Econ 712 Macroeconomics I

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Problem Set #4 - Due 10/01/20

Consider the following overlapping generations problem. Each period $t = 1, 2, 3, ...$ a new generation of 2 period lived households are born. The measure of identical households born in any period grows by $1 + n$. That is, we assume population growth of rate $n \geq 0$.

There is a unit measure of initial old who are endowed with $\overline{M}_1$ units of fiat money as well as w_2 units of consumption goods. Instead of a fixed money supply, now assume that the money supply increases at the rate $z \geq 0$. In other words, $\overline{M}_{t+1} = (1+z)\overline{M}_t$ for $t \geq 1$. The increase in money supply is handed out each period to old agents in direct proportion to the amount of money that they chose when young. In other words, if a young agent chooses $M^t_{t+1} \geq 0$, they will receive $(1+z)M^t_{t+1}$ units of money when old.

Each generation is endowed with w_1 in youth and w_2 in old age of nonstorable consumption goods where $w_1 > w_2$. The utility function of a household of generation $t \geq 1$ is

$$U\left(c^t_t, c^t_{t+1}\right) = \ln(c^t_t) + \ln(c^t_{t+1})$$

where $\left(c^t_t, c^t_{t+1}\right)$ is consumption of a household of generation t in youth (i.e. in period t) and old age (i.e in period $t+1$). The preferences of the initial old are given by $U(c^0_1) = \ln(c^0_1)$, where c^0_1 is consumption by a household of the initial old.

1. State and solve the planner's problem under the assumption of equal weights on each generation (i.e., on representative agents from each generation) so that the the objective of the social planner is to maximize $U(c^0_1) + \sum_{t=1}^{\infty} U\left(c^t_t, c^t_{t+1}\right)$. Since this objective may not be well defined (i.e. add up to infinity), we can apply the "overtaking" criterion to determine optimality.[1] Hence just go ahead and maximize away as usual.

[1] The "overtaking criterion" states that an allocation $\{c^{t-1}_t, c^t_t\}_{t=1}^T$ overtakes $\{\widehat{c}^{t-1}_t, \widehat{c}^t_t\}_{t=1}^T$ if

$$\lim\inf_{T\to\infty} \left[U(c^0_1) + \sum_{t=1}^{T} U(c^t_t, c^t_{t+1}) - U(\widehat{c}^0_1) + \sum_{t=1}^{T} U(\widehat{c}^t_t, \widehat{c}^t_{t+1})\right] > 0.$$

Since the finite sum is well defined, this sequence is well defined.

Answer: A social planner chooses $\{c_1^0, (c_t^t, c_{t+1}^t)_{t=1}^{\infty}\}$ to

$$\max_{c_1^0,(c_t^t,c_{t+1}^t)_{t=1}^{\infty}} \ln(c_1^0) + \sum_{t=1}^{\infty}\left[\ln(c_t^t) + \ln c_{t+1}^t\right]$$

$$\text{s.t. } c_t^{t-1} + (1+n)c_t^t = (1+n)w_1 + w_2 \quad \forall t \geq 1 \tag{S2.1}$$

$$c_t^{t-1}, c_t^t \geq 0 \quad \forall t \geq 1.$$

To solve the social planner's problem (SPP), write down the Lagrangian and take first order conditions (F.O.C) with respect to c_1^0, c_t^t, and c_{t+1}^t for $t \geq 1$.

$$L = \ln(c_1^0) + \sum_{t=1}^{\infty}\left[\ln(c_t^t) + \ln c_{t+1}^t\right] + \sum_{t=1}^{\infty}\lambda_t\left[(1+n)w_1 + w_2 - c_t^{t-1} - (1+n)c_t^t\right]$$

