Econ 711 – Fall 2019 – Problem Set 5 – Solutions

Question 1. The Consumer Problem

Solve the Consumer Problem and state the Marshallian demand $x(p, w)$ and indirect utility $v(p, w)$ for the following utility functions:

(a) $u(x) = x_1^\alpha + x_2^\alpha$ for $\alpha < 1$

Since $\alpha < 1$, the marginal utility of consuming either good i, $\frac{\partial u}{\partial x_i} = \alpha x_i^{\alpha-1}$, goes to $+\infty$ as $x_i \to 0$; this means it will never be optimal to consume 0 of either good. We can therefore ignore the non-negativity constraints $x \geq 0$ and solve the problem using the first-order conditions

$$\alpha x_i^{\alpha-1} \quad = \quad \lambda p_i$$

Solving for x_i,

$$x_i \quad = \quad \lambda^{\frac{1}{\alpha-1}} p_i^{\frac{1}{\alpha-1}} \alpha^{-\frac{1}{\alpha-1}}$$

and therefore

$$p_i x_i \quad = \quad \lambda^{\frac{1}{\alpha-1}} p_i^{\frac{\alpha}{\alpha-1}} \alpha^{-\frac{1}{\alpha-1}}$$

Since preferences are locally non-satiated, the budget constraint must hold with equality, so

$$w \quad = \quad p_1 x_1 + p_2 x_2 \quad = \quad \lambda^{\frac{1}{\alpha-1}} \left(p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}} \right) \alpha^{-\frac{1}{\alpha-1}}$$

Solving for $\lambda^{\frac{1}{\alpha-1}} \alpha^{-\frac{1}{\alpha-1}}$, since that shows up in the expression for x_i, gives

$$\lambda^{\frac{1}{\alpha-1}} \alpha^{-\frac{1}{\alpha-1}} \quad = \quad \frac{w}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}}$$

and therefore[1]

$$x_i(p, w) \quad = \quad \frac{w p_i^{\frac{1}{\alpha-1}}}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}}$$

Plugging these into the utility function gives

$$\begin{aligned} v(p, w) \quad = \quad u(x_1(p, w), x_2(p, w)) \quad &= \quad \left(w \frac{p_1^{\frac{1}{\alpha-1}}}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}} \right)^\alpha + \left(w \frac{p_2^{\frac{1}{\alpha-1}}}{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}} \right)^\alpha \\ &= \quad w^\alpha \frac{p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}}}{\left(p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}} \right)^\alpha} \\ &= \quad w^\alpha \left(p_1^{\frac{\alpha}{\alpha-1}} + p_2^{\frac{\alpha}{\alpha-1}} \right)^{1-\alpha} \end{aligned}$$

[1] Since there are just two goods, we could alternatively have taken the two first-order conditions $\alpha x_1^{\alpha-1} = \lambda p_1$ and $\alpha x_2^{\alpha-1} = \lambda p_2$, divided one by the other and simplified to get $x_1/x_2 = (p_1/p_2)^{1/(\alpha-1)}$, and then plugged in, say, $x_1 = (p_1/p_2)^{1/(\alpha-1)} x_2$ into the budget constraint $w = p_1 x_1 + p_2 x_2$ and solved for x_2, then found x_1 from the budget constraint. Doing it by solving the budget constraint for λ works better when there are more than two goods.

(b) $u(x) = x_1 + x_2$

This one's much simpler to solve, since it's almost always a corner solution: if $p_1 < p_2$, it's optimal to spend your entire budget on good 1, and if $p_1 > p_2$, to spend the whole budget on good 2. Thus,

$$x(p,w) \quad = \quad \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if} \quad p_1 < p_2 \\ \left(0, \frac{w}{p_2}\right) & \text{if} \quad p_1 > p_2 \\ \text{any bundle } (x_1, x_2) \text{ with } p_1 x_1 + p_2 x_2 = w & \text{if} \quad p_1 = p_2 \end{cases}$$

