Chpt. 8 Plane Wave

Contents

HPW in perfect dielectric medium

In the Medium we study:

source-free $\rho = 0 \quad \vec{J} = 0$

region of isotropic homogeneous: ε and μ are real constants.

Perfect dielectric $\boxed{\sigma = 0}$

Maxwell Eq.

$$\begin{cases} \nabla \times \vec{H} = j\omega\varepsilon\vec{E} \\ \nabla \times \vec{E} = -j\omega\mu\vec{H} \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases} \Rightarrow$$

Helmholtz eq.

$$\begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \\ k^2 = \omega^2 \mu\varepsilon \end{cases} \Rightarrow$$

HPW solutions

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k} \bullet \vec{r})}$$

$$\vec{H}(\vec{r}, t) = \vec{H}_0 e^{j(\omega t - \vec{k} \bullet \vec{r})}$$

$$\vec{H}(\vec{r}, t) = \frac{1}{\sqrt{\mu / \varepsilon}} \vec{e}_k \times \vec{E}(\vec{r}, t)$$

HPW in in conducting medium

In the Medium we study:

Conducting medium $\sigma \neq 0$

Isotropic homogeneous: (uniform) ε and μ are real constants.

source-free $\rho = 0$ $\vec{J} \neq 0$ $\vec{J} = \sigma\vec{E}$ Ohm's law

time-domain differential forms of Maxwell's equations

$$\begin{cases} \nabla\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t} \\ \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \nabla\bullet\vec{B} = 0 \\ \nabla\bullet\vec{D} = 0 \end{cases}$$

$$\begin{cases} \vec{D} = \varepsilon\vec{E} \\ \vec{B} = \mu\vec{H} \end{cases}$$

$$\begin{cases} \nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\,\partial\vec{E}/\partial t \\ \nabla\times\vec{E} = -\mu\,\partial\vec{H}/\partial t \\ \nabla\bullet\vec{H} = 0 \\ \nabla\bullet\vec{E} = 0 \end{cases}$$

How to get wave equations? $\nabla\times$

$\nabla \times$

$$\begin{cases} \nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \partial \vec{E} / \partial t \\ \boxed{\nabla \times \vec{E} = -\mu \partial \vec{H} / \partial t} \\ \nabla \bullet \vec{H} = 0 \\ \nabla \bullet \vec{E} = 0 \end{cases}$$

$$\nabla \times \left(\nabla \times \vec{E} \right) = \nabla \times [-\mu \partial \vec{H} / \partial t] = -\mu \partial (\nabla \times \vec{H}) / \partial t$$

$$\nabla \times \left(\nabla \times \vec{E} \right) = \nabla \left(\nabla \bullet \vec{E} \right) - \nabla^2 \vec{E}$$

$$= 0 - \nabla^2 \vec{E}$$

$$-\mu \partial (\nabla \times \vec{H}) / \partial t = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \bullet \vec{E} = 0$$

$$\boxed{\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

So, we get: **wave equations in conducting medium**

$\nabla\times$

$$\begin{cases} \nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\,\partial\vec{E}/\partial t \\ \nabla\times\vec{E} = -\mu\,\partial\vec{H}/\partial t \\ \nabla\bullet\vec{H} = 0 \\ \nabla\bullet\vec{E} = 0 \end{cases}$$

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

By similar way, we can also get:

$\nabla\times$

$$\begin{cases} \nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\,\partial\vec{E}/\partial t \\ \nabla\times\vec{E} = -\mu\,\partial\vec{H}/\partial t \\ \nabla\bullet\vec{H} = 0 \\ \nabla\bullet\vec{E} = 0 \end{cases}$$

$$\nabla^2\vec{H} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

wave equations in conducting medium

$$\begin{cases} \nabla^2 \vec{E} = \mu\sigma \dfrac{\partial \vec{E}}{\partial t} + \mu\varepsilon \dfrac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 \vec{H} = \mu\sigma \dfrac{\partial \vec{H}}{\partial t} + \mu\varepsilon \dfrac{\partial^2 \vec{H}}{\partial t^2} \end{cases}$$

Time-harmonic EM-Fields

$$\frac{\partial}{\partial t} = j\omega, \quad \frac{\partial^2}{\partial t^2} = -\omega^2$$

$$\begin{cases} \nabla^2 \vec{E} + \mu\varepsilon\omega^2 \vec{E} - \mu\sigma j\omega \vec{E} = 0 \\ \nabla^2 \vec{H} + \mu\varepsilon\omega^2 \vec{H} - \mu\sigma j\omega \vec{H} = 0 \end{cases}$$

$$\begin{cases} \nabla^2 \vec{E} + \omega^2 \mu(\varepsilon - j\dfrac{\sigma}{\omega})\vec{E} = 0 \\ \nabla^2 \vec{H} + \omega^2 \mu(\varepsilon - j\dfrac{\sigma}{\omega})\vec{H} = 0 \end{cases}$$

in perfect medium

$$\begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \\ k^2 = \omega^2 \mu\varepsilon \end{cases}$$

$$\begin{cases} \nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0 \\ \nabla^2 \vec{H} + \omega^2 \mu\varepsilon \vec{H} = 0 \end{cases}$$

wave equations in conducting medium

$$\begin{cases} \boxed{\nabla^2 \vec{E} + \mu\omega^2(\varepsilon - j\dfrac{\sigma}{\omega})\vec{E} = 0} \\ \boxed{\nabla^2 \vec{H} + \mu\omega^2(\varepsilon - j\dfrac{\sigma}{\omega})\vec{H} = 0} \end{cases}$$

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

ε_c complex permittivity of the medium

$$\begin{cases} \nabla^2 \vec{E} + \omega^2 \mu \varepsilon_c \vec{E} = 0 \\ \nabla^2 \vec{H} + \omega^2 \mu \varepsilon_c \vec{H} = 0 \end{cases}$$

$$\boxed{\begin{cases} \nabla^2 \vec{E} + k_c{}^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k_c{}^2 \vec{H} = 0 \\ k_c{}^2 = \omega^2 \mu \varepsilon_c \\ \varepsilon_c = \varepsilon - j\sigma/\omega \end{cases}}$$

HPW in perfect dielectric medium

Maxwell Eq.

$$\begin{cases} \nabla\times\vec{H} = j\omega\varepsilon\vec{E} \\ \nabla\times\vec{E} = -j\omega\mu\vec{H} \\ \nabla\bullet\vec{H} = 0 \\ \nabla\bullet\vec{E} = 0 \end{cases}$$

Helmholtz eq.

$$\Rightarrow \begin{cases} \nabla^2\vec{E} + k^2\vec{E} = 0 \\ \nabla^2\vec{H} + k^2\vec{H} = 0 \\ k^2 = \omega^2\mu\varepsilon \end{cases}$$

wave solutions

$$\Rightarrow \boxed{\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})} \\ \vec{H}(\vec{r},t) &= \vec{H}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})} \\ \vec{H}(\vec{r},t) &= \frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k \times \vec{E}(\vec{r},t) \end{aligned}}$$

