Econ 712 Macroeconomics I

Fall 2020, University of Wisconsin-Madison

Instructor: Dean Corbae, TAs: Katya Kazakova and Duong Dang

Problem Set #3

1. Consider the following overlapping generations problem. In each period $t = 1, 2, 3, ...$ a new generation of 2 period lived households are born. Each generation has a unitary mass. There is a unit measure of initial old who are endowed with $\bar{M} > 0$ units of fiat money. Each generation is endowed with w_1 in youth and w_2 in old age of non-storable consumption goods where $w_1 > w_2$. There is no commitment technology to enforce trades. The utility function of a household of generation $t \geq 1$ is

$$U\left(c_t^t, c_{t+1}^t\right) = \ln(c_t^t) + \ln(c_{t+1}^t)$$

where $\left(c_t^t, c_{t+1}^t\right)$ is consumption of a household of generation t in youth (i.e. in period t) and old age (i.e in period $t+1$). The preferences of the initial old are given by $U(c_1^0) = \ln(c_1^0)$ where c_1^0 is consumption by a household of the initial old.

(a) State and solve the planner's problem.[1]

Answer: A social planner chooses $\{c_1^0, (c_t^t, c_{t+1}^t)_{t=1}^{\infty}\}$ to

$$\max_{c_1^0, \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty}} \ln(c_1^0) + \sum_{t=1}^{\infty} \left[\ln(c_t^t) + \ln c_{t+1}^t\right] \tag{1}$$

$$\text{s.t. } c_t^{t-1} + c_t^t = w_1 + w_2 \quad \forall t \geq 1 \tag{2}$$

$$c_t^{t-1}, c_t^t \geq 0 \quad \forall t \geq 1. \tag{3}$$

To solve the social planner's problem (SPP), write down the Lagrangian and take first order conditions (F.O.C) with respect to c_1^0, c_t^t, and c_{t+1}^t for $t \geq 1$.

$$L = \ln(c_1^0) + \sum_{t=1}^{\infty} \left[\ln(c_t^t) + \ln c_{t+1}^t\right] + \sum_{t=1}^{\infty} \lambda_t \left[w_1 + w_2 - c_t^{t-1} - c_t^t\right] \tag{4}$$

[1] Don't worry about the finiteness of the objective function. We will learn more about solutions to this problem later.

F.O.C's:

$$\frac{1}{c_1^0} = \lambda_1 \tag{5}$$
$$\frac{1}{c_t^t} = \lambda_t \tag{6}$$
$$\frac{1}{c_{t+1}^t} = \lambda_{t+1} \tag{7}$$

We can combine the last two FOCs to get (just move condition 7 one period 'back' to see this):

$$\frac{1}{c_t^t} = \frac{1}{c_t^{t-1}} \tag{8}$$
$$\implies \quad c_t^{t-1} = c_t^t \tag{9}$$

Substituting (9) into the resource constraint (2) gives the planner's optimal consumption choices for $t \geq 1$.

$$c_t^{t*} = \frac{w_1 + w_2}{2} \tag{10}$$
$$c_t^{t-1*} = \frac{w_1 + w_2}{2} \tag{11}$$

And these two last equations determine $\{c_1^0, (c_t^t, c_{t+1}^t)_{t=1}^{\infty}\}$ as a function of the endowments.

(b) State the representative household's problem in period $t \geq 0$. Try to write the budget constraints in real terms.

Initial old solve:

$$\max_{c_1^0} \ln(c_1^0) \tag{12}$$
$$\text{s.t. } c_1^0 = w_2 + \bar{M}/p_1, \quad c_1^0 \geq 0. \tag{13}$$

While generations born in $t \geq 1$ solve:

$$\max_{\{c_t^t, c_{t+1}^t\}} \ln(c_t^t) + \ln c_{t+1}^t \tag{14}$$
$$\text{s.t. } c_t^t + \frac{M_t}{p_t} = w_1 \tag{15}$$
$$c_{t+1}^t = w_2 + \frac{M_t}{p_{t+1}} \tag{16}$$
$$c_t^{t-1}, c_t^t, M_t \geq 0. \tag{17}$$

Finally, if money demand is positive we can combine the budget constraints for the household, using $\Pi_t \equiv \frac{p_{t+1}}{p_t}$:

$$c_t^t + \Pi_t c_{t+1}^t = w_1 + \Pi_t w_2. \tag{18}$$

We can use this budget constraint instead of 15 and 16 in the household's problem.

(c) Define and solve for an autarkic equilibrium, assuming that it exists.

