# Problem Set #4

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## Question 1

Let $X$ be a choice set and $\succsim$ be a complete and transitive preference relation on $X$. Let

$$C(A, \succsim) = \{x \in A | x \succsim y \forall y \in A\}$$

Be the choice rule induced by $\succsim$.

1. Assume $A \subseteq X$ and $B \subseteq X$, where $x, y \in A \bigcap B$, $x \in C(A)$, and $y \in C(B)$
2. $y \in A \bigcap B \Rightarrow y \in A$
3. $x \in C(A) \rightarrow x \succsim z \forall z \in A$. Thus, $x \succsim y$
4. $x \in A \bigcap B \Rightarrow x \in B$
5. $y \in C(B) \Rightarrow y \succsim z \forall z \in B$
6. Since $\succsim$ is transitive, $x \succsim y$ and $y \succsim z \forall z \in B$ implies that $x \succsim z \forall z \in B$. Thus, $x \in C(B)$
7. Since $y \in C(B)$ and $x \in B$, $y \succsim x$. $y \in A$ and, since $x \in C(A)$, $x \succsim z \forall z \in A$. Thus, by the transitivity of $\succsim$, $y \succsim z \forall z \in A$. Therefore, $y \in C(A)$

$\therefore$ If $A, B \subseteq X$ where $x, y \in A \bigcap B$, then $x \in C(A) \wedge y \in C(B) \Rightarrow x \in C(B) \wedge y \in C(A)$ $\blacksquare$

## Question 2

Let $X$ be a choice set and $C : \mathcal{P}(X) \to \{(X)$ be a nonempty choice rule that satisfies WARP. Define the preference relation defined on $X$, $\succsim_C$, as

$$x \succsim_C y \iff \exists A \subseteq X \text{ s.t. } x, y \in A \wedge x \in C(A)$$

**Completeness.**

1. Let $x, z \in A \subseteq X$, where $x \in C(A)$
2. Suppose $\neg(x \succsim_C z) \wedge \neg(z \succsim_C z)$
3. By the definition of $\succsim_C$, $A \subseteq X \wedge x, z \in A \wedge x \in C(A) \to x \succsim_C z$
4. By 2 and 3, $\neg(x \succsim_C z) \wedge (x \succsim_C z)$

   $\therefore$ by contradiction, $\succsim_C$ has complete preferences on $X$

**Transitivity.**

1. Suppose $x \succsim_C y$ and $y \succsim_C z$.
2. By the definition of $\succsim_C$, $\exists A \subseteq X$ s.t. $x, y \in A \wedge x \in C(A)$ and $\exists B \subseteq X$ s.t. $x, z \in B \wedge x \in C(B)$
3. Clearly, $y \in A \bigcap B$. By WARP, if $x \in B$, then $x \in C(B)$. By the definition of $\succsim_C$, since $B \subseteq X \wedge x, z, \in B \wedge x \in C(B)$, it must be the case that $x \succsim_C Z$
4. Thus, $x \succsim_C y \wedge y \succsim_C z \Rightarrow x \succsim_C z$

   $\therefore \succsim_C$ has transitive preferences on $X$

$\mathbf{C}\left(\cdot, \succsim_{\mathbf{C}}\right) = \mathbf{C}$

1. Suppose $A \subseteq X$ is nonempty, where $x, y \in A$, $x \in C(A)$, and $y \in C(A, \succsim_C$ ).
2. By the definition of $\succsim_C$, $x \succsim_C y\ \forall y \in A$. Thus, $x \in C\left(A, \succsim_C\right)$. Thus,

$$x \in C(A) \Rightarrow x \in C\left(A, \succsim_C\right)$$

3. $y \in C\left(A, \succsim_C\right) \Rightarrow y \succsim_C x \forall x \in A$. By the definition of $\succsim_C$, $y \in C(A)$. Thus,

$$y \in C\left(A, \succsim_C\right) \Rightarrow y \in C(A)$$

4. By (2) and (3), $x \in C(A) \iff x \in C\left(A, \succsim_C\right)$

   $\therefore C\left(\cdot, \succsim_C\right) = C$

$\therefore \succsim_C$ is complete and transitive, and $C\left(\cdot, \succsim_C\right) = C$ ■

# Question 3

Let $X$ be finite and $\succsim$ be a complete and transitive relation on $X$.

