

# Problem Set #6

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## Questions 1

The monetary policy authority faces the following problem:

$$\min_{\{x_t, \pi_t, i_t\}} \frac{1}{2} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t (x_t^2 + \alpha \pi_t^2) \right], \text{ s.t. } \quad \sigma \mathbb{E}_t [\Delta x_{t+1}] = i_t - \mathbb{E}_t [\pi_{t+1}] - r_t^n$$
$$\pi_t = \kappa x_t + \beta \mathbb{E}_t [\pi_{t+1}] + u_t$$

Using the primal approach, we can optimize the Lagrangian, considering only the NKPC constraint:

$$\mathcal{L} = -\mathbb{E} \left[ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t (x_t^2 + \alpha \pi_t^2) - \lambda_t (\pi_t - \kappa x_t - \beta \pi_{t+1} - u_t) \right]$$

Which has the following first order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_t} = -\beta^t x_t + \kappa \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial \pi_t} = \begin{cases} -\beta^t \alpha \pi_t - \lambda_t + \beta \lambda_{t-1} = 0, & t \geq 1 \\ -\beta^t \alpha \pi_t - \lambda_t = 0, & t = 0 \end{cases}$$

Combining these FOCs enables us to derive an optimal policy rule:

$$\alpha \kappa \pi_t + \Delta x_t = 0, \quad t \geq 1 \qquad \alpha \kappa \pi_0 + x_0 = 0$$

Let  $x_{-1} = p_{-1} = 0$ ; then, we can represent the optimal rule as a single equation:

$$\alpha \kappa \pi_t + \Delta x_t = 0$$

Since this holds for all  $t$ , we can prove via induction that  $\alpha\kappa p_t + x_t = 0$ :

$$\begin{aligned}\alpha\kappa(p_0 - p_{-1}) + (x_0 - x_{-1}) &= \alpha\kappa p_0 + x_0 = 0 \\ \alpha\kappa(p_t - p_{t-1}) + (x_t - x_{t-1}) &= \alpha\kappa p_t + x_t - (\alpha\kappa p_{t-1} + x_{t-1}) = 0\end{aligned}$$

We can use this optimal policy rule and the NKPC (adjusted to use  $p_t - p_{t-1}$  instead of  $\pi$ ) to construct a linear system from which to solve for equilibrium dynamics:

$$\begin{aligned}-\beta\mathbb{E}[p_{t+1}] + p_t - p_{t-1} - \kappa(-\alpha\kappa p_t) &= u_t \\ -\beta\mathbb{E}[p_{t+1}] &= -(1 + \beta + \alpha\kappa^2)p_t + p_{t-1} + u_t \\ \Rightarrow \begin{pmatrix} \mathbb{E}[p_{t+1}] \\ p_t \end{pmatrix} &= \begin{pmatrix} 1 + \frac{1}{\beta} + \frac{\alpha\kappa^2}{\beta} & -\frac{1}{\beta} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{\beta} \\ 0 \end{pmatrix} u_t\end{aligned}$$

To determine equilibrium dynamics in this model, we must find the eigenvalues of the matrix in this linear system:

$$\begin{aligned}(1 + \frac{1}{\beta} + \frac{\alpha\kappa^2}{\beta} - \lambda)(-\lambda) + \frac{1}{\beta} &= 0 \\ \lambda^2 - (1 + \frac{1}{\beta} + \frac{\alpha\kappa^2}{\beta})\lambda + \frac{1}{\beta} &= 0\end{aligned}$$

Because this system has one state and one choice variable,  $\lambda_1 > 1$  and  $\lambda_2 < 1$ , where  $\lambda_1$  is the eigenvalue associated with  $\mathbb{E}[p_{t+1}]$ . Without paying too much mind to the exact values of  $\lambda_1$  and  $\lambda_2$  and omitting intermediate (and tedious) steps, we can find:

$$\begin{aligned}\lambda &= \frac{1}{2\beta} \left[ 1 + \beta + \alpha\kappa^2 \pm \sqrt{(1 + \beta + \alpha\kappa^2)^2 - 4\beta} \right] \\ \lambda_1\lambda_2 &= \frac{1}{4\beta^2} \left[ (1 + \beta + \alpha\kappa^2)^2 - (1 + \beta + \alpha\kappa^2)^2 + 4\beta \right] \\ &= \frac{1}{\beta}\end{aligned}$$

