Problem Set #7

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Exercise 13.1

We can use the moment condition $\mathbb{E}[Xe] = 0$ to obtain a consistent estimator for β, which we can then use to obtain a consistent estimator for e, which we can combine with the moment condition $\mathbb{E}[Z\eta] = 0$ to obtain an estimator for γ:

$$\begin{aligned}
\mathbb{E}[X(Y - X\beta)] &= 0 \\
\beta\mathbb{E}[X'X] &= \mathbb{E}[Y] \\
\Rightarrow \hat{\beta} &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i \\
\mathbb{E}\left[Z\left((Y - X'\hat{\beta})^2 - Z'\gamma\right)\right] &= 0 \\
\gamma\mathbb{E}[Z'Z] &= \mathbb{E}\left[Z(Y - X'\hat{\beta})^2\right] \\
\Rightarrow \hat{\gamma} &= \left(\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right)^{-1} Z_i\left(Y_i - X_i\hat{\beta}\right)^2
\end{aligned}$$

Exercise 13.2

The GMM estimator with weight matrix W is:

$$\hat{\beta} = (X'ZWZ'X)^{-1}(X'ZWZ'Y)$$

Thus, letting $W = (ZZ')^{-1}$,

$$\sqrt{n}\left(\hat{\beta}-\beta\right) = \left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right]^{-1}\left[\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'e\right)\right]$$

Then, let $M = \mathbb{E}\left[ZZ'\right]$ and $Q = \mathbb{E}\left[ZX'\right]$:

$$\begin{aligned}
\sqrt{n}\left(\hat{\beta}-\beta\right) &\to_d \left[Q'M^{-1}Q\right]^{-1}Q'M^{-1}\mathcal{N}\left(0, Ze^2Z'\right) = Q^{-1}\mathcal{N}\left(0, M\sigma^2\right)\\
\sqrt{n}\left(\hat{\beta}-\beta\right) &\to_d \mathcal{N}\left(0, \sigma^2(Q'M^{-1}Q)^{-1}\right)
\end{aligned}$$

Exercise 13.3

By the weak law of large numbers and the law of iterated expectation, and by recognizing that, since $\widetilde{\beta}$ is consistent, $\widetilde{\beta}-\beta \to_p 0$:

$$\begin{aligned}
\hat{W} &\to_p \mathbb{E}\left[ZZ'\widetilde{e}^2\right]^{-1} = \mathbb{E}\left[ZZ'(Y-X'\widetilde{\beta})^2\right]^{-1}\\
&= \mathbb{E}\left[ZZ'(X'\beta+e-X'\widetilde{\beta})^2\right]^{-1} = \mathbb{E}\left[ZZ'(X'(\beta-\widetilde{\beta})+e)^2\right]^{-1}\\
&= \mathbb{E}\left[ZZ'\left(\mathbb{E}\left[XX'(\beta-\widetilde{\beta})^2|Z\right]+\mathbb{E}\left[2X'(\beta-\widetilde{\beta})e|Z\right]+\mathbb{E}\left[e^2|Z\right]\right)\right]^{-1} \quad = \mathbb{E}\left[ZZ'e^2\right]^{-1}
\end{aligned}$$

Exercise 13.4

(a)

$$\begin{aligned}
V_0 &= (Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}\Omega\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1} = Q^{-1}\Omega(Q')^{-1}Q'\Omega^{-1}QQ^{-1}\Omega(Q')^{-1}\\
&= Q^{-1}\Omega(Q')^{-1} = \left(Q'\Omega^{-1}Q\right)^{-1}
\end{aligned}$$

(b) In the process of answering (a), we found that $B = (Q')^{-1}$. Simply looking at V, we can define:

$$A = WQ\left(Q'WQ\right)^{-1}$$

(c)

$$\begin{aligned}
B'\Omega A &= Q^{-1}\Omega WQ\left(Q'WQ\right)^{-1} = Q^{-1}\Omega WQQ^{-1}W^{-1}(Q')^{-1} = Q^{-1}\Omega(Q')^{-1}\\
&= B'\Omega B
\end{aligned}$$

Thus, $B'\Omega(A-B) = 0$. This also implies $(A-B)'\Omega B = 0$.

