

Problem Set #3

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Question 1

For this problem, we will consider an Arrow-Debreu market structure.

- (a) Each consumer faces the following problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E} [\beta^t \ln(c_t^i)] \quad \text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) e_t^i(s^t)$$

Where $s^t \in \{s_1, s_2\}$ and $s^0 = s_1$ such that $P(s_1|s^0) = (1 - \delta)^t$.

A competitive equilibrium in this Arrow-Debreu market is a price system, $\{q_t^0(s^t)\}$, and an allocation, $\{c_t^i(s^t)\}$, that:

- (i) Solves both consumers' problem
- (ii) Clears markets: $c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t) \quad \forall t$

Where, in this market, $e_t^1(s^t) + e_t^2(s^t) = 1 \quad \forall t$.

- (b) We begin by solving each consumer's problem separately. Since utility functions and budget constraints are symmetric, we can solve a representative problem, then determine each consumer's maximization conditions based on their endowments in each state. The first-order condition of consumer i 's Lagrangian function is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t^i(s^t)} &= \beta^t \frac{P(s^t|s^0)}{c_t^i(s^t)} - \lambda^i q_t^0(s^t) = 0 \\ \frac{1}{\lambda^i} \beta^t \frac{P(s^t|s^0)}{c_t^i(s^t)} &= q_t^0(s^t) \end{aligned}$$

Since there is no aggregate uncertainty, each consumer is able to perfectly smooth their consumption across time (but not across states, where

there is idiosyncratic uncertainty). Thus, $c_t^i(s^t) = \bar{c}^i(s^t)$ for all t . We also know that endowments vary only by states, but not by time, such that $e_t^i(s^t) = e^i(s^t)$ for all t . Then, using the first-order condition for an arbitrary $c_t^i(s^t)$, we can use the consumer's budget constraint (which holds with equality at the solution) to derive:

$$\begin{aligned} \sum_{t=0}^{\infty} \sum_{s^t} \left(\frac{1}{\lambda^i} \beta^t \frac{P(s^t|s^0)}{\bar{c}^i(s^t)} \right) \bar{c}^i(s^t) &= \sum_{t=0}^{\infty} \sum_{s^t} \left(\frac{1}{\lambda^i} \beta^t \frac{P(s^t|s^0)}{\bar{c}^i(s^t)} \right) e^i(s^t) \\ \sum_{t=0}^{\infty} \sum_{s^t} \beta^t Pr(s^t|s^0) &= \sum_{t=0}^{\infty} \sum_{s^t} \frac{e^i(s^t)}{\bar{c}^i(s^t)} \beta^t Pr(s^t|s^0) \\ \sum_{t=0}^{\infty} \beta^t (1-\delta)^t + \beta^t (1-(1-\delta)^t) &= \sum_{t=0}^{\infty} \frac{e^i(s^t)}{\bar{c}^i(s_1)} \beta^t (1-\delta)^t + \frac{e^i(s^t)}{\bar{c}^i(s_2)} \beta^t (1-(1-\delta)^t) \\ \frac{1}{1-\beta} &= \sum_{t=0}^{\infty} \frac{e^i(s^t)}{\bar{c}^i(s_1)} \beta^t (1-\delta)^t + \frac{e^i(s^t)}{\bar{c}^i(s_2)} \beta^t (1-(1-\delta)^t) \end{aligned}$$

To simplify the algebra, we can consider that $e^1(s_1) = e^2(s_2) = 1$ and $e^1(s_2) = e^2(s_1) = 0$:

$$\begin{aligned} \bar{c}^1(s_1) &= (1-\beta) \sum_{t=0}^{\infty} \beta^t (1-\delta)^t = \frac{1-\beta}{1-\beta(1-\delta)} = \frac{1-\beta}{1-\beta+\beta\delta} \\ \bar{c}^2(s_2) &= (1-\beta) \sum_{t=0}^{\infty} \beta^t (1-(1-\delta)^t) = (1-\beta) \left(\frac{1}{1-\beta} - \frac{1}{1-\beta+\beta\delta} \right) = \frac{\beta\delta}{1-\beta+\beta\delta} \end{aligned}$$

