Problem Set #3

Danny Edgel
Econ 709: Economic Statistics and Econometrics I
Fall 2020

September 23, 2020

Collaborated with Sarah Bass, Emily Case, Michael Nattinger, and Alex Von Hafften

Question 1

A random point (X, Y) is distributed uniformly on the square with vertices $(1, 1)$, $(1, -1)$, (-1.1), and $(-1, -1)$. That is, the joint PDF is $f(x, y) = 1/4$ on the square and $f(x, y) = 0$ outside the square. Determine the probability of:

(a) $\mathbf{X^2 + Y^2 < 1}$

$P(X^2 + Y^2 < 1)$ is the area of the circle inscribed within the square. Therefore, $P(X^2 + Y^2 < 1) = \frac{\pi}{4}$.

(b) $|\mathbf{X} + \mathbf{Y}| < \mathbf{2}$

$P(|X + Y| < 2) = P(-2 < X + Y < 2)$. Note that $|X + Y| = 2$ only if $X = Y = -1$ or $X = Y = 1$. Since X and Y are continuous, $P(X = 1) = P(Y = 1) = P(X = -1) = P(Y = -1) = 0$. Therefore, $P(|X + Y| < 2) = 0$.

Question 2

Let the joint PDF of X and Y be given by $f(x, y) = g(x)h(y)\ \forall x, y \in \mathbb{R}$. Let a denote $\int_{-\infty}^{\infty} g(x)dx$ and b denote $\int_{-\infty}^{\infty} h(x)dx$.

(a) **What conditions should a and b satisfy in order for $f(x, y)$ to be a bivariate PDF?**

If f is a bivariate PDF, then $\int_{-\infty}^{\infty} f(x, y)dxdy = 1$. Then,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)dxdy = \left(\int_{-\infty}^{\infty} g(x)dx\right)\left(\int_{-\infty}^{\infty} h(y)dy\right) = ab$$

Thus, $ab = 1$ if f is a bivariable PDF.

(b) **Find the marginal PDF of X and Y.**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} g(x)h(y)dy = bg(x)$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} g(x)h(y)dx = ah(y)$$

(c) **Show that X and Y are independent.**

X and Y are independent if and only if $f(x,y) = f_X(x)f_Y(y)$. From (a) and (b), we can derive:

$$f_X(x)f_Y(y) = ag(x)bh(y) = g(x)h(y) = f(x,y)$$

Thus, X and Y are independent.

Question 3

Let the joint PDF of X and Y be given by

$$\mathbf{f(x,y)} = \begin{cases} cxy & \text{if } x, y \in [0,1],\ x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) **Find the value of c such that $f(x,y)$ is a joint PDF.**

If f is a PDF, then $\int_{-\infty}^{\infty} f(x,y)dxdy = 1$. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= 1 \\ \int_0^1\int_0^{1-x} cxydydx &= 1 \\ \int_0^1 cx\frac{1}{2}[y^2]_0^{1-x}dx &= 1 \\ \int_0^1 cx\frac{1}{2}(1-x)^2dx &= 1 \\ c\left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^1 &= 2 \\ \frac{1}{12}c &= 2 \\ c &= 24 \end{aligned}$$

(b) **Find the marginal distributions of X and Y.**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \int_0^{1-x} cxydydx = \frac{1}{2}cx(1-x)^2$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \int_0^{1-y} cxydxdy = \frac{1}{2}cy(1-y)^2$$

(c) **Are X and Y independent? Compare your answer to Problem 2 and discuss.**

X and Y are independent if and only if $f(x,y) = f_X(x)f_Y(y)$. From (a) and (b), we can derive:

$$f_X(x)f_Y(y) = \frac{1}{4}c^2x(1-x)^2y(1-y)^2 \neq cxy$$

Thus, X and Y are **not** independent.

Question 4

Show that any random variable is uncorrelated with a constant.

Let X be a random variable and a be a constant. Then,

$$Cov(X,a) = E[Xa] - E[X]E[a] = aE[X] - aE[X] = 0$$

Thus, $Corr(X,a) = Cov(X,a)/\sqrt{Var(X)Var(a)} = 0$, so X and a are uncorrelated.

Question 5

Let X and Y be independent random variables with means μ_X, μ_Y, and variances σ_X^2, σ_Y^2. Find an expression for the correlation of XY and Y in terms of these means and variances.

Given the definition of correlation and covariance, we have:

$$Corr(XY, Y) = \frac{E(XY^2) - E(XY)E(Y)}{\sqrt{Var(XY)Var(Y)}}$$

Separately, since X and Y are independent, we can solve:

$$\begin{aligned}
E(XY^2) - E(XY)E(Y) &= E(X)E(Y^2) - E(X)E(Y)^2 = E(X)(E(Y^2) - E(Y)^2) = \mu_X \sigma_Y^2 \\
Var(XY)Var(Y) &= (E(X^2Y^2) - E(XY)^2)\sigma_Y^2 \\
&= ((\sigma_X^2 - \mu_X^2)(\sigma_Y^2 - \mu_Y^2) - E(X)^2E(Y)^2)\sigma_Y^2 \\
&= (\sigma_X^2\sigma_Y^2 - \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \mu_X^2\mu_Y^2 - \mu_X^2\mu_Y^2)\sigma_Y^2 \\
&= (\sigma_X^2\sigma_Y^2 - \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2)\sigma_Y^2
\end{aligned}$$

Thus, we can write the correlation as:

$$Corr(XY, Y) = \frac{\mu_X \sigma_Y^2}{\sqrt{(\sigma_X^2\sigma_Y^2 - \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2)\sigma_Y^2}} = \frac{\mu_X \sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2 - \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2}}$$

Question 6

Prove the following: For any random vector $(X_1, X_2, ..., X_n)$,

$$\mathbf{Var}\left(\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{X_i}\right) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{Var(X_i)} + \mathbf{2} \sum_{\mathbf{i \leq i < j \leq n}} \mathbf{Cov(X_i, Y_j)}$$

Question 7

Suppose that X and Y are joint normal, i.e., they have the joint PDF:

$$\mathbf{f(x,y)} = \frac{\mathbf{1}}{\mathbf{2}\pi\sigma_\mathbf{X}\sigma_\mathbf{Y}\sqrt{\mathbf{1}-\rho^\mathbf{2}}}\exp(-(\mathbf{2}(\mathbf{1}-\rho^\mathbf{2}))^{-\mathbf{1}}(\mathbf{x^2}/\sigma_\mathbf{X}^\mathbf{2} - \mathbf{2}\rho\mathbf{xy}/\sigma_\mathbf{X}\sigma_\mathbf{Y} + \mathbf{y^2}/\sigma_\mathbf{Y}^\mathbf{w}))$$

(a) **Derive the marginal distribution of X and Y, and observe that both are normal distributions.**

(b) **Derive the conditional distribution of Y given $X = x$, Observe that it is also a normal distribution.**

(c) **Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_)$, and then show that X and Z are independent.**

Question 8

Consider a function $g : \mathbb{R} \to \mathbb{R}$. Recall that the inverse image of a set A, denoted $g^{-1}(A)$, is $g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$.Let there be two functions, $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$. Let X and Y be two random variables that are independent. Suppose that g_1 and g_2 are both Borel-measurable, which means that $g_1^{-1}(A)$ and $g_2^{-1}(A)$ are both in the Borel σ-field whenenver A is in the Borel σ-field. Show that the two random variables $Z := g_1(X)$ and $W := g_2(Y)$ are independent. (Hint: use the 1st or the 2nd definition of independence.)