Advanced Transform Methods

Sampling and the Discrete Fourier Transform (DFT)

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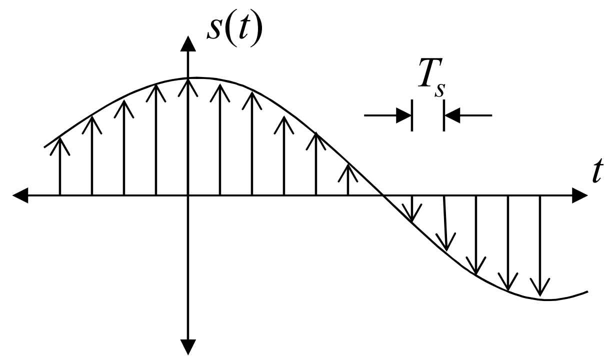
Sampling: Time Domain

- Many signals (all real-world signals) originate as continuous-time signals, e.g. conventional music or voice
- By sampling a continuous-time signal at isolated, equally-spaced points in time, we obtain a sequence of numbers (a discrete signal)

$$s[k] = s(k\,T_s)$$

$k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

T_s is the sampling period.

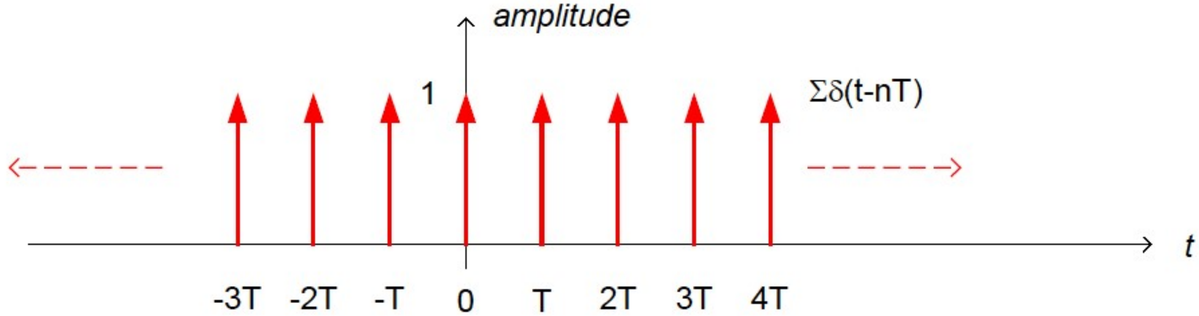


Sampled analog waveform

$$s_{sampled}(t) = s(t) \underbrace{\sum_{k=-\infty}^{\infty} \delta(t - k\,T_s)}_{\text{impulse train } \delta_{T_s}(t)} = \sum_{k=-\infty}^{\infty} \underbrace{s(k\,T_s)}_{s[k]} \delta(t - k\,T_s)$$

“Comb” of Delta Functions

$$f(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



Sampling = multiplication by comb of delta functions

In frequency domain -> convolve by “FT of comb of delta functions”.

So - What is “FT of comb of delta functions”?

Repeats with interval T, so can use Fourier Series expansion:

$$f(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi k f_0 t} \quad \text{where} \quad f_0 = 1/T$$

So – need to solve for a_k

FT of Comb of Delta Functions

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j2\pi f_0 t}dt$$ Fourier series term

$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t-nT)e^{-j2\pi f_0 t}dt$$ by expanding $f(t)$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-j2\pi f_0 t}dt$$ only $\delta(t-nT)$ in $-T/2 < t < T/2$

$$= \frac{1}{T}e^{-j2\pi f_0 0} = \frac{1}{T}$$ by action of delta function

so $$f(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi k f_0 t} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j2\pi k f_0 t}$$ sum of sinusoids

$$F(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty}\delta(f - kf_0)$$ or $$F(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_0)$$

