Problem Set #1

Danny Edgel
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Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

Let $(Y, X')'$ be a random vector, where $Y = X'\beta_0 \cdot U$, where $\mathbb{E}\left[U \mid X\right] = 1$, $\mathbb{E}\left[XX'\right]$ is invertible, and $\mathbb{E}\left[Y^2 + ||X||^2\right] < \infty$.

(i) Since $\mathbb{E}\left[U \mid X\right] = 1$, the expectation of Y, conditional on X, is $X'\beta_0$. Then, $\frac{\partial}{\partial X}Y = \beta_0$.

(ii) Define $V = U - 1$. Then, $\mathbb{E}\left[V \mid X\right] = \mathbb{E}\left[U - 1 \mid X\right] = 0$, and:

$$Y = X'\beta_0(V+1) = X'\beta_0 V + X'\beta_0 = X'\beta_0 + \widetilde{U}$$

Where:

$$\mathbb{E}\left[\widetilde{U} \mid X\right] = \mathbb{E}\left[X'\beta_0 V \mid X\right] = X'\beta_0\mathbb{E}\left[V \mid X\right] = 0$$

Thus, $Y = X'\beta_0 + \widetilde{U}$, where $\mathbb{E}\left[\widetilde{U} \mid X\right] = 0$.

(iii) Let $\beta = \beta_0$. Then,

(iv) Define $V = U - 1$. Then, $\mathbb{E}\left[V \mid X\right] = \mathbb{E}\left[U - 1 \mid X\right] = 0$, and:

$$\begin{aligned}\mathbb{E}\left[X(Y - X'\beta)\right] &= \mathbb{E}\left[X(X'\beta_0 \cdot U - X'\beta_0)\right] = \mathbb{E}\left[\mathbb{E}\left[X(X'\beta_0 \cdot U - X'\beta_0)|X\right]\right] \\ &= \mathbb{E}\left[XX'\beta_0\mathbb{E}\left[(U-1)|X\right]\right] = 0\end{aligned}$$

Thus, $\beta = \beta_0 \Rightarrow \mathbb{E}\left[X(Y - X'\beta)\right] = 0$. Now, Suppose $\mathbb{E}\left[X(Y - X'\beta)\right] = 0$. Then,

$$\mathbb{E}\left[X(Y - X'\beta)\right] = \mathbb{E}\left[X(X'\beta \cdot U - X'\beta_0)\right] = 0$$
$$\mathbb{E}\left[XX'\mathbb{E}\left[\beta \cdot U - \beta_0|X\right]\right] = (\beta - \beta_0)\mathbb{E}\left[XX'\right] = 0$$

We know that $\mathbb{E}[XX']$ is invertible, so $\mathbb{E}[XX'] \neq 0$. Thus, $\mathbb{E}[X(Y - X'\beta)] = 0 \Rightarrow \beta = \beta_0$.

$\therefore \mathbb{E}[X(Y - X'\beta)] = 0 \iff \beta = \beta_0$ ■

Knowing this, we can derive the method of moments estimator for β:

$$\begin{aligned}
\mathbb{E}[X(Y - X'\beta)] &= \mathbb{E}[XY] - \mathbb{E}[XX']\beta = 0 \\
\mathbb{E}[XX']\beta &= \mathbb{E}[XY] \\
\beta &= \mathbb{E}[XX']^{-1}\mathbb{E}[XY] \\
\Rightarrow \hat{\beta} &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i = \hat{\beta}_{OLS}
\end{aligned}$$

(v) We can simplify the final equation in (iii) to show:

$$\begin{aligned}
\mathbb{E}\left[\hat{\beta} \mid X_1, ..., X_n\right] &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i Y_i \mid X_1, ..., X_n\right] \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_i \mathbb{E}\left[X_i'\beta_0 \cdot U \mid X_1, ..., X_n\right] \\
&= \beta_0 \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} X_i X_i'\right) \mathbb{E}\left[U \mid X_1, ..., X_n\right] \\
&= \beta_0
\end{aligned}$$

Thus, $\hat{\beta}$ is unbiased.

(vi) According to the weak law of large numbers (WLLN), random variables converge in probability to their expected value. Thus,

$$\hat{\beta} \to_p \mathbb{E}\left[\hat{\beta}\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{\beta} \mid X_1, ..., X_n\right]\right] = \mathbb{E}[\beta_0] = \beta_0$$

Thus, $\hat{\beta}$ is consistent.

