Game Theory Review

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1 Normal Form Games

1.1 Characterization

A normal form game has:

1. Players: $N = \{1, 2, ..., n\}$
2. Action profiles, A_i, for each player $i \in \{1, ..., n\}$, with associated pure strategy sets, S_i
3. Payoff functions, u_i, for each player

A mixed strategy in a normal form game is denoted with $\sigma_i \in \Delta S_i$, where Δ is a probability distribution for each pure strategy in the mixed strategy. For example, if player i can play A or B, and σ_i is a mixed strategy in which she plays A 30% of the time, then $\sigma_i(A) = .3$ and $\sigma_i(B) = .7$.

Payoff functions, $u_i(\sigma_i, s_{-i})$, are functions of each player's behavior, where (σ_i) is the behavior of the player in question, and s_{-i} is the action of every player other than i.

1.2 Rationalizability and Strict Dominance

1.2.1 Dominance

A pure strategy, $s_i \in S_i$ is strictly dominated if any other strategy (pure or mixed) yields higher payoffs than s_i regardless of other players' actions:

$$\exists \sigma_i \text{ s.t. } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i} \in S_{-i}$$

Identifying **strictly dominated strategies** is straightforward when they're dominated by pure strategies, but it's tougher with strategies dominated by mixed strategies. Take the following game as an example:

	a	b	c	d
A	(2,2)	(3,1)	(4,3)	(3,3)
B	(7,3)	(3,5)	(3,2)	(0,0)

a is not dominated by any other pure strategy, but $u(A, c) > u(A, a)$ and $u(B, b) > u(B, a)$ for the column player, so there exists a mixed strategy of b and c that yields higher payoffs than a **regardless of the row player's strategy**.

Strictly dominated strategies will never be played by rational players in a normal form game, so they can be deleted from the action set for the purposes of rationalizability and equilibrium determination. It may be the case that some strategies that were not previously dominated are dominated once other dominated strategies are deleted. These are also considered dominated by the process of **iterated strict dominance** (ISD).

ISD_k is the set of strategies that survive k rounds of deletion of all possible strictly dominated strategies, Where $ISD_1 \supseteq ISD_2 \supseteq ... \supseteq ISD_k$, where any equilibrium must necessarily be in ISD_∞. Note that **strict** dominance is required for IDS. Deleting weakly-dominated strategies does not yield consistent results.[1]

1.2.2 Best Reply

A **best reply** is a strategy (pure or mixed) that maximizes a player's payoff, conditional on other players' strategies. Formally, σ_i is a best response to σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \; \forall \sigma_i' \in \Delta S_i$$

A best response is denoted as $\sigma_i \in B_i(\sigma_{-i})$.

Proposition: In a two-player game, σ_i is strictly dominated if and only if it is never a best response.

1.2.3 Rationalizability

A 1-rationalizable ($\mathbb{Q}_1$) strategy for player i is an element of S_i that is a best reply to an (independent) probability distribution over other players' strategies. A k-rationalizable ($\mathbb{Q}_k$) strategy for player i is an element of S_i that is a best reply to an (independent) probability distribution over other players' $(k-1)$-rationalizable strategies, for $k = 2, 3, ...$ A *rationalizable* ($\mathbb{Q}_\infty$) strategy is an element of S that is k-rationalizable for all players i and all $k = 1, 2, ...$

Theorem: In a two-player game, a strategy survives iterated strict dominance if an only if it is rationalizable.

[1] You should make yourself believe that this is the case.

1.3 Nash Equilibrium

Definition: $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ is a Nash equilibrium if:

$$\sigma_i \in B_i(\sigma_{-i})\ \forall i \in \{1, 2, ..., n\}$$

In other words, a Nash equilibrium is an outcome in which all players are optimizing. There are two strategies for computing a Nash equilibrium:

Option 1

1. Eliminate pure strategies that are not rationalizable
2. For each strategy profile σ whose supports involve rationalizable strategies, check for each player i that $\sigma_i \in B_i(\sigma_{-i})$

Option 2

1. Use ISD to eliminate all non-rationalizable strategies
2. Final all "closed rationalizable cycles"
3. Look for Nash equilibria on the suppose of each cycle:
   (i) Fore pure strategies, just check for pure best responses
   (ii) For mixed strategies, solve using indifference between all pure strategies in the support

For two-player games, step 2 of option 2 is done by first looking at each rationalizable pure strategy, s_i, for each i, and finding the best response for the other player and determining whether one of them can make s_i a best response.

Next, look at each possible mixed strategy support[2] of each player i, then use optimality conditions to restrict what other players' strategies may be (by changing the payoffs of each pure strategy), then determining whether it is optimal for the other player to do the same based on player i's mixed strategy.

1.3.1 Nash Equilibrium with a Continuum of Players

A continuum of players often simplifies a problem by foregoing the issue of thinking about which player plays which strategy. Also, a mixed-strategy equilibrium in a two-player game can be a pure strategy one when a continuum of players is used. For example, in rock paper scissors with two players, the unique NE is for each player to fully randomize with equal weight on each move. With a continuum of players, each player can either fully randomize, or each move (rock, paper, or scissors) can be played as a pure strategy by one third of the continuum.

