Chpt. 7 Time-Varying EM Fields

Contents

§ 7.1.2 Review

- What we have learnt are **static fields**
  - Static E-field
  - Static M-field
  - Steady-Current E-field
- Features of static fields

Summary of Fundamental Equs.

		Integral form	Diff. form
Static E-field	Div. equs.	$\oint \vec{D} \bullet d\vec{S} = Q$	$\nabla \bullet \vec{D} = \rho$
	Curl equs.	$\oint \vec{E} \bullet d\vec{l} = 0$	$\nabla \times \vec{E} = 0$
	Material equs.	$\vec{D} = \varepsilon \vec{E}$	
SC E-field	Div. equs.	$\oint_S \vec{J} \bullet d\vec{S} = 0$	$\nabla \bullet \vec{J} = 0$
	Curl equs.	$\oint_C \vec{E} \bullet d\vec{l} = 0$	$\nabla \times \vec{E} = 0$
	Material equs.	$\vec{J} = \sigma \vec{E}$	
Static M-field	Div. equs.	$\oint_s \vec{B} \cdot d\vec{S} = 0$	$\nabla \bullet \vec{B} = 0$
	Curl equs.	$\oint_C \vec{H} \bullet d\vec{l} = I$	$\nabla \times \vec{H} = \vec{J}$
	Material equs.	$\vec{B} = \mu \vec{H}$	

§ 7.1.1 Get to know what we will learn

Maxwell's Equations (in integral form)

1st $$\oint_C \vec{H} \bullet d\vec{l} = \int_S (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \bullet d\vec{S}$$

2nd $$\oint_C \vec{E} \bullet d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{S}$$

Two closed line integrals

3rd $$\oint_S \vec{D} \bullet d\vec{S} = Q$$

4th $$\oint_S \vec{B} \bullet d\vec{S} = 0$$

Two closed surface integrals

Maxwell's Equations (in diff. form)

1st $$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

2nd $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

3rd $$\nabla \bullet \vec{D} = \rho_{FC}$$

4th $$\nabla \bullet \vec{B} = 0$$

Manifesting relationship between fields & sources.

Development of Electromagnetics

- **Static** E- and M-field are **independent** of each other.
- However, **time-varying** E- and M-field are **interdependent.**
  - In 1820, Oersted (奥斯特) found the M-field around the current.
  - In also 1820, Ampere found the force among currents.
  - In 1831, Faraday presented Law of Magnetic Induction, that is E-field can be generated by changing M-field.
- Our conclusion: **a time-varying E-field creates a time-varying M-field, which in turn produces a time-varying E-field.**

As to static fields

- Static E-field is simulated by static charges
- Static M-field is generated by steady currents
- Static E- or M-fields are independent

As to time-varying EM-fields

- Time-varying charges or currents will create time-varying E-field and M-field
- These fields are functions of both time and space
- Time-varying E-field and M-field are interdependent. They are the reasons for each other, the sources of each other (互相激发,互相为源), hence the name electromagnetic fields

四种源

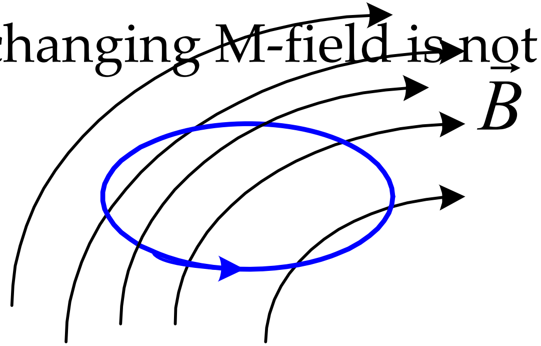
- **The questions are**
  - How to describe the time-varying E- and M-fields?
  - And how to describe their derivative fields --- electromagnetic fields?
- **In 1873, Maxwell answered these questions**
  - He modified the Ampere’s circuital law
  - He proposed the concept of displacement current
  - He summed up the div. and curl equs for electromagnetic fields, i.e. Maxwell’s Equs.
- **We’ll get to know these equs by way of**
  - Faraday’s Law, Ampere’s Law, Gauss’s Law
  - 2nd, 1st, 3rd, 4th

§ 7.2 Faraday's Law & Maxwell's 2nd equ.

Faraday's Law of E-M Induction

- By experiments, Faraday found that
  - If M-flux linked by a closed circuit changes, induced-potential thus generated within this circuit will simulate an induced current.
  - The induced current will prevent the change of the original M-field.
  - Thus we know E-field in a changing M-field is not conservative.

$$\oint_C \vec{E}_{in} \bullet d\vec{l} \neq 0$$



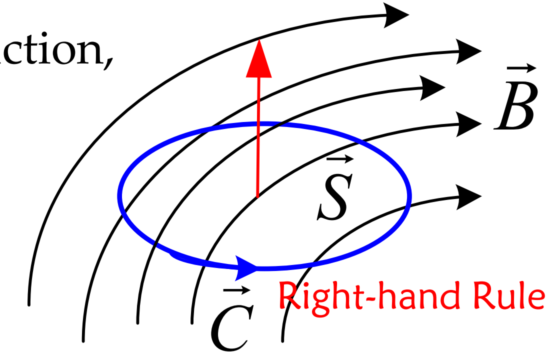
E-field in a changing M-field is non-conservative.

The induced potential is the line integral of the non-conservative along the closed circuit.

$$\mathcal{E}_{in} = \oint_C \vec{E}_{in} \bullet d\vec{l} \neq 0$$

- According to the Law of EM induction,

$$\mathcal{E}_{in} = -\frac{d\Phi}{dt}$$



A flash for Faraday's Induction Law

induced potential: $\varepsilon_{in} = \oint_C \vec{E}_{in} \bullet d\vec{l} = -\frac{d\Phi}{dt} \neq 0$

Beside the induced E-field, if there exits static E-field,

$$\oint_C \vec{E} \bullet d\vec{l} = \oint_C \vec{E}_{in} \bullet d\vec{l} + \oint_C \vec{E}_c \bullet d\vec{l} = \oint_C \vec{E}_{in} \bullet d\vec{l} + 0 = \varepsilon_{in}$$

$$\boxed{\varepsilon_{in} = \oint_C \vec{E} \bullet d\vec{l} = -\frac{d\Phi}{dt}}$$

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d\Phi}{dt}$$

$$\Phi = \int_S \vec{B} \bullet d\vec{S}$$

$$\Rightarrow \boxed{\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{S}}$$

§ 7.2 Faraday's Law & Maxwell's 2nd equ.

$$\boxed{\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \bullet d\vec{S}}$$

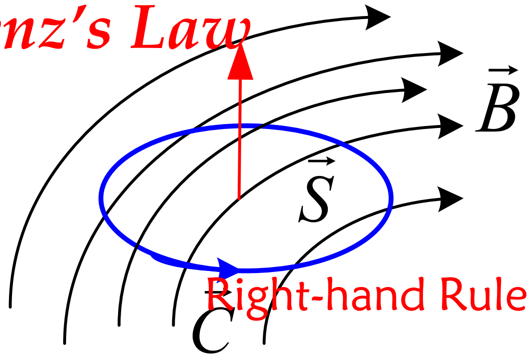
2nd equation

- Description 1:
  - If *C* is a closed path and *S* is its corresponding open surface, the induced-potential within *C* is just the decreasing rate of M-flux passing through *S*.
- Description 2:
  - If *C* is a closed curve and *S* is its corresponding open surface, the line integral of E along *C* is just the decreasing rate of M-flux passing through *S*.