Noting that you get utility of $\frac{w}{p_1}$ if $p_1 < p_2$ and $\frac{w}{p_2}$ if $p_2 < p_1$, the indirect utility is

$$v(p,w) \quad = \quad \frac{w}{\min\{p_1, p_2\}}$$

(c) $u(x) = x_1^\alpha + x_2^\alpha$ for $\alpha > 1$

Once again, it's optimal to spend the whole budget on whichever good is cheaper. It's easiest to see this by noting that $x_1^\alpha + \left(\frac{w - p_1 x_1}{p_2}\right)^\alpha$ is strictly convex in x_1, and therefore can't have an interior local maximum. Marshallian demand is now

$$x(p,w) \quad = \quad \begin{cases} \left(\frac{w}{p_1}, 0\right) & \text{if} \quad p_1 < p_2 \\ \left(0, \frac{w}{p_2}\right) & \text{if} \quad p_1 > p_2 \\ \text{either of those} & \text{if} \quad p_1 = p_2 \end{cases}$$

and indirect utility is

$$v(p,w) \quad = \quad \left(\frac{w}{\min\{p_1, p_2\}}\right)^\alpha$$

(d) $u(x) = \min\{x_1, x_2\}$ (Leontief utility)

Since Leontief preferences are monotone, and therefore locally non-satiated, it's never optimal to spend less than your entire budget. Further, if $x_1 \neq x_2$, one could strictly increase utility by consuming a little less of the good that's being consumed more, to consume a little more of the good being consumed less. Thus, the optimal consumption bundle is

$$x(p,w) \quad = \quad \left(\frac{w}{p_1 + p_2}, \frac{w}{p_1 + p_2}\right)$$

and indirect utility is

$$v(p,w) \quad = \quad \frac{w}{p_1 + p_2}$$

(e) $u(x) = \min\{x_1 + x_2, x_3 + x_4\}$

You'll consume either good 1 or good 2, whichever is cheaper; either good 3 or good 4, whichever is cheaper; and you'll consume equal amounts of the two goods you consume. Thus, you'll demand $w/(\min\{p_1, p_2\} + \min\{p_3, p_4\})$ each of two goods: the cheaper of 1 and 2, and the cheaper of 3 and 4. Indirect utility will be $w/(\min\{p_1, p_2\} + \min\{p_3, p_4\})$.

(f) $u(x) = \min\{x_1, x_2\} + \min\{x_3, x_4\}$

This time, you'll consume either goods 1 and 2 (in equal amounts), or goods 3 and 4 (in equal amounts), whichever is combination us cheaper. Demand will be

$$x(p, w) \quad = \quad \begin{cases} \left(\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}, 0, 0\right) & \text{if} \quad p_1 + p_2 < p_3 + p_4 \\ \\ \left(0, 0, \frac{w}{p_3+p_4}, \frac{w}{p_3+p_4}\right) & \text{if} \quad p_1 + p_2 > p_3 + p_4 \end{cases}$$

and indirect utility will be $w / \min\{p_1 + p_2, p_3 + p_4\}$.

Question 2. CES Utility

(a) *For each of the following utility functions, solve the consumer problem, and state $x(p, w)$.*

i. *linear utility* $u(x) = x_1 + x_2 + \ldots + x_k$

ii. *Cobb-Douglas utility* $u(x) = x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$

iii. *Leontief utility* $u(x) = \min\left\{\frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_k}{a_k}\right\}$

For linear utility, there's no need to solve via a Lagrangian; since the goods are perfect substitutes, the consumer will spend their entire budget on whichever is cheapest, so

$$x_i(p, w) \quad = \quad \begin{cases} \frac{w}{p_i} & \text{if} \quad p_i < \min_{j \neq i} p_j \\ \\ 0 & \text{if} \quad p_i > \min_{j \neq i} p_j \end{cases}$$

(You were told you could ignore the cases where two or more prices were the same.)

