Homework 3

Due SUNDAY night, November 15, at midnight.

Feel free to work together on these problems, but each student needs to write up his/her own answers at the end, rather than directly copying from one master solution.

1. (**Arms race with market power**) Two (expected payoff maximizing) gangs are competing in an arms race. Each of them likes having more weapons w_i but dislikes having a different amount than the other agent. Obtaining weapons is costly, with the price of weapons increasing in the average quantity of weapons purchased. The price of weapons is

$$P(\bar{w}) = \rho + \alpha\bar{w}$$

where $\bar{w}$ is the average amount of weapons purchased. Each gang's payoff is

$$u_i(w_i, w_j) = \gamma w_i - \beta(w_i - w_j)^2 - P(\bar{w})w_i$$

All parameters α, β, γ and ρ are strictly positive, and $\gamma > \rho$.

(a) Explain the economic intuition of the assumption that $\gamma > \rho$. What does this assumption guarantee?

(b) Under what condition(s) is this game supermodular?

(c) Find the symmetric pure-strategy Nash equilibrium.

(d) In the Nash equilibrium you found in part (c), how does the equilibrium quantity of weapons change with each parameter? Provide intuition for each effect.

(e) Does there exist an equilibrium in which one or both gangs choose to have no weapons? If so, find such an equilibrium; if not, show why not.

(f) Can this game support a mixed strategy Nash equilibrium? If so, find such an equilibrium. If not, explain why it cannot exist.

(g) Suppose both gangs have the equilibrium quantity of weapons from part (c). A horde of goblins suddenly invades the area. Because weapons can be used to fight goblins, the inherent value γ of weapons increases. However, the goblins also steal all the money of gang 2, leaving that gang unable to purchase new weapons. How does gang 1's weapons quantity respond to this shock? How does the magnitude of this response compare to the magnitude if both gangs were able to respond?

2. Consider the game from question 1, but now with a continuum of gangs. Let the payoffs now depend on the average weapons quantity $\bar{q}$, rather than that of any single other gang:

$$u_i(w_i, \bar{w}) = \gamma w_i - \beta(w_i - \bar{w})^2 - P(\bar{q})w_i$$

Find the symmetric pure strategy Nash equilibrium. How and why does the equilibrium quantity of weapons differ from the equilibrium quantity in the two-gang game?

3. A continuum of agents plays a guessing game, in which each agent i guesses a number $x_i \in [0, 1]$. Agents prefer to guess numbers closer to some commonly known constant $\alpha \in (0, 1)$, and dislike guessing further from the average value $\bar{x}$. Each agent's payoff is

$$u_i(x_i; \bar{x}, \alpha) = (x_i - \alpha)^2 - (x_i - \bar{x})^2$$

   (a) Find all symmetric pure strategy Nash equilibria.
   (b) Recall that with a continuum of agents, a Nash equilibrium can be characterized by a quantile function, regardless of whether it is achieved using asymmetric pure strategies or with randomization. Describe all non-degenerate quantile functions that are Nash equilibria in this game, excluding the equilibria found in part (a).
   (c) Now suppose agents can choose any $x \in \mathbb{R}$. Describe all Nash equilibria.

4. Two players choose numbers in $\mathbb{R}$. Their payoffs are

$$u_i(q_i, q_j) = q_i + q_i(q_j - 1)^{1/3} - \frac{1}{2}q_i^2$$

Find all Nash equilibria of this game.

5. Two players play a game of Rock, Paper, Scissors. In this game, each player simultaneously chooses a strategy from the set {Rock, Paper, Scissors}. Paper beats Rock: if one player chooses Paper and the other chooses Rock, the player choosing Paper receives a payoff of 10, and the other player receives a payoff of 10. Similarly, Scissors beats Paper, and Rock beats Scissors. If both players choose the same pure strategy, each receives a payoff of 0. Players can also choose mixed strategies. However, in order to implement a mixed strategy, a player must rent a randomizing device. The rental costs 1 unit of payoffs. Find all Nash equilibria of this game.