Mathematical Modeling in Physics

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CONTENTS

Under Construction

Overview of PHY 415

Historically, PHY 415 has focused on developing specialized analytical solutions to common physics problems and situations. However, the work of physicists involves a variety of tools that we use to **model the behavior of physical systems**. Hence, this course will emphasize how we go about building mathematical models of physical phenomenon and how we can investigate them through a variety of lenses: analytical mathematics, graphics and visualization, as well as computational modeling using Python. By emphasizing the critical aspects of the modeling process, this course aims to develop your expertise in the process of doing theoretical physics rather than specific tools or techniques. That being said, you will also learn many differently analytical and computational tools to use for your own aims.

Learning Objectives

In this course, you will learn to:

- investigate physical systems using a variety of tools and approaches,
- construct and document a reproducible process for those investigations,
- use analytical, computational, and graphical approaches to answer specific questions in those investigations,
- provide evidence of the quality of work using a variety of sources, and
- collaborate effectively and contribute to a inclusive learning environment

Table of Contents

The rest of this JupyterBook is (currently) organized as follows:

CHAPTER

ONE

OVERVIEW OF PHY 415

In designing this course, I plan to emphasize more independent learning on your part and greater agency for you in determining what you learn and how you demonstrate you have learned. So you should expect:

- to read a variety of pieces of information to coordinate information
- to present your ideas publicly and to discuss them
- to learn new approaches and novel techniques on your own
- to become more expert than me in the areas of your interest
- to learn more about scientists that you have not learned about

This is not to say that you are on your own. Here's what you can expect from me:

- resources, information, and tools to help you learn
- support and scaffolding to move you towards more independence in your learning
- timely and detailed feedback to help you along
- a commitment to an inclusive classroom

1.1 Contact Information

- Instructor: Prof. Danny Caballero (he/him/his)
- Class Meetings: Tuesdays and Thursdays 10:20am-12:10pm (Location: TBD)
- Email: caball14@msu.edu, office: 1310-A BPS
- Office hrs: To be scheduled, but I also have an open door policy. I enjoy visiting and talking with you about physics.
- Web page: dannycab.github.io/phy415msu/

1.1.1 Slack Team

This term we will be using Slack for class discussion. The system is highly catered to getting you help fast and efficiently from classmates and myself. Rather than emailing questions, I encourage you to post your questions on there. You should be added to the Slack team by the first week of class. Email Danny if you are not part of the team.

Link: https://phy415fall2022.slack.com/

1.2 Grading

Details about *course activities are here* and *information regarding asessment is here*. Your grade will be comprised of several projects that you will complete in the form of a Jupyter notebook (a "computational essay", which we will discuss later). Your grade on each project is split between completion (50%) and quality (50%). We will collectively define quality in class. Your final grade will be scaled based on your best performances; there will be slightly more projects than what comprises your grade. How your grade is calculated appears below.

Activity	Percent of Grade
Best Project Grade	40%
2nd Best Project Grade	30%
3rd Best Project Grade	20%
4th Best Project Grade	10%

While attendance is not required, you are unlikely to succeed with your projects without regular attendance and engagement.

1.3 Course Objectives

This course emphasizes making models of physical phenomenon and how we use various tools at our disposal to investigate those models. Hence, we have learning objectives for making models of these systems and for learning specific tools.

Principle Learning Objectives

Students will demonstrate they can:

- investigate physical systems of their choosing using a variety of tools and approaches
- construct and document a reproducible process for those investigations
- use analytical, computational, and graphical approaches to answer specific questions in those investigations
- provide evidence of the quality of their work using a variety of sources
- collaborate effectively and contribute to a inclusive learning environment

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Each of these learning objectives contributes to your development as a physicist. I recognize that these are **big** ideas to think about. What I mean is that the objectives above are quite broad and you might be able to see a little about what or why they are included. But, below, I added more detail about each one along with a smaller scale list of objectives that you will engage with. Throughout our course, you will have opportunities to demonstrate these objectives in your work. My aim is to make what you are assessed on in this course something you are interested in, so these objectives reflect that.

1.3.1 Investigate physical systems

Clearly, one of our central goals is learning how to make models of physical systems. This means learning about and developing fluency with a wide variety of mathematical and computational tools. In this courses, we will make extensive use of Jupyter notebooks for homework and projects. In fact, what you are reading is a set of Jupyter notebooks! Below, you will see the list of objectives for this principal objective.

Investigating Physical Systems Learning Objectives

Students will demonstrate they can:

- use mathematical techniques to predict or explain some physical phenomenon
- employ computational models and algorithms to investigate physical systems
- compare analytical and computational approaches to these investigations
- provide coherent explanations for their investigations buttressed by physical, mathematical, and/or computational knowledge and principles

1.3.2 Construct and document a reproducible process

A critical element of physics work is making sure that with the same setup and approach, others can reproduce the work you have done. This provides validity to your work and evidences how we develop collective understanding of physics. Physics is a social enterprise and the ensuring the reproducibility of work supports that enterprise. Below are the learning objectives for this principal objective.

Reproducibility Learning Objectives

Students will demonstrate they can:

- document their work and analysis such that others can reproduce their work
- consistently reproduce their work and results in a variety of contexts
- provide an explanation for why certain work or results are not (or should not be) reproducible

1.3.3 Use analytical, computational, and graphical approaches

The main approaches that we use to make models are mathematical, computational, and graphical. In this class, we will aim to leverage the benefits of each to learn more about the physical systems that we are investigating. Indeed, much of the "knowledge" that you are going to develop will be about specific analytical, computational, or graphical approaches to investigate physical systems. Below are the learning objectives for this principal objective.

Modeling Approaches Learning Objectives

Students will demonstrate they can:

- Use a wide variety of modeling techniques to investigate different physical systems
- Choose and employ appropriate approaches to modeling physical systems of their choosing
- Explain how those approaches lead to different results or conclusions

1.3.4 Provide evidence of the quality of their work

The definition of the quality of a piece of science is a collective decision by the scientific community. In established communities, like physics, there are commonly-accepted ways of defining the quality of work (norms, customs, and rules all play a role). But that is not to mean those ways can't change; papers describing quantum physics and relativity brushed up hard against this issue of quality and were both dismissed and celebrated. Newer disciplines are still establishing those norms and rules. And in some cases, disciplines are pushing back against Western norms of quality. In our class, we will collectively decide what we mean by "high quality" work. Below are the learning objectives for this principal objective.

