Problem Set #6

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Questions 1

The monetary policy authority faces the following problem:

$$\min_{\{x_t,\pi_t,i_t\}} \frac{1}{2}\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t\left(x_t^2+\alpha\pi_t^2\right)\right], \text{ s.t.} \qquad \sigma\mathbb{E}_t\left[\Delta x_{t+1}\right] = i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n$$

$$\pi_t = \kappa x_t + \beta\mathbb{E}_t\left[\pi_{t+1}\right] + u_t$$

Using the primal approach, we can optimize the Lagrangian, considering only the NKPC constraint:

$$\mathcal{L} = -\mathbb{E}\left[\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left(x_t^2+\alpha\pi_t^2\right) - \lambda_t\left(\pi_t - \kappa x_t - \beta\pi_{t+1} - u_t\right)\right]$$

Which has the following first order conditions:

$$\frac{\partial\mathcal{L}}{\partial x_t} = -\beta^t x_t + \kappa\lambda_t = 0$$

$$\frac{\partial\mathcal{L}}{\partial \pi_t} = \begin{cases} -\beta^t\alpha\pi_t - \lambda_t + \beta\lambda_{t-1} = 0, & t\geq 1 \\ -\beta^t\alpha\pi_t - \lambda_t = 0, & t = 0 \end{cases}$$

Combining these FOCs enables us to derive an optimal policy rule:

$$\alpha\kappa\pi_t + \Delta x_t = 0,\ t\geq 1 \qquad\qquad \alpha\kappa\pi_0 + x_0 = 0$$

Let $x_{-1} = p_{-1} = 0$; then, we can represent the optimal rule as a single equation:

$$\alpha\kappa\pi_t + \Delta x_t = 0$$

Since this holds for all t, we can prove via induction that $\alpha\kappa p_t + x_t = 0$:

$$\begin{aligned}
\alpha\kappa(p_0 - p_{-1}) + (x_0 - x_{-1}) &= \alpha\kappa p_0 + x_0 = 0 \\
\alpha\kappa(p_t - p_{t-1}) + (x_t - x_{t-1}) &= \alpha\kappa p_t + x_t - (\alpha\kappa p_{t-1} + x_{t-1}) = 0
\end{aligned}$$

We can use this optimal policy rule and the NKPC (adjusted to use $p_t - p_{t-1}$ instead of π) to contruct a linear system from which to solve for equilibrium dynamics:

$$\begin{aligned}
-\beta\mathbb{E}\left[p_{t+1}\right] + p_t - p_{t-1} - \kappa(-\alpha\kappa p_t) = u_t& \\
-\beta\mathbb{E}\left[p_{t+1}\right] &= -(1+\beta+\alpha\kappa^2)p_t + p_{t-1} + u_t
\end{aligned}$$

$$\Rightarrow \begin{pmatrix} \mathbb{E}\left[p_{t+1}\right] \\ p_t \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{\beta} + \frac{\alpha\kappa^2}{\beta} & -\frac{1}{\beta} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{\beta} \\ 0 \end{pmatrix} u_t$$

To determine equilibrium dynamics in this model, we must find the eigenvalues of the matrix in this linear system:

$$\begin{aligned}
(1 + \frac{1}{\beta} + \frac{\alpha\kappa^2}{\beta} - \lambda)(-\lambda) + \frac{1}{\beta} &= 0 \\
\lambda^2 - (1 + \frac{1}{\beta} + \frac{\alpha\kappa^2}{\beta})\lambda + \frac{1}{\beta} &= 0
\end{aligned}$$

Because this system has one state and one choice variable, $\lambda_1 > 1$ and $\lambda_2 < 1$, where λ_1 is the eigenvalue associated with $\mathbb{E}\left[p_{t+1}\right]$. Without paying too much mind to the exact values of λ_1 and λ_2 and omitting intermediate (and tedious) steps, we can find:

$$\begin{aligned}
\lambda &= \frac{1}{2\beta}\left[1 + \beta + \alpha\kappa^2 \pm \sqrt{(1+\beta+\alpha\kappa^2} - 4\beta\right] \\
\lambda_1\lambda_2 &= \frac{1}{4\beta^2}\left[(1+\beta+\alpha\kappa^2)^2 - (1+\beta+\alpha\kappa^2)^2 + 4\beta\right] \\
&= \frac{1}{\beta}
\end{aligned}$$

