Foreword

- In static field
  - E-potential is a scalar.
  - E-intensity is a vector.
  - It is much easier to determine E-potential than to calculate E-intensity.
  - Once potential is known, its negative grad. is intensity.
- How to determine E-potential?
  - If the charge distribution is typical, we can write out the potential directly, as in former sections.
  - However in most cases, we need to set up differential equations for E-potential.
  - These differential equations are of 2nd order, and set up according to fundamental equ. of electrostatics.

§ 3.11 Two Differential Equations

1. *Poisson's Equation* 泊松方程

2. *Laplace's Equation* 拉普拉斯方程

Derivation of the Equations

From fundamental electrostatic equations

1. Electrostatic Conservation Law

$$\nabla \times \vec{E} = 0 \quad \Longrightarrow \quad \vec{E} = -\nabla u$$

2. E-Gauss's Law in differential form

$$\nabla \bullet \vec{D} = \rho$$

3. Material Equation

$$\vec{D} = \varepsilon \vec{E}$$

$$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon}$$

——Poisson's Equation

$$\begin{cases} \nabla \bullet \vec{E} = \dfrac{\rho}{\varepsilon} \\ \vec{E} = -\nabla u \end{cases} \Rightarrow \nabla \bullet \nabla u = \nabla^2 u = -\frac{\rho}{\varepsilon}$$

$$\boxed{\nabla^2 u = -\frac{\rho}{\varepsilon}}$$ ——*Poisson's Equation*

∇^2 *Laplacian* Or in Chinese 拉普拉斯算符

It's a part differential function of 2nd order.

Laplace's Equation

At the source-free point or in the source-free region, there is no charge scattered and the charge volume density is 0.

$$\boxed{\nabla^2 u = -\rho / \varepsilon} \Longrightarrow \boxed{\nabla^2 u = 0}$$

Poisson's Equation $\Longrightarrow$ Laplace's Equation

About Laplacian

∇^2 refers to *div. of a grad*.

$$\nabla^2 = \nabla \bullet \nabla$$

$$\nabla^2(u) = \nabla \bullet (\nabla u)$$

Laplacian u is a scalar operator.

- Laplacian in Cartesian Coordinates should be remembered.

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Its a second-order differential operator.

In Cartesian Coordinates

$$\nabla \bullet \vec{X} = (\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}) \bullet \vec{X}$$

$$\nabla \psi = \vec{a}_x \frac{\partial \psi}{\partial x} + \vec{a}_y \frac{\partial \psi}{\partial y} + \vec{a}_z \frac{\partial \psi}{\partial z}$$

$\nabla^2 \psi$ is just the **dot product of del with gradient**

$$\nabla^2 \psi = \nabla \bullet \nabla \psi = \frac{\partial}{\partial x} \psi_x + \frac{\partial}{\partial y} \psi_y + \frac{\partial}{\partial z} \psi_z$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

- Expressions are more complicate in cylindrical and spherical coordinates.

In Cylindrical Coordinates

$$\nabla^2 u(r,\varphi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

In Spherical Coordinates

$$\nabla^2 u(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

Example 1.

- A ***conductor*** ball, with Radius of *a* & E-potential of *U*.
- Please determine E-potential outside the ball.
- Analysis:
  - Any Symmetry? —— Yes, Spherical symmetry.
  - How many approaches to determine E-potential?
    - Via differential equations
    - Via E-intensity
    - Direct solution——via integral or sum

Solution 1, Laplace's Equation.

Because ??? we obtain $\nabla^2 u = 0$

Because ??? we infer $u = u(r)$

Express Laplace's Equ. in spherical coordinates $\nabla^2 u = \frac{1}{r^2}\frac{d}{dr}r^2\frac{du}{dr} = 0$

Through integral of above equ. $u = -\frac{C_1}{r} + C_2$

By boundary conditions

$$u \xrightarrow{r=a} U$$

$$u \xrightarrow{r=\infty} 0$$

$$u = \begin{cases} r > a & \frac{a}{r}\cdot U \\ r = a & U \\ r < a & U \end{cases}$$

Solution 2，E-intensity

Via E-intensity

$$u(r)=\int_{\text{point A}}^{\infty}\vec{E}\bullet d\vec{l}=\int_{r}^{\infty}E_r\cdot dr$$

Assuming the ball is charged by $Q \Rightarrow \vec{E}=\vec{a}_r E_r=?$

Recall that we have calculated the E-intensity outside a conductor ball.

