Advanced Transform Methods

Short-Time Fourier Transform
STFT

Andy Watson

Limitation of Fourier Transform

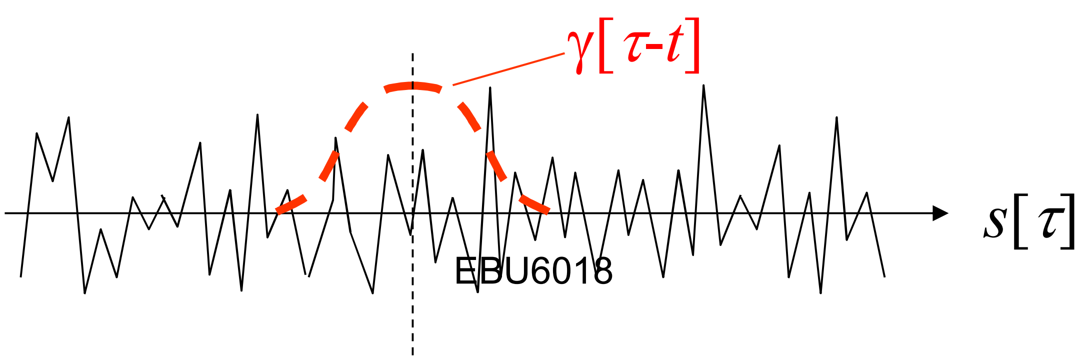
We know that:

- The basis functions (complex sinusoids) are spread over the entire time domain.
- Loses all time information since integration is performed over all times.
- Not possible to discriminate signals that have the same frequency but occurring at different times.
- Signals suitable for Fourier transform are time invariant (stationary).
- Applications suitable for the Fourier transform are those which concern frequency only.

So what can we about it for non-stationary signals?

Short-Time Fourier Transform

- For non-stationary signals, it is desirable to have an estimate of the input signal spectrum for short intervals of time.
- Want to see changes in spectrum with time.
- We could find a spectral “snapshot” by calculating the Fourier Transform of a short interval of the signal.
- If we assume the signal is stationary in certain time slots (window) (quasi-stationary), we can perform a Fourier Transform on this part of signal and obtain frequency information as well as time information.
- This is the “Short Time Fourier Transform”



Features of STFT

- Unlike FT, the STFT will give the information of frequency at different time intervals.
- Uncertainty Principle holds- The product of time resolution and frequency resolution is always greater than a minimum value.

$$\Delta_t \Delta_\omega \geq 1/2$$

- It is not possible to know exactly what frequency components exist at a particular point in time.
- What we can know are the time intervals in which certain bands of frequencies exist.
- Narrow window means good time resolution but poor frequency resolution and wide window means poor time resolution but good frequency resolution.
- For a given window, time resolution is fixed.

Features of STFT

- How could we choose the width of the window function?
- We could perhaps have an idea of the duration of a feature we are looking for….
- ….or we could perform an FT of the signal first to get the range of frequencies and so estimate a suitable window width.
- One area of application of the STFT is in the processing of audio signals, so in that application a knowledge of audio characteristics allows window widths to be estimated.
- In practice, the evaluation of the FT during the window will be performed using a DFT (or FFT).

The Windowed Fourier Transform

- **Harmonic wave e^{-jwt}**

(to perform the FT)