(d) First, note that $A \geq B$, so $A - B$ is positive semi-definite. Then,

$$\begin{aligned} V &= A'\Omega A = [B + (A-B)]'\Omega A = B'\Omega A + (A-B)'\Omega A = B'\Omega B + [A'\Omega(A-B)]' \\ &= V_0 + [(A-B)'\Omega(B + (A-B))] = V_0 + (A-B)'\Omega(A-B) \\ &\geq V_0 \end{aligned}$$

Exercise 13.11

The model in question is $Y = X\beta + e$, where X and β are scalars. The efficient GMM estimator is:

$$\hat{\beta}_{GMM} = \left(X'Z\Omega^{-1}Z'X\right)^{-1}\left(X'Z\Omega^{-1}Z'Y\right)$$

First, we must obtain a consistent estimator for Ω. To do so, consider the 2SLS estimator for β. Since X is also an instrument, 2SLS and OLS are the same. Then,

$$\hat{\beta}_{2SLS} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

So, letting $\hat{e}_i = y_i - \hat{\beta}_{2SLS} x_i$, we can calculate:

$$\hat{\Omega} = \frac{1}{n}\sum_{i=1}^n Z_i Z_i' \hat{e}_i^2 = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n x_i^2\hat{e}_i^2 & \frac{1}{n}\sum_{i=1}^n x_i^3\hat{e}_i^2 \\ \frac{1}{n}\sum_{i=1}^n x_i^3\hat{e}_i^2 & \frac{1}{n}\sum_{i=1}^n x_i^4\hat{e}_i^2 \end{pmatrix}$$

Then,

$$\begin{aligned} \hat{\beta}_{GMM} \to_p &\left(X(X\ X^2)\frac{1}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]}\begin{pmatrix} \mathbb{E}\left[x_i^4 e_i^2\right] & -\mathbb{E}\left[x_i^3 e_i^2\right] \\ -\mathbb{E}\left[x_i^3 e_i^2\right] & \mathbb{E}\left[x_i^2 e_i^2\right] \end{pmatrix}\begin{pmatrix} X \\ X^2 \end{pmatrix} X \right)^{-1}\left(X'Z\Omega^{-1}Z'Y\right) \\ &= \left((X^2\ X^3)\frac{1}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]}\begin{pmatrix} X^2\mathbb{E}\left[x_i^4 e_i^2\right] - X^3\mathbb{E}\left[x_i^3 e_i^2\right] \\ X^3\mathbb{E}\left[x_i^2 e_i^2\right] - X^2\mathbb{E}\left[x_i^3 e_i^2\right] \end{pmatrix}\right)^{-1}\left(X'Z\Omega^{-1}Z'Y\right) \\ &= \left(\frac{X^4\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^5\mathbb{E}\left[x_i^3 e_i^2\right] + X^6\mathbb{E}\left[x_i^2 e_i^2\right]}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]}\right)^{-1}\left(X'Z\Omega^{-1}Z'Y\right) \\ &= \left(\frac{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]}{X^4\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^5\mathbb{E}\left[x_i^3 e_i^2\right] + X^6\mathbb{E}\left[x_i^2 e_i^2\right]}\right)\left(\frac{X^3\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^4\mathbb{E}\left[x_i^3 e_i^2\right] + X^5\mathbb{E}\left[x_i^2 e_i^2\right]}{\mathbb{E}\left[x_i^3 e_i^2\right]^2 - \mathbb{E}\left[x_i^2 e_i^2\right]\mathbb{E}\left[x_i^4 e_i^2\right]}Y\right) \\ &= \frac{\mathbb{E}\left[x_i^4 e_i^2\right] - 2X\mathbb{E}\left[x_i^3 e_i^2\right] + X^2\mathbb{E}\left[x_i^2 e_i^2\right]}{X\mathbb{E}\left[x_i^4 e_i^2\right] - 2X^3\mathbb{E}\left[x_i^3 e_i^2\right] + X^4\mathbb{E}\left[x_i^2 e_i^2\right]}Y \end{aligned}$$

This is not, in general, equal to the OLS and 2SLS estimators for β.