About the "–" Symbol

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{S}$$

--- *Lenz's Law*



- It describe the behavior of static conductor in varying M-field.
- Induced M-field (generated by induced current) will **prevent** the change of original M-field.
- Must the loop be a circuit loop?

If the loop is an abstract one

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{S}$$

- The loop in Faraday's Law of EM Induction may be abstract. Namely, it is not necessarily a circuit loop; it may be any closed path within the dielectric or space.
- If the loop is extended to be such an abstract closed path, Faraday's Law of EM Induction turns into Maxwell's 2nd Equ..

Maxwell's 2nd equ in diff. form

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{S}$$

$$\oint_C \vec{E} \bullet d\vec{l} + \int_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{S} = 0$$

$$\oint_C \vec{E} \bullet d\vec{l} = \int_S (\nabla \times \vec{E}) \bullet d\vec{S}$$

$$\int_S (\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t}) \bullet d\vec{S} = 0$$

$$\therefore \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

§ 7.3 Ampere's Law & Maxwell's 1st Equ.

$$\frac{\partial \rho}{\partial t} \neq 0$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\boxed{\nabla \cdot \left(\nabla \times \vec{H} \right) \equiv 0 = \nabla \cdot \vec{J}}$$

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot \vec{J} + \frac{\partial \nabla \cdot \vec{D}}{\partial t} = 0$$

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\boxed{\nabla \cdot \left(\vec{J} + \vec{J}_d \right) \equiv 0}$$

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d$$

Concept of *Displacement Current*

Maxwell assumed that across the plates of the capacitor there existed a current, called the displacement current (位移电流).

The displacement current is **generated by time-varying E-field.**

As the conduction current does, displacement current can also stimulate M-field.

- **A comparison between 2 currents**
  - **Common points: both have the unit of Ampere and both can generate M-field.**
  - **Differences: conduction current is the motion of free charges, while the displacement current is not and it stands for the variation of E-field.**

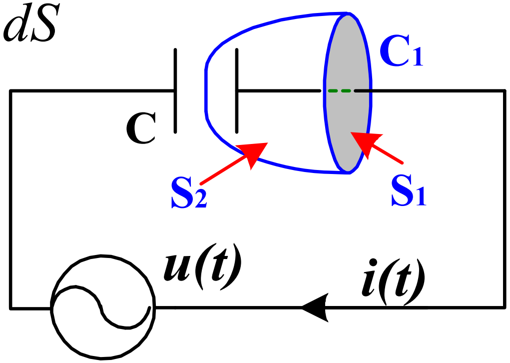
Let's examine the charges on the plates --- $q(t)$:

If $\vec{J}$ stands for the conduction current.

$$\oint_S \vec{J} \bullet d\vec{S} = \int_{S_1} \vec{J} \bullet d\vec{S} + \int_{S_2} \vec{J} \bullet d\vec{S} = i + 0 = -\frac{\partial q}{\partial t}$$

$$\because \oint_S \vec{D} \bullet d\vec{S} = q$$

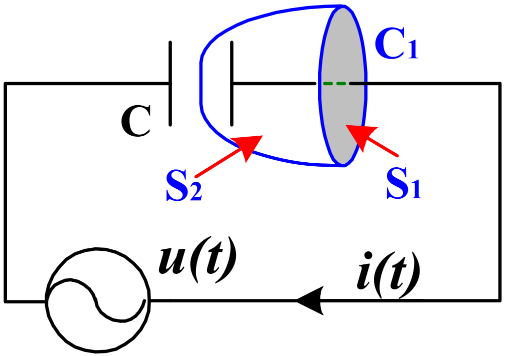
$$\therefore \oint_S \vec{J} \bullet d\vec{S} = -\frac{d}{dt}\oint_S \vec{D} \bullet d\vec{S} = -\oint_S \frac{\partial \vec{D}}{\partial t} \bullet d\vec{S}$$



Definition by Maxwell: $$\boxed{\vec{J}_d = \frac{\partial \vec{D}}{\partial t}}$$

Unit of J_d (A/m^2)

Density of
Displacement Current $\vec{J}_d = \dfrac{\partial \vec{D}}{\partial t}$



$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \vec{J} \bullet d\vec{S}$$

$$\boxed{\oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\vec{J}_T + \vec{J}_d\right) \bullet d\vec{S}}$$

$\vec{J}_T$ is density of conduction current.

The question is now answered satisfactorily

This is in fact Maxwell's 1st Equ..

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\vec{J}_T + \vec{J}_d\right) \bullet d\vec{S}$$

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\nabla \times \vec{H}\right) \bullet d\vec{S}$$

Since S is any surface surrounded by C

$$\nabla \times \vec{H} = \vec{J}_T + \vec{J}_d$$

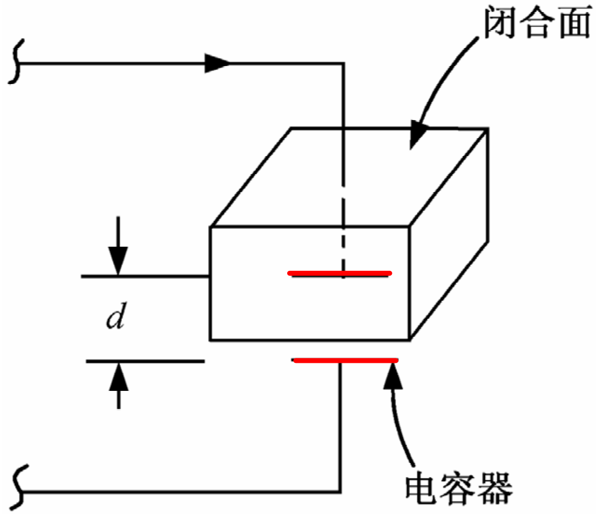
Maxwell's 1st Equ. in differential form

Example of Displacement Current

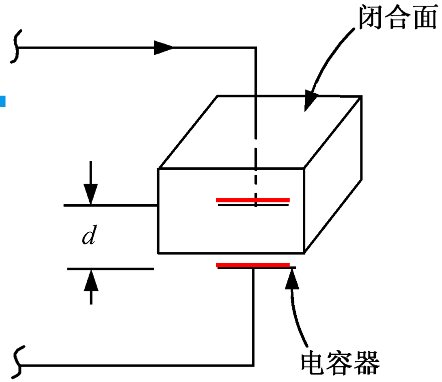
A closed surface (box) contains one plate of a parallel-plate capacitor. Between the plates is air and the voltage difference is $U=U_0\sin\omega t$. Areas of both plates (S) are rather large and the distance between them (d) is rather short, and thus E-field within 2 plates is assumed to be uniformly distributed.

