Chapter 4. Steady Electric Current

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Contents

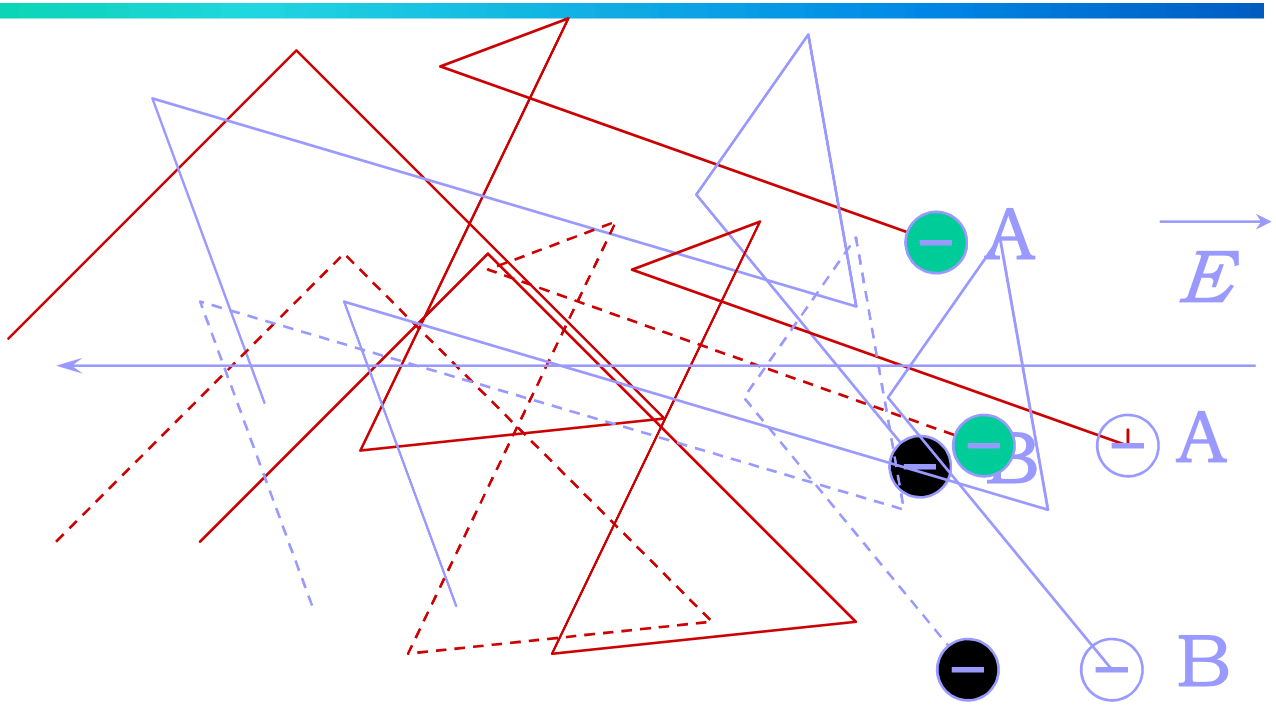
- Current Density
- Fundamental Equations
  - Current Continuity Equation (Div. Equation)
  - Conservative Equation (Curl Equation)
  - Intrinsic Equation (Ohm's Law)
- Boundary Conditions
- Analogy between *Static* & *Steady-Current* E-Field
- Examples on Resistance

Conceptions of steady electric current

- Conceptions of steady electric current:

1) **Generates** by power source;
2) ***Exists*** *in steady electric current space;*
3) **Current density is not zero, but charge density does not change with time.**

- Categories of Steady Currents
    - *Conduction current:* steady motion of charges in **conductors**, the object of this chapter
    - *Convection current(运流电流)*: steady motion of charges **in free space**, neither requiring a conductor nor obey Ohm's law



$10^{-4}\,m\cdot s^{-1}$ $10^{5}\,m\cdot s^{-1}$ $10^{8}\,m\cdot s^{-1}$

Drift velocity

Random thermal motion

Conduction current velocity

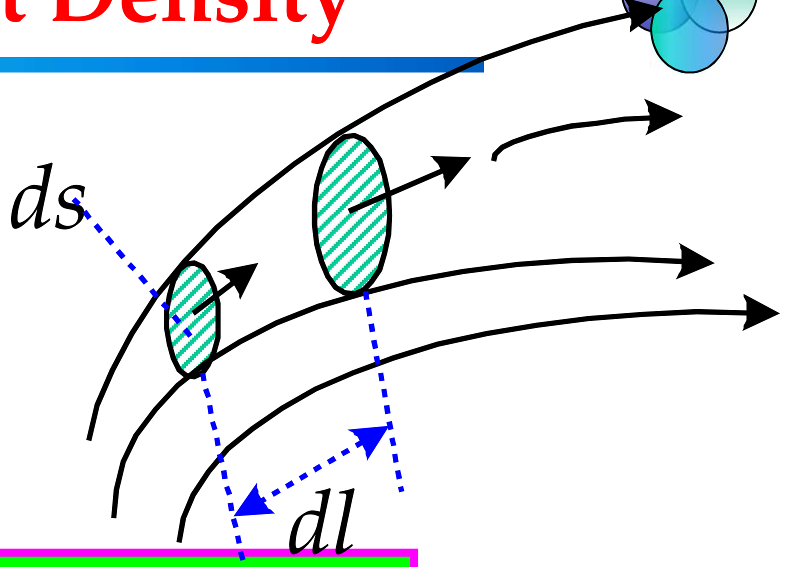
Features of steady current E-field

- **In static E-field**
  - E-intensity in conductor is 0, the conductor is an equipotential substance and its surface is an equipotential surface;
- **In steady-current E-field (SC E-field)**
  - E-intensity in the conductor is not 0, the conductor is not an equipotential substance and its surface is not an equipotential surface;
  - While the distribution of charges keeps unchanged.

