Problem Set #7

Danny Edgel
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Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

Let $X \subset \mathbb{R}^n$ be convex. We can prove that, for any $k \in \mathbb{N}$, $\lambda_1, ..., \lambda_k \geq 0$, $\sum_{i=1}^k \lambda_i = 1$, if $x_1, ..., x_k \in X$, then $\sum_{i=1}^k \lambda_i x_1 \in X$.

Proof.

1. *Base step.* Suppose $x_1, x_2 \in X$. Since X is convex, $(1-\lambda)x_1 + \lambda x_2$ is also in X for all $\lambda \in [0,1]$

2. *Induction Step.* Assume that, for some $k \in \mathbb{N}$, $\sum_{i=1}^k \lambda_i x_i \in X$, where $\sum_{i=1}^k = 1$. Let $x_{k+1} \in X$ and $\lambda' \in [0,1]$. Then, since X is convex,

$$(1-\lambda')x_{k+1} + \lambda' \sum_{i=1} \lambda_i x_i$$

is also in X. Now, define

$$\lambda'_i = \begin{cases} \lambda' \lambda_i, & i \in \{1, ..., k\} \\ 1 - \lambda', & i = k+1 \end{cases}$$

Then, $\sum_{i=1}^{k+1} \lambda'_i x_i \in X$ and $\sum_{i=1}^{k+1} \lambda'_i = 1$

$\therefore \sum_{i=1}^k \lambda_i x_i \in X$ for any $k \in \mathbb{N}$ ■

Question 2

Define C as the set of all convex combinations of S.

1. Suppose $x \in C$
   (a) By definition, $\exists s_1, ..., s_n \in S$, $\lambda_1, ..., \lambda_n \in [0,1]$, $\sum_{i=1}^{n} \lambda_i = 1$ such that $\sum_{i=1}^{n} \lambda_i s_i = x$
   (b) Let $X \supset S$ be a convex set. Since $x \in S$, $x \in X$. Since $\text{co}S = \bigcap_{\alpha \in \Omega} X_\alpha$, where Ω is the set of all convex sets that contain S, $x \in S \wedge x \in X \Rightarrow x \in \text{co}S$
   (c) Thus, $x \in C \Rightarrow x \in \text{co}S$

   $\therefore C \subseteq \text{co}S$

2. Suppose $x \in \text{co}S$
   (a) Any intersection of convex sets is also convex, so $\text{co}S$ is convex, so $\exists y_1, ..., y_m \in \text{co}S$, $\lambda_1, ..., \lambda_m \in [0,1]$, $\sum_{i=1}^{m} \lambda_i = 1$ such that $\sum_{i=1}^{m} \lambda_i y_i = x$
   (b) It is clearly apparent that $\text{co}S \subseteq S$, so $y_1, ..., y_m \in S$. Then, x is a convex combination of elements of S. so $x \in C$

   $\therefore \text{co}S \subseteq C$

$\therefore C = \text{co}S$ ■

Question 3

Suppose X is convex.

1. Let $x, y \in \text{cl}X$ and suppose $\exists z = (1-\lambda)x + \lambda y$, $z \notin \text{cl}X$
2. If $x, y \in X$, then, since X is convex, $(1-\lambda)x + \lambda y \in X$ $\forall \lambda$. Thus, $x, y \in X \Rightarrow z \in \text{cl}X$
3. If $x \in \text{cl}X$, $x \notin X$, and $y \in X$, then x is a limit point of X. Then, $\forall x' = (1-\lambda')x + \lambda' y$, $x' \in X$ or $x' = x$. Thus, either $z \in X$ or z is a limit point of x. Thus, $z \in \text{cl}X$.
4. If $x, y \in \text{cl}X$ and $x, y \notin X$, then both x and y are limit points of X. Thus, $\forall \varepsilon > 0$, $\exists x' \in B_\varepsilon(x)$, $y' \in B_\varepsilon(y)$ such that x' and y' are both in X and are convex combinations of x and y. Then, either z is equal to x or y, or $\exists \varepsilon$ such that $x' \in B_\varepsilon(x)$, $y' \in B_\varepsilon(y)$, and $z = (1-\lambda')x' + \lambda' y'$ for some $\lambda' \in [0,1]$. Thus, $z \in \text{cl}X$

$\therefore$ by contradiction, $\text{cl}X$ is convex ■

Question 4

1. Let $f : X \to \mathbb{R}$ be a concave function where $X \subseteq \mathbb{R}^n$.
   (a) Let $x_1, x_2 \in X$ and define $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then, $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are both in hyp$f$
   (b) Since X is convex and f is concave, $\forall \lambda \in [0,1]$, $f((1-\lambda)x_1 + \lambda x_2) \geq (1-\lambda)f(x_1) + \lambda f(x_2)$
   (c) Thus,
   $$(1-\lambda)x_1 + \lambda x_2 \in X \text{ and } (1-\lambda)y_1 + \lambda y_2 \leq f((1-\lambda)x_1 + \lambda x_2)$$
   so $(1-\lambda)z_1 + \lambda z_2 \in \text{hyp}f$

