Game Theory Review

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1 Normal Form Games

1.1 Characterization

A normal form game has:

1. Players: $N = \{1, 2, ..., n\}$
2. Action profiles, A_i, for each player $i \in \{1, ..., n\}$, with associated pure strategy sets, S_i
3. Payoff functions, u_i, for each player

A mixed strategy in a normal form game is denoted with $\sigma_i \in \Delta S_i$, where Δ is a probability distribution for each pure strategy in the mixed strategy. For example, if player i can play A or B, and σ_i is a mixed strategy in which she plays A 30% of the time, then $\sigma_i(A) = .3$ and $\sigma_i(B) = .7$.

Payoff functions, $u_i(\sigma_i, s_{-i})$, are functions of each player's behavior, where (σ_i) is the behavior of the player in question, and s_{-i} is the action of every player other than i.

1.2 Rationalizability and Strict Dominance

1.2.1 Dominance

A pure strategy, $s_i \in S_i$ is strictly dominated if any other strategy (pure or mixed) yields higher payoffs than s_i regardless of other players' actions:

$$\exists \sigma_i \text{ s.t. } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i} \in S_{-i}$$

Identifying **strictly dominated strategies** is straightforward when they're dominated by pure strategies, but it's tougher with strategies dominated by mixed strategies. Take the following game as an example:

	a	b	c	d
A	(2,2)	(3,1)	(4,3)	(3,3)
B	(7,3)	(3,5)	(3,2)	(0,0)

a is not dominated by any other pure strategy, but $u(A, c) > u(A, a)$ and $u(B, b) > u(B, a)$ for the column player, so there exists a mixed strategy of b and c that yields higher payoffs than a **regardless of the row player's strategy**.

Strictly dominated strategies will never be played by rational players in a normal form game, so they can be deleted from the action set for the purposes of rationalizability and equilibrium determination. It may be the case that some strategies that were not previously dominated are dominated once other dominated strategies are deleted. These are also considered dominated by the process of **iterated strict dominance** (ISD).

ISD_k is the set of strategies that survive k rounds of deletion of all possible strictly dominated strategies, Where $ISD_1 \supseteq ISD_2 \supseteq ... \supseteq ISD_k$, where any equilibrium must necessarily be in ISD_∞. Note that **strict** dominance is required for IDS. Deleting weakly-dominated strategies does not yield consistent results.[1]

1.2.2 Best Reply

A **best reply** is a strategy (pure or mixed) that maximizes a player's payoff, conditional on other players' strategies. Formally, σ_i is a best response to σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ \forall \sigma_i' \in \Delta S_i$$

A best response is denoted as $\sigma_i \in B_i(\sigma_{-i})$.

Proposition: In a two-player game, σ_i is strictly dominated if and only if it is never a best response.

1.2.3 Rationalizability

A 1-rationalizable ($\mathbb{Q}_1$) strategy for player i is an element of S_i that is a best reply to an (independent) probability distribution over other players' strategies. A k-rationalizable ($\mathbb{Q}_k$) strategy for player i is an element of S_i that is a best reply to an (independent) probability distribution over other players' $(k-1)$-rationalizable strategies, for $k = 2, 3, ...$ A *rationalizable* ($\mathbb{Q}_\infty$) strategy is an element of S that is k-rationalizable for all players i and all $k = 1, 2, ...$

Theorem: In a two-player game, a strategy survives iterated strict dominance if an only if it is rationalizable.

[1] You should make yourself believe that this is the case.

