§ 2.10 Capacitors & Capacitance

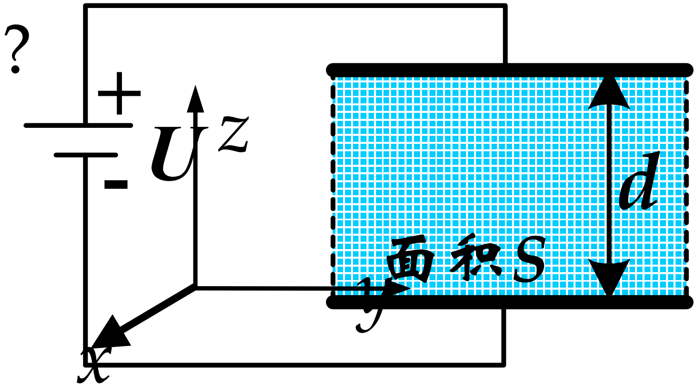
- Categories and definitions
  1. Generally, a capacitor consists of two isolated conductors, charged by q and $-q$, and with an E-potential of U. $C=q/U$
  2. **Self-capacitance**: an isolated conductor charged by q and with E-potential ψ, can be of self-C. $C= q/\psi$
  3. **Distributed Capacitance**: in fact, in a system of multi-conductors with a complex distribution of charges, there exists distributed capacitances between any 2 conductors.

Capacitors & Capacitance

- In fact, capacitance exists between 2 conductors of any shape adjacent to each other.
- The capacitance depends on its size and material, independent of whether it is charged or not.

Example 1. parallel-plate capacitor

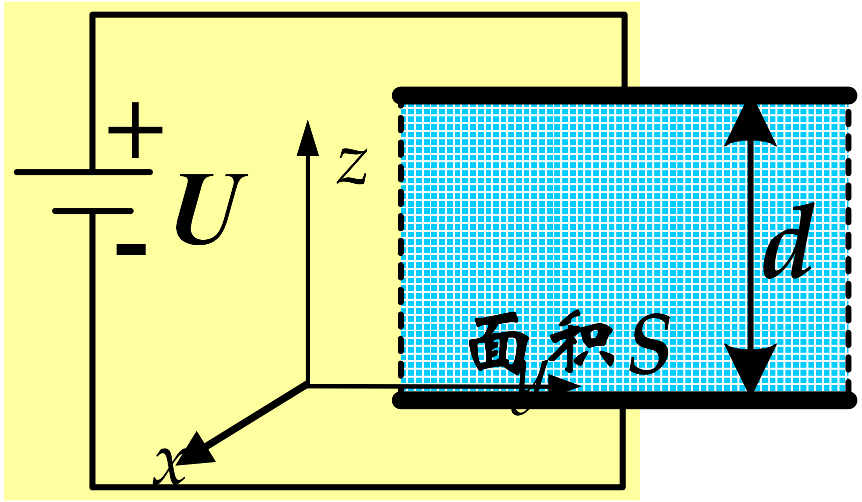
- Two parallel plates, each of area S, separated by a distance of d, with the dielectric of ε between them. Please determine the capacitance?



- The separation between the plates is very small compared to their other dimension, and thus we **neglect the edge effects** and assume E-field is uniformly distributed between 2 plates.
- Assume the E-potential difference U, and we set up the Laplace's Equ. since there is no charge between 2 plates.

$$\boxed{\nabla^2\psi = \nabla^2\psi(z) = \frac{d^2\psi}{dz^2} = 0}$$

$$\nabla^2\psi = \nabla^2\psi(z) = \frac{d^2\psi}{dz^2} = 0 \qquad \psi = C_1 z + C_2$$



Boundary conditions:

