Midterm Review

Danny Edgel
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1 Lecture 1

1.1 Logical Operators

And/or statements ($\wedge/\vee$), if/then statements ($\Rightarrow$, $\iff$), and negation ($\neg$)

1.2 Methods of proof

- Direct (aka Deductive): directly show that if P is true, then we can "deduce" that Q is also true
- Contrapositive: Instead of showing $P \Rightarrow Q$, show $\neg Q \Rightarrow \neg P$ (direct proof of the negation of the modus ponens)
- Contradiction: Show that if P is true, then Q being false yields a contradiction
- Induction: Proving that a statement P holds for all natural numbers, performed in two steps: the base step and the induction step. Example: $\forall n \in \mathbb{N}, \sum_1^n n = \frac{n(n+1)}{2}$
  1. The base step: $P(1)$ holds: $\frac{1(1+1)}{2} = \frac{2}{2} = 1$
  2. The induction step: $P(n+1) = P(n)$:

$$P(n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n^2+3n+2)}{2} = \frac{(n+1)(n+2)}{2}$$

1.3 Set Operations

Union ($\bigcup$), Intersection ($\bigcap$), Subset ($\subset$), complement (^{c}), and difference ($\setminus$)

De Morgan's Laws:

$$(A \bigcap B)^c = A^c \bigcup B^c$$
$$(A \bigcup B)^c = A^c \bigcap B^c$$

1.4 Cardinality

Cardinality: The size of a set

Numerical Equivalence: The elements of each set can be uniquely matched up and paired off

Finite: Being numerically equivalent to $J_n = \{1, 2, ..., n\}$. J_n's cadinality is n

Infinite: Not being finite

Countable: Being numerically equivalent to $\mathbb{N}$

Uncountable: Being infinite and not countable

Common Examples:

Countable sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z} \setminus \mathbb{N}$, $\mathbb{Q}$

Uncountable: $2^{\mathbb{N}}$ (Set of all subsets of $\mathbb{N}$)

2 Lecture 2

2.1 Metric Spaces

Distance/matrix functions: let X be a metric space with distance function d. Then the following holds $\forall x, y, z \in X$:

- Nonnegativity: $d(x, y) \geq 0$, where $d(x, y) = 0 \iff x = y$
- Symmetry: $d(x, y) = d(y, x)$
- Triangle Inequality:$d(x, y) \leq d(x, z) + d(z, y)$

2.2 Convergence in Metric Spaces

Sequences: $x : \mathbb{N} \to X$ is a sequence in metric space (X, d), written as $\{x_n\}$, where $x_n = s(n)$. $\{x_n\}$ **converges** to $x \in X$ if:

$$\forall \varepsilon > 0, \exists N(\varepsilon) > 0 \text{ s.t. } \forall n > N(\varepsilon),\ d(x_n, x) < \varepsilon$$

This is written as $x_n \to x$ or $\lim\limits_{x \to \infty} x_n = x$.

Convergence Theorems

- A sequence $\{x_n\}$ in a metric space (X, d) has at most one limit
- If, for some $\{x_n\}$, $x_n \to x$, then any **subsequence** $\{x_{n_k}\}$ also converges to x as $k \to \infty$. (If $\{x_n\}$ does not converge, then this theorem doesn't tell us anything about its subsequences)
- Every convergent sequence in a metric space is **bounded**. A subset $S \subset X$ in a metric space (X, d) is bounded if

$$\exists x \in X, \beta \in \mathbb{R} \text{ s.t. } \forall s \in S,\ d(x, s) < \beta$$

- Limits preserve weak inequality: If $x_n \to x \in \mathbb{R}$, $y_n \to y \in \mathbb{R}$ and $x_n \leq y_n$ $\forall n \in \mathbb{N}$, then $x \leq y$
- If $x_n \to x \in \mathbb{R}$ and $y_n \to y \in \mathbb{R}$, then:
  - $x_n + y_n \to x + y$, $x_n - y_n \to x - y$
  - $x_n y_n \to xy$
  - $\frac{x_n}{y_n} \to \frac{x}{y}$ (if $y \neq 0$ and $y_n \neq 0$ $\forall n$
- The last two theorems hold for elements of $\mathbb{R}^m$

2.3 Bolzano-Weierstrass Theorem

Every bounded real sequence contains at least one convergent subsequence.

