Problem Set #6

Danny Edgel
Econ 703: Mathematical Economics I
Fall 2020

September 29, 2020

Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

The time Bob takes to walk to Happy Cow Farm[1] is given by:

$$T = \frac{1}{5}D_R + \frac{1}{3}D_F$$

Where D_R is the distance Bob travels on the road, and D_F is the distance he travels through the forest. Letting x represent the difference between D_R and the maximim distance Bob would travel on the road, we can solve the problem as:

$$\begin{aligned} \min_x \ \frac{1}{5}(12 - x) + \frac{1}{6}\sqrt{25 + x^2} \\ \frac{dT}{dx} &= 0 \\ \frac{x}{3\sqrt{25 + x^2}} &= \frac{1}{5} \\ 25x^2 &= 9(25 + x^2) \\ x^2 &= \frac{225}{16} \end{aligned}$$

Since x cannot be negative, and $\frac{x}{3\sqrt{25+x^2}} - \frac{1}{5}$ is non-decreasing for $x \geq 0$, we know that only $x = \frac{15}{4}$ minimizes T. Thus,

$$T = \frac{1}{5}(12 - \frac{15}{4}) + \frac{1}{6}\sqrt{25 + \left(\frac{15}{4}\right)^2} = \frac{56}{15}$$

Thus, the shortest amount of time it will take Bob to walk to Happy Cow Farm is 224 minutes, or 3 hours and 44 minutes.

[1] Whether "Happy Cow" is an appropriate name for a place that exists primarily to harvest cows is an ethical one and thus beyond the scope of this question.

Question 2

It is not possible for x_0 to be a local optimum of f. Suppose $f'(x_0) = 0$. Then, x_0 is an inflection point of f but not a local optimum.

Proof.

1. Let $x_1 < x_0 < x_2 \in B_\varepsilon(x_0)$.[2] By assumption, $f'(x) < 0$ for all $x \in B_\varepsilon(x_0)$.
2. Since f is differentiable for all $x \in B_\varepsilon(x_0)$, f is also continuous $\forall x \in B_\varepsilon(x_0)$. Thus, $f(x_1) > f(x_2)$ and, by the mean value theorem, $f(x_1) > f(x_0) > f(x_2)$

$\therefore$ x_0 is not a local optimum of f $\blacksquare$

Question 3

By the chain rule, for $\beta \in \{r, s, t\}$, $\frac{\partial w}{\partial \beta} = \sum_{\alpha \in \{x,y,z\}} \frac{\partial w}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \beta}$. Thus, (r+2s+t) (2r+3s+t) (3r+s+t)

$$\begin{aligned}
\frac{\partial w}{\partial r} &= y^2 z + 4xyz + 3xy^2 \\
&= (2r+3s+t)^2(3r+s+t) + 4(r+2s+t)(2r+3s+t)(3r+s+t) + 3(r+2s+t)(2r+3s+t)^2 \\
\frac{\partial w}{\partial s} &= 2y^2 z + 6xyz + xy^2 \\
&= (2r+3s+t)^2(3r+s+t) + 4(r+2s+t)(2r+3s+t)(3r+s+t) + 3(r+2s+t)(2r+3s+t)^2 \\
\frac{\partial w}{\partial t} &= y^2 z + 2xyz + xy^2 \\
&= (2r+3s+t)^2(3r+s+t) + 4(r+2s+t)(2r+3s+t)(3r+s+t) + 3(r+2s+t)(2r+3s+t)^2
\end{aligned}$$

Question 4

Let $f : X \to \mathbb{R}^n$ be continuously differentiable on $X \subset \mathbb{R}^n$. Then, for any $x, y \in X$ and $i, j \in \{1, ..., n\}$,

$$\frac{\partial f^i}{\partial x_j}(x) = \lim_{y_j \to x_j} \frac{f^i(y_j) - f^i(x_j)}{y_j - x_j}$$

Removing the limit and taking absolute values enables us to derive, when $y \in B_\varepsilon(x)$ for some $\varepsilon > 0$,

$$|f^i(y_j) - f^i(x_j)| \leq k|y_j - x_j|$$

[2] Since the domain of f is $\mathbb{R}$, the existence of x_1 and x_2 are trivially proven by the fact that $B_\varepsilon(x_0)$ contains an infinite number of elements for all $\varepsilon > 0$.

