Topic (8) – POPULATION DISTRIBUTIONS

So far:

We've seen some ways to summarize a set of data, including numerical summaries.

We've heard a little about how to sample a population effectively in order to get good estimates of the population quantities of interest (e.g. taking a good sample and calculating the sample mean as a way of estimating the true but unknown population mean value)

We've talked about the ideas of probability and independence.

Now we need to start putting all this together in order to do **Statistical Inference**, the methods of analyzing data and interpreting the results of those analyses with respect to the population(s) of interest.

The Probability Distribution for a random variable can be

a table or
a graph or
an equation.

Let’s start by reviewing the ideas of frequency distributions for populations using categorical variables.

QUALITATIVE (NON-NUMERIC) VARIABLES

For a random variable that takes on values of categories, the Probability distribution is a table showing the likelihood of each value.

EXAMPLE Tree species found in a boreal forest. For each possible species there would a probability associated with it. E.g. suppose there are 4 species and three are very rare and one is very common. A probability table might look like:

Species	**Probability**
1	0.01
2	0.03
3	0.08
4	0.88
All	1.00

We interpret these values as the probability that a random selection would result in observing that species.

We could also draw a bar chart but it would be fairly non-informative in this instance since one value is so much larger than the others!
An equation cannot be developed since the values that the variable takes on are non-numeric.

QUANTITATIVE (NUMERICAL) VARIABLES

A) Discrete Random Variables

Recall that a discrete random variable is one that takes on values only from a set of isolated (specific) numbers.

The relative frequency distribution for a discrete random variable (also sometimes called a probability mass function) is a list of probabilities for each possible value that the variable can take on.

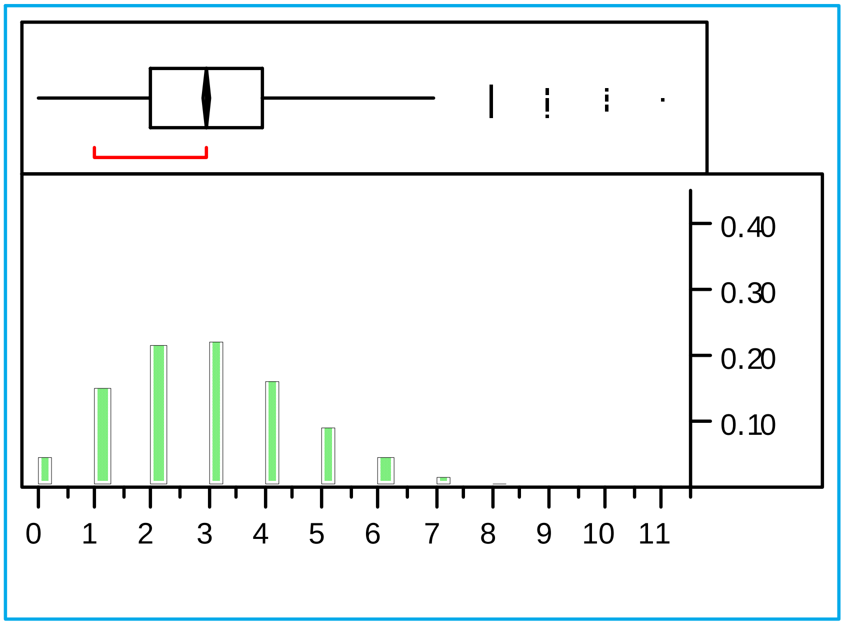
BERNOULLI DISTRIBUTION Suppose the scientist studying the tree species overlaid a grid of square quadrats over the region of interest and then recorded whether any tree was in the quadrat or not. Hence, the random variable is binary, i.e. only two outcomes presence (1) or absence (0). The Bernoulli distribution describes the probability of each outcome:

$$\Pr(X=1) = \pi$$
$$\Pr(X=0) = 1 - \pi$$

The mean for a Bernoulli variable is π and the variance is $\pi(1\text{-}\pi)$.

