

hw6

July 7, 2025

1 Homework 6 (Due 21 Mar)

Due 21 Mar (midnight)

Total points: 100

```
[5]: import numpy as np
from math import *
import matplotlib.pyplot as plt
import pandas as pd
%matplotlib inline
plt.style.use('seaborn-v0_8-colorblind')
```

1.1 Introduction to Homework 6

This week's exercises focus on oscillators and how to approximate the solution to the equations of motion using the SHO. The relevant reading background is: 1. chapter 5 of Taylor

2. chapter 6.6-6.8 of Boas
3. chapters 2.0-2.3 and 5.0-5.2 of Strogatz

In both textbooks there are many nice worked out examples.

1.1.1 Practicalities about homeworks and projects

1. You can work in groups (optimal groups are often 2-3 people) or by yourself. If you work as a group you can hand in one answer only if you wish. **Remember to write your name(s)!**
2. Homeworks are available ten days before the deadline.
3. How do I(we) hand in? You can hand in the paper and pencil exercises as a **single scanned PDF document**. For this homework this applies to exercises 1-5. Your jupyter notebook file should be converted to a **PDF** file, attached to the same PDF file as for the pencil and paper exercises. All files should be uploaded to Gradescope.

Instructions for submitting to Gradescope.

1.1.2 Exercise 1 (10pt) Morse Potential as an SHO

If the potential has a local minimum, we can often find SHO approximation for that potential near the local minimum.

The [Morse potential](#) is a convenient model for the potential energy of a diatomic molecule. The potential is a radial one and thus one-dimensional. It is given by,

$$U(r) = A \left[\left(e^{(R-r)/S} - 1 \right)^2 - 1 \right]$$

where the distance between the centers of the two atoms is r , and the constants A , R , and S are all positive. Here $S \ll R$.

- 1a (2pt) Sketch (or plot) the potential as a function of r .
- 1b (3pt) Find the equilibrium position of the potential, i.e. the position where the potential is at a minimum. We will call this r_e .
- 1c (3pt) Rewrite the potential in terms of the displacement from equilibrium, $r = r_e + x$. Expand the potential to second order in x .
- 1d (2pt) Find the effective spring constant, k , for the potential near the minimum. What is the frequency of small oscillations about the minimum?

1.1.3 Exercise 2 (10pt), Time Averaging and the SHO

Time Averaging is a common tool to use with periodic systems. It also us to discuss what happens to different properties of the system over one period.

An SHO has a period τ . We can find the time average of a variable $f(t)$ over one period, $\langle f \rangle$, by averaging over the period,

$$\langle f \rangle = \frac{1}{\tau} \int_0^\tau f(t) dt$$

- 2a (2pt) Show that the time average of the position of the SHO is zero.
- 2b (2pt) Show that the time average of the velocity of the SHO is zero.
- 2c (6pt) Show that the time average of the kinetic energy of the SHO is equal to the time average of the potential energy (and importantly, non-zero). If the total energy is E , these time averages are equal to $E/2$. You might need to show that the very useful trigonometric identity,

$$\langle \sin^2(\omega t - \delta) \rangle = \langle \cos^2(\omega t - \delta) \rangle = \frac{1}{2}$$

1.1.4 Exercise 3 (10pt), Toy Potential

Consider a toy potential of the form,

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right)$$

where U_0 , R , and λ are all positive constants and the domain of the potential is $0 < r < \infty$.

- 3a (2pt) Sketch (or plot) the potential as a function of r .
- 3b (3pt) Find the equilibrium position of the potential, i.e. the position where the potential is at a minimum. We will call this r_e .

- 3c (5pt) Rewrite the potential in terms of the displacement from equilibrium, $r = r_e + x$. Expand the potential to second order in x . What is the effective spring constant, k , for the potential near the minimum? What is the frequency of small oscillations about the minimum?

