IDSC 6490

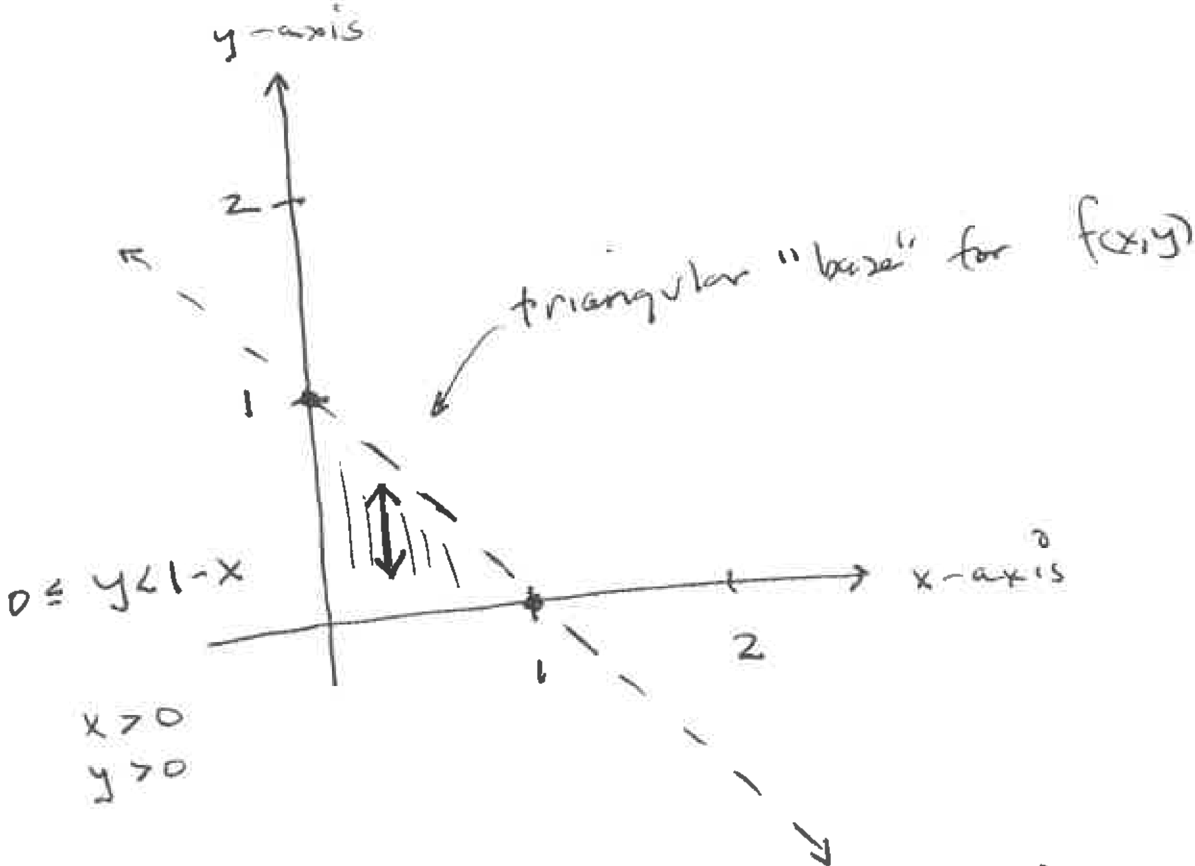
workbook part 3

Find $Cov(X,Y)$ and $\rho(X,Y)$.

same $y<1-x$ or $x<1-y$

Suppose $f(x,y) = \begin{cases} 2 & x+y<1 \quad x\geq 0, y\geq 0 \\ 0 & \text{otherwise} \end{cases}$

Let's have a picture ☺



Does $\iint f(x,y)\,dx\,dy = 1$?

Is this a proper distribution function?

