Econ 711 – Fall 2019 – Problem Set 6 – Solutions

Question 1. Rationalizing Demand

Suppose you observe the following data on prices, wealth, and chosen consumption bundles for a certain consumer at four points in time:

w	p	x
100	$(5, 5, 5)$	$(12, 4, 4)$
100	$(7, 4, 5)$	$(9, 3, 5)$
100	$(2, 4, 1)$	$(27, 9, 10)$
150	$(7, 4, 5)$	$(15, 5, 5)$

(a) *Are the data consistent with Walras law?*

Let (w_i, p^i, x^i) refer to the i^{th} observation. Walras' Law requires $p^i \cdot x^i = w_i$ for each i; it's mechanical to check that this holds for each observation.

(b) *Can these data be rationalized by a continuous, monotonic and concave utility function?*

We know from Afriat that the data can be rationalized by a continuous, monotonic, concave utility function if and only if it satisfies GARP. We can use the hint to reduce the set of costs we need to calculate: since $x^3 > \{x^1, x^2, x^4\}$ and $x^4 > \{x^1, x^2\}$, we know that $p \cdot x^3 > p \cdot x^4 > \{p \cdot x^1, p \cdot x^2\}$ for any $p \gg 0$.

Looking at the data, then:

- $p^1 \cdot x^2 = 85 < p^1 \cdot x^1$, so $x^1 \succ^D x^2$;
  since $x^3 \gg x^1$ and $x^4 \gg x^1$, x^3 and x^4 were not affordable at (w_1, p^1),
  so the choice of x^1 does not reveal anything about x^3 or x^4
- $p^2 \cdot x^1 = 120 > p^2 \cdot x^2$, and $x^3 > x^2$ and $x^4 > x^2$,
  so x^2 was not revealed preferred to any other bundle
- $x^3 > \{x^1, x^2, x^4\}$, so $p^3 \cdot x^i < p^3 \cdot x^3$ $(i \in \{1, 2, 4\})$, so $x^3 \succ^D \{x^1, x^2, x^4\}$
- $x^3 > x^4 > \{x^1, x^2\}$, so $x^4 \succ^D \{x^1, x^2\}$

Putting it together, we see that

$$x^3 \quad \succ \quad x^4 \quad \succ \quad x^1 \quad \succ \quad x^2$$

and there are no cycles; so the data satisfy GARP, and are therefore rationalizable.

Question 2. Aggregating Demand

Suppose there are n consumers, and consumer $i \in \{1, 2, \ldots, n\}$ has indirect utility function

$$v^i(p, w_i) \quad = \quad a_i(p) + b(p) w_i$$

where $\{a_i\}_{i=1}^n$ and b are differentiable functions from $\mathbb{R}^k_+$ to $\mathbb{R}$.

(a) *Use Roy's Identity to calculate each consumer's Marshallian demand $x^i(p, w_i)$.*

From Roy's identity, consumer i's demand for product j is

$$x^i_j(p, w) \quad = \quad -\frac{\frac{\partial v^i}{\partial p_j}}{\frac{\partial v^i}{\partial w}} \quad = \quad -\frac{1}{b(p)}\left(\frac{\partial a_i}{\partial p_j} + w_i \frac{\partial b}{\partial p_j}\right)$$

(b) *Calculate the Marshallian demand $X(p, W)$ of a "representative consumer" with wealth W and indirect utility function*

$$V(p, W) \quad = \quad \sum_{i=1}^n a_i(p) + b(p) W$$

and show that $X(p, \sum_{i=1}^n w_i) \quad = \quad \sum_{i=1}^n x^i(p, w_i)$.

Again applying Roy's Identity,

$$X_j(p, W) \quad = \quad -\frac{\frac{\partial V}{\partial p_j}}{\frac{\partial V}{\partial W}} \quad = \quad -\frac{1}{b(p)}\left(\sum_i \frac{\partial a_i}{\partial p_j} + W \frac{\partial b}{\partial p_j}\right)$$

Plugging in $W = \sum_i w_i$,

$$X_j(p, W) \quad = \quad -\frac{1}{b}\left(\sum_i \frac{\partial a_i}{\partial p_j} + \sum_i w_i \frac{\partial b}{\partial p_j}\right) \quad = \quad \sum_i \left(-\frac{1}{b}\left(\frac{\partial a_i}{\partial p_j} + w_i \frac{\partial b}{\partial p_j}\right)\right) \quad = \quad \sum_i x^i_j(p, w_i)$$

so demand aggregates.