1. Conduction Current Density

Current intensity

$$i = \lim_{\Delta t \to 0}\left(\frac{\Delta q}{\Delta t}\right) = \frac{dq}{dt}$$



Current density-

Volume Density:

current per unit area

$$\boxed{J = \lim_{\Delta S_{\perp} \to 0}\left(\frac{\Delta I}{\Delta S_{\perp}}\right) \quad \left(\frac{A}{m^2}\right)}$$

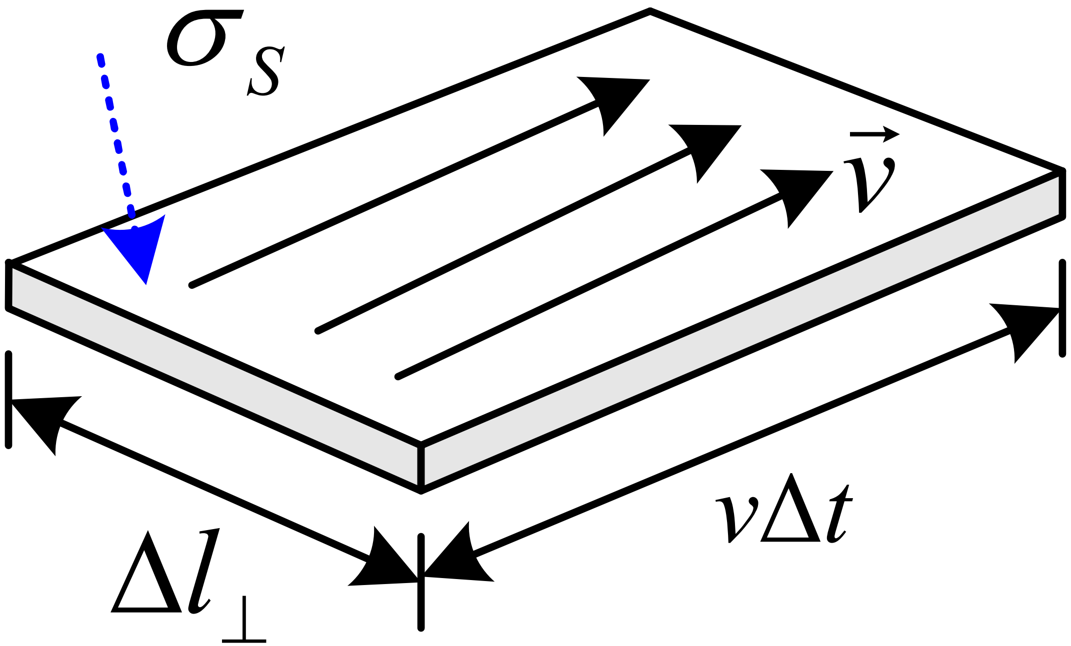
$$\boxed{\vec{J}_V = \rho\vec{v}}$$

1. Conduction Current Density

Surface Density

$$J_S = \lim_{\Delta l_\perp \to 0} \frac{\Delta I}{\Delta l_\perp} \qquad (A/m)$$

$$\vec{J}_S = \sigma_S \vec{v}$$



Fundamental Equations

(1) Continuity of Current

The principle of conservation of charge

- Charge can never be generated or destroyed.
- Charge ***flowing out*** of a closed surface in per unit time must equal the ***decreasing*** rate of the charge in that closed surface.

$$\oint_S \vec{J} \bullet d\vec{S} = -\frac{\partial q}{\partial t}$$

$$\oint_S \vec{J} \bullet d\vec{S} = \int_V \nabla \bullet \vec{J} \cdot dV$$

$$-\frac{\partial q}{\partial t} = -\frac{\partial}{\partial t}\int_V \rho \cdot dV$$

$$\nabla \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

True for any type of current

$$\boxed{\oint_S \vec{J} \bullet d\vec{S} = -\frac{\partial q}{\partial t}}$$

$$\boxed{\nabla \bullet \vec{J} = -\frac{\partial \rho}{\partial t}}$$

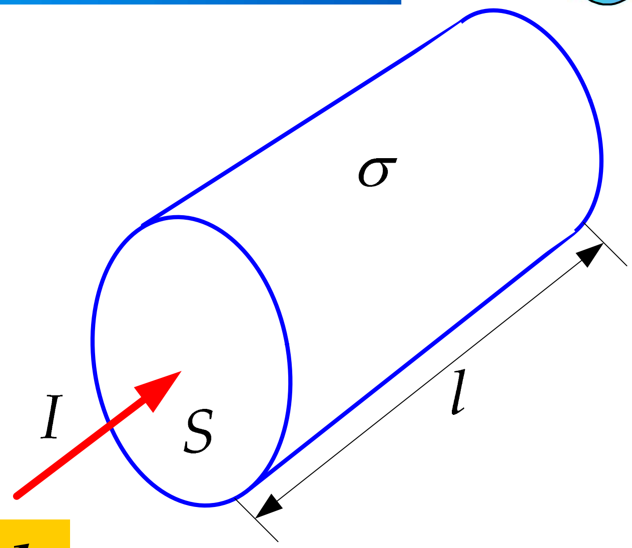
For steady current $\boxed{\frac{\partial \rho}{\partial t} \equiv 0}$

So we have ——

$$\boxed{\oint_S \vec{J} \bullet d\vec{S} = 0} \longleftarrow \boxed{\nabla \bullet \vec{J} = 0}$$

Ohm's Law (in microscopic view)

- Differential form: $\vec{J} = \sigma\vec{E}$
- Assume the current across a section of conductor is *I*, and the voltage is *U*. We discover the relationship between *U* & *I* from a microscopic point of view.



$$I = \int_S \vec{J} \bullet d\vec{S} = JS$$

$$U = \int_l \vec{E} \bullet d\vec{l} = El$$

$$\boxed{R = \frac{U}{I} = \frac{El}{JS}}$$

$$\boxed{R = \frac{l}{\sigma S}}$$

$$\boxed{\vec{J} = \sigma\vec{E}}$$

Conductivity (examples)