Quality Control Learning Objectives

Students will demonstrate they can:

- describe what it means to have high quality work in our class
- look for and evaluate when work meets those standards
- provide suggestions (or act on suggestions) to improve the quality of their work

1.3.5 Collaborate effectively

Physics is a social enterprise that relies on effective and productive collaborations. Very little (if any) science is done alone; the scale of science is too grand for individuals to effectively work – everyone needs a team. In this spirit, in this classroom, we deeply encourage collaboration. We will try to develop effective collaboration through your work on projects and our in-class activities. Below are the learning objectives for this principal objective.

Collaboration Learning Objectives

Students will demonstrate they can:

- Collaborate on a variety of activities in and out of class
- Document the contributions in these collaborations and make changes if contributions are unbalanced
- Develop personally effective strategies for collaboration

1.4 Course Design

For most of you, 4415 is an elective course that you are taking to learn more about how we use mathematical techniques in physics. As such, this course is designed under several different principles than a standard course. Below, I provide those principles and their rationale.

- 415 should help you learn the central tenets of modeling physical systems
  - The sheer volume of mathematical and computational physics knowledge out there is immense and impossible for any one person to learn. However, the central elements of making models, how to learn about specific techniques, and how to debug your approaches are things we can learn and employ broadly as well as to specific problems.
- 415 should be a celebration of your knowledge
  - For most of you, this course is part of your senior level coursework. What you have achieved in the last three to four years should be celebrated and enjoyed. This course will provide ample opportunities for you to share what things you know and what things you are learning with me and with each other.
- 415 should give you opportunities to engage in professional practice
  - As you start towards your professional career, it's important to learn what professional scientists do. You have probably already begun this work in advanced lab and research projects that you have worked on. We will continue developing your professional skills in this course through the use of course projects.
- 415 will illustrate that we can learn from each other
  - Even though I've been learning physics for almost 20 years, I don't know everything. I am excited to learn from you and I hope that you are excited to learn from me and each other.

1.4.1 Optional purchases:

The core readings and work for this course will be this jupyterbook. I will find resources online, make my own, and provide as much organized free material as possible. If you want to have a textbook that helps you organize your readings, please obtain copies of:

1. Mary Boas, *Mathematical Methods in the Physical Sciences* (Wiley; 2005). This book is the definitive text on mathematical approaches, written by Dr. Boas originally in 1966! Any 3rd edition will be useful and I will put the section numbers from Boas in the online readings.
2. Mark Newman, *Computational Physics* (CreateSpace Independent Publishing Platform; 2012). This book is a great introduction to a variety of computational physics techniques, written by UMich professor Mark Newman for a computational physics course. I will put section numbers from Newman in the online readings.

Additional sources

In addition, I will draw from the following books. I have copies of them if you want or need scans of sections. But they can found online in Google Books and other places as well. no need to purchase unless you want a copy for your personal library.

Mechanics

- Edwin Taylor, Mechanics
- Jerry Marion and Stephen Thornton, Classical Dynamics of Particles and Systems
- Charles Kittel, Walter D. Knight, Malvin A. Ruderman, A. Carl Helholtz, and Burton J. Moyer, Mechanics

Electromagnetism

- Edward Purcell, Electricity and Magnetism
- David J. Grriffths, Introduction to Electromagnetism

Quantum Mechanics

- David McIntyre, Quantum Mechanics
- David J. Griffiths, Introduction to Quantum Mechanics

Waves and Thermal Physics

- Frank S. Crawford, Waves
- Charles Kittel, Thermal Physics
- Ashley Carter, Classical and Statistical Thermodynamics
- Daniel Schroeder, Thermal Physics

Additional Physics Topics

- Steven H. Strogatz, Nonlinear Dynamics and Chaos
- B Lautrup, Physics of Continuous Matter
- Frank L. Pedrotti and Leno S. Pedrotti, Introduction to Optics

Mathematics

- Susan M. Lea, Mathematics for Physicists
- William E. Boyce and Richard C. DiPrima, Elementary Differential Equations
- James Brown and Ruel Churchill, Complex Variables and Applications
- Jerrold Marsden and Anthony Tromba, Vector Calculus
- Sheldon Ross, A First Course in Probability

Presenting (Visual) Information

- Edward Tufte, The Visual Display of Quantitative information
- Albert Cairo, The Truthful Art
- Stephen E. Toulmin, The Uses of Argument

1.5 Course Activities

1.5.1 "Readings"

"Reading" is an essential part of 415! Reading the notes before class is very important. I use "reading" in quotes, because in our class this idea goes beyond just reading text and includes understanding figures and watching videos. These should help inform the basis of your understating that we will draw on in class to clarify your understanding and to help you make sense of the material. I will assume you have done the required readings in advance! It will make a huge difference if you spend the time and effort to carefully read and follow the resources posted. The calendar has the details on readings that will be updated.

1.5.2 Class Meetings

Classroom Etiquette: Please silence your electronic devices when entering the classroom. I don't mind you using them (in fact, see below, we will use them). But, sometimes, they can very distracting to your neighbors, so use your judgement. I appreciate that you might have questions or comments about things in class. If you and your neighbors are confused, just raise your hand and ask. If you are confused, you are likely not the only one and it's better to chat about it, then move on. Questions in lecture are always good, and are strongly encouraged!

Computing Devices: Please bring some sort of computing device to class everyday. You might be researching information online, reviewing work you have done, or actively building models of systems together. This device can be a computer, a tablet, or a phone. You can also partner up with folks because we will use them in groups.

In-Class Activities: We will also use a variety of in-class activities and worksheets that help you construct an understanding of a particular topic or concept. These will not be collected or graded, but we will discuss the solutions in class.

