EBU6018
Advanced Transform Methods

Haar Functions

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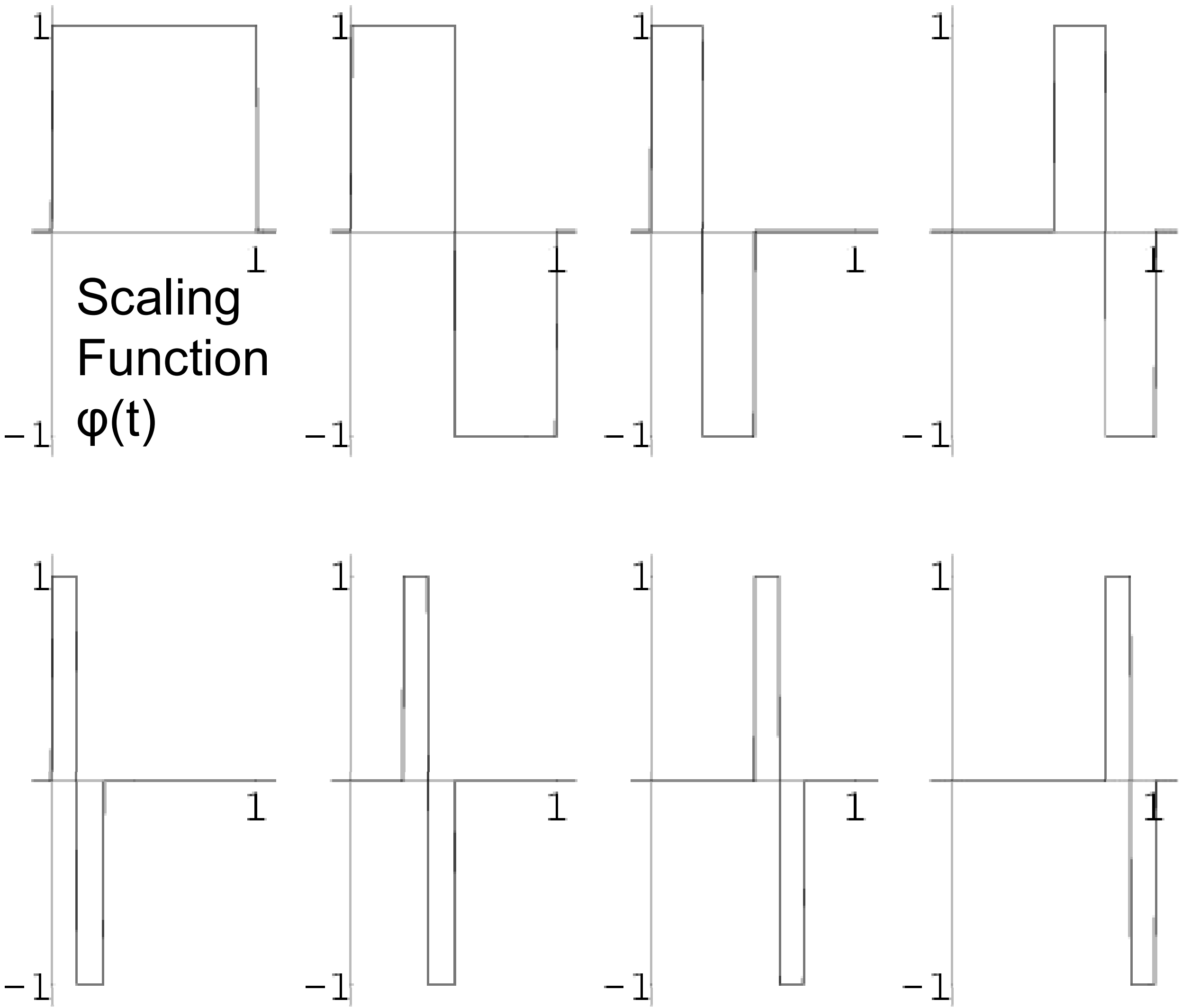
Haar Functions

- The Haar sequence of functions was first proposed by Alfred Haar in 1909 as an example of an orthonormal system.
- Although the concept of wavelets did not exist at that time, they can be considered to be the simplest wavelet function.
- Wavelet functions can be used as basis functions in the analysis of signals, similar to Fourier analysis but with the advantage of giving time-frequency analysis.
- Although Haar Functions are not continuous, they have some useful properties.

