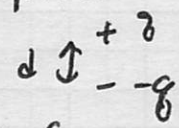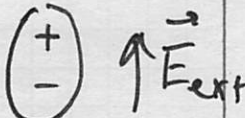# Electric Fields + Non-Conducting Materials

Up to now, we focused on finding the electric field in vacuum or in metals ($E=0$). While this helps us understand many of the techniques and principles surrounding the electric field, non-conducting (or weakly conducting) matter is everywhere. It would behoove us to try to understand how electric fields in these kinds of materials work and what tools we need to understand them well.

Clicker Question: ⊕ [insulator diagram] — insulator.

To help us make sense of "insulating materials" (the term we use for these broad classes of non/weak conductors), we will need to bring to bear our understanding of the dipole again.
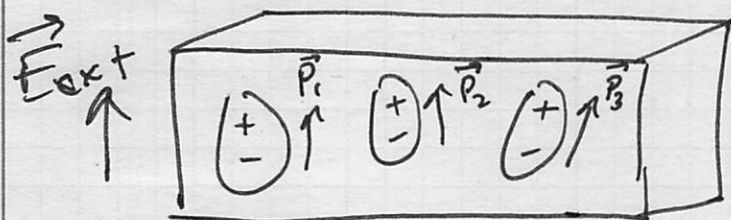
We argued that the physical dipole has a dipole moment, $\vec{p} = q\vec{d}$ $d\updownarrow \begin{smallmatrix}+q\\-q\end{smallmatrix}$

(and that in general we have $\vec{p} = \int_V \rho \vec{r} \, d\tau$).

In insulating materials, the electron cloud is fairly tightly bound to the nucleus (compared to metals where one or more electrons is pretty free to move.)

We model these atoms as a dipole: $\begin{pmatrix}+\\-\end{pmatrix}$ $\uparrow \vec{E}_{ext}$

Dielectric Material

Our model for a dielectric material (another way to refer to these weakly conducting materials) is a collection of these dipoles.

In an external field, these dipoles will tend to align with the electric field.

+ In this situation, we refer to the material as being polarized. Note: this effect isn't nearly as dramatic as it is for metals where the electrons are completely free (in our model).

+ Not all the dipoles line up exactly the same way. At the atomic level, there's quite a bit of variation between the dipole alignments. So, we consider the effects at slightly larger scales 1000s - 10000s of atoms in some volume ⟹ the average polarization in this volume.

Polarization, $\vec{P}$

Polarization, $\vec{P}$, is the average dipole moment over some volume of the material.

$$\vec{P} \equiv \frac{1}{\text{volume}} \sum_i \vec{P_i}$$

Small fluctuations will wash out by doing this averaging.

What are the units of $\vec{P}$?

$$\vec{P} = \frac{1}{V} \sum_i \vec{P_i} \qquad [\vec{P_i}] = [C \, m]$$

$$[\vec{P}] = \left[\frac{C}{m^2}\right] \longleftarrow \left[\frac{1}{V}\right] = \left[\frac{1}{m^3}\right]$$

The polarization, $\vec{P}$, has units of $charge/length^2$.

An object (macroscopically-sized) can develop or be forced to develop a polarization, $\vec{P}$. In that case, it can make sense to find the total dipole moment of the object, $\vec{P}_{tot}$.

One way to do that is to add all the little dipoles, $\vec{P}_{tot} = \sum_i \vec{P_i}$

But an equivalent way (or nearly so), is to use the average polarization, $\vec{P}$.

$$\vec{P}_{tot} = \int_V \vec{P} \, d\tau \qquad \text{where } \vec{P} \text{ might depend on location.}$$

Clicker Questions: (111) and [111]

Why are focused on dipoles anyway?

Well, for these materials (unless we add free charges to them, which we eventually will), it is the case that they are typically neutral. from our study of the multipole expansion we know,
$$V(\vec{r}) \simeq \frac{Q_{tot}}{4\pi\epsilon_0} \frac{1}{r} + \frac{\vec{P} \cdot \hat{r}}{4\pi\epsilon_0 r^2} + \text{higher terms}$$

