# **Mathematics 3345:** Foundations of Higher Mathematics – Homework 12

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## Question 1

Let $A$, $B$, and $C$ be sets. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

### Theorem 1

Suppose that $A$, $B$, and $C$ are sets. We would like to prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

### Proof of Theorem 1

Suppose that $A$, $B$, and $C$ are sets. We would like to prove that

$$\boxed{A \times (B \cup C) = (A \times B) \cup (A \times C)}$$

Let $z \in A \times (B \cup C)$. Then, $z = (x, y)$ with $x \in A$ and $y \in B \cup C$. So, $y \in B$ or $y \in C$. If $y \in B$, then $z = (x, y) \in A \times B$. Similarly, if $y \in C$, then $z = (x, y) \in A \times C$. So, $(x, y) \in (A \times B) \cup (A \times C)$. Therefore, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Let $z \in (A \times B) \cup (A \times C)$. Then, either $z \in A \times B$ or $z \in A \times C$. If $z \in A \times B$, then $z = (x, y)$, with $x \in A$ and $y \in B$. Since $B \subseteq B \cup C$, we have that $y \in B \cup C$. So, $z = (x, y) \in A \times (B \cup C)$. Similarly, If $z \in A \times C$, then $z = (x, y)$, with $x \in A$ and $y \in C$. Since $C \subseteq B \cup C$, we have that $y \in B \cup C$. So, $z = (x, y) \in A \times (B \cup C)$. This means that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Finally, we can conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. $\square$

## Question 2

Let $A$, $B$, and $C$ be sets. Prove that if $A \times C = B \times C$ and $C \neq \emptyset$, then $A = B$.

### Theorem 2

Suppose that $A$, $B$, and $C$ are sets. We would like to prove that if $A \times C = B \times C$ and $C \neq \emptyset$, then $A = B$.

**Proof of Theorem 2**

Suppose that $A$, $B$, and $C$ are sets. We would like to prove that if $A \times C = B \times C$ and $C \neq \emptyset$, then $A = B$. Suppose that $A \times C = B \times C$ and $C \neq \emptyset$, but $A \neq B$ for a contradiction. If $A \neq B$, there exists an element, $a$, such that $a \in A$ and $a \notin B$ or there exists and element, $b$, such that $b \notin A$ and $b \in B$. Since $C \neq \emptyset$, there exists and element, $c$, such that $c \in C$. $z = (a, c) \in A \times C$ since $a \in A$ and $c \in C$, but $z = (a, c) \notin B \times C$ since $a \notin B$. So, $A \times C \neq B \times C$, which is clearly a contradiction to our original assumption that $A \times C = B \times C$. Therefore, we can reasonably conclude that if $A \times C = B \times C$ and $C \neq \emptyset$, then $A = B$.

$\square$

## Question 3

Are these sets the graph of a function from $\mathbb{R}$ to $\mathbb{R}$? Justify your answer.

- $G = \{(x, e^x), x \in \mathbb{R}\}$

  **Solution**
  Yes, $G = \{(x, e^x), x \in \mathbb{R}\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$ since for every input value of $x$ in $\mathbb{R}$, there is a unique output value $e^x$ in $\mathbb{R}$. It is the function $f(x) = e^x$.

- $G = \{(e^x, x), x \in \mathbb{R}\}$

  **Solution**
  Yes, $G = \{(e^x, x), x \in \mathbb{R}\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$ since for every input value of $e^x$ in $\mathbb{R}$, there is a unique output value $x$ in $\mathbb{R}$.

- $G = \{(x, -3x), x \in \mathbb{R}\}$

  **Solution**
  Yes, $G = \{(x, -3x), x \in \mathbb{R}\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$ since for every input value of $x$ in $\mathbb{R}$, there is a unique output value $-3x$ in $\mathbb{R}$. It is the function $f(x) = -3x$.

