Problem Set #4

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Discussed and/or compared answers with Sarah Bass, Emily Case, Katherine Kwok, Michael Nattinger, and Alex Von Hafften

Questions 1

The static household cost-minimization problem is given by:

$$\min_{\{C_{ik}\}} \int \left(\sum_{i=1}^{N_k} P_{ik} C_{ik} \right) dk \text{ s.t. } \left(\int C_k^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} = C, \; \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} = C_k$$

This problem is represented by the Lagrangian:

$$\mathcal{L} = -\int \left(\sum_{i=1}^{N_k} P_{ik} C_{ik} \right) dk + P \left[\left(\int C_k^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} - C \right] + \int P_k \left[\left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} - C_k \right] dk$$

An implicit assumption with this setup is that households minimize their expenditures while setting an overall utility threshold across industries, then optimize consumption utility within each industry (i.e. across firms). This problem has two first order conditions, with respect to C_{ik} and C_k:

$$\frac{\partial \mathcal{L}}{\partial C_{ik}} = -P_{ik} + P_k \left(\frac{\theta}{\theta - 1} \right) \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} \left(\frac{\theta - 1}{\theta} \right) C_{ik}^{\frac{-1}{\theta}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial C_k} = P \left(\frac{\rho - 1}{\rho} \right) \left(\int C_k^{\frac{\rho}{\rho-1}} \right)^{\frac{1}{\rho-1}} \left(\frac{\rho - 1}{\rho} \right) C_k^{\frac{-1}{\rho}} - P_k = 0$$

We can then solve each equation to isolate our choice variables:

$$C_{ik} = \left(\frac{P_{ik}}{P_k} \right)^{-\theta} C_k \qquad\qquad C_k = \left(\frac{P_k}{P} \right)^{-\rho} C$$

Substituting these definitions back into the constraints enable us to define the price indexes:

$$C_k = \left[\sum_{i=1}^{N_k}\left(\left(\frac{P_{ik}}{P_k}\right)^{-\theta} C_k\right)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \qquad C = \left[\int \left(\left(\frac{P_k}{P}\right)^{-\rho} C\right)^{\frac{\rho-1}{\rho}} dk\right]^{\frac{\rho}{\rho-1}}$$

$$1 = P_k^{\theta}\left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{\theta}{\theta-1}} \qquad 1 = P^{\rho}\left(\int P_k^{1-\rho} dk\right)^{\frac{\rho}{\rho-1}}$$

$$P_k = \left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} \qquad P = \left(\int P_k^{1-\rho} dk\right)^{\frac{1}{1-\rho}}$$

Finally, we can substitute these price indices back into the consumption equation to delineate optimal demand for each good, as a function only of overall consumption utility and relative prices, P_{ik}:

$$C_{ik} = P_{ik}^{-\theta}\left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} \left[\int \left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{1-\rho}{1-\theta}} dk\right]^{\frac{\rho}{1-\rho}} C$$

Question 2

Assuming that firms engage in Bertrand competition, then firm i in industry k maximizes its profit by only choosing its own price, P_{ik}, taking the prices of all other firms as a given, rather than treating them as a function of P_{ik}. Firms also take the going wage, W, as given and seek to match output to demand, $Y_{ik} = C_{ik}$. Then, the firm's problem is:

$$\max_{P_{ik}, L_{ik}, Y_{ik}} P_{ik}Y_{ik} - WL_{ik} \text{ s.t. } Y_{ik} = A_{ik}L_{ik},\ Y_{ik} = C_{ik}$$

Substituting the demand for the firm's good for Y_{ik} and rearranging the production function yields an objective function that depends only on P_{ik}:

$$\begin{aligned} P_{ik}Y_{ik} - WL_{ik} &= C_{ik}\left(P_{ik} - \frac{W}{A_{ik}}\right) \\ &= P_{ik}^{-\theta}\left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C \left(P_{ik} - \frac{W}{A_{ik}}\right) \\ &= P^{\rho} C\left[P_{ik}^{1-\theta}\left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} - \frac{W}{A_{ik}} P_{ik}^{-\theta}\left(\sum_{i=1}^{N_k} P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}}\right] \end{aligned}$$

Recall our equations for P, which is technically a function of P_{ik}, but the firm occupies zero mass with respect to P. Therefore, P_{ik} does not influence its value.

The firm also does not influence C. Then, for the purposes of optimation, our objective function is:

$$P_{ik}^{1-\theta}\left(\sum_{i=1}^{N_k}P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} - \frac{W}{A_{ik}}P_{ik}^{-\theta}\left(\sum_{i=1}^{N_k}P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}}$$

