Problem Set 2 Solutions

1. **Problem 3 in Lecture 2**
   We can use the definition of CDF to find the distribution of Y.

$$F_Y(y) = \Pr(X^3 \leq y) = \Pr(X \leq y^{\frac{1}{3}}) = F_X(y^{\frac{1}{3}}), x \in (0,1) \to y \in (0,1)$$
$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(y^{1/3})\frac{1}{3}y^{-2/3} \text{ if } y \in (0,1) \text{ and zero otherwise}$$

   Now we can verify that its integral equals to zero on the support $y \in (0,1)$ since

$$\int_0^1 f_Y(y)dy = \int_0^1 14y(1-y^{1/3})dy = [7y^2 - 6y^{7/3}]_0^1 = 1$$

2. **Problem 4 in Lecture 2**
   First, we can see that the CDF is continuous since

$$\lim_{x\uparrow 0.5} F_X(x) = \lim_{x\downarrow 0.5} F_X(x) = 0.6$$

   Then we can check the value of $\int_0^x$ for three cases:

   - if $x \in [0, 0.5)$, then $\int_0^x f_X(t)dt = 1.2x = F_X(x)$
   - if $x = 0.5$, then $\int_0^x f_X(t)dt = 0.6 = F_X(0.5)$
   - if $x \in (0.5, 1)$, then $\int_0^x f_X(t)dt = \int_0^{0.5} 1.2dt + \int_{0.5}^x 0.8dt = 0.6 + 0.8x - 0.4 = 0.2 + 0.8x = F_X(x)$

   This is straightforward since the value of a function at a certain point does not change the result of the integral.

3. **Problem 5 in Lecture 2**
   First, see that we have a transformation from $x \in [-1, 2]$ to $y \in [0, 4]$, so this is not one-to-one since $x \in [-1, 1]$ maps into $y \in [0, 1]$. So we cannot use apply the theorem directly, so let's try to find out the CDF of Y.
   (You can also try to use the theorem for one-to-one case by separating your support into $x \in [-1, 0]$, $x \in (0, 1]$, and $x \in (1, 2]$ and sum up the results for each point of Y.)

$$F_Y(y) = \Pr(X^2 \leq y) = \begin{cases} \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}), & \text{if } y \in [0,1] \\ \Pr(-1 \leq X \leq \sqrt{(y)}) = F_X(\sqrt{y}), & \text{if } y \in (1,4] \end{cases}$$

   Then by differentiation we have

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} = \frac{2}{9\sqrt{y}}, & \text{if } y \in [0,1] \\ \frac{f_X(\sqrt{y})}{2\sqrt{y}} = \frac{1}{9} + \frac{1}{9\sqrt{y}}, & \text{if } y \in (1,4] \\ 0, & \text{otherwise} \end{cases}$$

4. **Problem 6 in Lecture 2**
To find the median, we need to find some m s.t. $\Pr(X \leq m) = \Pr(X \geq m) = 0.5$. Consider integral by substitution using $x = \tan\theta$. Then note that the region of $x \in (-\infty, m)$ becomes $\theta \in (-\frac{\pi}{2}, \tan^{-1}(m))$ where $\tan^{-1}(m)$ has the value between $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$\begin{aligned}
\int_{-\infty}^{m} \frac{1}{\pi(1+x^2)} dx &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\tan^{-1}(m)} \frac{1}{1+\tan^2(\theta)} \frac{1}{\cos^2(\theta)} d\theta \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\tan^{-1}(m)} \frac{1}{\cos^2(\theta)+\sin^2(\theta)} d\theta \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\tan^{-1}(m)} 1 d\theta \\
&= \frac{1}{\pi}(\tan^{-1}(m) + \frac{\pi}{2})
\end{aligned}$$

For this value to be equal to 0.5, we need $\tan^{-1}(m) = 0$, which implies $m = 0$. You need to consider the change of support when you are using integral by substitution. Indeed, as this PDF is symmetric around zero, the median should be zero.

5. **Problem 7 in Lecture 2**
We can express $E|X-a|$ using integral :

$$E|X-a| = \int_{-\infty}^{\infty} |X-a| f_X(x) dx = \int_{-\infty}^{a} (a-X) f_X(x) dx + \int_{a}^{\infty} (X-a) f_X(x) dx$$

Then we have the first order condition as:

$$\frac{dE|X-a|}{da} = \int_{-\infty}^{a} f_X(x) dx + \int_{a}^{\infty} -f_X(x) dx = 0$$

which implies $F_X(a) = 1 - F_X(a)$, so that minimizer a is a median(m). Still, you have to verify the second order condition to complete the proof, which comes from the fact that f_X is positive on any point included in its support:

$$\frac{d^2E|X-a|}{da^2} = f_X(a) + f_X(a) = 2f_X(a) > 0$$

This shows that f_X is convex, and it has a global minimum at $a = m$.

Remark: Refer to Leibniz integral rule if you are not used to this kind of differentiation.

