Problem Set #2

Danny Edgel
Econ 712: Macroeconomics I
Fall 2020

November 19, 2020

Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

(a) In a balanced growth path, households' utility and firms' profits are maximized. Thus, to find a system of equations that C_0, N_0, w_0, and r_0 must solve in a balanced growth path, we must characterize and solve the household and firm problems. To start, the household problem is given by

$$\max_{\{C_t, K_{t+1}^s, N_t^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_t u(C_t, 1-N_t) \text{ s.t. } \sum_{t=0}^{\infty} p_t \left(C_t + K_{t+1}\right) \leq \sum_{t=0}^{\infty} p_t \left(r_t K_t + W_t N_t\right) + \pi_0$$

And the firm's problem is:

$$\max_{\{K_t^d, N_t^d\}_{t=0}^{\infty}} \pi_0 = \sum_{t=0}^{\infty} p_t \left(Y_t - r_t K_t^d - W_t N_t^d\right) \text{ s.t. } Y_t \leq F(K_t^d, A_t N_t^d)$$

Where, recognizing that π_0 is monotone in Y_t, the firm's problem can be reduced to:

$$\max_{\{K_t^d, N_t^d\}_{t=0}^{\infty}} \pi_0 = \sum_{t=0}^{\infty} p_t \left(F(K_t^d, A_t N_t^d) - r_t K_t^d - W_t N_t^d\right)$$

Since the firm is a price-taker whose decision in each period has no bearing on its conditions in any other period, the firm's problem can be solved by solving a single arbitrary period, t, using first-order conditions. For all choice variables, X, let $\frac{X_t}{A_t} = x_t$:

$$\max_{\{k_t^d, N_t^d\}_{t=0}^{\infty}} \pi_0 = \sum_{t=0}^{\infty} A_t p_t \left(F(k_t^d, N_t^d) - r_t k_t^d - w_t N_t^d\right)$$

$$K_t^d: \qquad A_t p_t F_K(k_t^d, N_t^d) - A_t p_t r_t = 0$$

$$F_K(k_t^d, N_t^d) = r_t$$

$$N_t^d: \qquad p_t F_N(k_t^d, N_t^d) - p_t w_t = 0$$

$$F_N(k_t^d, N_t^d) = w_t$$

This solution ensures that that $\pi_0 = 0$. Assuming that the utility function is concave, strictly increasing, and differentiable, the constrating of the household problem holds with equality and an interior solution exists. Given the firm's solution, we can solve the constraint as:

$$c_t = (r_t + 1 - \delta)k_t^s + w_t N_t^s - k_{t+1}^s$$

Such that the household's problem has the Lagrangian function:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t u(A_t c_t, N_t) - p_t \lambda \left(c_t + k_{t+1} - (r_t + 1 - \delta)k_t - w_t N_t \right) \right]$$

Which has the first-order conditions:

$$c_t: \qquad \beta^t A_t u_c(A_t c_t, N_t) = p_t \lambda$$

$$c_{t+1}: \qquad \beta^{t+1} A_{t+1} u_c(A_{t+1} c_{t+1}, N_{t+1}) = p_{t+1} \lambda$$

$$N_t: \qquad -\beta^t \frac{u_N(A_t c_t, N_t)}{w_t} = p_t \lambda$$

$$N_{t+1}: \qquad \beta^{t+1} \frac{-u_N(A_{t+1} c_{t+1}, N_{t+1})}{w_{t+1}} = p_{t+1} \lambda$$

$$k_{t+1}: \qquad \frac{p_t}{p_{t+1}} = r_{t+1} + 1 - \delta$$

Where the growth of households' optimal level of consumption is dependent on wage growth:

$$-u_N(A_t c_t, N_t)/w_t = A_t u_c(A_t c_t, N_t)$$

In equilibrium, all markets clear:

$$K_t^s = K_t^d \qquad \qquad \text{(Capital)}$$

$$N_t^s = N_t^d \qquad \qquad \text{(Labor)}$$

$$C_t + K_{t+1} - (1 - \delta)K_t = F(K_t, A_t N_t) \qquad \text{(Goods)}$$

And each initial choice variable, N_0 and C_0, and price, w_0 and r_0, satisfies the firm and household optimization conditions for a balanced growth path:

$$F_K(k_0, N_0) = r_0$$

$$F_N(k_0, N_0) = w_0$$

$$-\frac{u_N(A_0 c_0, N_0)}{u_c(A_0 c_0, N_0)} = w_0 A_0$$

$$c_0 + k_1 - (1 - \delta)K_0 = F(k_0, N_0)$$

(b) Let $u(C, N) = \frac{C^{1-\gamma}}{1-\gamma} h(1 - N)$, where $\gamma > 0$, $\gamma \neq 1$. Then, the first-order conditions, provided in (a), can be used to derive:

$$\beta(1+g)^{1-\gamma} \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \left(\frac{h(1 - N_{t+1})}{h(1 - N_t)}\right) = \frac{p_{t+1}}{p_t} \tag{1}$$

$$\beta(1+g)^{1-\gamma} \left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} \left(\frac{h'(1 - N_{t+1})}{h'(1 - N_t)}\right) = \left(\frac{w_{t+1}}{w_t}\right)\left(\frac{p_{t+1}}{p_t}\right) \tag{2}$$

$$r_{t+1} + 1 - \delta = \frac{p_t}{p_{t+1}} \tag{3}$$

In a balanced growth path, capital is rented at a constant $\bar{r}$. Thus, by equation (3), the price level grows at constant rate $\bar{\pi}$:

$$\bar{\pi} = \frac{1}{\bar{r} + 1 - \delta}$$

Further, in a balanced growth path, $N_t = N_{t+1} = \overline{N}$. Thus, equations (1) and (2) become:

$$\beta(1+g)^{1-\gamma} \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \bar{\pi} \tag{4}$$

$$\beta(1+g)^{1-\gamma} \left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} = \left(\frac{w_{t+1}}{w_t}\right)\bar{\pi} \tag{5}$$

By dividing (5) by (4), solve for the wage growth rate:

$$\frac{c_{t+1}}{c_t} = \frac{w_{t+1}}{w_t}$$

So consumption growth is equal to wage growth. In a balanced growth path, standardized variables (those scaled by A_t) are stationary, so we can solve for $\overline{pi}$ and $\bar{r}$ using (4):

$$\beta(1+g)^{1-\gamma} = \bar{\pi}$$
$$(\bar{r} + 1 - \delta)\beta(1+g)^{1-\gamma} = 1$$
$$\bar{r} = \frac{1 - (1 - \delta)\beta(1+g)^{1-\gamma}}{\beta(1+g)^{1-\gamma}}$$

Knowing that, when scaled by A_t, capital, consumption, and output will all be constant, we can also solve for those values using the law of motion:

$$\bar{c} + (g + \delta)\bar{k} = F(\bar{k}, \overline{N})$$

Thus, there will be a balanced growth path for $\gamma > 0$, $\gamma \neq 1$. For $\gamma = 1$, we can see that the first-order conditions will yield the same result, but with a different inflation (and thus interest) rate:

$$\beta(1+g)\left(\frac{c_t}{c_{t+1}}\right) = \bar{\pi}$$

$$\beta\left(\frac{h'(1 - N_{t+1})}{h'(1 - N_{t+1})}\right) = \left(\frac{w_{t+1}}{w_t}\right)\bar{\pi}$$

(c) We cannot characterize this system with a phase diagram because the optimal level of capital and consumption depend not only on each other, but also on labor supply, which adds a third dimension to the system. Since we don't know the specific relationship between labor supply and utility, we cannot say whether an increase in, say, δ, will ultimately lead to a higher or lower level of consumption and capital because a higher marginal rate of substitution (MRS) of consumption for liesure could lead to greater consumption, while a lower MRS could lead to higher liesure levels but lower capital and consumption.

(d) If h is a constant function, then, since liesure is not valued, $N_t = 1$ for all t, and, letting $F(k, 1) = f(k)$, each of the system's variables can be solved by equations with known parameterized values:

$$\overline{\pi} = \beta(1 + g)^{1-\gamma} \qquad \text{(consumption FOC)}$$

$$\overline{r} = \frac{1}{\overline{\pi}} - (1 - \delta) \qquad \text{(capital FOC)}$$

$$\overline{w} = f(\overline{k}) - \overline{r}\overline{k} \qquad \text{(zero profit)}$$

$$\overline{k} = f'(f^{-1}(\overline{r})) \qquad \text{(firm FOC)}$$

$$\overline{c} = f(\overline{k}) - (g + \delta)\overline{k} \qquad \text{(law of motion)}$$

$$\overline{y} = f(\overline{k}) \qquad \text{(production function)}$$

(e) Suppose there is a decrease in the rate of technology, g. This causes a decrease in the rate of inflation, $\overline{pi}$, which increases the interest rate, $\overline{r}$. Long-run $\overline{k}$ is equal to the derivative of the production function at the inverse of the production function at $\overline{r}$. Since f is increasing but concave, an increase in $\overline{r}$ causes a decrease in the long-run value of $\overline{k}$.