F.O.C

$$\frac{1}{c_1^0} = \lambda_1$$

$$\frac{1}{c_t^t} = (1+n)\lambda_t \tag{S2.2}$$

$$\frac{1}{c_{t+1}^t} = \lambda_{t+1} \tag{S2.3}$$

(S2.2) and (S2.3) imply

$$\frac{1}{c_t^t} = (1+n)\frac{1}{c_t^{t-1}} \tag{S2.4}$$

$$\implies \quad c_t^{t-1} = (1+n)c_t^t \tag{S2.5}$$

Substituting (S2.5) into the resource constraint (S2.1) gives the planner's optimal consumption choices for $t \geq 1$.

$$c_t^{t*} = \frac{w_1 + \frac{w_2}{1+n}}{2}$$

$$c_t^{t-1*} = \frac{(1+n)w_1 + w_2}{2}$$

2. Let p_t be the price of consumption goods in terms of money at time t. Define a competitive equilibrium.

Answer: A competitive equilibrium is a sequence of consumptions, money holdings, and prices $\left\{c_1^{0*}, (c_t^{t*}, c_{t+1}^{t*}, M_{t+1}^{t*}, p_t^*)_{t=1}^{\infty}\right\}$ such that

(a) c_1^{0*} solves the initial old's optimization problem

$$\max_{c_1^0 \geq 0} \ln(c_1^0)$$
$$\text{s.t.} \qquad p_1^* c_1^0 = p_1^* w_2 + \overline{M}_1$$

(b) $(c_t^{t*}, c_{t+1}^{t*}, M_{t+1}^{t*})$ solves generation t households' problem given p_t^* for all $t \geq 1$.

$$\max_{c_t^t \geq 0, c_{t+1}^t \geq 0, M_{t+1}^t \geq 0} \ln(c_t^t) + \ln(c_{t+1}^t)$$
$$\text{s.t.} \qquad p_t^* c_t^t + M_{t+1}^t = p_t^* w_1 \tag{1}$$
$$p_{t+1}^* c_{t+1}^t = p_{t+1}^* w_2 + (1+z) M_{t+1}^t \tag{2}$$

(c) Markets clear:

$$\text{demand} = \text{supply}$$
$$c_t^{t-1*} + (1+n) c_t^{t*} = (1+n) w_1 + w_2 \qquad \text{for } t \geq 1 \tag{3}$$
$$N_t M_{t+1}^{t*} = \overline{M}_t \qquad \text{for } t \geq 1 \tag{4}$$

where $N_t = (1+n)^t$ is the population size at time t.

3. Solve for an autarkic equilibrium.

Answer:

In a steady state autarkic equilibrium $\left\{(c_1^{0*})_A, ((c_t^{t*})_A, (c_{t+1}^{t*})_A, (M_{t+1}^{t*})_A, (p_t^*)_A)_{t=1}^{\infty}\right\}$, each generation consume their endowment for each period $t \geq 1$,

$$(c_t^{t*})_A = w_1$$
$$(c_{t+1}^{t*})_A = w_2$$

Consolidating two period budjet constraints (1) and (2) gives

$$(1+z) p_t^* c_t^t + p_{t+1}^* c_{t+1}^t = (1+z) p_t^* w_1 + p_{t+1}^* w_2 \tag{5}$$

The household optimization requires

$$MRS(c_t^t, c_{t+1}^t) = \frac{(1+z) p_t^*}{p_{t+1}^*}$$
$$\Rightarrow \frac{p_t^*}{p_{t+1}^*} = \frac{(c_{t+1}^{t*})_A}{(1+z)(c_t^{t*})_A} = \frac{w_2}{(1+z) w_1} \tag{6}$$

By the budget constraints for the initial old and generation t and market clearing condition for money $M_{t+1}^{t*} = \frac{\overline{M}_t}{(1+n)^t} = \frac{\overline{M}_1(1+z)^{t-1}}{(1+n)^t}$,

$$\frac{\overline{M}_1}{(p_1^*)_A} = (c_1^{0*})_A - w_2 = 0$$
$$\frac{\overline{M}_1(1+z)^{t-1}}{(1+n)^t(p_t^*)_A} = w_1 - (c_t^{t*})_A = 0$$
$$\frac{\overline{M}_1(1+z)^{t-1}}{(1+n)^t(p_{t+1}^*)_A} = (c_{t+1}^{t*})_A - w_2 = 0$$