6. **Problem 8 in Lecture 2**
Let's go through each of the small questions here:

   a. As long as $\mu_2 \neq 0$, it is sufficient to show that $\mu_3 = 0$.
   Note that if a density is symmetric around a, then $f_X(x) = f_X(2a - 2x + x) =$

$f_X(2a-x)$. Now as $F_X(a)=\frac{1}{2}$,

$$\begin{aligned}E(X)=\int_{-\infty}^{\infty}xf_X(x)dx &= \int_{-\infty}^{a}xf_X(x)dx+\int_{a}^{\infty}xf_X(x)dx\\ &= \int_{-\infty}^{a}xf_X(2a-x)dx+\int_{a}^{\infty}xf_X(x)dx\\ &= \int_{a}^{\infty}(2a-x)f_X(x)dx+\int_{a}^{\infty}xf_X(x)dx\\ &= \int_{a}^{\infty}2af_X(x)dx\\ &= 2a-2aF_X(a)=2a-a=a\end{aligned}$$

Then we can see that, by using integration by substitution:

$$\begin{aligned}\mu_3 &= \int_{-\infty}^{a}(x-a)^3f_X(x)dx+\int_{a}^{\infty}(x-a)^3f_X(x)dx\\ &= \int_{-\infty}^{a}(x-a)^3f_X(x)dx+\int_{-\infty}^{a}(a-x)^3f_X(2a-x)dx\\ &= \int_{-\infty}^{a}(x-a)^3f_X(x)dx-\int_{-\infty}^{a}(x-a)^3f_X(x)dx=0\end{aligned}$$

b. We can calculate $E(X)$, μ_2, μ_3 as following:

$$E(X)=\int_{0}^{\infty}xf_X(x)dx=[-xe^{-x}]_0^{\infty}+\int_0^{\infty}e^{-x}dx=1$$

$$\begin{aligned}\mu_2=\int_0^{\infty}(x-1)^2e^{-x}dx &= [-(x-1)^2e^{-x}]_0^{\infty}+\int_0^{\infty}2(x-1)e^{-x}dx\\ &= 1+[-2(x-1)e^{-x}]_0^{\infty}+\int_0^{\infty}2e^{-x}dx\\ &= 1-2+2=1\end{aligned}$$

$$\begin{aligned}\mu_3=\int_0^{\infty}(x-1)^3e^{-x}dx &= [-(x-1)^3e^{-x}]_0^{\infty}+\int_0^{\infty}3(x-1)^2e^{-x}dx\\ &= -1+3\mu_2=2\end{aligned}$$

Then we can see that $\alpha_3=\frac{\mu_3}{\mu_2^{3/2}}$=2

It is useful to know that in terms of the convergence speed, a^x or a^{-x} is faster than x^a. For instance, $(x-1)^2e^{-x}$ evaluated at $x=\infty$ is zero because e^{-x} converges to zero faster than $(x-1)^2$ converges to ∞.

c. Given those PDFs, all of them are symmetric around zero, so $\mu_1=\mu_3=0$.

(1)

$$\begin{aligned}\mu_2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} ([-xe^{-x^2/2}]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1\end{aligned}$$

The last line comes from the property of PDF.

$$\mu_4 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^4 e^{-x^2/2} dx = [\frac{1}{\sqrt{2\pi}} (-x^3) e^{-x^2/2}]_{-\infty}^{\infty} + 3\mu_2 = 3$$

Then we have $\alpha_4 = \frac{\mu_4}{\mu_2^2} = 3$

(2)

$$\mu_2 = \int_{-1}^{1} \frac{x^2}{2} dx = \frac{1}{3}$$
$$\mu_4 = \int_{-1}^{1} \frac{x^4}{2} dx = \frac{1}{5}$$

Then we have $\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{9}{5}$

(3)

$$\begin{aligned}\mu_2 &= \int_{-\infty}^{0} \frac{1}{2} x^2 e^x dx + \int_{0}^{\infty} \frac{1}{2} x^2 e^{-x} dx \\ &= [\frac{1}{2} x^2 e^x]_{-\infty}^{0} - \int_{-\infty}^{0} x e^x dx + [-\frac{1}{2} x^2 e^{-x}]_{0}^{\infty} + \int_{0}^{\infty} x e^{-x} dx \\ &= [-xe^x]_{-\infty}^{0} + \int_{-\infty}^{0} e^x dx + [-xe^{-x}]_{0}^{\infty} + \int_{0}^{\infty} e^{-x} dx \\ &= 2 \int_{0}^{\infty} e^{-x} dx = 2\end{aligned}$$

Since e^{-x} is a PDF from (b).

$$\begin{aligned}\mu_4 &= \int_{-\infty}^{0} \frac{1}{2} x^4 e^x dx + \int_{0}^{\infty} \frac{1}{2} x^4 e^{-x} dx \\ &= [\frac{1}{2} x^4 e^x]_{-\infty}^{0} + \int_{-\infty}^{0} 2x^3 e^x dx + [-\frac{1}{2} x^4 e^{-x}]_{0}^{\infty} + \int_{0}^{\infty} 2x^3 e^{-x} dx \\ &= [2x^3 e^x]_{-\infty}^{0} + \int_{-\infty}^{0} 6x^2 e^x dx + [-2x^3 e^{-x}]_{0}^{\infty} + \int_{0}^{\infty} 6x^2 e^{-x} dx \\ &= 12\mu_2 = 24\end{aligned}$$

Then we have $\alpha_4 = \frac{\mu_4}{\mu_2^2} = 6$

Then we can see that peakness of the third case is the highest, and of the second case is the lowest.