LARGE CHARACTER SUMS WITH MULTIPLICATIVE COEFFICIENTS

ZIKANG DONG¹, YUTONG SONG², WEIJIA WANG³, HAO ZHANG⁴, AND SHENGBO ZHAO²

ABSTRACT. In this paper, we investigate large values of Dirichlet character sums with multiplicative coefficients $\sum_{n\leq N} f(n)\chi(n)$. We prove a new Omega result in the region $\exp((\log q)^{\frac{1}{2}+\delta}) \leq N \leq \sqrt{q}$, where q is the prime modulus.

1. Introduction

Character sums play an important role and appear in many central problems in analytic number theory. A classical well-known result of Pólya–Vinogradov shows that for any non-principal χ modulo large prime q,

$$\sum_{n \le N} \chi(n) \ll \sqrt{q} \log q.$$

For shorter sums, the celebrated method of Burgess gives the upper bound

$$\sum_{n < N} \chi(n) \ll N^{1 - \frac{1}{r}} q^{\frac{r+1}{4r^2}} \log q$$

for any integer $r \geq 1$, which provides nontrivial stronger estimate as soon as $N \gg q^{1/4+o(1)}$.

It is also natural to see how lower bounds can be achieved for both large or short character sums. Paley showed that for infinitely many q and quadratic characters χ modulo q,

$$\max_{N \le q} \Big| \sum_{n < N} \chi(n) \Big| \gg \sqrt{q} \log \log q$$

^{1.} School of Mathematical Sciences, Soochow University, Suzhou 215006, P. R. China

^{2.} School of Mathematical Sciences, Key Laboratory of Intelligent Computing and Applications (Ministry of Education), Tongji University, Shanghai 200092, P. R. China

^{3.} Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China

^{4.} School of Mathematics, Hunan University, Changsha 410082, P. R. China

E-mail addresses: zikangdong@gmail.com, 99yutongsong@gmail.com,

weijiawang@amss.ac.cn, zhanghaomath@hnu.edu.cn, shengbozhao@hotmail.com.

Date: September 12, 2025.

²⁰²⁰ Mathematics Subject Classification. Primary 11L40, 11M06.

Bateman and Chowla strengthened this by proving for an infinite sequence of moduli q and primitive quadratic characters χ modulo q, it holds that

$$\max_{N \le q} \left| \sum_{n \le N} \chi(n) \right| \ge \left(\frac{e^{\gamma}}{\pi} + o(1) \right) \sqrt{q} \log \log q,$$

where γ is the Euler–Mascheroni constant.

If we consider the maximum size

$$\max_{\chi \neq \chi_0 \pmod{q}} \Big| \sum_{n \le N} \chi(n) \Big|,$$

Granville and Soundararajan's celebrated work [4] shows several lower bounds as N varies in different ranges according to q, which provide much evidence for their conjecture: $\max_{\chi \neq \chi_0 \pmod{q}} |\sum_{n \leq N} \chi(n)|$ increases on $N \leq q$.

The aim of this paper is study the large values of the mixed sums $\sum_{n\leq N} f(n)\chi(n)$ which are ubiquitous in the regime $\exp((\log q)^{1/2}) \leq N \leq q^{1/2}$. We do so by adapting the resonance method to incorporate the twist f directly into the resonator. This leads to two Omega results of the Dirichlet character sums. The first guarantees large values when $\log N$ is taken in the size of $(\log q \log_2 q)^{\frac{1}{2}}(\log_2 q)^{O(1)}$.

Theorem 1.1. Let $\log N = (\log q \log_2 q)^{\frac{1}{2}} \tau$ with $\tau = (\log_2 q)^{O(1)}$. Let f be any completely multiplicative function such that |f(n)| = 1 for any integer n. Then we have

$$\max_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} \left| \sum_{n \le N} f(n)\chi(n) \right| \ge \sqrt{N} \exp\bigg((1 + o(1))A(\tau + \tau') \sqrt{\frac{\log q}{\log_2 q}} \bigg),$$

where $A, \tau' \in \mathbb{R}$ such that

$$\tau = \int_{\Delta}^{\infty} \frac{\mathrm{e}^{-u}}{u} du, \qquad \tau' = \int_{\Delta}^{\infty} \frac{\mathrm{e}^{-u}}{u^2} du.$$

This generalize Theorem 3.2 of Hough [6]. When $\log N$ is larger than $(\log q)^{\frac{1}{2}+\delta}$ for some $\delta > 0$, we have the second theorem.

