Homework #4

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Question 1

(a) This Bayesian game has two players, $N = \{1, 2\}$, with symmetric action spaces, $A_i = \{b_i \in \mathbb{R}_+\}$, and identically-distributed types, $\Theta_i = v_i$, with CDF $F(v_i) = v_i$. Note that, in the case where bids are exactly equal, the payoff reduces to:

$$\frac{1}{2}(v_i - b_j)^1 \tag{0}$$

Each player has the payoff function:

$$u_i(b_i, b_j; v_i) = \begin{cases} (1 - p - q)(-b_i), & b_i < b_j \\ \frac{1}{2}(v_i - b_j), & b_i = b_j \\ p(v_i - b_i) + q(v_i - b_j) + (1 - p - q)v_i, & b_i > b_j \end{cases}$$

(b) Note that $Pr(b_j(v_i) < b_j(v_j)) = F(b_j^{-1}(b_j(v_i))) = v_i$, and that the probability of any two values being equal on a uniform distribution is zero.

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[1]The expected payoff from the coin flip is

$$\frac{1}{2}\left[p(v_j - b_i) + q(v_i - b_j) + (1 - p - q)v_i\right] + \frac{1}{2}\left[(1 - p - q)(-b_j)\right]$$

Which reduces as follows:

$$\frac{1}{2}\left[(p + q)(v_i - b_j) + (1 - p - q)v_i\right] + \frac{1}{2}\left[(1 - p - q)(-b_j)\right]$$

$$\frac{1}{2}(p + q)(v_i - b_j) + \frac{1}{2}(1 - p - q)(v_i - b_j)$$

$$\frac{1}{2}(v_i - b_j)$$

Then, player i's expected payoff from betting with $b_j(v_i)$ is:

$$\mathbb{E}\left[u_i(b_j, b_j; v_i)\right] = Pr(b_j(v_j) > b_i)(1 - p - q)(-b_i) + Pr(b_i = b_j(v_j))\frac{1}{2}(v_i - b_j(v_j))$$

$$+ \int_0^{b_j^{-1}(b_i(v_i))} \left[p(v_i - b_i) + q(v_i - b_j) + (1 - p - q)v_i\right] dv_j$$

$$= (1 - v_i)(1 - p - q)(-b_i) + p\int_0^{v_i}(v_i - b_i)dv_j$$

$$+ q\int_0^{v_i}(v_i - b_j)dv_j + (1 - p - q)\int_0^{v_i} v_i dv_j$$

(c) Recall that the derivative of an inverse function is the reciprocal of the derivative at the inverse. Then, we can calculate player i's best response function using first-order conditions:

$$\frac{\partial}{\partial b_i}\left[(1 - b_j^{-1}(b_i))(1 - p - q)(-b_i) + b_j^{-1}(b_i)\left[p(v_i - b_i) + q(v_i - b_j) + (1 - p - q)v_i\right]\right] = 0$$

$$-\frac{(1 - p - q)b_i}{b_j'\left(b_j^{-1}(b_i)\right)} - (1 - p - q)b_j^{-1}(b_i) - p - \frac{pv_i}{b_j'\left(b_j^{-1}(b_i)\right)} + pv_i + \frac{pb_i}{b_j'\left(b_j^{-1}(b_i)\right)}$$

$$-\frac{q(v_i - b_j) + (1 - p - q)v_i}{b_j'\left(b_j^{-1}(b_i)\right)} = 0$$

Recognizing that the players' betting functions are symmetric, $b_j^{-1}(b_i) = v_i$ and $b_i = b_j$, so:

$$-(1 - p - q)v_i - \frac{(1 - p - q)b_i}{b'(v_i)} - p - \frac{pv_i}{b'(v_i)} + pv_i + \frac{pb_i}{b'(v_i)} - \frac{q(v_i - b_j) + (1 - p - q)v_i}{b'(v_i)} = 0$$

$$-(1 - p - q)v_i b'(v_i) - (1 - p - q)b_i - pb'(v_i) - pv_i + pv_i b'(v_i) + pb_i - q(v_i - b_j) + (1 - p - q)v_i = 0$$

