in $WVD_s(t,\omega)$ is equal to energy in original signal $s(t)$.

Time-shift & Freq-moduln. invariant

If the WVD of $s(t)$ is $WVD_s(t,\omega)$,

then the WVD of time-shifted signal $s_0(t) = s(t - t_0)$

is a time-shifted WVD : $WVD_{s_0}(t,\omega) = WVD_s(t - t_0, \omega)$

Further, the WVD of frequency-modulated signal

$s_1(t) = s(t)e^{j\omega_1 t}$ is a frequency-shifted WVD :

$WVD_{s_1}(t,\omega) = WVD_s(t, \omega - \omega_1)$

(Both follow immediately from the formulas for WVD)

WVD of multiple signals: Cross-terms

Wigner-Ville Distribution has many useful properties, and better resolution than STFT spectrogram. BUT

Applications are limited due to *cross-term interference.*

Consider composite signal $s(t) = s_1(t) + s_2(t)$. Then

$$WVD_s(t,\omega) = WVD_{s_1}(t,\omega) + WVD_{s_2}(t,\omega) + 2\,\mathrm{Re}\left\{WVD_{s_1,s_2}(t,\omega)\right\}$$

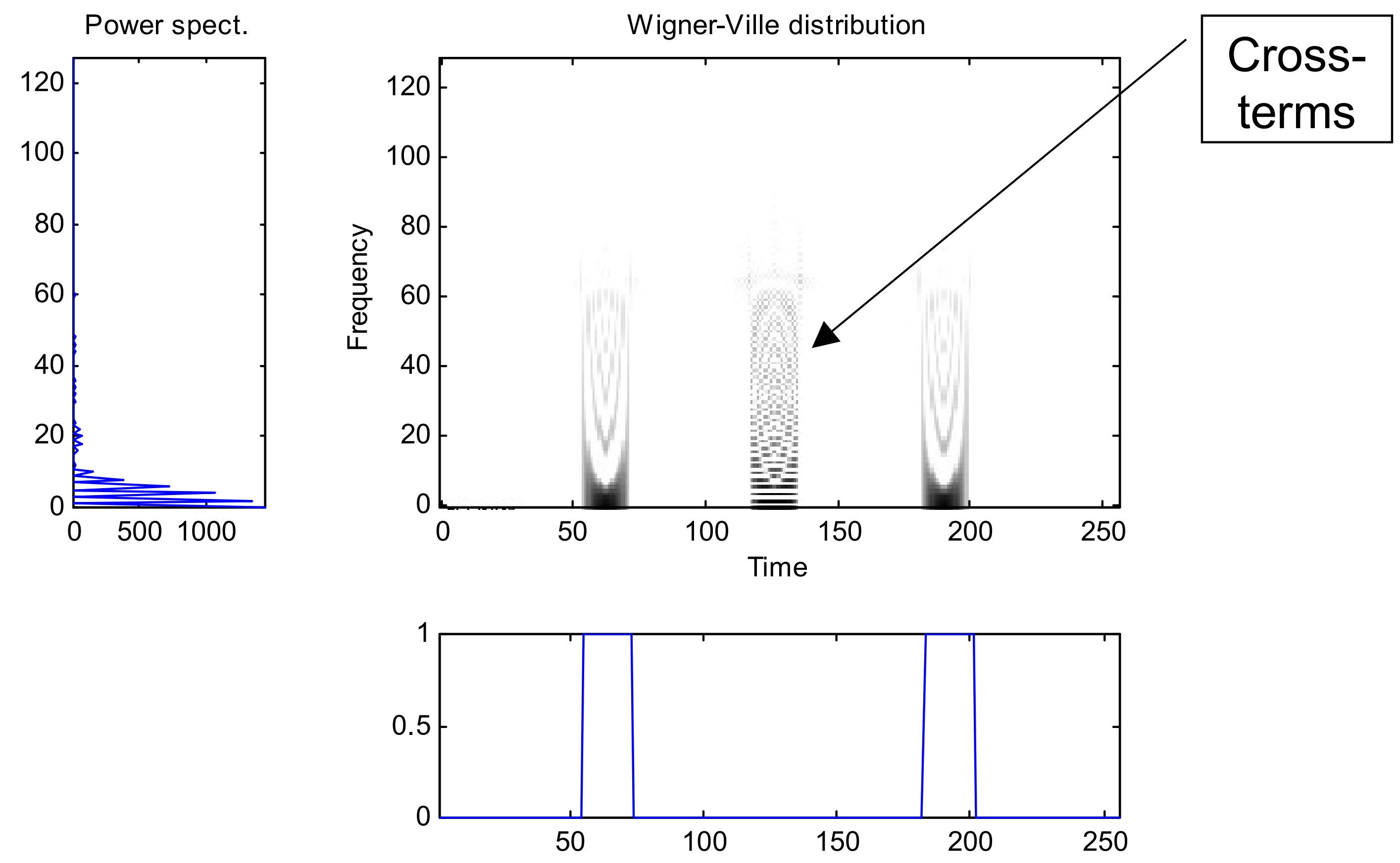
i.e. not only the sum of WVDs, but also the cross-term