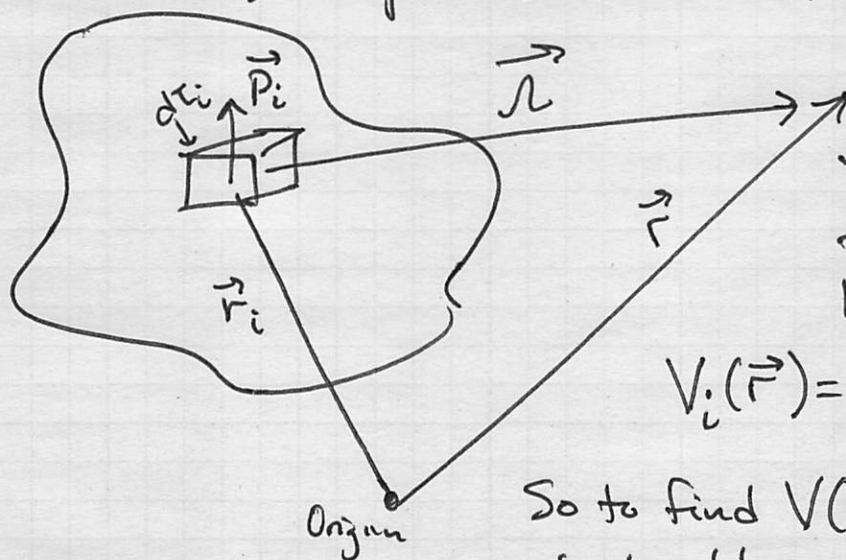So the monopole term will be zero because $Q_{tot} = 0$ for neutral materials,

$$V_{mono} = \frac{\cancel{Q_{tot}}^{0}}{4\pi\epsilon_0 r} = 0$$

Thus, the leading order term in our expansion will be the dipole term,

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\vec{P} \cdot \hat{r}}{r^2}$$

So by polarizing a neutral material, we find that V_{dip} leads!

Let's take a chunk of neutral, but polarized material

The potential from that little chunk will be,

$$V_i(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{P}_i \cdot \hat{\iota}}{\iota^2}$$

So to find $V(\vec{r})$ we could just add up V_i's. $V = \sum_i V_i$

But, instead we could use the average polarization, $\vec{P}$ and integrate over the volume. Here, $\vec{P}_i = \vec{P} d\tau_i$ which would suggest that,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}(\vec{r}') \cdot \hat{\iota} \, d\tau'}{\iota^2}$$

This form of $V(\vec{r})$ allows $\vec{P}$ to vary with location.

We can (and will later in the notes)
Show that our description of $V(\vec{r})$
Can be rewritten as the sum of two
integrals,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}\cdot\hat{n}\, d\tau'}{\imath^2}$$

$$\Downarrow$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_b(\vec{r}')\, da'}{\imath} + \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_b(\vec{r}')\, d\tau'}{\imath}$$

- The first term (surface integral) is the potential
  due to ordinary surface changes (bound to the surface)
- The second term (volume integral) is the potential
  due to ordinary volume changes (bound inside the
  material)

From this proof (which appears later in the
notes), we find that σ_b & ρ_b,
are related to the polarization $\vec{P}$.

$$\sigma_b \equiv \vec{P}\cdot\hat{n} = \text{the normal component of the}$$
average polarization at the surface.

$$\rho_b \equiv -\nabla\cdot\vec{P} = \text{the divergence of the average}$$
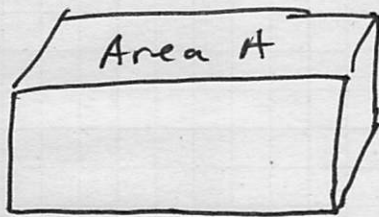polarization.

Clicker Questions: ↑↑↑ & ↑↑↑
                    ↑↑↑   ↑↑↑

So these charge distributions, σ_b & ρ_b, ~~and~~ are
bound to the object. To the outside world,
the object looks like an ordinarly charged
object with these specific distributions, so
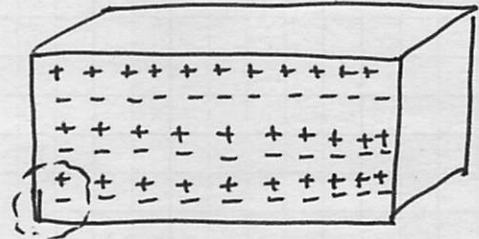we can bring what we know from elsewhere on this.

Before we try to do any calculations, let's understand the meaning and origin of these two terms (conceptually).

What's the origin of $\vec{P} \cdot \hat{n} = \sigma_b$?

