Problem Set #7

Danny Edgel
Econ 711: Microeconomics I
Fall 2020

October 22, 2020

(a) If u is linear, then then marginal utility is constant, so the agent maximizes utility by maximizing expected wealth. The agent's optimization problem, then, is

$$\max_a p(w+2a)+(1-p)(w-a)=\max_a a(1-3p)+w$$

Thus, expected wealth is maximized by choosing the maximum value of a if $p>\frac{1}{3}$ and by choosing the maximum value of w (i.e. $a=0$) if $p>\frac{1}{3}$

(b) If the marginal utility of investing is strictly positive at $a=0$, then the optimal level of investment is strictly positive. Then,

$$\frac{\partial U(a)}{\partial a}=2pu'(w+2a)-(1-p)u'(w-a)$$
$$\frac{\partial U(a)}{\partial a}\Big|_{a=0}=2pu'(w)-(1-p)u'(w)=3pu'(w)-u'(w)=u'(w)(3p-1)$$

Where $u'>0$ since u is strictly increasing, and $p>\frac{1}{3}$, so $U'(0)=u'(w)(3p-1)>0$.

(c) Continuing from the last problem's calculation, the second derivative of the utility function is

$$\frac{\partial^2 U(a)}{\partial a^2}=5pu''(w+2a)+(1-p)u''(w-a)$$

Where, by assumption, $u''<0$. Thus, $U''<0$, so $U(a)$ is strictly concave in a, and the FOC is necessary and sufficient for finding a^*.

(d) When all wealth is invested, $u'(w-a)=u'(0)$ and $u'(w+2a)=u'(3w)$. It would be optimal to invest all wealth if $U'(w)>0$:

$$\frac{\partial U(a)}{\partial a}=2pu'(3w)-(1-p)u'(0)>0$$
$$2pu'(3w)>(1-p)u'(0)$$
$$\frac{u'(3w)}{u'(0)}>\frac{1-p}{2p}$$

If $u'(0) \to \infty$, then the left side of the inequality is zero. Since $p \leq 1$, it is not possible for the right side of the inequality to be negative. Thus, it cannot be optimal to invest all wealth. If $u'(0)$ is finite, then we can solve for $\overline{p}$, the probability level above which the agent will invest all of their wealth:

$$\begin{aligned}
\frac{u'(3w)}{u'(0)} &> \frac{1-p}{2p} \\
2p\left(\frac{u'(3w)}{u'(0)}\right) &> 1-p \\
p\left(1+2\left(\frac{u'(3w)}{u'(0)}\right)\right) &> 1 \\
\overline{p} &= \frac{1}{1+2\left(\frac{u'(3w)}{u'(0)}\right)}
\end{aligned}$$

If $p \geq \overline{p}$, then $U'(w) > 0$, so the agent will invest all of their wealth.

(e) Given CARA utility, U becomes $U(a) = p\left(1-e^{-c(w+2a)}\right) + (1-p)\left(1-e^{-c(w-a)}\right)$, where solving the FOC yields:

$$\begin{aligned}
\frac{\partial U(a)}{\partial a} = p\left(e^{-c(w+2a)}\right)(-2c) + (1-p)\left(-e^{-c(w-a)}\right)(c) &= 0 \\
2pce^{-c(w+2a)} &= (1-p)ce^{-c(w-a)} \\
e^{-c(w+2a)+c(w-a)} &= \frac{1-p}{2p} \\
c(w-a-w-2a) &= \log\left(\frac{1-p}{2p}\right) \\
a^* &= \frac{-1}{3c}\log\left(\frac{1-p}{2p}\right)
\end{aligned}$$

Thus, optimal investment, a^*, does not depend on w.

(f) Assume $A(x) = -\frac{u''(x)}{u'(x)}$ is decreasing in x and recall that

$$\frac{\partial U(a)}{\partial a} = 2pu'(w+2a) - (1-p)u'(w-a)$$

Now, let $a^* = \text{argmax } U(a)$ be a function of w such that $a^* = a(w)$ (for syntactical simplicity, assume that $a(w)$ always refers to the optimal value of a and that a, inside of any utility function or derivative thereof, is a function of w). Then, since $U'(a^*) = 0$, we can solve:

$$\begin{aligned}
\frac{\partial}{\partial w}\left[2pu'(w+2a(w)) - (1-p)u'(w-a(w))\right] &= \frac{\partial}{\partial w}(0) \\
2pu''(w+2a)(1+2a'(w)) - (1-p)u''(w-a)(1-a'(w)) &= 0
\end{aligned}$$

$$2pu''(w+2a)) + a'(w)4pu''(w+2a) - (1-p)u''(w-a) + a'(w)(1-p)u''(w-a)) = 0$$

$$a'(w)(4pu''(w+2a) + (1-p)u''(w-a))) = (1-p)u''(w-a) - 2pu''(w+2a))$$
$$a'(w) = \frac{(1-p)u''(w-a) - 2pu''(w+2a)}{4pu''(w+2a) + (1-p)u''(w-a)}$$

$a'(w) > 0$ if the right side of the equality is greater than zero. Since $u'' < 0$, this is only true if:

$$2pu''(w+2a) > (1-p)u''(w-a)$$
$$\frac{u''(w+2a)}{u''(w-a)} < \frac{1-p}{2p}$$

Recall that, at a^*, $\frac{1-p}{2p} = \frac{u'(w+2a)}{u'(w-a)}$.[1] Thus, the condition for $a'(w) > 0$ is

$$\frac{u''(w+2a)}{u''(w-a)} < \frac{u'(w+2a)}{u'(w-a)}$$
$$\frac{u''(w+2a)}{u'(w+2a)} > \frac{u''(w-a)}{u'(w-a)}$$

Where, by assumption, $-\frac{u''(x)}{u'(x)}$ is decreasing in x. Since $w+2a > w-a$, this inequality holds. Therefore, $\frac{\partial a^*}{\partial w} > 0$. Thus, if the agent is wealthier, they will invest more in the start-up regardless of $p \in (\frac{1}{3}, \overline{p})$.

