

Introduction to Algorithms and Data Structures

Lecture 17: Introduction to Dynamic Programming

Mary Cryan

School of Informatics
University of Edinburgh

Optimization

Sem1: data structures, searching, sorting, graphs and graph algorithms.

Sem2: harder computational questions; “optimization” not just yes/no, applications to grammar problems.

In optimization, we don't just want to ask yes/no questions ... we want to find the “best” (in some way) solution.

- ▶ Can we do this efficiently?
(maybe - depends on the specifics of the problem)
- ▶ “Efficient” usually means “polynomial-time”
($\Theta(n)$, or $\Theta(n \cdot \lg(n))$, or $\Theta(n^3)$ or ...)
- ▶ Some optimization problems have polynomial-time algorithms via the method of **Dynamic Programming**

Divide and Conquer

The **Divide and Conquer** technique is when we design an algorithm to solve a problem by taking an **instance (or input)** I (of size n), then

1. Doing some preprocessing with I to construct some number of smaller sub-problems on smaller instances I_1, \dots, I_k ;
2. Making k **recursive calls** to compute the answer for the sub-problems;
3. Take the answers from 2. and do some computation to get the overall answer for the original input I .

Details (of the number of subproblems k , how to combine answers etc) will **vary from problem to problem**.

In some cases, Divide-and-Conquer can directly give an efficient (polynomial-time) algorithm - for example Mergesort, Quicksort.

Master theorem often features in the analysis.

But the recursive method is not always efficient.

Fibonacci numbers - a toy example

The **Fibonacci numbers** are defined as

$$\begin{aligned}F_0 &= 0, \\F_1 &= 1, \\F_n &= F_{n-1} + F_{n-2} \quad (\text{for } n \geq 2).\end{aligned}$$

There is an immediate **recursive algorithm**:

Algorithm Rec-Fib(n)

1. **if** $n = 0$ **then**
2. **return** 0
3. **else if** $n = 1$ **then**
4. **return** 1
5. **else**
6. **return** REC-FIB($n - 1$) + REC-FIB($n - 2$)

Ridiculously slow: **exponentially many** repeated computations of REC-FIB(j) for small values of j .

Fibonacci numbers (cont'd)

Dynamic Programming Approach

Algorithm Dyn-Fib(n)

1. $F[0] = 0$
2. $F[1] = 1$
3. **for** $i \leftarrow 2$ **to** n **do**
4. $F[i] \leftarrow F[i - 1] + F[i - 2]$
5. **return** $F[n]$

Build “from the bottom up”.

We are “turning recursion upside down”.

Running Time is $\Theta(n)$

Very fast in practice - just need an array (of linear size) to store the $F(i)$ values (in fact don't even need that array ...)

Implementing in



The plain recursive implementation: SLOW!

```
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n-1)+fib(n-2)
```

Dynamic programming implementation with a 1-dimensional array

```
def fibDP(n):
    F = [0]*(n+1)
    F[1] = 1
    # The range will be empty if n is 0 or 1
    for i in range(n-1):
        F[i+2] = F[i+1]+F[i]
    return F[n]
```

Test these on the value 44 (say) to see the difference.

Decorators in



Can get the benefit of the Dynamic Programming via ad-hoc [memoization](#).

The plain recursive implementation:

```
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n-1)+fib(n-2)
```

```
def memoize(f):
    memo = {}
    def check(s):
        if s not in memo:
            memo[s]=f(s)
        return memo[s]
    return check
```

```
# now make 'memoize' a Decorator for 'fib'
fib = memoize(fib)
```


The coin-changing problem



In the UK coins have denominations 1p, 2p, 5p, 10p, 20p, 50p, £1 and £2.

A frequently-executed task in the retail sector involves taking an input value (say 88p) and calculating a collection of coins (may include duplicates) which will sum to that value.

We assume an unlimited supply of coins of each value.

The coin-changing problem



The *coin changing* problem is the problem, given an input value v ($v \in \mathbb{N}_0$) of calculating a collection of coins (of minimum cardinality) that will sum to v .

This is an *optimization* problem, as we want a solution with as few individual coins as possible (we want to *minimize* the number of coins handed back).

Want to do this for *arbitrary systems of coin denominations*.

The coin-changing problem

Given: A value $v \in \mathbb{N}_0$, plus a sequence of coin values $c_1, c_2, \dots, c_k \in \mathbb{N}_0$ (these representing the denominations of the relevant system).

Output: A **multiset** S of coins whose values sum to v , whose cardinality (size) of S is the minimum possible for v in this coin system.

Return solution as an array S of length k with $S[i]$ being the number of coins of value c_{i+1} for this optimal solution, for each $0 \leq i \leq k - 1$.

We will now develop a **recurrence** for the (optimal) solution.

We assume some solution definitely exists
(assuming $c_1 = 1$ is enough to ensure this)

The coin-changing problem

Let $C(v)$ denote the **number of coins** in an optimal solution for value $v \in \mathbb{N}_0$ (with respect to denominations $c_1, c_2, \dots, c_k \in \mathbb{N}_0$).

Observation:

Suppose we have an optimal solution S for our given value v (with respect to our coin values c_1, \dots, c_k), with optimal count $C(v)$.

