Problem Set #2

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Question 1

Let $f : \mathbb{R}^m_+ \to \mathbb{R}_+$ be a production function for a single-output firm.

(a) Let $(q, -z)$ and $(q', -z')$ be in Y, where Y is convex such that $t(q, -z) + (1-t)(q', -z') \in Y$ for any $t \in (0, 1)$. Then,

$$f(tz + (1-t)z') \geq tq + (1-t)q' = tf(z) + (1-t)f(z')$$

Where $tq + (1-t)q'$ is in Y. $\therefore f$ is concave $\blacksquare$

(b) Fix a vector of input prices, w. Then, for any two input vectors z and z', let $c(q)$ be the minimum cost of producing $f(z) \geq q$ and $c(q')$ be the minimum cost of producing $f(z') \geq q'$. The cost of input vector $tz + (1-t)z'$ is:

$$w \cdot (tz + (1-t)z') = t(w \cdot z) + (1-t)(w \cdot z') = tc(q) + (1-t)c(q')$$

Where $tf(z) + (1-t)f(z') \geq tq + (1-t)q'$. However, we know from the concavity of our output function that:

$$f(tz + (1-t)z') \geq tf(z) + (1-t)f(z')$$

Thus, the convex combination of the costs of two separate outputs is at least as great as the cost of a convex combination of those two output quantities. Put another way,

$$c(tq + (1-t)q', w) \leq tc(q, w) + (1-t)c(q', w)$$

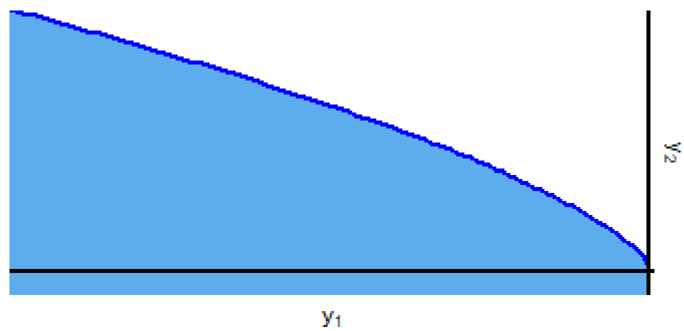
$\therefore$ the cost function is convex in q $\blacksquare$

Question 2

Define the following production set for $k = 2$, price vector $p = (p_1, p_2)$, and $B > 0$ being a known constant:

$$Y = \left\{ (y_1, y_2) \mid y_1 \leq 0 \text{ and } y_2 \leq B(-y_1)^{\frac{2}{3}} \right\}$$

(a) The production set is displayed below. A larger value of B will increase the steepness of the curve.



(b) Let $z = -y_1$ and define $f(z) = Bz^{(2/3)}$. Then,

$$\pi(p) = \max_z \; p_2 f(z) - p_1 z$$
$$\frac{d\pi}{dz} = p_2 f'(z) - p_1 = 0$$
$$\frac{2}{3} B p_2 z^{-1} 3 = p_1$$
$$z = \left(\frac{2Bp_2}{3p_1}\right)^3$$
$$\pi(p) = p_2 B \left(\left(\frac{2Bp_2}{3p_1}\right)^3 \right)^{(2/3)} - p_1 \left(\frac{2Bp_2}{3p_1}\right)^3 = \frac{4}{27} B^3 \frac{p_2^3}{p_1^2}$$
$$Y^*(p) = \left(-\left(\frac{2Bp_2}{3p_1}\right)^3, B^3 \left(\frac{2p_2}{3p_1}\right)^2 \right)$$

(c) We can verify the homogeneity of $\pi(\cdot)$ and $y(\cdot)$ by solving:

$$\pi(\lambda p) = \frac{4}{27} B^3 \frac{\lambda p_2^3}{\lambda p_1^2} = \lambda \frac{4}{27} B^3 \frac{p_2^3}{p_1^2} = \lambda \pi(p)$$
$$Y^*(\lambda p) = \left(-\left(\frac{2B\lambda p_2}{3\lambda p_1}\right)^3, B^3 \left(\frac{2\lambda p_2}{3\lambda p_1}\right)^2 \right) = Y^*(p)$$

