Problem Set #4

Danny Edgel
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Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

Let $L(X, Y)$ be the space of all $T : X \to Y$, where $\dim X = n$ and $\dim Y = m$. Define $A = \text{mtx}_{X,Y}(T) \in M_{mxn}$ such that each element of A is indexed a_{ij}, where $i \in \{1, ..., m\}$ refers to the row of A and $j \in \{1, ..., n\}$ refers to the column of A where a_{ij} is located.

Let $\{x_1, ..., x_n\}$, $x_i \in X$ $\forall i = \{1, ..., n\}$ be a basis for X and $\{y_1, ..., y_m\}$. $y_i \in Y$ $\forall i = \{1, ..., m\}$ be a basis for Y. Then, for $c_i, d_i \in \mathbb{R}$ $\forall i = \{1, ..., n\}$

1. $c_1x_1 + ... + c_nx_n$ spans X
2. $d_1y_1 + ... + d_my_m$ spans Y
3. For an arbitrary $T \in L(X, Y)$ and $x \in X$, $T(x) = d_1y_1 + ... + d_ny_n$
4. Thus, if we let $A = \text{mtx}_{X,Y}(T)$,

$$\begin{aligned}
T(x) &= T(c_1x_1 + ... + c_nx_n) \\
&= c_1T(x_1) + ... + c_nT(x_n) \\
&= c_1(a_{11}y_1 + a_{12}y_2 + ... + a_{1m}y_m) + ... + c_n(a_{n1}y_1 + a_{n2}y_2 + ... + a_{nm}y_m \\
&= \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}
\end{aligned}$$

Where:

$$A = a_{11} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + a_{21} \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + ... + a_{nm} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Thus, the set of nm transformations, represented by a set of matrices where each matrix equals one for some a_{ij} and zero for all others (but no two matrices have the same element equal to one) form a basis for $L(X, Y)$.

Question 2

(a)

Proof.

- Let λ be an eigenvalue of T and let $A = \text{mtx}T$.
- *Theorem*: λ is an eigenvalue of T if and only if λ is an eigenvalue of A. Thus, $\exists v \in X$ s.t. $Av = \lambda v$
- Then, for some $k \in \mathbb{N}$, $A^k v = A^{k-1}Av = A^{k-1}(\lambda v) = \lambda A^{k-2}Av = ... = \lambda^k v$. Thus, λ^k is an eigenvalue of A^k
- $\therefore \lambda^k$ is an eigenvalue of T^k $\blacksquare$

(b)

Proof.

- Let A^{-1} be the inverse of A. Then, $A^{-1}Av = Iv = v$
- Then, if λ is an eigenvalue of A:

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= A^{-1}(\lambda v) \\ Iv &= \lambda A^{-1}v \\ \lambda^{-1}v &= \lambda^{-1}\lambda A^{-1}v \\ \lambda^{-1}v &= A^{-1}v \end{aligned}$$

  Thus, λ^{-1} is an eigenvalue of A^{-1}

- $\therefore \lambda^{-1}$ is an eigenvalue of T^{-1} $\blacksquare$

(c)

Define $S : X \to X$ as $S(x) = T(x) - \lambda x$, $\forall x = \in X$. Let $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then, since T is linear by definition:

$$\begin{aligned} S(\alpha_1 x_1 + \alpha_2 x_2) &= T(\alpha_1 x_1 + \alpha_2 x_2) + \lambda(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \alpha_1 T(x_1) + \alpha_2 T(x_2) + \alpha_1 \lambda x_1 + \alpha_2 \lambda x_2 \\ &= \alpha_1(T(x_1) + \lambda x_1) + \alpha_2(T(x_2) + \lambda x_2) \\ &= \alpha_1 S(x_1) + \alpha_2 S(x_2) \end{aligned}$$

Thus, S is linear. Since $\ker S$ is defined as the set of all $x \in X$ s.t. $T(x) = \lambda x$, it encompasses all multiples of the eigenvector associated with λ.

Fix $v \in X$ s.t. $T(v) = \lambda v$. Then, for $x, y \in \ker S$, $\beta_1, \beta_2 \in \mathbb{R}$, $x = \beta_1 v$ and $y = \beta_2 v$. Then, for $x, y \in \ker S$ and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\begin{aligned}(\alpha_1+\alpha_2)(x+y) &= (\alpha_1+\alpha_2)(\beta_1 v+\beta_2 v)\\ &= (\alpha_1+\alpha_2)\beta_1 v + (\alpha_1+\alpha_2)\beta_2 v\\ &= (\alpha_1+\alpha_2)x + (\alpha_1+\alpha_2)y\\ (\alpha_1+\alpha_2)\beta_1 v + (\alpha_1+\alpha_2)\beta_2 v &= \alpha_1\beta_1 v + \alpha_2\beta_1 v + \alpha_1\beta_2 v + \alpha_2\beta_2 v\\ &= \alpha_1(\beta_1 v+\beta_2 v) + \alpha_2(\beta_1 v + \beta_2 v)\\ &= \alpha_1(x+y) + \alpha_2(x+y)\end{aligned}$$

Thus, properties 1, 2, 5, 6, and 7 of vector spaces are satisfied. The zero vector is also in $\ker S$:

$$S(\vec{0}) = T(\vec{0}) + \lambda\vec{0} = \vec{0} + \vec{0} = \vec{0}$$

Where, for $x \in \ker S$, $\vec{0} + x = \vec{0} + \beta_1 v = \beta_1 v = x$. Finally, the additive inverse and multiplicative identity conditions are satisfied:

$$\text{Let } \beta_2 = -\beta_1. \text{ Then,} x + y = \beta_1 v + \beta_2 v = \beta_1 v - \beta_1 v = \vec{0}$$
$$1x = 1\beta_1 v = \beta_1 v = x$$

Thus, $\ker S$ is a vector space.