- $G = \{(-3x, x), x \in \mathbb{R}\}$

  **Solution**
  Yes, $G = \{(-3x, x), x \in \mathbb{R}\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$ since for every input value of $-3x$ in $\mathbb{R}$, there is a unique output value $x$ in $\mathbb{R}$. It is the function $f(x) = \frac{-x}{3}$.

## Question 4

Sketch the set $G = \{(x^2, x), x \in \mathbb{R}\} \subset \mathbb{R}^2$. Does it define the graph of a function from $[0, +\infty]$ to $\mathbb{R}$? Justify your answer.

### Solution to Question 4

Please refer to Figure 1 for the sketch of the set $G = \{(x^2, x), x \in \mathbb{R}\} \subset \mathbb{R}^2$. The set $G = \{(x^2, x), x \in \mathbb{R}\} \subset \mathbb{R}^2$ does define a function from $[0, +\infty]$ to $\mathbb{R}$ since for every input value of $x^2$ in $[0, +\infty]$, there is only one unique output value $x$ in $\mathbb{R}$. Since the domain is restricted to $[0, +\infty]$, we get that $f(x) = \sqrt{x}$. We do not have to consider $f(x) = -\sqrt{x}$. If the domain would have been the set containing all real numbers, $G = \{(x^2, x), x \in \mathbb{R}\} \subset \mathbb{R}^2$ would not have been a function since it would not pass the vertical line test (VLT). By restricting the domain to $[0, +\infty]$, we ensure that there is only one unique output value for each input value.

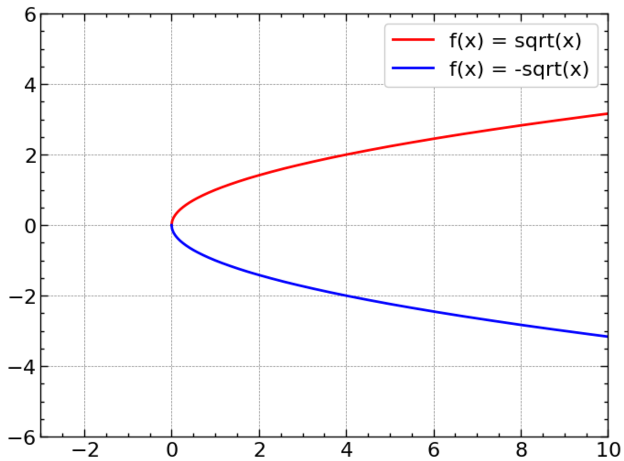


Figure 1: Sketch of $G = \{(x^2, x), x \in \mathbb{R}\} \subset \mathbb{R}^2$

## Question 5

Let $f : A \to B$ be a function, and let $Y, Z$ be subsets of $B$. Prove that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

### Theorem 3

Suppose that $f : A \to B$ is a function, and consider $Y, Z$ as subsets of $B$. We would like to prove that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

### Proof of Theorem 3

Suppose that $f : A \to B$ is a function, and consider $Y, Z$ as subsets of $B$. We would like to prove that:

$$\boxed{f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)}$$

Let $x \in f^{-1}(Y \cap Z)$. Then, $f(x) \in Y \cap Z$. So, $f(x) \in Y$ and $f(x) \in Z$. This means that $x \in f^{-1}(Y)$ and $x \in f^{-1}(Z)$, implying that $f^{-1}(Y) \cap f^{-1}(Z)$ is true. Therefore, $f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$. Let $x \in f^{-1}(Y) \cap f^{-1}(Z)$. Then, both:

- $x \in f^{-1}(Y)$
- $x \in f^{-1}(Z)$

If $x \in f^{-1}(Y)$, then $f(x) \in Y$. If $x \in f^{-1}(Z)$, then $f(x) \in Z$, implying that $f(x) \in Y \cap Z$, which means that $x \in f^{-1}(Y \cap Z)$. Therefore, $f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$. Finally, we conclude that $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

$\square$