Chpt. 5 MagnetoStatics

- Introduction --- Current and its Density
- Fundamental Equations
- M-Vector Potential
- Materials in MagnetoStatics
- Boundary Conditions
- Inductance
- Energy & Force in MagnetoStatics*

What's static magnetic field?

- **Current:** macroscopical & directional motion of charges
- **Steady current:** current that does not vary with time.
- **Static magnetic field:** ***M-field around the steady current which is changeless with time.***
- **Magnetostatics & electrostatics**, though ***different in nature***, possess much similar properties and correspond to similar method of analysis.
- Similar to chapter 3, in this chapter we start with related experiment laws, and then present general mathematical expressions, i.e. fundamental equs, and thereafter go on to study the other knowledge points.

Introduction --- Current & its Density

- Current: the rate at which the charges are transported.

$$i = \lim_{\Delta t \to 0} \left(\frac{\Delta q}{\Delta t} \right) = \frac{dq}{dt}$$

- Unit: A (Ampere, in memory of a French physicist)
- The current is a scalar.

For Steady Current $\quad \dfrac{dq}{dt} \equiv I \text{ (constant)}$

$$\Rightarrow q \propto t$$

It means the number of charges passing a given surface in given time is proportional to the time.

Current Volume Density

1. Take a differential surface element (ΔS) normal to the current.
2. Assume the current across ΔS is ΔI.
3. Define the current volume density as $\vec{J}$
   - in direction coinciding the motion of +q
   - with quantity as the current per unit area

$$\boxed{J = \lim_{\Delta S \to 0} \left(\frac{\Delta I}{\Delta S} \right) = ?\left({}^{A}\!/_{m^2} \right) \qquad \left({}^{安}\!/_{米^2} \right)}$$

Distance of +q in Δt is $\vec{v}\Delta t$

Charges across ΔS: $\Delta Q = \rho \cdot \left((\vec{v}\Delta t) \bullet \Delta\vec{S} \right)$

Current across ΔS: $\Delta I = \dfrac{\Delta Q}{\Delta t} = \rho \cdot (\vec{v} \bullet \Delta\vec{s})$

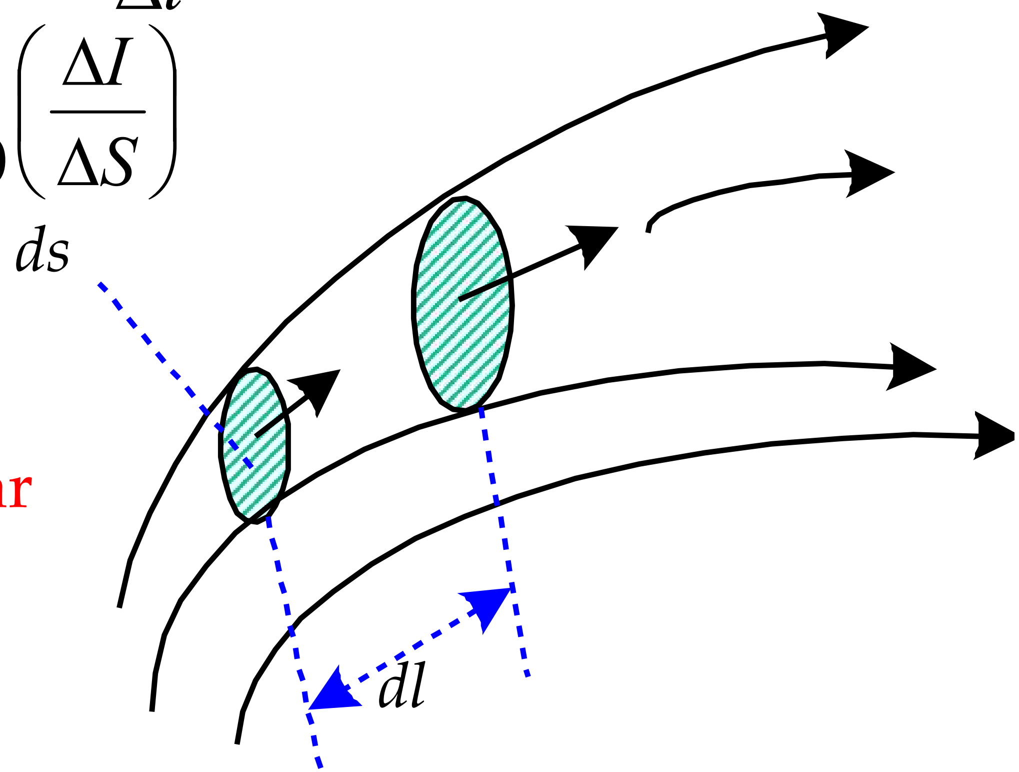
Then the density $J = \lim\limits_{\Delta S \to 0} \left(\dfrac{\Delta I}{\Delta S} \right)$

$$\boxed{\vec{J} = \rho\vec{v}}$$

A Vector

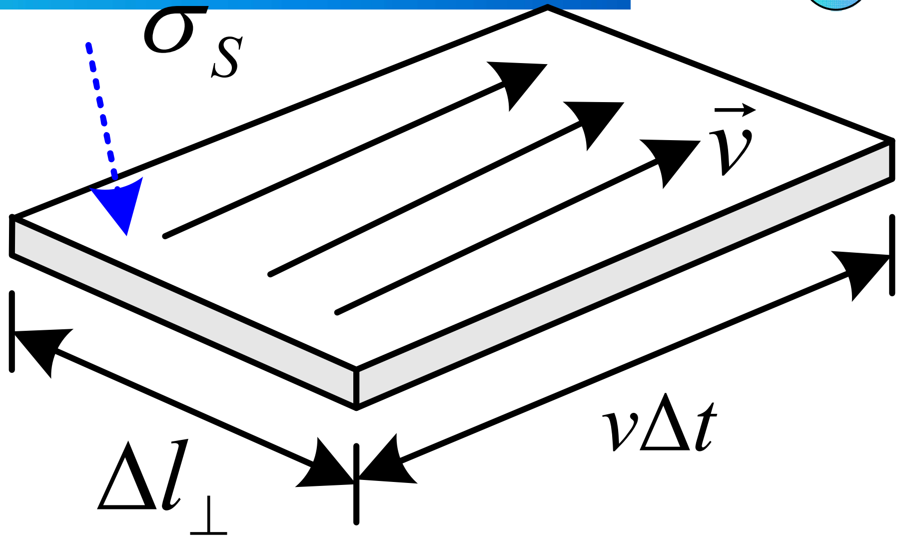
$$\boxed{\therefore I = \int_S \vec{J} \bullet d\vec{S}}$$

A Scalar



Current Surface Density

In the case that the current flows within a thin film.