   $\therefore$ f concave $\Rightarrow$ hyp$f$ convex

2. Let hyp$f$ be a convex hypograph of $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$.
   (a) Let $z_1, z_2 \in \text{hyp}f$. Then, $\forall \lambda \in [0,1]$, $(1-\lambda)z_1 + \lambda z_2 \in \text{hyp}f$. Then,
   $$(1-\lambda)x_1 + \lambda x_2 \in X \text{ and } (1-\lambda)y_1 + \lambda y_2 \leq f((1-\lambda)x_1 + \lambda x_2)$$
   Thus, X is a convex set and, $\forall x_1, x_2 \in X$, $(1-\lambda)f(x_1) + \lambda f(x_2)$
   $$f((1-\lambda)x_1 + \lambda x_2) \geq (1-\lambda)f(x_1) + \lambda f(x_2)$$

   $\therefore$ hyp$f$ convex $\Rightarrow$ f concave

$\therefore$ f is convave if and only if hyp$f$ is convex ■

Question 5

1. Let X and Y be closed, convex sets, and let X be compact
2. Fix $y_0 \in Y$ and define $A = \{y - x | x \in X\}$. Let a_n be a sequence in A such that $a_n = y_n - x_n$ for all n
3. Since X is compact, $\exists x_{n_k} \to x \in X$, and $y_n \to y \in \mathbb{R}^n$. Thus, $a_{n_k} \to a \in A$. Therefore, A is closed and, given that X and Y are convex, A is also convex.
4. Since X and Y are disjoint, $0 \notin A$. Thus $\exists H(p, \beta)$ that strictly separates A and $\{0\}$. Then, we can solve:
$$\begin{aligned} 0 <& \beta < p \cdot a \\ 0 <& \beta < p \cdot (y - x) \\ 0 <& \beta < p \cdot y - p \cdot x \\ p \cdot x <& \beta + p \cdot x < p \cdot y \end{aligned}$$

$\therefore$ $\exists H(p, \alpha)$ that strictly separates X and Y ■

Question 6

1. Suppose $\exists x \in \mathbb{R}^n$ such that $\inf\limits_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i(x_i) > 0$ and $\inf\limits_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i(-x_i) > 0$

   (a) Further suppose $\exists \pi' \in \Pi_A \bigcap \Pi_B$

   (b) If $\sum_{i=1}^n \pi'_i x_i > 0$, then $\sum_{i=1}^n \pi'_i(-x_i) < 0$. If $\sum_{i=1}^n \pi'_i(-x_i) > 0$, then $\sum_{i=1}^n \pi'_i x_i < 0$

   (c) Thus, $\exists \pi' \in \Pi_A \bigcap \Pi_B \Rightarrow \nexists x \in \mathbb{R}^n$ such that $\inf\limits_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i(x_i) > 0$ and $\inf\limits_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i((-x_i)) > 0$

   $\therefore$ By contradition, if $\exists x \in \mathbb{R}^n$ such that $\inf\limits_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i(x_i) > 0$ and $\inf\limits_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i((-x_i)) > 0$, then $\Pi_A \bigcap \Pi_B = \emptyset$

2. Suppose $\Pi_A \bigcap \Pi_B = \emptyset$, Therefore, Π_A and Π_B are disjoint.

   (a) Since Π_A and Π_B are disjoint and both compact, convex sets in $\mathbb{R}^n$, $A = \Pi_A - \Pi_B$ is a compact, convex set where $0 \notin A$. Then, by the second theorem for a lecture,

   $$\exists H(x^*, \beta) \text{ s.t. } 0 < \beta < a^* \cdot x^*$$

   Where $a^* = \inf\limits_{\pi_A \in \Pi_A, \pi_B \in \Pi_B} \{\pi_A \cdot x^* - \pi_B \cdot x^*\}$ Then, $a^* = \pi_A^* \cdot x^* - \pi_B^* \cdot x^*$, where:

   $$\pi_A^* = \inf_{\pi_A \in \Pi_A} \{\pi_A \cdot x^*\}, \text{ and } \pi_B^* = \sup_{\pi_B \in \Pi_B} \{\pi_B \cdot x^*\}$$

   Then, we can solve $0 < \beta < \underline{a^* \cdot x^*}$ to derive $\pi_B^* \cdot x^* \underline{< \beta} + \pi_B^* \cdot x^* < \pi_A^* \cdot x^*$. Define $\alpha = \beta + \pi_B^* \cdot x^*$ and $\overline{alpha} \in \mathbb{R}^n$ such that $\overline{alpha}_i = \alpha \; \forall i \in \{1, ..., n\}$. Thus, $\forall \pi_A \in \Pi_A, \pi_B \in \Pi_B$,

   $$\pi_B \cdot (x^* - \overline{\alpha}) \leq \pi_B^* \cdot x^* - \alpha < 0 < \pi_A^* \cdot x^* - \alpha \leq \pi_A \cdot (x^* - \overline{\alpha})$$

   So $(x^* - \overline{\alpha})$ is an agreeable trade.

   $\therefore \Pi_A \bigcap \Pi_B = \emptyset \Rightarrow \exists x \in \mathbb{R}^n$ such that $\inf\limits_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i(x_i) > 0$ and $\inf\limits_{\pi \in \Pi_B} \sum_{i=1}^n \pi_i(-x_i) > 0$

$\therefore$ There is an agreeable trade if and only if there is no common prior ■