Problem Set #7

Danny Edgel
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Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Consider a two-period overlapping generations model where agents earn y when young and 0 when old. Housing supply is fixed at $H^s = 1$ and preferences are given by

$$U(c_t^t, h_t, c_{t+1}^t) = \ln\left(c_t^t\right) + \alpha h_t + \beta c_{t+1}^t$$

Assume that the initial old hold the housing stock and $1 + \alpha > \beta y$.

1. The social planner's problem (SPP) is:

$$\max_{\{c_t^t, h_t, c_t^{-t}\}_{t=1}^{\infty}} \ln\left(c_t^t\right) + \alpha h_t + \beta c_t^{t-1} \text{ s.t. } h_t = 1,\ x_t^t + c_t^{t-1} = y$$

   The using the budget constraint to solve for the consumption of the old generation as a function of the consumption of the young generation and setting $h_t = 1$ in each period, the SPP can be re-written as:

$$\max_{\{c_t^t\}_{t=1}^{\infty}} \ln\left(c_t^t\right) + \alpha + \beta(y - c_t^t)$$

   Where the FOC for c_t^t can be used to solve for the social planner's allocation in each period:

$$\frac{1}{x_t^t} - \beta = 0$$

$$c_t^t = \frac{1}{\beta}$$

$$c_t^{t-1} = y - \frac{1}{\beta}$$

$$b_t = 1$$

2. Let p_t be the price of a house in period t.

(a) The young agent's problem is

$$\max_{\{c_t^t, h_t, c_{t+1}^t\}_{t=1}^{\infty}} \ln\left(c_t^t\right) + \alpha h_t + \beta c_{t+1}^t \text{ s.t. } c_t^t + p_t h_t = y,\ c_{t+1}^t = p_{t+1} h_t$$

(b) The market clearing conditions are:

$$\begin{aligned} c_t^t + c_t^{t-1} &= y && \text{(Goods market)} \\ h_t &= 1 && \text{(Housing market)} \end{aligned}$$

(c) A competitive general equilibrium is an allocation, $\{c_t^t, h_t, c_t^{t-1}\}_{t=1}^{\infty}$ and set of prices, $\{p_t\}_{t=1}^{\infty}$ that solve every agent's problem in each period and allow markets to clear.

(d) Using each period's budget constraint, the young agent's problem can be rewritten as a choice of only housing:

$$\max_{\{h_t\}_{t=1}^{\infty}} \ln\left(y - p_t h_t\right) + \alpha h_t + \beta p_{t+1} h_t$$

Using the FOC for housing, we can solve for the agent's optimal housing and consumption rules:

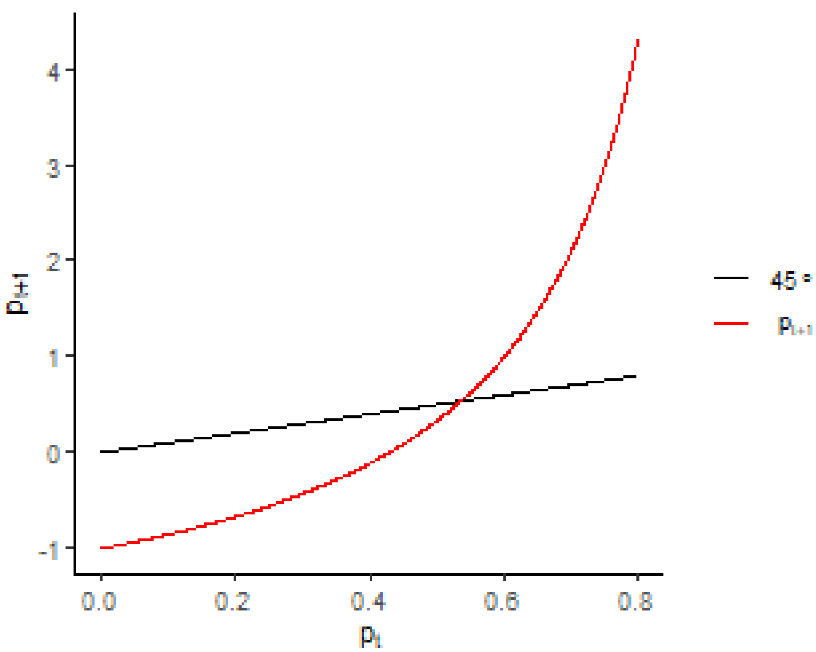
$$\begin{aligned} \frac{p_t}{y - p_t h_t} + \alpha + \beta p_{t+1} &= 0 \\ (y - p_t h_t)(\alpha + \beta p_{t+1}) &= p_t \\ p_t h_t &= y - \frac{p_t}{\alpha + \beta p_{t+1}} \\ h_t &= \frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}} \\ c_t^t &= y - p_t \left(\frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}}\right) \\ &= y - y + \frac{p_t}{\alpha + \beta p_{t+1}} = \frac{p_t}{\alpha + \beta p_{t+1}} \\ c_{t+1}^t &= p_{t+1}\left(\frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}}\right) \\ &= \frac{p_{t+1}}{p_t} y - \frac{p_{t+1}}{\alpha + \beta p_{t+1}} \\ &= \frac{p_{t+1}}{p_t}\left(y - \frac{p_t}{\alpha + \beta p_{t+1}}\right) \end{aligned}$$

