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Now, it's multivariate integration, but this is best done simply by doing it! Ha!

Consider the joint probability distribution function $f_{XY}(x,y) = \frac{1}{16}xy$ on the region $0 \leq x \leq 2$ and $0 \leq y \leq 4$.

(a) Please show that $f_{XY}(x,y) = \frac{1}{16}xy$ is a proper probability function on the stated interval by showing that $\iint_R f_{XY}(x,y)\text{dxdy} = 1$.

$$\int_0^4 \left[\int_0^2 \frac{xy}{16}\,dx \right] dy =$$

$$① \int_0^2 \frac{xy}{16}\,dx = \frac{yx^2}{32}\Big|_0^2 = \frac{y(2)^2}{32} - \frac{\cancel{y(0)^2}}{\cancel{32}} = \frac{4y}{32} = \boxed{\frac{y}{8}}$$

$$② \int_0^4 \frac{y}{8}\,dy = \frac{y^2}{16}\Big|_0^4 = \frac{(4)^2}{16} - \frac{\cancel{(0)^2}}{\cancel{16}} = \frac{16}{16} = \boxed{1}$$ ✓

$$\boxed{f_{XY}(x,y)\,dxdy = \frac{16}{16} = 1}$$

(b) If the joint probability density function of a continuous random variables X and Y is given by

$f_{XY}(x,y)$, we know the marginal probability density functions of X and Y are

$$f_X(x) = \int_y f_{XY}(x,y)dy \quad \text{and} \quad f_Y(y) = \int_x f_{XY}(x,y)dx.$$

Please find $f_X(x)$.

$$\int_0^4 \frac{xy}{16}\,dy = \int_0^4 \frac{y^2 x}{32}\Big|_0^4 = \frac{(4)^2 x}{32} - \cancel{\frac{(0)^2 x}{32}} = \frac{16x}{32} = \boxed{\frac{x}{2}}$$

$$\boxed{f_X(x) = \frac{x}{2}}$$

(b) continued

Please find $f_Y(y)$.

$$\int_0^2 \frac{xy}{16}\,dx = \int_0^2 \frac{yx^2}{32}\Big|_0^2 = \frac{y(2)^2}{32} - \cancel{\frac{y(0)^2}{32}} = \frac{4y}{32} = \boxed{\frac{y}{8}}$$

$$\boxed{f_Y(y) = \frac{y}{8}}$$

$f(x,y) = \frac{xy}{16}$ $\qquad xy \cdot \frac{xy}{16} = \frac{x^2y^2}{16}$

$\uparrow$

(c) Please compute $E(XY) = \iint_R xyf_{XY}(x,y)\mathrm{d}x\mathrm{d}y$

$$\int_0^4 \left[\int_0^2 \frac{x^2y^2}{16}\,dx\right]dy = ① \int_0^2 \frac{y^2x^3}{48}\Big|_0^2 = \frac{y^2(2)^3}{48} - \cancel{\frac{y^2(0)^3}{48}} = \frac{8y^2}{48} = \boxed{\frac{y^2}{6}}$$

$$② \int_0^4 \frac{y^2}{6}\,dy = \int_0^4 \frac{y^3}{18}\Big|_0^4 = \frac{(4)^3}{18} - \cancel{\frac{(0)^3}{18}} = \frac{64}{18} = \boxed{\frac{32}{9}}$$

$$\boxed{E(X,Y) = \frac{32}{9}}$$

$xf(xy) = \frac{x^2y}{16}$

(d) Please compute $\mu_x = E(X) = \iint_R xf_{XY}(x,y)\mathrm{d}x\mathrm{d}y$

$$\int_0^4 \left[\int_0^2 \frac{x^2y}{16}\,dx\right]dy = ① \int_0^2 \frac{x^2y}{16}\,dx = \int_0^2 \frac{yx^3}{48}\Big|_0^2 \frac{y(2)^3}{48} - \cancel{\frac{y(0)^3}{48}} = \frac{8y}{48}$$

$$= \boxed{\frac{y}{6}}$$

$$② \int_0^4 \frac{y}{6}\,dy = \int_0^4 \frac{y^2}{12}\Big|_0^4 \frac{(4)^2}{12} - \cancel{\frac{(0)^2}{12}} = \frac{16}{12} = \boxed{\frac{4}{3}}$$