HPW in in conducting medium

Maxwell Eq.

$$\begin{cases} \nabla\times\vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E} \\ \nabla\times\vec{E} = -j\omega\mu\vec{H} \\ \nabla\bullet\vec{H} = 0 \\ \nabla\bullet\vec{E} = 0 \end{cases}$$

Helmholtz eq.

$$\Rightarrow \begin{cases} \nabla^2\vec{E} + k_c^{\ 2}\vec{E} = 0 \\ \nabla^2\vec{H} + k_c^{\ 2}\vec{H} = 0 \\ k_c^{\ 2} = \omega^2\mu\varepsilon_c \\ \varepsilon_c = \varepsilon - j\,\sigma/\omega \end{cases}$$

wave solutions

$$\Rightarrow \boxed{\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}_0 e^{j(\omega t - \vec{k}_c\bullet\vec{r})} \\ \vec{H}(\vec{r},t) &= \vec{H}_0 e^{j(\omega t - \vec{k}_c\bullet\vec{r})} \\ \vec{H}(\vec{r},t) &= \frac{1}{\sqrt{\mu/\varepsilon_c}}\vec{e}_k \times \vec{E}(\vec{r},t) \end{aligned}}$$

HPW in perfect dielectric medium

wave solutions

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\omega t - \vec{k}\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}}\vec{e}_k \times \vec{E}(\vec{r},t)$$

$$\vec{E} = E_x \vec{e}_x$$

$$\vec{H} = H_y \vec{e}_y$$

$$\vec{k} = k_z \vec{e}_k$$

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - kz)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon}} E_x(\vec{r},t)$$

HPW in in conducting medium

wave solutions

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}_c\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{j(\omega t - \vec{k}_c\bullet\vec{r})}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon_c}}\vec{e}_k \times \vec{E}(\vec{r},t)$$

$$\vec{E} = E_x \vec{e}_x$$

$$\vec{H} = H_y \vec{e}_y$$

$$\vec{k}_c = k_{cz} \vec{e}_k$$

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon_c}} E_x(\vec{r},t)$$

HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu/\varepsilon_c}} E_x(\vec{r},t)$$

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

$$k_c^2 = \omega^2\mu\varepsilon_c = \omega^2\mu\varepsilon - j\omega\mu\sigma$$

If we define $k_c = \beta - j\alpha$

$$k_c^2 = \beta^2 - \alpha^2 - j2\beta\alpha$$

$\gamma = \alpha + j\beta = jk_c$ **Propagation constant**
传输线理论中惯用

$$\begin{cases} \beta^2 - \alpha^2 = \omega^2\mu\varepsilon \\ 2\alpha\beta = \omega\mu\sigma \end{cases} \Longrightarrow \begin{cases} \alpha = \omega\sqrt{\dfrac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \\ \beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} \end{cases}$$

HPW in in conducting medium

wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

1. **Propagation direction still in z direction, still TEM wave, still HPW**

   Constant phase plane: $\omega t - \beta z = const\ . \quad z = z_{const}$

   Constant amplitude plane: $E_m e^{-\alpha z} \cos(\omega t - \beta z) = const\ . \ z = z_{const.}$

2. **E and H are not in phase, because**

   is complex constant $k_c = \beta - j\alpha$

HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

3. β is phase constant, just like k in perfect dielectric medium

In conducting medium

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}$$

In perfect dielectric

$$k = \omega\sqrt{\mu\varepsilon}$$

It is related with σ , ε , μ ,*and* ω

HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = \frac{k_c}{\omega\mu} E_x(\vec{r},t)$$

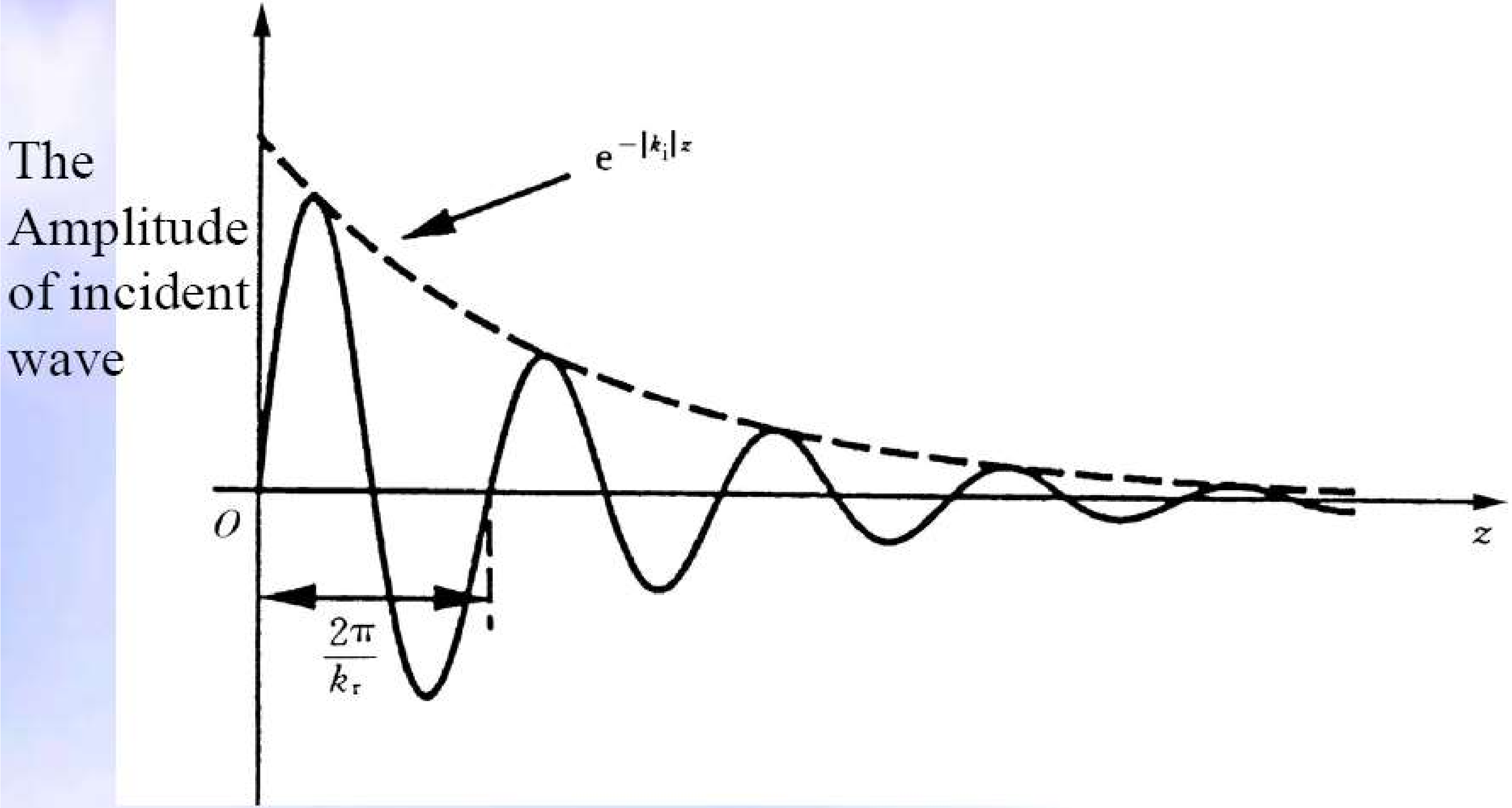
4. Attenuation of plane wave

Amplitude of wave attenuates as it proceeds in z direction by factor $e^{-\alpha z}$

The α is attenuation constant.

