3D Graphics Programming Tools

Geometric Transformations

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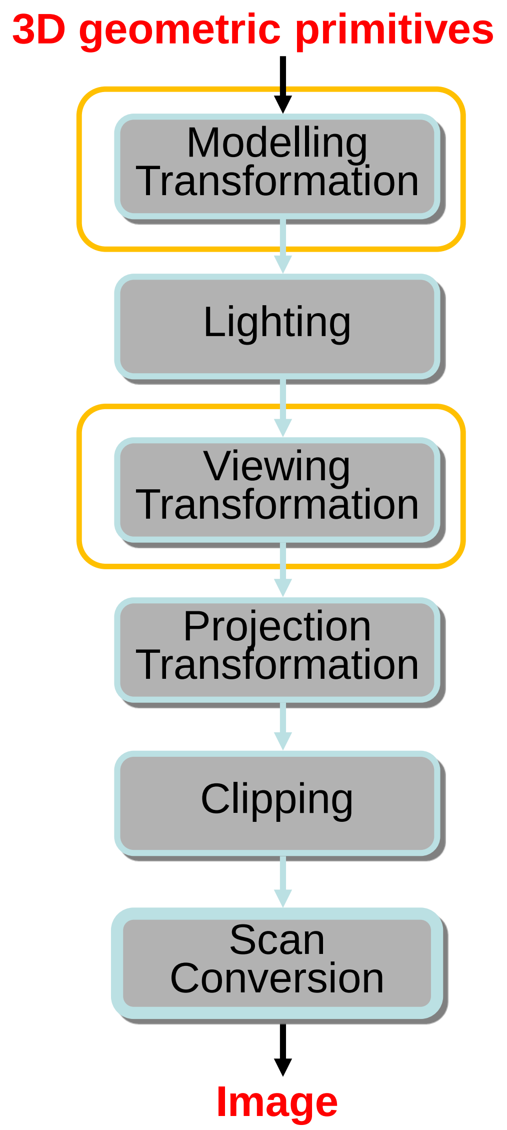
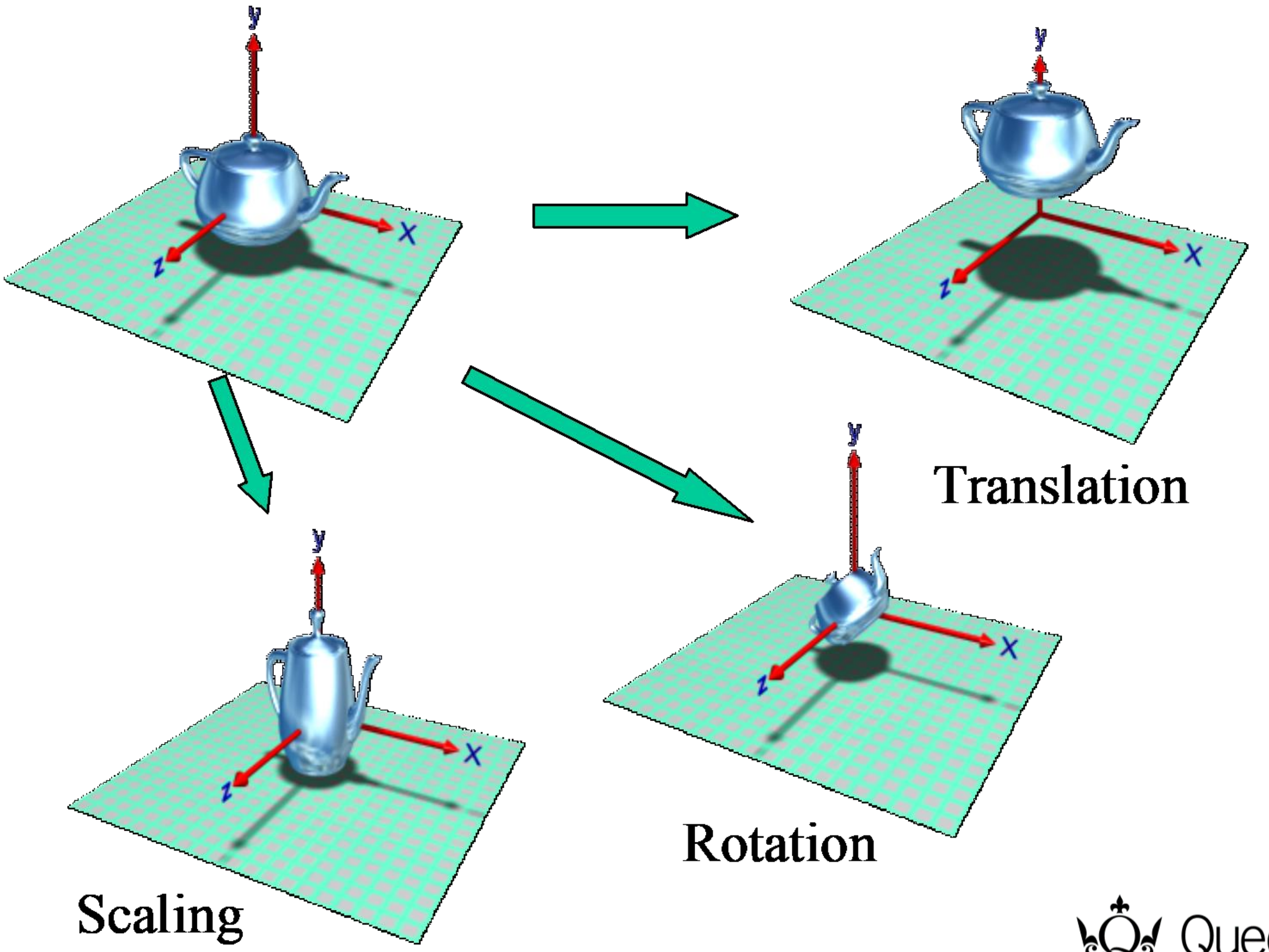
3D Computer Graphics Pipeline



Geometric transformations



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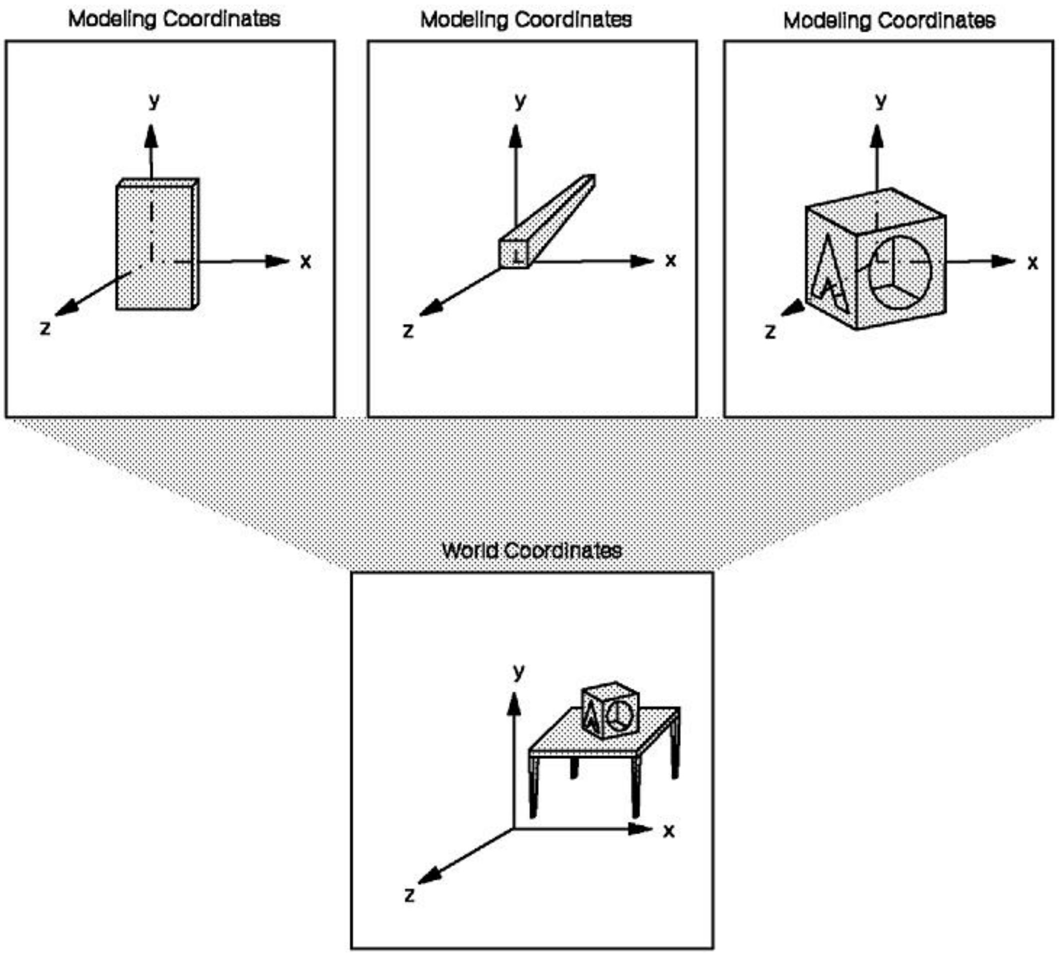
Topics

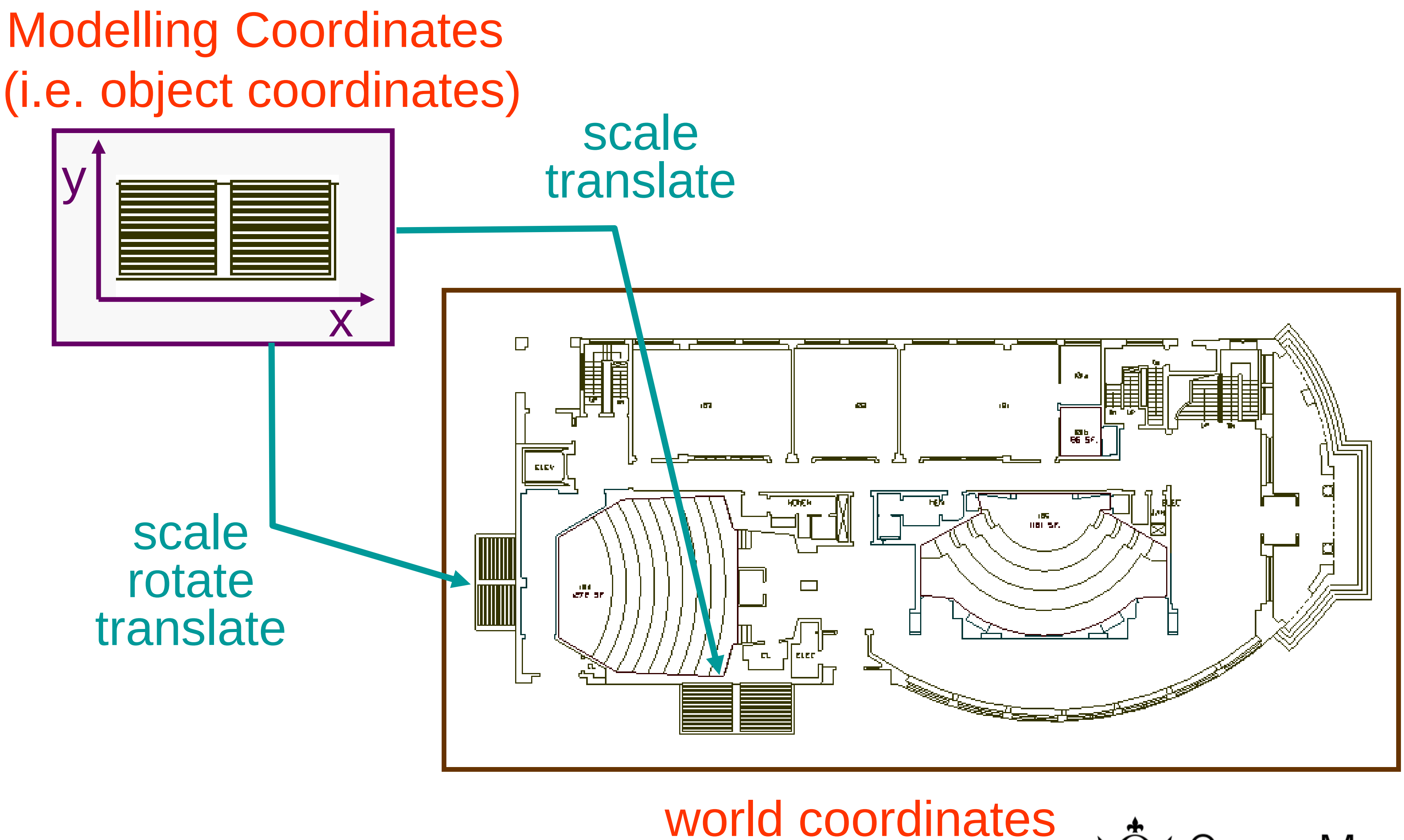
- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations

Modelling transformations

- Specify transformations for objects
  - definitions of objects in own coordinate systems
  - use of object definition multiple times in a scene

