Problem Set #3

Danny Edgel
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Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Haffen

Question 1

We are given:

$$U(c_t, c_{t+1}^t) = \ln\left(c_t^t\right) + \ln\left(c_{t+1}^t\right)$$
$$U(c_1^0) = \ln\left(c_1^0\right)$$

Where each generation is endowed with w_1 when young and $w_2 < w_1$ when old, and the initial old generation is endowed with $\overline{M} > 0$ units of fiat currency. Each generation has a unit measure of agents.

(a) In each period, the social planner faces the problem:

$$\max_{\{c_t^t, c_t^{t-1}\}} \ln\left(c_t^t\right) + \ln\left(c_t^{t-1}\right) \text{ s.t. } c_t^t + c_t^{t-1} \leq w_1 + w_2$$

Note that this includes the initial old geneation's allocation in time $t = 1$. Since the objective function is increasing in both c_t^t and c_t^{t-1}, the budget constraint is an equality. Solving for $c_t^{t-1} = w_1 + w_2 - c_t^t$, we can solve:

$$\max_{\{c_t^t\}} \ln\left(c_t^t\right) + \ln\left(w_1 + w_2 - c_t^t\right)$$
$$\text{F.O.C. } \frac{1}{c_t^t} - \frac{1}{w_1 + w_2 - c_t^t} = 0$$
$$c_t^t = w_1 + w_2 - c_t^t$$
$$c_t^t = \frac{1}{2}(w_1 + w_2)$$

Therefore, the social planner's problem is solved by the allocation:

$$\{c_1^0, \{c_t^t, c_t^{t=1}\}_{t=1}^\infty\} = \left\{\frac{1}{2}(w_1 + w_2), \left\{\frac{1}{2}(w_1 + w_2), \frac{1}{2}(w_1 + w_2)\right\}_{t=1}^\infty\right\}$$

(b) In this model, there are two representative households: the initial old generation, and each generation $t = \{1, 2, ...\}$. Then, the household problem is:

$$\max_{\{c_1^0\}} \ln\left(c_1^0\right) \text{ s.t. } c_1^0 \leq w_2 + \frac{\overline{M}}{p_1}$$

$$\max_{\{\{c_t^t, c_{t+1}^t, M_{t=1}^t\}\}} \ln\left(c_t^t\right) + \ln\left(c_{t+1}^t\right) \text{ s.t. } c_t^t \leq w_1 - \frac{M_{t+1}^t}{p_t}$$

$$c_{t+1}^t \leq w_2 + \frac{M_{t+1}^t}{p_{t+1}}$$

(c) An autarkik equilibrium for this model is a set of prices, $\{p_1, \{p_t, p_{t+1}\}_{t=1}^{\infty}\}$, and an allocation, $\{c_1^0, \{c_t^t, c_t^{t=1}\}_{t=1}^{\infty}\}$ such that in every period, each agent consumes their own endowment and all markets clear:

$$M_t = \overline{M} \qquad \text{(Money Market)}$$

$$c_t^t + c_t^{t-1} = w_1 + w_2 \qquad \text{(Goods Market)}$$

There is no savings technology and agents do not trade in autarky, so to solve for the equilibrium allocation, we simply set each agent's consumption equal to their endowment in each period:

$$\{c_1^0, \{c_t^t, c_t^{t=1}\}_{t=1}^{\infty}\} = \{w_2, \{w_1, w_2\}_{t=1}^{\infty}\}$$

Now, we can use our allocation and the market-clearing conditions to solve for the market-clearing prices and the money allocation. Since each agent's objective function is increasing in consumption, they meet their budget constraint in each period. Then, setting consumption equal to endowment for each agent in each period:

$$w_2 = w_2 + \frac{\overline{M}}{p_1}$$

$$w_1 = w_1 - \frac{M_{t+1}^t}{p_t}$$

$$w_2 = w_2 + \frac{M_{t+1}^t}{p_{t+1}}$$

Therefore, $\frac{\overline{M}}{p_1} = \frac{M_{t+1}^t}{p_t} = \frac{M_{t+1}^t}{p_{t+1}} = 0$. The equilibrium set of prices in autarky is $\left\{\frac{1}{p_t}\right\} = 0$.

