Econ 711 – Fall 2020 – Problem Set 1 – Solutions

Question 1. The Law of Supply

Suppose $k = 3$, and a firm uses goods one and two as inputs and produces good three as output, i.e., $y \in Y$ requires $y_1, y_2 \leq 0$. For each of the following, either give an example showing it's possible or prove that it's impossible. (Feel free to use examples where Y contains only a few points.)

(a) *If p_3 falls and p_1 and p_2 stay the same, can the firm's output y_3 go up?*

No, it can't. This is an application of the Law of Supply, which says that $\Delta p \cdot \Delta y \geq 0$. If we let $p = (p_1, p_2, p_3)$ and $p' = (p_1, p_2, p_3')$, with $p_3' > p_3$, then $\Delta p = p' - p = (0, 0, p_3' - p_3)$, and so

$$\Delta p \cdot \Delta y \quad = \quad (p_3' - p_3)(y_3' - y_3) \quad \geq \quad 0$$

so if $p_3' > p_3$, it must be that $y_3' \geq y_3$.

(b) *If p_1 rises and p_2 and p_3 stay the same, can the firm's output y_3 go up?*

Yes, it can. The simplest illustration is if Y contains just two production plans. Suppose that

$$Y \quad = \quad \{(-10, -5, 50), (-5, -15, 52)\}$$

At prices $p = (1, 1, 1)$, the firm maximizes profit by setting $y = (-10, -5, 50)$; but at prices $p = (3, 1, 1)$, the firm maximizes profit by setting $y = (-5, -15, 52)$. Thus, an increase in p_1 can lead to an increase in y_3.

(c) *(Harder:) If p_1 and p_2 both increase and p_3 stays the same, can the firm's output y_3 go up? What if p_1 and p_2 both increase by 10%?*

If p_1 and p_2 increase by different amounts, y_3 can still go up. If we stick with the same example as before, at $p = (1, 1, 1)$, the firm sets $y = (-10, -5, 50)$, but at $p = (3, 1.5, 1)$, it sets $y = (-5, -15, 52)$.

However, if p_1 and p_2 increase by the same relative amounts, y_3 *cannot* go up. This is because the firm's optimal production is homogeneous of degree 0, and so proportional increases in all prices except for p_3 has the same effect as a decrease in p_3, which (by the Law of Supply) can't lead to an increase in y_3. More formally, if we let $p' = (1.1p_1, 1.1p_2, p_3)$, then

$$Y^*(p') \quad = \quad Y^*(1.1p_1, 1.1p_2, p_3) \quad = \quad Y^*\left(p_1, p_2, \frac{p_3}{1.1}\right)$$

Since y_3 cannot increase in response to a decrease in p_3, it also cannot increase in response to 10% increases in both p_1 and p_2.

Question 2. Rationalizability

Consider the following two "datasets":

Dataset 1		Dataset 2	
p	$y(p)$	p	$y(p)$
$(7,4)$	$(-20,40)$	$(7,4)$	$(-20,40)$
$(5,5)$	$(-50,60)$	$(5,5)$	$(-40,70)$
$(4,8)$	$(-70,90)$	$(4,8)$	$(-70,90)$

For each one, determine whether the three observations are consistent with a profit-maximizing firm. If not, explain why not. If so, draw or describe:

(a) *the smallest production set that can rationalize the data*

(b) *the smallest* convex *production set* with free disposal and the shutdown property *that can rationalize the data*

(c) *the* largest *production set that can rationalize the data*

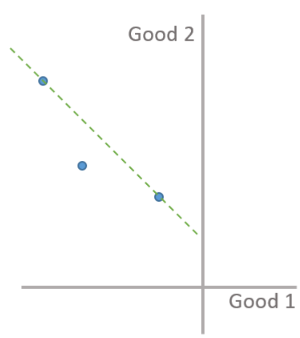
A dataset is rationalizable if (and only if) it satisfies the Weak Axiom of Profit Maximization,

$$p \cdot y(p) \quad \geq \quad p \cdot y(p')$$

for every pair of observations $(p, y(p))$ and $(p', y(p'))$.

The first dataset is not rationalizable, because $p^2 \cdot y^2$ is strictly less than both $p^2 \cdot y^1$ and $p^2 \cdot y^3$. (The firm could not have been maximizing profits by setting $y = (-50, 60)$ at $p = (5,5)$ if either $(-20, 40)$ or $(-70, 90)$ were in its production set.)

(In fact, it's worth noticing that the three production plans in the first dataset could not be rationalized if they had occurred at *any* price vectors. This is because $(-50, 60)$ is strictly below the line connecting $(-20, 40)$ and $(-70, 90)$, which implies that at *any* positive price vector, one of the two other production plans will be strictly more profitable than $(-50, 60)$.)



