Problem Set #6

Danny Edgel
Econ 711: Microeconomics I
Fall 2020

October 19, 2020

Question 1: Rationalizing Demand

Suppose we observe the following "data":

w	p	x
100	(5, 5, 5)	(12, 4, 4)
100	(7, 4, 5)	(9, 3, 5)
100	(2, 4, 1)	(27, 9, 10)
150	(7, 4, 5)	(15, 5, 5)

(a) Under Walras's law, $p_i \cdot x_i = w \ \forall i$. Then, we can calulate:

$$5 * 12 + 5 * 4 + 4 * 5 = 100 = w$$
$$7 * 9 + 4 * 3 + 5 * 5 = 100 = w$$
$$2 * 27 + 4 * 9 + 10 = 100 = w$$
$$7 * 15 + 4 * 5 + 5 * 5 = 150 = w$$

Thus, the data are consistent with Walras's Law.

(b) Given that Walras's Law is satisfied for each observation, $x^i > x^j \Rightarrow p \cdot x^i > p \cdot x^j$ for any $p >> 0$, and all price vectors in our data are strictly positive, we can conclude the following:

i. $x^3 > x^i \ \forall i \neq 3$ implies that: 1) all other goods bundles were affordable at p^3, and 2) x^3 was unaffordable at all $p^i \neq p^3$. Thus, $x^3 \succ^D x^i \ \forall i \neq 3$.

ii. $x^4 > x^1$ implies that 1) x^1 was affordable at p^4, and 2) x^4 was not affordable at p^1. Thus, $x^4 \succ^D x^1$.

iii. Since $p^4 = p^2$ and $w^4 > w^2$, we know 1) x^2 was affordable when x^4 was chosen, and 2) x^4 was not affordable when x^2 was chosen. Thus, $x^4 \succ^D x^2$.

iv. $p^1 \cdot x^2 = 85 < 100$, so $p^1 \cdot x^2 < p^2 \cdot x^2$, and x^2 was chosen at p^2. Therefore, $x^2 \succ^D x^1$

v. $p^2 \cdot x^1 = 120 > w^2$, so x^1 was not affordable when x^2 was chosen. Thus, $\neg(x^2 \succsim^D x^1)$

Taken together, these preference relations indicate:

$$x^3 \succ^D x^4 \succ^D x^1 \succ^D x^2$$

Where it is not possible to have any preference relation "loops". Therefore, these data satisfy GARP. By Afrias's theorem, satisfying GARP is a sufficient condition for conluding that these data can be rationalized by a continuous, monotonic, and concave utility function.

Question 2: Aggregating Demand

Suppose there are n consumers, where consumer $i \in \{1, 2, ..., n\}$ has the indirect utility function

$$v^i(p, w_i) = a_i(p) + b(p)w_i$$

where $\{a_i\}_{i=1}^n$ and b are differentiable functions from $\mathbb{R}_+^k$ to $\mathbb{R}$.

(a) Assuming that $b(p) > 0$ and $(p, w_i) >> 0\ \forall i$, then by Roy's identity,

$$\begin{aligned} x^i(p, w_i) &= \left(-\frac{\partial v^i(p, w_i)/\partial p_1}{\partial v^i(p, w_i)/\partial w_i}, ..., -\frac{\partial v^i(p, w_i)/\partial p_k}{\partial v^i(p, w_i)/\partial w_i}\right) \\ &= \left(-\frac{\frac{\partial a_i(p)}{\partial p_1} + \frac{\partial b(p)}{\partial p_1}w_i}{b(p)}, ..., -\frac{\frac{\partial a_i(p)}{\partial p_k} + \frac{\partial b(p)}{\partial p_k}w_i}{b(p)}\right) \end{aligned}$$

(b) Using Roy's identity on the representative consumer, we get

$$X(p, W) = \left(-\frac{\left(\sum_{i=1}^n \frac{\partial a_i(p)}{\partial p_1}\right) + \frac{\partial b(p)}{\partial p_1}W}{b(p)}, ..., -\frac{\left(\sum_{i=1}^n \frac{\partial a_i(p)}{\partial p_k}\right) + \frac{\partial b(p)}{\partial p_k}w_i}{b(p)}\right)$$

