EBU6018
Advanced Transform Methods

Wavelet Transform from Filter Banks

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Note

The full derivation and proof of the following recursive equations for calculating the coefficients of the approximation and fine detail of the resulting filter bank are given in the book “Introduction to Time-frequency and Wavelet Transforms” by Shie Qian, Chapter 5.

The detailed proof is NOT required for this module, what is important is the fact that wavelet transforms can be applied using low pass and high pass filters.

These filters are based on the scaling (and wavelet) functions of the wavelet being used.

Wavelet transform from Filter Banks

From Multiresolution Analysis (MRA),

if signal $s(t)$ is in V_m for fininte m, then $s(t)$ determined by

$$s(t) = \sum_{n=-\infty}^{\infty} c_{m,n}\phi_{m,n}(t) \qquad \left[\sum_k c_k^j \phi(2^j t - k) \quad \text{in MRA lecture} \right]$$

Since $V_m = V_{m-1} \oplus W_{m-1}$ this can be written

$$s(t) = \sum_n c_{m_0,n}\phi_{m_0,n}(t) + \sum_{k=m_0}^{m-1} \sum_n d_{k,n}\psi_{k,n}(t) \qquad m > m_0$$

where coefficients $d_{m,n}$ and $c_{m,n}$ are

inner products between $\psi_{m,n}(t)$ and $\phi_{m,n}(t)$ respectively.

Approximating the signal

Using Parseval, we have

$$c_{m,n} = 2^{m/2}\int_{-\infty}^{\infty} s(t)\phi^*(2^m t-n)dt$$

$$= \frac{1}{2\pi}2^{-m/2}\int_{-\infty}^{\infty} S(\omega)\Phi^*(2^{-m}\omega)e^{-j2^{-m}\omega n}d\omega$$

For large scale m and $\Phi(0)=1$ (i.e. $\phi(t)$ normalised), have

$$c_{m,n} \approx \frac{1}{2\pi}2^{-m/2}\int_{-\infty}^{\infty} S(\omega)e^{-j2^{-m}\omega n}d\omega = 2^{-m/2}s(2^{-m}n)$$

(since $\Phi(\omega/2^m)\approx\Phi(0)=1$ over the range where $S(\omega)$ exists).

Thus $c_{m,n}$ approximates

$s(t)$ at $t=2^{-m}n$ with a scaling factor of $2^{-m/2}$.

Recursive computation of coeffs

Dilation equation $\phi(t) = \sum_k p_k \phi(at-k)$ for integer k

From MRA, if φ(t/2) ∈ V_{-1} then φ(t/2) ∈ V_0

And define a filter h_0[n]
Such that we get the
refinement equation:

Let us define

$c_{m,n} \equiv s[n] \equiv s(t)\big|_{t=2^{-m}n}$ for large m.

Using the dilation equation $\phi(t/2) = 2\sum_n h_0[n]\phi(t-n)$ we get

$$c_{m-1,n} = \int_{-\infty}^{\infty} s(t)\phi^*{}_{m-1,n}(t)dt = 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)\phi^*\left(\frac{2^m t - 2n}{2}\right)dt$$

$$= 2^{(m-1)/2}\int_{-\infty}^{\infty} s(t)2\sum_i h_0[i]\phi^*(2^m t - 2n - i)dt$$

$$= \sqrt{2}\sum_i h_0[i]\int_{-\infty}^{\infty} s(t)\phi^*{}_{m,2n+i}(t)dt = \sqrt{2}\sum_i h_0[i]c_{m,2n+i}$$

i.e. $$c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$$ ***

Filtering and downsampling

Given this eqn $c_{m-1,n} = \sqrt{2}\sum_i h_0[i-2n]c_{m,i}$

once $c_{m,n}$ is known, we can compute $c_{k,n}$ for $k < m$,

using a low pass filter $H_0{}^*(\omega)$ and downsampling $2n = i$.