1.5.3 Homework

There will be a homework due every Friday by 5pm. Late homework can't be accepted once solutions are posted - but, your lowest score will be dropped. Homework is exceedingly important for developing an understanding of the course material, not to mention building skills in complex physical and mathematical problem solving. They will require considerable time and personal effort this term! *Your lowest homework grade will be dropped.*

There are four kinds of homework problems in this class:

Standard Homework Problems: These are regular back-of-the-book type homework problems that involve derivations, calculations, figures, and graphs. If you took 481, there will be fewer of these in 482. Each question will be coarsely graded for "completion":

- 10 pts. complete
- 8 pts. right idea, but incomplete
- 4 pts. relatively incomplete
- 0 pts. not turned in

Computational Homework Problems: There will be *some use of computation in this course* on homework problems. I will encourage and support the use of Python (through Jupyter notebooks). You do not need any computational experience for this course as you will learn some fundamentals early on and keep using them throughout the course. You are welcome to use any environment of your choosing (e.g., Octave/MATLAB, Mathematica, C++), but I will only provide support for Python. Python is in use across the sciences, but it is becoming much used in physics, so learning it will serve you well in your future work. I suggest downloading the Anaconda distribution of Python as it comes with all the packages you will need to get up and running with Jupyter notebooks. These will be graded on the same 10-8-4-0 scale as standard homework problems. Here are instructions for installing Jupyter Notebooks.

Project Homework Problems: These are homework problems to support your working towards completing your individual and paired projects (see below). *Projects are difficult to complete, so having some regular check-ins on how those projects are going, setting milestones to complete, and producing a semi-complete piece of a project are all important aspects of research!* These homework problems are meant to help you make that progress each week. They count twice as much as normal homework problems, but follow a similar grading scale:

- 20 pts. complete
- 16 pts. right idea, but incomplete
- 8 pts. relatively incomplete
- 0 pts. not turned in

You will be given detailed feedback on these project homework problems as you are working on bigger projects throughout this class. You should read and be responsive to this feedback as it will help you develop a strong project.

Self-reflection Homework Problems: These are homework problems that ask you to evaluate your progress on your projects and how you and your partner are working together. *Evaluating how well you understand something and what you need to do to move forward is a hard thing to learn. So is working on a team (or with a partner).* These homework problems are meant to help you do both and get feedback from me on how things are going. These problems are graded out of 10 points like regular homework problems on the same 10-8-4-0 scale. You will also be given detailed feedback on these homework problems.

I strongly encourage collaboration, an essential skill in science and engineering (and highly valued by employers!) Social interactions are critical to scientists' success – most good ideas grow out of discussions with colleagues, and essentially all physicists work as part of a group. Find partners and work on homework together. However, it is also important that you OWN the material. I strongly suggest you start homework by yourself (and that means really making an extended effort on every problem). Then work with a group, and finally, finish up on your own – write up your own work, in your own way. There will also be time for peer discussion during classes – as you work together, try to help your partners get over confusions, listen to them, ask each other questions, critique, teach each other. You will learn a lot this way! For all assignments, the work you turn in must in the end be your own: in your own words, reflecting your own understanding. (If, at any time, for any reason, you feel disadvantaged or isolated, contact me and I can discretely try to help arrange study groups.)

Help Session

Help sessions/office hours are to facilitate your learning. We encourage attendance - plan on working in small groups, our role will be as learning coaches. The sessions are homework-centric, but we will not be explicitly telling anyone how to do the homework (how would that help you learn?) I strongly encourage you to start all problems on your own. If you come to help sessions “cold”, the value of homework to you will be greatly reduced.

1.6 Assessments

1.6.1 In-Class Quizzes

Every other Friday, we will have a short, in-class quiz that covers the material discussed in the previous two weeks. The quiz will take the form of a typical exam-style question – more straight-forward than your homework questions with not much substantive calculations. I will inform you of the type and topic of the in-class quiz on the Friday prior to the quiz, so you will have a week to prepare should you want. There will seven of these quizzes. *Your lowest quiz grade will be dropped.*

1.6.2 Projects

In lieu of examinations, which are not at all representative of professional physics practice, you will produce two projects.

Individual Project

The first project is an individual research project and is meant to mimic the kind of literature review that is needed to understand a topic that is new to you. In a nutshell, you will select a topic of active research in electromagnetism, read several journal articles pertaining to the topic, and write a 4-5 page summary with references about the topic. You will be working to answer the following questions in your paper:

- What is the phenomenon and why is it interesting?
- What the relevant electrodynamic models that are used to make sense of this phenomenon?
- What are the assumptions that go into these models?
- What mathematical models and mathematical tools are used to make predictions?
- What are some of the major predictions?
- What experimental work has gone into validating these predictions?
- What the challenges in connecting the experimental and theoretical predictions?

There will be seven homework questions that help you develop your individual project. *Having deadlines and milestones for such a project is important, so that you don’t get behind.* Here’s a *preliminary* listing of the homework questions that will appear (note these are subject to change!):

- **Homework 1** (What is interesting?) - Define your phenomenon; what is it? Why is it interesting to you? What are a few papers that you can use to start your background research (give actual references). 2-3 paragraphs along with a listing of at least 4 relevant journal articles.

- **Homework 2** (Towards an annotated bibliography I) - Summarize 2 of the 4 relevant journal articles. What do they say about your phenomenon? How are the theoretical models constructed? What are the predictions and implications? 2-3 paragraphs per article.
- **Homework 3** (Towards an annotated bibliography II)- Summarize 2 more relevant journal articles (can be the remaining 2 or 2 new ones if the direction of your research has changed). What do they say about your phenomenon? How are the theoretical models constructed? What are the predictions and implications? 2-3 paragraphs per article.
- **Homework 4** (Building your paper I) - Summarize your background research so far (3-4 paragraphs on what you have learned so far) with references. What are people saying about this phenomenon? What are the relevant models? How are the models described?
- **Homework 5** (Building your paper II) - Summarize the models used to describe your phenomenon (3-4 more paragraphs on what you have learned so far) with references. What are the predictions of these models? What mathematical tools are used to make these predictions?
- **Homework 6** (Building your paper III) - Summarize the experimental work that validates the theoretical predictions (3-4 more paragraphs on what you have learned so far). What are the difficulties or successes connecting theory and experiment?
- **Homework 7** (Constructing an abstract) - Summarize your entire paper in a single paragraph - You are writing an abstract. It should be self-contained and describe the entire paper in just a few sentences.

After Homework 7 (Fri. Feb. 28), your paper is due. It should be 4-5 pages long not counting figures, equations, and references! It should fully describe all the aspects of the phenomenon that are being modeled including how the phenomenon is modeled, what predictions/implications they are, how it is connected to experiments, and what limitations there are in the modeling of it. There should be about 6-10 references to articles appearing in your paper. You have been continually reading about the topic, right?