Question 2

(i) We can use the continuous mapping theorem and law of large numbers to show:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^3 \rightarrow_p \mathbb{E}\left[X^3\right] \;\; , \; \frac{\sum_{i=1}^{n} X_i^3}{\sum_{i=1}^{n} X_i^2} = \frac{\mathbb{E}\left[X^3\right]}{\mathbb{E}\left[X^2\right]}$$

We cannot show convergence for the other two statistics.

(ii) You can use continuous mapping and the central limit theorem to show convergence in distribution of W_n, since:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i^2 - \mathbb{E}\left[X_i^2\right]\right) = \frac{1}{\sqrt{n}} n\left(\overline{X}_n^2 - \mathbb{E}\left[X_i^2\right]\right) = \sqrt{n}\left(\overline{X}_n^2 - \mathbb{E}\left[X_i^2\right]\right)$$

The other statistic cannot be shown to converge, as it simplifies to $\sqrt{n}\left(\overline{X}_n^2 - \overline{X}_n^2\right)$

(iii) The CDF of X is simply $f(x) = x$. The CDF for the maximum value, x, of n observations of X_i, then, is x^n. Thus, we can define for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
Pr\left(|\max_{1\leq i\leq n} X_i - 1| \leq \varepsilon\right) &= Pr\left(\max_{1\leq i\leq n} X_i - 1 \geq -\varepsilon\right) \\
&= Pr\left(\max_{1\leq i\leq n} X_i \geq 1-\varepsilon\right) \\
&= 1 - Pr\left(\max_{1\leq i\leq n} X_i \geq 1-\varepsilon\right) \\
&= 1-(1-\varepsilon)^n \\
1-(1-\varepsilon)^n \rightarrow_p 1 &
\end{aligned}$$

(iv) The CDF of X is $f(x) = 1 - e^{-x}$. Thus, Thus, we can define for any $\varepsilon > 0$ and $M \geq 0$,

$$Pr\left(\max_{1\leq i\leq n} X_i \leq M\right) = (1-e^{-M})^n(1-e^{-M})^n \qquad \rightarrow_p 1$$

Question 3

(i) Since $\{X_i\}_{i=1}^n$ is an i.i.d. sequence of a random variable $X \sim \mathcal{N}(0,1)$, by the CLT, $\sqrt{n}\overline{X}_n \rightarrow_d \mathcal{N}(0,1)$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \rightarrow_d \mathcal{N}(0,1)$$

(ii) We can show this by determining the convergence of the first and second central moments of Y_i in probability:

$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \to_p \mathbb{E}\left[Y_i\right] = \mathbb{E}\left[X_i W_i\right] = \mathbb{E}\left[X_i\right]\mathbb{E}\left[W_i\right] = 0$$

$$\begin{aligned}\hat{\sigma}_Y^2 &= \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2 \to_p \mathbb{E}\left[(Y_i - \mathbb{E}\left[Y_i\right])^2\right] = \mathbb{E}\left[(X_i W_i)^2\right] \\ &= \mathbb{E}\left[X_i^2\right] = \sigma_X^2 = 1\end{aligned}$$

Thus, by the CLT, $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i \to_d \mathcal{N}(0,1)$

(iii) Given the results in (ii), we can calculate,

$$Cov(X_i, Y_i) = \mathbb{E}\left[\left(X_i - \mathbb{E}\left[X_i\right]\right)\left(Y_i - \mathbb{E}\left[Y_i\right]\right)\right] = \mathbb{E}\left[X_i Y_i\right] = \mathbb{E}\left[X_i^2 W_i\right] = \mathbb{E}\left[X_i^2\right]\mathbb{E}\left[W_i\right] = 0$$

(iv) No. Given the results from (i)-(iii), we can show:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i \end{pmatrix} \to_d \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, V\right)$$

$$\text{Where } V = \begin{pmatrix} \sigma_X^2 & Cov(X_i, Y_i) \\ Cov(X_i, Y_i) & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

However, the answer to (iv) shows that this is not the case.

(v) Applying the Cramer-Wold device, V converges in distribution to $\mathcal{N}(0, I_2)$ if and only if $t'V$ converges, for some $t \in \mathbb{R}^2$ with $||t|| = 1$. If we let each entry of t be $1/\sqrt{2}$, we can see that $t'V$ cannot have a continuous distribution, let alone a normal one.