Econ 711 – Fall 2020 – Problem Set 4 – Solutions

Question 1. Choice rules from preferences

Let X be a choice set and $\succsim$ a complete and transitive preference relation on X. Show that the choice rule induced by $\succsim$,

$$C(A, \succsim) \quad = \quad \{x \in A : x \succsim y \;\; \forall y \in A\}$$

must satisfy the Weak Axiom of Revealed Preference (WARP).

We want to show that if $x, y \in A \cap B$, with $x \in C(A, \succsim)$ and $y \in C(B, \succsim)$, then $x \in C(B, \succsim)$ and $y \in C(A, \succsim)$. To show $x \in C(B, \succsim)$:

- If $x \in C(A, \succsim)$ and $y \in A$, then by definition of $C(\cdot, \succsim)$, $x \succsim y$
- If $y \in C(B, \succsim)$, then $y \succsim z$ for every $z \in B$
- By transitivity, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$ for every $z \in B$ (including y)
- Since in addition we know $x \in B$, this implies $x \in C(B, \succsim)$.

Similarly, for $y \in C(A, \succsim)$:

- Similarly, if $y \in C(B, \succsim)$ and $x \in B$, then $y \succsim x$;
  if $x \in C(A, \succsim)$, then $x \succsim z$ for every $z \in A$,
  so by transitivity, $y \succsim z$ for every $z \in A$,
  and therefore (since $y \in A$) $y \in C(A, \succsim)$.

Question 2. Preferences from choice rules

Let X be a choice set and $C : \mathcal{P}(X) \to \mathcal{P}(X)$ a nonempty choice rule. Show that if C satisfies WARP, then the preference relation $\succsim_C$ defined on X by

$$x \succsim_C y \qquad \text{if and only if} \qquad \text{there exists a set } A \subseteq X \text{ such that } x, y \in A \text{ and } x \in C(A)$$

is complete and transitive, and that the choice rule it induces, $C(\cdot, \succsim_C)$, is equal to C.

For completeness, note that since C is nonempty, for any $x, y \in X$, $C(\{x, y\}) \neq \emptyset$, and therefore either $x \in C(\{x, y\})$ or $y \in C(\{x, y\})$ (or both). In the first case, that establishes that $x \succsim_C y$; in the second case, $y \succsim_C x$; thus, at least one of these must be true, so preferences are complete.

For transitivity, suppose that $x \succsim_C y$ and $y \succsim_C z$; we want to show that $x \succsim_C z$, meaning we need to find a set D containing x and z such that $x \in C(D)$. Let $D = \{x, y, z\}$. Since C is non-empty, $C(D)$ must contain either x, y, or z (or more than one of them). Consider these three cases:

- If $C(D)$ contains x, our proof is done – D would be a set containing x and z with $x \in C(D)$, proving $x \succsim_C z$.
- Suppose $C(D)$ contains y. Let A be a set containing x and y with $x \in C(A)$; since $x \succsim_C y$, we know such a set must exist. So x and y are both in D and both in A, with $y \in C(D)$ and $x \in C(A)$. Since C satisfies WARP, this implies $x \in C(D)$, which proves $x \succsim_C z$.

- Finally, suppose $C(D)$ contains z. Let B be a set containing y and z with $y \in C(B)$, which must exist because $y \succsim_C z$. y and z are both in D and B, with $y \in C(B)$ and $z \in C(D)$; by WARP, $y \in C(D)$. We showed in the last step that if $y \in C(D)$ then $x \in C(D)$ as well; so $x \in C(D)$, proving $x \succsim_C z$.

Finally, we need to show that for any set A, $C(A, \succsim_C) = C(A)$, meaning $x \in C(A, \succsim_C)$ implies $x \in C(A)$ and vice versa:

- First, fix $A \subseteq X$, and pick $x \in C(A)$; we'll show $x \in C(A, \succsim_C)$.
  For any other $z \in A$, A is a set containing x and z with $x \in C(A)$, so $x \succsim_C z$;
  so $x \succsim_C z$ for every $z \in A$, implying $x \in C(A, \succsim_C)$.
- Second, fix $A \subseteq X$, and pick $x \in C(A, \succsim_C)$; we'll show $x \in C(A)$.
  We know $C(A)$ is nonempty, so pick $y \in C(A)$. If $y = x$, we're done.
  If not, we know $x \in C(A, \succsim_C)$, so by definition, $x \succsim_C z$ for every $z \in A$,
  meaning $x \succsim_C y$, meaning there exists a set B with $x, y \in B$ and $x \in C(B)$;
  since x and y are both in A and B, with $x \in C(B)$ and $y \in C(A)$,
  WARP implies $x \in C(A)$.