Theorem 1.2. Let $\delta \in (0, \frac{1}{100})$ be any fixed small number. Let q be a sufficiently large prime, and N satisfy $\exp((\log q)^{\frac{1}{2}+\delta}) \leq N \leq q^{\frac{1}{2}}$. Let f be any completely multiplicative function such that |f(n)| = 1 for any integer n. If we additionally assume $\operatorname{Re} f(m)\overline{f}(n) \geq c$ holds for any integers m, n and some absolute constant c > 0, then we have

$$\max_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} \Big| \sum_{n \le N} f(n)\chi(n) \Big| \ge \sqrt{N} \exp\bigg((\sqrt{2} + o(1)) \sqrt{\frac{\log(q/N)\log_3(q/N)}{\log_2(q/N)}} \bigg).$$

This improves previous result of [3], at a cost of assuming an additional condition for $f(\cdot)$.

Recently, Harper [5] showed some moments results for the mixed character sums. He conjectured that, when N goes beyond the size of $\exp(\sqrt{\log q})$, the influence of $f(\cdot)$ will surpass $\chi(\cdot)$, which can lead to a new upper bound for low moments of $\sum_{n\leq N} f(n)\chi(n)$. See [5] for more details.

2. Proof of Theorem 1.1

The Proof of Theorem 1.1 relies on Hough's work [6], which is essentially the resonance method of Soundararajan [7]. Let Y = q/N be large and $\lambda = \sqrt{\log Y \log_2 Y}$. Define the multiplicative function r for any prime p:

$$r(p) = \begin{cases} \frac{\lambda}{\sqrt{p \log p}}, & \lambda \le p \le \exp((\log \lambda)^2), \\ 0, & \text{otherwise.} \end{cases}$$

Denote

$$S_{\chi}(N) := \sum_{n \le N} f(n) \chi(n).$$

We define the resonator

$$R_{\chi} = \sum_{n \le Y} r(n) f(n) \chi(n),$$

and

$$M_1(R,q) := \sum_{\chi \neq \chi_0 \pmod{q}} |R_{\chi}|^2,$$

$$M_2(R,q) := \sum_{\chi \neq \chi_0 \pmod{q}} S_{\chi}(N) |R_{\chi}|^2.$$

then we have

$$\max_{\chi \neq \chi_0 (\text{mod } q)} |S_\chi(N)| \geq \frac{|M_2(R,q)|}{M_1(R,q)}.$$

By the orthogonal relation, we have that

$$\frac{1}{\varphi(q)} \sum_{\text{(mod } q)} \chi(n) \overline{\chi}(m) = \begin{cases} 1 & \text{if } q | n - m, \\ 0 & \text{otherwise.} \end{cases}$$

So for M_1 we have

$$M_1(R,q) \le \sum_{\chi \pmod{q}} |R_{\chi}|^2 \le \varphi(q) \sum_{m \le Y} r(m)^2.$$

For M_2 we firstly have

$$|S_{\chi_0}(N)||R_{\chi_0}|^2 \le N\Big(\sum_{m \le Y} r(m)\Big)^2 \le NY\sum_{m \le Y} r(m)^2 = q\sum_{m \le Y} r(m)^2.$$

Thus we have

$$M_{2}(R,q) + O\left(q \sum_{m \leq Y} r(m)^{2}\right)$$

$$\geq \sum_{\chi \pmod{q}} S_{\chi}(N) |R_{\chi}|^{2}$$

$$= \varphi(q) \sum_{m,n \leq Y} r(m) r(n) \sum_{\substack{k \leq N \\ mk = n}} f(mk) \overline{f}(n)$$

$$= \varphi(q) \sum_{k \leq N} r(k) \sum_{m \leq Y/k} r(m)^{2}.$$

By the proof of Page 105 in [6], we have

$$\sum_{m \le Y/k} r(m)^2 = (1 + o(1)) \sum_{m \ge 1} r(m)^2.$$

Combined with M_1 we have

$$\frac{M_2(R,q)}{M_1(R,q)} \ge (1+o(1)) \sum_{k \le N} r(k) + O(1).$$

Then we completes the proof by following Page 105–107 of [6].

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is highly based on the work of La Bretèche and Tenenbaum [2], which combines sharp evaluation of GCD sums and resonance method. This kind of technique can date back to Aistleitner's work [1].

Lemma 3.1. Let \mathcal{M} be any set of positive integers with $|\mathcal{M}| = K$. Then as $K \to \infty$, we have

$$\max_{|\mathcal{M}|=K} \sum_{m,n\in\mathcal{M}} \sqrt{\frac{(m,n)}{[m,n]}} = K \exp\bigg((2\sqrt{2} + o(1)) \sqrt{\frac{\log K \log_3 K}{\log_2 K}} \bigg).$$

Proof. This is Theorem 1.1 of [2].

