Problem Set #4

Danny Edgel
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Question 1

To set up the Ramsey problem, we must first solve for the resource constraint and implementability contraint of this economy. The resource constraint is simply:

$$c_t + g_t + k_{t+1} = F(k_t, 1 - l_t) + (1 - \delta)k_t$$

We can derive the implementability constraint by solving the household problem:

$$\max_{c_t, l_t, k_{t+1}, b_{t+1}} \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t) \right] \text{ s.t. } (1+\tau_t)c_t + k_{t+1} + b_{t+1} = w_t(1-l_t) + (1-\delta+r_t)k_t + R_{bt}b_t$$

Let p_t be the Langrangian multiplier for the household budget contraint. Then, the FOC of the household problem are:

$$\begin{aligned} \beta^t c_t^{-\sigma} - p_t(1+\tau_t) &= 0 && (c_t) \\ \beta^t v'(l_t) - p_t w_t &= 0 && (l_t) \\ [(1-\delta+r_{t+1})p_{t+1} - p_t]k_t &= 0 && (k_{t+1}) \\ (R_{bt+1}p_{t+1} - p_t)b_t &= 0 && (b_{t+1}) \end{aligned}$$

Multiplying each size of the budget constraint by the lagrangian multiplier and rearranging, we get:

$$\begin{aligned} p_t(1+\tau_t)c_t - p_t w_t(1-l_t) &= p_t(1-\delta+r_t)k_t - p_t k_{t+1} + p_t R_{bt} b_t - p_t b_{t+1} \\ \beta^t \left(c_t^{1-\sigma} - v'(l_t)(1-l_t) \right) &= p_t(1-\delta+r_t)k_t - p_t k_{t+1} + p_t R_{bt} b_t - p_t b_{t+1} \end{aligned}$$

Then, summing each side across all t yields the implementability constraint:

$$\sum_{t=1}^{\infty} \beta^t \left(c_t^{1-\sigma} - v'(l_t)(1-l_t) \right) = \frac{c_0^{-\sigma}}{1+\tau_0} \left[(1-\delta+r_0)k_{-1} + R_{b0}b{-}1 \right]$$

Then, the Ramsey problem is to maximize household utility subject to the resource and implementability constraints:

$$\max_{c_t, l_t, k_{t+1}, b_{t+1}} \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t) \right]$$

$$\text{s.t. } c_t + g_t + k_{t+1} = F(k_t, 1-l_t) + (1-\delta)k_t$$

$$\sum_{t=1}^{\infty} \beta^t \left(c_t^{1-\sigma} - v'(l_t)(1-l_t) \right) = \frac{c_0^{-\sigma}}{1+\tau_0} \left[(1-\delta+r_0)k_{-1} + R_{b0}b{-}1 \right]$$

This problem can be written by augmenting the objective function with the implementability constraint, as follows:

$$\max_{c_t, l_t, k_{t+1}, b_{t+1}} \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \nu(l_t) + \lambda \left(c_t^{1-\sigma} - v'(l_t)(1-l_t) \right) \right] - \lambda \frac{c_0^{-\sigma}}{1+\tau_0} \left[(1-\delta+r_0)k_{-1} + R_{b0}b{-}1 \right]$$

Denote the first two terms of the objective function (excluding the discount factor) as $w(c_t, l_t, \lambda)$. Then, the Ramsey problem is represented by the following Lagrangian function:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t w(c_t, l_t, \lambda) - \lambda \frac{c_0^{-\sigma}}{1+\tau_0} \left[(1-\delta+r_0)k_{-1} + R_{b0}b{-}1 \right] \\ - \mu_t \left(c_t + g_t + k_{t+1} - F(k_t, 1-l_t) - (1-\delta)k_t \right)$$

Which has the following first-order conditions:

$$\beta^t w_1(c_t, l_t, \lambda) - \mu_t = 0 \qquad (c_t)$$

$$\beta^t w_2(c_t, l_t, \lambda) - \mu_t F_2(k_t, 1-l_t) = 0 \qquad (l_t)$$

$$-\mu_{t-1} + \mu_t [F_1(k_t, 1-l_t) + 1 - \delta] = 0 \qquad (k_t)$$

Combining the first two FOCs, we have the intratemporal optimization condition:

$$\frac{w_2(c_t, l_t, \lambda)}{w_1(c_t, l_t, \lambda)} = F_2(k_t, 1-l_t)$$

Combining the first and third FOCs give the intertemporal optimization condition:

$$\frac{w_1(c_t, l_t, \lambda)}{w_1(c_{t+1}, l_{t+1}, \lambda)} = \beta \left[F_1(k_{t+1}, 1-l_{t+1}) + 1 - \delta \right]$$

