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Econ 714
Problem Set #2

Problem 1

$$c(s^t) + g(s^t) + K(s^t) = F(K(s^{t-1}), l(s^t)) + (1-\delta)K(s^{t-1})$$

is satisfied simply by the goods market clearing conditions which must hold in any CE.

To prove that the other equation holds, let us begin with the households' FOC:

$$c(s^t): \beta^t \mu(s^t) U_c(s^t) = p(s^t)$$
$$l(s^t): \beta^t \mu(s^t) U_l(s^t) = -p(s^t)(1 - \tau(s^t))w(s^t)$$
$$b(s^t): \Big[p(s^t) - \sum_{s^{t+1}} p(s^{t+1}) R_b(s^{t+1})\Big] b(s^t) = 0$$
$$K(s^t): \Big[p(s^t) - \sum_{s^{t+1}} p(s^{t+1}) R_k(s^{t+1})\Big] K(s^t) = 0$$

Subject to the HH budget constraint:

$$p(s^t)\big[c(s^t) + K(s^t) + b(s^t)\big] = p(s^t)\big[(1+\tau(s^t))w(s^t)l(s^t) + R_k(s^t)K(s^{t-1}) + R_b(s^t)b(s^{t-1})\big]$$

From FOC, we get:

$$p(s^t)(1+\tau(s^t))w(s^t)l(s^t) = -\beta^t \mu(s^t) U_l(s^t) l(s^t)$$
$$p(s^t) = \beta^t \mu(s^t) U_c(s^t)$$

Rearranging and substituting these values, we get:

$$\beta^t \mu(s^t)\big[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)\big] = p(s^t)\big[R_k(s^t)K(s^{t-1}) - K(s^t)\big] + p(s^t)\big[R_b(s^t)b(s^{t-1}) - b(s^t)\big]$$

From the FOC for $K(s^t)$ (and, by extension, $b(s^t)$), we can derive:

$$p(s^t)R_k(s^t)K(s^{t-1}) - p(s^t)K(s^t) =$$

Where

$$\sum_{t,s^t} p(s^t)R_k(s^t)K(s^{t-1}) - \sum_{s^{t+1}} p(s^{t+1})R_k(s^{t+1})K(s^t) = p(s^t)R_k(s^t)K(s^{t-1})$$

Thus,

$$\sum_{t,s^t} \beta^t \mu(s^t)\big[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)\big] = p(s_0)\big[R_k(s_0)K_{-1} + R_b(s_0)b_{-1}\big]$$

Where, by the FOC for $c(s^t)$, $p(s_0) = U_c(s^0)$.

Given an allocation that satisfies the resource constraint (RC) and implementability constraint (IC), we can construct prices that implement the allocation in a CE by first choosing $w(s^t)$ and $R_k(s^t)$ $\forall s^t$. The RC is implemented by setting:

$$w(s^t) = F_\ell(K(s^{t-1}), \ell(s^t)) \quad \forall s^t$$
$$R_k(s^t) = F_k(K(s^{t-1}), \ell(s^t)) \quad \forall s^t$$

From the HH FOC for $c(s^t)$ and $\ell(s^t)$, we can retrieve $\tau(s^t)$:

$$\frac{U_c(s^t)}{U_\ell(s^t)} = \frac{-w(s^t)}{1-\tau(s^t)} \Rightarrow 1-\tau(s^t) = -w(s^t)\frac{U_\ell(s^t)}{U_c(s^t)}$$
$$\Rightarrow \underline{\tau(s^t) = 1 + w(s^t)\frac{U_\ell(s^t)}{U_c(s^t)}}$$

Now we need only solve for $R_b(s^t)$. From the HH BC, we have:

$$b(s^t) = (1+\tau(s^t))w(s^t)\ell(s^t) + R_k(s^t)K(s^{t-1}) + R_b(s^t)b(s^{t-1})$$

Which is one equation in one unknown for each t and s^t.