[2]A support is the set of all pure strategies that are played in a mixed strategy–i.e. those with a nonzero probability assigned to them.

1.4 Nash Equilibrium with a Continuum of Actions

Most games involve continuous action sets (e.g., pricing in oligopolistic competition). Key concepts:

- Glicksbeg Fixed Point Theorem: If all player action spaces are compact, convex subsets of $\mathbb{R}^k$ and each payoff function is continuous, then the game has at least one (possibly mixed) Nash equilibrium
- We usually assume that payoff functions are strictly quasi-concave in one's own action in order to ensure that mixed NEs cannot be optimal.
- If there is an interior solution, the best response function is obtained using the first-order condition of the payoff function w/r/t one's own action. The NE is then obtained by inputting one player's best response function into the other player's and solving

1.4.1 Timing Games: war of attrition vs. pre-emption

Suppose there is a continuum of players *and* a continuum of actions, such as customers choosing when to go to the grocery store. Then there is a payoff from going at a certain time, $u(t)$ and a payoff from going at a certain place in the distribution of shoppers, $v(q)$, where q is the quantile of the distribution. The solution to this problem is a CDF, $Q(t)$, which gives the share of the shoppers who have already gone to the grocery store at each t. In equilibrium, all shoppers must be indifferent to going at any time t. Thus, if t^* optimizes u, then, at the equilibrium:

$$v(q) \text{ increasing in q} \Rightarrow u(t)v\left(Q(t)\right) = u(t^*)v(0)$$
$$v(q) \text{ decreasing in q} \Rightarrow u(t)v\left(Q(t)\right) = u(t^*)v(1)$$

Once $Q(t)$ is obtained, we can determine when people go to the store by solving for the domain of $Q(t)$, i.e. by setting $Q(t) = 0$ and $Q(t) = 1$ and solving for t.

To determine whether it's possible to have an initial (or terminal) rush, we find the quantile, $\tilde{q}$ such that the payoff of going just before the rush is equal to the average payoff of going during the rush to determine what the size of the rush would be. This is done with the equation:

$$\text{Initial rush: } \frac{1}{\tilde{q}}\int_0^{\tilde{q}} v(x)dx = v(\tilde{q})$$
$$\text{Terminal rush: } \frac{1}{1-\tilde{q}}\int_{\tilde{q}}^1 v(x)dx = v(\tilde{q})$$

If the $\tilde{q} = 0$ (or 1, in the terminal case), then the rush has a size of 0, meaning that there cannot be a rush.

See discussion handout 9, question 7 and/or HW2, question 5

1.5 Supermodular and Submodular Games

- A **supermodular game** is one that has *strategic complementarities*, i.e. players' best reponse functions are increasing in other players' actions.
- **Definition:** A game with $S = \{S_1, ..., S_n\}$ and payoff functions u_i is supermodular is, for all i,
  - $S_i \subseteq \mathbb{R}$ is compact
  - u_i is upper semi-continuous in s_i, s_{-i}
  - u_i has increasing differences in s_i, s_{-i}
- **Theorem**: Suppose (S, u) is a supermodular game, and let $BR_i(s_{-i}) = \underset{s_i \in S_i}{\operatorname{argmax}}\, u_i(s_i, s_{-i})$. Then,
  (i) $BR_i(s_{-i})$ has a greatest and least element $\overline{BR}_i(s_{-i})$ and $\underline{BR}_i(s_{-i})$
  (ii) If $s'_{-i} \geq s_{-i}$, then $\overline{BR}_i(s'_{-i}) \geq \overline{BR}_i(s_{-i})$ and $\underline{BR}_i(s'_{-i}) \geq \underline{BR}_i(s_{-i})$
- **Theorem (maximum and minimum equilibrium)**: Consider a supermodular game with continuous payoff functions $u_i(s)$ on a compact domain for all i. Then $\exists$ a maximum and minimum equilibrium.
- **Submodular games**: $f(x, \theta)$ has decreasing differences if $f(x, -\theta)$ has increasing differences. A submodular game (aka a **game of strategic substitutes**) is one whose payoffs $u_i(s_i, s_{-i})$ have decreasing differences for all i

2 Bayesian Games

2.1 Characterization

A Bayesian game has:

1. Players: $N = \{1, 2, ..., n\}$
2. Action profiles, A_i, for each player $i \in \{1, ..., n\}$, with associated pure strategy sets, S_i
3. A joint type space Θ with player type spaces Θ_i and player types $\theta_i \in \Theta_i$
4. A probability, $p \in \Delta\Theta$ of any particular type
5. Payoff functions, u_i, for each player

Each player has an expected utility from each strategy profile:

$$u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left(\prod_{j \neq i} \sigma_j(a_j|\theta_j) \right) \sigma_i(a_i) u_i(a, \theta)$$

Conditional on knowing the type space, the probability of any outcome, (a, θ), is

$$\left(\prod_{j \neq i} \sigma_j(a_j|\theta)\right)$$

In a Bayesian game, **each player optimizes conditional on their own type**. A pure strategy is a function, $s_i(\theta_i)$ that acts as a decision rule, e.g., "If I'm type A, I play a. If I'm type B, I play b" and so on.