Lemma 1 (Monotone Convergence Theorem). Every increasing sequence of real numbers that is bounded above converges. Every decreasing sequence of real numbers that is bounded below converges.

Lemma 2. Every real swquence contains either a decreasing subsequence of increasing subsequence (and possible both).

2.4 Infinite Sums

Given a real sequence $\{x_n\}$, the infinite sum of its terms is *well-defined* if the sequence of partial sums, $\{S_n\} = \sum_{i=1}^{n} x_i$, converges.

If $S_n \to S$, we write:

$$\sum_{i=1}^{\infty} x_i = S$$

3 Lecture 3

3.1 Open & Closed Sets

Let (X, d) be a metric space.

A set $A \subset X$ is **open** if:

$$\forall x \in A, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subset A$$

Or, put another way, each point has an open ball centered at each $x \in A$ with a radius ε with is contained in A.

A set $C \subset X$ is **closed** if its complement, $C^c = X \setminus C$, is open.

Examples:

- Any open ball, $B_\varepsilon(x)$, is an open set
- Any closed call, $B_\varepsilon[x]$, is a closed set

- Most sets are neither open, nor closed (e.g. $[0, 1)$ in $(\mathbb{R}, d_E)$)
- The same set can be open in one metric space but closed in another. e.g.:
  - $[0, 1]$ is not open in $(\mathbb{R}, d_E)$
  - $[0, 1]$ is open in $([0, 1], d_E)$

3.2 Intersections and Unions of Open and Closed Sets

Theorem: Let (X, d) be a metric space. Then (note that finite is important: see caveat sub-bullets),

- $\emptyset$ and X are simultaneously open and closed in X
- The union of an arbitrary collection of open sets is open
- The intersection of a finite collection of open sets is open
  - $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0, 1)$, is an open set
- The union of a finite collection of closed sets is closed
  - $\bigcup_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1)\frac{1}{n}\right) = \{1\}$, is an open set
- The intersection of an arbitrary collection of closed sets is closed

Equivalent definition of a **closed set**: $A \subset X$, where (X, d) is a metric space, is closed if and only if every convergence sequence $\{x_n\}$ contained in A has its limit in A

3.3 Limits of Functions

Let (X, d) be a metrix space and $A \subset X$.
$x_L \in X$ is a **limit point** of A if $\forall \varepsilon > 0$,

$$(B_\varepsilon(x_L) \setminus \{x_L\}) \bigcap A \neq \emptyset$$

(i.e. every neighborhood of x_L has a point in A that is not x_L)

Let (X, d) and (Y, ρ) be two metric spaces, $A \subset X$, $f : A \to Y$, where x^0 is a limit point of A. A function f has a **limit** y^0 as x approaches x^0 if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. if} x \in A \text{ and } 0 < d(x, x^0) < \delta, \text{ then } \rho(f(x), y^0) < \varepsilon$$

We write the limit as $\lim\limits_{x \to x^0} f(x) = y^0$

Theorems of limits: Let x^0 be a limit point of X. Then,

- $\lim\limits_{x \to x_n} f(x) = y^0 \iff \forall \{x_n\} \in X : x_n \to x \cap x_n \neq x^0$, $\{f(x_n)\}$ converges to y^0
- $\exists \lim\limits_{x \to x_n} f(x) \Rightarrow \exists! \lim\limits_{x \to x_n} f(x)$

