

Problem Set #2

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Question 1

- (a) To determine $Pr\left(\sum_{i=1}^n (X_i^* - \bar{X}_i^*)^2 = 0 | \{W_i\}\right)$, we need only find $Pr\left(\sum_{i=1}^n X_i^* - \bar{X}_i^* = 0 | \{W_i\}\right)$. Since $X_i \in \{0, 1\}$, $X_i^* \in \{0, 1\}$, so:

$$Pr\left(X_i^* - \bar{X}_i^* = 0 | \{W_i\}\right) = Pr\left(X_i^* = X_j^* \forall i, j | \{W_i\}\right)$$

Where $Pr(X_i^* = X_j^*) \geq \frac{1}{2} \forall i, j$. Then,

$$Pr\left(X_i^* = X_j^* \forall i, j | \{W_i\}\right) = \left(\frac{1}{2}\right)^n$$

Where, for finite n , $2^{-n} > 0$.

- (b) If the conditions for the Lindeberg CLT are satisfied, then $\hat{\theta}_n^*$ converges to the same distribution as $\hat{\theta}_n$. Thus, we must show that

$$\sup \frac{1}{n} \sum_{i=1}^n |X_i(s)|^{2+\delta} < \infty$$

For some $\delta > 0$. By the domain of X , this is necessarily satisfied.

- (c) Since $\Phi^{-1}(\cdot)$ is a constant,

$$\frac{F_{\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n) | \{W_i\}}^{-1}(0.75 | \{W_i\}) - F_{\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n) | \{W_i\}}^{-1}(0.25 | \{W_i\})}{\Phi^{-1}(0.75) - \Phi^{-1}(0.25)} \rightarrow_p \frac{F_{\mathcal{N}(0, \sigma_u^2 / \sigma_X^2)}^{-1}(0.75) - F_{\mathcal{N}(0, \sigma_u^2 / \sigma_X^2)}^{-1}(0.25)}{\Phi^{-1}(0.75) - \Phi^{-1}(0.25)}$$

Thus,

$$v_n^{igr} \rightarrow_{a.s.} \frac{\frac{\sigma_u}{\sigma_X} (\Phi^{-1}(0.75) - \Phi^{-1}(0.25))}{\Phi^{-1}(0.75) - \Phi^{-1}(0.25)} = \sigma_u / \sigma_X$$

(d) First, we can solve for $Var(\hat{\theta}_n^\dagger|\{W_i\})$:

$$\begin{aligned}
\mathbb{E} \left[\hat{\theta}_n^\dagger | \{W_i\} \right] &= \hat{\theta}_n + \frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \mathbb{E} [\varepsilon_i | \{W_i\}] \\
&= \hat{\theta}_n + \frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
\mathbb{E} \left[\hat{\theta}_n^\dagger | \{W_i\} \right]^2 &= \hat{\theta}_n^2 + 2\hat{\theta}_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} + \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \right)^2 \\
(\hat{\theta}_n^\dagger)^2 &= \hat{\theta}_n^2 + 2\hat{\theta}_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i \varepsilon_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} + \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i \varepsilon_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \right)^2 \\
Var(\hat{\theta}_n^\dagger | \{W_i\}) &= \mathbb{E} \left[(\hat{\theta}_n^\dagger)^2 | \{W_i\} \right] - \mathbb{E} \left[\hat{\theta}_n^\dagger | \{W_i\} \right]^2 \\
&= \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \right)^2 \mathbb{E} [\varepsilon_i^2 | \{W_i\}] \\
&= \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \right)^2
\end{aligned}$$

By the law of large numbers, this converges to its unconditional expectation. By assumption,

$$\left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n) \hat{u}_i}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \right)^2 \xrightarrow{a.s.} \frac{\mathbb{E} [(X - \mathbb{E}[X])^2]}{\mathbb{E} [(X - \mathbb{E}[X])^2]^2} \mathbb{E} [u^2] = \sigma_u^2 / \sigma_X^2$$

(e) The results of the simulation are below. See `edgel.ps2.jl` and `functions.jl` for the code that generates them.

	Mean	SD	Coverage
<i>s.e. asy</i>	0.322	0.089	0.907
<i>s.e. bt</i>	0.152	0.075	0.595
<i>s.e. IQR</i>	0.366	0.071	0.964
<i>s.e. bt-r</i>	0.052	0.027	0.218
<i>s.e. sbt-normal</i>	0.111	0.070	0.443
<i>s.e. sbt-rad</i>	0.111	0.070	0.441

Question 2

(a) The inequalities for this data generating process are:

$$\begin{aligned}
[g(0,0) - g(1,0)](-1) &\geq 0 \\
[g(0,0) - g(1,1)](-1 - \theta) &\geq 0 \\
[g(0,0) - g(0,1)](-\theta) &\geq 0 \\
[g(1,0) - g(0,0)] &\geq 0 \\
[g(1,0) - g(0,1)](1 - \theta) &\geq 0 \\
[g(1,0) - g(1,1)](-\theta) &\geq 0 \\
[g(1,1) - g(1,0)](\theta) &\geq 0 \\
[g(1,1) - g(0,0)](1 + \theta) &\geq 0 \\
[g(1,1) - g(0,1)] &\geq 0 \\
[g(0,1) - g(0,0)](\theta) &\geq 0 \\
[g(0,1) - g(1,1)](-1) &\geq 0
\end{aligned}$$

Note that the trivial inequalities ($0 \geq 0$) are excluded. The unique set of inequalities, letting $g(x_1, x_2) = F(x_1 + \theta x_2)$, is:

$$[F(\theta) - F(0)]\theta \geq 0 \quad (1)$$

$$[F(1) - F(0)] \geq 0 \quad (2)$$

$$[F(1 + \theta) - F(0)](1 + \theta) \geq 0 \quad (3)$$

$$[F(1) - F(\theta)](1 - \theta) \geq 0 \quad (4)$$

$$[F(1 + \theta) - F(1)]\theta \geq 0 \quad (5)$$

$$[F(1 + \theta) - F(\theta)] \geq 0 \quad (6)$$

Note that this set of inequalities holds for all values of θ .

(b) If $\theta = 1$, then

$$g(1,0) = g(0,1) = F(1)$$

$$g(0,0) = F(0)$$

$$g(1,1) = F(2)$$

Thus, the only useful moment inequalities for identification yield:

$$\hat{\Theta}_n = \{\theta : \theta \geq 0\}$$

(c) In this case, any variation in Y is due entirely to variation in ε . Then, for each (x_1, x_2) , $Var(Y) = Var(\varepsilon)$ and $\mathbb{E}[Y|(x_1, x_2)] = F(x_1 + \theta x_2)$. Thus, the standard error of $\hat{\theta}_n$ is $\frac{1}{n} \widehat{Var(Y)}$, where $\widehat{Var(Y)}$ is asymptotically normal.