EBU6018
Advanced Transform Methods

Karhunen-Loeve Transform
(An example of Linear Transform Coding)

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Linear Transform Coding (LTC)

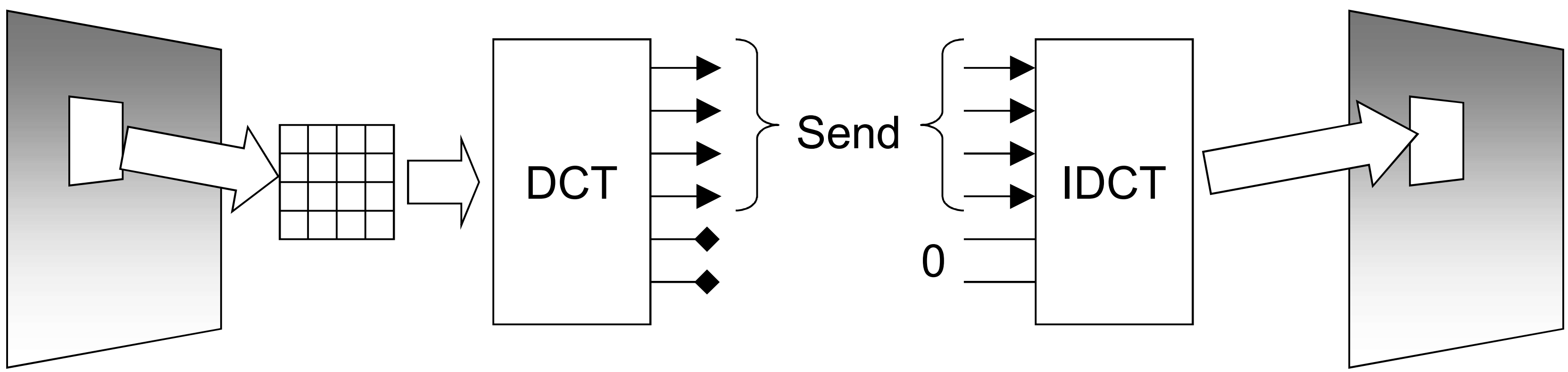
Met Discrete Cosine Transform (DCT) in a previous lecture.

An advantage of DCT:

– most energy concentrated in a few coefficients, so

– can discard some coeffs, while keep most of signal

E.g. image compression:



Fourier, Wavelets also do this, depending on the signal.

But - what is the “best” transform for this?

Let’s first look at the principle of LTC.

Linear Transform Coding

Divide image (or signal) into P blocks of N pixels (samples).

The kth block is now an N-dimensional vector :

$$\mathbf{x}_k = (x_{1,k}, x_{2,k}, \ldots, x_{N,k})^T$$

The image (signal) is now a sequence of vectors $\{\mathbf{x}_k\}$.

We now transform each $\mathbf{x}$ by multiplying by a linear matrix

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

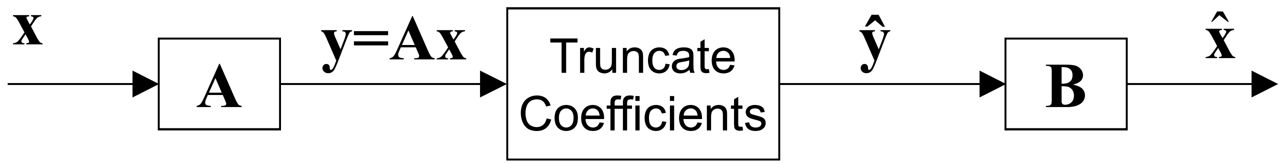
transmit the first M coefficients $\hat{\mathbf{y}} = (y_1, \ldots, y_M)^T$,

discarding the remaining $N - M$ coeffs $y_{M+1} \cdots y_N$

We then reconstruct the image block using another matrix

$$\hat{\mathbf{x}} = \mathbf{B}\hat{\mathbf{y}}$$

Linear Transform Coding



We measure the error introduced in $\hat{\mathbf{x}}$ as

$J = E(|\mathbf{x} - \hat{\mathbf{x}}|^2)$ *mean squared error* (MSE)

where $E(v)$ is the expected value (mean) of v.

We want to choose $\mathbf{A}$ and $\mathbf{B}$ to minimize J.