Please prove that current into the upper surface equals displacement current out of the lower surface of the box.

Refer to Problem 7.36 in page 347 of textbook.



$U=U_0\sin\omega t$



Solution: the conduction current I is:

$$i=\frac{\mathrm{d}q}{\mathrm{d}t}=C\frac{\mathrm{d}U}{\mathrm{d}t}=c\omega U_0\cos\omega t$$

How to get the displacement current i_d?

$$i_d \leftarrow \vec{J}_d \qquad \vec{J}_d \leftarrow \partial\vec{D}/\partial t \qquad \vec{D} \leftarrow \vec{E} \qquad \vec{E} \leftarrow U$$

E-field btw 2 plates $\qquad E=\dfrac{U}{d}=\dfrac{U_0}{d}\sin\omega t$

Density of D-current $\qquad \dfrac{\partial \boldsymbol{D}}{\partial t}=\varepsilon_0\dfrac{\partial \boldsymbol{E}}{\partial t}=\varepsilon_0\dfrac{U_0}{d}\omega\cos\omega t$

$$i_d=\left(\frac{\partial \boldsymbol{D}}{\partial t}\right)S=\left(\frac{\varepsilon_0 S}{d}\right)\omega V_0\cos\omega t=c\omega U_0\cos\omega t$$

$$i=i_d$$

Example 2.

- Example 7.13 in textbook page 307

§ 7.4 Gauss's Law & Maxwell's 3rd/4th Equs.

Gauss's Law for static E-field is still true for time-varying EM-fields.

$$\oint \vec{D} \bullet d\vec{S} = Q \qquad\qquad \nabla \bullet \vec{D} = \rho$$

- *D*, *Q* and ρ in upper equs are all time-varying.
- E-field described by upper equs is generated by both time-varying charges and time-varying M-field.
  - Div of E-field by time-varying M-field is 0;
  - Div of E-field by time-varying charges is ρ.

Maxwell's 4th Equ.

Div equs for static M-field still holds water for time-varying EM-fields.

$$\oint_S \vec{B} \cdot d\vec{S} = 0 \qquad\qquad \nabla \bullet \vec{B} = 0$$

- M-field described by upper equs is generated by both conduction current and time-varying E-field.
  - Div and flux of M-field by time-varying E-field are 0.
  - Div and flux of M-field by conduction current are also 0.
  - Lines of M-flux are always closed.

§ 7.5 Maxwell's Equations

Integral Form

$$\begin{cases} \oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\vec{J} + \partial \vec{D} / \partial t\right) \bullet d\vec{S} \\ \oint_C \vec{E} \bullet d\vec{l} = -\int_S \partial \vec{B} / \partial t \bullet d\vec{S} \\ \oint_S \vec{B} \bullet d\vec{S} = 0 \\ \oint_S \vec{D} \bullet d\vec{S} = Q \end{cases}$$

2 closed line integral

2 closed surface integral

Maxwell's Equations in Diff Form

$$\begin{cases} \nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t \\ \nabla \times \vec{E} = -\partial \vec{B} / \partial t \\ \nabla \bullet \vec{B} = 0 \\ \nabla \bullet \vec{D} = \rho_{FC} \end{cases}$$

Rotation Source of M-field

Rotation Source of E-field

Div. Source of EM-fields

$$\begin{cases} \oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \bullet d\vec{S} \\ \oint_C \vec{E} \bullet d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{S} \\ \oint_S \vec{B} \bullet d\vec{S} = 0 \\ \oint_S \vec{D} \bullet d\vec{S} = Q \end{cases}$$

Material Equations

——auxiliary equations of Maxwell's Equs.

$$\begin{cases} \vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \\ \vec{B} = \mu_0 \mu_r \vec{H} = \mu \vec{H} \approx \mu_0 \vec{H} \\ \vec{J} = \sigma \vec{E} \end{cases}$$

$$\begin{cases} \nabla \times \vec{H} = \vec{J} + \varepsilon \partial \vec{E} / \partial t \\ \nabla \times \vec{E} = -\mu \partial \vec{H} / \partial t \end{cases}$$

E- & M-fields are mutually induced, which forms a kind of wave.

How about direction of EM-wave?

Observe the curl operation and we know curl refers to the variation in direction of …?

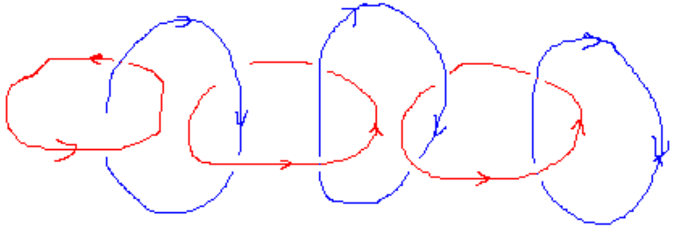
旋度关系

Direction of M-field lies normal to direction of E-field.

Direction of E-field lies normal to direction of M-field.

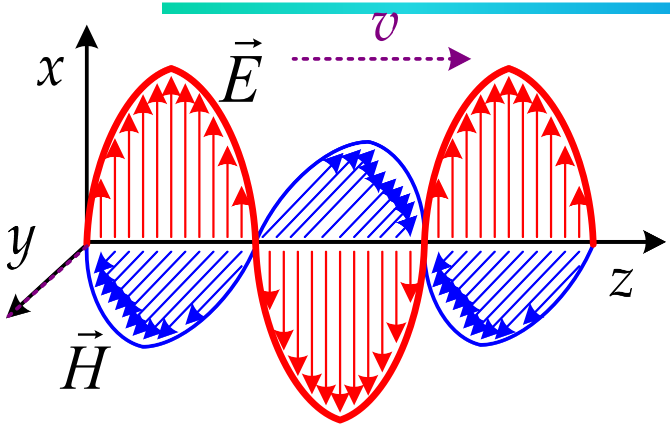
Directions of mutually induced E- & M-fields are perpendicular to each other.

Directions of EM-wave is perpendicular to both E & M.



Flash of HPW

Direction of EM-wave



- Don't walk in front of me,
- I may not follow.
- Don't walk behind me;
- I may not lead.
- Walk just beside me,
- And be my friend.

