Chapter 1 Free Oscillations of Simple Systems

1.1 Introduction

The world is full of things that move. Their motions can be broadly categorized into two classes, according to whether the thing that is moving stays near one place or travels from one place to another. Examples of the first class are an oscillating pendulum, a vibrating violin string, water sloshing back and forth in a cup, electrons vibrating (or whatever they do) in atoms, light bouncing back and forth between the mirrors of a laser. Parallel examples of traveling motion are a sliding hockey puck, a pulse traveling down a long stretched rope plucked at one end, ocean waves rolling toward the beach, the electron beam of a TV tube; a ray of light emitted at a star and detected at your eye. Sometimes the same phenomenon exhibits one or the other class of motion (i.e., standing still on the average, or traveling) depending on your point of view: the ocean waves travel toward the beach, but the water (and the duck sitting on the surface) goes up and down (and also forward and backward) without traveling. The displacement pulse travels down the rope, but the material of the rope vibrates without traveling.

We begin by studying things that stay in one vicinity and oscillate or vibrate about an average position. In Chaps. 1 and 2 we shall study many examples of the motion of a closed system that has been given an initial excitation (by some external disturbance) and is thereafter allowed to oscillate freely without further influence. Such oscillations are called *free* or *natural oscillations.* In Chap. 1 study of these simple systems having one or two moving parts will form the basis for our understanding of the free oscillations of systems with many moving parts in Chap. 2. There we shall find that the motion of a complicated system having many moving parts may always be regarded as compounded from simpler motions, called *modes,* all going on at once. No matter how complicated the system, we shall find that each one of its modes has properties very similar to those of a simple harmonic oscillator. Thus for motion of any system in a single one of its modes, we shall find that each moving part experiences the same return force per unit mass per unit displacement and that all moving parts oscillate with the same time dependence $\cos(\omega t + \varphi)$, i.e., with the same frequency ω and the same phase constant φ.

Each of the systems that we shall study is described by some physical quantity whose displacement from its equilibrium value varies with position in the system and time. In the mechanical examples (involving moving parts which are point masses subject to return forces), the physical quantity is the displacement of the mass at the point x,y,z from its equilibrium position. The displacement is described by a vector $\psi(x,y,x,t)$. Sometimes we call this vector function of x, y, z, t a *wave function.* (It is only a continuous function of x, y, and z when we can use the continuous approximation, i.e., when near neighbors have essentially the same motion.) In some of the electrical examples, the physical quantity may be the current in a coil or the charge on a capacitor. In others, it may be the electric field $\mathbf{E}(x,y,z,t)$ or the magnetic field $\mathbf{B}(x,y,z,t)$. In the latter cases, the waves are called electromagnetic waves.

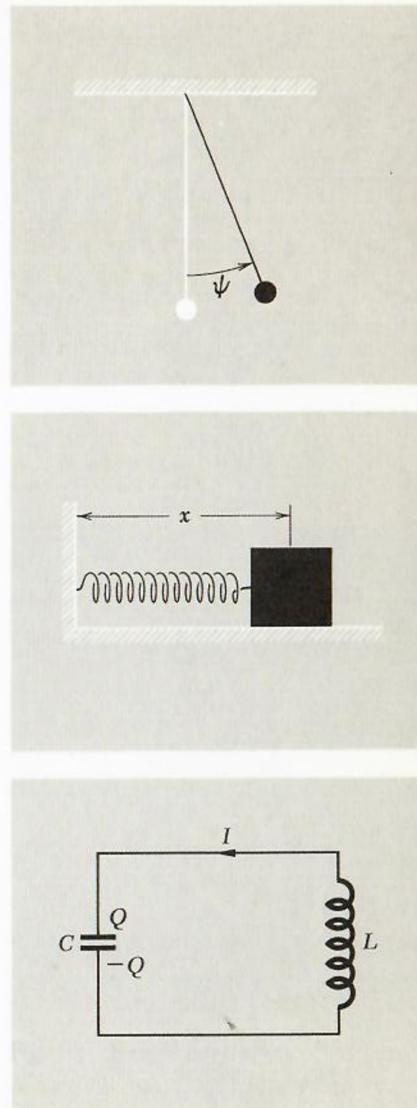

Fig. 1.1 Systems with one degree of freedom. (The pendulum is constrained to swing in a plane.)

