24. Solve Problems 9 and 10 by using an integrating factor as discussed in Problem 2

25. An equation of the form $y' = f(x)y^2 + g(x)y + h(x)$ is called a *Riccati* equation. If we know one particular solution y_p, then the substitution $y = y_p + \frac{1}{z}$ gives a linear first-order equation for z. We can solve this for z and substitute back to find a solution of the y equation containing one arbitrary constant (see Problem 2 Following this method, check the given y_p, and then solve

(a) $y' = xy^2 - \dfrac{2}{x}\,y - \dfrac{1}{x^3}, \quad y_p = \dfrac{1}{x^2};$

(b) $y' = \dfrac{2}{x}\,y^2 + \dfrac{1}{x}\,y - 2x, \quad y_p = x;$

(c) $y' = e^{-x}y^2 + y - e^x, \quad y_p = e^x.$

26. Show that the substitution given in Problem 25 does in general give a solution of the Riccati equation. *Hints:* First show that the substitution $y = y_p + u$ yields the following equation for u: $u' - (g + 2fy_p)u = fu^2$. Note by text equation (4.1) that this is a Bernoulli equation with $n = 2$, so by equation (4.2) we let $z = u^{-1}$. Show that the z equation is the linear first-order equation $z' + (g + 2fy_p)z = -f$. Note that we could have obtained the z equation in one step by substituting $y = y_p + \frac{1}{z}$ in the original equation as claimed in Problem 25.

5. SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE

Because of their importance in applications, we are going to consider carefully the solution of differential equations of the form

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0, \tag{5.1}$$

where a_2, a_1, a_0 are constants; also we shall consider (Section 6) the corresponding equation when the right-hand side of (5.1) is a function of x. Equations of the form (5.1) are called *homogeneous* because every term contains y or a derivative of y. Equations of the form (6.1) are called *inhomogeneous* because they contain a term which does not depend on y. (Note, however, that this use of the term homogeneous is completely unrelated to its use in Section 4.) Although we shall concentrate on second-order equations, which are the ones that occur most frequently in applications, most of our discussion can be extended immediately to linear equations of higher order with constant coefficients (see Problems 21 to 30).

These problems are pretty simple by hand; you may be able to write down answers faster than you can type the problem into a computer! Remember that a computer may not give an answer in the form you need. To use computer solutions effectively, you need to know what to expect, and you can learn this by studying the following methods and doing some problems by hand. Let us consider an equation of the form (5.1).

Example 1. Solve the equation

$$y'' + 5y' + 4y = 0. \tag{5.2}$$

It is convenient to let D stand for d/dx; then

$$Dy = \frac{dy}{dx} = y', \qquad D^2y = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''. \tag{5.3}$$

Expressions involving D, such as $D+1$ or D^2+5D+4, are called *differential operators*. (See Problem 31.) In this notation (5.2) becomes

$$D^2y + 5Dy + 4y = 0 \quad \text{or} \quad (D^2+5D+4)y = 0. \tag{5.4}$$

The *algebraic* expression D^2+5D+4 can be factored as $(D+1)(D+4)$ or $(D+4)(D+1)$. You should satisfy yourself that

$$(D+1)(D+4)y = (D+4)(D+1)y = (D^2+5D+4)y \tag{5.5}$$

when $D = d/dx$, and, in fact, that a similar statement is true for $(D-a)(D-b)$ where a and b are any *constants*. (This is not necessarily true if a and b are functions of x; see Problem 31.) Then we can write (5.2) or (5.4) as

$$(D+1)(D+4)y = 0 \quad \text{or} \quad (D+4)(D+1)y = 0. \tag{5.6}$$

To solve (5.4) [or (5.6) which is the same equation rewritten], we shall first solve the simpler equations

$$(D+4)y = 0 \quad \text{and} \quad (D+1)y = 0. \tag{5.7}$$

These are separable equations (Section 2) with solutions

$$y = c_1e^{-4x}, \quad y = c_2e^{-x}. \tag{5.8}$$

Now if $(D+4)y = 0$, then

$$(D+1)(D+4)y = (D+1)\cdot 0 = 0,$$

so any solution of $(D+4)y = 0$ is a solution of the differential equation (5.6) or (5.4). Similarly, any solution of $(D+1)y = 0$ is a solution of (5.6) or (5.4). Since the two solutions (5.8) are linearly independent [Problem 13; also see Chapter 3, equation (8.5)], a linear combination of them contains two arbitrary constants and so is the general solution. Thus

$$y = c_1e^{-4x} + c_2e^{-x} \tag{5.9}$$

is the general solution of (5.4). Note that we can think of the two solutions e^{-4x} and e^{-x} as basis vectors of a 2-dimensional linear vector space (see Chapter 3, Section 14). Then the general solution (5.9) gives all the vectors of that space. (See Problem 21.)