Example

- Components of M-field are known as H_x=0, H_y=H_0sin$k'y$·sin(wt-kz), where k' and k are constants.
- Please find H_z.
- Solution:

$$\nabla \cdot \vec{\boldsymbol{B}} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial H_z}{\partial z} = -\frac{\partial H_y}{\partial y} = -H_0 k' \cos k'y \sin(\omega t - kz)$$

$$H_z = -H_0 k' \cos k'y \int \sin(\omega t - kz) \mathrm{d}z$$

$$= -H_0 k' \cdot \frac{1}{k} \cdot \cos k'y \cdot \cos(\omega t - kz) + C$$

Maxwell's Eqs in Complex Form

For **time harmonic field**,

$$\vec{E} = \vec{E}_0(x, y, z)\sin(\omega t + \varphi) \qquad \vec{B} = \vec{B}_0(x, y, z)\sin(\omega t + \phi)$$

In complex form, they can be expressed as,

$$\vec{E} = \vec{E}_e(x, y, z)e^{jwt} \qquad \vec{B} = \vec{B}_e(x, y, z)e^{jwt}$$

Therefore, we have $\dfrac{\partial}{\partial t} = j\omega, \ \dfrac{\partial^2}{\partial t^2} = -\omega^2$

e.g. $\dfrac{\partial \vec{E}}{\partial t} = j\omega\vec{E}, \ \dfrac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$

Maxwell's Eqs in Complex Form

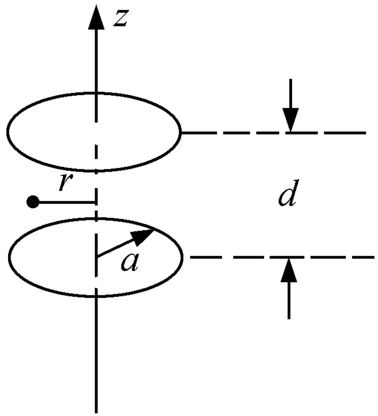
For time harmonic field, $\dfrac{\partial}{\partial t} = j\omega, \quad \dfrac{\partial^2}{\partial t^2} = -\omega^2$

$$\begin{cases} \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \bullet \vec{B} = 0 \\ \nabla \bullet \vec{D} = \rho_{FC} \end{cases} \Longrightarrow \begin{cases} \nabla \times \vec{\tilde{H}} = \vec{\tilde{J}} + j\omega \vec{\tilde{D}} \\ \nabla \times \vec{\tilde{E}} = -j\omega \vec{\tilde{B}} \\ \nabla \bullet \vec{\tilde{B}} = 0 \\ \nabla \bullet \vec{\tilde{D}} = \rho_{FC} \end{cases}$$

- Purpose
  - For simplicity
  - For convenience of analysis in frequency domain

Example 1.

- For a parallel-plate capacitor, both plates are disc. Between them is filled with dielectric (σ, ε, μ_0). Areas of both disc are so large that we neglect the edge effects. If the capacitor is charged with a voltage of $U=U_0\cos\omega t$, please find M-field intensity between the plates, $\boldsymbol{H}$.



- Solution: $\oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\vec{J}_T + \vec{J}_d\right) \bullet d\vec{S}$

$$\vec{J}_T = \sigma\vec{E} \qquad \vec{\boldsymbol{J}}_d = \frac{\partial \vec{\boldsymbol{D}}}{\partial t} = \varepsilon \frac{\partial \vec{\boldsymbol{E}}}{\partial t}$$

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{e}}_z \frac{\boldsymbol{U}}{d} = \vec{\boldsymbol{e}}_z \frac{\boldsymbol{U}_0}{d} \cos\omega t$$

$$H_\phi = \frac{\pi r^2}{2\pi r}\left(J + J_d\right) = \frac{r\boldsymbol{U}_0}{2d}\left[\sigma\cos\omega t - \omega\varepsilon\sin\omega t\right]$$

Example 2.

- Example 7.14 in page 313 of textbook

§ 7.6 Boundary Conditions

1. Boundary Conditions for E-Fields

2. Boundary Conditions for M-Fields

Boundary of Ideal Conductor $\boxed{\sigma = \infty}$

- Conductivity σ is infinity $\sigma = \infty$
- If there existed static or time-varying E-field, very weak E-field would stimulate infinite current. $\vec{J} = \sigma \vec{E}$
- And thus infinite M-field.
- It is obviously impossible.
- **Conclusion: no E-field in ideal conductor**

Boundary of Ideal Dielectrics $\boxed{\sigma = 0}$

Including 2 kinds of *lossless* boundaries, i.e. boundaries containing no free charges and no conduction current.

$$\rho_{FC} = 0 \qquad \vec{J}_{ST} = 0$$

Boundary Conditions for E-Field

In normal direction

$$\oint_S \vec{D} \bullet d\vec{S} = Q$$

$$\boxed{\vec{a}_n \bullet \left(\vec{D}_1 - \vec{D}_2\right) = \rho_{S_{FC}}}$$

$$\boxed{D_{1n} - D_{2n} = \rho_{S_{FC}}}$$

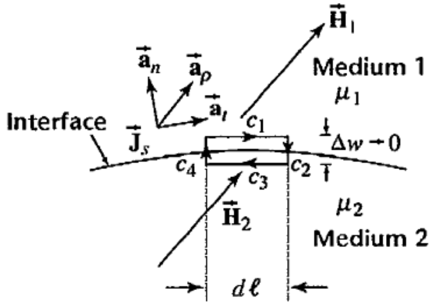
For boundary of ideal dielectrics $D_{1n} = D_{2n}$

For boundary of ideal conductor $D_{1n} = \rho_{S_{FC}}$

Boundary Conditions 2. (in tangential direction)

Construct a closed rectangular path

$$\oint_C \vec{H} \bullet d\vec{l} = I$$



$$\int_{c_1} (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) \cdot \vec{\mathbf{a}}_t \, d\ell = \int_{c_1} \vec{\mathbf{J}}_s \cdot \vec{\mathbf{a}}_\rho \, d\ell$$

$$\vec{a}_t = \vec{a}_\rho \times \vec{a}_n \qquad \boxed{\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})}$$

$$\vec{a}_\rho \bullet [\vec{a}_n \times (\vec{H}_1 - \vec{H}_2)] = J_{sFree} \qquad \boxed{\vec{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_{sFree}}$$

In case of no free surface current, tangential M-intensity is continuous.

$$\vec{a}_n \times \vec{H}_1 = \vec{a}_n \times \vec{H}_2 \qquad \boxed{H_{1t} = H_{2t}}$$