POISSON DISTRIBUTION Suppose the scientist studying the tree species overlaid a grid of square quadrats over the region of interest and then counted the number of hickory trees in each quadrat. The histogram of the number of trees per quadrat for all of the quadrats might look like

Tree Count



Quantiles

maximum	100.0%	11.000
	99.5%	8.000
	97.5%	7.000
	90.0%	5.000
quartile	75.0%	4.000
median	50.0%	3.000
quartile	25.0%	2.000
	10.0%	1.000
	2.5%	0.000
	0.5%	0.000
minimum	0.0%	0.000

Moments

Mean	2.999
Std Dev	1.750
Std Error Mean	0.025
Upper 95% Mean	3.048
Lower 95% Mean	2.951
N	5000.000
Sum Weights	5000.000

Since we have sampled the entire population (the set of counts for every quadrat in the region), this histogram represents the probability distribution of the random variable X = ”number of trees/quadrat”. In general, the Poisson distribution is a common probability distribution for counts per unit time or unit area or unit volume.

The graph can also be described using an equation known as the Poisson Distribution Probability Mass Function. It gives the probability of observing a specific count (x) in any randomly selected quadrat as

$$\Pr(X = x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

where $x! = x(x-1)(x-2)\ldots(3)(2)(1)$ and $x = 0, 1, 2, \ldots$.

In order for this distribution to be a valid probability distribution, we require that the total probability for all possible values equal 1 and that every possible value have a probability associated with it.

$$\sum_{X=0,1,2,\ldots}\Pr(X = x) = \sum_{X=0,1,2,\ldots}\frac{e^{-\mu}\mu^{x}}{x!} = 1$$

$$\text{and } \Pr(X = x) = \frac{e^{-\mu}\mu^{x}}{x!} \geq 0$$

The mean of the Poisson distribution is μ and the variance is μ as well.

DISCRETE UNIFORM DISTRIBUTION: every discrete value that the random variable can take on has the same probability of occurring.

For example, suppose a researcher is interested in whether the number of setae on the first antennae of an insect is random or not. Further, the researcher believes that there must be at least 1 seta and at most 8. Then s/he is postulating that every value between 1 and 8 are equally likely to be observed in a random draw of an insect from the population (or equivalently, that there are equal numbers of insects with 1, 2, …, or 8 setae in the population). Such a distribution is known as the Discrete Uniform Distribution.

Let K be the total number of distinct values that the random variable can take on (e.g. the set {1, 2, …, 8} contains K = 8 distinct values). Then,

$$\Pr(X = x) = \frac{1}{K} \text{ for x = 1, 2, …, 8}$$

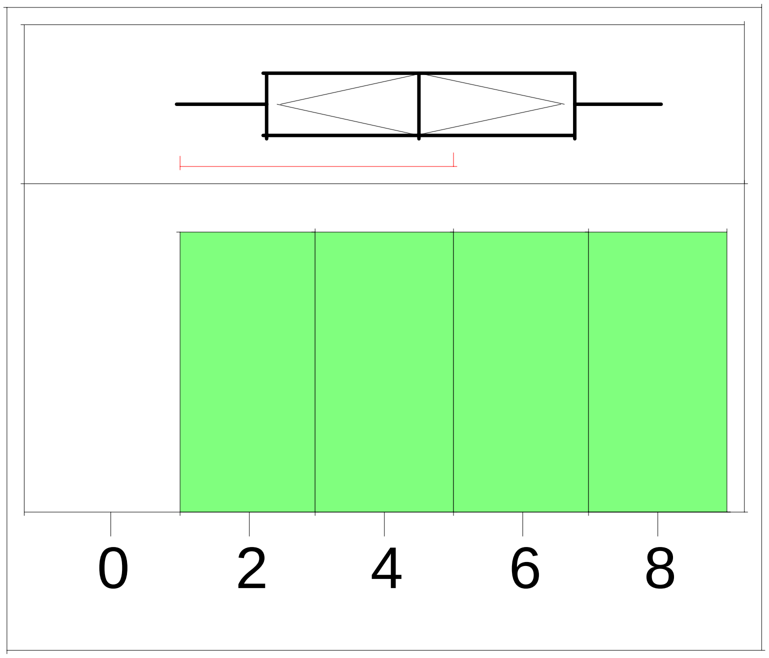
In addition, the mean for this particular discrete uniform is