Now we must investigate whether we can solve all second-order linear equations with constant coefficients (and zero right-hand side) by this method. We first wrote the differential equation using D for d/dx, and then factored the D expression to get (5.5). In this last step, we treated D as if it were an algebraic letter instead of d/dx; this is justified by checking the result (5.5) when $D = d/dx$. Recall from algebra that saying that the algebraic expression D^2+5D+4 has the factors $(D+4)$ and $(D+1)$ is equivalent to saying that the quadratic equation

$$D^2+5D+4 = 0 \tag{5.10}$$

has roots -4 and -1. The equation (5.10) is called the *auxiliary* (or characteristic) equation for the given differential equation (5.2). From equations (5.6) to (5.9), we see that to solve a linear second-order equation with constant coefficients, we should first solve the auxiliary equation; if the roots of the auxiliary equation are a and b $(a \neq b)$, the general solution of the differential equation is a linear combination of e^{ax} and e^{bx}.

(5.11)	$y = c_1 e^{ax} + c_2 e^{bx}$ is the general solution of $(D-a)(D-b)y = 0, \quad a \neq b.$

(If $a = b$, we get only one solution this way; we shall consider this case shortly.) Recall from algebra that the roots of a quadratic equation (with real coefficients; see Problem 19) can be real and unequal, real and equal, or a complex conjugate pair. The equation (5.2) which we have solved is an example in which the roots are real and unequal. Let us consider the other two cases.

Equal Roots of the Auxiliary Equation If the two roots of the auxiliary equation are equal, then the differential equation can be written

(5.12) $$(D-a)(D-a)y = 0,$$

where a is the value of the two equal roots. From our previous discussion (5.5) to (5.11), we know that one solution of (5.12) is $y = c_1 e^{ax}$. But our previous second solution $y = c_2 e^{bx}$ in (5.11) is not a second solution here since $b = a$. To find the second solution for this case, we let

(5.13) $$u = (D-a)y.$$

Then (5.12) becomes

$$(D-a)u = 0,$$

from which we get

(5.14) $$u = Ae^{ax}.$$

We substitute (5.14) into (5.13) to get

$$(D-a)y = Ae^{ax} \quad \text{or} \quad y' - ay = Ae^{ax}.$$

This is a first-order linear equation which we solve as in Section 3:

$$ye^{-ax} = \int e^{-ax} A e^{ax}\, dx = \int A\, dx = Ax + B.$$

Thus

(5.15)	$y = (Ax + B)e^{ax}$ is the general solution of (5.12).

This is the general solution of (5.1) for the case of equal roots of the auxiliary equation. The solution e^{ax} we already know; what is new here is the fact that xe^{ax} is a second (linearly independent; see Problem 14) solution of the differential equation when a is a double root of the auxiliary equation. Equations (5.11) and (5.15) then give the general solution of (5.1) for both unequal and equal roots of the auxiliary equation.

Complex Conjugate Roots of the Auxiliary Equation Suppose the roots of the auxiliary equation are $\alpha \pm i\beta$. These are unequal roots, so by (5.11) the general solution of the differential equation is

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} = e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}). \tag{5.16}$$

There are two other very useful forms of (5.16). If we substitute $e^{\pm i\beta x} = \cos\beta x \pm i\sin\beta x$ [see Chapter 2, equation (9.3)] into (5.16), then the parenthesis becomes a linear combination of $\sin\beta x$ and $\cos\beta x$ and we can write (5.16) as

$$y = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x), \tag{5.17}$$

where c_1 and c_2 are new arbitrary constants. We can also write (5.17) in the form

$$y = ce^{\alpha x}\sin(\beta x + \gamma), \tag{5.18}$$

where c and γ are now the arbitrary constants. An easy way to see that this is correct is to expand $\sin(\beta x + \gamma)$ by the trigonometric addition formula; this gives a linear combination of $\sin\beta x$ and $\cos\beta x$ as in (5.17). Although it is not hard to express any one of the sets of arbitrary constants [A, B in (5.16); c_1, c_2 in (5.17); and c, γ in (5.18)] in terms of either of the other sets, there is seldom any need to do this. In solving actual problems we simply write whichever one of the three forms seems best for the problem at hand and then determine the arbitrary constants in that form from the given data.

