EBU6018
Advanced Transform Methods

DFT, DCT, DWT
Transform Matrix

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Content

This lecture introduces the Fourier Matrix (the Discrete Fourier Transform having been preciously covered) and provides a summary of three Discrete Transforms covered in this module:

1. The Discrete Fourier Transform
2. The Discrete Cosine Transform
3. The Discrete Wavelet Transform, using the simplest wavelet function, the Haar function

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1. Discrete Fourier Transform (DFT)

The continuous Fourier Transform is defined as:

$$X(f) = \int_{t=-\infty}^{t=\infty} e^{-j\omega t} \cdot x(t)\,dt$$

However ANY continuous transform is not practical (infinite number of values across infinite time) so transforms need to be implemented discretely.

The Discrete Fourier Transform (DFT) is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} = X(\omega)\Big|_{\omega=\frac{2\pi}{N}k} \quad \text{for } k = 0, 1, \ldots, N\text{-}1$$

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DFT

This can be written:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{-nk}$$

where $W_N = e^{\frac{j2\pi}{N}}$

An N-point DFT can be written as X = Wx

where x is the N-point input sequence of samples of a continuous signal, W is the N-by-N DFT matrix and X is the DFT of the signal

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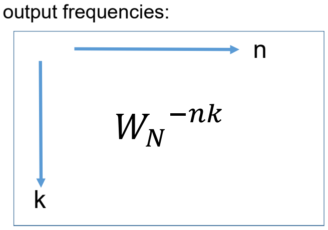
Fourier Matrix

We can produce an N-by-N Fourier Matrix where n are input samples and k are output frequencies:



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General Fourier Matrix

Call the N-by-N Fourier Matrix F_n (N = 0…..(n-1)):

$$F_n = \begin{bmatrix} 1 & 1 & 1 \dots \dots & 1 \\ 1 & \omega & \omega^2 \dots . & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 \dots . & \omega^{2(n-1)} \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{(n-1)^2} \end{bmatrix}$$

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2x2 Fourier Matrix

Now, consider $\omega = e^{\frac{j2\pi}{N}} = [\cos\left(\frac{2\pi}{N}\right) + j\sin(\frac{2\pi}{N})]$ $so\ \omega^N = e^{j2\pi} = 1$

All the entries of F_n are on the unit circle in the complex plane and raising each one to the n^{th} power gives 1.

For example, if N = 2, $e^{j2\pi/2}$ = -1

Then $F_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, normalised.

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4x4 Fourier Matrix

For N = 4 (N = 0.....3), $\omega = e^{j2\pi/4}$ = I and ω^{-1} = -I

So the normalised 4x4 Fourier Matrix is:

$$F_4 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

Each row corresponds to an increasing frequency.

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Summary of Fourier Matrix

- Each row of the Fourier Matrix corresponds to a cosine wave and a sine wave of the same frequency.
- Even if the input data is a sequence of real numbers, the output sequence will be complex numbers
- Multiplying an input sequence by the Fourier Matrix gives an output which is the correlation between the input data and a series of cosine and sine waves of increasing frequency.
- This is equivalent to what a Fourier Series does for a periodic input waveform.
- The 2x2 and 4x4 Fourier Matrices are relatively trivial. The 8x8 is less trivial and will not be considered here.

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2. Discrete Cosine Transform (DCT)

- If the input sequence applied to a DFT contains only real values from an even function, then the imaginary (sine) values of the DFT output are 0.
- We are then left with the real (cosine) output values of the DFT.
- So we now have a Discrete Cosine Transform DCT.
- As with the DFT, we assume that the input sequence is periodic in order to obtain an accurate Fourier Transform.
- For the DCT, the sequence is assumed to be even and periodic.

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1-Dimensional DCT

$$DCT[k] = c(k)\sum_{n=0}^{N-1} s[n]\cos\frac{\pi(2n+1)k}{2N} \qquad k = 0,1,2 \ldots N-1$$

$$DCT[k] = \langle s, \psi_k \rangle \qquad c(k) = \begin{cases} \sqrt{1/N} & k = 0 \\ \sqrt{2/N} & k \neq 0 \end{cases}$$

c(k) is the normalisation factor.