1.1.5 Exercise 4 (10pt), Defining Periodicity

A common issue with oscillators is determining their periodicity. For the SHO, we can show that the period is $2\pi/\omega_0$ where ω_0 is the natural frequency of the SHO. How might we define a periodicity more generally? Let's start with the damped harmonic oscillator. Consider a weakly damped oscillator ($\beta < \omega_0$). The motion of the oscillator is given by,

$$x(t) = Ae^{-\beta t} \cos(\omega_1 t - \delta)$$

where A is the amplitude, β is the damping constant, $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and δ is the phase.

The motion decays with time, but we can still define a periodicity, τ_1 , which is the time between peaks in the motion.

- 4a (4pt) Sketch (or plot) the motion of the oscillator as a function of time. Show that you can find $\tau_1 = 2\pi/\omega_1$ from looking at successive maxima.
- 4b (3pt) Show that an equivalent definition of τ_1 is twice the time between zero crossings of the motion.
- 4c (3pt) If the damping β is half the natural frequency (ω_0), how does the amplitude of the motion decay in one period?

1.1.6 Exercise 5 (10pt), Planning your Final Project

The point of this exercise is to get you thinking about what you want to do for a final project and how you will accomplish it. Research and project planning are important skills to develop and it takes time to do so. We are trying to scaffold that development and give you the opportunity to practice these skills. So, you will be responsible for developing a plan for your final project, and the timeline under which you will accomplish it. Our job is to provide feedback and guidance on your plan, and to offer support in the development and construction of your final project.

Remember there is no right or wrong way to do this, it is about developing a plan that works for you.

Your final project is your chance to bring all the tools we have developed to bear on a question that interests you. You have reviewed the information on computational essays before, but here it is again in case it's useful for this exercise:

1. Wolfram's [What is a Computational Essay?](#)
2. Tor and my short paper: [Computational Essays: An Avenue for Scientific Creativity in Physics](#)
3. Wolfram's [Steps to Writing a Computational Essay](#)
4. [University of Oslo's Computational Essay Showroom](#)

For your final project, you may work alone, in pairs, or in groups of up to three people. The expectation for your work will be higher if you have more people in your group; your explorations will need to be more comprehensive and detailed, and your essay will be expected to be longer and more polished if you have more people in your group. You can also develop more interesting ways

of presenting your work that demonstrate the creativity and depth of your explorations if you have more people in your group. If you have any questions, just ask.

For this exercise, we want you to get started on planning your final project by sending us a brief proposal with a timeline. To help you, we have provided a schedule where you can see the checkpoints for your project, and when we will have a comprehensive work week. You should take advantage of the work week to make significant progress on your project; especially if there are technical aspects you are unsure about.

Schedule for Final Project

- 28 March - HW 7 will have a request for your 1st project update.
- 11 Apr - Midterm Project 2 will have a request for your 2nd project update.
- 18 Apr - HW 8 will have a request for your 3rd project update.
- Week of 20-24 Apr - Project work week in-class.
- 28 Apr - Final project due.

Your Proposal Your proposal should be a one to two page, single spaced, document that outlines the following:

1. The question you are interested in exploring. The motivation and interest to you and your group mates should be made clear. That is, motivate the question, why is it interesting?
2. References that provide background for this work. This should include the sites, resources, and references you have consulted to get started.
3. The approach you think you need to take to answer this question. Include the concepts and tools you plan to use to explore the question. That is, what have we developed in class that you will use to explore the question?
4. The timeline for your project. This should include the checkpoints in the schedule above, and any additional milestones you think are important. Consider making a [Gantt chart](#) to help you plan your project. This is a useful tool for planning and tracking your progress. It's almost expected for proposals in science and engineering. Here, We suggest you take advantage of the work week to make significant progress on or to polish your project.

Your writing You should consider this a formal proposal. It should be well written, clear, and concise. You should also consider the audience for this proposal. You should write this proposal for someone who is not in this class, but does know physics. Think about writing this to another student, a graduate student, or a professor. Someone that has a good understanding of physics, but not necessarily the details of your physics. You need to explain the motivation, the physics background, the approach, and the timeline in a way that is compelling and clear.

