

## Problem Set #2

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### Question 1

The social planner in this problem seeks to maximize utility subject to the production function, resource constraint, and law of motion:

$$\max_{\{C_t, K_t, I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(C_t), \text{ s.t. } Y_t = AK_t^\alpha, K_{t+1} = K_t^{1-\delta} I_t^\delta, Y_t = C_t + I_t$$

Combining the production function, resource constraint, and law of motion gives the following Lagrangian function:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log(C_t) - \lambda_t (K_{t+1} - K_t^{1-\delta} (AK_t^\alpha - C_t)^\delta)$$

Which has the following first-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= \frac{\beta^t}{C_t} - \lambda_t \delta K_t^{1-\delta} (AK_t^\alpha - C_t)^{\delta-1} = 0 \\ \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= -\lambda_t + \lambda_{t+1} \left[ (1-\delta) K_{t+1}^{-\delta} (AK_{t+1}^\alpha - C_{t+1})^\delta + \delta K_{t+1}^{1-\delta} \alpha AK_{t+1}^{\alpha-1} (AK_{t+1}^\alpha - C_{t+1})^{\delta-1} \right] = 0 \\ \Rightarrow \frac{C_{t+1}}{C_t} &= \beta (K_t^{1-\delta} I_t^{\delta-1}) (K_{t+1}^{\delta-1} I_{t+1}^{1-\delta}) (K_{t+1}^{-\delta} I_{t+1}^\delta) (1 - \delta + \delta \alpha AK_{t+1}^\alpha I_{t+1}^{-1}) \end{aligned}$$

Which simplifies to the following Euler equation:

$$\frac{C_{t+1}}{C_t} = \beta (1 - \delta + \delta \alpha AK_{t+1}^\alpha I_{t+1}^{-1})$$

### Question 2

The two equations that pin down the steady-state of this model are the Euler equation from (1) and the combined production function, law of motion, and

resource constraint (which we will simply refer to as the consolidated resource constraint):

$$\begin{aligned} C_{t+1} &= \beta C_t \left( 1 - \delta + \delta \alpha A K_{t+1}^\alpha (A K_{t+1}^\alpha - C_{t+1})^{-1} \right) \\ K_{t+1} &= K_t^{1-\delta} (A K_t^\alpha - C_t)^\delta \end{aligned}$$

### Question 3

Before log-linearizing this system, let us first simplify the steady-state values of the model's variables using the two equations we have. First, take the law of motion:

$$\begin{aligned} \bar{K} &= \bar{K}^{1-\delta} \bar{I}^\delta \\ \bar{K}^\delta &= \bar{I}^\delta \Rightarrow \bar{K} = \bar{I} \end{aligned}$$

Then, if we plug this equality into the resource constraint, we get:

$$A\bar{K} = \bar{C} + \bar{K} \Rightarrow \bar{C} = \bar{K} (A\bar{K}^{\alpha-1} - 1)$$

To simplify notation, denote  $\phi = A\bar{K}^{\alpha-1}$ . Finally, using the law of motion equality and the Euler equation, we can determine:

$$\bar{C} = \beta \bar{C} (1 - \delta + \delta \alpha A \bar{K}^\alpha \bar{I}^{-1}) \Rightarrow 1 = \beta (1 - \delta + \delta \alpha \phi)$$

Or,  $\beta^{-1} = 1 - \delta + \delta \alpha \phi$ .

*The Euler Equation:* We can log-linearize the Euler equation in stages. Let  $I_{t+1} = A K_{t+1}^\alpha - C_{t+1}$  and  $Z_{t+1} = 1 - \delta + \delta \alpha A K_{t+1}^\alpha I_{t+1}^{-1}$ . Then:

$$\begin{aligned} c_{t+1} - c_t &= (\alpha k_{t+1} - i_{t+1}) \left( \frac{\delta \alpha A \bar{K}^\alpha \bar{I}^{-1}}{\bar{Z}} \right) \\ \frac{1}{\bar{Z}} &= \beta \\ i_{t+1} &= \left( \frac{A \bar{K}^\alpha}{\bar{I}} \right) \alpha k_{t+1} - \left( \frac{\bar{C}}{\bar{I}} \right) c_{t+1} \\ \Rightarrow c_{t+1} - c_t &= \beta \delta \alpha^2 \phi (1 - \phi) k_{t+1} - \beta \delta \alpha \phi (1 - \phi) c_{t+1} \\ (1 + \beta \delta \alpha \phi (1 - \phi)) c_{t+1} &= \beta \delta \alpha^2 \phi (1 - \phi) k_{t+1} + c_t \\ c_{t+1} &= \frac{\beta \delta \alpha^2 \phi (1 - \phi)}{1 + \beta \delta \alpha \phi (1 - \phi)} k_{t+1} + \frac{1}{1 + \beta \delta \alpha \phi (1 - \phi)} c_t \end{aligned}$$

