ADT: Priority Queue

- The following operations:
  - find element with highest priority
  - delete element with highest priority
  - insert element with assigned priority
- Many implementations possible:
  - Sorted list
  - Binary search tree
  - …

Binary Heap Implementation

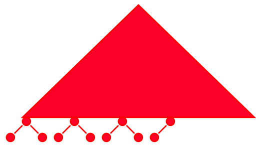
- Binary Heap data structure
  - Find – O(1)
  - Delete and insert – O(log n)
- Enhance with "change priority"
  - Delete a given element
  - Change priority for a given element –need for some important applications
- Goal: Find the element in the heap in constant time! Maintain an additional array for all the entities of interest e.g. vertices in a graph, that contains their position in the (heap) priority queue (handle)

Binary Heap

Operations
- BinaryHeap() creates a new, empty, binary heap.
- insert(k) adds a new item to the heap.
- find_min() returns the item with the minimum key value, leaving item in the heap.
- del_min() returns the item with the minimum key value, removing the item from the heap.
- is_empty() returns true if the heap is empty, false otherwise.
- size() returns the number of items in the heap.
- Build_heap(list) builds a new heap from a list of keys.

Heaps and Heapsort

- Definition A heap is a binary tree with keys at its nodes (one key per node) such that:
- It is essentially complete (note online text is slightly different), i.e., all its levels are full except possibly the last level, where only some rightmost keys may be missing – **shape property**.

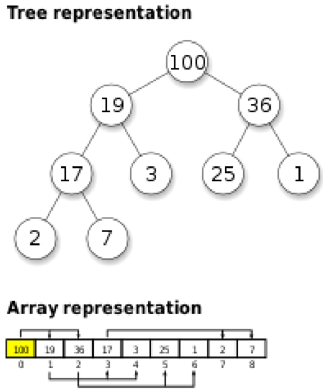


- The key at each node is ≤ keys at its children (MinHeap)– **heap order (structure) property** (≥ MaxHeap)

Illustration of the heap’s definition

Max Heap

Note: Heap’s elements are ordered top down (along any path down from its root), but they are not ordered left to right

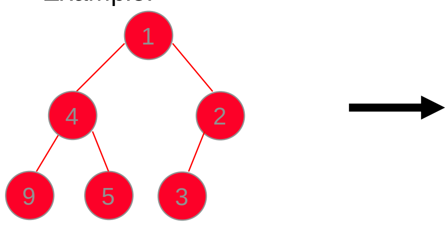


Some Important Properties of a Heap (MaxHeap)

- Given *n*, there exists a unique binary tree with *n* nodes that is *__essentially complete__*, with $h = \lfloor \log_2 n \rfloor$
- The root contains the largest key
- The subtree rooted at any node of a heap is also a heap
- A heap can be represented as an array

Heap's Array Representation

- Store heap's elements in an array (whose elements indexed, for convenience, 1 to n) in top-down left-to-right order
- Example:



1	2	3	4	5	6
1	4	2	9	5	3

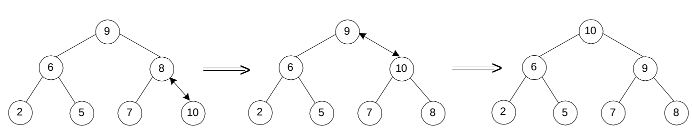
- Left child of node j is at 2j   Right child of node j is at 2j+1
- Parent of node j is at $\lfloor j/2 \rfloor$
- Parental nodes are represented in the first $\lfloor n/2 \rfloor$ locations

Insertion of a New Element into a Heap

- Insert the new element at last position in heap.
- Compare it with its parent and, if it violates heap condition, exchange them (Drift up)
- Continue comparing the new element with nodes up the tree until the heap condition is satisfied

Example: Insert key 10

Efficiency: O(log *n*)



Insertion into heap: perc_up (drift_up, sift_up)

1: Put new element into first open position, this maintains the structure property

```
def insert(self,k):
    self.heapList.append(k)
    self.currentSize = self.currentSize + 1
    self.percUp(self.currentSize)
```

2: Drift the element up until the heap property is restored

```
def percUp(self,i):
    while i // 2 > 0:
      if self.heapList[i] < self.heapList[i // 2]:
         tmp = self.heapList[i // 2]
         self.heapList[i // 2] = self.heapList[i]
         self.heapList[i] = tmp
      i = i // 2
```


Top-down heap construction

- Start with empty heap and repeatedly insert elements
- Performance O(n* log n)

Remove max from heap

IDEA:

1. Store root (maximum element) for return
2. Swap last entry in heap with first entry in heap
3. Reduce heapsize by 1
4. *perc_down* the new root (first element) until the heap property is restored

Note: There are only two basic “moves” in the heap.

