Econ 712, Q2: Problem set 1 - Solution

Q1

(a)

Bellman equation:

$$v(A,b) = \max_{A' \in \Gamma[\mathbb{A}]} u(A - \frac{A'}{R}, b) + \beta v(A', A - \frac{A'}{R})$$

where we define $b = c_{-1}$. For existence and uniqueness, strict monotonicity and concavity of v^* we need:

- Set $\mathbb{A}$ is convex, $\mathbb{B}$ is convex, $\mathbb{C} = \mathbb{A} \times \mathbb{B}$ is also convex,
- $\Gamma[A]$ is non-empty, compact-valued, continuous and convex,
- $u(.)$ is continuous, bounded, strictly increasing and strictly concave, $0 < \beta < 1$.
- $\exists W(A,b)$ which is concave, $W(A_0,b) = V(A_0,b)$ for all $A \in \mathbb{A}, B \in \mathbb{B}$, and $W(A,b) < V(A,b)$ for all $A \in \mathbb{A}, B \in \mathbb{B}$, $A \neq A_0$. Last assumption ensures differentiability of v^*.

T maps continuous bounded functions into a space of continuous bounded functions $T : C(\mathbb{C}) \rightarrow C(\mathbb{C})$. Assumptions stated above result in T mapping be contraction with modulus β as Blackwell sufficient conditions are satisfied. We know that $C(\mathbb{A})$ with the supnorm is a complete metric space. Therefore, by the Contraction Mapping Theorem, there exists a unique v^*. Corollary to the CMT suggests that v^* is strictly increasing and strictly concave.

FOC for A':

$$\frac{u_1(A - \frac{A'}{R}, b)}{R} = \beta v_1(A', A - \frac{A'}{R}) - \beta \frac{v_2(A', A - \frac{A'}{R})}{R}$$

Envelope condition for A:

$$v_1(A,b) = u_1(A - \frac{A'}{R}, b) + \beta v_2(A', A - \frac{A'}{R})$$

Envelope condition for b:

$$v_2(A,b) = u_2(A - \frac{A'}{R}, b)$$

Combining all three conditions we obtain Euler equation:

$$u_1(A - \frac{A'}{R}, b) = R\beta(u_1(A' - \frac{A''}{R}, A - \frac{A'}{R}) + \beta u_2(A'' - \frac{A'''}{R}, A' - \frac{A''}{R})) - \beta u_2(A' - \frac{A''}{R}, A - \frac{A'}{R}) \tag{1}$$

(b)

Using (1) the Euler equation for the case with separable utility,

$$\frac{1}{c_t} = R\beta\Big(\frac{1}{c_{t+1}} + \frac{\beta}{c_{t+1}}\Big) - \frac{\beta}{c_t}$$

Or,

$$R\beta = \frac{c_{t+1}}{c_t} \tag{2}$$

Guess that $c_t = (1-s)A_t$. It implies that $\frac{A_{t+1}}{R} = sA_t$. Substitute the guess into (2). It gives,

$$A_{t+1} = R\beta A_t$$

this gives $s = \beta < 1$ - the optimal policy function is to save a constant fraction of current assets.

(c)

From equation (1) it is clear that we may not get the result as marginal utilities have multiplicative c_t and c_{t+1} terms.

Q2

(a)

Sequence:

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0} \delta^t F(x_{t-1}, x_t)$$

Bellman equation:

$$V(x) = \max_y \{F(x,y) + \delta V(y)\}$$

(b)

Since $F(x,y) = ax - \frac{1}{2}bx^2 - \frac{1}{2}c(y-x)^2$, for $y \to \infty$ or $y \to -\infty$, the adjustment cost blows up, so F is unbounded below. F is concave in x, y, so FOC gives us the maximum at $y = x = \frac{a}{b}$. Plugging this back in gives $\bar{F} = \frac{a^2}{2b}$. Plugging this into the sequence problem will give the upper bound on $V : V < \frac{a^2}{2b(1-\delta)}$.

(c)

Apply the Bellman operator on $\hat{v}$, noting that $\hat{v}(x) = \frac{a^2}{2b(1-\delta)}\ \forall x$:

$$T\hat{v}(x) = \max_y ax - \frac{1}{2}bx^2 - \frac{1}{2}c(y-x)^2 + \delta\frac{a^2}{2b(1-\delta)}$$

FOC gives $x = y$, plugging this back gives $T\hat{v}(x) = ax - \frac{1}{2}bx^2 + \delta\frac{a^2}{2b(1-\delta)}$. Clearly $T\hat{v}(x) \leq \hat{v}(x)$, with equality for only $x = \frac{a}{b}$.

