Problem Set 1 Solutions

Problem 1 in Lecture 1

There are at least two ways to prove it. The first one:

Proof. The proof has two parts:

1) First, show $A \cup B \subseteq (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c))$. For any $x \in A \cup B$, $x \in A$ or $x \in B$, which implies that only one of the following three scenario must happen: i) $x \in A$ and $x \in B$; ii) $x \in A$ and $x \notin B$; iii) $x \notin A$ and $x \in B$.
   - If it is case 1), $x \in A$ and $x \in B$, then $x \in A \cap B$, so $x \in A \cap B \cup ((A \cap B^c) \cup (B \cap A^c))$.
   - If it is case ii), $x \in A$ and $x \notin B$, then $x \in A \cap B^c$, so $x \in A \cap B \cup ((A \cap B^c) \cup (B \cap A^c))$.
   - If it is case iii), $x \notin A$ and $x \in B$, then $x \in A^c \cap B$, so $x \in A \cap B \cup ((A \cap B^c) \cup (B \cap A^c))$.

   Thus, for any $x \in A \cup B$, x also $\in A \cap B \cup ((A \cap B^c) \cup (B \cap A^c))$, so $A \cup B \subseteq (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c))$.
2) Second, show $(A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c)) \subseteq A \cup B$. For any $x \in (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c))$, $x \in A \cap B$ or $x \in ((A \cap B^c) \cup (B \cap A^c))$, so $x \in A \cap B$ or $x \in A \cap B^c$ or $x \in B \cap A^c$.
   - If $x \in A \cap B$, $x \in A$ and $x \in B$, then $x \in A \cup B$.
   - $x \in A \cap B^c$, $x \in A$ and $x \notin B$, then $x \in A \cup B$.
   - If $x \in B \cap A^c$, $x \notin A$ and $x \in B$, then $x \in A \cup B$.

   Thus, for any $x \in (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c))$, x also $\in A \cup B$, so $(A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c)) \subseteq A \cup B$.

Hence, by 1) and 2), we have $A \cup B = (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c))$.

□

Alternatively, you can prove it using rules of set operations:

$$
\begin{aligned}
(A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c)) &= ((A \cap B) \cup (A \cap B^c)) \cup (B \cap A^c) \quad \text{(associativity)} \\
&= ((A \cap (B \cup B^c)) \cup (B \cap A^c) \quad \text{(distributive laws)} \\
&= A \cup (B \cap A^c) \\
&= (A \cup B) \cap (A \cup A^c) \quad \text{(distributive laws)} \\
&= A \cup B
\end{aligned}
$$

Problem 2 in Lecture 1

Proof. First, show the identity $A \cup B = A \cup (B \cap A^c)$ holds. Since $B = (B \cap A) \cup (B \cap A^c)$, $A \cup B = A \cup [(B \cap A) \cup (B \cap A^c)] = [A \cup (B \cap A)] \cup (B \cap A^c) = [(A \cup B) \cap A] \cup (B \cap A^c) = A \cup (B \cap A^c)$.

Then, since A and $B \cap A^c$ are disjoint (since A and A^c are), we have

$$P(A \cup B) = P(A) + P(B \cap A^c)$$

Notice that $B = (B \cap A) \cup (B \cap A^c)$ implies $P(B) = P(A \cap B) + P(B \cap A^c)$ (since $A \cap B$ and $B \cap A^c$ are disjoint), so $P(B \cap A^c) = P(B) - P(B \cap A)$. Plug it back into the equation, we have

$$P(A \cup B) = P(A) + P(B \cap A^c) = P(A) + P(B) - P(B \cap A)$$

$\square$

Problem 3 in Lecture 1

Let A be the event of having the disease, and B be the event of having a positive test result. Then $P(A) = 0.0025$, $P(B|A) = 0.9$, $P(B|A^c) = 0.01$. By Bayes rule, we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.9 * 0.0025}{0.9 * 0.0025 + 0.01 * 0.9975} \approx 0.184$$

(Although the false positive rate looks low, the Bayes rule tells us the screening test is nonetheless quite useless, because false positive rate are way too large compared with the extreme rarity of the disease.)