$$\psi\,\big|_{z=0} = 0 \qquad \psi\,\big|_{z=d} = U$$

Dirichlet Problems

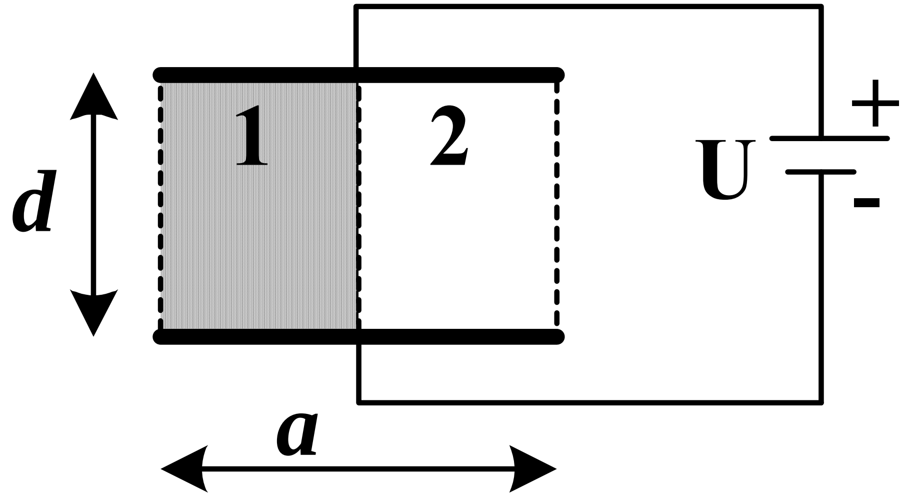
$$\psi(z) = \frac{z}{d}\cdot U \quad \Rightarrow \vec{E} = -\nabla\psi = (-U/d)\vec{e}_z \quad \Rightarrow Q = \sigma_s \bullet S = ?$$

$$\sigma_s = ?$$

$$\boxed{\therefore C = Q/U = \frac{S}{d}\cdot\varepsilon}$$

Example 2.

Parallel-plate capacitor,
area of each plate --- $a \times b$
E-potential difference --- U



Solution 1. parallel connection of capacitors, $C=C_1+C_2$

Solution 2. via the definition, C=Q/U

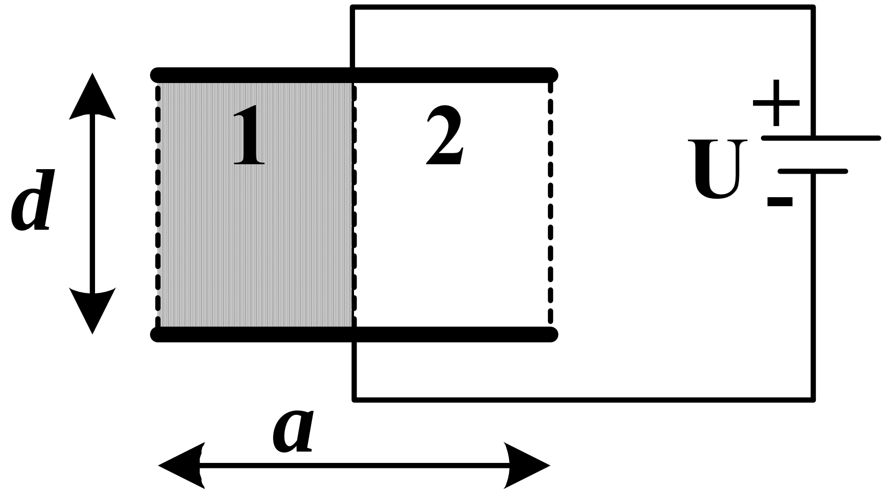
Solution 2. via the definition, C=Q/U

$C=Q/U \qquad Q=Q_1+Q_2$

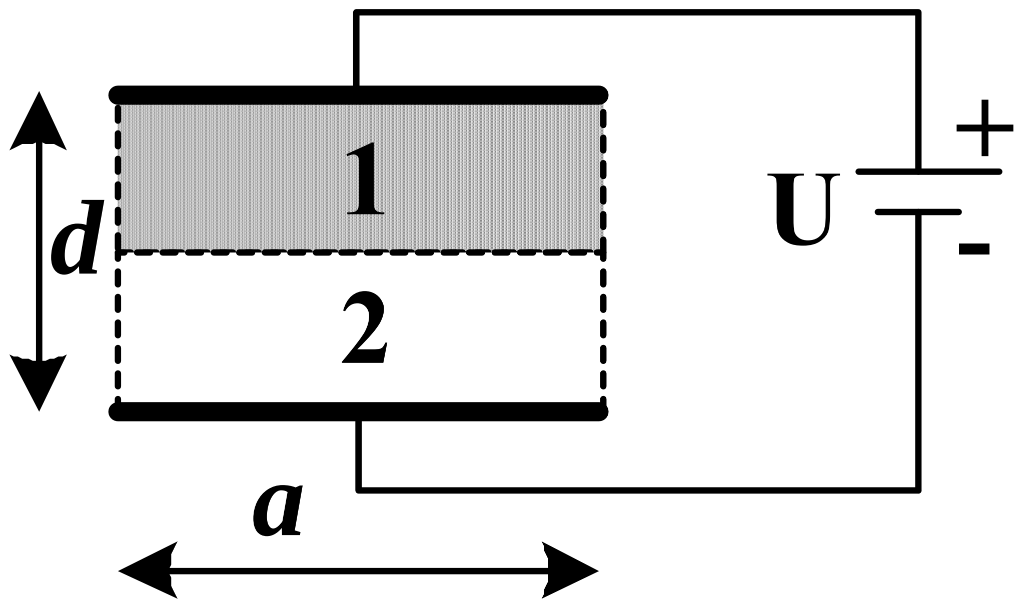
$$Q_1=\sigma_1\cdot\left(\tfrac{a}{2}\cdot b\right)$$