$WVD_{s_1,s_2}(t,\omega)$

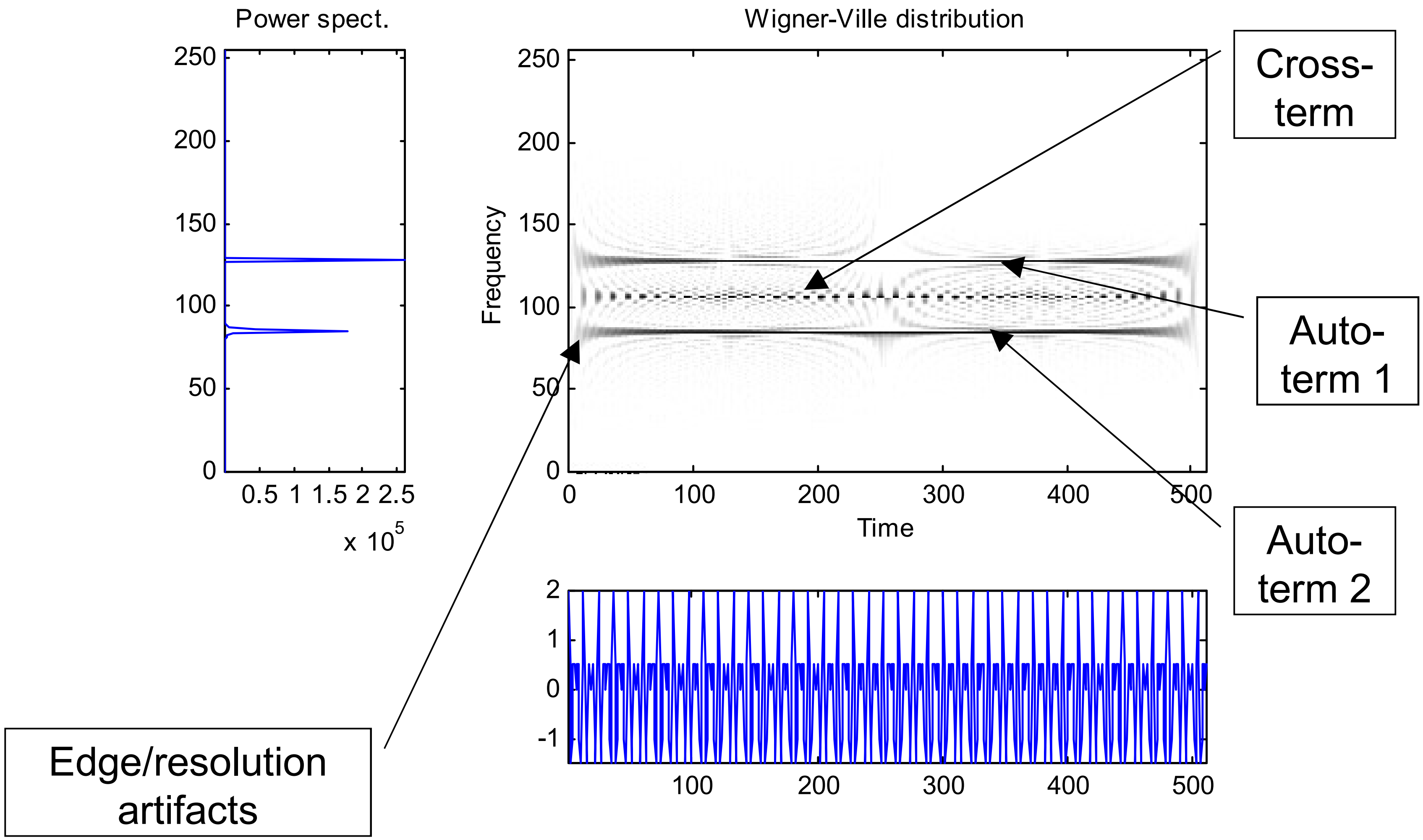
Also, cross-term is included at double the magnitude of the auto-terms, so often obscures useful patterns.

