Econ 711 – Fall 2020 – Problem Set 2 – Solutions

Question 1. Concave production functions and convex costs

Consider a production function $f : \mathbb{R}^m_+ \rightarrow \mathbb{R}_+$ for a single-output firm.

(a) *Prove that if the production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function f is concave.*

We want to show that

$$f(tz + (1-t)z') \quad \geq \quad tf(z) + (1-t)f(z')$$

for any two input vectors z and z' and any $t \in (0,1)$. Let $q = f(z)$ and $q' = f(z')$. By definition, $(q, -z)$ and (q', z') are both in Y. If Y is convex, then the point

$$t(q, -z) + (1-t)(q', -z') \quad = \quad \big(tq + (1-t)q', -tz - (1-t)z'\big)$$

is in Y as well. By the definition of Y, this implies that

$$f(tz + (1-t)z') \quad \geq \quad tq + (1-t)q' \quad = \quad tf(z) + (1-t)f(z')$$

which is what we wanted to show.

(b) *Prove that if f is concave, the cost function*

$$c(q, w) \quad = \quad \min w \cdot z \quad \textit{subject to} \quad f(z) \geq q$$

is convex in q.

Fix an input price vector w, and (for simplicity) drop w from the cost function, using $c(q)$ as shorthand for $c(q, w)$. We want to show that

$$c(tq + (1-t)q') \quad \leq \quad tc(q) + (1-t)c(q')$$

At input prices w, let z be a cost-minimizing input bundle producing q, so that $w \cdot z = c(q)$; and let z' be a cost-minimizing bundle producing q', so that $w \cdot z' = c(q')$. By definition, $f(z) \geq q$ and $f(z') \geq q'$; by the concavity of f,

$$f(tz + (1-t)z') \quad \geq \quad tf(z) + (1-t)f(z') \quad \geq \quad tq + (1-t)q'$$

so the input bundle $tz + (1-t)z'$ produces output at least $tq + (1-t)q'$. That means the lowest-cost way to produce output $tq + (1-t)q'$ is at most the cost of that bundle, which is

$$w \cdot \big(tz + (1-t)z'\big) \quad = \quad t(w \cdot z) + (1-t)(w \cdot z') \quad = \quad tc(q) + (1-t)c(q')$$

and therefore

$$c(tq + (1-t)q') \quad \leq \quad tc(q) + (1-t)c(q')$$

proving the cost function is convex.

Question 2. Solving for the profit function given technology...

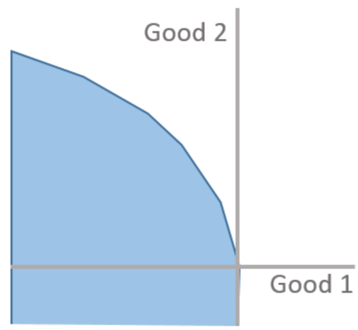
Let $k = 2$, and let the production set be

$$Y \quad = \quad \{(y_1, y_2) \quad : \quad y_1 \leq 0 \quad \textit{and} \quad y_2 \leq B(-y_1)^{\frac{2}{3}}\}$$

where $B > 0$ is a known constant. Assume both prices are strictly positive.

(a) *Draw Y, or describe it clearly.*

It looks like this:



(b) *Solve the firm's profit maximization problem to find $\pi(p)$ and $Y^*(p)$.*

(It may help to set $z = -y_1$ as the amount of input used, explain why a profit-maximizing firm will set $y_2 = Bz^{\frac{2}{3}}$, and solve a single-dimensional maximization problem for z, but be sure to state your solution $Y^*(p) \in \mathbb{R}^2$.)

