Homework 4

Due Monday night, November 30, at midnight.

Feel free to work together on these problems, but each student needs to write up his/her own answers at the end, rather than directly copying from one master solution.

1. Two agents are bidding for a good in an auction. Their valuations v_1 and v_2 for the good are private information to them, and are drawn from the $U[0,1]$ distribution. The players submit simultaneous bids. Whoever offers the higher bid wins the auction, and ties are broken by a coin flip. Once a winner is declared, the payment is randomly determined. With probability p, the winner pays his own bid, and the loser pays nothing; with probability q, the winner pays the loser's bid, and the loser pays nothing; with probability $1-p-q$, the winner pays nothing, but the loser pays their own bid. Assume $1/2 < p+q \leq 1$.

   (a) Formally define this as a Bayesian game.

   (b) Suppose that player i conjectures that player j bids $b_j = b(v_j)$, where $b(\cdot)$ is a continuously differentiable, strictly increasing function. Write out player i's expected payoff from bidding b_j, given his own valuation v_i.

   (c) Solve for the symmetric linear Bayesian Nash equilibrium.

   (d) How does the equilibrium bidding strategy from part (c) change as p and q change? How does it change as $p \to 1$ (and $q \to 0$) and as $q \to 1$ (and $p \to 0$)? How does this equilibrium bidding strategy behave as $q \to 1/2$ and $p \to 0$?

2. Let players $N = \{1, 2\}$ play a game in which each agent has the pure strategy set $A_1 = A_2 = \{A, B\}$, with payoffs in the following table. β_1 and β_2 are constants, and may be positive, negative or zero.

1 \ 2	A	B
A	1,1	0,2
B	2,0	β_1, β_2

(a) Suppose $\beta_1 = \beta_2 = \beta$, and this is common knowledge. For every value of β, find all rationalizable strategies and all Nash equilibria. (*Hint: You only need consider the cases of* $\beta > 0$, $\beta < 0$ *and* $\beta = 0$.)

(b) Now suppose β_1 and β_2 are private information, randomly drawn from some distribution F with support on the entire real line. Each player i observes their own β_i, but not that of their opponent. Formally define this as Bayesian game, and find the Bayesian Nash equilibrium.

(c) Suppose a mediator wants to induce a correlated equilibrium. If $\beta_1, \beta_2 > 0$, what correlated equilibrium (or equilibria) can be achieved?

(d) Suppose a mediator wants to induce a correlated equilibrium that randomizes over outcomes (A,B), (B,A) and (A,A) with probabilities p, p and $1 - 2p$ respectively, where $p \in [0, 1/2]$. For what values of β_1 and β_2 is this desired randomization a correlated equilibrium? How does this constraint change with p?

3. Fifty clever game theorists are sitting at a conference (pre-pandemic). Out of them, ten have bad breath eyes and 40 do not. Per usual, no one is able to tell if their own breath is bad, but all can tell anyone else's bad breath. If someone knows his breath is bad, it is a dominant strategy to take the walk of shame to the elevator leading to the mouthwash and water fountain. An elevator arrives precisely every minute.

   A visitor arrives, sniffs, and leaves, announcing to all, "Wow, what bad breath!" Everyone correctly interprets this as "At least one game theorist has bad breath."

   What will happen after the announcement? Precisely specify the time it happens.

4. Arthur and Beatrix compete in a race. At the start of the race, both players are 6 steps away from the finish line. Who gets the first turn is determined by a toss of a fair coin; the players then alternate turns, with the results of all previous turns being observed before the current turn occurs.

   During a turn, a player chooses from these four options:

   (I) Do nothing at cost 0.

   (II) Advance 1 step at cost 2.

(III) Advance 2 steps at cost 7.

(IV) Advance 3 steps of at cost 15.

The race ends when the first player crosses the finish line. The winner of the race receives a prize payoff of 20, while the loser gets no prize. Finally, there is discounting: after each turn, payoffs are discounted by a factor of δ, where δ is less than but very close to 1.

(a) Find all subgame perfect equilibria of this game. (Hint: In all subgame perfect equilibria, a player's choice at a decision node only depends on the number of steps he has left and on the number of steps his opponent has left. To help take advantage of this you might want to write down a table.)

(b) Suppose that Arthur wins the coin toss. Compare his equilibrium behavior with his optimal behavior in the absence of competition. Provide intuition for any similarities or differences you find.

5. Prove directly (i.e., without appealing to the minmax theorem) that in a finite two-player zero-sum game of perfect information, there is a unique subgame perfect equilibrium payoff vector.

6. Two profit maximizing firms, A and B, are engaged in price competition against each other in a market. The demand curve for each firm $i \in \{A, B\}$ is

$$D_i(p_i, p_j) = d - p_i + \alpha p_j$$

where $\alpha \in (0, 1)$. Both firms choose prices to maximize profits, and have zero marginal or fixed costs.

(a) Suppose both firms must choose their prices simultaneously. Formalize this as a strategic normal form game and find the Nash equilibrium.

(b) Now suppose that firm A chooses its price, then firm B observes A's decision and chooses its own price. Using backward induction, solve for the subgame perfect equilibrium.

(c) How do prices, the profits of each firm, and total profits compare across these two games in equilibrium?