Example of cross-terms

WVD gives cross-terms for all but simplest signals, e.g.:



Example: sum of two sinusoids



Example: sum of sinusoids (cont)

If $s(t) = \exp(j\omega_0 t)$ then

$$WVD_s(t,\omega) = \int_{-\infty}^{\infty} \exp\left\{j\omega_0\left(t + \frac{t}{2} - t + \frac{t}{2}\right)\right\} e^{-j\omega t} d\tau = 2\pi\delta(\omega - \omega_0)$$

i.e. the WVD is a "ridge" along frequency ω_0.

Now let $s(t) = \exp(j\omega_1 t) + \exp(j\omega_2 t)$. The power spectrum is

$$|S(\omega)|^2 = 2\pi\delta(\omega_1) + 2\pi\delta(\omega_2)$$

while the WVD is

$$WVD_s(t,\omega) = 2\pi\delta(\omega - \omega_1) + 2\pi\delta(\omega - \omega_2) + 4\pi\delta(\omega - \omega_\mu)\cos(\omega_d t)$$

where $\omega_\mu = \frac{\omega_1 + \omega_2}{2}$ and $\omega_d = \omega_1 - \omega_2$.

Example (cont): cross-term

Get a large cross-term $4\pi\delta(\omega-\omega_{\mu})\cos(\omega_d t)$

which varies as $\cos\left((\omega_1-\omega_2)t\right)$ at ω_d,

mid-way between the auto-terms.

While the auto-terms are $+$ve, this oscillates $+$ve & -ve

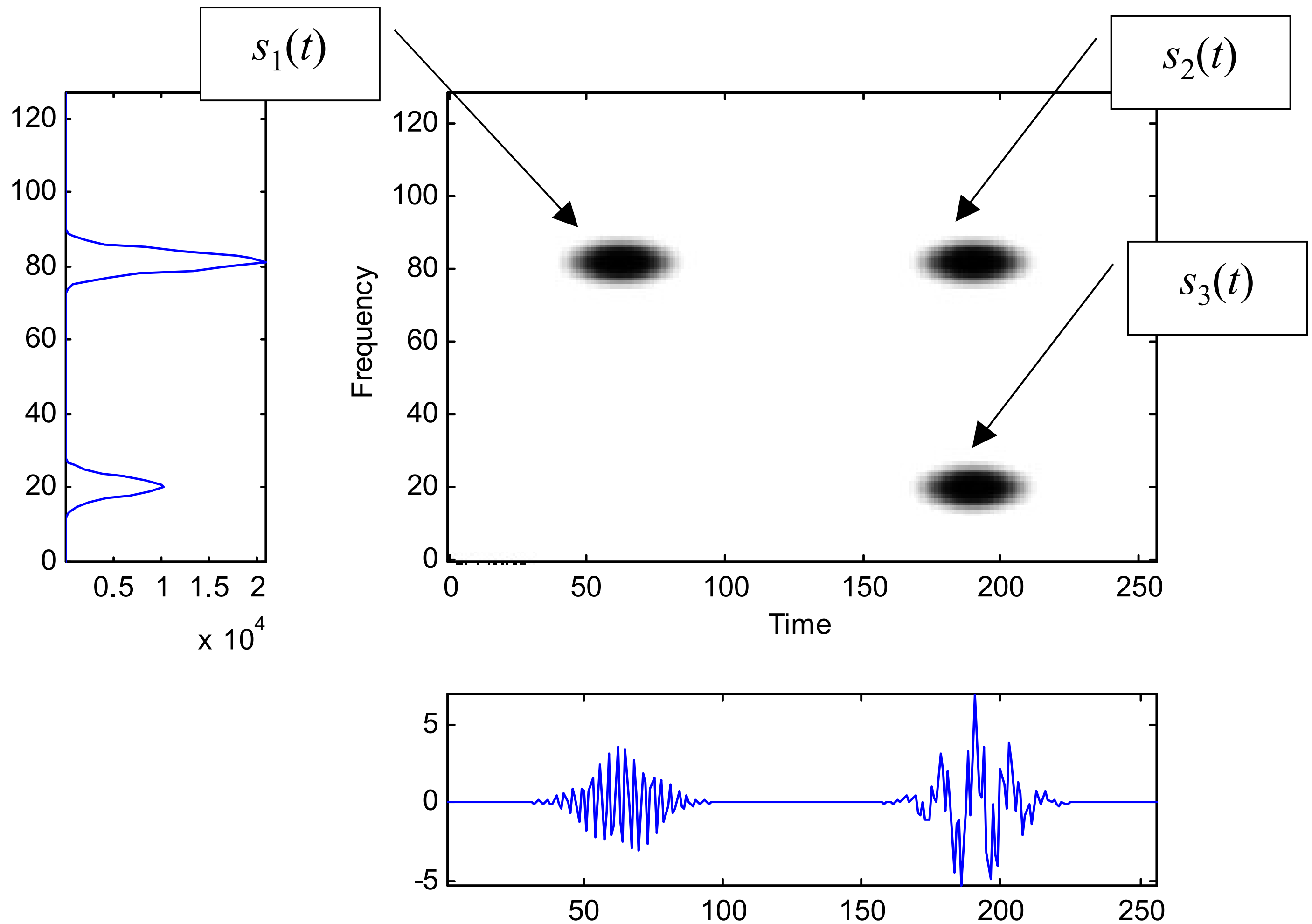
Average of the cross-term is zero:

$$\int_{-\infty}^{\infty} 4\pi\delta(\omega-\omega_{\mu})\cos(\omega_d t)dt = 0 \qquad \omega_d \neq 0$$

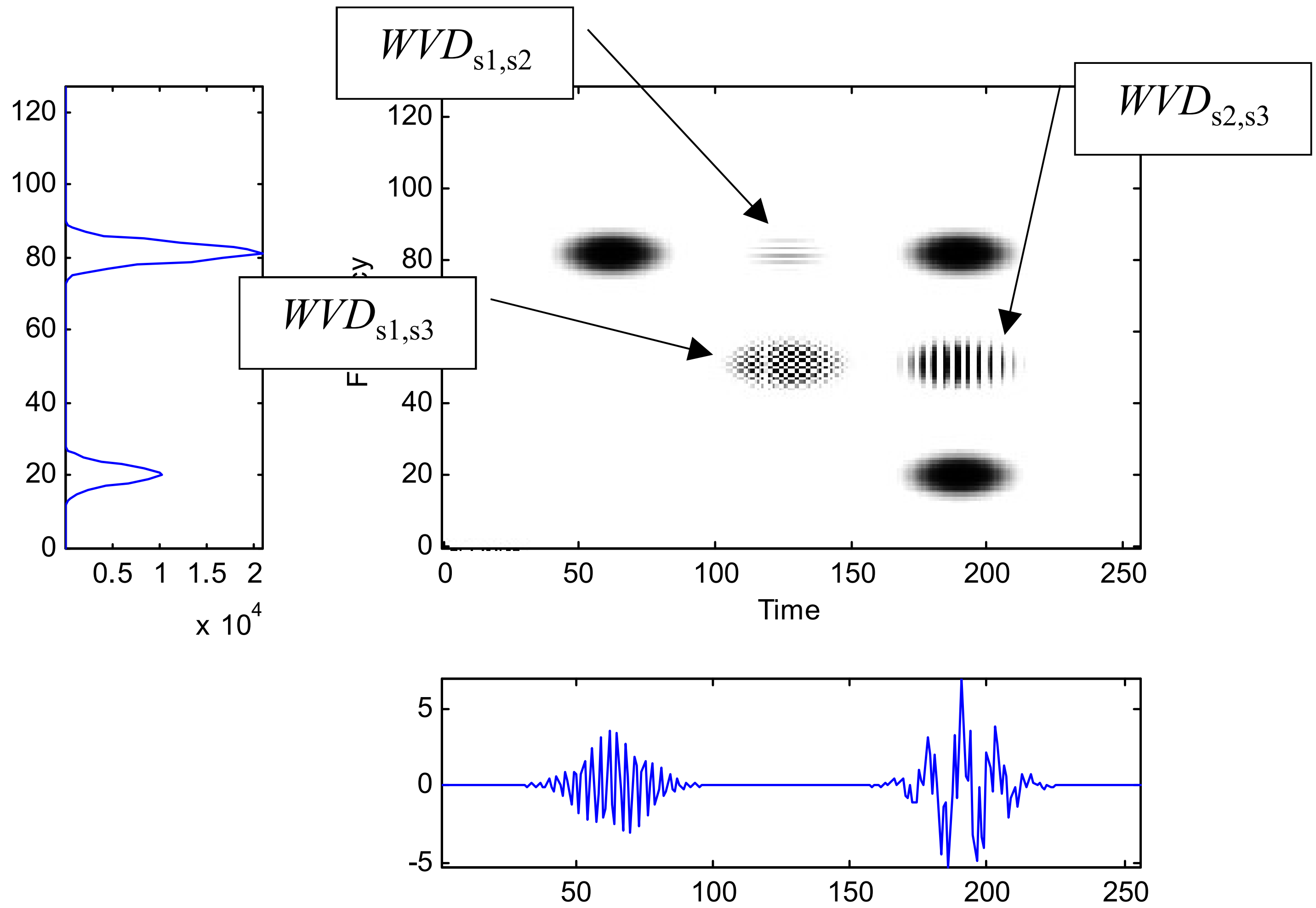
This suggests we may be able to remove these