$$\mu = \frac{\sum x}{K} = \frac{36}{8} = 4.5$$

and the variance is

$$\sigma^2 = \frac{\sum (x - 4.5)^2}{K} = 5.25.$$

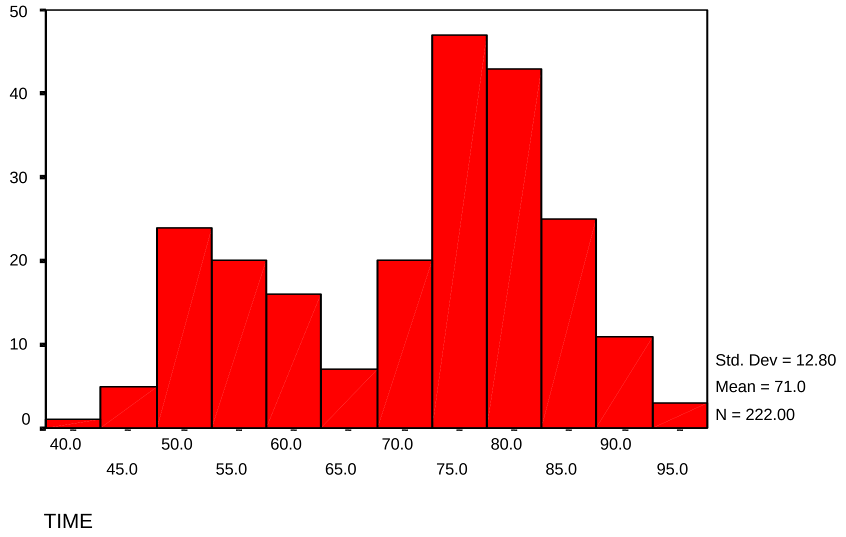
Also, it is easy to see that the probabilities sum to 1 as required. Finally, the graph of the distribution looks like a rectangle:



B) Continuous Random Variables

Recall that a continuous random variable is one that can take on any value from an interval on the number line. Now, for relative frequency distributions:

Fact 1*:* They show the frequencies of the values of the variable of interest in a set of data:

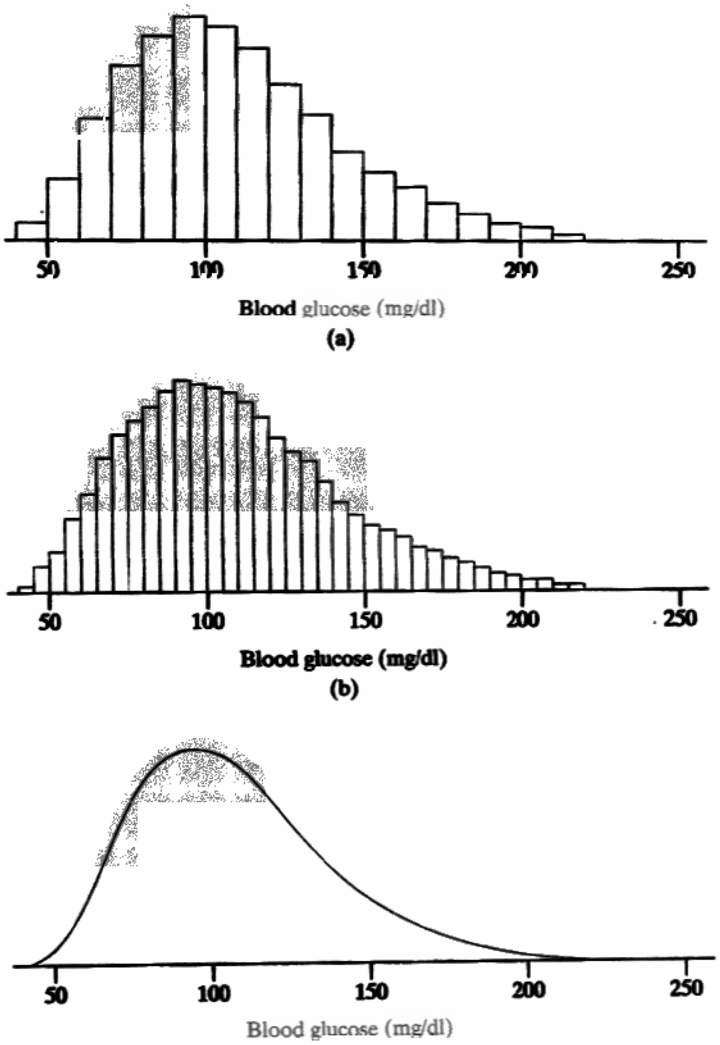


where the data have been assigned to specific groupings (bins or categories). The height of each bar is proportional to the relative frequency in the data set of the group it represents.