$$(1 - q)b_i = pv_i b'(v_i) - pv_i - pb'(v_i) - q(v_i - b_j) + (1 - p - q)v_i$$

$$(1 - q)b_i = (1 - q)v_i - q(v_i - b_j) - pb'(v_i)(1 - v_i)$$

$$b_i = v_i - \frac{q}{1 - q}(v_i - b_j) - \frac{p}{1 - q}b'(v_i)(1 - v_i)$$

$$b_i = v_i - \frac{q}{1 - q}\left(v_i - v_i + \frac{q}{1 - q}(v_i - b_i) + \frac{p}{1 - q}b'(v_i)(1 - v_i)\right) - \frac{p}{1 - q}b'(v_i)(1 - v_i)$$

$$\left(1 - \left(\frac{q}{1 - q}\right)^2\right)b_i = \left(1 - \left(\frac{q}{1 - q}\right)^2\right)v_i - \left(\frac{1}{1 - q}\right)\frac{p}{1 - q}b'(v_i)(1 - v_i)$$

$$b_i = v_i - \frac{pb'(v_i)(1 - v_i)}{(1 - q^2)^2 - q^2}$$

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Under the linearity assumption, $b(v_i) = \alpha + \beta v_i$, so $b'(v_i) = \beta$, where $\beta > 0$ (since b_i is strictly increasing in v_i), so player i's equilibrium bet becomes:

$$b_i = v_i - \frac{p\beta(1 - v_i)}{(1 - q^2)^2 - q^2}$$

Thus, the symmetric Bayesian Nash equilibium is:

$$\left(v_1 - \frac{p\beta(1 - v_1)}{(1 - q^2)^2 - q^2}, v_2 - \frac{p\beta(1 - v_2)}{(1 - q^2)^2 - q^2} \right)$$

(d) As $p \to 1$ and $q \to 0$, the equilibrium bidding strategy converges to $v_i - (1 - v_i)\beta$. This makes sense, given that $p = 1$ is a first-price auction, where the incentive is to bet just below your valuation in order to maximize your payoff. As $q \to 1$, the equilibrium converges to v_i, which is similarly intuitive; $q = 1$ is a second-price auction, where there is no incentive to bet differently than your valuation. As $q \to \frac{1}{2}$ and $p \to 0$, the optimal bid still tends toward v_i.

Question 2

(a) If $\beta_1 = \beta_2 = \beta$, then there are three cases of β that determine the Nash equilibria and rationalizable strategies of this game:

    (i) $\beta > 0$

        In this case, A is a strictly-dominated strategy, so (B, B) is a unique, pure-strategy Nash equilibrium, and B is the only rationalizable strategy for either player.

    (ii) $\beta = 0$

        Now, B is weakly dominated, so there are three pure-strategy Nash equilibria at (A, B), (B, A), and (B, B). There are no mixed-strategy Nash equilibria because the best response for either player to a strategy that places a positive weight on A is a pure strategy of B. Thus, any pure strategy is rationalizable, but no mixed strategy is.

    (iii) $\beta < 0$

        In this case, (A, B) and (B, A) are the only pure-strategy Nash equilibria, and there exists a single, symmetric, mixed-strategy Nash equilibrium. This equilibrium occurs where each player weights each move such that the other player is indifferent between A and B. To find this weighting for each $\beta < 0$, let π represent the frequency with which player j plays B. Then, player i is indifferent between A and B if:

$$1(1 - \pi) + 0\pi = 2(1 - \pi) + \beta\pi$$
$$1 - \pi = 2 - 2\pi + \beta\pi$$
$$-1 = (\beta - 1)\pi$$
$$\pi = \frac{1}{1 - \beta}$$

Thus, there exists a mixed-strategy Nash equilibrium at

$$\left(\left(\frac{\beta}{1-\beta}\right)A + \left(\frac{1}{1-\beta}\right)B, \left(\frac{\beta}{1-\beta}\right)A + \left(\frac{1}{1-\beta}\right)B\right)$$

This mixed strategy is rationalizable, as is any pure strategy.

