Econ 711 – Fall 2020 – Problem Set 2

Due online Monday night September 21 at midnight.

Please feel free to work together on these problems (and all homeworks), but each student needs to write up his/her own answers at the end, rather than directly copying from one master solution.

Question 1. Convex production sets, concave production functions, convex costs

Consider a production function $f : \mathbb{R}^m_+ \rightarrow \mathbb{R}_+$ for a single-output firm.

(a) Prove that if the production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function f is concave.[1]

(b) Prove that if f is concave, the cost function

$$c(q, w) \quad = \quad \min w \cdot z \quad \text{subject to} \quad f(z) \geq q$$

is convex in q.

(Note that "convexity of the cost function when the production set is convex" is the result used in Weitzman's "economic proof" of the second part of the Separating Hyperplanes Theorem.)

Question 2. Solving for the profit function given technology...

Let $k = 2$, and let the production set be

$$Y \quad = \quad \{(y_1, y_2) \quad : \quad y_1 \leq 0 \quad \text{and} \quad y_2 \leq B(-y_1)^{\frac{2}{3}}\}$$

where $B > 0$ is a known constant. Assume both prices are strictly positive.

(a) Draw Y, or describe it clearly.

(b) Solve the firm's profit maximization problem to find $\pi(p)$ and $Y^*(p)$.

(It may help to set $z = -y_1$ as the amount of input used, explain why a profit-maximizing firm will set $y_2 = Bz^{\frac{2}{3}}$, and solve a single-dimensional maximization problem for z, but be sure to state your solution $Y^*(p) \in \mathbb{R}^2$.)

Since $Y^*(p)$ is single-valued, I'll refer to it below as $y(p)$.

(c) Verify that $\pi(\cdot)$ is homogeneous of degree 1, and $y(\cdot)$ is homogeneous of degree 0.

(d) Verify that $y_1(p) = \frac{\partial \pi}{\partial p_1}(p)$ and $y_2(p) = \frac{\partial \pi}{\partial p_2}(p)$.

(e) Calculate $D_p y(p)$, and verify it is symmetric, positive semidefinite,[2] and $[D_p y]p = 0$.

[1] Recall a function f is concave if $f(tx + (1-t)x') \geq tf(x) + (1-t)f(x')$ for any points x and x' and any $t \in (0, 1)$.
[2] A two-by-two matrix is positive semidefinite if its first element and its determinant are both non-negative.

Question 3. ...and recovering technology from the profit function

Finally, suppose we didn't know a firm's production set Y, but did know its profit function was

$$\pi(p) \quad = \quad A p_1^{-2} p_2^3$$

for all $p_1, p_2 > 0$ and $A > 0$ a known constant.

(a) What conditions must hold for this profit function to be rationalizable? (You don't need to check them.)

(b) Recall that the outer bound was defined as

$$Y^O \quad = \quad \{y \quad : \quad p \cdot y \quad \leq \quad \pi(p) \qquad \text{for all} \qquad p \in P\}$$

In this case, this is

$$Y^O \quad = \quad \left\{(y_1, y_2) \quad : \quad p_1 y_1 + p_2 y_2 \quad \leq \quad A p_1^{-2} p_2^3 \qquad \text{for all} \quad (p_1, p_2) \in \mathbb{R}^2_{++}\right\}$$

Show that any $y \in Y^O$ must have $y_1 \leq 0$, i.e., that good 1 must be an input only.

(c) Dividing both sides by p_2 and moving $\frac{p_1}{p_2} y_1$ to the right-hand side, we can rewrite Y^O as

$$Y^O \quad = \quad \left\{(y_1, y_2) \quad : \quad y_2 \quad \leq \quad A p_1^{-2} p_2^2 - \tfrac{p_1}{p_2} y_1 \qquad \text{for all} \quad (p_1, p_2) \in \mathbb{R}^2_{++}\right\}$$

Since the expression on the right depends only on the price *ratio* $\frac{p_2}{p_1}$ rather than the two individual prices, we can let $r \equiv \frac{p_2}{p_1} > 0$ denote this ratio, and write Y^O as

$$\begin{aligned} Y^O \quad &= \quad \left\{(y_1, y_2) \quad : \quad y_2 \quad \leq \quad A r^2 - \tfrac{y_1}{r} \qquad \text{for all} \quad r \in \mathbb{R}_{++}\right\} \\ &= \quad \left\{(y_1, y_2) \quad : \quad y_2 \quad \leq \quad \min_{r>0} \left(A r^2 - \tfrac{y_1}{r}\right)\right\} \end{aligned}$$

Solve this minimization problem, and describe the production set Y^O.

(d) Verify that a production set Y equal to the set Y^O you just calculated would generate the "data" $\pi(p) = A p_1^{-2} p_2^3$ that we started with.

If you feel you've already done enough algebra on this assignment, you may use the fact that $(2^{-2/3} + 2^{1/3})^3 = \frac{27}{4}$ without calculating it.

If you already did question 2, you can use what you found there, and this question should not take you very long.