Example 2. Solve the differential equation

$$y'' - 6y' + 9y = 0. \tag{5.19}$$

We can write the equation as

$$(D^2 - 6D + 9)y = 0 \quad\text{or}\quad (D-3)(D-3)y = 0. \tag{5.20}$$

Since the roots of the auxiliary equation are equal, we know that the solution is of the form (5.15) and we simply write the result

$$y = (Ax + B)e^{3x}. \tag{5.21}$$

▸ **Example 3.** In Section 16, Chapter 2, we discussed the differential equation for the motion of a mass m oscillating at the end of a spring, and we solved it by guessing the solution. Now let's solve it by the methods of this chapter. The differential equation is [see Chapter 2, equation (16.21)]

$$m\frac{d^2y}{dt^2} = -ky \quad \text{or} \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y = -\omega^2 y \quad \text{if} \quad \omega^2 = \frac{k}{m}. \tag{5.22}$$

We can write this differential equation as

$$D^2y + \omega^2 y = 0 \quad \text{or} \quad (D^2 + \omega^2)y = 0 \tag{5.23}$$

where $D = d/dt$. The roots of the auxiliary equation are $D = \pm i\omega$; the solution may be written in any of the three forms, (5.16), (5.17), or (5.18):

$$\begin{aligned} y &= Ae^{i\omega t} + Be^{-i\omega t} \\ &= c_1 \sin\omega t + c_2 \cos\omega t \\ &= c\sin(\omega t + \gamma). \end{aligned} \tag{5.24}$$

An object whose displacement from equilibrium satisfies (5.22) or (5.24) is said to be executing *simple harmonic motion.* (Recall Chapter 7, Section 2.)

Equations (5.24) are general solutions of (5.22), each containing two arbitrary constants. Let us find a particular solution corresponding to given initial conditions.

▸ **Example 4.** Suppose the mass is held at rest at a distance 10 cm below equilibrium and then suddenly let go. If we agree to call y positive when m is above the equilibrium position, then at $t = 0$, we have $y = -10$, and $dy/dt = 0$. Using the second solution in (5.24), we get

$$\frac{dy}{dt} = c_1\omega\cos\omega t - c_2\omega\sin\omega t,$$

so the initial conditions give

$$\begin{aligned} -10 &= c_1 \cdot 0 + c_2 \cdot 1, \\ 0 &= c_1\omega \cdot 1 - c_2\omega \cdot 0. \end{aligned}$$

Thus we find

$$c_1 = 0, \qquad c_2 = -10,$$

and the particular solution we wanted is

$$y = -10\cos\omega t. \tag{5.25}$$

You can verify that either of the other solutions in (5.24) gives the same particular solution (5.25) for the same initial conditions (Problem 32).

This solution is pretty unrealistic from the practical viewpoint. Equations (5.24) and (5.25) imply that the mass m, once started, will simply oscillate up and down forever! This is certainly not true; what *will* happen is that the oscillations will gradually die down. The reason for the discrepancy between the physical facts and our mathematical answer is that we have neglected "friction" forces.