Thus, $C(A) = C(A, \succsim_C)$, or $\succsim_C$ induces the choice rule $C(\cdot)$.

Question 3. Choice over finite sets

Let X be a **finite** *set, and $\succsim$ a complete and transitive preference relation on X.*

(a) *Show that the induced choice rule $C(\cdot, \succsim)$ is nonempty – that $C(A, \succsim) \neq \emptyset$ if $A \neq \emptyset$.*

We want to show that if $A \neq \emptyset$, then $C(A, \succsim) \neq \emptyset$; and we'll show this by induction on $|A|$, the number of elements in A. When $|A| = 1$, $A = \{x\}$; by completeness, $x \succsim x$, and therefore $x \succsim y$ for every $y \in A$, so $x \in C(A)$, proving the "base case".

So suppose we know that $C(\cdot, \succsim)$ is nonempty on all sets with k or fewer elements, and let $|A| = k+1$. Pick some element $z \in A$, and let $B = A - \{z\}$. Note that $|B| = k$, and therefore we know $C(B, \succsim) \neq \emptyset$; choose $x \in C(B, \succsim)$. We know $x \succsim y$ for every $y \in B$.

By completeness, either $x \succsim z$ or $z \succsim x$. If $x \succsim z$, then $x \succsim y$ for every $y \in A$, and therefore $x \in C(A, \succsim)$. If $z \succsim x$, then by transitivity, $z \succsim y$ for every $y \in B$, and therefore $z \succsim y$ for every $y \in A$, and therefore $z \in C(A, \succsim)$. So either way, $C(A, \succsim)$ is nonempty, proving the inductive step.

(b) *Show that a utility representation exists.*

We'll prove a slightly stronger result: that complete, transitive preferences over a finite set X can be represented by a utility function whose range is $\{1, 2, 3, \ldots, |X|\}$. We will prove this by induction on the size of the set X. If $|X| = 1$, then $X = \{x\}$, and defining $u(x) = 1$ allows us to represent the preferences $x \succsim x$ (since $u(x) = u(x)$).

So suppose we know that complete, transitive preferences over a set of size k or less can be represented by a utility function with range $\{1, 2, \ldots, k\}$. Let X be a set of size $k+1$. We just proved in part (a) that $C(X, \succsim)$ is nonempty, which means that $Y = X - C(X, \succsim)$ is a set with at most k elements. By our inductive assumption, $\succsim$ has a utility representation on Y with range $\{1, 2, \ldots, k\}$; let v be such a utility function. Now define a new function u on X by

$$u(x) \quad = \quad \begin{cases} k+1 & \text{if } x \in C(X, \succsim) \\ v(x) & \text{otherwise} \end{cases}$$

We need to show that $u(\cdot)$ represents the preferences $\succsim$, meaning $u(x) \geq u(y)$ if and only if $x \succsim y$.

First, suppose $x \succsim y$; we'll show this implies $u(x) \geq u(y)$

- We'll consider three cases: $y \in C(X, \succsim)$, $x \in C(X, \succsim)$, and $x, y \notin C(X, \succsim)$.
- If $y \in C(X, \succsim)$, then $y \succsim z$ for every $z \in X$;
  since $x \succsim y$, by transitivity, $x \succsim z$ for every $z \in X$, so $x \in C(X, \succsim)$.
  In this case, $u(x) = u(y) = k + 1$, so $u(x) \geq u(y)$.
- If $x \in C(X, \succsim)$, then $u(x) = k + 1$; since $u(y) \leq k + 1$, $u(x) \geq u(y)$.
- Finally, if $x, y \notin C(X, \succsim)$, then $u(x) = v(x)$ and $u(y) = v(y)$.
  Since $v(\cdot)$ represents the preferences $\succsim$ and $x \succsim y$,
  $v(x) \geq v(y)$, and therefore $u(x) \geq u(y)$.

Thus, $x \succsim y$ implies $u(x) \geq u(y)$. Now suppose $u(x) \geq u(y)$; we'll show this implies $x \succsim y$.

- If $u(x) = k + 1$, then $x \in C(X, \succsim)$, meaning $x \succsim z$ for every $z \in x$, meaning $x \succsim y$.
- If $u(x) < k + 1$, then $u(y) \leq u(x) < k + 1$ as well, so $u(x) = v(x)$ and $u(y) = v(y)$.
  Since $v(\cdot)$ represents preferences $\succsim$, $v(x) \geq v(y)$ implies $x \succsim y$.

Thus, $u(x) \geq u(y)$ implies $x \succsim y$.

So $u(x) \geq u(y)$ if and only if $x \succsim y$; so the utility function $u(\cdot)$ represents the preferences $\succsim$.