(b) This Bayesian game has two players, $N = \{1, 2\}$, with symmetric action spaces, $A_i = \{A, B\}$, and types, $\Theta_i = \{A, B\}$, which are determined by the decision rule, $S_i : \beta_i \to \Theta_i$, which determines the players type based on β_i, drawn from the CDF, F. Each player's payoff function is:

$$u_i(A_i, A_j; \Theta_i) = \begin{cases} 0, & \Theta_i = A \wedge \Theta_j = B \\ 1, & \Theta_i = \Theta_j = A \\ 2, & \Theta_i = B \wedge \Theta_j = A \\ \beta_i, & \Theta_i = \Theta_j = B \end{cases}$$

Where each player plays A if they're type $\Theta_i = A$, and, according to S_i, $\exists \beta^*$ such that $\Theta_i = A$ if $\beta_i < \beta^*$ and $\Theta_i = B$ if $\beta_i = geq\beta^*$. β^* is determined by finding the payoff, β_i, at which player i is indifferent to either pure strategy:

$$(0)(1 - F(\beta^*)) + (1)F(\beta^*) = (2)F(\beta^*) + (\beta^*)(1 - F(\beta^*))$$

$$\beta^* = -\frac{F(\beta^*)}{1 - F(\beta^*)}$$

(c)

(d)

Question 3

Upon the visitor's announcement that *at least one person* has bad breath, 40 game theorists know that 10 game theorists, and possibly 11 (themselves), have bad breath, while 10 game theorists know that 9 (and possibly 10) have bad breath. Each game theorist among the game theorists with bad beath (who witness 9 bad-breathed game theorists) knows that, if they themselves don't have bad breath, then those 9 that they witness having bad breath would each witness 8 other game theorists with bad breath. Thus, if a piece of information suggesting that more than 8 game theorists have bad breath emerged and still no game theorists board the elevator, then all 10 game theorists would board the elevator at once.

Assume that realizations do not occur while the elevator doors are open–i.e., the information that comes from the number of people boarding or not boarding the elevator occurs only when the elevator leaves, so decisions to board or not board are made simultaneously with each elevator arrival. Upon the visitor's announcement, the only way for a game theorist to know that they have bad

breath is if they don't witness any other game theorist with bad breath. Thus, nobody boards the elevator when it first arrives. It then becomes common knowledge that at least two game theorists have bad breath. However, each game theorist witnesses 9 or 10 game theorists with bad breath, so no game theorists board the elevator when it arrives in minute 2.

This continues for 8 minutes, when the elevator comes for the eighth time and no game theorists board. At this point, at least 9 game theorists must have bad breath, and each game theorist with bad breath sees 9 game theorists with bad breath. 9 minutes in, the elevator comes, and still no game theorists board, making each game theorist with bad breath certain of the grim reality that they've crossed themselves off of the invite list of all future cocktail parties.[2] 10 minutes after the guest's announcement, all 10 game theorists with bad breath board the elevator, which is sure to smell like a dairy farm by the time they make it to the mouth wash.

Question 4

(a) We can eliminate option (I) for both players, as it has no fundamental or strategic benefit but costs them by discounting their possible payoff. Therefore, it is strictly dominated. Beyond leading to the strict domination of doing nothing, δ does not affect play, since it is very close to 1 and thus does not meaningfully impact payoffs when raised to the 11th power (which is the maximum possible discount in this game). Thus, at any turn, if a player faced, say, winning with costs of 21 and losing with a cost of 2, the player will chose to win, as $20\delta^k - 21 > -2$ for any $k = 1, 2, ..., 11$. For simplicity of notation, then, I will calculate payoffs as though $\delta = 1$.

At any turn, each player optimizes by either forcing a win (if they can) or minimizing their costs if their opponent can force a win. For example, the first mover in this game has the option to force a win by moving three steps with each turn, resulting in a payoff of -10. If they take three steps as their first choice, the equilibrium of the subgame that follows is for each player to take two steps until the first-moving player wins (see table below). If the first-moving player takes fewer than three steps as a first move, then the second-moving player can similarly force a win by taking three steps. Thus, there is a unique SPNE at $(3, 1, 1, 1, 1, 1, 1)$.

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[2]Little do they know that no such cocktail party will occur for the next year due to COVID, and by the time that happens, their social *faux pas* will be long forgotten.