$$\boxed{J_S = \left(\frac{\Delta I}{\Delta l_\perp}\right) = ?(A/m) \qquad (安/米)}$$

$$\boxed{\Delta Q = \sigma_S \cdot (v \cdot \Delta t) \cdot \Delta l_\perp}$$

$$\boxed{\Delta I = \sigma_S \cdot v \cdot \Delta l_\perp}$$

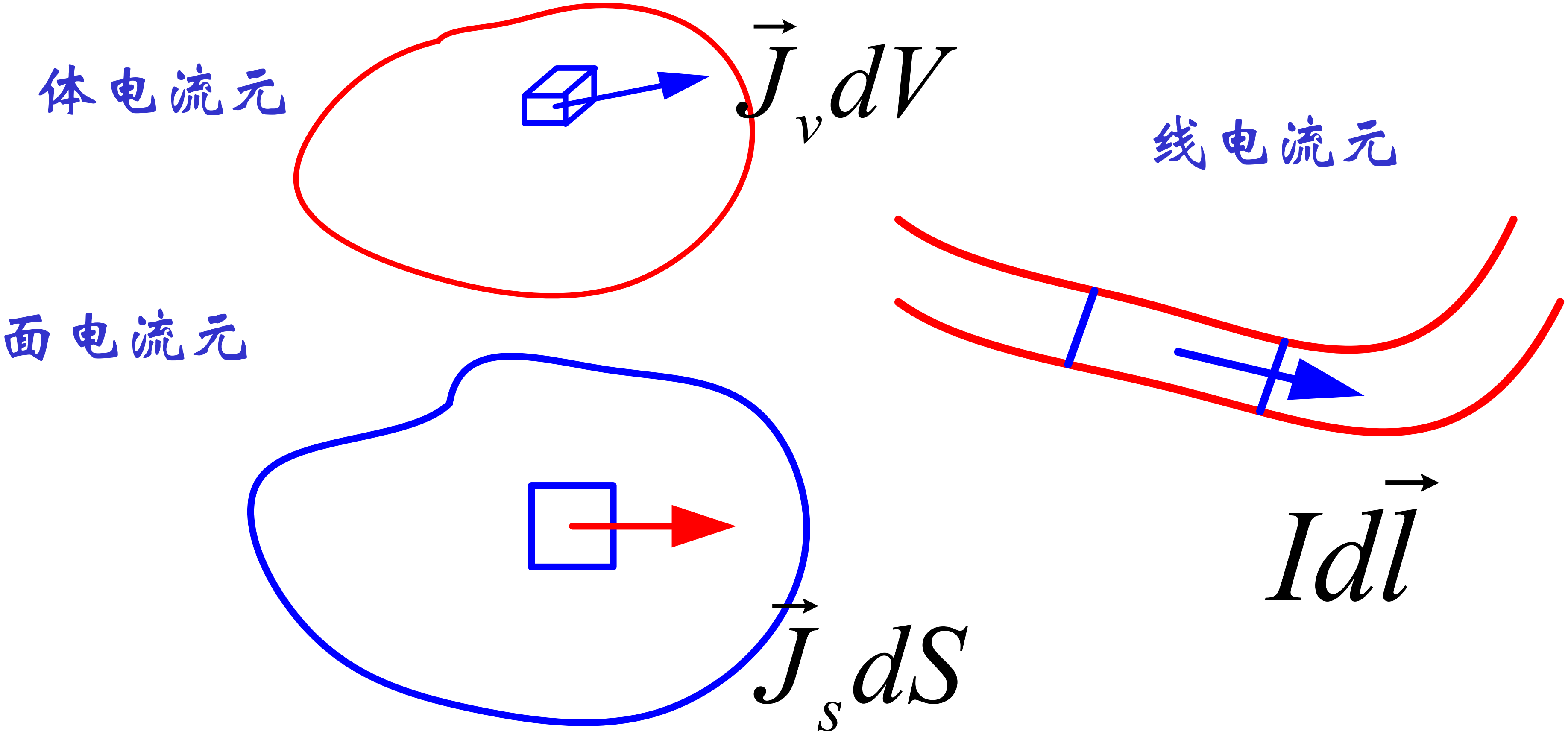
$$\boxed{\vec{J}_S = \sigma_S \vec{v} = ?(A/m) \quad (安/米)}$$

Current Line Density

NO!

There is no such a parameter.

电流元-单位长度的电流元



§ 5.1 Fundamental Equations

Contents

- **Old Contents: Experimental Laws**
  - Ampere's Force Law
  - Biot-Savart Law
  - Ampere's Circuital Law
- **New Contents**
  - Variables for Magnetostatics
  - M-Flux Density, or M-Induction Intensity
  - M-Field Intensity
  - Fundamental Equs

Variables for Magnetostatics

磁通： *Magnetic Flux* Φ

磁通密度： *Magnetic Flux Density* $\vec{B}$

M-Induction Intensity

磁场强度： *Magnetic Field Intensity* $\vec{H}$

$\vec{B}$ Unit: (1)Wb/m^2 (2)*Tesla* (T) (3)Gauss

$$1 Wb/m^2 = 1\text{Tesla} = 1\times10^4\text{Gauss}$$

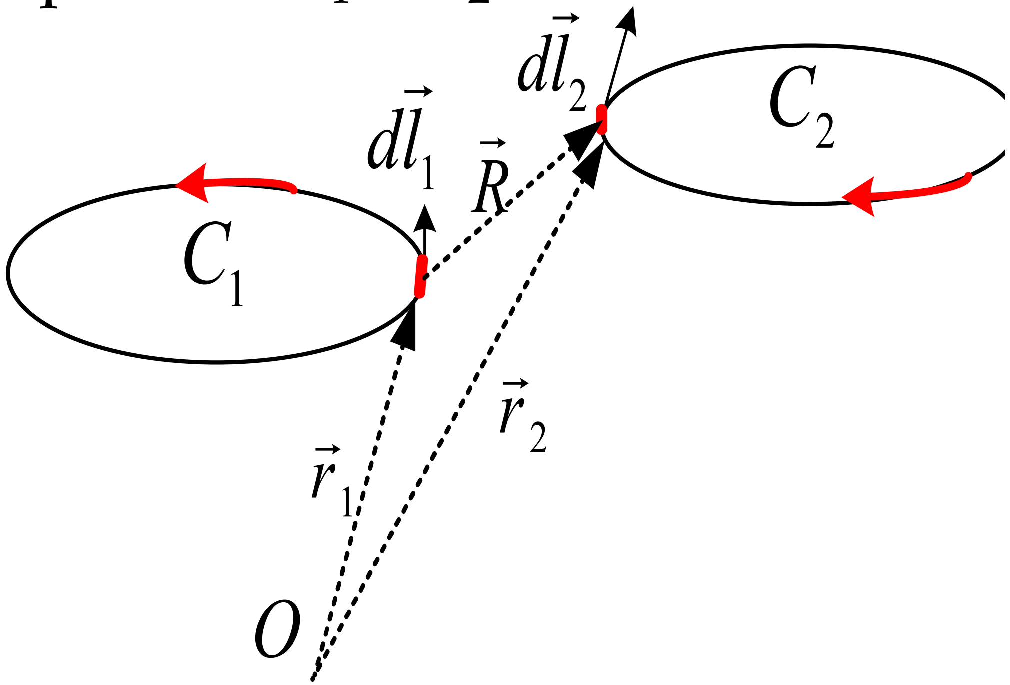
Clue of our Study

- From experiments to math expressions
  - Ampere's Force Law ➔ M-Flux Density
  - Biot-Savart Law ➔ Div. Equations
  - Ampere's Circuital Law ➔ Curl Equations
- Examples