Grading the individual project

A rubric for the individual project appears here. Notice that the rubric emphasizes several aspects of the paper with different weights. Each category will be scored on the following scale (4.0, 3.5, 3.0, 2.0, 0.0) and then averaged together using the weights for each category. Your final grade will be this averaged score converted to a 100 point scale.

Pair Project

The second project can (but doesn't have to) build on this first project. It is a team project that you will complete with a partner. It is meant to mimic the common practice of poster preparation and presentation. In a nutshell, you will conduct an original modeling project where you analytically and computationally model some E&M phenomenon of your choosing, prepare a poster of the project, and present it to your classmates and me. In working on this project, you will be trying to answer the following questions:

- What is the area of E&M that you are doing research on?
- What are the questions that you are trying to answer about this area?
- What theoretical models can be used to answer those questions?
- What analytical and computational work did you do to answer those questions?
- What were the resulting predictions that your work produced?
- What are the limitations of what you have done? What are some remaining open questions?
- What did each member of your pair contribute?

There will be six homework questions (10 if you count the 4 self-reflection homework questions) to help your team develop your poster project, Here is a *preliminary* listing of the homework questions that will appear (note that these are subject to change!):

- **Homework 8** - What are you and your partner proposing to do? What area of E&M will you be conducting original calculations for? What source material are you drawing from? What has been done so far and what are you going to do? It's ok if it's a solved problem, but you will need to reproduce what has been done and extend it beyond what your reference material offers.
- **Homework 9** - What is the plan for the next 5 weeks? How do you intend to structure the work? Explain the details of what will be done and who will be doing what. I expect 2-3 paragraphs describing the work and a detailed timeline.
- **Homework 10** - Provide a detailed explanation of the models and theoretical calculations needed to set up your work. This should be presented as a "graphic" that would appear in a poster under "background or model." There will also be a self-reflection homework problem - Who did what? What questions do you need to answer to continue to move forward? What help do you need from me or others?
- **Homework 11** - There should be some sample calculations and figures produced by your code. This can be a notebook (Mathematica; Jupyter; etc.), but the work needs to be explained inline (i.e., what are you doing and why?). This is the work that is the meat of your original contribution. It need not be complete yet. There will also be a self-reflection homework problem - Who did what? What questions were you able to answer last week? What questions do you need to answer to continue to move forward? What help do you need from me or others?
- **Homework 12** - At this point, you should produce draft figures for poster with captions. You should have pressed onward with your calculations and produced appropriate figures for your poster with captions. The "graphic" and caption should be turned in. These need not be complete in the sense that you should continue working on your calculations and models until the poster is turned in. There will also be a self-reflection homework problem - Who did what? What questions were you able to answer last week? What questions do you need to answer to continue to move forward? What help do you need from me or others?
- **Homework 13** - Finally, you should produce an abstract for your poster. You should have a self-contained paragraph on what you will present in your poster. There will also be a self-reflection homework problem - Who did what? What questions were you able to answer last week? What questions do you need to answer to continue to move forward? What help do you need from me or others?

After Homework 13 (Tues. Apr 28 - during our final exam period), your poster is due. You and your teammate will present your poster to your classmates and myself. Your poster and presentation will be graded by me. But, you will also be given evaluation sheets for your classmates' posters, which I'll ask that you share with them (anonymously if you like). Your participation in the evaluation of your classmates' posters counts towards your grade on your poster. There will also be a self-reflection/evaluation component to this assignment that asks you: Who did what? What did you learn? What did you want to learn more about? What was straight-forward? What was more difficult? Completion of this self-reflection/evaluation will also count towards the overall grade on your poster.

Grading the pair project

A rubric for the individual project appears here. Notice that the rubric emphasizes several aspects of the poster with different weights. Each category will be scored on the following scale (4.0, 3.5, 3.0, 2.0, 0.0) and then averaged together using the weights for each category. Your final grade will be this averaged score converted to a 100 point scale.

1.7 Classroom Environment

1.7.1 Commitment to an Inclusive Classroom

I am committed to creating an inclusive classroom - one where you and your classmates feel comfortable, intellectually challenged, and able to speak up about your ideas and experiences. This means that our classroom, our virtual environments, and our interactions need to be as inclusive as possible. Mutual respect, civility, and the ability to listen and observe others are central to creating a classroom that is inclusive. I will strive to do this and I ask that you do the same. If I can do anything to make the classroom a better learning environment for you, please let me know.

If you observe or experience behaviors that violate our commitment to inclusivity, please let me know as soon as possible.

If I violate this principle, please let me know or please tell the undergraduate department chair, Stuart Tessmer (tessmer@pa.msu.edu), who I have informed to tell me about any such incidents without conveying student information to me.

1.7.2 Comments on preparation:

Physics 482 covers material you might have seen before (Many of the topics stem from PHY 184/294H material) but at a higher level of conceptual and mathematical sophistication.

Therefore you should expect:

- a large amount of material covered quickly.
- no recitations, and few examples covered in lecture. Most homework problems are not similar to examples from class.
- long, hard homework problems that usually cannot be completed by one individual alone.
- challenging projects.

Physics 482 is a challenging, upper-division physics course. Unlike more introductory courses, you are fully responsible for your own learning. In particular, you control the pace of the course by asking questions in class. I tend to speak quickly, and questions are important to slow down the lecture. This means that if you don't understand something, it is your responsibility to ask questions. Attending class and the homework help sessions gives you an opportunity to ask questions. I am here to help you as much as possible, but I need your questions to know what you don't understand.

Physics 482 covers some of the most important physics and mathematical methods in the field. Your reward for the hard work and effort will be learning important and elegant material that you will use over and over as a physics major. Here is what I have experienced, and heard from other faculty teaching upper division physics in the past:

- most students reported spending a minimum of 10 hours per week on the homework (!!)
- students who didn't attend the homework help sessions often did poorly in the class.
- students reported learning a tremendous amount in this class.

The course topics that we will cover in Physics 482 are among the greatest intellectual achievements of humans. Don't be surprised if you have to think hard and work hard to master the material.

