Homework 6

Due Monday night, December 14, at midnight.

Feel free to work together on these problems, but each student needs to write up his/her own answers at the end, rather than directly copying from one master solution.

1. Consider an infinite repetition of the normal form game below, in which both players discount the future at rate $\delta \in (0, 1)$.

		2: C	2: D
1	C	2,2	0,8
	D	8,0	1,1

For what values of δ can the play path $\{(C, C), (C, C), ...\}$ be supported in a subgame perfect equilibrium?

♠ Solution: Consider the grim trigger strategy: cooperate until anyone defects; then defect forever. If both players choose this strategy, there are two future play paths to consider: those after histories in which both players have always chosen C, and those after histories in which some player has chosen D. In the former case, we use the one shot deviation principle to show that following the equilibrium is optimal whenever $2 \geq (1 - \delta)8 + \delta(1)$, or equivalently whenever $\delta \geq 6/7$. In the latter case, following the equilibrium is optimal whenever $1 \geq (1 - \delta)0 + \delta(1)$, which is always true.

2. Consider an infinite repetition of the following two-player normal form game: Consider the following repeated game strategy profile σ:

		2: C	2: D	2: P
	C	2,2	0,3	0,0
1	D	3,0	1,1	1,0
	P	0,0	0,1	0,0

(I) Play (C,C) initially, or if (C,C) was played last period.

(II) If there is a deviation from (I), play P once and then restart (I).

(III) If there is a deviation from (II), then restart (II).

Now answer the following questions:

(a) For what values of δ is strategy profile $\sigma = (\sigma_1, \sigma_2)$ a subgame perfect equilibrium?

♠ Solution: There is no profitable one-shot deviation from the equilibrium phase (I) if

$$(1-\delta)\sum_{t=0}^{\infty} 2\delta^t \geq (1-\delta)\left(3 + 0\delta + \sum_{t=2}^{\infty} 2\delta^t\right)$$

$$\iff \qquad 2 + 2\delta \geq 3 \qquad \iff \qquad \delta \geq \frac{1}{2}$$

There is no profitable one-shot deviation from the punishment phase (II) if

$$(1-\delta)\left(0 + \sum_{t=1}^{\infty} 2\delta^t\right) \geq (1-\delta)\left(1 + 0\delta + \sum_{t=2}^{\infty} 2\delta^t\right)$$

$$\iff \qquad 0 + 2\delta \geq 1 \qquad \iff \qquad \delta \geq \frac{1}{2}$$

Thus, the strategy profile is subgame perfect whenever $\delta \geq 1/2$.

(b) Suppose that in the stage game, action profile (P, P) results in both players receiving a payoff of $1/2$ rather than a payoff of 0. In this case, what are the values of δ for which strategy profile $\sigma = (\sigma_1, \sigma_2)$ is a subgame perfect equilibrium?

♠ Solution: There is no profitable one-shot deviation from the equilibrium phase (I) if

$$(1-\delta)\sum_{t=0}^{\infty} 2\delta^t \geq (1-\delta)\left(3 + \frac{1}{2}\delta + \sum_{t=2}^{\infty} 2\delta^t\right)$$

$$\iff \qquad 2 + 2\delta \geq 3 + \frac{1}{2}\delta \qquad \iff \qquad \delta \geq \frac{2}{3}$$

There is no profitable one-shot deviation from the punishment phase (II) if

$$(1-\delta)\left(\frac{1}{2} + \sum_{t=1}^{\infty} 2\delta^t\right) \geq (1-\delta)\left(1 + \frac{1}{2}\delta + \sum_{t=2}^{\infty} 2\delta^t\right)$$

$$\iff \qquad \frac{1}{2} + 2\delta \geq 1 + \frac{1}{2}\delta \qquad \iff \qquad \delta \geq \frac{1}{3}$$

Thus, the strategy profile is subgame perfect whenever $\delta \geq 2/3$.

(c) Give intuitive explanations for any differences in the results of your analyses of parts (a) and (b).

