3D Graphics Programming Tools

Revision – Key Concepts
Projection
(Past Exam Questions Review)

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Perspective Projection

Question 4

a) This question is about projection.

[16 marks]

i) Consider Figure 3. Give the perspective projection transformation matrix for point P(x,y,z) projecting to the viewplane at z=2 if the centre of projection is the origin.

(6 marks)

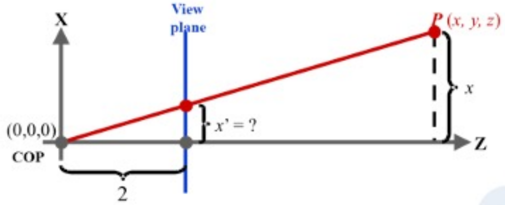


Figure 3.

Solution:

The relations between the point coordinates (x,y,z) and the projected coordinates (x',y',z') are:

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \quad z' = d = \frac{z}{z/d}$$ where d=2. (3 marks)

Use w to substitute z/d, $w = z/d$, we have

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d} = \frac{x}{w}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d} = \frac{y}{w}, \quad z' = d = \frac{z}{z/d} = \frac{z}{w}$$ (1 mark)

Then using homogeneous coordinate system, the relations can be formulated as matrix form:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}$$ (1 mark)

The transformation matrix is: $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$ (1 mark)

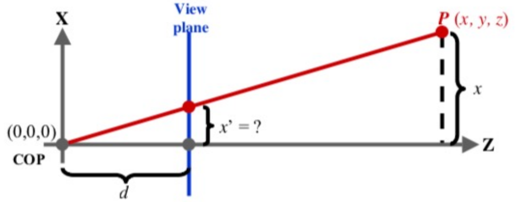
(total 6 *marks*)

Projection

Perspective Projection

Perspective projection

- Desired result for a point $[x, y, z, 1]^T$ projected onto the view plane:



$$\frac{x'}{d} = \frac{x}{z}, \quad \frac{y'}{d} = \frac{y}{z}$$

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \quad z' = d = \frac{z}{z/d}$$

What could a matrix look like to do this?

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Perspective projection matrix

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \quad z' = d = \frac{z}{z/d}$$

We use: $w = z/d$

We have:
$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d} = \frac{x}{w},$$
$$y' = \frac{d \cdot y}{z} = \frac{y}{z/d} = \frac{y}{w},$$
$$z' = d = \frac{z}{z/d} = \frac{z}{w}$$

Perspective projection matrix

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \quad z' = d = \frac{z}{z/d}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}$$

- in 3-D coordinates: $\left(\frac{x}{z/d}, \frac{y}{z/d}, d \right)$

Perspective projection matrix

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \quad z' = d = \frac{z}{z/d}$$

$$P_{perspective} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

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Projection

Perspective Projection

- Prerequisite: Homogeneous Coordinates

  - Homogeneous coordinates
    - represent coordinates in 2 dimensions with a 3D vector
    - seem unintuitive, but they make graphics operations much easier

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- How can we represent translation as a 3x3 matrix?
  - Using the rightmost column

$$x' = x + t_x \qquad\qquad \textit{Translation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$
$$y' = y + t_y$$

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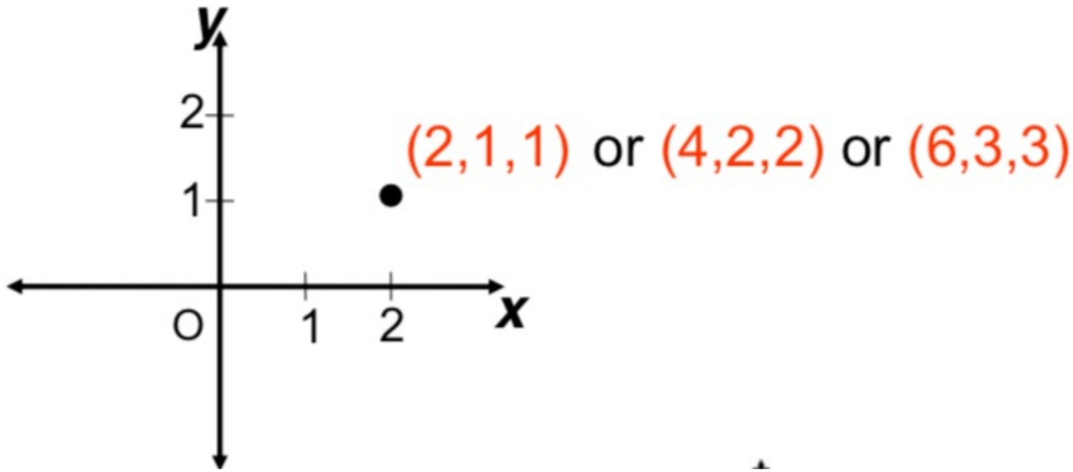
Projection

Perspective Projection

- Prerequisite: Homogeneous Coordinates
  - Homogeneous coordinates
    - add a 3rd coordinate to every 2D point
      - (x, y, w) represents a point at location (x/w, y/w)
      - (x, y, 0) represents a point at infinity
      - (0, 0, 0) is not allowed

Convenient coordinate system to represent many useful transformations



Projection

Perspective Projection

- Prerequisite: Homogeneous Coordinates
  - **Similar to 2D => 3D**
  - **Homogenization**

    3D coordinates:

    $$(x, y, z) \rightarrow (x, y, z, 1) \rightarrow (wx, wy, wz, w)$$

    Homogeneous:

    $$(x, y, z, w) \rightarrow (x/w, y/w, z/w, 1) \rightarrow (x/w, y/w, z/w)$$

    3D transformation matrices: 4x4 matrices

  - **Translation**

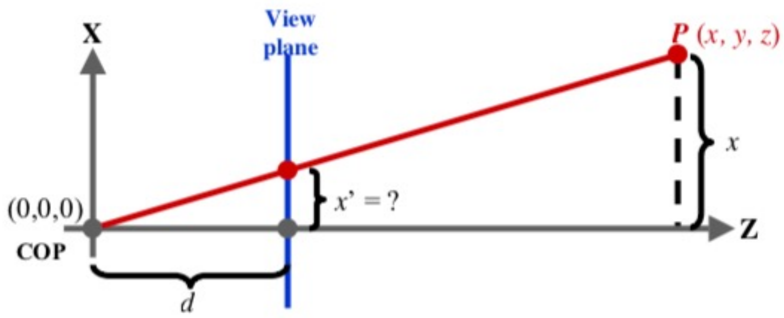
$$T(dx, dy, dz) = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection

Perspective Projection

- COP at the origin

- Desired result for a point $[x, y, z, 1]^T$ projected onto the view plane:



$$\frac{x'}{d} = \frac{x}{z}, \qquad \frac{y'}{d} = \frac{y}{z}$$

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \qquad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \qquad z' = d = \frac{z}{z/d}$$

We use: $w = z/d$

We have: $x' = \frac{d \cdot x}{z} = \frac{x}{z/d} = \frac{x}{w},$

$$y' = \frac{d \cdot y}{z} = \frac{y}{z/d} = \frac{y}{w},$$

$$z' = d = \frac{z}{z/d} = \frac{z}{w}$$

Projection

Perspective Projection

- COP at the origin

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \quad z' = d = \frac{z}{z/d}$$

$$(x', y', z') = \left(\frac{x}{z/d}, \frac{y}{z/d}, \frac{z}{z/d}\right) \xrightarrow{\text{Homogeneous Coordinates}} \left(\frac{x}{z/d}, \frac{y}{z/d}, \frac{z}{z/d}, 1\right) \longrightarrow \boxed{(x, y, z, z/d)}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}}$$

$$P_{perspective} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

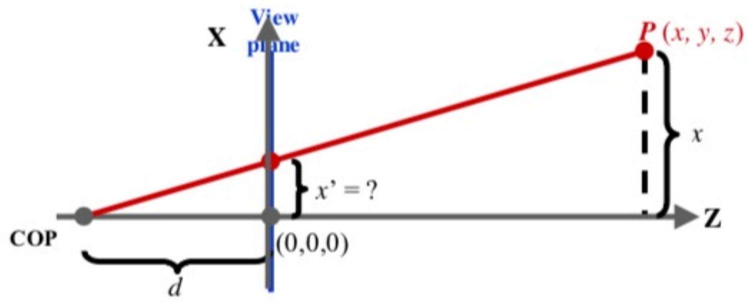
- in 3-D coordinates: $\left(\frac{x}{z/d}, \frac{y}{z/d}, d\right)$