Where $k = |\frac{\partial f^i}{\partial x_j}(x)|$. Now let $k = \max_{j \in \{1,\dots,n\}} \left\{\frac{\partial f^i}{\partial x_j}(x)\right\}$. Then, for every $i \in \{1, ..., n\}$,

$$|f^i(y) - f^i(x)| \leq k|y_i - x_i|$$

Thus, if we let $k = \max_{i,j \in \{1,\dots,n\}} \left\{\frac{\partial f^i}{\partial x_j}(x)\right\}$, we can conclude that:

$$\sqrt{\sum_{i=1}^{n} \left(f(y_i) - f(x_i)\right)^2} \leq \sqrt{\sum_{i=1}^{n} k^2 \left(y_i - x_i\right)^2}$$

$$d(f(x), f(y)) \leq \sqrt{n}kd(x, y)$$

$\therefore f$ is locally Lipschitz on X $\blacksquare$

Question 5

We know that $f(1, 1) = 0$. $D_x f(x, y) = 5x - 2x + 1$, so $D_X f(1, 1) \neq 0$. Then the implicit function theorem applies, and we can calcluate:

$$\frac{\partial x(y)}{\partial y}\Big|_{y=1} = -(D_x f(1, 1))^{-1}(D_y(1, 1)) = -\frac{-3-2}{5-2+1} = \frac{5}{4}$$

Question 6

First, we must solve for the critical points of $f(x, y) = 2x^4 + y^2 - xy + 1$:

$$\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 8x^3 - y \\ 2y - x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By substituting $y = 8x^3$ into $x = 2y$, we can solve that this is satisfied when $x(16x^2 - 1) = 0$. Thus, the critical points are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1/4 \\ 1/8 \end{pmatrix}$ and $\begin{pmatrix} -1/4 \\ -1/8 \end{pmatrix}$. To determine whether these are maxima, minima, or saddle points, we must calculate the function's Hessian matrix:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial y \partial x} \\ \frac{\partial f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 24x^2 & -1 \\ -1 & 2 \end{pmatrix}$$

Then the determinant of H is $48x^2 - 1$. For each of our critical points, we can solve:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} : \ |H| = -1 < 0$$

$$\begin{pmatrix} 1/4 \\ 1/8 \end{pmatrix} : \ |H| = \frac{48}{16} - 1 = 2 > 0$$

$$\begin{pmatrix} -1/4 \\ -1/8 \end{pmatrix} : \ |H| = \frac{48}{16} - 1 = 2 > 0$$

Thus, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a saddle point. Since $\frac{\partial^2 f}{\partial x^2} > 0 \ \forall x \in \mathbb{R}$, $\begin{pmatrix} 1/4 \\ 1/8 \end{pmatrix}$ and $\begin{pmatrix} -1/4 \\ -1/8 \end{pmatrix}$ are local minima. To determine whether either of these points are global minima, we must determine the function's behavior at its limits:

$$\lim_{x \to \infty, y \to \infty} f(x, y) = \infty$$
$$\lim_{x \to -\infty, y \to \infty} f(x, y) = \infty$$
$$\lim_{x \to \infty, y \to -\infty} f(x, y) = \infty$$
$$\lim_{x \to -\infty, y \to -\infty} f(x, y) = \infty$$

Further, $f(1/4, 1/8) = f(-1/4, -1/8)$. Thus, both $\begin{pmatrix} 1/4 \\ 1/8 \end{pmatrix}$ and $\begin{pmatrix} -1/4 \\ -1/8 \end{pmatrix}$ are global minima.