In tangential direction

$$\oint_C \vec{E} \bullet d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{S}$$

$$\vec{a}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0$$

$$\boxed{E_{1t} - E_{2t} = 0}$$

For boundary of ideal dielectrics $E_{1t} = E_{2t}$

For boundary of ideal conductor $E_{1t} = 0$

Boundary Conditions for M-Field

In normal direction

$$\oint_S \vec{B} \bullet d\vec{S} = 0$$

$$\vec{a}_n \bullet \left(\vec{B}_1 - \vec{B}_2\right) = 0$$

$$\boxed{B_{1n} - B_{2n} = 0}$$

For boundary of ideal dielectrics $B_{1n} - B_{2n} = 0$

For boundary of ideal conductor $B_{1n} = B_{2n} = 0$

In tangential direction

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \bullet d\vec{S}$$

$$(H_{2t} - H_{1t})\Delta l \approx Jh \cdot \Delta l + \frac{\partial \vec{D}}{\partial t} h \cdot \Delta l$$

$$\vec{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_{ST}$$

$$H_{2t} - H_{1t} = i$$

$$H_{1t} - H_{2t} = J_{ST}$$

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{i}$$

For boundary of ideal dielectrics $H_{1t} = H_{2t}$

For boundary of ideal conductor $H_{1t} = J_{ST}$

A Comparison of BCs (E and M)

$$\vec{a}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0$$

$$\vec{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_{S_{FC}}$$

$$\vec{a}_n \bullet \left(\vec{D}_1 - \vec{D}_2\right) = \rho_{S_{FC}}$$

$$\vec{a}_n \bullet \left(\vec{B}_1 - \vec{B}_2\right) = 0$$

$$\begin{cases} \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \bullet \vec{B} = 0 \\ \nabla \bullet \vec{D} = \rho_{FC} \end{cases}$$

电磁场的边界条件

电位移法向条件

$$\vec{a}_n \bullet \left(\vec{D}_1 - \vec{D}_2\right) = \rho_{S_{FC}}$$

$$D_{1n} - D_{2n} = \rho_{S_{FC}}$$

电场强度切向连续

$$\vec{a}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0$$

$$E_{1t} - E_{2t} = 0$$

磁通密度法向条件

$$\vec{a}_n \bullet \left(\vec{B}_1 - \vec{B}_2\right) = 0$$

$$B_{1n} - B_{2n} = 0$$

磁场强度切向条件

$$\vec{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_{ST}$$

$$H_{1t} - H_{2t} = J_{ST}$$

理想导体边界 $\sigma = \infty$

理想导体内部内无交变电磁场

$$E_{1t} = 0 \qquad D_{1n} = \rho_{S_{FC}}$$

$$H_{1t} = J_{ST} \qquad B_{1n} = B_{2n} = 0$$

理想介质边界 $\sigma = 0$

边界无自由电荷、无面电流 $\rho_{FC} = 0$ $\vec{J}_{ST} = 0$

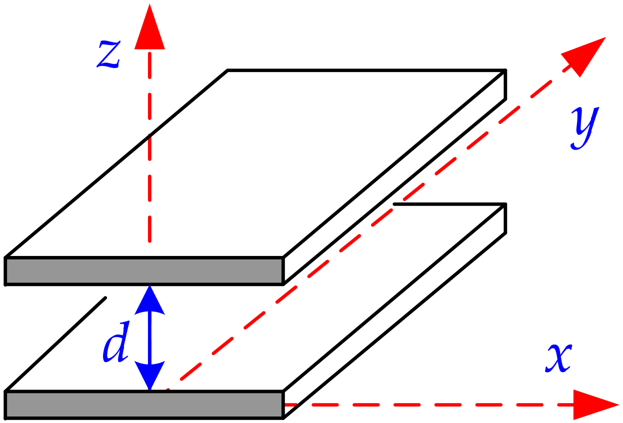
$$E_{1t} = E_{2t} \qquad D_{1n} = D_{2n}$$

$$H_{1t} = H_{2t} \qquad B_{1n} = B_{2n}$$

Example

- EM-wave propagates in the air between 2 parallel conductor plates, as shown in the following figure. Also E-field intensity is given as below, where k_x is a constant. Please find ***H*** between 2 plates and ***J*** on plate surface.

$$\vec{E} = \vec{a}_y E_0 \sin(\pi z/d)\cos(wt - k_x x)$$



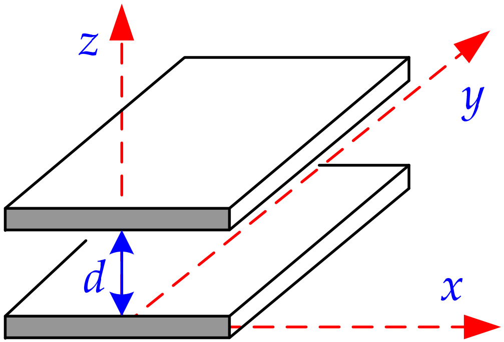
Solution to 1st question

Give $\vec{E} = \vec{a}_y E_0 \sin(\pi z/d)\cos(wt - k_x x)$

Find $\vec{H}$.

?

$$\begin{cases} \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \bullet \vec{B} = 0 \\ \nabla \bullet \vec{D} = \rho_{FC} \end{cases}$$



$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \Longrightarrow \boxed{\vec{H} = -\frac{1}{\mu_0}\int \nabla \times \vec{E} \cdot dt}$$

Solution to 2nd question

Give $\vec{E}$ and $\vec{H}$. Find $\vec{J}_{ST}$. ?

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$?

$$\vec{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_{ST}$$

$$H_{1t} - H_{2t} = J_{ST}$$?

Please finish it after class.

Homework

A rectangular waveguide made of ideal conductor is given as the following figure. E- & M-field intensities within waveguide are given also as below.

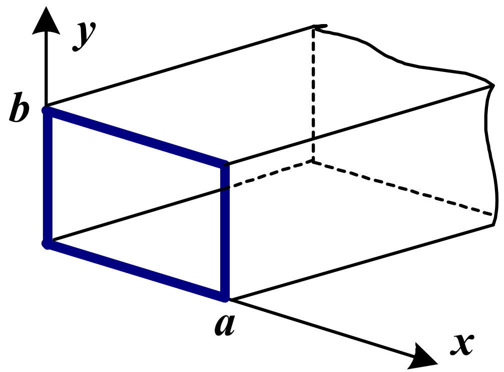
$$\vec{E} = \vec{a}_y E_y \qquad \vec{H} = \vec{a}_x H_x + \vec{a}_z H_z$$

$$E_y = -j\omega\mu \cdot \tfrac{\pi}{a} \cdot H_0 \cdot \sin\left(\tfrac{\pi}{a} \cdot x\right)$$

$$H_x = j\beta \cdot \tfrac{\pi}{a} \cdot H_0 \cdot \sin\left(\tfrac{\pi}{a} \cdot x\right)$$

$$H_z = H_0 \cdot \cos\left(\tfrac{\pi}{a} \cdot x\right)$$

H_0、ω、μ、β constants



Please find charge density and current density on surface of waveguide.