Projection

Perspective Projection

- Origin in the view plane

- Desired result for a point $[x, y, z, 1]^T$ projected onto the view plane:



$$\frac{x'}{d} = \frac{x}{z+d}, \quad \frac{y'}{d} = \frac{y}{z+d}, \quad z' = 0$$

$$x' = \frac{d \cdot x}{z+d} = \frac{x}{z/d+1}, \quad y' = \frac{d \cdot y}{z+d} = \frac{y}{z/d+1}, \quad z' = 0 = \frac{0}{z/d+1}$$

$$w = z/d+1: \quad x' = \frac{x}{w}, \quad y' = \frac{y}{w}, \quad z' = \frac{0}{w}$$

$$(x', y', z') = \left(\frac{x}{z/d+1}, \frac{y}{z/d+1}, \frac{0}{z/d+1}\right) \xrightarrow{\text{Homogeneous Coordi0ates}} \left(\frac{x}{z/d+1}, \frac{y}{z/d+1}, \frac{0}{z/d+1}, 1\right)$$

$$\rightarrow \boxed{(x, y, 0, z/d+1)}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} x \\ y \\ 0 \\ z/d+1 \end{bmatrix}}$$

$$P'_{perspective} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix}$$

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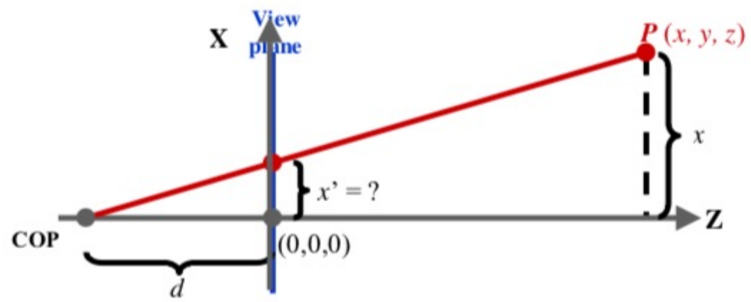
Projection

Perspective Projection

- Revisit the question

- Desired result for a point $[x, y, z, 1]^T$ projected onto the view plane:



Solution:

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Use w to substitute z/d, $w = z/d$, we have

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d} = \frac{x}{w}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d} = \frac{y}{w}, \quad z' = d = \frac{z}{z/d} = \frac{z}{w}$$ (1 mark)

Then using homogeneous coordinate system, the relations can be formulated as matrix form:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}$$ (1 mark)

The transformation matrix is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$ (1 mark)

(total 6 *marks*)

Projection

View Reference Coordinate System

ii) In the world coordinate system, if the view plane normal is [4,3,0], the view-up vector is [0,0,1], the view reference point is [1,2,3], find the transformation matrix for parallel projection.

(10 marks)

Solution:
The translation matrix to the view reference point:

$$T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(1 mark)}$$

The 3 axis vectors are (1 mark for n, 2 marks for u and 2 marks for v, up to 5 marks)

$$n = \frac{VPN}{|VPN|} = \frac{[4,3,0]}{|[4,3,0]|} = \frac{[4,3,0]}{5} = [\frac{4}{5}, \frac{3}{5}, 0]$$

$$u = \frac{VUP \times VPN}{|VUP \times VPN|} = \frac{[0,0,1] \times [4,3,0]}{|VUP \times VPN|} = \frac{[-3,4,0]}{5} = [-\frac{3}{5}, \frac{4}{5}, 0]$$

$$v = n \times u = [\frac{4}{5}, \frac{3}{5}, 0] \times [-\frac{3}{5}, \frac{4}{5}, 0] = [0,0,1]$$

The rotation matrix is

$$R = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(2 marks)}$$

The composite transformation matrix is

$$R \cdot T = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & -1 \\ 0 & 0 & 1 & -3 \\ \frac{4}{5} & \frac{3}{5} & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(2 marks)}$$

(total 10 *marks*)

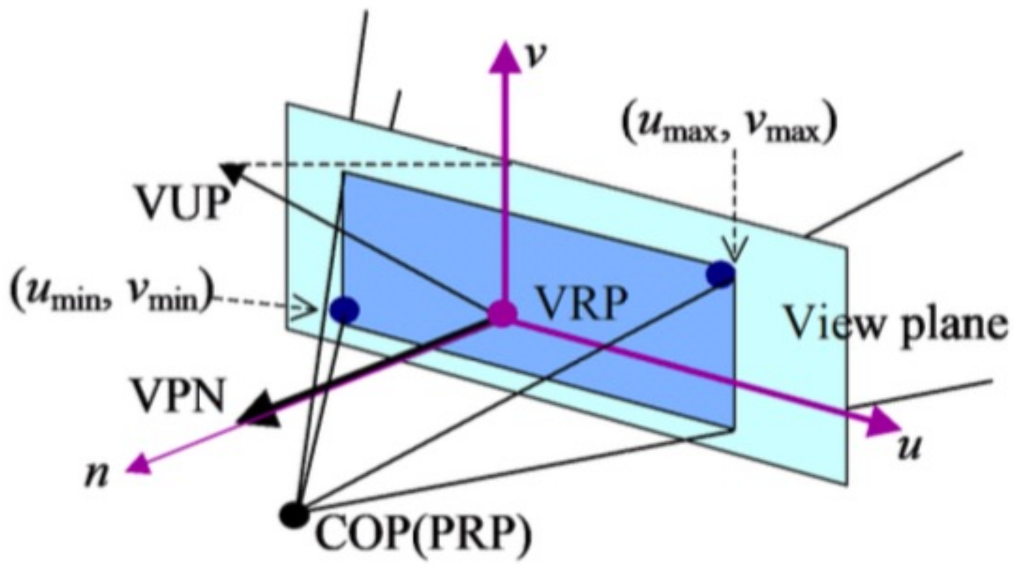
Projection

View Reference Coordinate System

View Reference Coordinate System

View Reference Coordinate (VRC) System

- **Three orthogonal axes:**
  - **VPN is one axis (*n*-axis).**
  - **The second axis (*v*-axis): projection of *view-up vector* (VUP) onto the view plane.**
  - **The third axis (*u*-axis) can be easily found in the right-handed coordinate system.**



$$n = \frac{\text{VPN}}{|\text{VPN}|}$$

$$u = \frac{\text{VUP} \times \text{VPN}}{|\text{VUP} \times \text{VPN}|}$$

$$v = n \times u$$

Projection

View Reference Coordinate System

Transform world coordinate into VRC

1. **Translation of the coordinate system to the origin in homogeneous matrix form is:**

$$T = \begin{bmatrix} 1 & 0 & 0 & -VRP_x \\ 0 & 1 & 0 & -VRP_y \\ 0 & 0 & 1 & -VRP_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. **Using the unit vectors of the coordinate axes, the resulting rotation matrix is:**

$$R = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. **Combination for the single transformation matrix (parallel):**

$$M = R \cdot T = \begin{bmatrix} u_x & u_y & u_z & -(u_x \cdot VRP_x + u_y \cdot VRP_y + u_z \cdot VRP_z) \\ v_x & v_y & v_z & -(v_x \cdot VRP_x + v_y \cdot VRP_y + v_z \cdot VRP_z) \\ n_x & n_y & n_z & -(n_x \cdot VRP_x + n_y \cdot VRP_y + n_z \cdot VRP_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Projection