It is related with σ, ε, μ, *and* ω

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$



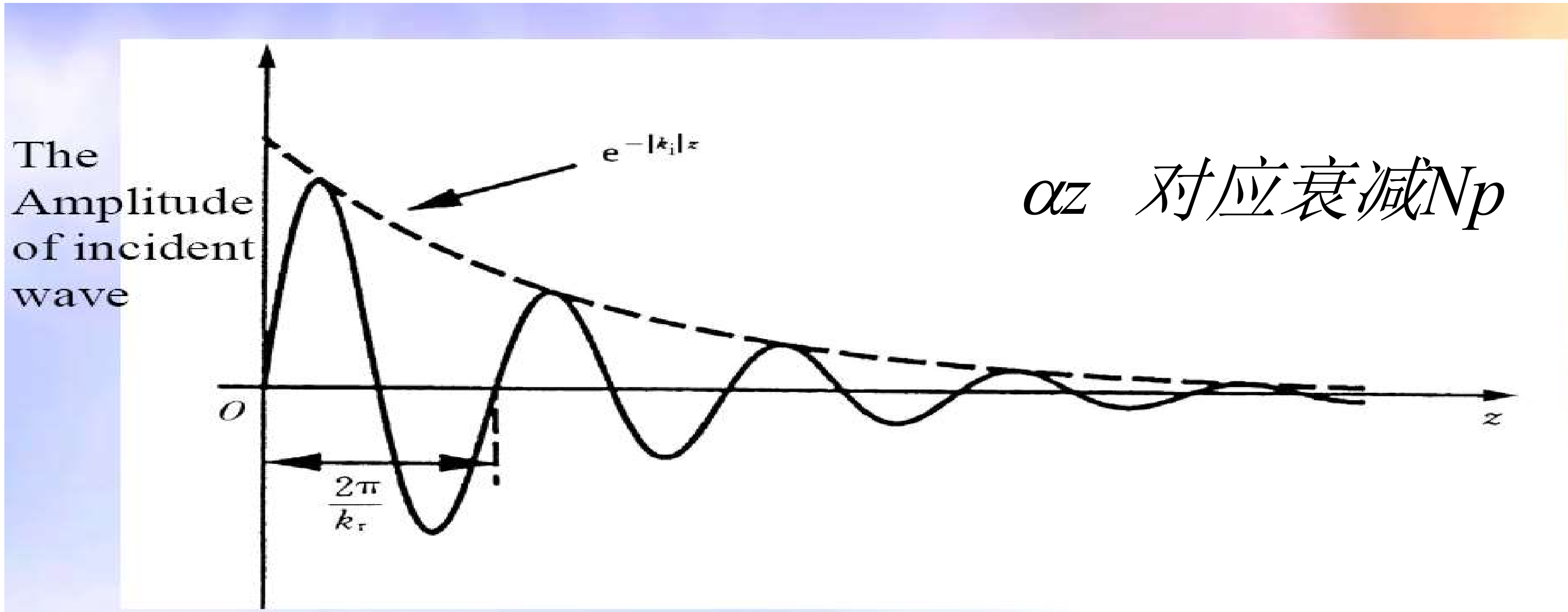
The
Amplitude
of incident
wave
$e^{-|k_i|z}$
O
z
$\frac{2\pi}{k_r}$

$$E(z) = E_0 e^{-\alpha z}$$ We know :

$$\alpha = -\frac{1}{z}\ln\frac{E(z)}{E_0}$$ measured in nepers per meter (Np/m).

1Np/m 的意指电磁波传输 **1m** 后，其幅值衰减到初始值的 **e^{-1}** 倍。

Neper (['ni:pə]) 奈培



HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$
$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$
$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

5. Phase velocity

Velocity of Constant-phase plane

Constant phase plane:

$$\omega t - \beta z = const \ .$$

$$z = z_{const}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

Phase velocity is related with frequency, the shape of a wave comprising many different frequencies will keep on changing as it progresses; that is, the signal is distorted by the time it reaches its destination. This phenomenon is called dispersion. A conducting medium is dispersive medium in general.

HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$
$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$
$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

6. Intrinsic impedance :

$$\begin{cases} \vec{E} = \vec{a}_x E_m e^{-\alpha z} e^{j(\omega t - \beta z)} \\ \vec{H} = \vec{a}_y \dfrac{1}{|\eta_c|} E_m e^{-\alpha z} e^{j(\omega t - \beta z - \frac{1}{2}\delta_c)} \end{cases}$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}} = |\eta_c| e^{j\frac{1}{2}\delta_c} = |\eta_c| e^{j\frac{1}{2}\arctan\frac{\sigma}{\omega\varepsilon}}$$