1.8 Resources

1.8.1 Confidentiality and Mandatory Reporting

College students often experience issues that may interfere with academic success such as academic stress, sleep problems, juggling responsibiities, life events, relationship concerns, or feelings of anxiety, hopelessness, or depression. As your instructor, one of my responsibilities is to help create a safe learning environment and to support you through these situations and experiences. I also have a mandatory reporting responsibility related to my role as a University employee. It is my goal that you feel able to share information related to your life experiences in classroom discussions, in written work, and in one-on-one meetings. I will seek to keep information you share private to the greatest extent possible. However, under Title IX, I am required to share information regarding sexual misconduct, relationship violence, or information about criminal activity on MSU's campus with the University including the Office of Institutional Equity (OIE).

Students may speak to someone confidentially by contacting MSU Counseling and Psychiatric Service (CAPS) (caps.msu.edu, 517-355-8270), MSU's 24-hour Sexual Assault Crisis Line (endrape.msu.edu, 517-372-6666), or Olin Health Center (olin.msu.edu, 517-884-6546).

1.8.2 Spartan Code of Honor Academic Pledge

As a Spartan, I will strive to uphold values of the highest ethical standard. I will practice honesty in my work, foster honesty in my peers, and take pride in knowing that honor is worth more than grades. I will carry these values beyond my time as a student at Michigan State University, continuing the endeavor to build personal integrity in all that I do.

1.8.3 Handling Emergency Situations

In the event of an emergency arising within the classroom, Prof. Caballero will notify you of what actions that may be required to ensure your safety. It is the responsibility of each student to understand the evacuation, "shelter-in-place," and "secure-in-place" guidelines posted in each facility and to act in a safe manner. You are allowed to maintain cellular devices in a silent mode during this course, in order to receive emergency SMS text, phone or email messages distributed by the university. When anyone receives such a notification or observes an emergency situation, they should immediately bring it to the attention of Prof. Caballero in a way that causes the least disruption. If an evacuation is ordered, please ensure that you do it in a safe manner and facilitate those around you that may not otherwise be able to safely leave. When these orders are given, you do have the right as a member of this community to follow that order. Also, if a shelter-in-place or secure-in-place is ordered, please seek areas of refuge that are safe depending on the emergency encountered and provide assistance if it is advisable to do so.

CHAPTER

TWO

WHAT IS MATHEMATICAL MODELING?

Nature reveals itself to us through interactions. We can tell from observations that it is nature's interactions that lead to its evolution. How nature is changing and predicting how it will change in the future is the work of science. In this work, we observe nature and its interactions to make models of those observations. We aim to predict and explain our observations of nature through this building of models.

In physics, our goals are typically to explain and predict observations of physical phenomenon. Here, we focus ourselves to those canonical things that physicists concern themselves with: motion, fields, waves, atoms, nuclei, and so on.

2.1 The Simple Harmonic Oscillator

2.1.1 The SHO model

From these observations we find that a reasonable force model for a 1D spring system is given by:

$$F_{spring} = -k|x - x_0|$$

where k is the spring constant for the spring and x_0 is the equilibrium location. As we know from Newton's second law, if we can argue that this force is the only force or the dominant behavior we are trying to model in a mechanical situation, we have the following ordinary differential equation:

$$F_{net} = F_{spring}$$

$$m\ddot{x} = -k|x - x_0|$$

Without much loss, we can recast the problem in terms of the distance from equilibrium ($s = |x - x_0|$). *As you will learn this is not a problem from vertically hanging springs near the surface of the Earth because the graviational force is constant and depends on the mass of the hanging weight.*

The SHO mathematical model

$$m\ddot{s} = -ks$$

Finding the general solution

We can solve this using our typical approach to 2D linear ODEs; guess the form of the solution and check it.

$$\ddot{s} = -\frac{k}{m}s$$

Assume: $s(t) = A\exp(i\omega t)$ As we will learn this form is a good guess for linear ODEs. This gives:

$$\ddot{s} = -\omega^2\ A\exp(i\omega t)$$

So that,

$$-\omega^2\ A\exp(i\omega t) = -\frac{k}{m}A\exp(i\omega t)$$

This solution is consistent with the model (i.e., solves the differential equation) if we set ω and notice it is exclusively positive:

$$\omega^2 = \frac{k}{m} > 0$$

Thus, $s(t) = A\exp(i\sqrt{\frac{k}{m}}t)$ is **a solution** to this problem. Notice we have to set A using initial conditions for the problem. This is called a **initial value problem** in mathematics. You likely have heard about the Euler formulation; it is one of the most important relationships in mathematics [Nahin, 2011, Stipp, 2017]. We will get into it more later. It appears below:

Leonhard Euler's famous formula

$$\exp i\theta = \cos\theta + i\sin\theta$$

Thus, we know that our solution gives rise to real sinusoidal solutions of the form:

$$s(t) = A\left(cos\sqrt{\frac{k}{m}}t + i\sin\sqrt{\frac{k}{m}}t\right)$$

This probably doesn't look quite right…there's an i in the solution for real physical motion! We will get to that, but we need the initial conditions to do so. More importantly, there's only one free parameter in our current solution, A. Given that this equation is **a solution, is it the only solution?**

The order of your differential equation matters

You probably already know that our general solution for the SHO is:

$$s(t) = A\exp(i\sqrt{\frac{k}{m}}t) + B\exp(-i\sqrt{\frac{k}{m}}t)$$

Or written in a way that is more often presented:

$$s(t) = A'\cos(\sqrt{\frac{k}{m}}t) + B'\sin(\sqrt{\frac{k}{m}}t)$$

where we have made clear that the free parameters A and B are not equal to A' and B', respectively. Regardless, notice there's two free parameters in these formula. That is because we are solving a **two dimensional** linear ODE. Because there's two derivatives on s (i.e., $\ddot{s}$), we expect to have two free parameters that are set by the intial conditions. This is characteristic of an **initial value problem**.

Important Tip

The number of free parameters that are set by problem conditions in an initial value problem is determined by the order of the differential equation.

Second order? Two conditions needed. Two free parameters to determine. Fourth order? Four conditions needed. Four free parameters to determine.