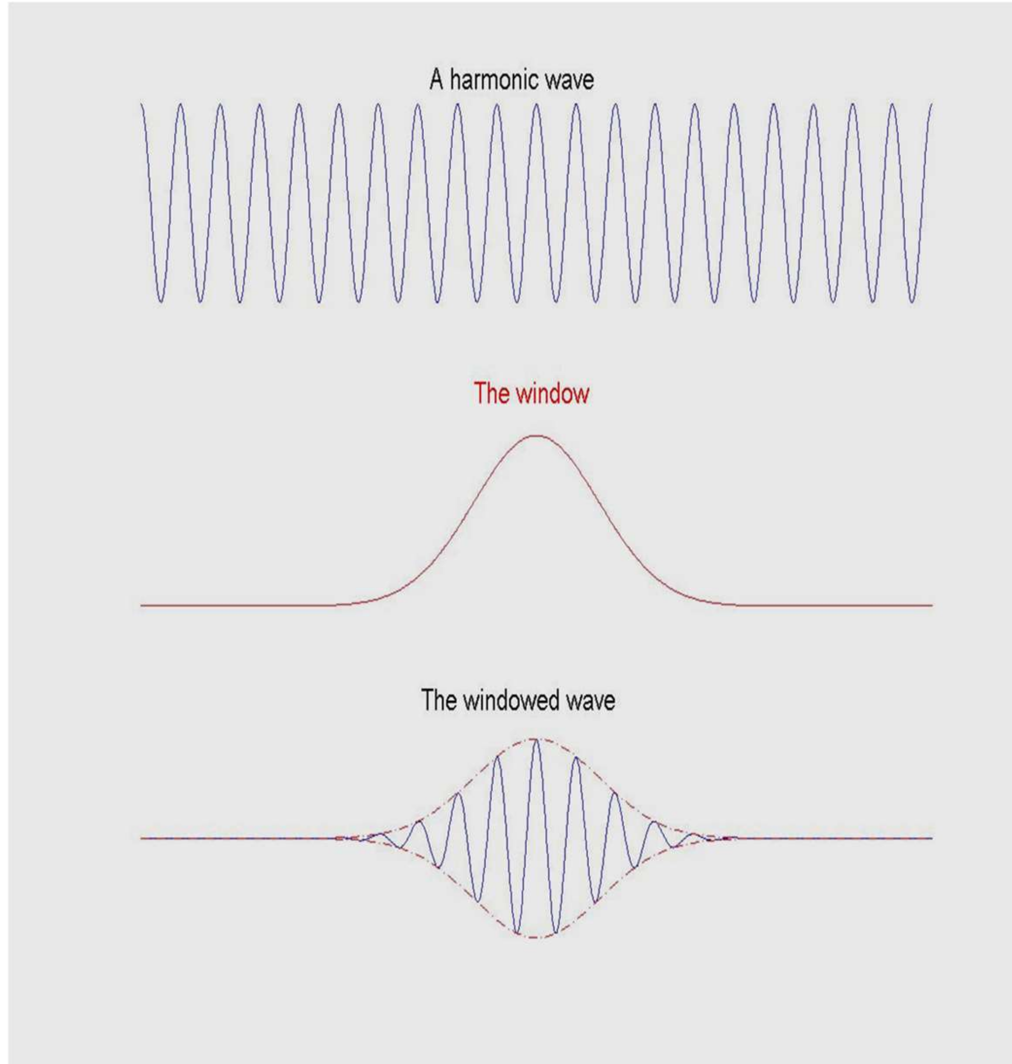
- **A window $\gamma(t)$**

(this will be moved across the signal)

- **A windowed wave**

  $\gamma(\tau - t)\, e^{-jwt}$

(the basis function)



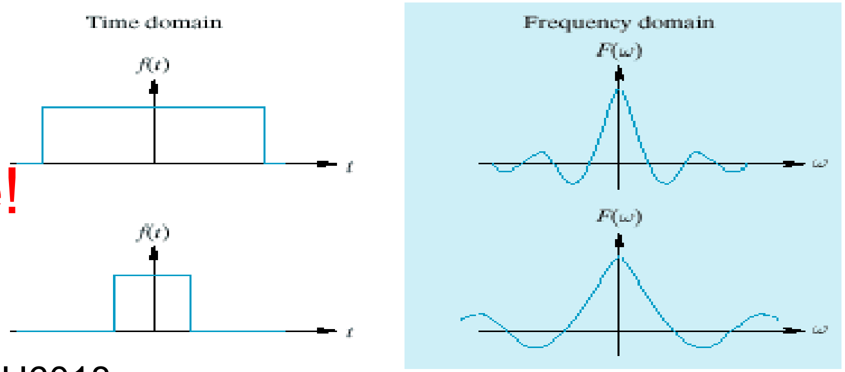
The Windowed Fourier Transform

- We see that the basis function for the STFT is the product of the window function and the complex exponential of the FT.
- This is then multiplied by the signal to be transformed in order to perform the STFT.