$$\delta_c = \arctan\frac{\sigma}{\omega\varepsilon}$$

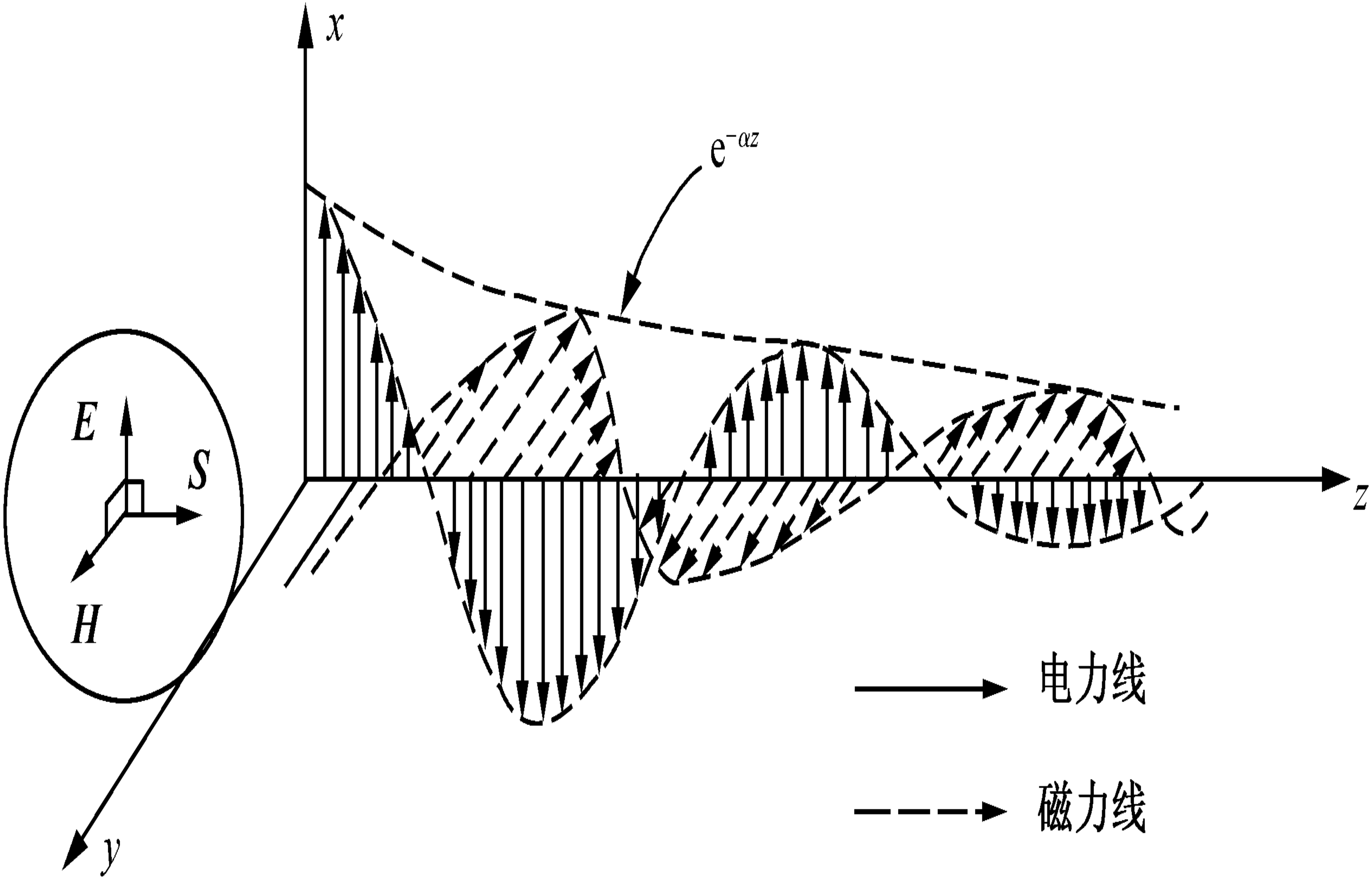
Complex quantity

The *electric field* of traveling wave in conducting medium leads the *magnetic field* by $\frac{1}{2}\delta_c$

HPW in in conducting medium

$$\eta^e = \sqrt{\frac{\mu}{\varepsilon^e}} = \sqrt{\frac{\mu}{\varepsilon\left(1 - \mathrm{j}\frac{\sigma}{\omega\varepsilon}\right)}}$$

$$= \left\{\sqrt{\frac{\mu}{\varepsilon}} \Bigg/ \left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{\frac{1}{4}}\right\} \exp\left(\mathrm{j}\frac{1}{2}\operatorname{arctg}\frac{\sigma}{\omega\varepsilon}\right)$$



x
$e^{-\alpha z}$
E
S
H
z
y
电力线
磁力线

HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

7. *Loss tangent*

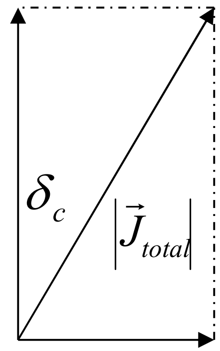
$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\frac{\sigma}{\omega}}} = |\eta_c| e^{j\frac{1}{2}\delta_c} = |\eta_c| e^{j\frac{1}{2}\arctan\frac{\sigma}{\omega\varepsilon}}$$

Displacement current density $\quad \vec{J}_d = \partial\vec{D} / \partial t = j\omega\varepsilon\vec{E}$

Conducting current density $\quad \vec{J}_c = \sigma\vec{E}$

Loss Tangent $\quad \tan\delta_c = \frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\varepsilon}$

$$\left|\vec{J}_d\right| = \left|j\omega\varepsilon\vec{E}\right|$$



$$\left|\vec{J}_c\right| = \sigma\left|\vec{E}\right|$$

The ratio of conducting current density and displacement current density

Where δ_c *Is loss tangent angle*

HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$

$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

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Wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = \frac{1}{|\eta_c|} E_x(\vec{r},t) e^{-j\frac{1}{2}\delta_c}$$

7. *Energy Density*

For HPW in conducting media $w_e < w_m$

$$w_m = \frac{1}{2}\mu H_y^2 = \frac{1}{2}\mu\left(\frac{E_x}{|\eta_c|}\right)^2 = \frac{1}{2}\mu\left(\left|\sqrt{\varepsilon_c / \mu}\right|\right)^2 E_x^2$$

$$= \frac{1}{2}E_x^2\left|\varepsilon - j\frac{\sigma}{\omega}\right| = \frac{1}{2}\varepsilon E_x^2\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} = w_e\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}$$

For HPW in perfect dielectrics $w_e = w_m$

$$\frac{1}{2}\varepsilon E_x^2 = \frac{1}{2}\varepsilon(\eta H_y)^2 = \frac{1}{2}\varepsilon\left(\sqrt{\frac{\mu}{\varepsilon}}\right)^2 H_y^2 = \frac{1}{2}\mu H_y^2$$

HPW in in conducting medium

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = H_0 e^{j(\omega t - k_c z)}$$
$$H_y(\vec{r},t) = \frac{1}{\sqrt{\mu / \varepsilon_c}} E_x(\vec{r},t)$$

$$k_c = \beta - j\alpha$$

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

Wave solutions

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$
$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$
$$H_y(\vec{r},t) = \frac{1}{|\eta_c|} E_x(\vec{r},t) e^{-j\frac{1}{2}\delta_c}$$

For HPW in conducting media $w_e < w_m$

8. Average energy density :

$$\vec{S}_{avg} = \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right] = \vec{a}_z \frac{E_0^2}{2|\eta_c|} e^{-2\alpha z} \cos\frac{\delta_c}{2}$$

HPW in in conducting medium

9. Skin depth :

The factor $e^{-\alpha z}$ signifies that the wave attenuates as it proceeds in the z direction

$$E_x(\vec{r},t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

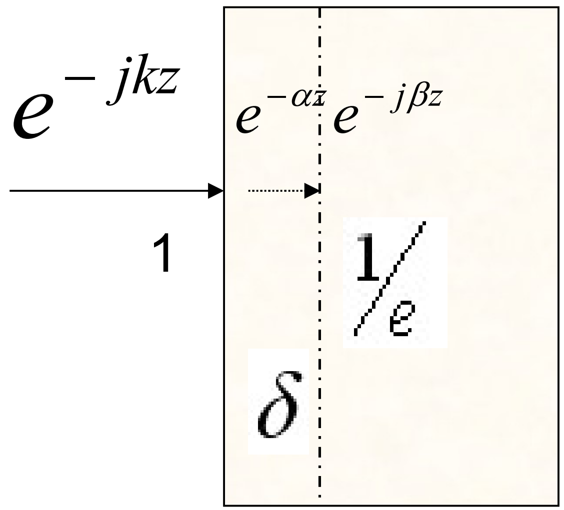
$$H_y(\vec{r},t) = H_0 e^{-\alpha z} e^{j(\omega t - \beta z)}$$

$$H_y(\vec{r},t) = \frac{1}{|\eta_c|} E_x(\vec{r},t) e^{-j\frac{1}{2}\delta_c}$$

The **skin depth** is the distance traveled by the wave in a conducting medium at which its amplitude falls to $1/e$ of its value on the surface of that conducting medium

