Numerically solving coupled Lippmann-Schwinger equations

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Our goal is to numerically solve coupled Fredholm integral equations of the second kind:

$$u_i(x) = f_i(x) + \sum_j \int_{-\infty}^{\infty} dt\, K_{i,j}(x,t) u_j(t), \tag{1}$$

where $\{f_i(x)\}$ are known functions and $\{K_{i,j}\}$ is a set of known integral kernels. If we discretize the problem in x and t, this is just becomes a system of linear equations. Our goal is then to solve,

$$u_{i,x} = f_{i,x} + \sum_{j,t} w_{j,t}\, k_{i,j,x,t}\, u_{j,t}, \tag{2}$$

where w_t is a weight that depends on the quadrature method used.

Now let's put this in the language of the Lippmann-Schwinger equations we wish to solve. For a given incoming momentum k' and channel β, we need to solve for $t_{\alpha,\beta}(k,k';E)$ in,

$$\begin{aligned} t_{\alpha,\beta}\left(k,k';E\right) &= V_{\alpha,\beta}\left(k,k'\right) \\ &+ \sum_\gamma \int_0^\infty k''^2 dk''\, V_{\alpha,\gamma}\left(k,k''\right) \frac{1}{E-\omega_{\gamma_1}\left(k''\right)-\omega_{\gamma_2}\left(k''\right)+i\epsilon} t_{\gamma,\beta}\left(k'',k';E\right) \end{aligned} \tag{3}$$

where $V_{\alpha,\beta}\left(k,k'\right) = \sum_i g^*_{i,\alpha}(k) \frac{1}{E-m_i^0} g_{i,\beta}\left(k'\right) + v_{\alpha,\beta}\left(k,k'\right)$ is known, and $\omega_{\gamma_i}(\mathbf{k}) = \sqrt{m^2_{\gamma_i}+\mathbf{k}^2}$. Since we are integrating numerically, we choose some upper cutoff, Λ, for k''. The linear system arising from the discretization of the full coupled case looks like,

$$\begin{aligned} t_{\alpha,\beta,k,k',E} &= V_{\alpha,\beta,k,k'} \\ &+ \sum_{\gamma,k''} w_{\gamma,k''}\, k''^2 V_{\alpha,\gamma,k,k''} \frac{1}{E-\omega_{\gamma_1}\left(k''\right)-\omega_{\gamma_2}\left(k''\right)+i\epsilon} t_{\gamma,\beta,k'',k',E}. \end{aligned} \tag{4}$$

Attempting to approach the problem by taking $i\epsilon \to 0$ could be numerically unstable without using methods to deal with the principle value. Recall that for $a < x_0 < b$, the Sokhotski-Plemelj theorem states:

$$\lim_{\epsilon\to 0^+} \int_a^b dx\, \frac{f(x)}{x-x_0+i\epsilon} = -i\pi f(x_0) + \mathcal{P}\int_a^b dx\, \frac{f(x)}{x-x_0}. \tag{5}$$

However, using this theorem complicates the problem for several reasons, so we will attempt to take the limit as ϵ becomes small.

In order to deal with the singular kernel, we will make use of a quadrature scheme where we factor out the Lippmann-Schwinger propagator and use it as a weight function, since we can integrate several of its moments analytically.

Ideally when attempting a solution, adaptive refinement of the integration domain should be used. This is because as we take the limit where Λ becomes large, there will be areas of the domain where a fine mesh is needed, and areas where it will not be needed. Using a uniform mesh spacing will cause the system to be larger than it needs to be. For now though, we're going to use a uniform mesh for simplicity.

The parametrizations of the 1-to-1 interaction $g_{\sigma,\alpha}(k)$ and the 2-to-2 interaction $v_{\alpha,\beta}(k,k')$ we use are based on [1] and are given below. Following the notation of [1], $g_{\sigma,\alpha}$ is confusingly used both as a constant and a function.

$$g_{\sigma,\alpha}(k) = \frac{g_{\sigma,\alpha}}{\sqrt{m_\pi}} \frac{1}{\left[1 + (c_\alpha \times k)^2\right]} \tag{6}$$

$$v_{\alpha,\beta}(k,k') = \frac{G_{\alpha,\beta}}{m_\pi^2} \times \frac{1}{\left[1 + (d_\alpha \times k)^2\right]^2} \times \frac{1}{\left[1 + (d_\beta \times k')^2\right]^2} \tag{7}$$

References

[1] Wu, Jia-Jun, Lee, T.-S H, Thomas, A W, Young, R D, **Phys. Rev. C 90, 055206 (2014)**