Problem Set #5

Danny Edgel
Econ 703: Mathematical Economics I
Fall 2020

September 16, 2020

Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

Let X and Y be normed vector spaces, where $T \in L(X, Y)$.

(a) **Show that if there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$, then T is one-to-one.**

T is one-to-one if and only if $T(x) = \vec{0}$ has only the trivial solution (i.e. $x = \vec{0}$). If $\exists m > 0$ s.t. $m||x|| \leq ||T(X)||$, then

$$\begin{aligned} m||0|| &\leq ||T(\vec{0})|| \\ 0 &\leq ||T(\vec{0})|| \end{aligned}$$

Since $|| \cdot || \geq 0$, $T(\vec{0}) = 0$. We can likewise show that this inequality requires $x = \vec{0}$ if $T(\vec{0}) = 0$:

$$\begin{aligned} m||x|| &\leq ||T(\vec{0})|| \\ m||x|| &\leq 0 \end{aligned}$$

Thus, $T(\vec{0}) = \vec{0} \iff x = \vec{0}$, so T is one-to-one.

$\therefore$ if $\exists m > 0$ s.t. $m||x|| \leq ||T(X)||$, then T is one-to-one.

(b) **Use the theorem with five equivalent properties to show that $T^{-1}(\cdot)$ is continuous on $T(X)$**

We know that, for any linear $T \in L(X, Y)$, where X and Y are normed vector spaces, that "T is Lipschitz" and "T is continuous" are equivalent statements. $T^{-1} \in L(T(X), X)$, where X and $T(X) \subseteq Y$ are normed vector spaces, so this theorem applies to T^{-1}. From $m||x|| \leq ||T(X)||$, $m > 0$, we can derive:

$$m||T^{-1}(T(x))|| \leq ||T(x)||$$
$$||T^{-1}(T(x))|| \leq \frac{1}{m} \leq ||T(x)||$$

So $\exists \beta = \frac{1}{m} \in \mathbb{R}$ such that $||T^{-1}(T(x))|| \leq \beta \leq ||T(x)|| \; \forall x \in \text{Im}T(X)$. Thus, T^{-1} is bounded on $T(X)$. $\therefore$, T^{-1} is continuous on $T(X)$.

(c) **Use the same theorem to show that if T^{-1} is continuous on $T(X)$, then there exists some $m > 0$ such that $m||x|| \leq ||T(x)||$**

Since the continuity of T^{-1} implies that T^{-1} is also Lipschitz, then if T^{-1} is continuous, if $a, \vec{0} \in \text{Im}T(X)$, $T(x) = a$, and $k > 0$:

$$||a - \vec{0}|| \leq k||T(x) - T(\vec{0})||$$
$$||T^{-1}(T(x))|| \leq k||T(x)||$$
$$\frac{1}{k}||x|| \leq ||T(x)||$$

Thus, $\exists m = \frac{1}{k} > 0$ such that $m||x|| \leq ||T(x)||$.

Question 2

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ as $T(x, y) = (x + 5y, 8x + 7y)$.

(a) **Calculate $||T||$ given the norm $||(x, y)||_1 = |x| + |y|$ in $\mathbb{R}^2$**

$||T||$ is the supremum of $||T||$ using the norm function $||(x, y)||_1 = |x| + |y|$, where $||(x, y)|| = |x| + |y| = 1$. Thus, we can rewrite the problem as:

$$||T|| = \max_{|x|+|y|=1} \{|x + 5y| + |8x + 7y|\}$$

Since there is no multiplicative interaction between x and y in $||T||$, the (x, y) that maximizes $||T||$ will have one zero element and one element equal to one. y clearly maximizes $||T||$ relative to x, so:

$$||T|| = ||T(0, 1)|| = |5| + |7| = 12$$

(b) **Calculate $||T||$ given the norm $||(x, y)||_\infty = \max\{|x|, |y|\}$ in $\mathbb{R}^2$**

Since both x and y contribute positively to $||T||$ and our constraint is $\max\{|x|, |y|\} = 1$, the vector that maximizes $||T||$ is $(1, 1)$. Therefore,

$$||T|| = ||T(1, 1)|| = \max\{|1 + 5|, |8 + 7|\} = 15$$

Question 3

Define $V = \{(a_1, a_2), (b_1, b_2)\}$ as an orthonormal basis of R^2. Define W as the standard basis of $\mathbb{R}^2$. For some $x = (x, y)$, define $x = [x]_W$. Then, for some orthogonal matrix P, $x = P[x]_V$. Since $P'P = I$,

$$\begin{aligned} x &= P[x]_V \\ P'x &= (P'P)[x]_V \\ P'x &= [x]_V \end{aligned}$$

Let $v = [x]_V$. Then, we can derive:

$$||[x]_V|| = ||v|| = \sqrt{v'v} = \sqrt{(Px)'(Px)} = \sqrt{(x'P'Px)} = \sqrt{x'x} = ||x||$$

$\therefore$ the length of x does not depend on the choice of orthonormal basis $\blacksquare$

Question 4

Define:

$$\frac{d}{dt}y(t) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} y(t),\ y(0) \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Then, to solve for $y(t) = Pdiag\{e^{t\lambda_1}, ..., e^{t\lambda_n}\}P^{-1}y(0)$, we begin by finding A's eigenvalues and eigenvectors:

$$\begin{aligned} |A - \lambda I| &= 0 \\ (1-\lambda)(-1-\lambda) - 3 &= 0 \\ \lambda^2 - 4 &= 0 \end{aligned}$$

Thus, $\lambda_1 = 2$ and $\lambda_2 = -2$.

$$(A - \lambda_1 I)v_1 = 0 \qquad (A - \lambda_2 I)v_2 = 0$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Now, define:

$$A = Pdiag\{\lambda_1, \lambda_2\}P^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Letting $y(t) = Pdiag\{e^{t\lambda_1}, ..., e^{t\lambda_n}\}P^{-1}$, we can solve:

$$y(t) = \frac{1}{4}\begin{pmatrix}1 & 3\\ 1 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}4 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 3\end{pmatrix}$$
$$y(t) = \frac{1}{4}\begin{pmatrix}1 & 3\\ 1 & -1\end{pmatrix}\begin{pmatrix}3e^{2t} & e^{2t}\\ -e^{-2t} & e^{-2t}\end{pmatrix}\begin{pmatrix}1\\ 3\end{pmatrix}$$
$$y(t) = \frac{1}{4}\begin{pmatrix}3e^{2t} - 3e^{-2t} & e^{2t} + 3e^{-2t}\\ 3e^{2t} + e^{-2t} & e^{2t} - e^{-2t}\end{pmatrix}\begin{pmatrix}1\\ 3\end{pmatrix}$$
$$y(t) = \frac{1}{4}\begin{pmatrix}6e^{2t} + 6e^{-2t}\\ 6e^{2t} - 2e^{-2t}\end{pmatrix}$$
$$y(t) = \frac{1}{2}\begin{pmatrix}3e^{2t} + 3e^{-2t}\\ 3e^{2t} - e^{-2t}\end{pmatrix}$$

Question 5

The solution to question 4 is not stable because $\lambda_1 = 2 > 0$, so the solution:

$$y(t) = \frac{1}{2}\begin{pmatrix}3e^{2t} + 3e^{-2t}\\ 3e^{2t} - e^{-2t}\end{pmatrix}$$

Is not stable. A small change in $y(0)$ will cause infinitely large swings in e^{2t} as $t \to \infty$.