$$\sigma_1=D_{1n}=D_1=\varepsilon_1 E_1$$

$$\vec{E}=? \qquad \vec{E}_1=\vec{E}_2\,?$$



Homework



Please determine the capacitance.

- Expressions are more complicate in cylindrical and spherical coordinates.

In Cylindrical Coordinates

$$\nabla^2 u(r,\varphi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$$

In Spherical Coordinates

$$\nabla^2 u(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

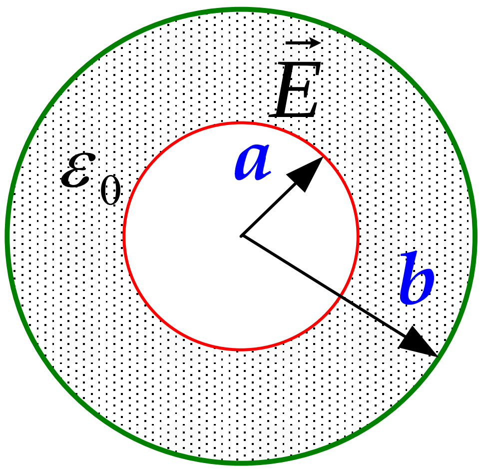
Example 3. spherical capacitor

- Formed by a metallic ball and a concentric metallic sphere.
- Please determine the capacitance.
- In general, the first step is to assume Q or U.

$$\vec{E} = ?$$

$$U = \int_{?}^{?} \vec{E} \cdot d\vec{r}$$

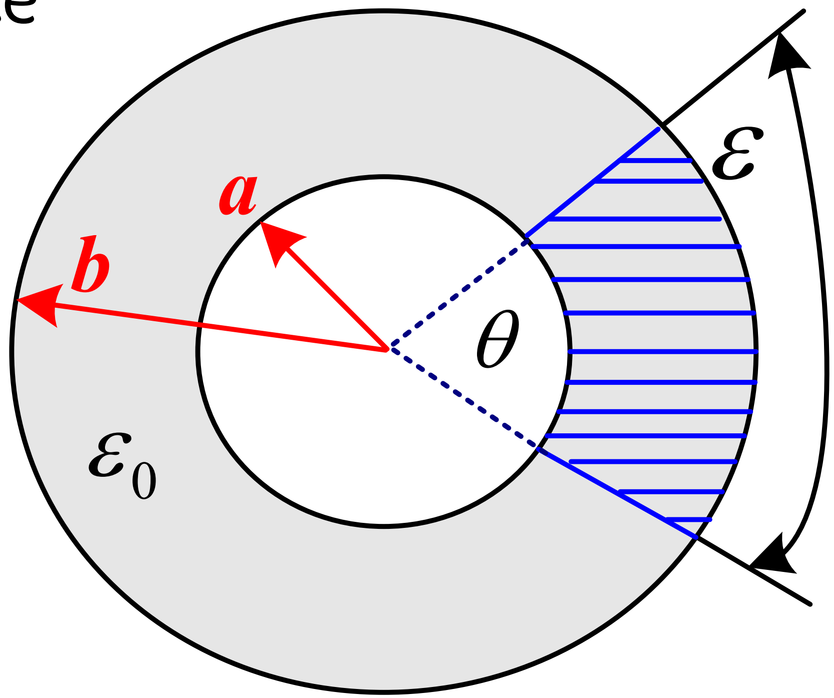
$$C = Q / U$$



Refer to Example 3.20 in textbook page 110

Example 4. special coaxial lines

Please calculate the capacitance per unit length.