Which we can combine with each state's market clearing conditions:

$$\begin{aligned} \bar{c}^1(s_1) + \bar{c}^2(s_1) &= e^1(s_1) + e^2(s_1) = 1 \\ \bar{c}^1(s_2) + \bar{c}^2(s_2) &= e^1(s_2) + e^2(s_2) = 1 \end{aligned}$$

Now we have four equations with four unknowns, which allows us to solve for the equilibrium consumption allocation for each consumer in each state:

$$\begin{aligned} \bar{c}^1(s_1) &= \frac{1-\beta}{1-\beta+\beta\delta}, \bar{c}^1(s_2) = \frac{1-\beta}{1-\beta+\beta\delta} \\ \bar{c}^2(s_1) &= \frac{\beta\delta}{1-\beta+\beta\delta}, \bar{c}^2(s_2) = \frac{\beta\delta}{1-\beta+\beta\delta} \end{aligned}$$

To determine the pricing kernel for claims to the endowment in each state,

we can use the first-order condition for two different periods:

$$\begin{aligned}
q_t^0(s^t) &= \frac{1}{\lambda^t} \beta^t \frac{P(s^t|s^0)}{\bar{c}^t(s^t)} \\
q_{t+1}^0(s^{t+1}) &= \frac{1}{\lambda^t} \beta^t \frac{P(s^{t+1}|s^0)}{\bar{c}^t(s^{t+1})} \\
q_t^0(s^t) &= q_{t+1}^0(s^{t+1}) \frac{1}{\beta} \frac{P(s^t|s^0)}{P(s^{t+1}|s^0)} \\
q_t^0(s_1) &= \frac{q_{t+1}^0(s_1)}{\beta(1-\delta)} \\
q_t^0(s_2) &= q_{t+1}^0(s_2) \frac{1}{\beta} \left(\frac{1 - (1-\delta)^t}{1 - (1-\delta)^{t+1}} \right)
\end{aligned}$$

With each iteration of the recursive pricing function for s_2 , the numerator and denominator of the probability of s_2 cancel out, providing the closed-form solution:

$$q_t^0(s_2) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{\beta} \left(\frac{1 - (1-\delta)^t}{1 - (1-\delta)^n} \right) \right\} = \frac{1 - (1-\delta)^t}{\beta}$$

The price of a claim to the endowment in s_1 , however, blows up to infinity in each period in the closed form. Let $\bar{q}_t^0(s_1)$ represent the price of a claim to the endowment in s_1 , normalized by the price in the first period. Then,

$$\bar{q}_t^0(s_1) = \frac{[\beta(1-\delta)]^{t-n}}{[\beta(1-\delta)]^{-n}} = [\beta(1-\delta)]^t$$

- (c) The price of a claim to the endowment in each period depends on the state in that period. Prices and allocations for the whole time horizon, and for each possible state in the horizon, are determined in period 0.¹ For this reason, the price of a claim to the endowment in s_2 increases for later dates (when s_2 is likelier), as the price of a claim to the endowment in s_1 declines (when s_1 gets increasingly unlikely). Thus, the price of a claim to the endowment in period t is represented by:

$$q_t(s^t) = \begin{cases} [\beta(1-\delta)]^t, & s^t = s_1 \\ \frac{1 - (1-\delta)^t}{\beta}, & s^t = s_2 \end{cases}$$

A risk-free asset would hold one claim in each state, with the price $\bar{q}_t^0(s_1) + q_t^0(s_2)$.

¹Note that this means that each consumer ends up holding unused claims in each period, which were purchased as insurance *for if the other state occurred in that period*.

Setting this equal to the reciprocal of the interest rate, we can solve:

$$\begin{aligned}\overline{q}_t^0(s_1) + q_t^0(s_2) &= \frac{(\beta^{t+1}(1-\delta)^t + 1 - (1-\delta)^t)}{\beta} = \frac{1}{R^f} \\ R^f &= \frac{\beta}{(\beta^{t+1}(1-\delta)^t + 1 - (1-\delta)^t)}\end{aligned}$$

- (d) Since there is no aggregate uncertainty in this economy, the social planner has no need to maximize *expected* utility and can simply allocate the aggregate endowment (which is constant in each period, regardless of state) across the two consumers however she pleases. Thus, the social planner's problem is:

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\lambda \ln(c_t^1) + (1-\lambda) \ln(c_t^2)] \quad \text{s.t.} \quad c_t^1 + c_t^2 \leq 1$$