“电导” — 导电....电阻 — “阻电”

Conductors: good ones $(\sigma > 10^7)$ ideals $(\sigma \to \infty)$

Silver: $\sigma = 6.17 \times 10^7 (S/m)$

Copper: $\sigma = 5.8 \times 10^7 (S/m)$

Dielectrics: common ones $(\sigma > 0)$ ideals $(\sigma \to 0)$

Seawater: $\sigma = 5 \quad (S/m)$

Rubber: $\sigma = 1 \times 10^{-15} \quad (S/m)$

(2) SC E-field stimulated by the steady current

- **Divergence:**

$$\nabla \bullet \vec{J} = 0$$

$$\vec{J} = \sigma\vec{E}$$

$$\nabla \bullet (\sigma\vec{E}) = \sigma\nabla \bullet \vec{E} + \vec{E} \bullet \nabla\sigma = 0$$

$$\nabla \bullet \vec{E} = -\vec{E} \bullet \frac{\nabla\sigma}{\sigma}$$

For homogeneous media $\nabla\sigma = 0$

$$\boxed{\nabla \bullet \vec{E} = 0}$$

Curl:

- **The SC E-field is conservative field.**
  - This field is derived by external power so as to maintain a steady current.
  - Work by this field is similar to that by static E-field, since both are conservative fields.

$$\oint_C \vec{E} \bullet d\vec{l} = 0$$

$$\nabla \times \vec{E} = 0$$

- **Power Source:**
  - An Power-source is an equipment to transform external power into E-power and thus maintain the current.
  - Power-source acts upon the charges via a force from external power, and this force is called electromotive force (e.m.f)
  - Joule's Law & Kirchhoffs Laws are thus derived.

For detail, refer to the book by 毕德显.

SC E-field just on the conductor surface

- The tangential static E-field on a conductor surface is always 0, but it is not true for the tangential SC E-field, since it has to exist to stimulate a current.
- However, the tangential SC E-field on conductor surface is rather tiny, the reason is that:
  - In good conductor, σ>>1 and $J=\sigma E$, therefore a tiny tangential E-field may stimulate a huge current.
  - For safety, the current shall not be too large and thus the tangential E-field on conductor surface is usually very tiny.
  - Usually the lines of SC E-flux is approximately normal to the conductor surface.
- As to the E-potential around a steady current, it satisfy the Laplace's Equation.

2. Fundamental Equations---conclusions

	Integral Form	Differential Form
Continuity of current	$\oint_S \vec{J} \bullet d\vec{S} = 0$	$\nabla \bullet \vec{J} = 0$
Conservation of SC E-field	$\oint_C \vec{E} \bullet d\vec{l} = 0$	$\nabla \times \vec{E} = 0$

Material Equation for Conductors $\vec{J} = \sigma \vec{E}$ Ohm's Law

σ is *Conductivity* (电导率), unit: A/(v.m) or S/m

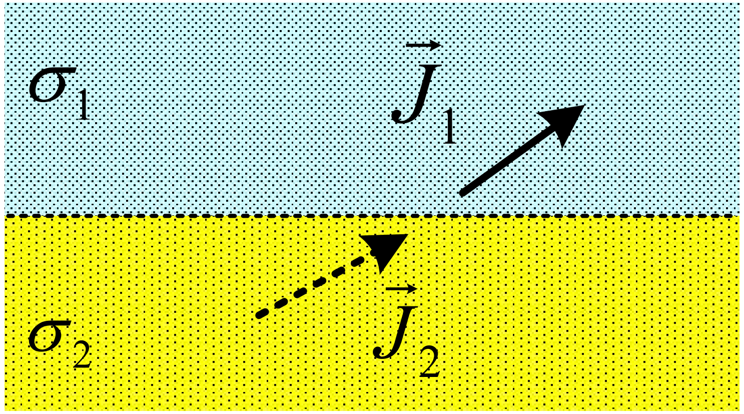
Note that: γ is *Resistivity* (电阻率) $\gamma = 1/\sigma$ $\Omega \cdot m$

Summary of Fundamental Equs.

		Integral form		Diff. form
Static E-field	Div. equs.	$\oint \vec{D} \bullet d\vec{S} = Q$		$\nabla \bullet \vec{D} = \rho$
	Curl equs.	$\oint \vec{E} \bullet d\vec{l} = 0$		$\nabla \times \vec{E} = 0$
	Material equs.		$\vec{D} = \varepsilon \vec{E}$	
SC E-field	Div. equs.	$\oint_S \vec{J} \bullet d\vec{S} = 0$		$\nabla \bullet \vec{J} = 0$
	Curl equs.	$\oint_C \vec{E} \bullet d\vec{l} = 0$		$\nabla \times \vec{E} = 0$
	Material equs.		$\vec{J} = \sigma \vec{E}$	
Static M-field	Div. equs.	$\oint_s \vec{B} \cdot d\vec{S} = 0$		$\nabla \bullet \vec{B} = 0$
	Curl equs.	$\oint_C \vec{H} \bullet d\vec{l} = I$		$\nabla \times \vec{H} = \vec{J}$
	Material equs.		$\vec{B} = \mu \vec{H}$	