As the hint suggests, it's easiest to let $z = -y_1$. Since prices are positive, the firm will always produce as much of the output good as possible, so it will set $y_2 = Bz^{\frac{2}{3}}$ rather than a smaller amount; so the firm's problem amounts to

$$\max_{z \geq 0} \left\{ -p_1 z + p_2 B z^{\frac{2}{3}} \right\}$$

The maximizer is characterized by the first-order condition

$$-p_1 + \frac{2}{3} p_2 B z^{-\frac{1}{3}} \quad = \quad 0 \qquad \longrightarrow \qquad z^{\frac{1}{3}} \quad = \quad \frac{2}{3}\frac{p_2}{p_1} B \qquad \longrightarrow \qquad z \quad = \quad \left(\frac{2}{3}B\right)^3 \frac{p_2^3}{p_1^3}$$

giving

$$Y^*(p) \quad = \quad \left(-z,\ Bz^{\frac{2}{3}}\right) \quad = \quad \left(-\left(\frac{2}{3}B\right)^3 \frac{p_2^3}{p_1^3},\ B\left(\frac{2}{3}B\right)^2 \frac{p_2^2}{p_1^2}\right) \quad = \quad \left(-\frac{8}{27}B^3\frac{p_2^3}{p_1^3},\ \frac{4}{9}B^3\frac{p_2^2}{p_1^2}\right)$$

Plugging these into $p \cdot y$ gives

$$\pi(p) \quad = \quad p_1 y_1(p) + p_2 y_2(p) \quad = \quad -\frac{8}{27}B^3\frac{p_2^3}{p_1^2} + \frac{4}{9}B^3\frac{p_2^3}{p_1^2} \quad = \quad \frac{4}{27}B^3\frac{p_2^3}{p_1^2}$$

(c) *Verify that $\pi(\cdot)$ is homogeneous of degree 1, and $y(\cdot)$ is homogeneous of degree 0.*

Well,

$$\pi(\lambda p) = \frac{4}{27} B^3 \frac{(\lambda p_2)^3}{(\lambda p_1)^2} = \lambda \frac{4}{27} B^3 \frac{p_2^3}{p_1^2} = \lambda \pi(p)$$

and

$$y(\lambda p) = \left(-\frac{8}{27} B^3 \frac{(\lambda p_2)^3}{(\lambda p_1)^3}, \frac{4}{9} B^3 \frac{(\lambda p_2)^2}{(\lambda p_1)^2} \right) = \left(-\frac{8}{27} B^3 \frac{p_2^3}{p_1^3}, \frac{4}{9} B^3 \frac{p_2^2}{p_1^2} \right) = y(p)$$

so everything is homogeneous the way it should be.

(d) *Verify that $y_1(p) = \frac{\partial \pi}{\partial p_1}(p)$ and $y_2(p) = \frac{\partial \pi}{\partial p_2}(p)$.*

Differentiating,

$$\frac{\partial \pi}{\partial p_1}(p) = \frac{\partial}{\partial p_1} \frac{4}{27} B^3 \frac{p_2^3}{p_1^2} = -\frac{8}{27} B^3 \frac{p_2^3}{p_1^3} = y_1(p)$$

and

$$\frac{\partial \pi}{\partial p_2}(p) = \frac{\partial}{\partial p_2} \frac{4}{27} B^3 \frac{p_2^3}{p_1^2} = \frac{4}{9} B^3 \frac{p_2^2}{p_1^2} = y_2(p)$$

so, yeah, Hotelling was right.

(e) *Calculate $D_p y(p)$, and verify it is symmetric, positive semidefinite, and $[D_p y] p = 0$.*

First, a correction. Footnote 2 on the homework claimed that a two-by-two matrix is positive semidefinite if its first element and determinant are both non-negative. This is false. First, I meant to claim this applies only to symmetric matrices; but even then, the statement is false, as the matrix $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ has a first element and determinant which are both zero, but is not positive semidefinite. The correct statement is, a symmetric two-by-two matrix is positive semidefinite if *both* diagonal elements and the determinant are non-negative.

(It's true that a symmetric two-by-two matrix is *positive definite* if the first element and determinant are both *strictly* positive, but the analogous result does not hold for positive semidefiniteness. This is apparently widely misunderstood – Kerr (1990), "Misstatements of the test for positive semidefinite matrices," *Journal of Guidance, Control and Dynamics* 13(3), available here, has more information if you're interested. Thanks to one of your classmates for bringing this to my attention!)

