Problem Set #7

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Question 1

Let $X \subset \mathbb{R}^n$ be convex. We can prove that, for any $k \in \mathbb{N}$, $\lambda_1, ..., \lambda_k \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, if $x_1, ..., x_k \in X$, then $\sum_{i=1}^k \lambda_i x_1 \in X$.

Proof.

1. *Base step.* Suppose $x_1, x_2 \in X$. Since X is convex, $(1-\lambda)x_1 + \lambda x_2$ is also in X for all $\lambda \in [0, 1]$
2. *Induction Step.* Assume that, for some $k \in \mathbb{N}$, $\sum_{i=1}^k \lambda_i x_i \in X$, where $\sum_{i=1}^k = 1$. Let $x_{k+1} \in X$ and $\lambda' \in [0, 1]$. Then, since X is convex,

   $$(1-\lambda')x_{k+1} + \lambda' \sum_{i=1} \lambda_i x_i$$

   is also in X. Now, define

   $$\lambda'_i = \begin{cases} \lambda' \lambda_i, & i \in \{1, ..., k\} \\ 1 - \lambda', & i = k+1 \end{cases}$$

   Then, $\sum_{i=1}^{k+1} \lambda'_i x_i \in X$ and $\sum_{i=1}^{k+1} \lambda'_i = 1$

$\therefore \sum_{i=1}^k \lambda_i x_i \in X$ for any $k \in \mathbb{N}$ ■

Question 2

Question 3

Suppose X is convex.

1. Let $x, y \in \mathrm{cl}X$ and suppose $\exists z = (1-\lambda)x + \lambda y$, $z \notin \mathrm{cl}X$
2. If $x, y \in X$, then, since X is convex, $(1-\lambda)x + \lambda y \in X$ $\forall \lambda$. Thus, $x, y \in X \Rightarrow z \in \mathrm{cl}X$
3. If $x \in \mathrm{cl}X$, $x \notin X$, and $y \in X$, then x is a limit point of X. Then, $\forall x' = (1-\lambda')x + \lambda' y$, $x' \in X$ or $x' = x$. Thus, either $z \in X$ or z is a limit point of x. Thus, $z \in \mathrm{cl}X$.
4. If $x, y \in \mathrm{cl}X$ and $x, y \notin X$, then both x and y are limit points of X. Thus, $\forall \varepsilon > 0$, $\exists x' \in B_\varepsilon(x)$, $y' \in B_\varepsilon(y)$ such that x' and y' are both in X and are convex combinations of x and y. Then, either z is equal to x or y, or $\exists \varepsilon$ such that $x' \in B_\varepsilon(x)$, $y' \in B_\varepsilon(y)$, and $z = (1-\lambda')x' + \lambda' y'$ for some $\lambda' \in [0, 1]$. Thus, $z \in \mathrm{cl}X$

$\therefore$ by contradiction, $\mathrm{cl}X$ is convex $\blacksquare$