Problem Set #2

Danny Edgel
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Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

3.24) The table below displays the results of each of the regressions that must be estimated for this question. The first column displays the estimates for equation (3.50), and the fourth column displays the estimates for the residual approach, using the residuals from the regressions displayed in columns (2) and (3).

VARIABLES	(1) log(wage)	(2) log(wage)	(3) education	(4) $\hat{\varepsilon}_{wage}$
Education	0.144***			
	(0.0116)			
Experience	0.0426***	0.0448***	0.0150	
	(0.0122)	(0.0153)	(0.0643)	
Experience^2	-0.0951***	-0.116***	-0.143	
	(0.0349)	(0.0438)	(0.184)	
$\hat{\varepsilon}_{educ}$				0.144***
				(0.0116)
Constant	0.531***	2.679***	14.89***	-2.33e-09
	(0.190)	(0.0973)	(0.409)	(0.0341)
Observations	267	267	267	267
R-squared	0.389	0.033	0.014	0.369
Sum-of-squared Errors	82.50	130.7	2314	82.50

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

(a) As shown in the table above, $R^2 = 0.389$ and the sum of squared errors is 82.50.

(b) Comparing the coefficients from the "Education" row in column (1) and the $\varepsilon_{\hat{educ}}$ row of column (4) shows that the coefficient on education is the same using either a partition regression or a residual regression. They each equal 0.144.

(c) The bottom three rows of the table above provide summary statistics for each regression. These show that the sum-of-squared errors for the regressions in (a) and (b) are the same, but the R^2 is slightly smaller when the residual regression approach is used than when a partition regression is used. This is to be expected, as using a regression with more independent variables always weakly increases R^2, and the residual regression uses two fewer independent variables than the partition regression.

3.25) Each of the values below is rounded to the nearest thousandth.

(a) $\sum_{i=1}^{n} \hat{e}_i = 0$

(b) $\sum_{i=1}^{n} X_{1i}\hat{e}_i = 0$

(c) $\sum_{i=1}^{n} X_{2i}\hat{e}_i = 0$

(d) $\sum_{i=1}^{n} X_{1i}^2\hat{e}_i = 133.133$

(e) $\sum_{i=1}^{n} X_{2i}^2\hat{e}_i = 0$

(f) $\sum_{i=1}^{n} \hat{Y}_i\hat{e}_i = 0$

(g) $\sum_{i=1}^{n} \hat{e}_i^2 = 82.505$

Question 2

7.2) To find the limit of $\hat{\beta}$ as $n \to \infty$, we can first rewrite $\hat{\beta}$ in terms of expectation:

$$
\begin{aligned}
\hat{\beta} &= \left(\sum_{i=1}^{n} X_i X_i' + \lambda I_k \right)^{-1} \left(\sum_{i=1}^{n} X_i Y_i \right) \\
&= \left(\sum_{i=1}^{n} X_i X_i' + \lambda I_k \right)^{-1} n \left(\frac{1}{n} \sum_{i=1}^{n} X_i Y_i \right) \\
&= \left(\frac{1}{n} \sum_{i=1}^{n} X_i X_i' + \frac{1}{n} \lambda I_k \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} X_i Y_i \right)
\end{aligned}
$$

And, recognizing $Y_i = X_i\beta + \varepsilon_i$,

$$\begin{aligned}\hat{\beta} \to_p & \mathbb{E}\left(X_iX_i' + 0I_k\right)^{-1}\mathbb{E}\left(X_i(X_i\beta + \varepsilon_i)\right)\\ =& \mathbb{E}\left(X_iX_i'\right)^{-1}\left[\mathbb{E}\left(X_iX_i'\beta\right) + \mathbb{E}\left(X_i\varepsilon_i\right)\right]\\ =& \beta\,\mathbb{E}\left(X_iX_i'\right)^{-1}\mathbb{E}\left(X_iX_i'\right)\\ =& \beta\end{aligned}$$

