PSO.

1(a) $f(x) = \frac{1}{2}x^TAx + b^Tx$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix}$$

$$x^TAx = [x_1 \cdots x_n] \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \cdots x_n] \begin{bmatrix} A_{11}x_1 + \cdots + A_{1n}x_n \\ \vdots \\ A_{n1}x_1 + \cdots + A_{nn}x_n \end{bmatrix}$$

$$= x_1(A_{11}x_1 + \cdots + A_{1n}x_n) + \cdots + x_n(A_{n1}x_1 + \cdots + A_{nn}x_n)$$

$$= A_{11}x_1^2 + A_{12}x_1x_2 + \cdots + A_{1n}x_1x_n + A_{21}x_1x_2 + A_{31}x_1x_3 + \cdots + A_{n1}x_1x_n + \cdots$$

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}x^TAx\right) = A_{ii}x_i + \frac{1}{2}A_{1i}x_1 + \frac{1}{2}A_{2i}x_2 + \cdots + \frac{1}{2}A_{ni}x_n + \frac{1}{2}A_{i1}x_1 + \frac{1}{2}A_{i2}x_2 + \cdots + \frac{1}{2}A_{in}x_n$$

Since matrix A is symmetric,

$$\therefore \frac{\partial}{\partial x_i}\left(\frac{1}{2}x^TAx\right) = A_{1i}x_1 + A_{2i}x_2 + \cdots + A_{ni}x_n = \sum_{j=1}^{n} A_{ji}x_j$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}x^TAx\right) = \begin{bmatrix} \sum_{j=1}^{n} A_{j1}x_j \\ \vdots \\ \sum_{j=1}^{n} A_{jn}x_j \end{bmatrix} = Ax$$

$$b^Tx = [b_1 \cdots b_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b_1x_1 + \cdots + b_nx_n$$

$$\therefore \frac{\partial}{\partial x_i}(b^Tx) = b_i, \quad \frac{\partial}{\partial x}(b^Tx) = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b$$

$$\therefore \nabla f(x) = Ax + b$$

(b)
$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} g(h(x_1)) \\ \vdots \\ \frac{\partial}{\partial x_n} g(h(x_n)) \end{bmatrix} = \begin{bmatrix} \frac{d}{du} g(h) \cdot \frac{\partial}{\partial x_1} h(x) \\ \vdots \\ \frac{d}{du} g(h) \cdot \frac{\partial}{\partial x_n} h(x) \end{bmatrix}$$

(c)
$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f(x) & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} f(x) \\ \frac{\partial}{\partial x_2 \partial x_1} f(x) & \ddots & \vdots \\ \vdots & & \\ \frac{\partial}{\partial x_n \partial x_1} f(x) & \cdots & \frac{\partial^2}{\partial x_n^2} f(x) \end{bmatrix}$$

$$\frac{\partial}{\partial x_j \partial x_i}\left(\frac{1}{2}x^TAx\right) = \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{n} A_{ki}x_k\right) = \frac{\partial}{\partial x_j}(\cancel{A_{1i}x_k + A_{2i}x_k} \; A_{1i}x_1 + A_{2i}x_2 + \cdots + A_{ji}x_j + \cdots) = A_{ji}$$

$$\frac{\partial}{\partial x_j \partial x_i}(b^Tx) = \frac{\partial}{\partial x_j}(b_i) = 0$$

$$\nabla^2 f(x) = A$$

(d) $f(x) = g(a^Tx)$

$$\nabla f(x) = \begin{bmatrix} g'(a^Tx) \cdot \frac{\partial}{\partial x_1}(a^Tx) \\ \vdots \\ g'(a^Tx) \cdot \frac{\partial}{\partial x_n}(a^Tx) \end{bmatrix} = \begin{bmatrix} g'(a^Tx)\, a_1 \\ \vdots \\ g'(a^Tx)\, a_n \end{bmatrix} = g'(a^Tx) \cdot a$$

$$[a_1 \cdots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + \cdots + a_nx_n$$