1.2 *Free Oscillations of Systems with One Degree of Freedom*

We shall begin with things that stay in one vicinity, oscillating or vibrating about an average position. Such simple systems as a pendulum oscillating in a plane, a mass on a spring, and an LC circuit, whose configuration at any time can be completely specified by giving a single quantity, are said to have one degree of freedom—loosely speaking, one moving part (see Fig. 1.1). For example, the swinging pendulum can be described by the angle that the string makes with the vertical, and the LC circuit by the charge on the capacitor. (A pendulum free to swing in any direction, like a bob on a string, has not one but two degrees of freedom; it takes two coordinates to specify the position of the bob. The pendulum on a grandfather clock is constrained to swing in a plane, and thus has only one degree of freedom.)

For all these systems with one degree of freedom, we shall find that the displacement of the "moving part" from its equilibrium value has the same simple time dependence (called *harmonic oscillation*),

$$\psi(t) = A\cos(\omega t + \varphi). \tag{1}$$

For the oscillating mass, ψ may represent the displacement of the mass from its equilibrium position; for the oscillating LC circuit, it may represent the current in the inductor or the charge on the capacitor. More precisely, we shall find Eq. (1) gives the time dependence provided the moving part does not move too far from its equilibrium position. [For large-angle swings of a pendulum, Eq. (1) is a poor approximation to the motion; for large extensions of a real spring, the return force is not proportional to the extension, and the motion is not given by Eq. (1); a large enough charge on a capacitor will cause it to "break down" by sparking between the plates, and the charge will not satisfy Eq. (1).]

Nomenclature. We use the following nomenclature with Eq. (1): A is a positive constant called the *amplitude;* ω is the *angular frequency,* measured in radians per second; $\nu = \omega/2\pi$ is the *frequency,* measured in cycles per second, or hertz (abbreviated cps, or Hz). The inverse of ν is called the *period* T, which is given in seconds per cycle:

$$T = \frac{1}{\nu}. \tag{2}$$

The *phase constant* φ corresponds to the choice of the zero of time. Often we are not particularly interested in the value of the phase constant. In these cases we can always "reset the clock," so that φ becomes zero, and then we write $\psi = A \cos \omega t$ or $\psi = A \sin \omega t$, instead of the more general Eq. (1).

Return force and inertia. The oscillatory behavior represented by Eq. (1) always results from the interplay of two intrinsic properties of the physical system which have opposite tendencies: *return force* and *inertia*. The "return force" tries to return ψ to zero by imparting a suitable "velocity" $d\psi/dt$ to the moving part. The greater ψ is, the stronger the return force. For the oscillating LC circuit, the return force is due to the repulsive force between the electrons, which makes the electrons prefer not to crowd onto one of the capacitor plates, but rather to distribute themselves equally on each plate, giving zero charge. The second property, "inertia," "opposes" any change in $d\psi/dt$. For the oscillating LC circuit, the inertia is due to the inductance L, which opposes any change in the current $d\psi/dt$ (letting ψ stand for the charge on the capacitor).

Oscillatory behavior. If we start with ψ positive and $d\psi/dt$ zero, the return force gives an acceleration which induces a negative velocity. By the time ψ returns to zero, the negative velocity is maximum. The return force is zero at $\psi = 0$, but the negative velocity now induces a negative displacement. Then the return force becomes positive, but it must now overcome the inertia of the negative velocity. Finally, the velocity $d\psi/dt$ is zero, but by that time the displacement is large and negative, and the process reverses. This cycle goes on and on: the return force tries to restore ψ to zero; in so doing, it induces a velocity; the inertia preserves the velocity and causes ψ to "overshoot." The system oscillates.