View Reference Coordinate System

Transform world coordinate into VRC

4. Parallel or Perspective projection:

$$P_{perspective} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \text{ or } P'_{perspective} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix} \text{ or } P''_{parallel} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. Combination for the single transformation matrix :

$$M = P \cdot R \cdot T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix} \cdot \begin{bmatrix} u_x & u_y & u_z & -(u_x \cdot VRP_x + u_y \cdot VRP_y + u_z \cdot VRP_z) \\ v_x & v_y & v_z & -(v_x \cdot VRP_x + v_y \cdot VRP_y + v_z \cdot VRP_z) \\ n_x & n_y & n_z & -(n_x \cdot VRP_x + n_y \cdot VRP_y + n_z \cdot VRP_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_x & u_y & u_z & -(u_x \cdot VRP_x + u_y \cdot VRP_y + u_z \cdot VRP_z) \\ v_x & v_y & v_z & -(v_x \cdot VRP_x + v_y \cdot VRP_y + v_z \cdot VRP_z) \\ n_x & n_y & n_z & -(n_x \cdot VRP_x + n_y \cdot VRP_y + n_z \cdot VRP_z) \\ \frac{n_x}{d} & \frac{n_y}{d} & \frac{n_z}{d} & -\frac{n_x \cdot VRP_x + n_y \cdot VRP_y + n_z \cdot VRP_z}{d} \end{bmatrix}$$

(perspective projection with origin at d distance from in the view plane)

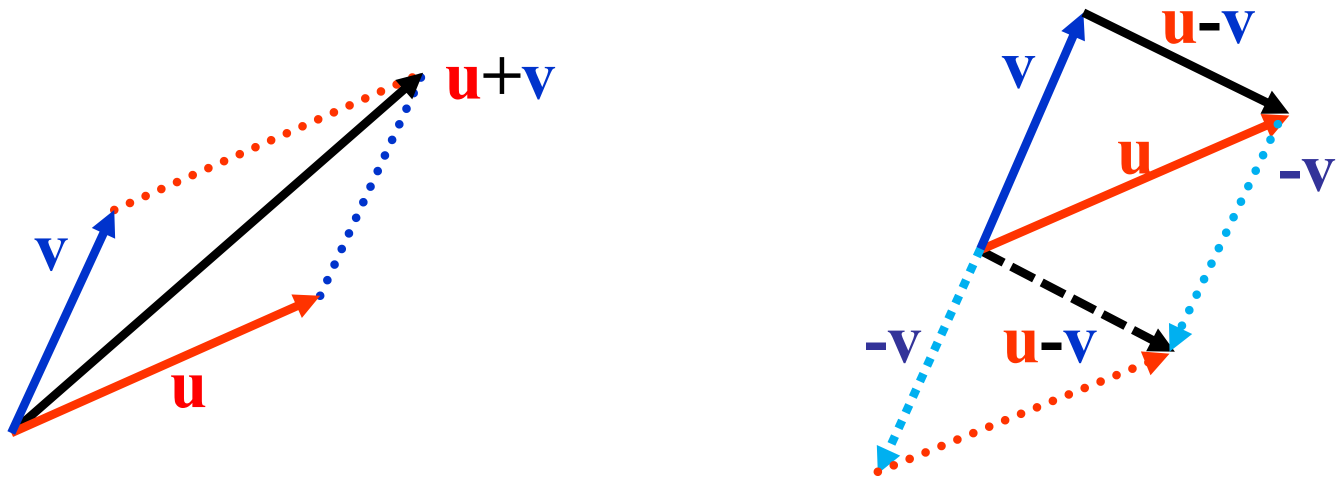
$or\ M' = P' \cdot R \cdot T$ (perspective projection with origin in the view plane)

$or\ M'' = P'' \cdot R \cdot T$ (parallel projection)

Projection

View Reference Coordinate System

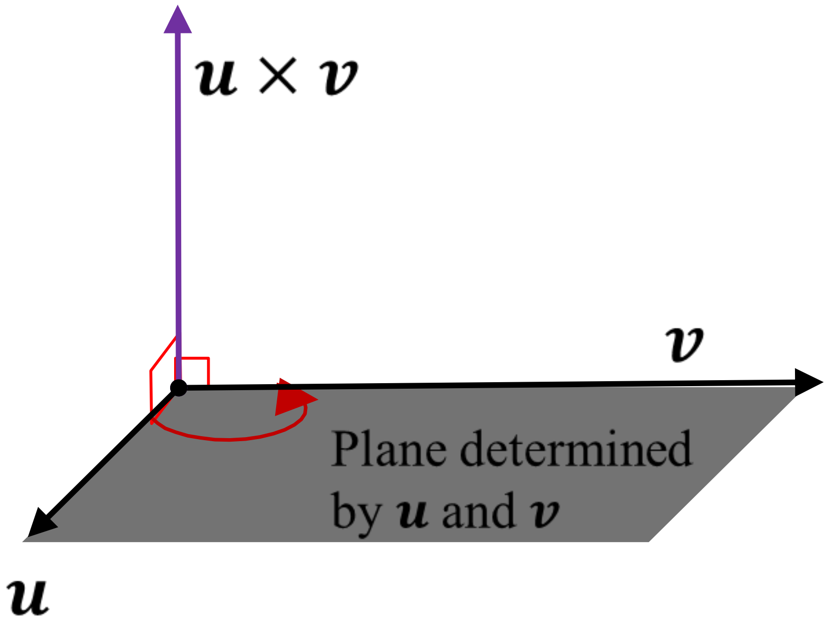
- Prerequisite: Vector Addition & Subtraction

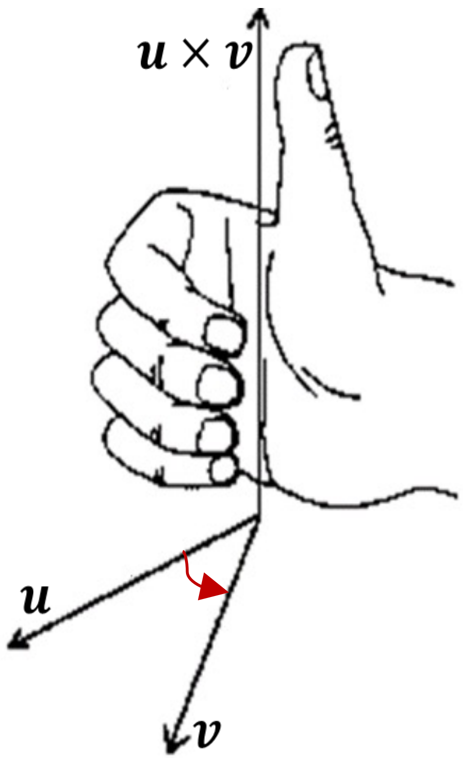


Projection

View Reference Coordinate System

- Prerequisite: Vector Cross Product
  - The *cross product* or *vector product* of two vectors is a vector orthogonal to both
  - Direction: Right hand rule





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Projection

View Reference Coordinate System

- Prerequisite: Vector Cross Product

Definition of Cross Product of Two Vectors in Space

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

$\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are unit standard basis vectors

$$\boldsymbol{u}\times\boldsymbol{v} = (u_1, u_2, u_3)\times(v_1, v_2, v_3) = \det \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{array}$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} - \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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Projection

View Reference Coordinate System

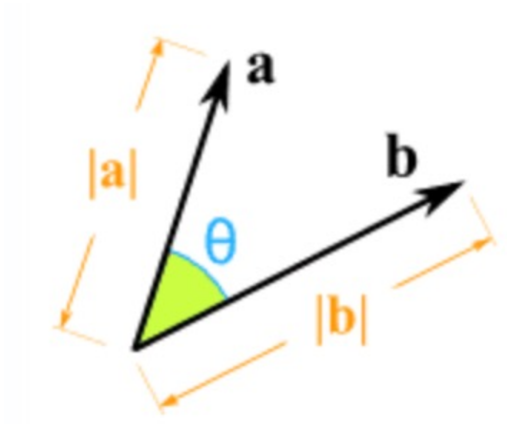
- Prerequisite: Vector Dot Product

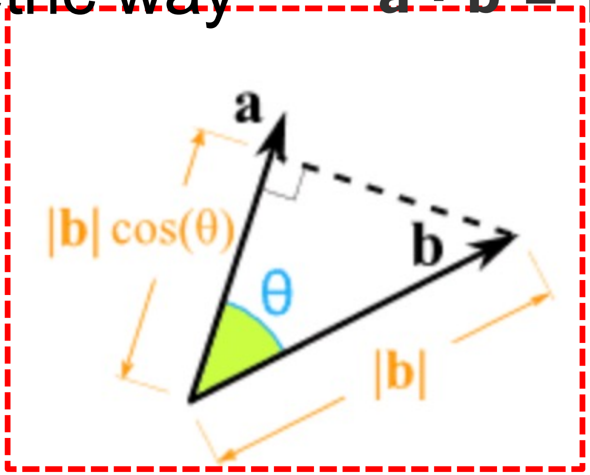
a • b

Calculate in an algebraic way $\mathbf{a} \cdot \mathbf{b} = a_x \times b_x + a_y \times b_y + a_z \times b_z$

The fact that we know **a · b** can be calculated in two ways could be useful!