♠ Solution: When the payoffs to (P, P) increase from $(0, 0)$ to $(1/2, 1/2)$, the punishment phase (II) becomes less unpleasant. Because of this, the punishment is only enough to prevent deviations from phase I if the players are relatively patient; this is why the the phase (I) cutoff increases from 1/2 to 2/3. On the other hand, since the punishment is not as harsh, the agents are less reluctant to endure it; this is why the phase (II) cutoff declines from 1/2 to 1/3. Since only the higher cutoff matters, the end result is that a higher discount rate is needed to ensure thatσ is a subgame perfect equilibrium.

(d) What is the set of payoffs supportable as a subgame perfect equilibrium (if possible, using a public randomizing device) for some $\delta < 1$?

♠ Solution: Per the Folk Theorem, we can support any payoffs vectors that strictly exceed the mixmax payoffs for each player. Because D is strictly dominant in the stage game, the mixmax payoff vector is $(1, 1)$, which is produced by the (D, D) Nash equilibrium. Plotting the payoffs and restricting them to weakly exceed the mixmax payoffs, we can support payoffs using randomizations between (C, C), (C, D), (D, C) and (D, D) that put sufficiently low weight on the asymmetric actions to ensure expected payoffs greater than 1. This is the space defined by

$$F^* = \left\{ v \in \mathbb{R}^2_+ : v_1 \geq 1, v_2 \geq 1, v_1 + \frac{1}{2}v_2 \leq 3, \frac{1}{2}v_2 + v_2 \leq 3 \right\}$$

3. In the three-player normal form game G, each player's pure strategy set is $S_i = \{A, B, C, D\}$. Payoffs in G are described as follows: If any player plays D, all players obtain a payoff of 0. If one player plays A, one B, and one C, then the A player's payoff is 2, the B player's payoff is 0, and the C player's payoff is -1. Under any other strategy profile, all players obtain -2.

(a) Let $G^\infty(\delta)$ be the infinite repetition of G at discount rate $\delta \in (0, 1)$. Construct a pure strategy profile whose equilibrium play path is

(I) $(A, B, C), (B, C, A), (C, A, B), (A, B, C), (B, C, A), (C, A, B), \ldots$

and that is a subgame perfect equilibrium of G^∞ for large enough values of δ. For which values of δ is the strategy profile you constructed a subgame perfect equilibrium?

♠ Solution: Suppose each player follows a grim trigger strategy under which any deviation from the play path leads to perpetual play of (D, D, D). Since (D, D, D) is a Nash equilibrium of G, to determine whether all players following this strategy constitutes a subgame perfect equilibrium, it is enough to check for one-shot deviations from the equilibrium path. At each point on the equilibrium path, a player's best one-shot deviation is to D, leading to a continuation payoff of 0. In periods in which a player is supposed to play A, his payoff on the equilibrium path is

$$(1-\delta)\left(2-\delta^2+2\delta^3-\delta^5+\dots\right) = (1-\delta)\sum_{k=0}^{\infty}\delta^{3k}(2-\delta^2) = \frac{(1-\delta)(2-\delta^2)}{1-\delta^3} = \frac{2-\delta^2}{1+\delta+\delta^2}$$

Since this is positive there is no profitable one-shot deviation. Proceeding similarly, for periods in which a player is supposed to play B, his payoff on the equilibrium path is

$$\frac{-\delta+2\delta^2}{1+\delta+\delta^2}$$

There is no profitable one-shot deviation when this is nonnegative, which is the case when $\delta \geq 1/2$. In periods in which a player is supposed to play C, his payoff on the equilibrium path is

$$\frac{-1+2\delta}{1+\delta+\delta^2}$$

which is also nonnegative when $\delta \geq 1/2$. Thus the grim trigger strategy is subgame perfect when $\delta \geq 1/2$.

(b) Now consider the play path

(II) $\quad (A, B, C), (C, A, B), (B, C, A), (A, B, C), (C, A, B), (B, C, A), \dots$

Is play path (II) attainable in a subgame perfect equilibrium for a smaller or larger set of discount rates than play path (I)? Provide intuition for your answer.