Your proposal need not be long, but it should be complete. Here are some additional resources for you to develop a good proposal:

- [NIH, How to write a research proposal?](#)
- [University of Sheffield, How to write a research proposal](#)

Reference pages do not count towards the length, so if you have a lot of references, that's fine. But the main text should be one to two pages. This is how it works with most proposals in science and engineering.

1.1.7 Exercise 6 (50 pt), Find your own 1D Oscillator

We have built all the tools to study 1D unforced oscillators. Now you get to pick your own potential and study it. You can pick any 1D potential you like, but it should have a local minimum. Make sure it is not a driven oscillator (i.e., no explicit time dependence in the equations of motion). To earn full credit for this exercise, you must:

- 6a (5pt) Present the potential and describe its origin, why it is interesting, where it comes from, etc. Educate us about it.
- 6b (5pt) Sketch (or plot) the potential as a function of it's argument (and chosen variables) and find the equilibrium position of the potential, i.e. the position where the potential is at a minimum.
- 6c (10pt) Rewrite the potential in terms of the displacement from equilibrium. Expand the potential to second order to find the effective spring constant, k , for the potential near the minimum. What is the frequency of small oscillations about the minimum?
- 6d (10pt) Construct the equations of motion for the potential and solve them numerically. Choose initial conditions and parameters that give oscillatory motion. Note it doesn't have to be SHO (In fact, it probably won't be). Plot the position as a function of time. Make sure we can see the oscillations.
- 6e (10pt) Plot the phase diagram of the trajectory (you don't have to produce a phase diagram, but just plot the trajectory in phase space). What does the phase diagram tell you about the motion?
- 6f (10pt) Find the period of your motion. Here you might have to make some definitions of what periodicity means for your potential.

Note: this might seem similar to your midterm, but notice we expect you to do some research on the potential in 6a, and we are going into more depth with questions 6e and 6f. This is also a scaffold for your final project in terms of practicing aspects that should appear.

Examples of 1D potentials

Simple Pendulum (Nonlinear Small Angle Approximation)

$$V(\theta) = mgh(1 - \cos(\theta))$$

where m is the mass, g is the acceleration due to gravity, h is the length of the pendulum, and θ is the angular displacement.

Nonlinear Spring (Hardening or Softening)

$$V(x) = \frac{k}{2}x^2 + \frac{\beta}{3}x^3$$

where k and β are constants. Depending on the sign of β , the spring can exhibit hardening ($\beta > 0$) or softening ($\beta < 0$) nonlinearity.

Lennard-Jones Potential Oscillator (for a diatomic molecule model):

$$V(r) = 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^6 \right]$$

where ϵ is the depth of the potential well, σ is the finite distance at which the inter-particle potential is zero, and r is the distance between particles.

Morse Potential (for molecular vibrations)

$$V(x) = D_e \left(1 - e^{-a(x-x_0)}\right)^2$$

where D_e is the depth of the potential well, a is a constant related to the width of the well, x is the displacement from equilibrium, and x_0 is the equilibrium bond length. This potential models the energy of a diatomic molecule as a function of the distance between atoms, showing oscillatory behavior that represents molecular vibrations.

Double Well Potential

$$V(x) = -\frac{\mu}{2}x^2 + \frac{\lambda}{4}x^4$$

where μ and λ are positive constants. This system exhibits bistability with two stable equilibria, leading to interesting nonlinear dynamics and potential oscillations between the wells under certain conditions.

1.1.8 Extra Credit - Integrating Classwork With Research

This opportunity will allow you to earn up to 5 extra credit points on a Homework per week. These points can push you above 100% or help make up for missed exercises. In order to earn all points you must:

1. Attend an MSU research talk (recommended research oriented Clubs is provided below)
2. Summarize the talk using at least 150 words
3. Turn in the summary along with your Homework.

Approved talks: Talks given by researchers through the following clubs: * Research and Idea Sharing Enterprise (RAISE): Meets Wednesday Nights Society for Physics Students (SPS): Meets Monday Nights

- Astronomy Club: Meets Monday Nights
- Facility For Rare Isotope Beam (FRIB) Seminars: Occur multiple times a week