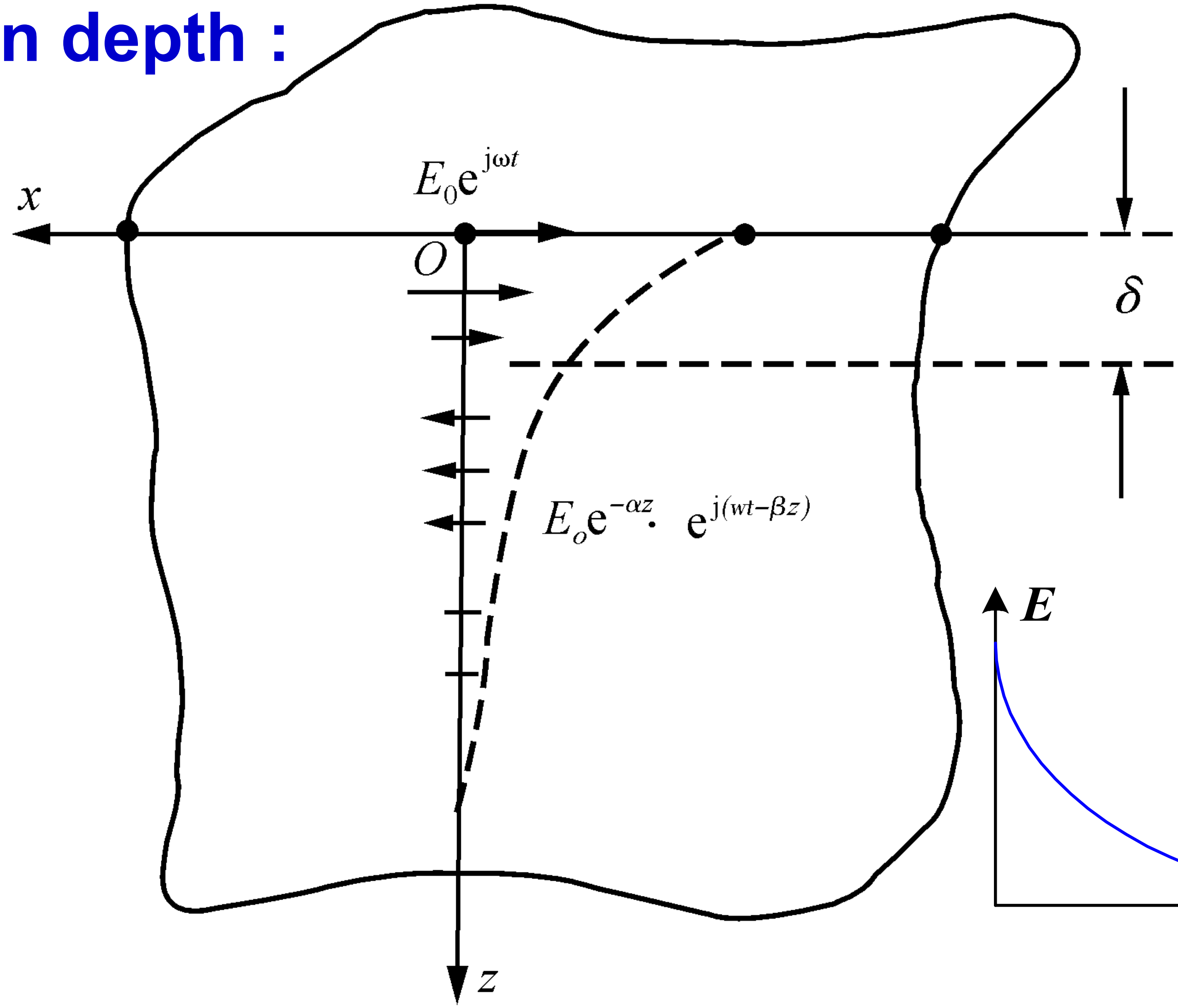
We denote the **skin depth** by $\delta = \frac{1}{\alpha}$

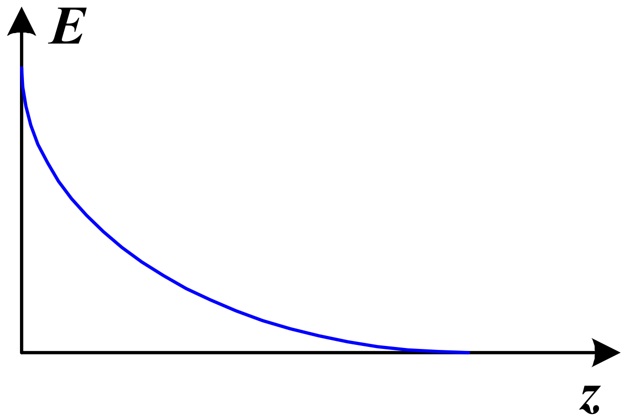
$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}}$$



HPW in in conducting medium

➢ **Skin depth :**





HPW in in conducting medium

$$\frac{\sigma}{\omega\varepsilon}\begin{cases} >>1 & \text{良导体} \\ <<1 & \text{弱导体, 良介质} \\ \approx 1 & \text{半导体} \end{cases}$$

HPW in in conducting medium

- **For EM wave, the conducting ability of medium sorts by the value of loss tangent**

$$\tan \delta_c = \frac{|J_c|}{|J_d|} = \frac{\sigma}{\omega\varepsilon}$$

- **Poorly conducting medium, with** $\frac{\sigma}{\omega\varepsilon} << 1$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \qquad \beta \approx \omega\sqrt{\mu\varepsilon}$$

$$\begin{cases} \alpha = \omega\sqrt{\dfrac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} \\ \beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} \end{cases}$$

In poorly conducting medium, there still exists attenuation of EM energy, the phase constant is nearly same with perfect dielectric medium

The intrinsic impedance

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}} = \sqrt{\frac{\mu}{\varepsilon}}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{-\frac{1}{2}} \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right)$$

HPW in in conducting medium

- **Good conducting medium** $\dfrac{\sigma}{\omega\varepsilon} >> 1$

$$\begin{cases} \alpha = \omega\sqrt{\dfrac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \\ \beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} \end{cases}$$

$$\alpha = \beta \approx \sqrt{\frac{1}{2}\omega\mu\sigma} = \sqrt{\pi f\mu\sigma}$$