Question 3. Homothetic Preferences

Complete, transitive preferences $\succsim$ on $\mathbb{R}^k_+$ are called homothetic *if for all $x, y \in \mathbb{R}^k_+$ and all $t > 0$,*

$$x \succsim y \qquad \leftrightarrow \qquad tx \succsim ty$$

(a) *Show that if preferences are homothetic, Marshallian demand is homogeneous of degree 1 in wealth: for any $t > 0$, $x(p, tw) = tx(p, w)$.*

Suppose $x \in x(p, w)$; we'll show that $tx \in x(p, tw)$.

- To show $tx \in x(p, tw)$, we need $tx \in B(p, tw)$ and $tx \succsim y$ for any $y \in B(p, tw)$
- Since $x \in x(p, w)$, $x \in B(p, w)$, so $p \cdot x \leq w$;
  so $p \cdot (tx) = t(p \cdot w) \leq tw$, and therefore $tx \in B(p, tw)$

- For any $y \in B(p, tw)$, we know $p \cdot y \leq tw$, or $p \cdot \frac{y}{t} \leq w$, so $\frac{y}{t} \in B(p, w)$
- Since $x \in x(p, w)$, we know that $x \succsim z$ for any $z \in B(p, w)$;
  since $\frac{y}{t} \in B(p, w)$, this means $x \succsim \frac{y}{t}$
- If preferences are homothetic, then $x \succsim \frac{y}{t}$ implies $tx \succsim y$
- So $tx \in B(p, tw)$, and $tx \succsim y$ for every $y \in B(p, tw)$, so $tx \in x(p, tw)$, finishing the proof.

(b) *Show that if preferences are homothetic, monotone, and continuous, they can be represented by a utility function which is homogeneous of degree 1. (Hint: try the utility function we used to prove existence of a utility function in class!)*

Recall the proof of existence of a utility representation in class, where we defined $\alpha(x)$ as the unique value of α for which $x \sim (\alpha, \alpha, \ldots, \alpha)$, and showed that if preferences were continuous and monotone, $u(x) = \alpha(x)$ was well-defined and represented the preferences $\succsim$. With homothetic preferences, this utility function is homogeneous of degree 1:

- Suppose that $u(x) = \alpha$, which means that $x \sim \alpha e$.
- If preferences are homothetic, then $x \sim \alpha e$ implies $tx \sim t(\alpha e) = (t\alpha)e$
  - if $x \sim \alpha e$, then $x \succsim \alpha e$ and $\alpha e \succsim x$
  - homothetic preferences means $tx \succsim t\alpha e$ and $t\alpha e \succsim tx$, so $tx \sim t\alpha e$
- So if $u(x) = \alpha$, then $u(tx) = t\alpha$, meaning $u(tx) = tu(x)$,
  proving the utility function is homogeneous of degree 1.

(c) *Show that given (a) and (b), the indirect utility function takes the form $v(p, w) = b(p)w$ for some function b.*

For each (p, w), let $\hat{x}(p, w)$ be some arbitrary selection from $x(p, w)$ (i.e., $\hat{x}$ is a single-valued choice from Marshallian demand), so that $v(p, w) = u(\hat{x}(p, w))$. Then

$$v(p, w) \quad = \quad u(\hat{x}(p, w)) \quad = \quad u(w\hat{x}(p, 1)) \quad = \quad wu(\hat{x}(p, 1)) \quad = \quad wv(p, 1)$$

so if we let $b(p) = v(p, 1)$, then $v(p, w) = b(p)w$.

(To put it in words: first solve the consumer problem with wealth normalized to 1, and let $b(p)$ be the resulting indirect utility. By (a), if we multiply wealth by a constant w, Marshallian demand gets multiplied by w; by (b), if we consume a constant w times x, utility gets multiplied by the same constant w, so indirect utility is w times $b(p)$.)

Question 4. Quasilinear Utility

Let $X = \mathbb{R} \times \mathbb{R}_+^{k-1}$ (allow positive or negative consumption of the first good), suppose utility

$$u(x) \quad = \quad x_1 + U(x_2, \ldots, x_k)$$

is quasilinear, and fix the price of the first good $p_1 = 1$.

