Econ 712 Macroeconomics I

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Problem Set[1] #1

Problem 1: A first look at difference equations in economics

Here we are interested in how the stock market may react to a future FOMC announcement of an increase in short term interest rates. To do so, consider the following simple perfect foresight model of stock price dynamics given by the following equation

$$p_t = \frac{d + p_{t+1}}{(1+r)} \tag{1}$$

where p_t is the price of a share at the beginning of period t before constant dividend d is paid out, and r is the short term risk free interest rate. The left hand side of (1) is the cost of buying a share while the right hand side is the benefit of buying the share (the owner receives a dividend and capital gain or loss from the sale of the share). Assume $r > 0$.

1. Solve for the steady state stock price $p^* = p_t = p_{t+1}$.

   Since $r > 0$, substituting p_t and p_{t+1} with p^* on either side of equation (1) and solving p^* gives
   $$p^* = \frac{d}{r} \tag{2}$$

   Assume the initial price, p_0, is given. Solve the closed form solution to the first order linear difference equation in (1). Explain how price evolves over time (i.e. what if $p_0 > p^*$, $p_0 < p^*$, $p_0 = p^*$) using both a phase diagram (i.e p_{t+1} against p_t) as well as a graph of p_t against time t. If the initial stock price is away from the steady state, does it converge or diverge from the steady state. Explain why.

   Equation (1) is a first-order autonomous equation to which a general solution is

   $$p_t^g = c(1+r)^t + p_t^p \tag{3}$$

[1] **Based on previous problem sets by Anton Babkin, Fu Tan and Eirik Brandsås**

where the first term on the right hand side of (3) is the complementary solution to the associated homogeneous equation and the second term is the particular solution. As we have solved the steady state, p^*, in part 1, the general solution to equation (1) becomes

$$p_t^g = p^* + c(1+r)^t = \frac{d}{r} + c(1+r)^t \tag{3}$$

To solve c, we use the initial price as a boundary condition

$$p_0 = \frac{d}{r} + c \implies c = p_0 - \frac{d}{r} \tag{4}$$

Thus the closed form solution to equation (1) is

$$p_t = \frac{d}{r} + \left(p_0 - \frac{d}{r}\right)(1+r)^t \tag{5}$$

If the initial price is away from the steady state, it would diverge from the steady state. If we rearrange equation (1) to obtain the following relation

$$\frac{p_{t+1} + d}{p_t} = 1 + r \tag{6}$$

Now suppose $p_0 > p^*$, in order for equation (6) to hold, it must be that the price at $t = 1$ increases so that capital gains compensate for a dividend that is low relative to the price

$$\frac{d}{p_0} < r \implies \frac{p_1}{p_0} > 1 \tag{7}$$

A higher price at $t = 1$ requires an even greater capital gain/price afterwards. If $p_0 < p^*$, in order for equation (6) to hold, the price at $t = 1$ has to drop to offset the excess dividend price ratio to risk-free return rate. A lower price at $t = 1$ then requires an even bigger capital loss thereafter. The only way to interpret an explosive path is as a speculative bubble in which unreasonable expectations become self-fulfilling. Though we have seen assets bubbles from time to time, we may wish to rule out such expectations and price behavior under normal circumstances and focus on the fundamental solution which reflects the precent value of the underlying stream of dividends.

To attain the fundamental solution, we set $c = 0$ to eliminate the bubble term, making the initial stock price endogenous. In equilibrium, the stock price p jumps immediately to p^* and remains there forever until the system is disturbed in some way.

From the closed form solution (5), we can see that how price evolves over time depends on the initial price since $r > 0$. If we take the initial value of p as given, unless $p_0 = p^*$, the stock price will explode as the bubble term $\left(p_0 - \frac{d}{r}\right)(1+r)^t$ goes to either plus or minus infinity as t goes to infinity.

The left panel of Figure 1 is a phase diagram for price and right panel is a graph for price dynamics with different initial values for price. When the stock price is above the steady state level, $p_0 > p^*$, the price would increase forever. When the stock price is below the steady state, $p_0 < p^*$, the price would decrease forever[2]. The price will be at the steady state forever if it starts at the steady state.