Wavelet Functions

- Each wavelet function forms a set of functions, which have a SCALING FUNCTION and a WAVELET FUNCTION
- These functions can be SCALED (i.e. Compressed) and TRANSLATED, forming a family of functions
- The Scaling is DYADIC (power of 2)
- The basic wavelet function is often called the MOTHER WAVELET, and the scaled and translated functions called DAUGHTER WAVELETS

Haar Functions



Wavelet Function:

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_{jk}(x) \equiv \psi(2^j x - k),$$

$\phi_{00} = \phi(x)$

$\psi_{00} = \psi(x)$

$\psi_{10} = \psi(2x)$

$\psi_{11} = \psi(2x-1)$

$\psi_{20} = \psi(4x)$

$\psi_{21} = \psi(4x-1)$

$\psi_{22} = \psi(4x-2)$

$\psi_{23} = \psi(4x-3)$

Orthogonal

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_{jk}(x) \equiv \psi(2^j x - k),$$

$$\int_0^1 \psi_{jk}(x)\psi_{lm}(x)dx = 0$$

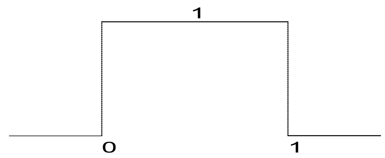
Orthonormal

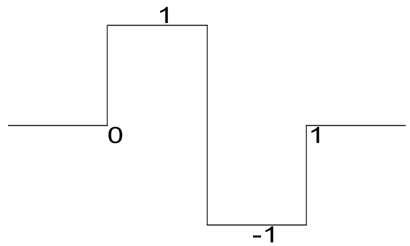
$$\psi_{jk}(x) = 2^{-j/2}\psi(2^j x - k)$$

$$\int_0^1 \psi_{jk}(x)\psi_{jk}(x)dx = 1$$

Orthogonal

For example, the Haar Scaling Function and the Haar Wavelet Function are Orthogonal:





$\varphi_{00} = [1 \quad 1]$ $\psi_{00} = [1 \quad -1]$

$$< \varphi_{00}, \psi_{00} > = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Orthogonal

And all members of the Haar Function family are mutually orthogonal.

For example, consider ψ_{10} and ψ_{11}

ψ_{10}= [1, -1, 0, 0]

ψ_{11}= [0, 0, 1, -1]

$$< \psi_{10}, \psi_{11} > = [1, -1, 0, 0] \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

Orthonormal

$$\psi_{jk}(x) = 2^{-j/2}\psi(2^j x - k)$$

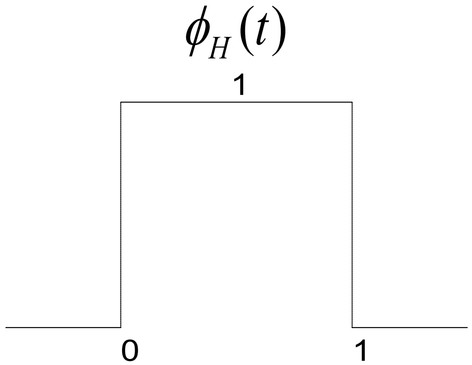
For example, ψ_{10}= [1, -1, 0, 0]
Norm is $\sqrt{2}$

So normalised, ψ_{10}= $\frac{1}{\sqrt{2}}$[1, -1, 0, 0]

