Problem Set #7

Danny Edgel
Econ 710: Economic Statistics and Econometrics II
Spring 2021

March 20, 2021

Discussed and/or compared answers with Sarah Bass, Emily Case, Katherine Kwok, Michael Nattinger, and Alex Von Hafften

Exercise 13.1

We can use the moment condition $\mathbb{E}[Xe] = 0$ to obtain a consistent estimator for β, which we can then use to obtain a consistent estimator for e, which we can combine with the moment condition $\mathbb{E}[Z\eta] = 0$ to obtain an estimator for γ:

$$\begin{aligned}
\mathbb{E}[X(Y - X\beta)] &= 0 \\
\beta\mathbb{E}[X'X] &= \mathbb{E}[Y] \\
\Rightarrow \hat{\beta} &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i \\
\mathbb{E}\left[Z\left((Y - X'\hat{\beta})^2 - Z'\gamma\right)\right] &= 0 \\
\gamma\mathbb{E}[Z'Z] &= \mathbb{E}\left[Z(Y - X'\hat{\beta})^2\right] \\
\Rightarrow \hat{\gamma} &= \left(\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right)^{-1} Z_i\left(Y_i - X_i\hat{\beta}\right)^2
\end{aligned}$$

Exercise 13.2

The GMM estimator with weight matrix W is:

$$\hat{\beta} = (X'ZWZ'X)^{-1}(X'ZWZ'Y)$$

Thus, letting $W = (ZZ')^{-1}$,

$$\sqrt{n}\left(\hat{\beta} - \beta\right) = \left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right]^{-1}\left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'e\right)\right]$$

Then, let $M = \mathbb{E}[ZZ']$ and $Q = \mathbb{E}[ZX']$:

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d \left[Q'M^{-1}Q\right]^{-1} Q'M^{-1}\mathcal{N}\left(0, Ze^2Z'\right) = Q^{-1}\mathcal{N}\left(0, M\sigma^2\right)$$
$$\sqrt{n}\left(\hat{\beta} - \beta\right) \to_d \mathcal{N}\left(0, \sigma^2(Q'M^{-1}Q)^{-1}\right)$$

Exercise 13.3

By the weak law of large numbers and the law of iterated expectation, and by recognizing that, since $\widetilde{\beta}$ is consistent, $\widetilde{\beta} - \beta \to_p 0$:

$$\begin{aligned}\hat{W} &\to_p \mathbb{E}\left[ZZ'\widetilde{e}^2\right]^{-1} = \mathbb{E}\left[ZZ'(Y - X'\widetilde{\beta})^2\right]^{-1} \\ &= \mathbb{E}\left[ZZ'(X'\beta + e - X'\widetilde{\beta})^2\right]^{-1} = \mathbb{E}\left[ZZ'(X'(\beta - \widetilde{\beta}) + e)^2\right]^{-1} \\ &= \mathbb{E}\left[ZZ'\left(\mathbb{E}\left[XX'(\beta - \widetilde{\beta})^2|Z\right] + \mathbb{E}\left[2X'(\beta - \widetilde{\beta})e|Z\right] + \mathbb{E}\left[e^2|Z\right]\right)\right]^{-1} \quad = \mathbb{E}\left[ZZ'e^2\right]^{-1}\end{aligned}$$

Exercise 13.4

(a)

$$\begin{aligned}V_0 &= (Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}\Omega\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1} = Q^{-1}\Omega(Q')^{-1}Q'\Omega^{-1}QQ^{-1}\Omega(Q')^{-1} \\ &= Q^{-1}\Omega(Q')^{-1} = \left(Q'\Omega^{-1}Q\right)^{-1}\end{aligned}$$

(b) In the process of answering (a), we found that $B = (Q')^{-1}$. Simply looking at V, we can define:

$$A = WQ\left(Q'WQ\right)^{-1}$$

(c)

$$\begin{aligned}B'\Omega A &= Q^{-1}\Omega WQ\left(Q'WQ\right)^{-1} = Q^{-1}\Omega WQQ^{-1}W^{-1}(Q')^{-1} = Q^{-1}\Omega(Q')^{-1} \\ &= B'\Omega B\end{aligned}$$

Thus, $B'\Omega(A - B) = 0$. This also implies $(A - B)'\Omega B = 0$.