The autarky equilibrium is a sequence of allocations $\{c_1^0, \{c_t^{t-1}, c_t^t, M_t\}_{t=1}^{\infty}\}$ and prices $\{p_t\}_{t=1}^{\infty}$ such that the goods market and the money market clears in each period:

$$M_t = \bar{M} \quad \forall t \geq 1 \tag{19}$$

$$c_t^t + c_t^{t-1} = w_1 + w_2 \quad \forall t \geq 1 \tag{20}$$

and where all agents consume their endowments: $c_t^{t-1} = w_2, c_t^t = w_1 \quad \forall t \geq 1$.

Notice that we are not solving the problem without money, we are simply finding the equilibrium where the prices are such that there is no trade. There are two ways to solve the problem. The first is to solve each households optimization problem and go through those steps. However, in this case we can do it easier, simply by looking at the budget constraints.

By the budget constraints for the initial old and generation t and market clearing condition for money $M_t = \bar{M}$,

$$\frac{\bar{M}}{p_1} = (c_1^0) - w_2 = 0 \tag{21}$$

$$\frac{\bar{M}}{p_t} = w_1 - (c_t^t) = 0 \tag{22}$$

$$\frac{\bar{M}}{p_{t+1}} = (c_{t+1}^t) - w_2 = 0 \tag{23}$$

which can only be true if $\frac{1}{p_t} = 0$ for $t \geq 1$. One way to interpret the autarkic equilibrium is when the first generation in youth does not value money, i.e. $\frac{1}{p_1} = 0$, neither will the following generations. Therefore, $\frac{1}{p_t} = 0$ for $t \geq 1$ will be sustained in the equilibrium.

(d) Define and solve for a competitive equilibrium assuming valued money but with $w_2 = 0$.

As before the C.E. is given by a sequence of allocations $\{c_1^0, \{c_t^{t-1}, c_t^t, M_t\}_{t=1}^{\infty}\}$ and prices

$\{p_t\}_{t=1}^{\infty}$ such that the goods market and the money market clears in each period:

$$M_t = \bar{M} \quad \forall t \geq 1 \tag{24}$$

$$c_t^t + c_t^{t-1} = w_1 + w_2 \quad \forall t \geq 1. \tag{25}$$

The initial old's problem is straightforward, and optimization is given by their budget constraint: $c_1^0 = \frac{\bar{M}}{p_1}$. In other words they only consume if money is valued. To solve the representative household's problem we insert for the two budget constraints in the utility function. Next, we can ignore the non-negativity constraints on consumption and money demand since utility is strictly increasing and it is clear that money demand will be weakly positive.[2] The HH problem is then:

$$\max_{\{M_t\}} \ln(w_1 - M_t/p_t) + \ln(M_t/p_{t+1}) \tag{26}$$

FOC:

$$\frac{-1/p_t}{w_1 - M_t/p_t} + \frac{1}{M_t} = 0 \tag{27}$$

$$\implies M_t(p_t) = \frac{w_1 p_t}{2} \tag{28}$$

$$(\text{if } w_2 \neq 0\text{: } M_t(p_t, p_{t+1}) = \frac{w_1 p_t}{2} - \frac{w_2 p_{t+1}}{2}) \tag{29}$$

We then have the policy function for money demand as a function of price and endowments. Next, we insert this into the budget constraints to find the policy functions for consumption:

$$c_t^t(p_t) = \frac{w_1}{2} \tag{30}$$

$$c_{t+1}^t(p_t, p_{t+1}) = \frac{w_1}{2}\frac{p_t}{p_{t+1}} = \frac{w_1}{2}\frac{1}{\Pi_t} \tag{31}$$

Next we insert this into the market clearing for the goods market:

$$c_{t+1}^{t+1} + c_{t+1}^t = \frac{w_1}{2} + \frac{w_1}{2}\frac{1}{\Pi_t} = w_1 \tag{32}$$

$$\implies \Pi_t = 1 \tag{33}$$

$$\implies p_t = p_{t+1} \equiv \bar{p} \tag{34}$$

We have then pinned down the consumption allocation in equilibrium. However to pin down money demand we need to pin down prices. From the money market clearing

[2] Or we can just assume that they are non-binding, and then verify that this is true in our solution.

Table 1: Consumption

	c_1^0	c_t^t	c_{t+1}^t
SPP	$\frac{w_1}{2}$	$\frac{w_1}{2}$	$\frac{w_1}{2}$
Autarky	0	w_1	0
Monetary	$\frac{w_1}{2}$	$\frac{w_1}{2}$	$\frac{w_1}{2}$

condition and money demand we get:

$$\bar{M} = M_t(p_t) = \frac{w_1 p_t}{2} = \frac{w_1 \bar{p}}{2} \tag{35}$$

$$\implies \bar{p} = \frac{2\bar{M}}{w_1} \tag{36}$$

$$\tag{37}$$

Notice finally that consumption is non-negative for all agents in all periods and that money demand is strictly positive, so the non-negativity constraints all hold.