Multiplying the heights by the widths of the bars and adding all the areas gives the total area under in the bars (red). The area under any one bar divided by the total area equals *Pr*(*an observation falls in that grouping*)

Fact 2: For a continuous variable and an extremely large population, the number of bars is very large and the heights of the bars approach a smooth curve. This curve is often referred to as a **DENSITY CURVE** or the probability distribution.

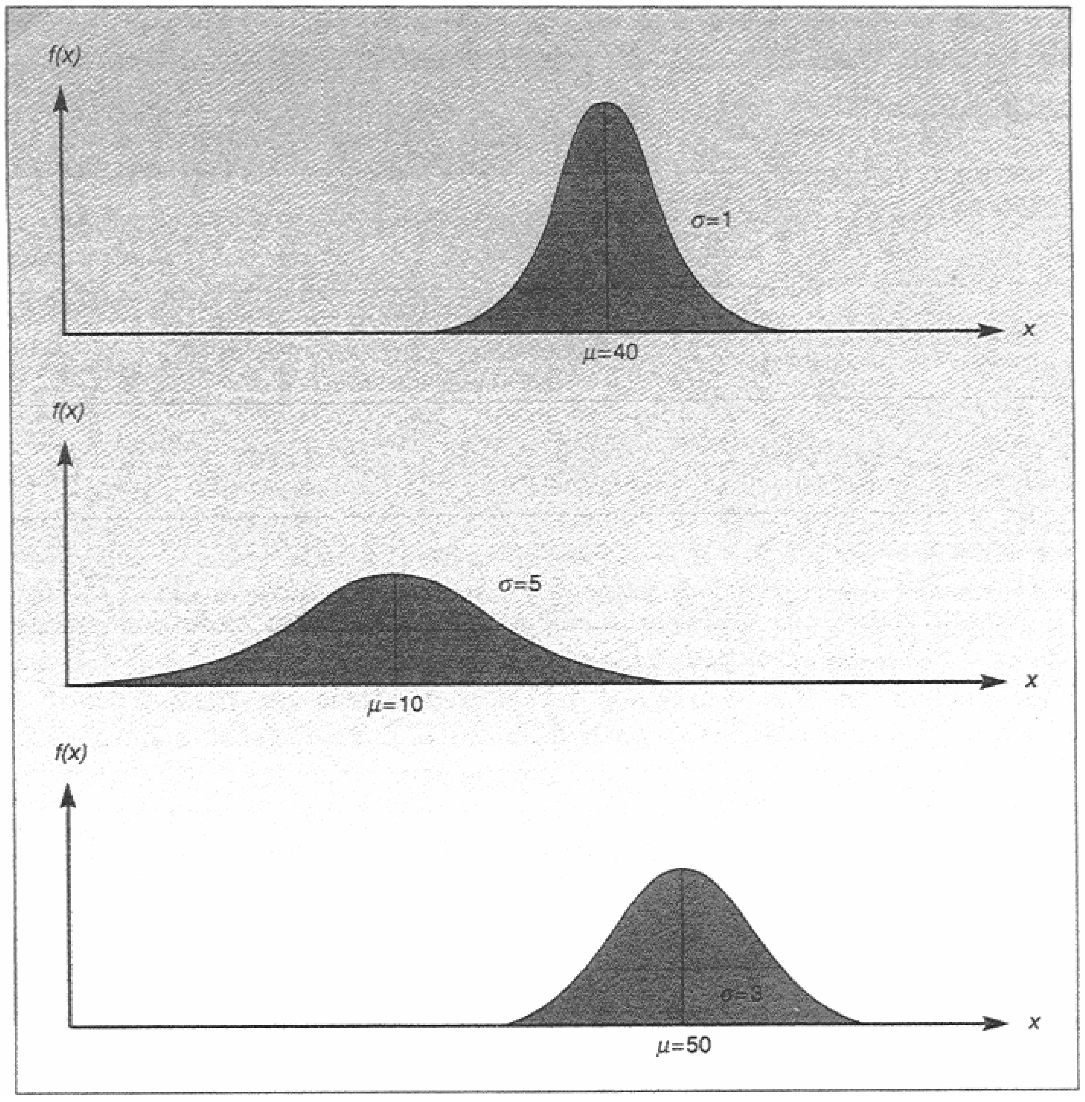
The curve describes the ***shape*** of the distribution and also depends on the mean and standard deviation of the population under study.



