Problem Set #2

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Question 1

Suppose that $Y = X^3$ and $f_X(x) = 42x^5(1-x)$, $x \in (0,1)$. Find the PDF of Y, and show that the PDF integrates to 1.

We know that the CDF of Y, $F_Y(y)$ is equal to $F_X(f^{-1}(x))$. So we can solve for the PDF of Y by first finding its CDF:

$$\begin{aligned}
f^{-1}(y) &= \sqrt[3]{y} \\
F_X(x) &= \int 42x^5(1-x) = 42\int x^5 - x^6 = 42(\frac{1}{6}x^6 - \frac{1}{7}x^7) \\
F_X(f^{-1}(y)) &= 42(\frac{1}{6}y^{6/3} - \frac{1}{7}y^{7/3}) = 42y^2(\frac{1}{6} - \sqrt[3]{y}) = F_Y(y) \\
f_Y(y) &= \frac{d}{dy}F_Y(y) = 14y - 14y\sqrt[3]{y}
\end{aligned}$$

Since we already know $F_Y(y)$, we can easily show that the PDF of Y integrates to 1:

$$\int_0^1 f_Y(y)dy = F_Y(1) - F_Y(0) = 1^2(7 - 6\sqrt[3]{1}) - 0 = 7 - 6 = 1$$

Question 2

Consider the CDF $F_X(x) = \begin{cases} 1.2x & \textbf{if } x \in [0, 0.5) \\ 0.2 + 0.8x & \textbf{if } x \in [0.5, 1] \end{cases}$, and the function

$$f_X(x) = \begin{cases} 1.2 & \text{if } x \in [0, 0.5) \\ a & \text{if } x = 0.5 \\ 0.8 & \text{if } x \in (0.5, 1] \end{cases}$$

Show that f_X is the density function of F_X as long as $a \geq 0$. That is, show that for all $x \in [0,1]$, $F_X(x) = \int_0^x f_X(t)dt$.

We can define $F_X(x) = \int_0^x f_X(t)dt$ on a case-by-case basis:

$$
\begin{aligned}
x \in [0, 0.5) : \int_0^x f_X(t)dt &= \int_0^x 1.2dt = [1.2t]_0^x = 1.2x \\
x = 0.5 : \int_0^x f_X(t)dt &= \int 0^0 .51.2dt + \int_0 .5^x adt = [1.2t]^0 .5_0 + [at]_0^x .5 \\
&= 1.2(0.5) - 0 + ax - 0.5a = 0.6 + 0.5x - 0.5x = 0.6 \\
x \in (0.5, 1) : \int_0^x f_X(t)dt &= \int 0^0 .51.2dt + \int_0 .5^x adt + \int_0 .5^x 0.8dt = [1.2t]^0 .5_0 + [at]^0 .5_0 .5 + [0.8]_0^x .5 \\
&= 0.6 + 0.8x - 0.4 = 0.8x + 0.2
\end{aligned}
$$

Since $0.6 = 1.2x$ when $x = 0.5$, then $\int_0^x 1.2dt = \int 0^0 .51.2dt + \int_0 .5^x adt$ when $x = 0.5$. Thus, $\forall x \in [0,1]$, $F_X(x) = \int_0^x f_X(t)dt$.

Question 3

Let X have the PDF $F_X(x) = \frac{2}{9}(x+1)$, $x \in [-1, 2]$. Find the PDF of $Y = X^2$. Note that this is a bit different from the exercise in the lecture note.

Question 4

A median of a distribution is a value m such that $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$. Find the median of the distribution $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$.

Question 5

Show that if X is a continuous random variable, then $\min_a E|X - a| = E|X - m|$, where m is the median of X.

(hint: work out the integral expression of $E|X - a|$ and notice that it is differentiable)

Question 6

Let μ_n denote the nth central moment of a random variable X. Two quantities of interest, in addition to the mean and variance are

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}} \text{ and } \alpha_4 = \frac{\mu_4}{\mu_2^2}$$

The value α_3 is called the skewness and α_4 is called the kurtosis. The skewness measures the lack of symmetry in the density function. The kurtosis measures the peakedness or flatness of the density function.

1. **Show that if a density function is symmetric about a point a, then $\alpha_3 = 0$**

2. **Calculate α_3 for $f(x) = e^{-x}$, $x \geq 0$, a density function that is skewed to the right.**

3. **Calculate α_4 for the following density functions and commend on the peakedness of each:**

   (a) $\mathbf{f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x \in \mathbb{R}}$

   (b) $\mathbf{f(x) = 1/2, x \in (-1, 1)}$

   (c) $\mathbf{f(x) = \frac{1}{2} e^{-|x|}, x \in \mathbb{R}}$