Homework #1

Danny Edgel
Econ 713: Microeconomics II
Spring 2021

March 22, 2021

Question 1

a) The auction is a Bayesian game where each player $i \in \{1, ..., N\}$ observes type $v_i \sim F(x)$ and chooses a continuous action $b_i \geq 0$ to maximize their expected payoff, $\mathbb{E}[u_i]$, where:

$$u_i(v_i, b_i, b_{-i}) = \begin{cases} v_i - b_i, & b_i > b_j \forall j \neq i \\ \frac{1}{k}(v_i - b_i), & b_i \in A = \max\{b_1, ..., b_N\}, \text{ where } |A| = k \\ 0 & b_i < \max\{b_1, ..., b_N\} \end{cases}$$

On any continuous distribution, the probability that any two draws are exactly equal is zero. Therefore, the knife's-edge case is excluded from the reminder of the analysis for brevity.

b) Player i's expected payoff is:

$$\mathbb{E}[u_i] = (v_i - b_i)\left[\prod_{j \neq i} Pr(b_i > b_j)\right] + 0\left[1 - \prod_{j \neq i} Pr(b_i > b_j)\right] = (v_i - b_i)\left[\prod_{j \neq i} Pr(b_i > b_j)\right]$$

Assume that all players bet according to a linear function of their valuation: $b_j = d + cv_j$. Then, player i's expected payoff becomes:

$$\mathbb{E}[u_i] = (v_i - b_i)\left[\prod_{j \neq i} Pr(b_i > d + cv_j)\right] = (v_i - b_i)\left[\prod_{j \neq i} Pr\left(v_j < \frac{b_i - d}{c}\right)\right]$$

$$= (v_i - b_i)\left[\prod_{j \neq i}\left(\frac{b_i - d}{c}\right)^a\right] = (v_i - b_i)\left(\frac{b_i - d}{c}\right)^{Na}$$

Player i's optimal bet, then, can be derived from the first-order condition of their expected payoff:

$$\frac{v_i Na}{c}\left(\frac{b_i - d}{c}\right)^{Na-1} - \left(\frac{b_i - d}{c}\right)^{Na} - \frac{b_i Na}{c}\left(\frac{b_i - d}{c}\right)^{Na-1} = 0$$

$$Nav_i - b_i + d - b_iNa = 0$$
$$b_i = \left(\frac{Na}{1+Na}\right)v_i + \frac{d}{Na+1}$$

Assuming every player uses the same formula, then, we can solve:

$$c = \frac{Na}{Na+1}d \qquad = \frac{d}{Na+1} \Rightarrow d = 0$$

Thus, the Bayesian Nash equilibrium is for all players to bet:

$$b = \left(\frac{Na}{Na+1}\right)v$$

c)

d) The limit of b as $a \to \infty$ is v, so the payoff of the winner converges to zero. Therefore, bidding becomes more competitive as a increases.

e)

Question 2

a)

b)

c)

d)

Question 3

a) The auction is a Bayesian game where each player $i \in \{1, 2, 3\}$ observes type $v_i \sim U[0, 1]$ and chooses a continuous action $b_i \geq 0$ to maximize their expected payoff, $\mathbb{E}[u_i]$, where, for $i \neq j \neq k$:

$$u_i(v_i, b_i, b_j, b_k) = \begin{cases} v_i - b_i, & b_i > b_j \geq b_k \\ \frac{1}{2}(v_i - b_i), & b_i = b_j > b_k \\ \frac{1}{3}(v_i - b_i), & b_i = b_j = b_k \\ 0 & b_i < b_j \leq b_k \end{cases}$$

On any continuous distribution, the probability that any two draws are exactly equal is zero. Therefore, the knife's-edge cases are excluded from the reminder of the analysis for brevity.

b)

c) To determine expected revenue, we must first determine the expected value of the third-highest valuation. Begin by solving the pdf of the third-highest price, f^3, using the order statistic pdf formula:

$$f^3(x) = (fracn!(2!(n-3)!)\, x^2(1-x)^{n-3} = \left(\frac{n(n-1)(n-2)}{2}\right) x^2(1-x)^{n-3}$$

Then, the expected revenue is the expected value of the third-lowest bid:

$$R_s = \frac{n(n-1)(n-2)}{2} \int_0^1 v^2(1-v)^{n-3}dv = \frac{n(n-1)(n-2)}{2} \int_1^0 (1-2y+y^2)y^{n-3}dy$$
$$= \frac{n(n-1)(n-2)}{2} \int_1^0 y^{n-3} - 2y^{n-2} + y^{n-1}dy$$

Thus, the expected revenue of the seller is:

$$\mathbb{E}\left[\frac{n-1}{n-2}v\right] = \frac{n-1}{(n-2)(n+1)}$$

d)