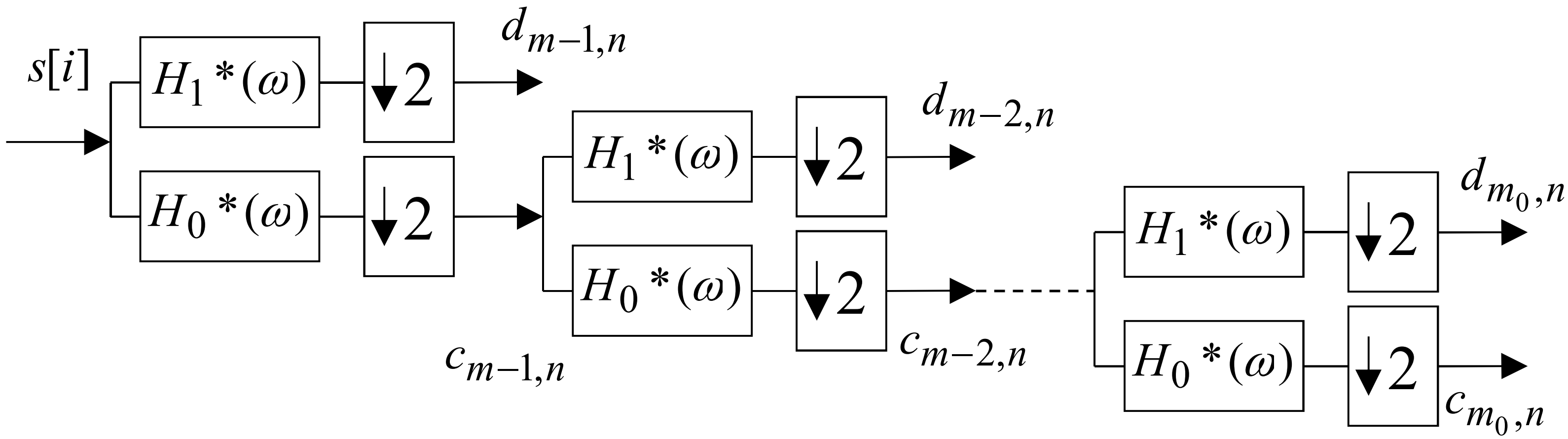
$$c_{m,n} \longrightarrow \boxed{H_0{}^*(\omega)} \longrightarrow \boxed{\downarrow 2} \longrightarrow c_{m-1,n}$$

Similarly, we can show $d_{m-1,n} = \sqrt{2}\sum_i h_1[i-2n]c_{m,i}$ ***

i.e. a high pass filter $H_1{}^*(\omega)$ and downsampling.

$$c_{m,n} \longrightarrow \boxed{H_1{}^*(\omega)} \longrightarrow \boxed{\downarrow 2} \longrightarrow d_{m-1,n}$$