- The intrinsic impedance

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}} = \sqrt{\frac{\mu}{\varepsilon}}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{-\frac{1}{2}} \approx \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\frac{(1+j)}{\sqrt{2}} = \sqrt{\frac{\omega\mu}{\sigma}}e^{j\frac{\pi}{4}}$$

*In good conducting medium, the electric field of traveling wave leads the magnetic field by **45^0***

- **Skin effect of conducting medium**

Skin depth $\delta = \dfrac{1}{\alpha} = \sqrt{\dfrac{1}{\pi f\mu\sigma}}$

In good conductors, the wave attenuates very rapidly and the fields are confined to the region near the surface of the conductor. This phenomenon is called **Skin Effect.**

HPW in in conducting medium

- **Good conducting medium, with**

$$\alpha = \beta \approx \sqrt{\frac{1}{2}\omega\mu\sigma} = \sqrt{\pi f \mu\sigma}$$

Phase velocity

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\pi f \mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Velocity of EM-wave in good conductor is a function of w, and thus we call this phenomenon Dispersion.

Notice that better conductor yields slower velocity (due to a larger σ).

Skin Effect & Skin Depth

EM fields or the induced current attenuates along z-axis according to the rule of $e^{-\alpha z}$.

Therefore, the time-varying fields or alternating current concentrates only in a thin layer next to the outer surface of the conductor. This phenomenon is called skin effect.

When the magnitude attenuates to 1/*e* of its original value, we call the depth as the skin depth or the penetration depth (趋肤深度 δ).

$$E_x = E_0 e^{-\alpha z} \cdot e^{j(\omega t-\beta z)}$$

$$E_{xm} = E_0 e^{-\alpha z}$$

$$E_0 e^{-\alpha\delta} = E_0 / e$$

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}}}$$

skin depth — Good conducting medium

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^{2}}-1\right)}} \xrightarrow{\text{Good conductor}} \approx \frac{1}{\sqrt{\pi f\mu\sigma}}$$

For good conductor

$$\alpha = \beta = \sqrt{\pi f\mu\sigma} \qquad \beta = 2\pi/\lambda \qquad \delta = 1/\alpha \qquad \therefore \delta = \frac{\lambda}{2\pi}$$

Skin depth is the depth that an EM-wave penetrates into the good conductor effectively.

Better conductor, σ□, δ□. Also, $f\uparrow$, $\delta\downarrow$.

交变电磁场进入导体表面后很快就衰减殆尽，"势力范围" 只在离表面很浅的导体中，顾名思义"趋肤深度"。

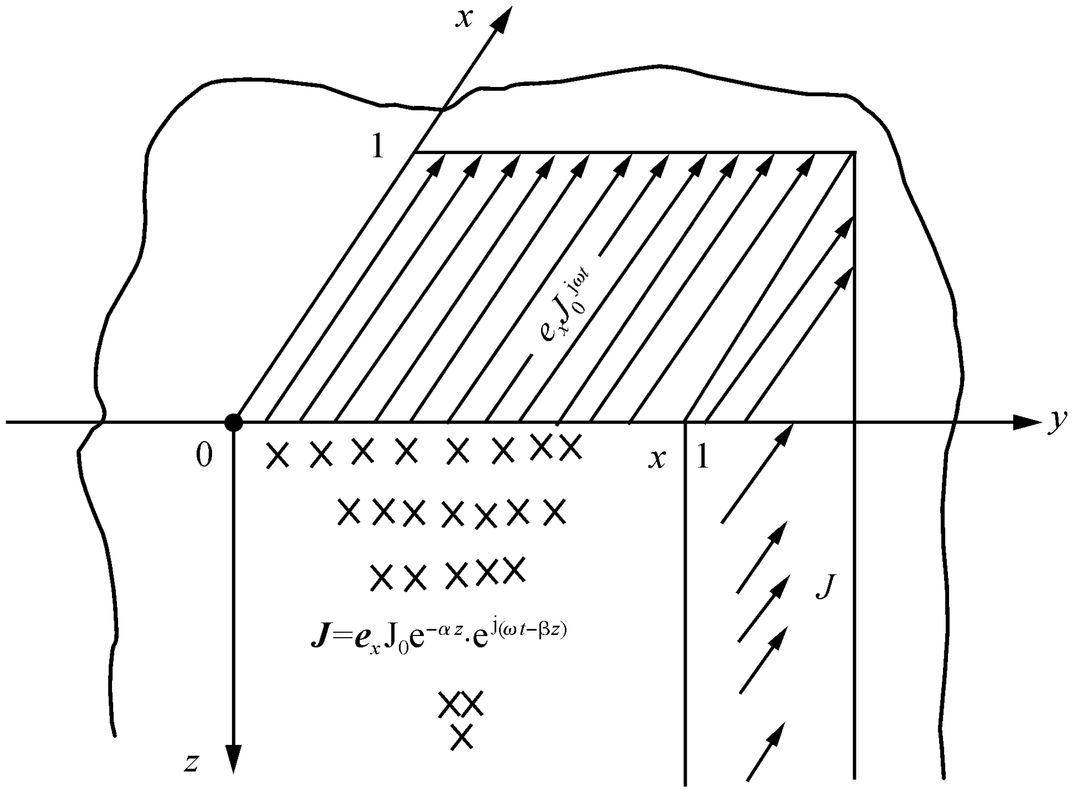
Some Examples

材料	σ	μ_r	趋肤深度δ			
			60Hz/cm	**1kHz/mm**	**1MHz/mm**	**3GHz/μm**
铝	3.54×10^7	1.00	1.1	2.7	0.085	1.6
铜	5.8×10^7	1.00	0.85	2.1	0.066	1.2
金	4.5×10^7	1.00	0.97	2.38	0.075	1.4
磁性铁	1.0×10^7	2×10^2	0.14	0.35	0.011	0.20
镍	1.3×10^7	1×10^2	0.18	4.4	0.014	0.26
银	6.15×10^7	1.00	0.83	2.03	0.064	1.17
锡	0.87×10^7	1.00	2.21	5.41	0.1714	3.12
锌	1.86×10^7	1.00	1.51	3.70	0.117	3.14

Surface Impedance or Internal Impedance (Z_S)