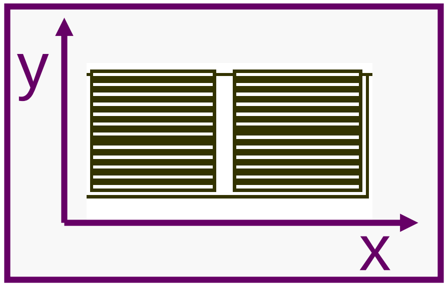


2D modelling transformations

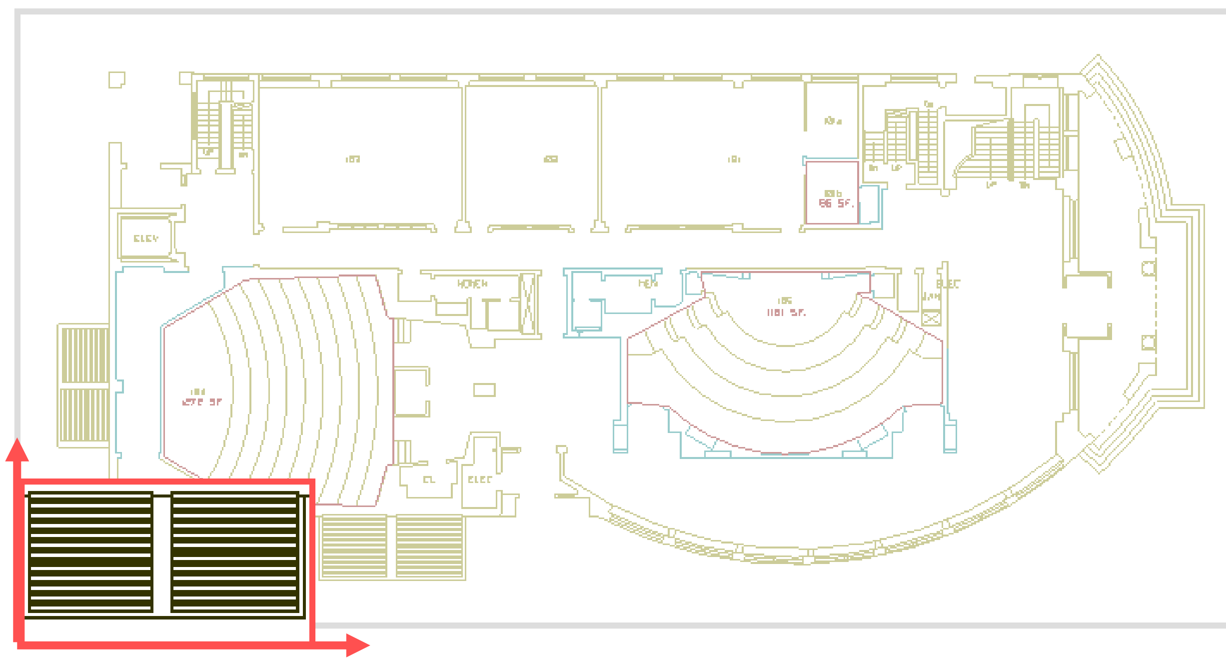


2D modelling transformations

modelling coordinates

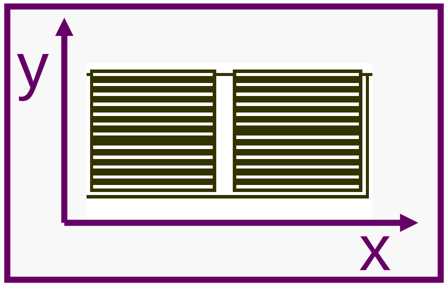


Initial location at (0, 0) with x- and y-axes aligned

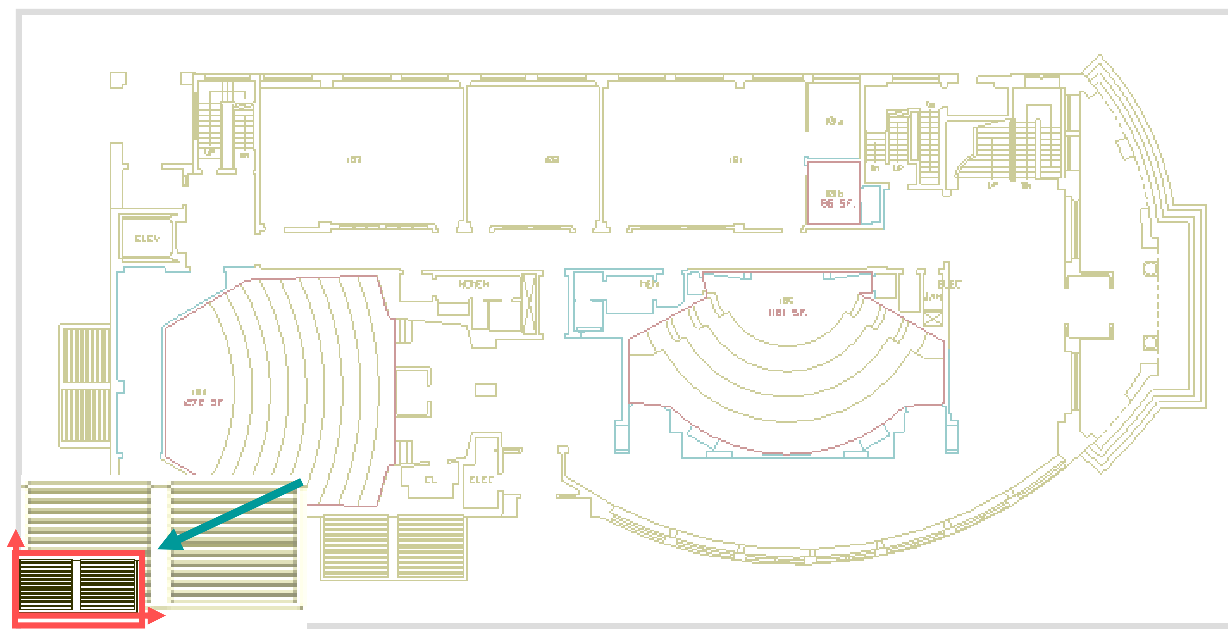


2D modelling transformations

modelling
coordinates

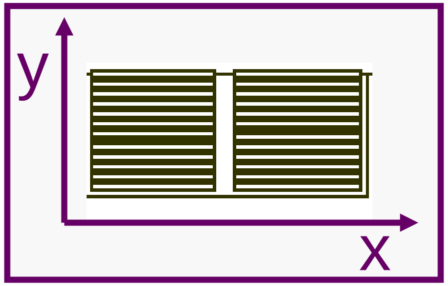


scale .3, .3

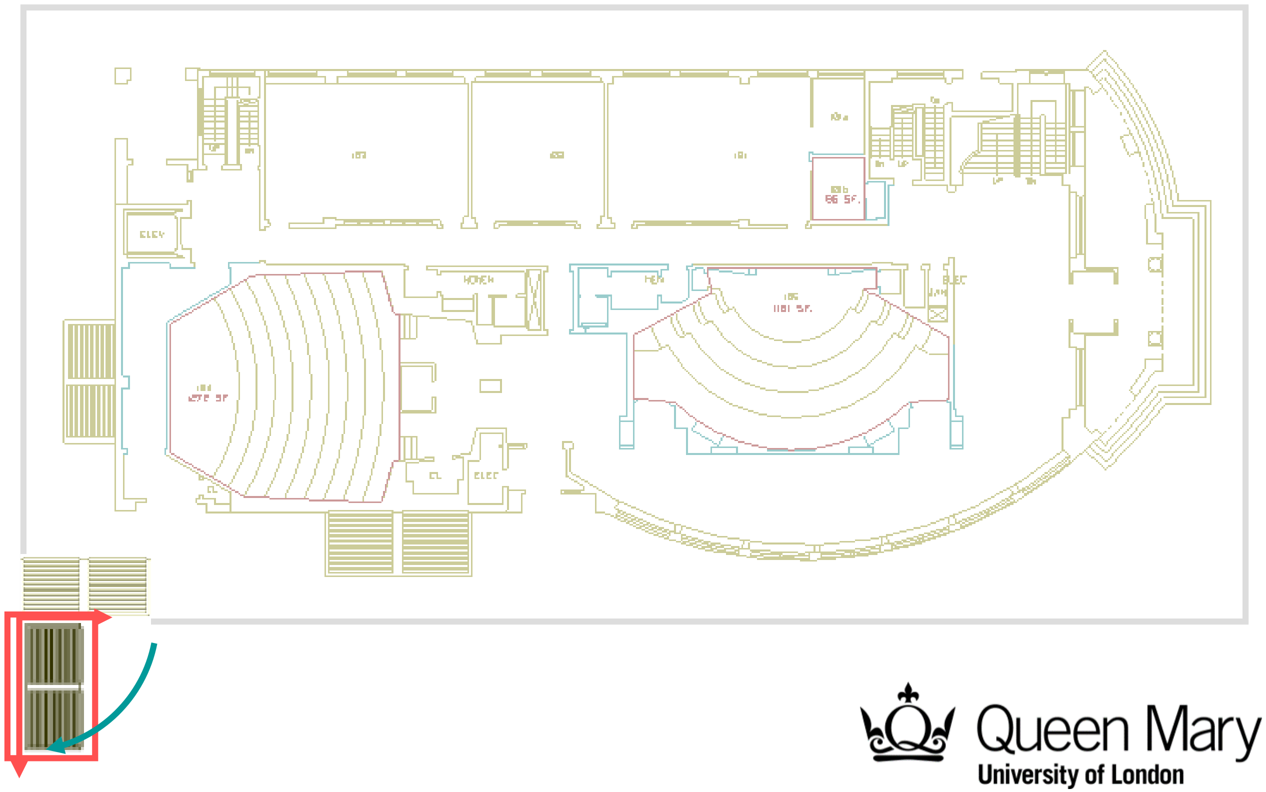


2D modelling transformations

modelling coordinates



scale .3, .3
rotate -90



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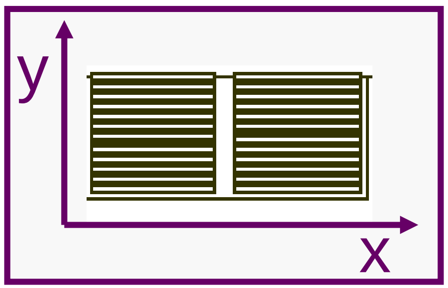
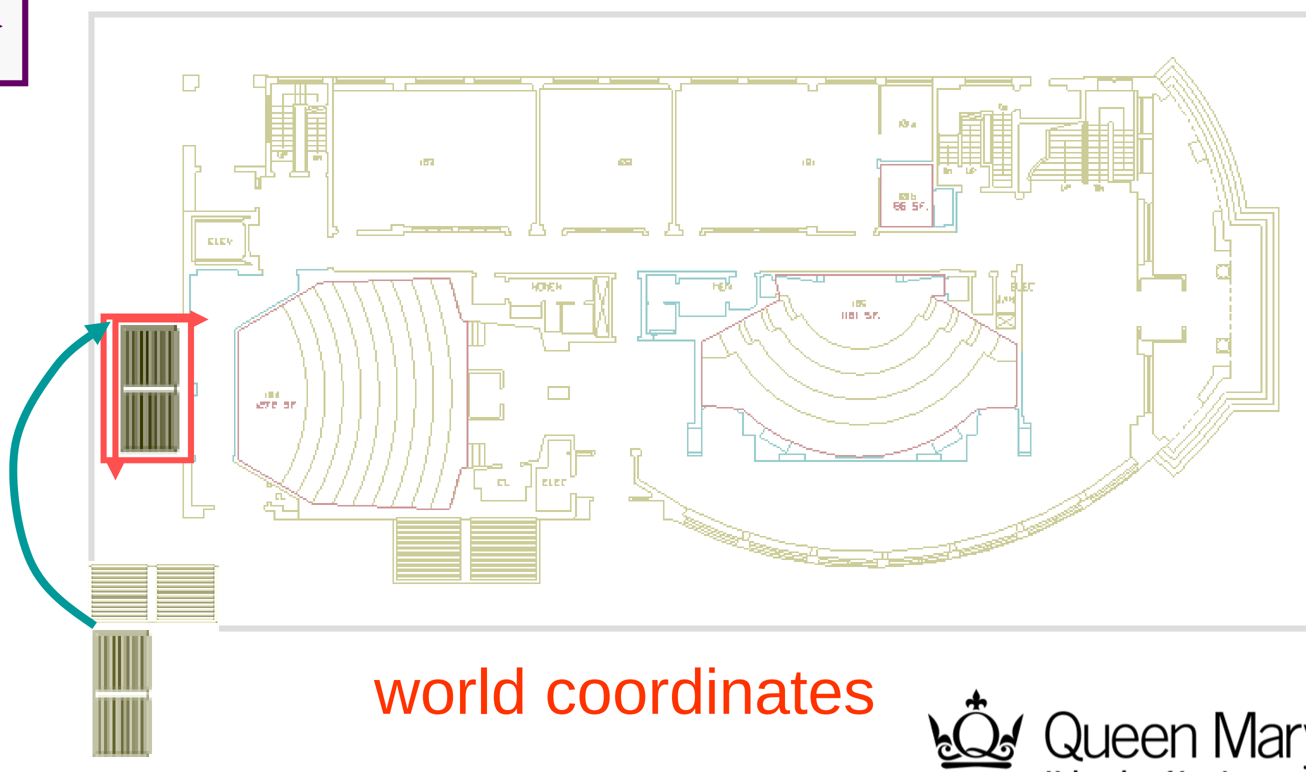
2D modelling transformations

modelling
coordinates



scale .3, .3
rotate -90
translate 5, 3



world coordinates

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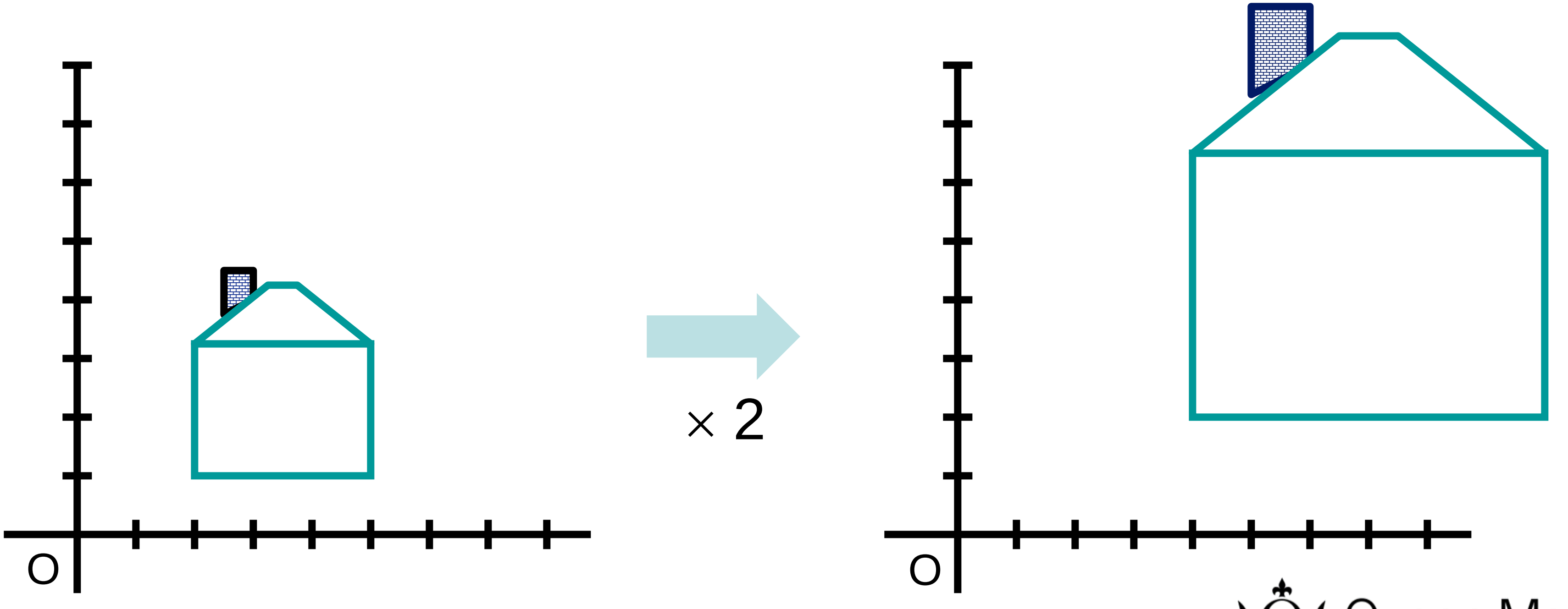
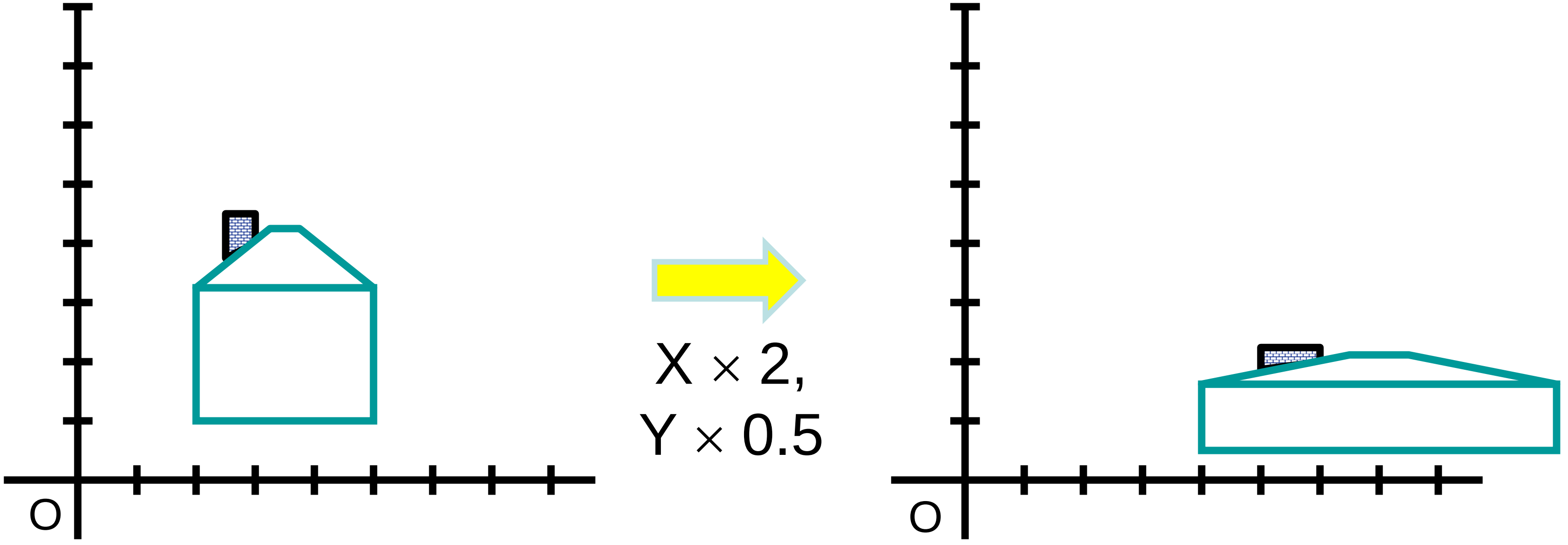
Scaling