Furthermore, we can see that $\beta(\lambda_1 + \lambda_2) = 1 + \beta + \alpha\kappa^2$. This enables us to write the NKPC with just our eigenvalues and lag operators:

$$\begin{aligned}
-\beta(1-\lambda_1 L)(1-\lambda_2 L)L^{-1}p_t = u_t& \\
\left(\beta\lambda_1 - \beta L^{-1}\right)(1-\lambda_2 L)p_t = u_t& \\
p_t - \lambda_2 p_{t-1} &= \left(\frac{1}{\lambda_2} - \beta L^{-1}\right)^{-1} u_t \\
p_t &= \left(\frac{\lambda_2}{1-\beta\lambda_2 L^{-1}}\right) u_t + \lambda_2 p_{t-1}
\end{aligned}$$

Since we are given the distribution of the markup shock u_t, we can determine solve for p_t at any given t, with past realizations accounted for in p_{t-1} and expected future realizations given by the distribution of u_t:

$$\begin{aligned}
p_t &= \lambda_2 p_{t-1} + \lambda_2 \mathbb{E}_t \left[\sum_{j=0}^{\infty} (\lambda_2 \beta)^j u_{t+j} \right] \\
&= \lambda_2 p_{t-1} + \lambda_2 \left(u_t + \sum_{j=1}^{\infty} (\lambda_2 \beta)^j \mathbb{E}_t \left[u_{t+j} \right] \right) \\
&= \lambda_2 p_{t-1} + \lambda_2 \left(u_t + \left(\frac{\lambda_2 \beta}{1 - \lambda_2 \beta} \right) \overline{u} \right) \\
p_t &= \lambda_2 (p_{t-1} + u_t) + \left(\frac{\lambda_2}{\lambda_1 - 1} \right) \overline{u}
\end{aligned}$$

Recalling our equation for the output gap, this equation can be used to describe the dynamics of x_t, as well:

$$x_t = \lambda_2 x_{t-1} - \lambda_2 \alpha \kappa u_t - \left(\frac{\lambda_2 \alpha \kappa}{\lambda_1 - 1} \right) \overline{u}$$

Question 2

Under a discretionary policy, the planner can ensure that $\alpha \kappa \pi_t + x_t = 0$ in every period. Then, since the NKPC holds each period, we can solve:

$$\begin{aligned}
\pi_t &= \kappa x_t + \beta \mathbb{E}_t \left[\pi_{t+1} \right] + u_t \\
\pi_t &= -\alpha \kappa^2 \pi_t + \beta \mathbb{E}_t \left[\pi_{t+1} \right] + u_t \\
\pi_t &= \frac{1}{1 + \alpha \kappa^2} \left(\beta \mathbb{E}_t \left[\pi_{t+1} \right] + u_t \right) \\
\pi_t &= \frac{1}{1 + \alpha \kappa^2} \sum_{j=0}^{\infty} \left(\frac{\beta}{1 + \alpha \kappa^2} \right)^j \mathbb{E} \left[u_{t+j} \right] \\
\pi_t &= \left(\frac{1}{1 + \alpha \kappa^2} \right) u_t + \sum_{j=0}^{\infty} \left(\frac{\beta}{1 + \alpha \kappa^2} \right)^j \mathbb{E} \left[u_{t+j} \right] \\
\pi_t &= \left(\frac{1}{1 + \alpha \kappa^2} \right) u_t + \left(\frac{\beta}{(1 + \alpha \kappa^2)(1 - \beta + \alpha \kappa^2)} \right) \overline{u}
\end{aligned}$$

Applying this to the optimal policy rule yields our equation for the output gap:

$$x_t = -\left(\frac{\alpha \kappa}{1 + \alpha \kappa^2} \right) u_t - \left(\frac{\beta \alpha \kappa}{1 - \beta + \alpha \kappa^2} \right) \overline{u}$$

Question 3

Under the $\pi_t = 0$ rule, the NKPC yields the equilibrium allocation:

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t \left[\pi_{t+1} \right] + u_t \Rightarrow x_t = -\frac{u_t}{\kappa}$$

