Problem Set 3 Solutions

Problem 1 in Lecture 3

(a) The joint density is

$$f(x,y) = \begin{cases} 1/4, & -1 < x, y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the support is $S = \{(x,y) : -1 < x, y < 1\}$. Denote $A = \{(x,y) : x^2 + y^2 < 1\}$. Since $x^2 + y^2 < 1$ implies that $x^2 < 1$ and $y^2 < 1$, i.e., $-1 < x, y < 1$, then $A \subseteq S$. Therefore,

$$P(X^2 + Y^2 < 1) = \int_A \frac{1}{4} dxdy = \frac{1}{4} \int_A dxdy = \frac{\pi}{4}.$$

Here, A is an unit circle and $\int_A dxdy$ is the area of A, which is π.

(b) Denote $B = \{(x,y) : |x+y| < 2\}$. For any $(x,y) \in S$, then $-1 < x, y < 1$. Thus, $|x+y| \le |x| + |y| < 2$. That means $S \subseteq B$. Therefore,

$$P(|X+Y| < 2) = P((X,Y) \in B) \ge P((X,Y) \in S) = 1.$$

i.e., $P(|X+Y| < 2) = 1$.

Problem 2 in Lecture 3

(a) For $f(x,y)$ to be a valid bivariate PDF, its integral on $\mathbb{R}^2$ should be 1, i.e.,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dxdy = 1$$

Plugging in $f(x,y) = g(x)h(y)$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) dxdy &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x) dx \right) h(y) dy \\ &= \int_{-\infty}^{\infty} ah(y) dy \\ &= a \int_{-\infty}^{\infty} h(y) dy = ab = 1 \end{aligned}$$

So a and b should satisfy $ab = 1$.

(b) By definition of marginal distribution,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_{-\infty}^{\infty} g(x)h(y) dy = g(x) \int_{-\infty}^{\infty} h(y) dy = bg(x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx = \int_{-\infty}^{\infty} g(x)h(y) dx = h(y) \int_{-\infty}^{\infty} g(x) dx = ah(x)$$

(c) *Proof.* From part (a) and (b),

$$f(x,y) = g(x)h(y) = ab \cdot g(x)h(y) = (ah(x))\,(bg(x)) = f_X(x)f_Y(y), \forall x, y \in \mathbb{R}$$

which satisfies the third definition of independence. $\square$

Problem 3 in Lecture 3

(a) For $f(x,y)$ to be a valid joint PDF, its integral on $\mathbb{R}^2$ should be 1, i.e.,

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^1\int_0^{1-x} cxy\ dydx \\
&= c\int_0^1 x\left(\int_0^{1-x} y\ dy\right)dx \\
&= c\int_0^1 x\frac{1}{2}(1-x)^2dx \\
&= \frac{1}{2}c\int_0^1 \left(x^3 - 2x^2 + x\right)dx \\
&= \frac{1}{2}c\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{c}{24} = 1
\end{aligned}$$

So $c = 24$.

(b) For $x \in [0,1]$,

$$\begin{aligned}
f_X(x) &= \int_0^{1-x} 24xy\ dy \\
&= 12x\int_0^{1-x} 2y\ dy = 12x(1-x)^2
\end{aligned}$$

For $x \notin [0,1]$, $f_X(x) = 0$.
Since x and y are symmetric, the marginal distribution of y is derived in the same way, and we have $f_Y(y) = 12y(1-y)^2$ for $y \in [0,1]$; $f_Y(y) = 0$ otherwise.

(c) No. Note that the support of marginal distributions of x and y are both $[0,1]$, but the support of joint distribution is not the square $[0,1]\times[0,1]$; its the lower triangle of that square. That means, if you pick a point like $(\frac{1}{2},\frac{3}{4})$, $f_{XY}(\frac{1}{2},\frac{3}{4}) = 0$ while $f_X(\frac{1}{2})$ and $f_Y(\frac{3}{4})$ are both strictly positive. So $f_{XY} \neq f_X(x)f_Y(y)$ at that point, which is enough to disprove the independence.