Consider a rectangular chunk of dielectric material that you polarize (maybe, by putting it into a big capacitor). It's neutral to begin with.

polarize

↳ Each atom will produce a dipole with $\vec{p}_i = q\vec{d}$ (d here is quite small)

You can probably see that there's a "net" charge on the top and bottom face of the block $+\sigma$ on top and $\bullet -\sigma$ on the bottom.
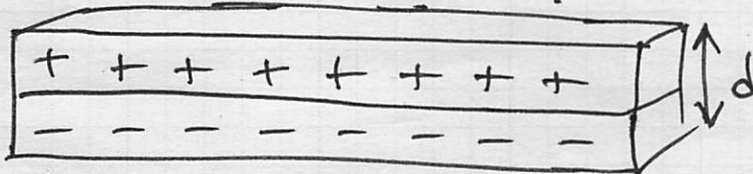
- Let's define the number of atoms/volume.
  if the block has volume, V, and total number of atoms, N, then the number of atoms/volume is just, n.

$$n = N/V$$

(this will come into our determination of the value of P.)

We intend to find how big σ is on the top surface, so let's take a thin slice at the top of width, d.

Because of the polarization some amount $+Q$ has crossed the mid line of this slice and landed on top.

Now this layer is very thin (on the order
of an atomic length scale), so for the
sake of the argument we are making
the volume charge density we seek $\rho = Q/V$,
can be thought of as a surface density, σ,
but we need convert $\rho \rightarrow \sigma$ using d.

maybe
↳ <u>Clicker Question</u>: $\rho \rightarrow \sigma$?

So, Q on the top of the block is
equal to the total number of atoms time
the charge of each.

$$Q = nAdq \qquad \text{where} \quad n \equiv N/V \text{ (remember?)}$$

So the charge density σ on the top of
the block is,

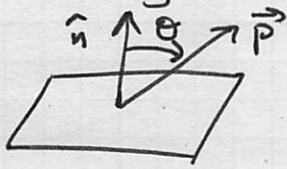$$\sigma = Q/A = nqd = n\rho$$

So the charge density that accumulates at the
top surface is determined simply by the
dipole moment of each (or any) atom times
the number density.

Let's compare that to the average Polarization,

$$\underline{P} = \frac{\text{polarization}}{\text{volume}} = \frac{(\text{number of atoms})(\text{dipole moment}/\text{atom})}{(\text{volume})}$$

$$= \frac{(nV)(qd)}{(V)} = nqd = n\rho \;!!$$

So $\qquad \sigma_{\text{on top}} = \underline{P}$ $\left(\begin{array}{l}\text{so } \sigma \text{ is equal to} \\ \text{the amount of charge} \\ \text{crossing area A when} \\ \text{the material is polarized}\end{array}\right)$

What if $\vec{p}$ was not parallel to the normal of my area?

The amount of charge passing through the area A depends on $\cos\theta$. So,

$$\sigma = P\cos\theta = \vec{P}\cdot\hat{n}$$

- This is a conceptual way to think about how σ_b shows up on the surfaces of a polarized dielectric.

- Note: Throughout the bulk of the material, every area has Q passing through it due to the polarization. But everywhere Q leaving (i.e. going up) is equal to Q entering (from below), and this has no impact anywhere except right at the surface, where the charge "builds up". $\sigma_b = \vec{P}\cdot\hat{n} = $ "bound surface charge"

- This bound charge is real physical charge; there is a an excess of $+\sigma$ on top and $-\sigma$ on the bottom.

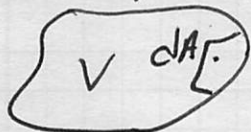- these charges can create external electric fields

What's the origin of $\rho_b = -\vec{\nabla}\cdot\vec{P}$?

If $\vec{P}$ is uniform all we get is surface charge, σ_b.

- But, if $\vec{P}(\vec{r})$ varies ~~vary~~ then we will find that a bound charge distribution in the bulk can develop.

- Locally, it could be the case that the charge leaving some region is not equal to the charge entering (any more).

We just argued that $\dfrac{Q}{Area} = \vec{P}\cdot\hat{n}$

This is charge that leaves and passes through this dA

So the total charge leaving is,

$$\oint_S \vec{P}\cdot\hat{n}\, dA = \int_V \vec{\nabla}\cdot\vec{P}\, d\tau \quad \left(\begin{array}{c}\text{Divergence}\\\text{theorem!}\end{array}\right)$$

Q leaving through some surface, S $\quad\equiv\quad$ Q lost by the enclosed Volume, V

So after polarizing this dielectric a given volume, V will have had Q leave it,

$$Q = \int_V \nabla\cdot\vec{P}\, d\tau \qquad \text{so the charge}$$

remaining (conservation of charge) must be $Q_{left} = -\int_V \vec{\nabla}\cdot\vec{P}\, d\tau$