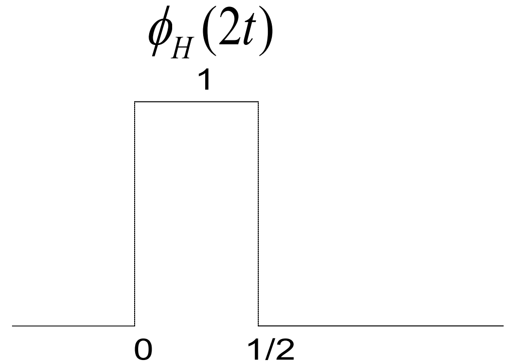
Two-Scale Relations

- A useful property of Haar Functions is that both the Scaling Function and the Wavelet Function can be constructed from scaled and translated versions of the Scaling Function only.
- This property can be used when Haar Functions are applied to Wavelet Transforms.
- In wavelet transform theory this property is called the "Two-Scale Relations"

Two-Scale Relations



$=$



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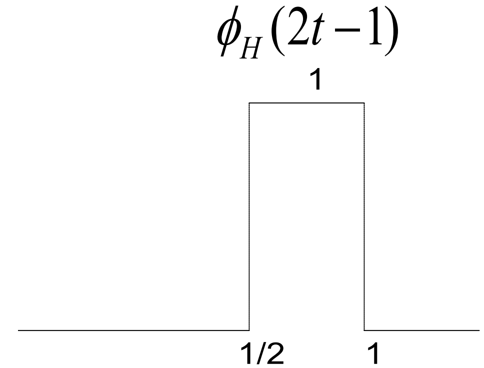
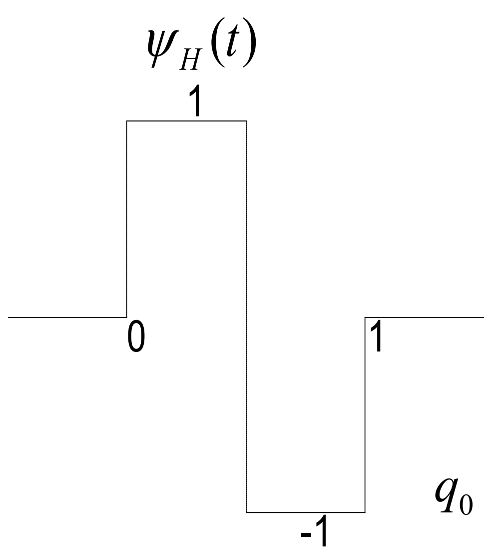


$p_0 = p_1 = 1$, $p_k = 0$ for integer k (except $1, 0$)



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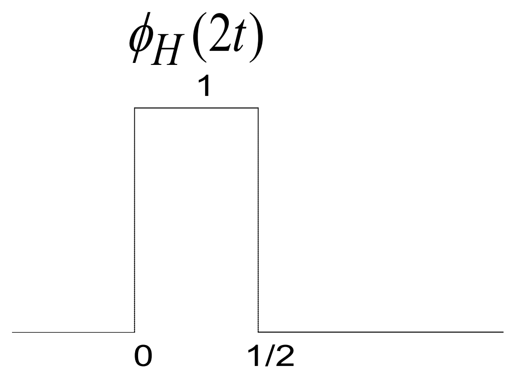

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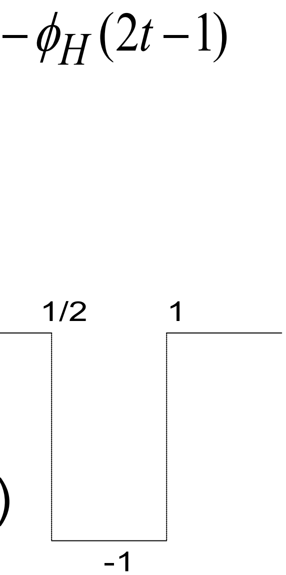


$q_0 = 1$, $q_1 = -1$, $q_k = 0$ for integer k (except $1, 0$)

