Problem Set #2

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Econ 712: Macroeconomics I
Fall 2020

November 18, 2020

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Question 1

(a) In a balanced growth path, households' utility and firms' profits are maximized. Thus, to find a system of equations that C_0, N_0, w_0, and r_0 must solve in a balanced growth path, we must characterize and solve the household and firm problems. To start, the household problem is given by

$$\max_{\{C_t, K_{t+1}^s, N_t^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_t u(C_t, 1-N_t) \text{ s.t. } \sum_{t=0}^{\infty} p_t \left(C_t + K_{t+1}\right) \leq \sum_{t=0}^{\infty} p_t \left(r_t K_t + w_t A_t N_t\right) + \pi_0$$

And the firm's problem is:

$$\max_{\{K_t^d, N_t^d\}_{t=0}^{\infty}} \pi_0 = \sum_{t=0}^{\infty} p_t \left(Y_t - r_t K_t^d - w_t A_t N_t^d\right) \text{ s.t. } Y_t \leq F(K_t^d, A_t N_t^d)$$

Where, recognizing that π_0 is monotone in Y_t, the firm's problem can be reduced to:

$$\max_{\{K_t^d, N_t^d\}_{t=0}^{\infty}} \pi_0 = \sum_{t=0}^{\infty} p_t \left(F(K_t^d, A_t N_t^d) - r_t K_t^d - w_t A_t N_t^d\right)$$

Since the firm is a price-taker whose decision in each period has no bearing on its conditions in any other period, the firm's problem can be solved by solving a single arbitrary period, t, using first-order conditions. For all choice variables, X, let $\frac{X_t}{A_t} = x_t$:

$$\max_{\{k_t^d, N_t^d\}_{t=0}^{\infty}} \pi_0 = \sum_{t=0}^{\infty} A_t p_t \left(F(k_t^d, N_t^d) - r_t k_t^d - w_t N_t^d\right)$$

$$\begin{aligned}
K_t^d: \quad & A_t p_t F_K(k_t^d, N_t^d) - A_t p_t r_t = 0 \\
& F_K(k_t^d, N_t^d) = r_t \\
N_t^d: \quad & p_t F_N(k_t^d, N_t^d) - p_t w_t = 0 \\
& F_N(k_t^d, N_t^d) = w_t
\end{aligned}$$

This solution ensures that that $\pi_0 = 0$. Assuming that the utility function is concave, strictly increasing, and differentiable, the constrating of the household problem holds with equality and an interior solution exists. Given the firm's solution, we can solve the constraint as:

$$c_t = (r_t + 1 - \delta)k_t^s + w_t N_t^s - k_{t+1}^s$$

Such that the household's problem has the Lagrangian function:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t u(A_t c_t, N_t) - p_t \lambda \left(c_t + k_{t+1} - (r_t + 1 - \delta)k_t - w_t N_t\right)\right]$$

Which has the first-order conditions:

$$\begin{aligned}
c_t: \quad & \beta^t A_t u_c(A_t c_t, N_t) = p_t \lambda \\
c_{t+1}: \quad & \beta^{t+1} A_{t+1} u_c(A_{t+1} c_{t+1}, N_{t+1}) = p_{t+1} \lambda \\
N_t: \quad & -\beta^t \frac{u_N(A_t c_t, N_t)}{w_t} = p_t \lambda \\
N_{t+1}: \quad & \beta^{t+1} \frac{-u_N(A_{t+1} c_{t+1}, N_{t+1})}{w_{t+1}} = p_{t+1} \lambda \\
k_{t+1}: \quad & \frac{p_t}{p_{t+1}} = r_{t+1} + 1 - \delta
\end{aligned}$$

