第三章

3.1

(1) $E\{Y(t)\} = E\{X(t)\}\, E\{\cos(2\pi f_c t+\theta)\} = 0$

$R_Y(t, t+\tau) = E\{X(t)X(t+\tau)\} \cdot E\{\cos(2\pi f_c t+\theta)\cos(2\pi f_c(t+\tau)+\theta)\}$

$= R_X(\tau) \cdot E\{\frac{1}{2}\cos(4\pi f_c t + \underbrace{2\pi f_c \tau + \theta}_{0}) + \frac{1}{2}\cos(2\pi f_c \tau)\}$

$= \frac{1}{2} R_X(\tau) \cdot \cos(2\pi f_c \tau)$

$P_Y(f) = \mathcal{F}\{R_Y(\tau)\} = \frac{1}{2} P_X(f) \otimes \frac{1}{2}(\delta(f+f_c) + \delta(f-f_c))$

$= \frac{1}{4}(P_X(f+f_c) + P_X(f-f_c))$

(2) $E\{Y(t)\} = E\{X(t)\} \cdot \cos(2\pi f_c t+\theta) = 0$

$R_Y(t, t+\tau) = \underbrace{E\{X(t)X(t+\tau)\}}_{= R_X(\tau)} \cdot \cos(2\pi f_c t+\theta) \cdot \cos(2\pi f_c(t+\tau)+\theta)$

此时 $R_Y(t, t+\tau)$ 不仅与时间差 τ 有关，还与时刻 t 有关。

因此 $Y(t)$ 不平稳.

3.2

解：$P_x(f) = P_n(f) \cdot |H(f)|^2$

$$P_n(f) = \frac{N_0}{2}$$

$$P_x(f) = \begin{cases} \frac{N_0 T_s}{4}\left(1 + \cos\frac{2\pi f T_s}{2}\right) & |f| \le \frac{1}{T_s} \\ 0, & \text{others} \end{cases}$$

$$P_x = \int_{-\frac{1}{T_s}}^{\frac{1}{T_s}} P_x(f)\, df$$

$$= \frac{N_0 T_s}{4} \int_{-\frac{1}{T_s}}^{\frac{1}{T_s}} \left(1+\cos\frac{2\pi f T_s}{2}\right) df$$

$$= \frac{N_0 T_s}{4}\left[2 \cdot \frac{1}{T_s} + \frac{1}{\pi T_s} \cdot \sin \pi f T_s \Big|_{-\frac{1}{T_s}}^{\frac{1}{T_s}}\right]$$

$$= \frac{N_0 T_s}{4}\left[\frac{2}{T_s} + \frac{1}{\pi T_s} \cdot \sin\pi - \sin(-\pi)\right]$$

$$= \frac{N_0}{2}$$

3.3

时域微分 $\frac{d}{dt}(\cdot)$ ⟶ 频域 $: \times j2\pi f$

整个系统的传输函数 $H_S(f) = j2\pi f H(f)$

$$P_Y(f) = P_n(f) \cdot |H_S(f)|^2 = \frac{N_0}{2} \cdot |j2\pi f|^2 |H(f)|^2$$

$$= \frac{N_0}{2} \cdot 4\pi^2 f^2 \; |H(f)|^2$$

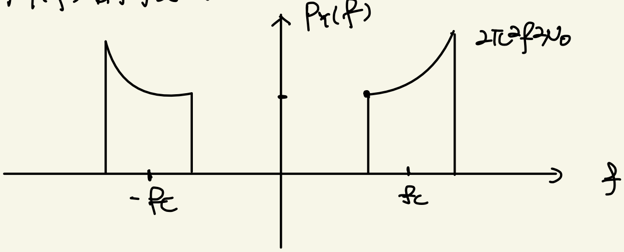
$$= 2\pi^2 f^2 N_0 \; |H(f)|^2$$

$$P_{Y_c}(f) = P_{Y_S}(f) = P_Y(f+f_c) + P_Y(f-f_c)$$

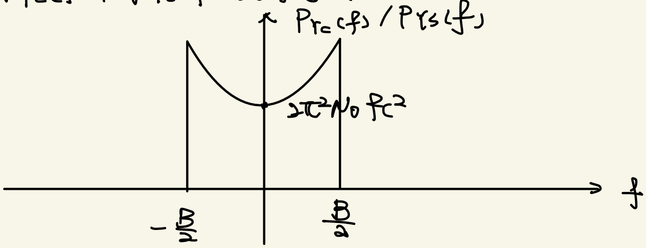
$$= 2\pi^2 (f-f_c)^2 N_0 + 2\pi^2 (f+f_c)^2 N_0$$

