Econ 711 – Fall 2020 – Problem Set 3 – Solutions

Question 1. Monotone Selection Theorems

Consider a single-output firm facing a tax τ on revenue *(not profit). The firm is not a price-taker in input markets, but its technology is still characterized by a weakly-increasing cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$, with $c(q)$ the cost of producing q units of output.*

(a) *Suppose the firm is a price taker in its output market. Show that its objective function $(1-\tau)pq - c(q)$ has* strictly *increasing differences in q and $-\tau$. Prove that this implies a* monotone selection rule*: an increase in τ can never result in an increase in output. Explain why this is a stronger result than "baby Topkis".*

It's easiest to define a new parameter $t = 1 - \tau$, and let $g(q,t) = tpq - c(q)$. The difference

$$g(q',t) - g(q,t) \quad = \quad tp(q'-q) - c(q') + c(q)$$

is strictly increasing in t for $q' > q$, hence increasing in $-\tau$.

Again let $t = 1 - \tau$. If q is optimal at t and q' is optimal at t', this implies

$$g(q,t) \quad \geq \quad g(q',t) \qquad \text{and} \qquad g(q',t') \quad \geq \quad g(q,t')$$

which implies

$$g(q,t) - g(q',t) \quad \geq \quad 0 \quad \geq \quad g(q,t') - g(q',t')$$

If $t' > t$ but $q > q'$, this would violate strictly increasing differences, as $g(q,\cdot) - g(q',\cdot)$ must be strictly increasing. Thus, if g has strictly increasing differences and $t' > t$, then any optimal choice at t' must be at least as big as any optimal choice at t.

This is stronger than the result we proved in class because "baby Topkis" (with just regular increasing differences) allows for the possibility that two points q and q' are both in both $q^*(t)$ and $q^*(t')$, and therefore that there's a point in $q^*(t')$ which is strictly less than a point in $q^*(t)$; this is ruled out by strictly increasing differences.

Now suppose the firm is not a price-taker in the output market, but faces an inverse demand function $P(\cdot)$, where $P(q)$ is the price at which the firm can sell q units of output.

(b) *Show that the firm's objective function $(1-\tau)P(q)q - c(q)$ does* not *necessarily have increasing differences in q and $-\tau$.*

Well,

$$\frac{\partial g}{\partial(-\tau)} \quad = \quad -\frac{\partial g}{\partial \tau} \quad = \quad P(q)q$$

so g only has increasing differences when $P(q)q$ is increasing in q, which need not always be true. (For most products, if price keeps increasing, revenue will eventually begin to fall.)

(c) *Show that if* $c(\cdot)$ *is strictly increasing, the firm's objective function still has* strictly single-crossing differences*; prove that an increase in* τ *cannot result in an increase in output.*

Strictly single-crossing differences means

$$g(q',t) - g(q,t) \quad \geq \quad 0 \qquad \longrightarrow \qquad g(q',t') - g(q,t') \quad > \quad 0$$

for any $q' > q$ and $t' > t$. Continue to let $t = 1 - \tau$. If $g(q',t) - g(q,t) \geq 0$ with $q' > q$, this means

$$tP(q')q' - c(q') \quad \geq \quad tP(q)q - c(q)$$

or

$$t\left[P(q')q' - P(q)q\right] \quad \geq \quad c(q') - c(q)$$

If $c(\cdot)$ is strictly increasing, then the right-hand side is strictly positive, and this therefore requires $P(q')q' > P(q)q$. This means that for $t' > t$,

$$t'\left[P(q')q' - P(q)q\right] \quad > \quad t\left[P(q')q' - P(q)q\right] \quad \geq \quad c(q') - c(q)$$

and therefore

$$t'P(q')q' - c(q') \quad > \quad t'P(q)q - c(q)$$

as required, so the objective function has strictly single-crossing differences.

To show that this suffices to prove monotone selection, recall that just like above, if q is optimal at t and q' at t', then

$$g(q,t) - g(q',t) \quad \geq \quad 0 \quad \geq \quad g(q,t') - g(q',t')$$

If $q > q'$, then this would violate strictly single-crossing differences, as the left-hand side being weakly positive would imply that the right-hand side must be strictly positive. Thus, strictly single-crossing differences implies that if $t' > t$, then $q' \in q^*(t')$ must be weakly greater than $q \in q^*(t)$.

Question 2. Robot Carwashes

A firm provides car washes using four inputs: unskilled labor (ℓ), managers (m), robots (r), and engineers (e). Managers are required to supervise unskilled labor, and engineers are required to keep the robots running; the firm's output is

$$q \quad = \quad f(\ell,m,r,e) \quad = \quad \left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)^z$$

with $z = 1.1$. *Input costs are* w_ℓ, w_m, w_r, *and* w_e, *so the firm's problem is*

$$\max_{\ell,m,r,e\geq 0} \left\{pf(\ell,m,r,e) - w_\ell \ell - w_m m - w_r r - w_e e\right\}$$

Suppose at each input price vector, the firm's problem has a unique solution.