For Cobb-Douglas, first note that utility is 0 if any $x_i = 0$, and strictly positive otherwise; so if $w > 0$, it's always optimal to consume some of each good, and we can ignore the non-negativity constraints. It's easiest to maximize the natural log of utility, using

$$\mathcal{L} \quad = \quad \sum_i a_i \log x_i + \lambda(w - p \cdot x)$$

giving first-order conditions

$$\frac{a_i}{x_i} \quad = \quad \lambda p_i$$

Rearranging, $p_i x_i = a_i / \lambda$; summing over the k goods, Walras' Law gives

$$w \quad = \quad \sum_j p_j x_j \quad = \quad \frac{\sum_j a_j}{\lambda}$$

or $\lambda = (\sum_j a_j)/w$. Plugging this into the original first-order condition and solving for x_i gives

$$x_i \quad = \quad \frac{w}{p_i} \frac{a_i}{\sum_j a_j}$$

or $p_i x_i = a_i w / \sum_j a_j$ – the consumer spends a fraction $a_i / \sum_j a_j$ of their budget on each good i, regardless of prices.

Finally, with Leontief utility, the consumer always optimizes by setting $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_k}{a_k}$. If we let $\gamma = \frac{x_i}{a_i}$, then Walras' Law is

$$w = \sum_j p_j x_j = \sum_j a_j p_j \gamma$$

so $\gamma = w / \sum_j a_j p_j$, giving

$$x_i = a_i \gamma = w \frac{a_i}{\sum_j a_j p_j}$$

(b) *Consider the Constant Elasticity of Substitution (CES) utility function*

$$u(x) = \left(\sum_{i=1}^k a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}} \right)^{\frac{s}{s-1}}$$

with $s \in (0,1) \cup (1, +\infty)$. *Solve the consumer problem and state* $x(p,w)$.

First, suppose $s > 1$, so $\frac{s}{s-1} > 0$; rather than maximizing $\left(\sum_{i=1}^k a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}} \right)^{\frac{s}{s-1}}$, we can maximize $\sum_{i=1}^k a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}}$. Since $\frac{s-1}{s} < 1$, the marginal utility of each good i is infinite when $x_i = 0$, so the consumer will always opt to consume every good, and we can ignore the non-negativity constraints. We therefore use the Lagrangian

$$\mathcal{L} = \sum_{i=1}^k a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}} + \lambda(w - p \cdot x)$$

giving first-order condition

$$\frac{s-1}{s} a_i^{\frac{1}{s}} x_i^{-\frac{1}{s}} = \lambda p_i$$

Solving for x_i,

$$\left(\frac{s-1}{s} \right)^s a_i \left(\frac{1}{\lambda p_i} \right)^s = x_i$$

giving

$$w = \sum_j p_j x_j = \left(\frac{s-1}{s} \right)^s \lambda^{-s} \sum_j a_j p_j^{-(s-1)}$$

which implies that

$$w \frac{1}{\sum_j a_j p_j^{-(s-1)}} = \left(\frac{s-1}{s} \right)^s \lambda^{-s}$$

Plugging this back into the original first-order condition and simplifying,

$$x_i = w \frac{a_i p_i^{-s}}{\sum_j a_j p_j^{-(s-1)}}$$

For the case where $s < 1$, maximizing $u(x)$ is instead equivalent to *minimizing* $\sum_{i=1}^k a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}}$, or maximizing its negative, so we would want to use the Lagrangian

$$\mathcal{L} = -\sum_{i=1}^k a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}} + \lambda(w - p \cdot x)$$

but everything would turn out the same. (λp_1 would now be $\frac{1-s}{s} a_i^{\frac{1}{s}} x_i^{-\frac{1}{s}}$, rather than $\frac{s-1}{s}$, which is good because we need $\lambda \geq 0$, but we would still get the same expression for x_i.)