4. Solve for a steady state (non-autarkic) monetary equilibrium. As $w_1 > w_2$, we know this corresponds to the Samuelson case with $\beta = 1$. Verify the non-negativity constraint on money is not binding in the equilibrium. What is the rate of return on money in the monetary equilibrium? Give intuition why the rate of return is at the level you find. Hints:

   (a) Solve the initial old and generation t's problems given goods prices (p_t, p_{t+1}) for $t \geq 1$. That is, solve for $(c_t^t, c_{t+1}^t, M_{t+1}^t)$ given (p_t, p_{t+1}).

   (b) Denote $q_t \equiv \frac{p_t}{p_{t+1}}$ and use goods market clearing with the optimal consumption functions $c_{t+1}^{t+1}(p_{t+1}, p_{t+2})$, $c_{t+1}^t(p_t, p_{t+1})$ you derived in part (3) to get a first-order difference equation in q_t (i.e. just update $c_t^t(p_t, p_{t+1})$).

   (c) Solve for steady states $\overline{q}$. Show that one steady state of $\overline{q}$ corresponds to a steady state monetary equilibrium with inter-generational trade. (Hint: plug the steady state into $c_{t+1}^{t+1}(p_{t+1}, p_{t+2}), c_{t+1}^t(p_t, p_{t+1})$).

Answer:

(a) A stationary monetary equilibrium exists. Let it be $\left\{(c_1^{0*})_M, ((c_t^{t*})_M, (c_{t+1}^{t*})_M, (M_{t+1}^{t*})_M, (p_t^*)_M)_{t=1}^{\infty}\right\}$.

For the initial old, the problem is easy. By the budget constraint, the initial old consume the endowment and any real value of the endowed money in terms of consumption good.

$$c_1^0 = w_2 + \frac{\overline{M}_1}{p_1}$$

For generation t households, their optimization problem can be rewritten as

$$\max_{M_{t+1}^t \geq 0} \ln\left(w_1 - \frac{M_{t+1}^t}{p_t}\right) + \ln\left(w_2 + \frac{(1+z)M_{t+1}^t}{p_{t+1}}\right)$$

by plugging in

$$c_t^t = w_1 - \frac{M_{t+1}^t}{p_t} \tag{7}$$

$$c_{t+1}^t = w_2 + \frac{(1+z)M_{t+1}^t}{p_{t+1}} \tag{8}$$

Taking first order condition with respect to M_{t+1}^t gives the optimal money holding as a function of (p_t, p_{t+1}),

$$\text{F.O.C} \qquad -\frac{1}{p_t\left(w_1 - \frac{M_{t+1}^t}{p_t}\right)} + \frac{1+z}{p_{t+1}\left(w_2 + \frac{(1+z)M_{t+1}^t}{p_{t+1}}\right)} = 0 \qquad [M_{t+1}^t]$$

$$\Longrightarrow \qquad M_{t+1}^t(p_t, p_{t+1}) = \frac{(1+z)p_t w_1 - p_{t+1}w_2}{2(1+z)} \tag{9}$$