- Scaling a coordinate
  - means multiplying each of its components by a scalar
- Uniform scaling
  - means this scalar is the same for all components



Scaling

- Non-uniform scaling
  - different scalars per component



How can we represent this in matrix form?

Scaling

- Scaling operation:

$$\boxed{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}}$$

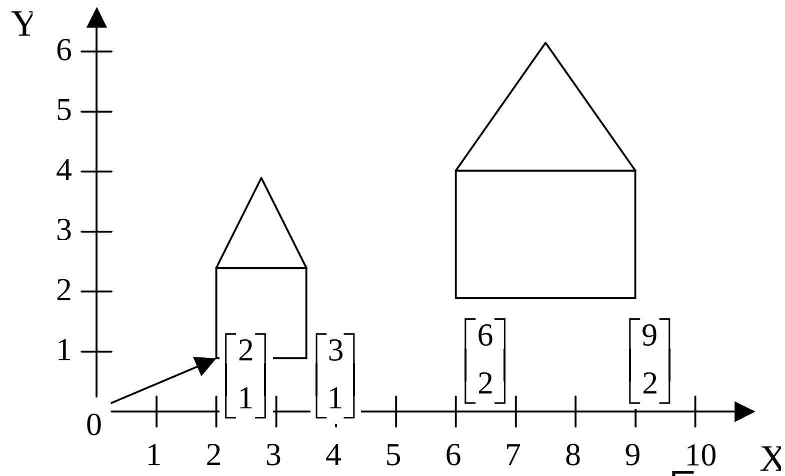
- Or, in matrix form:

$$\boxed{\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix}} \begin{bmatrix} x \\ y \end{bmatrix}}$$

Multiplying a point (or a vector) by a matrix (a transformation) yields a new transformed point (or a new vector)

Scaling

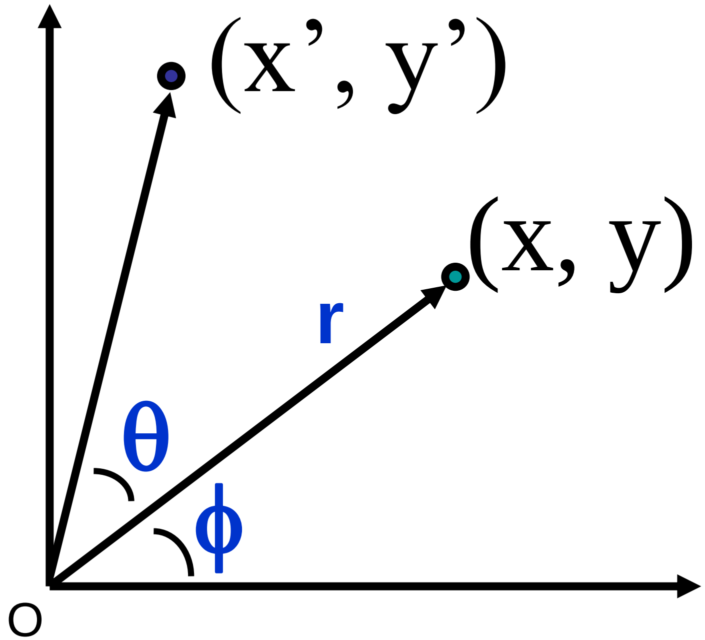
$$s_x = 3$$
$$s_y = 2$$



$$\boldsymbol{v}' = \boldsymbol{S}\boldsymbol{v} \text{ where } \boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \ \boldsymbol{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \text{ and } \begin{cases} x' = s_x x \\ y' = s_y y \end{cases}$$

- Scaling about the origin
- Negative scaling is reflection
- Scaling needn't be uniform, differential scaling
- Does not preserve lengths
- Does not preserve angles (except uniform scaling)
- Not a rigid body transformation

2D rotation about origin



$x = r\cos(\phi)$

$y = r\sin(\phi)$

$x' = r\cos(\phi + \theta)$

$y' = r\sin(\phi + \theta)$

trigonometric identity…

$x' = r\cos(\phi)\cos(\theta) - r\sin(\phi)\sin(\theta)$

$y' = r\sin(\phi)\cos(\theta) + r\cos(\phi)\sin(\theta)$

substitute…

$x' = \mathbf{x}\cos(\theta) - \mathbf{y}\sin(\theta)$

$y' = \mathbf{x}\sin(\theta) + \mathbf{y}\cos(\theta)$

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2D rotation

- Or, in matrix form:

$$\boxed{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$$

- Even though sin(θ) and cos(θ) are nonlinear functions of θ,
  - x' is a linear combination of x and y
  - y' is a linear combination of x and y

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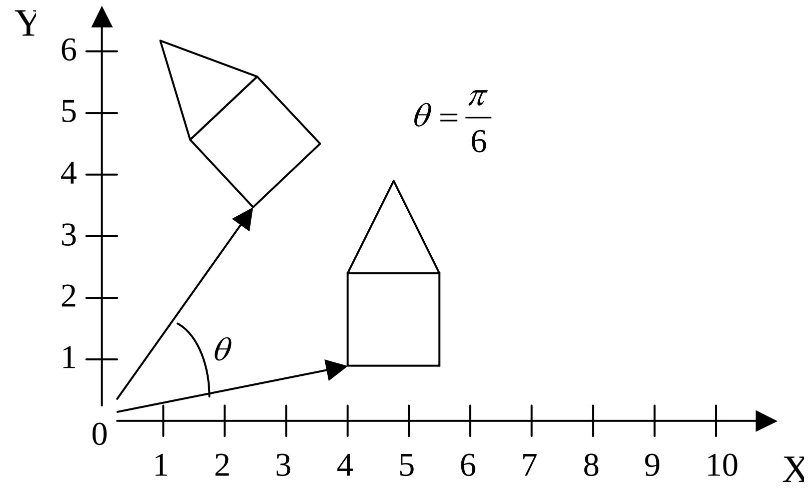
2D rotation

- **Rotation about the origin**
- **Preserves lengths and angles**
- **Rigid body transformation**



$$v' = Rv, \text{ where } v = \begin{bmatrix} x \\ y \end{bmatrix}, v' = \begin{bmatrix} x' \\ y' \end{bmatrix}, R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases}$$

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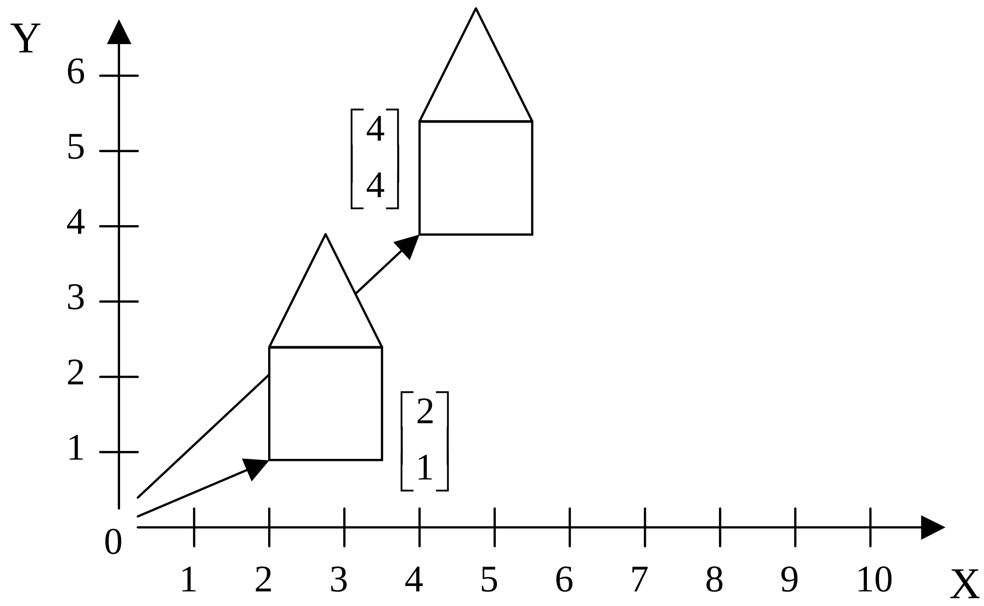
2D Translation



dx = 2
dy = 3

$$\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{t}, \text{ where } \boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \boldsymbol{t} = \begin{bmatrix} dx \\ dy \end{bmatrix}, \text{ and } \begin{cases} x' = x + dx \\ y' = y + dy \end{cases}$$

- **Preserves lengths (isometric)**
- **Preserves angles (conformal)**
- **Rigid body transformation**

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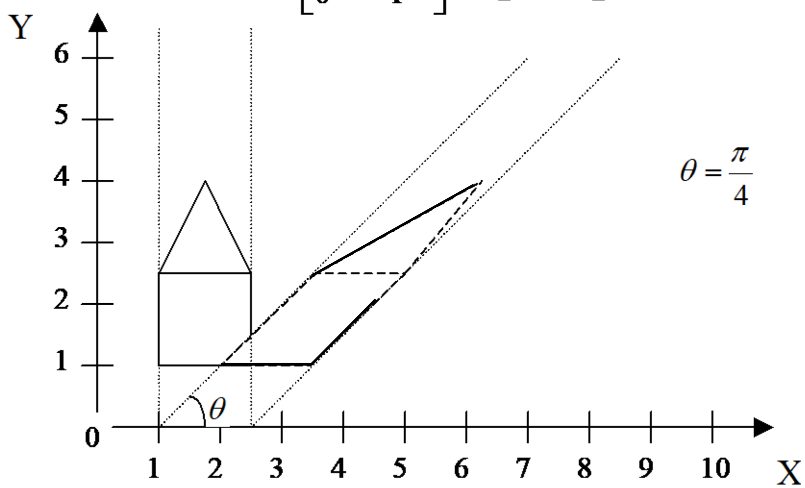
2D Shearing

$$Shear_\theta = \begin{bmatrix} 1 & \frac{1}{\tan\theta} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & dx \\ 0 & 1 \end{bmatrix}$$



$\theta = \frac{\pi}{4}$

Not rigid body transformation

$$\begin{cases} x' = x + sh_x * y \\ y' = y \end{cases}$$

$$\begin{cases} x' = x \\ y' = sh_y * x + y \end{cases}$$

- **Shear transformations are also affine transformations**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ **(shear along x axis by using y)**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ **(shear along y axis by using x)**

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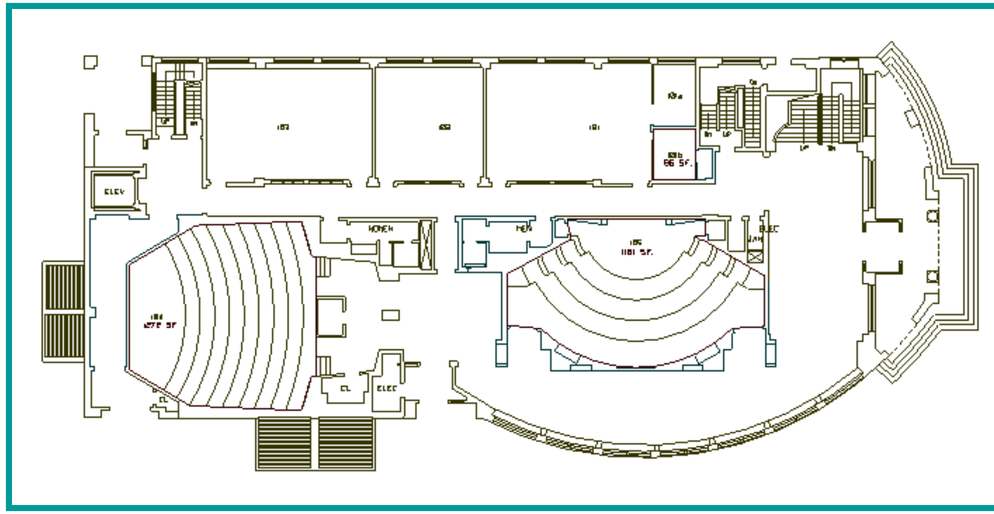
Basic 2D transformations

- Translation
  - $x' = x + t_x$
  - $y' = y + t_y$
- Scale
  - $x' = x * s_x$
  - $y' = y * s_y$
- Shear
  - $x' = x + h_x * y$
  - $y' = y + h_y * x$
- Rotation
  - $x' = x * \cos\Theta - y * \sin\Theta$
  - $y' = x * \sin\Theta + y * \cos\Theta$