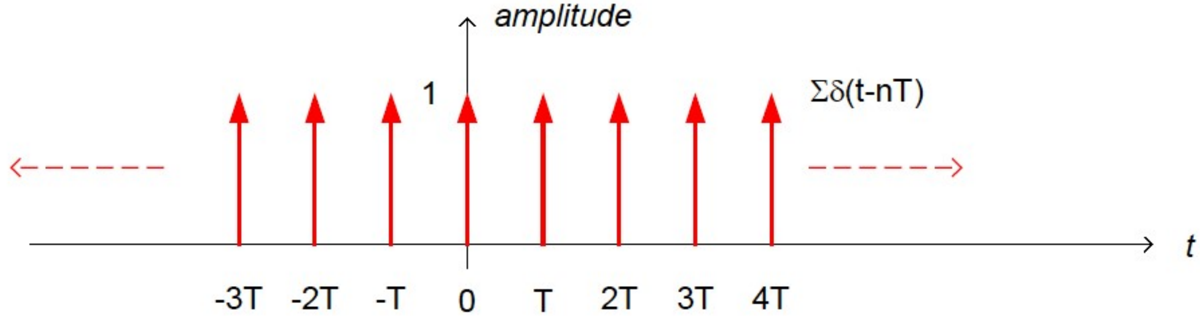
$$\omega = 2\pi f$$

Comb of Delta Functions: Summary

In Time Domain

$$f(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

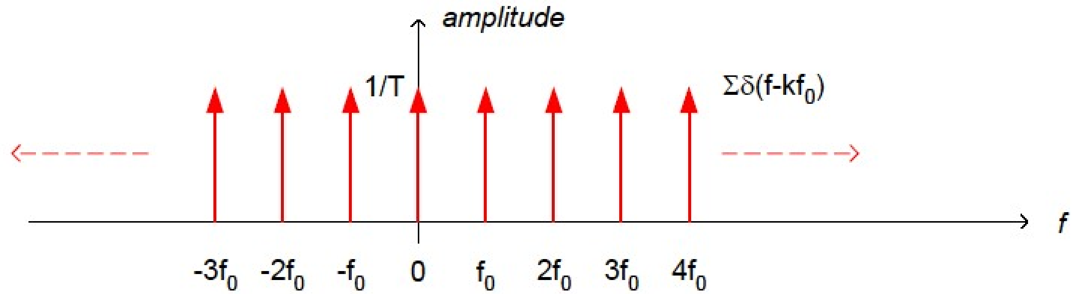
$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{j2\pi k f_0 t}$$



In Frequency Domain

$$F(f) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} \delta(f - kf_0)$$

$$F(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$



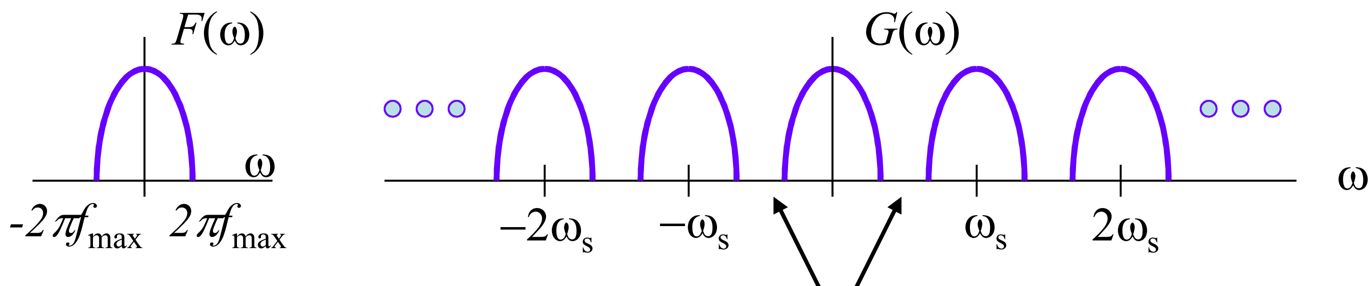
So: sampling in time domain = convolution with comb of deltas in freq domain

Sampling: Frequency Domain

- Sampling replicates spectrum of continuous-time signal at integer multiples of sampling frequency
- Fourier series of impulse train where ω_s **= 2** π $\boldsymbol{f_s}$

