Introduction to Algorithms and Data Structures

Lecture 22: Parsing for context-free languages

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The parsing problem

Last time, we saw what a context-free grammar was.

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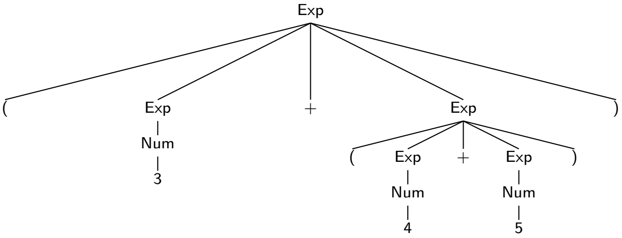
. . . to a tree



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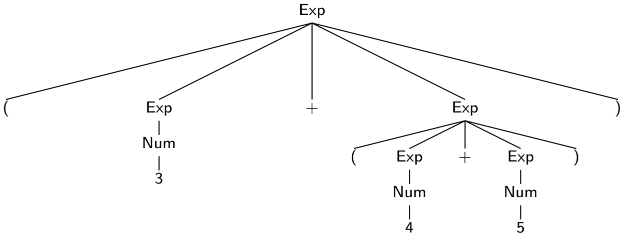

Often an essential prelude to other tasks (e.g. evaluating an expression!)

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- CYK parses inputs of length n in time $\Theta(n^3)$. Fine for short sentences, but not practical for long computer programs. Next time, we'll look at parsing algorithms better suited to computer languages: less general, but faster.

What's Chomsky normal form?

Recall that in a CFG, the right-hand side of each production is a (possibly empty) string of terminals and non-terminals. E.g.

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- *or* just one terminal (e.g. $X \rightarrow +$).

We'll see soon what this curious restriction buys us.
Most important point is that RHSs with ≥ 3 symbols are forbidden.

Chomsky normal form: example

The following grammar is in CNF.

Terminals:	book, orange, heavy, my, very
Non-terminals:	NP, Nom, AP, A, Det, Adv
Start symbol:	NP

$$
\begin{array}{rcl}
\mathsf{NP} & \rightarrow & \mathsf{Det\ Nom} \\
\mathsf{Nom} & \rightarrow & \mathsf{book} \mid \mathsf{orange} \mid \mathsf{AP\ Nom} \\
\mathsf{AP} & \rightarrow & \mathsf{heavy} \mid \mathsf{orange} \mid \mathsf{Adv\ A} \\
\mathsf{A} & \rightarrow & \mathsf{heavy} \mid \mathsf{orange} \\
\mathsf{Det} & \rightarrow & \mathsf{my} \\
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Generates noun phrases like:

my very heavy orange   my very heavy orange book

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\text{A} & \rightarrow & \text{heavy} \mid \text{orange} \\
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(N.B. CNF grammars often involve some duplication!
Writing AP $\rightarrow$ A would be simpler, but not CNF.)

CYK parsing: the idea

Let's insert 'position markers' in the input string we wish to parse:

$_0$ my $_1$ very $_2$ heavy $_3$ orange $_4$ book $_5$

We can then talk about substrings of the input: e.g. the pair (2,4) indicates the substring 'heavy orange'.

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Broadly speaking, we work our way from shorter to longer substrings (some flexibility re precise ordering of subproblems).

Filling out the CYK chart: example

NP	$\rightarrow$	Det Nom	A	$\rightarrow$	heavy | orange
Nom	$\rightarrow$	book | orange | AP Nom	Det	$\rightarrow$	my
AP	$\rightarrow$	heavy | orange | Adv A	Adv	$\rightarrow$	very

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CYK: The general algorithm

```
CYK (s,G):                    # s=input string, G=CNF grammar
    n = length(s)
    allocate table[0,...,n−1][1,...,n]
    for j = 1 to n            # columns
        for (X → t) ∈ G
            if t = s[j−1]
                add X to table[j−1,j]     # diagonal cell
        for i = j−2 downto 0              # rows
            for k = i+1 to j−1            # possible splits
                for (X → YZ) ∈ G
                    if Y ∈ table[i,k] and Z ∈ table [k,j]
                        add X to table[i,j]     # non-diagonal cell
    return table
```

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- Changing this to a parser requires recording which existing constituents were combined to make each new constituent.