Easy case: if we keep all the coefficients, we have

$\hat{\mathbf{y}} = \mathbf{y}$ so $\hat{\mathbf{x}} = \mathbf{B}\hat{\mathbf{y}} = \mathbf{B}\mathbf{y} = \mathbf{B}\mathbf{A}\mathbf{x}$

so $\hat{\mathbf{x}} = \mathbf{x}$ (making $J = 0$) if $\mathbf{B}\mathbf{A} = \mathbf{I}$ i.e. $\mathbf{B} = \mathbf{A}^{-1}$

Principal Components Analysis (PCA) -The basis for the KLT

- Multivariate procedure
- Main use of PCA is to reduce dimensionality of a data set while retaining as much information as is possible.
- Finds a projection of the observations onto orthogonal axes contained in the space defined by the original variables.
- Correlated variables transformed into uncorrelated variables
  - Ordered by reducing variability.
  - Uncorrelated variables are linear combinations of original variables
- Computes compact, optimal description of data set.
- Rotates data so that maximum variabilities projected onto the axes
- Rotation of existing axes to new positions in the space defined by the original variables.

The new components/ axes/ variables

1. First principal component is the combination of variables that explains the greatest amount of variation
   - ✓ Contains the maximum amount of variation
2. Second principal component defines the next largest amount of variation and is independent to the first principal component
   - ✓ Contains the maximum amount of variation unexplained by and orthogonal to the first
3. Third axis contains the maximum amount of variation orthogonal to the first and second axis
   - ✓ Contains the maximum amount of variation unexplained by and orthogonal to the first and second axes

… Last new axis which is the last amount of variation left

- ✓ Can be removed with minimum loss of real data

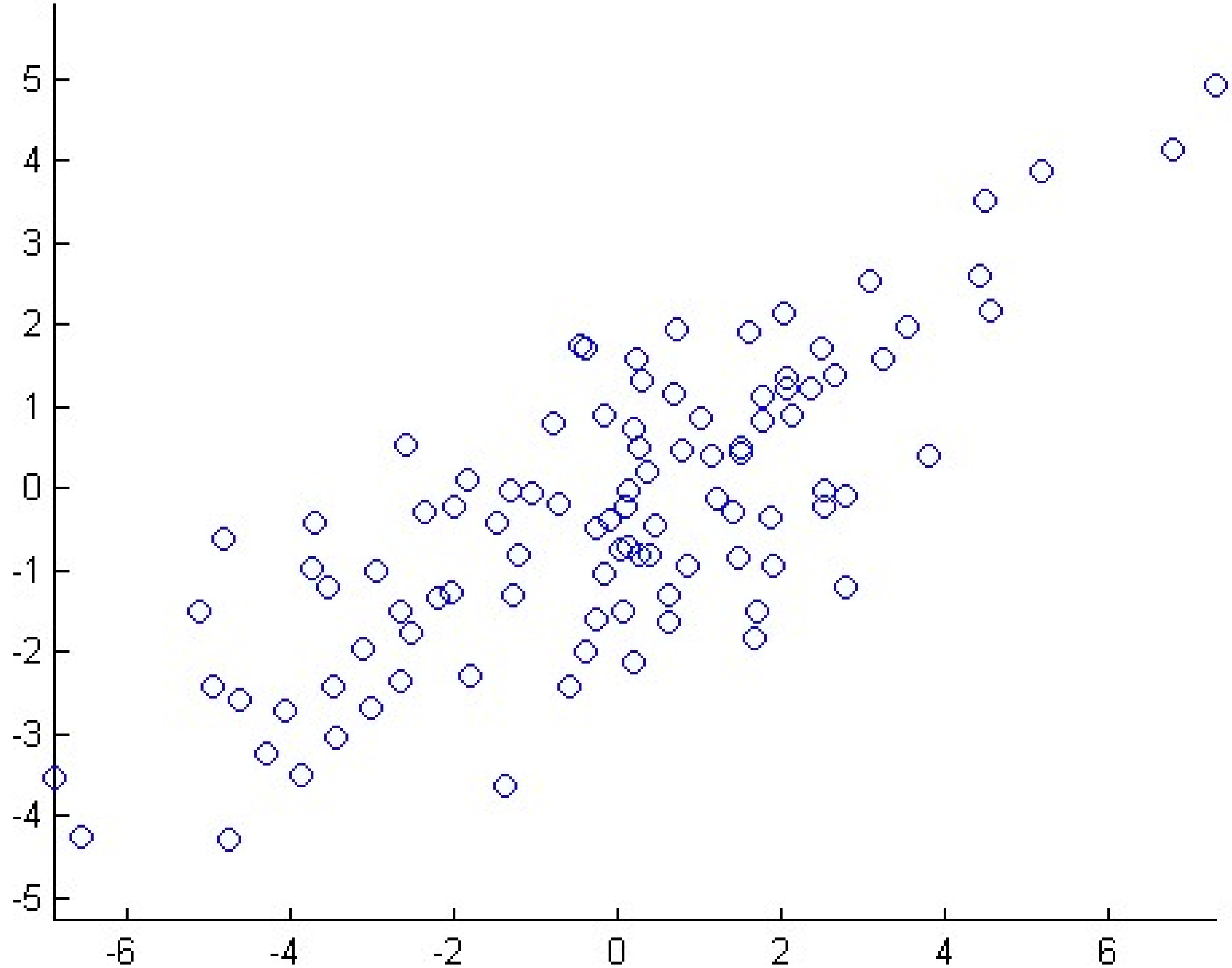
- ❑ Can be as many principal components as there are variables
- ❑ No correlation between the new variables defined by rotation