2.2 Numerically Integrating the SHO model

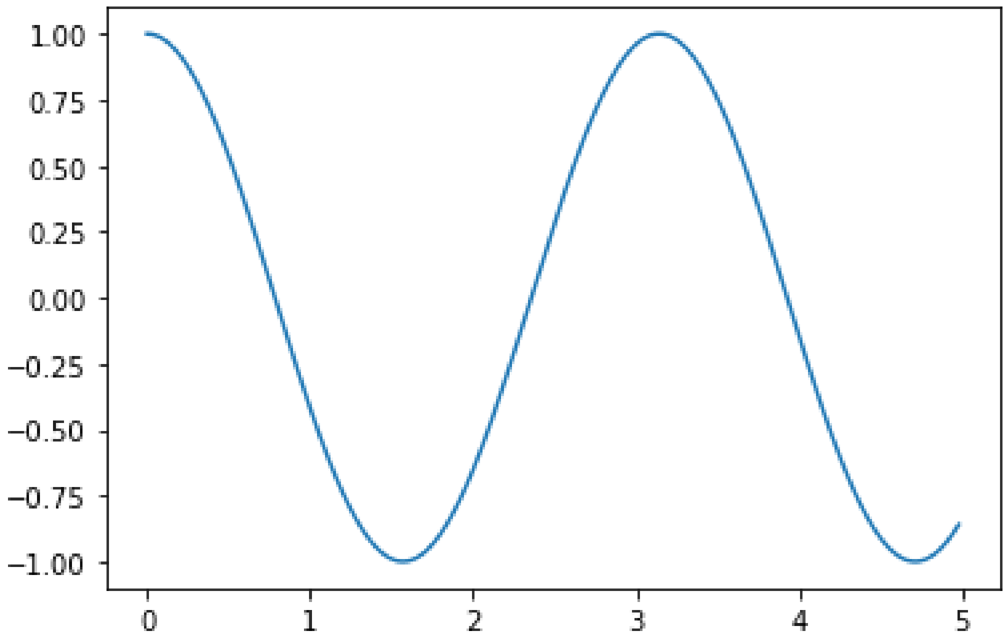
```
import numpy as np
import matplotlib.pyplot as plt

%matplotlib inline
```

```
x0 = 1.0 ## Initial position
v0 = 0.0 ## Initial velocity

omega = 2 ## Angular freq. of SHO

tf = 5 ## Model time
```

```
def AnalyticalSolutionSHO(tf, deltat, amp, omega, t0 = 0, phase = 0):

    if t0 < tf:

        t = np.arange(t0,tf,deltat) ## equal steps
        x = amp*np.cos(omega*t+phase)

        return t,x

    else:

        raise ValueError('Final time is before start time.')
```

```
t,x = AnalyticalSolutionSHO(tf, 0.02, x0, omega)
plt.plot(t,x)
```

```
[<matplotlib.lines.Line2D at 0x1132ebaf0>]
```



$$\ddot{x} = -\omega x$$

$$\frac{d^2x}{dt^2} = -\omega x$$

Introduce $u = \dot{x}$, so that $\dot{u} = \ddot{x}$. We can produce two coupled, linear, first order, ODEs:

$$\dot{u} = -\omega x$$

$$\dot{x} = u$$

Imagine we allow ourselves to take a small step in time Δt, how would u and x change in that time?

$$\dot{x} = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = u$$

So that,

$$x(t + \Delta t) \approx u\Delta t + x(t)$$

Similarly, \$$u(t + \Delta t) \approx -\omega x \Delta t + u(t)$\$

2.2.1 Developing a Numerical Routine

Notice we have two equations that describe how to obtain new values of location (x) and velocity (u) at a time $t + \Delta t$ given information about the system at some earlier time, t, (or at the least, considering the location and velocity at time t), which we take as a pair of **update equations** where the equality holds:

$$x(t + \Delta t) = u\Delta t + x(t)$$

$$u(t + \Delta t) = -\omega x \Delta t + u(t)$$

That is, they can potentially tell us at least in a short time Δt that we can estimate the velocity (u) and location (x) of the oscillator. *We have not shown these can be used repeatedly to produce an estimated trajectory yet.* As you might expect, these are better update equations when Δt is small. But there's another ambiguity:

$$x(t+\Delta t) = \underbrace{u}_{?}\Delta t + x(t)$$

$$u(t+\Delta t) = -\omega\underbrace{x}_{?}\Delta t + u(t)$$

The quantities with the underbrace are ambiguous. Do we use the values of x and u at a time t, $t+\Delta t$, or something else?!

The choice matters

Let's illustrate this with making different choices using the following routine:

Basic ODE Integration Routine

```
initialCond0 = VAL0
initialCond1 = VAL1
...
initialCondN = VALN

startTime = START
stopTime = STOP
steps =  STEPS
deltaT = (stopTime-startTime)/steps

t = startTime

while t < stopTime:

    updatedVal0 = updateEqn0()
    updatedVal1 = updateEqn1()
    ...
    updatedValN = updateEqnN()

    store(updatedVals)

    t += deltaT
```

This might seem quite abstract, so let's make a table of choices for our integration routines:

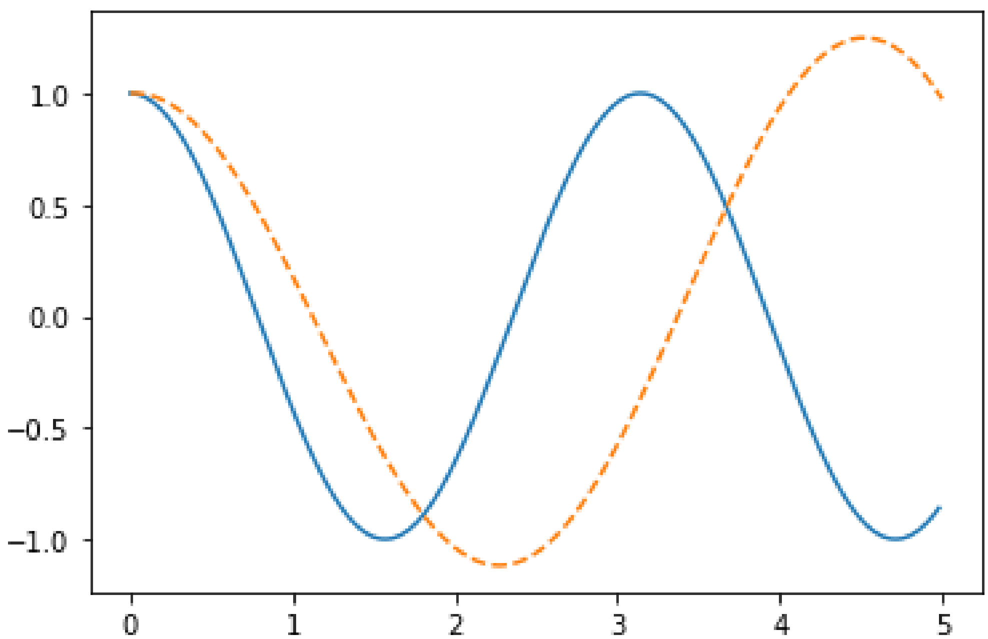
Approach	Value of x	Value of u	Considerations
1	$x(t)$	$u(t)$	We have both of these values to start
2	$x(t)$	$u(t+\Delta t)$	For this, we will need a $u(t+\Delta t)$ estimate first
3	$x(t+\Delta t)$	$u(t)$	Hmm…we will need a $x(t+\Delta t)$ estimate first
4	$x(t+\Delta t)$	$u(t+\Delta t)$	Well this doesn't seem possible to get both estimates at the same time!