- Orthonormal $\langle \psi_m, \psi_n \rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$

The Basis Functions Ψ_k are the cosine terms in the definition. They are calculated for each value of k, with n = 0….N-1

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1-Dimensional DCT

Example. N = 4 (input is a 4-point sequence)

For each value of k = 0….N-1, insert n = 0….N-1:

$$\psi_0 = (1,1,1,1)/2$$

$$\psi_1 = \sqrt{1/2}(\cos(\pi/8), \cos(3\pi/8), \cos(5\pi/8), \cos(7\pi/8))$$

$$\psi_2 = \sqrt{1/2}(\cos(\pi/4), \cos(3\pi/4), \cos(5\pi/4), \cos(7\pi/4))$$

$$\psi_3 = \sqrt{1/2}(\cos(3\pi/8), \cos(9\pi/8), \cos(15\pi/8), \cos(5\pi/8))$$

$$DCT[0] = \frac{1}{\sqrt{N}}\sum_{n=0}^{3} s[n] \qquad DCT[1] = \sqrt{\frac{2}{N}}\sum_{n=0}^{3} s[n]\cos\frac{\pi(2n+1)}{8}$$

$$DCT[2] = \sqrt{\frac{2}{N}}\sum_{n=0}^{3} s[n]\cos\frac{\pi(2n+1)}{4} \qquad DCT[3] = \sqrt{\frac{2}{N}}\sum_{n=0}^{3} s[n]\cos\frac{\pi(2n+1)3}{8}$$

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Example

These 4 Basis Functions can be written in Matrix format.

- Calculate the elements of the 4x4 Basis Function Matrix.
- Then determine the output sequence if the input sequence is s[n] = [2, 3, 1, 4]

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4x4 DCT Basis Matrix

$$\Psi = \begin{bmatrix} 1/2 & & & \\ \frac{cos(\frac{\pi}{8})}{\sqrt{2}} & & & \\ & & & \\ & & & \end{bmatrix}$$

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4x4 DCT Basis Matrix

$$\Psi = \begin{bmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{bmatrix}$$

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4x4 DCT Transform

$$\text{DCT}=\begin{bmatrix}0.50 & 0.50 & 0.50 & 0.50\\ 0.65 & 0.27 & -0.27 & -0.65\\ 0.50 & -0.50 & -0.50 & 0.50\\ 0.27 & -0.65 & 0.65 & -0.27\end{bmatrix}\begin{bmatrix}2\\3\\1\\4\end{bmatrix}=\begin{bmatrix}5.00\\-2.10\\1.00\\-1.84\end{bmatrix}$$

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1-Dimensional DCT

- The DCT is used to perform image compression to produce jpeg format.
- For this format, an image is sub-divided into 8x8 blocks of data.
- The transform is then the dot-product of the basis function matrix with an 8-point input sequence to produce an 8-point output sequence.
- This is effectively correlation of the input data with a range of cosine waves of different frequency.

So an 8x8 Basis function is required……

The blocks in the following slide are the 8 samples of each cosine wave, for n = 0……7.

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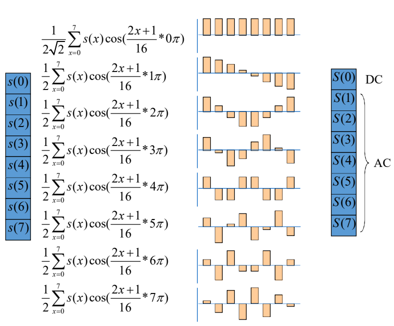
DCT Basis Functions



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8x8 DCT Matrix

For each row of the basis function matrix, take k=0, 1, 2.......7, and for each value of k take x=0, 1, 2.......7

$$\Psi = \frac{1}{2}\begin{bmatrix} .71 & .71 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cos\frac{\pi}{16} & \cos\frac{3\pi}{16} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cos\frac{15\pi}{16} \\ \cos\frac{2\pi}{16} & \cos\frac{6\pi}{16} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cos\frac{30\pi}{16} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cos\frac{7\pi}{16} & \cos\frac{21\pi}{16} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cos\frac{105\pi}{16} \end{bmatrix}$$

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8x8 DCT Transform Matrix

The coefficients in each row of the transform matrix are the amplitudes of 8 samples of a cosine wave. The first row is a cosine wave of 0Hz (DC), then the frequencies of each cosine wave are increasing (AC).

