1.4 Free Oscillations of Systems with Two Degrees of Freedom

In nature there are many fascinating examples of systems having two degrees of freedom. The most beautiful examples involve molecules and elementary particles (the neutral K mesons especially); to study them requires quantum mechanics. Some simpler examples are a double pendulum (one pendulum attached to the ceiling, the second attached to the bob of the first); two pendulums coupled by a spring; a string with two beads; and two coupled LC circuits. (See Fig. 1.6.) It takes two variables to describe the configuration of such a system, say ψ_a and ψ_b. For example, in the case of a simple pendulum free to swing in any direction, the "moving parts" ψ_a and ψ_b would be the positions of the pendulum in the two perpendicular horizontal directions; in the case of coupled pendulums, the moving parts ψ_a and ψ_b would be the positions of the pendulums; in the case of two coupled LC circuits, the "moving parts" ψ_a and ψ_b would be the charges on the two capacitors or the currents in the circuits.

The general motion of a system with two degrees of freedom can have a very complicated appearance; no part moves with simple harmonic motion. However, we will show that for two degrees of freedom and for linear equations of motion the most general motion is a *superposition* of two independent simple harmonic motions, both going on simultaneously. These two simple harmonic motions (described below) are called *normal modes* or simply *modes*. By suitable starting conditions (suitable initial values of ψ_a, ψ_b, $d\psi_a/dt$, and $d\psi_b/dt$), we can get the system to oscillate in only one mode or the other. Thus the modes are "uncoupled," even though the moving parts are not.

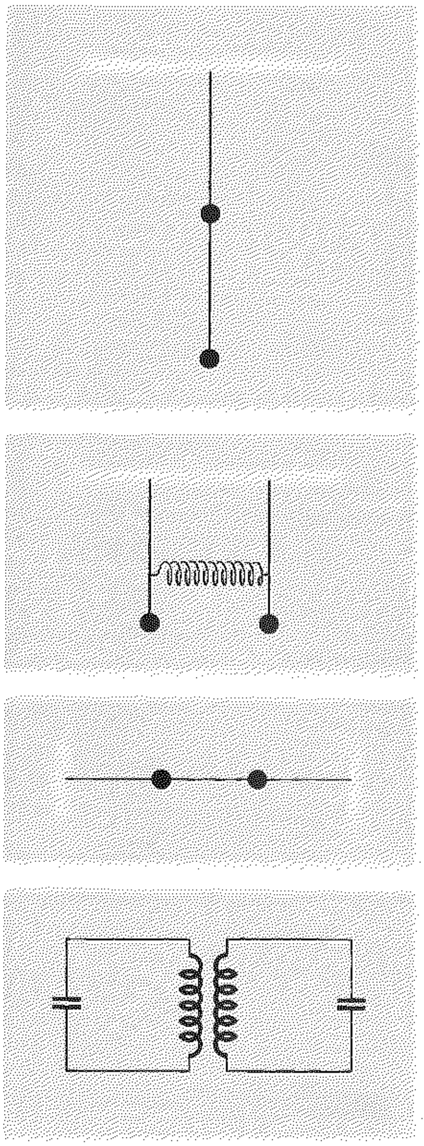


Fig. 1.6 Systems with two degrees of freedom. (The masses are constrained to remain in the plane of the figure.)

Properties of a mode. When only one mode is present, each moving part undergoes simple harmonic motion. All parts oscillate with the same frequency. All parts pass through their equilibrium positions (where ψ is zero) simultaneously. Thus, for example, one never has in a single mode, $\psi_a(t) = A \cos \omega t$ and $\psi_b(t) = B \sin \omega t$ (different phase constants) or $\psi_a(t) = A \cos \omega_1 t$ and $\psi_b(t) = B \cos \omega_2 t$ (different frequencies). Instead one has, for one mode (which we call mode 1),

$$\begin{aligned} \psi_a(t) &= A_1 \cos (\omega_1 t + \varphi_1), \\ \psi_b(t) &= B_1 \cos (\omega_1 t + \varphi_1) = \frac{B_1}{A_1}\psi_a(t), \end{aligned} \tag{41}$$

with the *same frequency and phase constant* for both degrees of freedom (moving parts). Similarly, for mode 2, the two degrees of freedom a and b move according to

$$\begin{aligned} \psi_a(t) &= A_2 \cos (\omega_2 t + \varphi_2), \\ \psi_b(t) &= B_2 \cos (\omega_2 t + \varphi_2) = \frac{B_2}{A_2}\psi_a(t). \end{aligned} \tag{42}$$

Each mode has its own characteristic frequency: ω_1 for mode 1, ω_2 for mode 2. In each mode the system also has a characteristic "configuration" or "shape," given by the ratio of the amplitudes of motion of the moving parts: A_1/B_1 for mode 1 and A_2/B_2 for mode 2. Note that in a mode the ratio $\psi_a(t)/\psi_b(t)$ is constant, independent of time. It is given by the appropriate ratio A_1/B_1 or A_2/B_2, which can be either positive or negative.