(e) Compare the solutions to the planners problem, the autarky equilibrium and the stationary monetary competitive equilibrium with valued money, all with $w_2 = 0$

The table shows to allocations of consumption in each equilibrium. The key point is that the monetary equilibrium achieves the first best allocation, just as the social planner does, while in autarky there is no trade so the old in each period gets zero consumption (and $-\infty$ utility).

Another important point is that there is a unique equilibrium in the planners economy, while in the competitive market version of the model we have two equilibriums, one with trade and one without. Note that the one with trade is Pareto optimal, but this does not guarantee that it is the equilibrium "chosen" by the economy. Our theory so far cannot tell us which of the equilibrium the economy will go to. A third point is that this an example of a rational bubble, since the price of money in trade is positive, even though it's intrinsic value is zero.

(f) What happens to consumption, money demand and prices in a competitive equilibrium with valued money if the initial money supply is halved, i.e. $\bar{M}' = \frac{\bar{M}}{2}$. Keep the assumption that $w_2 = 0$.

Notice that the only thing that changes now is one parameter, the $\bar{M}$ from the earlier exercises. Thus we simply look at our explicit solutions for consumption, and conclude that consumption is unchanged. From 36 we see that the new price level is $\bar{p}' = \frac{2\bar{M}'}{w_1} = \frac{\bar{M}}{2}$, i.e. exactly half of the old price level. Since money demand only depends on the price level, we get that money demand is also halved (which we already knew from the market clearing condition for money).

The interesting(?) thing is that changes in the money supply has no impact on the allocation of real resources in the competitive equilibrium. Note that real money demand ($\frac{M_t}{p_t}$) is unchanged. This is related to money neutrality, which we will study further in Econ 714.

2. Plot the trade offer curves for the following utility functions where the endowment is (w_1, w_2) for goods 1 and 2, respectively.

   (a) $U = 10c_1 - 4c_1^2 + 4c_2 - c_2^2$, $(w_1, w_2) = (0, 2)$

   (b) $U = min\{2c_1 + c_2, c_1 + 2c_2\}$, $(w_1, w_2) = (1, 0)$

   (c) $U = min\{2c_1 + c_2, c_1 + 2c_2\}$, $(w_1, w_2) = (1, 10)$

Figure 1: Solution to a)

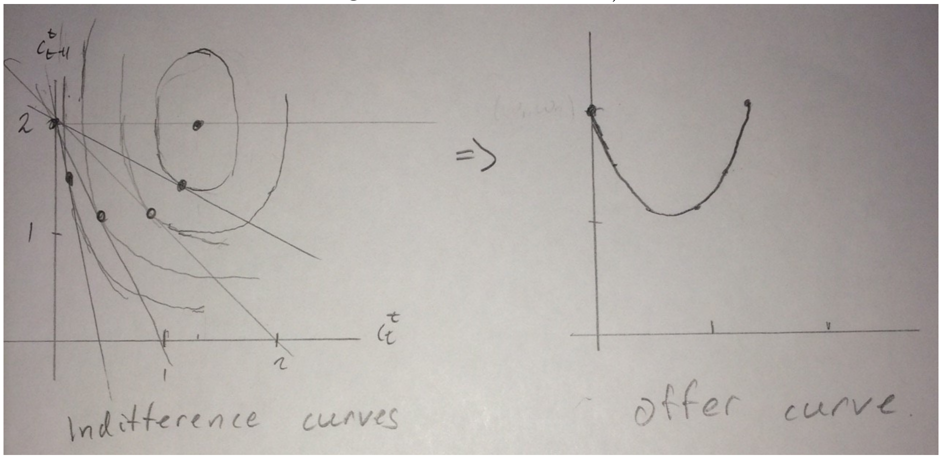

Figure 2: Solution to b)

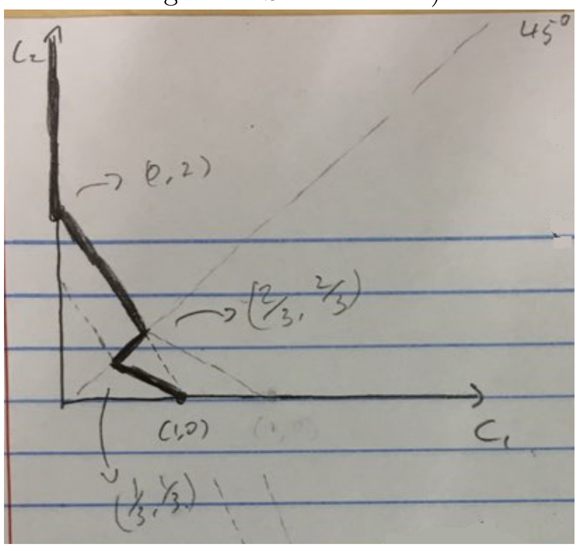

Figure 3: Solution to c)