Caution is advised here to use the F.O.C. above as it requires an interior solution. However, we will verify that money is positively demanded in the equilibrium later. Plugging (9) back to (7) and (8) gives

$$c_t^t(p_t, p_{t+1}) = \frac{(1+z)p_t w_1 + p_{t+1}w_2}{2(1+z)p_t}$$

$$c_{t+1}^t(p_t, p_{t+1}) = \frac{(1+z)p_t w_1 + p_{t+1}w_2}{2p_{t+1}}$$

(b) Substituting $q_t = \frac{p_t}{p_{t+1}}$ into the optimal consumption functions gives

$$c_{t+1}^{t+1} = \frac{w_1}{2} + \frac{w_2}{2(1+z)} \cdot \frac{1}{q_{t+1}}$$

$$c_{t+1}^t = \frac{(1+z)w_1}{2} \cdot q_t + \frac{w_2}{2}$$

By market clearing condition for goods market for $t \geq 1$,

$$(1+n)c_{t+1}^{t+1} + c_{t+1}^t = (1+n)w_1 + w_2$$

we have a first order difference equation for q_t

$$\frac{(1+n)w_1}{2} + \frac{(1+n)w_2}{1+z} \cdot \frac{1}{q_{t+1}} + \frac{w_2}{2} + \frac{(1+z)w_1}{2} \cdot q_t = (1+n)w_1 + w_2 \tag{10}$$

(c) In the steady state equilibrium, q_t is constant over time. Let it be $\bar{q}$. Solve $\bar{q}$ by setting

$q_t = q_{t+1} = \overline{q}$ in the first order difference equation (10).

$$
\begin{aligned}
(1+z)w_1\overline{q}^2 - ((1+n)w_1 + w_2)\overline{q} + \frac{(1+n)w_2}{1+z} = 0 \\
\implies \overline{q} = \frac{((1+n)w_1 + w_2) \pm \sqrt{((1+n)w_1 + w_2)^2 - 4(1+n)w_1 w_2}}{2(1+z)w_1} \\
\implies \overline{q} = \frac{(1+n)w_1 + w_2 \pm ((1+n)w_1 - w_2)}{2(1+z)w_1} \\
\implies \overline{q} = \frac{1+n}{1+z} \quad \text{or} \quad \frac{w_2}{(1+z)w_1}
\end{aligned}
$$

Observe that the second solution to the difference equation is the price ratio for autarkic equilibrium. Similarly to the lecture, the price ratio associated with the monetary equilibrium is not stable. Any deviation from this price ratio would cause the system diverge.

When $\frac{p_t}{p_{t+1}} = \overline{q} = \frac{1+n}{1+z}$, generation t households consume $c_t^t = \frac{(1+n)w_1+w_2}{2(1+n)}$, $c_{t+1}^t = \frac{(1+n)w_1+w_2}{2}$. Young households use money as a storage technology to save $w_1 - c_t^t = \frac{(1+n)w_1-w_2}{2(1+n)} > 0$ for their old age. Thus, the non-negativity constraint on money is not binding. The initial price is pinned down by the budget constraint of the initial old and money supply in the first period as $c_1^0 = c_{t+1}^t = \frac{(1+n)w_1+w_2}{2}$ in the steady state equilibrium,

$$
p_1 = \frac{\overline{M}_1}{c_1^0 - w_2} = \frac{\overline{M}_1}{\frac{(1+n)w_1+w_2}{2} - w_2} = \frac{2\overline{M}_1}{(1+n)w_1 - w_2}
$$

This is the stationary monetary equilibrium.

Therefore, in the stationary monetary equilibrium $\left\{(c_1^{0*})_M, ((c_t^{t*})_M, (c_{t+1}^{t*})_M, (M_{t+1}^{t*})_M, (p_t^*)_M)_{t=1}^{\infty}\right\}$,

$$
\begin{aligned}
(c_{t+1}^{t*})_M &= \frac{(1+n)w_1 + w_2}{2} && \text{for } t \geq 0 \\
(c_t^{t*})_M &= \frac{(1+n)w_1 + w_2}{2(1+n)} && \text{for } t \geq 1 \\
(M_{t+1}^{t*})_M &= (1+z)^t \overline{M}_1 && \text{for } t \geq 1 \\
(p_t^*)_M &= \left(\frac{1+z}{1+n}\right)^{t-1} \frac{2\overline{M}_1}{(1+n)w_1 - w_2} && \text{for } t \geq 1
\end{aligned}
$$