Calculate in a geometric way $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos(\theta)$





$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} \in [-1, 1]$$

Similarity between two vectors

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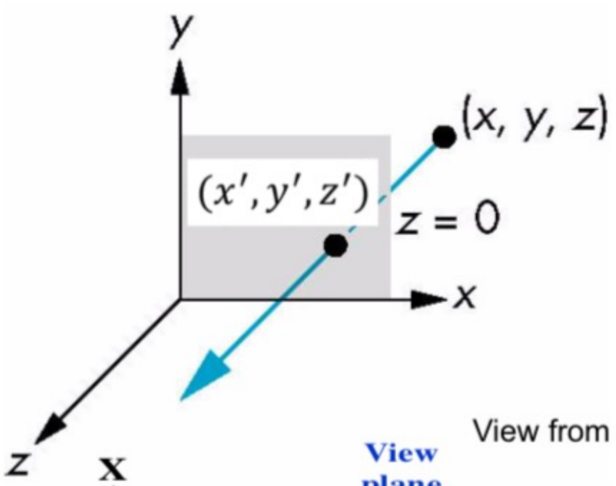
Projection

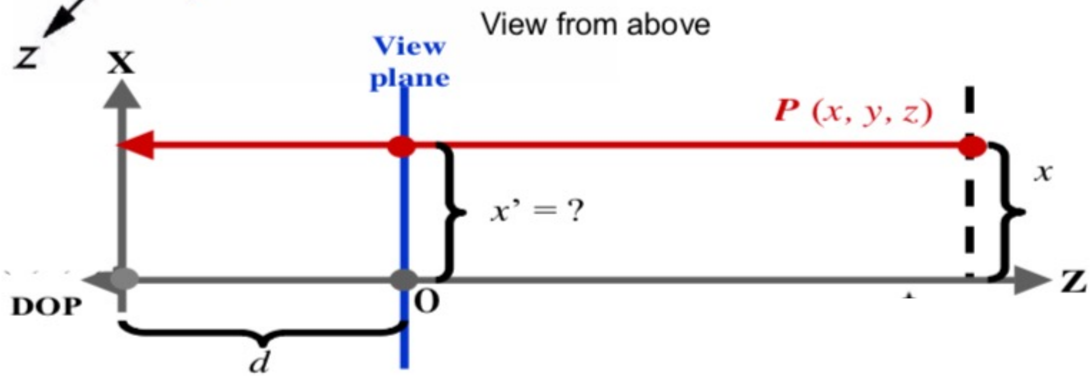
View Reference Coordinate System

- Prerequisite: Parallel Projection



$$\begin{cases} x' = x \\ y' = y \\ z' = 0 \end{cases}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$\boxed{P''_{parallel} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Projection

View Reference Coordinate System

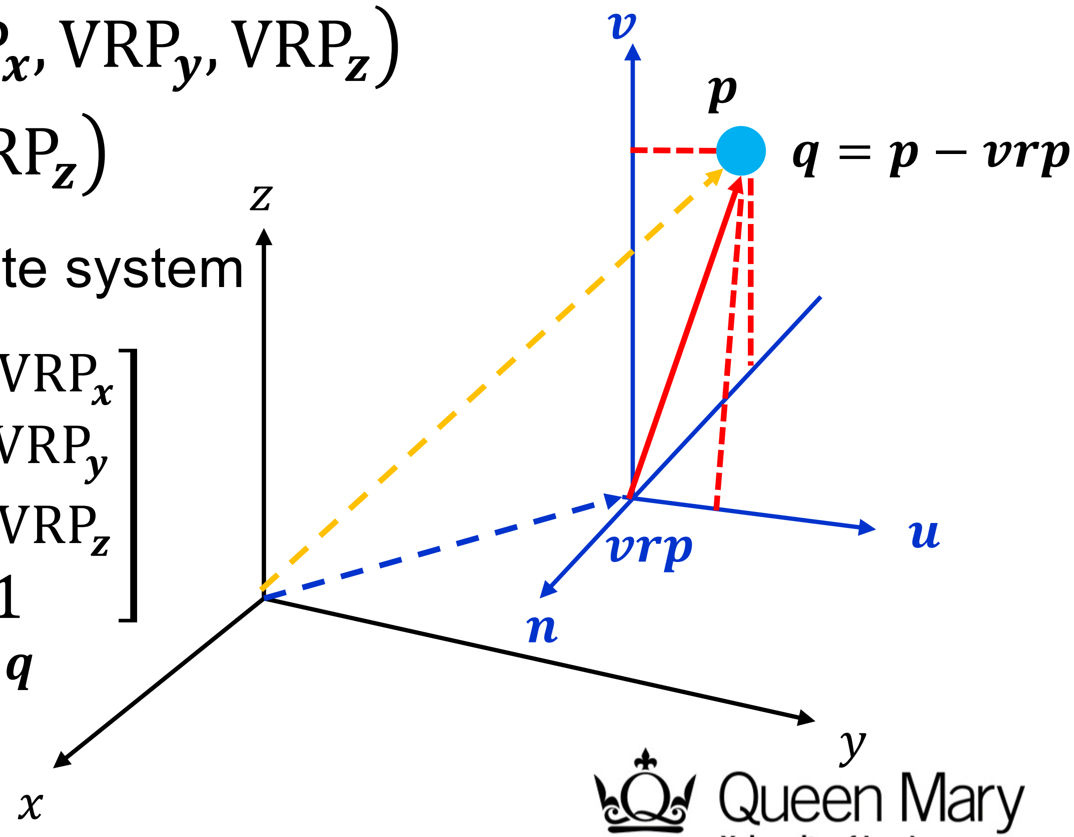
- Our job is to convert $\boldsymbol{p} = (x, y, z)$ in the world coordinate system (WCS) to $\boldsymbol{p}' = (u_p, v_p, n_p)$ in the view reference coordinate system (VRCS)

Vector $\boldsymbol{q}$ (still in the WCS) can be written as

$$\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{vrp} = (x, y, z) - (\mathrm{VRP}_{\boldsymbol{x}}, \mathrm{VRP}_{\boldsymbol{y}}, \mathrm{VRP}_{\boldsymbol{z}})$$
$$= (x - \mathrm{VRP}_{\boldsymbol{x}},\ y - \mathrm{VRP}_{\boldsymbol{y}},\ z - \mathrm{VRP}_{\boldsymbol{z}})$$

If we put $\boldsymbol{q}$ in homogenous coordinate system

$$\boldsymbol{q} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -\mathrm{VRP}_{\boldsymbol{x}} \\ 0 & 1 & 0 & -\mathrm{VRP}_{\boldsymbol{y}} \\ 0 & 0 & 1 & -\mathrm{VRP}_{\boldsymbol{z}} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{T}} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\boldsymbol{p}} = \underbrace{\begin{bmatrix} x - \mathrm{VRP}_{\boldsymbol{x}} \\ y - \mathrm{VRP}_{\boldsymbol{y}} \\ z - \mathrm{VRP}_{\boldsymbol{z}} \\ 1 \end{bmatrix}}_{\boldsymbol{q}}$$