(a)

(b)

Normal Distributions with Different Means and Standard Deviations



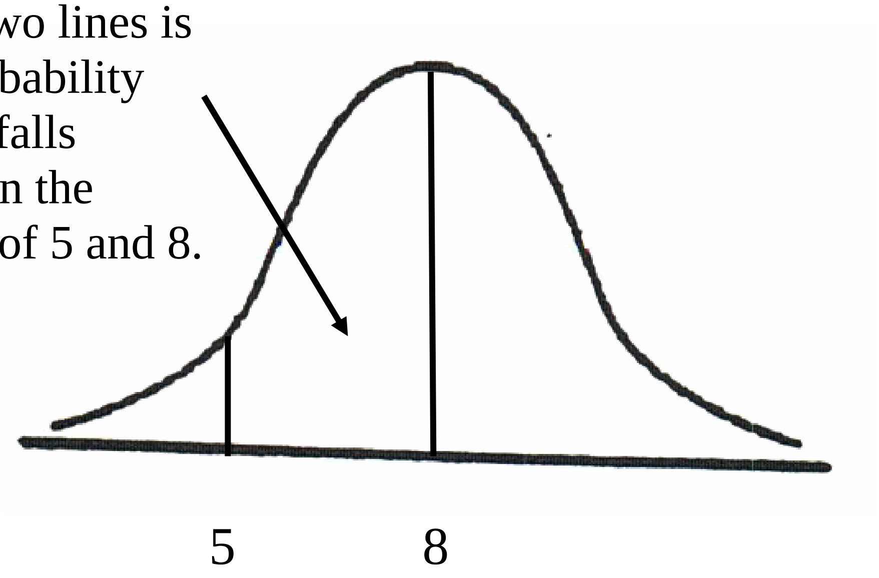
Fact 3: When the curve is describing frequency distribution of the population, every observation must fall within the limits of the distribution. Hence, 100% of the observations are listed.

When we combine these three facts, we get that the density curve describing the frequency distribution of values of a quantitative variable

1) has a total area under the curve of 1 (analogous to 100%) **and**

2) the area over a range of values equals the relative frequency of that range in the population,

i.e. the area equals the probability of observing a value within that range



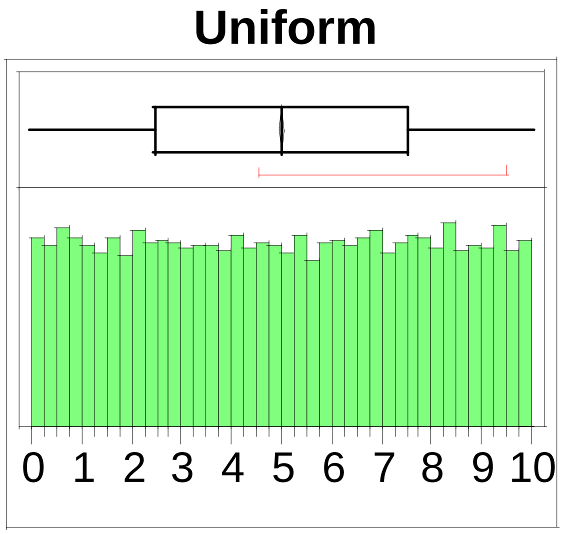
Area in between these two lines is the probability that X falls between the values of 5 and 8.

There are many standard (common) density curves:

UNIFORM DISTRIBUTION – every subset interval of the same length is equal likely. For example, suppose we randomly selected a number from the number line [0, 10]. Then the Probability distribution is given by