It looks like we can try approach 1, 2, and 3 without much fuss. Let's write a few functions.

```
N = 100
steps = np.arange(0, N-1)
deltaT = tf/N
time = np.linspace(0,tf,N)
```

```
##Approach 1
xVals1 = np.zeros(N)
uVals1 = np.zeros(N)

xVals1[0] = x0
uVals1[0] = v0

for i in steps:

    uVals1[i+1] = uVals1[i] + xVals1[i]*deltaT
    xVals1[i+1] = xVals1[i] - omega*uVals1[i]*deltaT

plt.plot(t, x)
plt.plot(time, xVals1, '--')
```

```
[<matplotlib.lines.Line2D at 0x1134a2500>]
```




```
xVals2 = np.zeros(N)
uVals2 = np.zeros(N)

xVals2[0] = x0
uVals2[0] = v0

for i in steps:

    uVals2[i+1] = uVals2[i] + xVals2[i]*deltaT
```

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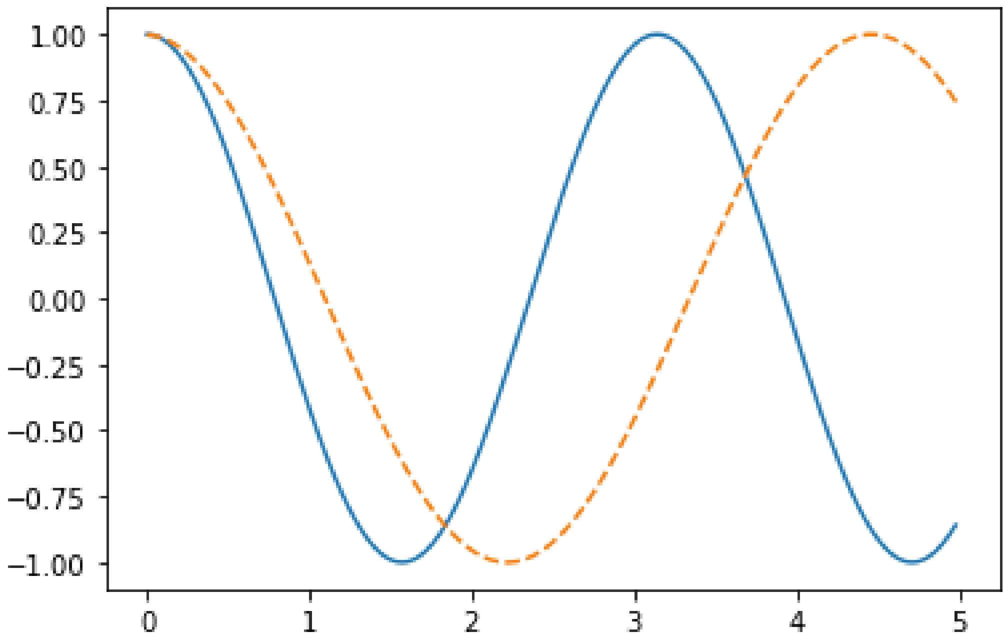
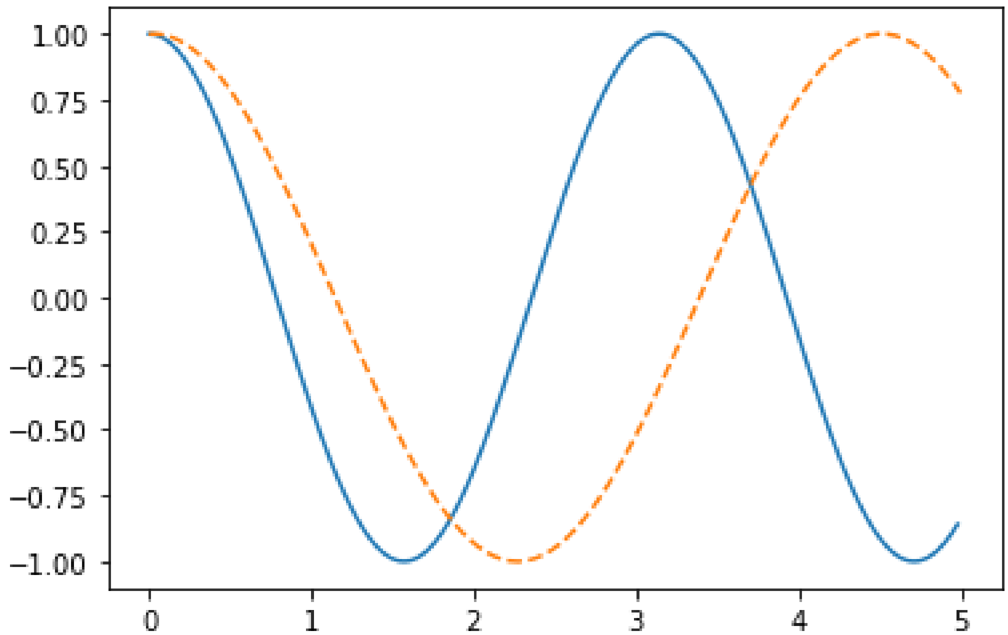
```
    xVals2[i+1] = xVals2[i] - omega*uVals2[i+1]*deltaT

plt.plot(t, x)
plt.plot(time, xVals2, '--')
```

```
[<matplotlib.lines.Line2D at 0x1135253c0>]
```