Furthermore, we can see that  $\beta(\lambda_1 + \lambda_2) = 1 + \beta + \alpha\kappa^2$ . This enables us to write the NKPC with just our eigenvalues and lag operators:

$$\begin{aligned}-\beta(1 - \lambda_1 L)(1 - \lambda_2 L)L^{-1}p_t &= u_t \\ (\beta\lambda_1 - \beta L^{-1})(1 - \lambda_2 L)p_t &= u_t \\ p_t - \lambda_2 p_{t-1} &= \left( \frac{1}{\lambda_2} - \beta L^{-1} \right)^{-1} u_t \\ p_t &= \left( \frac{\lambda_2}{1 - \beta\lambda_2 L^{-1}} \right) u_t + \lambda_2 p_{t-1}\end{aligned}$$

Since we are given the distribution of the markup shock  $u_t$ , we can determine solve for  $p_t$  at any given  $t$ , with past realizations accounted for in  $p_{t-1}$  and expected future realizations given by the distribution of  $u_t$ :

$$\begin{aligned}
p_t &= \lambda_2 p_{t-1} + \lambda_2 \mathbb{E}_t \left[ \sum_{j=0}^{\infty} (\lambda_2 \beta)^j u_{t+j} \right] \\
&= \lambda_2 p_{t-1} + \lambda_2 \left( u_t + \sum_{j=1}^{\infty} (\lambda_2 \beta)^j \mathbb{E}_t [u_{t+j}] \right) \\
&= \lambda_2 p_{t-1} + \lambda_2 \left( u_t + \left( \frac{\lambda_2 \beta}{1 - \lambda_2 \beta} \right) \bar{u} \right) \\
p_t &= \lambda_2 (p_{t-1} + u_t) + \left( \frac{\lambda_2}{\lambda_1 - 1} \right) \bar{u}
\end{aligned}$$

Recalling our equation for the output gap, this equation can be used to describe the dynamics of  $x_t$ , as well:

$$x_t = \lambda_2 x_{t-1} - \lambda_2 \alpha \kappa u_t - \left( \frac{\lambda_2 \alpha \kappa}{\lambda_1 - 1} \right) \bar{u}$$

## Question 2

Under a discretionary policy, the planner can ensure that  $\alpha \kappa \pi_t + x_t = 0$  in every period. Then, since the NKPC holds each period, we can solve:

$$\begin{aligned}
\pi_t &= \kappa x_t + \beta \mathbb{E}_t [\pi_{t+1}] + u_t \\
\pi_t &= -\alpha \kappa^2 \pi_t + \beta \mathbb{E}_t [\pi_{t+1}] + u_t \\
\pi_t &= \frac{1}{1 + \alpha \kappa^2} (\beta \mathbb{E}_t [\pi_{t+1}] + u_t) \\
\pi_t &= \frac{1}{1 + \alpha \kappa^2} \sum_{j=0}^{\infty} \left( \frac{\beta}{1 + \alpha \kappa^2} \right)^j \mathbb{E} [u_{t+j}] \\
\pi_t &= \left( \frac{1}{1 + \alpha \kappa^2} \right) u_t + \sum_{j=0}^{\infty} \left( \frac{\beta}{1 + \alpha \kappa^2} \right)^j \mathbb{E} [u_{t+j}] \\
\pi_t &= \left( \frac{1}{1 + \alpha \kappa^2} \right) u_t + \left( \frac{\beta}{(1 + \alpha \kappa^2)(1 - \beta + \alpha \kappa^2)} \right) \bar{u}
\end{aligned}$$

Applying this to the optimal policy rule yields our equation for the output gap:

$$x_t = - \left( \frac{\alpha \kappa}{1 + \alpha \kappa^2} \right) u_t - \left( \frac{\beta \alpha \kappa}{1 - \beta + \alpha \kappa^2} \right) \bar{u}$$

## Question 3

Under the  $\pi_t = 0$  rule, the NKPC yields the equilibrium allocation:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t [\pi_{t+1}] + u_t \Rightarrow x_t = -\frac{u_t}{\kappa}$$

