5 LINEAR SYSTEMS

5.0 Introduction

As we've seen, in one-dimensional phase spaces the flow is extremely confined—all trajectories are forced to move monotonically or remain constant. In higher-dimensional phase spaces, trajectories have much more room to maneuver, and so a wider range of dynamical behavior becomes possible. Rather than attack all this complexity at once, we begin with the simplest class of higher-dimensional systems, namely *linear systems in two dimensions*. These systems are interesting in their own right, and, as we'll see later, they also play an important role in the classification of fixed points of *nonlinear* systems. We begin with some definitions and examples.

5.1 Definitions and Examples

A ***two-dimensional linear system*** is a system of the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where a, b, c, d are parameters. If we use boldface to denote vectors, this system can be written more compactly in matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Such a system is ***linear*** in the sense that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Notice that $\dot{\mathbf{x}} = \mathbf{0}$ when $\mathbf{x} = \mathbf{0}$, so $\mathbf{x}^* = \mathbf{0}$ is always a fixed point for any choice of A.

The solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualized as trajectories moving on the (x, y) plane, in this context called the ***phase plane***. Our first example presents the phase plane analysis of a familiar system.

EXAMPLE 5.1.1:

As discussed in elementary physics courses, the vibrations of a mass hanging from a linear spring are governed by the linear differential equation

$$m\ddot{x} + kx = 0 \qquad (1)$$

where m is the mass, k is the spring constant, and x is the displacement of the mass from equilibrium (Figure 5.1.1). Give a phase plane analysis of this ***simple harmonic oscillator***.

meta messages

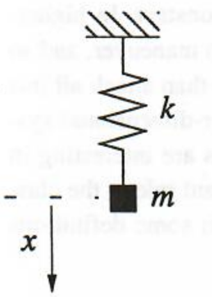

Figure 5.1.1

Solution: As you probably recall, it's easy to solve (1) analytically in terms of sines and cosines. But that's precisely what makes linear equations so special! For the *nonlinear* equations of ultimate interest to us, it's usually impossible to find an analytical solution. We want to develop methods for deducing the behavior of equations like (1) *without actually solving them.*

The motion in the phase plane is determined by a vector field that comes from the differential equation (1). To find this vector field, we note that the ***state*** of the system is characterized by its current position x and velocity v; if we know the values of *both* x and v, then (1) uniquely determines the future states of the system. Therefore we rewrite (1) in terms of x and v, as follows:

$$\dot{x} = v \qquad (2a)$$
$$\dot{v} = -\tfrac{k}{m}x. \qquad (2b)$$

Equation (2a) is just the definition of velocity, and (2b) is the differential equation (1) rewritten in terms of v. To simplify the notation, let $\omega^2 = k/m$. Then (2) becomes

$$\dot{x} = v \qquad (3a)$$
$$\dot{v} = -\omega^2 x. \qquad (3b)$$

The system (3) assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point (x, v), and therefore represents a ***vector field*** on the phase plane.

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shown in Figure 5.1.2.

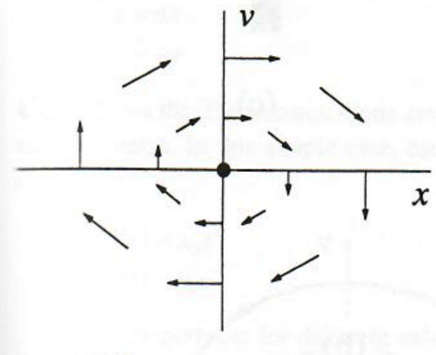

Figure 5.1.2

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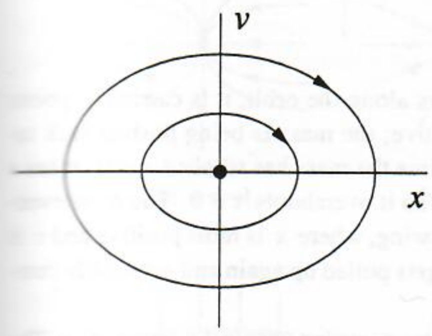

Figure 5.1.3

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Figure 5.1.4).

For example, let's see what the vector field looks like when we're on the x-axis. Then $v = 0$ and so $(\dot{x}, \dot{v}) = (0, -\omega^2 x)$. Hence the vectors point vertically downward for positive x and vertically upward for negative x (Figure 5.1.2). As x gets larger in magnitude, the vectors $(0, -\omega^2 x)$ get longer. Similarly, on the v-axis, the vector field is $(\dot{x}, \dot{v}) = (v, 0)$, which points to the right when $v > 0$ and to the left when $v < 0$. As we move around in phase space, the vectors change direction as shown in Figure 5.1.2.