Now consider the effect on consumption. A decrease in $\overline{k}$ causes a decrease in $\overline{k}$ but decreases $(g + \delta)\overline{k}$ (which is also shrunk by the first-order effect of a decrease in g). Thus, the effect of g's decrease in consumption in the long term depends on the specification of f.

In the short run, capital levels are fixed. Thus, in order to move from the initial steady state to the new steady state, consumption will spike, followed by a gradual decrease in both c and k until the new steady state is reached.

(f) Define $\bar{s}$ as the gross output that is invested in capital in the steady state and $\bar{\theta}$ be the fraction of output saved. Then, using the law of motion, we can solve,

$$\bar{s} = f(\bar{k}) - \bar{c}$$

$$\bar{\theta} = \frac{\bar{s}}{f(\bar{k}}$$

$$= 1 - \frac{\bar{c}}{f(\bar{k}}$$

$$\frac{\partial \bar{\theta}}{\partial g} = -\frac{f(\bar{k})\frac{\partial \bar{c}}{\partial g} - \bar{c}f'(\bar{k})\frac{\partial \bar{k}}{\partial g}}{f(\bar{k})^2}$$

$$\frac{\partial \bar{c}}{\partial g} = f'(\bar{k})\frac{\partial \bar{k}}{\partial g} - \frac{\partial \bar{k}}{\partial g}$$

$$\frac{\partial \bar{k}}{\partial g} = \left(\frac{f''(f^{-1}(\bar{r}))}{f'(f^{-1}(\bar{r}))}\right)\frac{\partial \bar{r}}{\partial g}$$

$$\frac{\partial \bar{r}}{\partial g} = -\frac{1}{\bar{\pi}}\left(\frac{\partial \bar{\pi}}{\partial g}\right)$$

$$\frac{\partial \bar{\pi}}{\partial g} = (1-\gamma)\beta(1+g)^{-\gamma}$$

$$\Rightarrow \frac{\partial \bar{\theta}}{\partial g} = \left(\frac{\partial \bar{k}}{\partial g}\right)\frac{\bar{c}f'(\bar{k}) - f(\bar{k})(f'(\bar{k})-1)}{f(\bar{k})^2}$$

$$= \left(\frac{f''(f^{-1}(\bar{r}))}{f'(f^{-1}(\bar{r}))}\right)\left(\frac{(\gamma-1)\beta(1+g)^{-\gamma}}{\bar{\pi}^2}\right)\frac{\bar{c}f'(\bar{k}) - f(\bar{k})(f'(\bar{k})-1)}{f(\bar{k})^2}$$

$$= \left(\frac{f''(f^{-1}(\bar{r}))}{f'(f^{-1}(\bar{r}))}\right)\left(\frac{(\gamma-1)\beta(1+g)^{-\gamma}}{\bar{\pi}^2}\right)\frac{f'(\bar{k})\left(\bar{c} - f(\bar{k}) + \frac{f(\bar{k})}{f'(\bar{k})}\right)}{f(\bar{k})^2}$$

So the change in fraction of output saved, with respect to g, comes down to a product of three terms. Recall that $\gamma > 1$ and f is strictly increasing and concave. Thus, the first term is negative and the second term is positive. Further, since savings are always positive, $\bar{c} < f(\bar{k})$, so the third term is negative. Therefore, the overall product is positive, and we can conclude that the fraction of output saved increases when g increases.

Specialzed to Cobb-Douglas production, $f(k) = k^\alpha$, so the derivative from above is:

$$\frac{\partial \bar{\theta}}{\partial g} = \left((\alpha-1)\bar{r}^{-\frac{1}{\alpha}}\right)\left(\frac{(\gamma-1)\beta(1+g)^{-\gamma}}{\bar{\pi}^2}\right)\frac{\alpha\bar{k}^{\alpha-1}\left(\bar{c} - \bar{k}^\alpha + \frac{1}{\alpha}\bar{k}\right)}{f(\bar{k})^2}$$

Question 2

(a) The consumer's problem is:

$$\max_{\{c_t, s_t\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}\right] \quad \text{s.t. } x_{t+1} = A_t s_t, \ s_t \leq x_t - c_t$$

The expectations operator in this case averages utility in each period across states of the world, weighting by that state's probability. Specifically, for each t, letting $c_t = A_{t-1} s_{t-1} - s_t$:

$$\mathbb{E}\left[\beta^t \frac{(A_{t-1} s_{t-1} - s_t)^{1-\gamma}}{1-\gamma}\right] = \pi \left[\beta^t \frac{(A_h s_{t-1} - s_t)^{1-\gamma}}{1-\gamma}\right] + (1-\pi) \left[\beta^t \frac{(A_l s_{t-1} - s_t)^{1-\gamma}}{1-\gamma}\right]$$

