C H A P T E R

3

OSCILLATIONS

3.1 INTRODUCTION

We begin by considering the oscillatory motion of a particle constrained to move in one dimension. We assume that a position of stable equilibrium exists for the particle, and we designate this point as the origin (see Section 2.6). If the particle is displaced from the origin (in either direction), a certain force tends to restore the particle to its original position. An example is an atom in a long molecular chain. The restoring force is, in general, some complicated function of the displacement and perhaps of the particle's velocity or even of some higher time derivative of the position coordinate. We consider here only cases in which the restoring force F is a function only of the displacement: $F = F(x)$.

We assume that the function $F(x)$ that describes the restoring force possesses continuous derivatives of all orders so that the function can be expanded in a Taylor series:

$$F(x) = F_0 + x\left(\frac{dF}{dx}\right)_0 + \frac{1}{2!}x^2\left(\frac{d^2F}{dx^2}\right)_0 + \frac{1}{3!}x^3\left(\frac{d^3F}{dx^3}\right)_0 + \cdots \qquad \textbf{(3.1)}$$

where F_0 is the value of $F(x)$ at the origin ($x = 0$), and $(d^nF/dx^n)_0$ is the value of the nth derivative at the origin. Because the origin is defined to be the equilibrium point, F_0 must vanish, because otherwise the particle would move away from the equilibrium point and not return. If, then, we confine our attention to displacements of the particle that are sufficiently small, we can normally neglect all terms involving x^2 and higher powers of x. We have, therefore, the approximate relation

$$\boxed{F(x) = -kx} \tag{3.2}$$

where we have substituted $k \equiv -(dF/dx)_0$. Because the restoring force is always directed toward the equilibrium position (the origin), the derivative $(dF/dx)_0$ is negative, and therefore k is a positive constant. Only the first power of the displacement occurs in $F(x)$, so the restoring force in this approximation is a *linear* force.

Physical systems described in terms of Equation 3.2 obey **Hooke's Law**.* One of the classes of physical processes that can be treated by applying Hooke's Law is that involving elastic deformations. As long as the displacements are small and the elastic limits are not exceeded, a linear restoring force can be used for problems of stretched springs, elastic springs, bending beams, and the like. But we must emphasize that such calculations are only approximate, because essentially every real restoring force in nature is more complicated than the simple Hooke's Law force. Linear forces are only useful approximations, and their validity is limited to cases in which the amplitudes of the oscillations are small (but see Problem 3-8).

Damped oscillations, usually resulting from friction, are almost always the type of oscillations that occur in nature. We learn in this chapter how to design an efficiently damped system. This damping of the oscillations may be counteracted if some mechanism supplies the system with energy from an external source at a rate equal to that absorbed by the damping medium. Motions of this type are called **driven** (or **forced**) **oscillations**. Normally sinusoidal, they have important applications in mechanical vibrations as well as in electrical systems.

The extensive discussion of linear oscillatory systems is warranted by the great importance of oscillatory phenomena in many areas of physics and engineering. It is frequently permissible to use the linear approximation in the analysis of such systems. The usefulness of these analyses is due in large measure to the fact that we can usually use *analytical* methods.

When we look more carefully at physical systems, we find that a large number of them are *nonlinear* in general. We will discuss nonlinear systems in Chapter 4.

3.2 SIMPLE HARMONIC OSCILLATOR

The equation of motion for the simple harmonic oscillator may be obtained by substituting the Hooke's Law force into the Newtonian equation $F = ma$. Thus

$$-kx = m\ddot{x} \tag{3.3}$$

* Robert Hooke (1635–1703). The equivalent of this force law was originally announced by Hooke in 1676 in the form of a Latin cryptogram: CEIIINOSSSTTUV. Hooke later provided a translation: *ut tensio sic vis* [the stretch is proportional to the force].

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If we define

$$\omega_0^2 \equiv k/m \tag{3.4}$$

Equation 3.3 becomes

$$\boxed{\ddot{x} + \omega_0^2 x = 0} \tag{3.5}$$

According to the results of Appendix C, the solution of this equation can be expressed in either of the forms

$$x(t) = A \sin(\omega_0 t - \delta) \tag{3.6a}$$

$$x(t) = A \cos(\omega_0 t - \phi) \tag{3.6b}$$

where the phases* δ and ϕ differ by $\pi/2$. (An alteration of the phase angle corresponds to a change of the instant that we designate $t = 0$, the origin of the time scale.) Equations 3.6a and b exhibit the well-known sinusoidal behavior of the displacement of the simple harmonic oscillator.

We can obtain the relationship between the total energy of the oscillator and the amplitude of its motion as follows. Using Equation 3.6a for $x(t)$, we find for the kinetic energy,

$$\begin{aligned} T = \frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t - \delta) \\ &= \frac{1}{2}kA^2 \cos^2(\omega_0 t - \delta) \end{aligned} \tag{3.7}$$

The potential energy may be obtained by calculating the work required to displace the particle a distance x. The incremental amount of work dW necessary to move the particle by an amount dx against the restoring force F is

$$dW = -F\,dx = kx\,dx \tag{3.8}$$

Integrating from 0 to x and setting the work done on the particle equal to the potential energy, we have

$$U = \frac{1}{2}kx^2 \tag{3.9}$$

Then

$$U = \frac{1}{2}kA^2 \sin^2(\omega_0 t - \delta) \tag{3.10}$$

* The symbol δ is often used to represent phase angle, and its value is either assigned or determined within the context of an application. Be careful when using equations within this chapter because δ in one application may not be the same as the δ in another. It might be prudent to assign subscripts, for example, δ_1 and δ_2, when using different equations.