7.3) Let $\lambda = cn$ where $c > 0$. Then,

$$\begin{aligned}\hat{\beta} =& \left(\frac{1}{n}\sum_{i=1}^{n} X_iX_i' + \frac{1}{n}(cn)I_k\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_iY_i\right)\\ \hat{\beta} \to_p & \mathbb{E}\left(X_iX_i' + cI_k\right)^{-1}\mathbb{E}\left(X_i(X_i\beta + \varepsilon_i)\right)\\ =& \mathbb{E}\left(X_iX_i' + cI_k\right)^{-1}\left[\mathbb{E}\left(X_iX_i'\beta\right) + \mathbb{E}\left(X_i\varepsilon_i\right)\right]\\ =& \mathbb{E}\left(X_iX_i' + cI_k\right)^{-1}\beta\,\mathbb{E}\left(X_iX_i'\right)\\ =& \mathbb{E}\left(X_iX_i' + cI_k\right)^{-1}\beta\left[\mathbb{E}\left(X_iX_i'\right) + cI_k - cI_k\right]\\ =& \mathbb{E}\left(X_iX_i' + cI_k\right)^{-1}\left[\mathbb{E}\left(X_iX_i'\right) + cI_k\right]\beta - \lambda\beta\\ =& \beta - c\,\mathbb{E}\left(X_iX_i' + cI_k\right)^{-1}\beta\end{aligned}$$

7.4) (a) $\mathbb{E}(X_1) = \frac{4}{8}(-1) + \frac{4}{8}(1) = 0$

(b) $\mathbb{E}(X_1^2) = \frac{4}{8}(-1)^2 + \frac{4}{8}(1)^2 = 1$

(c) $\mathbb{E}(X_1X_1) = \frac{3}{8}(1)(1) + \frac{3}{8}(-1)(-1) + \frac{1}{8}(1)(-1) + \frac{1}{8}(-1)(1) = \frac{6}{8} - \frac{2}{8} = \frac{1}{2}$

(d) $\mathbb{E}(e^2) = \left(\frac{3}{4}\right)\left(\frac{5}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{15}{16} + \frac{1}{16} = 1$

(e) $\mathbb{E}(X_1^2e^2) = \left(\frac{3}{4}\right)\left(\frac{5}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = 1$

(f) $\mathbb{E}(X_1X_1e^2) = \left(\frac{3}{4}\right)\left(\frac{5}{4}\right)(1) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(-1) = \frac{15}{16} - \frac{1}{16} = \frac{7}{8}$

Question 3

7.8 Using the notation from 6.8, where $Z_n \to_p 0 \equiv Z_n = o_p(a_n)$

$$\sqrt{N}\left(\hat{\sigma}^2 - \sigma^2\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 - \sigma^2\right)$$

Question 4

I show below that both estimators, $\hat{\beta}$ and $\widetilde{\beta}$ are consistent for β.

$$\begin{aligned}
\hat{\beta} =& \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2} = \frac{n^{-1}\sum_{i=1}^n X_i(X_i'\beta + e_i)}{n^{-1}\sum_{i=1}^n X_i^2} \\
\rightarrow_p & \frac{\mathbb{E}(X_i(X_i'\beta + e_i))}{\mathbb{E}(X_i X_i')} = \frac{\mathbb{E}(X_i X_i'\beta) + \mathbb{E}(X_i' e_i))}{\mathbb{E}(X_i X_i')} \\
=& \beta \frac{\mathbb{E}(X_i X_i')}{\mathbb{E}(X_i X_i')} + \frac{\mathbb{E}(X_i' e_i))}{\mathbb{E}(X_i X_i')} = \beta(1) + 0
\end{aligned}$$

$$\therefore \hat{\beta} \rightarrow_p \beta$$

$$\begin{aligned}
\widetilde{\beta} =& \frac{1}{n}\sum_{i=1}^n \frac{Y_i}{X_i} = \frac{1}{n}\sum_{i=1}^n \frac{X_i'\beta + e_i}{X_i} = \frac{1}{n}\sum_{i=1}^n \beta\left(1 + \frac{e_i}{X_i}\right) \\
\rightarrow_p & \beta\,\mathbb{E}\left(1 + \frac{e_i}{X_i}\right) = \beta\left(1 + \mathbb{E}\left(\frac{1}{X_i}\,\mathbb{E}(e_i|X)\right)\right) = \beta(1+0)
\end{aligned}$$

$$\therefore \widetilde{\beta} \rightarrow_p \beta$$

Question 5

7.10

Question 6

7.13)

7.14)

7.15)

Question 7

7.17

Question 8

7.19

Question 9

1.
2.

3.

4.

5.

6.