4 Lecture 4

4.1 Continuity

Let (X,d) and (Y,ρ) be two metric spaces, $A \subset X$, $f : A \to Y$ is **continuous** at x^0 if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } d(x,x^0) < \delta \Rightarrow \rho(f(x), f(x^0)) < \varepsilon$$

This requires:

- $f(x)$ to be defined
- Either:
  - x^0 to be an isolated point of X ($\exists \varepsilon > 0$ s.t. $B_\varepsilon(x^0) = \{x^0\}$)
  - $\exists \lim\limits_{x \to x_n} f(x) = f(x^0)$

Alternative definition (theorem): f is continuous at x^0 if and only if either of the following statements is true:

- $f(x^0)$ is defined and either x^0 is an isolated point or x^0 is a limit point of X, where $\lim\limits_{x \to x_n} f(x) = f(x^0)$
- For any sequence $\{x_n\}$ s.t. $x_n \to x^0$, the sequence $\{f(x_n)\}$ converges to $f(x^0)$

Continuity vis-a-vis open and closed sets: A function, f is continous if it is continuous at every point of its domain.

Define the *preimage* (pre-image) of a function as:

$$f^{-1}(A) = \{x \in X | f(x) \in A$$

Then, for metric spaces (X,d) and (Y,ρ) and function $fX \to Y$, we have the theorems:

- f is continuous $\iff$ for any closed set $C \subset Y$, $f^{-1}(C)$ is closed in X
- f is continuous $\iff$ for any open set $A \subset Y$, $f^{-1}(A)$ is open in X

4.2 Uniform Continuity

Let (X,d) and (Y,ρ) be metric spaces. $f : X \to Y$ is **uniformly continuous** if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } d(x,x^0) < \delta \Rightarrow \rho(f(x), f(x^0)) < \varepsilon$$

A critical feature of uniform continuity is that δ depends *only* on ε and *cannot* depend on x^0.

f is uniformly continuous $\Rightarrow$ f is continuous

4.3 Lipschitz

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $E \subset X$. f is **Lipschitz** on E if

$$\exists K > 0 \text{ s.t. } \rho(f(x), f(y)) \leq K d(x, y) \ \forall x, y \in E$$

f is **locally Lipschitz** on E if

$$\forall x \in E \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x) \bigcap E$$

Lipschitz is a stronger form of continuity than uniform continuity.

5 Lecture 5

5.1 Supremum and Infimum

Let $X \subset \mathbb{R}$. Then $u \in \mathbb{R}$ is an **upper bound** for X if

$$x \leq u \ \forall x \in X$$

and $l \in \mathbb{R}$ is a **lower bound** for X if

$$l \leq x \ \forall x \in X$$

If X has a lower bound, then X is **bounded below**. If X has an upper bound, then X is **bounded above**.

Suppose X is both bounded above and bounded below. The **supremum** of X, written as $\sup X$, is the smallest upper bound for X and satisfies:

- $\sup X \geq x \ \forall x \in X$ ($\sup X$ is an upper bound)
- $\forall y < \sup X \ \exists x \in X$ s.t. $x > y$ (there is no smaller upper bound)

The **infimum** of X, written as $\inf X$, is the largest lower bound of X and satisfies:

- $\inf X \leq x \ \forall x \in X$ ($\inf X$ is a lower bound)
- $\forall y > \inf X \ \exists x \in X$ s.t. $x < y$ (there is no larger lower bound)

Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum. This supremum is a real number.

5.2 Extreme Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then F attains its maximum and minimum on $[a, b]$:

$$f(x_M) = \sup_{x \in [a,b]} f(x)$$
$$f(x_m) = \inf_{x \in [a,b]} f(x)$$
$$x_M, x_m \in [a, b]$$

5.3 Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any $\gamma \in [f(a), f(b)]$, $\exists c \in [a, b]$ s.t. $f(c) = \gamma$

5.4 Monotonic Functions

$f : \mathbb{R} \to \mathbb{R}$ is **monotonically increasing** if $\forall x, y,\ x < y \Rightarrow f(x) < f(y)$.