(d)

$T^n(\hat{v})$ takes this form for $n = 1$ (part c). Suppose it takes this form for n. Now

$$T^{n+1}\hat{v}(x) = \max_y \{ax - \frac{1}{2}bx^2 - \frac{1}{2}c(y-x)^2$$
$$+ \delta(\alpha_n y - \frac{1}{2}\beta_n y^2 + \gamma_n)\}$$

FOC: $-c(y-x) + \delta\alpha_n - \beta_n y = 0 \to y = \frac{cx+\delta\alpha_n}{c+\delta\beta_n}$. Plugging this back in:

$$T^{n+1}\hat{v}(x) = ax - \frac{1}{2}bx^2 - \frac{1}{2}c(\frac{\delta\alpha_n - \delta\beta_n x}{c+\delta\beta_n})^2$$
$$+ \delta(\alpha_n \frac{cx+\delta\alpha_n}{c+\delta\beta_n} - \frac{1}{2}\beta_n \frac{cx+\delta\alpha_n}{c+\delta\beta_n}^2 + \gamma_n)$$

Breaking out the quadratics and collecting terms give

$$\alpha_{n+1} = a + \frac{\delta\alpha_n c}{c+\delta\beta_n}$$
$$\beta_{n+1} = b + \frac{\delta\beta_n c}{c+\delta\beta_n}$$
$$\gamma_{n+1} = \frac{1}{2}\frac{\delta^2\alpha_n^2}{c+\delta\beta_n} + \delta\gamma_n$$

(e)

For β_n, since $\frac{\delta\beta_n}{c+\delta\beta_n} < 1$ and is increasing in β_n, the sequence $\{\beta_n\}$ is a bounded increasing sequence, so has a limit β where $b < \beta < b + c$. Now $\frac{\delta c}{c+\delta\beta_n} < \frac{\delta c}{c+\delta b} < 1$, so $\alpha_n \to \alpha$. Then $\gamma_n \to \gamma$.

Then $T^\infty\hat{v}(x) = \alpha x - \frac{1}{2}\beta x^2 + \gamma$. This satisfies the Bellman equation by construction.

Alternatively, you could solve for a fixed point of the system of 1st order difference equation above, then briefly argue for the stability of the system.

Q3

(a)

$$V(k) = \max_{I\geq 0}\{\pi(k) - \gamma(I) + \frac{1}{R}V((1-\delta)k + I)\}$$

FOC: $\gamma'(I) = \frac{1}{R}V'((1-\delta)k + I)$

Envelope theorem: $V'(k) = \pi'(k) + \frac{1-\delta}{R}V'((1-\delta)k+I)$. Subbing in from FOC: $V'(k) = \pi'(k) + (1-\delta)\gamma'(I)$. Changing k to $k' = (1-\delta)k + I$ and subbing into the envelope theorem:

$$\gamma'(I) = \frac{1}{R}[\pi'(k') + (1-\delta)\gamma'(I')]$$

Interpretation: LHS: Cost of investment today; RHS: Returns to investment, consisting of discounted gain to profits tomorrow, and decrease in the need to invest tomorrow.

(b)

Steady state capital and investment (if exists): k, I satisfies

$$\begin{aligned}\gamma'(I) &= \frac{1}{R}[\pi'(k) + (1-\delta)\gamma'(I)] \\ k &= (1-\delta)k + I\end{aligned}$$

The second eq gives $I = \delta k$. Subbing into the first eq gives $\gamma'(\delta k)(R-1+\delta) = \pi'(k)$. LHS increases from 0 to ∞ and RHS decreases from ∞ to $-\infty$ as k increases from 0 to ∞, hence there is an unique steady state.

If R increase, LHS increases. Now k has to decrease to decrease LHS and increase RHS. I then also decreases. Intepretation: R increase amounts to more discounting. Investment returns are realized in the future, so more discounting means less returns, hence less investment. This results in less steady state capital.

(c)

$$\begin{aligned}I_t &= \frac{1}{R}[-k_{t+1} + k^* + (1-\delta)I_{t+1}] \\ k_{t+1} &= (1-\delta)k_t + I_t\end{aligned}$$

First eq: $\frac{1-\delta}{R}(I_t - I_{t+1}) = \frac{1}{R}[-k_{t+1} + k^* + I_t(1-\delta-R)]$. I_t increases iff $-k_{t+1} + k^* + I_t(1-\delta-R) < 0 \iff (1-\delta)k_t + I_t(\delta+R) > k^*$

Second eq: $k_{t+1} - k_t = I_t - \delta k_t$. k_t increases iff $I_t - \delta k_t > 0$

Refer to Figure 1 for phase diagram and transition.