by *smoothing* (see later.)

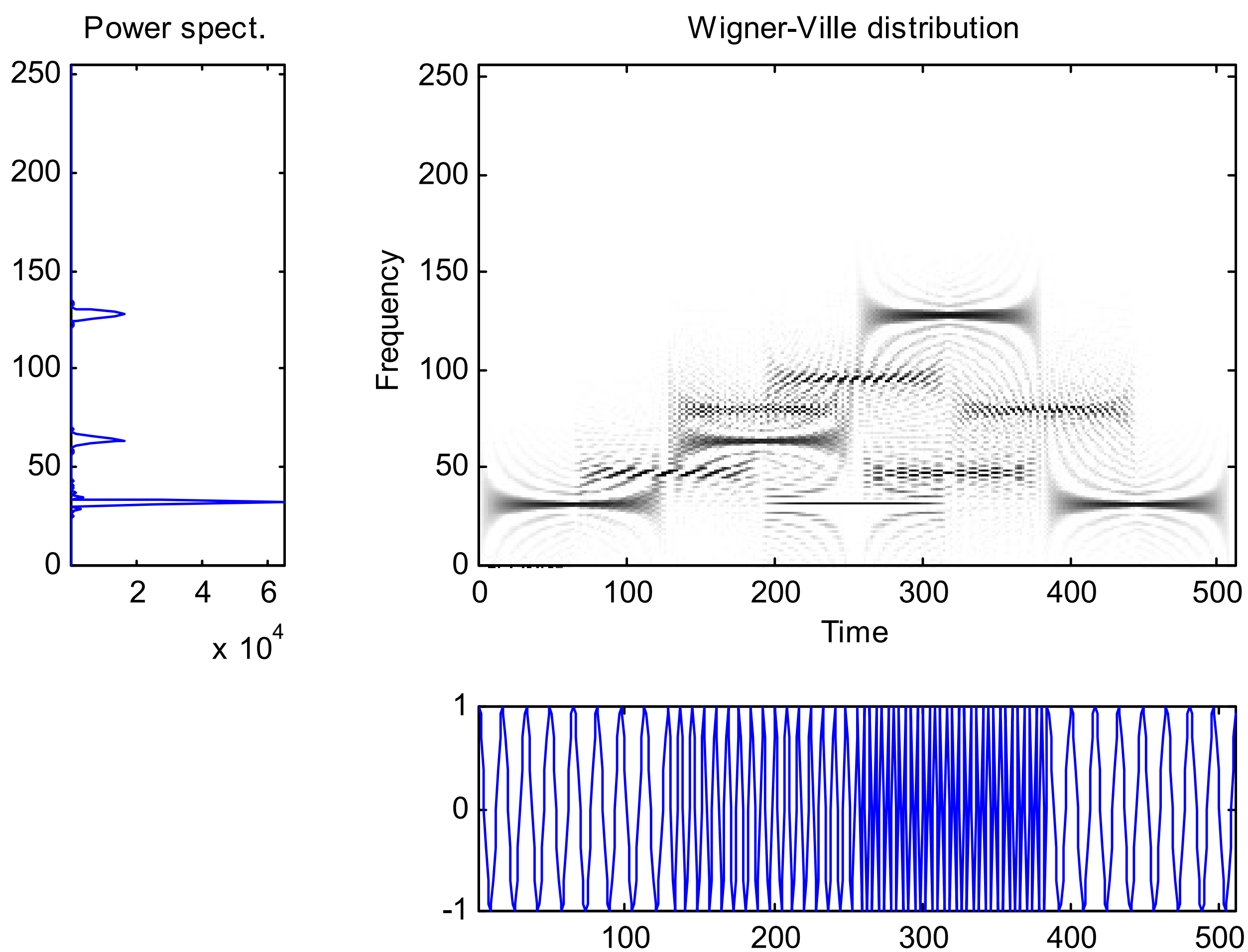
Example: 3-tone test signal



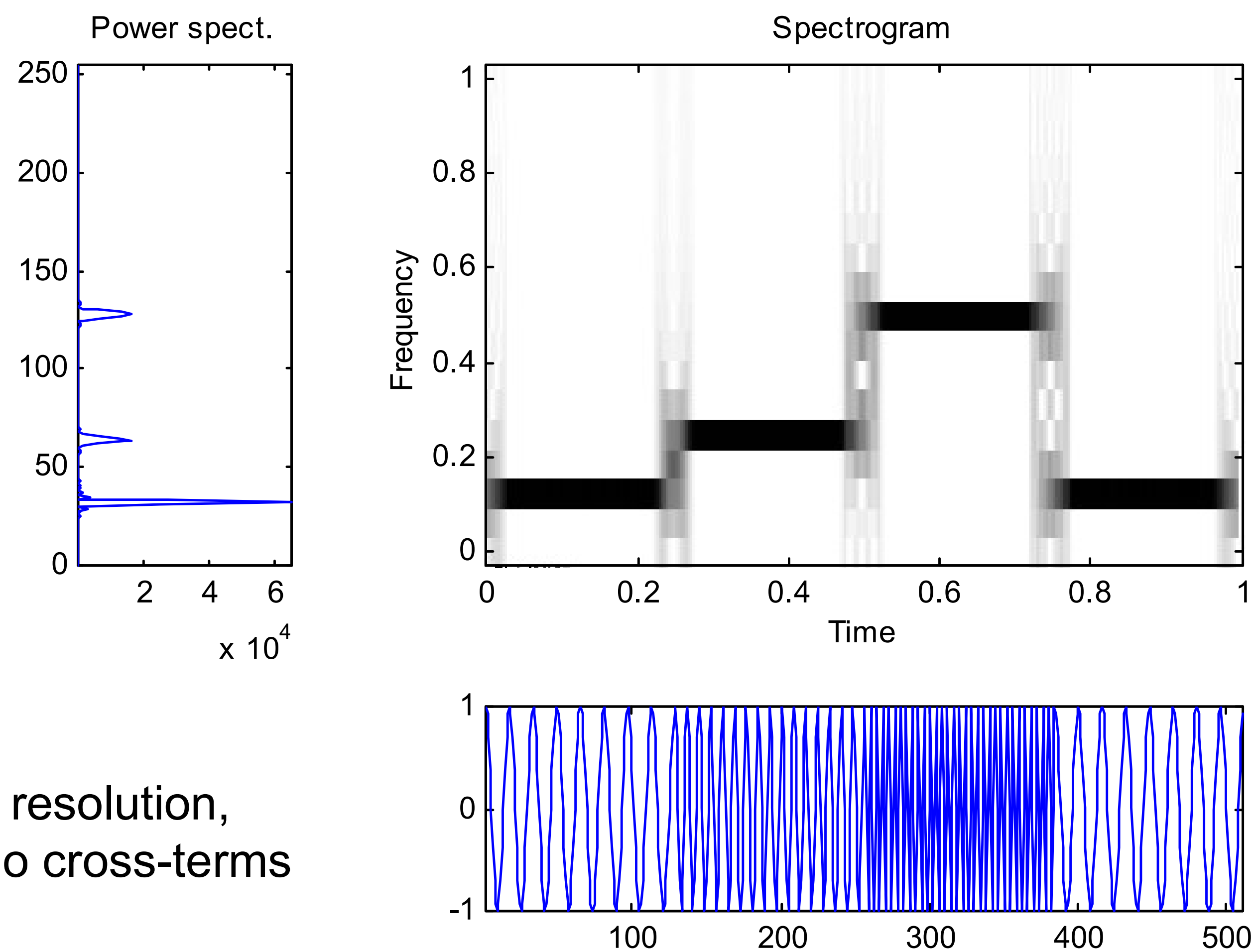
Wigner-Ville Distribution



Another example: Frequency pulses



Compare spectrogram



Less resolution,
but no cross-terms

Wigner-Ville Distribution

Since the cross - terms WVD are usually strongly oscillating, try removing them by using 2D low - pass filtering, to give a "smoothed Wigner - Ville distribution" (SWVD) :

$$SWVD_s(t,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y) WVD_s(t-x, \omega-y)\,dxdy$$

where $\phi(x,y)$ is a 2D low - pass filter.

Example : 2D Gaussian

$$\phi(x,y) = e^{-\alpha t^2 - \beta\omega^2} \qquad \alpha, \beta > 0$$

We have a trade - off :

more smoothing $\rightarrow$ less cross - terms, BUT

more smoothing $\rightarrow$ reduced resolution

STFT Spectrogram from WVD

In general, WVD can have -ve values.

But if $\alpha\beta \geq 1$ in the Gausian 2D filter,