$$\vec{E}=\frac{1}{4\pi\varepsilon_0\cdot r^2}\cdot Q\vec{a}_R$$

Recall:

Applying *E-Gauss's Law*

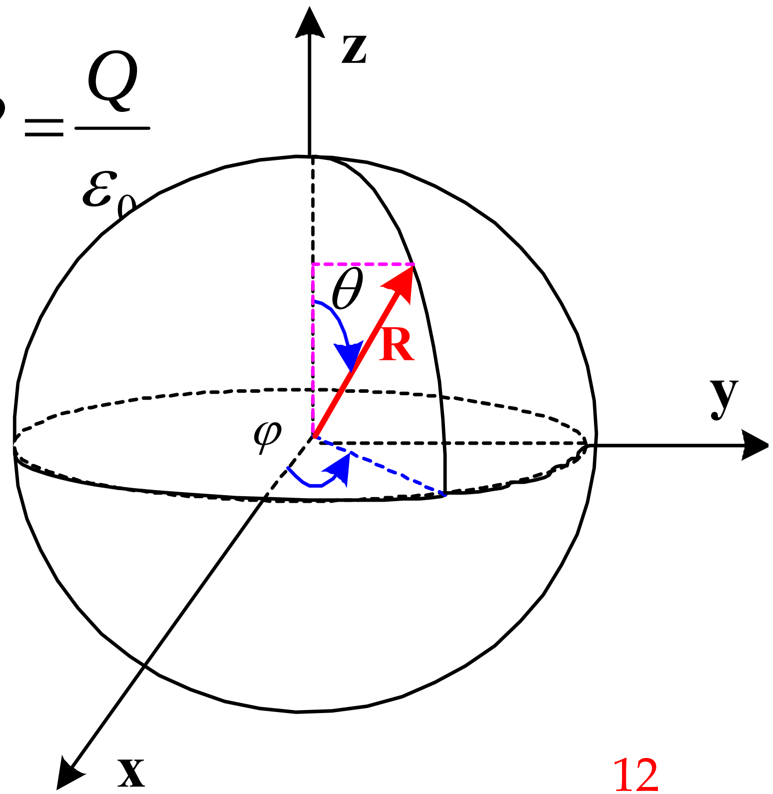
$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_0}\int_V \rho dV = \frac{Q}{\varepsilon_0}$$

Inside the ball ($r<a$): $\because \frac{1}{\varepsilon_0}\int_V \rho dV = 0 \qquad \therefore \vec{E} = 0$

Outside the ball ($r>a$): $\because \frac{1}{\varepsilon_0}\int_V \rho dV = ? = \frac{Q}{\varepsilon_0}$

$$\oint_S \vec{E} \bullet d\vec{S} = E \cdot (4\pi \cdot r^2)$$

$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0 \cdot r^2} \cdot Q\vec{a}_R$$



$$\vec{E} = \frac{1}{4\pi\varepsilon_0 \cdot r^2} \cdot Q\vec{a}_R$$

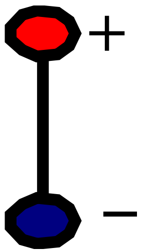
$$\psi(r) = \int_r^{\infty} \vec{E} \bullet d\vec{l} = \int_r^{\infty} E_r \cdot dr = ?$$

$$\because \psi(r)\,\big|_{r=a} = U \Rightarrow Q = ? \qquad \psi(r) = ?$$

§ 3.6 Electric Dipole

A pair of equal charges of opposite signs that are very close together.

Two charges of equal charge but of opposite polarity and separated by a small distance.