$$\nabla^2 f(x) \Rightarrow \frac{\partial^2}{\partial x_j \partial x_i} f(x) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i} f(x)\right)$$

$$= \frac{\partial}{\partial x_j}\left(g'(a^T x)\, a_i\right)$$

$$= a_i \frac{\partial}{\partial x_j}\left(g'(a^T x)\right)$$

$$= a_i \cdot g''(a^T x) \cdot a_j$$

$$\frac{\partial^2}{\partial x_j \partial x_i} f(x) = a_i a_j \cdot g''(a^T x)$$

$$\nabla^2 f(x) = g''(a^T x) \cdot \begin{bmatrix} a_1 a_1 & a_1 a_2 & \cdots & a_1 a_n \\ \vdots & \vdots & \cdots & \vdots \\ a_n a_1 & a_n a_2 & \cdots & a_n a_n \end{bmatrix} = g''(a^T x)\, a a^T$$

2. (a) $A = z z^T$

$A^T = (z z^T)^T = (z^T)^T \cdot z^T = z \cdot z^T$

① Hence $A = A^T$

$x^T A x = \cancel{(z z^T) A (z z^T)} \quad x^T \cdot (z z^T) x$

$= x^T z \cdot z^T x$

$= (\cancel{x^T z}) \cdot (z^T x)^T (z^T x)$

$= \|z^T x\|^2 \geq 0$

② Hence $x^T A x \geq 0$

Hence, A is positive semi-definite (PSD)

(b) Null space $N(A) = [x_1, \cdots, x_n]$

$Ax = 0$

$$A = \begin{bmatrix} z_1^2 & z_1 z_2 & \cdots & z_1 z_n \\ \vdots & & & \vdots \\ z_n z_1 & z_n z_2 & \cdots & z_n^2 \end{bmatrix} \longrightarrow \begin{bmatrix} z_1^2 & z_1 z_2 & \cdots & z_1 z_n \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$z_1^2 x_1 + z_1 z_2 x_2 + \cdots + z_1 z_n x_n = 0$

$z_1 x_1 + z_2 x_2 + \cdots + z_n x_n = 0, \quad * \sum_{i=1}^{n} z_i x_i = 0$

rank $(A) = 1$

(c) A is PSD, so $A = A^T$, $x^T A x \geq 0$

① $(BAB^T)^T = (B^T)^T A^T B^T = BA^T B^T$

$= BAB^T$

② $BAB^T \Rightarrow x^T BAB^T x = (x^T B) A (B^T x)$

$= (B^T x)^T A (B^T x) \geq 0$

Hence BAB^T is PSD

3.

(a) $A = T\Lambda T^{-1}$

$AT = T\Lambda$

$At^{(i)} = T\Lambda^{(i)}$

$$\Lambda^{(i)} = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix}$$

$T\Lambda^{(i)} = \lambda_i \cdot t^{(i)}$

Hence $At^{(i)} = \lambda_i \cdot t^{(i)}$ $\Leftrightarrow$ ~~eigenpairs~~ eigenpairs of A is $(t^{(i)}, \lambda_i)$

(b) $A = U\Lambda U^T$

If U is orthogonal. then $U^TU = I$

$AU = U\Lambda U^TU$

$AU = U\Lambda$

$Au^{(i)} = U\Lambda^{(i)} = \lambda_i \cdot u^{(i)}$

Hence $u^{(i)}$ is an eigenvector of A.

(c) If A is PSD. then $A = A^T$, $x^TAx \geq 0$ for all $x \in \mathbb{R}^n$

If A is PSD. A is symmetric $(A = A^T)$.

then $U^TAU = \Lambda$, where U is an orthogonal matrix.

Λ is a diagonal matrix

let $u^{(i)}$ represents the ith coloumn of U,

$$u^{(i)T} A u^{(i)} = \underbrace{\lambda^{(i)}}_{\lambda_i(A)} \geq 0$$

Hence for each i. $\lambda_i(A) \geq 0$.

2022. 5. 25