Problem Set #1

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Discussed and/or compared answers with Sarah Bass, Emily Case, Katherine Kwok, Michael Nattinger, and Alex Von Hafften

Question 1

Since both u and w are increasing, strinctly concave, and twice differentiable, the solution to the consumer problem comes from the first-order conditions of the Lagrangian function:

$$\mathcal{L} = \theta u(c^1) + (1-\theta)w(c^2) - \lambda(c^1 + c^2 - c)$$

$$\frac{\partial \mathcal{L}}{\partial c^1} = \theta u'(c^1) - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial c^2} = (1-\theta)w'(c^2) - \lambda = 0$$

$$\Rightarrow \theta u'(c^1) = (1-\theta)w'(c^2)$$

Using the envelope condition, we can show the change in total utility given a change in c:

$$v'_\theta(c) = \theta u'(c^1)\frac{\partial c^1}{\partial c} + (1-\theta)w'(c^2)\frac{\partial c^2}{\partial c}$$

$$= \theta u'(c^1)\frac{\partial c^1}{\partial c} + \theta u'(c^1)\frac{\partial c^2}{\partial c}$$

$$= \theta u'(c^1)\frac{\partial(c^1 + c^2)}{\partial c} = \theta u'(c^1) = (1-\theta)w'(c^2)$$

To show that $v_\theta(c)$ is concave, we must prove that, for any $c < c'$ and $\lambda \in (0,1)$,

$$v_\theta(\lambda c + (1-\lambda)c') \geq \lambda v_\theta(c) + (1-\lambda)v_\theta(c')$$

Let $f(c) = c^1$ and $g(c) = c^2$ determine the value of c^1 and c^2 that maximize v_θ such that $f(c) + g(c) = c$. Then,

$$v_\theta(\lambda c + (1-\lambda)c') = \theta u(f(\lambda c + (1-\lambda)c')) + (1-\theta)w(g(\lambda c + (1-\lambda)c'))$$

$$\lambda v_\theta(c) + (1-\lambda)v_\theta(c') = \theta\left[\lambda u(f(c)) + (1-\lambda)u(f(c'))\right] + (1-\theta)\left[\lambda w(g(c)) + (1-\lambda)w(g(c'))\right]$$

Where, since u and w are concave,

$$u(f(\lambda c + (1 - \lambda)c')) \geq \lambda u(f(c)) + (1 - \lambda)u(f(c'))$$
$$w(g(\lambda c + (1 - \lambda)c')) \geq \lambda w(g(c)) + (1 - \lambda)w(g(c'))$$

Thus, $v_\theta(\lambda c + (1 - \lambda)c') - (\lambda v_\theta(c) + (1 - \lambda)v_\theta(c')) \geq 0$, so v_θ is concave.

Question 2

a. A competitive equilibrium in this economy is a price system, $\{q_t\}_{t=0}^{\infty}$, and an allocation, $\{c_t^1, c_t^2\}_{t=0}^{\infty}$, that solves each consumer's problem and clears both the goods and claims markets in every period, t:

$$c_t^1 + c_t^2 = y_t^1 + y_t^2 \qquad\qquad q_t^1 + q_t^2 = 0$$

b. Each agent solves their utility maximization problem in period zero, which is represented by the following Lagrangian function and features a single budget constraint:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t^i) - \lambda^i \left(\sum_{t=0}^{\infty} q_t c_t^i - q_t y_t^i \right)$$

Then optimal consumption in each period, for each agent, has the following first order condition:

$$\frac{\partial \mathcal{L}}{\partial c_t^i} = \beta^t u'(c_t^i) - \lambda^i (q_t) = 0$$

Thus, we can determine that relative consumption across consumers is constant in every state and time period:

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\lambda^1}{\lambda^2}$$

Furthermore, since markets are complete with full commitment, we know that consumers with perfectly insure, such that $c_t^i(s^t) = c^i$ for all t and s^t. Thus, using each consumer's first order condition, we can solve:

$$\frac{\beta^t u'(c_t^i)}{u'(c_0^i)} = \frac{q_t}{q_0} \Rightarrow q_t = \beta^t q_0$$