Thus, the optimal rules for each variable and their associated nonnegativity conditions are (assuming prices are weakly positive):[1]

$$h_t = \frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}}, \qquad y > \frac{p_t}{\alpha + \beta p_{t+1}}$$
$$c_t^t = \frac{p_t}{\alpha + \beta p_{t+1}}, \qquad \alpha > -\beta p_{t+1}$$
$$c_{t+1}^t = \frac{p_{t+1}}{p_t}\left(y - \frac{p_t}{\alpha + \beta p_{t+1}}\right), \qquad y > \frac{p_t}{\alpha + \beta p_{t+1}}$$

(e) In equilibrium, $h_t = 1$, so we can solve:

$$\begin{aligned}
h_t &= 1 \\
\frac{y}{p_t} - \frac{1}{\alpha + \beta p_{t+1}} &= 1 \\
\frac{1}{\alpha + \beta p_{t+1}} &= \frac{y}{p_t} - 1 \\
\alpha + \beta p_{t+1} &= \frac{1}{\frac{y}{p_t} - 1} \\
p_{t+1} &= \frac{p_t}{\beta(y - p_t)} - \frac{\alpha}{\beta}
\end{aligned}$$



[1]As housing cannot go negative and positively relates to utility, this is a reasonable assumption.

(f) In the steady state, $p_t = p_{t+1} = \overline{p}\ \forall t$. So, using the law of motion from 2(e):

$$\begin{aligned}
\overline{p} &= \frac{\overline{p}}{\beta(y-\overline{p})} - \frac{\alpha}{\beta} \\
\beta\overline{p} + \alpha &= \frac{\overline{p}}{y-\overline{p}} \\
(y-\overline{p})(\beta\overline{p}) + \alpha &= \overline{p} \\
y\beta\overline{p} + \alpha y - \beta\overline{p}^2 - \alpha\overline{p} - \overline{p} &= 0 \\
\beta\overline{p}^2 + (\alpha - y\beta + 1)\overline{p} - \alpha y &= 0 \\
\overline{p} &= \frac{-\alpha + y\beta - 1 \pm \sqrt{(\alpha - y\beta + 1)^2 - 4\beta\alpha y}}{2\beta}
\end{aligned}$$

Since p cannot be negative, the unique, steady-state value of p is

$$\overline{p} = \frac{-\alpha + y\beta - 1 + \sqrt{(\alpha - y\beta + 1)^2 - 4\beta\alpha y}}{2\beta}$$

(g) Given the steady-state price of housing and optimal choice of consumption and housing, the competitive allocation is

$$\begin{aligned}
h_t &= \frac{y}{\frac{-\alpha+y\beta-1+\sqrt{(\alpha-y\beta+1)^2-4\beta\alpha y}}{2\beta}} - \frac{1}{\alpha+\beta p_{t+1}} \\
c_t^t &= \left(\frac{-\alpha + y\beta - 1 + \sqrt{(\alpha - y\beta + 1)^2 - 4\beta\alpha y}}{2\beta}\right) \\
&\quad \left(\alpha + \left(\frac{-\alpha + y\beta - 1 + \sqrt{(\alpha - y\beta + 1)^2 - 4\beta\alpha y}}{2\beta}\right)\right)^{-1} \\
c_{t+1}^t &= y - \left(\frac{-\alpha + y\beta - 1 + \sqrt{(\alpha - y\beta + 1)^2 - 4\beta\alpha y}}{2\beta}\right) \\
&\quad \left(\alpha + \left(\frac{-\alpha + y\beta - 1 + \sqrt{(\alpha - y\beta + 1)^2 - 4\beta\alpha y}}{2\beta}\right)\right)^{-1}
\end{aligned}$$

Which clearly does not simplify to the social planner's allocation.