Recall also our equation for P_k, which was not used in defining the objective function but can be used to simplify the final result. The first-order condition for the firm, then, is:

$$\begin{aligned}
(1-\theta)P_{ik}^{-\theta}P_k^{\theta-\rho} + P_{ik}^{1-\theta}(\theta-\rho)P_k^{2\theta-\rho-1}P_{ik}^{-\theta} &= \frac{W}{A_{ik}}\left[-\theta P_{ik}^{-\theta-1}P_k^{\theta-\rho} + P_{ik}^{-\theta}(\theta-\rho)P_k^{2\theta-\rho-1}P_{ik}^{-\theta}\right] \\
P_{ik}^{-\theta}P_k^{\theta-\rho}\left[1-\theta+(\theta-\rho)P_{ik}^{1-\theta}P_k^{\theta-1}\right] &= \frac{W}{A_{ik}}P_{ik}^{-\theta}P_k^{\theta-\rho}\left[-\theta P_{ik}^{-1}+(\theta-\rho)P_{ik}^{-\theta}P_k^{\theta-1}\right] \\
1-\theta+(\theta-\rho)P_{ik}^{1-\theta}P_k^{\theta-1} &= \frac{W}{A_{ik}}\left(-\theta P_{ik}^{-1}+(\theta-\rho)P_{ik}^{-\theta}P_k^{\theta-1}\right) \\
1-\theta+(\theta-\rho)\left(\frac{P_{ik}}{P_k}\right)^{1-\theta} &= \frac{W}{A_{ik}}P_{ik}^{-1}\left[-\theta+(\theta-\rho)\left(\frac{P_{ik}}{P_k}\right)^{1-\theta}\right]
\end{aligned}$$

Let s_{ik} represent firm i's share of the price of one unit of utility in its industry—i.e., $s_{ik}=\frac{P_{ik}}{P_k}$. Then,

$$P_{ik}=\frac{W}{A_{ik}}\left(\frac{(\theta-\rho)s_{ik}^{1-\theta}-\theta}{(\theta-\rho)s_{ik}^{1-\theta}+1-\theta}\right)$$

Finally, the elasticity of demand for the firm, $\varepsilon(P_{ik})$ reuses much of the alegebra from above:

$$\begin{aligned}
\frac{dC_{ik}}{dP_{ik}} &= CP^{\rho}P_{ik}^{-\theta-1}P_k^{\theta-\rho}\left[-\theta+(\theta-\rho)s_{ik}^{1-\theta}\right] \\
\frac{P_{ik}}{C_{ik}} &= \left[CP^{\rho}P_{ik}^{-\theta-1}P_k^{\theta-\rho}\right]^{-1} \\
\Rightarrow \varepsilon(P_{ik}) = \frac{dC_{ik}}{dP_{ik}}\frac{P_{ik}}{C_{ik}} &= (\theta-\rho)s_{ik}^{1-\theta}-\theta
\end{aligned}$$

Question 3

The total costs of firm i in industry k are given by WL_{ik}, where $L_{ik} = A_{ik}^{-1}Y_{ik}$. The marginal cost of production for such a firm, then, is represented by WA_{ik}^{-1}. Recall that we solved the price charged by firm i in industry k as:

$$P_{ik} = \frac{W}{A_{ik}} \left(\frac{(\theta - \rho)s_{ik}^{1-\theta} - \theta}{(\theta - \rho)s_{ik}^{1-\theta} + 1 - \theta} \right)$$

Then, the markup of this firm is

$$\begin{aligned}
\eta_{ik} &= \frac{(\theta - \rho)s_{ik}^{1-\theta} - \theta}{(\theta - \rho)s_{ik}^{1-\theta} + 1 - \theta} \\
&= 1 - \frac{1}{(\theta - \rho)\left(\frac{P_{ik}}{P_k}\right)^{1-\theta} + 1 - \theta} \\
&= 1 - \left[(\theta - \rho)\left(WA_{ik}^{-1}\eta_{ik}\right)^{1-\theta} P_k^{\theta-1} + 1 - \theta\right]^{-1} \\
&= 1 - \left[(\theta - \rho)W^{1-\theta}A_{ik}^{\theta-1}\eta_{ik}^{1-\theta}P_k^{\theta-1} + 1 - \theta\right]^{-1}
\end{aligned}$$

This is markup function is recursive, so its derivative with respect to A_{ik} cannot be exactly determined, but we can determine the effect of A_{ik} on η_{ik} to a first approximation:

$$\frac{\partial \eta_{ik}}{\partial A_{ik}} \approx \frac{(\theta - 1)(\theta - \rho)s_{ik}^{1-\theta}}{A_{ik}\left[(\theta - \rho)s_{ik}^{1-\theta} + 1 - \theta\right]^2}$$

This value is positive, since $\theta > \rho \geq 1$ by assumption. Thus, firms with higher A_{ik} charge higher markups, *ceteris paribus.*

Question 4

See the attached code, which is also pasted at the end of this document, for the solution to this model. The solution simply requires assuming a value of s_{ik} for each firm, calculating P_{ik} and P_k, then recalculating s_{ik} and repeating until the absolute difference between iterations of s_{ik} fall below some threshold. The solution yields the following real wage, rounded to the nearest thousandth:

Question 5

The real wage in the model, $\frac{W}{P}$, is equal to C. The first-best value for C (i.e. the one that a social planner would choose) would come from all firms charging their marginal cost: $P_{ik} = \frac{W}{A_{ik}}$. Each of these values from my solution to the model are given below, rounded to the neareast thousandth.

$$C = 4.609 \qquad\qquad C^{fb} = 7.232$$

It is immediately apparent that household wages are meaningfully higher in the social planner's allocation than the competitive equilibirum. Then is to be expected, as the assumptions of the first fundamental theorem of welfare are not satisfied. Namely, the market is not perfectly competitive.