Problem 1 (13 credits)

HW1

####

February 02, 2020

```
suppressWarnings(suppressPackageStartupMessages({
  library(TSA)
  library(forecast)
  library(ggplot2)
  library(dplyr)
}))
```

Binary Random Walk

Assume Y_t is a ***binary*** random walk as we illustrated in the class, such that

$$\begin{cases} Y_1 = e_1 \\ Y_2 = e_1 + e_2 \\ \vdots \\ Y_t = \sum_{i=1}^{t} e_i \end{cases}$$

where $e_i \in \{-1, 1\}$ i.i.d

Note that this is not the Gaussian random walk where $e_t \sim N(0,1)$ but a binary one where e_t can only take two values: -1 and 1 each with probability 50%.

Question 1

a) (1 credit)

Please compute analytically:

$E[Y_3] =$

```
E_Y3 <- 0
```

(please include only the numerical answer above not the computation. An acceptable format for your answer is such as `E_Y3 <- 5`. Please do not rename this variable)

b) (1 credit)

Please compute analytically:

$\text{Var}[Y_3] =$

```
Var_Y3 <- 3
```

c) (1 credit)

Please compute analytically:

$\mathrm{Cov}(Y_2, Y_3) =$

```
Cov_Y2Y3 <- 2
```

d) (1 credit)

Please compute analytically:

$\mathrm{Cov}(Y_5, Y_{10}) =$

```
Cov_Y5Y10 <- 5
```

Question 2

a) (1 credit)

Please generate one sample path of length $T = 100$ for this binary random walk.

- Please save it into a data.frame `df2a` column `df2a$Y`

```
set.seed(42) # Please do not change the seed

T <- 100L

df2a <- data.frame(Y = rep(sample(c(-1, 1), T, TRUE, prob = c(0.5, 0.5))))
```

b) (2 credits)

Please generate $N = 100$ sample paths of length $T = 100$ for this binary random walk.

- Please save the results into a data.frame `df2b` where:
  - column `df2b$Y` has the values of the process
  - column `df2b$id` has the id of the sample path
  - column `df2b$t` has the time

```
set.seed(42) # Please do not change the seed

N <- 100L
T <- 100L

df2b <- data.frame(Y = rep(sample(c(-1, 1), N*T, TRUE, prob = c(0.5, 0.5))),
                   id = rep(1:N, each = T),
                   t = rep(1:T, N))
```

```r
df2b$Yt = do.call(c, tapply(df2b$Y, df2b$id, FUN = cumsum))

head(df2b)
```

```
##    Y id t Yt
## 1 -1  1 1 -1
## 2 -1  1 2 -2
## 3  1  1 3 -1
## 4 -1  1 4 -2
## 5 -1  1 5 -3
## 6 -1  1 6 -4
```

c) (1 credit)

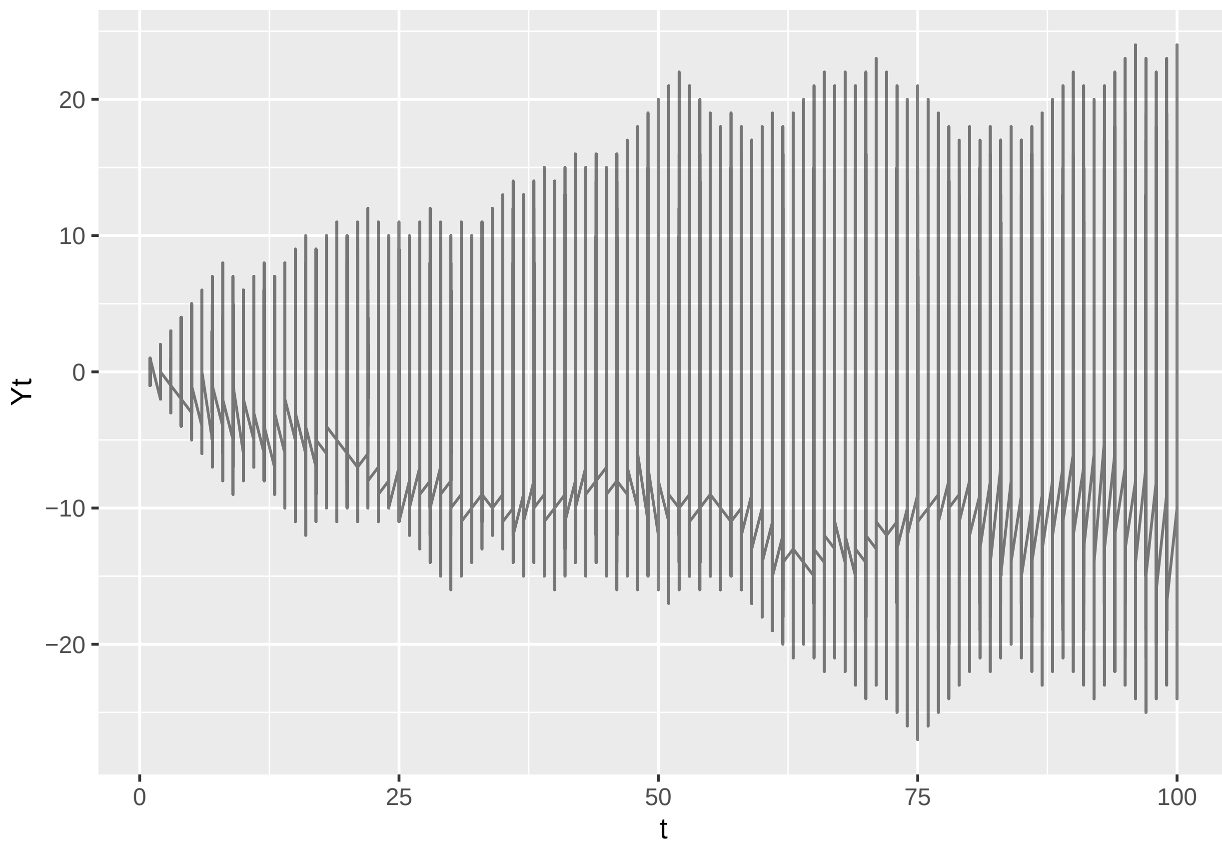
Please plot the sample paths that you generated in the previous question