1. Ampere's Force Law --- for M-Induction

- In **E-field**, work is done by **Coulomb Force**.
- In **M-field,** work is done by **Ampere's Force**
- Two current loops in free space --- C_1、C_2
- Force on C_2 by C_1 is…$F_{1\text{-}2}$

$$\vec{F}_{1-2} = \frac{\mu_0}{4\pi}\oint_{C_2}\oint_{C_1}\frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{a}_R)}{R^2}$$



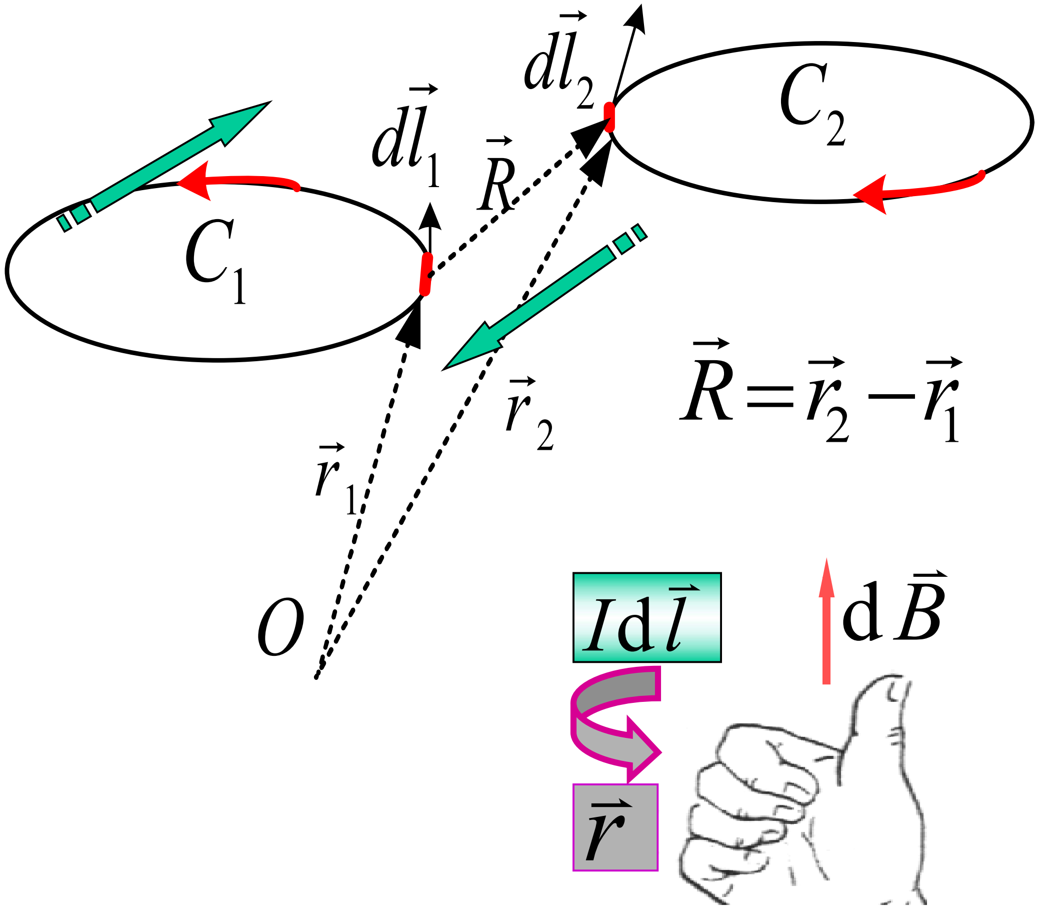
$$\vec{F}_{1-2} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{a}_R)}{R^2}$$

- Direction of the force?

1. Use cross product
2. Generic rule

同向电流相吸；

反向电流相斥。



$$\vec{F}_{1-2} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{a}_R)}{R^2}$$

Permeability in space (真空中<u>磁导率</u>):

$$\mu_0 = 4\pi \cdot 10^{-7} (H/m)$$

Dielectric Constant in space (真空中<u>介电常数</u>):

$$\varepsilon_0 = \frac{1}{4\pi \cdot 9 \times 10^9} = 8.85 \times 10^{-12} (F/m)$$

$$\frac{1}{\sqrt{\varepsilon_0 \cdot \mu_0}} = ?$$

$$\vec{F}_{1-2} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{a}_R)}{R^2}$$

- Direction of the force?

1. Use cross product
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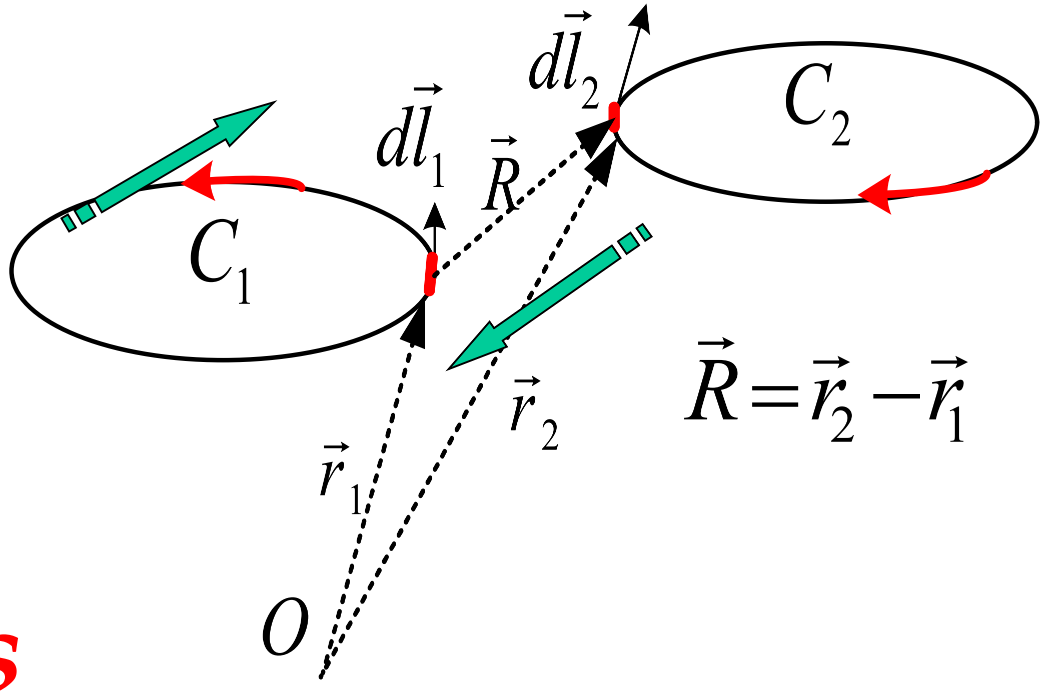
   同向电流相吸；