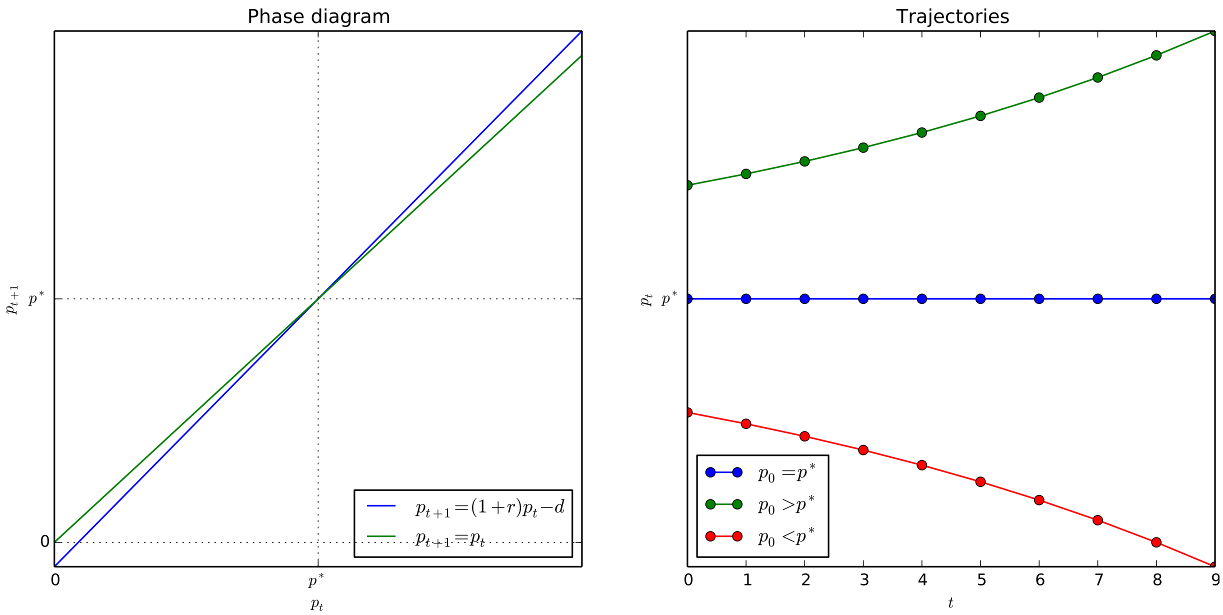

Figure 1: Phase diagram and price dynamics

[Matlab] Suppose the risk free rate is $r = 1\%$ and the stock pays constant dividend $d = 1$ per share per period. Open the Matlab code we provided. The code generates and plots the price dynamics given the first-order difference equation in part 2 given initial share price $p_0 = 100$ at time $t = 0$. Modify the code (i.e. simply replace 100) with three different initial prices which respectively are below, at, and above the steady state price level implied by these

[2]In reality, the price should not be below zero. However, let us ignore the non-negativity for a moment.

parameters to plot the price dynamics over 100 periods.

This is a Matlab exercise. With the Matlab code provided, you only need to try three different initial values for price. The steady state price is 100. The following graph plots the price trajectories for 100 periods when the initial price level is at 75, 100, 125 respectively which confirms what we get in part 2.