♠ Solution: Again suppose that each player plays the grim trigger strategy, this time to support path (II). In this case the payoffs on the equilibrium path when a player is supposed to play A, C and B are $\frac{1}{1+\delta+\delta^2}(-1+2\delta)$, which is always positive, $\frac{1}{1+\delta+\delta^2}(-1+2\delta^2)$, which is nonnegative when $\delta \geq 1/\sqrt{2}$, and $\frac{1}{1+\delta+\delta^2}(2\delta-\delta^2)$, which is always positive. Thus the strategy profile above is subgame perfect when $\delta \geq 1/\sqrt{2}$.

This requirement is more demanding than the one needed to support (I). The reason is that under (II), the player who is supposed to play C and obtain -1 will not be rewarded for two periods, when he will play A and obtain 2, while under (I) the reward comes in the next period. The point is that a player who is not willing to wait two periods to obtain his reward may be willing to wait just one.

4. (Bill PS3 Q1) In the game Γ, player 1 moves first, choosing between actions A and B. If he chooses B, then player 2 chooses between actions C and D. If she chooses D, then player 1 moves again, choosing between actions E, F, and G.

   (a) Find a behavior strategy which is equivalent to the mixed strategy

   $$\sigma_1 = \Big(\sigma_1(AE), \sigma_1(AF), \sigma_1(AG), \sigma_1(BE), \sigma_1(BF), \sigma_1(BG)\Big) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{12}, \frac{1}{12}\right)$$

   ♠ Solution:

   $$\bigg(\Big(\beta_1(A), \beta_1(B)\Big), \Big(\beta_1(E), \beta_1(F), \beta_1(G)\Big)\bigg) = \left(\left(\frac{5}{6}, \frac{1}{6}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)\right)$$

   (b) Describe all mixed strategies which are equivalent to the behavior strategy

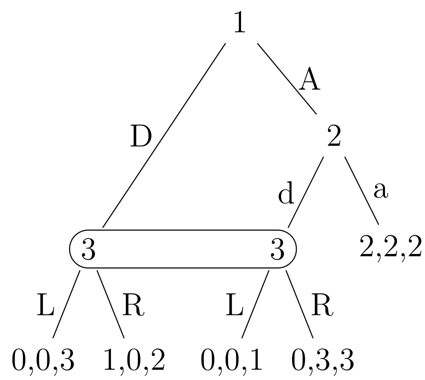
   $$\beta_1 = \bigg(\Big(\beta_1(A), \beta_1(B)\Big), \Big(\beta_1(E), \beta_1(F), \beta_1(G)\Big)\bigg) = \left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)\right)$$

   ♠ Solution: The requirements are:

   $$\sigma_1(AE)+\sigma_1(AF)+\sigma_1(AG) = \frac{1}{3} \qquad \sigma_1(BE) = \frac{1}{3} \qquad \sigma_1(BF) = \sigma_1(BG) = \frac{1}{6}$$

5. (Bill PS4 Q1) Compute all sequential equilibria of the following game.

   ♠ Solution: It is easy to check that if 3's information set isn't reached, consistency places no restriction on his beliefs. Let x denote 3's left decision node (following D). I will consider all possible behaviors of players 1 and 2:



- (D, d) Implies that 3 plays L. This is a Nash equilibrium but is not sequential because if reached, 2 would deviate to a.
- (D, a) Implies that 3 plays L, which implies that 1 prefers to deviate to A.
- (D, mix) Implies that 3 plays L, but again, 2 should deviate to a.
- (A, d) Implies that 3 plays R, which implies that 1 prefers to deviate to D.
- (A, a) 1 and 2 are willing to do this if $\sigma_2(L) \geq 1/3$. 3 is willing to play L if $\mu(x) \geq 2/3$ and to choose $\sigma_3(L) \geq 1/3$ if $\mu(x) = 2/3$. These are sequential equilibria.
- (A, mix) 2 only mixes if $\sigma_3(L) = 1/3$; this is only possible if $\mu(x) = 2/3$, but given the behavior of 1 and 2, $\mu(x)$ should be zero, a contradiction.
- (mix, a) Implies that 3 plays L, which implies that 1 strictly prefers A.
- (mix, d) For 1 to be willing to mix, 3 must play L, but then 2 would deviate to a.
- (mix, mix) For 2 to be indifferent, $\sigma_3(L) = 1/3$, which implies that $\mu(x) = 2/3$. Then by Bayes rule, we need