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\,T_s) = \frac{1}{T_s}\left(1 + 2\cos(\omega_s\, t) + 2\cos(2\,\omega_s\, t) + \cdots\right)$$

$$g(t) = f(t)\,\delta_{T_s}(t) = \frac{1}{T_s}\left(f(t) + \underbrace{2 f(t)\cos(\omega_s\, t)}_{\text{Modulation by } \cos(\omega_s\, t)} + \underbrace{2 f(t)\cos(2\,\omega_s\, t)}_{\text{Modulation by } \cos(2\,\omega_s\, t)} + \cdots \right)$$



$\boxed{\text{gap if and only if } 2\pi f_{\max} < 2\pi f_s - 2\pi f_{\max} \Leftrightarrow f_s > 2 f_{\max}}$

Amplitude Modulation by Cosine

- Multiplication in time: convolution in Fourier domain

$$y(t) = f(t)\cos(\omega_0 t)$$

$$Y(\omega) = \frac{1}{2\pi} F(\omega) * \pi\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right)$$

- Sifting property of Dirac delta functional

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau) x(t - \tau) d\tau = x(t)$$

$$x(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t - \tau) d\tau = x(t - t_0)$$

- Fourier transform property for modulation by a cosine

$$Y(\omega) = \frac{1}{2} F(\omega + \omega_0) + \frac{1}{2} F(\omega - \omega_0)$$

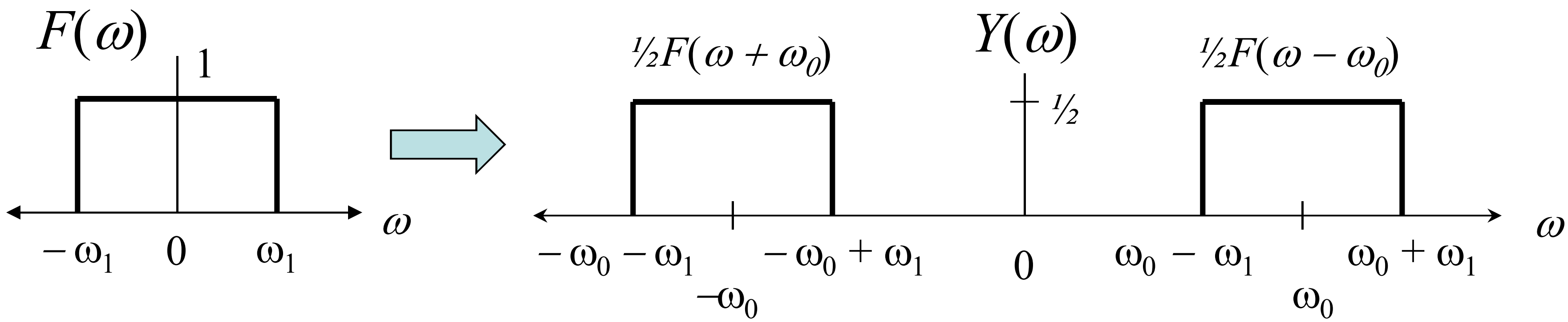
Amplitude Modulation by Cosine

- Example: $\boldsymbol{y(t) = f(t)\cos(\omega_0 t)}$

  Assume $f(t)$ is ideal lowpass signal with bandwidth ω_1

  Assume $\omega_1 << \omega_0$

  $Y(\omega)$ is real-valued if $F(\omega)$ is real-valued



- Demodulation: modulation then lowpass filtering
- Similar derivation for modulation with $\mathbf{sin(\omega_0\, t)}$

Shannon Sampling Theorem

- Continuous-time signal $\boldsymbol{x(t)}$ with frequencies no higher than $\boldsymbol{f_{max}}$ can be reconstructed from its samples $\boldsymbol{x(k\,T_s)}$ if samples taken at rate $\boldsymbol{f_s > 2\,f_{max}}$

  Nyquist rate = $2\,f_{max}$

  Nyquist frequency = $f_s / 2$

Critical Sampling if $f_s = 2\,f_{max}$

- Example: Sampling audio signals

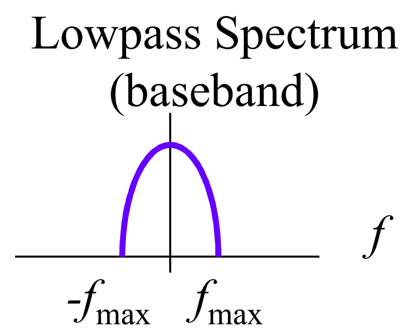
  Human hearing is from about 20 Hz to 20 kHz

