Problem Set #1

Danny Edgel
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Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

Let $(Y, X')'$ be a random vector, where $Y = X'\beta_0 \cdot U$, where $\mathbb{E}\left[U \mid X\right] = 1$, $\mathbb{E}\left[XX'\right]$ is invertible, and $\mathbb{E}\left[Y^2 + ||X||^2\right] < \infty$.

(i) Since $\mathbb{E}\left[U \mid X\right] = 1$, the expectation of Y, conditional on X, is $X'\beta_0$. Then, $\frac{\partial}{\partial X} Y = \beta_0$.

(ii) Define $V = U - 1$. Then, $\mathbb{E}\left[V \mid X\right] = \mathbb{E}\left[U - 1 \mid X\right] = 0$, and:

$$Y = X'\beta_0(V+1) = X'\beta_0 V + X'\beta_0 = X'\beta_0 + \widetilde{U}$$

Where:

$$\mathbb{E}\left[\widetilde{U} \mid X\right] = \mathbb{E}\left[X'\beta_0 V \mid X\right] = X'\beta_0 \mathbb{E}\left[V \mid X\right] = 0$$

Thus, $Y = X'\beta_0 + \widetilde{U}$, where $\mathbb{E}\left[\widetilde{U} \mid X\right] = 0$.

(iii) Let $\beta = \beta_0$. Then,

(iv) Define $V = U - 1$. Then, $\mathbb{E}\left[V \mid X\right] = \mathbb{E}\left[U - 1 \mid X\right] = 0$, and:

$$\begin{aligned}\mathbb{E}\left[X(Y - X'\beta)\right] &= \mathbb{E}\left[X(X'\beta_0 \cdot U - X'\beta_0)\right] = \mathbb{E}\left[\mathbb{E}\left[X(X'\beta_0 \cdot U - X'\beta_0)|X\right]\right] \\ &= \mathbb{E}\left[XX'\beta_0 \mathbb{E}\left[(U-1)|X\right]\right] = 0\end{aligned}$$

Thus, $\beta = \beta_0 \Rightarrow \mathbb{E}\left[X(Y - X'\beta)\right] = 0$. Now, Suppose $\mathbb{E}\left[X(Y - X'\beta)\right] = 0$. Then,

$$\mathbb{E}\left[X(Y - X'\beta)\right] = \mathbb{E}\left[X(X'\beta \cdot U - X'\beta_0)\right] = 0$$
$$\mathbb{E}\left[XX'\mathbb{E}\left[\beta \cdot U - \beta_0|X\right]\right] = (\beta - \beta_0)\mathbb{E}\left[XX'\right] = 0$$

We know that $\mathbb{E}[XX']$ is invertible, so $\mathbb{E}[XX'] \neq 0$. Thus, $\mathbb{E}[X(Y - X'\beta)] = 0 \Rightarrow \beta = \beta_0$.

$\therefore \mathbb{E}[X(Y - X'\beta)] = 0 \iff \beta = \beta_0$ ■

Knowing this, we can derive the method of moments estimator for β:

$$\begin{aligned}
\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[XY] - \mathbb{E}[XX']\beta = 0 \\
\mathbb{E}[XX']\beta &= \mathbb{E}[XY] \\
\beta &= \mathbb{E}[XX']^{-1}\mathbb{E}[XY] \\
\Rightarrow \hat{\beta} &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i = \hat{\beta}_{OLS}
\end{aligned}$$

(v) We can simplify the final equation in (iii) to show:

$$\begin{aligned}
\mathbb{E}\left[\hat{\beta} \mid X_1, ..., X_n\right] &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i \mid X_1, ..., X_n\right] \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i \mathbb{E}\left[X_i'\beta_0 \cdot U \mid X_1, ..., X_n\right] \\
&= \beta_0 \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right) \mathbb{E}\left[U \mid X_1, ..., X_n\right] \\
&= \beta_0
\end{aligned}$$

Thus, $\hat{\beta}$ is unbiased.

(vi) According to the weak law of large numbers (WLLN), random variables converge in probability to their expected value. Thus,

$$\hat{\beta} \to_p \mathbb{E}\left[\hat{\beta}\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{\beta} \mid X_1, ..., X_n\right]\right] = \mathbb{E}[\beta_0] = \beta_0$$

Thus, $\hat{\beta}$ is consistent.

Question 2

Question 3