Note that in the proof of the above lemma, the choice for the set \mathcal{M} satisfies $y_{\mathcal{M}} := \max_{m \in \mathcal{M}} P(m) \leq (\log K)^{1+o(1)}$.

Let \mathcal{M} be a set of positive integers with $|\mathcal{M}| = \lfloor q/N \rfloor$ such that

$$\frac{1}{|\mathcal{M}|} \sum_{m,n \in \mathcal{M}} \sqrt{\frac{(m,n)}{[m,n]}} = \exp\left((2\sqrt{2} + o(1))\sqrt{\frac{\log(q/N)\log_3(q/N)}{\log_2(q/N)}}\right).$$

For any $1 \le j \le q - 1$, let

$$\mathcal{M}_j := \{ m \in \mathcal{M} : \ q | m - j \},\$$

$$\mathcal{J} := \{ 1 \le j \le q - 1 : \ \mathcal{M}_j \ne \emptyset \},\$$

and

$$\mathcal{M}' := \{ \min \mathcal{M}_j : j \in \mathcal{J} \}.$$

Define for $m \in \mathcal{M}'$

$$r(m') := \sqrt{|\mathcal{M}_j|}, \quad (m' \in \mathcal{M}_j)$$

$$R_{\chi} := \sum_{m' \in \mathcal{M}'} r(m') f(m') \chi(m'),$$

and

$$M_1(R,q) := \sum_{\chi \neq \chi_0 \pmod{q}} |R_\chi|^2,$$

$$M_2(R,q) := \sum_{\chi \neq \chi_0 \pmod{q}} |S_{\chi}(N)|^2 |R_{\chi}|^2.$$

Then

$$\max_{\chi \neq \chi_0 \pmod{q}} |S_{\chi}(N)| \ge \sqrt{\frac{M_2(R, q)}{M_1(R, q)}}.$$
 (3.1)

For M_1 we have

$$M_{1}(R,q) \leq M_{1}(R,q) + |R_{\chi_{0}}|^{2}$$

$$= \sum_{\chi \pmod{q}} \sum_{m',n' \in \mathcal{M}'} f(m') \overline{f}(n') \chi(m') \overline{\chi}(n')$$

$$= \varphi(q) \sum_{m \in \mathcal{M}'} f(m') \overline{f}(m')$$

$$= \varphi(q) |\mathcal{M}'|$$

$$< q^{2}/N.$$

For M_2 , firstly we have

$$|S_{\chi_0}(N)|^2 |R_{\chi_0}|^2 \le N^2 |\mathcal{M}'|^2 \le q^2.$$

Then

$$\begin{split} &M_2(R,q) + O(q^2) \\ &= M_2(R,q) + |S_{\chi_0}(N)|^2 |R_{\chi_0}|^2 \\ &= \sum_{\chi \pmod{q}} \sum_{m',n' \in \mathcal{M}'} r(m') r(n') \sum_{k,\ell \leq N} f(m'k) \overline{f}(n'\ell) \chi(m'k) \overline{\chi}(n'\ell) \\ &= \varphi(q) \sum_{m',n' \in \mathcal{M}'} r(m') r(n') \sum_{\substack{k,\ell \leq N \\ q \mid m'k - n'\ell}} f(m'k) \overline{f}(n'\ell) \\ &= \varphi(q) \sum_{m',n' \in \mathcal{M}'} r(m') r(n') \sum_{\substack{k,\ell \leq N \\ m'k = n'\ell}} 1 \\ &+ \varphi(q) \sum_{m',n' \in \mathcal{M}'} r(m') r(n') \sum_{\substack{k,\ell \leq N \\ q \mid m'k - n'\ell > 0}} 2 \operatorname{Re} f(m'k) \overline{f}(n'\ell) \\ &\gg \varphi(q) \sum_{m',n' \in \mathcal{M}'} r(m') r(n') \sum_{\substack{k,\ell \leq N \\ q \mid m'k - n'\ell}} 1 \\ &= \varphi(q) \sum_{k,\ell \leq N} \sum_{\substack{m',n' \in \mathcal{M}' \\ q \mid m'k - n'\ell}} r(m') r(n'). \end{split}$$

For the inner sum with $k, \ell \leq N$ fixed, we have

$$\sum_{m',n'\in\mathcal{M}'\atop q|m'k-n'\ell} r(m')r(n') \ge \sum_{m',n'\in\mathcal{M}'\atop q|m'k-n'\ell} \min\{r(m')^2,r(n')^2\} \ge \sum_{m,n\in\mathcal{M}\atop mk=nl} 1.$$

So we get

$$M_2(R,q) + O(q^2) \gg \varphi(q) \sum_{k,\ell \leq N} \sum_{\substack{m,n \in \mathcal{M} \\ mk = nl}} 1 = \varphi(q) \sum_{m,n \in \mathcal{M}} \sum_{\substack{k,\ell \leq N \\ mk = nl}} 1.$$