Question 4

Similar to in question 3, we can determine the equilibrium allocation by setting $x_t = 0$ in the NKPC:

$$\pi_t = \beta \mathbb{E}_t\left[\pi_{t+1}\right] + u_t = \sum_{j=0}^{\infty} \beta^j \mathbb{E}\left[u_{t+j}\right]$$
$$\pi_t = u_t + \left(\frac{\beta}{1-\beta}\right)\overline{u}$$

Question 5

To determine under which circumstances one policy is preferable to the other, we must first determine the expected welfare losses under each policy:, letting W_π and W_d denote welfare losses under an inflation-targeting and discretionary policy, respectively:

$$W_\pi = \frac{1}{2}\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(-\frac{u_t}{\kappa}\right)^2\right] = \frac{\sigma^2 + \overline{u}^2}{2\kappa^2(1-\beta)}$$

$$\begin{aligned}
W_d &= \frac{1}{2}\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left((-\alpha\kappa\pi_t)^2 + \alpha\pi_t^2\right)\right] \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2}\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(\left(\frac{1}{1+\alpha\kappa^2}\right)u_t + \left(\frac{\beta}{(1+\alpha\kappa^2)(1-\beta+\alpha\kappa^2)}\right)\overline{u}\right)^2\right] \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2(1-\beta)}\left[\left(\frac{1}{(1+\alpha\kappa^2)^2}\right)\mathbb{E}_t\left[u_t^2\right] + 2\left(\frac{\beta}{(1+\alpha\kappa^2)^2(1-\beta+\alpha\kappa^2)}\right)\mathbb{E}_t\left[u_t\right]\overline{u} + \left(\frac{\beta^2}{(1+\alpha\kappa^2)^2(1-\beta+\alpha\kappa^2)^2}\right)\overline{u}^2\right] \\
&= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)^2}\left[\sigma^2 + \overline{u}^2 + \left(\frac{2\beta}{1-\beta+\alpha\kappa^2} + \frac{\beta^2}{(1-\beta+\alpha\kappa^2)^2}\right)\overline{u}^2\right] \\
&= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)}\left[\sigma^2 + \left(1 + \frac{2\beta}{1-\beta+\alpha\kappa^2} + \frac{\beta^2}{(1-\beta+\alpha\kappa^2)^2}\right)\overline{u}^2\right] \\
&= \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)}\left[\sigma^2 + \left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1-\beta+\alpha\kappa^2)^2}\right)\overline{u}^2\right]
\end{aligned}$$

The social planner prefers an inflation-targeted policy if $W_\pi \leq W_d$:

$$\begin{aligned}
\frac{\sigma^2 + \overline{u}^2}{2\kappa^2(1-\beta)} &\leq \frac{\alpha}{2(1-\beta)(1+\alpha\kappa^2)}\left[\sigma^2 + \left(\frac{1+2\alpha\kappa^2+\alpha^2\kappa^4}{(1-\beta+\alpha\kappa^2)^2}\right)\overline{u}^2\right] \\
\left(\frac{1+\alpha\kappa^2}{\kappa^2}\right)\sigma^2 - \alpha\sigma^2 &\leq \left[\frac{\alpha(1+2\alpha\kappa^2+\alpha^2\kappa^4)}{(1-\beta+\alpha\kappa^2)^2} - \frac{1+\alpha\kappa^2}{\kappa^2}\right]\overline{u}^2 \\
\sigma^2 &\leq \left[\frac{\alpha\kappa^2(1+2\alpha\kappa^2+\alpha^2\kappa^4)}{(1-\beta+\alpha\kappa^2)^2} - 1 - \alpha\kappa^2\right]\overline{u}^2
\end{aligned}$$

At the limit as $\beta \to 1$, this inequality simplifies cleanly:

$$\begin{aligned}
\sigma^2 &\leq \left[\frac{\alpha\kappa^2(1+2\alpha\kappa^2+\alpha^2\kappa^4)}{(\alpha\kappa^2)^2} - 1 - \alpha\kappa^2\right]\overline{u}^2 \\
&\leq \left[\frac{(\alpha\kappa^2)^2((\alpha\kappa^2)^{-2}+2+\alpha\kappa^2)}{(\alpha\kappa^2)^2} - 1 - \alpha\kappa^2\right]\overline{u}^2 \\
&\leq \left((\alpha\kappa^2)^{-2}2+\alpha\kappa^2 - 1 - \alpha\kappa^2\right)\overline{u}^2 \\
\sigma^2 &\leq \frac{\overline{u}^2}{1+\alpha^2\kappa^4}
\end{aligned}$$