The most general motion of the system is (as we will show) simply a superposition with both modes oscillating at once:

$$\begin{aligned} \psi_a(t) &= A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2), \\ \psi_b(t) &= B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2). \end{aligned} \tag{43}$$

Let us consider some specific examples.

Example 6: Simple spherical pendulum

This example is almost too simple, for it does not reveal the full richness of complexity of the general motion that corresponds to Eqs. (43) because the two modes, corresponding respectively to oscillation in the x and in the y direction, have the same frequency, given by $\omega^2 = g/l$. Rather than the superpositions of Eq. (43), corresponding to two different frequencies, we have the simpler results obtained in Eqs. (39) and (40)

$$\begin{aligned} x(t) \equiv \psi_a(t) &= A_1 \cos(\omega_1 t + \varphi_1), \qquad \omega_1 = \omega, \\ y(t) \equiv \psi_b(t) &= B_2 \cos(\omega_2 t + \varphi_2), \qquad \omega_2 = \omega_1 = \omega, \end{aligned} \tag{44}$$

where we have forced Eqs. (44) to appear to resemble Eqs. (43). For the two modes to have the same frequency is unusual; the two modes are then said to be "degenerate."

Example 7: Two-dimensional harmonic oscillator

In Fig. 1.7 we show a mass M that is free to move in the xy plane. It is coupled to the walls by two unstretched massless springs of spring constant K_1 oriented along x and by two unstretched massless springs of spring constant K_2 oriented along y. In the small-oscillations approximation, where we neglect x^2/a^2, y^2/a^2, and xy/a^2, we shall show that the x component of return force is due entirely to the two springs K_1. Similarly, the y component of return force is entirely due to the springs K_2. You can prove this by writing out the exact F_x and F_y and then discarding nonlinear terms. Here is an easier way to see it: Start at the equilibrium position of Fig. 1.7a. Mentally make a small displacement x of M in the $+x$ direction. The return force at this stage in the argument is given by inspection of Fig. 1.7:

$$F_x = -2K_1 x, \qquad F_y = 0.$$

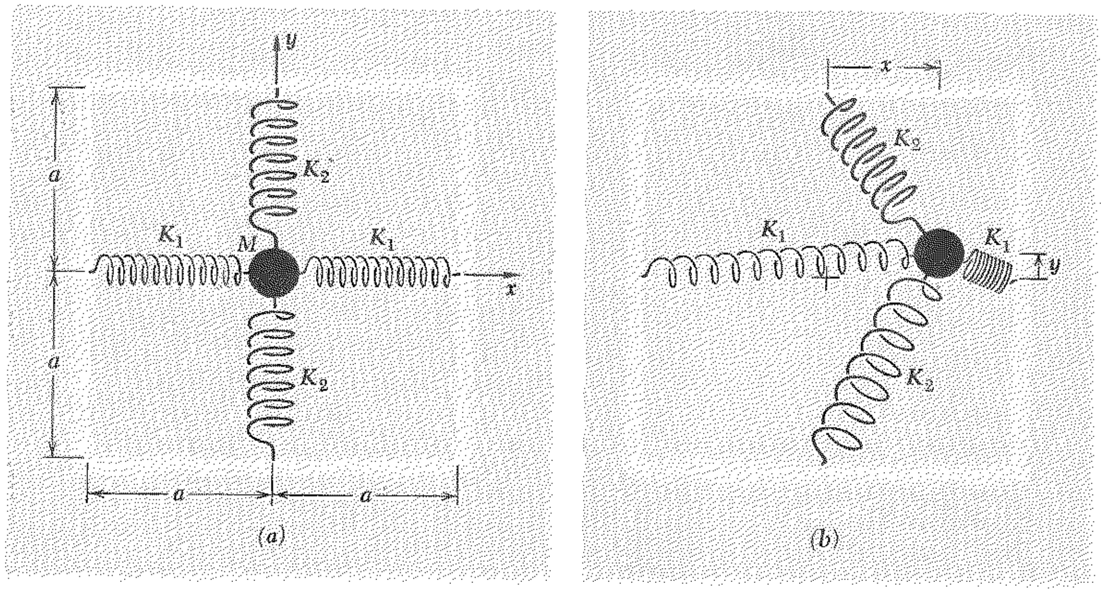


Fig. 1.7 Two-dimensional harmonic oscillator. (a) Equilibrium. (b) General configuration.

Next make a second small displacement y (starting at the terminus of the first displacement), this time in the $+y$ direction. The question of interest is whether F_x changes. The K_1 springs get longer by a small amount proportional to y^2. We neglect that. The K_2 springs change their length by an amount proportional to y (one gets shorter, the other longer), but the projection of their force on the x direction is proportional also to x. We neglect the product yx. Thus F_x is unchanged. A similar argument applies to F_y. Thus we obtain the two linear equations

$$M\frac{d^2x}{dt^2} = -2K_1x, \qquad \text{and} \qquad M\frac{d^2y}{dt^2} = -2K_2y, \tag{45}$$

which have the solutions

$$\begin{aligned} x &= A_1 \cos(\omega_1 t + \varphi_1), & \omega_1{}^2 &= \frac{2K_1}{M}, \\ y &= B_2 \cos(\omega_2 t + \varphi_2), & \omega_2{}^2 &= \frac{2K_2}{M}. \end{aligned} \tag{46}$$

We see that the x motion and y motion are uncoupled, and that each is a harmonic oscillation with its own frequency. Thus the x motion corresponds to one normal mode of oscillation, the y motion to the other. The x mode has amplitude A_1 and phase constant φ_1 that depend only on the initial values $x(0)$ and $\dot{x}(0)$, i.e., the x displacement and velocity at time $t = 0$. Similarly the y mode has amplitude B_2 and phase constant φ_2 that depend only on the initial values $y(0)$ and $\dot{y}(0)$.