Back to the problem. Taking derivatives,

$$D_p y(p) = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{bmatrix} = \begin{bmatrix} \frac{8}{9} B^3 \frac{p_2^3}{p_1^4} & -\frac{8}{9} B^3 \frac{p_2^2}{p_1^3} \\ -\frac{8}{9} B^3 \frac{p_2^2}{p_1^3} & \frac{8}{9} B^3 \frac{p_2}{p_1^2} \end{bmatrix}$$

The two off-diagonal terms are equal, so $D_p y$ is symmetric. Both diagonal terms are positive, and the determinant turns out to be 0, so $D_p y$ is positive semidefinite. And if we right-multiply by a column vector of prices, we get

$$[D_p y(p)]p \quad = \quad \frac{8}{9}B^3 \begin{bmatrix} \frac{p_2^3}{p_1^4} & -\frac{p_2^2}{p_1^3} \\ & \\ -\frac{p_2^2}{p_1^3} & \frac{p_2}{p_1^2} \end{bmatrix} \begin{bmatrix} p_1 \\ \\ p_2 \end{bmatrix} \quad = \quad \frac{8}{9}B^3 \begin{bmatrix} \frac{p_2^3}{p_1^3} - \frac{p_2^3}{p_1^3} \\ \\ -\frac{p_2^2}{p_1^2} + \frac{p_2^2}{p_1^2} \end{bmatrix} \quad = \quad 0$$

so $[D_p y]p = 0$.

Question 3. ...and recovering technology from the profit function

Finally, suppose we didn't know a firm's production set Y, but did know its profit function was

$$\pi(p) \quad = \quad A p_1^{-2} p_2^3$$

for all $p_1, p_2 > 0$ and $A > 0$ a known constant.

(a) *What conditions must hold for this profit function to be rationalizable?*

A differentiable profit function is rationalizable if it's homogeneous of degree 1 and convex.

(I didn't ask you to check them, but π is homogeneous of degree 1, since $\pi(\lambda p) = A(\lambda p_1)^{-2}(\lambda p_2)^3 = \lambda A p_1^{-2} p_2^3 = \lambda \pi(p)$. As for convexity, showing $D_p^2 \pi$ is positive definite is exactly the same calculations as we used in question 2 to show $D_p y$ is positive definite after first showing that $y_i = \frac{\partial \pi}{\partial p_i}$.)

(b) *Recall that the outer bound was defined as*

$$Y^O \quad = \quad \{y \quad : \quad p \cdot y \quad \leq \quad \pi(p) \qquad \text{for all} \qquad p \in P\}$$

In this case, this is

$$Y^O \quad = \quad \left\{(y_1, y_2) \quad : \quad p_1 y_1 + p_2 y_2 \quad \leq \quad A p_1^{-2} p_2^3 \qquad \text{for all} \quad (p_1, p_2) \in \mathbb{R}^2_{++}\right\}$$

Show that any $y \in Y^O$ must have $y_1 \leq 0$, i.e., that good 1 must be an input only.

Pick any $y = (y_1, y_2)$ with $y_1 > 0$. Fix p_2, and make p_1 arbitrarily large; then $p_1 y_1 + p_2 y_2$ grows unboundedly, while $A p_1^{-2} p_2^3$ goes to 0. Thus, $p_1 y_1 + p_2 y_2 > A p_1^{-2} p_2^3$ for price vectors with p_1 sufficiently large; which means y would be excluded from the set Y^O. Y^O therefore only contains points with $y_1 \leq 0$.

(c) *Dividing both sides by p_2 and moving $\frac{p_1}{p_2} y_1$ to the right-hand side, we can rewrite Y^O as*

$$Y^O \quad = \quad \left\{(y_1, y_2) \quad : \quad y_2 \quad \leq \quad A p_1^{-2} p_2^2 - \frac{p_1}{p_2} y_1 \qquad \text{for all} \quad (p_1, p_2) \in \mathbb{R}^2_{++}\right\}$$