Problem 4 in Lecture 3

Proof. Let a be a constant. $Cov(a,X) = E(aX) - Ea\cdot EX = aE(X) - aE(X) = 0$. $\square$

Problem 5 in Lecture 3

$$
\begin{aligned}
Cov(XY, Y) &= E(XY^2) - E(XY)E(Y) \\
&= E(X)E(Y^2) - E(X)\left(E(Y)\right)^2 \\
&= \mu_X(\sigma_Y^2 + \mu_Y^2) - \mu_X\mu_Y^2 \\
&= \mu_X\sigma_Y^2
\end{aligned}
$$

$$
\begin{aligned}
Var(XY) &= E(X^2Y^2) - \left(E(XY)\right)^2 \\
&= E(X^2)E(Y^2) - \mu_X^2\mu_Y^2 \\
&= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 \\
&= \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2
\end{aligned}
$$

Plugging in the formula for $\rho_{XY,Y}$, we have

$$
\rho_{XY,Y} = \frac{Cov(XY, Y)}{\sqrt{Var(XY)}\sqrt{Var(Y)}} = \frac{\mu_X\sigma_Y^2}{\sigma_Y\sqrt{\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2}} = \frac{\mu_X\sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2}}
$$

Problem 6 in Lecture 3

For simplicity of notation, let $\mu_i = E(X_i)$, then

$$
\begin{aligned}
Var\left(\sum_{i=1}^n X_i\right) &= E\left[\left(\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i\right)^2\right] = E\left[\left(\sum_{i=1}^n X_i - \mu_i\right)^2\right] \\
&= E\left[\sum_{i=1}^n (X_i - \mu_i)^2 + 2\sum_{1\le i\le j\le n}(X_i - \mu_i)(X_j - \mu_j)\right] \\
&= \sum_{i=1}^n E\left[(X_i - \mu_i)^2\right] + 2\sum_{1\le i\le j\le n} E\left[(X_i - \mu_i)(X_j - \mu_j)\right] \\
&= \sum_{i=1}^n Var(X_i) + 2\sum_{1\le i\le j\le n} Cov(X_i, X_j)
\end{aligned}
$$

Problem 7 in Lecture 3

a. We will compute the marginal of X. The calculation for Y is similar. Start with

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \\ &\times \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}\right] \end{aligned}$$

and compute

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\omega^2-2\rho\omega z+z^2\right)}\sigma_Y dz,$$

where we make the substitution $z = \frac{y-\mu_Y}{\sigma_Y}$, $dy = \sigma_Y dz$, $\omega = \frac{x-\mu_X}{\sigma_X}$. Now the part of the exponent involving ω^2 can be removed from the integral, and we complete the square in z to get

$$\begin{aligned} f_X(x) &= \frac{e^{-\frac{\omega^2}{2(1-\rho^2)}}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[(z^2-2\rho\omega z+\rho^2\omega^2)-\rho^2\omega^2\right]}dz \\ &= \frac{e^{-\omega^2/2(1-\rho^2)}e^{\rho^2\omega^2/2(1-\rho^2)}}{2\pi\sigma_X\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(z-\rho\omega)^2}dz. \end{aligned}$$

The integrand is the kernel of normal pdf with $\sigma^2 = (1-\rho^2)$, and $\mu = \rho\omega$, so it integrates to $\sqrt{2\pi}\sqrt{1-\rho^2}$. Also note that $e^{-\omega^2/2(1-\rho^2)}e^{\rho^2\omega^2/2(1-\rho^2)} = e^{-\omega^2/2}$. Thus,

$$f_X(x) = \frac{e^{-\omega^2/2}}{2\pi\sigma_X\sqrt{1-\rho^2}}\sqrt{2\pi}\sqrt{1-\rho^2} = \frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2},$$

the pdf of $\mathrm{n}(\mu_X, \sigma_X^2)$.

b.