Where the growth of households' optimal level of consumption is dependent on wage growth:

$$-u_N(A_t c_t, N_t)/w_t = A_t u_c(A_t c_t, N_t)$$

In equilibrium, all markets clear:

$$\begin{aligned}
K_t^s &= K_t^d && \text{(Capital)} \\
N_t^s &= N_t^d && \text{(Labor)} \\
C_t + K_{t+1} - (1-\delta)K_t &= F(K_t, A_t N_t) && \text{(Goods)}
\end{aligned}$$

And each initial choice variable, N_0 and C_0, and price, w_0 and r_0, satisfies the firm and household optimization conditions for a balanced growth path:

$$\begin{aligned}
F_K(k_0, N_0) &= r_0 \\
F_N(k_0, N_0) &= w_0 \\
-\frac{u_N(A_0 c_0, N_0)}{u_c(A_0 c_0, N_0)} &= w_0 A_0 \\
c_0 + k_1 - (1-\delta)K_0 &= F(k_0, N_0)
\end{aligned}$$

(b) Let $u(C, N) = \frac{C^{1-\gamma}}{1-\gamma} h(1-N)$, where $\gamma > 0$, $\gamma \neq 1$. Then, the first-order conditions, provided in (a), can be used to derive:

$$\beta(1+g)^{1-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{h(1-N_{t+1})}{h(1-N_t)}\right) = \frac{p_{t+1}}{p_t} \tag{1}$$

$$\beta(1+g)^{1-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}\left(\frac{h'(1-N_{t+1})}{h'(1-N_t)}\right) = \left(\frac{w_{t+1}}{w_t}\right)\left(\frac{p_{t+1}}{p_t}\right) \tag{2}$$

$$r_{t+1} + 1 - \delta = \frac{p_t}{p_{t+1}} \tag{3}$$

In a balanced growth path, capital is rented at a constant $\overline{r}$. Thus, by equation (3), the price level grows at constant rate $\overline{\pi}$:

$$\overline{pi} = \frac{1}{\overline{r} + 1 - \delta}$$

Further, in a balanced growth path, $N_t = N_{t+1} = \overline{N}$. Thus, equations (1) and (2) become:

$$\beta(1+g)^{1-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} = \frac{1}{\overline{\pi}} \tag{4}$$

$$\beta(1+g)^{1-\gamma}\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma} = \left(\frac{w_{t+1}}{w_t}\right)\left(\frac{1}{\overline{\pi}}\right) \tag{5}$$

By (4), we can derive a constant growth rate of $\left(\overline{\pi}\beta(1+g)^{1-\gamma}\right)^{1/\gamma}$ for consumption and, by dividing (5) by (4), solve for the wage growth rate:

$$\frac{c_{t+1}}{c_t} = \frac{w_{t+1}}{w_t}$$

So consumption growth is equal to wage growth.

(c)

(d)

(e)

(f)

Question 2

(a) (I think this is what the state-contingent problem is)

$$\max_{\{c_t, s_t\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}\right] \text{ s.t. } x_{t+1} = A_t s_t,\ s_t \leq x_t - c_t$$

The expectations operator in this case averages utility in each period across states of the world, weighting by that state's probability. Specifically, for each t, letting $c_t = A_{t-1}s_{t-1} - s_t$:

$$\mathbb{E}\left[\beta^t \frac{(A_{t-1}s_{t-1} - s_t)^{1-\gamma}}{1-\gamma}\right] = \pi\left[\beta^t \frac{(A_h s_{t-1} - s_t)^{1-\gamma}}{1-\gamma}\right] + (1-\pi)\left[\beta^t \frac{(A_l s_{t-1} - s_t)^{1-\gamma}}{1-\gamma}\right]$$

(b) In each period, the consumer's consumption-savings problem is constrained by her savings in the last period, multiplied by draw of A she received that period. Thus, her relevant state variable is $A_{t-1}s_{t-1}$. The Bellman for this problem, letting A^{-1} and s^{-1} be the values for A and s in the prior period, is:

$$V(A^{-1}s^{-1}) = \max_s \left\{\frac{\left(A^{-1}s^{-1} - s\right)^{1-\gamma}}{1-\gamma} + \beta\left[\pi V(A_h s) + (1-\pi)V(A_l s)\right]\right\}$$

Given that $\gamma \in (0,1)$, $\frac{\left(A^{-1}s^{-1}-s\right)^{1-\gamma}}{1-\gamma}$ is clearly concave, increasing, and continuous in s^{-1}. [how do I prove this?]