(d)

$$\frac{\partial \pi}{\partial p_1}(p) = \frac{-8}{27}B^3 p_2^3 p_1^{-3} = -\left(\frac{2Bp_2}{3p_1}\right)^3 = y_1(p)$$
$$\frac{\partial \pi}{\partial p_2}(p) = \frac{(4)(3)}{27}B^3 p_2^2 p_1^{-2} = B^3\left(\frac{2Bp_2}{3p_1}\right)^2 = y_2(p)$$

(e)

$$D_p y(p) = \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} = \begin{pmatrix} \frac{8(Bp_2)^3}{9p_1^4} & \frac{-8B^3p_2^2}{9p_1^3} \\ \frac{-8B^3p_2^2}{9p_1^3} & \frac{8B^3p_2}{9p_1^2} \end{pmatrix}$$
$$|D_p y(p)| = \left(\frac{8}{9}\right)^2 \frac{B^6 p_2^4}{p_1^6} - \left(\frac{8}{9}\right)^2 \frac{B^6 p_2^4}{p_1^6} = 0 \geq 0$$
$$\frac{\partial y_1}{\partial p_1} = \frac{8(Bp_2)^3}{9p_1^4} \geq 0$$
$$[D_p y]p = \begin{pmatrix} \frac{8(Bp_2)^3}{9p_1^4} & \frac{-8B^3p_2^2}{9p_1^3} \\ \frac{-8B^3p_2^2}{9p_1^3} & \frac{8B^3p_2}{9p_1^2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{8}{9}\left(\frac{Bp_2}{p_1}\right)^3 - \frac{8}{9}\left(\frac{Bp_2}{p_1}\right)^3 \\ -\frac{8}{9}B^3\left(\frac{p_2}{p_1}\right)^2 + \frac{8}{9}B^3\left(\frac{p_2}{p_1}\right)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Question 3

Let $\pi(p) = Ap_1^{-2}p_2^3$, where $A > 0$ is known and $p_1, p_2 > 0$.

(a) If $\pi(\cdot)$ is differentiable and convex, then it is rationalizable.

(b) Suppose $y = (y_1, y_2) \in Y^0$, where $y_2 > 0$. Then, given the definition of the outer bound,
$$p_2 y_2 \leq Ap_1^{-2}p_2^3 - p_1 y_1$$
Where $\lim_{p_1 \to \infty} Ap_1^{-2}p_2^3 = 0$. Thus, if p_1 is arbitrarily large,
$$p_2 y_2 \leq -p_1 y_1$$
Where p_1, p_2, and y_2 are all strictly positive. Thus, for this inequality to hold, y_1 must be non-negative. In the case of $y_1 = 0$, then y_2 must also equal zero (the shutdown case).

(c) Starting with the first order condition of the problem, Y^0 is derived below by solving $\min\limits_{r} Ar^2 - \frac{y_1}{r}$:

$$\begin{aligned}
2Ar + \frac{y_1}{r^2} &= 0 \\
y_1 &= -2Ar^3 \\
r &= \sqrt[3]{\frac{-y_1}{2A}} \\
\therefore y_2 &\leq A\left(\frac{-y_1}{2A}\right)^{(2/3)} + (-y_1)\left(\frac{-y_1}{2A}\right)^{-(1/3)} \\
y_2 &\leq A^{\frac{-1}{3}}(-y_1)^{\frac{2}{3}}\left(2^{\frac{-2}{3}} + 2^{\frac{1}{3}}\right) \\
y_2 &\leq A^{\frac{-1}{3}}(-y_1)^{\frac{2}{3}}\left(\frac{27}{4}\right)
\end{aligned}$$

Thus, the production set encompasses the line $y_2 = \frac{27}{4}\sqrt[3]{\frac{(-y)^2}{A}}$ and everything below it, for $y_1 \leq 0$. A visual representation is provided in my answer to question 2(a).

(d) Let $B = \frac{27}{4}A^{-(1/3)}$ and $z = -y_1$. Then, since profit is increasing on y_2, $y_2 = \frac{27}{4}\sqrt[3]{\frac{(-y)^2}{A}}$, and:

$$\pi(p) = p_2 y_2 + p_1 y_1 = p_2 B z^{(2/3)} - p_1 z$$

Notice that this is the same profit function that was determined in question 2 to have a symmestric and positive semidefifinite Jacobian matrix in question 2, which satisfied the law of supply. Therefore, this profit function would indeed generate the "data" that we started with.