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^{*}(\tau - t)e^{-j\omega t}\,d\tau$$

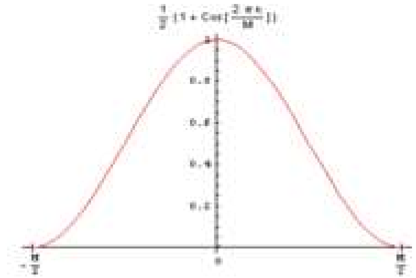
About the Shape of the Window

- If the window was rectangular, there would be discontinuities at the edges of the window, resulting in additional frequency components in the FT causing distortion of the spectrum.
- It would also require to be precisely translated so as to avoid overlap or gaps.
- Multiplication in the time domain is equivalent to convolution in the frequency domain, and the energy in the spectrum of the rectangular window is spread over its sinc function in frequency.
- So a rectangular window would be a very poor choice!



About the Shape of the Window

- A better shape of window would be "bell" shaped, such as Hamming, Hann, Gaussian etc.



- The energy in the window is then concentrated near the middle and tapers towards the edges.
- This allows the window to overlap when moved and minimises distortion.
- The frequency spectrum of the "bell" shaped window is concentrated in the centre (approximating to an impulse in frequency).

Time-Frequency Window

Windowing a function *s*(*t*) near *τ* =*t* :

$$s_b(\tau) \equiv s(\tau)\gamma^*(\tau - t)$$

where *γ*(*t*) is a suitable time-window, such as $\gamma(t) = \chi_{[0,1)}(t)$ **with**

$$\chi_{[0,1)}(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & t < 0, t > 1 \end{cases}$$

[Note: this is our poor choice of rectangular window!]

$s_b(\tau)$ **contains the information of the original function *s*(*t*) within the time-window**

$$[t + \tau_0 - \Delta_\tau / 2, t + \tau_0 + \Delta_\tau / 2]$$

where τ_0 is the center and Δ_τ is the time duration of the window function *γ*(*t*).

Time-Frequency Window

In general, for a windowing function $\gamma(t)$, the energy distribution can be defined by:

Center $\langle t \rangle_\gamma \equiv \frac{1}{\|\gamma\|^2} \int_{-\infty}^{\infty} t \left|\gamma(t)\right|^2 dt$

Radius $\Delta_\gamma \equiv \frac{1}{\|\gamma\|} \left[\int_{-\infty}^{\infty} (t - \langle t \rangle_\gamma)^2 \left|\gamma(t)\right|^2 dt \right]^{1/2}$

Width $= 2\Delta_\gamma$

Where, $\|W\|$ is the norm of $W(t)$ defined as

$$\|W(t)\|^2 = \langle W, W \rangle = \int_{-\infty}^{\infty} \left|W(t)\right|^2 dt$$

Short-Time Fourier Transform (STFT)

The Short-time Fourier transform $STFT(t,\omega)$ of a function $s(t)$ with respect to a window function γ evaluated at a point (t,ω) in the t-ω plane is defined as

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}\,d\tau = \int_{-\infty}^{\infty} s(\tau)\gamma_{t,\omega}{}^*(\tau)d\tau$$

$$s(t)\gamma^*(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} STFT_\gamma(\tau - t,\omega)e^{j\omega\tau}\,d\omega$$

The STFT $STFT(t_0,\omega_0)$ provides *local spectral information* of the function $s(t)$ around the point t_0. More precisely, it offers information in the time-frequency window

$$[t+t_0-\Delta_t/2, t+t_0+\Delta_t/2]\times[\omega+\omega_0-\Delta_\omega/2, \omega+\omega_0+\Delta_\omega/2]$$

That is, it characterises the signal's behaviour in the vicinity of t_0,ω_0 in the time-frequency domain.