Example

- Set of 2-D data, the table is a sample of a population

x=

X_1	X_2
-3.0139	-2.6748
0.1810	-2.1227
-6.5559	-4.2424
1.5960	1.9161
2.7825	-1.2017
2.0292	2.1421
-2.5200	-1.7787
0.2340	-0.8122
0.8263	-0.9407
-4.6227	-2.5955
-4.3052	-3.2307
-2.6013	0.5435
3.5409	1.9755
5.1704	3.8629
.	.
.	.
.	.



Covariance Matrix

- Find covariance matrix

$$\operatorname{cov}[\mathbf{X}] = \begin{bmatrix} 7.5649 & 3.8464 \\ 3.8464 & 3.2451 \end{bmatrix} \quad \begin{bmatrix} v_{X_1} & cov_{X_{12}} \\ cov_{X_{21}} & v_{X_2} \end{bmatrix}$$

- Find principal components

$$\mathbf{pc} = \begin{bmatrix} 0.8630 & -0.5052 \\ 0.5052 & 0.8630 \end{bmatrix}$$

The columns are the eigenvectors of cov[X]

- Are they orthogonal?

$$\langle \psi_1, \psi_2 \rangle = \langle (0.8630, 0.5052), (-0.5052, 0.8630) \rangle$$

$$= 0.8630 \cdot (-0.5052) + 0.5052 \cdot 0.8630 = 0$$ **Yes**

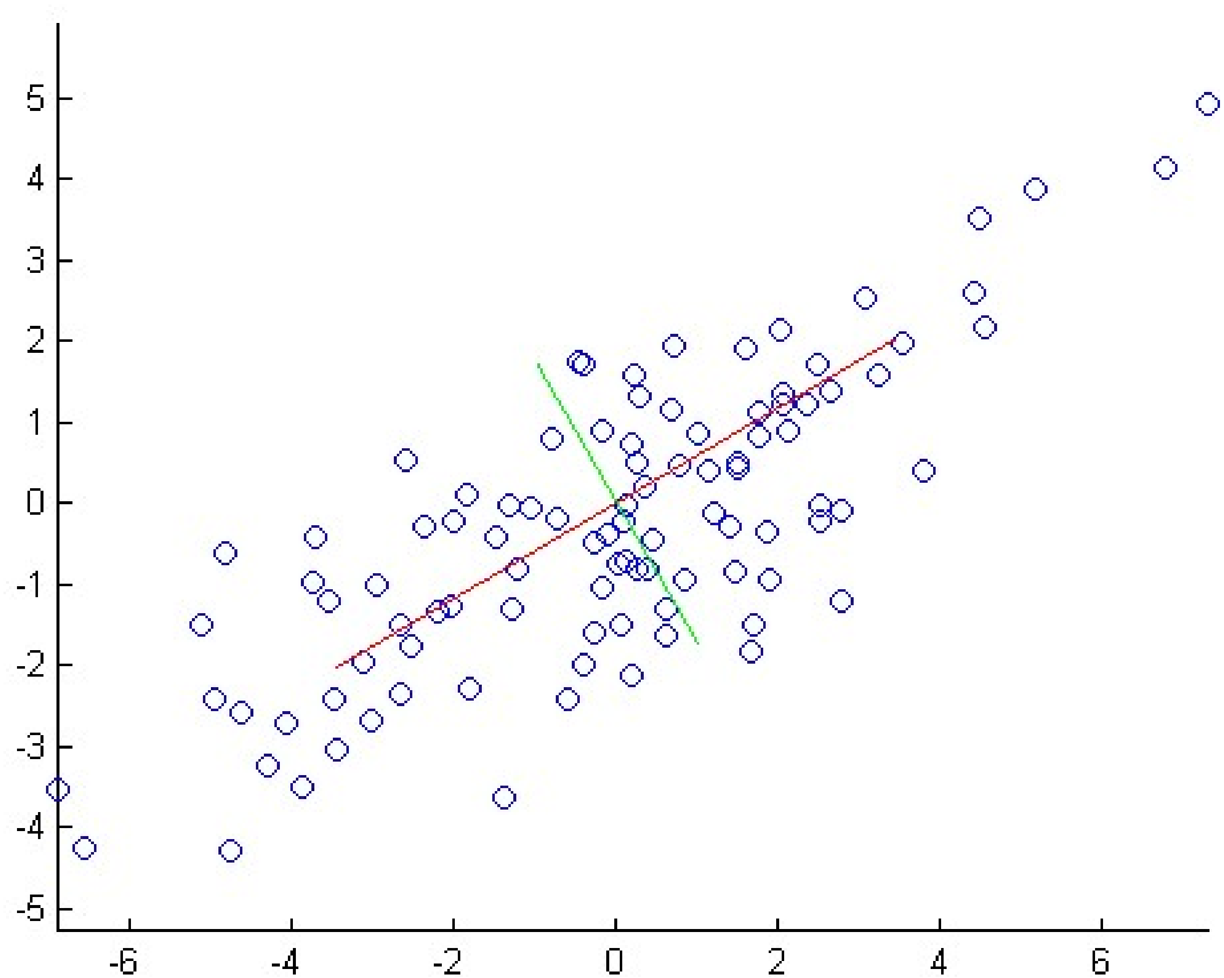
- Are they normalised?