(g) Let $u(x) = \frac{1}{1-\rho}x^{1-\rho}$, where $\rho \leq 1$ and $\rho \neq 0$. Define investment in terms of t, where $a = tw$. Then, the utility function is

$$U(t) = p\frac{1}{1-\rho}((1_t)w)^{1-\rho} + (1-p)\frac{1}{1-\rho}(w(1-t))^{1-\rho}$$
$$= \frac{w^{1-\rho}}{1-\rho}\left[p(1+2t)^{1-rho} + (1-p)(1-t)^{1-\rho}\right]$$

[1] This comes from solving the first order condition:

$$2pu'(w+2a) - (1-p)u'(w-a) = 0$$

We can find t^* by solving the first-order condition:

$$
\begin{aligned}
\frac{\partial U(t)}{\partial t} = \frac{w^{1-\rho}}{1-\rho}\left[2p(1-\rho)(1+2t)^{-rho} - (1-p)(1-\rho)(1-t)^{-\rho}\right] &= 0 \\
2p(1+2t)^{-rho} - (1-p)(1-t)^{-\rho} &= 0 \\
2p(1+2t)^{-rho} &= (1-p)(1-t)^{-\rho} \\
\left(\frac{1-t}{1+2t}\right)^{\rho} &= \frac{1-p}{2p} \\
1-t &= \left(\frac{1-p}{2p}\right)^{\frac{1}{\rho}}(1+2t) \\
t &= 1 - \left(\frac{1-p}{2p}\right)^{\frac{1}{\rho}} - 2t\left(\frac{1-p}{2p}\right)^{\frac{1}{\rho}} \\
\left[1+2\left(\frac{1-p}{2p}\right)^{\frac{1}{\rho}}\right] t &= 1 - \left(\frac{1-p}{2p}\right)^{\frac{1}{\rho}} \\
t^* &= \frac{1-\left(\frac{1-p}{2p}\right)^{\frac{1}{\rho}}}{1+2\left(\frac{1-p}{2p}\right)^{\frac{1}{\rho}}}
\end{aligned}
$$

Thus, t^* does not depend on w.

(h) Suppose $R(x) = -\frac{xu''(x)}{u'(x)}$ is increasing. The first-order condition for a general u is:

$$
\begin{aligned}
\frac{\partial U(t)}{\partial t} &= pu'((1+2t)w)(2w) + (1-p)u'(w(1-t))(-w) = 0 \\
& 2pu'((1+2t)w) - (1-p)u'(w(1-t)) = 0
\end{aligned}
$$

Which gives the relation:

$$\frac{u'((1+2t)w)}{u'(w(1-t))} = \frac{1-p}{2p}$$

Let $t = t^* = t(w)$ and assume that all references to t below are to the optimal value of t as a function of w. Taking the partial derivative of the first-order condition at $t = t^*$ yields:

$$\frac{\partial}{\partial w}[2pu'((1+2t)w) - (1-p)u'(w(1-t))] = \frac{\partial}{\partial w}(0)$$

$$2pu''((1+2t)w)[2t'(w)w + 1 + 2t] - (1-p)u''(w(1-t))[w(-t'(w)) + 1 - t] = 0$$

$$[4pu''((1+2t)w)w + (1-p)u''(w(1-t))]t'(w) = (1-p)u''(w(1-t))(1-t) - 2pu''((1+2t)w)(1+2t)$$

$$t'(w) = \frac{(1-p)u''(w(1-t))(1-t) - 2pu''((1+2t)w)(1+2t)}{4pu''((1+2t)w)w + (1-p)u''(w(1-t))}$$

For $t'(w) < 0$, the following inequality must be satisfied:

$$(1-p)u''(w(1-t))(1-t) > 2pu''((1+2t)w)(1+2t)$$

Recall the relation derived from the first-order condition, $\frac{u'((1+2t)w)}{u'(w(1-t))} = \frac{1-p}{2p}$. Then, we can simplify the above relation as:

$$\begin{aligned}
\frac{1-p}{2p} &< \frac{u''((1+2t)w)(1+2t)}{u''(w(1-t))(1-t)} \\
\frac{u'((1+2t)w)}{u'(w(1-t))} &< \frac{u''((1+2t)w)(1+2t)}{u''(w(1-t))(1-t)} \\
\frac{u''(w(1-t))(1-t)}{u'(w(1-t))} &> \frac{u''((1+2t)w)(1+2t)}{u'((1+2t)w)}
\end{aligned}$$

Which, if we let $w(1-t) = x_0$ and $w(1_2 t) = x_1$, then multiplying each side of the inequality by $-w < 0$, we have:

$$-\frac{u''(x_0)x_0}{u'(x_0)} < -\frac{u''(x_1)x_1}{u'(x_1)}$$

Since $x_0 < x_1$ and, by assumption, $-\frac{xu''(x)}{u'(x)}$ is increasing in x, this inequality holds. Thus, if $R(x)$ is increasing, the agent invests a smaller fraction of their welath as w increases.