Problem Set #5

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Discussed and/or compared answers with Sarah Bass, Emily Case, Katherine Kwok, Michael Nattinger, and Alex Von Hafften

Question 1

1. A utilitarian planner chooses an allocation, $\left\{c_1^l, c_1^h, c_2^l, c_2^h, y^l, y^h\right\}$ that maximizes total utility:

$$\frac{1}{2}\left[u(c_1^h) + \beta u(c_2^h) - \nu(y^h)\right] + \frac{1}{2}\left[u(c_1^l) + \beta u(c_2^l)\right]$$

Subject to the (simplified[1]) resource constraint:

$$c_1^l + c_1^h + \frac{c_2^l + c_2^h}{R} = y^h$$

And each type of agent's incentive compatability constraint:

$$u(c_1^h) + \beta u(c_2^h) - \nu(y^h) \geq u(c_1^l) + \beta u(c_2^l) - \nu(y^l) \quad \text{(high-productivity IC)}$$
$$u(c_1^l) + \beta u(c_2^l) \geq u(c_1^h) + \beta u(c_2^h) \quad \text{(low-productivity IC)}$$

Note, however, that $y^l = 0$ regardless of hours worked. Then the low type cannot report as the high type, but the high type can report as the low type, since the planner can observe output. Thus, the Lagrangian of the planner's problem is:

$$\mathcal{L} = \frac{1}{2}\left[u(c_1^h) + \beta u(c_2^h) - \nu(y^h)\right] + \frac{1}{2}\left[u(c_1^l) + \beta u(c_2^l)\right] - \lambda\left[c_1^l + c_1^h + \frac{c_2^l + c_2^h}{R} - y^h\right]$$
$$+ \mu\left[u(c_1^h) + \beta u(c_2^h) - \nu(y^h) - u(c_1^l) - \beta u(c_2^l)\right]$$

[1] By recognizing that each period's constraint binds at the optimal allocation, be can eliminate the savings variable and consolidate the period-specific constraints into a single resource constraint.

Then, the planner's FOC are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_1^l} &= .5u'(c_1^l) - \lambda - \mu u'(c_1^l) = 0 \\
\frac{\partial \mathcal{L}}{\partial c_1^h} &= .5u'(c_1^h) - \lambda + \mu u'(c_1^h) = 0 \\
\frac{\partial \mathcal{L}}{\partial c_2^l} &= .5\beta u'(c_2^l) - \frac{\lambda}{R} - \beta\mu u'(c_2^l) = 0 \\
\frac{\partial \mathcal{L}}{\partial c_2^h} &= .5\beta u'(c_2^h) - \frac{\lambda}{R} + \beta\mu u'(c_2^h) = 0 \\
\frac{\partial \mathcal{L}}{\partial y^h} &= -.5\nu'(y^h) + \lambda - \mu\nu'(y^h) = 0
\end{aligned}$$

Let us begin with the intertemporal optimization condition. This yields the same result for either type, but for illustrative purposes, consider each period's consumption FOC for the low type (recall that $\beta = 1/R$):

$$\lambda = u'(c_1^l)\left(\frac{1}{2} - \mu\right) = R\beta u'(c_2^l)\left(\frac{1}{2} - \mu\right)$$

$$\Rightarrow \frac{u'(c_1^l)}{u'(c_2^l)} = 1$$

Thus, $c_1^l = c_2^l$. It can be trivially shown that the same is true for the high type. Thus, there is perfect consumption smoothly across periods. Then, for comparing consumption across agents, we need only consider one period's consumption. Consider the FOC for each agent's consumption in the first period:

$$\lambda = u'(c_1^l)\left(\frac{1}{2} - \mu\right) = u'(c_1^h)\left(\frac{1}{2} + \mu\right)$$

$$\Rightarrow \frac{u'(c_1^h)}{u'(c_1^l)} = \frac{1-2\mu}{1+2\mu}$$

Since the IC constraint binds, $\mu > 0$ by complementary slackness. Since u is concave, this ratio being less than one implies that $c_t^h > c_t^l$. This is clear from the IC constraint, since the high type needs to be compensated with additional consumption in order to have any incentive to report as a high type and be forced to work. Finally, we can use the FOC for y^h solve for the optimal level of output:

$$\nu'(y^h) = \frac{\lambda}{1/2 + \mu}$$

Note that this can be combined with the first-period consumption FOC for the high type to derive:

$$\nu'(y^h) = u'(c_1^h)$$

In other words, there is no distortion for the productive agent.

2. Let $\left\{c_1^{l*}, c_1^{h*}, c_2^{l*}, c_2^{h*}, y^*\right\}$ be the optimal allocation from part 1. Further, let b_t be the transfer paid by the government in period t to all houses. Since only the productive agents work, we do not need to define type-specific transfers and taxes. Instead, let $b_t = c_t^{l*}$ If the tax, τ, can both fund the transfers and satisfy the implementability constraint, then the government will be able to implement the optimal allocation.

   The government budget constraint pins down the tax rate:

$$\tau y^* = b_1 + \frac{b_2}{R} = c_1^{l*} + \frac{c_2^{l*}}{R} = c^{l*}\left(1 + \frac{1}{R}\right) = \left(\frac{1+R}{R}\right) c^{l*}$$
$$\Rightarrow \tau = \left(\frac{1+R}{R}\right) \frac{c^{l*}}{y^*}$$

   To satisfy the implementability constraint and ensure that the productive agents work the optimal amount, the govenment can undertake the following transfer scheme in the second period (assuming that transfers must be linear in the first period):

$$b_2 = \begin{cases} c^{l*}, & y = 0 \\ -c_1, & y \neq y^* \end{cases}$$

Question 2

1. Let c_i^j and y_i^j denote the consumption and output, respectively, of an agent of type i that sends a signal that they are of type j. The planner has to solve the following problem:

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\begin{aligned}
\max\ & pq\left[u(c_H^H) - v\left(\frac{y_H^H}{\theta_H}\right)\right] + p(1-q)\left[u(c_H^L) - v\left(\frac{y_H^L}{\theta_H}\right)\right] + \\
& \quad (1-p)q\left[u(c_L^L) - v\left(\frac{y_L^L}{\theta_L}\right)\right] + (1-p)(1-q)\left[u(c_L^H) - v\left(\frac{y_L^H}{\theta_L}\right)\right] \\
\text{s.t. } & (c_H - y_H)[pq + (1-p)(1-q)] + (c_L - y_L)[p(1-q) + q(1-p)] \leq 0 \\
& u(c_H^H) - v\left(\frac{y_H^H}{\theta_H}\right) \geq u(c_L^H) - v\left(\frac{y_L^H}{\theta_H}\right) \\
& u(c_H^L) - v\left(\frac{y_H^L}{\theta_H}\right) \geq u(c_L^L) - v\left(\frac{y_L^L}{\theta_H}\right) \\
& u(c_L^L) - v\left(\frac{y_L^H}{\theta_L}\right) \geq u(c_H^L) - v\left(\frac{y_H^H}{\theta_L}\right) \\
& u(c_L^L) - v\left(\frac{y_L^H}{\theta_L}\right) \geq u(c_H^L) - v\left(\frac{y_H^H}{\theta_L}\right)
\end{aligned}
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