$$\langle \psi_1, \psi_1 \rangle = 0.8630 \cdot 0.8630 + 0.5052 \cdot 0.5052 = 1$$

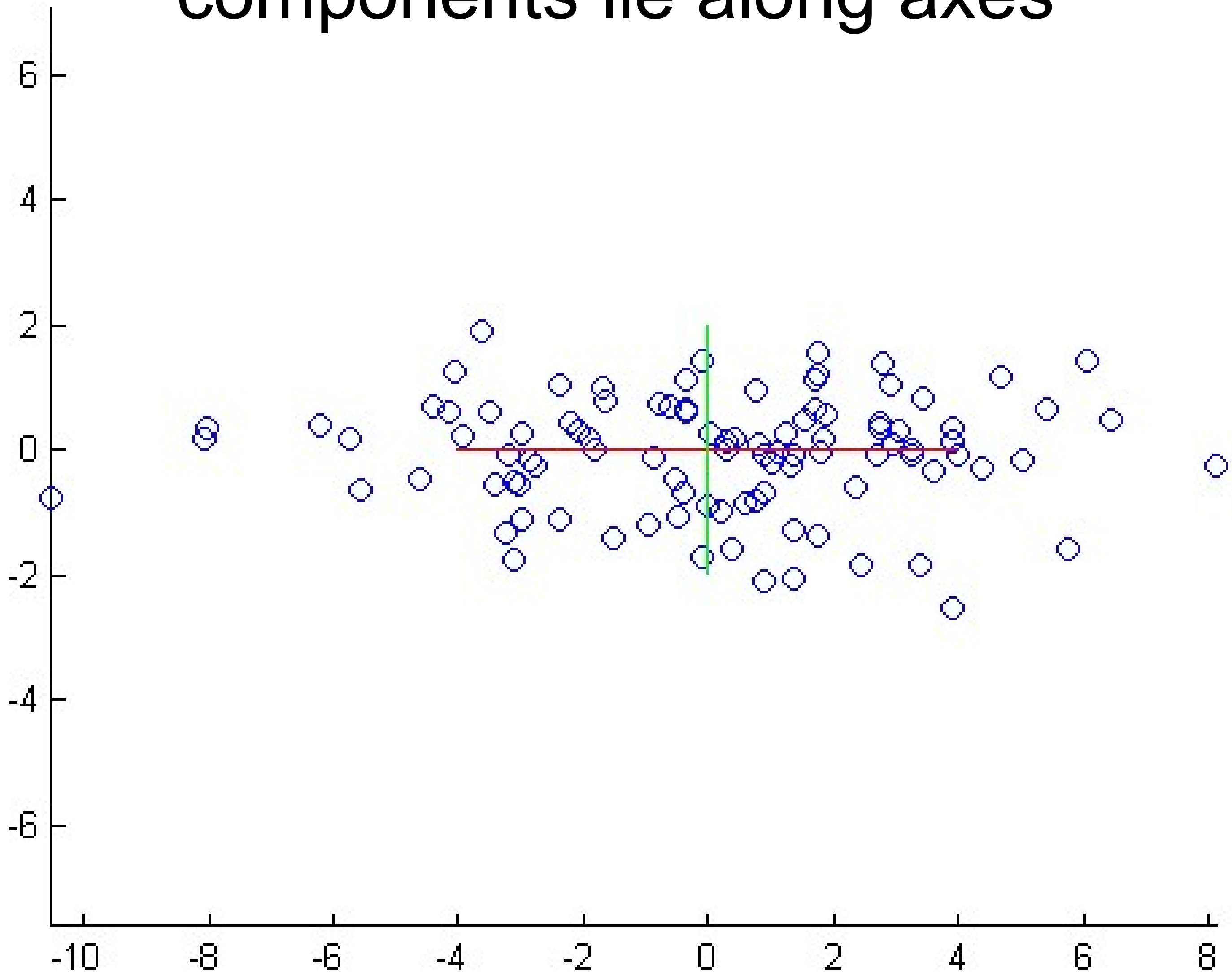
$$\langle \psi_2, \psi_2 \rangle = (-0.5052) \cdot (-0.5052) + 0.8630 \cdot 0.8630 = 1$$ **Yes**

