Econ 711 – Fall 2020 – Problem Set 7

Question. A Risky Investment

You have wealth $w > 0$ and preferences over lotteries represented by a von Neumann-Morgenstern expected utility function with Bernoulli utility u which is strictly increasing, twice differentiable, and weakly concave. Your friend wants you to invest in his startup; you can choose any amount $a \leq w$ to invest, and your investment will either triple in value (with probability p) or become worthless (with probability $1-p$). Your expected utility if you invest a is therefore

$$U(a) \quad = \quad pu(w-a+3a)+(1-p)u(w-a) \quad = \quad pu(w+2a)+(1-p)u(w-a)$$

(a) *Show that if u is linear, then you invest all your wealth if $p > \frac{1}{3}$, and nothing if $p < \frac{1}{3}$.*

Since u is differentiable, we can calculate

$$U'(a) \quad = \quad 2pu'(w+2a)-(1-p)u'(w-a) \tag{1}$$

which we'll use a lot. If u is linear, then $u'(\cdot) = b$ a positive constant, so

$$U'(a) \quad = \quad 2pb-(1-p)b \quad = \quad (3p-1)b$$

So if $p > \frac{1}{3}$, U is strictly increasing in a and you'll invest as much as you can; if $p < \frac{1}{3}$, U is strictly decreasing in a, and you'll invest as little as you can (0).

From here on, assume $p > \frac{1}{3}$, so the expected value of the investment is positive; and assume that you are strictly risk-averse ($u'' < 0$).

(b) *Show that it's optimal to invest a strictly positive amount. (You can do this by showing that $U'(0) > 0$ – the marginal expected utility of increasing a is positive when $a = 0$.)*

Returning to (1), we can plug in $a = 0$ and get

$$U'(0) \quad = \quad 2pu'(w)-(1-p)u'(w) \quad = \quad (3p-1)u'(w)$$

If $p > \frac{1}{3}$, this is strictly positive, so you'll always invest strictly more than zero.

(c) *Show that $U(a)$ is strictly concave in a, so that except at a corner solutions, the first-order condition is necessary and sufficient to find a^*.*

Differentiating (1) a second time,

$$U''(a) \quad = \quad 4pu''(w+2a)+(1-p)u''(w-a) \quad < \quad 0$$

since by assumption $u'' < 0$.

(d) *Show that if $u'(0)$ is infinite, it's not optimal to invest all your wealth; and that if $u'(0)$ is finite, then there's a cutoff $\bar{p}$ such that it's optimal to invest all your wealth if $p \geq \bar{p}$.*

Again return to (1), and plug in $a = w$ to get

$$U'(w) \quad = \quad 2pu'(3w) - (1-p)u'(0)$$

Since U is strictly concave in a, if this is weakly positive, then U' is strictly positive on $[0, w)$ and $a = w$ is optimal; and if this is strictly negative, $a < w$ is optimal.

If $u'(0)$ is infinite, then $U'(w)$ is unboundedly negative, so $a < w$ is optimal. If $u'(0)$ is finite, then $U'(w)$ is weakly positive if and only if

$$\begin{aligned} 2pu'(3w) &\geq (1-p)u'(0) \\ \frac{2u'(3w)}{u'(0)} &\geq \frac{1-p}{p} \\ 1 + \frac{2u'(3w)}{u'(0)} &\geq \frac{1}{p} \\ \frac{1}{1+\frac{2u'(3w)}{u'(0)}} &\leq p \end{aligned}$$

and therefore investing all your wealth is optimal when $p \geq 1 \Big/ \left(1 + \frac{2u'(3w)}{u'(0)}\right)$.

From here on, assume that either $u'(0)$ is infinite or $p \in \left(\frac{1}{3}, \bar{p}\right)$, so the optimal level of investment a^* is strictly positive but below w.

(e) *Show that if $u(x) = 1 - e^{-cx}$ (the Constant Absolute Risk Aversion or CARA utility function), your optimal investment a^* does not depend on w.*

With CARA utility, $u'(x) = ce^{-cx}$; plugging this into (1) gives

$$U'(a) \quad = \quad 2pce^{-c(w+2a)} - (1-p)ce^{-c(w-a)} \quad = \quad ce^{-cw}\left(2pe^{-2ca} - (1-p)e^{ca}\right)$$

Since the first-order condition determines a^*, the optimal investment level is the solution to $2pe^{-2ca} = (1-p)e^{ca}$, which does not depend on w.