§ 7.7 Poynting's Law & Poynting Vector

- Power within *R*, *C* and *L* can be calculated in terms of *U* and *I*.
- **Power within E- & M-fields can be calculated in terms of …?**
  - ***E* and *H***
- Power is conservative. Power may be static (stored) or dynamic (transmitted).
- How to describe the dynamic power?
  - We define a **vector of power flow**, named as **Poynting's Vector**, to describe the **density of power flow,** **is** **the power flows across per unit area.**

- 电阻、电容、电感中的能量可由电压和电流这两个参数来描述。
- 电磁场中的能量由谁来描述?
  - 电场强度和磁场强度
- 能量是守恒的
  - 能量可是静止的（存储）也可是流动的（传送）
- 如何描述电磁场中流动的能量?
  - 定义一个能流矢量——坡印廷矢量——来描述能流密度。
  - 坡印廷定理是用场的观点,描述在电磁场中的能量守恒关系

Poynting's Law

(1) Using Maxwell's Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\begin{cases} \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \bullet \vec{B} = 0 \\ \nabla \bullet \vec{D} = \rho_{FC} \end{cases}$$

(2) $\vec{H} \bullet \left(\nabla \times \vec{E}\right) - \vec{E} \bullet \left(\nabla \times \vec{H}\right) = ?$

a. $\vec{H} \bullet \left(\nabla \times \vec{E}\right) - \vec{E} \bullet \left(\nabla \times \vec{H}\right)$

$$= -\vec{H} \bullet \partial \vec{B} / \partial t - \vec{E} \bullet \left(\vec{J} + \partial \vec{D} / \partial t\right)$$

$$= -\vec{H} \bullet \partial\left(\mu \vec{H}\right) / \partial t - \vec{E} \bullet \vec{J} - \vec{E} \bullet \partial\left(\varepsilon \vec{E}\right) / \partial t$$

b. Vector characters:

$$\boxed{\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \times \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \times \boldsymbol{b}}$$

$$\vec{H} \bullet \left(\nabla \times \vec{E}\right) - \vec{E} \bullet \left(\nabla \times \vec{H}\right) = \nabla \bullet \left(\vec{E} \times \vec{H}\right)$$

simplification

$$\vec{H}\bullet\left(\nabla\times\vec{E}\right)-\vec{E}\bullet\left(\nabla\times\vec{H}\right)=\nabla\bullet\left(\vec{E}\times\vec{H}\right)$$

$$=-\vec{H}\bullet\partial\left(\mu\vec{H}\right)/\partial t-\vec{E}\bullet\vec{J}-\vec{E}\bullet\partial\left(\varepsilon\vec{E}\right)/\partial t$$

$$\vec{H}\bullet\frac{\partial(\mu\vec{H})}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)\qquad \vec{E}\bullet\vec{J}=\sigma\cdot E^2\qquad \vec{E}\bullet\frac{\partial(\varepsilon\vec{E})}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)$$

Since parameters do not vary with time,

$$\therefore\nabla\bullet\left(\vec{E}\times\vec{H}\right)=-\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)-\sigma E^2$$

(3) Volume integral

$$\int_V\nabla\bullet\left(\vec{E}\times\vec{H}\right)dV=-\int_V\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)dV-\int_V\sigma E^2 dV$$

(4) Gauss's Law

$$\oint_S\left(\vec{E}\times\vec{H}\right)\bullet d\vec{S}=-\int_V\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)dV-\int_V\sigma\cdot E^2 dV$$

$$-\oint_S \left(\vec{E}\times\vec{H}\right)\bullet d\vec{S} = \int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dV + \int_V \sigma\cdot E^2 dV$$

— —Poynting's Law

$$-\oint_S \left(\vec{E}\times\vec{H}\right)\bullet d\vec{S} = \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dV + \int_V (\sigma\cdot E^2)dV$$

E-engergy Density

M-energy Density

Variation Rate of Heat energy density

$$\oint_S \left(\vec{E}\times\vec{H}\right)\bullet d\vec{S} = -\frac{d}{dt}\left(W_e + W_m\right) - P_\Omega$$

E-power M-power Heat-power

Physical Meaning of Poynting's Law

$$\oint_S \left(\vec{E}\times\vec{H}\right)\bullet d\vec{S} = -\frac{d}{dt}\left(W_e + W_m\right) - P_\Omega$$

Power of energy flow out of closed surface

Energy Dissipating rate in closed surface

Define $\vec{S}_P = \vec{E}\times\vec{H}$ as Power Flow Density.

Poynting's vector, *in honor of John H. Poynting*

Unit: W/m²

$$\oint_S \vec{S}_P \bullet d\vec{S} = -\frac{d}{dt}\left(W_e + W_m\right) + \left(-P_\Omega\right)$$

Refer to power flow per cross area in EM-fields

(1) quantity: power density

(2) direction: normal to plane containing $\vec{\boldsymbol{E}}$ & $\vec{\boldsymbol{H}}$

Summary: Poynting's Vector and Poynting's Law

Poynting's Vector $\boxed{\vec{S}_P = \vec{E} \times \vec{H}}$ **Unit** $\boxed{\mathrm{W/m^2}}$

Physical meaning of Poynting vector

The Poynting vector means the instantaneous flow of power per unit area (or power flow density).

Poynting's Law

$$\boxed{-\oint_S \left(\vec{E} \times \vec{H}\right) \bullet d\vec{S} = \frac{d}{dt}\left(W_e + W_m\right) + P_\Omega}$$

Physical meaning of Poynting law

The net power must flow into a closed surface (or the volume bounded by it) in order to account for (a) the power dissipation in the region as heat and (b) the increase in the energy stored in both the electric & magnetic fields.