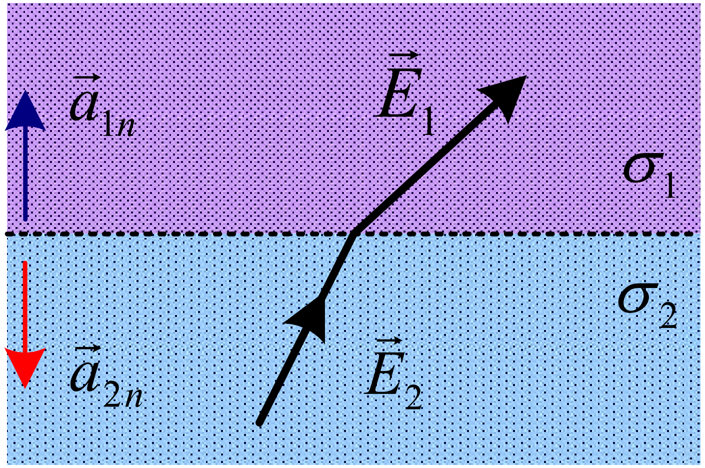
3. Boundary Conditions for SC E-field

- In normal direction
  - Construct a flat box
  - Apply Gauss's Law $\oint_S \vec{J} \bullet d\vec{S} = 0$



$$\boxed{J_{1n} = J_{2n}} \qquad \boxed{\sigma_1 E_{1n} = \sigma_2 E_{2n}}$$

- In tangential direction
  - Construct a rectangular loop
  - Apply conservation $\oint_C \vec{E} \bullet d\vec{l} = 0$

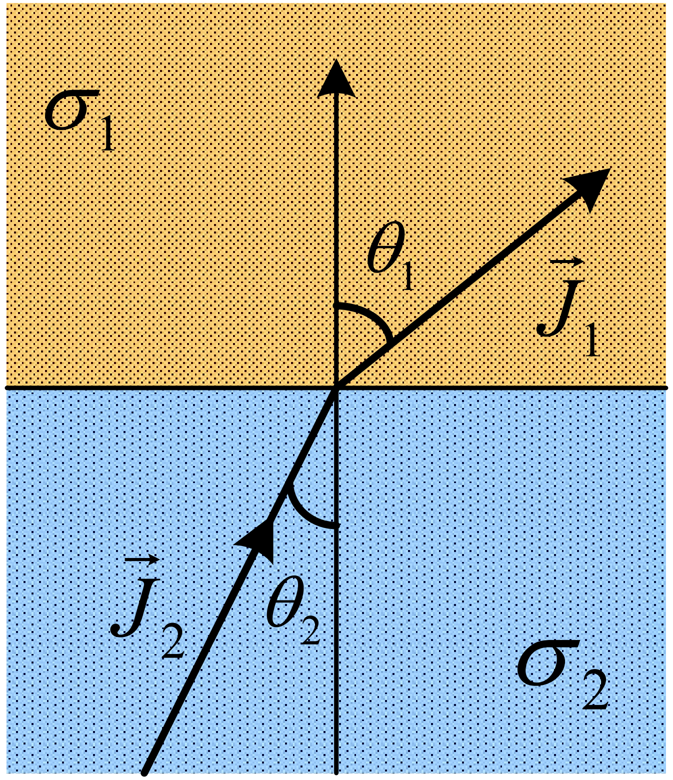


$$\boxed{E_{1t} = E_{2t}} \qquad \boxed{J_{1t} / \sigma_1 = J_{2t} / \sigma_2}$$

Application of Boundary Conditions

$$J_{1n} = J_{2n} \qquad J_{1t} / \sigma_1 = J_{2t} / \sigma_2$$

$$\therefore \begin{cases} J_1 \cos\theta_1 = J_2 \cdot \cos\theta_2 \\ \sigma_2 J_1 \sin\theta_1 = \sigma_1 J_2 \sin\theta_2 \end{cases}$$



Thus $$\frac{tg\theta_1}{tg\theta_2} = \frac{\sigma_1}{\sigma_2}$$

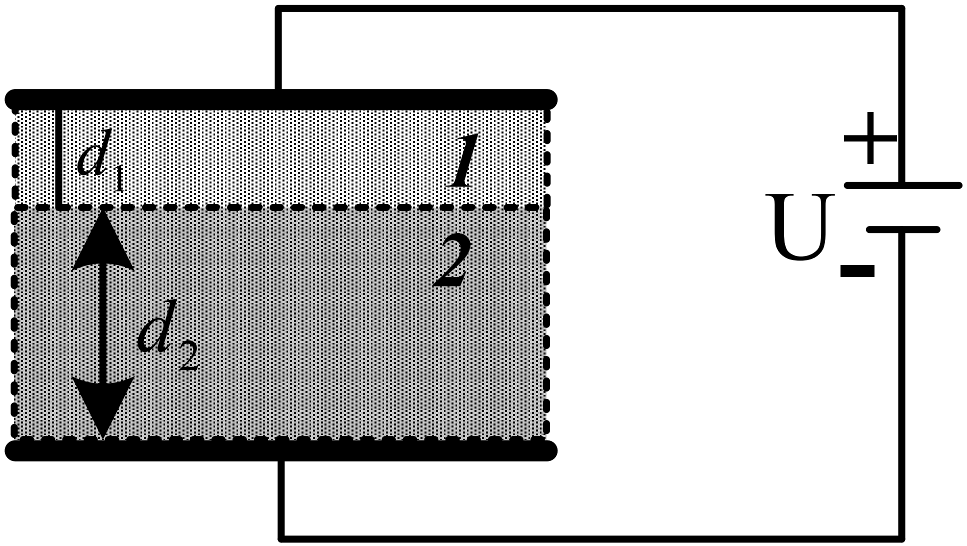
By comparison

For static E-field $$\frac{tg\theta_1}{tg\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

For Static M-field $$\frac{tg\theta_1}{tg\theta_2} = \frac{\mu_1}{\mu_2}$$

Example 2. 已知非理想介质1、2，电压U

Parallel-plate capacitor,
Nonideal dielectrics 1 & 2,
Conductivities σ_1 & σ_2,
Dielectric const. ε_1 & ε_2,
Please determine the charge density on boundary of 1&2.