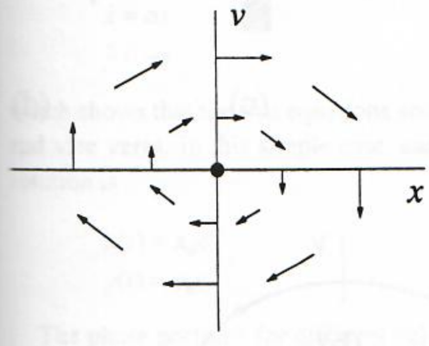


Figure 5.1.2

Just as in Chapter 2, it is helpful to visualize the vector field in terms of the motion of an imaginary fluid. In the present case, we imagine that a fluid is flowing steadily on the phase plane with a local velocity given by $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$. Then, to find the trajectory starting at (x_0, v_0), we place an imaginary particle or ***phase point*** at (x_0, v_0) and watch how it is carried around by the flow.

Concept + practice

The flow in Figure 5.1.2 swirls about the origin. The origin is special, like the eye of a hurricane: a phase point placed there would remain motionless, because $(\dot{x}, \dot{v}) = (0,0)$ when $(x, v) = (0,0)$; hence the origin is a ***fixed point***. But a phase point starting anywhere else would circulate around the origin and eventually return to its starting point. Such trajectories form ***closed orbits***, as shown in Figure 5.1.3. Figure 5.1.3 is called the ***phase portrait*** of the system—it shows the overall picture of trajectories in phase space. they should interact

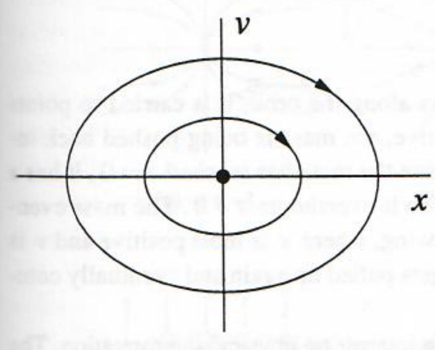
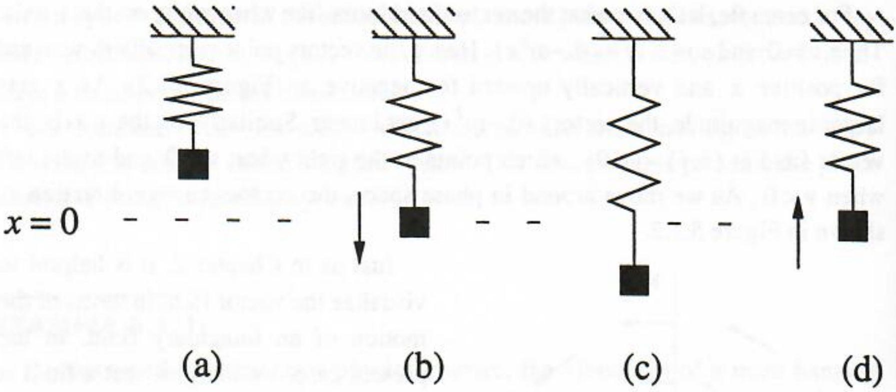


Figure 5.1.3

What do fixed points and closed orbits have to do with the original problem of a mass on a spring? The answers are beautifully simple. The fixed point $(x, v) = (0,0)$ corresponds to static equilibrium of the system: the mass is at rest at its equilibrium position and will remain there forever, since the spring is relaxed. The closed orbits have a more interesting interpretation: they correspond to periodic motions, i.e., oscillations of the mass. To see this, just look at some points on a closed orbit (Figure 5.1.4). When the displacement x is most negative, the velocity v is zero; this corresponds to one extreme of the oscillation, where the spring is most compressed (Figure 5.1.4).



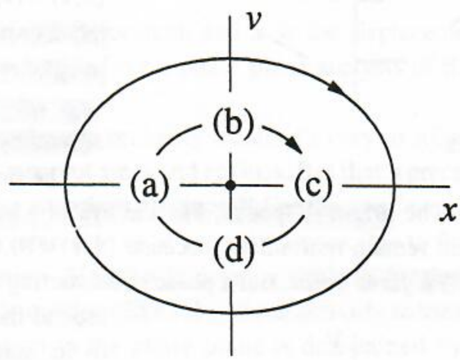


Figure 5.1.4

In the next instant as the phase point flows along the orbit, it is carried to points where x has increased and v is now positive; the mass is being pushed back toward its equilibrium position. But by the time the mass has reached $x = 0$, it has a large positive velocity (Figure 5.1.4b) and so it overshoots $x = 0$. The mass eventually comes to rest at the other end of its swing, where x is most positive and v is zero again (Figure 5.1.4c). Then the mass gets pulled up again and eventually completes the cycle (Figure 5.1.4d).

The shape of the closed orbits also has an interesting physical interpretation. The orbits in Figures 5.1.3 and 5.1.4 are actually *ellipses* given by the equation $\omega^2 x^2 + v^2 = C$, where $C \geq 0$ is a constant. In Exercise 5.1.1, you are asked to derive this geometric result, and to show that it is equivalent to conservation of energy. ■

EXAMPLE 5.1.2:

Solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$. Graph the phase portrait