Theorem: Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then $\forall x \in (a, b)$, the following **one-sided limits** exist:

$$f(x^+) := \lim_{y \to x} f(y),\ f(x^-) := \lim_{y \to x^-} f(y)$$

Moreover,

$$\sup f(s)|a < s < x = f(x^-) \leq f(x) \leq f(x^+) = \inf f(s)|x < s < b$$

6 Lecture 6

6.1 Complete Metric Spaces

Cauchy Sequences: A sequence $\{x_n\}$ in a metric space (X, d) is Cauchy if

$$\forall \varepsilon > 0 \exists N > 0 \text{ s.t. } m, n > N \Rightarrow d(x_n, x_m) < \varepsilon$$

Theorem: Every convergent sequence in a metric space is Cauchy. The converse is only true if the metric space is **complete**.

A metric space (X, d) is **complete** if every Cauchy sequence contained in X converges to a point in X. E.g. Euclidean space $(\mathbb{R}^m, d_E)$ is complete for any m. If (X, d) is a complete metric space and $Y \subset X$, then (Y, d) is complete if and only if Y is closed.

6.2 Contraction Mapping Theorem

Any function $T : X \to X$ from a metric space to itself is called an **operator**.

An operator $T : X \to X$ is a **contraction of modulus** β if $\beta < 1$ and

$$d(T(x), T(y)) \leq \beta d(x, y)\ \forall x, y \in X$$

Every contraction is uniformly continuous.

A **fixed point** of an operator T is an element $x^* \in X$ s.t. $T(x^*) = x^*$.

Contraction Mapping Theorem: Let (X, d) be a nonempty, complete metric space where $T : X \to X$ is a contraction with modulus $\beta < 1$. Then:

- T has a unique fixed point x^*

- $\forall x_0 \in X$, the sequence $\{x_n\}$, where $x_n = T^n(x_0) = T(T(...T(x_0)...))$, converges to x^*

Continuous dependence of the fixed point on parameters: Let (X, d) and (Ω, ρ) be metric spaces and $T : X \times \Omega \to X$. For each $\omega \in \Omega$, let $T_\omega : X \to X$ be defined by $T_\omega(x) = T(x, \omega)$.

Suppose X is complete, T is continuous in ω, and $\exists \beta < 1$ such that T_ω is a contraction of modulus β $\forall \omega \in \Omega$. Then the fixed point function $x^* : \Omega \to X$ defined by $x^*(\omega) = T_\omega(x^*(\omega))$ is continuous.

Blackwell's Sufficient Conditions: Let $B(X)$ be the set of all bounded functions from X to $\mathbb{R}$ with metric $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$. Let $T : B(X) \to B(X)$ satisfy:

$$\begin{aligned} \text{Monotonicity: } & f(x) \leq g(x)\ \forall x \in X \Rightarrow (T(f))(x) \leq (T(g))(x)\ \forall x \in X \\ \text{Discounting: } & \exists \beta \in (0,1) \text{ s.t. } \forall a \geq 0, x \in X,\ (T(f + a))(x) \leq (T(f))(x) + \beta a \end{aligned}$$

Then T is a contraction with modulus β.

7 Lecture 7

7.1 Compactness

Open Covers: A collection of sets $\mathcal{U} = \{U_\lambda | \lambda \in \Lambda\}$ in a metric space (X, d) is an **open cover** of the set A if U_λ is open for all $\lambda \in \Lambda$ and $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$ (Note: Λ can be finite, countable, or uncountable) A set A in a metric space is **compact** if every open cover of A contains a finite subcover of A. i.e., if $\{U_\lambda | \lambda \in \Lambda\}$ is an open cover of A, then

$$\exists n \in \mathbb{N} \text{ s.t. } A \subset U_{\lambda_1} \bigcup U_{\lambda_2} \bigcup ... \bigcup U_{\lambda_n}$$

