We have shown that the Hamiltonian can be written of the QHO as, (1)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

And we seek eigenvalues and eigenstates for

$$H|E\rangle = E|E\rangle \quad \text{or}$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_E(x) + \frac{1}{2}m\omega^2 x^2\varphi_E(x) = E\varphi_E(x)$$

Introducing the operator approach

Instead of seeking a solution directly to the Diffy Q, we will introduce a new approach that relies on operators and commutation relations → why?

⇒ Because it is more widely applicable to future QM systems than brute forcing the Diffy Q.

Notice:

$$\underbrace{\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)}_{\text{Sum of squares!}} \varphi_E = E\varphi_E$$

when things commute nicely,

$$(u^2+v^2) = (u^2 \underbrace{- iuv + iuv}_{\text{commute so } =0} + v^2) = (u+iv)(u-iv)$$

Now this approach will work for us even when things don't commute like $[\hat{x},\hat{p}] = i\hbar$, but we need to pay attention to order.

Raising & Lowering (Ladder) Operators

Let's first rewrite $\hat{H}$ to have some dimensionless parts,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{1}{2}m\omega^2\left[\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right]$$

$$= \frac{\hbar\omega}{\hbar\omega}\frac{1}{2}m\omega^2\left[\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right] = \underbrace{\hbar\omega}_{\text{units of E}}\underbrace{\left[\frac{m\omega}{2\hbar}\left\{\hat{x}^2 + \frac{\hat{p}^2}{\omega^2 m^2}\right\}\right]}_{\text{unitless}}$$

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$$\boxed{\hat{H} = \hbar\omega\left\{\frac{m\omega}{2\hbar}\left[\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right]\right\}}$$

Now our goal is to factor the Hamiltonian.
That is the key to this method.
(Like $u^2+v^2 = (u+iv)(u-iv)$ we want to factor H.)

We introduce a & $a^\dagger$ ("a" and "a-dagger")

Non-Hermitian operators

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}^\dagger - i\frac{\hat{p}^\dagger}{m\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$

So that

$$a^\dagger a = \frac{m\omega}{2\hbar}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$= \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + i\frac{\hat{x}\hat{p}}{m\omega} - i\frac{\hat{p}\hat{x}}{m\omega}\right)$$

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$$a^{\dagger}a = \frac{m\omega}{2\hbar}\Big(\underbrace{\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}}_{\text{our squared terms in } H} + \frac{i}{m\omega}\big[\underbrace{\hat{x}\hat{p} - \hat{p}\hat{x}}_{[\hat{x},\hat{p}]!\ = i\hbar}\big]\Big)$$

$\Leftarrow$ result from lack of commuting

$$a^{\dagger}a = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}[\hat{x},\hat{p}]\right)$$

So,

$$a^{\dagger}a = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\right)$$

$$\boxed{a^{\dagger}a = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right) - \frac{1}{2}}$$

with

$$\hat{H} = \hbar\omega\left\{\frac{m\omega}{2\hbar}\left[\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right]\right\}$$

$$\boxed{\hat{H} = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right)}$$

QHO Hamiltonian in terms of a & $a^{\dagger}$

We can show $aa^{\dagger} = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right) + \frac{1}{2}$

So that

$$\boxed{[a, a^{\dagger}] = aa^{\dagger} - a^{\dagger}a = 1}$$

How does this relate back to our original objective? $H|E\rangle = E|E\rangle$? 5

using this form of H we can see how H acts on $a|E\rangle$ and we will uncover the energy spectrum.

$$[H,a] = Ha - aH$$

$$= \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)a - a\hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

$$[H,a] = \hbar\omega\left(a^\dagger a a - a a^\dagger a\right)$$

note $aa^\dagger = 1 + a^\dagger a$ from $[a,a^\dagger] = 1$

$$[H,a] = \hbar\omega\left(a^\dagger a a - (1 + a^\dagger a)a\right)$$

$$= \hbar\omega\left(a^\dagger a a - a - a^\dagger a a\right) = -\hbar\omega a$$

$$\boxed{[H,a] = -\hbar\omega a}$$

Similarly

$$\boxed{[H,a^\dagger] = +\hbar\omega a^\dagger}$$

Let's operate with H on $a|E\rangle$, ⑥

$$H(a|E\rangle) = Ha|E\rangle$$

$$Ha = aH - \hbar\omega a \quad \text{from } [H, a] = -\hbar\omega a$$

$$Ha|E\rangle = aH|E\rangle - \hbar\omega a|E\rangle$$

$H|E\rangle = E|E\rangle$ as $|E\rangle$ is assumed to be an energy eigenstate

$$Ha|E\rangle = aE|E\rangle - \hbar\omega a|E\rangle$$

$$Ha|E\rangle = (E-\hbar\omega)(a|E\rangle)$$

So $a|E\rangle$ is an unnormalized energy eigenstate with eigenvalue $E-\hbar\omega$!

What about $a^\dagger$?

$$H(a^\dagger|E\rangle) = Ha^\dagger|E\rangle$$

$$Ha^+ = a^+H + \hbar\omega a^+$$

$$Ha^+|E\rangle = a^+H|E\rangle + \hbar\omega a^+|E\rangle$$

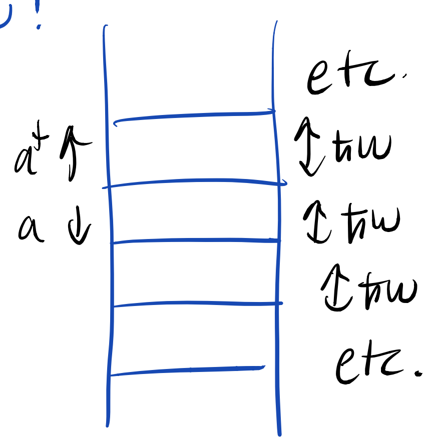
$$= a^+E|E\rangle + \hbar\omega a^+|E\rangle$$

$$Ha^+|E\rangle = (E+\hbar\omega)(a^+|E\rangle)$$

So! $a^+|E\rangle$ is an unnormalized eigenstate of H with eigenvalue $E+\hbar\omega$!

Now we can see how these are "raising & lowering" or ladder operators.

The energy rungs (the spectrum) are spaced by $\hbar\omega$!



We can find the full spectrum by realizing there's some ground state where ⑧

$$a|E_{ground}\rangle = 0$$ no more lower states!

ladder termination condition

so with H,

$$\hat{H}|E_{ground}\rangle = \hbar\omega\left(a^{+}a + \tfrac{1}{2}\right)|E_{ground}\rangle$$

$$= \hbar\omega\, a^{+}\underbrace{a|E_{ground}\rangle}_{0\,!} + \tfrac{1}{2}\hbar\omega|E_{ground}\rangle$$

$$\hat{H}|E_{ground}\rangle = \tfrac{1}{2}\hbar\omega\,|E_{ground}\rangle$$

$$\boxed{E_{ground} = \tfrac{1}{2}\hbar\omega\,!}$$

Thus the spectrum is,

$$\boxed{E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots}$$

We can write some of the results compactly by assuming $|n\rangle$ is the eigenstate.

$$\boxed{\begin{gathered} H|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle \\ \langle n|n\rangle = 1 \qquad \langle m|n\rangle = \delta_{mn} \end{gathered}}$$

We will explore position representations later.