Analysis:

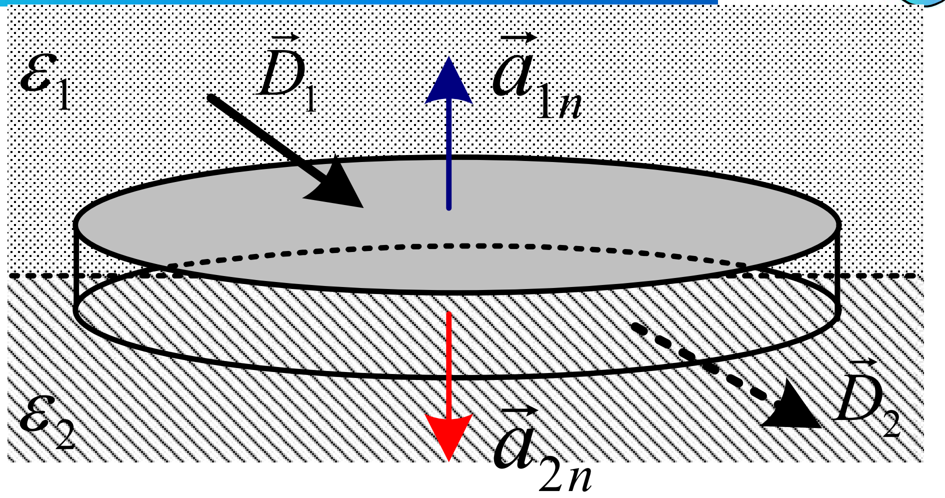
- Nonideal dielectrics → leaky current across the capacitor
- Does there exist free charge on the boundary?
- What's the continuous parameter?

$$D_{1n} ? D_{2n} \qquad E_{1n} ? E_{2n} \qquad \cdots \qquad \boxed{J_{1n} = J_{2n}}$$

For SC E-field we also have the following boundary condition in normal direction.

- Construct a flat box
- Apply Gauss's Law

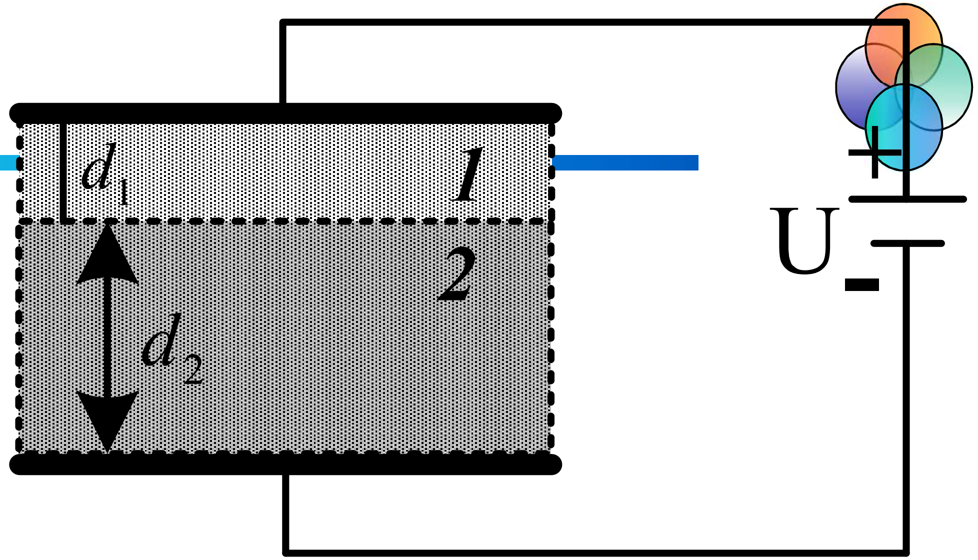
$$\oint_S \vec{D} \bullet d\vec{S} = \rho_s \cdot \Delta S$$



$$\boxed{\therefore D_{2n} - D_{1n} = \rho_s \qquad (c/m^2)}$$

$$J_{1n} = J_{2n}$$

$$\sigma_1 E_1 = \sigma_2 E_2$$



$$U = d_1 E_1 + d_2 E_2$$

$$E_1 = \frac{\sigma_2 U_0}{\sigma_2 d_1 + \sigma_1 d_2}$$

$$E_2 = \frac{\sigma_1 U_0}{\sigma_1 d_2 + \sigma_2 d_1}$$

$$\rho_s = \left(\frac{\sigma_1 \varepsilon_2 - \sigma_2 \varepsilon_1}{\sigma_2 d_1 + \sigma_1 d_2} \right) \cdot U$$

There do exist free charges on the boundary.

A Summary of Boundary Conditions

	normal	tangential
Static E-field	$\boxed{D_{1n} - D_{2n} = \sigma_{fc}}$ $\varepsilon_1 \frac{\partial \psi_1}{\partial n} = \varepsilon_2 \frac{\partial \psi_2}{\partial n} \quad (\text{if } \sigma_S = 0)$	$\boxed{E_{1t} = E_{2t}}$ $\psi_1 = \psi_2$
SC E-field	$\boxed{J_{1n} = J_{2n}}$ $\sigma_1 E_{1n} = \sigma_2 E_{2n}$ $\sigma_1 \frac{\partial \psi_1}{\partial n} = \sigma_2 \frac{\partial \psi_2}{\partial n}$	$\boxed{E_{1t} = E_{2t}}$ $J_{1t} / \sigma_1 = J_{2t} / \sigma_2$ $\psi_1 = \psi_2$

再次小结

在边界上

- 与保守性有关，则切向连续。

静电场介质边界 $E_{1t} = E_{2t}$

恒定电场导体边界 $E_{1t} = E_{2t}$

- 与连续性有关，则法向连续。

恒定电场导体边界 $J_{1n} = J_{2n}$

静磁场介质边界 $B_{1n} = B_{2n}$

4. Analogy between Two Steady E-Fields

- Fundamental equs. of SC E-field in conductor is the **same in form** as those of static E-field in source-free region. Moreover, boundary conditions for both cases are the same. Thus both E-field are analogous to each other.
- 好处？可以从一种情况的解导出另一种情况的解。

$$\boxed{\begin{cases} \nabla\times\vec{E}=0 \\ \nabla\cdot\vec{J}=0 \\ \vec{J}=\sigma\vec{E} \end{cases}} \xleftrightarrow{\text{mapping}} \boxed{\begin{cases} \nabla\times\vec{E}=0 \\ \nabla\cdot\vec{D}=0 \\ \vec{D}=\varepsilon\vec{E} \end{cases}}$$