(a) *In an effort to encourage STEM education, a politician proposes subsidizing the wage of engineers. From the firm's point of view, this simply reduces the cost of engineers, w_e. What effect will this have on the firm's demand for each input?*

With $z = 1.1$, the firm's objective function

$$g \quad = \quad p\left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)^{1.1} - (w_\ell, w_m, w_r, w_e) \cdot (\ell, m, r, e)$$

is supermodular in (ℓ, m, r, e), and has increasing differences in (ℓ, m, r, e) and $-w_e$. To see this, we can calculate

$$\frac{\partial g}{\partial \ell} \quad = \quad p \cdot 1.1\left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)^{0.1} \cdot 0.5\ell^{-0.5}m^{0.3} - w_\ell$$

and note that it is increasing in m, r, and e, and weakly decreasing in w_e (since w_e does not appear). Similarly,

$$\frac{\partial g}{\partial m} \quad = \quad p \cdot 1.1\left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)^{0.1} \cdot 0.3\ell^{0.5}m^{-0.7} - w_m$$

$$\frac{\partial g}{\partial r} \quad = \quad p \cdot 1.1\left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)^{0.1} \cdot 0.7r^{-0.3}e^{0.1} - w_r$$

$$\frac{\partial g}{\partial e} \quad = \quad p \cdot 1.1\left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)^{0.1} \cdot 0.1r^{0.7}e^{-0.9} - w_e$$

which are each increasing in the other three choice variables and either weakly or strictly decreasing in w_e. Thus, we can apply Topkis' Theorem; a decrease in w_e must increase the set of optimal production plans (ℓ, m, r, e). With the additional information that the firm's problem always has a unique solution, this means all four choice variables ℓ, m, r, and e must weakly increase when w_e falls.

(b) *Over time, the firm's technology shifts, with z changing from 1.1 to 0.9. With $z = 0.9$, what effect would the subsidy on engineers' wages have on the firm's demand for each input?*

This time, the firm's objective function turns out to be supermodular in $(-\ell, -m, r, e)$, with increasing differences in $(\ell, m, -r, -e)$ and w_e.[1] To see this, we can calculate

$$\frac{\partial g}{\partial \ell} \quad = \quad p \cdot 0.9\left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)^{-0.1} \cdot 0.5\ell^{-0.5}m^{0.3} - w_\ell$$

Given the negative exponent on $\left(\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}\right)$, it's clear this is now decreasing in r and e. To see it's still increasing in m, we can rewrite it as

$$\begin{aligned}\frac{\partial g}{\partial \ell} \quad &= \quad p \cdot 0.9\left(\frac{\ell^{0.5}m^{0.3} + r^{0.7}e^{0.1}}{m^{0.3}}\right)^{-0.1}\left(m^{0.3}\right)^{-0.1} \cdot 0.5\ell^{-0.5}m^{0.3} - w_\ell \\ &= \quad p \cdot 0.9\left(\ell^{0.5} + r^{0.7}e^{0.1}m^{-0.3}\right)^{-0.1} \cdot 0.5\ell^{-0.5}\left(m^{0.3}\right)^{0.9} - w_\ell\end{aligned}$$

[1] Or, equivalently, supermodular in $(-\ell, -m, r, e)$, with increasing differences in $(-\ell, -m, r, e)$ and $-w_e$.

which is clearly increasing in m. In the same way, we can show that $\frac{\partial g}{\partial m}$ is increasing in ℓ but decreasing in r and e; $\frac{\partial g}{\partial r}$ is decreasing in ℓ and m and increasing in e; and $\frac{\partial g}{\partial e}$ is decreasing in ℓ and m and increasing in r. Further, $\frac{\partial g}{\partial e}$ is decreasing in w_e, and the other three partial derivatives don't change with w_e (so we can claim them as weakly increasing or weakly decreasing, as needed). Thus, if we consider the choice variables to be $(\ell, m, -r, -e)$, the objective function is supermodular; and it has increasing differences in the choice variables $(\ell, m, -r, -e)$ and w_e. Thus, a decrease in w_e leads to increases in r and e, and decreases in ℓ and m. (The intuition: human labor and robots are now substitutes, so when robot-wranglers get cheaper, the firm uses more robots, but less unskilled labor and managers.)

(c) *If the supply of managers is fixed in the short-run, would the subsidy's effect on unskilled labor be larger in the short-run or the long-run? Explain.*

The LeChatelier Principle tells us the effect will always be larger in the long-run. In the short run, when w_e falls, e and r go up, and ℓ goes down while m is "stuck". In the longer run, all of these effects reduce the returns to managers, so m falls; e and r rise further in response, and ℓ falls further in response. So the long-run decrease in demand for unskilled labor is larger than the short-run decrease.

(To show this more formally, let (ℓ_0, m_0, r_0, e_0) be the original (pre-subsidy) level of inputs, and $(\ell_{LR}, m_{LR}, r_{LR}, e_{LR})$ the long-run (post-subsidy) level of inputs, after m has been allowed to adjust. We saw above that in the long-run, the subsidy leads to increases in r and m and decreases in ℓ and m, so we know that $m_{LR} < m_0$. We can think of the problem as a problem with three choice variables – ℓ, r, e – and treat m (along with w_e) as a parameter. Note the problem is supermodular in $(\ell, -r, -e)$, and has increasing differences in $(\ell, -r, -e)$ and (m, w_e). In the short run, w_e falls (and m stays the same), so ℓ goes down (and r and e go up), while m stays fixed at m_0. In between the "short run" and the "long run," we can think of m falling from m_0 to m_{LR}; this leads to a further decrease in ℓ, and further increases in r and e. This is all based on the intuition that if (ℓ^*, m^*, r^*, e^*) is a solution to the firm's problem at some parameter level, then (ℓ^*, r^*, e^*) is a solution to the firm's problem when m is held fixed at m^*.)