then the smoothed WVD will be non-negative.

Special case: when $\alpha\beta = 1$ in $\phi(x, y) = e^{-\alpha t^2 - \beta\omega^2}$

then $\phi(x, y)$ is actually a WVD of a Gaussian function.

The STFT spectrogram is a smoothed WVD, with

the WVD of the analysis function $\gamma(t)$ doing the smoothing:

$$|STFT_s(t,\omega)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} WVD_\gamma(x,y)WVD_s(t-x,\omega-y)dxdy$$

(Quan p163)

Convolution of the WVD of s(t) and the WVD of the STFT window function

Wavelet Scalogram from WVD

The scalogram (square of the wavelet transform) can be written in terms of the WVD :

$$SCAL_s(a,b) = |CWT_s(a,b)|^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} WVD_s(x,y)\, WVD_\psi\left(\frac{x-b}{a}, ay\right) dx dy$$

where $WVD_s(x,y)$ is the WVD of the signal $s(t)$ and $WVD_\psi\left(\frac{x-b}{a}, ay\right)$ is the WVD of the mother wavelet $\psi(t)$.

This operation is known as *affine correlation*.

From WVD to …?

$|s(t)|^2$ $\qquad$ $|S(\omega)|^2$

Integrate over ω $\qquad$ Integrate over t

$WVD_s(t,\omega)$

Affine correlation $\qquad$ Convolution

$|CWT_s(a,b)|^2$ $\qquad$ $|STFT_s(t,\omega)|^2$

Both the STFT spectrogram and the WT scalogram are smoothed versions of the WVD, explaining why the WVD has the best time-frequency resolution

Summary: Wigner-Ville Distribution

- **Kind of Decomposition**

  Time-Frequency
- **Analyzing Function**

  Uses the signal itself. Motivated by time-frequency energy density (c.f. a probability density).
- **Variable**

  Time and Frequency. Has high resolution in both time and frequency.
- **Suited for**

  Simple signals (non-stationary), e.g. linear chirp, gaussian pulse
- **Notes**

  More complex signals lead to undesired “cross-terms”. Can be suppressed with smoothing, but lose high resolution in the process.