SC E-field　　　　static E-field

Corresponding Pairs

$$\vec{E} \leftrightarrow \vec{E} \quad \vec{J} \leftrightarrow \vec{D} \quad \sigma \leftrightarrow \varepsilon$$

$$I \leftrightarrow q \quad \varphi \leftrightarrow \varphi$$

$$\begin{cases} \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{J} = 0 \\ \vec{J} = \sigma \vec{E} \end{cases}$$

SC E-field

mapping

$$\begin{cases} \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{D} = 0 \\ \vec{D} = \varepsilon \vec{E} \end{cases}$$

static E-field

C ←→ G

$$R=\frac{U}{I}=\frac{\int_l E\bullet d\vec{l}}{\int_S \vec{J}\bullet d\vec{S}}=\frac{\int_l E\bullet d\vec{l}}{\int_S \sigma\vec{E}\bullet d\vec{S}}$$

$$G=\frac{I}{U}=\frac{\int_s \vec{J}\cdot d\vec{s}}{\int_l \vec{E}\cdot d\vec{l}}=\frac{\sigma\int_s \vec{E}\cdot d\vec{s}}{\int_l \vec{E}\cdot d\vec{l}}$$

$$C=\frac{q}{U}=\frac{\oint_S \vec{D}\cdot d\vec{s}}{\int_l \vec{E}\cdot d\vec{l}}=\frac{\varepsilon\oint_S \vec{E}\cdot d\vec{s}}{\int_l \vec{E}\cdot d\vec{l}}$$

- 如果电极的电导率比周围媒质的电导率大的多，则电极表面近似为等位面，如果电极的形状也相同，则两电极之间的电导与电容、电阻与电容存在下列关系

$$G-\sigma---C-\varepsilon$$

$$R\cdot C=\frac{\varepsilon}{\sigma}$$

5. Examples: on calculating of resistors

Example 1. Copper wire in length of 1km, diameter of 2mm. Please calculate R.

Solution: via its definition

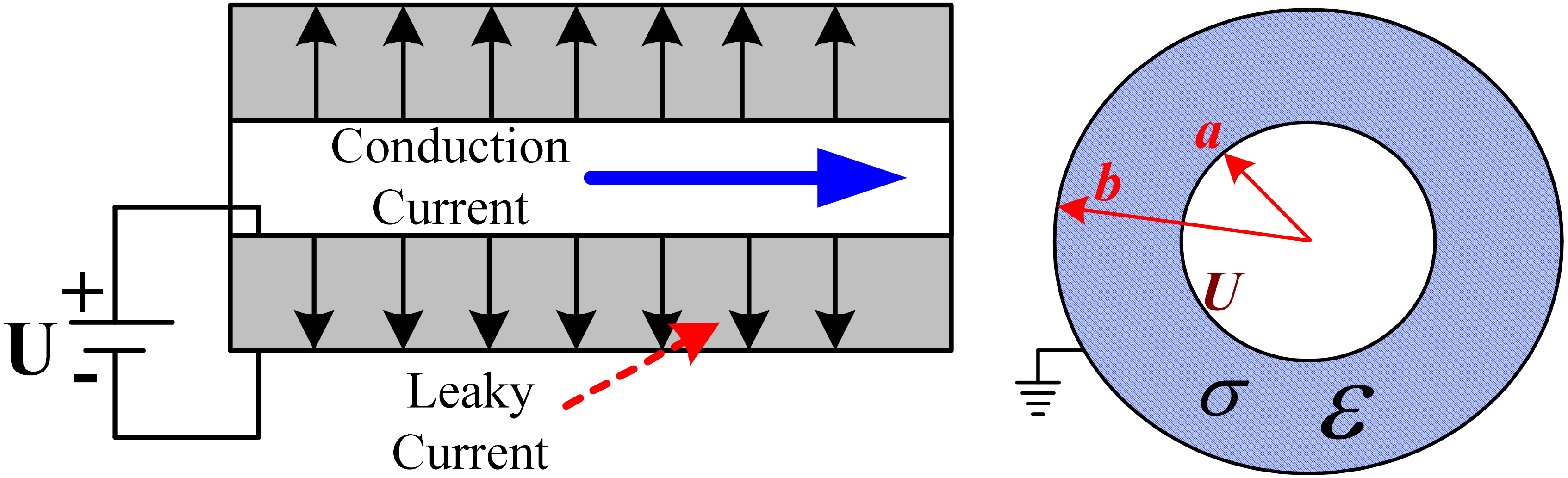
$$\boxed{R = \frac{l}{\sigma S}}$$

For copper $\sigma = 5.8 \times 10^{7} (S/m)$

$$R = 5.49(\Omega)$$

Example 2. nonideal diecletric

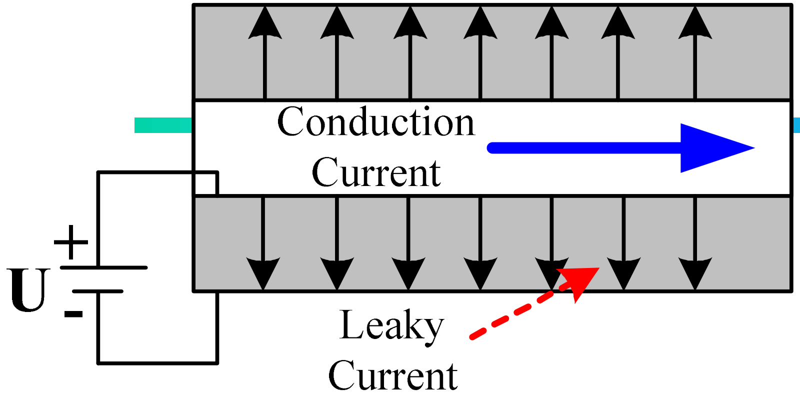
Coaxial line as in the figure, filled with nonideal dielectrics ε and $\sigma \neq 0$. Please determine leaky conductance @ unit length.