   反向电流相斥。



More discuss

2. M-Flux Density $\vec{B}$

$$\vec{F}_{1-2} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{I_2 d\vec{l}_2 \times (I_1 d\vec{l}_1 \times \vec{a}_R)}{R^2}$$

$$= \oint_{C_2} I_2 d\vec{l}_2 \times \left[\frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 d\vec{l}_1 \times \vec{a}_R}{R^2} \right]$$

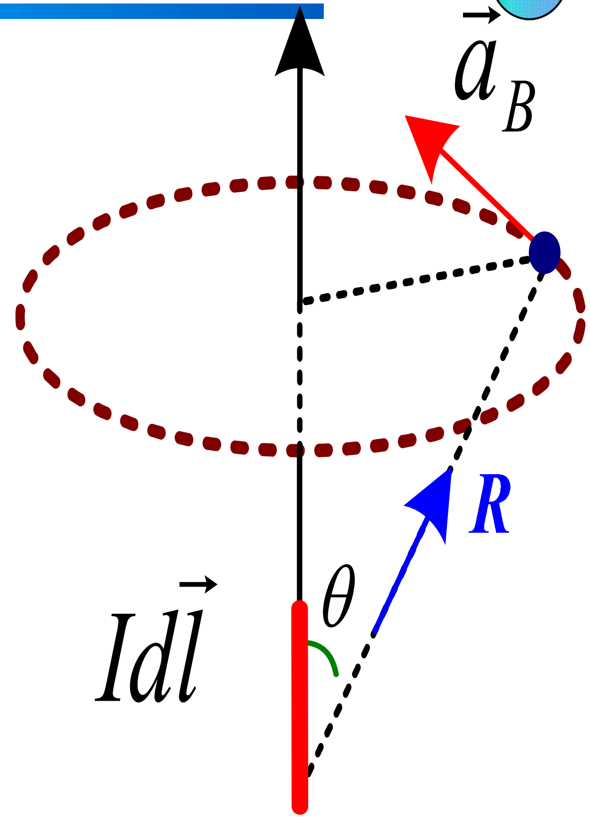
$$= \oint_{C_2} I_2 d\vec{l}_2 \times \vec{B}$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 d\vec{l}_1 \times \vec{a}_R}{R^2}}$$

M-flux density around any current

$$\vec{B} = \oint_S d\vec{B} = \frac{\mu_0}{4\pi} \oint_C \frac{I_{\text{source}} d\vec{l}_{\text{source}}}{R^2_{\text{Source-Spot}}} \times \vec{a}_R$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \left(\frac{I_{\text{source}} d\vec{l}_{\text{source}}}{R^2_{\text{Source-Spot}}} \times \vec{a}_R \right)$$



A Comparison

$$d\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{I_{\text{source}}\, d\vec{l}_{\text{source}}}{R^2_{\text{Source-Spot}}} \times \vec{a}_R\right)$$

M-Flux Density
(M-induction intensity)
by current element

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{dq_{\text{source}}}{R^2_{\text{Source-Spot}}}\vec{a}_R\right)$$

E-field intensity by
charge element

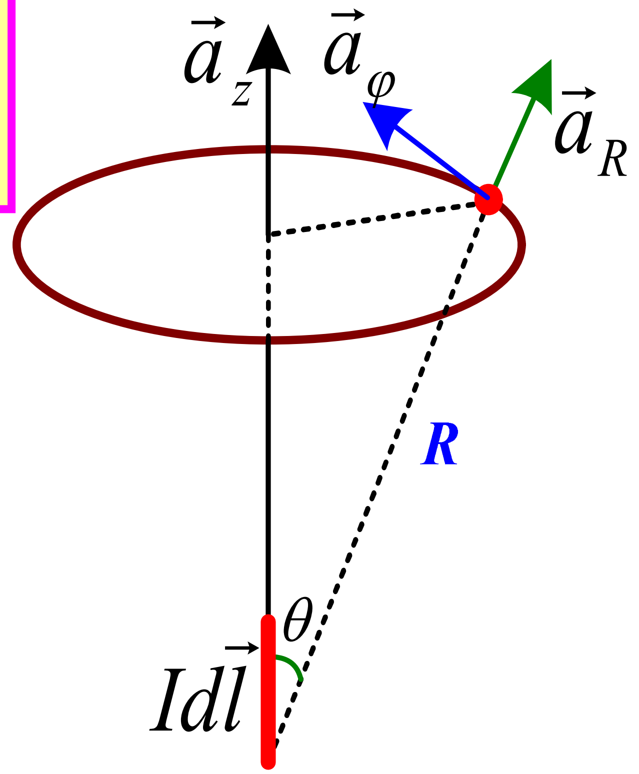
Quantity & Direction in Spherical Coordinates

Quantity

$$d\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{I_{\text{source}} d\vec{l}_{\text{source}}}{R^2_{\text{Source-Spot}}} \times \vec{a}_R\right)$$

Direction --- Right-Hand Rule

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Idl}{R^2}(\vec{a}_z \times \vec{a}_R) = \frac{\mu_0}{4\pi}\left(\frac{Idl \cdot \sin\theta}{R^2}\right)\vec{a}_\varphi$$



——Differential Form of Biot-Savart Law

3. Biot-Savart's Law

1. Differential Form

$$d\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{Idl' \cdot \sin\theta}{R^2}\right)\vec{a}_\varphi$$

2. Integral Form

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\vec{l}\,' \times \vec{a}_R}{R^2}$$

For line current

For volume current

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}_v \times \vec{a}_R}{R^2} dV'$$

For surface current

$$\vec{B} = \frac{\mu_0}{4\pi}\int_S \frac{\vec{J}_s \times \vec{a}_R}{R^2} dS'$$

5. Div. Equation for Magnetostatics

- The flux of $\vec{B}$ passing through a surface.

$$\oiint_S (\boldsymbol{n}\times\boldsymbol{a})\mathrm{d}S = \int_V \nabla\times\boldsymbol{a}\cdot\mathrm{d}V$$

$$\Phi = \int_S \vec{B}\bullet d\vec{S}$$

Unit: Wb (Weber)