$$= 2\pi^2 N_0 (f^2 + f_c^2) \quad |f| \le \frac{B}{2}$$

$P_Y(f)$ 的示意图



$P_{Y_c}(f)$ / $P_{Y_S}(f)$ 的示意图



3.4

$$E\{\xi_1\xi_2\} = E\left\{\int_0^T\int_0^T n(t_1)n(t_2)\,\underbrace{\varphi_1(t_1)\,\varphi_2(t_2)}_{\text{确定信号}}\,dt_1dt_2\right\}$$

$$= \int_0^T\int_0^T E\{n(t_1)n(t_2)\}\,\varphi_1(t_1)\,\varphi_2(t_2)\,dt_1dt_2$$

$$E\{n(t_1)n(t_2)\} = \frac{N_0}{2}\,\delta(t_2-t_1)$$

当 $t_1 \neq t_2$ 时，积分项为零

当 $t_1 = t_2$ 时，积分项为 $\int_0^T \frac{N_0}{2}\varphi_1(t)\varphi_2(t)\,dt$

因此 $E\{\xi_1\xi_2\} = \int_0^T \frac{N_0}{2}\varphi_1(t)\varphi_2(t)\,dt$

对于高斯分布，统计独立 $\Leftrightarrow$ 二者不相关

由于 $E\{\xi_1\} = E\{\xi_2\} = 0$，若 $E\{\xi_1\xi_2\} = E\{\xi_1\}E\{\xi_2\} = 0$

则二者统计独立。因此只需满足 $\int_0^T \varphi_1(t)\varphi_2(t)\,dt = \langle\varphi_1, \varphi_2\rangle = 0$

即 $\varphi_1(t)$，$\varphi_2(t)$ 正交时，ξ_1，ξ_2 统计独立

3.5

$Y(t) = LPF\{X(t)\cdot 2\cos\omega_c t\} = X_c(t)$

由于 $X(t)$ 为窄带高斯平稳过程，则 $E\{X(t)\} = 0 \Rightarrow E\{X_c(t)\} = 0$

$\delta_Y^2 = R_{X_c}(0) = R_X(0) = \delta_X^2$

因此 $E[Y^2] = \delta_X^2$

$f_Y(y)$ 仍然服从高斯分布且 $\sim N(0, \delta_X^2)$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\delta_X}\cdot e^{-\frac{x^2}{2\delta_X^2}}$$

3.6

(1) $P_Y(f) = |H(f)|^2 P_n(f) = \frac{N_0}{2}\cdot 4\pi^2 f^2 = 2\pi^2 f^2 N_0$

$H(f) = j2\pi f$ (微分器) $\quad (N_0 = 2\times 10^{-6}\ W/Hz)$

(2) $P_{Y_0}(f) = |H_2(f)|^2\cdot P_Y(f) = \begin{cases} 2\pi^2 f^2 N_0, & |f|\le 10\,Hz \\ 0, & \text{others} \end{cases}$

$H_2(f) = rect\left(\frac{f}{20}\right)$

$$P_{Y_0} = \int_{-10}^{10} 2\pi^2 f^2 N_0\, df$$

$$= 2\pi^2 N_0\cdot \frac{1}{3}f^3\Big|_{-10}^{10}$$

$$= \frac{2\pi^2 N_0}{3}\times 2000 = \frac{4000\pi^2}{3}\times 2\times 10^{-6} = \frac{8}{3}\pi^2\times 10^{-3}\ (W)$$

