Prove Beta-Binomial conjugation.

Def. Let Φ be a prametric family. A prior $P(\theta)$ belonging to Φ is said to be conjugate for the likelihood $p(x|\theta)$ if and only if the posterior $p(\theta|x)$ belongs to Φ

Pf. Suppose the prior $P(\theta) = Beta(\mu | a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}$,

where a, b are hyperparameters and $\Gamma(x) = \int_0^{\infty} p^{x-1} e^{-p} dp$.

and the likelihood $P(x|\theta) = Bin(x|N, \mu)$, where

N is the number of samples

By Bayes theorem, $P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)}$

$$\therefore P(\theta|x) = \frac{P(x|\theta)\, P(\theta)}{\int_\theta P(x|\theta)\, P(\theta)\, d\theta}$$

$$= \frac{\binom{N}{x}\theta^x(1-\theta)^{N-x} \frac{1}{B(a,b)} \theta^{a-1}(1-\theta)^{b-1}}{\int_0^1 \binom{N}{x}\theta^x(1-\theta)^{N-x}\frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1} d\theta}, \quad \frac{1}{B(a,b)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{\binom{N}{x}\frac{1}{B(a,b)}\, \theta^{x+a-1}(1-\theta)^{N+b-x-1}}{\binom{N}{x}\frac{1}{B(a,b)} \int_0^1 \theta^{x+a-1}(1-\theta)^{N+b-x-1} d\theta}$$

$$= Beta(x+a, N+b-x)$$

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