$$\Psi = \frac{1}{2}\begin{bmatrix} .71 & .71 & .71 & .71 & .71 & .71 & .71 & .71 \\ .98 & .83 & .56 & .19 & -.19 & -.56 & -.83 & -.98 \\ .92 & .38 & -.38 & -.92 & -.92 & -.38 & .38 & .92 \\ .83 & -.19 & -.98 & -.56 & .56 & .98 & .19 & -.83 \\ .71 & -.71 & -.71 & .71 & .71 & -.71 & -.71 & .71 \\ .56 & -.98 & .19 & .83 & -.83 & -.19 & .98 & -.56 \\ .38 & -.92 & .92 & -.38 & -.38 & .92 & -.92 & .38 \\ .19 & -.56 & .83 & -.98 & .98 & -.83 & .56 & -.19 \end{bmatrix}$$

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8x8 EXAMPLE 1

Suppose we have an 8-point input sequence, this could be a row of pixel values:

S[n] = [2.1, 2.0, 2.11, 1.99, 1.98, 2.02, 2.08, 1.98]

The values in this sequence do not change much.

The DCT is:

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8x8 EXAMPLE 1

$$\text{DCT} = \frac{1}{2}\begin{bmatrix} .71 & .71 & .71 & .71 & .71 & .71 & .71 & .71 \\ .98 & .83 & .56 & .19 & -.19 & -.56 & -.83 & -.98 \\ .92 & .38 & -.38 & -.92 & -.92 & -.38 & .38 & .92 \\ .83 & -.19 & -.98 & -.56 & .56 & .98 & .19 & -.83 \\ .71 & -.71 & -.71 & .71 & .71 & -.71 & -.71 & .71 \\ .56 & -.98 & .19 & .83 & -.83 & -.19 & .98 & -.56 \\ .38 & -.92 & .92 & -.38 & -.38 & .92 & -.92 & .38 \\ .19 & -.56 & .83 & -.98 & .98 & -.83 & .56 & -.19 \end{bmatrix}\begin{bmatrix} 2.1 \\ 2.0 \\ 2.11 \\ 1.99 \\ 1.98 \\ 2.02 \\ 2.08 \\ 1.98 \end{bmatrix}$$

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8x8 EXAMPLE 1

Transposing back to a row, the output sequence is:

$S[k] = \frac{1}{2}[11.54, 0.10, 0.08, 0.02, -0.11, 0.17, 0.09, 0.13]$

- Only the first element is big, all the others are small.
- This shows that there is a high correlation between the input data and the first row of the transform matrix (the lowest frequency, 0Hz).
- That is, the input data has little variation.

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8x8 EXAMPLE 2

Suppose we have another 8-point input sequence, this could be another row of pixel values:

$S[n] = [2.1, 9.6, -11.2, 7.9, -10.1, 8.6, -6.7, 8.3]$

The values in this sequence change a great deal from pixel to pixel.

The DCT is:

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8x8 EXAMPLE 2

$$\text{DCT} = \frac{1}{2}\begin{bmatrix} .71 & .71 & .71 & .71 & .71 & .71 & .71 & .71 \\ .98 & .83 & .56 & .19 & -.19 & -.56 & -.83 & -.98 \\ .92 & .38 & -.38 & -.92 & -.92 & -.38 & .38 & .92 \\ .83 & -.19 & -.98 & -.56 & .56 & .98 & .19 & -.83 \\ .71 & -.71 & -.71 & .71 & .71 & -.71 & -.71 & .71 \\ .56 & -.98 & .19 & .83 & -.83 & -.19 & .98 & -.56 \\ .38 & -.92 & .92 & -.38 & -.38 & .92 & -.92 & .38 \\ .19 & -.56 & .83 & -.98 & .98 & -.83 & .56 & -.19 \end{bmatrix}\begin{bmatrix} 2.1 \\ 9.6 \\ -11.2 \\ 7.9 \\ -10.1 \\ 8.6 \\ -6.7 \\ 8.3 \end{bmatrix}$$

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8x8 EXAMPLE 2

Transposing back to a row, the output sequence is:

$S[k] = \frac{1}{2}[6.04, -0.22, 13.68, 1.08, 5.61, -8.27, -0.27, -44.38]$

- Only the last element is very big, all the others are relatively small.
- This shows that there is a high correlation between the input data and the last row of the transform matrix (i.e. the highest frequency).
- That is, the input data has very large variation.