Transformations can be combined
(with simple algebra)

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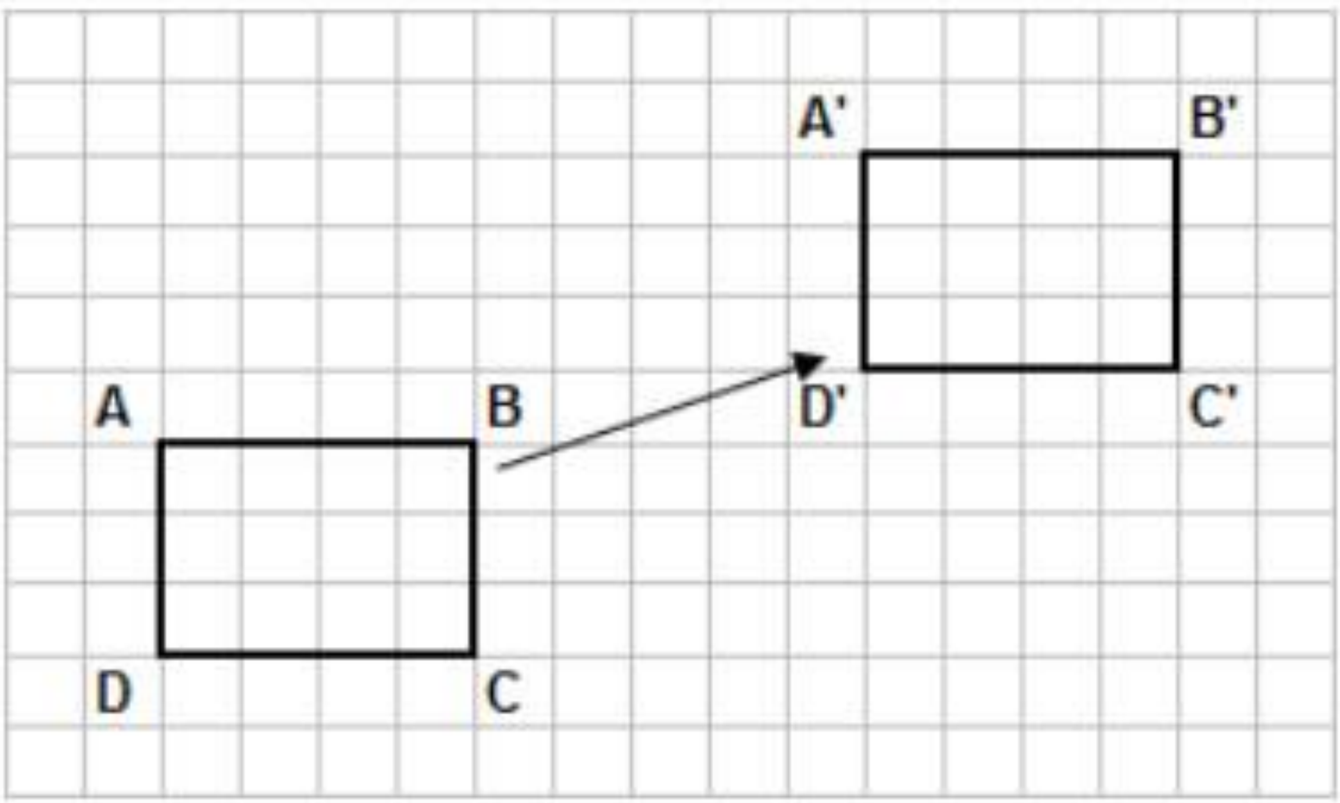
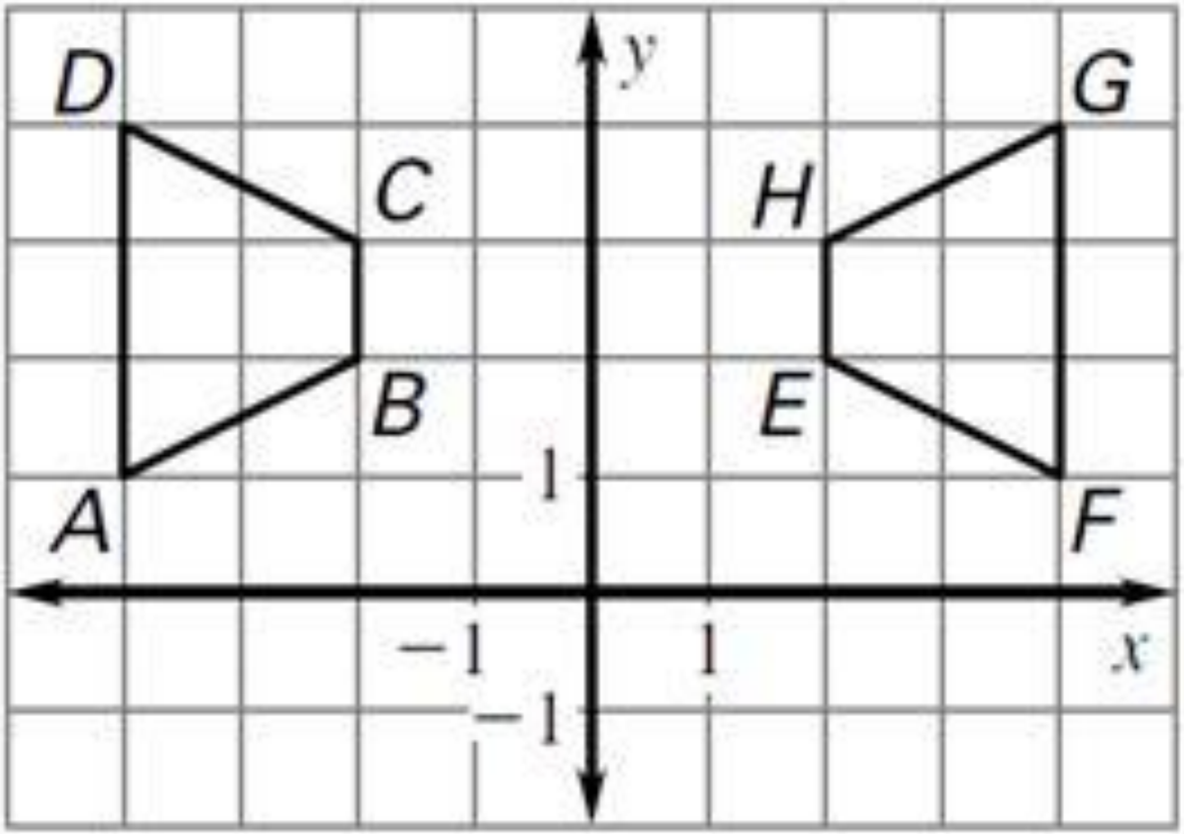
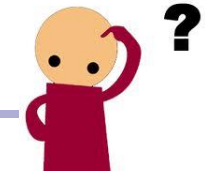
Name the transformation!

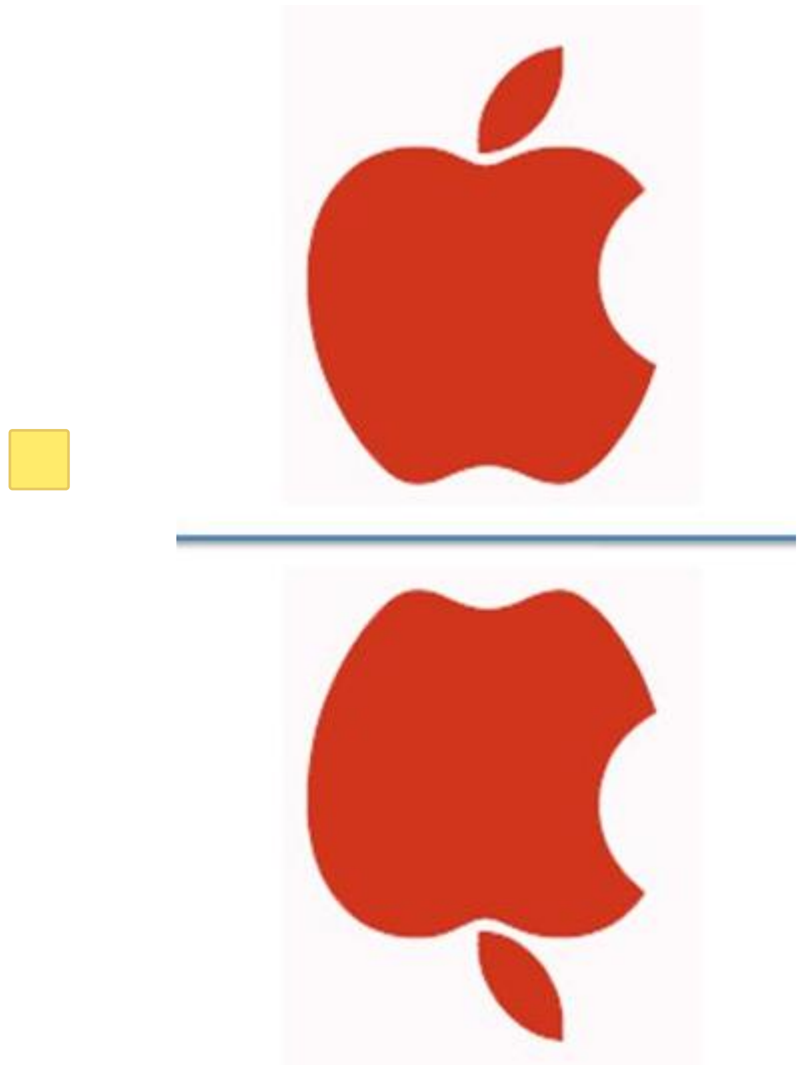


Name the transformation!



Name the transformation!





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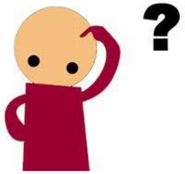
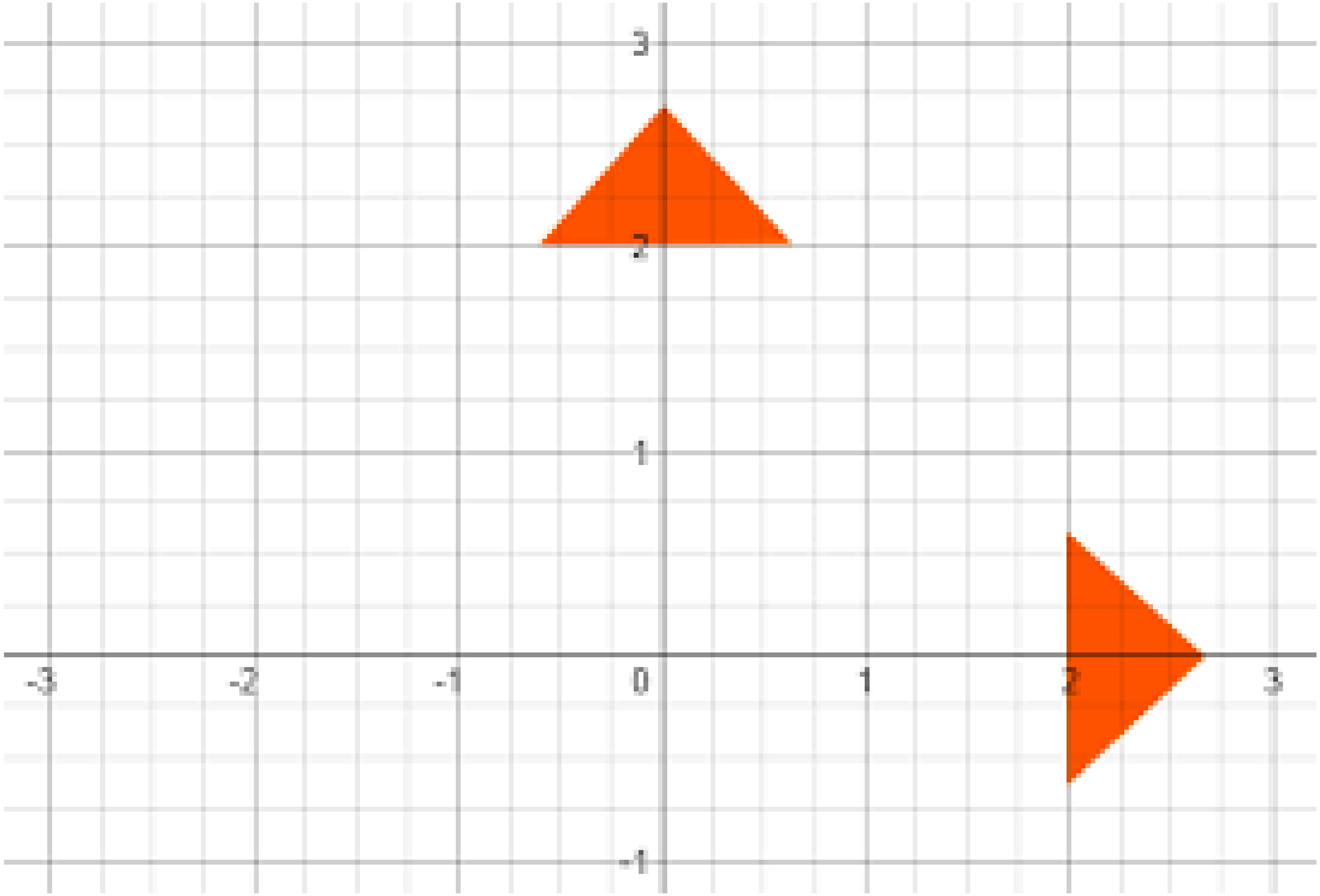
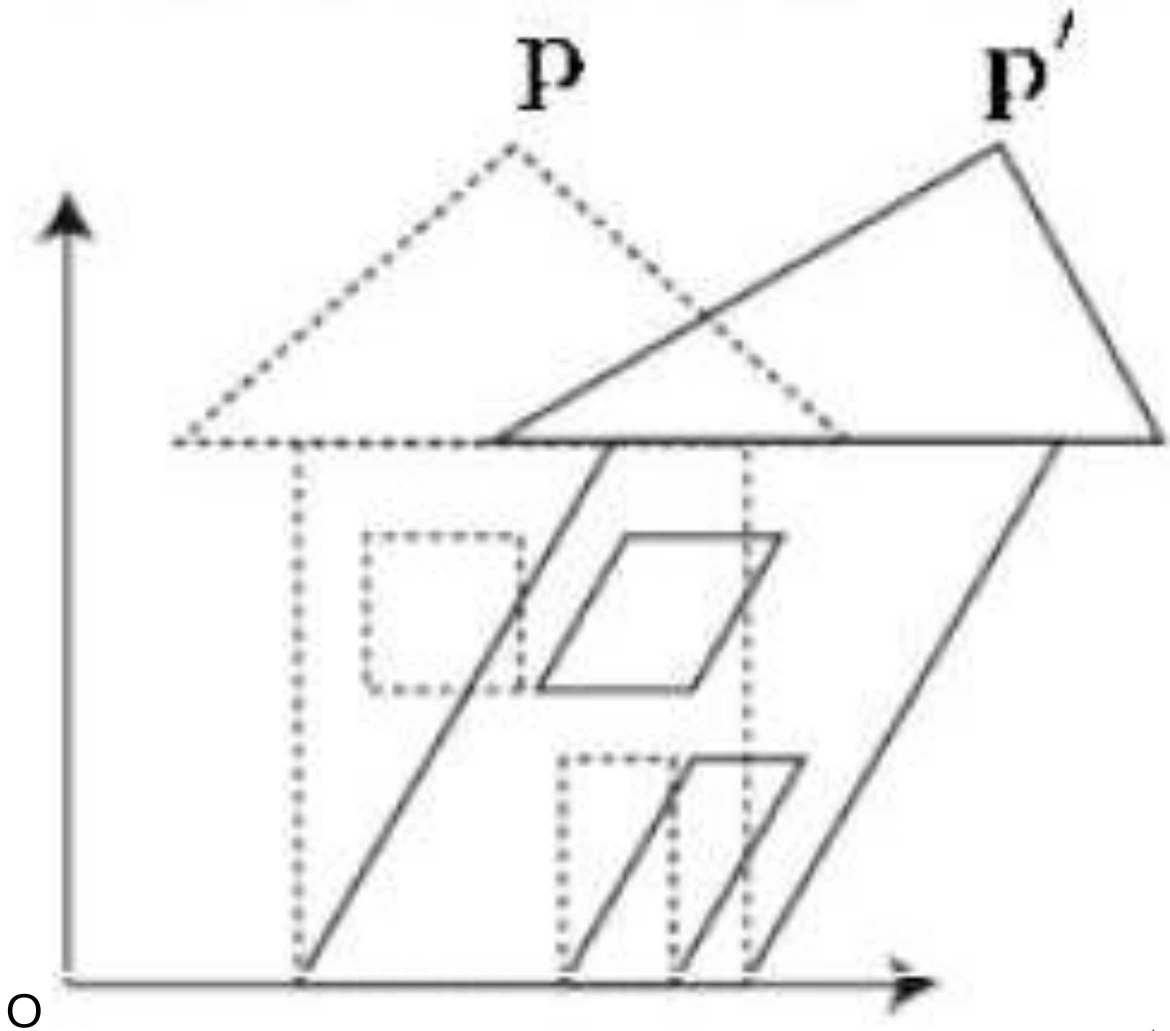
Name the transformation!