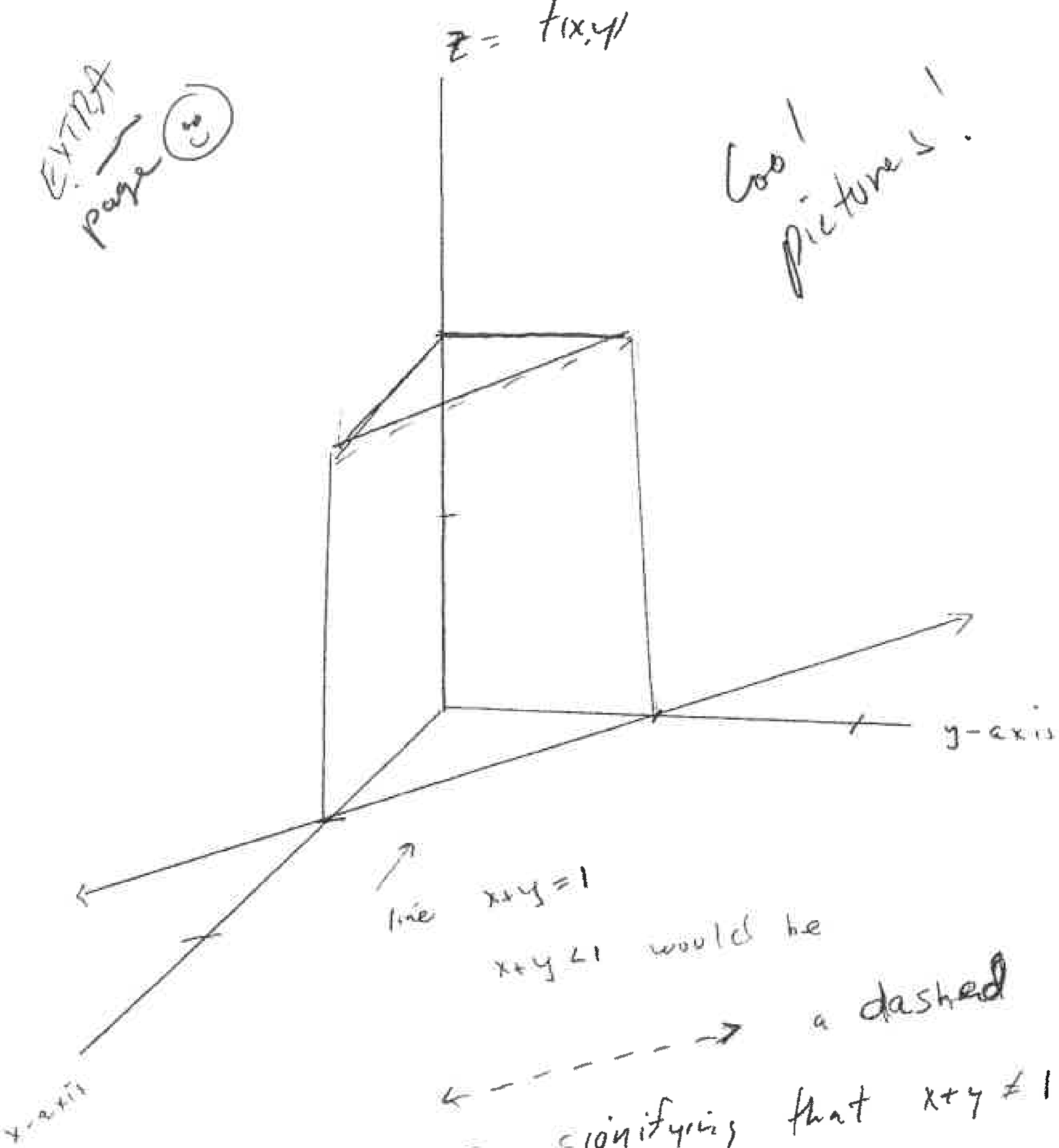
$\int_0^1 \int_0^{1-x} 2\,dy\,dx$ or $\int_0^1 \int_0^{1-y} 2\,dx\,dy$ These are the same - why?

$\boxed{16}$

$$F_{(x,y)} = \begin{cases} 2 & x+y<1 \quad x \geq 0 \quad y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

EXTRA page ☺

Cool pictures!



$x+y<1$ would be a dashed

line signifying that $x+y \neq 1$

$$\int_0^1 \int_0^{1-x} 2 \, dy \, dx =$$

$$\int_0^1 \left(\int_0^{1-x} 2 \, dy \right) dx = \int_0^1 \left(2y \Big|_0^{1-x} \right) dx$$

$$= \int_0^1 \left[2(1-x) - 2 \cdot 0 \right] dx$$

Square seemed cooler LOL*

$$= \int_0^1 2 - 2x \, dx = 2x - x^2 \Big|_0^1 = 2(1) - (1)^2 - (2 \cdot 0 - 0^2)$$

$$= 1$$

HA! yes!

we did it!

Next we need

$f_X(x)$ and $f_Y(y)$ the marginals

$f_{\underline{X}}(x)$ looks better than $f_X(x)$ in my mind.