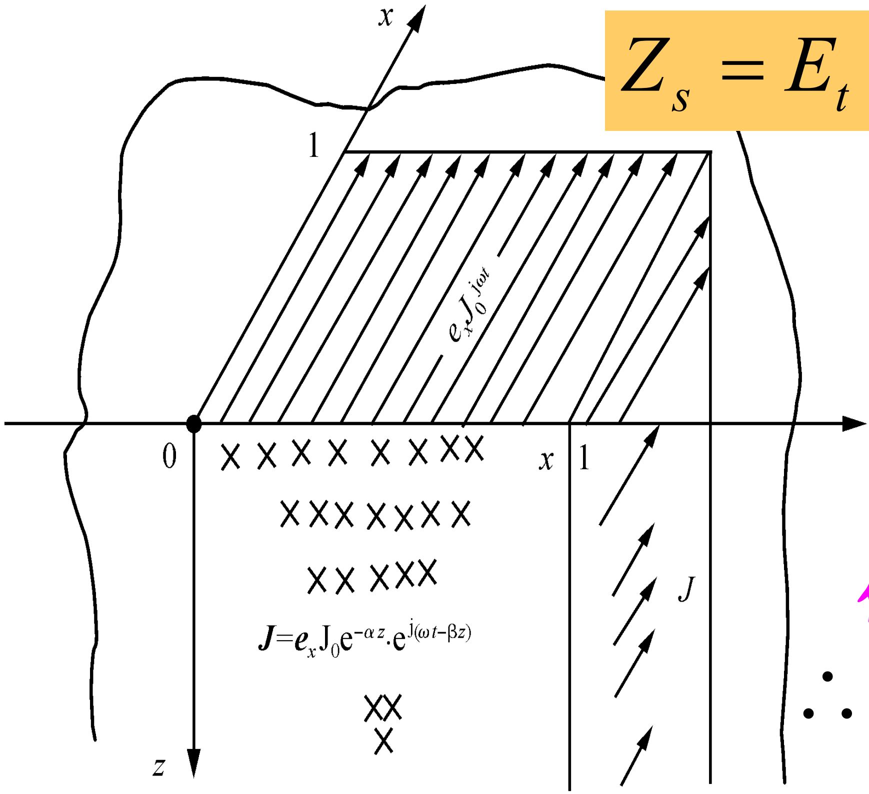
Since the induced alternating current concentrates largely in the thin layer in a depth of δ, we define an **effective impedance per unit length along direction of current** to describe the Ohm's Law for the skin current.

$$\sigma^{-1} = \frac{\vec{E}}{\vec{J}} \Longrightarrow \boxed{Z_s = \frac{E_t}{J_s}}$$



E_t is tangential E-field on inner surface.

J_S is current density per unit width.



$$Z_s = E_t / J_s$$

$$\gamma = \alpha + j\beta = jk_c$$

$$k_c = \beta - j\alpha$$

Wave equations

$$k_c = \omega\sqrt{\mu\varepsilon_c}$$

$$\because \nabla^2 E_x - \gamma^2 E_x = 0$$

$$\therefore \nabla^2 \sigma E_x - \gamma^2 \sigma E_x = 0$$

$$\therefore \nabla^2 J_x - \gamma^2 J_x = 0$$

体电流密度

$$\therefore \quad J_x = J_0 e^{-\alpha z} \cdot e^{j(\omega t - \beta z)} = J_0 e^{-\gamma \cdot z} e^{j\omega t}$$

E_t is tangential E-field on inner surface.

$$E_t = E_x\big|_{z=0} = \frac{1}{\sigma} J_x\Big|_{z=0} = \frac{1}{\sigma} J_0 e^{j\omega t}$$

J_S is current density per unit width.

$$J_S = \int_0^\infty J_x \mathrm{d}z = \int_0^\infty J_0 e^{-\gamma \cdot z} \cdot e^{j\omega t} \mathrm{d}z = \frac{J_0}{\gamma} e^{j\omega t}$$

$$\sigma^{-1} = \frac{\vec{E}}{\vec{J}}$$

$$Z_s = \frac{E_t}{J_s} = \gamma / \sigma$$

Good conducting medium

$$Z_S = \gamma / \sigma$$

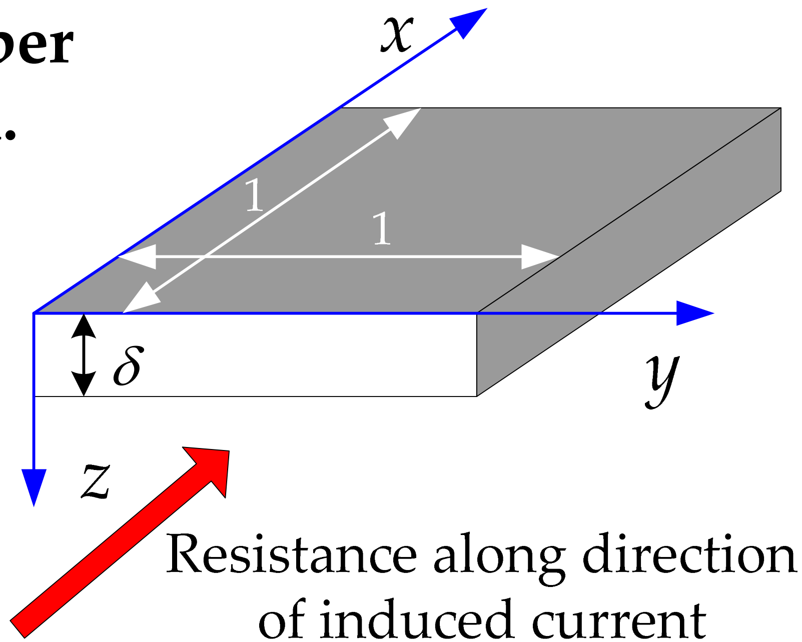
$$Z_S = \frac{\gamma}{\sigma} = \frac{\alpha}{\sigma} + j\frac{\beta}{\sigma} \approx \frac{\sqrt{\pi f \mu \sigma}}{\sigma} + j\frac{\sqrt{\pi f \mu \sigma}}{\sigma} = R_S + jX_S$$

Surface Resistivity 表面电阻率

$$R_S = \frac{\alpha}{\sigma} = \frac{\sqrt{\pi f \mu \sigma}}{\sigma} = \frac{1}{\sigma \delta}$$

单位长度单位宽度的等效电阻

- **It is the effective resistance per unit width & per unit length.**



- 对于同一块导体，其交流电阻率$(1/\sigma\delta)$比直流电阻率$(1/\sigma)$大，这是趋肤效应所造成的。
  - 解释为：对高频电流，由于趋肤效应，与均匀分布在导体中的直流电流相比较，其有效的导电面积大大的减少，电阻增大。
- R_S是在假定导体的厚度为无穷大的条件下得到的，
  - 对于厚度有限的实际导体，上式精确度很高；
  - 对于圆柱形导体，把圆柱纵向视为长度，圆周视为宽度，径向视为厚度，上式依然适用.

例： 书P489 例9.19

Example

1. Perfect conductor.

For perfect conductor, $\sigma \to \infty$ no electric field is allowed to exist.

Examining Ohm's laws $\vec{J} = \sigma \vec{E}$

We see that, if E is zero, then $\sigma \to \infty$ in order to have finite current. The skin depth for a perfect conductor is zero.

Ordinary metals such as copper, aluminum, gold, silver, etc., can be regarded as perfect conductors in solving electromagnetic wave problems.