*Then there is **some** initial coin i (maybe, take the lowest one in the solution) which contributes to this solution.*

Then $C(v) = 1 + C(v - c_i)$ for this coin.

We will not know which coin c_i is definitely in the optimal solution.

However we can write

$$C(v) = \begin{cases} 1 & v = c_i \text{ for some } 1 \leq i \leq k \\ 1 + \min\{C(v - c_i) : 1 \leq i \leq k, c_i < v\} & \text{otherwise} \end{cases}$$

coin-changing: the algorithm

The recurrence helps me describe the solution (for v) in terms of other values, but that only helps if I already know the value of $C(v - c_i)$ for the various c_i . Would need to have precomputed those.

Solution:

- ▶ We will **expand** our Objective to computing $C(w)$ for **every w from 1 to v** .
- ▶ We will have an array C of length $v + 1$, and $C[w]$ will be computed as the “minimum number of coins to make w ” for each w .
- ▶ We will **compute the solution for small values of w first**.
- ▶ It will help to have an extra array P to store the “coin used to get the best answer” for each w (to know how to reconstruct).
- ▶ At the end we will also use the arrays C and P to build the list of coin values for v (smaller array S of length k).

coin-changing by dynamic programming: example

Consider the case of coin values 1, 5, 7, and the change-value $v = 18$.

Dynamic programming algorithm

Algorithm Dyn-Coins($v; c_1, \dots, c_k$)

1. initialise array c of length k to hold the c_i values
2. initialise array S of length k (to 0s)
3. initialise arrays C, P of length $v + 1$ (to ∞)
4. $C[0] \leftarrow 0, C[1] \leftarrow 1, P[1] \leftarrow 0$ // Assume $c_1 = c[0] = 1$
5. **for** $w \leftarrow 2$ **to** v // We work “bottom-up”
6. **for** $i = 0$ **to** $k - 1$ // We try all coin values
7. **if** ($c[i] \leq w$) **and** ($C[w - c[i]] + 1 < C[w]$)
8. $C[w] \leftarrow 1 + C[w - c[i]]$
9. $P[w] \leftarrow i$
10. **while** $v > 0$ // Now we work back to build S
11. $i \leftarrow P[v]$
12. $S[i] \leftarrow S[i] + 1; v \leftarrow v - c[i]$
13. **return** $C[v]$ “is the number of coins. The solution is in array S ”.

Other options?

Recursive implementation:

- ▶ A straightforward recursive implementation will show repeated subproblems, as in the case of Fibonacci (though it is less immediate)
- ▶ Even with some simple optimizations (like putting an order on considering the “next coin”), still we will get repetitions.
- ▶ So there is redundancy in a naïve implementation of the recurrence on slide 10.

A “greedy” algorithm:

- ▶ The greedy algorithm always chooses the coin of max-value (less than remaining value)
- ▶ A very natural heuristic which will work on many coin systems.
- ▶ This is *not* guaranteed to be optimal for all systems - try the system with coin values 1, 5, 7 for the value $v = 18$

Dynamic Programming principles (in general)

4 (related) features we need in order to design an (efficient) Dynamic Programming algorithm:

(dp1) Need is to see that computing the optimum solution for our original instance can be achieved by finding solutions to (smaller) problems of the same type, and combining them.

(Sometimes we will have need to **generalise** the way we define the problem for this; eg, with **coin changing**, we think about computing the best solution for every value from 1 to v).

(dp2) Closely related to (dp1), we need the solution to an instance of the problem to be expressible in terms of a **recurrence**, where the right-hand side contains one or more recursive calls for smaller instances of the same problem.

(for **coin-changing**, this was the recurrence on slide 11)

Dynamic Programming principles (in general)

(dp3) We need to be able to organise storage for the results for all possible subproblems (identified in dp1/dp2) which will be solved. It should be possible to store the subproblem results in a table with meaningful indexing - hopefully, 1-dimensional or 2-dimensional. (there are some problems which rely on 3-dimensional (or greater) tables. That's ok, as long as the space is "polynomially bounded").

(for [coin-changing](#), we use 1-dim arrays of length w and k)

(dp4) We need an algorithm to control the *order* in which subproblems are solved (and results stored in the appropriate cell of the table). This must be done so that all of the subproblems appearing on the right-hand side of the recurrence must be computed and in-the-table *in advance* of computing the left-hand side.

(for [coin-changing](#), all of the sub-problems on the rhs of the recurrence have $w < v$, so the solution is already pre-computed)

Reading and Working

Reading:

- ▶ Stuart Dreyfus' 2002 article on the history of Dynamic Programming and why it was named "Dynamic Programming":
<https://pubsonline.informs.org/doi/pdf/10.1287/opre.50.1.48.17791>
- ▶ Dynamic Programming Algorithm for coin-changing is due to J.W. Wright, "The Change Making Problem", *Journal of the ACM*, 1975.
<https://dl.acm.org/doi/10.1145/321864.321874>
- ▶ Section 16.4 of "Algorithms Illuminated" by Tom Roughgarden discusses the Principles of Dynamic programming in a similar way to us.
- ▶ Section 15.4 of [CLRS] gives a dynamic programming algorithm for the *longest common subsequence* problem. This is similar to the *edit distance* problem, which we will cover in Lecture 18.