Q4

(a)

Denote $\bar{G}_t$ as the stream of gov. spending from t to ∞.

$$V(k_t, \bar{G}_t) = \max_{c_t \in [0,(1-\delta)k_t + f(k_t)]} \{\frac{(c_t G_t^{\eta})^{1-\gamma}}{1-\gamma} + \beta V((1-\delta)k_t + f(k_t) - c_t, \bar{G}_{t+1})\}$$

FOC: $G_t^{\eta(1-\gamma)} c_t^{-\gamma} = \beta V_1(k_{t+1}, \bar{G}_{t+1})$

Envelope theorem: $V_1(k_t, \bar{G}_t) = \beta V_1(k_{t+1}, \bar{G}_{t+1})(1-\delta+f'(k_t))$. Sub into FOC gives $V_1(k_t, \bar{G}_t) = (1-\delta+f'(k_t)) G_t^{\eta(1-\gamma)} c_t^{-\gamma}$. Changing $k_t, \bar{G}_t$ to $k_{t+1}, \bar{G}_{t+1}$ and subbing back into envelope theorem gives the Euler eq:

$$G_t^{\eta(1-\gamma)} c_t^{-\gamma} = G_{t+1}^{\eta(1-\gamma)} c_{t+1}^{-\gamma} \beta(1-\delta+f'(k_{t+1}))$$

Along with $k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$, this system of diff eq gives the evolution of consumption and capital.

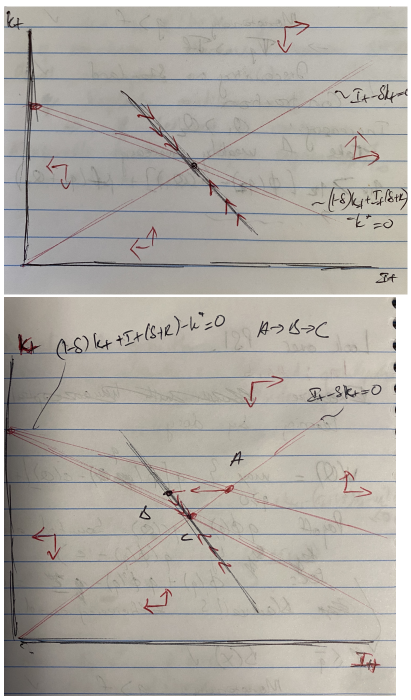

Figure 1: Phase diagram (Top) and Transition (Bot)-3c

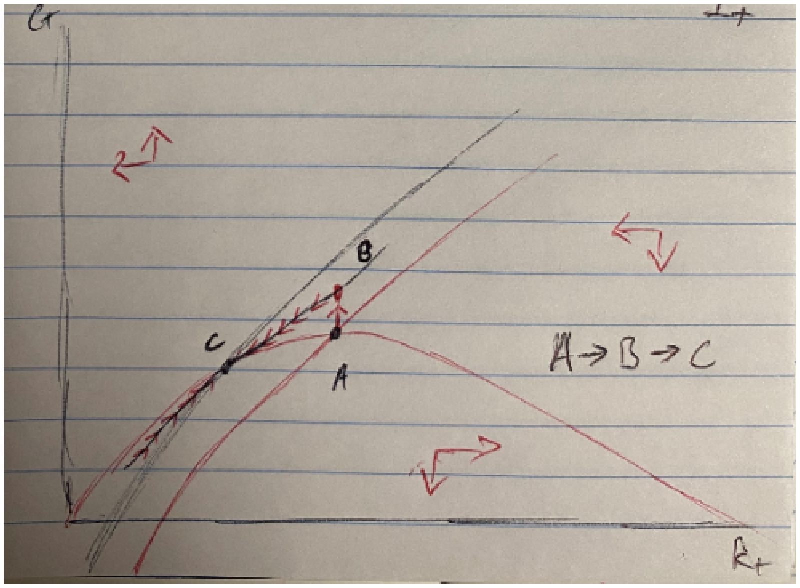

Figure 2: Transition-4c

(b)

Suppose steady state c, k. Let g be the gross growth rate. Then $c = f(k) - \delta k$; and $\beta(1 - \delta + f'(k)) = (\frac{1}{g})^{\eta(1-\gamma)} \rightarrow f'(k) = g^{\eta(\gamma-1)}\frac{1}{\beta} - 1 + \delta$. Note that g acts like an decrease in the time discount factor (more discounting). Higher G tomorrow decreases the marginal utility of consuming tomorrow, and we value future consumption less. Our assumption on f guarantees uniqueness.

(c)

A higher g implies a higher RHS, hence k has to decrease since f' is decreasing. The size of the effect increases in η and γ. This is consistent with the time discounting intepretation.

The transition comprises of an initial jump to the new saddle path, and then movement along the saddle path to the new steady state. $k_{t+1} - k_t = f(k_t) - c_t - \delta k_t$ and $(\frac{c_{t+1}}{c_t})^{\gamma} = \frac{\beta}{g^{\eta(1-\gamma)}}(1 - \delta + f'((1-\delta)k_t + f(k_t) - c_t))$.

Refer to Figure 2 for transition