- ✓ Distance: l
- ✓ Point charges: $q_1 = q$, $q_2 = -q$

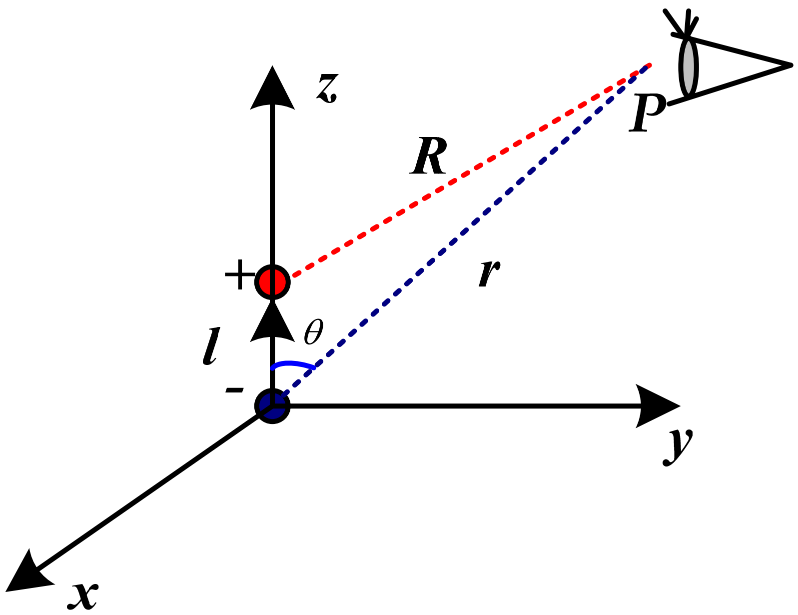
$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$\vec{E}_{-} = -\frac{-q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{|\vec{r}|}\right)$$

$$\vec{E}_{+} = -\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{|\vec{R}|}\right)$$

Cosine Theorem $\quad \dfrac{1}{|\vec{R}|} = \dfrac{1}{R} = \dfrac{1}{\sqrt{r^2 + l^2 - 2\cdot r\cdot l\cos\theta}}$

Taylor Series (l<<r) $\quad \frac{1}{R} = R^{-1} \approx \frac{1}{r} + \frac{1}{r^2}\cdot l\cdot\cos\theta$



In Spherical Coordinates

$$\vec{E}_{-} = -\frac{-q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r}\right) \qquad \vec{E}_{+} \approx -\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r}+\frac{l}{r^2}\cdot\cos\theta\right)$$

$$\vec{E} = \vec{E}_{+} + \vec{E}_{-} = -\frac{q}{4\pi\varepsilon_0}\left[\nabla(?) - \nabla(?)\right]$$

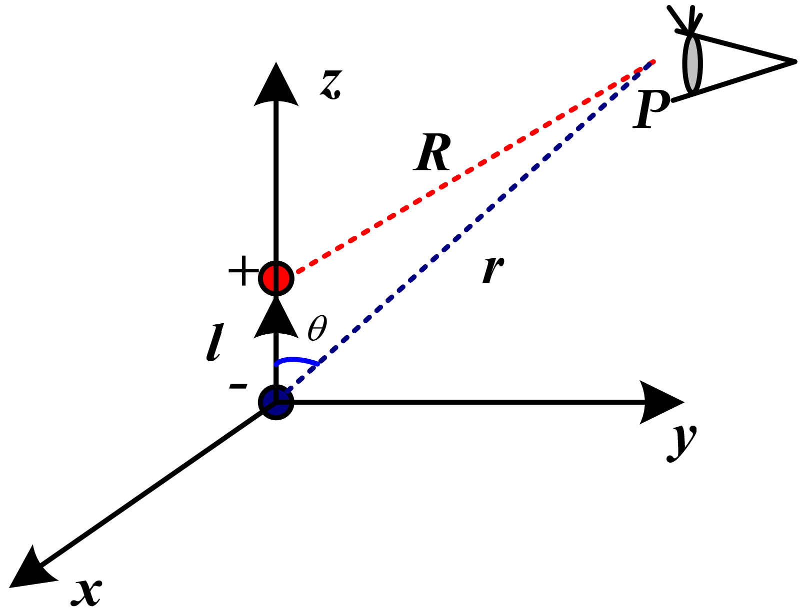
$$\vec{E} = -\frac{q}{4\pi\varepsilon_0}\left[\nabla\left(\frac{1}{r}+?\right)-\nabla\left(\frac{1}{r}\right)\right] = -\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{l\cdot\cos\theta}{r^2}\right)$$

Dipole Moment Vector（电偶极距）

$$\vec{E} = -\frac{q}{4\pi\varepsilon_0}\left[\nabla\left(\frac{1}{r} + ?\right) - \nabla\left(\frac{1}{r}\right)\right] = -\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{l\cdot\cos\theta}{r^2}\right)$$

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{(ql)\cdot r\cdot\cos\theta}{r^3}\right)$$

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\frac{(q\vec{l}\,)\bullet\vec{r}}{r^3}\right)$$



Let $\boxed{\vec{p} = q\vec{l}}$ The quantity? The direction ?