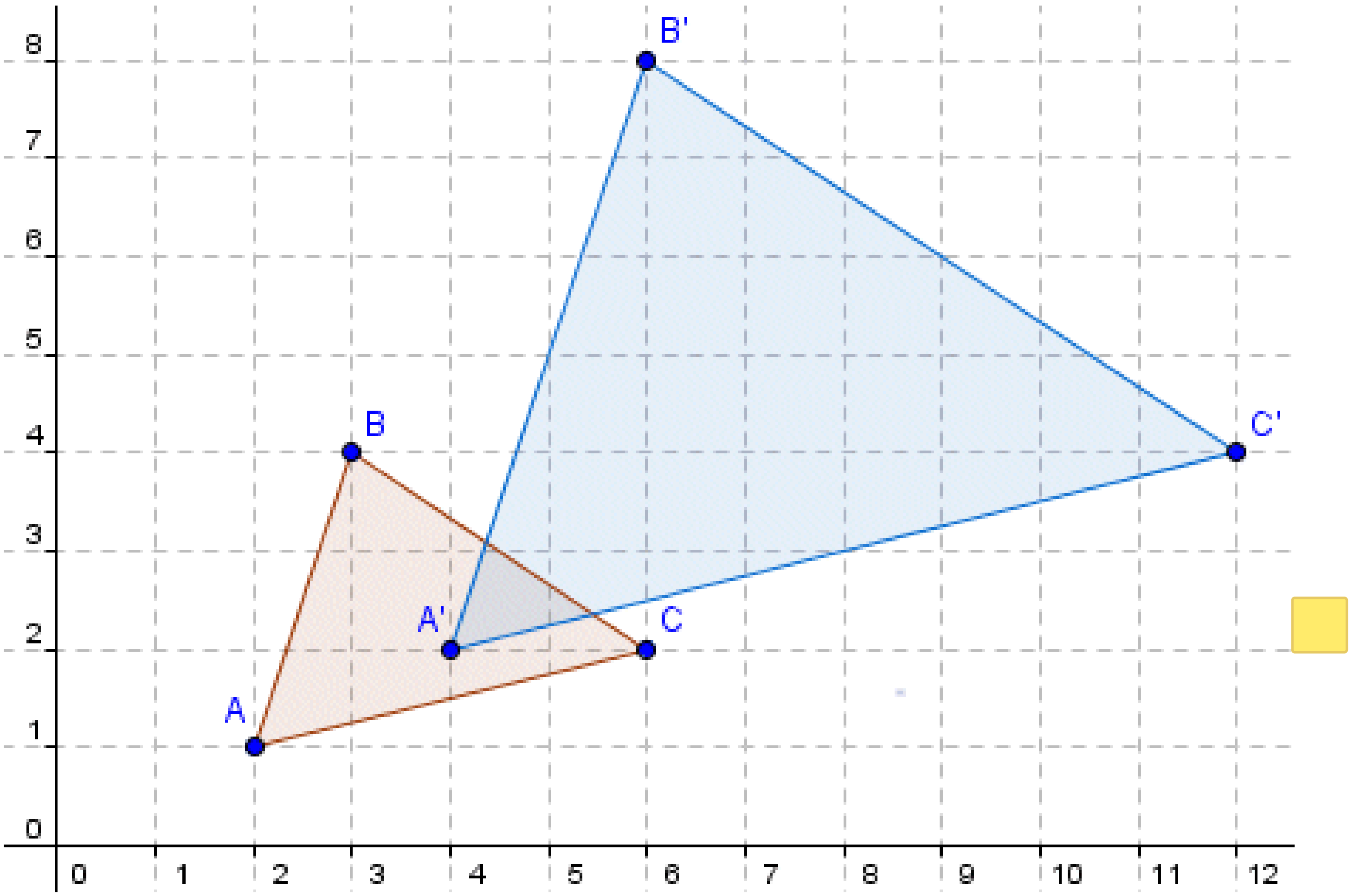




Name the transformation!



Name the transformation!



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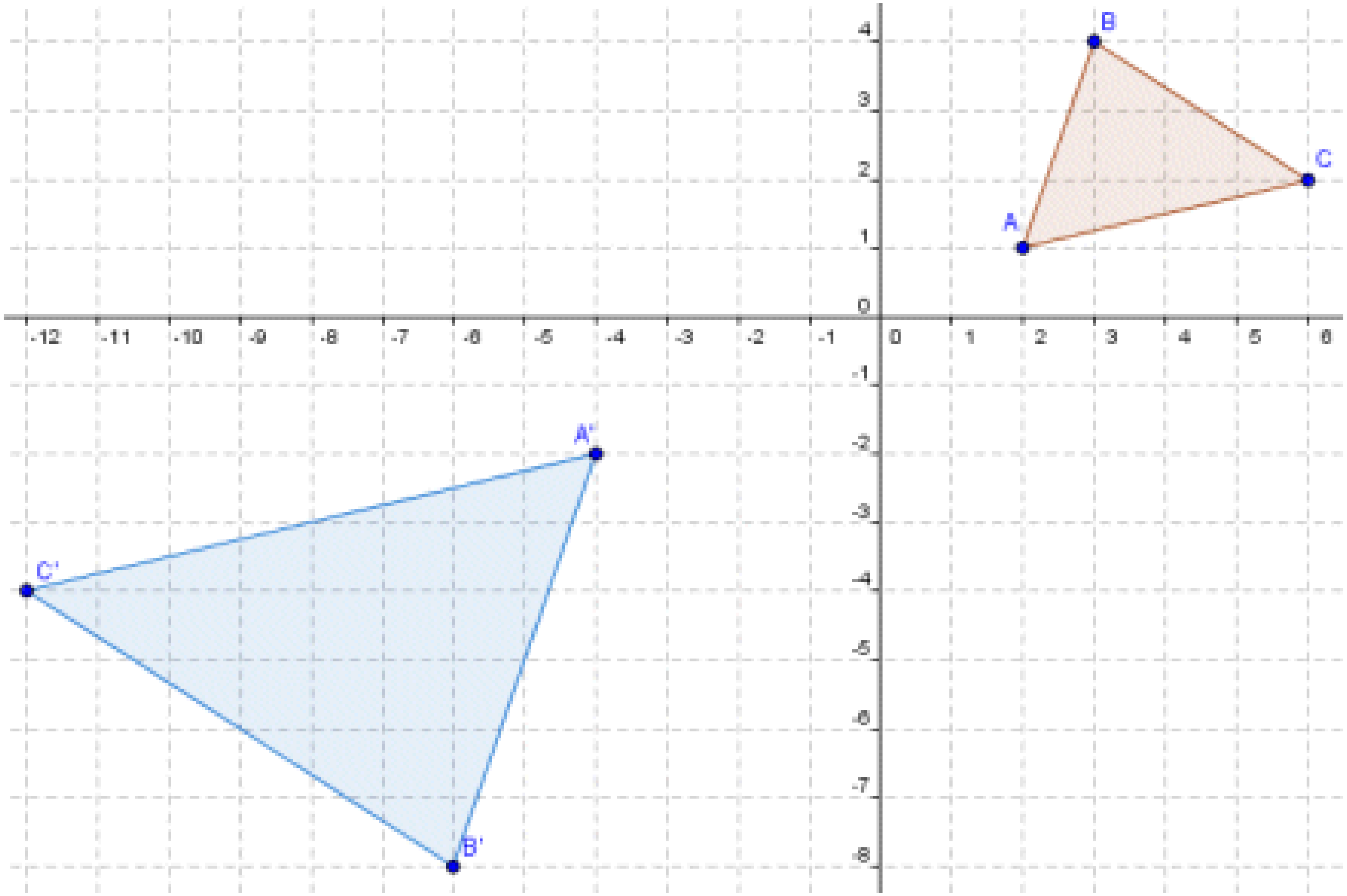
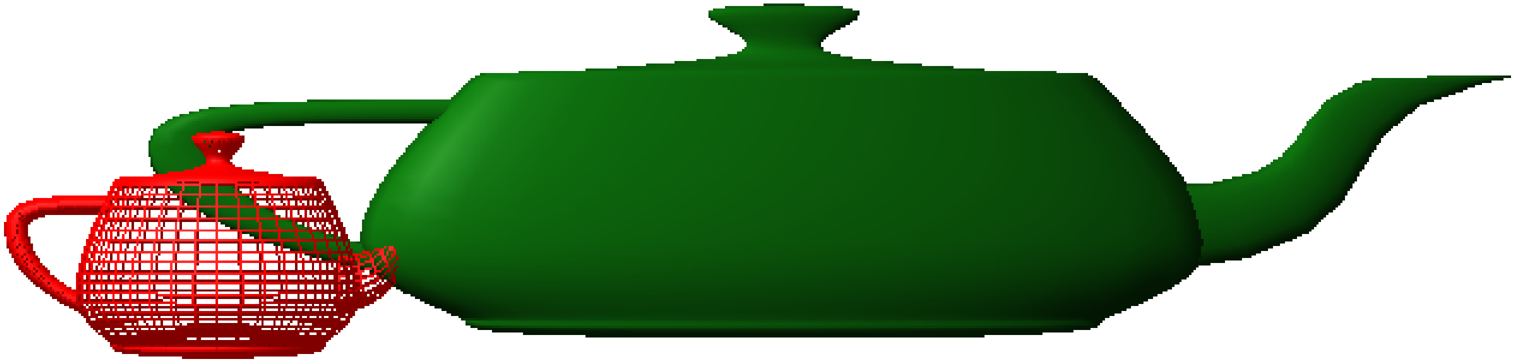
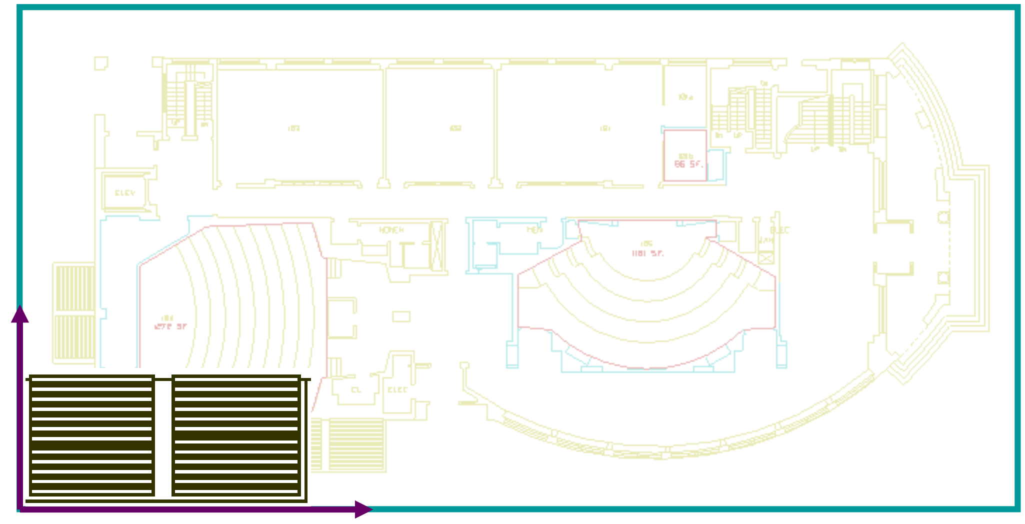
Name the transformation!



Name the transformation!