  $$\frac{2}{3} = \frac{\sigma_1(D)}{\sigma_1(D) + (1 - \sigma_1(D))\sigma_2(d)}$$

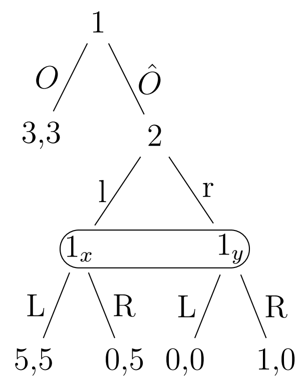
  Moreover, for 1 to be willing to mix, we must have $\sigma_2(d) = 2/3$. This implies that $\sigma_1(D) = 4/7$. This is a sequential equilibrium.

So there are two components of sequential equilibria:

$$\left((A, a, \sigma_3(L) \geq 1/3), \mu(x) = \frac{2}{3}\right) \text{ and } \left((A, a, L), \mu(x) \geq \frac{2}{3}\right)$$

$$\left((\frac{3}{7}A + \frac{4}{7}D, \frac{1}{3}a + \frac{2}{3}d, \frac{1}{3}L + \frac{2}{3}R), \mu(x) = \frac{2}{3} \right)$$

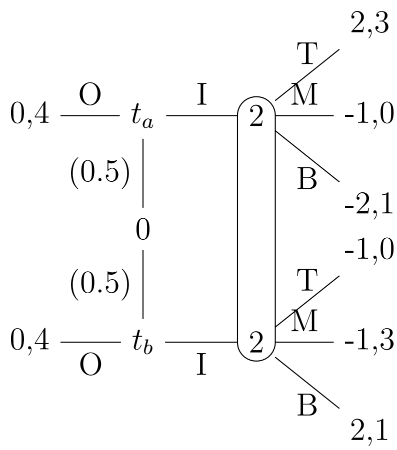
6. (Bill PS4 Q3) For the game in the figure below, specify an assessment (i.e., a strategy profile and a belief profile) with these three properties: (i) beliefs are Bayesian; (ii) no player has a profitable one-shot deviation at any information set; (iii) the assessment is not a weak sequential equilibrium.



♠ Solution: Consider the assessment $((O, R), l), \mu_1(y) = 1$. Since player 1 plays O, his other information set is unreached, so all beliefs are Bayesian. It is clear that no player has a profitable one-shot deviation. But the assessment is not a weak sequential equilibrium, since player 1 can profitably deviate to $(\hat{O}, L)$. Notice that player 1's beliefs are do not satisfy consistency, so the one-shot deviation principle need not (and does not) hold.

7. (Bill PS4 Q6 modified) Compute all sequential equilibria of the game in the figure below. For each equilibrium, identify whether it is a pooling or separating equilibrium, and whether it satisfies the intuitive criterion.

♠ Solution: To compute the sequential equilibria, note that B is dominated for player 2 (by $\frac{1}{2}T + \frac{1}{2}M$) at her information set, and that once B is removed, I is dominated for t_b. Hence, in every sequential equilibrium, t_b must play O. There are two components of sequential equilibria.
In the first, $\sigma_{1a}(I) = 1$, $\sigma_{1b}(O) = 1$, and $\sigma_2^I(T) = 1$; this gives $\mu_2^I(t_a) = 1$. This is a separating equilibrium, in which all information sets are reached, so the resulting Bayesian beliefs are consistent and the equilibrium satisfies the intuitive criterion.



In the other, $\sigma_{1a}(O) = \sigma_{1b}(O) = 1$, and either $\sigma_2^I(M) = 1$ with $\mu_2^I(t_b) \geq 1/2$ or $\sigma_2^I(M) \geq \frac{2}{3}$ (and $\sigma_2^I(B) = 0$) with $\mu_2^I(t_b) = \frac{1}{2}$. This second component consists of pooling equilibria, in which player 2's information set is unreached, so these equilibria are the only ones which might be eliminated by the intuitive criterion. Once player 2 is required to play a best response (i.e. T or M), t_b's equilibrium payoff of 0 dominates the payoffs he might obtain by deviating to I. Therefore, player 2 must not believe that she is facing t_b, and so must play T. In this case, t_a benefits by deviating to I, breaking the equilibrium.