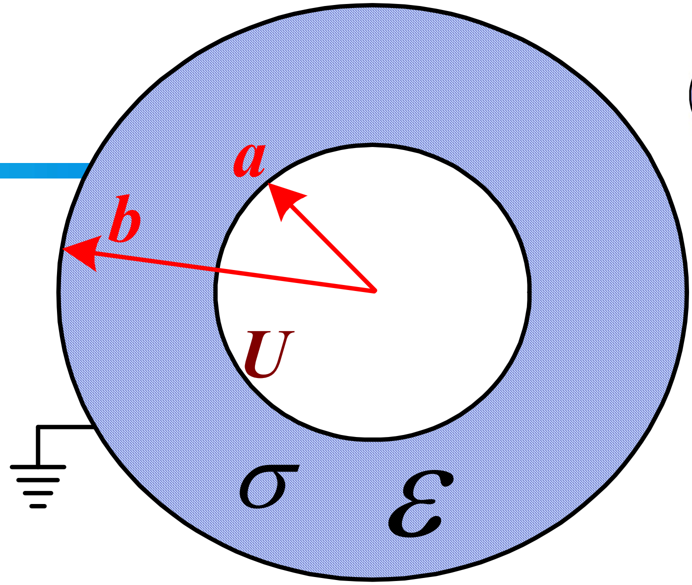


Analysis: leaky G can be obtained via leaky R

A hint: leaky current I → E → U

$$\frac{1}{G} = R = \frac{"U"}{"I"}$$

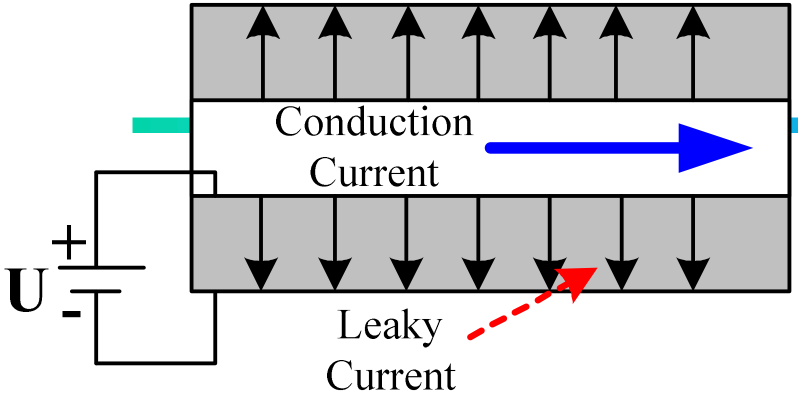


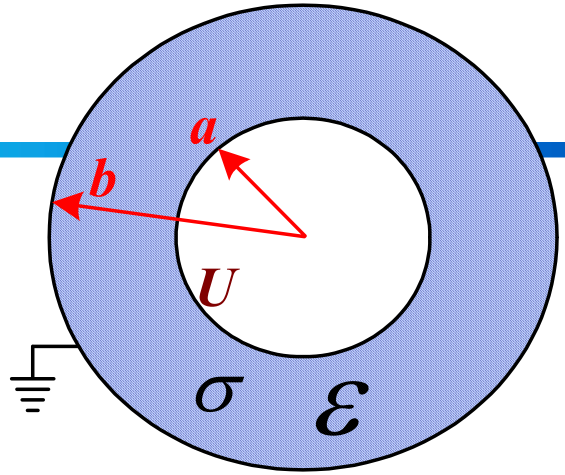


$$J = I/(2\pi\, r)$$

$$U = \int_a^b \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{r} = \int_a^b \frac{J\mathrm{d}r}{\sigma} = \frac{I}{2\pi\,\sigma}\ln\left(\frac{b}{a}\right) = \frac{Jr}{\sigma}\ln\left(\frac{b}{a}\right)$$

$$J = \frac{U\sigma}{r\ln\left(\frac{b}{a}\right)} \qquad \frac{1}{G} = R = \frac{"U"}{"I"} \qquad R = \frac{\ln\frac{b}{a}}{2\pi\,\sigma}$$





Solution 2. via analogy

We have obtained C @ unit length as $C = \dfrac{Q}{U} = \dfrac{2\pi \cdot \varepsilon}{\ln(\frac{b}{a})}$

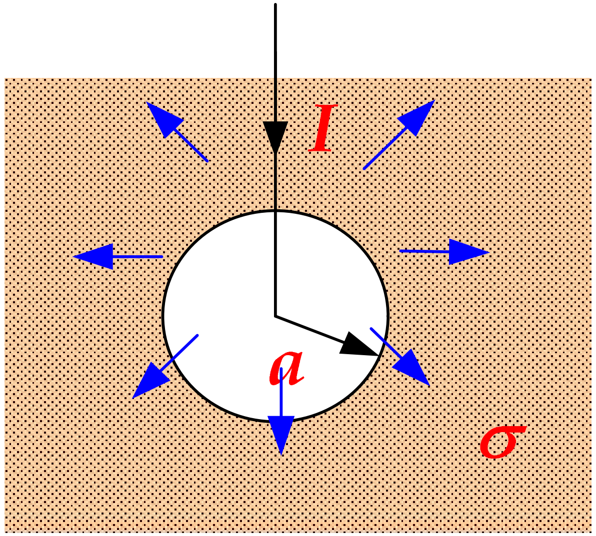
$$\vec{E} \leftrightarrow \vec{E} \quad \vec{J} \leftrightarrow \vec{D} \quad \sigma \leftrightarrow \varepsilon$$

$$I \leftrightarrow q \quad \varphi \leftrightarrow \varphi$$

G=?