At room temperature the conductivity for copper is $\sigma = 5.8 * 10^7 \, s/m$

Example

Radio communication in submarines:

The main difficulty in radio communication between submarines is the high attenuation of electromagnetic wave propagating in the ocean.

The relative permittivity (相对介电常数) of seawater is approximately 81, and its average conductivity is approximately 4s/m. The attenuation constant is given by:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$

As frequency increase, the attenuation constant increases steadily, at very high frequency:

$$\alpha = \frac{\sigma}{2}\sqrt{\mu_0/\varepsilon} = 83.8N/m = 728dB/m$$

The attenuation is extremely high because power is reduced by half for each 4mm it travels.

To keep the attenuation low, the operating frequency must be low. But even at 1kHz, the attenuation is still appreciable $\alpha = 0.126N/m = 1.1dB/m$

Thus an EM wave at 1kHz will be attenuated 110dB when it has traveled a distance of 100m.

Example

2. Using microwave oven to heating foods.

To estimate the depth of the microwave penetration, we use the permittivity for bottom around steak $\varepsilon_c = 40(1 - j0.3)\varepsilon_0$, to calculate complex wave

number k_c. We find that at 3GHz, $k_c = 402 - j59 = \beta - j\alpha$, the penetration depth is equal to

$$\delta = 1/\alpha = 1.7cm$$

Homework

The electric field intensity of a linearly polarized uniform plane wave propagating in the +z-direction in seawater is $\vec{E} = \vec{e}_x 2017\cos\left(10^7 \pi t\right)$ (V/m) at z=0. The intrinsic parameters of seawater are $\varepsilon_r = 72$, $\mu_r = 1$, $\sigma = 4$(S/m).

a) Determine the frequency of the plane wave.

b) Is it a good conductor? And give the reasons.

c) Determine the skin depth.

A uniform plane wave propagates in seawater ($\varepsilon_r = 64, \mu_r = 1, \sigma = 4$). Determine:

a) If the loss tangent is 0.01 (i.e. $\tan\delta << 1$), calculate the frequency and phase velocity.

b) If the loss tangent is 100 (i.e. $\tan\delta >> 1$), calculate the skin depth, wave length and impedance.

Homework

7.1 Let the instantaneous electric field intensity $\boldsymbol{E}$ of the electromagnetic wave in free space be

$$\boldsymbol{E} = \boldsymbol{e}_y 37.7 \cos(6\pi \times 10^8 t + 2\pi z)\mathrm{V/m}$$

Solve the following problems:
(1) the direction of the wave propagation;
(2) the frequency, wavelength, phase constant and phase velocity;
(3) the instantaneous magnetic field intensity $\boldsymbol{H}$. Is it a uniform plane wave?

7.15 Show that the attenuation of field intensity amplitude is approximately 55dB per wavelength as the electromagnetic propagates in conducting medium.

7.19 The wavelength of the uniform linear polarization plane wave in the air is 60m. $\boldsymbol{E} = \boldsymbol{e}_x \cos \omega t \mathrm{V/m}$ at the place where it is 1 meters below sea level as the wave enters into the sea along the z axis and propagates down vertically. Find the instantaneous $\boldsymbol{E}$ and $\boldsymbol{H}$, the phase velocity and the wavelength at any point below the sea level. For the sea water, $\sigma = 4\mathrm{S/m}$ and $\varepsilon_r = 80$, $\mu_r = 1$.

7.20 The conductivity of sea water is $\sigma = 4\mathrm{S/m}$ and $\varepsilon_r = 8$. Find the attenuation constant, wavelength and wave impedance of the electromagnetic wave in the sea with frequencies of 10kHz, 1MHz, 10MHz and 1GHz.

三、良介质（弱导电媒质)中的平面波

良介质意味着复介电常数 $\varepsilon_c = \varepsilon - i\dfrac{\sigma}{\omega}$ 中有：

$$\frac{\sigma}{\omega} << \varepsilon \quad \text{即} \quad \frac{\sigma}{\omega\varepsilon} << 1$$

则各特性参量分别为：

$$k_c = \omega\sqrt{\varepsilon_c \mu} = \omega\sqrt{\mu\varepsilon\left(1 - i\frac{\sigma}{\omega\varepsilon}\right)} \approx \omega\sqrt{\mu\varepsilon}\left(1 - i\frac{\sigma}{2\omega\varepsilon}\right)$$

$$\beta \approx \omega\sqrt{\mu\varepsilon} \qquad \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

相速 $$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon\mu}}$$

波阻抗

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + i\frac{\sigma}{2\omega\varepsilon}\right)$$

Example： 电路板常用材料 **FR4**，其相对介电常数ε_r＝**4.5**，频率为 **1MHz** 时材料的损耗角正切为 **0.001**。求频率为 **100MHz**、**1GHz** 及 **10GHz** 情况下平面波的衰减常数、相移常数及相速。

解：在时变场情况下，介质的损耗(无论损耗的起因)通过引入复介电常数的虚部来表达，即

$$\varepsilon_e = \varepsilon - i\frac{\sigma}{\omega}$$

对于导电媒质，有

$$\varepsilon_e = \varepsilon - i\frac{\sigma}{\omega} \Rightarrow tg\delta = \frac{\sigma}{\omega\varepsilon}$$

对于 **FR4** 板，有

$$tg\delta_e = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{\sigma \cdot 4\pi\times 9\times 10^9}{2\pi\times 10^6\times 4.5} = 0.001 \Rightarrow$$

$$\sigma = 2.5\times 10^{-7}(S/m)$$

由于 $tg\delta_e = \frac{\sigma}{\omega\varepsilon} = 0.001 << 1$，因此 **FR4** 可视为弱导电媒质，则衰减常数

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} = \frac{2.5\times 10^{-7}}{2}\frac{377}{\sqrt{4.5}} = 2.2\times 10^{-5}(Np/m)$$

对衰减常数单位(***Np/m***)的解释：

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} = \frac{\sigma}{2}\frac{\eta}{\sqrt{\varepsilon_r}}$$

$$\sigma[S/m]\cdot\eta[\Omega] = \sigma[1/\Omega m]\cdot\eta[\Omega] \propto \alpha[1/m]$$

由 $E(z)=E_0e^{-\alpha z}$ 可知：

$$\alpha=-\frac{1}{z}\ln\frac{E(z)}{E_0}$$

1Np/m 的意指电磁波传输 **1m** 后，其幅值衰减到初始值的 **e^{-1}** 倍。

相移常数为：

$$\beta=\omega\sqrt{\varepsilon\mu}=\frac{\omega}{c/\sqrt{\varepsilon_r}}=\frac{2\pi\times10^6\cdot\sqrt{4.5}}{c}=0.044(rad/m)$$

波长： $\lambda=\frac{2\pi}{\beta}=\frac{c}{10^6\cdot\sqrt{4.5}}=141(m)$

相速： $v_p=\frac{\omega}{\beta}=\frac{1}{\sqrt{\varepsilon\mu}}=\frac{c}{\sqrt{\varepsilon_r}}=1.4\times10^8(m/s)$