Problem Set #1

Danny Edgel
Econ 761: Industrial Organization Theory
Fall 2021

September 26, 2021

1. Consider a market in which the goods are homogenous.

   (a) The elasticity of demand, $\varepsilon < 0$, can be written as:

   $$\varepsilon = (P'(Q))^{-1}\frac{P(Q)}{Q}$$

   Thus, letting ε remain constant, we can derive:

   $$\begin{aligned} P(Q) &= QP'(Q)\varepsilon \\ P'(Q) &= (QP''(Q) + P'(Q))\,\varepsilon \end{aligned}$$

   Thus, with $P'(Q) < 0$ and $\varepsilon < 0$, $QP''(Q) + P'(Q) > 0$ for all Q.

   (b) Under Cournot competition, each firm, i, solves the following problem:

   $$\max_{q_i} \Pi_i = P(Q)q_i - c(q_i), \qquad Q = \sum_{j=1}^{N} q_j$$

   Which yields the following FOC, which is identical for all firms:

   $$P(Q) + P'(Q)q_i = c'(q_i) \Rightarrow q_i = (P'(Q))^{-1}\left(c'(q_i) - P(Q)\right)$$

   Since cost functions are identical by assumption, $q_i = q_j = q\ \forall i, j$ in equilibrium, so we use the implicit function theorem to solve:[1]

   $$\begin{aligned} qP'(Nq) + c'(q) &= P(Nq) \\ \frac{\partial q}{\partial N}\left[P'(Nq) + c''(q)\right] &= \left(q + N\frac{\partial q}{\partial N}\right)\left[P'(Nq) - qP''(Nq)\right] \\ \frac{\partial q}{\partial N} &= \frac{q\left[P'(Nq) - qP''(Nq)\right]}{P'(Nq) + c''(q) + N\left[qP''(Nq) - P'(Nq)\right]} \end{aligned}$$

[1] Due to algebraic errors, I had to redo this several times, spending a long time on it. As a result, many intermediate steps are omitted below.

By assumption (A1), we know $c''(q) \geq P'(Nq)$, and by assumption (A2), we know $-qP''(Nq) \geq P'(Nq)$. Thus, we can reduce the equation as follows:

$$\frac{\partial q}{\partial N} \leq \frac{q^2 P''(Nq)}{c''(q) - NP'(Nq)}$$
$$\frac{\partial q}{\partial N} \leq \frac{q^2 c''(q)}{c''(q) - Nc''(q)}$$
$$\frac{\partial q}{\partial N} \leq \frac{q^2}{1-N} < 0 \quad \forall N > 1$$

Since demand slopes downward and $Q = Nq$, an increase in q necessarily increases Q, decreasing price. Thus, equilibrium price and price per firm quantity are decreasing in N.

2. (a) Each player, $i \in \{1, 2\}$ chooses $b_i \in \mathbb{R}_+$ to maximize:

$$\pi_i(b_i, b_j) = \begin{cases} V - b_i, & b_i > b_j \\ \frac{1}{2}(V - b_i), & b_i = b_j \\ 0, & b_i < b_j \end{cases}$$

Since payoffs and valuations are symmetric, $b_i = b_j$ in equilibrium. For all $b_i = b_j < V$, each player has an incentive to raise their bid. Thus, the unique equilibrium is:

$$b_1^* = b_2^* = v \qquad\qquad \pi_1^* = \pi_2^* = 0$$

(b) In all all-pay auction, player 1's payoff function is:

$$\pi_1(b_1, b_2) = \begin{cases} V - b_1, & b_1 > b_2 \\ \frac{1}{2}V - b_1, & b_1 = b_2 \\ -b_1, & b_1 < b_2 \end{cases}$$

(c) suppose $\exists$ a pure-strategy equilibrium with bids (b_1^*, b_2^*). Since Payoffs and valuations are identical, any pure strategy equilibrium has $b_1^* = b_2^* = b^*$. Then, $\pi^* = \frac{1}{2}V - b$. Thus, either player could improve their payoff by deviating to $b_i = b^* + \varepsilon$ for $\varepsilon > 0$. Thus, $b_1 = b_2$ cannot be a pure-strategy Nash equilibrium.[2]

(d) A mixed-strategy Nash equilibrium is a pair of distribution functions, $(F_1(b), F_2(b))$, from which each player draws their bid. Since bids must be weakly positive, $F_i(0) = 0$. Since payoffs are negative for all

[2]The nonexistence of a $b_1 \neq b_2$ equilibrium is trivial.