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Projection

View Reference Coordinate System

- Our job is to convert $\boldsymbol{p} = (x, y, z)$ in the world coordinate system (WCS) to $\boldsymbol{p}' = (u_p, v_p, n_p)$ in the view reference coordinate system (VRCS)

Next, we obtain $\boldsymbol{q}$ in VRCS: $\boldsymbol{p}' = (u_p, v_p, n_p)$

The $u - axis$ coordinate of $\boldsymbol{q}$ is the projection of $\boldsymbol{p}$ onto $\boldsymbol{u}$

$$u_p = \boldsymbol{u} \cdot \boldsymbol{q} = u_x(x - \mathrm{VRP}_{\boldsymbol{x}}) + u_y(y - \mathrm{VRP}_{\boldsymbol{y}}) + u_z(z - \mathrm{VRP}_{\boldsymbol{z}})$$

$$v_p = \boldsymbol{v} \cdot \boldsymbol{q}$$

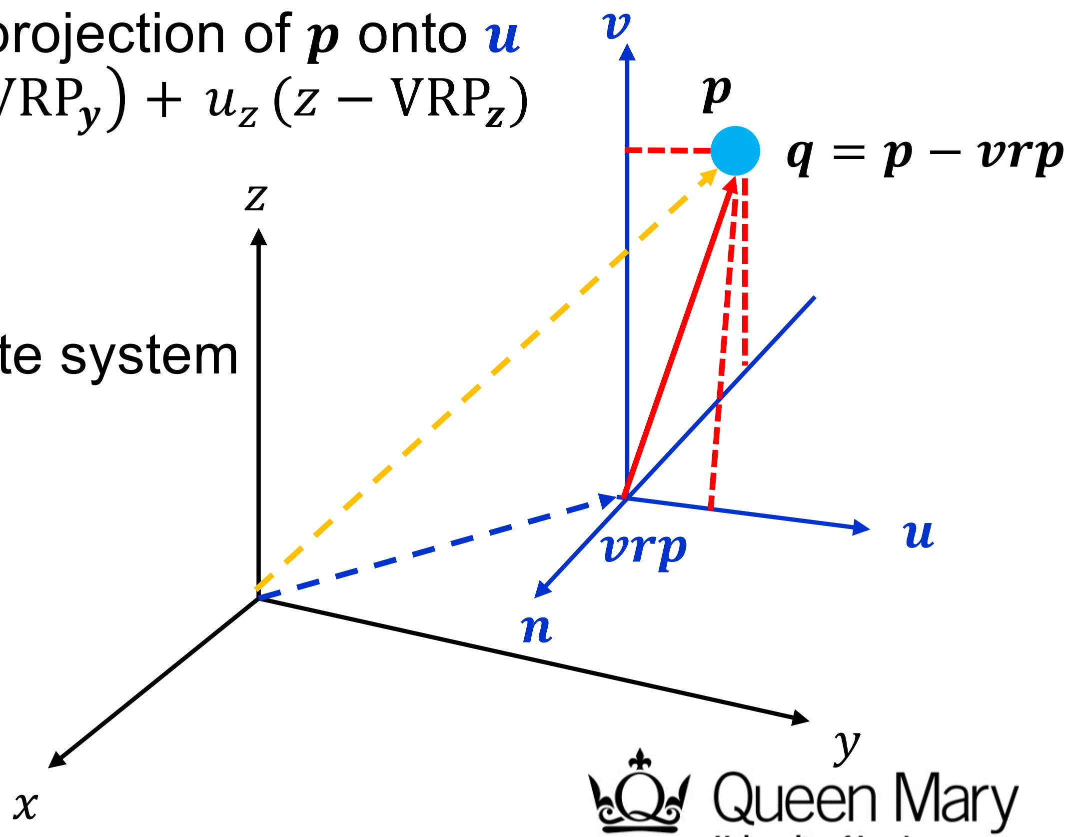
$$n_p = \boldsymbol{n} \cdot \boldsymbol{q}$$

If we put $\boldsymbol{q}$ in homogenous coordinate system

$$\boldsymbol{p}' = \begin{array}{c} \boldsymbol{u} \\ \\ \boldsymbol{n} \\ \\ \end{array} \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - \mathrm{VRP}_{\boldsymbol{x}} \\ y - \mathrm{VRP}_{\boldsymbol{y}} \\ z - \mathrm{VRP}_{\boldsymbol{z}} \\ 1 \end{bmatrix}$$

$\boldsymbol{R} \quad \cdot \quad \boldsymbol{q} = \boldsymbol{T} \cdot \boldsymbol{p}$

$$\boxed{\boldsymbol{p}' = \boldsymbol{R} \cdot \boldsymbol{T} \cdot \boldsymbol{p}}$$



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Projection

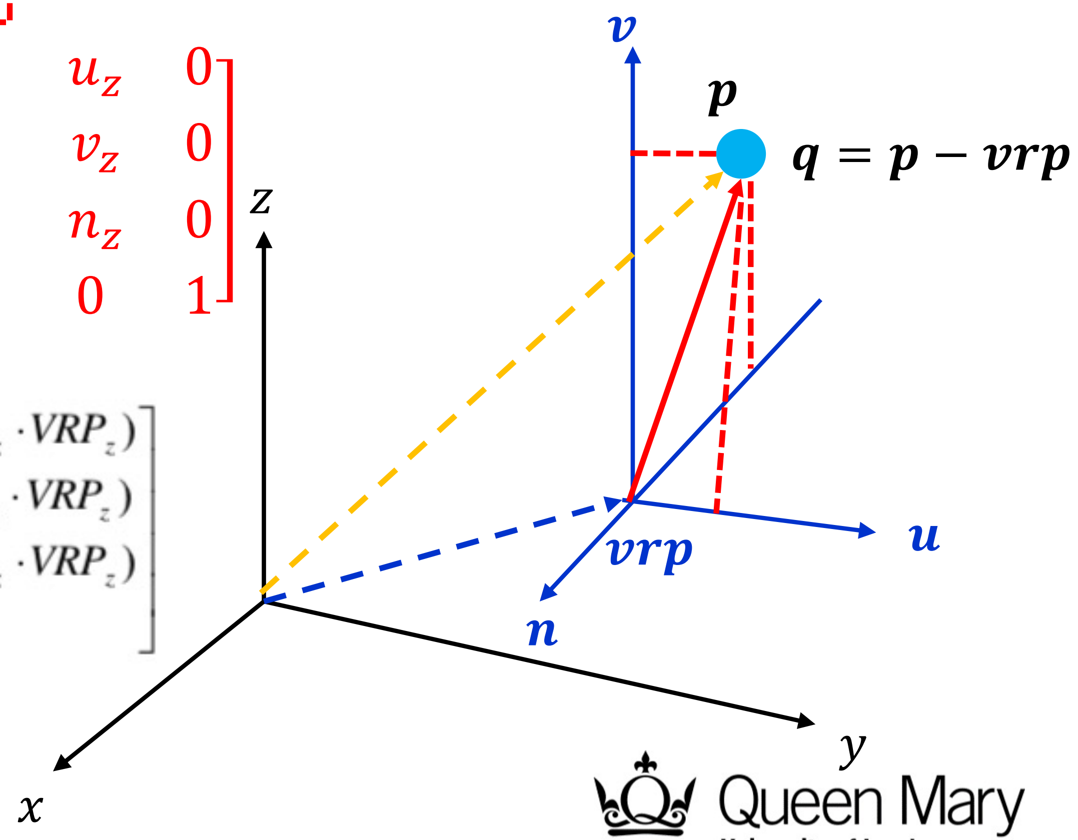
View Reference Coordinate System

- Our job is to convert $\boldsymbol{p} = (x, y, z)$ in the world coordinate system (WCS) to $\boldsymbol{p}' = (u_p, v_p, n_p)$ in the view reference coordinate system (VRCS)

Summary $\boxed{\boldsymbol{p}' = \boldsymbol{R} \cdot \boldsymbol{T} \cdot \boldsymbol{p}}$

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & -\mathrm{VRP}_{\boldsymbol{x}} \\ 0 & 1 & 0 & -\mathrm{VRP}_{\boldsymbol{y}} \\ 0 & 0 & 1 & -\mathrm{VRP}_{\boldsymbol{z}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{R} = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

