Problem Set #4

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We have an overlapping generations problem with an infinite number of discrete periods and households that live for two periods. Each generation has a measure of households of population $(1+n)^t$, assuming that the inital old generation has a unit measure population. The inital old are endowed with $\overline{M_1}$ units of fiat currency and w_2 units of consumption goods. The money supply increases at a rate of $z \geq 0$, with new fiat money distributed to each period's old generation in proportion to their money holdings such that M_{t+1}^t chosen when young becomes $(1+z)M_{t+1}^t$ when old.

Consumption goods are non-storable and each young generation is endowed with w_1, where $w_1 > w_2$. The utility of households in generation $t \geq 1$ is represented by

$$U(c_t^t, c_{t+1}^t) = \ln\left(c_t^t\right) + \ln\left(c_{t+1}^t\right)$$

Where $U(c_1^0) = \ln\left(c_1^0\right)$ represents the utility of the initial old generation.

1. An allocation in this environment is defined as $\{c_0^1, \{c_t^t, c_t^{t-1}\}_{t=1}^\infty\}$, so the social planner's problem, weighting all generations equally, is:

$$\max_{\{c_0^1, \{c_t^t, c_t^{t-1}\}_{t=1}^\infty\}} U(c_1^0) + \sum_{t=1}^\infty U(c_t^t, c_{t+1}^t)$$
$$\text{s.t. } (1+n)^t c_t^t + (1+n)^{t-1} c_t^{t-1} \leq (1+n)^t w_1 + (1+n)^{t-1} w_2 \ \forall t \geq 1$$

$$\max_{\{c_0^1, \{c_t^t, c_t^{t-1}\}_{t=1}^\infty\}} \ln\left(c_1^0\right) + \sum_{t=1}^\infty \ln\left(c_t^t\right) + \ln\left(c_t^{t-1}\right)$$
$$\text{s.t. } (1+n)^t c_t^t + (1+n)^{t-1} c_t^{t-1} \leq (1+n) w_1 + w_2 \ \forall t \geq 1$$

The Lagrangian for this problem is

$$\mathcal{L} = \sum_{t=1}^\infty \ln\left(c_t^t\right) + \ln\left(c_t^{t-1}\right) - \lambda_t\left(c_t^{t-1} + c_t^{t-1} - (1+n)w_1 - w_2\right)$$

Taking first-order conditions and letting t represent any $t \geq 1$, we get:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t^t} &= \frac{1}{c_t^t} - (1+n)\lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial c_t^{t-1}} &= \frac{1}{c_t^{t-1}} - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= (1+n)c_t^t + c_t^{t-1} - (1+n)w_1 - w_2 = 0
\end{aligned}$$

Since c_1^0 is the consumption of the old generation in time t, we need only solve for c_t^{t-1} and c_t^t to derive c_0^1:

$$\frac{1}{c_t^t} = \frac{1+n}{c_t^{t-1}} \qquad\qquad c^{t-1} = (1+n)(w_1 - c_t^t) + w_2$$

$$c_t^{t-1} = (1+n)c_t^t = (1+n)(w_1 - c_t^t) + w_2$$

$$2c_t^t = w_1 + \frac{w_2}{1+n}$$

$$c_t^t = \frac{1}{2}\left(w_1 + \frac{w_2}{1+n}\right) \qquad\qquad c_t^{t-1} = \frac{1}{2}\left((1+n)w_1 + w_2\right)$$

Thus, the social planner's optimal allocation, $\{c_0^1, \{c_t^t, c_t^{t-1}\}_{t=1}^{\infty}\}$, is

$$\left\{\frac{1}{2}\left((1+n)w_1 + w_2\right), \left\{\frac{1}{2}\left(w_1 + \frac{w_2}{1+n}\right), \frac{1}{2}\left((1+n)w_1 + w_2\right)\right\}_{t=1}^{\infty}\right\}$$