Examples:

- $(0, 1)$ is not compact in $\mathbb{R}$:
  - $U_n = \left(\frac{1}{n}, 1\right)$, $n \in \mathbb{N}$
  - U_n is open for any $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} U_n = (0, 1)$
  - However, $\nexists n_1, ..., n_k \in \mathbb{N}$ s.t. $U_{n_1} \bigcup U_{n_2} \bigcup ... \bigcup U_{n_k} \supset (0, 1)$
- $[0, \infty)$ is closed but not compact:
  - $U_n = (-1, n)$, $n \in \mathbb{N}$
  - U_n is open for any $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} U_n = (-1, \infty) \supset [0, \infty)$
  - However, $\nexists n_1, ..., n_k \in \mathbb{N}$ s.t. $U_{n_1} \bigcup U_{n_2} \bigcup ... \bigcup U_{n_k} \supset [0, \infty)$

Properties of Compact Sets:

- Any closed subset of a compact space is compact
- If A is a compact subset of a metric space, then A is closed
- If A is a compact subset of a metric space, then A is bounded

Compact Subsets in $\mathbb{R}^m$

Heine-Borel Theorem: If $A \subset \mathbb{R}^m$, then A is compact if and only if A is closed and bounded.

NOTE: It is *not* enough for A to be closed and bounded in a subset of $\mathbb{R}^m$. It must be closed and bounded **in** $\mathbb{R}^m$.

Sequential Compactness: A set A in a metric space (X, d) is compact if and only if it is sequentially compact. A is **sequentially compact** if every sequence of elements of A contains a convergent subsequence whose limit lies in A.

7.2 Extreme Value Theorem, Revisited

Theorems relating to functions on compact subsets:

- Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact set in (X, d), then $f(C)$ is compact in (Y, ρ)
- (Extreme Value Theorem) If (1) C is a compact set in a metric space, and (2) $f : C \to \mathbb{R}$ is continuous, then f is bounded on C and attains its maximum and minimum
- Let (X, d) and (Y, ρ) be metric spaces, where $C \subset X$ is compact and $f : C \to Y$ is continuous. Then f is uniformly continuous on C

8 Lecture 8

8.1 Vector Spaces

Closed under linear compositions, identity elements, null elements, well-behaved operations

Def: V is a collection of objects called vectors, which may be added together and multiplied by real numbers, called scalars, satisfying:

- $\forall x, y, z \in V, (x + y) + z = x + (y + z)$
- $\forall x, y \in V, x + y = y + x$
- $\exists! \vec{0} \in V$ s.t. $\forall x \in V, x + \vec{0} = \vec{0} + x = 0$
- $\forall x \in V \exists! (-x) \in V$ s.t. $x + (-x) = \vec{0}$

- $\forall \alpha \in \mathbb{R}, x, y \in V, \alpha(x+y) = \alpha x + \alpha y$
- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha+\beta)x = \alpha x + \beta x$
- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha\beta)x = \alpha(\beta x)$
- $\forall x \in V, 1 \cdot x = x$

Let V be a vector space. A **linear combination** of $x_1, ..., x_n \in V$ is a vector of the form

$$y = \sum_{i=1}^{n} \alpha_i x_i, \text{ where } \alpha_1, ..., \alpha_n \in \mathbb{R}$$

α is called the **coefficient** of x_i in the linear combination Let W be a subset of V. A **span** of W is the set of all linear combinations of elements of W,

$$\text{span}W = \left\{ \sum_{i=1}^{n} \alpha_i x_i | n \in \mathbb{N}, \alpha_1, ..., \alpha_n \in \mathbb{R}, x_1, ..., x_n \in W \right\}$$

The set $W \subset V$ spans V if $V = \text{span}W$.