$$
\begin{aligned}
&f_{Y|X}(y|x)\\
&= \frac{\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}}{\frac{1}{\sqrt{2\pi\sigma_X}} e^{-\frac{1}{2\sigma_X^2}(x-\mu_X)^2}}\\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - (1-\rho^2)\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}\\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}\\
&= \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_Y^2\sqrt{(1-\rho^2}}\left[(y-\mu_Y)-\left(\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)\right]^2},
\end{aligned}
$$

which is the pdf of $\mathrm{n}\Big((\mu_Y - \rho(\sigma_Y/\sigma_X)(x-\mu_X)\,, \sigma_Y\sqrt{1-\rho^2}\Big)$.

c.

$$
\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y/\sigma_Y - \rho X/\sigma_X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho/\sigma_X & 1/\sigma_Y \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}
$$

This transformation is one-to-one, and the inverse of the transformation is

$$
\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ \sigma_Y(Z + \rho X/\sigma_X) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho\sigma_Y/\sigma_X & \sigma_Y \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}
$$

Since it is a linear transformation, the Jacobian is the matrix itself, and its determinant $|J| = \sigma_Y$. Now applying the multivariate transformation formula, we have

$$
\begin{aligned}
f_{X,Z}(x,z) &= \sigma_Y f_{XY}\big(x, \sigma_Y\left(z + \rho x/\sigma_X\right)\big)\\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2/\sigma_X^2 - 2\rho x\sigma_Y\left(z+\rho x/\sigma_X\right)/\sigma_X\sigma_Y + \sigma_Y^2\left(z+\rho x/\sigma_X\right)^2/\sigma_Y^2}{2(1-\rho^2)}\right\}\\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2/\sigma_X^2 - 2\rho x\left(z+\rho x/\sigma_X\right)/\sigma_X + \left(z+\rho x/\sigma_X\right)^2}{2(1-\rho^2)}\right\}\\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2/\sigma_X^2 - 2\rho xz/\sigma_X - 2\rho^2x^2/\sigma_X^2 + z^2 - 2\rho xz/\sigma_X + 2\rho^2x^2/\sigma_X^2}{2(1-\rho^2)}\right\}\\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2/\sigma_X^2 + z^2}{2(1-\rho^2)}\right\}\\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2/\sigma_X^2}{2(1-\rho^2)}\right\} \exp\left\{-\frac{z^2}{2(1-\rho^2)}\right\}
\end{aligned}
$$

which means that it can be factored out into two parts, one is about x and the other is about z. By our conclusion in question 2, X and Z are independent.

Problem 8 in Lecture 2

Proof. For $\forall A, B$ in the Borel σ-field,

$$Z \in A \Longleftrightarrow g_1(X) \in A \Longleftrightarrow X \in g_1^{-1}(A)$$

$$W \in B \Longleftrightarrow g_2(Y) \in B \Longleftrightarrow Y \in g_2^{-1}(B)$$

where the second equivalence comes from the definition of the inverse image. Since g^{-1} is Borel-measurable,

$$P(Z \in A) = P\left(X \in g_1^{-1}(A)\right) \tag{1}$$

$$P(W \in B) = P\left(Y \in g_2^{-1}(B)\right) \tag{2}$$

We also have

$$Z \in A, W \in B \Longleftrightarrow g_1(X) \in A, g_2(Y) \in B \Longleftrightarrow X \in g_1^{-1}(A), Y \in g_2^{-1}(B)$$

which implies that

$$P(Z \in A, W \in B) = P\left(X \in g_1^{-1}(A), Y \in g_2^{-1}(B)\right) \tag{3}$$

By the independence of X and Y, we have

$$P\left(X \in g_1^{-1}(A), Y \in g_2^{-1}(B)\right) = P\left(X \in g_1^{-1}(A)\right) P\left(Y \in g_2^{-1}(B)\right)$$

Then by equation (1)(2)(3), we have

$$P(Z \in A, W \in B) = P(Z \in A)P(W \in B), \text{ for } \forall A, B \text{ in the Borel } \sigma\text{-field}$$

which satisfies the first definition of independence. □