Exercise 13.13

(a) Since Ω is positive definite and square, we can define some orthonormal Q and diagonal matrix of eigenvalues Λ such that $\Omega = Q'\Lambda Q$. Then, we can define $C = Q\Lambda^{-1/2}$:

$$\Omega = Q'\Lambda Q = Q'\Lambda^{1/2}\Lambda^{1/2}Q = [(Q\Lambda^{-1/2})']^{-1}(Q\Lambda^{-1/2})^{-1} = (C')^{-1}C^{-1}$$

Thus, $\Omega^{-1} = CC'$

(b) We can demonstrate a simple equality:

$$n\left(C'\overline{g}_n(\hat{\beta})\right)'\left(C'\hat{\Omega}C\right)^{-1}C'\overline{g}_n(\hat{\beta}) = n\overline{g}_n(\hat{\beta})'CC^{-1}\hat{\Omega}^{-1}(C')^{-1}C'\overline{g}_n(\hat{\beta}) = n\overline{g}_n(\hat{\beta})'\hat{\Omega^{-1}}\overline{g}_n(\hat{\beta}) = J$$

(c) Letting $\overline{g}_n(\hat{\beta}) = \frac{1}{n}Z\hat{e}$, note that:

$$\begin{aligned}\hat{e} &= Y - X\hat{\beta} = X\beta + e - X\hat{\beta} = (\beta - \hat{\beta})X + e \\ &= e - X\left(X'Z\hat{\Omega^{-1}}Z'X\right)^{-1}\left(X'Z\hat{\Omega^{-1}}Z'e\right)\end{aligned}$$

Then,

$$\begin{aligned}C'\overline{g}_n(\hat{\beta}) &= C'\frac{1}{n}Z'e - C'\frac{1}{n}Z'X\left(X'Z\hat{\Omega^{-1}}Z'X\right)^{-1}\left(X'Z\hat{\Omega^{-1}}Z'e\right) \\ &= \left(I_\ell - C'\left(\frac{1}{n}Z'X\right)\left(\left(\frac{1}{n}X'Z\right)\hat{\Omega^{-1}}\left(\frac{1}{n}Z'X\right)\right)^{-1}\left(\frac{1}{n}X'Z\right)\hat{\Omega^{-1}}(C')^{-1}\right)C'\frac{1}{n}Z'e \\ &= D_nC'\overline{g}_n(\beta)\end{aligned}$$

(d) Recall that $\Omega = (C')^{-1}C^{-1}$. By the law of large numbers and the continuous mapping theorem,

$$D_n \to I_\ell - C'\mathbb{E}\left[Z'X\right]\left(\mathbb{E}\left[X'Z\right]CC'\mathbb{E}\left[Z'X\right]\right)^{-1}\mathbb{E}\left[X'Z\right]CC'(C')^{-1} = I_\ell - R\left(R'R\right)^{-1}R'$$

(e) It is apparent that $I_\ell - R\left(R'R\right)^{-1}R' = 0$, so we are left to demonstrate that the asymptotic variance of $C'\overline{g}_n(\beta)$ is I_ℓ. We can begin by noting, by the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'e \to_d \mathscr{N}(0, \Omega)$$

Then,

$$\frac{1}{\sqrt{n}}C'Z'e \to_d C'\mathscr{N}\left(0, (C')^{-1}C^{-1}\right) = \mathscr{N}\left(0, C'(C')^{-1}C^{-1}C\right) = \mathscr{N}(0, I_\ell)$$

(f) Note that $I_\ell - R\left(R'R\right)^{-1}R'$ is idempotent. From our equivalences above (and denoting the asymptotic distribution of $C'\overline{g}_n(\beta)$ as u), we can solve:

$$\begin{aligned}
J &= n\left(C'\overline{g}_n(\hat{\beta})\right)'\left(C'\hat{\Omega}C\right)^{-1}C'\overline{g}_n(\hat{\beta}) \\
&= \left(\sqrt{n}C'\overline{g}_n(\beta)\right)'D_n'\left(C'\hat{\Omega}C\right)^{-1}D_n\sqrt{n}C'\overline{g}_n(\beta) \\
&\to_d u'\left(I_\ell - R\left(R'R\right)^{-1}R'\right)'\left(C'(C')^{-1}C^{-1}C\right)\left(I_\ell - R\left(R'R\right)^{-1}R'\right)u \\
&= u'\left(I_\ell - R\left(R'R\right)^{-1}R'\right)'\left(I_\ell - R\left(R'R\right)^{-1}R'\right)u \\
&= u'\left(I_\ell - R\left(R'R\right)^{-1}R'\right)u
\end{aligned}$$

(g) The asymptotic distribution of J is chi-squared by dint of the fact that u is normally distributed with variance one and $I_\ell - R\left(R'R\right)^{-1}R'$ is a projection matrix. We can solve for the degrees of freedom of the distribution using the trace of the projection matrix:

$$tr\left(I_\ell - R\left(R'R\right)^{-1}R'\right) = \ell - tr\left(R'R\left(R'R\right)^{-1}\right) = \ell - tr(I_k) = \ell - k$$