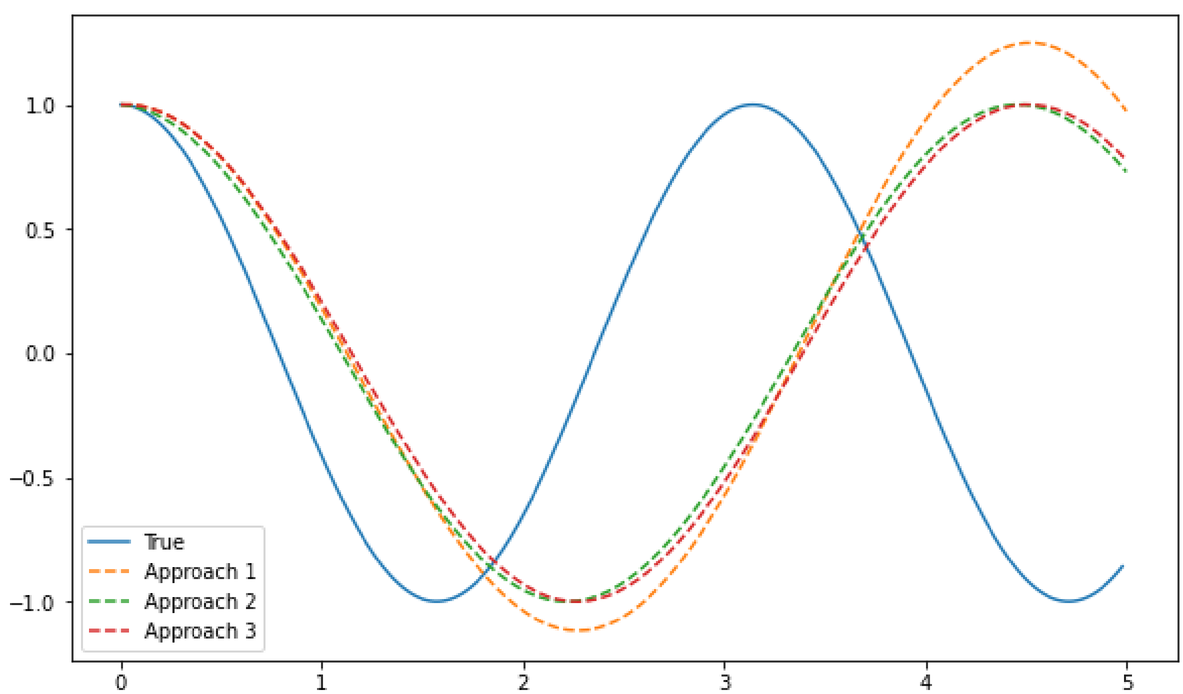

```
xVals3 = np.zeros(N)
uVals3 = np.zeros(N)

xVals3[0] = x0
uVals3[0] = v0

for i in steps:

    xVals3[i+1] = xVals3[i] - omega*uVals3[i]*deltaT
    uVals3[i+1] = uVals3[i] + xVals3[i+1]*deltaT

plt.plot(t, x)
plt.plot(time, xVals3, '--')
```

```
[<matplotlib.lines.Line2D at 0x113591630>]
```



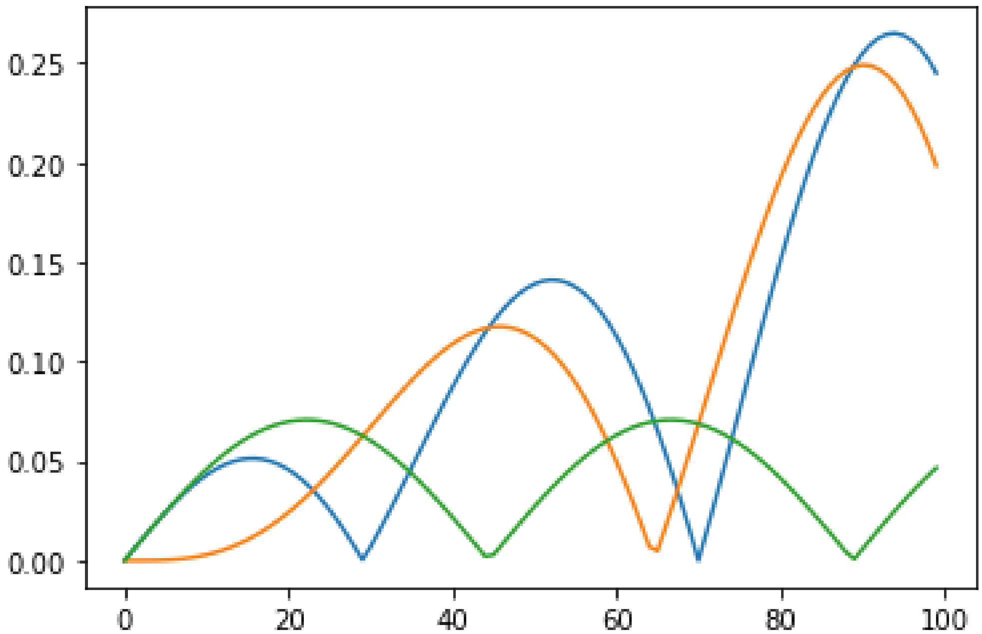

```
plt.figure(figsize=(10,6))
plt.plot(t, x)
plt.plot(time, xVals1, '--')
plt.plot(time, xVals2, '--')
plt.plot(time, xVals3, '--')
plt.legend(['True', 'Approach 1', 'Approach 2', 'Approach 3'])
```

```
<matplotlib.legend.Legend at 0x1135bb2e0>
```



```
plt.plot(np.abs(xVals1-xVals2))
plt.plot(np.abs(xVals1-xVals3))
plt.plot(np.abs(xVals2-xVals3))
```

```
[<matplotlib.lines.Line2D at 0x1136724d0>]
```



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[1] Paul J Nahin. *Dr. Euler's fabulous formula*. Princeton University Press, 2011.

[2] David Stipp. *A most elegant equation: Euler's formula and the beauty of mathematics*. Hachette UK, 2017.