Normal coordinates. Notice that our solution (46), which is completely general, is still not as general in appearance as Eqs. (43). That is because we were lucky! Our natural choice for x and y along the springs gave us the uncoupled equations (45), each of which corresponds to one of the modes. In terms of Eq. (43), we came out with ψ_a luckily chosen so that A_2 came out identically zero and with ψ_b chosen so that B_1 came out identically zero. Our fortunate choice of coordinates gave us what are called *normal coordinates;* in this example the normal coordinates are x and y.

Suppose we had not been so lucky or so wise. Suppose we had used a coordinate system x' and y' related to x and y by a rotation through angle α, as shown in Fig. 1.8. By inspection of the figure we see that the normal coordinate x is a linear combination of the coordinates x' and y', as is the other normal coordinate, y. If we had used the "dumb" coordinates x' and y' instead of the "smart" coordinates x and y, we would have obtained two "coupled" differential equations, with both x' and y' appearing in each equation, rather than the uncoupled equations (5).

In most problems involving two degrees of freedom it is not easy to find the normal coordinates "by inspection," as we did in the present example. Thus the equations of motion of the different degrees of freedom are usually coupled equations. One method of solving these two coupled differential equations is to search for new variables that are linear combinations of the original "dumb" coordinates such that the new variables satisfy uncoupled equations of motion. The new variables are then called "normal coordinates." In the present example we know how to find the

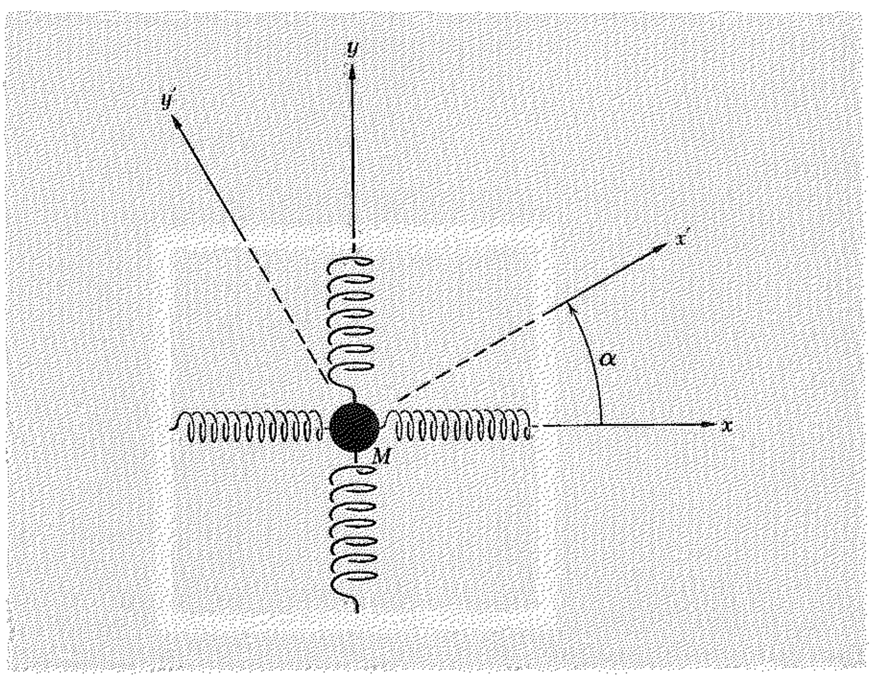


Fig. 1.8 Rotation of coordinates.

normal coordinates, given the "dumb" coordinates x' and y'. Simply rotate the coordinate system so as to obtain x and y, each of which is a linear combination of x' and y'. In a more general problem, we would have to use a more general linear transformation of coordinates than can be obtained by a simple rotation. That would be the case if, for example, the pairs of springs in Fig. 1.7 were not orthogonal.