Question 6

Following the same steps and logic as in question 5, we can determine the necessary relationship between σ^2 and $\overline{u}^2$ such that an output-targeting montary policy is optimal. First, we must find the welfare loss from an output policy, W_x:

$$\begin{aligned}
W_x &= \frac{1}{2}\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t\alpha\left(u_t+\left(\frac{\beta}{1-\beta}\right)\overline{u}\right)^2\right] \\
&= \frac{\alpha\beta}{2(1-\beta)}\left[\mathbb{E}_t\left[u_t^2\right]+2\left(\frac{\beta}{1-\beta}\right)\overline{u}\mathbb{E}_t\left[u_t\right]+\left(\frac{\beta}{1-\beta}\right)^2\overline{u}^2\right] \\
&= \frac{\alpha\beta}{2(1-\beta)}\left[\sigma^2+\left(\frac{2\beta}{1-\beta}+\frac{\beta^2}{(1-\beta)^2}\right)\overline{u}^2\right] \\
W_x &= \frac{\alpha\beta}{2(1-\beta)}\left[\sigma^2+\left(\frac{\beta(2-\beta)}{(1-\beta)^2}\right)\overline{u}^2\right]
\end{aligned}$$

Then,

$$\begin{aligned}
W_x &\leq W_\pi \\
\frac{\alpha\beta}{2(1-\beta)}\left[\sigma^2+\left(\frac{\beta(2-\beta)}{(1-\beta)^2}\right)\overline{u}^2\right] &\leq \frac{\sigma^2+\overline{u}^2}{2\kappa^2(1-\beta)} \\
\alpha\beta\left[\sigma^2+\left(\frac{\beta(2-\beta)}{(1-\beta)^2}\right)\overline{u}^2\right] &\leq \frac{\sigma^2+\overline{u}^2}{\kappa^2} \\
(\alpha\beta\kappa^2-1)\sigma^2 &\leq \left[1-\frac{2\kappa^2\beta(2-\beta)}{(1-\beta)^2}\right]\overline{u}^2 \\
\sigma^2 \leq \left(\frac{1-\frac{2\kappa^2\beta(2-\beta)}{(1-\beta)^2}}{\alpha\beta\kappa^2-1}\right)\overline{u}^2
\end{aligned}$$

At the limit of $\beta \to 1$, this converges to:

$$\sigma^2 \leq \frac{\overline{u}^2}{\alpha\kappa^2-1}$$

Thus, if $\alpha\kappa^2 < 1$, output targeting is never preferred to inflation targeting.

Question 7

Recall the final two equations from question 1, which decribe the dynamics of inflation and output in this model:

$$
\begin{aligned}
p_t &= \lambda_2(p_{t-1} + u_t) + \left(\frac{\lambda_2}{\lambda_1 - 1}\right)\overline{u} \\
x_t &= \lambda_2 x_{t-1} - \lambda_2 \alpha\kappa u_t - \left(\frac{\lambda_2 \alpha\kappa}{\lambda_1 - 1}\right)\overline{u}
\end{aligned}
$$

Setting $u_t = \overline{u} = 0$ for all t, the value of p and x in any period t are:

$$p_t = \lambda_2^t p_{-1} \qquad\qquad x_t = \lambda_2^t x_{-1}$$

Then $p_{-1} = x_{-1} = 0$ would give a first-best allocation. From the NKPC, we know that $\Delta x_t = -\alpha\kappa\pi_t$. Then, by the NKIS, we can determine π_t under the Taylor rule:

$$
\begin{aligned}
\sigma\mathbb{E}_t\left[\Delta x_{t+1}\right] &= i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n \\
-\alpha\kappa\sigma\mathbb{E}_t\left[\pi_{t+1}\right] &= \phi\pi_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n \\
\pi_t &= \frac{1}{\phi}\left[(1 - \alpha\kappa\sigma)\mathbb{E}_t\left[\pi_{t+1}\right] + r_t^n\right]
\end{aligned}
$$

Thus, as $\phi \to \infty$, $\pi_t \to 0$ for all t, indicating that the first-best allocation is achieved.