$$G = \frac{2\pi\sigma}{\ln\left(\frac{b}{a}\right)}$$

Example 3. 接地电阻



- 深埋地下的球形电极，求接地电阻
- 电流到达金属球之后，以地为电阻作球面扩散，直到无穷远.
- “深埋”暗指不计地表影响.
- 未知金属电导率，暗示不必考虑导体的电阻.
- 方法一：

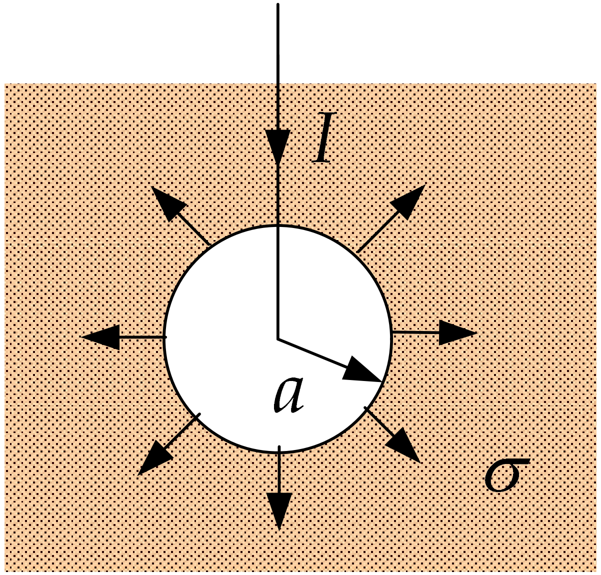
$$\text{假设} I \to J_r \to E_r \to U \to \frac{U}{I} = R$$

- 方法二：

$$\text{假设球表} Q \to E_r \begin{matrix} \to U \\ \to J_r \to I \end{matrix} \to \frac{U}{I} = R$$

- 深埋地下的球形电极，求接地电阻
- 方法三：根据R定义

$$dR=\frac{dl}{\sigma S}=\frac{dr}{\sigma\cdot 4\pi r^2}\qquad r\in[0,\infty)$$

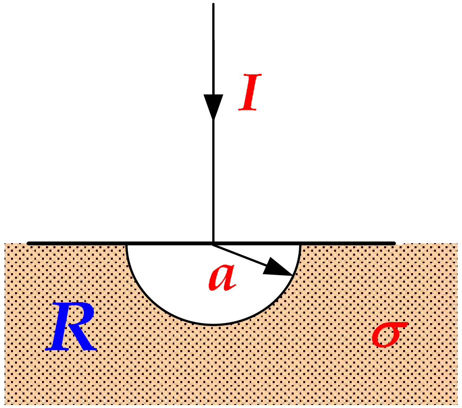


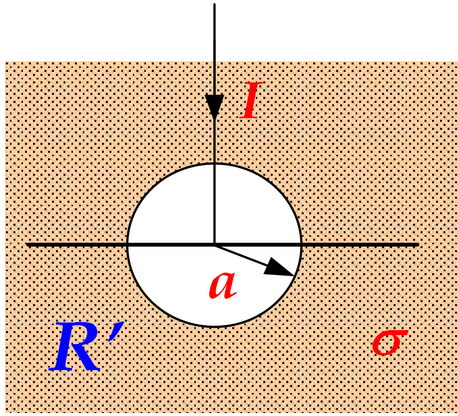
- 方法四：$G\rightarrow R$
- 方法五：
  - 同心球壳电容—— $C=\dfrac{4\pi\varepsilon_0}{\frac{1}{a}-\frac{1}{b}}$
  - b → ∞时的C = ?
  - 根据静电比拟

$$\boxed{R\cdot C=\frac{\varepsilon}{\sigma}}$$

$$R=\frac{\varepsilon_0}{\sigma}\cdot\frac{1}{C}=\frac{\varepsilon_0}{\sigma}\cdot\frac{1}{4\pi\varepsilon_0 a}$$

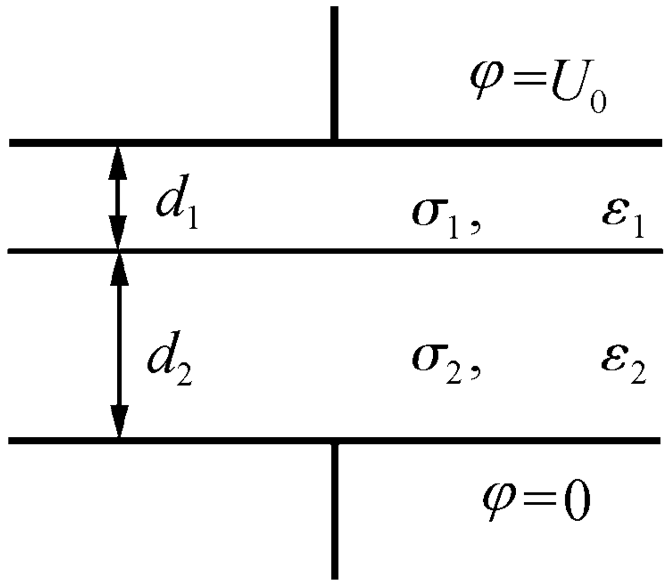
Example 4. 半球电极的接地电容





- 只提示一点——
  - 先求出金属球的接地电阻 R'
  - 再去考虑半球的接地电阻 R
- 那么R'相当于2个R的串联还是并联？？
- $R=2R'$

- 如下图所示,在平行板电容器的两极板之间,填充两导电介质片,若在电极之间外加电压U_0,求:(1)两种介质片中的电场强度和电流密度;(2)每种介质片上的电压;(3)上、下极板和介质分界面上的自由电荷面密度。



Homework

- E4.20
- P4.28
- P4.31