Systematic solution for modes. Without considering any specific physical system, we assume that we have found two coupled first-order linear homogeneous equations in the "dumb" coordinates x and y:

$$\frac{d^2x}{dt^2} = -a_{11}x - a_{12}y \tag{47}$$

$$\frac{d^2y}{dt^2} = -a_{21}x - a_{22}y. \tag{48}$$

Now we simply *assume* that we have oscillation in a single normal mode. That means we assume that both degrees of freedom, namely x and y, oscillate with harmonic motion with the *same frequency and same phase constant.* Thus we *assume* we have

$$x = A\cos(\omega t + \varphi), \qquad y = B\cos(\omega t + \varphi), \tag{49}$$

with ω unknown and B/A unknown at this stage. Then we have

$$\frac{d^2x}{dt^2} = -\omega^2 x, \qquad \frac{d^2y}{dt^2} = -\omega^2 y. \tag{50}$$

Substituting Eq. (50) into Eqs. (47) and (48) and rearranging, we obtain two homogeneous linear equations in x and y:

$$(a_{11} - \omega^2)x + a_{12}y = 0, \tag{51}$$

$$a_{21}x + (a_{22} - \omega^2)y = 0. \tag{52}$$

Equations (51) and (52) each give the ratio y/x:

$$\frac{y}{x} = \frac{\omega^2 - a_{11}}{a_{12}}, \tag{53}$$

$$\frac{y}{x} = \frac{a_{21}}{\omega^2 - a_{22}}. \tag{54}$$

For consistency, we need to have Eqs. (53) and (54) give the same result. Thus we need the condition

$$\frac{\omega^2 - a_{11}}{a_{12}} = \frac{a_{21}}{\omega^2 - a_{22}},$$

i.e.,

$$(a_{11} - \omega^2)(a_{22} - \omega^2) - a_{21}a_{12} = 0. \tag{55}$$

Another way to write Eq. (55) is to say that the determinant of coefficients of the linear homogeneous equations (51) and (52) must vanish:

$$\begin{vmatrix} a_{11}-\omega^2 & a_{12} \\ a_{21} & a_{22}-\omega^2 \end{vmatrix} \equiv (a_{11}-\omega^2)(a_{22}-\omega^2) - a_{21}a_{12} = 0. \tag{56}$$

Equation (55) or (56) is a quadratic equation in the variable ω^2. It has two solutions, which we call $\omega_1{}^2$ and $\omega_2{}^2$. Thus we have found that if we assume we have oscillation in a single mode, there are exactly two ways that that assumption can be realized. Frequency ω_1 is the frequency of mode 1; ω_2 is that of mode 2. The shape or configuration of x and y in mode 1 is obtained by substituting $\omega^2 = \omega_1{}^2$ back into either one of Eqs. (53) and (54). [They are equivalent, because of Eq. (56).] Thus

$$\left(\frac{y}{x}\right)_{\text{mode 1}} = \left(\frac{B}{A}\right)_{\text{mode 1}} = \frac{B_1}{A_1} = \frac{\omega_1{}^2 - a_{11}}{a_{12}}. \tag{57a}$$

Similarly,

$$\left(\frac{y}{x}\right)_{\text{mode 2}} = \left(\frac{B}{A}\right)_{\text{mode 2}} = \frac{B_2}{A_2} = \frac{\omega_2{}^2 - a_{11}}{a_{12}}. \tag{57b}$$

Once we have found the mode frequencies ω_1 and ω_2 and the amplitude ratios B_1/A_1 and B_2/A_2, we can write down the most general superposition of the two modes as follows:

$$x(t) = x_1(t) + x_2(t) = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2), \tag{58}$$

$$\begin{aligned} y(t) &= \frac{B_1}{A_1} A_1 \cos(\omega_1 t + \varphi_1) + \frac{B_2}{A_2} A_2 \cos(\omega_2 t + \varphi_2) \\ &= B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2). \end{aligned} \tag{59}$$

Notice that, whereas we chose A_1, φ_1, A_2, and φ_2 with complete freedom in Eq. (58), we had no freedom at all left when we came to write the constants in Eq. (59), because φ_1 and φ_2 were already fixed and because we had to satisfy Eqs. (57).

The most general solution of Eqs. (47) and (48) consists of a superposition of any two independent solutions which satisfies the four initial conditions given by $x(0)$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$. A superposition of the two normal modes, with the four constants A_1, φ_1, A_2, and φ_2 determined by the four initial conditions, is such a solution. Thus the general solution can be (although it need not be) written as a superposition of the modes.

Example 8: Longitudinal oscillations of two coupled masses

The system is shown in Fig. 1.9. The two masses M slide on a frictionless table. The three springs are massless and identical, each with spring constant K. We will let the reader do the systematic solution (Prob. 1.23), but here let us try to *guess* the normal modes. We know there must be two

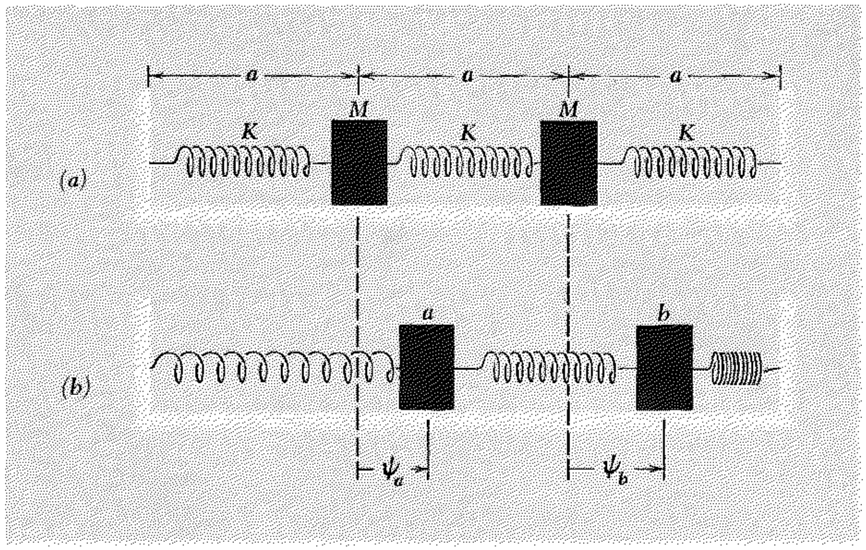


Fig. 1.9 Longitudinal oscillations. (a) Equilibrium. (b) General configuration.