(f) (HARD) *For general u, show that if your Coefficient of Absolute Risk Aversion $A(x) = -\frac{u''(x)}{u'(x)}$ is decreasing, you invest more as w increases.*

As noted in the hint, it's enough to show that $U'(a)$ is increasing in w at $a = a^*(w)$. We can calculate

$$\frac{d}{dw}(U'(a)) \quad = \quad 2pu''(w+2a) - (1-p)u''(w-a)$$

and divide and multiply by $-u'$ terms to get

$$\frac{d}{dw}(U'(a)) \quad = \quad -2pu'(w+2a)\left(-\frac{u''(w+2a)}{u'(w+2a)}\right) + (1-p)u'(w-a)\left(-\frac{u''(w-a)}{u'(w-a)}\right)$$

Next, we note that at $a = a^*(w)$, the first-order condition holds, or

$$2pu'(w+2a) \quad = \quad (1-p)u'(w-a)$$

and therefore

$$\begin{aligned}\frac{d}{dw}(U'(a))\bigg|_{a=a^*(w)} &= -(1-p)u'(w-a)\left(-\frac{u''(w+2a)}{u'(w+2a)}\right) + (1-p)u'(w-a)\left(-\frac{u''(w-a)}{u'(w-a)}\right) \\ &= (1-p)u'(w-a)\left[-A(w+2a)+A(w-a)\right]\end{aligned}$$

If A is decreasing, then $A(w+2a) < A(w-a)$, and therefore the term in square brackets is positive; since u is increasing, $u' > 0$, so the whole expression is positive, and therefore $U'(a)$ is strictly increasing in w at the optimum, which gives the result.

Now reframe the question as deciding what fraction t of your wealth to invest; writing $a = tw$,

$$U(t) = pu(w(1+2t)) + (1-p)u(w(1-t))$$

(g) *Show that if $u(x) = \frac{1}{1-\rho}x^{1-\rho}$, with $\rho \leq 1$ and $\rho \neq 0$ (the Constant Relative Risk Aversion or CRRA utility function), you invest the same fraction of your wealth regardless of w.*

Calculating the first-order condition,

$$U'(t) = 2pwu'(w(1+2t)) - (1-p)wu'(w(1-t)) \tag{2}$$

Differentiating the CRRA utility function, $u'(x) = x^{-\rho}$, giving

$$U'(t) = 2pw\,(w(1+2t))^{-\rho} - (1-p)w\,(w(1-t))^{-\rho} = w^{1-\rho}\left[2p(1+2t)^{-\rho} - (1-p)(1-t)^{-\rho}\right]$$

As before, w has dropped out of the first-order condition – $a^*(w)$ is the solution to $2p(1+2t)^{-\rho} - (1-p)(1-t)^{-\rho} = 0$, which does not depend on w, so you invest the same fraction of your wealth regardless of w.

(h) *(HARD) For general u, show that if your Coefficient of Relative Risk Aversion $R(x) = -\frac{xu''(x)}{u'(x)}$ is increasing, you invest a smaller fraction of your wealth as w increases.*

This time, we'll show that $U'(t)$ is decreasing in w at the optimum t^*. Differentiating (2) w.r.t. w gives

$$\begin{aligned}\frac{d}{dw}\left(U'(t)\right) &= 2pu'(w(1+2t)) + 2pw(1+2t)u''(w(1+2t)) \\ &\quad -(1-p)u'(w(1-t)) - (1-p)w(1-t)u''(w(1-t))\end{aligned}$$

At $t = t^*$, the first-order condition holds, so $2pwu'(w(1+2t)) = (1-p)wu'(w(1-t))$, which means $2pu'(w(1+2t)) = (1-p)u'(w(1-t))$, so the first and third terms cancel each other; we can multiply and divide by u' terms and get

$$\begin{aligned}\frac{d}{dw}\left(U'(t)\right)\bigg|_{t=t^*} &= -2pu'(w(1+2t))\left(-w(1+2t)\frac{u''(w(1+2t))}{u'(w(1+2t))}\right) \\ &\quad +(1-p)u'(w(1-t))\left(-w(1-t)\frac{u''(w(1-t))}{u'(w(1-t))}\right)\end{aligned}$$

Once again using the fact that $2pu'(w(1+2t)) = (1-p)u'(w(1-t))$ at the optimum, this is

$$\frac{d}{dw}\left(U'(t)\right)\bigg|_{t=t^*} = (1-p)u'(w(1-t))\left[-R(w(1+2t)) + R(w(1-t))\right]$$

If $R = -x\frac{u''(x)}{u'(x)}$ is increasing, then $R(w(1+2t)) > R(w(1-t))$, so the whole expression is negative. So $U'(t)$ is decreasing in w at the optimum, and so t^* is decreasing in w: if you have increasing relative risk aversion, you invest a smaller fraction of your wealth as w grows.