$$\boxed{\mu_x = E(X) = \frac{4}{3}}$$

$$Yf(xy) = \frac{Y^2 x}{16}$$

(e) Please compute $\mu_y = E(Y) = \iint_R y f_{XY}(x, y) \mathrm{dxdy}$

$$\int_0^4 \left[\int_0^2 \frac{y^2 x}{16} dx \right] dy = \text{①} \int_0^2 \frac{y^2 x}{16} dx = \int_0^2 \frac{y^2 x^2}{32} \Big|_0^2 = \frac{(y^2)(2)^2}{32} - \frac{\cancel{(y^2)(0)}}{\cancel{32}} = \frac{4y^2}{32}$$

$$= \frac{y^2}{8}$$

$$\text{②} \int_0^4 \frac{y^2}{8} dy = \int_0^4 \frac{y^3}{24} \Big|_0^4 = \frac{(4)^3}{24} - \frac{\cancel{(0)^3}}{\cancel{24}} = \frac{64}{24} = \boxed{\frac{8}{3}}$$

$$\text{③} \boxed{\mu_y = E(Y) = \frac{8}{3}}$$

(f) The covariance between random variables X and Y, denoted as cov(X,Y) or σ_{XY}, is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \boxed{E(XY) - \mu_X\mu_Y.}$$

Recall that $E(XY) - \mu_X\mu_Y$ will be $\iint_R xyf_{XY}(x,y)\text{dxdy} - \mu_X\mu_Y$ which you have in theory already computed ☺.

$E(X,Y) = \frac{32}{9}$ $\mu_X = \frac{4}{3}$

Please compute cov(X,Y) =

$\mu_Y = \frac{8}{3}$

$\text{cov}(X,Y) = E(X,Y) - \mu_X\mu_Y$

$= \frac{32}{9} - \left(\frac{4}{3}\right)\left(\frac{8}{3}\right) = \frac{32}{9} - \frac{32}{9} = \boxed{0}$

$$\boxed{\text{cov}(X,Y) = 0}$$

(g) The correlation between the random variables X and Y, denoted by ρ_{XY} is given by

$$\rho_{XY} = \frac{cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

where Var(X) = $\int_R (x - \mu_x)^2 f_X(x)dx$ and

Var(Y) = $\int_R (y - \mu_y)^2 f_Y(y)dy$.

$f_X(x) = \frac{x}{2}$ $\quad E(x) = \frac{4}{3}$

Please compute ρ_{XY}.

$Var(x) = \int_0^2 x^2 f_X(x)\,dx = \int_0^2 \frac{x^3}{2} - [E(x)]^2$

$\int_0^2 \frac{x^3}{2}dx = \int_0^2 \frac{x^4}{8}\Big|_0^2 = \frac{(2)^4}{8} - \frac{(0)^4}{8} = \frac{16}{8} = 2$

$f_Y(y) = \frac{y}{8}$ $\quad 2 - \left[\frac{4}{3}\right]^2 = 2 - \frac{16}{9} = \frac{18}{9} - \frac{16}{9} = \boxed{\frac{2}{9}}$ $\boxed{Var(x) = \frac{2}{9}}$

$Var(y) = \int_0^4 y^2 f_Y(y)\,dy = \int_0^4 \frac{y^3}{8} - [E(y)]^2$

$\int_0^4 \frac{y^3}{8}dy = \int_0^4 \frac{y^4}{32}\Big|_0^4 = \frac{(4)^4}{32} - \frac{(0)^4}{32} = \frac{256}{32} = 8$

$Var(y) = 8 - \left[\frac{8}{3}\right]^2 = 8 - \frac{64}{9} = \frac{72}{9} - \frac{64}{9} = \boxed{\frac{8}{9}}$ $\boxed{Var(y) = \frac{8}{9}}$

Holy Hot Buckets Batman!! We survived ☺ YESSSSS

$P_{xy} = \frac{cov(x,y)}{\sqrt{Var(x)}\sqrt{Var(y)}} = \frac{0}{\sqrt{\frac{2}{9}} \cdot \sqrt{\frac{8}{9}}} = 0$

$\boxed{P_{xy} = 0}$

THANK YOU FOR EVERYTHING THIS SEMESTER