modes, since there are two degrees of freedom. In a mode, each moving part (each mass) oscillates with harmonic motion. This means that each moving part oscillates with the same frequency, and thus *the return force per unit displacement per unit mass is the same for both masses.* (We learned in Sec. 1.2 that ω^2 is the return force per unit displacement per unit mass. That holds for each moving part, whether it is a single isolated system with one degree of freedom or is part of a larger system. The only requirement is that the motion be harmonic motion with a single frequency.)

In the present example the masses are equal. We need therefore only search for configurations that have the same return force per unit displacement for both masses. Let us guess that the displacements may be the same, and see if that works: Suppose we start at the equilibrium position and then displace both masses by the same amount to the right. Is the return force the same on each mass? Notice that the central spring has the same length as it had at equilibrium, so that it exerts no force on either mass. The left-hand mass is pulled to the left because the left-hand spring is extended. The right-hand mass is pushed to the left with the *same* force, because the right-hand spring is compressed by the same amount. We have therefore discovered one mode!

$$\text{Mode 1:} \qquad \psi_a(t) = \psi_b(t), \qquad \omega_1^2 = \frac{K}{M}. \tag{60}$$

The frequency $\omega_1^2 = K/M$ in Eq. (60) follows from the fact that each mass oscillates just as it would if the central spring were removed.

Now let us try to guess the second mode. From the symmetry, we guess that if a and b move oppositely we may have a mode. If a moves a distance ψ_a to the right and b moves an equal distance to the left, each has the same return force. Thus the second mode has $\psi_b = -\psi_a$. The frequency ω_2 can be found by considering a single mass and finding its return force per unit displacement per unit mass. Consider the left-hand mass a. It is pulled to the left by the left-hand spring with a force $F_z = -K\psi_a$. It is pushed to the left by the middle spring with a force $F_z = -2K\psi_a$. (The factor of two occurs because the central spring is compressed by an amount $2\psi_a$.) Thus the net force for a displacement ψ_a is $-3K\psi_a$, and the return force per unit displacement per unit mass is $3K/M$:

$$\text{Mode 2:} \qquad \psi_a = -\psi_b, \qquad \omega_2{}^2 = \frac{3K}{M}. \tag{61}$$

The modes are shown in Fig. 1.10.

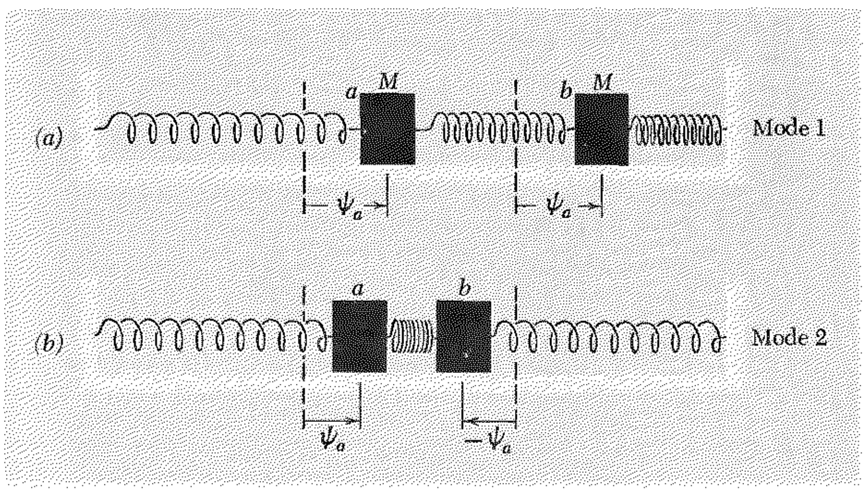


Fig. 1.10 Normal modes of longitudinal oscillation. (a) Mode with lower frequency. (b) Mode with higher frequency.