(b) In each period, the consumer's consumption-savings problem is constrained by her savings in the last period, multiplied by draw of A she received that period. Thus, her relevant state variable is $A_{t-1} s_{t-1}$. The Bellman for this problem, letting A^{-1} and s^{-1} be the values for A and s in the prior period, is:

$$V(A^{-1} s^{-1}) = \max_s \left\{ \frac{(A^{-1} s^{-1} - s)^{1-\gamma}}{1-\gamma} + \beta \left[\pi V(A_h s) + (1-\pi) V(A_l s)\right]\right\}$$

Given that $\gamma \in (0,1)$, $\frac{(A^{-1} s^{-1} - s)^{1-\gamma}}{1-\gamma}$ is clearly concave, increasing, and continuous in s^{-1}. Though the utility function itself is unbounded, the set of feasible choices for s is bounded (and convex, continuous, and nonempty), since β is bounded above by 1 and A is bounded above by $1/\beta$. Thus, we can treat u as if it is bounded, since there exists a maximum level of utility that could feasibly be reached by the consumer.

(c) Given the conditions shown in (b), we can obtain a solution to the Bellman equation by taking first order conditions:

$$- \left(A^{-1} s^{-1} - s\right)^{-\gamma} + \beta \left(\pi V'(A_h s) + (1-\pi) V'(A_l s)\right) = 0 \qquad \text{(FOC w/r/t } s)$$

$$V'(As) = (As - s')^{-\gamma} \qquad \text{(envelope condition)}$$

$$\Rightarrow \beta \left[\pi (A_h - s')^{-\gamma} + (1-\pi)(A_l s - s')^{-\gamma}\right] = \left(A^{-1} s^{-1} - s\right)^{-\gamma}$$

Guess that the optimal policy function is to save a constant fraction of wealth, such that $A^{-1} s^{-1} - s = \alpha A^{-1} s^{-1}$. Then, solving for the optimization condition yields:

$$A^{-1} s^{-1} - s = \alpha A^{-1} s^{-1}$$

$$s = (1-\alpha) A^{-1} S^{-1}$$

$$\Rightarrow \beta \left[\pi (\alpha A_h s)^{-\gamma} + (1-\pi)(\alpha A_l s)^{-\gamma}\right] = (\alpha A^{-1} s^{-1})^{-\gamma}$$

$$\beta \left[\pi (\alpha A_h (1-\alpha) A^{-1} s^{-1})^{-\gamma} + (1-\pi)(\alpha A_l (1-\alpha) A^{-1} s^{-1})^{-\gamma}\right] = \alpha^{-\gamma} (A^{-1} s^{-1})^{-\gamma}$$

$$(\alpha - \alpha^2)^{-\gamma} (A^{-1} s^{-1})^{-\gamma} \beta \left[\pi A_h^{-\gamma} + (1-\pi) A_l^{-\gamma}\right] = \alpha^{-\gamma} (A^{-1} s^{-1})^{-\gamma}$$

$$(1-\alpha)^{-\gamma} = \left(\beta \left[\pi A_h^{-\gamma} + (1-\pi)A_l^{-\gamma}\right]\right)^{-1}$$

$$\alpha = 1 - \left(\beta \left[\pi A_h^{-\gamma} + (1-\pi)A_l^{-\gamma}\right]\right)^{1/\gamma}$$

Thus, the optimal savings are a constant fraction of current wealth that does not depend on current wealth, previous period savings, or previous period's shock.

(d) We know that this policy function is the unique solution to the sequence problem because of all of the conditions specified in (b), that the utility function is strictly increasing, continuous and (effectively) bounded; the feasible set is convex, continuous, and nonempty; and $0 < \beta < 1$.

Question 3

To begin, we can analytically determine the steady-state of this system, which can be used to check the policy function derived by the program. Letting α represent the exponent on capital for Cobb-Douglas production, the Bellman of the social planner's problem is:

$$V(K) = \max_{K'} \left\{ \frac{(zK^\alpha + (1-\delta)K - K')^{1-\gamma}}{1-\gamma} + \beta V(K') \right\}$$