The gross real rate of return on money is $\frac{1+n}{1+z}$ as $\frac{(p_t^*)_M}{(p_{t+1}^*)_M} = \frac{1+n}{1+z}$. A positive net real rate of return on money arises with a higher population growth than money growth in the steady state monetary equilibrium. As money is valued, a higher population growth implies fewer money units held by each household of the next generation. To achieve the same consumption when young for the next generation, the price of goods in terms of money has to decrease at the same rate of population growth rate net of money growth.

5. Does the stationary monetary equilibrium Pareto dominate autarky? Can you use your answer in part 1 to establish that? If so, how can the government implement it?

   Answer:

   Observe that consumption allocations in the steady state monetary equilibrium are the same as in the solution to SPP in part 1. As the solution to SPP is Pareto optimal, the steady state monetary equilibrium pareto dominates autarkic equilibrium regardless of money growth.

   You can also conclude this result directly comparing the utility of each generation in the autarkic and monetary case. Table 1 compares consumptions in autarkic equilibrium and steady state monetary equilibrium,

Equilibrium	Autarkic	Monetary	SPP
c_1^0	w_2	$\frac{(1+n)w_1+w_2}{2}$	$\frac{(1+n)w_1+w_2}{2}$
c_t^t	w_1	$\frac{(1+n)w_1+w_2}{2(1+n)}$	$\frac{(1+n)w_1+w_2}{2(1+n)}$
c_{t+1}^t	w_2	$\frac{(1+n)w_1+w_2}{2}$	$\frac{(1+n)w_1+w_2}{2}$

Table 1

   As $w_1 > w_2$ and log utility is strictly is strictly concave, the steady state monetary equilibrium strictly dominates autarkic equilibrium.

$$\frac{(1+n)w_1+w_2}{2} > w_2$$

$$\implies \ln(c_1^{0*})_M > \ln(c_1^{0*})_A$$

$$\ln\frac{(1+n)w_1+w_2}{2(1+n)} + \ln\frac{(1+n)w_1+w_2}{2} > \ln w_1 + \ln w_2$$

$$\implies \ln(c_t^{t*})_M + \ln(c_{t+1}^{t*})_M > \ln(c_t^{t*})_A + \ln(c_{t+1}^{t*})_A$$

   You can also show it as the way in the lecture notes. For all generation $t \geq 1$, stationary monetary equilibrium pareto dominates the autarkic steady state equilibrium since autarky is always feasible. The initial old are also better off at non-autarky since $(p_1^*)_M < (p_1^*)_A$.

   The government can implement the stationary monetary equilibrium with money growth by coordinate public belief on the initial price level to be $(p_1^*)_M = \frac{2\overline{M}_1}{(1+n)w_1-w_2}$.

6. Does money exhibit super-neutrality (i.e. the level of inflation does not change equilibrium consumption allocations)?

Answer:

Yes, we have money super-neutrality here. The equilibrium consumption allocations are not affected by the level of inflation rate. We can see this from the optimality condition for households, who equate their marginal rate of substitution between today's consumption and tomorrow's consumption to the real return on money faced by households.

$$MRS(c_t^t, c_{t+1}^t) = (1+z)\frac{p_t}{p_{t+1}} \tag{11}$$

The gross real rate of return on money $(1+z)\frac{p_t}{p_{t+1}} = (1+z) \cdot \frac{1+n}{1+z} = 1+n$, which is independent of money growth in the steady state monetary equilibrium. This is because the effect of money growth on inflation rate is exactly offset by its effect on the gross nominal return on money (i.e. 1+z) and the extra money supply is given entirely to the old generation.

Therefore, we have money super-neutrality which has no impact on real allocations. However, the money super-neutrality does not necessarily hold for overlapping generations models in more general settings (e.g. with production, leisure choices).