(d) First, note that $A \geq B$, so $A - B$ is positive semi-definite. Then,

$$\begin{aligned} V &= A'\Omega A = [B + (A-B)]'\Omega A = B'\Omega A + (A-B)'\Omega A = B'\Omega B + [A'\Omega(A-B)]' \\ &= V_0 + [(A-B)'\Omega(B + (A-B))] = V_0 + (A-B)'\Omega(A-B) \\ &\geq V_0 \end{aligned}$$

Exercise 13.11

The model in question is $Y = X\beta + e$, where X and β are scalars. The efficient GMM estimator is:

$$\hat{\beta}_{GMM} = \left(X'Z\Omega^{-1}Z'X\right)^{-1}\left(X'Z\Omega^{-1}Z'Y\right)$$

First, we must obtain a consistent estimator for Ω. To do so, consider the 2SLS estimator for β. Since X is also an instrument, 2SLS and OLS are the same. Then,

$$\hat{\beta}_{2SLS} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

So, letting $\hat{e}_i = y_i - \hat{\beta}_{2SLS} x_i$, we can calculate:

$$\hat{\Omega} = \frac{1}{n}\sum_{i=1}^n Z_i Z_i' \hat{e}_i^2 = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_i^2 \hat{e}_i^2 & \frac{1}{n}\sum_{i=1}^n x_i^3 \hat{e}_i^2 \\ \frac{1}{n}\sum_{i=1}^n x_i^3 \hat{e}_i^2 & \frac{1}{n}\sum_{i=1}^n x_i^4 \hat{e}_i^2 \end{pmatrix}$$

Then,

$$\begin{aligned} \hat{\beta}_{GMM} \to_p &\left(X(X\ X^2)\frac{1}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]} \begin{pmatrix} \mathbb{E}\left[x_i^4 e_i^2\right] & -\mathbb{E}\left[x_i^3 e_i^2\right] \\ -\mathbb{E}\left[x_i^3 e_i^2\right] & \mathbb{E}\left[x_i^2 e_i^2\right] \end{pmatrix} \begin{pmatrix} X \\ X^2 \end{pmatrix} X \right)^{-1} \left(X'Z\Omega^{-1}Z'Y\right) \\ &= \left((X^2\ X^3)\frac{1}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]} \begin{pmatrix} X^2\mathbb{E}\left[x_i^4 e_i^2\right] - X^3\mathbb{E}\left[x_i^3 e_i^2\right] \\ X^3\mathbb{E}\left[x_i^2 e_i^2\right] - X^2\mathbb{E}\left[x_i^3 e_i^2\right] \end{pmatrix} \right)^{-1} \left(X'Z\Omega^{-1}Z'Y\right) \\ &= \left(\frac{X^4\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^5\mathbb{E}\left[x_i^3 e_i^2\right] + X^6\mathbb{E}\left[x_i^2 e_i^2\right]}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]} \right)^{-1} \left(X'Z\Omega^{-1}Z'Y\right) \\ &= \left(\frac{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]}{X^4\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^5\mathbb{E}\left[x_i^3 e_i^2\right] + X^6\mathbb{E}\left[x_i^2 e_i^2\right]} \right) \left(\frac{X^3\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^4\mathbb{E}\left[x_i^3 e_i^2\right] + X^5\mathbb{E}\left[x_i^2 e_i^2\right]}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]} Y \right) \\ &= \frac{\mathbb{E}\left[x_i^4 e_i^2\right] - 2X\mathbb{E}\left[x_i^3 e_i^2\right] + X^2\mathbb{E}\left[x_i^2 e_i^2\right]}{X\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^3\mathbb{E}\left[x_i^3 e_i^2\right] + X^4\mathbb{E}\left[x_i^2 e_i^2\right]} Y \end{aligned}$$

This is not, in general, equal to the OLS and 2SLS estimators for β.

Exercise 13.13

(a)

(b)

(c)

(d)

(e)

(f)

(g)

Exercise 13.18

Exercise 13.19

Exercise 13.28

(a)

(b)

(c)

Exercise 17.5

To show this statement, recognize that, since M_D is idempotent, the inequality can be reduced as follows:

$$\begin{aligned}
\sigma_\varepsilon^2 \left(\sum_{i=1}^{n} \dot{X}_i' \dot{X}_i\right)^{-1} &\geq \sigma_\varepsilon^2 \left(\sum_{i=1}^{n} X_i' X_i\right)^{-1} \\
\left(\sum_{i=1}^{n} X_i' X_i\right) &\geq \left(\sum_{i=1}^{n} \dot{X}_i' \dot{X}_i\right) \\
\sum_{i=1}^{n} X_i' X_i &\geq \sum_{i=1}^{n} (M_D X_i)'(M_D X_i) \\
\sum_{i=1}^{n} X_i' X_i &\geq \sum_{i=1}^{n} (M_D X_i)'(M_D X_i) \\
\sum_{i=1}^{n} X_i' X_i &\geq \sum_{i=1}^{n} X_i' M_D X_i \\
\sum_{i=1}^{n} X_i' X_i &\geq \sum_{i=1}^{n} X_i' X_i - X_i' D(D'D)^{-1} X_i
\end{aligned}$$

Where $X_i' D(D'D)^{-1} X_i$ is positive semi-definite.