- *perc_down* – used in bottom up construction and removal
- *perc_up* – used in insertion and top-down construction

perc_down

```
def percDown(self, pos):
   while (pos * 2) <= self.currentSize:
      minchd = self.minChild(pos)   // index of the min child
      if self.heapList[i] > self.heapList[minchd]:  // if min child smaller
         tmp = self.heapList[i]                       // swap
         self.heapList[i] = self.heapList[minchd]
         self.heapList[minchd] = tmp
      pos = minchd
```

min_child

```
def minChild(self,pos):
    if pos * 2 + 1 > self.currentSize:      // if no right child then return left child pos
        return pos * 2
    else:
        if self.heapList[pos *2] < self.heapList[pos *2+1]:
            return pos * 2
        else:
            return pos * 2 + 1
```


Bottom-up heap construction

1. Insert elements into array respecting the shape property. Tree is essentially complete.

2. Rearrange elements to enforce the heap order (structure) property

Heap Construction (bottom-up)

High level pseudo-code

Initialize the array structure with keys in the order given
(shape property)

Loop: node = rightmost parental node to root
if node doesn't satisfy the heap condition:
loop: exchange it with its smallest child until the heap condition holds ("perc_down")

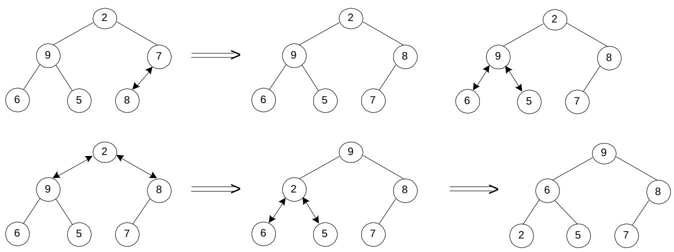
Bottom-up heap construction

1. Insert elements into array respecting the shape property
2. Rearrange elements to enforce the heap order property

```
def buildHeap(self,alist):
    i = len(alist) // 2
    self.currentSize = len(alist)
    self.heapList = [0] + alist[:]
    while (i > 0):
        self.percDown(i)
        i = i - 1
```

Example of Heap Construction

Construct a maxheap for the list 2, 9, 7, 6, 5, 8



Which is better Top-down vs. Bottom-up heap construction?

- Bottom up is O(n)
  - Most nodes are near the bottom of the tree, why
  - Almost have the nodes are leaves!
  - Driftdown drifts entries down, thus most nodes do not have far to travel
- Top down is O(n log n)
  - Half the nodes are near the bottom
  - Thus in the worst case, those n/2 nodes may need to move to the top
  - Since the height of the tree is log n
  - That could require O(n log n) comparisons

Priority Queue

- A priority queue is the ADT of a set of elements with numerical priorities and the following operations:
  - find element with highest priority
  - delete element with highest priority
  - insert element with assigned priority
- Heap is a very efficient way for implementing priority queues
- Applications determine what is a priority ordering!
  Sometimes want largest, sometimes smallest element to be found/deleted.
- Many implementation options : list, sorted list, …

Heapsort

Construct a heap for a given list of *n* keys

Repeat operation of root removal *n*-1 times:
- Exchange keys in the root and in the last (rightmost) leaf
- Decrease heap size by 1
- If necessary, swap new root with larger child until the heap condition holds (Drift Down)

Example of Sorting by Heapsort

Sort the list 2, 9, 7, 6, 5, 8 in ascending order using Max Heap

Stage 1 (bottom up heap construction)

2 9 7 6 5 8	7↔8 swap
2 9 8 6 5 7	9↔6,5 no swap
2 9 8 6 5 7	2↔9,8 swap
9 2 8 6 5 7	2↔6,5 swap
9 6 8 2 5 7	2 no children

Stage 2 (root/max removal)

9 6 8 2 5 7	swap 9,7
7 6 8 2 5 | 9	↓ size
8 6 7 2 5 | 9	swap 8,5
5 6 7 2 | 8 9	↓ size
7 6 5 2 | 8 9	…
2 6 5 | 7 8 9	
6 2 5 | 7 8 9	
5 2 | 6 7 8 9	
5 2 | 6 7 8 9	
2 | 5 6 7 8 9	sorted order