

Problem Set #1

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Question 1

Let $(Y, X')'$ be a random vector, where $Y = X'\beta_0 \cdot U$, where $\mathbb{E}[U | X] = 1$, $\mathbb{E}[XX']$ is invertible, and $\mathbb{E}[Y^2 + \|X\|^2] < \infty$.

- (i) Since $\mathbb{E}[U | X] = 1$, the expectation of Y , conditional on X , is $X'\beta_0$. Then, $\frac{\partial}{\partial X} Y = \beta_0$.
- (ii) Define $V = U - 1$. Then, $\mathbb{E}[V | X] = \mathbb{E}[U - 1 | X] = 0$, and:

$$Y = X'\beta_0(V + 1) = X'\beta_0 V + X'\beta_0 = X'\beta_0 + \tilde{U}$$

Where:

$$\mathbb{E}[\tilde{U} | X] = \mathbb{E}[X'\beta_0 V | X] = X'\beta_0 \mathbb{E}[V | X] = 0$$

Thus, $Y = X'\beta_0 + \tilde{U}$, where $\mathbb{E}[\tilde{U} | X] = 0$.

- (iii) Let $\beta = \beta_0$. Then,
- (iv) Define $V = U - 1$. Then, $\mathbb{E}[V | X] = \mathbb{E}[U - 1 | X] = 0$, and:

$$\begin{aligned}\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[X(X'\beta_0 \cdot U - X'\beta_0)] = \mathbb{E}[\mathbb{E}[X(X'\beta_0 \cdot U - X'\beta_0) | X]] \\ &= \mathbb{E}[XX'\beta_0 \mathbb{E}[(U - 1) | X]] = 0\end{aligned}$$

Thus, $\beta = \beta_0 \Rightarrow \mathbb{E}[X(Y - X'\beta)] = 0$. Now, Suppose $\mathbb{E}[X(Y - X'\beta)] = 0$. Then,

$$\begin{aligned}\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[X(X'\beta \cdot U - X'\beta_0)] = 0 \\ \mathbb{E}[XX' \mathbb{E}[\beta \cdot U - \beta_0 | X]] &= (\beta - \beta_0) \mathbb{E}[XX'] = 0\end{aligned}$$

We know that $\mathbb{E}[XX']$ is invertible, so $\mathbb{E}[XX'] \neq 0$. Thus, $\mathbb{E}[X(Y - X'\beta)] = 0 \Rightarrow \beta = \beta_0$.

$$\therefore \mathbb{E}[X(Y - X'\beta)] = 0 \iff \beta = \beta_0 \blacksquare$$

Knowing this, we can derive the method of moments estimator for β :

$$\begin{aligned}\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[XY] - \mathbb{E}[XX']\beta = 0 \\ \mathbb{E}[XX']\beta &= \mathbb{E}[XY] \\ \beta &= \mathbb{E}[XX']^{-1} \mathbb{E}[XY] \\ \Rightarrow \hat{\beta} &= \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^n X_i Y_i = \hat{\beta}_{OLS}\end{aligned}$$

(v) We can simplify the final equation in (iii) to show:

$$\begin{aligned}\mathbb{E}[\hat{\beta} \mid X_1, \dots, X_n] &= \mathbb{E} \left[\left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^n X_i Y_i \mid X_1, \dots, X_n \right] \\ &= \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^n X_i \mathbb{E}[X_i' \beta_0 \cdot U \mid X_1, \dots, X_n] \\ &= \beta_0 \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n X_i X_i' \right) \mathbb{E}[U \mid X_1, \dots, X_n] \\ &= \beta_0\end{aligned}$$

Thus, $\hat{\beta}$ is unbiased.

(vi) According to the weak law of large numbers (WLLN), random variables converge in probability to their expected value. Thus,

$$\hat{\beta} \rightarrow_p \mathbb{E}[\hat{\beta}] = \mathbb{E}[\mathbb{E}[\hat{\beta} \mid X_1, \dots, X_n]] = \mathbb{E}[\beta_0] = \beta_0$$

Thus, $\hat{\beta}$ is consistent.

