Problem Set #2

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Question 1

The social planner in this problem seeks to maximize utility subject to the production function, resource constraint, and law of motion:

$$\max_{\{C_t, K_t, I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log\left(C_t\right), \text{ s.t. } Y_t = AK_t^{\alpha},\ K_{t+1} = K_t^{1-\delta} I_t^{\delta},\ Y_t = C_t + I_t$$

Combining the production function, resource constraint, and law of motion gives the following Lagrangian function:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \log\left(C_t\right) - \lambda_t \left(K_{t+1} - K_t^{1-\delta}(AK_t^{\alpha} - C_t)^{\delta}\right)$$

Which has the following first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{\beta^t}{C_t} - \lambda_t \delta K_t^{1-\delta}\left(AK_t^{\alpha} - C_t\right)^{\delta-1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_t + \lambda_{t+1}\left[(1-\delta)K_{t+1}^{-\delta}\left(AK_{t+1}^{\alpha} - C_{t+1}\right)^{\delta} + \delta K_{t+1}^{1-\delta} \alpha A K_{t+1}^{\alpha-1}\left(AK_{t+1}^{\alpha} - C_{t+1}\right)^{\delta-1}\right] = 0$$

$$\Rightarrow \frac{C_{t+1}}{C_t} = \beta\left(K_t^{1-\delta} I_t^{\delta-1}\right)\left(K_{t+1}^{\delta-1} I_{t+1}^{1-\delta}\right)\left(K_{t+1}^{-\delta} I_{t+1}^{\delta}\right)\left(1 - \delta + \delta\alpha A K_{t+1}^{\alpha} I_{t+1}^{-1}\right)$$

Which simplifies to the following Euler equation:

$$\frac{C_{t+1}}{C_t} = \beta\left(1 - \delta + \delta\alpha A K_{t+1}^{\alpha} I_{t+1}^{-1}\right)$$

Question 2

The two equations that pin down the steady-state of this model are the Euler equation from (1) and the combined production function, law of motion, and resource constraint (which we will simply refer to as the consolidated resource contraint):

$$\begin{aligned}
C_{t+1} &= \beta C_t \left(1 - \delta + \delta\alpha A K_{t+1}^{\alpha} \left(A K_{t+1}^{\alpha} - C_{t+1}\right)^{-1}\right) \\
K_{t+1} &= K_t^{1-\delta} \left(A K_t^{\alpha} - C_t\right)^{\delta}
\end{aligned}$$

Question 3

Before log-linearizing this system, let us first simplify the steady-state values of the model's variables using the two equations we have. First, take the law of motion:

$$\begin{aligned}
\overline{K} &= \overline{K}^{1-\delta}\overline{I}^{\delta} \\
\overline{K}^{\delta} &= \overline{I}^{\delta} \Rightarrow \overline{K} = \overline{I}
\end{aligned}$$

Then, if we plug this equality into the resource constraint, we get:

$$A\overline{K} = \overline{C} + \overline{K} \Rightarrow \overline{C} = \overline{K}\left(A\overline{K}^{\alpha-1} - 1\right)$$

To simply notation, denote $\phi = A\overline{K}^{\alpha-1}$. Finally, using the law of motion equality and the Euler equation, we can determine:

$$\overline{C} = \beta\overline{C}(1 - \delta + \delta\alpha A\overline{K}^{\alpha}\overline{I}^{-1} \rightarrow 1 = \beta(1 - \delta + \delta\alpha\phi)$$

Or, $\beta^{-1} = 1 - \delta + \delta\alpha\phi$.

The Euler Equation: We can log-linearize the Euler equation in stages. Let $I_{t+1} = AK_{t+1}^{\alpha} - C_{t+1}$ and $Z_t = 1 - \delta + \delta\alpha A K_{t+1}^{\alpha} I_{t+1}^{-1}$. Then:

$$\begin{aligned}
c_{t+1} &= c_t + z_t \\
z_t &= \alpha k_{t+1} - i_{t+1} \\
i_{t+1} &= \left(\frac{A\overline{K}^{\alpha}}{\overline{I}}\right)\alpha k_{t+1} - \left(\frac{\overline{C}}{\overline{I}}\right)c_{t+1} \\
\Rightarrow c_{t+1} &= c_t + \alpha k_{t+1} - \left(\frac{A\overline{K}^{\alpha}}{\overline{K}}\right)\alpha k_{t+1} + \left(\frac{\overline{K}\,(\phi - 1)}{\overline{K}}\right)c_{t+1} \\
(2 - \phi)c_{t+1} &= c_t + \alpha(1 - \phi)k_{t+1} \\
c_{t+1} &= \left(\frac{1}{2 - \phi}\right)c_t + \left(\frac{\alpha(1 - \phi)}{2 - \phi}\right)k_{t+1}
\end{aligned}$$

The Consolidated Resource Constraint: Once again letting $I_t = AK_t^\alpha - C_t$:

$$\begin{aligned}
k_{t+1} &= (1-\delta)k_t + \delta i_t \\
i_t &= \alpha\phi k_t - (\phi - 1)c_t \\
k_{t+1} &= (1-\delta)k_t + \delta\left(\alpha\phi k_t - (\phi-1)c_t\right) \\
k_{t+1} &= (1-\delta+\delta\alpha\phi)k_t - \delta(\phi-1)c_t \\
k_{t+1} &= \beta^{-1}k_t - \delta(\phi-1)c_t
\end{aligned}$$

Question 4

The log-linearized consolidated resource constraint from (3) is already given as a function of one state variable (k_t) and one choice variable (c_t). We can plug this function into k_{t+1} in the log-linearized Euler equation from (3) to transform it into a function with one state variable and one choice variable, as well:

$$\begin{aligned}
c_{t+1} &= \left(\frac{1}{2-\phi}\right)c_t + \left(\frac{\alpha(1-\phi)}{2-\phi}\right)\left[\beta^{-1}k_t - \delta(\phi-1)c_t\right] \\
c_{t+1} &= \left(\frac{1+\alpha\delta(1-\phi)^2}{2-\phi}\right)c_t + \left(\alpha\beta^{-1}\frac{1-\phi}{2-\phi}\right)k_t
\end{aligned}$$

We begin the Blanchard-Kahn method by writing the system in vector-matrix form:

$$x_{t+1} = \begin{pmatrix} c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1+\alpha\delta(1-\phi)^2}{2-\phi} & \alpha\beta^{-1}\frac{1-\phi}{2-\phi} \\ -\delta(\phi-1) & \beta^{-1} \end{pmatrix} \begin{pmatrix} c_t \\ k_t \end{pmatrix} = Bx_t$$

Next, we can find the eigenvalues of B by solving $B - \lambda I = 0$:

$$\left(\frac{1+\alpha\delta(1-\phi)^2}{2-\phi} - \lambda\right)(\beta^{-1}-\lambda) + \delta(\phi-1)\alpha\beta^{-1}\frac{1-\phi}{2-\phi} = 0$$

Now, let $B = Q\Lambda Q^{-1}$

Question 5

Question 6

Question 7

Question 8