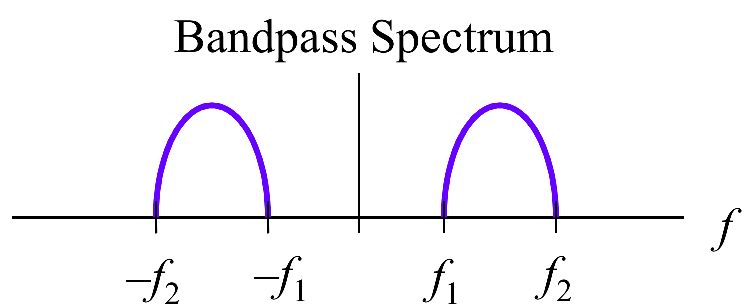
  Apply lowpass filter before sampling to pass frequencies up to 20 kHz and reject high frequencies

  Lowpass filter needs 10% of maximum passband frequency to roll off to a sufficiently small value (2 kHz rolloff in this case), hence high order filter.

Generalized Sampling Theorem

- Sampling rate must be greater than analog signal's bandwidth
  - Bandwidth is defined as non-zero extent of spectrum of the continuous-time signal in positive frequencies
  - Lowpass spectrum on right: bandwidth is f_{max}
  - Bandpass spectrum on right: bandwidth is $f_2 – f_1$



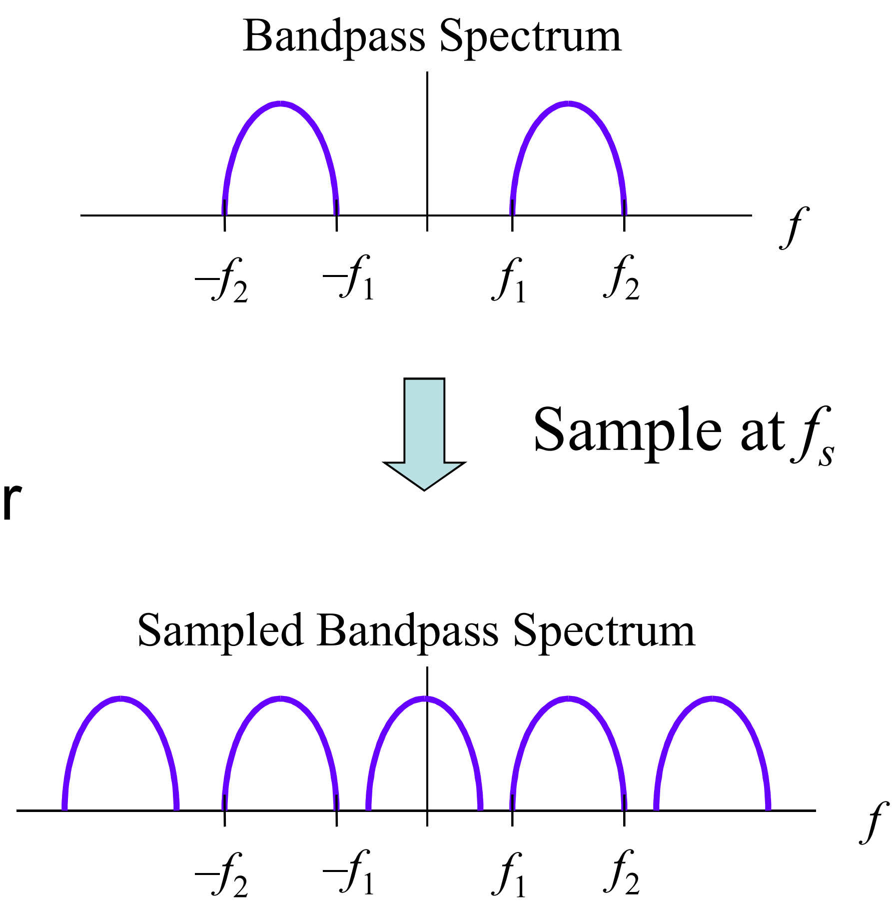


Bandpass Sampling

- Bandwidth: $\boldsymbol{f_2 - f_1}$
- Sampling rate $\boldsymbol{f_s}$ must greater than analog bandwidth