Analysis:

(1) How many approaches are there to calculate a capacitance?

(2) Is there symmetry? What coordinates shall we choose?

(1) Assume 0 E-potential to be at infinite

(2) Assume the boundary conditions: $\psi\,|_{r=a} = U \qquad \psi\,|_{r=b} = 0$

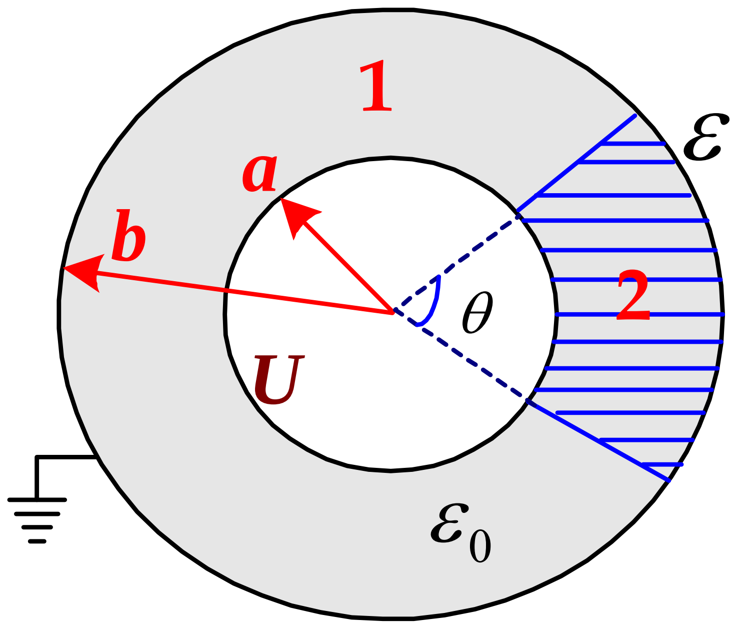
(3) Present Laplace's Equ.

(4) Due to axial symmetry, E-potential depends on only r.

$$\nabla^2 \psi_1 = 0$$

$$\psi_1 = \psi_1(r) = A \ln r + B$$

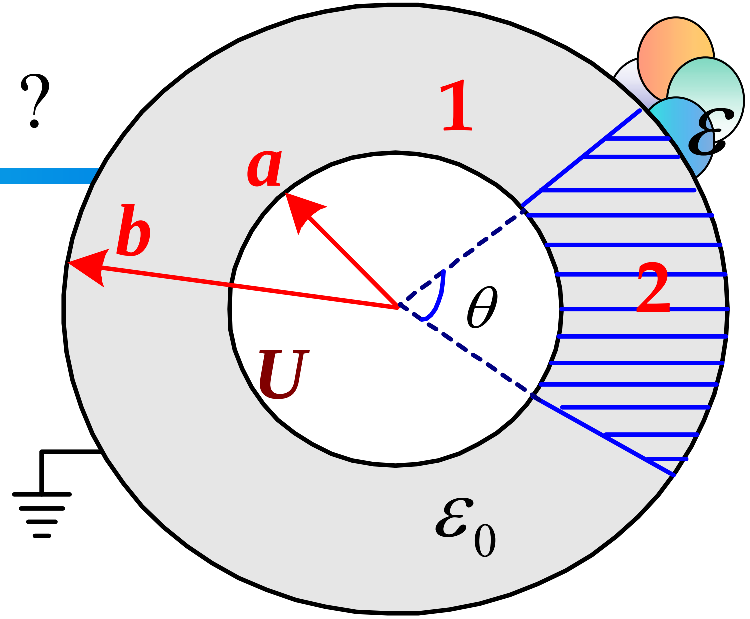
$$\psi_2 = \psi_2(r) = E \ln r + D$$



Apply boundary conditions

$$\psi_1 = \psi_2 = \frac{U}{\ln \frac{b}{a}} \ln \frac{b}{r}$$

$$\psi_1 = \psi_2 \qquad \vec{E}_1 = \vec{E}_2 \ ? \qquad \vec{D}_1 = \vec{D}_2 \ ?$$



$$\therefore \vec{E} = -\nabla\psi = \vec{a}_r \frac{U}{\ln\frac{b}{a}} \cdot \frac{1}{r}$$