Problem 2

The changes to the IC come from the updated HH BC:

$$p(s^t)\left[(1-\tau_c(s^t))\,c(s^t)+k(s^t)+b(s^t)\right]=p(s^t)\left[(1+\tau_\ell(s^t))\,w(s^t)\,\ell(s^t)+R_k(s^t)k(s^{t-1})+R_b(s^t)b(s^{t-1})\right]$$

Which yields the new FOC:

$$c(s^t):\ \beta^t\mu(s^t)U_c(s^t)=p(s^t)(1-\tau_c(s^t))$$

$$\ell(s^t):\ \beta^t\mu(s^t)U_\ell(s^t)=-p(s^t)(1-\tau_\ell(s^t))\,w(s^t)$$

$$b(s^t):\ \left[p(s^t)-\sum_{s^{t+1}}p(s^{t+1})R_b(s^{t+1})\right]b(s^t)=0$$

$$k(s^t):\ \left[p(s^t)-\sum_{s^{t+1}}p(s^{t+1})R_k(s^{t+1})\right]k(s^t)=0$$

Given the same steps as in question 1, we can see that the lefthand side of the IC has not changed. From the FOC for $c(s^t)$, we can see that

$$p(s^t)(1-\tau_c(s^t))\,c(s^t)=\beta^t\mu(s^t)U_c(s^t)\,c(s^t)$$

And the remaining derivation steps are unaffected by the introduction of a proportional consumption tax, except for the first term of the RHS of the IC:

$$\sum_{t,s^t}\beta^t\mu(s^t)\left[U_c(s^t)c(s^t)+U_\ell(s^t)\ell(s^t)\right]=\frac{U_c(s^0)}{1-\tau_c(s^0)}\left[R_k(s_0)k_{-1}+R_b(s_0)b_{-1}\right]$$

Problem 3

1. A CE in this economy is an allocation, $\{C(s^t), K(s^t), b(s^t), g(s^t)\}_{t,s^t}$, and price system, $\{w(s^t), R_K(s^t), R_b(s^t), \tau_c(s^t), \tau_\ell(s^t)\}_{t,s^t}$, such that households maximize expected utility subject to the HH BC, the government BC is satisfied:

$$b(s^t) = g(s^t) + R_b(s^t)b(s^{t-1}) - \tau_c(s^t)C(s^t) - \tau_\ell(s^t)w(s^t)\ell(s^t)$$

the firm problem is solved:

$$w(s^t) = F_\ell(K(s^{t-1}), \ell(s^t))$$
$$R_K(s^t) = F_K(K(s^{t-1}), \ell(s^t))$$

And the goods market clears:

$$C(s^t) + g(s^t) + K(s^t) = F(K(s^{t-1}), \ell(s^t)) - (1-\delta)K(s^{t-1})$$

2. We showed in problem 2 that the IC for this setting is:

$$\sum_{t,s^t} \beta^t \mu(s^t)\left[U_c(s^t)C(s^t) + U_\ell(s^t)\ell(s^t)\right] = \frac{U_c(s^0)}{1-\tau_c(s^0)}\left[R_K(s_0)K_{-1} + R_b(s_0)b_{-1}\right]$$

Which differs from the IC without consumption taxes only with the first term of the RHS, which comes from the FOC for $C(s^t)$. If the gov't used capital and labor taxes only, that FOC would go back to the one from problem 1 and every other FOC would be the same, except that $R_K(s^t)$ would now represent the post-tax return on capital:

$$R_K'(s^t) = 1 + [1-\tau_K(s^t)][r(s^t) - \delta]$$
$$\tau_K(s^t) = 0 \Rightarrow R_K'(s^t) = R_K(s^t) = 1 + r(s^t) - \delta$$

Thus for any $\tau_c(s^t)$, we can find $\tau_K(s^t)$ that equates the RHS of each IC:

$$\frac{U_c(s^0)}{1-\tau_c(s^0)}\left[(1+r(s^t)-\delta)K_{-1} + R_b(s_0)b_{-1}\right] = U_c(s^0)\left[\left(1+[1-\tau_K(s^t)][r(s^t)-\delta]\right)K_{-1} + R_b(s_0)b_{-1}\right]$$