2.2 Bayesian Nash equilibrium

A **Bayesian Nash Equilibrium** is a mixed strategy profile fuch that, for each player i and type $\theta_i \in \Theta_i$,

$$\sigma_i(\cdot|\theta_i) \in \arg\max_{\tilde{\sigma}_i \in \Delta a_i} \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left(\prod_{j \neq i} \sigma_j(a_j|\theta_j)\right) \tilde{\sigma}_i(a_i) u_i(a, \theta)$$

A more intuitive description, from MWG, 8.E.1: A profile of decision rules, $\{s_1(\cdot), ..., s_n(\cdot)\}$ is a BNE if and only if, for all i and $\overline{\theta}_i$,

$$\mathbb{E}_{\theta_{-i}}\left[u_i(s_i(\overline{\theta}_i), s_{-i}(\theta_{-i}), \overline{\theta}_i)|\overline{\theta}_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i(s_i', s_{-i}(\theta_{-i}), \overline{\theta}_i)|\overline{\theta}_i\right]$$

for all $s_i' \in S_i$, where the expectation is taken over realizations of the other players' random variables conditional on player i's realization of $\overline{\theta}_i$. This equation makes clear the solution to the BNE: the best response rule can be determined by finding a cutoff point equal to the probability weight that makes the two sides of this equation equal. This cutoff is used to assign pure strategies for each Bayesian type. This becomes clearer and easier to understand when looking at a two-player, two-action game. For an example of such, see 2(b) on problem set 4 or Section 8.E in Mas-Colell, Whinston, and Green; particularly example 8.E.1, which is set up on p.254 and solved on p.256.

2.3 Correlated Equilibrium

Suppose a randomizing device **privately** suggests an action to each player, who knows the probability of each outcome being suggested but does not know what has been suggested to each other player. If the payoffs and probability distribution of suggestions do not produce any incentive to ignore the suggested action, then this results in a **correlated equilibrium**.

More formally, a probability distribution $\rho \in \Delta S$ results in a correlated equilibrium if:

$$\sum_{s_{-i} \in S_{-i}} \frac{\rho(s_i, s_{-i})}{\rho(s_i)} u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \frac{\rho(s_i, s_{-i})}{u}{}_i(s_i', s_{-i})$$

This rule is actually quite simple when applied to a standard two-player, two-action game. For examples, review question 5 on discussion handout 11 or 2(d) on problem set 4.

2.4 Knowledge

Full disclosure: much of what I've written below is formal and technical, but I have no idea if we're required to get this formal/technical with these questions. I barely understand it.

- $\Omega \equiv$ set of finitely many states of nature, with $\omega \in \Omega$
- $H_i = \{h_1, h_2, ...\}$ is player i's information set
- H_i is a partition of Ω
- Each player i cannot distinguish between any two $\omega, \omega' \in h_i(\omega)$
- $E \subseteq \Omega$ is an event
- If ω is the true state of the world and $\omega \in E$, then E occurs
- For any E, $K(E)$ is the set of all states in which i knows E
- $K_i(E) = \{\omega \in \Omega | h_i(\omega) \subseteq E\}$
- If player i observes E, then they know that some $\omega \in E$ is the true state

Consider the question from Problem set 4, question 4, where 10 of 40 game theorists have bad breath and can observe the bad breath of others but not themselves. The true state of the world, ω is the assignment of game theorists with bad breath. The event, E, for each game theorist is the game theorists with bad breath that they observe with bad breath. Each game theorist's information set, H_i contains two elemnts of Ω: in one only the game theorists they smell have bad breath. In the other, they *also* have bad breath. One of these is the true state, but they cannot distinguish between them.

2.4.1 Mutual and Common Knowledge

- E is **mutual knowledge** in state ω if $\omega \in \bigcup_{i \in \{1,...,n\}} K_i(E)$. In other words, "everyone knows event E"
- E is **common knowledge** in state ω if $\omega \in \bigcup_{i \in \{1,...,n\}} K_i^m(E)$ for all $m = 1, 2, ...$ $m = 1$ is mutual knowledge, but all m means "everyone knows that everyone knows that..."
- **Theorem:** If two people have the same priors and their posteriors for an event E are common knowledge, then these posteriors are equal.[3]

Turn, again, to the smelly game theorist example. Since each economist with bad breath observes 9 with bad breath and each economist without bad breath observes 10, it is mutual knowledge that nine game theorists have bad breath, but this is not *common knowledge*. If everyone knew that everyone knew that

[3] I don't know what this means but I don't think it's important.

there were at least nine game theorists with bad breath, then each game theorist with bad breath would understand that they have bad breath. Because if only nine game theorists had bad breath, then it would be mutual knowledge that only 8 had bad breath, because then the 9 game theorists with bad breath would smell 8 but not know if they had it.

Note that, in this example, the elevators coming and going do not represent events. The elevators are signals, which are not formalized in this model.[4]

3 Extensive Form Games

[4] I know this is weird, but I asked Cody in office hours, and that is what he told me.