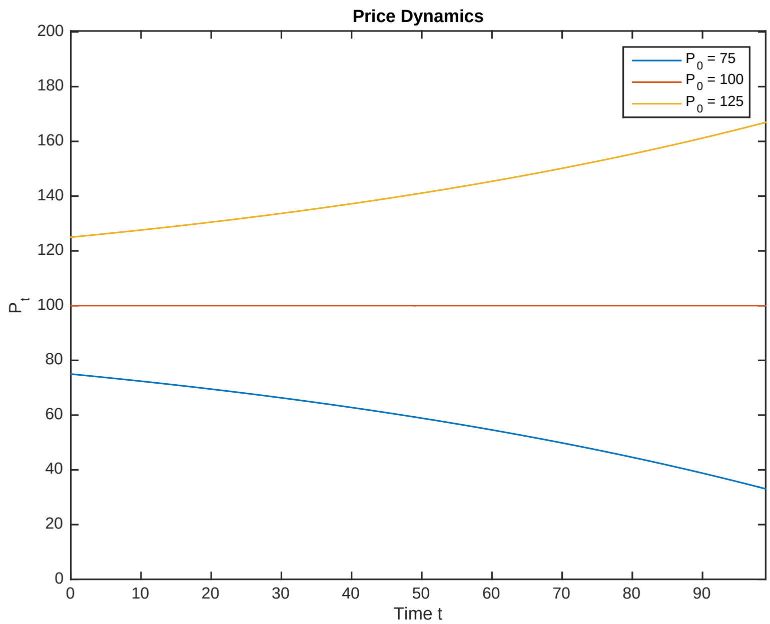

Figure 2: Price dynamics with different initial prices

[**Analytical and Matlab**] Suppose the Federal Reserve announces at $t = 20$ to raise the federal funds rate from 1% to 2% at $t = 50$ and remain at the new level forever. Using (1) with $d = 1$, how does the price respond to the policy announcement and the interest rate change over time? Plot the price dynamics from $t = 0$ to $t = 99$.

The rise in risk-free interest rate yields a lower steady state for price at

$$p^{*\prime} = \frac{d}{r'} = \frac{1}{0.02} = 50 < p^* = \frac{1}{0.01} = 100 \tag{8}$$

If we rule out bubbles, the price can be only be at the new steady state for $t \geq 50$. For $t < 50$, the risk-free interest rate remains at 1% and so the price dynamics still satisfy the old law of motion and the price path for $20 \leq t \leq 50$ follows

$$p_t = \frac{1}{1+r} p_{t+1} + \frac{d}{1+r} = \frac{1}{1.01} p_{t+1} + \frac{1}{1.01} \tag{9}$$

and the price stays at the initial steady state for $0 \leq t < 20$. Price drops immediately following the announcement at $t = 20$ and continue to decline until it reaches the new steady state price at the time of interest rate change, $t = 50$. Figure 3 illustrates the time path of stock price.

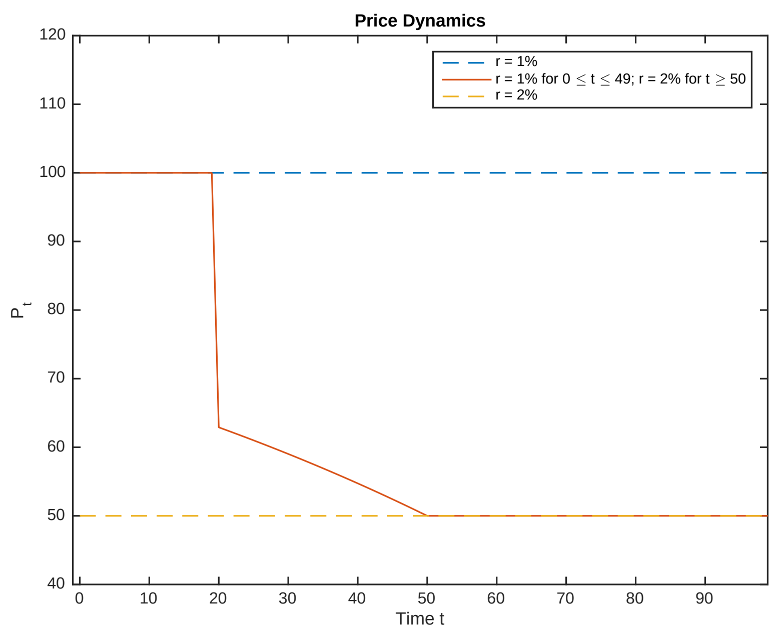

Figure 3: Price response to policy announcement and interest rate change

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