Principal components visualised

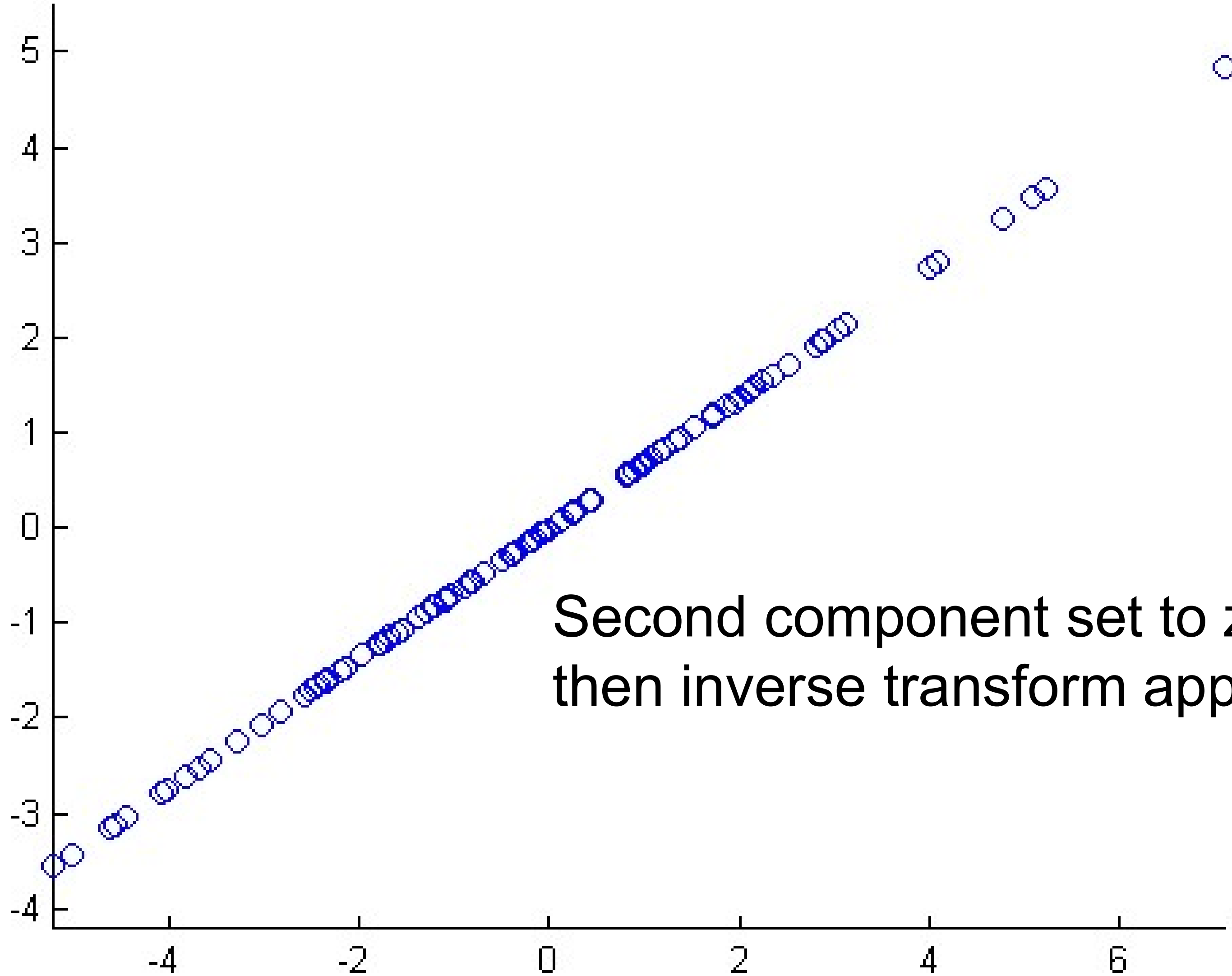
- red line represents direction of first principal component
  - line of greatest variation
- green line is direction of second principal component
  - perpendicular to red line.
- When there are more than 2 dimensions,
  - next component along line of next greatest variation