We shall solve this problem once more, using the method of searching for normal coordinates, i.e., "smart" coordinates. The "smart" coordinates are always a linear combination of ordinary "dumb" coordinates, such that instead of two coupled linear equations, one obtains two uncoupled equations. From Fig. 1.9b, we easily see that the equations of motion for a general configuration are

$$M\frac{d^2\psi_a}{dt^2} = -K\psi_a + K(\psi_b - \psi_a), \tag{62}$$

$$M\frac{d^2\psi_b}{dt^2} = -K(\psi_b - \psi_a) - K\psi_b. \tag{63}$$

By inspection of these equations of motion, we see that alternately adding and subtracting these equations will produce the desired uncoupled equations. Adding Eqs. (62) and (63), we obtain

$$M\frac{d^2}{dt^2}(\psi_a + \psi_b) = -K(\psi_a + \psi_b). \tag{64}$$

Subtracting Eq. (63) from Eq. (62), we obtain

$$M\frac{d^2(\psi_a - \psi_b)}{dt^2} = -3K(\psi_a - \psi_b). \tag{65}$$

Equations (64) and (65) are uncoupled equations in the variables $\psi_a + \psi_b$ and $\psi_a - \psi_b$. They have the solutions

$$\psi_a + \psi_b \equiv \psi_1(t) = A_1 \cos(\omega_1 t + \varphi_1), \qquad \omega_1{}^2 = \frac{K}{M}, \tag{66}$$

$$\psi_a - \psi_b \equiv \psi_2(t) = A_2 \cos(\omega_2 t + \varphi_2), \qquad \omega_2{}^2 = \frac{3K}{M}, \tag{67}$$

where A_1 and φ_1 are the amplitude and phase constant of mode 1 and where A_2 and φ_2 are the amplitude and phase constant of mode 2. We see that $\psi_1(t)$ corresponds to the motion of the center of mass, since $\frac{1}{2}(\psi_a + \psi_b)$ is the position of the center of mass. (We could have divided Eq. (64) by 2 and defined ψ_1 to be the position of the center of mass. The proportionality factor of $\frac{1}{2}$ is not of much interest.) We see that ψ_2 is the compression of the central spring, or (what amounts to the same thing) it is the relative displacement of the two masses. If we had been smart enough, we might have chosen ψ_1 and ψ_2 to start with, since the motion of the center of mass and the "internal motion" (relative motion of the two particles) are physically interesting variables. In many cases it is not so easy to find a simple physical meaning for the normal coordinates. Thus we shall usually stick with our original "dumb" coordinates even after finding the modes, simply because we understand them best.

In the present problem we have found the normal coordinates ψ_1 and ψ_2. Let us go back to our more familiar coordinates ψ_a and ψ_b. Solving Eqs. (66) and (67), we find

$$2\psi_a = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) \tag{68}$$

$$2\psi_b = A_1 \cos(\omega_1 t + \varphi_1) - A_2 \cos(\omega_2 t + \varphi_2). \tag{69}$$

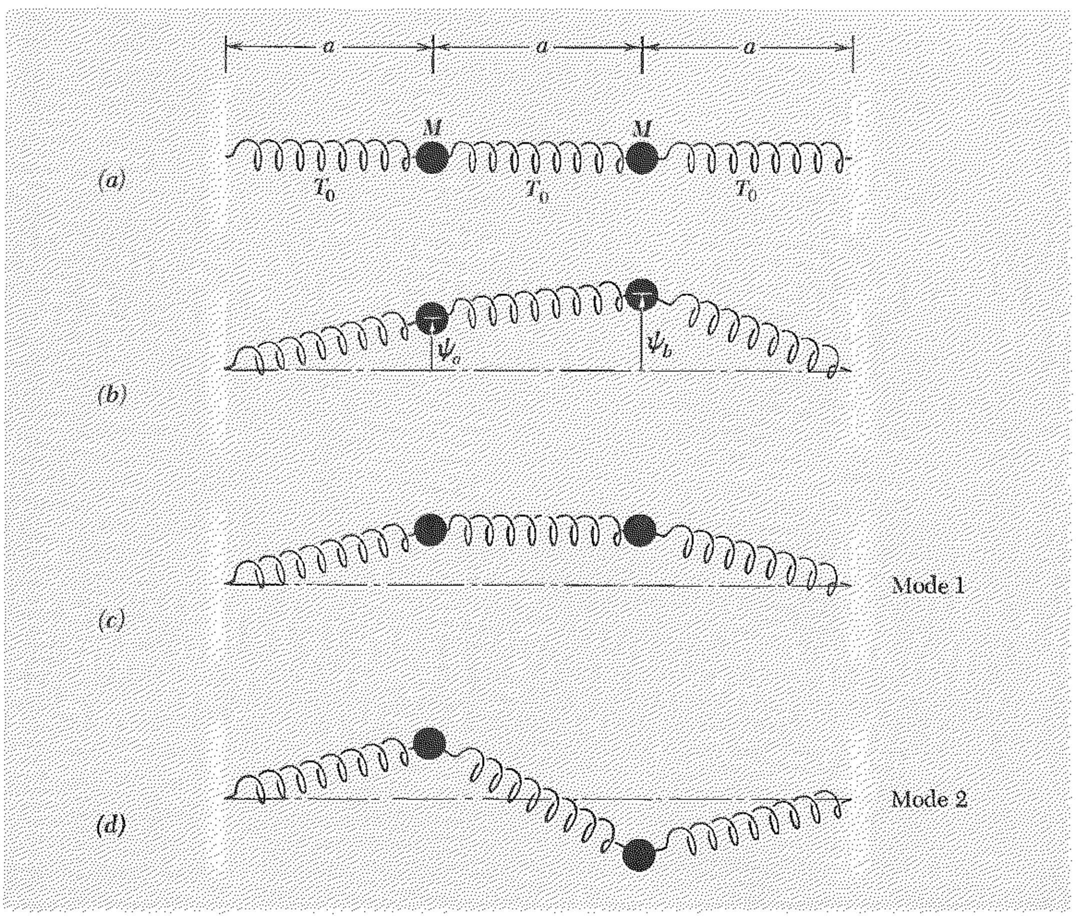
Notice that if we have a motion that is purely mode 1, then A_2 is zero, and, according to Eqs. (68) and (69), we have $\psi_b = \psi_a$. Similarly, in mode 2 we have $A_1 = 0$ and $\psi_b = -\psi_a$. That is what we found before [in Eqs. (60) and (61)].