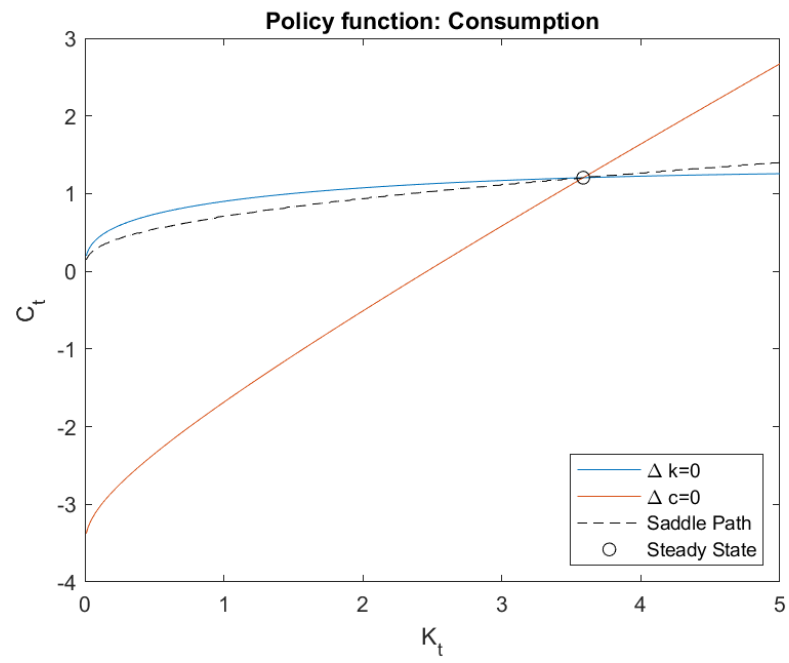Taking first-order conditions and applying the envelope condition, we get:

$$\beta \left(z\alpha K'^{\alpha-1} + 1 - \delta\right) c'^{-\gamma} = c^{-\gamma}$$

In the steady-state, $c = c' = \bar{c}$ and $K = K' = \overline{K}$, which allows us to solve:

$$\overline{K} = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} + \delta - 1\right)\right]^{-\frac{1}{\alpha-1}}$$

$$\bar{c} = z\overline{K}^\alpha - \delta\overline{K}$$

(a) The following chart displays the phase diagram for $\Delta c = 0$ and $\Delta K = 0$, alongside the optimal policy function, $c(K)$, which intersects the intersection of $\Delta c = 0$ and $\Delta K = 0$ at the steady state.

Policy function: Consumption

(b) When γ decreases to 1.01, the steady-state values of K and c do not change. This is unsurprising, given that γ does not appear in our analytical solution of the steady state. However, the saddle path gets steeper, meaning that one-time shocks will cause a larger change in consumption, and consumption will proceed to move to the steady state more quickly.
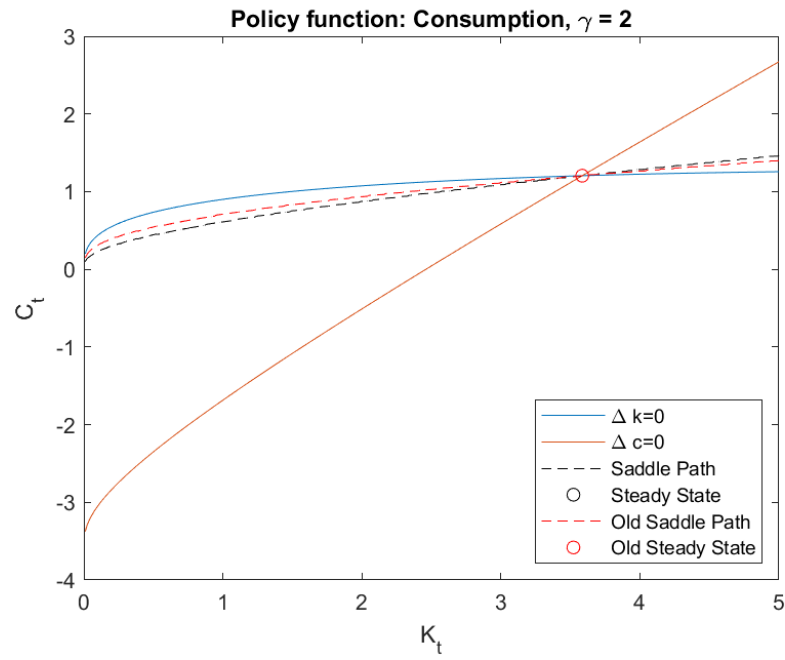
(c) Productivity, z, appears only in the steady-state value of consumption. Thus, a permanent, unexpected shock to productivity will change the steady-state value of consumption, but not that of capital. Since movements along the saddle path to the steady state are achieved by immediate changes to consumption but gradual accumulation or depreciation of capital, this shock to productivity will cause an immediate, one-time increase in consumption that moves the agent from the old steady-state to the new one. This is shown in the plot below.