Dipole Moment Vector

$$\vec{p} = q\,\vec{l}$$

Unit: $C \cdot m$

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\nabla\left(\left(\vec{p}\bullet\vec{r}\right)\cdot\frac{1}{r^3}\right)$$

$$\nabla(u \cdot v) = ?$$

$$\boxed{\nabla(u \cdot v) = u\nabla(v) + v\nabla(u) \qquad Vector}$$

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\nabla\left((\vec{p}\bullet\vec{r})\cdot\frac{1}{r^3}\right) = ?$$

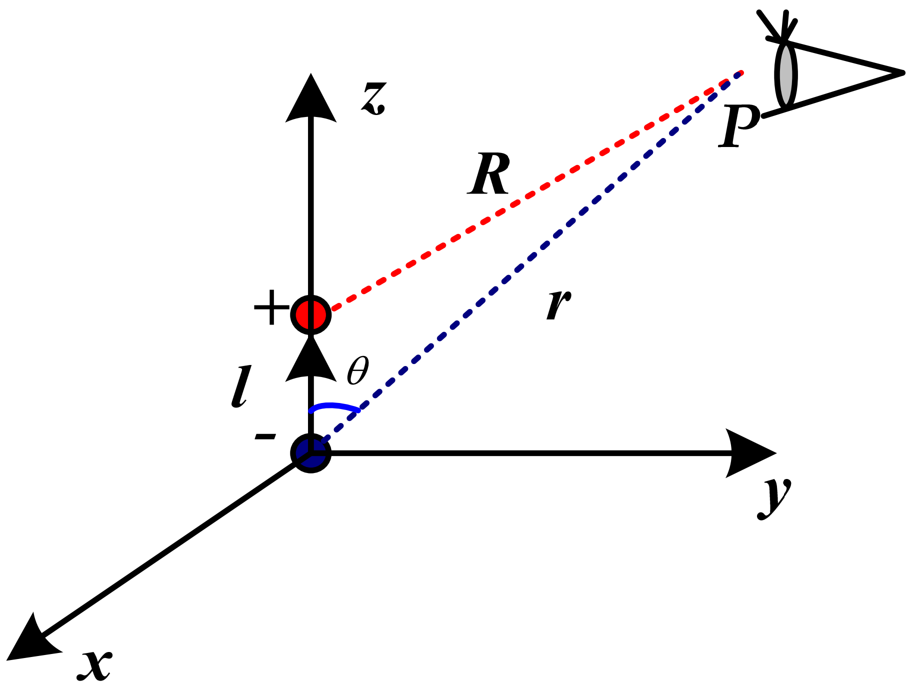
$$\vec{E} = ?\quad \nabla\left((\vec{p}\bullet\vec{r})\cdot\frac{1}{r^3}\right) = ?\left[(\vec{p}\bullet\vec{r})\nabla(\frac{1}{r^3}) + \frac{1}{r^3}\nabla(\vec{p}\bullet\vec{r})\right]$$

$$= ?\left[(\frac{-3\cdot(\vec{p}\bullet\vec{r})}{r^5})\vec{r} + \frac{1}{r^3}\vec{p}\right]$$

$$\boxed{\nabla(u\cdot v) = u\nabla(v) + v\nabla(u) \qquad Vector}$$

$$\vec{E}(\vec{p},\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\cdot(\vec{p}\bullet\vec{r})}{r^5}\vec{r} - \frac{1}{r^3}\vec{p}\right]$$