(d) Let $w_2 = 0$ and assume that money is valued. Then, the competitive equilibrium is, again, an allocation and set of prices such that the goods and money market clear. We can derive these values by solving each representative agent's household problem. Since the initial old generation consumes only in period $t = 1$ and their utility is increasing on c_1^0, we can derive that $c_1^0 = w_2 + \frac{\overline{M}}{p_1}$. Using the budget contraint of the generation $t \geq 1$ problem, we can solve:

$$\max_{\{M_{t+1}^t\}} \ln\left(w_1 - \frac{M_{t+1}^t}{p_t}\right) + \ln\left(\frac{M_{t+1}^t}{p_{t+1}}\right)$$

$$\text{F.O.C.:} \quad -\frac{1}{p_t\left(w_1 - \frac{M_{t+1}^t}{p_t}\right)} + \frac{1}{p_{t+1}\frac{M_{t+1}^t}{p_{t+1}}} = 0$$

$$p_{t+1}w_2 + M_{t+1}^t = p_t m_1 - M_{t+1}^t$$

$$M_{t+1}^t = \frac{1}{2}(p_t w_1 - p_{t+1}w_2) = \frac{1}{2}p_t w_1$$

Using our definition of M_{t+1}^t with respect to consumption in each period, we can derive:

$$c_t^t = w_1 - \frac{1}{2p_t}(p_t w_1 - p_{t+1}w_2) = \frac{1}{2}\left(w_1 + \frac{p_{t+1}}{p_t}w_2\right) = \frac{1}{2}w_1$$

$$c_{t+1}^t = w_2 + \frac{1}{2p_{t+1}}(p_t w_1 - p_{t+1}w_2) = \frac{1}{2}\left(\frac{p_t}{p_{t+1}}w_1 + w_2\right) = \frac{p_t}{2p_{t+1}}w_1$$

Thus, our goods market clearing condition enables us to solve:

$$\begin{aligned} c_t^t + c_t^{t-1} = \frac{1}{2}w_1 + \frac{p_{t-1}}{2p_t}w_1 &= w_1 \\ 1 + \frac{p_{t-1}}{p_t} &= 2 \\ \frac{p_{t-1}}{p_t} &= 1 \\ p_{t-1} &= p_t \end{aligned}$$

While the money market clearing condition combines with $c_1^0 = w_2 + \frac{\overline{M}}{p_1}$ and the equation for M_{t+1}^t to determine the initial old generation's consumption in t:

$$\begin{aligned} M_2^1 &= \overline{M} \\ \frac{1}{2}p_1 w_1 &= p_1 c_1^0 \\ c_1^0 &= \frac{1}{2}w_1 \end{aligned}$$

Which is consistent with our finding for c_t^{t-1} for an arbitrary $t \geq 1$ Thus, the allocation of the competitive equilibrium in this model is:

$$\{c_1^0, \{c_t^t, c_t^{t=1}\}_{t=1}^\infty\} = \left\{\frac{1}{2}w_1, \left\{\frac{1}{2}w_1, \frac{1}{2}w_1\right\}_{t=1}^\infty\right\}$$

(e) The table below displays the equilibrium allocations of initial old generation and the representative agents from generation t and $t-1$ at time t in each of the three scenarios (social planner's problem, autarky, monetary), each assuming that $w_2 = 0$.

	c_0^1	c_t^t	c_t^{t-1}
SPP	$\frac{1}{2}w_1$	$\frac{1}{2}w_1$	$\frac{1}{2}w_1$
Autarky	0	w_1	0
Monetary	$\frac{1}{2}w_1$	$\frac{1}{2}w_1$	$\frac{1}{2}w_1$