(c) *Show that CES utility gives the same demand as linear utility in the limit $s \to +\infty$, as Cobb-Douglas utility in the limit $s \to 1$, and as Leontief utility in the limit $s \to 0$.*

For the first limit, it's helpful to rewrite x_i as

$$x_i \quad = \quad \frac{w}{p_i} \frac{a_i}{\sum_j a_j \left(\frac{p_j}{p_i}\right)^{-(s-1)}} \quad = \quad \frac{w}{p_i} \frac{a_i}{\sum_j a_j \left(\frac{p_i}{p_j}\right)^{s-1}}$$

If $p_i > p_j$, then $\frac{p_i}{p_j} > 1$, so $\left(\frac{p_i}{p_j}\right)^{s-1} \to +\infty$ as $s \to +\infty$. Thus, if i is not the cheapest of the goods, the denominator goes to infinity and therefore $x_i \to 0$. On the other hand, if i is strictly cheaper than any other good, then $\left(\frac{p_i}{p_j}\right)^{s-1} \to 0$ for $j \neq i$, so $x_i \to \frac{w}{p_i}\frac{a_i}{a_i} = \frac{w}{p_i}$. As $s \to +\infty$, CES utility leads to consumers spending their entire budget on the cheapest good, the same as linear utility.

Plugging $s = 1$ into the expression for x_i gives $x_i = \frac{w}{p_i}\frac{a_i}{\sum_j a_j}$, same as Cobb-Douglas demand.

Plugging $s = 0$ into the expression for x_i gives $x_i = w\frac{a_i}{\sum_j a_j p_j}$, same as Leontief demand.

(d) *The Elasticity of Substitution between goods 1 and 2 is defined as*

$$\xi_{1,2} \quad = \quad -\frac{\partial \log(x_1/x_2)}{\partial \log(p_1/p_2)}$$

While this looks complicated, in the case of CES demand, we can actually write the ratio $\frac{x_1}{x_2}$ as a relatively simple function of the price ratio $\frac{p_1}{p_2}$, and calculate this elasticity without much difficulty. Calculate the elasticity of substitution for CES demand, and note its values as $s \to +\infty$, $s \to 1$, and $s \to 0$.

From our expression for x_i, letting $D = \sum_j a_j p_j^{-(s-1)}$ the denominator in the expression for x_i,

$$\frac{x_1}{x_2} \quad = \quad \frac{wa_1p_1^{-s}/D}{wa_2p_2^{-s}/D} \quad = \quad \frac{a_1}{a_2}\left(\frac{p_1}{p_2}\right)^{-s}$$

If we let $r = \frac{p_1}{p_2}$ be the price ratio, then

$$\frac{x_1}{x_2} \quad = \quad \frac{a_1}{a_2}r^{-s} \qquad \longrightarrow \qquad \log\left(\frac{x_1}{x_2}\right) \quad = \quad \log a_1 - \log a_2 - s\log r$$

giving

$$\xi_{1,2} \quad = \quad -\frac{\partial \log(x_1/x_2)}{\partial \log r} \quad = \quad -(-s) \quad = \quad s$$

so the parameter s in the CES utility function is exactly the elasticity of substitution. This is infinite as $s \to +\infty$ (when goods are perfect substitutes), 1 when $s = 1$ (when goods are neither complements nor substitutes – the price of good 2 doesn't change the demand for good 1), and 0 when $s = 0$ (perfect complements, as the goods are always consumed in the same ratio regardless of their prices).

Question 3. Exchange Economies

We've been considering the problem facing a consumer with wealth w at prices p. An "exchange economy" is a different model where instead of money, each consumer is instead endowed with an initial bundle of goods $e \in \mathbb{R}^k_+$, and can either buy or sell any quantity of the goods at market prices p. The consumer's problem is then

$$\max_{x \in \mathbb{R}^k_+} u(x) \qquad \textit{subject to} \qquad p \cdot x \leq p \cdot e$$

Suppose preferences are locally non-satiated and the consumer's problem has a unique solution $x(p,e)$. We'll say the consumer is a net buyer *of good i if $x_i(p,e) > e_i$, and a* net seller *if $x_i(p,e) < e_i$.*

(a) *Show that if p_i increases, the consumer cannot switch from being a net seller to a net buyer.*

Let p and p' be two price vectors, with $p'_i > p_i$ and $p'_j = p_j$ for $j \neq i$. Let $x = x(p,e)$ and $x' = x(p',e)$. The claim is that we cannot have $x'_i > e_i > x_i$.

