Problem Set #4

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Question 1

In order for Pr(Defying) $= 0$, Z must be monotonic in X. In order for Pr(Complying) > 0, it must be the case that, in a nonzero number of cases, $X(Z = 1) = 1$ where $X(Z = 0) = 0$. Thus, $U_1 > 0$ and $\Pr(U_1 > U_0) > 0$.

Question 2

(i) The autocovariance function is defined as:

$$\gamma(k) = Cov(Y_t, Y_{t-k}) = \mathbb{E}\left[(Y_t - \mathbb{E}[Y_t])(Y_{t-k} - \mathbb{E}[Y_{t-k}])\right] = \mathbb{E}[Y_t Y_{t-k}] - \mathbb{E}[Y_t]\mathbb{E}[Y_{t-k}]$$

Where:

$$\begin{aligned} \mathbb{E}[Y_t] &= \mu + \mathbb{E}[\varepsilon_t] + \theta_1 \mathbb{E}[\varepsilon_{t-1}] + ... + \theta_q \mathbb{E}[\varepsilon_{t-q}] = \mu = \mathbb{E}[Y_{t-k}], \ \forall k \\ Y_t Y_{t-k} &= \mu^2 + \mu\left(\varepsilon_{t-k} + \theta_1 \varepsilon_{t-k-1} + ... + \theta_q \varepsilon_{t-k-q}\right) + \varepsilon_t\left(\varepsilon_{t-k} + \theta_1 \varepsilon_{t-k-1} + ... + \theta_q \varepsilon_{t-k-q}\right) + ... \\ \mathbb{E}[Y_t Y_{t-k}] &= \mu^2 + \varepsilon_t^2 + ... + \varepsilon_{t-k}^2 \end{aligned}$$

Thus, letting $\varepsilon_t^2 = \sigma^2$ for all t and recognizing that $\theta_k = 0$ for all $k < t - q$,

$$\gamma(k) = \begin{cases} \left(\theta_k + ... + \theta_{q-k}\theta_q\right)\sigma^2, & k \leq q \\ 0, & k > q \end{cases}$$

(ii) If $q = 1$, then:

$$\gamma(k) = \begin{cases} \left(1 + \theta_1^2\right)\sigma^2 & k = 0 \\ \theta_1 \sigma^2, & k = 1 \\ 0, & k > 1 \end{cases} \Rightarrow \rho(k) = \begin{cases} 1 & k = 0 \\ \frac{\theta_1}{1+\theta_1^2}, & k = 1 \\ 0, & k > 1 \end{cases}$$

(iii) θ_1 is *not* identified from the autocorrelation function, since it only shows up in the $k = 1$ case, in which the solution is nonunique in most cases, since any value other than -1 or 1 has the same autocorrelation for its reciprocal.

(iv) In the case where $\theta_1 \in [-1, 1]$, we can rule out the solution with an absolute value greater than 1, in which case θ_1 is identified.

Question 3

(i) To set $\mu = \mathbb{E}[Y_t]$ such that $\mathbb{E}[Y_t]$ doesn't rely on t, we can use the first two observations of Y_t:

$$Y_1 = \alpha_0 + Y_0\rho + \varepsilon_1 + \theta\varepsilon_0 = \alpha_0 + (\mu + \varepsilon_0 + \nu)\rho + \varepsilon_1 + \theta\varepsilon_0$$

Thus,

$$\begin{aligned}\mathbb{E}[Y_0] &= \mathbb{E}[Y_1] \\ \mu &= \alpha_0 + \rho\mu \\ \mu &= \frac{\alpha_0}{1-\rho}\end{aligned}$$

Similarly, we can use Y_0 and Y_1 to determine τ:

$$\begin{aligned}Var(Y_0) &= Var(Y_1) = Var(\rho Y_0 + \varepsilon_1 + \theta\varepsilon_0) \\ \sigma^2 + \tau &= \rho^2(\sigma^2 + \tau) + \sigma^2 + \theta^2\sigma^2 \\ (1-\rho^2)\tau &= (\rho^2 + \theta^2)\sigma^2 \\ \tau &= \frac{(\rho^2 + \theta^2)\sigma^2}{1-\rho^2}\end{aligned}$$

(ii) In order for $(1, Y_{t-2})$ to be a valid instrument for $(1, Y_{t-1})$, it would need to satisfy (1) exogeneity, and (2) relevance:

$$\mathbb{E}[U_t | Y_{t-2}] = 0 \tag{1}$$

$$Cov(Y_{t-1}, Y_{t-2}) \neq 0 \tag{2}$$

Since we've established that $\{Y_t\}$ is stationary, we need only establish exogeneity and relevance for Y_0 relative to Y_1 and Y_2:

$$\begin{aligned}\mathbb{E}[U_2 | Y_0] &= \mathbb{E}[\varepsilon_1 + \theta\varepsilon_0 | \mu + \varepsilon_0 + \nu] = 0 \\ Cov(Y_1, Y_0) &= \mathbb{E}[Y_1 Y_0] - \mu^2 \\ &= \alpha_0\mu + \rho\mathbb{E}\left[(\mu + \varepsilon_0 + \nu)^2\right] + \mathbb{E}[(\varepsilon_1 + \theta\varepsilon_0)Y_0] - \mu^2 \\ &= \alpha_0\mu + \rho(\mu^2 + \sigma^2 + \tau) + \mathbb{E}[(\varepsilon_1 + \theta\varepsilon_0)Y_0] - \mu^2 \\ &= \alpha_0\mu + \rho\mu^2 + \rho\sigma^2 + \rho\tau + \theta\sigma^2 - \mu^2 \\ &= \frac{\alpha_0^2}{1-\rho} - \frac{\alpha_0^2}{1-\rho} + \rho\tau + (\rho + \theta)\sigma^2 \\ &= \rho\tau + (\rho + \theta)\sigma^2 > 0\end{aligned}$$