Where:

$$w_1(c_t, l_t, \lambda) = c_t^{-\sigma} \left(1 + \lambda(1-\sigma) \right)$$

Thus, the intertemporal condition becomes:

$$\left(\frac{c_t}{c_{t+1}} \right)^{-\sigma} = \beta \left[F_1(k_{t+1}, 1-l_{t+1}) + 1 - \delta \right]$$

In a competitive equilibrium, households have the following intertemporal optimization condition:

$$\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} = \beta\frac{1+\tau_t}{1+\tau_{t+1}}\left(\frac{p_t}{p_{t+1}}\right) = \beta\frac{1+\tau_t}{1+\tau_{t+1}}[1-\delta+r_{t+1}]$$

Where, since the production function satisfies the usual conditions,

$$r_{t+1} = F_1(k_{t+1}, 1-l_{t+1})$$

In a competitive equilibrium. Thus, the optimal policy is to set $\tau_t = \tau_{t+1}$ for all $t \geq 1$.

Question 2

1. A competitive equilibrium is a policy, (M_t, B_t); allocation, (c_{1t}, c_{2t}, n_t); and price system, (p_t, w_t, R_t), such that:

   (a) Given the policy and price system, the allocation solves the household problem

   (b) The allocation satisfies the government budget constraint

2. The first order conditions of the household problem are:

$$\begin{aligned}
&\frac{\beta^t}{c_{1t}} - \lambda_{t+1}p_t - \mu_t p_t = 0 && (c_{1t}) \\
&\frac{\alpha\beta^t}{c_{2t}} - \lambda_{t+1}p_t = 0 && (c_{2t}) \\
&-\frac{\gamma\beta^t}{1-n_t} + \lambda_{t+1}w_t = 0 && (n_t) \\
&-\lambda_t + \lambda_{t+1}R = 0 && (B_t) \\
&-\lambda_t + \lambda_{t+1} + \mu_t = 0 && (M_t)
\end{aligned}$$

From the FOCs for M_t, B_t, and c_{1t}, we can solve:

$$\begin{aligned}
&\frac{\lambda_t}{\lambda_{t+1}} = 1 + \frac{\mu_t}{\lambda_{t+1}} \\
\Rightarrow &R_t = \frac{\lambda_t}{\lambda_{t+1}} = 1 + \frac{\mu_t}{\lambda_{t+1}} \\
\Rightarrow &\frac{\beta^t}{\lambda_{t+1}c_{1t}} = p_t\left(1 + \frac{\mu_t}{\lambda_{t+1}}\right) \\
\Rightarrow &\frac{\beta^t}{c_{1t}} = \lambda_{t+1}p_t R
\end{aligned}$$

Combining this with the FOC for c_{2t} gives us:

$$\frac{c_{2t}}{\alpha c_{1t}} = R$$

Combining the FOCs for c_{2t} and n_t yields:

$$\frac{\gamma}{\alpha}\left(\frac{c_{2t}}{1-n_t}\right) = \frac{w_t}{p_t}$$

Since production is linear in labor, $w_t = p_t$, so the righthand side of the equation becomes 1. To observe the relationship between n_t and R, combine our two optimization conditions with the resource constraint to solve for n_t as a function of R:

$$\begin{aligned}
c_{1t} &= \frac{c_{2t}}{\alpha R} \\
c_{1t} + c_{2t} &= n_t \\
c_{2t} &= \frac{\alpha R n_t}{1 + \alpha R} \\
\frac{\gamma}{\alpha}\left[\frac{\frac{n_t}{1+\frac{1}{\alpha R}}}{1 - n_t}\right] &= 1 \\
\gamma n_t &= \left(\alpha + \frac{1}{R}\right)(1 - n_t) \\
n_t &= \frac{1 + \alpha R}{1 + (\gamma + \alpha) R}
\end{aligned}$$

Taking the derivative yields:

$$\frac{dn_t}{dR} = -\frac{\gamma}{\left[1 + (\gamma + \alpha) R\right]^2} < 0$$

Thus, labor decreases when R increases.