Physical meaning of ω^2. The angular frequency of oscillation ω is related to the physical properties of the system in every case (as we shall show) by the relation

$$\boxed{\omega^2 = \text{return force per unit displacement per unit mass.}} \tag{3}$$

Sometimes, as in the case of the electrical examples (LC circuit), the "inertial mass" may not actually be mass.

Damped oscillations. If left undisturbed, an oscillating system will continue to oscillate forever in accordance with Eq. (1). However, in any real physical situation, there are "frictional," or "resistive," processes which "damp" the motion. Thus a more realistic description of an oscillating system is given by a "damped oscillation." If the system is "excited" into oscillation at $t = 0$ (by giving it a bump or closing a switch or something), we find (see Vol. I, Chap. 7, page 209)

$$\psi(t) = Ae^{-t/2\tau} \cos(\omega t + \varphi), \tag{4}$$

for $t \geqq 0$, with the understanding that ψ is zero for $t < 0$. For simplicity we shall use Eq. (1) instead of the more realistic Eq. (4) in the examples that follow. We are thus neglecting friction (or resistance in the case of the LC circuit) by taking the decay time τ to be infinite.

Example 1: Pendulum

A simple pendulum consists of a massless string or rod of length l attached at the top to a rigid support and at the bottom to a "point" bob of mass M (see Fig. 1.2). Let ψ denote the angle (in radians) that the string makes with the vertical. (The pendulum swings in a plane; its configuration is given by ψ alone.) The displacement of the bob, as measured along the perimeter of the circular arc of its path, is $l\psi$. The corresponding instantaneous tangential velocity is $l\, d\psi/dt$. The corresponding tangential acceleration is $l\, d^2\psi/dt^2$. The return force is the tangential component of force. The string does not contribute to this force component. The weight Mg contributes the tangential component $-Mg \sin \psi$. Thus Newton's second law (mass times acceleration equals force) gives

$$\frac{Ml\, d^2\psi}{dt^2} = -Mg \sin \psi(t). \tag{5}$$

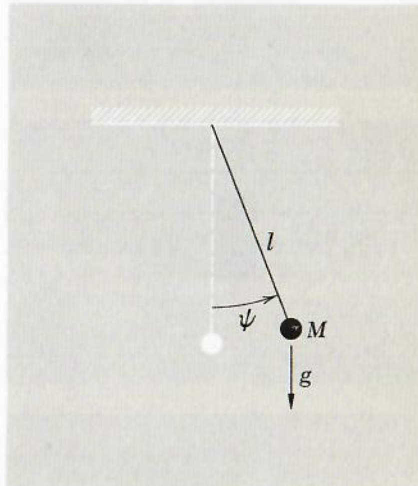

Fig. 1.2 Simple pendulum.

We now use the Taylor's series expansion [Appendix, Eq. (4)]

$$\sin \psi = \psi - \frac{\psi^3}{3!} + \frac{\psi^5}{5!} - \cdots, \tag{6}$$

where the ellipsis $(\cdots)$ denotes the rest of the infinite series. We see that for sufficiently small ψ (in radians, remember), we can neglect all terms in Eq. (6) except the first one, ψ. You may ask, "How small is 'sufficiently small'?" That question has no universal answer—it depends on how accurately you can determine the function $\psi(t)$ in the experiment you have in mind (this is physics, remember—nothing is perfectly measurable) and on how much you care. For example, for $\psi = 0.10$ rad (5.7 deg), $\sin \psi$ is 0.0998; in some problems "$0.0998 = 0.1000$" is a poor approximation. For $\psi = 1.0$ rad (57.3 deg), $\sin \psi$ is 0.841; in some problems "$0.8 = 1.0$" is an adequate approximation.