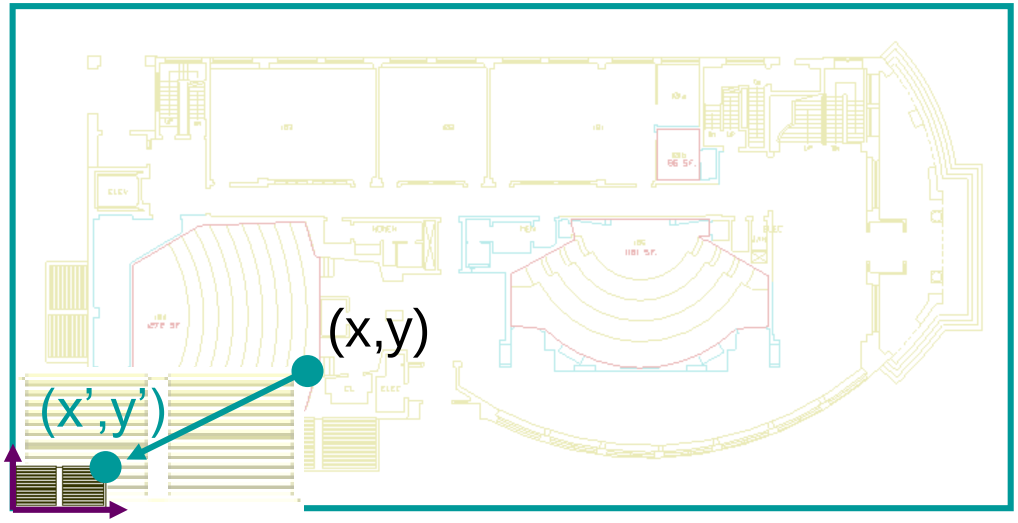


Basic 2D transformations (combination)



Basic 2D transformations (combination)

- Scale
  - $x' = x * s_x$
  - $y' = y * s_y$



$$x' = x * s_x$$
$$y' = y * s_y$$

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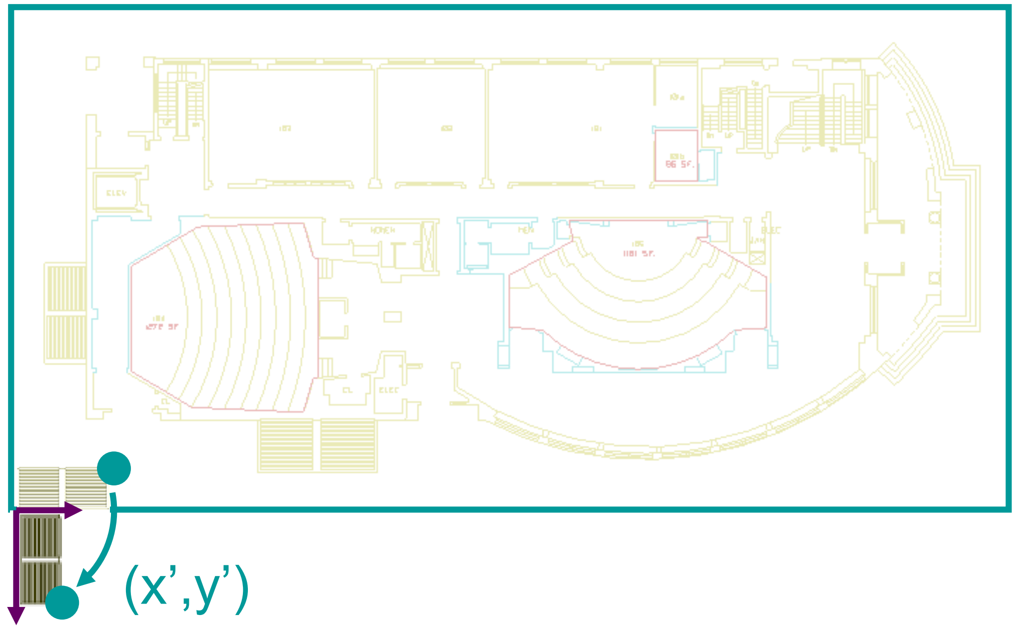
Basic 2D transformations (combination)

- Scale
  - $x' = x * s_x$
  - $y' = y * s_y$

- Rotation

  $x' = x * \cos\Theta - y * \sin\Theta$

  $y' = x * \sin\Theta + y * \cos\Theta$



$$x' = (x * s_x) * \cos\Theta - (y * s_y) * \sin\Theta$$
$$y' = (x * s_x) * \sin\Theta + (y * s_y) * \cos\Theta$$

Basic 2D transformations (combination)

- Scale
  - $x' = x * s_x$
  - $y' = y * s_y$

- Rotation

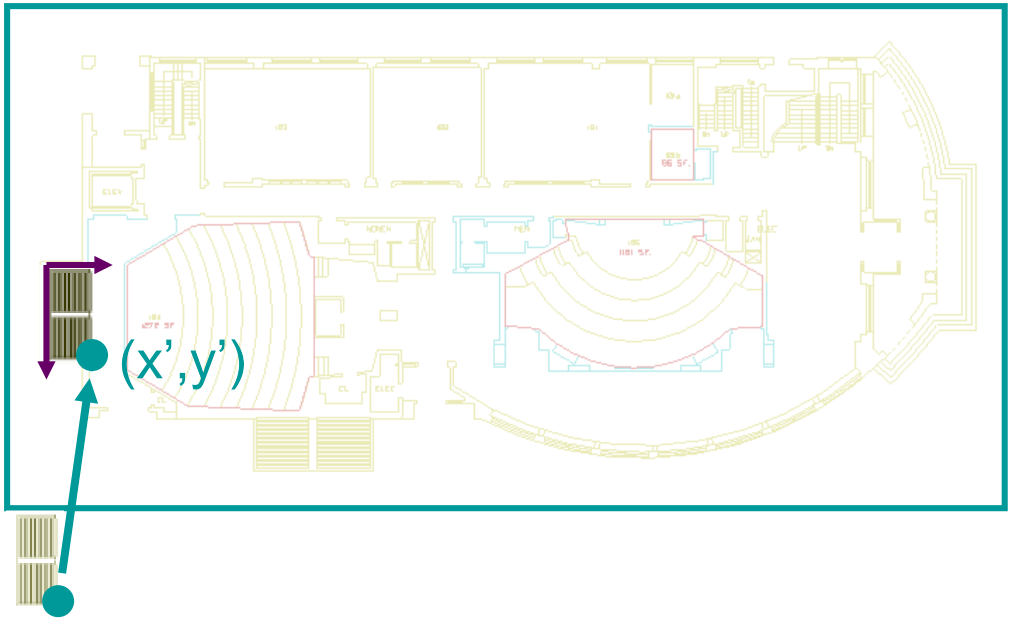
  $x' = x * \cos\Theta - y * \sin\Theta$

  $y' = x * \sin\Theta + y * \cos\Theta$

- Translation

  $x' = x + t_x$

  $y' = y + t_y$



$$x' = ((x * s_x) * \cos\Theta - (y * s_y) * \sin\Theta) + t_x$$

$$y' = ((x * s_x) * \sin\Theta + (y * s_y) * \cos\Theta) + t_y$$

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Today's agenda

- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations

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Matrix representation

- Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector
  $\Leftrightarrow$ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + by$$
$$y' = cx + dy$$

Matrix representation

- Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations

Matrix multiplication is not generally commutative !

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2x2 matrices

- What types of transformations can be represented with a 2x2 matrix?

2D identity

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D scale

$$x' = s_x * x$$
$$y' = s_y * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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2x2 matrices

- What types of transformations can be represented with a 2x2 matrix?

2D rotate around (0,0)

$$x' = \cos\Theta * x - \sin\Theta * y$$
$$y' = \sin\Theta * x + \cos\Theta * y$$

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

2D shears

$$\begin{cases} x' = x + sh_x * y \\ y' = y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x' = x \\ y' = sh_y * x + y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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2x2 matrices

- What types of transformations can be represented with a 2x2 matrix?

2D mirror about Y axis

$$x' = -x \\ y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D mirror over (0,0)

$$x' = -x \\ y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 matrices

- What types of transformations can be represented with a 2x2 matrix?

2D translation

$$x' = x + t_x$$

$$y' = y + t_y$$

NO!

Only linear 2D transformations
can be represented with a 2x2 matrix

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Linear transformations

- Linear transformations are combinations of
  - scale
  - rotation
  - shear and
  - mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations
  - origin maps to origin
  - lines map to lines
  - parallel lines remain parallel
  - ratios are preserved
  - closed under composition

$$T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1 T(\mathbf{p}_1) + s_2 T(\mathbf{p}_2)$$

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Homogeneous coordinates

- Homogeneous coordinates
  - represent coordinates in 2 dimensions with a 3D vector
  - seem unintuitive, but they make graphics operations much easier

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Homogeneous coordinates

- How can we represent translation as a 3x3 matrix?
  - Using the rightmost column

$$x' = x + t_x$$
$$y' = y + t_y$$

$$\mathbf{Translation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

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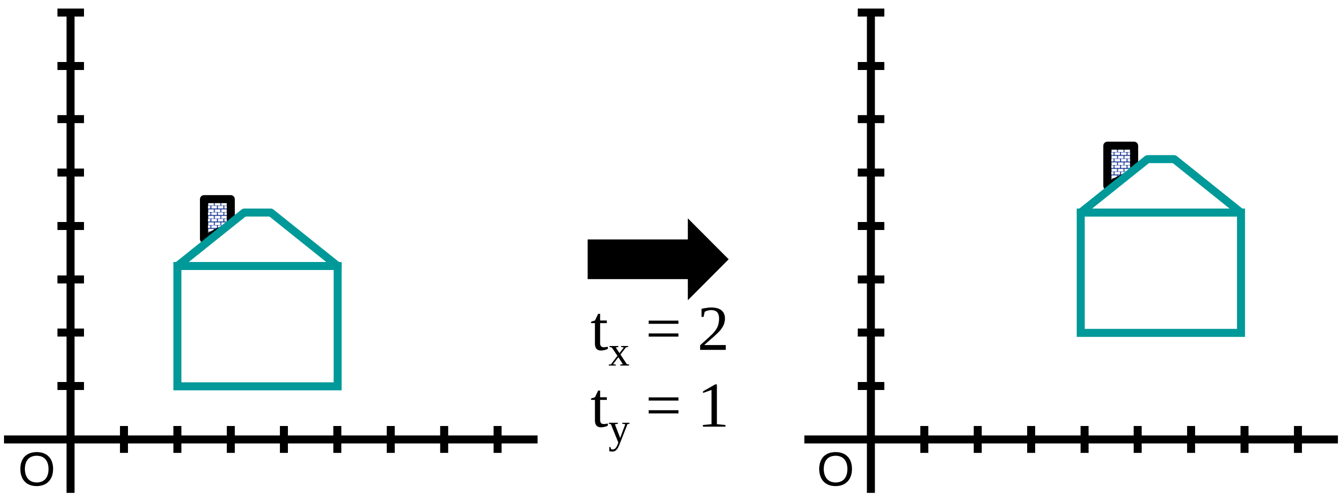
Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



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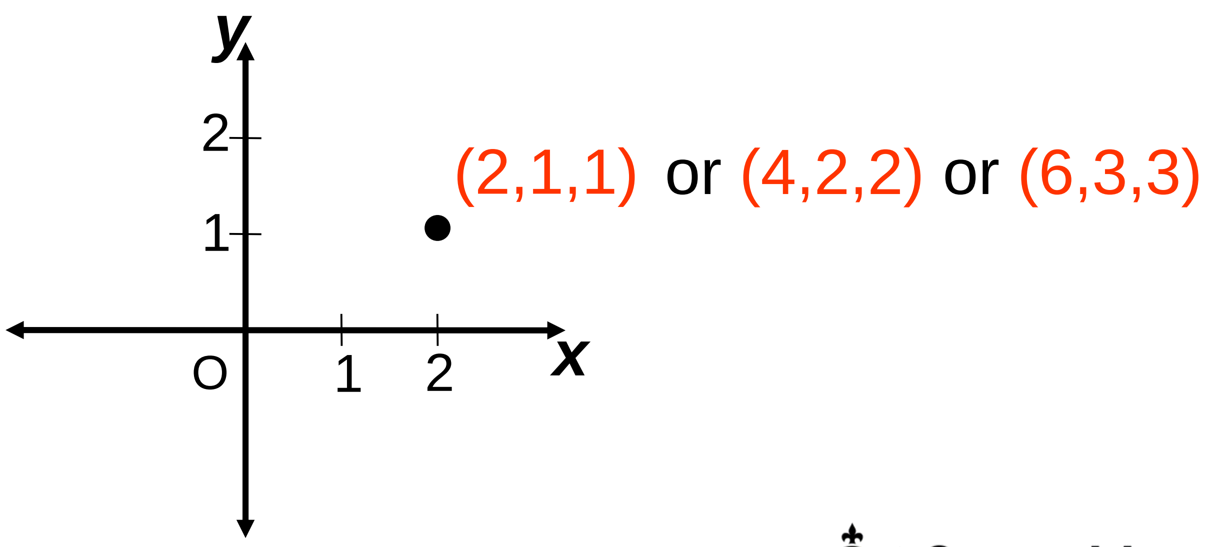
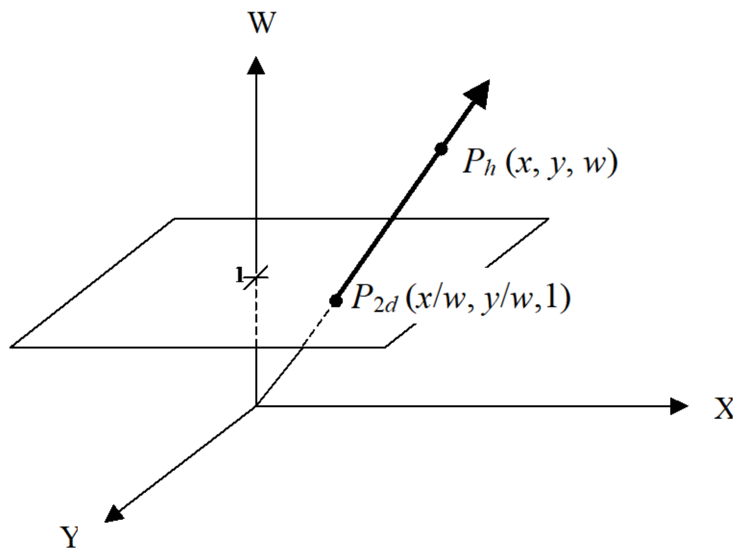
Homogeneous coordinates

- Homogeneous coordinates
  - add a 3rd coordinate to every 2D point
    - (x, y, w) represents a point at location (x/w, y/w)
    - (x, y, 0) represents a point at infinity
    - (0, 0, 0) is not allowed

Convenient coordinate system to represent many useful transformations



Homogeneous Coordinates



$$P_{2d}(x, y) \quad \rightarrow \quad P_h(wx, wy, w), \quad w \neq 0$$

$$P_h(x', y', w), \quad w \neq 0 \quad \rightarrow \quad P_{2d}(x, y) = P_{2d}\left(\frac{x'}{w}, \frac{y'}{w}\right)$$

Homogeneous coordinates allow translation, scaling and rotation to be expressed homogeneously, allowing composition via multiplication

Basic 2D transformations

- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \boldsymbol{t}_x \\ 0 & 1 & \boldsymbol{t}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

translate

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 & 0 \\ 0 & \boldsymbol{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotate

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{sh}_x & 0 \\ \boldsymbol{sh}_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

shear

Affine transformations

- Affine transformations are combinations of
  - Linear transformations, and
  - Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations
  - <u>origin does not necessarily map to origin</u>
  - lines map to lines
  - parallel lines remain parallel
  - ratios are preserved
  - closed under composition