By multiplying original data by principal components, data rotated so that principal components lie along axes



PCA is used to reduce data dimensionality while retaining the most information



Second component set to zero, then inverse transform applied

Now let's give the theory

- The Karhunen Loève Transform is based on PCA
- Also known as the Hotelling Transform

Karhunen Loève Transform (KLT)

- Basis functions are eigenvectors of the covariance matrix R_{XX} of the input signal.
  - This set of basis vectors is *not* fixed
  - Basis vectors depend on the data set
- KLT yields decorrelated transform coefficients (covariance matrix R_{YY} is diagonal).
- Optimal linear transform for keeping the subspace that has largest variance
  - Achieves optimum energy concentration.
- KLT maximizes coding gain

Procedure

1. Find mean vector for input data $\mathbf{X} = \left[\vec{x}_0, \vec{x}_1, \ldots \vec{x}_{N-1}\right]$

$$E(\vec{x}) = \frac{1}{N}\sum_{i=0}^{N-1} \vec{x}_i$$

2. Find covariance matrix

$$\mathbf{R}_{\mathbf{XX}} = \frac{1}{N-1}\sum_{i=0}^{N-1} (\vec{x}_i - E(\vec{x}))(\vec{x}_i - E(\vec{x}))^T$$

3. Find eigenvalues of the covariance matrix

$$\left|\mathbf{R}_{\mathbf{XX}} - \lambda \mathbf{I}\right| = 0$$

$|\mathbf{A}|$ means "Determinant of $\mathbf{A}$"

4. Find eigenvectors of the covariance matrix

$$\left[\mathbf{R}_{\mathbf{XX}} - \lambda_i \mathbf{I}\right]\vec{\phi}_i = 0$$

Procedure

5. Normalise the eigenvectors

$$\left\langle \vec{\phi}_i, \vec{\phi}_i \right\rangle = 1$$

6. Transform the input

$$\mathbf{Y} = \phi^T \mathbf{X}$$

7. *To check*, find covariance matrix of $\mathbf{Y}$

$$\mathbf{R}_{\mathbf{YY}} = \frac{1}{N-1} \sum_{i=0}^{N-1} (\vec{y}_i - E[\vec{y}])(\vec{y}_i - E[\vec{y}])^T$$