Exercise 13.18

Since X is exogenous, our instrument for this specification is $Z = (X, Q)'$. Then OLS and 2SLS are equivalent, so the variance matrix for 2SLS is:

$$\Omega = \mathbb{E}\left[Z_iZ_i'e_i^2\right] = \begin{pmatrix} \mathbb{E}\left[X_iX_i'e_i^2\right] & \mathbb{E}\left[X_iQ_i'e_i^2\right] \\ \mathbb{E}\left[X_iQ_i'e_i^2\right] & \mathbb{E}\left[Q_iQ_i'e_i^2\right] \end{pmatrix}$$

Let $\hat{\Omega}$ be an efficient estimator of Ω. Then, the efficient GMM estimator of β is:

$$\hat{\beta}_{GMM} = \left(X'Z\hat{\Omega}^{-1}Z'X\right)^{-1}X'Z\hat{\Omega}^{-1}Z'Y$$

Exercise 13.19

Our moment function for this estimator is:

$$g(\mu) = \begin{pmatrix} Y - \mu \\ X \end{pmatrix}$$

Efficient GMM uses the optimal weight matrix Ω^{-1}, where

$$\Omega = \begin{pmatrix} \sigma_Y^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_X^2 \end{pmatrix}$$

Then the error function is

$$J(\mu) = \overline{g}(\mu)'\Omega^{-1}\overline{g}(\mu) = \frac{\sigma_X^2(\overline{Y}-\mu)^2 - 2\sigma_{XY}\overline{X}(\overline{Y}-\mu) + \sigma_Y^2X^2}{\sigma_X^2\sigma_Y^2 - \sigma_{XY}^2}$$

Taking the first-order condition of this function gives us the optimal estimate for μ:

$$
\begin{aligned}
J'(\mu) &= \frac{2\sigma_{XY}\overline{X} - 2\sigma_X^2(\overline{Y}-\mu)}{\sigma_X^2\sigma_Y^2 - \sigma_{XY}^2} = 0 \\
\overline{Y} - \mu &= \frac{\sigma_{XY}\overline{X}}{\sigma_X^2} \\
\hat{\mu} &= \overline{Y} - \frac{\sigma_{XY}\overline{X}}{\sigma_X^2}
\end{aligned}
$$

Exercise 13.28

The table below displays the results of both 2SLS and GMM estimations of the model, with the J statistic for the GMM estimations reported at the bottom of the table.[1] Whether or not the results change "meaningfully" depends on what you consider meaningful, I guess. The model fit does not change when approximated to the nearest hundreth, and the coefficient of interest changes only slightly, with no discenable change in standard errors.

	(1)	(2)	(3)	(4)
VARIABLES	(a)-2SLS	(a)-GMM	(b)-2SLS	(b)-GMM
education	0.161***	0.162***	0.0825***	0.0839***
	(0.0405)	(0.0405)	(0.00622)	(0.00621)
experience	0.119***	0.120***	0.0871***	0.0876***
	(0.0182)	(0.0182)	(0.00705)	(0.00705)
experience^2/100	-0.231***	-0.232***	-0.225***	-0.225***
	(0.0368)	(0.0368)	(0.0320)	(0.0320)
south	-0.0950***	-0.0954***	-0.122***	-0.124***
	(0.0217)	(0.0218)	(0.0154)	(0.0154)
black	-0.102**	-0.101**	-0.181***	-0.177***
	(0.0440)	(0.0440)	(0.0180)	(0.0180)
urban	0.116***	0.115***	0.157***	0.153***
	(0.0263)	(0.0263)	(0.0153)	(0.0152)
Observations	3,010	3,010	3,010	3,010
R-squared	0.145	0.143	0.289	0.289
J		0.869		10.44

Robust standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

[1] See the attached .do file to see how it was generated. That file also includes the code for exercise 17.15.

Exercise 17.15

The table below displays the output for both (a) and (b). The results differ because K_t is too close to a random walk, as evidenced by the coefficient on K_{t-1} being near 1. Thus, the lagged values of K are too weak for a reliable Arellano-Bond estimate. The weak instrument problem is attenuated by the Blundell-Bond estimator, which explains the difference in estimates.

	(1)	(2)
VARIABLES	(a) Arellano-Bond	(b) Blundell-Bond
L.k	0.936***	1.101***
	(0.106)	(0.0234)
Observations	751	891
Number of id	140	140

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1