Question 3

(a)

$$\mathrm{mtx}_W(T) = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

(b)

$\mathrm{mtx}_V(T) = P\mathrm{mtx}_W(T)P^{-1}$, where, based on the basis vectors of V,

$$P^{-1} = \begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix}$$

Thus,

$$\mathrm{mtx}_V(T) = \begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} -15 & 29 \\ -10 & 19 \end{pmatrix}$$

(c)

Assuming $(1, 2)$ is given as W coordinates, since $P^{-1} = \{v_1, v_2\}$ and $\text{mtx}_W = P^{-1}\text{mtx}_V(T)P$, we can solve for $T(1, 2)$ in the coordinates in V by multiplying $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ by $\text{mtx}_V(T)P$:

$$\text{mtx}_V(T)P\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -15 & 29 \\ -10 & 19 \end{pmatrix}\begin{pmatrix} -7 & -2 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ -4 \end{pmatrix}$$

Thus, The coordinates of $T(1, -2)$ in $\left\{\begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 7 \end{pmatrix}\right\}$ are $\begin{pmatrix} -9 \\ -4 \end{pmatrix}$.

Question 4

Step 1:

$$\det(A - \lambda I) = \lambda^2 - 9 = 0$$

Thus, $\lambda_1 = 3$, $\lambda_2 = -3$.

$$A - 3I = \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}$$
$$A + 3I = \begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}$$

Thus, $v_1 = \begin{pmatrix} \sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

Step 2:

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}, P = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Step 3:

$$P^{-1} = \frac{1}{-1 - 1/2}\begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2}/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & -(2\sqrt{2})/3 \end{pmatrix}$$
$$PDP^{-1} = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} \sqrt{2}/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & -(2\sqrt{2})/3 \end{pmatrix}$$

Step 4:

$$A^t = (PDP^{-1})^t = PD^tP^{-1} = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} \sqrt{2}/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & -(2\sqrt{2})/3 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2}3^t & (-3)^t/\sqrt{2} \\ 3^t/\sqrt{2} & -(-3^t)/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2}/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & -(2\sqrt{2})/3 \end{pmatrix}$$

$$= \begin{pmatrix} 2(3^{t-1}) + (-1)^t 3^{t-1} & 2(3^{t-1}) - 2(-1)^t 3^{t-1} \\ 3^{t-1} - (-1)^t 3^{t-1} & 3^{t-1} + 2(-1)^t 3^{t-1} \end{pmatrix}$$

$$= 3^{t-1} \begin{pmatrix} 2 + (-1)^t & 2 - 2(-1)^t \\ -(-1)^t & 1 + 2(-1)^t 3^{t-1} \end{pmatrix}$$

Thus,

$$x_t = 3^{t-1} \begin{pmatrix} 2 + (-1)^t & 2 - 2(-1)^t \\ -(-1)^t & 1 + 2(-1)^t 3^{t-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_t = 3^{t-1} \begin{pmatrix} 2 + (-1)^t + 2 - 2(-1)^t \\ -(-1)^t + 1 + 2(-1)^t 3^{t-1} \end{pmatrix}$$

$$x_t = 3^{t-1} \begin{pmatrix} 4 - (-1)^t \\ 2 + (-1)^t \end{pmatrix}$$

Question 5

0.1 (a)

$A \in M_{nxn}$, such that:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

(b)

Let $t = 0$. Then,

$$x_{t+1} = \begin{pmatrix} z_0 \\ z_{-1} \\ \vdots \\ z_{-n} \end{pmatrix} = \begin{pmatrix} c_1\lambda_1^0 + c_2\lambda_2^0 + \cdots + c_n\lambda_n^0 \\ c_1\lambda_1^{-2} + c_2\lambda_2^{-2} + \cdots + c_n\lambda_n^{-2} \\ \vdots \\ c_1\lambda_1^{-n} + c_2\lambda_2^{-n} + \cdots + c_n\lambda_n^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^{-1} & \lambda_2^{-1} & \cdots & \lambda_n^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{-n} & \lambda_2^{-n} & \cdots & \lambda_n^{-n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

(c)

Let $n = 3$, $(a_1, a_2, a_3) = (2, 1, -2)$, and $(z_0, z_{-1}, z_{-2}) = (2, 2, 1)$. Then,

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then,

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)\det\begin{pmatrix} -\lambda & 0 \\ 1 & -\lambda \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix} - 2\det\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \\ &= (2-\lambda)\lambda^2 - \lambda - 2 \\ &= -(\lambda^3 - 2\lambda^2 + \lambda + 2) \\ &= -(\lambda+1)(\lambda-1)(\lambda-2) = 0 \end{aligned}$$

Thus, $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = 1$. Then, to solve for (c_1, c_2, c_3):

$$\begin{pmatrix} z_0 \\ z_{-1} \\ z_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1/2 \\ 1 & 1 & 1/4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1/2 & 2 \\ 1 & 1 & 1/4 & 1 \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 4/3 \end{array}\right]$$

$\therefore$ $(c_1, c_2, c_3) = (1, -1/3, 4/3)$, so $z_t = 2^t - \frac{1}{3}(-2)^t + \frac{4}{3}$, $\forall t$.