Mover:	i	j	i	j	i	j	i	j	(payoff)
	1 step (1)	1 step (4)	1 step (2)	1 step (5)	1 step (3)	1 step (6)			(-1,-6)
					2 steps (4)	1 step (6)			(-1,-11)
					3 steps (5)	1 step (6)			(-1,-19)
				2 steps (6)	(-4,-4)				
			2 steps (3)	1 step (5)	1 step (4)	1 step (6)			(-1,-11)
					2 steps (5)	1 step (6)			(-1,-16)
					3 steps (6)	(-19,-4)			
				2 steps (6)	(-4,-9)				
			3 steps (4)	1 step (5)	1 step (6)	(-1,-19)			
					2 steps (6)	(-19,4)			
				2 steps (6)	(-4,-17)				
		2 steps (5)	1 step (2)	1 step (6)	(-4,-4)				
			2 steps (3)	1 step (6)	(-4,-9)				
			3 steps (4)	1 step (6)	(-4,-17)				
		3 steps (6)	(-10,-2)						
3 steps (3)	2 steps (2)	1 step (4)	1 step (3)	1 step (5)	1 step (3)	1 step (6)			(-1,-11)
					2 steps (6)	(-4,-9)			
				1 step (5)	1 step (3)	1 step (6)			(-1,-16)
			2 steps (4)						
				2 steps (6)	(-4,-14)				
			3 steps (5)	1 step (5)	1 step (6)	(-19,-4)			
				2 steps (6)	(-4,-22)				
		2 steps (5)	1 step (4)	1 step (6)	(-4,-9)				
		3 steps (6)	(-10,-7)						
	3 steps (3)	1 step (4)	1 step (5)	1 step (5)	1 step (6)	(-1,-19)			
				2 steps (6)	(-19,-4)				
				2 steps (6)	(-4,-17)				
			2 steps (5)	1 step (5)	1 step (6)	(-19,-4)			
				2 steps (6)	(-4,-22)				
			3 steps (6)	(-17,-10)					
			1 step (4)	1 step (6)	(-4,-17)				
		2 steps (5)	2 steps (5)	1 step (6)	(-4,-22)				
			3 steps (6)	(-22,-10)					
	3 steps (6)	(-10,-15)							

(b) In the absence of competition, Arthur would maximize his net payoff by taking two steps at a time, resulting in a payoff of $20 - 2 - 2 - 2 - 2 - 2 - 2 = 8$, compared to his equilibrium behavior of taking three steps on his first turn, then one step at a time until reaching the finish line, for a payoff of -1. The intuition behind taking one step at a time is that taking one step costs \$2 per step, which is less than taking either two or three. The difference between Arthur's competitive and non-competitive behavior comes from his need to prevent Beatrix from maximizing her payoff (or, really, minimizing her losses) by competing with Arthur for the prize. Taking three steps on the first move puts the two players in a subgame where each player's best option is to only take one step at a time.

Question 5

In a two-player game of perfect information, each player knows precisely what their payoffs, and those of their opponent, are at each terminal node (if the game is finite, every play leads to a terminal node). Further, they know what their opponent will play, either totally (i.e. as a first play or as a play in a simultaneous game) or in response to their own move. Thus, for any decision node, each player can determine their terminal payoff from any move. In a zero-sum game, one player "wins" and the other loses at every terminal node, so the player who moves first will choose a move that enables them to win, taking into account the moves that they know they and their opponent will make, using the subgame equilibrium at each deicision node. If payoffs strictly differ

across terminal nodes, then the player who moves first will choose the move that leads to the terminal node with the highest payoff. If the node with the highest payoff for the first-moving player does not have a unique payoff, then the payoff vector will be the same across all terminal nodes with that payoff for the first-moving player, since the game is zero-sum. Thus, each finite, two-player, zero-sum game with perfect information has a unique, subgame perfect equilbibrium payoff vector, though the set of moves that produces this vector is not necessarily unique.

Question 6

(a) This game consists of two firms, $N = \{A, B\}$ with symmetric action sets, $A_i = \{p_i \in \mathbb{R}_+\}$, and payoff functions, $u_i = \{p_i(D_i(p_i, p_j))\}$. Firm i's best response funciton can be found using the first-order condition of its payoff function with respect to its action, p_i:

$$\frac{\partial u_i}{\partial p_i} = d - 2p_i + \alpha p_j = 0$$

$$p_i = \frac{1}{2}d + \frac{\alpha}{2}p_j$$

If both firms choose simultaneously, then, in equilibirum, each firm will input the other's best response funciton into its own and determine its action from there:

$$p_i = \frac{1}{2}d + \frac{\alpha}{2}\left(\frac{1}{2}d + \frac{\alpha}{2}p_i\right)$$

$$\left(\frac{4 - \alpha^2}{4}\right)p_i = \left(\frac{2 + \alpha}{4}\right)d$$

$$p_i = \left(\frac{2 + \alpha}{4 - \alpha^2}\right)d = \left(\frac{1}{2 - \alpha}\right)d$$

Thus, the symmetric Nash equilibrium is $(p_A, p_B) = \left(\left(\frac{1}{2-\alpha}\right)d, \left(\frac{1}{2-\alpha}\right)d\right)$.