1.3 Nash Equilibrium

Definition: $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ is a Nash equilibrium if:

$$\sigma_i \in B_i(\sigma_{-i}) \ \forall i \in \{1, 2, ..., n\}$$

In other words, a Nash equilibrium is an outcome in which all players are optimizing. There are two strategies for computing a Nash equilibrium:

Option 1

1. Eliminate pure strategies that are not rationalizable
2. For each strategy profile σ whose supports involve rationalizable strategies, check for each player i that $\sigma_i \in B_i(\sigma_{-i})$

Option 2

1. Use ISD to eliminate all non-rationalizable strategies
2. Final all "closed rationalizable cycles"
3. Look for Nash equilibria on the suppose of each cycle:
   (i) Fore pure strategies, just check for pure best responses
   (ii) For mixed strategies, solve using indifference between all pure strategies in the support

For two-player games, step 2 of option 2 is done by first looking at each rationalizable pure strategy, s_i, for each i, and finding the best response for the other player and determining whether one of them can make s_i a best response.

Next, look at each possible mixed strategy support[2] of each player i, then use optimality conditions to restrict what other players' strategies may be (by changing the payoffs of each pure strategy), then determining whether it is optimal for the other player to do the same based on player i's mixed strategy.

1.3.1 Nash Equilibrium with a Continuum of Players

A continuum of players often simplifies a problem by foregoing the issue of thinking about which player plays which strategy. Also, a mixed-strategy equilibrium in a two-player game can be a pure strategy one when a continuum of players is used. For example, in rock paper scissors with two players, the unique NE is for each player to fully randomize with equal weight on each move. With a continuum of players, each player can either fully randomize, or each move (rock, paper, or scissors) can be played as a pure strategy by one third of the continuum.

[2] A support is the set of all pure strategies that are played in a mixed strategy–i.e. those with a nonzero probability assigned to them.

1.4 Nash Equilibrium with a Continuum of Actions

Most games involve continuous action sets (e.g., pricing in oligopolistic competition). Key concepts:

- Glicksbeg Fixed Point Theorem: If all player action spaces are compact, convex subsets of $\mathbb{R}^k$ and each payoff function is continuous, then the game has at least one (possibly mixed) Nash equilibrium
- We usually assume that payoff functions are strictly quasi-concave in one's own action in order to ensure that mixed NEs cannot be optimal.
- If there is an interior solution, the best response function is obtained using the first-order condition of the payoff function w/r/t one's own action. The NE is then obtained by inputting one player's best response function into the other player's and solving

1.4.1 Timing Games: war of attrition vs. pre-emption

Suppose there is a continuum of players *and* a continuum of actions, such as customers choosing when to go to the grocery store. Then there is a payoff from going at a certain time, $u(t)$ and a payoff from going at a certain place in the distribution of shoppers, $v(q)$, where q is the quantile of the distribution. The solution to this problem is a CDF, $Q(t)$, which gives the share of the shoppers who have already gone to the grocery store at each t. In equilibrium, all shoppers must be indifferent to going at any time t. Thus, if t^* optimizes u, then, at the equilibrium:

$$v(q) \text{ increasing in q} \Rightarrow u(t)v\left(Q(t)\right) = u(t^*)v(0)$$
$$v(q) \text{ decreasing in q} \Rightarrow u(t)v\left(Q(t)\right) = u(t^*)v(1)$$

Once $Q(t)$ is obtained, we can determine when people go to the store by solving for the domain of $Q(t)$, i.e. by setting $Q(t) = 0$ and $Q(t) = 1$ and solving for t.

To determine whether it's possible to have an initial (or terminal) rush, we find the quantile, $\tilde{q}$ such that the payoff of going just before the rush is equal to the average payoff of going during the rush to determine what the size of the rush would be. This is done with the equation:

$$\text{Initial rush: } \frac{1}{\tilde{q}}\int_0^{\tilde{q}} v(x)dx = v(\tilde{q})$$
$$\text{Terminal rush: } \frac{1}{1-\tilde{q}}\int_{\tilde{q}}^{1} v(x)dx = v(\tilde{q})$$

If the $\tilde{q} = 0$ (or 1, in the terminal case), then the rush has a size of 0, meaning that there cannot be a rush.

See discussion handout 9, question 7 and/or HW2, question 5

2 Bayesian Games

3 Extensive Form Games