Example

Given E- & M-fields of a spherical EM-wave emitted by an antenna, please calculate the **power** send by this antenna.

$$E_\theta = A_0 \frac{\sin\theta}{r} \sin(\omega t - kr)$$

$$H_\phi = \frac{1}{\eta_0} A_0 \frac{\sin\theta}{r} \sin(\omega t - kr)$$

$$\vec{S}(t) = \vec{E} \times \vec{H} = \vec{e}_r E_\theta H_\phi = \vec{e}_r \frac{1}{\eta_0} A_0^2 \frac{\sin^2\theta}{r^2} \sin^2(\omega t - kr)$$

$$\mathrm{d}\vec{A} = \vec{e}_r r \mathrm{d}\theta r \sin\theta \mathrm{d}\phi$$

$$\oint_S \vec{S}(T) \cdot \mathrm{d}\vec{A} = \int_0^{2\pi} \int_0^{\pi} \frac{1}{\eta_0} A_0^2 \frac{\sin^2\theta}{r^2} \sin^2(\omega t - kr) \cdot r \mathrm{d}\theta r \sin\theta \mathrm{d}\phi$$

$$= \frac{A_0^2}{\eta_0} \sin^2(\omega t - kr) \int_0^{2\pi} \int_0^{\pi} \sin^3\theta \mathrm{d}\theta \mathrm{d}\phi = \frac{8\pi A_0^2}{3\eta_0} \sin^2(\omega t - kr)$$

$$P_{\mathrm{av}} = \frac{1}{T} \int_0^T \oint_S (\boldsymbol{S}(t) \cdot \mathrm{d}\boldsymbol{S}) \mathrm{d}t = \frac{4}{3} \pi \frac{A_0^2}{\eta_0}$$

Poynting's Law in Complex Form

- Review that
  - For time-harmonic AC circuit $I = I_0 e^{jwt}$ $U = U_0 e^{j(wt+\phi)}$
  - Average power and reactive power (有功功率和无功功率)

$$\tfrac{1}{2}UI^* = \tfrac{1}{2}U_0 e^{j(wt+\phi)} I_0 e^{-jwt} = \underbrace{\tfrac{1}{2}U_0 I_0 \cos\phi}_{\text{Dissipative Energy}} + j\underbrace{\tfrac{1}{2}U_0 I_0 \sin\phi}_{\text{Stored Energy}}$$

- **Poynting's Law in Complex Form**
  - For time harmonic EM-fields $\vec{E} = \vec{E}_0 e^{jwt-jkz}$ $\vec{H} = \vec{H}_0 e^{jwt-jkz}$
  - Direction of EM-wave propagation? +z

$$\boxed{-\tfrac{1}{2}\oint_S \left(\vec{E} \times \vec{H}^*\right) \bullet d\vec{S} = jw\int_V \tfrac{1}{2}\left(\mu H_0^2 - \varepsilon E_0^2\right) dV + \int_V \tfrac{1}{2}(\sigma \cdot E_0^2) dV}$$

A Comparison

$$-\oint_S \left(\vec{E}\times\vec{H}\right)\bullet d\vec{S} = \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dV + \int_V (\sigma\cdot E^2)dV$$

$$-\tfrac{1}{2}\oint_S \left(\vec{E}\times\vec{H}^*\right)\bullet d\vec{S} = jw\int_V \tfrac{1}{2}\left(\mu H_0^2 - \varepsilon E_0^2\right)dV + \int_V \tfrac{1}{2}(\sigma\cdot E_0^2)dV$$

Stored Energy Consumed Energy

- Poynting's Vector in Complex Form $\vec{S} = \frac{1}{2}(\vec{E}\times\vec{H}^*)$
- Average Poynting's Vector $\vec{S}_{av} = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)$

example

- Example 7.15, 716 in text book

Example

- Co-axial lines, with radii of a and b, voltage U and current I. Please find transmitted power consumed.
- Solution:

I → H (A-C Law)

$$H_\phi = \frac{I_0}{2\pi r} e^{j(\omega t - kz)}$$

H → E (Maxwell's 1st Equ.)

$$\vec{e}_r \left(\frac{1}{r} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) = \vec{e}_r \frac{\varepsilon \partial E_r}{\partial t}$$

U → determine E

$$E_r = \frac{U}{r \cdot \ln\left(\frac{b}{a}\right)}$$

E X H → S (in complex)

Integral S with respect to V → Transmitted Power

$$P = \frac{1}{2} \mathrm{Re} \int_S (\boldsymbol{E} \times \boldsymbol{H}^*) \cdot \mathrm{d}S = \frac{I_0 U_0}{2 \ln b/a} \ln r \,|_a^b = \frac{1}{2} I_0 U_0$$

§ 7.8 Scalar Potential & Vector Potential

- For static fields
  - Scalar potential $\vec{E} = -\nabla\psi$ Vector potential $\vec{B} = \nabla\times\vec{A}$
- For time-varying fields
  - Vector potential remains unchanged. However, curl of E is no longer 0 and scalar potential has to be modified.
  - Thus to define scalar potential we must find out a vector whose curl is 0.

$$\left.\begin{aligned}\nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}\\ \vec{B} &= \nabla\times\vec{A}\end{aligned}\right\} \Rightarrow \nabla\times\vec{E} = -\frac{\partial\nabla\times\vec{A}}{\partial t} = -\nabla\times\frac{\partial\vec{A}}{\partial t}$$

$$\Rightarrow \nabla\times(\vec{E}+\frac{\partial\vec{A}}{\partial t}) = 0$$

$$-\nabla\psi = \vec{E}+\frac{\partial\vec{A}}{\partial t}$$

Vector in brackets can be expressed by the gradient of a scalar

- Partial diff. Equ for vector potential

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

- Partial diff. Equ for scalar potential

$$\nabla^2 \psi - \mu\varepsilon \frac{\partial^2 \psi}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

- Lorentz Condition & Lorentz Gauge
  - Significance: determine div of a vector potential

$$\nabla \bullet \vec{A} = -jw\mu\varepsilon\psi$$

$$\psi = \frac{-\nabla \bullet \vec{A}}{jw\mu\varepsilon}$$

Coulomb's Gauge $\nabla \bullet \vec{A} = 0$

is just a special case.

$$\vec{E} = \frac{\nabla(\nabla \bullet \vec{A})}{jw\mu\varepsilon} - jw\vec{A}$$

Homework

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- Exercises 7.17
- Problems 7.29 7.30 7.33 7.39
- Optional: Problems 7.24 7.37

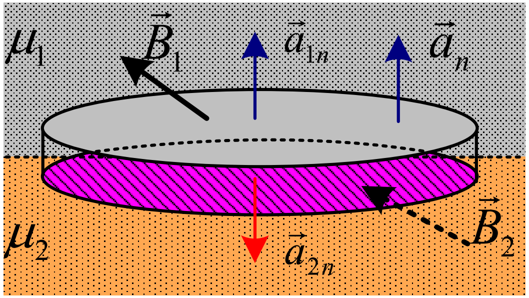
§ 5.5 Boundary Conditions

Boundary Condition 1. (in normal direction)

Make an auxiliary closed surface of a very very flat box.

from $$\oint_S \vec{B} \bullet d\vec{S} = 0$$

$$\int_{s_1} B_{n1}\, ds_1 - \int_{s_2} B_{n2}\, ds_2 = 0$$



$$\boxed{B_{1n} = B_{2n}}$$

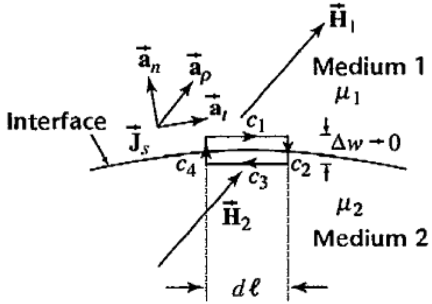
$$\boxed{\vec{B}_1 \bullet \vec{a}_n = \vec{B}_2 \bullet \vec{a}_n}$$

Normal components of M-flux density are equal at boundary.

