Problem Set #4

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Question 1

Suppose that another observation X_{n+1} becomes available. Show that:

(a) $\mathbf{\overline{X}_{n+1} = (n\overline{X}_n + X_{n+1})/(n+1)}$

$$\begin{aligned}\overline{X}_{n+1} &= \frac{1}{n+1}\sum_{i=1}^{n+1} X_i \\ &= \frac{1}{n+1}\left(\sum_{i=1}^{n} X_i + X_{n+1}\right) \\ &= \frac{1}{n+1}\left(n\overline{X}_n + X_{n+1}\right)\end{aligned}$$

(b) $\mathbf{s^2_{n+1} = \frac{1}{n}((n-1)s^2_n + (n/(n+1))(X_{n+1} - \overline{X}_n)^2)}$

Using the relation from (a), we can derive:

$$
\begin{aligned}
s_{n+1}^2 &= \frac{1}{n}\sum_{i=1}^{n+1}(X_i - \overline{X}_{n+1})^2 \\
&= \frac{1}{n}\sum_{i=1}^{n+1}\big((X_i - \overline{X}_n) + (\overline{X}_n - \overline{X}_{n+1})\big)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n+1}\big[(X_i - \overline{X}_n)^2 + 2(X_i - \overline{X}_n)(\overline{X}_n - \overline{X}_{n+1}) + (\overline{X}_n - \overline{X}_{n+1})^2\big] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 + (X_{n+1} - \overline{X}_n)^2 + 2(\overline{X}_n - \overline{X}_{n+1})\sum_{i=1}^{n+1}(X_i - \overline{X}_n) + \sum_{i=1}^{n+1}(\overline{X}_n - \overline{X}_{n+1})^2\right] \\
&= \frac{1}{n}\big[(n-1)s_n^2 + (X_{n+1} - \overline{X}_n)^2 + 2(n+1)(\overline{X}_n - \overline{X}_{n+1})(\overline{X}_{n+1} - \overline{X}_n) + (n+1)(\overline{X}_n - \overline{X}_{n+1})^2\big] \\
&= \frac{1}{n}\big[(n-1)s_n^2 + (X_{n+1} - \overline{X}_n)^2 - 2(n+1)(\overline{X}_n - \overline{X}_{n+1})^2 + (n+1)(\overline{X}_n - \overline{X}_{n+1})^2\big] \\
&= \frac{1}{n}\big[(n-1)s_n^2 + (X_{n+1} - \overline{X}_n)^2 - (n+1)(\overline{X}_n - \overline{X}_{n+1})^2\big] \\
&= \frac{1}{n}\left[(n-1)s_n^2 + (X_{n+1} - \overline{X}_n)^2 - (n+1)\left(\overline{X}_n - \frac{1}{n+1}(n\overline{X}_n + X_{n+1})\right)^2\right] \\
&= \frac{1}{n}\left[(n-1)s_n^2 + (X_{n+1} - \overline{X}_n)^2 - (n+1)\left(\frac{1}{n+1}\overline{X}_n - \frac{1}{n+1}X_{n+1}\right)^2\right] \\
&= \frac{1}{n}\left[(n-1)s_n^2 + (X_{n+1} - \overline{X}_n)^2 - (n+1)\left(-\frac{1}{n+1}\right)^2\big(X_{n+1} - \overline{X}_n)\big)^2\right] \\
&= \frac{1}{n}\left[(n-1)s_n^2 + \left(1 - \frac{1}{n+1}\right)(X_{n+1} - \overline{X}_n)^2\right] \\
&= \frac{(n-1)s_n^2 + \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2}{n}
\end{aligned}
$$

Question 2

For some integer k, set $\mu_k = E(X^k)$. Construct an unbiased estimator $\hat{\mu}_k$ for μ_k, and show its unbiasedness. Define $\hat{\mu}_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$. If the bias of this estimator is equal to zero, then it is unbiased:

$$E(\hat{\mu}_k) - \mu_k = 0$$
$$E(\frac{1}{n}\sum_{i=1}^{n} X_i^k) - E(X^k) = 0$$
$$\frac{1}{n}\sum_{i=1}^{n} X_i^k = X^k$$

Since $\{X_i\}_{i=1}^n$ is assumed to be a random sample, this is equality holds. Thus, $\hat{\mu}_k$ is an unbiased estimator.

Question 3

Consider the central moment $m_k = E((X-\mu)^k)$. Construct an estimator $\hat{m}_k$ for m_k without assuming a known μ. In general, do you expect $\hat{m}_k$ to be biased or unbiased?

Question 4

Calculate the variance of $\hat{\mu}_k$ that you proposed above, and call it $Var(\hat{\mu}_k)$.

Question 5

Show that $E(s_n) \leq \sigma$ using Jensen's inequality (CB Theorem 4.7.7).

Question 6

Show algebraically that $\hat{\sigma}^2 = n^{-1}\sum_{i-1}^{n}(X_i-\mu)^2 - (\overline{X}_n - \mu)^2$.

Question 7

Find the covariance of $\hat{\sigma}^2$ and $\overline{X}_n$. Under what condition is this zero? (See lecture question for hint)

Question 8

Suppose that X_i are independent but not necessarily identically distributed (i.n.i.d.) with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$.

(a) **Find $E(\overline{X}_n)$.**

(b) **Find $Var(\overline{X}_n)$.**

Question 9

Show that if $Q \sim \chi_r^2$, then $E(Q) = r$ and $Var(Q) = 2r$ (hint: use the representation $Q = \sum_{i=1}^n X_i^2$ with X_i being i.i.d $\mathcal{N}(0,1)$).

Question 10

Suppose that $X_i \sim \mathcal{N}(\mu_X, \sigma_X^2) : i = 1, ..., n_1$ and $Y_i \sim \mathcal{N}(\mu_Y, \sigma_Y^2), i = 1, ..., n_2$ are mutually independent. Set $\overline{X}_n = n_1^{-1} \sum_{i=1}^{n_2} Y_i$.

(a) **Find $E(\overline{X}_n - \overline{Y}_n)$.**

(b) **Find $Var(\overline{X}_n - \overline{Y}_n)$.**

(c) **Find the distribution of $\overline{X}_n - \overline{Y}_n$.**