Problem 4 in Lecture 1

Proof. Since A and B are mutually exclusive with $A \cap B =$, then $P(A \cap B) = 0$. If A and B are independent, then by definition $0 = P(A \cap B) = P(A)P(B)$. But this cannot be since $P(A) > 0$ and $P(B) > 0$. Thus A and B cannot be independent. $\square$

Problem 5 in Lecture 1

(a) *Proof.* A and B are independent unconditionally because

$$P(A \cap B) = P(\text{First die is 6, second die is 6}) = (1/6)^2 = 1/36 = P(A)P(B).$$

They are dependent given C because given you know the the two dice are exactly same, knowing the fist die is 6 implies the second die must also be 6. So

$$\begin{aligned} P(A \cap B|C) &= P(\text{First die is 6, second die is 6}|\text{Both dice are the same}) \\ &= P(\text{First die is 6}|\text{Both dice are the same}) = P(A|C) \neq P(A|C)P(B|C). \end{aligned}$$

$\square$

(b) Here this solutions uses the given example and shows A and B are dependent unconditionally but independent given C.

Proof. Suppose the proportion of black balls in the first urn is π_1, while in the second urn it's π_2 with $\pi_1 \neq \pi_2$. Then

$$P(A \cap B) = P(C)P(A \cap B|C) + P(C^c)P(A \cap B|C^c) = 0.5\pi_1^2 + 0.5\pi_2^2.$$

Since $P(A) = P(C)P(A|C) + P(C^c)P(A|C^c) = 0.5\pi_1 + 0.5\pi_2 = P(B)$, then

$$P(A)P(B) = 0.25(\pi_1 + \pi_2)^2.$$

Note that $(0.5\pi_1^2 + 0.5\pi_2^2) - 0.25(\pi_1 + \pi_2)^2 = 0.25(\pi_1 - \pi_2)^2 \neq 0$ since $\pi_1 \neq \pi_2$, so

$$P(A \cap B) = 0.5\pi_1^2 + 0.5\pi_2^2 \neq P(A)P(B) = 0.25(\pi_1 + \pi_2)^2$$

which means that A and B are dependent unconditionally.
On the other hand, if we condition on C, selecting the first urn, then by the logic of part (a), drawing a black ball in the first draw is conditionally independent with drawing a black ball in the second draw in that urn. So A and B are independent given C. □

Problem 1 in Lecture 2

Proof. For every t, $F_X(t) \leq F_Y(t)$. Thus we have

$$P(X > t) = 1 - P(X \leq t) = 1 - F_X(t) \geq 1 - F_Y(t) = 1 - P(Y \leq t) = P(Y > t).$$

And for some t^*, $F_X(t^*) < F_Y(t^*)$, then we have

$$P(X > t^*) = 1 - P(X \leq t^*) = 1 - F_X(t^*) > 1 - F_Y(t^*) = 1 - P(Y \leq t^*) = P(Y > t^*).$$

□

Problem 2 in Lecture 2

Proof. $F_X(x)$ defined above is valid CDF because
a) It is continuous, hence right-continuous;
b) It is non-decreasing, because $\frac{d}{dx}(1 - \exp(-x)) = \exp(-x) > 0$.
c) $\lim_{x \to -\infty}(1 - \exp(-x)) = 0$, $\lim_{x \to +\infty}(1 - \exp(-x)) = 1$.

□

Its PDF can be found by taking derivative of the CDF in the support:

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \exp(-x) & \text{if } x \geq 0 \end{cases}$$

$F_X^{-1}(y)$ is defined on $y \in [0, 1]$ with

$$F_X^{-1}(y) = \begin{cases} -\log(1 - y) & \text{if } y \in [0, 1) \\ +\infty & \text{if } y = 1 \end{cases}$$