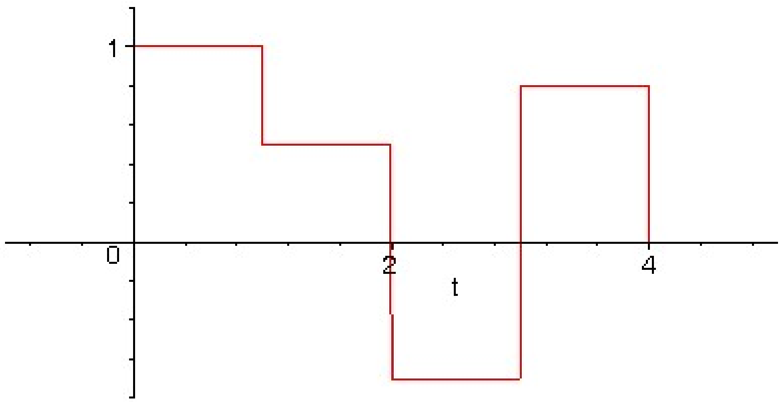
Linear Piecewise Approximation

- Another useful property of Haar Functions is that they can be used to approximate any continuous real function by linear combinations of scaled and translated members of the family of Haar Functions.

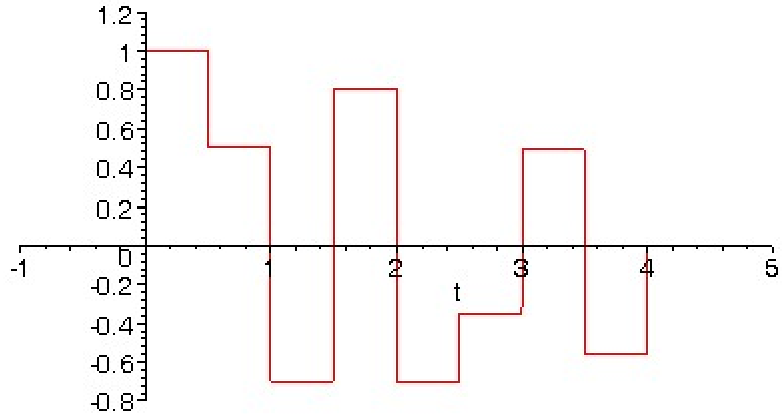
- This principle is illustrated on the following slide:

Synthesis using the Haar function

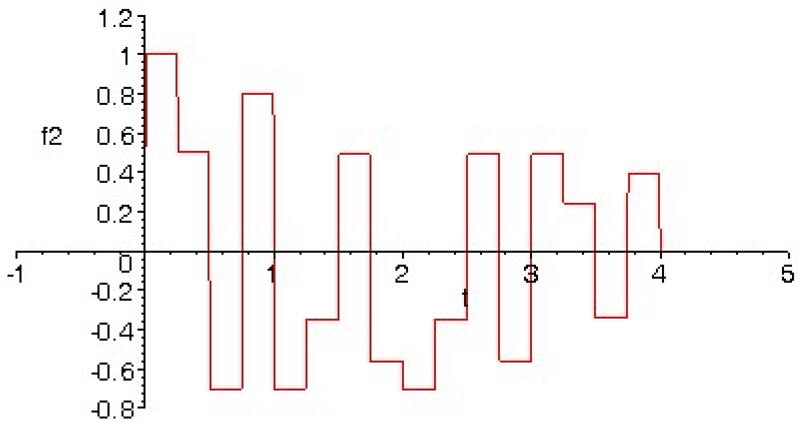
Using just ϕ(x)



Using ψ_{j0} and ϕ



Using ψ_{j1} , ψ_{j0} and ϕ



Exercise

- Using scaled and translated Haar Functions, construct the function given by:

$$1.25\varphi_{0,0}(t)-0.25\varphi_{0,2}(t)-0.25\psi_{0,0}(t)-0.125\psi_{0,2}(t)$$

Summary

Haar Functions are a relatively simple set of functions with some useful properties:

- They are orthogonal (and can be normalised)
- They can be used to approximate any continuous real function
- Both the Scaling Function and the Wavelet Function can be derived from the Scaling Function only
- The application to wavelet transforms will be discussed after the lecture on Wavelet Transforms