We'll show by contradiction. Suppose $x'_i > e_i > x_i$. Since $x_i - e_i < 0$, $p \cdot (x-e) > p' \cdot (x-e)$, so if the consumer could afford bundle x at prices p, they could afford x at p' and would still have had money left over. Since preferences are LNS, this is only possible if $x' \succ x$.

Similarly, since $x'_i - e_i > 0$, $p' \cdot (x'-e) > p \cdot (x'-e)$, so if the consumer could afford x' at p', they could have afforded x' and p and would still have had money left over, which is only possible if $x \succ x'$.

Thus, $x'_i > e_i > x_i$ would require both $x' \succ x$ and $x \succ x'$, so it's impossible.

(b) *Use the Lagrangian and the envelope theorem to show that $\frac{\partial v}{\partial p_i}$ is negative if the consumer is a net buyer of good i, and positive if the consumer is a net seller.*

As in lecture,

$$v \quad = \quad \min_{\lambda,\mu \geq 0} \max_x \mathcal{L}(x,\lambda,\mu)$$

where now

$$\mathcal{L} \quad = \quad u(x) + \lambda(p \cdot e - p \cdot x) + \mu \cdot x$$

since the budget constraint is now $p \cdot e - p \cdot x \geq 0$ instead of $w - p \cdot x \geq 0$. Continue to let $\Phi = \max_x \{u(x) + \lambda(p \cdot e - p \cdot x) + \mu \cdot x\}$, so that $v = \min_{\lambda,\mu \geq 0} \Phi(\lambda,\mu,p,w)$. Applying the envelope theorem to the outer problem

$$v(p,w) \quad = \quad \min_{\lambda,\mu \geq 0} \Phi(\lambda,,\mu,p,w)$$

gives

$$\frac{\partial v}{\partial p_i} \quad = \quad \left.\frac{\partial \Phi}{\partial p_i}\right|_{\lambda=\lambda^*,\mu=\mu^*}$$

Applying the envelope theorem to the inner problem

$$\Phi(\lambda^*,\mu^*,p,w) \quad = \quad \max_x \{u(x) + \lambda^*(p \cdot e - p \cdot x) + \mu^* \cdot x\}$$

gives

$$\frac{\partial \Phi}{\partial p_i} = \frac{\partial}{\partial p_i}(u(x) + \lambda^*(p \cdot e - p \cdot x) + \mu^* \cdot x)\bigg|_{x=x^*} = \lambda^*(e_i - x_i)|_{x=x^*} = \lambda^*(e_i - x_i^*)$$

where (x^*, λ^*, μ^*) is the saddle point of $\mathcal{L}$. Thus,

$$\frac{\partial v}{\partial p_i} = \lambda^*(e_i - x_i(p, e))$$

Since $\lambda^* \geq 0$, this is negative when the consumer is a net buyer of i and positive when the consumer is a net seller.

(While the problem asked you to make this argument using the Envelope Theorem and the Lagrangian, we could have shown this with a revealed-preference type argument as well. Suppose I'm a net buyer of good i, and the price of good i goes down. I can still afford my old consumption bundle; so I must be weakly better off. In fact, if preferences are LNS and I was a strict net buyer ($x_i > e_i$), I must now be strictly better off, since my old consumption bundle is now in the interior of my budget set, so I must be able to afford some bundle that's strictly preferred. Similarly, if I'm a net seller and the price goes up, I can still afford my old consumption bundle, so I'm weakly better off; and if preferences are LNS and I was a strict net seller, I'm again strictly better off.)

(c) *Consider the following statement. "If the consumer is a net buyer of good i and its price goes up, the consumer must be worse off." True or false? Explain.*

False. As long as the consumer has a positive endowment of the good in question, they could be a net buyer at initial prices, but if the price of good i goes up enough, they could switch to being a net seller and be better off. For example, suppose I had a small endowment of a low-priced good that I value moderately. Since it's cheap, I choose to consume more than I started with. However, if the price skyrocketed, I would switch over to selling my endowment of it, and could buy lots more of other stuff, making me better off.