$$\Pr(a < X < b) = \frac{b-a}{U-L}$$

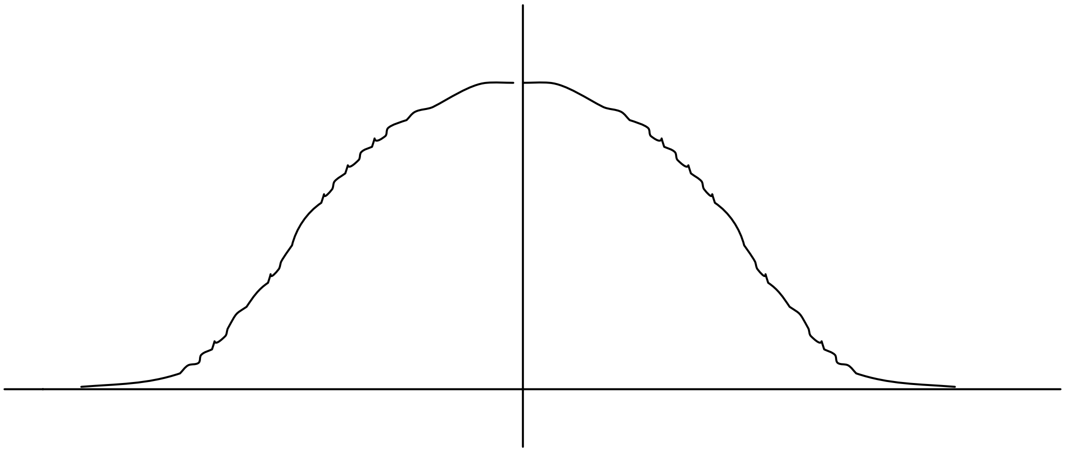
for $X \in [L,U]$ and $L,U > 0$.



e.g. Pr(3<X<4) =

The mean of a Uniform distribution is $\mu = \frac{U-L}{2}$ and the variance is

NORMAL DISTRIBUTION (Bell-Curve or Gaussian Distribution) – symmetric, unimodal and bell-shaped



Some interesting facts about the **NORMAL DISTRIBUTION**:

1. mean = median = mode
2. the shape is perfectly symmetric with equal sized tails
3. the Empirical Rule has an exact form:
   68.26 % of the values fall within $\mu \pm \sigma$
   95.44% of the values fall within $\mu \pm 2\sigma$
   99.74% of the values fall within $\mu \pm 3\sigma$
4. the endpoints of the interval $\mu \pm \sigma$ fall exactly at the inflection points of the curve
5. it's the most common distribution for natural phenomena that take on continuous values

Calculating Probabilities Of Events For A Normal Distribution:

EXAMPLE IQ as measured by the Stanford-Binet test has a mean of μ=100 and a standard deviation of σ=15.

1. What proportion of the US adult population has an IQ above 100? i.e. find *Pr(IQ>100)*.

2. What proportion of the population has an IQ between 85 and 115? i.e. find *Pr(85<IQ<115)*.

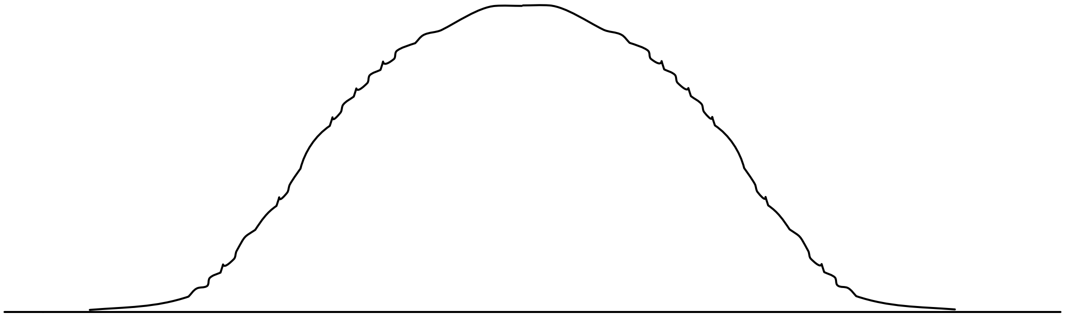
Question: What do we do when the value of interest in the probability phrase does NOT fall exactly at the standard deviation cutoffs? E.g. find *Pr(IQ<110)*?

Answer: Convert the value to a Z-score and use it and a look up table (or a computer program) to calculate the probability.