Where, if $W = \sum_{i=1}^n w_i$, we can solve, for each $j = 1, ..., k$:

$$\begin{aligned} X_j(p, W) &= -\frac{\left(\sum_{i=1}^n \frac{\partial a_i(p)}{\partial p_j}\right) + \frac{\partial b(p)}{\partial p_j}W}{b(p)} \\ &= -\frac{\sum_{i=1}^n \left(\frac{\partial a_i(p)}{\partial p_j} + \frac{\partial b(p)}{\partial p_j}w_i\right)}{b(p)} \\ &= \sum_{i=1}^n \left(-\frac{\frac{\partial a_i(p)}{\partial p_j} + \frac{\partial b(p)}{\partial p_j}w_i}{b(p)}\right) \\ X_j(p, W) &= \sum_{i=1}^n x_j^i(p, w_i) \end{aligned}$$

Question 3: Homothetic Proferences

Complete, transitive preferences, $\succsim$, are homothetic if, $\forall x, y \in \mathbb{R}^k_+, t > 0$,

$$x \succsim y \iff tx \succsim ty$$

(a) Let $x^* \in x(p, w)$ and define $Y = \{y \in \mathbb{R}^k_+ | y \notin x(p, w)\}$.

1. Suppose, for some $t > 0$, that $tx^*(p, w) \notin x(p, tw)$
   a. Since preferences are complete, there must exist some $y^* \in Y$ such that $ty^* \in x(p, tw)$
   b. $x^* \in x(p, w) \wedge y^* \notin x(p, w) \Rightarrow x^* \succsim y^*$. By homothetic preferences, this implies that $tx^* \succsim ty^*$ $\forall t > 0$
   c. $tx^* \succsim ty^* \Rightarrow u(tx) \geq u(ty)$. Since $x^* \in x(p, w)$, then $p \cdot x^* \leq w$. This implies also that $p \cdot (tx^*) \leq tw$
   d. Since $ty^* \in x(p, tw)$,

   $$ty^* = \underset{x}{\operatorname{argmax}}\ u(x) \text{ s.t. } p \cdot x \leq tw$$

   And by c., $u(tx^*) \geq u(ty^*)$, where $p \cdot (tx^*) \leq tw$. Thus, $tx^* \in x(p, tw)$

   $\therefore$ by contradiction, $x^* \in x(p, w) \Rightarrow tx^* \in x(p, tw)$ ■
2. Suppose $\exists y^* \in Y$ such that $ty^* \in x(p, tw)$
   a. By definition, $ty \succsim z$ $\forall z \in \mathbb{R}^k_+$ such that $p \cdot z \leq tw$
   b. $p \cdot (tx^*) = t(p \cdot x^*)$ where, by definition, $p \cdot x^* \leq w$. Then $p \cdot (tx^*) \leq tw$. Thus, $ty^* \succsim tx^*$
   c. Since preferences are homothetic, $ty^* \succsim tx^* \Rightarrow y^* \succsim x^*$. Thus, $y^* \in x(p, w)$

   $\therefore$ by contradiction, $tx^* \in x(p, tw) \Rightarrow x^* \in x(p, w)$ ■

$\therefore$ for any $t > 0$, $x(p, tw) = tx(p, w)$ ■

(b) If preferences are continuous and monotonic, then $\forall x \in \mathbb{R}^k_+, \exists! \alpha(x)$ s.t. $\alpha(x)e \sim x$, where $e \in \mathbb{R}^k$ is a vector of ones.[1] For each x, define $u(x) = \alpha(x)$.

1. By definition, $\alpha(x) \sim x \Rightarrow \alpha(x)e \succsim x \succsim \alpha(x)e$ Since preferences are homothetic,

   $$\alpha(x) \sim x \Rightarrow t\alpha(x)e \succsim tx \succsim t\alpha(x)e \equiv t\alpha(x)e \sim tx$$
2. By construction, $\alpha(tx)e \sim tx$. Paired with the result from 1, $\alpha(tx)e \sim tx \sim t\alpha(x)$. Thus, $\alpha(tx) = t\alpha(x)$.

$\therefore$ if preferences are monotone and homothetic, then $u(tx) = tu(x)$ for any $t > 0$ ■

[1] As this existence was proved in-lecture, I will take existence as a given.

(c) By (a), $x(p, tw) = tx(p, w)$. By (b), $u(tx) = tu(x)$. Taken together, $v(p, tw) = u(x(p, tw)) = tu(x(p, w)) = tv(p, w)$. Then,

$$v(p, tw) = b(p)(tw) = tb(p)w = tv(p, w)$$

So $v(p, w) = b(p)w$ is consistent with continuous, monotonic, and homothetic preferences.

