# Problem Set #2

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## Question 1

(i) Yes, $\hat{\beta}_1^{IV} \to_p \beta_1$. By the Weak Law of Large Numbers (WLLN) and the recognition that the law of iterated expectation (LIE) implies $\mathbb{E}[U] = \mathbb{E}[\mathbb{E}[U|Z]] = 2$,

$$\begin{aligned}
\hat{\beta}_1^{IV} \to_p & \frac{\mathbb{E}[(Z-\mathbb{E}[Z])(Y-\mathbb{E}[Y])]}{\mathbb{E}[(Z-\mathbb{E}[Z])(X-\mathbb{E}[X])]} \\
=& \frac{\mathbb{E}[(Z-\mathbb{E}[Z])(\beta_0 + X\beta_1 + U - \mathbb{E}[\beta_0 + X\beta_1 + U])]}{\mathbb{E}[(Z-\mathbb{E}[Z])(X-\mathbb{E}[X])]} \\
=& \frac{\mathbb{E}[(Z-\mathbb{E}[Z])(\beta_0 + X\beta_1 + U - \beta_0 - \beta_1\mathbb{E}[X] - \mathbb{E}[U])]}{\mathbb{E}[(Z-\mathbb{E}[Z])(X-\mathbb{E}[X])]} \\
=& \frac{\mathbb{E}[\beta_1(Z-\mathbb{E}[Z])(X-\mathbb{E}[X]) + (Z-\mathbb{E}[Z])(U-\mathbb{E}[U])]}{\mathbb{E}[(Z-\mathbb{E}[Z])(X-\mathbb{E}[X])]} \\
=& \frac{\beta_1\mathbb{E}[(Z-\mathbb{E}[Z])(X-\mathbb{E}[X])] + \mathbb{E}[ZU - U\mathbb{E}[Z] - Z\mathbb{E}[U] + \mathbb{E}[Z]\mathbb{E}[U]]}{\mathbb{E}[(Z-\mathbb{E}[Z])(X-\mathbb{E}[X])]} \\
=& \beta_1 + \frac{2\mathbb{E}[Z] - 2\mathbb{E}[Z] + 2\mathbb{E}[Z] - 2\mathbb{E}[Z]}{\mathbb{E}[(Z-\mathbb{E}[Z])(X-\mathbb{E}[X])]} \\
=& \beta_1
\end{aligned}$$

(ii) Yes, $\hat{\beta}_0^{IV} \to_p \beta_0$. Given (i), we can calculate:

$$\hat{\beta}_0^{IV} = \overline{Y} - \overline{X}\hat{\beta}_1 \to_p \mathbb{E}[Y] - \mathbb{E}[X]\beta_1 = \beta_0$$

## Question 2

(i) $Z$ is a valid instrument if $Cov(Z, X) \neq 0$, i.e., if $\pi_1 \neq 0$.

(ii) We can derive $\gamma_0$, $\gamma_1$ and $\varepsilon$ as functions of the structural parameters by first deriving the reduced form of the model:

$$
\begin{aligned}
Y &= \beta_0 + (\pi_0 + Z\pi_1 + V)\,\beta_1 + U \\
Y &= \beta_0 + \pi_0\beta_1 + Z\pi_1\beta_1 + V\beta_1 + U \\
Y &= \gamma_0 + Z\gamma_1 + \varepsilon
\end{aligned}
$$

Where:

$$\gamma_0 = \beta_0 + \pi_0\beta_1,\ \gamma_1 = \pi_1\beta_1,\ \varepsilon = V\beta_1 + U$$

(iii) The IV estimator of $\beta_1$ is

$$\hat{\beta}_1^{IV} = \frac{\widehat{Cov(Z, Y)}}{\widehat{Cov(Z, X)}}$$

And the two OLS estimators, of $\gamma_1$ and $\pi_1$ respectively, are

$$\hat{\pi}_1 = \frac{\sum_{i=1}^{n}(Z_i - \overline{Z})(X_i - \overline{X})}{\sum_{i=1}^{n}(Z_i - \overline{Z})^2},\ \hat{\gamma}_1 = \frac{\sum_{i=1}^{n}(Z_i - \overline{Z})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(Z_i - \overline{Z})^2}$$

Then the indirect least squares estimator of $\beta_1$ is

$$\frac{\hat{\gamma}_1}{\hat{\pi}_1} = \frac{\sum_{i=1}^{n}(Z_i - \overline{Z})(Y_i - \overline{Y})\sum_{i=1}^{n}(Z_i - \overline{Z})^2}{\sum_{i=1}^{n}(Z_i - \overline{Z})^2\sum_{i=1}^{n}(Z_i - \overline{Z})(X_i - \overline{X})} = \frac{\sum_{i=1}^{n}(Z_i - \overline{Z})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(Z_i - \overline{Z})(X_i - \overline{X})} = \hat{\beta}_1^{IV}$$

(iv)

(v)

## Question 3

(i)

(ii)

(iii)

(iv)

(v)

(vi)