$b > V$ but zero for a bid of zero, $F_i(V) = 1$. Each player i chooses F_i to maximizes expected payoff:[3]

$$\mathbb{E}\left[\pi_i(b_i, b_j)\right] = F_j(b_i)V - b_i$$

From the first-order condition of this problem, we can obtain the symmetric equilibrium distribution function:

$$\begin{aligned} V f_j(b_i) - 1 &= 0 \\ f_j(b_i) &= \frac{1}{V} \\ F_j(b) &= \int_0^b \frac{1}{V} dx = \frac{b}{V} \end{aligned}$$

Since payoffs and valuations are constant, $F_i^*(b) = F_j^*(b) = F^*(b)$. Thus, the mixed-strategy equilibrium is for each player to submit a uniformly random bid between 0 and V. The seller's expected revenue is:

$$R = 2\mathbb{E}\left[b^*\right] = 2\int_0^V \left(\frac{1}{V}\right) b db = \frac{1}{V}\left[b^2\right]_0^V = V$$

(e) If the seller sets some reserve price $R \in (0, V)$, then the lower bound of the equilibrium distribution will be truncated such that $F^*(b)$ is instead be a uniform distribution from R to V. Intuitively, this would increase the seller's revenue by increasing the mean of the equilibrium bid distribution.

[3] Using the same logic as in (c), we can rule out any mass points, since such mass points will exist in both players' distributions, and either player could improve their payoffs by shifting the mass to a slightly higher bid.

3. (a) The marginal consumers on either side of Esquires are indifferent to purchasing a cup of coffee from Starbucks and Esquires. Letting p_i represent the price from the nearest Starbucks for $i \in \{0, 1\}$ and $x_i \in [0, 1]$ represent the location of the consumer on Main Street, where $i = 1$ is the consumer closer to the Starbucks on the end of main street:

$$
\begin{aligned}
v - x_0^2 - p_0 &= v - (.5 - x_0)^2 - q \\
v - (1 - x_1)^2 - p_1 &= v - (x_1 - .5)^2 - q
\end{aligned}
$$

Solving for x_i yields:

$$
x_0 = q - p_0 + \frac{1}{4} \qquad\qquad x_1 = p_1 - q + \frac{3}{4}
$$

(b) Assuming Starbucks can set different prices at each location, the firms' optimization problems are:

$$
\begin{aligned}
q(p_0, p_1) &= \underset{q}{\operatorname{argmax}}\ q \left[p_1 - q + \frac{3}{4} - \left(q - p_0 + \frac{1}{4}\right)\right] \\
&= \frac{1}{4}\left(p_0 + p_1\right) + \frac{1}{8} \\
p(q) &= \underset{p_0, p_1}{\operatorname{argmax}}\ p_0 \left[q - p_0 + \frac{1}{4}\right] + p_1 \left[1 - \left(p_1 - q + \frac{3}{4}\right)\right] \\
&= \begin{pmatrix} \frac{1}{2}q + \frac{1}{8} \\ \frac{1}{2}q + \frac{1}{8} \end{pmatrix}
\end{aligned}
$$

(c) Since $p_0 = p_1$ in equilibrium, Esquires's best response function can be simplified as $\frac{1}{2}p + \frac{1}{8}$. Then, we can solve for the equilibrium as follows:

$$
\begin{aligned}
q = q(p(q)) &= \frac{1}{2}\left(\frac{1}{2}q + \frac{1}{8}\right) + \frac{1}{8} \\
\Rightarrow q^* &= \frac{1}{4} \\
p^* = p(q^*) &= \frac{1}{2}\left(\frac{1}{8}\right) + \frac{1}{8} = \frac{1}{4}
\end{aligned}
$$

(d) Assume that the Starbucks at the end of the street swaps with Esquires. Then, Starbucks's best response function becomes:

$$
p(q) = \underset{p_0, p_1}{\operatorname{argmax}}\ p_0 \left[p_1 - p_0 + \frac{1}{4}\right] + p_1 \left[1 - \left(q - p_1 + \frac{3}{4} - \left(p_1 - p_0 + \frac{1}{4}\right)\right)\right]
$$