8.2 Bases

Linear Dependence: A set $X \subset V$ is **linearly dependent** if $\exists x_1, ..., x_n \in X$, $\alpha_1, ..., \alpha_n \in \mathbb{R}$ s.t. $\sum_{i=1}^{n} \alpha_i^2 \neq 0$ and $\sum_{i=1}^{n} \alpha_i x_i = \vec{0}$.
A set $X \subset V$ is **linearly independent** if $\nexists x_1, ..., x_n \in X$, $\alpha_1, ..., \alpha_n \in \mathbb{R}$ s.t. $\sum_{i=1}^{n} \alpha_i^2 \neq 0$ and $\sum_{i=1}^{n} \alpha_i x_i = \vec{0}$
i.e. if $\sum_{i=1}^{n} \alpha_i x_i = \vec{0}$, then $\alpha_1 = ... = \alpha_n = 0$

A **basis** of a vector space V is a linearly independent set of vectors in V that spans V.

Basis Theorems:

- Let B be a basis for V and enumerate elements of B by a set Λ so that $B = \{v_\lambda | \lambda \in \Lambda\}$. Then every vector $x \in V$ has a unique representation as a linear combination of elements of B with finitely many nonzero coefficients
- Every vector space has a basis. Any two bases of a vector space V have hte same cardinality (i.e. are numerically equivalent)
- If V is a vector space and $W \subset V$ is linearly independent, then there exists a linearly independent set B such that $W \subset B \subset \text{span}B = V$

Dimension: Let V be a vector space. The **dimension** of V, denoted $\text{dim}V$, is the cardinality of any basis of V. If $\text{dim}V = n$ for some $n \in \mathbb{N}$, then V is finite-dimensional. Otherwise, V is infinite-dimensional.

Dimension theorems:

- Suppose $\dim V = n \in \mathbb{N}$. If $W \subset V$ and $|W| > n$, where $|W|$ denotes the cardinality of W, then W is linearly dependent
- Suppose $\dim V = n$ and $W \subset V$, $|W| = n$. Then
  - If W is linearly independent, then $\mathrm{span} W = V$, so W is a basis of V
  - If $\mathrm{span} W = V$, then W is linearly independent, so W is a basis of V

8.3 Linear Transformations

Let X and Y be vector spaces. We say that $T : X \to Y$ is a **linear transformation** if for all $x_1, x_2 \in X$, $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$

$L(X, Y)$ is the set of all linear transformations from X to Y

Linear transformation theorems:

- $L(X, Y)$ is a vector space
- If $R : X \to Y$ and $S : Y \to Z$ are linear transformations, then $S \circ R : X \to Z$ is a linear transformation

Image, Kernel, and Rank: Let $T \in L(X, Y)$. Then,

- The **image** of T is $\mathrm{Im} T := T(X) = \{T(x) | x \in X\}$
- The **kernel** of T is $\ker T := \{x \in X | T(x) = \vec{0}\}$
- The **rank** of T is $\mathrm{rank} T := \dim(\mathrm{Im} T)$

Theorems:

- If $T \in L(X, Y)$, then $\mathrm{Im} T$ and $\ker T$ are vector subspaces of Y and X, respectively
- Let X be a finite-dimensional vector space and $T \in L(X, Y)$. Then,

$$\dim X = \dim(\mathrm{Ker} T) + \mathrm{rank} T = \dim(\mathrm{Ker} T) + \dim(\mathrm{Im} T)$$

Invertible Linear Transformation: $T \in L(X, Y)$ is **invertible** if $\exists S : Y \to X$ s.t

$$S(T(x)) = x \ \forall x \in X, \ T(S(y)) = y \ \forall y \in Y$$

The transformation S is called the inverse of T and is denoted T^{-1}. If T is invertible, then:

- T is one-to-one: $\forall x_1 \neq x_2$, $T(x_1) \neq T(x_2)$
- T is onto: $\forall y \in Y$, $\exists x \in X$ s.t. $T(x) = y$

Invertible Theorems:

- If $T \in L(X, Y)$ is invertible, then $T^{-1} \in L(Y, X)$
- $T \in L(X, Y)$ is one-to-one if and only if $\ker T \equiv \{\vec{0}\}$

8.4 Isomorphisms

Two vector spaces X and Y are **isomorphic** if there exists an invertible linear function (i.e. one-to-one and onto) from X to Y. A function with these properties is called an isomorphism.