(c)

(d)

Question 3

To begin, we can analytically determine the steady-state of this system, which can be used to check the policy function derived by the program. Letting α represent the exponent on capital for Cobb-Douglas production, the Bellman of the social planner's problem is:

$$V(K) = \max_{K'} \left\{\frac{\left(zK^\alpha + (1-\delta)K - K'\right)^{1-\gamma}}{1-\gamma} + \beta V(K')\right\}$$

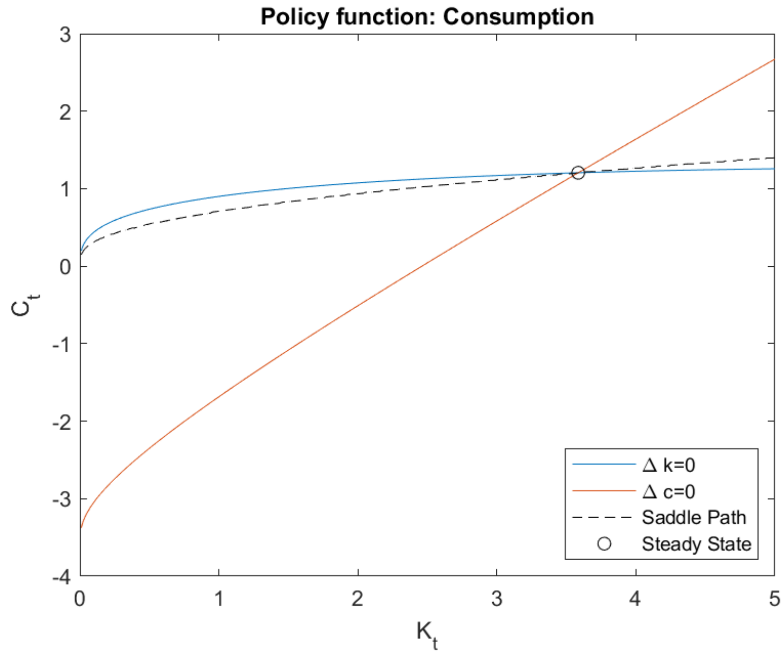
Taking first-order conditions and applying the envelope condition, we get:

$$\beta\left(z\alpha K'^{\alpha-1} + 1 - \delta\right)c'^{-\gamma} = c^{-\gamma}$$

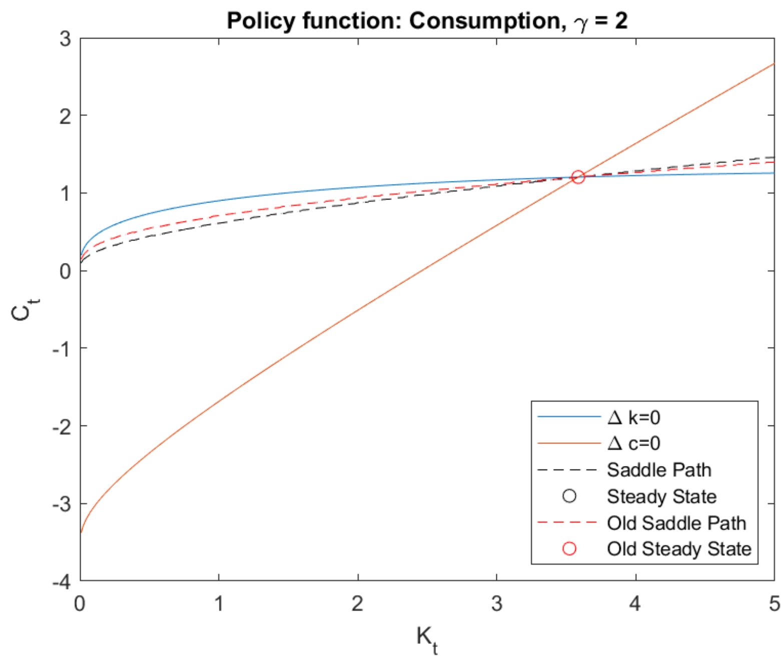
In the steady-state, $c = c' = \overline{c}$ and $K = K' = \overline{K}$, which allows us to solve:

$$\overline{K} = \left[\frac{1}{\alpha}\left(\frac{1}{\beta} + \delta - 1\right)\right]^{-\frac{1}{\alpha-1}}$$
$$\overline{c} = z\overline{K}^\alpha - \delta\overline{K}$$

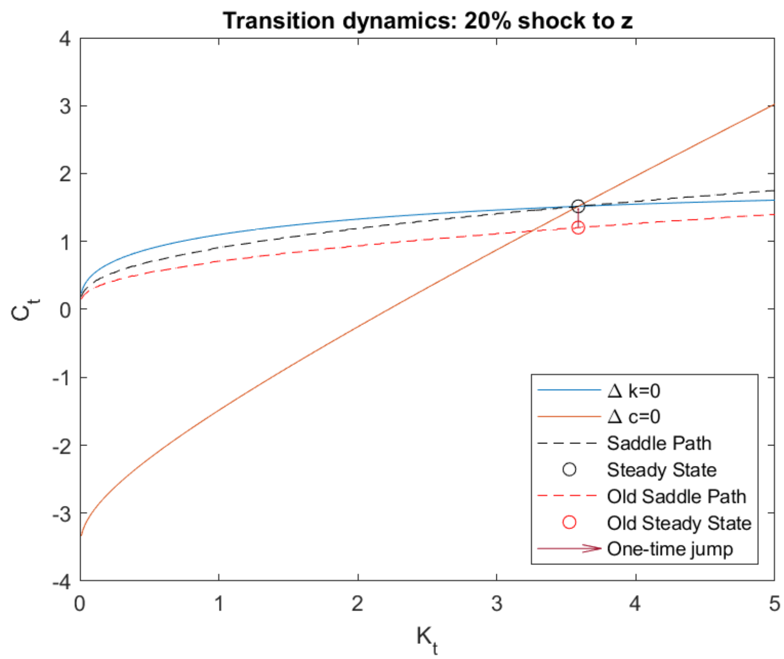
(a) The following chart displays the phase diagram for $\Delta c = 0$ and $\Delta K = 0$, alongside the optimal policy function, $c(K)$, which intersects the intersection of $\Delta c = 0$ and $\Delta K = 0$ at the steady state.



(b) When γ decreases to 1.01, the steady-state values of K and c do not change. This is unsurprising, given that γ does not appear in our analytical solution of the steady state. However, the saddle path gets steeper, meaning that one-time shocks will cause a larger change in consumption, and consumption will proceed to move to the steady state more quickly.



(c) Productivity, z, appears only in the steady-state value of consumption. Thus, a permanent, unexpected shock to productivity will change the steady-state value of consumption, but not that of capital. Since movements along the saddle path to the steady state are achieved by immediate changes to consumption but gradual accumulation or depreciation of capital, this shock to productivity will cause an immediate, one-time increase in consumption that moves the agent from the old steady-state to the new one. This is shown in the plot below.



Transition dynamics: 20% shock to z
C_t
K_t
4
3
2
1
0
-1
-2
-3
-4
0
1
2
3
4
5
Δ k=0
Δ c=0
Saddle Path
Steady State
Old Saddle Path
Old Steady State
One-time jump