### Question 4

Similar to in question 3, we can determine the equilibrium allocation by setting  $x_t = 0$  in the NKPC:

$$\pi_t = \beta \mathbb{E}_t [\pi_{t+1}] + u_t = \sum_{j=0}^{\infty} \beta^j \mathbb{E} [u_{t+j}]$$

$$\pi_t = u_t + \left( \frac{\beta}{1-\beta} \right) \bar{u}$$

### Question 5

To determine under which circumstances one policy is preferable to the other, we must first determine the expected welfare losses under each policy: letting  $W_\pi$  and  $W_d$  denote welfare losses under an inflation-targeting and discretionary policy, respectively:

$$W_\pi = \frac{1}{2} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t \left( -\frac{u_t}{\kappa} \right)^2 \right] = \frac{\sigma^2 + \bar{u}^2}{2\kappa^2(1-\beta)}$$

$$\begin{aligned} W_d &= \frac{1}{2} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t ((-\alpha\kappa\pi_t)^2 + \alpha\pi_t^2) \right] \\ &= \frac{\alpha(1+\alpha\kappa^2)}{2} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t \left( \left( \frac{1}{1+\alpha\kappa^2} \right) u_t + \left( \frac{\beta}{(1+\alpha\kappa^2)(1-\beta+\alpha\kappa^2)} \right) \bar{u} \right)^2 \right] \\ &= \frac{\alpha(1+\alpha\kappa^2)}{2(1-\beta)} \left[ \left( \frac{1}{(1+\alpha\kappa^2)^2} \right) \mathbb{E}_t [u_t^2] + 2 \left( \frac{\beta}{(1+\alpha\kappa^2)^2(1-\beta+\alpha\kappa^2)} \right) \mathbb{E}_t [u_t] \bar{u} + \left( \frac{\beta^2}{(1+\alpha\kappa^2)^2(1-\beta+\alpha\kappa^2)^2} \right) \bar{u}^2 \right] \\ &= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)^2} \left[ \sigma^2 + \bar{u}^2 + \left( \frac{2\beta}{1-\beta+\alpha\kappa^2} + \frac{\beta^2}{(1-\beta+\alpha\kappa^2)^2} \right) \bar{u}^2 \right] \\ &= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)} \left[ \sigma^2 + \left( 1 + \frac{2\beta}{1-\beta+\alpha\kappa^2} + \frac{\beta^2}{(1-\beta+\alpha\kappa^2)^2} \right) \bar{u}^2 \right] \\ &= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)} \left[ \sigma^2 + \left( \frac{1+2\alpha\kappa^2+\alpha^2\kappa^4-\beta^2}{(1-\beta+\alpha\kappa^2)^2} \right) \bar{u}^2 \right] \end{aligned}$$

The social planner prefers an inflation-targeted policy if  $W_\pi \leq W_d$ :

$$\begin{aligned} \frac{\sigma^2 + \bar{u}^2}{2\kappa^2(1-\beta)} &\leq \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)} \left[ \sigma^2 + \left( \frac{1+2\alpha\kappa^2+\alpha^2\kappa^4-\beta^2}{(1-\beta+\alpha\kappa^2)^2} \right) \bar{u}^2 \right] \\ \left( \frac{1+\alpha\kappa^2}{\kappa^2} \right) \sigma^2 - \alpha\sigma^2 &\leq \left[ \frac{\alpha(1+2\alpha\kappa^2+\alpha^2\kappa^4-\beta^2)}{(1-\beta+\alpha\kappa^2)^2} - \frac{1+\alpha\kappa^2}{\kappa^2} \right] \bar{u}^2 \\ \sigma^2 &\leq \left[ \frac{\alpha\kappa^2(1+2\alpha\kappa^2+\alpha^2\kappa^4-\beta^2)}{(1-\beta+\alpha\kappa^2)^2} - 1 - \alpha\kappa^2 \right] \bar{u}^2 \end{aligned}$$

At the limit as  $\beta \rightarrow 1$ , this inequality simplifies cleanly:

$$\begin{aligned}
 \sigma^2 &\leq \left[ \frac{\alpha\kappa^2(2\alpha\kappa^2 + \alpha^2\kappa^4)}{(\alpha\kappa^2)^2} - 1 - \alpha\kappa^2 \right] \bar{u}^2 \\
 &\leq \left[ \frac{(\alpha\kappa^2)^2(2 + \alpha\kappa^2)}{(\alpha\kappa^2)^2} - 1 - \alpha\kappa^2 \right] \bar{u}^2 \\
 &\leq (2 + \alpha\kappa^2 - 1 - \alpha\kappa^2) \bar{u}^2 \\
 \sigma^2 &\leq \bar{u}^2
 \end{aligned}$$

Thus, an inflation-targeted policy is preferable if the mean of mark-up shocks exceeds its standard deviation.

**Question 6**

**Question 7**