*The Consolidated Resource Constraint:* Once again letting  $I_t = AK_t^\alpha - C_t$ :

$$\begin{aligned} k_{t+1} &= (1 - \delta)k_t + \delta i_t \\ i_t &= \alpha \phi k_t - (\phi - 1)c_t \\ k_{t+1} &= (1 - \delta)k_t + \delta (\alpha \phi k_t - (\phi - 1)c_t) \\ k_{t+1} &= (1 - \delta + \delta \alpha \phi)k_t - \delta(\phi - 1)c_t \\ k_{t+1} &= \beta^{-1}k_t - \delta(\phi - 1)c_t \end{aligned}$$

#### Question 4

The log-linearized consolidated resource constraint from (3) is already given as a function of one state variable ( $k_t$ ) and one choice variable ( $c_t$ ). We can plug this function into  $k_{t+1}$  in the log-linearized Euler equation from (3) to transform it into a function with one state variable and one choice variable, as well:

$$\begin{aligned} c_{t+1} &= \frac{\beta \delta \alpha^2 \phi (1 - \phi)}{1 + \beta \delta \alpha \phi (1 - \phi)} [\beta^{-1}k_t - \delta(\phi - 1)c_t] + \frac{1}{1 + \beta \delta \alpha \phi (1 - \phi)} c_t \\ c_{t+1} &= \frac{\delta \alpha^2 \phi (1 - \phi)}{1 + \beta \delta \alpha \phi (1 - \phi)} k_t + \frac{\delta^2 \alpha^2 \phi (1 - \phi)^2}{1 + \beta \delta \alpha \phi (1 - \phi)} c_t \end{aligned}$$

To simplify the notation, denote  $\theta = \delta \alpha \phi (1 - \phi)$ . We begin the Blanchard-Kahn method by writing the system in vector-matrix form:

$$x_{t+1} = \begin{pmatrix} c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{\delta \alpha (1 - \phi) \theta}{1 + \beta \theta} & \frac{\alpha \theta}{1 + \beta \theta} \\ -\delta(\phi - 1) & \beta^{-1} \end{pmatrix} \begin{pmatrix} c_t \\ k_t \end{pmatrix} = Bx_t$$

Next, we can find the eigenvalues of  $B$  by solving  $B - \lambda I = 0$ :

$$\begin{aligned} &\left( \frac{\delta \alpha (1 - \phi) \theta}{1 + \beta \theta} - \lambda \right) (\beta^{-1} - \lambda) - \frac{\delta(\phi - 1) \alpha \theta}{1 + \beta \theta} = 0 \\ \Rightarrow \lambda^2 - \left( \frac{\beta^{-1} + \theta + \delta \alpha (1 - \phi) \theta}{1 + \beta \theta} \right) \lambda + \frac{(\beta^{-1} + 1) \delta \alpha (1 - \phi) \theta}{1 + \beta \theta} &= 0 \end{aligned}$$

Using the quadratic formula, we can solve:

$$\lambda = \frac{\beta^{-1} + \theta + \delta \alpha (1 - \phi) \theta}{2(1 + \beta \theta)} \pm \sqrt{\frac{(\beta^{-1} + \theta + \delta \alpha (1 - \phi) \theta)^2}{4(1 + \beta \theta)^2} - \frac{(\beta^{-1} + 1) \delta \alpha (1 - \phi) \theta}{1 + \beta \theta}}$$

Note that  $\phi > 1$ , since  $\bar{C}/\bar{K} = \phi - 1$ , and  $\bar{C}, \bar{K} \geq 0$ .

I'm going to be frank here. I can't solve this. I must have gotten something wrong in the algebra along the way. I have re-calculated the log-linearization multiple times over several hours, with a different approach each time, and I

keep getting the same result that I know is incorrect. I've run out of time to get this right, and I hope I can get partial credit. Here's how I would solve it if I found  $\lambda_1$  and  $\lambda_2$ :

First, I would note that  $B = Q\Lambda Q^{-1}$ , where  $Q = [v_1 \ v_2]$ , where  $v_i$  is the eigenvector that corresponds to  $\lambda_i$  and  $y_{t+1} = Q^{-1}x_{t+1}$ . Then, I would identify  $i$  such that  $|\lambda_i| > 1$ , then note that this implies that  $y_{it} = \lim_{j \rightarrow \infty} \lambda_i y_{i,t+j} = 0$ . Thus, the saddle path of the system is given by setting  $y_{it} = 0$  for all  $t$  and solving  $x_{t+1} = Q\Lambda y_t$ .

**Question 5**

**Question 6**

**Question 7**

**Question 8**