(b) If firm A moves first, then it knows that firm B will maximize its profit according to its best response function. Thus, firm A's payoff function is:

$$u_A = p_A\left(d - p_A + \alpha\left(\frac{1}{2}d + \frac{\alpha}{2}p_A\right)\right) = \frac{3\alpha}{2}dp_A + \left(\frac{\alpha^2 - 2}{2}\right)p_A^2$$

We can find the profit-maximizing price for firm A using the first-order condition of its modified payoff function:

$$\frac{\partial u_A}{\partial p_A} = \frac{3\alpha}{2}d + \left(\alpha^2 - 2\right)p_A = 0$$

$$\left(\alpha^2 - 2\right)p_A = -\frac{3\alpha}{2}d$$

$$p_A = \left(\frac{3\alpha}{4 - 2\alpha^2}\right)d$$

Which firm B will take as given, choosing p_B according to its best response function:

$$p_B = \frac{1}{2}d + \frac{\alpha}{2}\left(\frac{3\alpha}{4-2\alpha^2}\right)d = \left(\frac{3\alpha^2}{4-2\alpha^2}+1\right)\frac{1}{2}d$$

$$= \left(\frac{\alpha^2+4}{2-\alpha}\right)\frac{1}{4}d$$

Thus, the subgame perfect equilibrium is $\left(\left(\frac{3\alpha}{4-2\alpha^2}\right)d,\left(\frac{\alpha^2+4}{2-\alpha}\right)\frac{1}{4}d\right)$.

(c) Letting p_i^{sim} and u_i^{sim} be the equilibrium price chosen and payoff earned, respectively, by i under simultaneous play, with sequential play denoted without a superscript, we can solve [NOT FINISHED BELOW; am I missing a major cancellation or something? It can't actually be this messy]:

$$p_A = \left(\frac{3\alpha}{4-2\alpha^2}\right)d = \left(\frac{3\alpha}{2-\alpha^2}\right)\left(\frac{2-\alpha}{2}\right)\left(\frac{1}{2-\alpha}\right)d$$

$$= \left(\frac{3\alpha}{2}\right)\left(\frac{2-\alpha}{2-\alpha^2}\right)p_A^{sim}$$

$$p_B = \left(\frac{\alpha^2+4}{2-\alpha}\right)\frac{1}{4}d = \left(\frac{\alpha^2+4}{4}\right)\left(\frac{1}{2-\alpha}\right)d$$

$$= \left(\frac{\alpha^2}{4}+1\right)p_B^{sim}$$

$$u_A^{sim} = p_A^{sim}\left(d-p_A^{sim}+\alpha p_B^{sim}\right) = \left(\frac{1}{2-\alpha}d\right)\left(d-\frac{1}{2-\alpha}d+\frac{\alpha}{2-\alpha}d\right)$$

$$= \left(\frac{1}{2-\alpha}d\right)^2$$

$$u_B^{sim} = u_A^{sim} = \left(\frac{1}{2-\alpha}d\right)^2$$

$$u_A = p_A\left(d-p_A+\alpha p_B\right)$$

$$= \left(\frac{3\alpha}{2}\right)\left(\frac{2-\alpha}{2-\alpha^2}\right)p_A^{sim}\left(d-\left(\frac{3\alpha}{2}\right)\left(\frac{2-\alpha}{2-\alpha^2}\right)p_A^{sim}+\alpha\left(\frac{\alpha^2}{4}+1\right)p_B^{sim}\right)$$

$$= \left(\frac{3\alpha}{2(2-\alpha^2)}\right)d\left(d-\left(\frac{3\alpha}{4-2\alpha^2}\right)d+\alpha\left(\frac{\alpha^2+4}{4(2-\alpha)}\right)d\right)$$

$$= \left[\frac{1}{2-\alpha^2}-\frac{3\alpha}{2(2-\alpha^2)^2}+\alpha\left(\frac{\alpha^2+4}{4(2-\alpha)(2-\alpha^2)}\right)\right]\left(\frac{3\alpha}{2}\right)d^2$$

$$u_B = \left[1-\frac{\alpha^2+4}{4(2-\alpha)}+\alpha\left(\frac{3\alpha}{2(2-\alpha^2)}\right)\right]\left(\frac{\alpha^2+4}{4(2-\alpha)}\right)d^2$$