## (a)

Suppose $A \neq \emptyset$ and $A \subseteq X$.

1. *Base step.* Let $A = \{x\}$. $x \sim x$, so $x \succsim x$. Thus, $C(A, \succsim) = \{x\} \neq \emptyset$
2. *Induction step.* Let $A = \{x_1, ..., x_n\}$ and assume $C(A, \succsim) \neq \emptyset$. Then, $\exists x^* \in A$ s.t. $x^* \succsim y \forall y \in A$. Say $\exists x_{n+1} \in X$. Then, since $\succsim$ is complete, either $x_{n+1} \succsim x^*$ or $x^* \succsim x_{n+1}$ (or both).
   (a) If $x_{n+1} \succsim x^*$, then, since $\succsim$ is transitive and $x^* \succsim y \forall y \in A$, $x_{n+1} \in C(A \bigcup \{x_{n+1}\}, \succsim)$
   (b) If $x^* \succsim x_{n+1}$, then $x^* \succsim y \forall y \in A \bigcup \{x_{n+1}\}$. Thus, $x^* \in C(A \bigcup \{x_{n+1}\}, \succsim)$

$\therefore$ by induction, $A \neq \emptyset \Rightarrow C(A, \succsim) \neq \emptyset$ ■

## (b)

1. *Base step.* Let $A = \{x\}$. Let $u(x) = 1$. Then, $\forall x, y \in A$, $x \succsim y \Rightarrow u(x) \geq u(y)$
2. *Induction step.* Now let $|A| = n$. Since $\succsim$ is complete and transitive, $A$ can be sorted such that

   $$A = \{x_1, ..., x_n\}, \text{ where } x_n \succsim x_{n-1} \succsim ... \succsim x_2 \succsim x_1$$

   Define $u(x_i) = i$ for all $i \in \{1, ..., n\}$.

   Suppose $x_{n+1} \in X$. Since $\succsim$ is complete, then $x_{n+1} \succsim x_n$ or $x_n \succsim x_{n+1}$, or both.

   (a) If $x_{n+1} \succsim x_n$, define $u(x_{n+1}) = n + 1$. Then, $\forall x, y \in A \bigcup \{x_{n+1}\}$, $x \succsim y \Rightarrow u(x) \geq u(y)$.
   (b) If $\neg(x_{n+1} \succsim x_n)$ and $x_n \succsim x_{n+1}$, then set $u(x_n) = n+1$. If $x_{n+1} \succsim x_{n-1}$, then set $u(x_{n+1}) = n$ and leave the utility mappings for $i < n$ unchanged. If $\neg(x_{n+1} \succsim x_{n-1})$, then continue this reassigment process until, for some $i$, $x_{n+1} \succsim x_i$. Then, set $u(x_{n+1}) = i + 1$ and $u(x_j) = j + 1$ $\forall j > i$ and leave the utility mappings unchanged for all $x_k$, where $k <= i$. If $x_1 \succsim x_{n+1}$ and $\nexists i \in \{1, ..., n\}$ such that $x_{n+1} \succsim x_i$, then set $u(x_{n+1}) = 1$ and set $u(x_i) = i + 1$ for all $x_i$, where $i \in \{1, ..., n\}$. Then, $\forall x, y \in A \bigcup \{x_{n+1}\}$, $x \succsim y \Rightarrow u(x) \geq u(y)$

$\therefore$ When $X$ is finite, $\exists u : X \to \mathbb{R}$ such that $\forall x, y \in X$, $x \succsim y \Rightarrow u(x) \geq u(y)$ ■