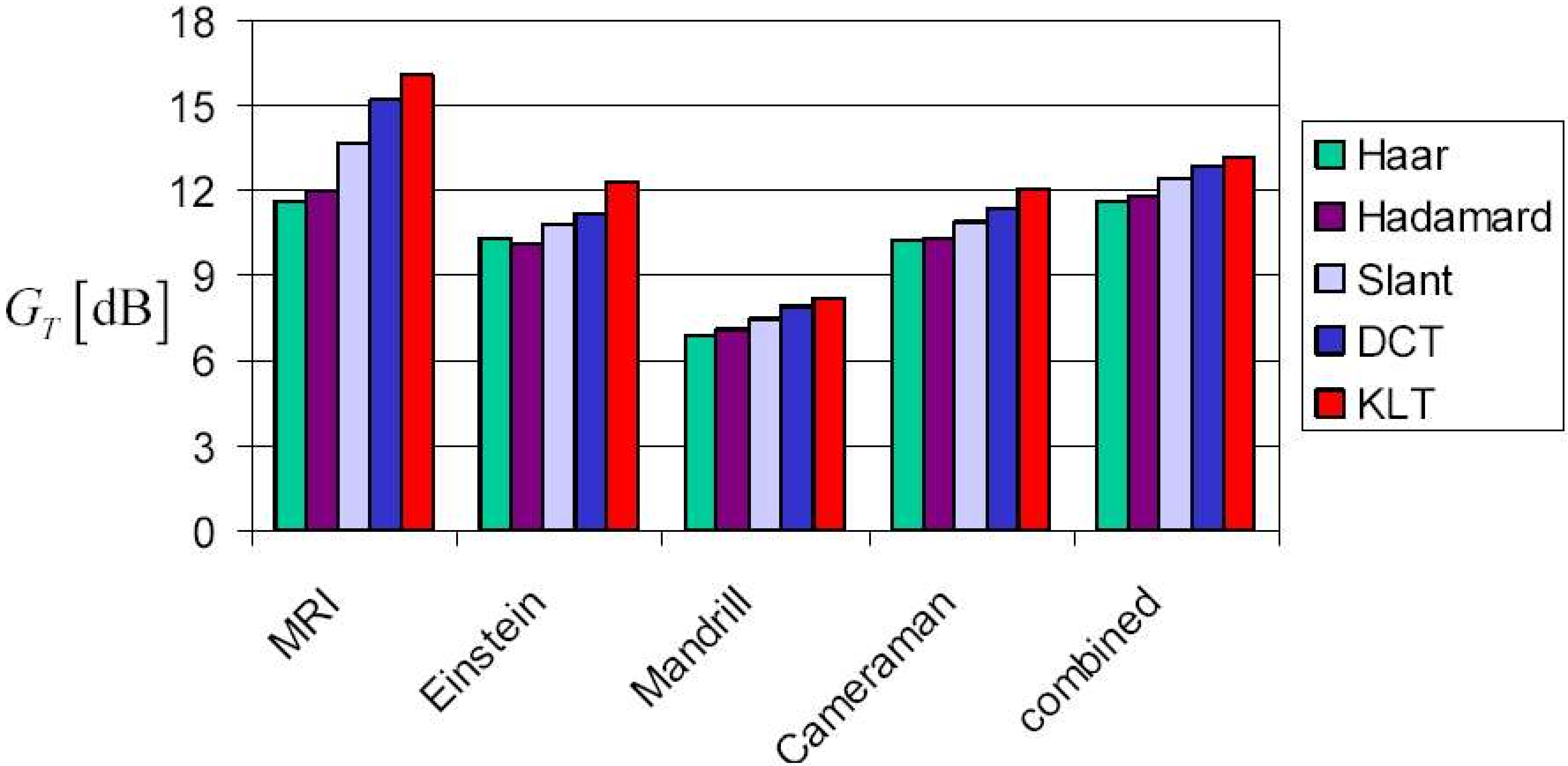
8. *Optional (for compression),*
    1. set last row vector(s) of $\mathbf{Y}$ to 0

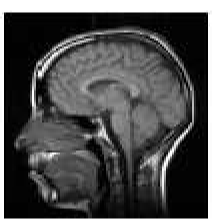
    $$\mathbf{Y} \rightarrow \mathbf{Y'}$$

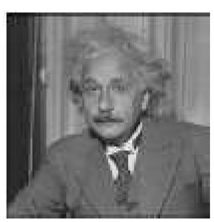
    2. Inverse transform this

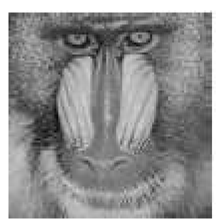
    $$\mathbf{X}' = \phi \mathbf{Y}'$$

Coding gain with 8x8 transforms









New Example

Determine the KLT of the following 2D data set.
Assume sample statistics.

$$\mathbf{X} = \begin{bmatrix} 2 & 4 & 5 & 5 & 3 & 2 \\ 2 & 3 & 4 & 5 & 4 & 3 \end{bmatrix}$$

Replace the elements of the least principal component of the output by zero and then perform the inverse KLT

New Example

$$\mathbf{X} = \begin{bmatrix} 2 & 4 & 5 & 5 & 3 & 2 \\ 2 & 3 & 4 & 5 & 4 & 3 \end{bmatrix}$$

Mean vector for input data

$$E(\vec{x}) = \frac{1}{6}\begin{bmatrix} 2+4+5+5+3+2 \\ 2+3+4+5+4+3 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$$

Covariance matrix

$$\mathbf{R_{XX}} = \frac{1}{5}\sum_{i=0}^{5}(\vec{x}_i - \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix})(\vec{x}_i - \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix})^T$$

$$= \frac{1}{5}\left\{\begin{bmatrix} -1.5 \\ -1.5 \end{bmatrix}\begin{bmatrix} -1.5 & -1.5 \end{bmatrix} + \ldots + \begin{bmatrix} -1.5 \\ -.5 \end{bmatrix}\begin{bmatrix} -1.5 & -.5 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 2.25 & 2.25 \\ 2.25 & 2.25 \end{bmatrix} + \ldots + \begin{bmatrix} 2.25 & 0.75 \\ 0.75 & 0.25 \end{bmatrix}\right\} = \begin{bmatrix} 1.9 & 1.1 \\ 1.1 & 1.1 \end{bmatrix}$$