Short Time Fourier Transform: Summary

- A window function $\gamma(t)$ is introduced.
- The transform is

$$STFT(t,\omega) = \int_{-\infty}^{\infty} s(\tau)\gamma^*(\tau - t)e^{-j\omega t}d\tau$$

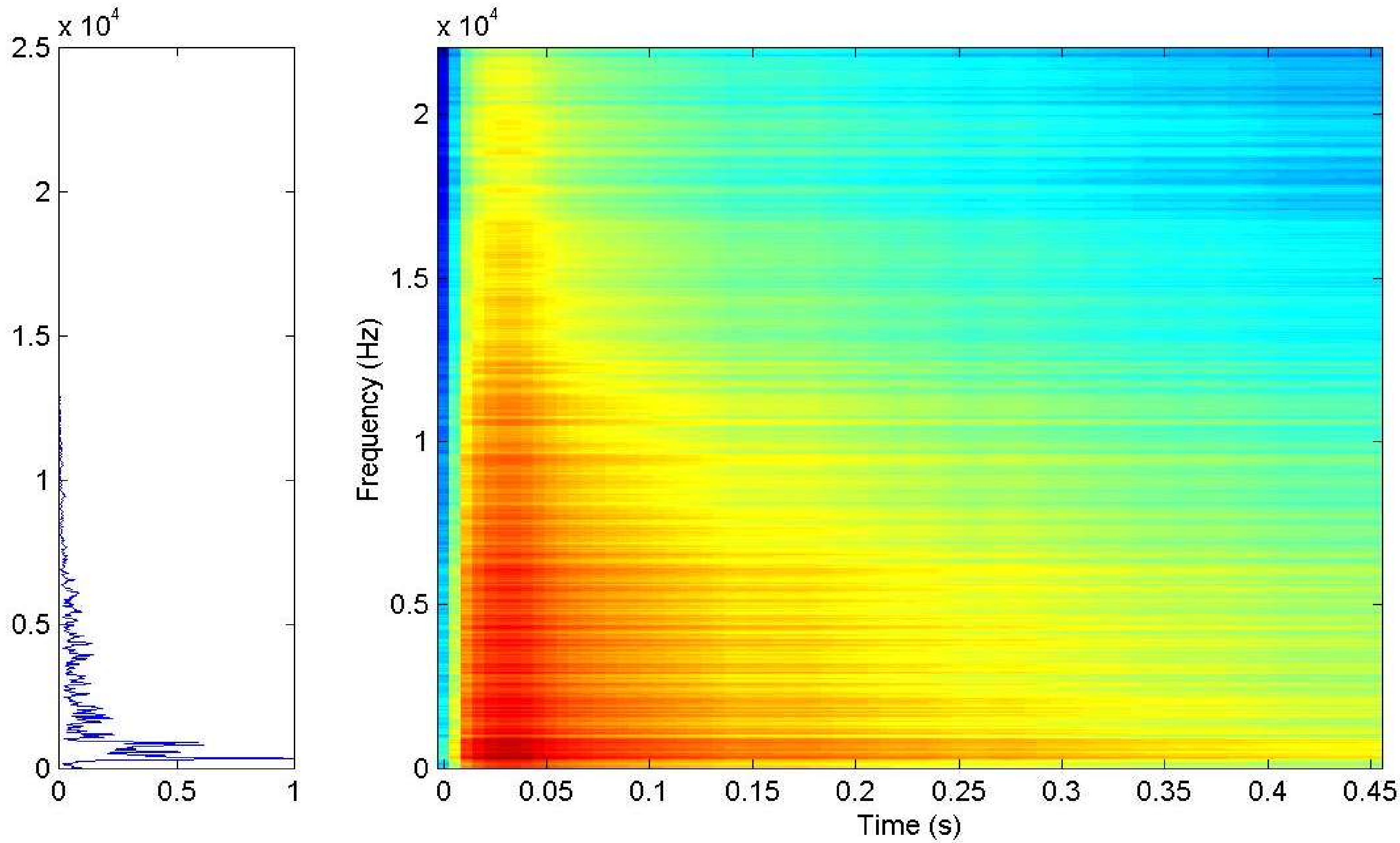
- Compare with the Fourier transform $S(\omega) = \int_{-\infty}^{\infty} s(t)\,\mathrm{e}^{-j\omega t}\,dt$

- For each time *t*, there is a Fourier transform. We obtain a time-frequency representation of the signal.
- Typically, we compute and plot the **Spectrogram**

(a plot of the energy in the signal as a function of time and frequency) $\left|STFT(t,\omega)\right|^2$

Spectrogram Example

Red means high energy, blue means low energy



Time-Freq Partition: STFT

STFT: Equal time and frequency resolution

$\Delta_t \Delta_\omega \geq 1/2$