In a small volume $d\tau$, there's dQ left behind,

$$dQ = -\nabla \cdot \vec{P} \, d\tau \quad \text{and} \quad \frac{dQ}{d\tau} = \rho_b \quad \text{charge density}$$

So

$$\rho_b = -\nabla \cdot \vec{P} \quad \Longleftarrow \quad \text{left behind}$$

This is real bound charge that is in $d\tau$

To summarize:

If we polarize a dielectric with average polarization $\vec{P}(\vec{r}) = $ dipole moment /volume.,
we get tiny shifts everywhere that results in the accumulation of charge on the surfaces. If more polarization occurs in some places in the bulk than others, then there might be local charge densities in the bulk.

$$\sigma_B = \vec{P} \cdot \hat{n} = \text{charge density on surface}$$

$$\rho_B = -\nabla \cdot \vec{P} = \text{volume charge density inside}$$

These are real charges that produce real $\vec{E}$ & V!

Clicker Questions: 　[↑↑↑]　a　(↑↑↑)　σ_b, ρ_b?

Maybe you find the conceptual description of how we get $\sigma_B + \rho_B$ too hand-wavy. Here's the formal proof.

Earlier, we derived $V(\vec{r})$ in terms of $\vec{P}(\vec{r})$,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}(\vec{r}') \cdot \hat{\imath}}{\imath^2} \, d\tau'$$

Recall that $\nabla \cdot \frac{1}{\imath} = -\frac{\hat{\imath}}{\imath^2}$

if $\vec{\imath} = \vec{r} - \vec{r}'$ then $|\vec{\imath}| = |\vec{r} - \vec{r}'|$ such that the gradient with respect to primed variables changes the sign of the result!

So

$$\nabla' \frac{1}{\imath} = +\frac{\hat{\imath}}{\imath^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}(\vec{r}') \cdot \hat{\imath}}{\imath^2} \, d\tau' = \frac{1}{4\pi\epsilon_0} \int_V \vec{P}(\vec{r}') \cdot \nabla' \frac{1}{\imath} \, d\tau'$$

We can expand the integral by noticing that $\nabla' \cdot \left(\frac{\vec{P}(\vec{r}')}{\imath} \right)$ can be computed using the product rule.

$$\nabla' \cdot \left(\frac{\vec{P}(\vec{r}')}{\imath} \right) = \vec{P}(\vec{r}') \cdot \nabla' \frac{1}{\imath} + \nabla' \cdot \vec{P}(\vec{r}') \left(\frac{1}{\imath} \right)$$

So,

$$\vec{P}(\vec{r}') \cdot \nabla' \frac{1}{\imath} = \nabla' \cdot \left(\frac{\vec{P}(\vec{r})}{\imath} \right) - \nabla' \cdot \vec{P}(\vec{r}') \frac{1}{\imath}$$

Our integral becomes

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{-\nabla' \cdot \vec{P}(\vec{r}')}{\imath} \, d\tau' + \frac{1}{4\pi\epsilon_0} \int_V \nabla' \cdot \left(\frac{\vec{P}(\vec{r})}{\imath} \right) d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{(-\nabla\cdot\vec{P}(\vec{r}'))}{\imath}d\tau' + \frac{1}{4\pi\epsilon_0}\oint_S \frac{\vec{P}(\vec{r}')\cdot d\vec{A}'}{\imath}$$

so if $\rho_B = -\nabla\cdot\vec{P}(\vec{r}')$ and $\vec{P}(\vec{r}')\cdot\hat{n} = \sigma_B$

with $d\vec{A} = \hat{n}\,dA$

then we recover,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_B}{\imath}d\tau' + \frac{1}{4\pi\epsilon_0}\oint_S \sigma_B\,dA/\imath$$

as expected.

Example: Consider a dielectric slab in a uniform electric field, $\vec{E}_{ext} = \vec{E}_0\,\hat{z}$
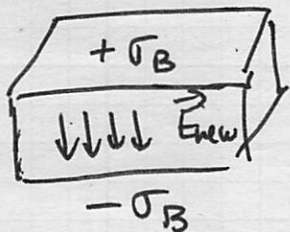
the dielectric will polarize. Let's assume it's uniform in this case,

$$\vec{P} = P_0\hat{z} \qquad \text{What's } \sigma_b \text{ \& } \rho_b?$$

$$\rho_b = -\nabla\cdot\vec{P} = 0 \qquad \text{there are no bound charges in the bulk.}$$

