Problem Set #5

Danny Edgel
Econ 710: Economic Statistics and Econometrics II
Spring 2021

February 22, 2021

Discussed and/or compared answers with Sarah Bass, Emily Case, Katherine Kwok, Michael Nattinger, and Alex Von Hafften

Question 1

(i) Each of the series is covariance stationary if their autocovariance can be represented by some constant function $\gamma(k)$:

$$\begin{aligned}
Cov(U_t, U_{t+k}) &= \mathbb{E}\left[(U_t - \mathbb{E}\left[U_t\right])(U_{t+k} - \mathbb{E}\left[U_{t+k}\right])\right] \\
&= \mathbb{E}\left[(\varepsilon_t\varepsilon_{t-1} - \mathbb{E}\left[\varepsilon_t\varepsilon_{t-1}\right])(\varepsilon_{t+k}\varepsilon_{t+k-1} - \mathbb{E}\left[\varepsilon_{t+k}\varepsilon_{t+k-1}\right])\right] \\
&= \mathbb{E}\left[\varepsilon_t\varepsilon_{t-1}\varepsilon_{t+k}\varepsilon_{t+k-1}\right] = 0\ \forall k \in \{1, T-t\} \\
\Rightarrow \gamma_U(k) &= \begin{cases} \sigma^4, & k = 0 \\ 0, & k \geq 1 \end{cases} \\
Cov(W_t, W_{t+k}) &= \mathbb{E}\left[(W_t - \mathbb{E}\left[W_t\right])(U_{W+k} - \mathbb{E}\left[W_{t+k}\right])\right] \\
&= \mathbb{E}\left[(\varepsilon_t\varepsilon_0 - \mathbb{E}\left[\varepsilon_t\varepsilon_0\right])(\varepsilon_{t+k}\varepsilon_0 - \mathbb{E}\left[\varepsilon_{t+k}\varepsilon_0\right))\right] \\
&= \mathbb{E}\left[\varepsilon_t\varepsilon_0\varepsilon_{t+k}\varepsilon_0\right] = 0\sigma^2 = 0\ \forall k \in \{1, T-t\} \\
\Rightarrow \gamma_W(k) &= \begin{cases} \sigma^4, & k = 0 \\ 0, & k \geq 1 \end{cases} \\
Cov(V_t, V_{t+k}) &= \mathbb{E}\left[(V_t - \mathbb{E}\left[V_t\right])(U_{V+k} - \mathbb{E}\left[V_{t+k}\right])\right] \\
&= \mathbb{E}\left[(\varepsilon_t^2\varepsilon_{t-1} - \mathbb{E}\left[\varepsilon_t^2\varepsilon_{t-1}\right])(\varepsilon_{t+k}^2\varepsilon_{t+k-1} - \mathbb{E}\left[\varepsilon_{t+k}^2\varepsilon_{t+k-1}\right])\right] \\
&= \mathbb{E}\left[\varepsilon_t^2\varepsilon_{t-1}\varepsilon_{t+k}^2\varepsilon_{t+k-1}\right] = 0\ \forall k \in \{1, T-t\} \\
\Rightarrow \gamma_V(k) &= \begin{cases} \sigma^6, & k = 0 \\ 0, & k \geq 1 \end{cases}
\end{aligned}$$

(ii) $\overline{U}$, $\overline{W}$, and $\overline{V}$ converge in probability if their variance, divided by T, converges to zero as $T \to \infty$. Since, as we showed in (i), each series' variance is constant, this is true. Therefore, each sample mean converges to its expectation.

(iii) As with (ii), the weak law of large numbers holds that any estimator will converge to its expectation if its variance, divided by sample size, converges to zero as sample size approaches infinity. By assumption, $\mathbb{E}\left[\varepsilon^8\right] < \infty$, where $\mathbb{E}\left[\varepsilon^8\right]$ is the variance of $\hat{\gamma}_U(0)$ and $\hat{\gamma}_W(0)$. Thus, the sample second moments of U and V converge in probability to their expectations. However, we do not have enough information to determine whether $\hat{\gamma}_W(0)$ has a finite second moment and thus converges to its expectation in probability.

(iv) Since each of the three series are mean zero with finite variance, drawn from a random sample and serially independent, the Central Limit Theorem holds that all three have asymptotically normal scaled sample means.

Question 2

(i) Table below displays the OLS estimates of a single simulation of the model, generated by the attached code.

T	ρ_1	α_0	δ_0	ρ_1
50	0.7	0.818 [-0.010,1.646]	0.015 [-0.009,0.039]	0.751 [0.576,0.926]
50	0.9	0.803 [0.254,1.352]	0.035 [-0.004,0.074]	0.859 [0.775,0.943]
50	0.95	2.151 [1.419,2.882]	0.098 [0.027,0.169]	0.753 [0.624,0.882]
150	0.7	1.172 [0.828,1.515]	0.008 [0.004,0.012]	0.678 [0.613,0.742]
150	0.9	1.207 [0.765,1.649]	0.011 [0.004,0.019]	0.888 [0.842,0.934]
150	0.95	1.020 [0.661,1.378]	0.010 [-0.000,0.019]	0.950 [0.922,0.979]
250	0.7	1.004 [0.735,1.273]	0.012 [0.009,0.015]	0.667 [0.610,0.724]
250	0.9	1.084 [0.758,1.411]	0.009 [0.005,0.013]	0.901 [0.866,0.937]
250	0.95	0.941 [0.576,1.305]	0.011 [0.005,0.016]	0.951 [0.929,0.974]

(ii) The table below displays the mean of the OLS coefficient from this model, across 10,000 simulations, as well as the coverage rate of 95% confidence intervals in each simulation.

T	ρ_1	α_0 Mean	α_0 Coverage	δ_0 Mean	δ_0 Coverage	ρ_1 Mean	ρ_1 Coverage
50	.7	1.097	0.923	1.007	0.945	0.657	0.936
50	.9	1.146	0.905	0.999	0.940	0.853	0.911
50	.95	1.100	0.880	0.990	0.943	0.896	0.898
150	.7	1.039	0.941	1.004	0.948	0.687	0.945
150	.9	1.089	0.929	1.003	0.948	0.886	0.937
150	.95	1.113	0.918	1.001	0.947	0.937	0.925
250	.7	1.025	0.946	1.003	0.950	0.692	0.950
250	.9	1.058	0.939	1.002	0.950	0.892	0.941
250	.95	1.086	0.931	1.001	0.943	0.942	0.934

(iii) As the table above shows, coverage decreases and bias increases with greater persistence in Y_t but increases with sample size. However, the bias and inconsistency from the persistence of Y_t is attenuated by larger samples, with δ_0 being more-or-less without error above 150 observations.