- If C is the current loop and S is an arbitrary closed path,

$$\Phi = \oint_S \left[\frac{\mu_0}{4\pi}\oint_C \frac{Id\vec{l}\,'\times\vec{a}_R}{R^2}\right]\bullet d\vec{S}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\vec{l}\,'\times\vec{a}_R}{R^2}$$

$$(\vec{A}\times\vec{B})\bullet\vec{C} = (\vec{B}\times\vec{C})\bullet\vec{A} = (\vec{C}\times\vec{A})\bullet\vec{B}$$

$$= \oint_C \frac{\mu_0 I}{4\pi}\oint_S \frac{\vec{a}_R\times d\vec{S}}{R^2}\bullet d\vec{l}\,'$$

$$= \oint_C \frac{\mu_0 Id\vec{l}\,'}{4\pi}\bullet\oiint_S -\nabla\frac{1}{R}\times d\vec{S}$$

由北邮教材p419倒数第3式

$$= \oint_C \frac{\mu_0 Id\vec{l}\,'}{4\pi}\bullet\int_V \nabla\times\nabla\frac{1}{R}dV$$

$$\boxed{\oint_C \frac{\mu_0 I d\vec{l}\,'}{4\pi} \bullet \int_V \nabla \times \nabla \frac{1}{R} dV}$$

Curl of a gradient is always 0

$$\boxed{\Phi = \oint_C \frac{\mu_0 I d\vec{l}\,'}{4\pi} \bullet \int_V \nabla \times \nabla \frac{1}{R} dV \equiv 0}$$

$$\boxed{\Phi = \oint_S \vec{B} \bullet d\vec{S} \equiv 0}$$

——Div. Equ. in integral form

From Gauss's Law

$$\boxed{\oiint_S \vec{B} \bullet d\vec{S} = \int_V \nabla \bullet \vec{B} dV \equiv 0}$$

$$\boxed{\nabla \bullet \vec{B} = 0}$$

——Div. Equ. in differential form

→→ or, we may study the case from another point of view…

From another point of view

- **According to Biot-Savart's Law**

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\vec{l}\,' \times \vec{a}_R}{R^2} = \frac{\mu_0 I}{4\pi}\oint_C d\vec{l}\,' \times (-)\nabla\frac{1}{R} = \frac{\mu_0 I}{4\pi}\oint_C \nabla\frac{1}{R} \times d\vec{l}\,'$$

$$\because \nabla \times \psi\vec{A} = \nabla\psi \times \vec{A} + \psi\nabla \times \vec{A}$$

$$\therefore \nabla\frac{1}{R} \times d\vec{l}\,' = \nabla \times \frac{d\vec{l}\,'}{R} - \frac{1}{R}\nabla \times d\vec{l}\,' \qquad = \nabla \times \frac{d\vec{l}\,'}{R}$$

This term equals 0

$\nabla \times$ acts upon the observation spots.

while $d\vec{l}\,'$ is in fact the element of the source.

$$\therefore \vec{B} = \frac{\mu_0}{4\pi}\oint_C \nabla \times \frac{I d\vec{l}\,'}{R} = \nabla \times \left[\frac{\mu_0}{4\pi}\oint_C \frac{I d\vec{l}\,'}{R}\right]$$

$$\vec{B} = \nabla \times \left[\frac{\mu_0}{4\pi} \oint_C \frac{I d\vec{l}\,'}{R} \right] = \nabla \times \vec{A}$$

Obviously, ***M-flux density is the curl of a vector.***

The divergence of a curl is always 0.

$$\nabla \bullet \vec{B} = 0$$

Div. Equ. in differential form for M-field

Spread the wings of your fancy

$$\nabla \bullet \vec{D} = \rho_{fc}$$

Div. Equ. in differential form for E-field

$\vec{B} = \nabla \times \vec{A}$, $\vec{A}$ is in fact the M - vector potential.

6. M-Field Intensity

$$\vec{H} = \frac{\vec{B}}{\mu}$$

From Biot-Savart's Law:

$$\vec{B} = \frac{\mu}{4\pi} \oint_C \frac{I_{\text{source}} d\vec{l}_{\text{source}}}{R^2_{\text{Source-Spot}}} \times \vec{a}_R$$

$$\vec{H} = \frac{I_{\text{source}}}{4\pi} \oint_C \frac{d\vec{l}_{\text{source}}}{R^2_{\text{Source-Spot}}} \times \vec{a}_R$$

7. Ampere's Circuital Law

- **The line integral of M-field intensity around a closed path equals the current enclosed.**

$$\oint_C \vec{H} \bullet d\vec{l} = I$$

- **According to Stoke's Law, we have the differential form:**

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \nabla \times \vec{H} \bullet d\vec{S} = I = \int_S \vec{J} \bullet d\vec{S}$$

$$\nabla \times \vec{H} = \vec{J}$$

- **They are in fact the Curl Equation for M-field, in integral and differential forms.**

补充说明：

- 安培环路实验定律的严格证明比较繁琐
  - 北邮教材P74给出了特殊情况下(无限长直导线)的证明
- 一般情况的证明：
  - 书(谢处方) 第2版pp.110-112, 第3版pp.106-107
  - 书(毕德显) pp.244-246, pp.254-255

Summary --- Fundamental Equations for Magnetostatics

Differential form:

$$\vec{B} = \mu_0 \vec{H}$$

- Div. Equation: $\boxed{\nabla \bullet \vec{B} = 0}$
- Curl Equation: $\boxed{\nabla \times \vec{H} = \vec{J}}$ $\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$

Integral form:

- Continuity of M-flux $\boxed{\oint_S \vec{B} \cdot d\vec{S} = 0}$
- Ampere's Circuital Law $\boxed{\oint_C \vec{H} \bullet d\vec{l} = I}$ $\boxed{\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I}$

Physical Meaning:

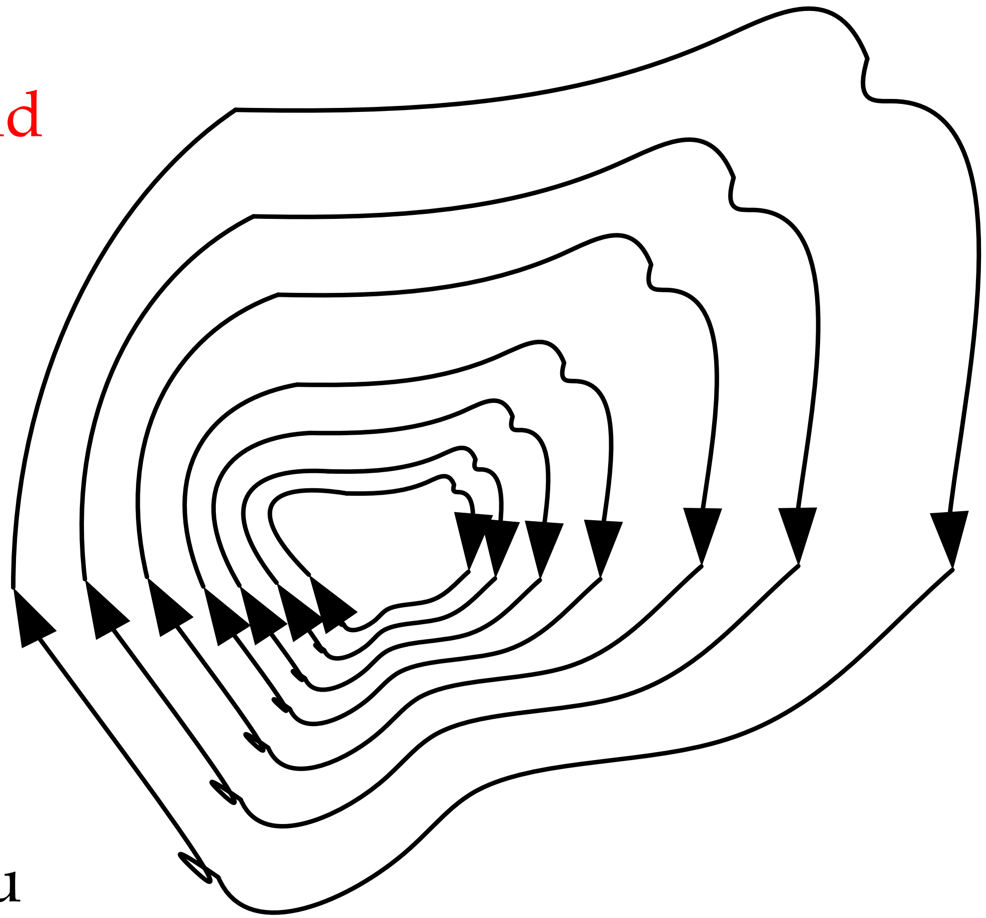
- Lines of M-flux are always continuous. They start and end at nowhere.
- There is no divergence but curl for static M-field.

Lines of M-flux are always continuous. They start and end at nowhere.

We have found no example that the magnetic poles do not exist in pairs.

No single magnetic pole has been found up to now.

If you may find such one, you should be awarded the Nobel Prize.



Comparisons

$$\begin{cases} \nabla \bullet \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \times \vec{E} = 0 \end{cases} \quad ? \quad \leftarrow \quad \begin{cases} \nabla \bullet \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{J} \end{cases}$$

$$\begin{cases} \oint_S \vec{E} \bullet d\vec{S} = \int_V (\nabla \bullet \vec{E}) dV = ? \\ \oint_C \vec{E} \bullet d\vec{l} = 0 \end{cases} \quad \rightarrow \quad ? \quad \begin{cases} \oint_S \vec{B} \bullet d\vec{S} = 0 \\ \oint_C \vec{B} \bullet d\vec{l} = \mu_0 I \end{cases}$$

Gauss's Law for Electrostatics

$$\oint_S \vec{E} \bullet d\vec{S} = \int_V (\nabla \bullet \vec{E}) dV = ?$$

$$\begin{cases} \oint_S \vec{B} \bullet d\vec{S} \equiv 0 & \text{No single M-pole!} \\ \oint_S \vec{E} \bullet d\vec{S} = \frac{Q}{\varepsilon_0} & \text{Single E-pole!} \end{cases}$$

Ampere's Circuital Law

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 \cdot I$$

$$\begin{cases} \oint_C \vec{E} \bullet d\vec{l} = 0 \\ \oint_C \vec{B} \bullet d\vec{l} = \mu_0 \cdot I \end{cases}$$

Note that I is current ringed by the closed path C.

Two Zero Integrals

$$\begin{cases} \oint_C \vec{E} \bullet d\vec{l} = 0 \\ \oint_S \vec{B} \bullet d\vec{S} = 0 \end{cases}$$

Physical Meaning:

(1) E-field, potential energy, conservative field

(2) M-field, no single M-pole, no divergence $\nabla \bullet \vec{B} = 0$

(3) We have only electric charge but not magnetic charge.

8. examples --- How to calculate $\vec{B}$ & $\vec{H}$?

- **In general**
  - Directly integral via Biot-Savart Law

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\vec{l}\,' \times \vec{a}_R}{R^2}$$

Line current

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}_v \times \vec{a}_R}{R^2} dV'$$

Volume current

$$\vec{B} = \frac{\mu_0}{4\pi}\int_S \frac{\vec{J}_s \times \vec{a}_R}{R^2} dS'$$

Surface current

- **In symmetrical cases**
  - Ampere's circuital law

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I \qquad \oint_C \vec{H} \bullet d\vec{l} = I$$

Example 1. M-field at axis of a circular current

Example 5.2 in page 181

Symmetry? Yes! But no proper closed path that …

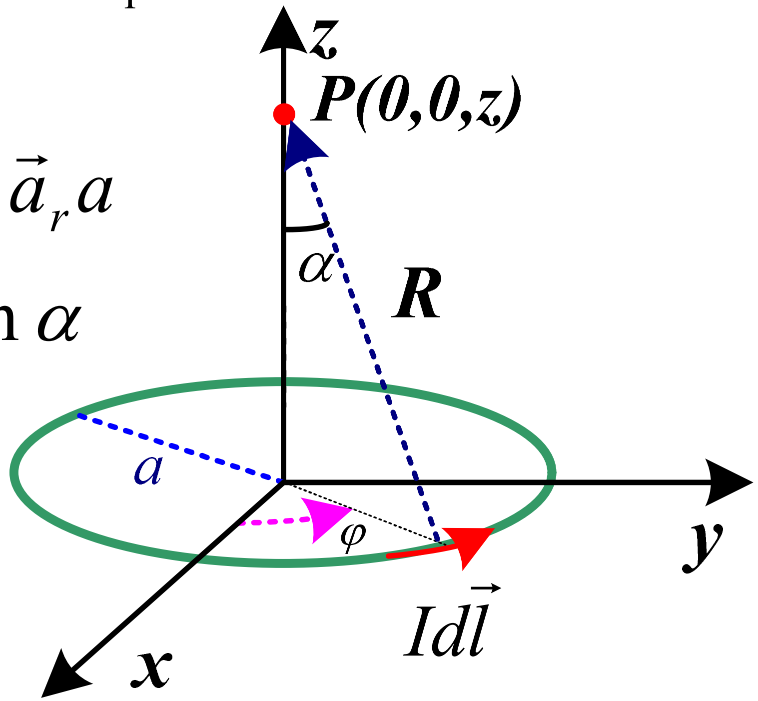
Direct solution $$\vec{B} = \oint_S d\vec{B} = \frac{\mu_0}{4\pi}\oint_C \frac{I_{\text{source}} d\vec{l}_{\text{source}}}{R^2_{\text{Source-Spot}}} \times \vec{a}_R$$