Today's agenda

- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations

Matrix composition

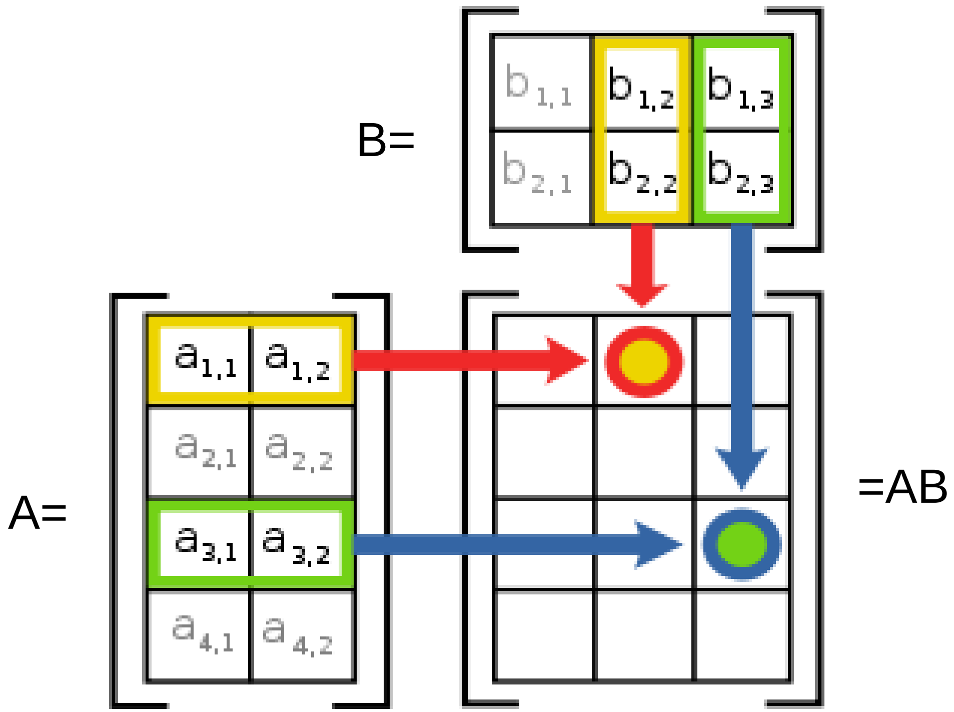
- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = $T(t_x,t_y)$ $R(\Theta)$ $S(s_x,s_y)$ **p**

Matrix multiplication (reminder)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$



Matrix composition

- Matrices are a convenient and efficient way to represent a sequence of transformations
  - general purpose representation
  - hardware matrix multiply

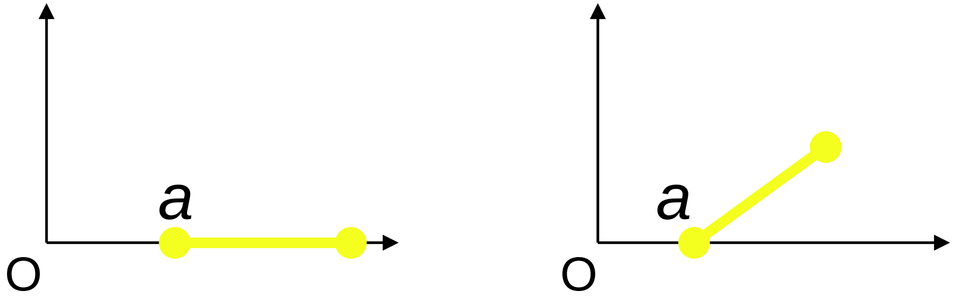
$$\mathbf{p}' = (T * (R * (S*\mathbf{p})\ )\ )$$

$$\mathbf{p}' = (T*R*S) * \mathbf{p}$$

  - Note: order of transformations matters
    - matrix multiplication is generally not commutative

Example

- What if we want to rotate and translate?
  - Ex: Rotate line segment by 45 degrees about endpoint *a*



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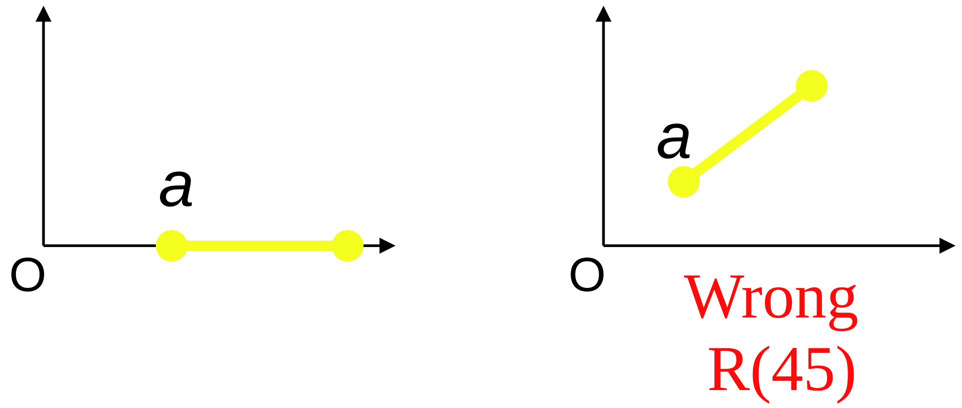
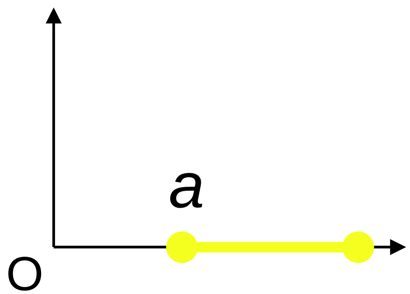
Multiplication order – wrong way

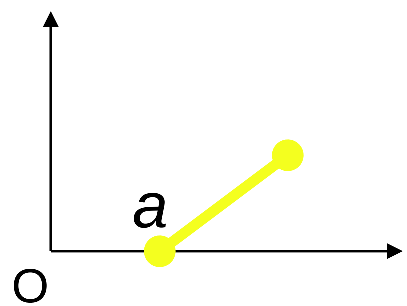
- The line segment is defined by two endpoints
  - Applying a rotation of 45 degrees, R(45), affects both points
  - We could try to translate both endpoints to return endpoint *a* to its original position, but by how much?

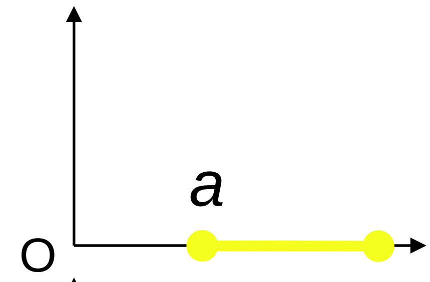


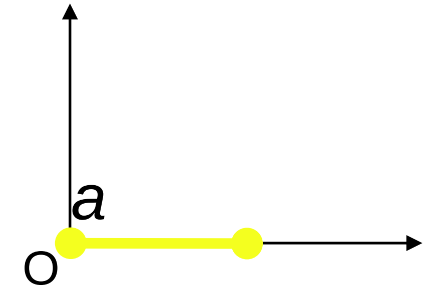
Multiplication order - correct

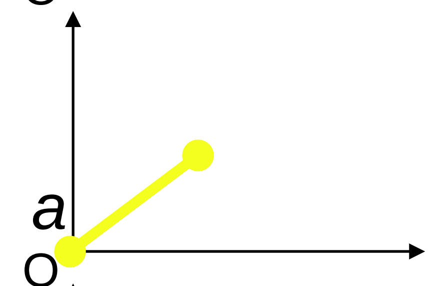
- Isolate endpoint a from rotation effects
  - First translate line so *a* is at origin: T (-3,0)
  - Then rotate line 45 degrees: R(45°)
  - Then translate back so *a* is where it was: T(3,0)

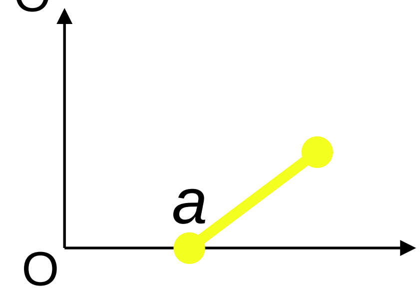












$$T(-3,0)=\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, R(45)=\begin{bmatrix}\cos(45) & -\sin(45) & 0\\ \sin(45) & \cos(45) & 0\\ 0 & 0 & 1\end{bmatrix}, T(3,0)=\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Example

Will this sequence of operations work?

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ 1 \end{bmatrix} = \begin{bmatrix} a'_x \\ a'_y \\ 1 \end{bmatrix}$$

$$T(-3,0) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R(45) = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}, T(3,0) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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No! Matrix multiplication is not commutative!

Example

Will this sequence of operations work?

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ 1 \end{bmatrix} = \begin{bmatrix} a'_x \\ a'_y \\ 1 \end{bmatrix}$$

$$T(-3,0) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R(45) = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}, T(3,0) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix composition

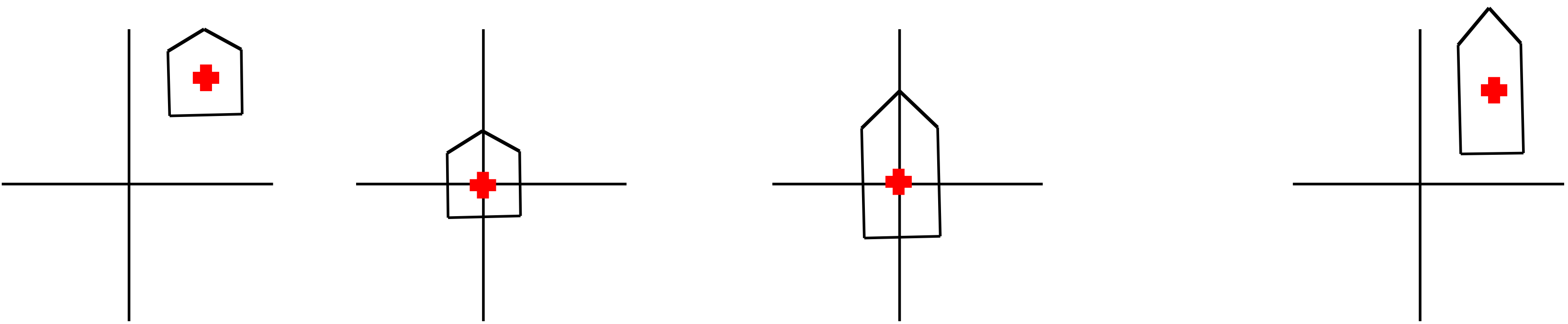
- After correctly ordering the matrices
- Multiply matrices together
- The results is just one matrix for the whole transformation
- Multiply this matrix by the vector of each vertex

→ All vertices easily transformed with one matrix to multiply

Composition of 2D Transformations

- **Scaling about a fixed point, not origin**

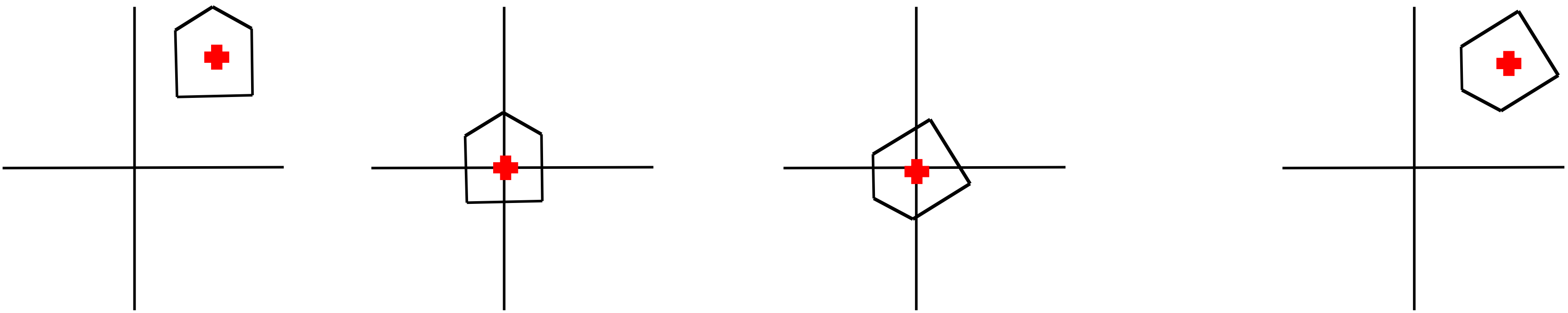
$House\ (H)$ $\qquad \boldsymbol{T}(dx, dy)H$ $\qquad \boldsymbol{S}(S_x, S_y)\boldsymbol{T}(dx, dy)H$ $\qquad \boldsymbol{T}(-dx, -dy)\boldsymbol{S}(S_x, S_y)\boldsymbol{T}(dx, dy)H$



Composition of 2D Transformations

- **Rotation about a fixed point, not origin**

$House\ (H)$ $\quad$ $\boldsymbol{T}(dx, dy)H$ $\quad$ $\boldsymbol{R}(\theta)\boldsymbol{T}(dx, dy)H$ $\quad$ $\boldsymbol{T}(-dx, -dy)\boldsymbol{R}(\theta)\boldsymbol{T}(dx, dy)H$



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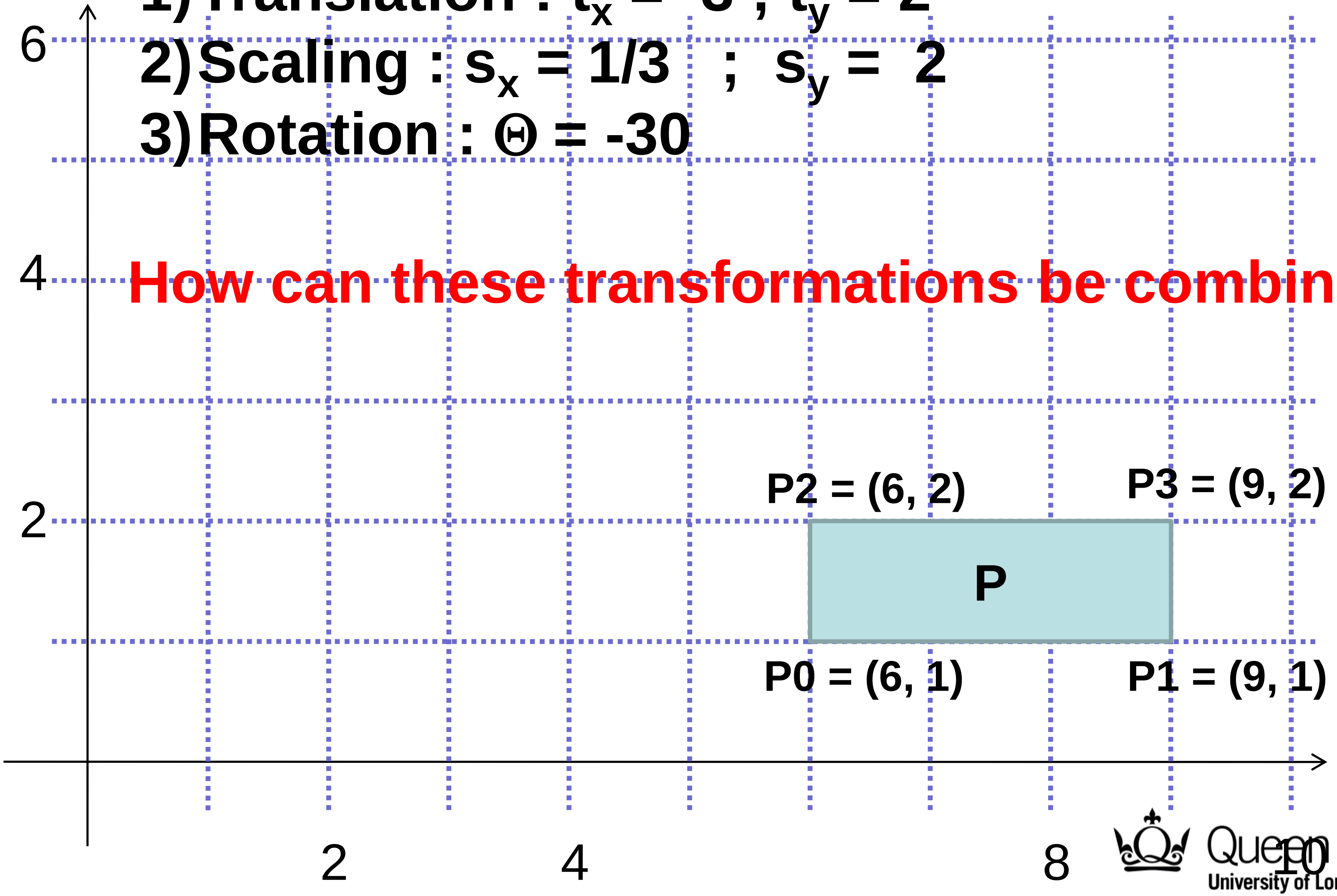
Exercise

1) **Translation : $t_x = -3$; $t_y = 2$**
2) **Scaling : $s_x = 1/3$; $s_y = 2$**
3) **Rotation : $\Theta = -30$**

How can these transformations be combined?