Example 9: Transverse oscillations of two coupled masses

The system is shown in Fig. 1.11. The oscillations are assumed to be confined to the plane of the paper. Therefore there are just two degrees of freedom. The three identical massless springs have a relaxed length a_0 that is less than the equilibrium spacing a of the masses. Thus they are all stretched. When the system is at its equilibrium configuration (Fig. 1.11a), the springs have tension T_0.

Because of the symmetry of the system, the modes are easy to guess. They are shown in Fig. 1.11. The lower mode (the one with the lower frequency, i.e., the one with the smaller return force per unit displacement per unit mass for each of the masses) has a shape (Fig. 1.11c) such that the center spring is never compressed or extended. The frequency is thus obtained by considering either one of the masses separately, with the return force provided only by the spring that connects it to the wall. For either the slinky approximation (unstretched spring length of zero) or the small-oscillations approximation (displacements very small compared with the spacing a), we shall show presently that a displacement ψ_a of the left-hand

Fig. 1.11 Transverse oscillations. (a) Equilibrium. (b) General configuration. (c) Mode with lower frequency. (d) Mode with higher frequency.



mass causes the left-hand spring to exert a return force of $T_0(\psi_a/a)$. Hence, in this mode the return force per unit displacement per unit mass, ω_1^2, is given by

$$\text{Mode 1:} \qquad \omega_1^2 = \frac{T_0}{Ma}, \qquad \frac{\psi_b}{\psi_a} = +1. \tag{70}$$

We see this as follows. First consider the slinky approximation (Sec. 1.2). In this approximation, the tension T is larger than T_0 by the factor l/a, where l is the spring length and a is the length at equilibrium (Fig. 1.11*a*). The spring exerts a transverse return force equal to the tension T times the sine of the angle between the spring and the equilibrium axis of the springs, i.e., the return force is $T(\psi_a/l)$. But $T = T_0(l/a)$. Thus the return force is $T_0(\psi_a/a)$, and this gives Eq. (70). Next consider the small-oscillations approximation (Sec. 1.2). In that approximation, the increase in length of the spring is neglected, because it differs from the equilibrium length a only by a quantity of order $a(\psi_a/a)^2$, and therefore the increase in tension also is neglected. The tension is thus T_0 when the displacement is ψ_a. The return force is equal to the tension T_0 times the sine of the angle between the spring and the equilibrium axis. This angle may be taken to be a "small" angle, since the oscillations are small. Then the angle (in radians) and its sine are equal, and both are equal to ψ_a/a. Thus the return force is $T_0(\psi_a/a)$. This gives Eq. (70).

Similarly, we can obtain the frequency for mode 2 (Fig. 1.11*d*) as follows: Consider the left-hand mass. The left-hand spring contributes a return force per unit displacement per unit mass of T_0/Ma, as we have just seen in considering mode 1. In mode 2 the center spring is "helping" the left-hand spring, and in fact it is providing twice as great a return force as is the left-hand spring. This is easily seen in the small-oscillations approximation: The spring tension is T_0 for both springs, but the center spring makes twice as large an angle with the axis as does the end spring, so that it gives twice as large a transverse force component. The total return force per unit displacement per unit mass, ω_2^2, is thus given by

$$\text{Mode 2:} \qquad \omega_2^2 = \frac{T_0}{Ma} + \frac{2T_0}{Ma} = \frac{3T_0}{Ma}, \qquad \frac{\psi_b}{\psi_a} = -1. \tag{71}$$

Notice that in the slinky approximation, where the relation $T_0 = K(a - a_0)$ becomes $T_0 = Ka$, the frequencies of the modes of transverse oscillation [Eqs. (70) and (71)] are the same as those for longitudinal oscillation [Eqs. (60) and (61)]. Thus we have a form of degeneracy. This degeneracy does not occur for the small-oscillation approximation, where a_0 is not negligible compared with a.

If the modes had not been so easy to guess, we would have written down the equations of motion of the two masses a and b and then proceeded with the equations, rather than with a mental picture of the physical system itself. We shall let you do that (Prob. 1.20).