  $f_s > f_2 - f_1$
- For replicas of bands to be centered at origin after sampling

  $f_c = ½ (f_1 + f_2) = k\, f_s$
- Lowpass filter to extract baseband



Aliasing

1. Analog sinusoid

   $x(t) = A\cos(2\pi f_0 t + \phi)$

2. Sample at $\boldsymbol{T_s = 1/f_s}$

   $x[n] = x(T_s n) =$
   $A\cos(2\pi f_0 T_s n + \phi)$

3. Keeping the sampling period same, sample

   $y(t) = A\cos(2\pi (f_0 + l f_s)\, t + \phi)$

   where $\boldsymbol{l}$ is an integer

4. $y[n]= y(T_s n)$

$$
\begin{aligned}
&= A\cos(2\pi(f_0 + lf_s)T_s n + \phi)\\
&= A\cos(2\pi f_0 T_s n + 2\pi l f_s T_s n + \phi)\\
&= A\cos(2\pi f_0 T_s n + 2\pi l n + \phi)\\
&= A\cos(2\pi f_0 T_s n + \phi)\\
&= x[n]
\end{aligned}
$$

Here, $f_s T_s = 1$

5. Since l is an integer,
   $\cos(x + 2\pi l) = \cos(x)$

6. $\boldsymbol{y[n]}$ indistinguishable from $\boldsymbol{x[n]}$

Aliasing

- Since l is any integer, a countable but infinite number of sinusoids will give same sequence of samples
- Frequencies $f_0 + l f_s$ for $l \neq 0$ are called aliases of frequency f_0 with respect to f_s

  All aliased frequencies appear to be the same as f_0 when sampled by f_s

Folding

- Second source of aliasing frequencies
- From negative frequency component of a sinusoid, $-f_0 + l f_s$,

$$w(t) = A\cos(2\pi(-f_0 + l\, f_s)\, t - \phi)$$

where l is any integer

f_s is the sampling rate

f_0 is sinusoid frequency

- Sampling $w(t)$ with a sampling period of $T_s = 1/f_s$

$$\begin{aligned} w[n] &= w(T_s\, n) \\ &= A\cos(2\pi(-f_0 + l\, f_s)\, T_s\, n - \phi) \\ &= A\cos(-2\pi\, f_0\, T_s\, n + 2\pi\, l\, f_s\, T_s\, n - \phi) \\ &= A\cos(-2\pi\, f_0\, T_s\, n + 2\pi\, l\, n - \phi) \\ &= A\cos(-2\pi\, f_0\, T_s\, n - \phi) \\ &= A\cos(2\pi\, f_0\, T_s\, n + \phi) \end{aligned}$$