$$I d\vec{l} = \vec{a}_\varphi (I \cdot a \cdot d\varphi)$$

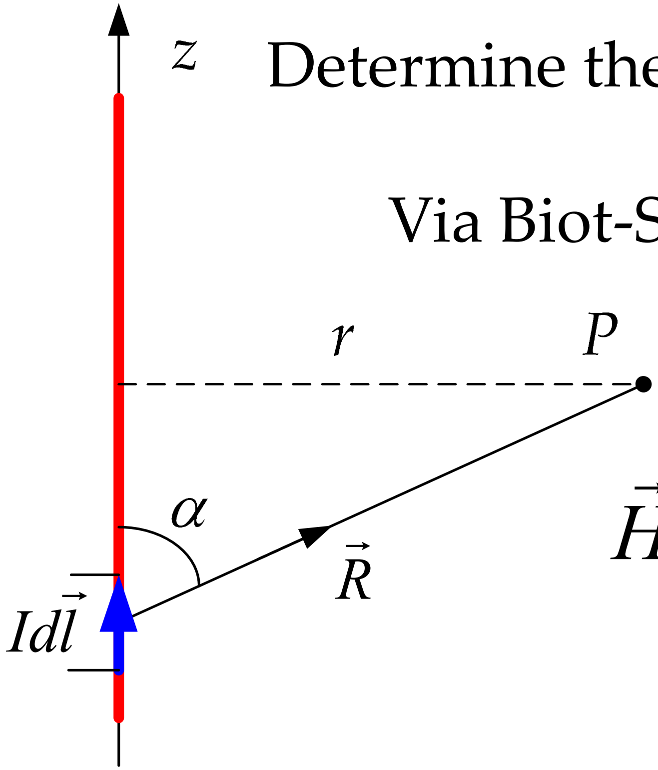
$$\vec{R} = \vec{a}_z R\cos\alpha - \vec{a}_r R\sin\alpha = \vec{a}_z z - \vec{a}_r a$$

unit vector : $\vec{a}_R = \vec{a}_z \cos\alpha - \vec{a}_r \sin\alpha$

$$\vec{B} = \vec{a}_z \frac{\mu_0 I a^2}{2\left(z^2 + a^2\right)^{\frac{3}{2}}} \quad (T)$$



Example 3. current in infinite filamentary wire



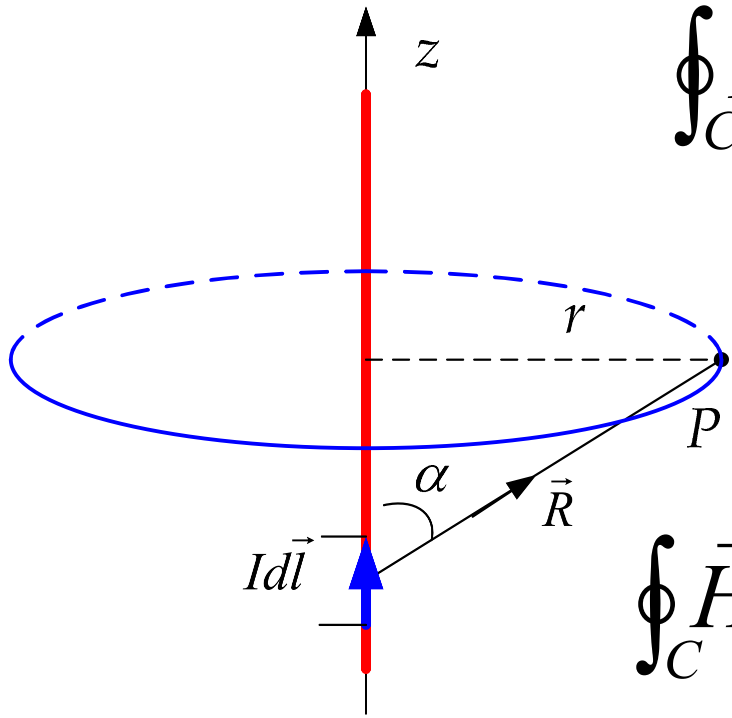
Determine the M-Intensity in space.

Via Biot-Savart law $$\vec{B} = \frac{\mu_0 I_{\text{source}}}{4\pi} \oint_C \frac{d\vec{l}_{\text{source}} \times \vec{a}_R}{R^2_{\text{Source-Spot}}}$$

$$\vec{H} = \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{d\vec{z} \times \vec{a}_R}{R^2} = \frac{\vec{a}_\varphi I}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{R^2} dz$$

$$= \frac{\vec{a}_\varphi I}{4\pi} \int_{-\infty}^{\infty} \frac{r / \sqrt{r^2 + z^2}}{r^2 + z^2} dz \boxed{= \vec{a}_\varphi \frac{I}{2\pi r}}$$

Solution 2. via Ampere's circuital law



$$\oint_C \vec{H} \bullet d\vec{l} = I$$

Passing through point *P*, construct an auxiliary line of a circular loop *C*, and *I* is normal to the plane of *C*.

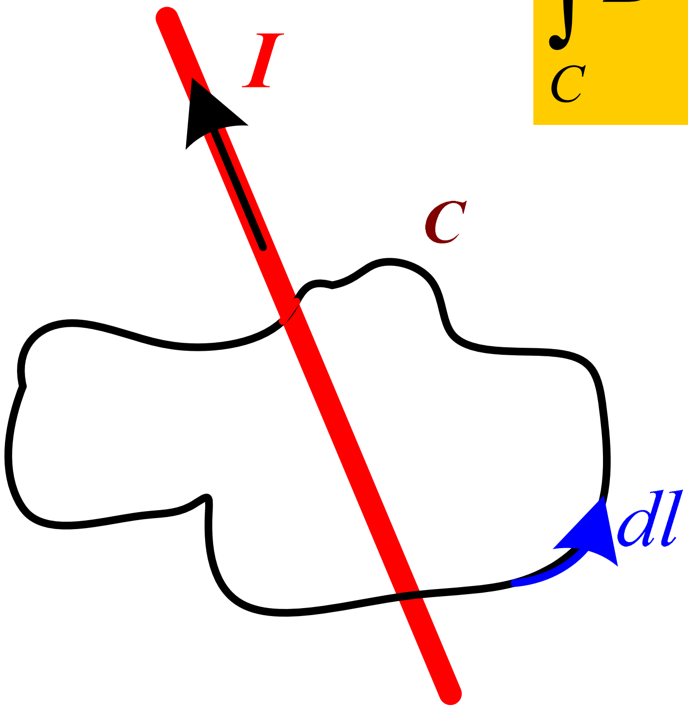
$$\oint_C \vec{H} \bullet d\vec{l} = \oint_C H_\varphi dl = H_\varphi \cdot 2\pi r = I$$