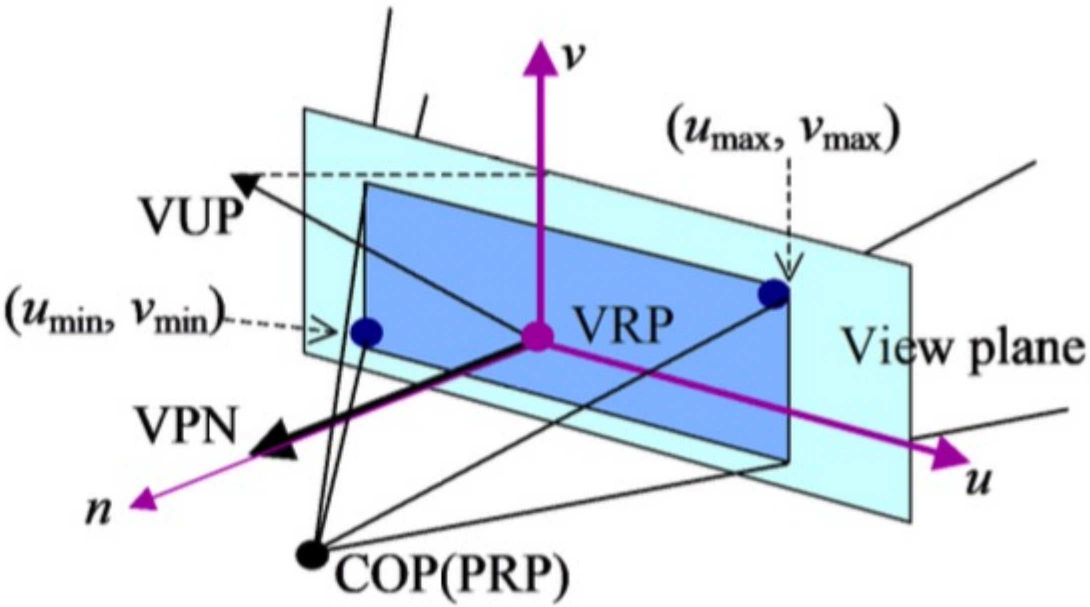
$$M = R \cdot T = \begin{bmatrix} u_x & u_y & u_z & -(u_x \cdot VRP_x + u_y \cdot VRP_y + u_z \cdot VRP_z) \\ v_x & v_y & v_z & -(v_x \cdot VRP_x + v_y \cdot VRP_y + v_z \cdot VRP_z) \\ n_x & n_y & n_z & -(n_x \cdot VRP_x + n_y \cdot VRP_y + n_z \cdot VRP_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Projection

View Reference Coordinate System

- **Translation to align the origin with VRP**
- **Find the axes of the view reference coordinate system: n, u, v**
- **Rotation to align the axes with (u, v, n)**
- **Composition of the 3D transformations: R.T**



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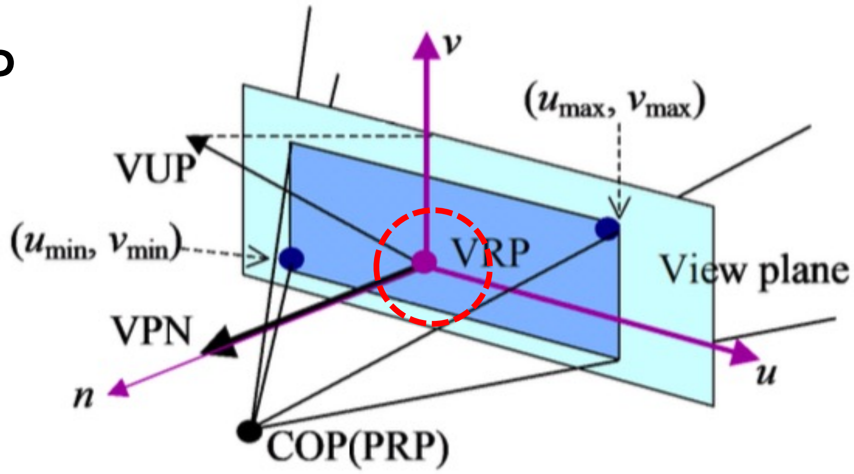
Projection

View Reference Coordinate System

In the world coordinate system, if the view plane normal is [4,3,0], the view-up vector is [0,0,1], the view reference point is [1,2,3], find the transformation matrix for parallel projection.

- Translation to align the origin with VRP

$$T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & -\mathrm{VRP}_x \\ 0 & 1 & 0 & -\mathrm{VRP}_y \\ 0 & 0 & 1 & -\mathrm{VRP}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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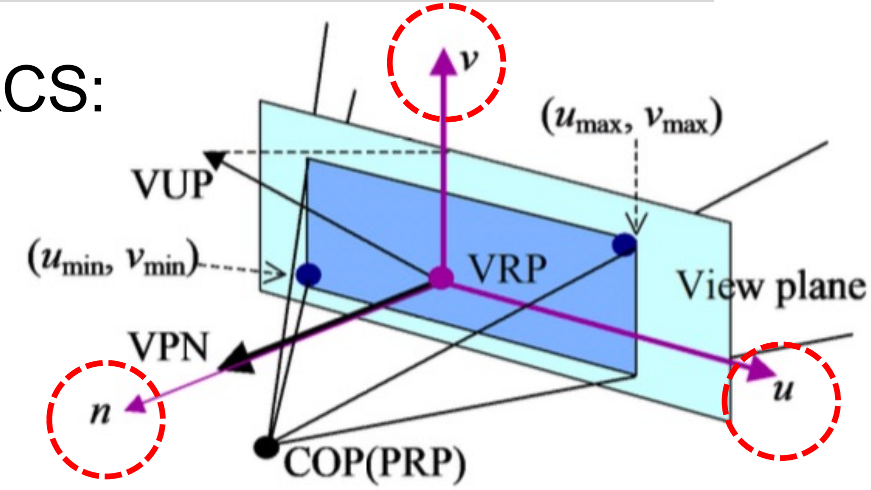
View Reference Coordinate System

In the world coordinate system, if the view plane normal is [4,3,0], the view-up vector is [0,0,1], the view reference point is [1,2,3], find the transformation matrix for parallel projection.

- Find the axes (basis vector) of the VRCS: u, v, n



$$n = \frac{VPN}{|VPN|} = \frac{[4,3,0]}{|[4,3,0]|} = \frac{[4,3,0]}{5} = [\frac{4}{5}, \frac{3}{5}, 0]$$

$$u = \frac{\boxed{VUP \times VPN}}{|VUP \times VPN|} = \frac{[0,0,1]\times[4,3,0]}{|VUP \times VPN|} = \frac{[-3,4,0]}{5} = [-\frac{3}{5}, \frac{4}{5}, 0]$$

$$v = n \times u = [\frac{4}{5}, \frac{3}{5}, 0] \times [-\frac{3}{5}, \frac{4}{5}, 0] = [0,0,1]$$

$$VUP \times VPN = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 1 \\ 4 & 3 & 0 \end{vmatrix} = \boldsymbol{i} \cdot \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} - \boldsymbol{j} \cdot \begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} + \boldsymbol{k} \cdot \begin{vmatrix} 0 & 0 \\ 4 & 3 \end{vmatrix}$$
$$= -\boldsymbol{3i} - \boldsymbol{4j} + \boldsymbol{0k} = (-3, 4, \boldsymbol{0})$$

Projection

View Reference Coordinate System

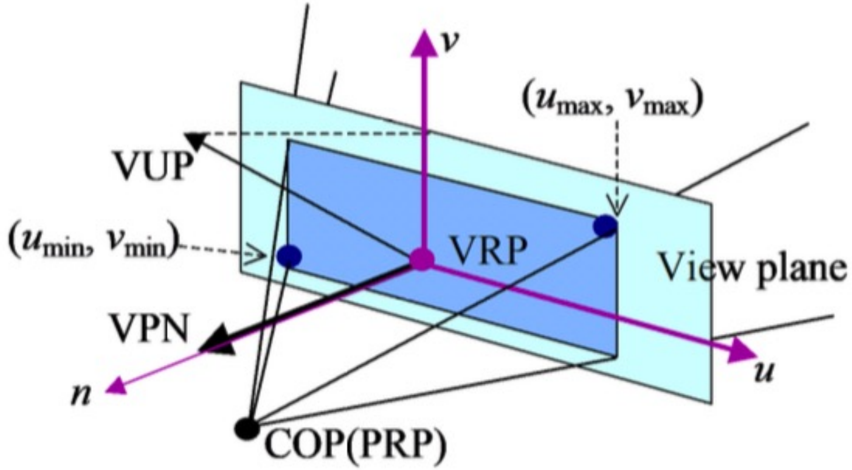
In the world coordinate system, if the view plane normal is [4,3,0], the view-up vector is [0,0,1], the view reference point is [1,2,3], find the transformation matrix for parallel projection.