Then each consumer's budget constraint implies:

$$\sum_{t=0}^{\infty} q_t(s^t)c_t^i = \sum_{t=0}^{\infty} q_t y_t^i$$

$$c^i q_0 \sum_{t=0}^{\infty} \beta^t = q_0 \sum_{t=0}^{\infty} \beta^t y_t^i$$

So the allocation is independent of date 0 prices and the right-hand side of this equation depends on each consumer's endowment series:

$$\sum_{t=0}^{\infty} \beta^t y_t^1 = 1 + \beta^3 + \beta^6 + \ldots = \sum_{t=0}^{\infty}(\beta^3)^t = \frac{1}{1-\beta^3}$$

$$\sum_{t=0}^{\infty} \beta^t y_t^2 = \beta + \beta^2 + \beta^4 + \beta^5 + \ldots = \beta \sum_{t=0}^{\infty}(\beta^3)^t + \beta^2 \sum_{t=0}^{\infty}(\beta^3)^t = \frac{\beta + \beta^2}{1-\beta^3}$$

Thus, the competitive equilibrium allocation is:

$$c_t^1 = \frac{1-\beta}{1-\beta^3} \qquad\qquad c_t^2 = \frac{\beta - \beta^3}{1-\beta^3}$$

c. The present discounted value of this asset is

$$p = \sum_{t=0}^{\infty} 0.05\beta^t = \frac{1}{20(1-\beta)}$$

So this would also be the price of the asset.

Question 3

Exercise 8.4 from Ljungqvist and Sargent

Part I

a. A competitive equilibrium in this economy is a price system, $\{\{q_t(s^t)\}_{s^t \in S}\}_{t=0}^{\infty}$, and an allocation, $\{\{c_t\}_{s^t \in S}\}_{t=0}^{\infty}$, that solves the consumer's problem and clears the goods in every period, t:

$$c_t(s^t) = d_t(s^t)$$

If there were a market for claims in period 0, then the claims market would also have to clear. However, since there is only one consumer in this economy, there will be no claims market. The consumer's problem is solved by the following first order condition:

$$\beta^t \pi_t(s^t) u'(c_t(s^t)) = \mu q_t(s^t)$$

Where $\pi_t(s^t)$ is the probability of s^t occuring in time t and μ is the La-grangian multiplier for the consumer problem. This condition will be used to price claims to consumption in each period, but the goods market clear-ing condition ensures that the consumer consumes only their endowment in each period. Let $q_0(s_0) = 1$ to normalize all prices according to date 0 consumption. Then, we can solve for the equilibrium price of consumption at any period, t:

$$q_t(s^t) = \beta^t \frac{\pi_t(s^t) u'(c_t(s^t))}{\pi_0(s_0) u'(c_0(s_0))} = (0.95)^t \frac{\pi_t(s^t) d_t(s^t)^{-2}}{1 d_0^{-2}} = (0.95)^t \pi_t(s^t) d_t(s^t)^{-2}$$

b. Using the pricing equation derived in part (a), we can calculate:

$$q_5(s^5) = (0.95)^5 \pi_5(s^5) d_5(s^5)^{-2}$$
$$d_5(s^5) = (0.97)(0.97)(1.03)(0.97)(1.03) \approx 0.968$$
$$\pi_5(s^5) = (0.8)(0.8)(0.2)(0.1)(0.2) = 0.00256$$
$$\Rightarrow q_5(s^5) \approx (0.774)(0.00256)(0.968)^{-2} \approx 0.00211$$

c. Using the same pricing system, we can calculate:

$$q_5(s^5) = (0.95)^5 (0.2 * 0.9 * 0.9 * 0.9 * 0.1)(1.03 * 1.03 * 1.03 * 1.03 * 0.97)^{-2} \approx 0.00946$$

d. The price of a claim to the entire endowment sequence, Q, is simply the sum of the price of a claim to each possible state in each period:

$$Q = \sum_{t=1}^{\infty} \sum_{s^t \in S} (0.95)^t \pi_t(s^t) d_t(s^t)^{-2}$$

e. The price of a claim to consumption in period 5 if λ is 0.97 regardless of the transtition sequence is the sum of the price of all claims in period 5 with sequences ending in 0.97:

$$Q_5^{\lambda=.97} = \sum_{s^5|s_5=0.97} (0.95)^5 \pi_5(s^5) d_5(s^5)^{-2}$$

Part II

f.

g.

h.

i.