

# Numerically solving coupled Lippmann-Schwinger equations

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Our goal is to numerically solve coupled Fredholm integral equations of the second kind:

$$u_i(x) = f_i(x) + \sum_j \int_{-\infty}^{\infty} dt K_{i,j}(x, t) u_j(t), \quad (1)$$

where  $\{f_i(x)\}$  are known functions and  $\{K_{i,j}\}$  is a set of known integral kernels. Let's simplify the problem by first considering the uncoupled case,

$$u(x) = f(x) + \int_{-\infty}^{\infty} dt K(x, t) u(t). \quad (2)$$

We discretize and make finite (2):

$$u_x = f_x + a \sum_t k_{x,t} u_t, \quad (3)$$

where  $x$  and  $t$  belong to some large discretized interval, and  $a$  is the spacing in our Riemann sum (easily generalized to trapezoidal integration or other forms of quadrature). This is just a system of linear equations. Now if we extend to consider the full coupled case of (1) and discretize it, we arrive at:

$$u_{i,x} = f_{i,x} + a \sum_{j,t} k_{i,j,x,t} u_{j,t}, \quad (4)$$

which is again just a system of linear equations.

Now let's put this in the language of the Lippmann-Schwinger equations we wish to solve. For a given incoming momentum  $k'$  and channel  $\beta$ , we need to solve for  $t_{\alpha,\beta}(k, k'; E)$  in,

$$t_{\alpha,\beta}(k, k'; E) = V_{\alpha,\beta}(k, k') + \sum_{\gamma} \int_0^{\infty} k''^2 dk'' V_{\alpha,\gamma}(k, k'') \frac{1}{E - \omega_{\gamma_1}(k'') - \omega_{\gamma_2}(k'') + i\epsilon} t_{\gamma,\beta}(k'', k'; E) \quad (5)$$

where  $V_{\alpha,\beta}(k, k') = \sum_i g_{i,\alpha}^*(k) \frac{1}{E - m_i^0} g_{i,\beta}(k') + v_{\alpha,\beta}(k, k')$  is known, and  $\omega_{\gamma_i}(\mathbf{k}) = \sqrt{m_{\gamma_i}^2 + \mathbf{k}^2}$ . Since we are integrating numerically, we choose some upper cutoff,  $\Lambda$ , for  $k''$ . The linear system arising from the discretization of the full coupled case looks like,

$$t_{\alpha,\beta,k,k',E} = V_{\alpha,\beta,k,k'} + a \sum_{\gamma,k''} k''^2 V_{\alpha,\gamma,k,k''} \frac{1}{E - \omega_{\gamma_1}(k'') - \omega_{\gamma_2}(k'') + i\epsilon} t_{\gamma,\beta,k'',k',E}. \quad (6)$$

Attempting to approach the problem by taking  $i\epsilon \rightarrow 0$  could be numerically unstable without using methods to deal with the principle value. Recall that for  $a < x_0 < b$ , the Sokhotski-Plemelj theorem states:

$$\lim_{\epsilon \rightarrow 0^+} \int_a^b dx \frac{f(x)}{x - x_0 + i\epsilon} = -i\pi f(x_0) + \mathcal{P} \int_a^b dx \frac{f(x)}{x - x_0}. \quad (7)$$

However, using this theorem complicates the problem for several reasons, so we will attempt to take the limit as  $\epsilon$  becomes small.

Note that in attempting a solution, adaptive refinement of the integration domain should be used. This is because as we take the limit where  $\Lambda$  becomes large, there will be areas of the domain where a fine mesh is needed, and areas where it will not be needed. Using a uniform mesh spacing will cause the system to become prohibitively large.

The parametrizations of the 1-to-1 interaction  $g_{\sigma,\alpha}(k)$  and the 2-to-2 interaction  $v_{\alpha,\beta}(k, k')$  we use are based on [1] and are given below. Following the notation of [1],  $g_{\sigma,\alpha}$  is confusingly used both as a constant and a function.

$$g_{\sigma,\alpha}(k) = \frac{g_{\sigma,\alpha}}{\sqrt{m_\pi}} \frac{1}{\left[1 + (c_\alpha \times k)^2\right]} \quad (8)$$

$$v_{\alpha,\beta}(k, k') = \frac{G_{\alpha,\beta}}{m_\pi^2} \times \frac{1}{\left[1 + (d_\alpha \times k)^2\right]^2} \times \frac{1}{\left[1 + (d_\beta \times k')^2\right]^2} \quad (9)$$

## References

- [1] Wu, Jia-Jun, Lee, T.-S H, Thomas, A W, Young, R D, **Phys. Rev. C** **90**, 055206 (2014)