- Please save your plot into variable `p2c` and plot it

Hints:

- use `ggplot` and take advantage of the long format of the data
- please don't change the color (keep the lines black) but do put `alpha=0.5` into your `geom_line` to make sample paths somewhat transparent.
- do not use `geom_points` just `geom_line` is fine
- As you will see from your plot:
  - the fainter the line the less likely the random walk would reach this spot

```r
p2c <- ggplot(data = df2b, aes(x = t, y = Yt)) +
            geom_line(alpha = 0.5)
p2c
```



d) (1 credit)

Please use the code that you wrote before to generate $N = 5000L$ sample paths.

- Please save the result into `df2d` data.frame following conventions of the prior question
  - column `df2d$Y` has the values of the process
  - column `df2d$id` has the id of the sample path
  - column `df2d$t` has the time

```
set.seed(42) # Please do not change the seed

N <- 5000L
T <- 100L

df2d <- data.frame(Y = rep(sample(c(-1, 1), N*T, TRUE, prob = c(0.5, 0.5))),
                   id = rep(1:N, each = T),
                   t = rep(1:T, N))

df2d$Yt = do.call(c, tapply(df2d$Y, df2d$id, FUN = cumsum))

head(df2d)
```

```
##    Y id t Yt
## 1 -1  1 1 -1
```

```
## 2 -1  1 2 -2
## 3  1  1 3 -1
## 4 -1  1 4 -2
## 5 -1  1 5 -3
## 6 -1  1 6 -4
```

e) (2 credits)

Use the data in `df2d` to numerically verify your analytical results in Question 1 (a,b,c,d)

```
#Please enter your code here first, AND then answer:

## Create Y3 variable by subsetting on t == 3
Y3 = df2d$Yt[df2d$t == 3]

E_Y3 <- mean(Y3)
E_Y3
```

```
## [1] 0.0068
```

```
Var_Y3 <- var(Y3)
Var_Y3
```

```
## [1] 3.013356
```

```
## Create Y2 variable by subsetting on t == 2
Y2 = df2d$Yt[df2d$t == 2]

Cov_Y2Y3 <- cov(Y2, Y3)
Cov_Y2Y3
```

```
## [1] 2.005196
```

```
## Create Y5 and Y10 vars by subsetting on t == 5 and t == 10
Y5 = df2d$Yt[df2d$t == 5]
Y10 = df2d$Yt[df2d$t == 10]

Cov_Y5Y10 <- cov(Y5, Y10)
Cov_Y5Y10
```

```
## [1] 5.225461
```

Hints:

- For Y_3, extract rows corresponding to `[df2d$t==3]`.
- Since 5000 sample paths are used, the results should be very close to the analytical ones

f) (1 credit)

Assume now that you just got a phone call from your friend telling you that this random walk has just been observed to pass through the point $Y_6 = 6$.

- What that means is you know that all first 6 steps of the random walk happened to be in the "up" direction not "down".

In other words, as this new information from your friend became you can refine your forecast: you can remove the sample paths that the process has not taken and only concentrate on the remaining sample paths that it could still take

- This idea is simulation-based forecasting

Please create data.frame `df2e` such that it only has the sample paths (and only those sample paths) from the previous answer that satisfy the condition that $Y_6 = 6$.

Hints:

- use `group_by`
- the cleanest way is to use dplyr's `do()`
- less clean but also doable: you could create a separate data.frame with acceptable sample path ids and then `inner_join` it to the main data.frame to keep only the good sample paths in it.

```
ids = df2d$id[ which(df2d$t == 6 & df2d$Yt == 6) ]

df2e <- df2d[df2d$id %in% ids, ]

head(df2e)
```

```
##      Y id t Yt
## 7601 1 77 1  1
## 7602 1 77 2  2
## 7603 1 77 3  3
## 7604 1 77 4  4
## 7605 1 77 5  5
## 7606 1 77 6  6
```

g) (1 credit)

Please use the data.frame from the previous question to plot your results.

- Please don't change the color (keep the lines black) but do put `alpha=0.5` into your `geom_line` to make sample paths somewhat transparent.
- Please do plot the first 6 time periods as well. Yes, the first 6 steps will be exactly the same for all sample paths (since all other sample paths have been eliminated).

Key takeaways:

- You should see from your plot why people say that random walks have memory and are not mean reverting.

- This idea of sequentially eliminating sample paths as you observe the process in order to forecast the future better is called *filtration*. This idea is fundamental in continuous-time stochastic processes but we will only touch upon it for discrete-time stochastic processes.

```
p2e <- ggplot(data = df2e, aes(x = t, y = Yt)) +
          geom_line(alpha = 0.5)
p2e
```