Problem Set #1

Danny Edgel
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Question 1

For two events, $A, B \in S$, prove that $A \bigcup B = (A \bigcap B) \bigcup ((A \bigcap B^c) \bigcup (B \bigcap A^c))$.

Proof.

1. $(A \bigcap B) \bigcup ((A \bigcap B^c) \bigcup (B \bigcap A^c)) = ((A \bigcap B) \bigcup (A \bigcap B^c)) \bigcup ((A \bigcap B) \bigcup (B \bigcap A^c))$
2. $B \bigcup B^c = S$, so $(A \bigcap B) \bigcup (A \bigcap B^c) = A$
3. $A \bigcup A^c = S$, so $(A \bigcap B) \bigcup (B \bigcap A^c) = B$
4. Given 2 and 3, $((A \bigcap B) \bigcup (A \bigcap B^c)) \bigcup ((A \bigcap B) \bigcup (B \bigcap A^c)) = A \bigcup B$

$\therefore A \bigcup B = (A \bigcap B) \bigcup ((A \bigcap B^c) \bigcup (B \bigcap A^c))$ ■

Question 2

Prove that $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$.

Proof.

1. $A \bigcup B = A \bigcup (B \bigcap A^c)$. A and $(B \bigcap A^c)$ are disjoint, so $P(A \bigcup B) = P(A) + P(B \bigcup A^c)$
2. $A = (A \bigcap B) \bigcup (A \bigcap B^c)$. Each of these are disjoint, so $P(A) = P(A \bigcap B) + P(A \bigcap B^c)$
3. Given 1 and 2,

$$\begin{aligned}
P(A \bigcup B) &= P(A \bigcap B) + P(A \bigcap B^c) + P(B \bigcap A^c) \\
P(A \bigcup B) + P(A \bigcap B) &= (P(A \bigcap B^c) + P(A \bigcap B)) + (P(B \bigcap A^c) + P(A \bigcap B)) \\
P(A \bigcup B) + P(A \bigcap B) &= P(A) + P(B) \\
P(A \bigcup B) &= P(A) + P(B) - P(A \bigcap B)
\end{aligned}$$

$\therefore P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$ ■

Question 3

Suppose that the unconditional probability of a disease is 0.0025. A screening test for this disease has a detection rate of 0.9, and has a false positive rate of 0.01. Given that the screening test returns positive, what is the conditional probability of having the disease?

Let A be the event of having the disease and P be the event of a positive test result. Then the conditional probability of having the disease in the event of a positive test result is given by:

$$P(A|P) = \frac{P(A \bigcap P)}{P(P)} = \frac{P(P|A)P(A)}{P(P|A)P(A) + P(P|A^c)P(A^c)}$$

Where:

- $P(P|A)$ is the probability of a positive test result conditional on having the disease. This is given as 0.9
- $P(P|A)P(A)$ is the probability of having the disease and getting a positive result. $P(A)$ is given as 0.0025
- $P(P|A^c)P(A^c)$ is the probability of not having the disease and getting a false positive. $P(P|A^c)$ is given as 0.01

Thus, we can derive:

$$P(A|P) = \frac{P(P|A)P(A)}{P(P|A)P(A) + P(P|A^c)P(A^c)} \frac{(0.9)(0.0025)}{(0.9)(0.0025) + (0.01)(1 - 0.0025)} \approx 0.1840491$$

Therefore, the probability of having the disease, conditional on a positive test result, is roughly 0.184.

Question 4

Suppose that a pair of events A and B are mutually exclusive, i.e., $A \bigcap B = \emptyset$, and that $P(A) > 0$ and $P(B) > 0$. Prove that A and B are not independent.

By definition of independence, if A and B are independent, then $P(A \bigcap B) = P(A)P(B)$. However, it is given that $A \bigcap B = \emptyset$, $P(A) > 0$, and $P(B) > 0$. Then $P(A)P(B) > 0$. Thus,

$$P(A \bigcap B) = P(\emptyset) = 0 \neq P(A)P(B)$$

$\therefore A$ and B are not independent ■

Question 5

Consider the experiment of tossing two dice. Let $A = \{\text{First die is 6}\}, B = \{\text{Second die is 6}\}$, and $C = \{\text{Both dice are the same}\}$**.**

(a)

Show that A and B are independent (unconditionally), but A and B are dependent given C.

Each die roll has one of six possible outcomes, so $P(A) = P(B) = \frac{1}{6}$. The probability that A and B both occur ($A \bigcap B$) is one of 36 possible outcomes when two die are rolled. Then,

$$P(A \bigcap B) = \frac{1}{36} = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = P(A)P(B)$$

Thus, A and B are independent.

Since $A \bigcap B$ is one of six possibilities in the event of C, so $P(A \bigcap B|C) = \frac{1}{6}$. However, C does not change the probability of A or B, so $P(A|C) = P(B|C) = \frac{1}{6}$. Thus, $P(A \bigcap B|C) \neq P(A|C)P(B|C)$, and A and B are dependent given C.