For $m, n \in \mathcal{M}$ fixed, we have for the inner sum

$$\sum_{k,\ell \leq N \atop mk=n\ell} 1 \geq \frac{N}{\max\{\frac{m}{(m,n)},\frac{n}{(m,n)}\}} \geq \frac{N}{\sqrt{2\frac{m}{(m,n)}\frac{n}{(m,n)}}} = \frac{N}{\sqrt{2}} \sqrt{\frac{(m,n)}{[m,n]}}.$$

Thus

$$M_{2}(R,q) + O(q^{2})$$

$$\gg \varphi(q)N \sum_{\substack{m,n \in \mathcal{M} \\ [m,n]/(m,n) \leq N^{2}/2}} \sqrt{\frac{(m,n)}{[m,n]}}$$

$$= \varphi(q)N \left(\sum_{m,n \in \mathcal{M}} \sqrt{\frac{(m,n)}{[m,n]}} - \sum_{\substack{m,n \in \mathcal{M} \\ [m,n]/(m,n) > N^{2}/2}} \sqrt{\frac{(m,n)}{[m,n]}}\right)$$

$$\gg \varphi(q)N|\mathcal{M}|\exp\left((2\sqrt{2} + o(1))\sqrt{\frac{\log(q/N)\log_{3}(q/N)}{\log_{2}(q/N)}}\right).$$

Here in the last step we used

$$\sum_{\substack{m,n \in \mathcal{M} \\ [m,n]/(m,n) > N^2/2}} \sqrt{\frac{(m,n)}{[m,n]}} \ll N^{-2\eta} \sum_{m,n \in \mathcal{M}} \left(\frac{(m,n)}{[m,n]}\right)^{\frac{1}{2}-\eta}$$

$$\ll N^{-2\eta} \prod_{p \leq y_{\mathcal{M}}} \left(1 + \frac{2}{p^{\frac{1}{2}-\eta} - 1}\right)$$

$$\ll N^{-2\eta} \exp\left(y_{\mathcal{M}}^{\frac{1}{2}+\eta}\right)$$

$$\ll N^{-2\eta} \exp\left((\log(q/N))^{\frac{1}{2}+\eta+o(1)}\right)$$

$$\ll \exp\left(-\frac{2}{3}\delta(\log q)^{\frac{1}{2}+\delta}\right) \exp\left((\log q)^{\frac{1}{2}+\frac{2}{3}\delta}\right)$$

$$\ll 1.$$

with $\eta = \delta/3$, $y_{\mathcal{M}} := \max_{m \in \mathcal{M}} P(m) \leq (\log(q/N))^{1+o(1)}$ and $N > \exp((\log q)^{\frac{1}{2}+\delta})$. At last we have

$$\frac{M_2(R,q)}{M_1(R,q)} \gg N \exp\left((2\sqrt{2} + o(1))\sqrt{\frac{\log(q/N)\log_3(q/N)}{\log_2(q/N)}}\right) + O(N).$$

Back to (3.1), we finish the proof.

ACKNOWLEDGEMENTS

Z. Dong is supported by the Shanghai Magnolia Talent Plan Pujiang Project (Grant No. 24PJD140) and the National Natural Science Foundation of China (Grant No. 1240011770). W. Wang is supported by the National Natural Science Foundation of China (Grant No. 12500). H. Zhang is supported by the Fundamental Research Funds for the Central Universities (Grant No. 531118010622), the National Natural Science Foundation of China (Grant No. 1240011979) and the Hunan Provincial Natural Science Foundation of China (Grant No. 2024JJ6120).

References

- [1] Aistleitner, C. Lower bounds for the maximum of the Riemann zeta function along vertical lines, *Math. Ann.*, **365** (2016), 473–496.
- [2] de la Bretèche, R.; Tenenbaum, G. Sommes de Gál et applications, *Proc. Lond. Math. Soc.*, **119** (2019), 104–134.
- [3] Dong, Z.; Li, Z.; Song, Y.; Zhao, S. Large values of character sums with multiplicative coefficients, preprint, arXiv:2508.09750.
- [4] Granville, A.; Soundararajan K. Large character sums, J. Amer. Math. Soc., 14 (2001), 365–397.
- [5] Harper, A. J. The typical size of character and zeta sums is $o(\sqrt{x})$, preprint, arXiv:2301.04390.
- [6] Hough, B. The resonance method for large character sums, Mathematika, 59,(2013), 87–118
- [7] Soundararajan, K. Extreme values of zeta and L-functions, Math. Ann., 342 (2008), 67–86.