Problem Set #6

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Question 1

Say $P(X = 1) = p$ and $P(X = 0) = 1 - p$, where $0 < p < 1$.

(a) Say $f(x) = p^x(1-p)^{1-x}$. Then,

$$\begin{aligned} f(0) &= p^0(1-p)^{1-0} = 1 - p = P(X = 0) \\ f(1) &= p^1(1-p)^{1-1} = p = P(X = 1) \end{aligned}$$

(b)

$$\ell_n = \sum_{i=1}^{n} \log\left(f(x_i)\right) = \sum_{i=1}^{n} x_i \log(p) + (1-x_i)\log(1-p) = n\log(p) + \log(1-p)\sum_{i=1}^{n} 1 - x_i$$

(c) To find $\hat{p}$, we simply maximize ℓ_n with repspect to p:

$$\begin{aligned} \frac{\partial \ell_n}{\partial p} &= \frac{1}{p}\sum_{i=1}^{n} x_i - \frac{1}{1-p}\sum_{i=1}^{n} 1 - x_i = 0 \\ \frac{n}{p}\overline{X}_n &= \frac{n}{1-p} - \frac{n}{1-p}\overline{X}_n \\ \frac{p-1}{p}\overline{X}_n &= 1 - \overline{X}_n \\ \left(\frac{p-1}{p} + 1\right)\overline{X}_n &= 1 \\ \frac{1}{p}\overline{X}_n &= 1 \\ \hat{p}_n &= \overline{X}_n \end{aligned}$$

Question 2

$X \sim f(x) = \frac{\alpha}{x^{1+\alpha}}$, $x \geq 1$

(a) The log-likelihood function is:

$$\ell_n = \sum_{i=1}^{n} \log\left(f(x_i)\right) = \sum_{i=1}^{n} \log(\alpha) - (1+\alpha)\log(x_i) = n\log(\alpha) - (1+\alpha)\sum_{i=1}^{n} \log(x_i)$$

(b) To find $\hat{\alpha}$, we simply maximize ℓ_n with repspect to α:

$$\frac{\partial \ell_n}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log(x_i) = 0$$

$$\frac{n}{\hat{\alpha}} = \sum_{i=1}^{n} \log(x_i)$$

$$\hat{\alpha}_n^{-1} = \frac{1}{n}\sum_{i=1}^{n} \log(x_i)$$

Question 3

$X \sim f(x) = \left[\pi(1 + (x-\theta)^2)\right]^{-1}$, $x \in \mathbb{R}$

(a) The log-likelihood function is:

$$\ell_n = \sum_{i=1}^{n} \log\left(f(x_i)\right) = \sum_{i=1}^{n} \log(\pi) + \log\left(1 + (x_i - \theta)^2\right) = -n\log(\pi) - \sum_{i=1}^{n} \log\left(1 + (x_i - \theta)^2\right)$$

(b) The first-order condition for the MLE $\hat{\theta}$ is:

$$\frac{\partial \ell_n}{\partial \theta} = \sum_{i=1}^{n} \frac{2(x_i - \hat{\theta}_n)}{1 + (x_i - \hat{\theta}_n)} = 0$$

Question 4

$X \sim f(x) = \frac{1}{2}\exp(-|x-\theta|)$, $x \in \mathbb{R}$

(a) The log-likelihood function is:

$$\ell_n = \sum_{i=1}^{n} \log\left(f(x_i)\right) = \sum_{i=1}^{n} \log\left(\frac{1}{2}\right) - |x_i - \theta| = n\log\left(\frac{1}{2}\right) - \sum_{i=1}^{n} -|x_i - \theta|$$

(b) The MLE will be $\hat{\theta}_n$ that minimizes $\sum_{i=1}^{n} |x_i - \hat{\theta}_n|$, so we want to choose theta that will minimize the sum of the absolute deviations from X_i. We already know that this value is $\frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}_n$. Thus,

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Question 5

Question 6

Question 7

Question 8