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Today's agenda

- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations

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3D Transformations

- **Similar to 2D => 3D**
- **Homogenization**

3D coordinates:

$$(x, y, z) \rightarrow (x, y, z, 1) \rightarrow (wx, wy, wz, w)$$

Homogeneous:

$$(x, y, z, w) \rightarrow (x/w, y/w, z/w, 1) \rightarrow (x/w, y/w, z/w)$$

3D transformation matrices: 4x4 matrices

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3D Transformations

- **Translation**

$$T(dx, dy, dz) = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1}(dx, dy, dz) = T(-dx, -dy, -dz)$$

- **Preserves lengths, angles, areas, and volumes**

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3D Transformations

- **Scaling**

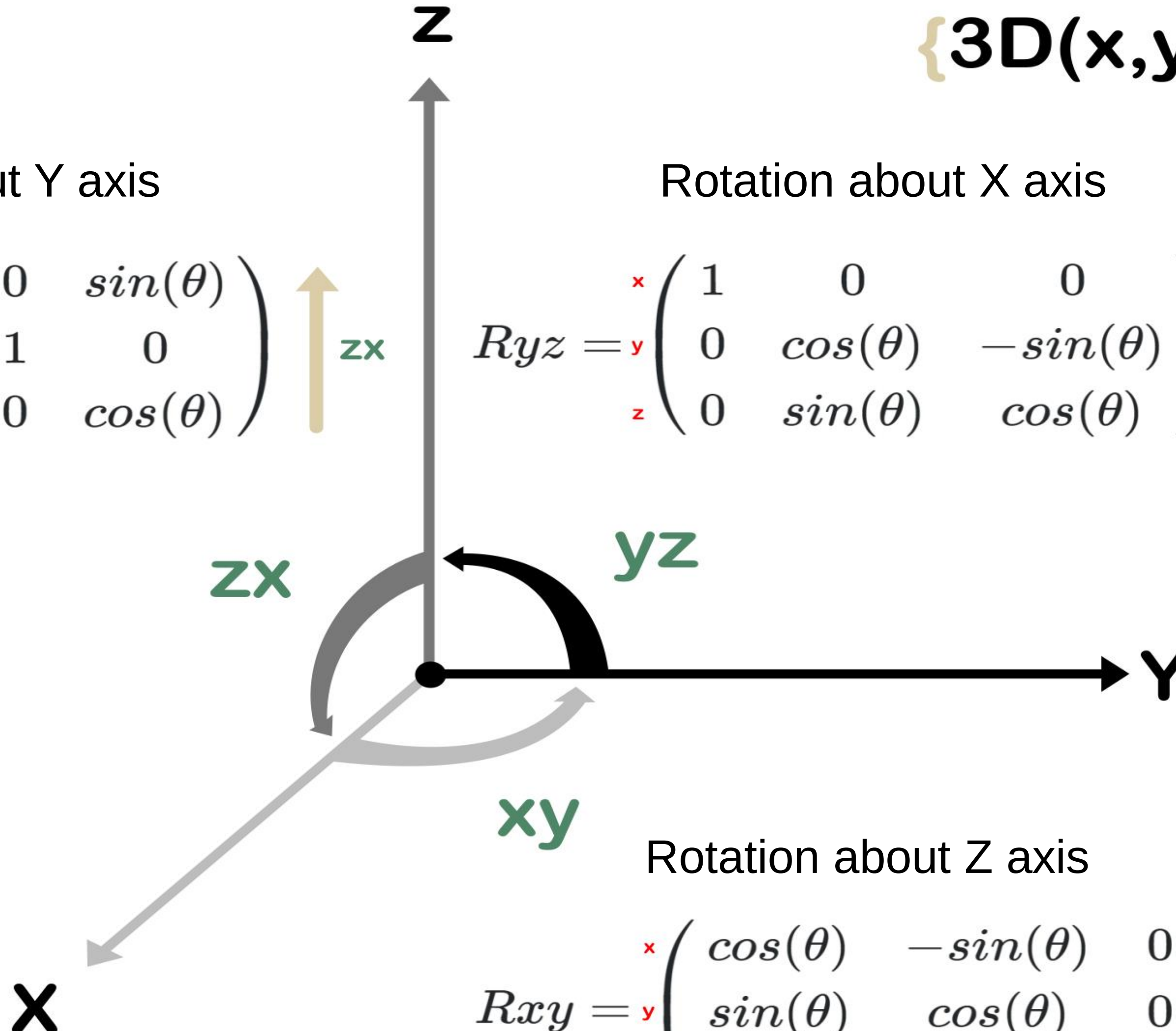
$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1}(s_x, s_y, s_z) = S(s_x^{-1}, s_y^{-1}, s_z^{-1})$$

- **A negative value on one or three of the components of scales results in a reflection.**
- **Does not preserve lengths, angles, areas, or volumes, except when scaling is uniform**

3D canonical rotations

{3D(x,y,z)}



Rotation about Y axis

$$Rzx = \begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{pmatrix} \uparrow zx$$

Rotation about X axis

$$Ryz = \begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix} \downarrow yz$$

Rotation about Z axis

$$Rxy = \begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \downarrow xy$$

3D Transformations

- **Rotation**

about x-axis: **y-axis:** **z-axis:**

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_u^{-1}(\theta) = R_u(-\theta)$$

- **Preserves lengths, angles, areas, and volumes.**
- **Rotations about arbitrary axis: Any 3D rotation is a composition of 3 rotations, one about each coordinate axis (Euler angles: roll, pitch, yaw).**

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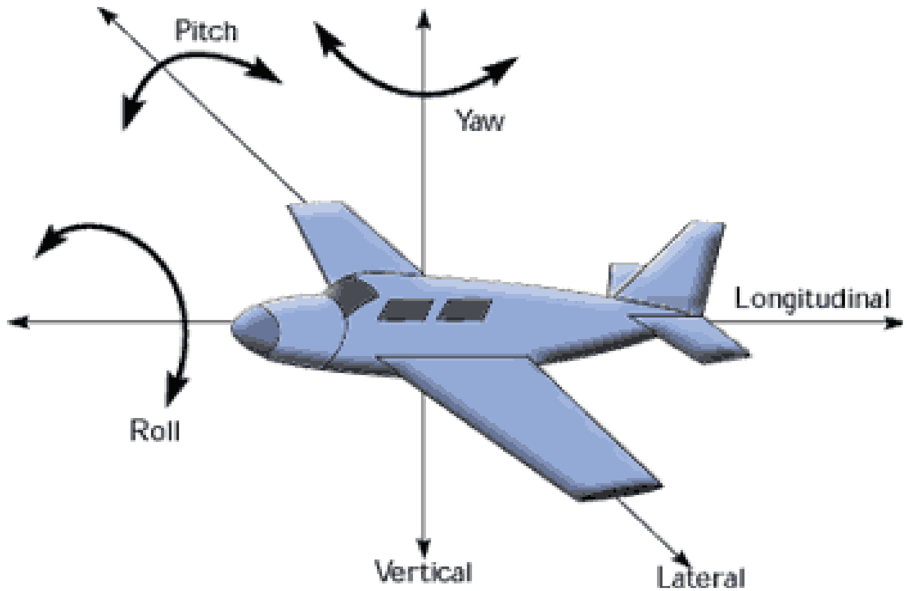
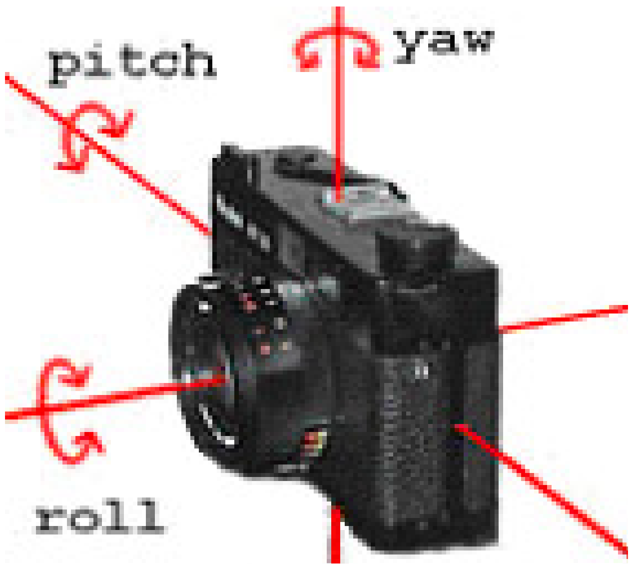
3D Transformations

- **Euler angles: roll, pitch, yaw**





- **Rotations about arbitrary axis: Any 3D rotation is a composition of 3 rotations, one about each coordinate axis (Euler angles: roll, pitch, yaw).**

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3D Transformations

- **Shearing**

$$SH = \begin{bmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Basic shears use one coordinate to shear another, use one coordinate to shear other two, or use two coordinates to shear the other one.**
- **Does not preserve lengths, angles, areas in all directions, but preserve volumes.**

3D transformations

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & dx \\ m_{21} & m_{22} & m_{23} & dy \\ m_{31} & m_{32} & m_{33} & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = M * \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

- **3D geometric transformations can be composed by matrix multiplication in reverse order .**
- **Generally, the transformations are order dependent, and composition of transforms is not commutative, even of rotations.**

Basic 3D transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

identity

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ \boldsymbol{z}' \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 & 0 & 0 \\ 0 & \boldsymbol{s}_y & 0 & 0 \\ 0 & 0 & \boldsymbol{s}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix}$$

scale

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ \boldsymbol{z}' \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \boldsymbol{t}_x \\ 0 & 1 & 0 & \boldsymbol{t}_y \\ 0 & 0 & 1 & \boldsymbol{t}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix}$$

translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

mirror about Y/Z plane

Reverse rotations

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- To undo a rotation of θ, $R(\theta)$
  - apply the inverse of the rotation $R^{-1}(\theta) = R(-\theta)$

- To construct $R^{-1}(\theta) = R(-\theta)$
  - Inside the rotation matrix: $\cos(-\theta) = \cos(\theta)$
- The cosine elements of the inverse rotation matrix are unchanged
  - The sign of the sine elements will flip $\sin(-\theta) = -\sin(\theta)$

$\rightarrow R^{-1}(\theta) = R(-\theta) = R^{T}(\theta)$

Rotation matrices are orthogonal.

3D rotation

General rotations in 3D

- require rotating about an arbitrary *axis of rotation*
- deriving the rotation matrix for such a rotation directly is a good exercise in linear algebra …
- standard approach
  - express general rotation as composition of canonical rotations
    - rotations about **X, Y, Z**

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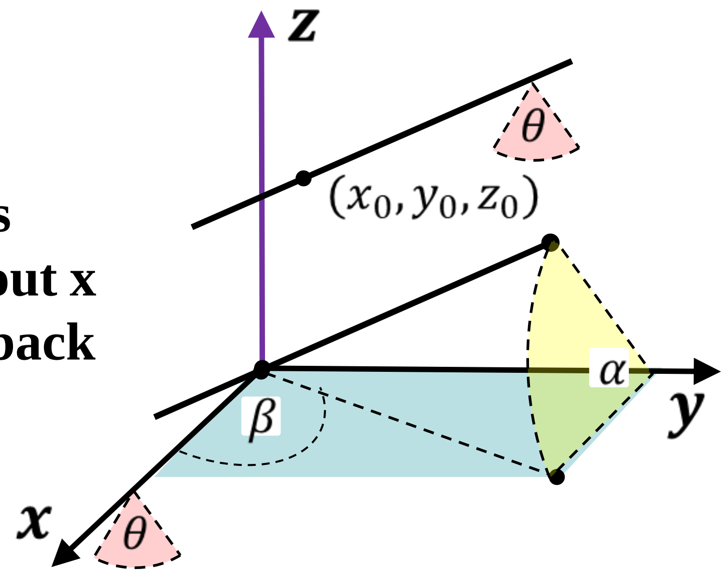
Composition of 3D Transformations

- **Rotation about an arbitrary axis**
  - **Step1. Translate the object to the origin**
  - **Step2. Rotate to align the axis with x-axis**
  - **Step3. Perform the specified rotation about x**
  - **Step4. Inverse rotations to turn the axis back**
  - **Step5. Inverse translation to move back**



$$R_v(\theta) = \underbrace{T(x_0, y_0, z_0)}_{\textbf{Step5}} \cdot \underbrace{R_y(-\alpha) \cdot R_z(-\beta)}_{\textbf{Step4}} \cdot \underbrace{R_x(\theta)}_{\textbf{Step3}} \cdot \underbrace{R_z(\beta) \cdot R_y(\alpha)}_{\textbf{Step2}} \cdot \underbrace{T(-x_0, -y_0, -z_0)}_{\textbf{Step1}}$$

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