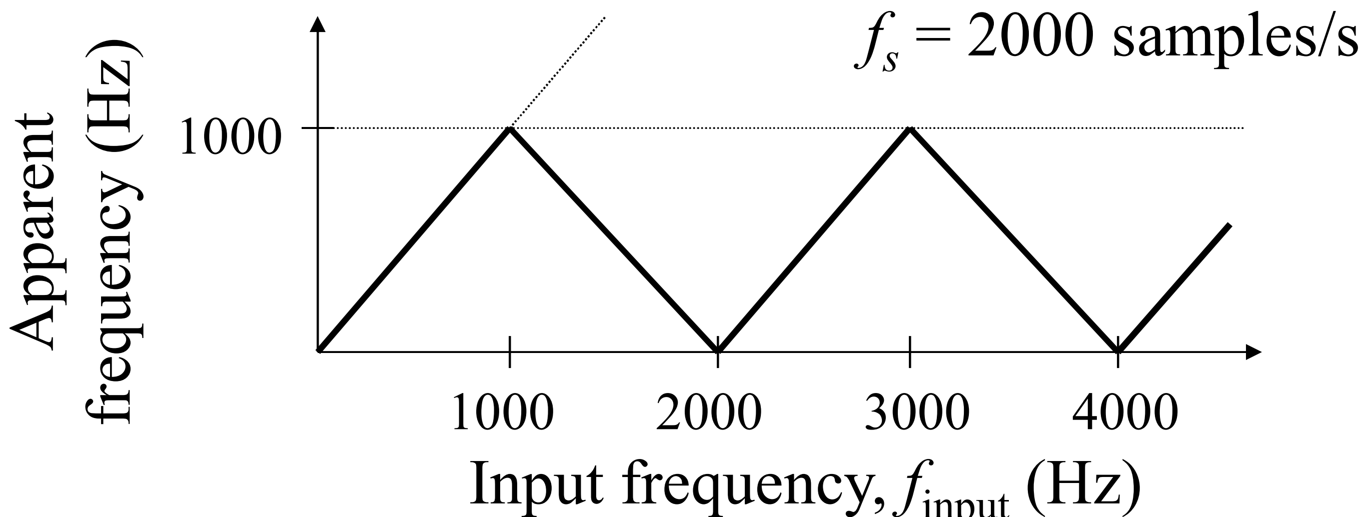
So

$$w[n] = x[n] = x(T_s\, n)$$

$$x(t) = A\cos(2\pi f_0\, t + \phi)$$

Aliasing and Folding

- Aliasing and folding of a sinusoid $\mathbf{sin(2\,\pi\, \boldsymbol{f}_{input}\, \boldsymbol{t})}$ sampled at $\boldsymbol{f_s} = \mathbf{2000}$ samples/s with $\boldsymbol{f}_{\mathbf{input}}$ varied



- Mirror image effect about $\boldsymbol{f}_{\mathbf{input}} = \mathbf{½}\, \boldsymbol{f_s}$ gives rise to name of folding

So if sampled at 2000 samples/s, 1000Hz appears as 1000Hz; 2000Hz appears as 0Hz; 3000Hz as 1000Hz; 3500Hz as 500Hz; etc.

Discrete-Time Fourier Transform

- Forward transform of discrete-time signal *x*[*n*]

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\,\omega\, n}$$

  – Assumes that *x*[*n*] is two-sided and infinite in duration
  – Produces *X*(ω) that is continuous and periodic in ω (in units of rad/sample) with period 2 π due to exponential term

- Inverse discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Discrete Fourier Transform

- Discrete Fourier transform (DFT) of a discrete-time signal $x[n]$ with finite extent $n \in [0, N\text{-}1]$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} = X(\omega)\Big|_{\omega=\frac{2\pi}{N}k} \quad \text{for } k = 0,\ 1,\ \ldots,\ N\text{-}1$$

  – $X[k]$ is periodic with period N due to exponential

  – DFT assumes $x[n]$ is also periodic with period N

- Inverse discrete Fourier transform

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk}$$

- "Twiddle factor" $W_N = e^{j\frac{2\pi}{N}} \Rightarrow x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{nk}$

Discrete Fourier Transform (con't)

- Forward transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{-nk}$$

  for k = 0, 1, …, N-1

  Exponent of W_N has period N
- Memory usage

  $x[n]$: N complex words of RAM

  $X[k]$: N complex words of RAM

  W_N: N complex words of ROM
- Halve memory usage

  Allow output array $X[k]$ to write over input array $x[n]$

  Exploit twiddle factor symmetry
- Computation

  N^2 complex multiplications

  N (N –1) complex additions

  N^2 integer multiplications

  N^2 modulo indexes into lookup table of twiddle factors
- Inverse transform

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{nk}$$

  for n = 0, 1, …, N-1

  Memory? Computation?

Watch out for sign of "twiddle factor" !!
Different texts use + or - !!

DFT in Practice

- The DFT is the only practical way to find the FT of a signal (any other version of the FT requires an infinite amount of calculation).
- The DFT assumes that the sequence to be transformed is periodic.
- Suppose we want to find the DFT of a piece of music.
- If we sample the whole piece of music we will find its accurate DFT.
- If we take a segment of the music, we can find its DFT.
- But the DFT assumes that the sequence is periodic and this is very unlikely to be the case for the music.
- So if we find the DFTs for each segment of the music, the result will not be an accurate FT of the music.
- Finding the DFT of the whole music is accurate but will take a long time because there are many samples.
- The shorter each segment (fewer samples) the faster the calculation but the less accurate the transform.