Question 4: Quasilinear Utility

Let $X = \mathbb{R} \times \mathbb{R}_{+}^{k-1}$ and suppose that utility is represented by

$$u(x) = x_1 + U(x_2, ..., x_k)$$

Where $p_1 = 1$ is fixed.

(a) Marshallian demand is obtained by choosing $x \in X$ that maximizes utility, where the consumer problem has the following Lagrangian:

$$\mathcal{L} = x_1 + U(x_2, ..., x_k) - \lambda \left(x_1 + \sum_{i=2}^{k} p_i x_i - w \right)$$

With the following first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_i} \Big|_{i=1,\ldots,k} = \frac{\partial U}{\partial x_i} - \lambda p_i = 0$$

At the Lagrangian saddle point, complementary slackness suggests that, since $\lambda > 0$, the budget constraint is met with equality. When utility is maximized, then demand for each good $i \in \{2, ..., k\}$ is determined by:

$$\frac{\partial U}{\partial x_i} = p_i$$

Since the consumer can choose to consume any real value of x_1, they will choose $(x_2, ..., x_k)$ such that the marginal utility of each good is equal to its price, then choose x_1 such that $\sum_{i=2}^{k} p_i x_i = w - x_1$.

(b) Another way to augment the budget contraint after recognizing x_1 as a residual is $x_1 = w - \sum_{i=2}^{k} p_i x_i$. Substituing this into the utility function gives the value function:

$$v(p, w) = \max_{x} u(x) = w - \sum_{i=2}^{k} p_i x_i + U(x_2, ..., x_k) = w + \tilde{v}(p)$$

(c) The expenditure function is defined as

$$e(p,u) = \min_x \; x_1 + \sum_{i=1}^{k} p_i x_i \text{ s.t. } x_1 + U(x_2, ..., x_k) \geq u$$

Where $x_1 + U(x_2, ..., x_k) = u$.[2] Substituting this equality for x_1 to eliminate the constraint shows

$$e(p,u) = u - U(x_2, ..., x_k) + \sum_{i=1}^{k} p_i x_i = u - f(p)$$

(d) The FOC for goods $\{x_2, ..., x_k\}$ in the Hicksian consumer problem is:

$$-\frac{\partial U(x_2, ..., x_k)}{\partial x_i} + p_i = 0$$

Thus, as with Marshallian demand, Hicksian demand for each good $i \neq 1$ depends only on the price of that good, which is set equal to the good's marginal utility, with x_i acting as a residual to satisfy any constraints. Therefore, $h_i(p,u)$ does not depend on target utility for good $i \neq 1$.

(e) Suppose $p_i' \neq p_i$ for some $i \neq 1$. The compensating variation (CV) and equivalent variation (EV) for this price change are:

$$\text{CV} = \int_{p_i'}^{p_i} h_i(p,u) dp_i = e(p,u) = e(p',u) = u - f(p) - (u - f(p')) = f(p') - f(p)$$

$$\text{EV} = \int_{p_i'}^{p_i} h_i(p,u)' dp_i = e(p,u') = e(p',u') = u' - f(p) - (u' - f(p')) = f(p') - f(p)$$

Where u' is the utility level reached by the consumer folllowing the price change. The consumer surplus from this change is the area between the new price and the old price along the Marshallian demand curve:

$$\text{CS} = \int_{p_i'}^{p_i} x_i(p,w) = v(p_i, w) - v(p_i', w) = w + \tilde{v}(p) - (w + \tilde{v}(p')) = \tilde{v}(p) - \tilde{v}(p')$$

Where, as we defined above,

$$f(p) = \sum_{i=2}^{k} p_i x_i - U(x_2, ..., x_k)$$

$$\tilde{v}(p) = U(x_2, ..., x_k) - \sum_{i=2}^{k} p_i x_i$$

Thus, $f(p) - \tilde{v}(p)$. Therefore,

$$\text{CS} = \tilde{v}(p) - \tilde{v}(p') = f(p') - f(p) = \text{CV} = \text{EV}$$

[2] This is not shown, but if we set up the Langrangian as we do in (a), we would find that, one again, $\lambda = 1$, so by complementary slackness, the utility constraint holds with equality.