Since the expression on the right depends only on the price ratio $\frac{p_2}{p_1}$ *rather than the two individual prices, we can write this as*

$$Y^O \quad = \quad \left\{(y_1, y_2) \quad : \quad y_2 \quad \leq \quad A p_1^{-2} p_2^2 - \frac{p_1}{p_2} y_1 \qquad \text{for all} \quad \frac{p_2}{p_1} \in \mathbb{R}_{++}\right\}$$

or, if we let $r \equiv \frac{p_2}{p_1} > 0$ denote this price ratio,

$$\begin{aligned} Y^O &= \left\{(y_1, y_2) \;:\; y_2 \;\leq\; Ar^2 - \tfrac{y_1}{r} \quad \text{for all} \quad r \in \mathbb{R}_{++}\right\} \\ &= \left\{(y_1, y_2) \;:\; y_2 \;\leq\; \min_{r>0}\left(Ar^2 - \tfrac{y_1}{r}\right)\right\} \end{aligned}$$

Solve this minimization problem, and describe the production set Y^O.

With $y_1 \leq 0$, $Ar^2 - \frac{y_1}{r}$ is convex, so the solution to

$$\min_{r>0}\left\{Ar^2 - \frac{y_1}{r}\right\}$$

is given by the first-order condition

$$2Ar + \frac{y_1}{r^2} \;=\; 0 \qquad \longrightarrow \qquad r \;=\; \left(\frac{-y_1}{2A}\right)^{\frac{1}{3}}$$

Plugging this into the minimand,

$$\min_{r>0}\left\{Ar^2 - \frac{y_1}{r}\right\} \;=\; A\left(\frac{-y_1}{2A}\right)^{\frac{2}{3}} - y_1\left(\frac{-y_1}{2A}\right)^{-\frac{1}{3}} \;=\; 2^{-\frac{2}{3}}A^{\frac{1}{3}}(-y_1)^{\frac{2}{3}} + 2^{\frac{1}{3}}A^{\frac{1}{3}}(-y_1)^{\frac{2}{3}}$$

If we let $B = (2^{-2/3} + 2^{1/3})A^{1/3}$, the production set Y^O is therefore

$$Y^O \;=\; \left\{(y_1, y_2) \;:\; y_1 \;\leq\; 0 \quad \text{and} \quad y_2 \;\leq\; B(-y_1)^{\frac{2}{3}}\right\}$$

(d) *Verify that a production set Y equal to the set Y^O you just calculated would generate the "data" $\pi(p) = Ap_1^{-2}p_2^3$ that we started with.*

We showed in question 2 that this production set yields $\pi(p) = \frac{4}{27}B^3p_1^{-2}p_2^3$. Plugging in $B = (2^{-2/3} + 2^{1/3})A^{1/3}$, our production set Y^O would give

$$\pi(p) \;=\; \frac{4}{27}\left((2^{-2/3} + 2^{1/3})A^{1/3}\right)^3 p_1^{-2}p_2^3 \;=\; \frac{4}{27}(2^{-2/3} + 2^{1/3})^3 Ap_1^{-2}p_2^3$$

All that's left is to calculate $(2^{-2/3} + 2^{1/3})^3$, which is

$$\begin{aligned} \left(2^{-\frac{2}{3}} + 2^{\frac{1}{3}}\right)^3 &= \left(2^{-\frac{2}{3}}\right)^3 + 3\left(2^{-\frac{2}{3}}\right)^2\left(2^{\frac{1}{3}}\right) + 3\left(2^{-\frac{2}{3}}\right)\left(2^{\frac{1}{3}}\right)^2 + \left(2^{\frac{1}{3}}\right)^3 \\ &= 2^{-2} + 3 \cdot 2^{-\frac{4}{3}} \cdot 2^{\frac{1}{3}} + 3 \cdot 2^{-\frac{2}{3}} \cdot 2^{\frac{2}{3}} + 2 \\ &= \frac{1}{4} + \frac{3}{2} + 3 + 2 \;=\; \frac{27}{4} \end{aligned}$$

(which I told you you could assume), and we therefore get

$$\pi(p) \;=\; \frac{4}{27} \cdot \frac{27}{4} \cdot Ap_1^{-2}p_2^3 \;=\; Ap_1^{-2}p_2^3$$

and we're right back where we started.