Filter Bank for Wavelet Series Coeffs

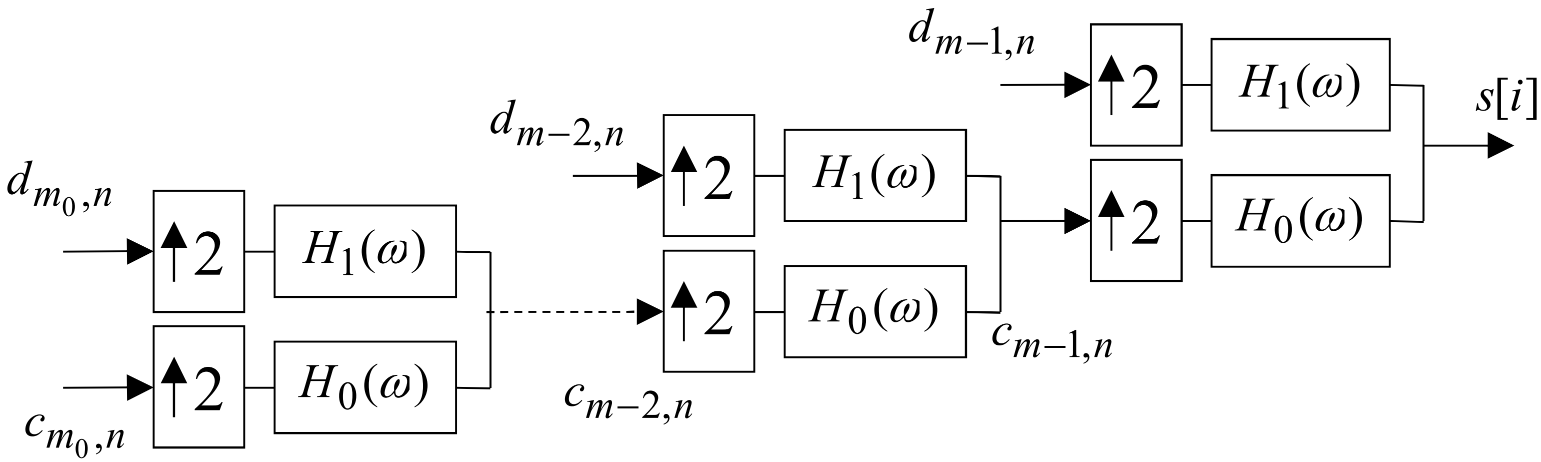


So for discrete-time samples $s[i]$, can compute wavelet transform directly by applying filter banks. No need to compute the mother wavelet $\psi(t)$.

Signal recovery filterbank

Can also compute high - res coeffs from low - res coeffs :

$$c_{m,n} = \sqrt{2}\left(\sum_i h_0[n-2i]c_{m-1,i} + \sum_i h_1[n-2i]d_{m-1,i}\right)$$



(For proof, see e.g. Qian)

So – don’t need scaling functions or wavelets, just filter banks!

Example 1

Suppose we have a Haar wavelet transform analysis filterbank which uses a low-pass filter

$$h_0[0] = h_0[1] = \tfrac{1}{2}$$

And a high pass filter

$$h_1[0] = \tfrac{1}{2} \quad h_1[1] = -\tfrac{1}{2}$$

Use the recursive equations:

$$\begin{aligned} c_{m-1,n} &= \sqrt{2} \cdot \frac{1}{2}(c_{m,2n} + c_{m,2n+1}) \\ &= \frac{1}{\sqrt{2}}(c_{m,2n} + c_{m,2n+1}) \\ d_{m-1,n} &= \frac{1}{\sqrt{2}}(c_{m,2n} - c_{m,2n+1}) \end{aligned}$$

to calculate the Haar wavelet transform for a sampled signal s[n] = [2, 5, -3, 7] after 1 and 2 stages of the transform filterbank.

Example 1….Solution

Start with the signal in the finest resolution coefficient,

$S[n] = [2, 5, -3, 7]$

First level:

$C_{1,0} = 1/\sqrt{2}(2+5) = 7/\sqrt{2}$

$C_{1,1} = 1/\sqrt{2}(-3+7) = 4/\sqrt{2}$

$D_{1,0} = 1/\sqrt{2}(2-5) = -3/\sqrt{2}$

$D_{1,1} = 1/\sqrt{2}(-3-7) = -10/\sqrt{2}$

Hence the first level of the wavelet transform is

$\frac{1}{\sqrt{2}}[7, 4, -3, -10]$

Second level:

$C_{0,0} = ½(7+4) = 11/2$

$D_{0,0} = ½(7-4) = 3/2$

Hence the second level of the wavelet transform is

$[11/2, 3/2, -3/\sqrt{2}, -10/\sqrt{2}]$

Example 2

A Haar wavelet transform is implemented using an analysis filterbank using normalised low-pass and high-pass filters:

$$h_0 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \quad h_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

Calculate the Haar Transform for 2 levels of decomposition for the input sequence:

s[n] = [1, 2, 3, 4]

Example 2….Solution