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$$f_X(x) = \int_y f(x,y)\,dy = \int_0^{1-x} 2\,dy = 2y\Big|_0^{1-x} = 2(1-x) - 2\cdot 0$$

The reigon of y

$$\boxed{f_X(x) = 2 - 2x}$$

$$f_Y(y) = \int_x f(x,y)\,dx = \int_0^{1-y} 2\,dx = 2x\Big|_0^{1-y}\ 2(1-y) - 2\cdot 0$$

Tha reigon of x

$f_Y(y) = 2 - 2y$ So $Cov(X,Y) = E(X,Y) - E(X)E$

$$Cov(X,Y) = \iint_{\mathbb{R}} xy\, f(x,y)\, dy\,dx - \left(\int_x x f_X(x)\,dx\right)\left(\int y f_Y(y)\,dy\right)$$

$$E(X,Y) = \int_0^1 \int_0^{1-x} xy\,2\,dy\,dx \quad \text{or}$$

I'll do this one 🙂

$$\int_0^1 \int_0^{1-y} xy\,2\,dx\,dy$$

$$E(X,Y) = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx =$$

$$\int_0^1 \left[2x\frac{y^2}{2} \Big|_0^{1-x} = x(1-x)^2 - x \cdot 0^2 \right] dx$$

$$= \int_0^1 x(1-x)^2 dx = \int_0^1 x(1-2x+x^2)\,dx$$

$$= \int_0^1 x - 2x^2 + x^3\,dx = \frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4} \Big|_0^1$$

$$= \frac{(1)^2}{2} - \frac{2}{3}(1)^3 + \frac{(1)^4}{4} - \left(\frac{0^2}{2} - \frac{2}{3}(0)^3 + \frac{0^4}{4} \right)$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$ Thus $\boxed{E(XY) = \frac{1}{12}}$

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(2-2x)\,dx$$

$$= \int_0^1 2x - 2x^2\,dx = x^2 - \frac{2}{3}x^3 \Big|_0^1$$

$$= (1)^2 - \frac{2}{3}(1)^3 - \left((0)^2 - \frac{2}{3}(0)^3\right) = 1 - \frac{2}{3} = \frac{1}{3}$$

So

$$E(X) = \frac{1}{3}$$

$$E(Y) = \int_0^1 y\, f_Y(y)\, dy = \int_0^1 y(2-2y)\, dy$$

$$\int_0^1 2y - 2y^2\, dy = y^2 - \frac{2}{3} y^3 \Big|_0^1$$

$$= (1)^2 - \frac{2}{3}(1)^3 - \left(0^2 - \frac{2}{3}(0)^3\right) \text{ also } = \frac{1}{3}$$

Thus $E(Y) = \frac{1}{3}$

$$Cov(XY) = \frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{12} - \frac{1}{9} = \boxed{-\frac{1}{36}}$$

Finally

$$\rho(XY) = \frac{Cov(XY)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$Var(X) = E(X^2) - E(X)^2$$

we already figured this out ☺

$$= \int_0^1 x^2(2-2x)\,dx - \left(\frac{1}{3}\right)^2$$

$$= \int_0^1 x^2(2-2x)\,dx - \left(\frac{1}{3}\right)^2$$

$$= \int_0^1 2x^2 - 2x^3\,dx - \frac{1}{9}$$

$$= \frac{2}{3}x^3 - \underbrace{\frac{2}{4}}_{1/2}x^4 \Big|_0^1 - \frac{1}{9}$$

$$= \frac{2}{3}(1)^3 - \frac{1}{2}(1)^4 - \left(\frac{2}{3}(0)^3 - \frac{1}{2}(0)^4\right) - \frac{1}{9}$$

$$= \frac{2}{3} - \frac{1}{2} - 0 - \frac{1}{9} = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$\boxed{Var(X) = \frac{1}{18}}$$

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$$

$$= \int_{y} y^2 f_Y(y)\, dy - \left(\frac{1}{3}\right)^2$$ already computed ☺

$$= \int_0^1 y^2 (2-2y)\, dy - \left(\frac{1}{3}\right)^2$$

Note: We already did

$$\int_0^1 x^2 (2-2x)\, dx - \left(\frac{1}{3}\right) = \frac{1}{18}$$ so

Clearly

$$\int_0^1 y^2 (2-2y)\, dy - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

Thus $\mathrm{Var}(Y) = \mathrm{Var}(X) = \dfrac{1}{18}$

So finally

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

rho

$$= \frac{-\frac{1}{36}}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}} = \frac{-\frac{1}{36}}{\frac{1}{18}} = -\frac{1}{2}$$

Yea! 

Fin