The second dataset is rationaliable. If we construct a table showing $p^i \cdot y^j$ for each (i, j),

Prices	Production plan		
	$y^1 = (-20, 40)$	$y^2 = (-40, 70)$	$y^3 = (-70, 90)$
$p^1 = (7, 4)$	**20**	0	-130
$p^2 = (5, 5)$	100	**150**	100
$p^3 = (4, 8)$	240	400	**440**

each diagonal term $p^i \cdot y^i$ is greater than the off-diagonal terms in the same row $p^i \cdot y^j$, so WAPM is satisfied and the data is rationalizable.

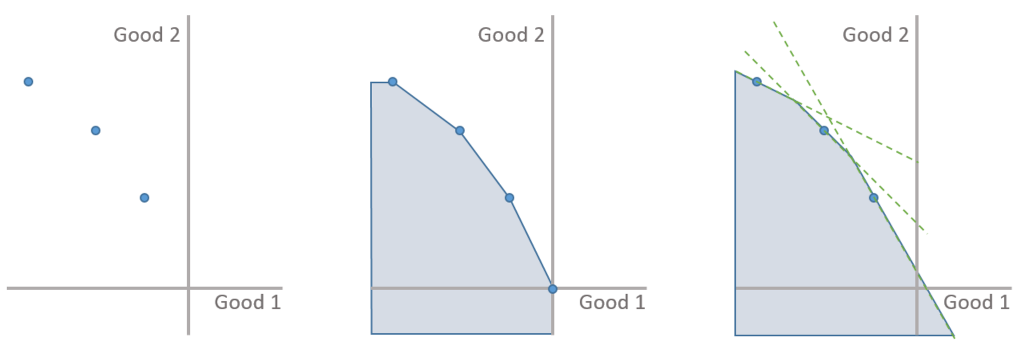
The smallest production plan that can rationalize dataset 2 is the inner bound, which is just the three observed production plans:

$$Y \quad = \quad Y^I \quad = \quad \{(-20, 40), (-40, 70), (-70, 90)\}$$

The smallest convex production set with free disposal and shutdown must contain the three observed production plans, the plan $y = 0$, the lines connecting these production plans, and every point "below" a point already in the set.

The largest production set will be the outer bound Y^O, which is $\mathbb{R}^2$ minus the half-spaces of points which would be more profitable than one of the observed production plans at the corresponding observed price vector. Specifically, it excludes all points (y_1, y_2) such that either $7y_1 + 4y_2 > 20$ or $5y_1 + 5y_2 > 150$ or $4y_1 + 8y_2 > 440$, and thus consists of the space below these three lines (including the boundary).

These three production sets (the answers to (a), (b), and (c) for dataset 2) are shown here:



Question 3. Aggregate Production

Suppose an industry consists of n profit-maximizing, price-taking firms, each with its own production set $Y_1, Y_2, \ldots, Y_n$*. You observe industry-level data at several price vectors: instead of observing individual firm production* $(y_1(p), y_2(p), \ldots, y_n(p))$*, you observe only the sum* $y_1(p) + \ldots + y_n(p)$*. Will this aggregate data satisfy the Weak Axiom? Can industry production be rationalized as if it were the choice of a single profit-maximizing firm? Explain. (You may find it helpful to use an example.)*

Let (p^j, Y^j) denote an observation of prices and industry-level production $\sum_i y_i(p^j)$. The data satisfy the Weak Axiom if

$$p^j \cdot Y^j \quad \geq \quad p^j \cdot Y^{j'}$$

for every pair of observations (p^j, Y^j) and $(p^{j'}, Y^{j'})$. Of course, since each individual firm i is maximizing profits, each individual firm's behavior must satisfy WAPM, $p^j \cdot y_i(p^j) \geq p^j \cdot y_i(p^{j'})$; if we sum over firms, we find

$$\sum_i \left(p^j \cdot y_i(p^j)\right) \quad \geq \quad \sum_i \left(p^j \cdot y_i(p^{j'})\right)$$

$$p^j \cdot \left(\sum_i y_i(p^j)\right) \quad \geq \quad p^j \cdot \left(\sum_i y_i(p^{j'})\right)$$

$$p^j \cdot Y^j \quad \geq \quad p^j \cdot Y^{j'}$$

so if each firm's behavior satisfies WAPM, aggregate-level industry data must also satisfy WAPM. Since any data satisfying WAPM can be rationalized, there is some production set Y which rationalizes the aggregate data.

(In fact, the data can be rationalized by the set

$$Y \quad = \quad Y_1 + Y_2 + \cdots + Y_n \quad = \quad \left\{ y \quad : \quad y = \sum_i y_i \quad \text{for some} \quad (y_1, y_2, \ldots, y_n) \in Y_1 \times Y_2 \times \cdots \times Y_n \right\}$$

that is, the set constructed by summing elements of the individual firms' production sets.)

Thus, "production aggregates" – optimizing behavior by individual firms looks just like optimizing behavior by a single "aggregate" firm. The same will *not* automatically be true when we look at consumer behavior; consumer demand only aggregates under fairly strong restrictions on individual preferences.