Problem Set #2

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Question 1

(a) To determine $Pr\left(\sum_{i=1}^{n}(X_i^* - \overline{X}_i^*)^2 = 0|\{W_i\}\right)$, we need only find $Pr\left(\sum_{i=1}^{n} X_i^* - \overline{X}_i^* = 0|\{W_i\}\right)$. Since $X_i \in \{0,1\}$, $X_i^* \in \{0,1\}$, so:

$$Pr\left(X_i^* - \overline{X}_i^* = 0|\{W_i\}\right) = Pr\left(X_i^* = X_j^* \forall i,j|\{W_i\}\right)$$

Where $Pr(X_i^* = X_j^*) \geq \frac{1}{2}\ \forall i,j$. Then,

$$Pr\left(X_i^* = X_j^* \forall i,j|\{W_i\}\right) = \left(\frac{1}{2}\right)^n$$

Where, for finite n, $2^{-n} > 0$.

(b) If the conditions for the Lindeberg CLT are satisfied, then $\hat{\theta}_n^*$ converges to the same distribution as $\hat{\theta}_n$. Thus, we must show that

$$\sup \frac{1}{n}\sum_{i=1}^{n}|X_i(s)|^{2+\delta} < \infty$$

For some $\delta > 0$. By the domain of X, this is necessarily satisfied.

(c) Since $\Phi^{-1}(\cdot)$ is a constant,

$$\frac{F^{-1}_{\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n)|\{W_i\}}(0.75|\{W_i\}) - F^{-1}_{\sqrt{n}(\hat{\theta}_n^* - \hat{\theta}_n)|\{W_i\}}(0.25|\{W_i\})}{\Phi^{-1}(0.75) - \Phi^{-1}(0.25)} \to_p \frac{F^{-1}_{\mathcal{N}(0,\sigma_u^2/\sigma_X^2)}(0.75) - F^{-1}_{\mathcal{N}(0,\sigma_u^2/\sigma_X^2)}(0.25)}{\Phi^{-1}(0.75) - \Phi^{-1}(0.25)}$$

Thus,

$$v_n^{iqr} \to_{a.s.} \frac{\frac{\sigma_u}{\sigma_X}\left(\Phi^{-1}(0.75) - \Phi^{-1}(0.25)\right)}{\Phi^{-1}(0.75) - \Phi^{-1}(0.25)} = \sigma_u/\sigma_X$$

(d) First, we can solve for $Var(\hat{\theta}_n^\dagger|\{W_i\})$:

$$\begin{aligned}
\mathbb{E}\left[\hat{\theta}_n^\dagger|\{W_i\}\right] &= \hat{\theta}_n + \frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2}\mathbb{E}\left[\varepsilon_i|\{W_i\}\right] \\
&= \hat{\theta}_n + \frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2} \\
\mathbb{E}\left[\hat{\theta}_n^\dagger|\{W_i\}\right]^2 &= \hat{\theta}_n^2 + 2\hat{\theta}_n\frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2} + \left(\frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2}\right)^2 \\
(\hat{\theta}_n^\dagger)^2 &= \hat{\theta}_n^2 + 2\hat{\theta}_n\frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i\varepsilon_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2} + \left(\frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i\varepsilon_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2}\right)^2 \\
Var(\hat{\theta}_n^\dagger|\{W_i\}) &= \mathbb{E}\left[(\hat{\theta}_n^\dagger)^2|\{W_i\}\right] - \mathbb{E}\left[\hat{\theta}_n^\dagger|\{W_i\}\right]^2 \\
&= \left(\frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2}\right)^2 \mathbb{E}\left[\varepsilon_i^2|\{W_i\}\right] \\
&= \left(\frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2}\right)^2
\end{aligned}$$

By the law of large numbers, this converges to its unconditional expectation. By assumption,

$$\left(\frac{\sum_{i=1}^n (X_i - \overline{X}_n)\hat{u}_i}{\sum_{i=1}^n (X_i - \overline{X}_n)^2}\right)^2 \to_{a.s.} \frac{\mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right]}{\mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right]^4}\mathbb{E}\left[u^2\right] = \sigma_u^2/\sigma_X^2$$