- Rotate to align the axes with (u, v, n)

$$n = \frac{VPN}{|VPN|} = \frac{[4,3,0]}{|[4,3,0]|} = \frac{[4,3,0]}{5} = [\frac{4}{5}, \frac{3}{5}, 0]$$

$$u = \frac{VUP \times VPN}{|VUP \times VPN|} = \frac{[0,0,1] \times [4,3,0]}{|VUP \times VPN|} = \frac{[-3,4,0]}{5} = [-\frac{3}{5}, \frac{4}{5}, 0]$$

$$v = n \times u = [\frac{4}{5}, \frac{3}{5}, 0] \times [-\frac{3}{5}, \frac{4}{5}, 0] = [0,0,1]$$



$$R = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

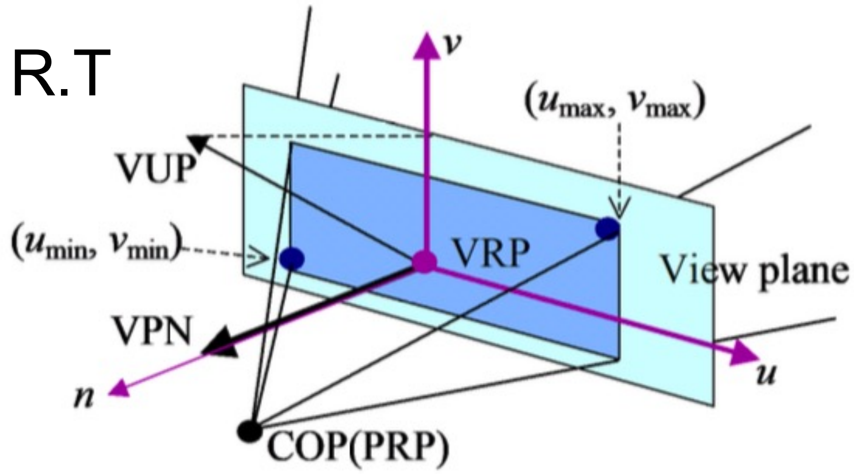
Projection

View Reference Coordinate System

In the world coordinate system, if the view plane normal is [4,3,0], the view-up vector is [0,0,1], the view reference point is [1,2,3], find the transformation matrix for parallel projection.

- Composition of the 3D transformation: R.T

$$R \cdot T = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & -1 \\ 0 & 0 & 1 & -3 \\ \frac{4}{5} & \frac{3}{5} & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



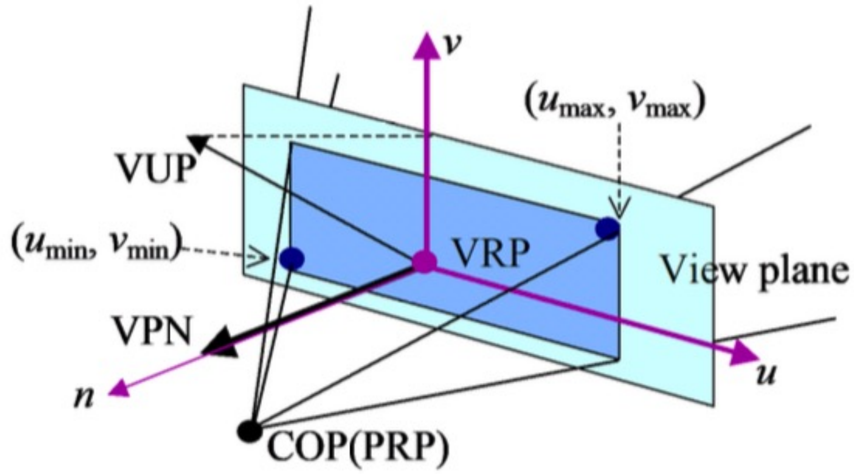
Projection

View Reference Coordinate System

In the world coordinate system, if the view plane normal is [4,3,0], the view-up vector is [0,0,1], the view reference point is [1,2,3], find the transformation matrix for parallel projection.

- Parallel projection: P

$$P''_{parallel} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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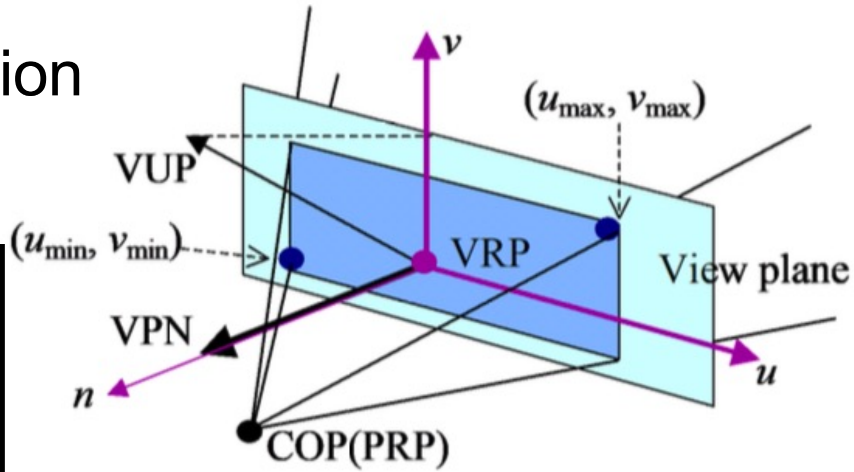
Projection

View Reference Coordinate System

In the world coordinate system, if the view plane normal is [4,3,0], the view-up vector is [0,0,1], the view reference point is [1,2,3], find the transformation matrix for parallel projection.

- Combination for the single transformation matrix



$$M = P \cdot R \cdot T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & -1 \\ 0 & 0 & 1 & -3 \\ \frac{4}{5} & \frac{3}{5} & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Geometric Transformation

b) This question is about geometric transformations.

[11 marks]

i) Consider Model A and Model B in Figure 1. Give a chain of the basic transformation matrices for translation, scaling and rotation which, when post-multiplied by the homogeneous coordinates of the vertices of Model A, will transform the vertices of Model A into their corresponding vertices of Model B, such that the vertex at $(-1,1)$ of Model A is transformed to the vertex at $(c,0)$ of Model B.

(7 marks)

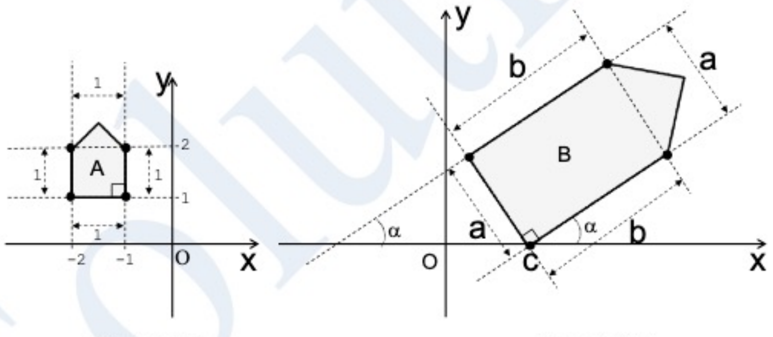


Figure 1.

ii) Compute the composite 2D transformation matrix for the transformations found in question i) above.