Question 2

- (i) We can use the continuous mapping theorem and law of large numbers to show:

$$\frac{1}{n} \sum_{i=1}^n X_i^3 \rightarrow_p \mathbb{E}[X^3], \quad \frac{\sum_{i=1}^n X_i^3}{\sum_{i=1}^n X_i^2} = \frac{\mathbb{E}[X^3]}{\mathbb{E}[X^2]}$$

We cannot show convergence for the other two statistics.

- (ii) You can use continuous mapping and the central limit theorem to show convergence in distribution of W_n , since:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i^2 - \mathbb{E}[X_i^2]) = \frac{1}{\sqrt{n}} n (\bar{X}_n^2 - \mathbb{E}[X_i^2]) = \sqrt{n} (\bar{X}_n^2 - \mathbb{E}[X_i^2])$$

The other statistic cannot be shown to converge, as it simplifies to $\sqrt{n} (\bar{X}_n^2 - \bar{X}_n^2)$

- (iii) The CDF of X is simply $f(x) = x$. The CDF for the maximum value, x , of n observations of X_i , then, is x^n . Thus, we can define for any $\varepsilon \in (0, 1)$,

$$\begin{aligned} Pr \left(\left| \max_{1 \leq i \leq n} X_i - 1 \right| \leq \varepsilon \right) &= Pr \left(\max_{1 \leq i \leq n} X_i - 1 \geq -\varepsilon \right) \\ &= Pr \left(\max_{1 \leq i \leq n} X_i \geq 1 - \varepsilon \right) \\ &= 1 - Pr \left(\max_{1 \leq i \leq n} X_i \leq 1 - \varepsilon \right) \\ &= 1 - (1 - \varepsilon)^n \\ &\rightarrow_p 1 \end{aligned}$$

- (iv) The CDF of X is $f(x) = 1 - e^{-x}$. Thus, Thus, we can define for any $\varepsilon > 0$ and $M \geq 0$,

$$Pr \left(\max_{1 \leq i \leq n} X_i \leq M \right) = (1 - e^{-M})^n (1 - e^{-M})^n \rightarrow_p 1$$

Question 3

- (i) Since $\{X_i\}_{i=1}^n$ is an i.i.d. sequence of a random variable $X \sim \mathcal{N}(0, 1)$, by the CLT, $\sqrt{n}\bar{X}_n \rightarrow_d \mathcal{N}(0, 1)$, where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Then,

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \rightarrow_d \mathcal{N}(0, 1)$$

- (ii) We can show this by determining the convergence of the first and second central moments of Y_i in probability:

$$\begin{aligned}\bar{Y}_n &= \frac{1}{n} \sum_{i=1}^n Y_i \rightarrow_p \mathbb{E}[Y_i] = \mathbb{E}[X_i W_i] = \mathbb{E}[X_i] \mathbb{E}[W_i] = 0 \\ \hat{\sigma}_Y^2 &= \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y}_n)^2 \rightarrow_p \mathbb{E}[(Y_i - \mathbb{E}[Y_i])^2] = \mathbb{E}[(X_i W_i)^2] \\ &= \mathbb{E}[X_i^2] = \sigma_X^2 = 1\end{aligned}$$

Thus, by the CLT, $\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \rightarrow_d \mathcal{N}(0, 1)$

- (iii) Given the results in (ii), we can calculate,

$$Cov(X_i, Y_i) = \mathbb{E}[(X_i - \mathbb{E}[X_i])(Y_i - \mathbb{E}[Y_i])] = \mathbb{E}[X_i Y_i] = \mathbb{E}[X_i^2 W_i] = \mathbb{E}[X_i^2] \mathbb{E}[W_i] = 0$$

- (iv) Yes. Given the results from (i)-(iii), we can show:

$$\begin{aligned}\frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} X_i \\ Y_i \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \\ \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \end{pmatrix} \rightarrow_d \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, V \right) \\ \text{Where } V &= \begin{pmatrix} \sigma_X^2 & Cov(X_i, Y_i) \\ Cov(X_i, Y_i) & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2\end{aligned}$$

- (v)