$$\approx 0.0263\ (W)$$

3.7

首先将 $\xi(t)$ 通过截止频率为 f_H 的 LPF、

此时 $P_\xi(f) = \mathrm{rect}\left(\frac{f}{2f_H}\right) \cdot \frac{N_0}{2}$

带通高斯白噪声

$R_\xi(\tau) = 2f_H \,\mathrm{Sa}(2f_H \cdot \tau) \cdot \frac{N_0}{2}$

当 $\tau = \frac{1}{2f_H}$ 时，$R_\xi(\tau) = 0$

现以 $f_s = 2f_H$ 的速率采样，则各采样点间时间间隔为 $\tau = \frac{1}{2f_H}$

因为 $R_\xi(\tau = \frac{1}{2f_H}) = 0$，因此取任意两采样点 $E[\xi t_n \xi t_m]$，

两两不相关 $\Longleftrightarrow$ 两两独立

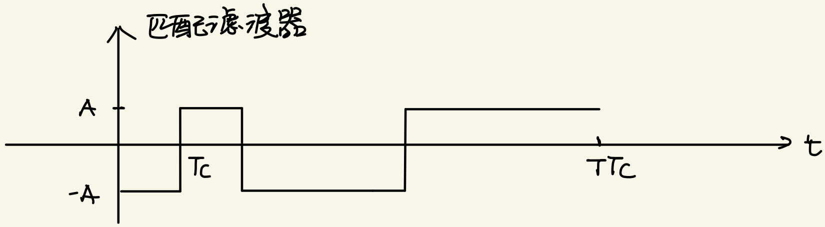
因此联合概率密度是各个随机变量概率密度相乘

$$f_{\xi_1 \cdots \xi_n}(\xi_1, \cdots, \xi_n) = \frac{1}{(\sqrt{2\pi\sigma_\xi^2})^n} \cdot \sum_{i=1}^{n} e^{-\frac{\xi_i^2}{2\sigma_\xi^2}}$$

$$\sigma_\xi^2 = R_\xi(0) = 2f_H \cdot \mathrm{Sa}(0) \cdot \frac{N_0}{2} = N_0 f_H$$

3.8

(1) 匹配滤波器 $b(t_0-t)$，$t_0 = TT_c$



(2) 输入 $b(t)+n(t)$ 时，其最大输出信噪比

当通过匹配滤波器 $h(t)=b(t_0-t)$ 时，获得最大信噪比

$$\frac{2E_b}{N_0},\quad E_b=\int_0^{TT_c} A^2\,dt = TA^2T_c$$

$$\frac{2E_b}{N_0} = \frac{14A^2T_c}{N_0}$$

(3) $f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_y}\cdot e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$

$$\mu_y = E\{y(t)\} = E\{(b(t)+n(t))\otimes h(t)\}$$

$$= E\{n(t)\otimes h(t)\} + b(t)\otimes h(t)$$

最大信噪比时，$t = TT_c$

$$b(t)\otimes h(t) = \int_0^{TT_c} b(\tau)\cdot b(t-t_0+\tau)\,d\tau$$

$t=TT_c$ 时

$$= \int_0^{TT_c} b^2(\tau)\,d\tau = TA^2T_c$$

$$\mu_y = TA^2T_c$$

$$\sigma_y^2 = \frac{N_0}{2}\cdot E_h = \frac{N_0}{2}\cdot TA^2T_c$$

(此时的 σ_y^2 主要是由 $n(t)$ 通过 $h(t)$ 产生，大小为 $\frac{N_0}{2}E_h$)

3.10

$E\{\xi_1(t)\} = E\{\xi_2(t)\} = 0$,

若 $\xi_1(t)$, $\xi_2(t)$ 两者独立, $\Longleftrightarrow$ 二者不相关

则必需满足 $E\{\xi_1(t_1)\,\xi_2(t_2)\} = R_{\xi_1\xi_2}(\tau) = 0$

$\mathcal{F}\{R_{\xi_1\xi_2}(\tau)\} = P_{\xi_1\xi_2}(f) = 0$

$P_{\xi_1\xi_2}(f) = \frac{N_0}{2}|H_1(f)|^2\,|H_2(f)|^2 = 0$

当 $|H_1(f)|^2\,|H_2(f)|^2 = 0$ 时 $\xi_1(t)$ 与 $\xi_2(t)$ 统计独立

×

$H_1^*(f)\,H_2(f) \equiv 0$