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j i	1 my	2 very	3 heavy	4 orange	5 book
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2 heavy			A,AP	Nom	Nom
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4 book					Nom

- The algorithm identifies all possible parses.
  There may also be phantom constituents that don't form part of any complete syntax tree (e.g. 'my very heavy orange').

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To fill a cell (i, j), we'd need to consider all possible three-way splits $(i, k), (k, l), (l, j)$ where $i < k < l < j$.
Number of these is quadratic in $j - i$.
So our overall runtime would go up to $\Theta(n^4)$.

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Number of these is quadratic in $j - i$.
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That's the main reason we like Chomsky normal form
(there are other minor benefits).

More on Chomsky normal form

Recall: a context-free grammar $G = (\Sigma, N, S, P)$ is in Chomsky normal form (CNF) if all productions are of the form

$$A \;\rightarrow\; BC \quad \text{or} \quad A \;\rightarrow\; a \qquad (A, B, C \in N,\; a \in \Sigma)$$

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Theorem: Disregarding the empty string, every CFG G is equivalent to a grammar G′ in Chomsky normal form. $(\mathbf{L}(\mathsf{G}') = \mathbf{L}(\mathsf{G}) - \{\epsilon\})$
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Key idea: To eliminate rules with ≥ 3 symbols on the RHS, we could replace e.g.

$$X \rightarrow ABCD \quad \text{by} \quad X \rightarrow AY,\; Y \rightarrow BZ,\; Z \rightarrow CD$$

where Y, Z are newly added nonterminals.

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Consider for example the grammar

$$S \rightarrow TT \mid [S] \qquad\qquad T \rightarrow \epsilon \mid (T)$$

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Step 1: Apply trick on last slide to rules with ≥ 3 symbols on RHS.
In this case, apply it to $S \to [S]$ and $T \to (T)$:

$$S \;\to\; TT \;\mid\; [W \qquad\qquad T \;\to\; \epsilon \;\mid\; (V$$

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In this case, $T \rightarrow \epsilon$ tells us $T \in E$. Then $S \rightarrow TT$ tells us $S \in E$. And that's all. So $E = \{S, T\}$.

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In general, E is the smallest set such that if $X \rightarrow Y_1 \ldots Y_r \in P$ and $Y_1, \ldots, Y_r \in E$ then $X \in E$ (allowing $r = 0$ here).

Converting to Chomsky Normal Form, ctd.

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Step 3: Delete all ϵ-productions.
To compensate, for each rule $X \to Y\alpha$ or $X \to \alpha Y$, where $Y \in E$ and $\alpha \neq \epsilon$, add a new rule $X \to \alpha$.

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To compensate, for each rule $X \rightarrow Y\alpha$ or $X \rightarrow \alpha Y$, where $Y \in E$ and $\alpha \neq \epsilon$, add a new rule $X \rightarrow \alpha$.

In this case, since $E = \{S, T\}$, we get:

$$S \rightarrow TT \mid T \mid [W \qquad T \rightarrow (V$$
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Step 4: Remove unit productions $X \rightarrow Y$.
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In this case, do this for $S \rightarrow T$:

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By this stage, all RHSs consist of 1 terminal or 2 symbols. So just need to get rid of terminals from the 'binary' rules.

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And rewrite the existing rules to:

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$$S \to TT \mid (V \mid [W \qquad T \to (V$$
$$W \to S] \mid] \qquad V \to T) \mid)$$

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The grammar is now in Chomsky Normal Form, and we're done.

Assorted remarks

- Given a CFG $\mathcal{G}$, we can do the above (once for all) to convert it to a CNF grammar $\mathcal{G}'$, then run CYK for $\mathcal{G}'$ (many times).

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- Unfortunately, this is an example of an NP-hard problem. No known polynomial-time algorithm (i.e. one with runtime $O(n^d)$ for some d).
- Versions of CYK are quite widely used in Natural Language context (where sentences typically have < 100 words). But $\Theta(n^3)$ parsing not good enough for computer languages.

Reading

Recommended: D. Jurafsky and J.H. Martin,
Speech and Language Processing, 3rd ed. (draft).
Chapter 13 (Constituency parsing), Sections 1 and 2.
Available at `https://web.stanford.edu/~jurafsky/slp3`