Econ 711 – Fall 2020 – Problem Set 7

Due online SUNDAY night October 25 at midnight.

Please feel free to work together on these problems (and all homeworks), but each student needs to write up his/her own answers at the end, rather than directly copying from one master solution.

There's just one question, but some parts of it are hard. There are hints for several parts on the next page. *This homework is due Sunday night so I can post solutions Monday morning and you'll have time to study for the exam Tuesday evening.*

Question. A Risky Investment

You have wealth $w > 0$ and preferences over lotteries represented by a von Neumann-Morgenstern expected utility function with Bernoulli utility u which is strictly increasing, twice differentiable, and weakly concave. Your friend wants you to invest in his startup; you can choose any amount $a \leq w$ to invest, and your investment will either triple in value (with probability p) or become worthless (with probability $1-p$). Your expected utility if you invest a is therefore

$$U(a) \quad = \quad pu(w - a + 3a) + (1-p)u(w-a) \quad = \quad pu(w + 2a) + (1-p)u(w-a)$$

(a) Show that if u is linear, then you invest all your wealth if $p > \frac{1}{3}$, and nothing if $p < \frac{1}{3}$.

From here on, assume $p > \frac{1}{3}$, so the expected value of the investment is positive; and assume that you are strictly risk-averse ($u'' < 0$).

(b) Show that it's optimal to invest a strictly positive amount. (You can do this by showing that $U'(0) > 0$ – the marginal expected utility of increasing a is positive when $a = 0$.)

(c) Show that $U(a)$ is strictly concave in a, so that except at a corner solutions, the first-order condition is necessary and sufficient to find a^*.

(d) Show that if $u'(0)$ is infinite, it's not optimal to invest all your wealth; and that if $u'(0)$ is finite, then there's a cutoff $\bar{p}$ such that it's optimal to invest all your wealth if $p \geq \bar{p}$.

From here on, assume that either $u'(0)$ is infinite or $p \in \left(\frac{1}{3}, \bar{p}\right)$, so the optimal level of investment a^* is strictly positive but below w.

(e) Show that if $u(x) = 1 - e^{-cx}$ (the Constant Absolute Risk Aversion or CARA utility function), your optimal investment a^* does not depend on w.

(f) (HARD) For general u, show that if your Coefficient of Absolute Risk Aversion $A(x) = -\frac{u''(x)}{u'(x)}$ is decreasing, you invest more as w increases. (You may use the fact that if $U'(a)$ is strictly increasing in w at $a = a^*(w)$, then a^* is strictly increasing in w. More hints on the next page.)

Now reframe the question as deciding what fraction t of your wealth to invest; writing $a = tw$,

$$U(t) \quad = \quad pu(w(1+2t)) + (1-p)u(w(1-t))$$

(g) Show that if $u(x) = \frac{1}{1-\rho}x^{1-\rho}$, with $\rho \leq 1$ and $\rho \neq 0$ (the Constant Relative Risk Aversion or CRRA utility function), you invest the same fraction of your wealth regardless of w.

(h) (HARD) For general u, show that if your Coefficient of Relative Risk Aversion $R(x) = -\frac{xu''(x)}{u'(x)}$ is increasing, you invest a smaller fraction of your wealth as w increases.

A Few Hints

Feel free to try the problem first without the hints, or feel free to use them from the beginning.

You can answer pretty much every part with reference to $U'(a)$, so calculate that first.

For **part (b)**, plus in $a = 0$ and show that if $p > \frac{1}{3}$, $U'(0) > 0$ – even if you're risk-averse, the marginal value of increasing your investment is strictly positive when you're not investing at all

For **part (e)**, since you just showed that the first-order condition determines a^*, plugging in the CARA utility function and showing that w drops out of the first-order condition is enough.

Part (f) is hard. First, note the following. For a parameterized maximization problem in one dimension, if $x^*(w) = \arg\max f(x, w)$, f is differentiable and strictly concave in x, and $\frac{\partial f}{\partial x}$ is strictly increasing in w wherever it's zero, then x^* is strictly increasing in w. Translated to our problem, this means that if $U'(a)$ is strictly increasing in w at $a = a^*(w)$, then a^* is strictly increasing in w.

(Why is this true? Since f is differentiable and strictly concave in x, $x^*(w)$ is unique and determined by the FOC $\frac{\partial f}{\partial x}(x, w) = 0$. If this is increasing in w whenever it's zero, then for $w' > w$, $\frac{\partial f}{\partial x}(x(w), w') > 0$; since $\frac{\partial f}{\partial x}$ is decreasing in x, $\frac{\partial f}{\partial x}(x(w'), w') = 0$ then requires $x(w') > x(w)$.)

Given that, the key step in part (f) is to calculate

$$\frac{d}{dw}(U'(a)) \quad = \quad 2pu''(w + 2a) - (1 - p)u''(w - a)$$

multiply and divide by some u' terms to rewrite it as

$$\frac{d}{dw}(U'(a)) \quad = \quad -2pu'(w + 2a)\left(-\frac{u''(w + 2a)}{u'(w + 2a)}\right) + (1 - p)u'(w - a)\left(-\frac{u''(w - a)}{u'(w - a)}\right)$$

and note that at the optimum, $2pu'(w + 2a) = (1 - p)u'(w - a)$ (since $U'(x^*(w)) = 0$), so that

$$\left.\frac{d}{dw}(U'(a))\right|_{a=a^*(w)} \quad = \quad (1 - p)u'(w - a^*)\left(-A(w + 2a^*) + A(w - a^*)\right)$$

which leads to the result.

Part (h) is similar to part (f). This time, calculate $\frac{\partial}{\partial w}(U'(t))$, again use the fact that $U'(t) = 0$ at $t = t^*(w)$, and again do some clever multiplying and dividing by u' (and whatever else you need to do) to express $\frac{\partial}{\partial w}(U'(t))$ at $t = t^*(w)$ in terms of the coefficients of relative risk aversion $-x\frac{u''(x)}{u'(x)}$, to show that it's (this time) negative if $R(x)$ is increasing.