Problem Set #1

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Question 1

Let $(Y, X')'$ be a random vector, where $Y = X'\beta_0 \cdot U$, where $\mathbb{E}\left[U \mid X\right] = 1$, $\mathbb{E}\left[XX'\right]$ is invertible, and $\mathbb{E}\left[Y^2 + ||X||^2\right] < \infty$.

(i) Since $\mathbb{E}\left[U \mid X\right] = 1$, the expectation of Y, conditional on X, is $X'\beta_0$. Then, $\frac{\partial}{\partial X}Y = \beta_0$.

(ii) Define $V = U - 1$. Then, $\mathbb{E}\left[V \mid X\right] = \mathbb{E}\left[U - 1 \mid X\right] = 0$, and:

$$Y = X'\beta_0(V + 1) = X'\beta_0 V + X'\beta_0 = X'\beta_0 + \widetilde{U}$$

Where:

$$\mathbb{E}\left[\widetilde{U} \mid X\right] = \mathbb{E}\left[X'\beta_0 V \mid X\right] = X'\beta_0\mathbb{E}\left[V \mid X\right] = 0$$

Thus, $Y = X'\beta_0 + \widetilde{U}$, where $\mathbb{E}\left[\widetilde{U} \mid X\right] = 0$.

(iii) Let $\beta = \beta_0$. Then,

(iv) Define $V = U - 1$. Then, $\mathbb{E}\left[V \mid X\right] = \mathbb{E}\left[U - 1 \mid X\right] = 0$, and:

$$\mathbb{E}\left[X(Y - X'\beta)\right] = \mathbb{E}\left[X(X'\beta_0 \cdot U - X'\beta_0)\right] = \mathbb{E}\left[\mathbb{E}\left[X(X'\beta_0 \cdot U - X'\beta_0)|X\right]\right]$$
$$= \mathbb{E}\left[XX'\beta_0\mathbb{E}\left[(U - 1)|X\right]\right] = 0$$

Thus, $\beta = \beta_0 \Rightarrow \mathbb{E}\left[X(Y - X'\beta)\right] = 0$. Now, Suppose $\mathbb{E}\left[X(Y - X'\beta)\right] = 0$. Then,

$$\mathbb{E}\left[X(Y - X'\beta)\right] = \mathbb{E}\left[X(X'\beta \cdot U - X'\beta_0)\right] = 0$$
$$\mathbb{E}\left[XX'\mathbb{E}\left[\beta \cdot U - \beta_0|X\right]\right] = (\beta - \beta_0)\mathbb{E}\left[XX'\right] = 0$$

We know that $\mathbb{E}\left[XX'\right]$ is invertible, so $\mathbb{E}\left[XX'\right] \neq 0$. Thus, $\mathbb{E}\left[X(Y - X'\beta)\right] = 0 \Rightarrow \beta = \beta_0$.

$\therefore \mathbb{E}\left[X(Y - X'\beta)\right] = 0 \iff \beta = \beta_0$ ∎

Knowing this, we can derive the method of moments estimator for β:

$$\mathbb{E}\left[X(Y - X'\beta)\right] = \mathbb{E}\left[XY\right] - \mathbb{E}\left[XX'\right]\beta = 0$$
$$\mathbb{E}\left[XX'\right]\beta = \mathbb{E}\left[XY\right]$$
$$\beta = \mathbb{E}\left[XX'\right]^{-1}\mathbb{E}\left[XY\right]$$
$$\Rightarrow \hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i = \hat{\beta}_{OLS}$$

(v) We can simplify the final equation in (iii) to show:

$$\mathbb{E}\left[\hat{\beta} \mid X_1, ..., X_n\right] = \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i \mid X_1, ..., X_n\right]$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i \mathbb{E}\left[X_i'\beta_0 \cdot U \mid X_1, ..., X_n\right]$$

$$= \beta_0 \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right) \mathbb{E}\left[U \mid X_1, ..., X_n\right]$$

$$= \beta_0$$

Thus, $\hat{\beta}$ is unbiased.

(vi) According to the weak law of large numbers (WLLN), random variables converge in probability to their expected value. Thus,

$$\hat{\beta} \to_p \mathbb{E}\left[\hat{\beta}\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{\beta} \mid X_1, ..., X_n\right]\right] = \mathbb{E}\left[\beta_0\right] = \beta_0$$

Thus, $\hat{\beta}$ is consistent.

Question 2

(i) We can use the continuous mapping theorem and law of large numbers to show:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^3 \to_p \mathbb{E}\left[X^3\right] \quad , \quad \frac{\sum_{i=1}^{n} X_i^3}{\sum_{i=1}^{n} X_i^2} = \frac{\mathbb{E}\left[X^3\right]}{\mathbb{E}\left[X^2\right]}$$

We cannot show convergence for the other two statistics.

(ii) You can use continuous mapping and the central limit theorem to show convergence in distribution of W_n, since:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i^2 - \mathbb{E}\left[X_i^2\right]\right) = \frac{1}{\sqrt{n}}n\left(\overline{X}_n^2 - \mathbb{E}\left[X_i^2\right]\right) = \sqrt{n}\left(\overline{X}_n^2 - \mathbb{E}\left[X_i^2\right]\right)$$

The other statistic cannot be shown to converge, as it simplifies to $\sqrt{n}\left(\overline{X}_n^2 - \overline{X}_n^2\right)$

(iii) The CDF of X is simply $f(x) = x$. The CDF for the maximum value, x, of n observations of X_i, then, is x^n. Thus, we can define for any $\varepsilon \in (0,1)$,

$$Pr\left(|\max_{1\leq i\leq n} X_i - 1| \leq \varepsilon\right) = Pr\left(\max_{1\leq i\leq n} X_i - 1 \geq -\varepsilon\right)$$

$$= Pr\left(\max_{1\leq i\leq n} X_i \geq 1 - \varepsilon\right)$$

$$= 1 - Pr\left(\max_{1\leq i\leq n} X_i \geq 1 - \varepsilon\right)$$

$$= 1 - (1-\varepsilon)^n$$

$$1 - (1-\varepsilon)^n \to_p 1$$

(iv) The CDF of X is $f(x) = 1 - e^{-x}$. Thus, Thus, we can define for any $\varepsilon > 0$ and $M \geq 0$,

$$Pr\left(\max_{1\leq i\leq n} X_i \leq M\right) = (1 - e^{-M})^n (1 - e^{-M})^n \qquad \to_p 1$$

Question 3

(i)

(ii)

(iii)

(iv)

(v)