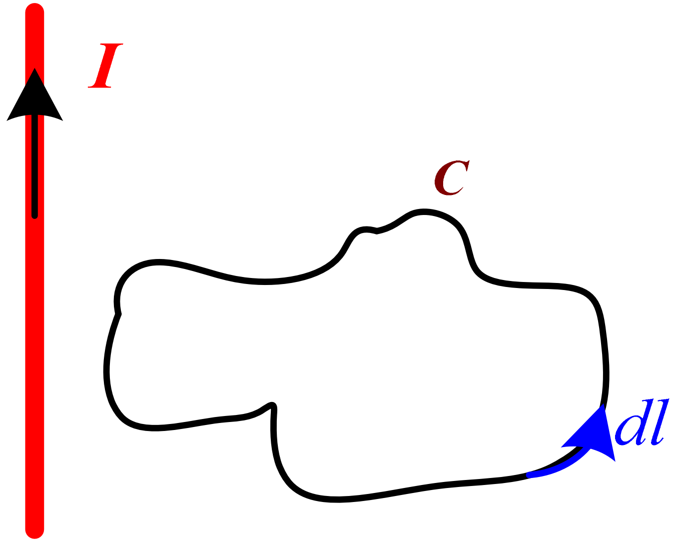
$$\therefore \quad H_\varphi = I/2\pi r$$

$$\therefore \quad \vec{H} = \vec{a}_\varphi I/2\pi r$$

Question: given a current I in infinite thin line, and a closed path C. please calculate the integral of M-intensity.

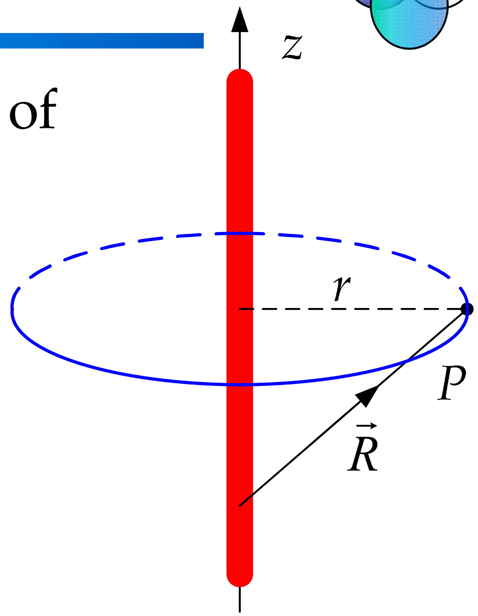
$$\oint_C \vec{B} \bullet d\vec{l} = ?$$





Example 4. cylindrical current with radius *a*

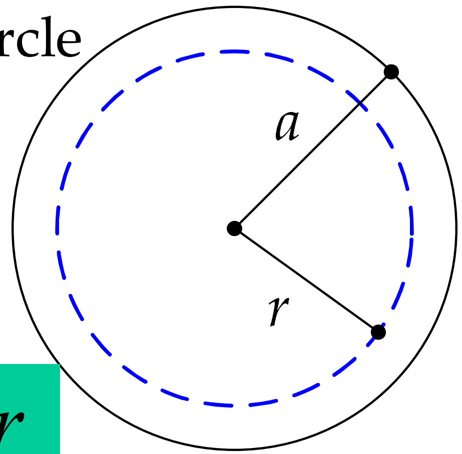
Exercise 5.16



- The cylindrical current I uniformly distributes, with of radius a. Please determine the M-intensity.
- Analysis: treat field inside/outside respectively
- M-intensity outside: same as Example 3.

$$\vec{H} = \vec{a}_\varphi \frac{I}{2\pi r} \quad (r > a)$$

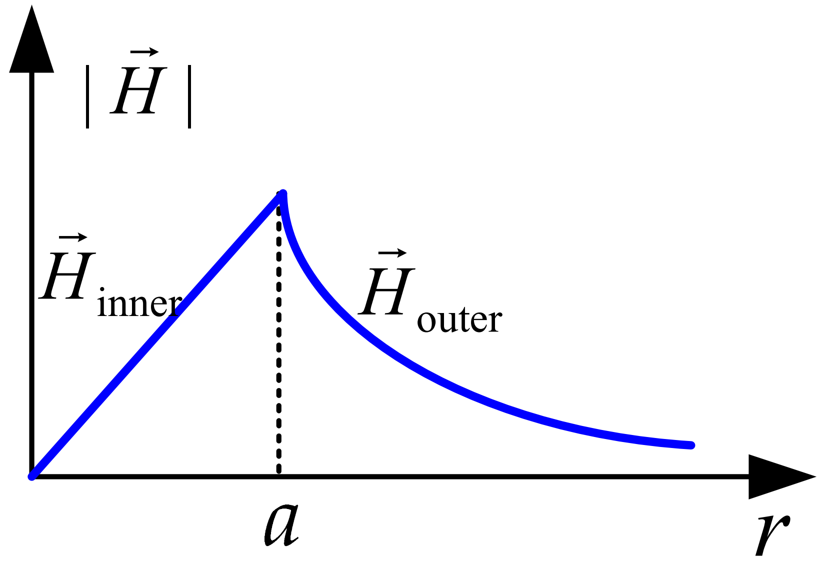
- Within the volume current: construct an auxiliary circle within the wire and perpendicular to the axis.



$$\oint_C \vec{H} \bullet d\vec{l} = \oint_C H_\varphi dl = H_\varphi \cdot 2\pi r = ?$$

$$I_{\text{ringed}} = J \cdot \pi r^2 = \frac{I}{\pi a^2} \cdot \pi r^2 = \frac{I r^2}{a^2}$$

$$\vec{H} = \frac{\vec{a}_\varphi I r}{2\pi a^2}$$



$$\vec{H}_{\text{inner}} = \frac{\vec{a}_{\varphi} I r}{2\pi a^2}$$

$$\vec{H}_{\text{outer}} = \vec{a}_{\varphi} \frac{I}{2\pi r}$$

Summary:

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 \cdot I$$

On applications of *A-C Law:*
(1) Construct an auxiliary curve. $(\vec{l}\,)$
(2) Quantity of $\vec{B}$ at the curve shall be constant.

$$\oint_C \vec{B} \bullet d\vec{l} = \oint_C B \cdot dl = "B \cdot l"$$

By comparison, in electrostatics

If the charges distributes symmetrically,
try *E-Gauss's Law* !

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{Q}{\varepsilon_0}$$

On applications of *E-G Law:*
(1) Construct an auxiliary surface. $(\vec{S})$
(2) Quantity of $\vec{E}$ on the surface shall be constant.