Boundary Conditions 2. (in tangential direction)

Construct a closed rectangular path

$$\oint_C \vec{H} \bullet d\vec{l} = I$$



$$\int_{c_1} (\vec{\mathbf{H}}_1 - \vec{\mathbf{H}}_2) \cdot \vec{\mathbf{a}}_t \, d\ell = \int_{c_1} \vec{\mathbf{J}}_s \cdot \vec{\mathbf{a}}_\rho \, d\ell$$

$$\vec{a}_t = \vec{a}_\rho \times \vec{a}_n$$

$$\boxed{\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})}$$

$$\vec{a}_s \bullet [\vec{a}_n \times (\vec{H}_1 - \vec{H}_2)] = J_{sFree}$$

$$\boxed{\vec{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_{sFree}}$$

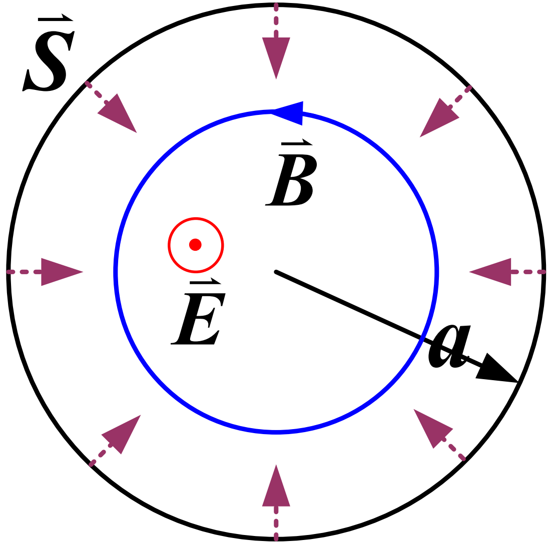
In case of no free surface current, tangential M-intensity is continuous.

$$\vec{a}_n \times \vec{H}_1 = \vec{a}_n \times \vec{H}_2$$

$$\boxed{H_{1t} = H_{2t}}$$

Example 2：单位长度直导线的功率损耗。

解：设导线中存在 z 方向的均匀电流 I，导线电阻为 R。



单位长度导线内的电场强度为：

$$\vec{E} = U\vec{e}_z = RI\vec{e}_z$$

电流在导线表面产生的磁场强度为：

$$\vec{H} = \frac{I}{2\pi a}\vec{e}_\varphi$$

能流密度矢量为：

$$\vec{S}=\vec{E}\times\vec{H}=\frac{I^2R}{2\pi a}\left(-\vec{e}_r\right)$$

由导线侧面流入的能量补偿了导线传输的热损耗：

$$P=\int\vec{S}\cdot d\vec{a}=\frac{I^2R}{2\pi a}2\pi a=I^2R$$

- 由场计算导体热损耗的结果与电路方法一致，但场的计算是更本质的方法。

一、定义

dynamic Vector potential scalar potential

$$\left.\begin{array}{l}\nabla\cdot\vec{B}=0\Rightarrow\ \vec{B}=\nabla\times\vec{A}\\ \nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}\end{array}\right\}\Rightarrow\nabla\times\vec{E}=-\frac{\partial}{\partial t}(\nabla\times\vec{A})$$

$$\Rightarrow\nabla\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0$$

令：$\nabla\varphi=-(\vec{E}+\dfrac{\partial\vec{A}}{\partial t})$ ， $\Rightarrow\quad\vec{E}=-(\nabla\varphi+\dfrac{\partial\vec{A}}{\partial t})$

故：
$$\boxed{\begin{cases}\vec{E}=-(\nabla\varphi+\dfrac{\partial\vec{A}}{\partial t})\\ \vec{B}=\nabla\times\vec{A}\end{cases}}$$

$\vec{A}(\vec{r},t)$：动态矢量位

$\varphi(\vec{r},t)$：动态标量位

- 时变场电场场量和磁场场量均为时间和空间位置的函数，因此动态矢量位和动态标量位也为时间和空间位置的函数。

- 由于时变场电场和磁场为统一整体，因此动态标量位和动态矢量位也是一个统一的整体。

二、洛伦兹规范条件

为了使时变电磁场场量和动态位之间满足一一对应关系，须引入额外的限定条件——规范条件。

$$\nabla \cdot \vec{A} = -\mu\varepsilon \frac{\partial \varphi}{\partial t}$$

洛伦兹规范条件

三、动态位满足的方程

$$\nabla\bullet\vec{E}=\frac{\rho}{\varepsilon}\Rightarrow\nabla\bullet(\nabla\varphi+\frac{\partial\vec{A}}{\partial t})=-\frac{\rho}{\varepsilon}$$

$$\Rightarrow\boxed{\nabla^2\varphi+\frac{\partial}{\partial t}(\nabla\bullet\vec{A})=-\frac{\rho}{\varepsilon}}$$

$$\boxed{\nabla\bullet\vec{A}=-\mu\varepsilon\frac{\partial\varphi}{\partial t}}$$

$$\left.\begin{aligned}\nabla\times\vec{H}&=\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}\\ \vec{H}&=\frac{1}{\mu}\nabla\times\vec{A}\end{aligned}\right\}\Rightarrow\frac{1}{\mu}\nabla\times\nabla\times\vec{A}=\vec{J}+\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow\nabla(\nabla\bullet\vec{A})-\nabla^2\vec{A}=\mu\vec{J}-\mu\varepsilon\frac{\partial}{\partial t}(\nabla\varphi+\frac{\partial\vec{A}}{\partial t})$$

$$\Rightarrow\boxed{\nabla^2\vec{A}-\mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\vec{J}+\nabla(\nabla\bullet\vec{A}+\mu\varepsilon\frac{\partial\varphi}{\partial t})}$$

引入洛伦兹规范条件，则方程简化为

$$\begin{cases}\nabla^2\varphi-\mu\varepsilon\dfrac{\partial^2\varphi}{\partial^2 t}=-\dfrac{\rho}{\varepsilon}\\[2ex]\nabla^2\vec{A}-\mu\varepsilon\dfrac{\partial^2\vec{A}}{\partial t^2}=-\mu\vec{J}\end{cases}$$

达朗贝尔方程

从达朗贝尔方程可以看出：

$\varphi(\vec{r},t)$的源是$\rho(\vec{r},t)$，$\vec{A}(\vec{r},t)$的源是$\vec{J}(\vec{r},t)$