Isomorphism Theorems:

- Let X and Y be two vector spaces, and let $V = \{v_\lambda | \lambda \in \Lambda\}$ be a basis for X. Then a linear transformation $T : X \to Y$ is completely defined by its value on V. That is:
  - Given any set $\{y_\lambda | \lambda \in \Lambda\} \subset Y$, $\exists T \in L(X,Y)$ s.t. $T(v_\lambda) = y_\lambda$ for all $\lambda \in \Lambda$
  - If $S, T \in L(X,Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$
- Two vector spaces X and Y are isomorphic if and only if $\dim X = \dim Y$ (Note: if $\dim X = n$, then X is isomorphic to $\mathbb{R}^n$

9 Lecture 9

9.1 Isomorphism between X and $\mathbb{R}^n$

$V = \{v_1, ..., v_n\} \in X$ is a basis of X. $\forall x \in X$, x has a unique representation, $x = \sum_{i=1}^{n} \alpha_i v_i$. The isomorphism from X to $\mathbb{R}^n$ is denoted:

$$\mathrm{crd}_V(x) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$$

9.2 Isomorphism between $L(X,Y)$ and $R^{n \times m}$

Let:

$$V = \{v_1, ..., v_n\} \in X \text{ be a basis of } X$$
$$W = \{w_1, ..., w_n\} \in Y \text{ be a basis of } Y$$

For example,

$$T(v_1) = \sum_{i=1}^{m} \alpha_{i1} w_i, ..., T(v_n) = \sum_{i=1}^{m} \alpha_{in} w_i$$

Thus, the isomorphism can be represented as:

$$\mathrm{mtx}_{W,V}(T) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \in M_{m \times n}$$

Matrix Representation of a Composition of Linear Functions:
Let

$$
\begin{aligned}
U \subset X &\text{ be a basis of } X \\
V \subset Y &\text{ be a basis of } Y \\
W \subset Z &\text{ be a basis of } Z \\
S \in L(X, Y), T \in L(Y, Z)&
\end{aligned}
$$

Then,

$$\text{mtx}_{W,V}(T) \cdot \text{mtx}_{V,U}(S) = \text{mtx}_{W,U}(T \circ S)$$

9.3 Change of Basis

- $\dim X = n$, $T \in L(X, X)$
- $\text{mtx}_V(T) \equiv \text{mtx}_{V,V}(T)$

If we change basis from V to W, then:

$$\text{mtx}_V(T) = \text{mtx}_{V,W}(id) \cdot \text{mtx}_W(T) \cdot \text{mtx}_{W,V}(id)$$

Where:

$$
\begin{aligned}
&\text{mtx}_{V,W}(id) \cdot \text{mtx}_{W,V}(id) = \text{mtx}_V(id) = I \\
&\text{mtx}_{V,W}(id) = [\text{mtx}_{W,V}(id)]^{-1}
\end{aligned}
$$

Thus, $\text{mtx}_V(T) = P^{-1} \cdot \text{mtx}_W(T) \cdot P$, where $P = \text{mtx}_{W,V}(id)$

9.4 Similarity

$A, B \in <_{n \times n}$ are **similar** if $A = P^{-1}BP$ for some invertible matrix P
Theorem: IF $\dim X = n$, then:

- If $T \in L(X, X)$, then any two matrix representations of T are similar
- Two similar matrices represent the same linear transformation T, relative to suitable bases