If we retain only the first term in Eq. (6), then Eq. (5) takes on the form

$$\frac{d^2\psi}{dt^2} = -\omega^2\psi, \tag{7}$$

where

$$\omega^2 = \frac{g}{l}. \tag{8}$$

The general solution of Eq. (7) is the harmonic oscillation given by

$$\psi(t) = A \cos(\omega t + \varphi).$$

Note that the angular frequency of oscillation, given by Eq. (8), can be written

$$\omega^2 = \text{return force per unit displacement per unit mass:}$$

$$\omega^2 = \frac{Mg\psi}{(l\psi)M} = \frac{g}{l},$$

using the approximation that $\sin\psi$ equals ψ.

The two constants A and φ are determined by the initial conditions, i.e., by the displacement and velocity at time $t = 0$. (Since ψ is an angular displacement, the corresponding "velocity" is the angular velocity $d\psi/dt$.) Thus we have

$$\psi(t) = A\cos(\omega t + \varphi),$$
$$\dot{\psi}(t) \equiv \frac{d\psi(t)}{dt} = -\omega A \sin(\omega t + \varphi),$$

so that

$$\psi(0) = A\cos\varphi,$$
$$\dot{\psi}(0) = -\omega A \sin\varphi.$$

These two equations may be solved for the positive constant A and for $\sin\varphi$ and $\cos\varphi$ (which determine φ).

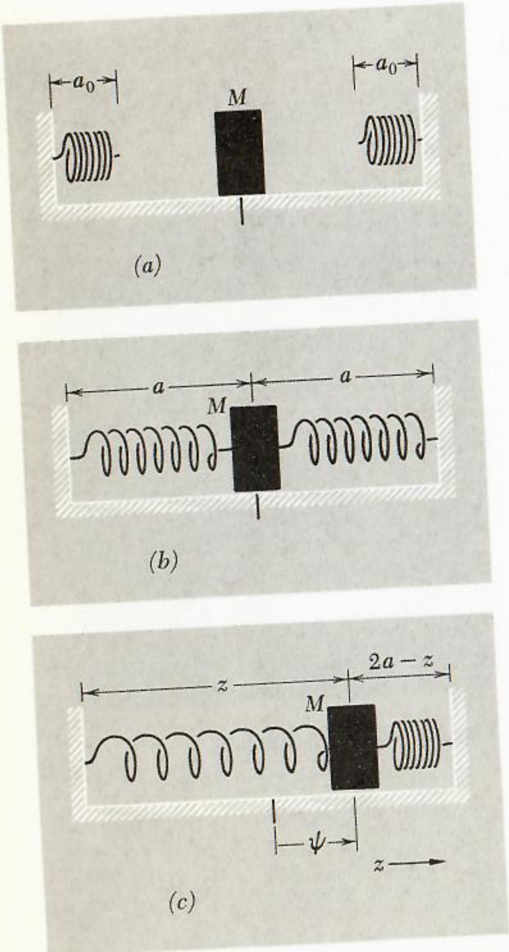

Fig. 1.3 Longitudinal oscillations. (a) Springs relaxed and unattached. (b) Springs attached, M at equilibrium

Example 2: Mass and springs—longitudinal oscillations

Mass M slides on a frictionless surface. It is connected to rigid walls by means of two identical springs, each of which has zero mass, spring constant K, and relaxed length a_0. At the equilibrium position, each spring is stretched to length a, and thus each spring has tension $K(a - a_0)$ at equilibrium (see Fig. 1.3a and b). Let z be the distance of M from the left-hand wall. Then its distance from the right-hand wall is $2a - z$ (see Fig. 1.3c). The left-hand spring exerts a force $K(z - a_0)$ in the $-z$ direction. The right-hand spring exerts a force $K(2a - z - a_0)$ in the $+z$ direction. The