Charges on the inner line per meter

$$Q = \sigma \cdot S = ?$$

$$Q = D_1[a(2\pi-\theta)] + D_2[a\theta]$$

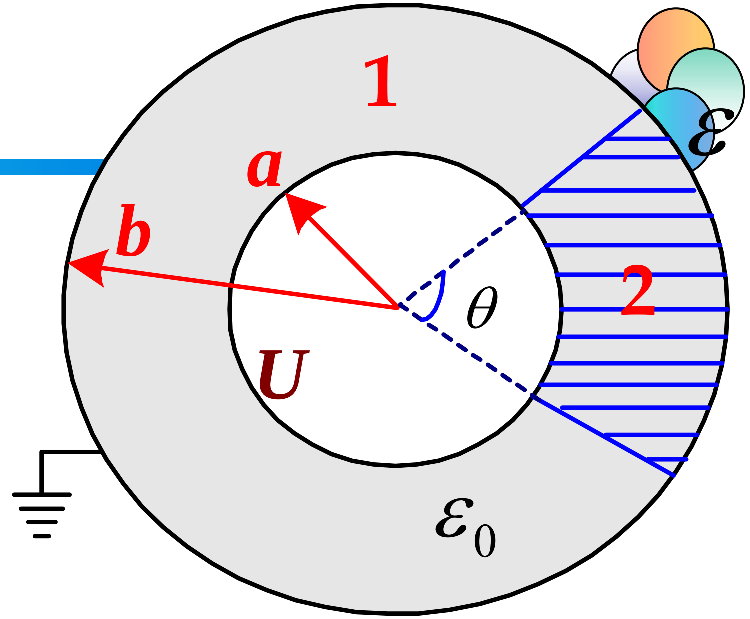
$$= \varepsilon_0 E_1 a(2\pi-\theta) + \varepsilon E_2 a\theta$$

$$= aE[\varepsilon_0(2\pi-\theta)+\varepsilon\theta] = \frac{U}{\ln\frac{b}{a}}[\varepsilon_0(2\pi-\theta)+\varepsilon\theta]$$

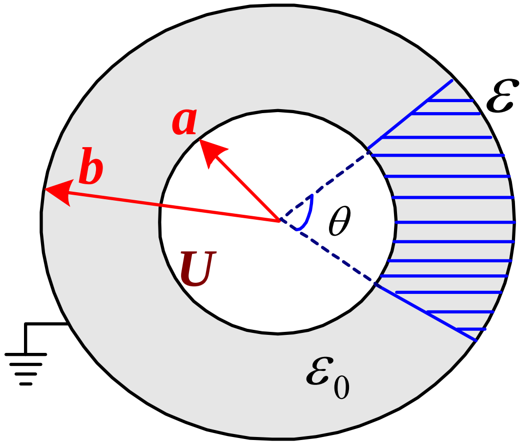
$$\psi_1 = \psi_2 = U$$

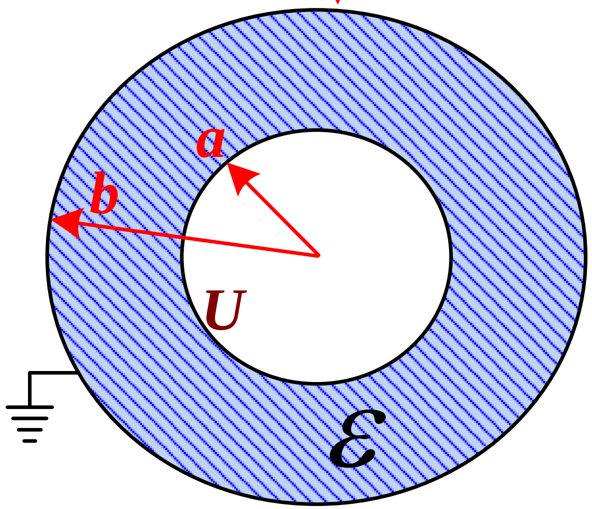
$$Q = \cdots$$

$$C = \frac{Q}{U} = \frac{\varepsilon\theta + \varepsilon_0(2\pi - \theta)}{\ln(\frac{b}{a})}$$



Capacitance of Common Coaxial Lines





$$C = \frac{Q}{U} = \frac{\varepsilon\theta + \varepsilon_0(2\pi - \theta)}{\ln(\frac{b}{a})}$$

$$\theta \to 2\pi$$

$$C = \frac{Q}{U} = \frac{2\pi \cdot \varepsilon}{\ln(\frac{b}{a})}$$

Homework

- **Exercises:**

3.31 3.37 3.38