(b)

Given the urn experiment (see 5(b)), Show that A and B are not independent, but are conditionally independent given C.

Urn 1 and urn 2 are chosen with equal probability $\left(\text{i.e. } P(C) = P(C^c) = \frac{1}{2}\right)$. If the first urn is chosen, two black balls are drawn in 81 of 100 possible outcomes. If urn 2 is chosen, two black balls are drawn in 19 of 100 outcomes. Thus,

$$P(A \bigcap B) = \frac{1}{2}\left(\frac{81}{100}\right) + \frac{1}{2}\left(\frac{19}{100}\right) = \frac{1}{2}$$

Meanwhile, drawing a black ball is one of nine possibilities if urn 1 is chosen and one of ten possibilities if urn two is chosen. This is true on either the first or second draw. Then,

$$P(A) = P(B) = \frac{1}{2}\left(\frac{9}{10}\right) + \frac{1}{2}\left(\frac{1}{10}\right) = \frac{1}{2}$$

Therefore, $P(A)P(B) = \frac{1}{4} \neq P(A \bigcap B)$, so A and B are not independent.

As I mentioned above, two consecutive draws of a black ball occurs in 81 of 100 possibilities if urn 1 is chosen. Thus, $P(A \bigcap B|C) = \frac{9}{10}$. Since each of the two draws yield a black ball in nine of outcomes, $P(A|C) = P(A|B) = \frac{9}{10}$. Then,

$$P(A|C)P(A|B) = \left(\frac{9}{10}\right)\left(\frac{9}{10}\right) = \frac{81}{100} = P(A \bigcap B|C)$$

So A and B are conditionally independent, given C.

Question 6

Prove that if $X \sim F_X$ and $Y \sim F_Y$, then $P(X > t) \geq P(Y > t)$, $\forall t$ and $P(X > t) > P(Y > t)$, for some t.

$P(X > t) = 1 - F_X(t)$ and $P(Y > T) = 1 - F_Y(t)$, so given that $F_X(t) \leq F_Y(t)$ $\forall t$, we can solve:

$$\begin{aligned} F_X(t) &\leq F_Y(t) \\ F_X(t) - 1 &\leq F_Y(t) - 1 \\ 1 - F_X(t) &\geq 1 - F_Y(t) \\ P(X > t) &\geq P(Y > t) \end{aligned}$$

Therefore, $P(X > t) \geq P(Y > t)$ for all t. We also know that $\exists t_0$ such that $F_X(t_0) < F_Y(t_0)$. Using the same process, we can derive that $P(X > t_0) > P(Y > t_0)$:

$$\begin{aligned} F_X(t_0) &< F_Y(t_0) \\ F_X(t_0) - 1 &< F_Y(t_0) - 1 \\ 1 - F_X(t_0) &> 1 - F_Y(t_0) \\ P(X > t_0) &> P(Y > t_0) \end{aligned}$$

Question 7

Show that the function

$$F_X = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$$

is a CDF, and find $f_X(x)$ and $F_X^{-1}(y)$.

I will show that F_X has each of the properties of a CDF:

1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
   $F(x) = 0$ for all $x < 0$, so $\lim_{x \to -\infty} F(x) = 0$. $\lim_{x \to \infty} e^{-x} = 0$, so $\lim_{x \to \infty} (1 - e^{-x}) = 1$

2. $F(x)$ is non-decreasing
   $F(0) = 1 - e^0 = 1 - 1 = 0$, and e^{-x} is a decreasing function, so $1 - e^{-x}$ is an increasing function for $x \geq 0$. Since $F(x) = 0$ $\forall x \in (-\infty, 0]$, $F(x)$ is non-decreasing on $(-\infty, \infty)$.

3. $F(x)$ is right-continuous
   $1 - e^{-x}$ is continuous for all x, and $\underset{x \to x_0^-}{F}(x) = \underset{x \to x_0^+}{F}(x) = 0$, so $F(x)$ is also continuous.

Thus, F_X is a CDF.

$$f_X(x) = \frac{d}{dX}F_X = \begin{cases} \frac{d}{dx}0 & \text{if } x < 0 \\ \frac{d}{dx}(1 - e^{-x}) & \text{if } x \geq 0 \end{cases}$$

Where $\frac{d}{dx}(1 - e^{-x}) = e^{-x}$, so:

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$$

Let $y = F_X$. Then, for $x \geq 0$,

$$\begin{aligned} y &= 1 - e^{-x} \\ y - 1 &= -e^{-x} \\ 1 - y &= e^{-x} \\ ln(1 - y) &= -x \\ -ln(1 - y) &= x \end{aligned}$$

Thus, $F_X^{-1}(y) = -ln(1 - y)$, $y \in [0, 1)$