$$\vec{p} = q\vec{l}$$



$$\left|\vec{E}\right| \propto \frac{1}{r^3}$$

E-Flux Lines of Electric Dipole

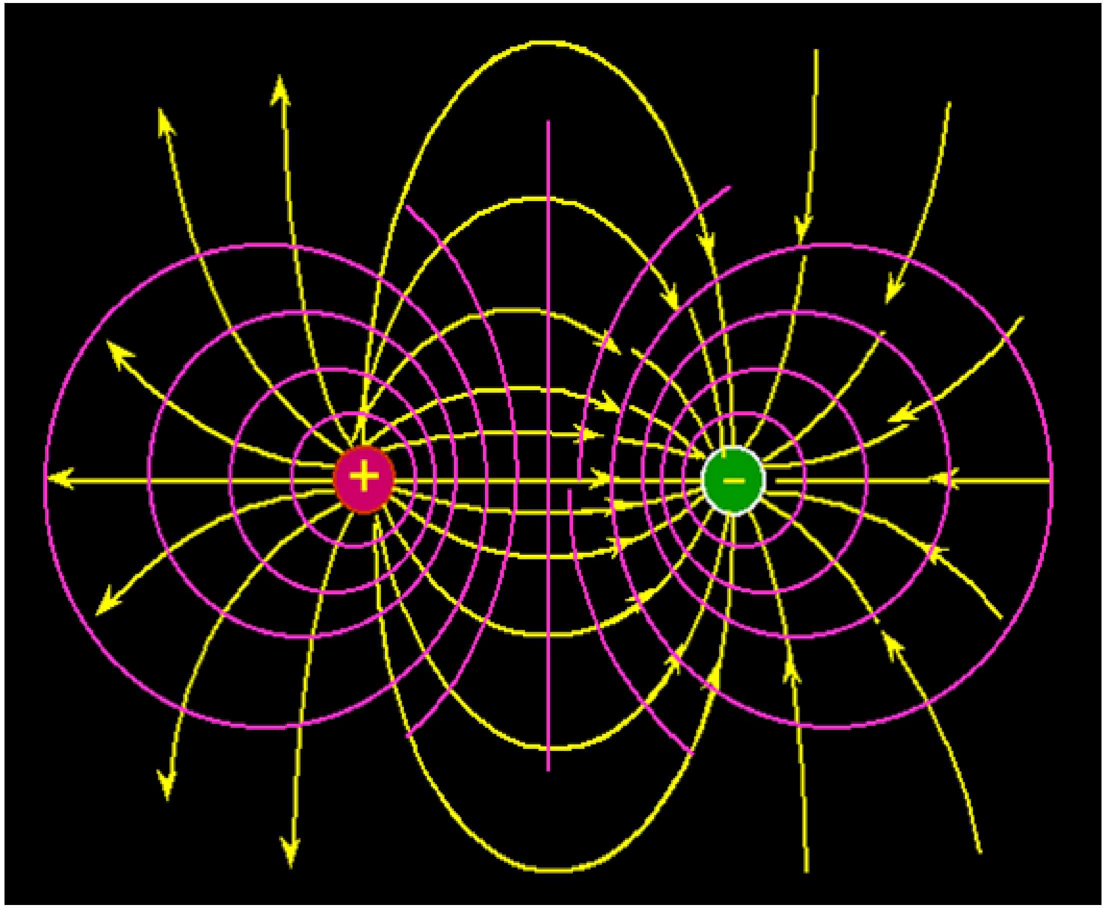
Equ. of E-Flux Lines: The lines shall be parallel to E-Field.

(1) $d\vec{l} \times \vec{E}(\vec{p},\vec{r}) = 0$ (2) $k d\vec{l} = \vec{E}(\vec{p},\vec{r})$

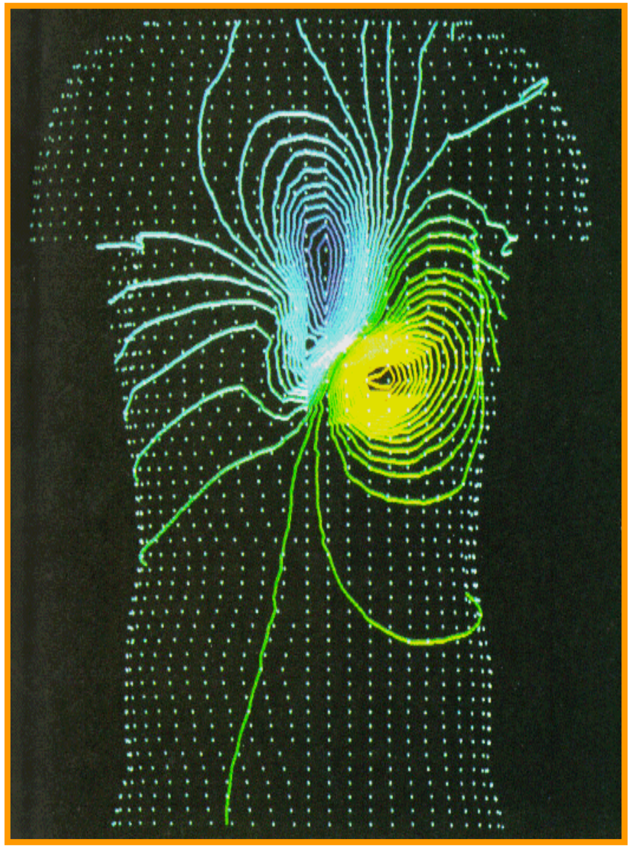
In spherical coordinates $\vec{E}(\vec{p},\vec{r}) = E_r \vec{a}_r + E_\theta \vec{a}_\theta + 0\vec{a}_\varphi$

$$d\vec{l} \times \vec{E}(\vec{p},\vec{r}) = ? + ? + ? = 0$$

Equ. of E-Field Lines $r = C \cdot \sin^2\theta$ The figure ?