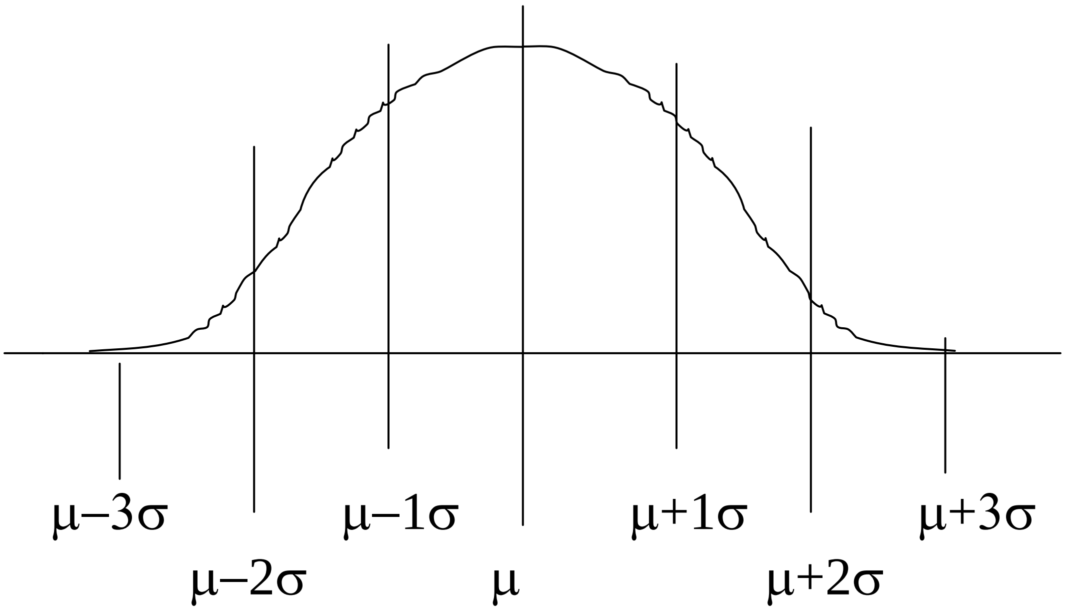
Recall the **Z-SCORE** for a value is the number of standard deviations that value is from the mean:

$$Z-score = z^* = \frac{x-\mu}{\sigma}$$

e.g. IQ of 110 ≡ $z^* = \frac{110-\mu}{\sigma} = \frac{110-100}{15} = 0.667$



Defn: When X is normally distributed, the Z-score has a **STANDARD NORMAL DISTRIBUTION**. The Standard normal distribution is a normal distribution with a mean of $\mu=0$ and a standard deviation of $\sigma=1$.



Original IQ score

55 70 85 100 115 130 145

Equivalent Z-score

-3 -2 -1 0 +1 +2 +3

So, the important point here is that we need to do the conversion

$$\Pr(X < a) = \Pr\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right) = \Pr(Z < z)$$

in order to find probabilities of events under a normal distribution

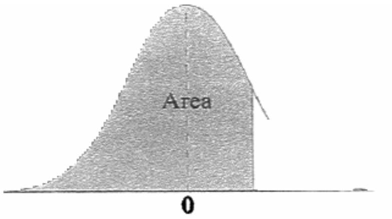
e.g.

$$\Pr(IQ < 110) = \Pr\left(\frac{IQ-\mu}{\sigma} < \frac{110-\mu}{\sigma}\right)$$