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Summary of the Discrete Cosine Transform

- The Discrete Cosine Transform correlates an input sequence of real samples of an input signal with a set of cosine waveforms of increasing frequency.
- The output is a set of real values.
- When applied to images, it transforms from the spatial domain to a rate-of-change domain.
- It is used in the production of jpeg image compression format.

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3. Discrete Wavelet Transform (DWT)

- Wavelets are a class of functions which are of short duration and are oscillatory.
- They are used for perform time-frequency analysis.
- They are of the form:

$$\psi(t) \rightarrow \psi\left(\frac{t-b}{a}\right)$$

They are used as basis functions by translation (b) and scaling (a)

$$CWT(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} s(t)\psi^{*}\left(\frac{t-b}{a}\right)dt$$

$$= \int_{-\infty}^{\infty} s(t)\psi_{a,b}^{*}(t)dt = < s, \psi_{a,b} >$$

As with any transform, they are implemented discretely.

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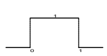
Haar Functions

The simple Haar Functions will be used to illustrate a discrete implementation of Wavelet Transforms.

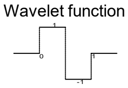
Haar Functions are a set of scaled and translated Haar Scaling Function and the Haar Wavelet Function:

Scaling function



$\varphi_{00} = [1 \quad 1]$

Wavelet function



$\psi_{00} = [1 \quad -1]$

These are orthogonal $< \varphi_{00}, \psi_{00} > = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

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Haar Matrix

These two functions can be written in matrix form:

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The normalised Haar Matrix is:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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4x4 Haar Matrix

The 4x4 Haar Matrix combines two stages of a Haar Wavelet transform:

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Normalised this is: $H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$

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4x4 Example

Apply the Haar Transform to the 4-point input sequence:
S[n] = [2, 5 -3, 7]

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Example…Solution

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -3 \\ 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 11 \\ 3 \\ (2-5)\sqrt{2} \\ (-3-7)\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ \frac{3}{2} \\ \frac{-3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

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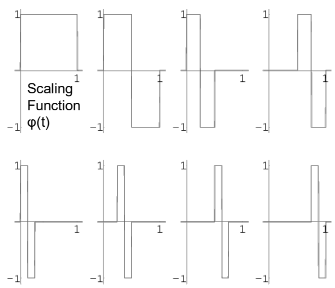
Haar Functions



Wavelet Function:

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_{jk}(x) \equiv \psi(2^j x - k),$$

$\phi_{00}=\phi(x)$
$\psi_{00}=\psi(x)$
$\psi_{10}=\psi(2x)$
$\psi_{11}=\psi(2x-1)$
$\psi_{20}=\psi(4x)$
$\psi_{21}=\psi(4x-1)$
$\psi_{22}=\psi(4x-2)$
$\psi_{23}=\psi(4x-3)$

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8x8 Haar Matrix

The un-normalised 8x8 Haar Matrix can be used to show how a Haar Matrix is derived:

$$H_8 = \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \begin{array}{l} \varphi_0(t) \\ \psi_0(t) \\ \psi_{1,0}(t) \\ \psi_{1,1}(t) \\ \psi_{2,0}(t) \\ \psi_{2,1}(t) \\ \psi_{2,2}(t) \\ \psi_{2,3}(t) \end{array} \end{array}$$

The first row gives the average value of the input sequence. Then subsequent rows correspond to increasing frequencies (similar to DCT)

The matrix would need to be normalised before it could be applied directly to a transform.

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Summary

- We have seen that a Haar Matrix can be constructed to perform Haar Transforms directly.
- The Haar Transform is fast because the matrix contains many zero terms.
- It can be used to identify frequency components in the signal to be analysed.
- It can be used for compression by reducing or eliminating the coefficients corresponding to high frequencies in the signal and then inverting the transform.

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Exercise

For a 4-point input sequence
S[n] = [6, 3, -2.5, 7]

Determine the output from each of the following discrete transforms:

a. DFT
b. DCT
c. DWT, using a Haar wavelet function.

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