Combining the expressions for T and U to find the total energy E, we have

$$E = T + U = \frac{1}{2}kA^2\,[\cos^2(\omega_0 t - \delta) + \sin^2(\omega_0 t - \delta)]$$

$$\boxed{E = T + U = \frac{1}{2}kA^2} \tag{3.11}$$

so that the total energy is proportional to the *square of the amplitude*; this is a general result for linear systems. Notice also that E is independent of the time; that is, energy is conserved. (Energy conservation is guaranteed, because we have been considering a system without frictional losses or other external forces.)

The period τ_0 of the motion is defined to be the time interval between successive repetitions of the particle's position and direction of motion. Such an interval occurs when the argument of the sine in Equation 3.6a increases by 2π:

$$\omega_0\tau_0 = 2\pi \tag{3.12}$$

or

$$\tau_0 = 2\pi\sqrt{\frac{m}{k}} \tag{3.13}$$

From this expression, as well as from Equation 3.6a, it should be clear that ω_0 represents the **angular frequency** of the motion, which is related to the **frequency** ν_0 by*

$$\boxed{\begin{aligned} \omega_0 &= 2\pi\nu_0 = \sqrt{\frac{k}{m}} && (3.14)\\ \nu_0 &= \frac{1}{\tau_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} && (3.15) \end{aligned}}$$

Note that the period of the simple harmonic oscillator is independent of the amplitude (or total energy); a system exhibiting this property is said to be **isochronous**.

For many problems, of which the simple pendulum is the best example, the equation of motion results in $\ddot{\theta} + \omega_0^2 \sin\theta = 0$, where θ is the displacement angle from equilibrium, and $\omega_0 = \sqrt{g/\ell}$, where ℓ is the length of the pendulum arm. We can make this differential equation describe simple harmonic motion by invoking the **small oscillation** assumption. If the oscillations about the equilibrium are small,

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* Henceforth we shall denote angular frequencies by ω (units: radians per unit time) and frequencies by ν (units: vibrations per unit time or Hertz, Hz). Sometimes ω will be referred to as a "frequency" for brevity, although "angular frequency" is to be understood.

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we expand $\sin\theta$ and $\cos\theta$ in power series (see Appendix A) and keep only the lowest terms of importance. This often means $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1 - \theta^2/2$, where θ is measured in radians. If we use the small oscillation approximation for the simple pendulum, the equation of motion above becomes $\ddot{\theta} + \omega_0^2\theta = 0$, an equation that does represent simple harmonic motion. We shall often invoke this assumption throughout this text and in its problems.

3.3 HARMONIC OSCILLATIONS IN TWO DIMENSIONS

We next consider the motion of a particle that is allowed two degrees of freedom. We take the restoring force to be proportional to the distance of the particle from a force center located at the origin and to be directed toward the origin:

$$\mathbf{F} = -k\mathbf{r} \tag{3.16}$$

which can be resolved in polar coordinates into the components

$$\left.\begin{aligned} F_x &= -kr\cos\theta = -kx \\ F_y &= -kr\sin\theta = -ky \end{aligned}\right\} \tag{3.17}$$

The equations of motion are

$$\left.\begin{aligned} \ddot{x} + \omega_0^2 x &= 0 \\ \ddot{y} + \omega_0^2 y &= 0 \end{aligned}\right\} \tag{3.18}$$

where, as before, $\omega_0^2 = k/m$. The solutions are

$$\left.\begin{aligned} x(t) &= A\cos(\omega_0 t - \alpha) \\ y(t) &= B\cos(\omega_0 t - \beta) \end{aligned}\right\} \tag{3.19}$$

Thus, the motion is one of simple harmonic oscillation in each of the two directions, both oscillations having the same frequency but possibly differing in amplitude and in phase. We can obtain the equation for the path of the particle by eliminating the time t between the two equations (Equation 3.19). First we write

$$y(t) = B\cos[\omega_0 t - \alpha + (\alpha - \beta)]$$

$$= B\cos(\omega_0 t - \alpha)\cos(\alpha - \beta) - B\sin(\omega_0 t - \alpha)\sin(\alpha - \beta) \tag{3.20}$$

Defining $\delta \equiv \alpha - \beta$ and noting that $\cos(\omega_0 t - \alpha) = x/A$, we have

$$y = \frac{B}{A}x\cos\delta - B\sqrt{1 - \left(\frac{x^2}{A^2}\right)}\sin\delta$$

or

$$Ay - Bx\cos\delta = -B\sqrt{A^2 - x^2}\sin\delta \tag{3.21}$$

On squaring, this becomes

$$A^2y^2 - 2ABxy\cos\delta + B^2x^2\cos^2\delta = A^2B^2\sin^2\delta - B^2x^2\sin^2\delta$$