Example 5. A fairly reasonable assumption for this problem and many other similar ones is that there is a retarding force proportional to the velocity; let us call this force $-l(dy/dt)$ $(l > 0)$. Then (5.22), revised to include this force, becomes

$$m\frac{d^2y}{dt^2} = -ky - l\frac{dy}{dt} \qquad (l > 0) \tag{5.26}$$

or with the abbreviations

$$\omega^2 = \frac{k}{m}, \qquad 2b = \frac{l}{m} \qquad (b > 0)$$

it is

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0. \tag{5.27}$$

To solve (5.27), we find the roots of the auxiliary equation

$$D^2 + 2bD + \omega^2 = 0, \tag{5.28}$$

which are

$$D = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2} = -b \pm \sqrt{b^2 - \omega^2}. \tag{5.29}$$

There are three possible types of answer here depending on the relative size of b^2 and ω^2, and there are three special names given to the corresponding types of motion. We say that the motion is

$$\begin{aligned} &\text{overdamped if} && b^2 > \omega^2,\\ &\text{critically damped if} && b^2 = \omega^2,\\ &\text{underdamped or oscillatory if} && b^2 < \omega^2. \end{aligned}$$

Let us discuss the corresponding general solutions of the differential equation for the three cases.

Overdamped Motion Since $\sqrt{b^2 - \omega^2}$ is real and less than b, both roots of the auxiliary equation are negative, and the general solution is a linear combination of two negative exponentials:

$$y = Ae^{-\lambda t} + Be^{-\mu t}, \qquad \text{where} \quad \begin{cases} \lambda = b + \sqrt{b^2 - \omega^2}, \\ \mu = b - \sqrt{b^2 - \omega^2}. \end{cases} \tag{5.30}$$

Critically Damped Motion Since $b = \omega$, the auxiliary equation has equal roots and the general solution is

$$y = (A + Bt)e^{-bt}. \tag{5.31}$$

In both overdamped and critically damped motion, the mass m is subject to such a large retarding force that it slows down and returns to equilibrium rather than oscillating repeatedly.

Underdamped or Oscillatory Motion In this case $b^2 < \omega^2$ so $\sqrt{b^2-\omega^2}$ is imaginary. Let $\beta = \sqrt{\omega^2 - b^2}$; then $\sqrt{b^2-\omega^2} = i\beta$ and the roots (5.29) of the auxiliary equation are $-b \pm i\beta$. The general solution in the form (5.17) is then

$$y = e^{-bt}(A\sin\beta t + B\cos\beta t) \tag{5.32}$$

This result is more in accord with what we know actually happens to the mass m; because of the factor e^{-bt}, the oscillations in this case decrease in amplitude as time goes on. Also note that the frequency of the damped vibrations, namely $\beta = \sqrt{\omega^2 - b^2}$, is less than the frequency ω of the undamped vibrations.

Although we have stated a rather special physical problem, the mathematics we have just discussed applies to a great variety of problems. First, there are many kinds of mechanical vibrations besides a mass attached to a spring. Think of a tuning fork, a pendulum, the needle on the scale of a measuring device, and as more involved examples, the vibrations of complicated structures such as bridges or airplanes, and the vibrations of atoms in a crystal lattice. In such problems, we need to solve differential equations similar to the ones we have discussed. Differential equations of the same form arise in electricity. Consider equations (1.2) and (1.3) when $V = 0$. Remembering that $I = dq/dt$, we can write (1.2) as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \tag{5.33}$$

and (1.3) as

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0. \tag{5.34}$$

Both these equations are of the form (5.27) which we have solved. Thus there is an analogy between a series circuit and the motion of a mass m described by (5.26); L corresponds to m, R to the "friction" constant l, and $1/C$ to the spring constant k.

▶ PROBLEMS, SECTION 5

Solve the following differential equations by the methods discussed above and compare computer solutions.

1. $y'' + y' - 2y = 0$ **2.** $y'' - 4y' + 4y = 0$

3. $y'' + 9y = 0$ **4.** $y'' + 2y' + 2y = 0$

5. $(D^2 - 2D + 1)y = 0$ **6.** $(D^2 + 16)y = 0$

7. $(D^2 - 5D + 6)y = 0$ **8.** $D(D + 5)y = 0$

9. $(D^2 - 4D + 13)y = 0$ **10.** $y'' - 2y' = 0$

11. $4y'' + 12y' + 9 = 0$ **12.** $(2D^2 + D - 1)y = 0$

Recall from Chapter 3, equation (8.5), that a set of functions is linearly independent if their Wronskian is not identically zero. Calculate the Wronskian of each of the following sets to show that in each case they are linearly independent. For each set, write the differential equation of which they are solutions. Also note that each set of functions is a set of basis functions for a linear vector space (see Chapter 3, Section 14, Example 2) and that the general solution of the differential equation gives all vectors of the vector space.