Mathematics 3345: Foundations of Higher Mathematics – Homework 11

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Question 1

Let A and B be sets. Prove that $A \subset A \cup B$.

Theorem 1

Suppose that A and B are sets. We would like to prove that $A \subset A \cup B$.

Proof of Theorem 1

Suppose that A and B are sets. We would like to prove that:

$$\boxed{A \subset A \cup B}$$

We begin with the definition of a subset. To prove that $A \subset A \cup B$, we have to show that every element of A is an element of $A \cup B$. Let $x \in A$ be arbitrary. By definition of the union of two sets, an element belongs to $A \cup B$ if it belongs to either A or B. Since $x \in A$, we have that $x \in A \cup B$. Therefore, any element in A is also an element in $A \cup B$, that is, $A \subset A \cup B$.

$\square$

Question 2

Let A, B, and C be sets. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 2

Suppose that A, B, and C are sets. We would like to prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof of Theorem 2

Suppose that A, B, and C are sets. We would like to prove that:

$$\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$$

First, we show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Let $x \in A \cup (B \cap C)$ be arbitrary. Then, either:

- $x \in A$.
- $x \in B \cap C$.

If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, which means $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$. This means $x \in A \cup B$ (when $x \in B$) and $x \in A \cup C$ (when $x \in C$), which means $x \in (A \cup B) \cap (A \cup C)$. In either case, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Next, we show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Let $x \in (A \cup B) \cap (A \cup C)$ be arbitrary. Then, both:

- $x \in A \cup B$.
- $x \in A \cup C$.

If $x \in A \cup B$, then $x \in A$ or $x \in B$. Similarly, if $x \in A \cup C$, then $x \in A$ or $x \in C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then $x \in B$ and $x \in C$. This means $x \in B \cap C$, which implies $x \in A \cup (B \cap C)$. Therefore, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Since $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$, we have that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$\square$

Question 3

Let A and B be sets. Prove that if $A \cup B = B$, then $A \backslash B = \emptyset$.

Theorem 3

Suppose that A and B are sets. We would like to prove that if $A \cup B = B$, then $A \backslash B = \emptyset$.

Proof of Theorem 3

Suppose that A and B are sets. We would like to prove that:

$$\boxed{A \cup B = B \implies A \backslash B = \emptyset}$$

We use a method of proof by contrapositive, that is, $A \backslash B \neq \emptyset \implies A \cup B \neq B$. First, assume that $A \backslash B \neq \emptyset$ is true, then show that $A \cup B \neq B$. Since $A \backslash B \neq \emptyset$, there exists an element x such that $x \in A$ and $x \notin B$. Since $x \in A$, we have that $x \in A \cup B$ by definition of the union of two sets. Similarly, since $x \notin B$, there exists some element in $A \cup B$ that is not in B, which means $A \cup B \neq B$. Therefore, we have proven the contrapositive and can reasonably conclude that if $A \cup B = B$, then $A \backslash B = \emptyset$.

$\square$

Question 4

Let A and B be sets. Prove that if $A \backslash B = \emptyset$, then $A \subset B$.

Theorem 4

Suppose that A and B are sets. We would like to prove that if $A\backslash B = \emptyset$, then $A \subset B$.

Proof of Theorem 4

Suppose that A and B are sets. We would like to prove that:

$$\boxed{A\backslash B = \emptyset \implies A \subset B}$$

We use a method of direct proof, that is, we assume that $A\backslash B = \emptyset$ is true, then show that $A \subset B$. Since $A\backslash B = \emptyset$, there is no element, x, such that $x \in A$ and $x \notin B$. In other words, there are no elements in set A not belonging to set B, which means for any arbitrary element in set A, it must also be contained in set B. Therefore, by definition of a subset, $A \subset B$. We may conclude that if $A\backslash B = \emptyset$, then $A \subset B$.

$\square$

Alternatively, we may use a method of proof by contradiction, that is, we assume that $A\backslash B = \emptyset$ is true, then show that $A \not\subset B$ would lead to a contradiction in our original assumption. If $A \not\subset B$, then there exists an element, x, such that $x \in A$ and $x \notin B$, which means $x \in A\backslash B$. Since $x \in A\backslash B$, we have that $A\backslash B \neq \emptyset$ as there is an element in set A not belonging to set B, which is clearly a contradiction. Therefore, we may reasonably conclude that if $A\backslash B = \emptyset$, then $A \subset B$.

$\square$