New Example

Find eigenvalues

$$0 = \left|\mathbf{R_{XX}} - \lambda \mathbf{I}\right| = \begin{vmatrix} 1.9 - \lambda & 1.1 \\ 1.1 & 1.1 - \lambda \end{vmatrix}$$

$$\lambda^2 - 3\lambda + 0.88 = 0$$

$$\lambda_1 = 2.67, \lambda_2 = 0.33$$

Find eigenvectors

$$\begin{bmatrix} -0.77 & 1.1 \\ 1.1 & -1.57 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = 0$$

$$\rightarrow \phi_{11} = 1.43\phi_{21}$$

$$\begin{bmatrix} 1.57 & 1.1 \\ 1.1 & 0.77 \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \end{bmatrix} = 0$$

$$\rightarrow \phi_{12} = -0.7\phi_{22}$$

New Example

$$\phi_{11} = 1.43\phi_{21}, \phi_{12} = -0.7\phi_{22}$$

- Normalise and solve

$$\left\langle \vec{\phi}_1, \vec{\phi}_1 \right\rangle = 1 \rightarrow \phi_{11}{}^2 + \phi_{21}{}^2 = 1$$

$$\left\langle \vec{\phi}_2, \vec{\phi}_2 \right\rangle = 1 \rightarrow \phi_{12}{}^2 + \phi_{22}{}^2 = 1$$

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} 0.82 & -0.57 \\ 0.57 & 0.82 \end{bmatrix}$$

- Transform the input

$$\mathbf{Y} = \phi^T \mathbf{X} = \begin{bmatrix} 0.82 & 0.57 \\ -0.57 & 0.82 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 & 5 & 3 & 2 \\ 2 & 3 & 4 & 5 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2.78 & 4.99 & 6.38 & 6.95 & 4.74 & 3.35 \\ 0.5 & 0.18 & 0.43 & 1.25 & 1.57 & 1.32 \end{bmatrix}$$

New Example

$$\mathbf{Y} = \begin{bmatrix} 2.78 & 4.99 & 6.38 & 6.95 & 4.74 & 3.35 \\ 0.5 & 0.18 & 0.43 & 1.25 & 1.57 & 1.32 \end{bmatrix}$$

- Covariance of Transformed Data

$$\mathbf{R}_{\mathbf{YY}} = \frac{1}{N-1}\sum_{i=0}^{N-1}(\vec{y}_i - E[\vec{y}])(\vec{y}_i - E[\vec{y}])^T = \begin{bmatrix} 2.67 & 0 \\ 0 & 0.33 \end{bmatrix}$$

- 89% (2.67/3 *100%) of energy along the first axis, 11% on the second.
- Now suppose we do dimensionality reduction and remove second coordinates.

$$\mathbf{Y'} = \begin{bmatrix} 2.78 & 4.99 & 6.38 & 6.95 & 4.74 & 3.35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

New Example

- Inverse transform of reduced data

$$\mathbf{X'} = \phi \mathbf{Y'}$$

$$= \begin{bmatrix} 0.82 & -0.57 \\ 0.57 & 0.82 \end{bmatrix} \begin{bmatrix} 2.78 & 4.99 & 6.38 & 6.95 & 4.74 & 3.35 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2.28 & 4.1 & 5.23 & 5.7 & 3.89 & 2.75 \\ 1.58 & 2.84 & 3.64 & 3.96 & 2.7 & 1.91 \end{bmatrix}$$

- Compare with

$$\mathbf{X} = \begin{bmatrix} 2 & 4 & 5 & 5 & 3 & 2 \\ 2 & 3 & 4 & 5 & 4 & 3 \end{bmatrix}$$

- Error

$$J = E(|\,\mathbf{X} - \mathbf{X'}\,|^2\,)$$

Drawbacks of KLT

KLT is theoretically optimal (in the MSE sense). (KLT maximises the coding gain, i.e. maximises the SNR after a given level of compression.)

BUT, it has practical difficulties:

- Estimate of correlation can be unwieldy
- Solution of eigenvector decomposition is computationally intensive (i.e. slow)
- Calculation of forward and inverse transforms is $O(MN)$ for each image block
- Transmission of data-dependent basis $\mathbf{A}$ is required
- The technique is linear, therefore any non-linear correlation between variables will not be captured.

In comparison, turns out that DCT is fixed, and

- a good approximation to KLT for typical images,
- needs no eigenvalue decomposition, and
- transform is $O(N \log N)$.