电偶极子的电场线和等势面



作心电图时人体的等势面分布

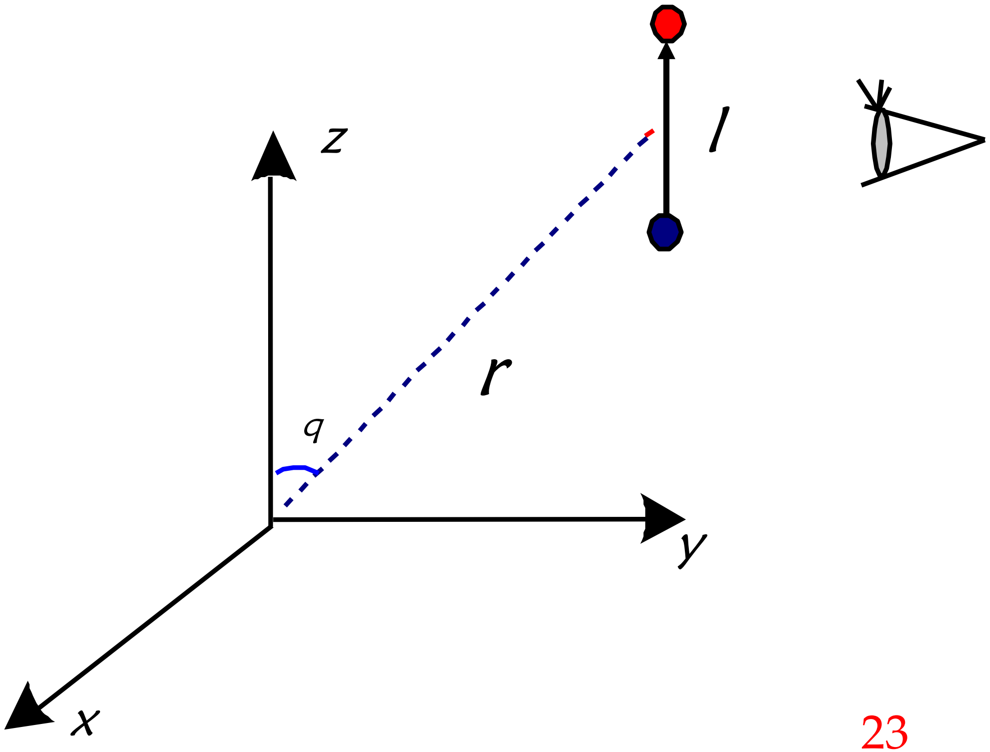
Force of -Dipole

$$\vec{f}_{\vec{p}}(\vec{r}) = q\vec{E}(\vec{r} + \frac{\vec{l}}{2}) + (-q)\vec{E}(\vec{r} - \frac{\vec{l}}{2})$$

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0}\nabla\left((\vec{p}\bullet\vec{r})\cdot\frac{1}{r^3}\right) = ?$$

In static E-field

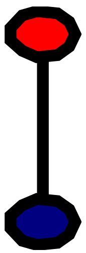
$$\vec{f}_{\vec{p}}(\vec{r}) = \nabla\left[\vec{p}\bullet\vec{E}(\vec{r})\right]$$



Moment of Force of E-Dipole

$$\vec{T}_{\vec{p}}(\vec{r}) = \frac{\vec{l}}{2} \times \left[q\vec{E}(\vec{r} + \tfrac{\vec{l}}{2}) \right] - \frac{\vec{l}}{2} \times \left[(-q)\vec{E}(\vec{r} - \tfrac{\vec{l}}{2}) \right]$$

$$= \frac{q\vec{l}}{2} \times \left[\underline{\vec{E}(\vec{r} + \tfrac{\vec{l}}{2}) + \vec{E}(\vec{r} - \tfrac{\vec{l}}{2})} \right] = \vec{p} \times \vec{E}(\vec{r})$$



- Distance: *l*
- Point Charges: $q_1 = q$, $q_2 = -q$