Example 10: Two coupled LC circuits

Consider the system shown in Fig. 1.12. Let us find the equations of "motion"—motion of the charges in this case. The electromotive force (emf) across the left-hand inductance is $L\,dI_a/dt$. A positive charge Q_1 on the left-hand capacitor gives an emf $C^{-1}Q_1$ that tends to increase I_a (with our sign conventions). A positive charge Q_2 on the middle capacitor gives an emf $C^{-1}Q_2$ that tends to decrease I_a. Thus we have for the complete contribution to $L\,dI_a/dt$

$$\frac{L\,dI_a}{dt} = C^{-1}Q_1 - C^{-1}Q_2. \tag{72}$$

Similarly,

$$\frac{L\,dI_b}{dt} = C^{-1}Q_2 - C^{-1}Q_3. \tag{73}$$

As in Sec. 1.2, we will express the configuration of the system in terms of currents rather than charges. To do this, we differentiate Eqs. (72) and (73) with respect to time and use conservation of charge. Differentiating gives

$$L\frac{d^2I_a}{dt^2} = C^{-1}\frac{dQ_1}{dt} - C^{-1}\frac{dQ_2}{dt}, \tag{74}$$

$$L\frac{d^2I_b}{dt^2} = C^{-1}\frac{dQ_2}{dt} - C^{-1}\frac{dQ_3}{dt}. \tag{75}$$

Charge conservation gives

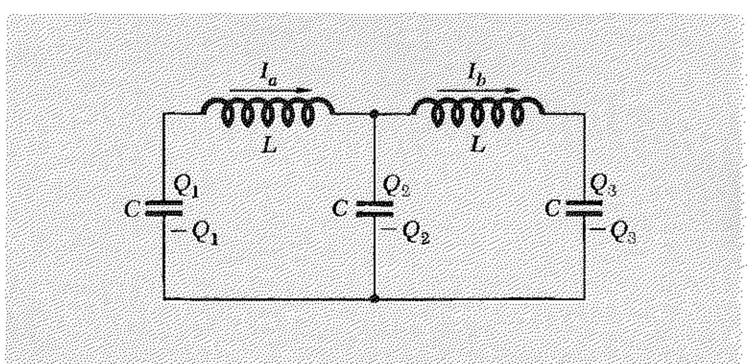
$$\frac{dQ_1}{dt} = -I_a, \qquad \frac{dQ_2}{dt} = I_a - I_b, \qquad \frac{dQ_3}{dt} = I_b. \tag{76}$$

Substituting Eqs. (76) into Eqs. (74) and (75), we obtain the coupled equations of motion

$$L\frac{d^2I_a}{dt^2} = -C^{-1}I_a + C^{-1}(I_b - I_a) \tag{77}$$

$$L\frac{d^2I_b}{dt^2} = -C^{-1}(I_b - I_a) - C^{-1}I_b. \tag{78}$$

Fig. 1.12 Two coupled LC circuits. General configuration of charges and currents. The arrows give sign conventions for positive currents.



Now that we have the two equations of motion we want to find the two normal modes. These can be found by searching for normal coordinates, by guessing, or by the systematic method (see Prob. 1.21). One finds

$$\begin{aligned} &\text{Mode 1:} \qquad I_a = I_b, \qquad \omega_1^2 = \frac{C^{-1}}{L}. \\ &\text{Mode 2:} \qquad I_a = -I_b, \qquad \omega_2^2 = \frac{3C^{-1}}{L}. \end{aligned} \tag{79}$$

Notice that in mode 1 the center capacitor never acquires any charge, and it could be removed without affecting the motion of the charges. Also, in mode 1 the charges Q_1 and Q_3 are always equal in magnitude and opposite in sign. In mode 2 the charges Q_1 and Q_3 are always equal in both magnitude and sign, and Q_2 has twice that magnitude, but opposite sign.

We purposely chose the three examples (8–10) of longitudinal oscillations (Fig. 1.9), transverse oscillations (Fig. 1.11), and coupled LC circuits (Fig. 1.12) to have the same spatial symmetry and to give equations of motion and normal modes with the same mathematical form. We also chose these examples to be the natural extensions (to two degrees of freedom) of the similar systems with one degree of freedom that we considered in Examples 2–4 in Sec. 1.2, as shown in Figs. 1.3, 1.4, and 1.5. In Chap. 2 we shall extend these same three examples to an arbitrarily large number of degrees of freedom.

1.5 Beats

There are many physical phenomena where the motion of a given moving part is a superposition of two harmonic oscillations having different angular frequencies ω_1 and ω_2. For example, the two harmonic oscillations may correspond to the two normal modes of a system having two degrees of freedom. As a contrasting example, the two harmonic oscillations may be due to driving forces produced by two independently oscillating uncoupled systems. This sort of situation is illustrated by two tuning forks of different frequencies. Each produces its own "note" by causing harmonically oscillating pressure variations at the fork, which radiate through the air as sound waves. The motion induced in your eardrum is a superposition of two harmonic oscillations.

In all these examples, the mathematics is the same. For simplicity we assume that the two harmonic oscillations have the same amplitude. We also assume that the two oscillations have the same phase constant, which we take to be zero. Then we write the superposition ψ of the two harmonic oscillations ψ_1 and ψ_2:

$$\psi_1 = A\cos\omega_1 t, \qquad \psi_2 = A\cos\omega_2 t, \tag{80}$$

$$\psi = \psi_1 + \psi_2 = A\cos\omega_1 t + A\cos\omega_2 t. \tag{81}$$