(Makes sense b/c $\vec{P}$ is uniform)

$$\sigma_b = \vec{P}\cdot\hat{n} \begin{cases} \hat{n} \text{ on top, } \hat{z} \longrightarrow +P_0 \\ \hat{n} \text{ on bottom, } -\hat{z} \longrightarrow -P_0 \end{cases}$$

"equal" polarization on top and bottom again, make sense b/c $\vec{P}$ is uniform!
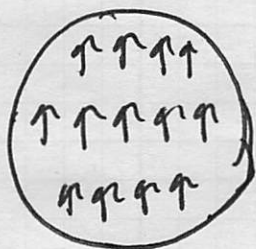
This created a field, a new one inside the dielectric

$$\vec{E}_{new} = \frac{\sigma_B}{\epsilon_0}(-\hat{z}) = -\frac{P_0}{\epsilon_0}\hat{z}$$

So the total field inside is now the superposition of $\vec{E}_{new}$ & $\vec{E}_{ext}$,

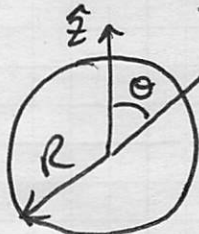$$\vec{E}_{inside} = \vec{E}_{ext} + \vec{E}_{new} = \left(E_0 - \frac{P_0}{\epsilon_0}\right)\hat{z}$$

So the field in the space occupied by the dielectric is reduced (but not entirely like a metal)

Example : Consider a sphere with uniform polarization, $\vec{P} = P_0 \hat{z}$. Let's find the potential inside and outside.

$$\nabla \cdot \vec{P} = 0 \qquad \rho_B = 0$$

But $\vec{P} \cdot \hat{n} = \sigma_B$ is non zero.

$$\vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{r} = P_0 \hat{z} \cdot \hat{r} = P_0 \cos\theta$$

where θ is the usual polar angle.

So $\sigma_B = P_0 \cos\theta$. We solved this problem (or one like it before) using Legendre Polynomials!

*Inside or Outside the Boundary Condition forces use to have pure cosine terms.

$$V(r,\theta) = \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell (\cos\theta)$$

outside A_ℓ's $\to 0$ B_ℓ's $\to 0$ except $\ell = 1$

inside B_ℓ's $\to 0$ A_ℓ's $\to 0$ except $\ell = 1$

$$V_{outside} = \frac{B_1}{r^2} \cos\theta \qquad V_{inside} = A_1 r \cos\theta$$

We have a condition on σ, so,

$$\left. \frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} \right|_{r=R} = -\frac{\sigma_B}{\epsilon_0}$$

$$\frac{\partial V_{out}}{\partial r} = -\frac{2B_1}{r^3} \cos\theta \qquad \frac{\partial V_{in}}{\partial r} = A_1 \cos\theta$$

$$-\frac{2B_1}{R^3} \cos\theta - A_1 \cos\theta = -\frac{P_0 \cos\theta}{\epsilon_0} \Rightarrow \boxed{+\frac{2B_1}{R^3} + A_1 = \frac{P_0}{\epsilon_0}}$$

V must be continuos at R also,

$$V_{out}(R,\theta) = V_{in}(R,\theta) \Rightarrow \frac{B_1}{R^2} \cos\theta = A_1 R \cos\theta \quad \boxed{B_1 = R^3 A_1}$$

$$+ \frac{2(R^3 A_1)}{R^3} + A_1 = \frac{P_0}{\varepsilon_0} \implies A_1 = \frac{P_0}{3\varepsilon_0} \quad B_1 = \frac{R^3 P_0}{3\varepsilon_0}$$

Thus,

$$V_{outside}(r,\theta) = \frac{P_0 R}{3\varepsilon_0}\left(\frac{R}{r}\right)^2 \cos\theta = \frac{\vec{P}\cdot\hat{r}}{r^2}\frac{R^3}{3\varepsilon_0}$$

$$V_{inside}(r,\theta) = \frac{P_0}{3\varepsilon_0} r\cos\theta$$

+ The form of $V_{outside} \sim \frac{\cos\theta}{r^2}$ is that of a pure dipole.

+ the form of V_{inside} is a bit trickier to see but its a uniform field!

Inside
$$V(r,\theta) = \frac{P_0}{3\varepsilon_0} r\cos\theta = \frac{P_0}{3\varepsilon_0} z \qquad \underline{z = r\cos\theta}$$

so
$$\vec{E} = -\nabla V = -\frac{P_0}{3\varepsilon_0}\hat{z} \ !$$

a perfectly uniform field.