(4 marks)

Geometric Transformation

Geometric Transformation

- Prerequisite: Transformation Matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

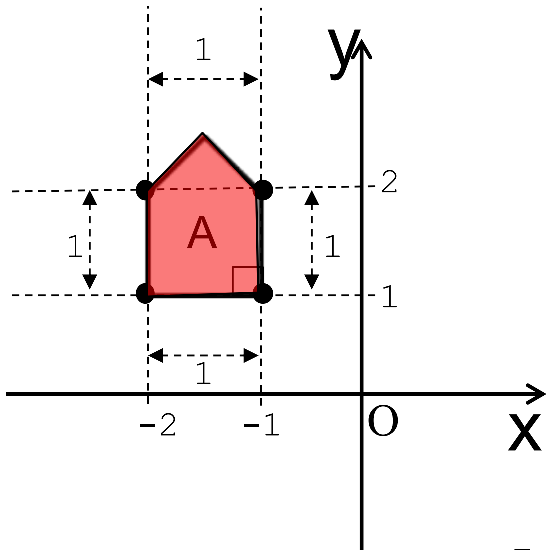
rotate

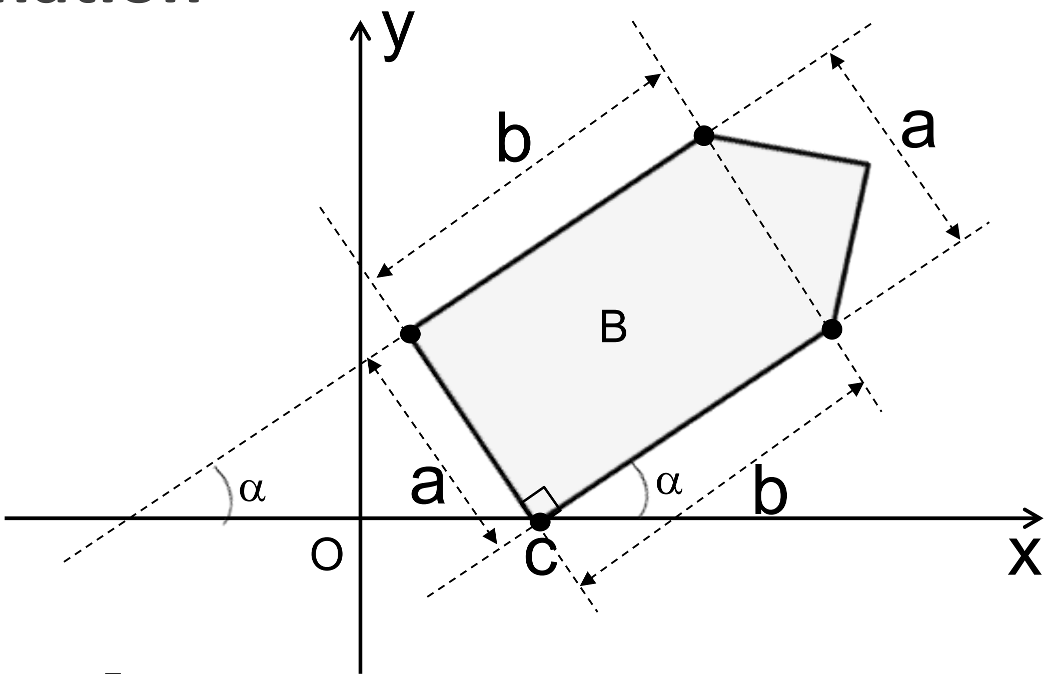
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shear

Geometric Transformation

Geometric Transformation





$$T1(1,-1)=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\qquad S1(a,b)=\begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$R1(\alpha-\pi/2)=\begin{bmatrix}\cos(\alpha-\pi/2) & -\sin(\alpha-\pi/2) & 0\\ \sin(\alpha-\pi/2) & \cos(\alpha-\pi/2) & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\sin(\alpha) & \cos(\alpha) & 0\\ -\cos(\alpha) & \sin(\alpha) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$T2(c,0)=\begin{bmatrix}1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

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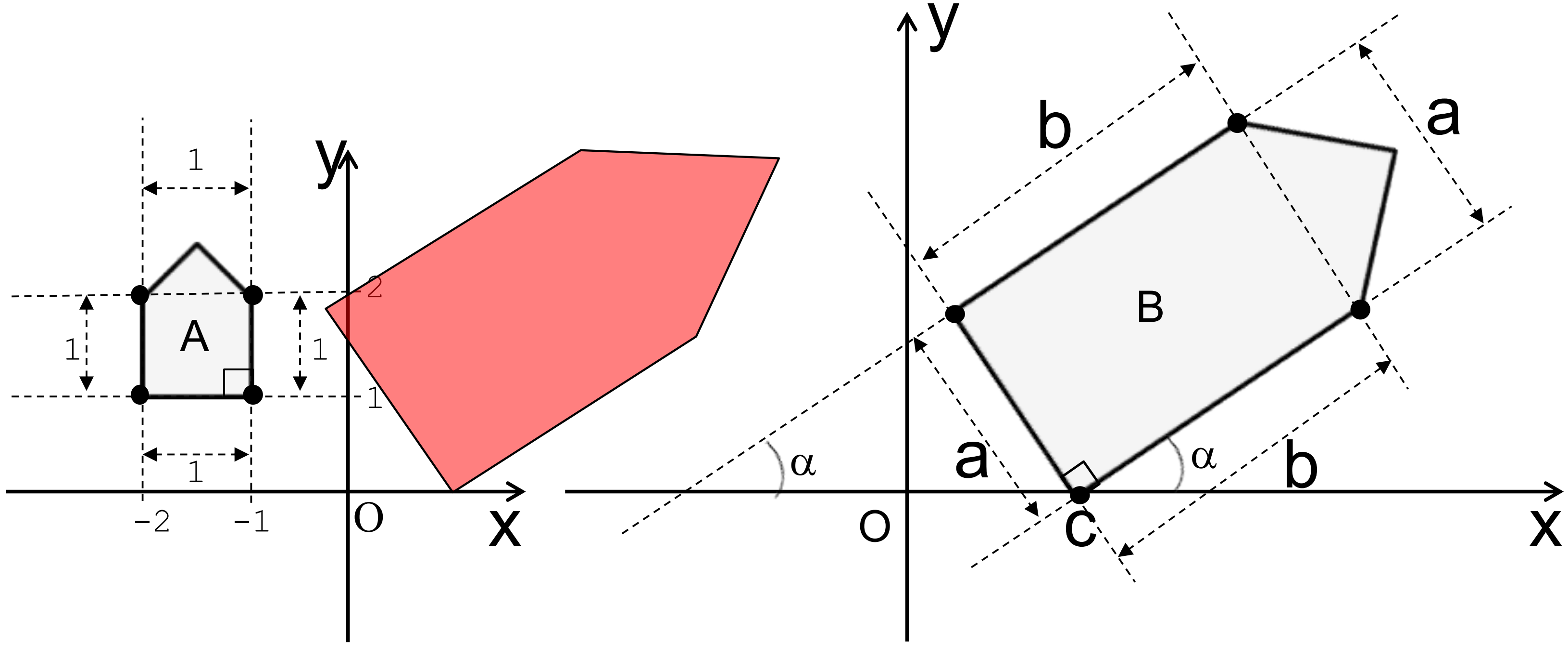
Geometric Transformation

Geometric Transformation



$$M1 = T2 \cdot R1 \cdot S1 \cdot T1 = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sin(\alpha) & \cos(\alpha) & 0 \\ -\cos(\alpha) & \sin(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin(\alpha) & \cos(\alpha) & c \\ -\cos(\alpha) & \sin(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 & a \\ 0 & b & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a\sin(\alpha) & b\cos(\alpha) & c + a\sin(\alpha) - b\cos(\alpha) \\ -a\cos(\alpha) & b\sin(\alpha) & -a\cos(\alpha) - b\sin(\alpha) \\ 0 & 0 & 1 \end{bmatrix}$$

Questions

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