(a) *Show that Marshallian demand for goods 2 through k do not depend on wealth.*

For simplicity, define $y = (x_2, \ldots, x_k)$, so that $x = (x_1, y)$, and let $p_y = (p_2, \ldots, p_k)$. Then given the normalization $p_1 = 1$, we can write Marshallian demand as

$$\begin{aligned} x(p, w) \quad &= \quad \arg\max_{x \in X} u(x) \quad \text{subject to} \quad p \cdot x \leq w \\ &= \quad \arg \max_{(x_1, y) \in X} \{x_1 + U(y)\} \quad \text{subject to} \quad x_1 + p_y \cdot y \leq w \end{aligned}$$

Since u is strictly increasing in x_1, preferences are locally non-satiated, so the budget constraint will hold with equality, and we can rewrite the consumer problem as

$$\begin{aligned} &\max_{(x_1, y) \in X} \{x_1 + U(y)\} \quad \text{subject to} \quad x_1 = w - p_y \cdot y \\ &= \quad \max_{(x_1, y) \in X} \{w - p_y \cdot y + U(y)\} \\ &= \quad w + \max_{(x_1, y) \in X} \{-p_y \cdot y + U(y)\} \end{aligned}$$

So demand for the last $k - 1$ goods is the solution to

$$\max_{y \geq 0} \{-p_y \cdot y + U(y)\}$$

which does not depend on w.

(If we let $y(p_y)$ denote the solution to this last problem, note that the result depends on the fact that the bundle $(w - p_y \cdot y(p_y), y(p_y))$ is feasible. In a standard problem with the constraint $x_1 \geq 0$, this won't always be true. Here, we eliminated the nonnegativity constraint on x_1 to get around the problem. If x_1 were constrained to be positive, then the demand for goods 2 through k would only be independent of wealth if w were high enough for the constraint on x_1 to not bind.)

(b) *Show that indirect utility can be written as $v(p, w) = w + \tilde{v}(p)$ for some function $\tilde{v}$.*

Well,

$$v(p, w) \quad = \quad \max_{(x_1, y) \in X} \{w - p_y \cdot y + U(y)\} \quad = \quad w + \max_{y \geq 0} \{-p_y \cdot y + U(y)\}$$

so we can simply define

$$\tilde{v}(p) \quad = \quad \max_{y \geq 0} \{-p_y \cdot y + U(y)\}$$

and we're done.

(c) *Show the expenditure function can be written as $e(p,u) = u - f(p)$ for some function f.*

Since we already have $v(p,w) = w + \tilde{v}(p)$, we can note that $v(p, e(p,u)) = u$, meaning

$$u = v(p, e(p,u)) = e(p,u) + \tilde{v}(p)$$

and therefore

$$e(p,u) = u - \tilde{v}(p)$$

(d) *Show that the Hicksian demand for goods 2 through k does not depend on target utility.*

Working from $e(p,u) = u - \tilde{v}(p)$, we get

$$h_i(p,u) = \frac{\partial e}{\partial p_i}(p,u) = \frac{\partial(u - \tilde{v}(p))}{\partial p_i} = -\frac{\partial \tilde{v}}{\partial p_i}(p)$$

which does not depend on u. Or alternatively, for good $i \geq 2$, we already showed that Marshallian demand x_i does not depend on wealth, and therefore

$$h_i(p,u) = x_i(p, e(p,u)) = x_i(p, e(p,u')) = h_i(p,u')$$

for any two target utility levels u and u'.

(e) *Show that Compensating Variation and Equivalent Variation are the same when the price of good $i \neq 1$ changes, and also equal to Consumer Surplus.*

For the first part,

$$CV = \int_{p_i^1}^{p_i^0} h_i(p,u^0) dp_i = \int_{p_i^1}^{p_i^0} h_i(p,u^1) dp_i = EV$$

since we just showed Hicksian demand for good $i \neq 1$ does not depend on u, and therefore $h_i(p,u^0) = h_i(p,u^1)$.

For the last part, since

$$x_i(p,w) = h_i(p, v(p,w))$$

and h_i does not depend on u,

$$x_i(p,w) = h_i(p, v(p,w)) = h_i(p,u^0)$$

(or $h_i(p,u^1)$), so

$$CS = \int_{p_i^1}^{p_i^0} x_i(p,w) dp_i = \int_{p_i^1}^{p_i^0} h_i(p,u^0) dp_i = CV = EV$$