This is a straightforward maximization problem that can be solved with first-order conditions, letting $c_t^1 = 1 - c_t^2$:

$$\begin{aligned}\beta^t \left[\frac{\partial u_t}{\partial c_t^2} = -\frac{\lambda}{1-c_t^2} + \frac{1-\lambda}{c_t^1} \right] &= 0 \\ \frac{1-\lambda}{c_t^1} &= \frac{\lambda}{1-c_t^2} \\ 1 - c_t^2 - \lambda + \lambda c_t^2 &= \lambda c_t^2 \\ c_t^2 &= 1 - \lambda \\ c_t^1 &= \lambda\end{aligned}$$

Thus, the social planner's allocation is $\{c_t^1, c_t^2\} = \{\lambda, 1 - \lambda\}$ for all t . Put another way, the Pareto optimal allocation for a given λ is to allocate a fraction of the aggregate endowment to each consumer in each period according to the consumer's weight, as determined by λ .

- (e) The competitive equilibrium is Pareto optimal, with $\lambda = \frac{1-\beta}{1-\beta+\beta\delta}$. Engineering a competitive solution to match any λ is more difficult, but still possible. For $\lambda = 0$ or $\lambda = 1$, δ can simply be set to give either consumer the endowment in every period. For $\lambda \in (0, 1)$, There are an infinite number of combinations of β and δ that result in the Pareto allocation given δ , with the relation between β and δ of:

$$\beta = \frac{1-\lambda}{1-(1-\delta)\lambda}$$

Question 2

- (a) A recursive competitive equilibrium is a continuous pricing function, $p(x)$, and a continuous, bounded value function, $V(a, x)$, such that:
- (i) $V(a, x)$ solves the Bellman equation
 - (ii) $\forall x$, $V(a, x)$ is attained by $c = x$, $x' = 1$
- (b) The stochastic endowment in this model, x_t , is the dividend. In equilibrium, $p_t = \mathbb{E} \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + x_{t+1}) \right]$ where $c_t = x_t$ and $x_{t+1} = 1$. Thus, with logarithmic utility and using a Taylor expansion around $\Delta c_{t+1} = 0$ and $p_{t+1} =$, we can solve:

$$\begin{aligned} p_t &= \mathbb{E}_t \left[\beta \frac{c_t}{c_{t+1}} (p_{t+1} + x_{t+1}) \right] \\ &= \mathbb{E}_t \left[\beta \frac{1}{1 + \Delta c_{t+1}} (p_{t+1} + 1) \right] \\ \frac{p_t}{x_t} &= \frac{1}{x_t} \mathbb{E}_t \left[\beta \frac{1}{1 + \Delta c_{t+1}} (p_{t+1} + 1) \right] \end{aligned}$$

(c)

(d)

Question 3

- (a) The Bellman equation for the consumer's problem is:

$$V(a, l) = \max_a \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \mathbb{E} [V(a', l')] \right\} \text{ s.t. } c + a' \leq wl + (1+r)a$$

Since the consumer does not value leisure, the budget constraint will hold with equality, and the Bellman becomes:

$$V(a, l) = \max_a \left\{ \frac{(wl + (1+r)a - a')^{1-\gamma}}{1-\gamma} + \beta \int V(a', l') Q(l, dl') \right\}$$

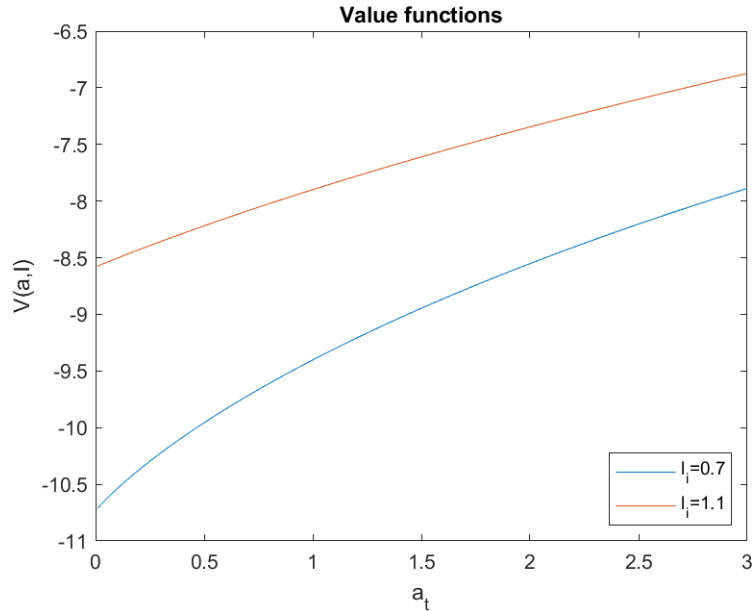
We can find the optimality conditions by taking the first order condition of the maximization problem and using the envelope condition:

$$\begin{aligned} (wl + (1+r)a - a')^{-\gamma} (1+r) + \beta \int V'(a', l') Q(l, dl') &= 0 \\ V'(a', l') &= (wl' + (1+r)a' - a'')^{-\gamma} (-1) \\ (wl + (1+r)a - a')^{-\gamma} (1+r) &= \beta \mathbb{E} \left[(wl' + (1+r)a' - a'')^{-\gamma} \right] \end{aligned}$$

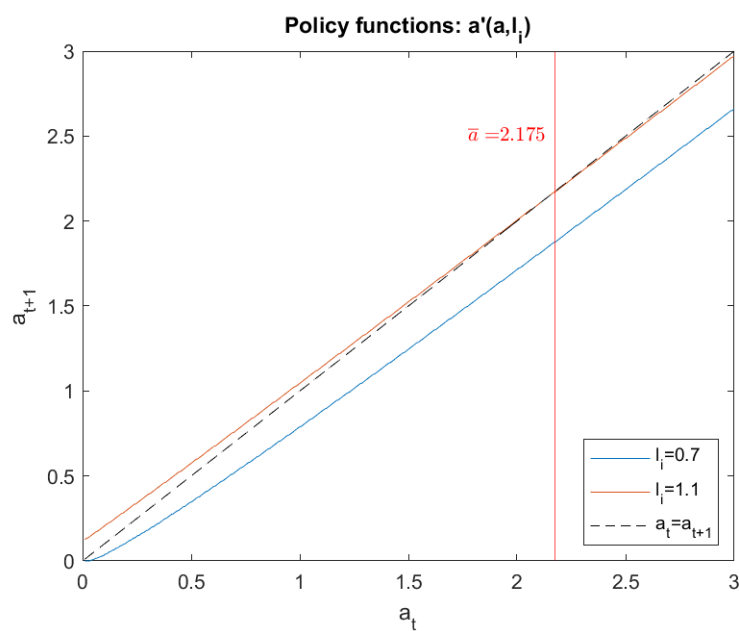
- (b) We can find the stationary distribution of the labor endowment by choosing an arbitrary starting point, l_0 , for the labor endowment, where l_0 is a row vector with one entry equal to one and the other equal to 0, then iteratively calculating $l = l_0 Q$, setting $l_0 = l$, then repeating until l converges to an unconditional probability of each labor value. This is done in the attached Matlab code, which outputs $l = [0.25 \ 0.75]$, where 0.25 is the probability mass of $l_t = 0.7$, and 0.75 is the probability mass of $l_t = 1.1$. The unconditional mean, then, is:

$$\mathbb{E}[l_t] = 0.25(0.7) + 0.75(1.1) = 1$$

- (c) The attached Matlab file conducts all of the necessary calculations for this problem.
- i. The chart below plots the value function, separated by labor states, against the asset values in the asset grid. The functions appear to be consistent with our assumptions: continuous, concave, differentiable, and strictly increasing. Furthermore, they appear as expected: at every level of asset holdings, the value function is lower for the lower labor supply state than for the higher labor supply state. It is also intuitive that the value function is steeper with respect to asset holdings in the lower labor supply state, as the consumer saves in order to smooth consumption from the high labor supply state to the low labor supply state.



- ii. The policy functions, $a'(a, l_i)$ are displayed below. There does appear to be a $\bar{a} \approx 2.175$ such that $\forall a \geq \bar{a}, a'(a, l_2) < a$, which is indicated on the chart. The existence of this value and the fact that $a'(a, l_1) < a$ for all a suggest that the long-run value of a_t never exceeds \bar{a} .



- (d) The chart below displays the marginal probability mass function for a_t , along with its mean, which is ≈ 1.02