8. Consider the following card game: Players 1 and 2 each bet \$1 by placing it on the table. Player 1 is dealt a card that only he sees. This card can be an Ace, King, or Queen, with each card being equally likely.

   After seeing his card, player 1 decides whether to raise the bet to \$2 (i.e., place another dollar on the table) or fold; if he folds, player 2 takes the money on the table. If player 1 raises, player 2 can call (i.e., place another dollar on the table) or fold; if she folds, player 1 takes the money on the table. If player 2 calls, then player 1 takes the money on the table if he has an Ace; otherwise, player 2 takes the money on the table.

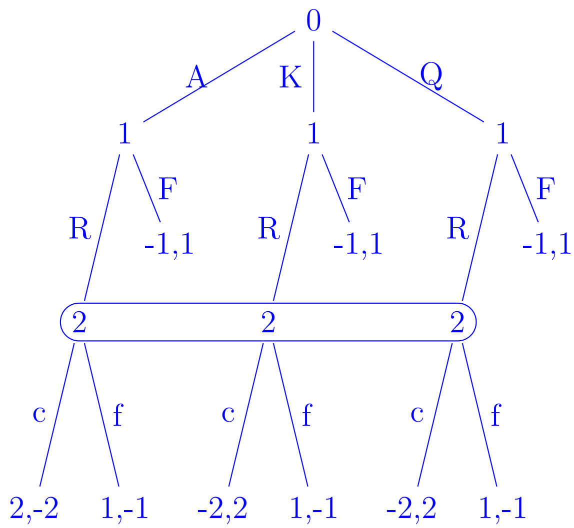
   (a) Draw an extensive form game Γ that represents this interaction.

   ♠ Solution: The extensive form game is below.

   (b) What is each player's pure strategy set in Γ?

   ♠ Solution: The pure strategy sets are

$$S_1 = \{RRR, RRF, RFR, RFF, FRR, FRF, FFR, FFF\}$$



and $S_2 = \{c, f\}$.

(c) Find all sequential equilibria of Γ.

♠ Solution: When player 1 has an Ace, it is optimal for him to choose R. Thus in any equilibrium, player 2's information set is reached with positive probability, and so her beliefs are given by

$$\mu_2(A) = \frac{\frac{1}{3}}{\frac{1}{3}\big(1 + \sigma_1(R|K) + \sigma_1(R|Q)\big)} = \frac{1}{1 + \sigma_1(R|K) + \sigma_1(R|Q)}$$

We now divide it into cases according to player 2's strategy. Note that player 2 chooses to call if

$$c \succsim f \iff -2\mu_2(A) + 2\big(1 - \mu_2(A)\big) \geq -1 \iff \mu_2(A) \leq \frac{3}{4}$$

- If $\sigma_2 = c$, then player's best response is RFF, so player 2's beliefs should be $\mu_2(A) = 1$, so player 2 prefers to play f. Contradiction.
- If $\sigma_2 = f$, then player 1's best response is RRR, so player 2's beliefs should be $\mu_2(A) = 1/3$, so player 2 prefers to play c. Contradiction.
- Thus player 2 mixes. This requires $\mu_2(A) = 3/4$. Setting $\mu_2(A) = 3/4$ in the equation above and rearranging gives the requirement that $\sigma_1(R|K) + \sigma_1(R|Q) = 1/3$. For player 1 to be willing to mix with a King or Queen, he must be indifferent. This implies that $-2\sigma_2(c) + 1 - \sigma_2(c) = -1$, or equivalently $\sigma_2(c) = 2/3$.

Thus the sequential equilibria are

$$\sigma = \left(\big(R, \alpha R + (1-\alpha)F, \beta R + (1-\beta)F\big), \frac{2}{3}c + \frac{1}{3}f\right) \qquad \mu_2 = \left(\frac{3}{4}, \frac{3\alpha}{4}, \frac{3\beta}{4}\right)$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta = \frac{1}{3}$.