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Economics 712
Macroeconomic Theory
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Problem Set 4

1. In Problem #3 on Problem Set 3 the wage and rate of return were given. Now suppose instead that they are determined from an aggregate production function:

$$Y = K^{\alpha} N^{1-\alpha}$$

and an aggregate law of motion for the capital stock:

$$K' = (1 - \delta)K + I.$$

Assume that $\alpha = 0.36$ and $\delta = 0.08$.

   (a) Find the equilibrium levels of the capital stock, the real wage, and the real interest rate.

   (b) Plot the equilibrium distributions of income and consumption distribution. Compare the relative degree of inequality in the two distributions.

   (c) Now suppose that the borrowing constraint is loosened so that households face the debt limit $a_t \geq -2$ (rather than $a_t \geq 0$. What happens to the equilibrium results from part (a)?

2. In Problem #3 on Problem Set 2 we calculated the solution of a deterministic optimal growth model. We now revisit that model, but add stochastic productivity shocks. Recall that we have the following specification: An infinitely-lived representative household owns a stock of capital which it rents to firms. The household's capital stock K depreciates at rate δ. Households do not value leisure and are endowed with one unit of time each period with which they can supply labor N to firms. They have standard time additive expected utility preferences with discount factor β and period utility $u(c)$. Firms produce output according to the production function $zF(K, N)$ where z is the level of technology . Set $\beta = 0.95$, $\delta = 0.1$, $u(c) = c^{1-\gamma}/(1 - \gamma)$ with $\gamma = 2$, and $F(K, N) = K^{0.35} N^{0.65}$. Before we set $z = 1$ but now we consider a Markov chain where for productivity where $z_t \in \{0.8, 1.2\}$ with transition matrix $P = [0.9, 0.1; 0.1, 0.9]$.

   (a) Now solve for the optimal policy function $K' = g(K, z)$ and implied policy $c = c(K, z)$. Compare them to the solution in the deterministic case where $z_t \equiv 1$. How do they differ?

   (b) Simulate the model to calculate time series of capital, output, consumption, investment, wages, and interest rates. What is the long run mean of consumption and capital? How do they compare to the deterministic steady state levels?

(c) How do the volatility of consumption and investment compare to that of output? What is the correlation between consumption and output? Between interest rates and output?

3. We now consider a recursive version of the neoclassical model with distorting taxes. An infinitely-lived representative household owns a stock of capital k which it rents to firms. The household's capital stock depreciates are rate δ, and denote aggregate capital by K. Households do not value leisure and are endowed with one unit of time each period with which they can supply labor N to firms. They have standard time additive expected utility preferences with discount factor β and period utility $u(c)$. Firms produce output according to the production function $zF(K, N)$ where z is the stochastic level of technology which is Markov with transition function $P(z', z)$. There is also a government which levies a proportional tax τ on households' capital income and labor income. The tax rate is the same for both types of income and constant over time. The government uses the proceeds of the taxes to provide households with a lump sum transfer payment T (which may vary over time), and suppose that in equilibrium the government must balance its budget every period.

   (a) Define a recursive competitive equilibrium for this economy. Be specific about all of the objects in the equilibrium and the conditions they must satisfy.

   (b) Characterize the equilibrium by finding a functional (Euler) equation which the household's optimal capital accumulation policy must satisfy (i.e. k' as a function of k, K, z).

   (c) Impose the equilibrium conditions and find a functional equation which the aggregate capital accumulation policy must satisfy.

   (d) Show that when $\delta = 1$, the recursive competitive equilibrium allocation coincides with the solution of a social planner's problem, but with a different discount factor. Interpret your result.

4. Consider an endowment economy where a representative agent has recursive preferences of the Epstein-Zin type. That is, the utility V_t of a consumption stream $\{c_s\}_{s=t}^{\infty}$ is evaluated recursively:

$$V_t = \left((1-\beta)c_t^{1-\rho} + \beta \left(E_t V_t^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}\right)^{\frac{1}{1-\rho}},$$

where $\rho > 0$ and $\alpha > 0$. Notice that this is a combination of a CES aggregate (with parameter ρ) of current utility of consumption and a risk-adjustment (with parameter α) of future utility.

   (a) Show that when $\alpha = \rho$ these preferences collapse to standard expected utility with a power utility function.

   (b) Epstein-Zin preferences allow us to disentangle risk aversion and intertemporal substitution. How are these properties characterized here?

(c) Find an expression for the intertemporal marginal rate of substitution (stochastic discount factor), which we can define here as:

$$S_t = \frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}}.$$

Now suppose that the endowment process (fruit from the Lucas tree) has i.i.d. growth rates, that is:

$$c_{t+1}/c_t = g + \sigma_c \varepsilon_{t+1}$$

where $g > 0$ and $\sigma_c > 0$ are constants and $\varepsilon_t \sim N(0,1)$.

(a) Conjecture a Markov pricing function, then write down the Bellman equation for the representative agent and find his optimality conditions.

(b) Define a recursive competitive equilibrium, being specific about the objects which make it up.

(c) Show that the value function can be written $V(c_t) = vc_t$ for some constant v, and find an expression for $\log S_t$.

(d) Find an expression for the risk-free rate. How does this differ from the standard CRRA case?

(e) Find an expression for the return on the Lucas tree. How does this differ from the standard CRRA case?