Problem Set #1

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Question 1

(a)

Question 2

(a) Letting $\widetilde{\theta}$ be some value between θ_0 and $\hat{\theta}$, the mean-value expansion of the first-order condition of the problem, at $\hat{\theta}$, is:

$$\begin{aligned}\frac{\partial \hat{Q}(\hat{\theta}_n)}{\partial \theta} &= \frac{1}{n}\sum_{i=1}^{n} \frac{\partial g(W_i, \hat{\theta}_n, \hat{\gamma}_n)}{\partial \theta} \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{\partial g(W_i, \theta_0, \hat{\gamma}_n)}{\partial \theta} + \frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 g(W_i, \widetilde{\theta}_n, \hat{\gamma}_n)}{\partial \theta \partial \theta'}(\hat{\theta}_n - \theta_0)\end{aligned}$$

Note that $\hat{\gamma}_n \to_p \gamma_0$, and since $\hat{\gamma}_n$ was acquired via a sample independent of $\{W_i\}$, $Cov(\hat{\gamma}_n, \hat{\theta}_n) = 0$. Then:

$$\begin{aligned}\sqrt{n}\frac{\partial \hat{Q}(\theta_0)}{\partial \theta} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{\partial g(W_i, \theta_0, \hat{\gamma}_n)}{\partial \theta} \to_d \mathcal{N}(0, \Omega_0) \\ \text{Where } \Omega_0 &= \mathbb{E}\left[\frac{\partial g(W_i, \theta_0, \gamma_0)}{\partial \theta}\frac{\partial g(W_i, \theta_0, \gamma_0)}{\partial \theta'}\right]\end{aligned}$$

Denote $B_n = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 g(W_i, \widetilde{\theta}_n, \hat{\gamma}_n)}{\partial \theta \partial \theta'}$, where, since the conditions for ULLN are satisfied:

$$B_n \to_p B_0 = \frac{\partial^2 g(W_i, \theta_0, \gamma_0)}{\partial \theta \partial \theta'}$$

Thus,

$$\sqrt{n}\frac{\partial\hat{Q}(\hat{\theta}_n)}{\partial\theta} = \sqrt{n}\frac{\partial\hat{Q}(\theta_0)}{\partial\theta} + \frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 g(W_i,\widetilde{\theta}_n,\hat{\gamma}_n)}{\partial\theta\partial\theta'}\sqrt{n}(\hat{\theta}_n-\theta_0) = 0$$

$$\sqrt{n}(\hat{\theta}_n-\theta_0) = -\hat{B}_n^{-1}\sqrt{n}\frac{\partial\hat{Q}(\theta_0)}{\partial\theta} \to_d \mathcal{N}\left(0, B_0^{-1}\Omega_0 B_0^{-1}\right)$$

Where B_0 and Ω_0 are known and given above.

(b) First, the additional conditions necessary to derive the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n-\theta_0)$ are the necessary assumptions for ULLN, which are the assumptions given in (f) and (g) for g, but instead for m.

Since $\hat{\gamma}_n$ and $\hat{\theta}_n$ were retrieved from the same sample, we can no longer assume that their asymptotic covariance is zero and therefore must account for the asymptotic variance of $\hat{\theta}_n$ in the asumptotic variance of $\hat{\theta}_n$. Since m does not depend on θ, we can rewrite Σ_γ as $A_0^{-1}\Omega_0^\gamma A_0^{-1}$, where:

$$A_0 = \mathbb{E}\left[\frac{\partial^2 m(W_i,\gamma_0)}{\partial\gamma\partial\gamma'}\right], \qquad \Omega_0^\gamma = \mathbb{E}\left[\frac{\partial m(W_i,\gamma_0)}{\partial\gamma}\frac{\partial m(W_i,\gamma_0)}{\partial\gamma'}\right]$$

Now, the Taylor expansion from (a) becomes:

$$\frac{\partial\hat{Q}(\hat{\theta}_n)}{\partial\theta} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial g(W_i,\theta_0,\gamma_0)}{\partial\theta} + \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix}\frac{\partial^2 g(W_i,\widetilde{\theta}_n,\gamma_0)}{\partial\theta\partial\theta'}\\ \frac{\partial^2 g(W_i,\theta_0,\widetilde{\gamma}_n)}{\partial\theta\partial\gamma'}\end{pmatrix}'\begin{pmatrix}\hat{\theta}_n-\theta_0\\ \hat{\gamma}_n-\gamma_0\end{pmatrix}$$

Thus, we can write:

$$\sqrt{n}\begin{pmatrix}\hat{\theta}_n-\theta_0\\ \hat{\gamma}_n-\gamma_0\end{pmatrix} = -C_n^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\partial g(W_i,\theta_0,\gamma_0)}{\partial\theta}$$

Where:

$$C_n = \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix}\frac{\partial^2 g(W_i,\widetilde{\theta}_n,\gamma_0)}{\partial\theta\partial\theta'}\\ \frac{\partial^2 g(W_i,\theta_0,\widetilde{\gamma}_n)}{\partial\theta\partial\gamma'}\end{pmatrix}' \to_p C_0 = \mathbb{E}\left[\begin{pmatrix}\frac{\partial^2 g(W_i,\theta_0,\gamma_0)}{\partial\theta\partial\theta'}\\ \frac{\partial^2 g(W_i,\theta_0,\gamma_0)}{\partial\theta\partial\gamma'}\end{pmatrix}'\right]$$

Then, by the Central Limit Theorem,

$$\sqrt{n}\begin{pmatrix}\hat{\theta}_n-\theta_0\\ \hat{\gamma}_n-\gamma_0\end{pmatrix} \to_d \mathcal{N}\left(0, C_0^{-1}\begin{pmatrix}\Omega_0^\theta\\ \Omega_0^\gamma\end{pmatrix}(C_0^{-1})'\right)$$