$$= \Pr\left(\frac{IQ-100}{15} < \frac{110-100}{15}\right) = \Pr(Z < 0.667)$$

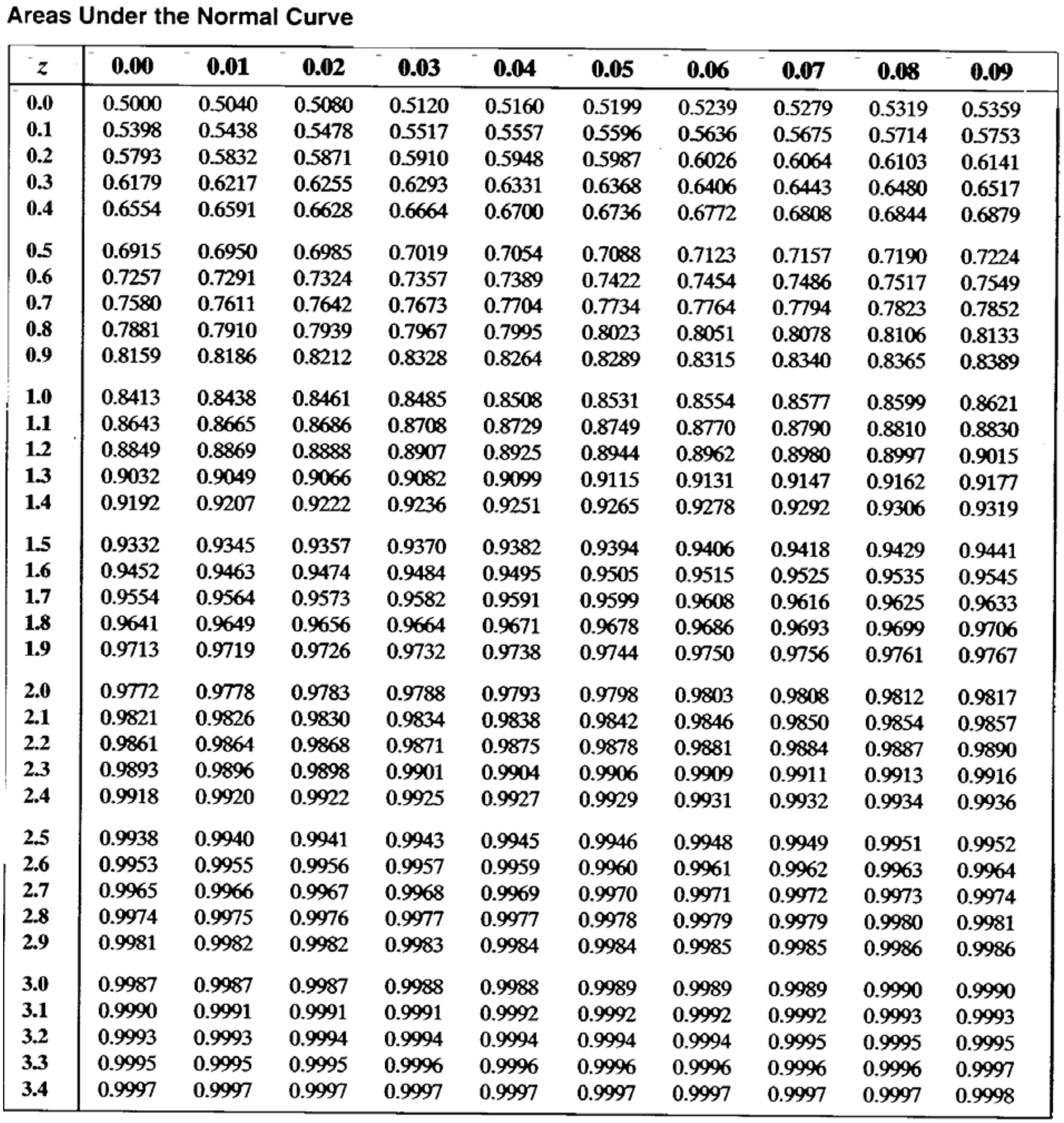
Next, look up the area (i.e. Probability) on a table:

$\Pr(Z < 0.667) = 0.7486$, so approximately 75% of the population has an IQ less than 110.



Areas Under the Normal Curve

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
− 3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
− 3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
− 3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
− 3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
− 3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
− 2.9	0.0019	0.0018	0.0017	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
− 2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
− 2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
− 2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
− 2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
− 2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
− 2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
− 2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
− 2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
− 2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
− 1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
− 1.8	0.0359	0.0352	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
− 1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
− 1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
− 1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
− 1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
− 1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
− 1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
− 1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
− 1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
− 0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
− 0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
− 0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
− 0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
− 0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
− 0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
− 0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
− 0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
− 0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
− 0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641



Areas Under the Normal Curve

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8328	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9278	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Some practice which also uses the rules for Probability that we learned earlier:

1. Find $Pr(IQ>92)$

2. Find $Pr(70<IQ<120)$.

Finding Quantiles for the Normal Distribution

Most often used to find extreme values in the very highest (or lowest) percentages

EXAMPLE Suppose adult male heights are normally distributed with a mean of 69” and a standard deviation of 3.5”. We have learned how to answer questions like: What proportion of the population are taller than 6’ (72”)?

How do we answer a question like: Find the range of likely heights for the shortest 5% of the male population, i.e. what height is the 5th percentile of the population?

Here we are being asked to find the value of *a* that makes the following probability statement true:

$$Pr(Height < a) = 0.05$$

We know that

$$Pr(Height < a) = Pr(Z < z^{*})$$

So we’ll start by solving

$$Pr(Z < z^{*})=0.05$$

for z^*.

Now, we’ll use the fact that $z^* = \frac{a - \mu}{\sigma}$ and our knowledge of the values of μ and σ to solve for a.