Problem Set #6

Danny Edgel
Econ 711: Microeconomics I
Fall 2020

October 18, 2020

Question 1: Rationalizing Demand

Suppose we observe the following "data":

w	p	x
100	(5, 5, 5)	(12, 4, 4)
100	(7, 4, 5)	(9, 3, 5)
100	(2, 4, 1)	(27, 9, 10)
150	(7, 4, 5)	(15, 5, 5)

(a) Under Walras's law, $p_i \cdot x_i = w \; \forall i$. Then, we can calulate:

$$5 * 12 + 5 * 4 + 4 * 5 = 100 = w$$
$$7 * 9 + 4 * 3 + 5 * 5 = 100 = w$$
$$2 * 27 + 4 * 9 + 10 = 100 = w$$
$$7 * 15 + 4 * 5 + 5 * 5 = 150 = w$$

Thus, the data are consistent with Walras's Law.

(b) Given that Walras's Law is satisfied for each observation, $x^i > x^j \Rightarrow p \cdot x^i > p \cdot x^j$ for any $p >> 0$, and all price vectors in our data are strictly positive, we can conclude the following:

i. $x^3 > x^i \; \forall i \neq 3$ implies that: 1) all other goods bundles were affordable at p^3, and 2) x^3 was unaffordable at all $p^i \neq p^3$. Thus, $x^3 \succ^D x^i \; \forall i \neq 3$.

ii. $x^4 > x^1$ implies that 1) x^1 was affordable at p^4, and 2) x^4 was not affordable at p^1. Thus, $x^4 \succ^D x^1$.

iii. Since $p^4 = p^2$ and $w^4 > w^2$, we know 1) x^2 was affordable when x^4 was chosen, and 2) x^4 was not affordable when x^2 was chosen. Thus, $x^4 \succ^D x^2$.

iv. $p^1 \cdot x^2 = 85 < 100$, so $p^1 \cdot x^2 < p^2 \cdot x^2$, and x^2 was chosen at p^2. Therefore, $x^2 \succ^D x^1$

v. $p^2 \cdot x^1 = 120 > w^2$, so x^1 was not affordable when x^2 was chosen. Thus, $\neg(x^2 \succsim^D x^1)$

Taken together, these preference relations indicate:

$$x^3 \succ^D x^4 \succ^D x^1 \succ^D x^2$$

Where it is not possible to have any preference relation "loops". Therefore, these data satisfy GARP. By Afrias's theorem, satisfying GARP is a sufficient condition for conluding that these data can be rationalized by a continuous, monotonic, and concave utility function.

Question 2: Aggregating Demand

Suppose there are n consumers, where consumer $i \in \{1, 2, ..., n\}$ has the indirect utility function

$$v^i(p, w_i) = a_i(p) + b(p)w_i$$

where $\{a_i\}_{i=1}^n$ and b are differentiable functions from $\mathbb{R}_+^k$ to $\mathbb{R}$.

(a) Assuming that $b(p) > 0$ and $(p, w_i) >> 0\ \forall i$, then by Roy's identity,

$$\begin{aligned} x^i(p, w_i) &= \left(-\frac{\partial v^i(p, w_i)/\partial p_1}{\partial v^i(p, w_i)/\partial w_i}, ..., -\frac{\partial v^i(p, w_i)/\partial p_k}{\partial v^i(p, w_i)/\partial w_i}\right) \\ &= \left(-\frac{\frac{\partial a_i(p)}{\partial p_1} + \frac{\partial b(p)}{\partial p_1} w_i}{b(p)}, ..., -\frac{\frac{\partial a_i(p)}{\partial p_k} + \frac{\partial b(p)}{\partial p_k} w_i}{b(p)}\right) \end{aligned}$$

(b) Using Roy's identity on the representative consumer, we get

$$X(p, W) = \left(-\frac{\left(\sum_{i=1}^n \frac{\partial a_i(p)}{\partial p_1}\right) + \frac{\partial b(p)}{\partial p_1} W}{b(p)}, ..., -\frac{\left(\sum_{i=1}^n \frac{\partial a_i(p)}{\partial p_k}\right) + \frac{\partial b(p)}{\partial p_k} w_i}{b(p)}\right)$$

Where, if $W = \sum_{i=1}^n w_i$, we can solve, for each $j = 1, ..., k$:

$$\begin{aligned} X_j(p, W) &= -\frac{\left(\sum_{i=1}^n \frac{\partial a_i(p)}{\partial p_j}\right) + \frac{\partial b(p)}{\partial p_j} W}{b(p)} \\ &= -\frac{\sum_{i=1}^n \left(\frac{\partial a_i(p)}{\partial p_j} + \frac{\partial b(p)}{\partial p_j} w_i\right)}{b(p)} \\ &= \sum_{i=1}^n \left(-\frac{\frac{\partial a_i(p)}{\partial p_j} + \frac{\partial b(p)}{\partial p_j} w_i}{b(p)}\right) \\ X_j(p, W) &= \sum_{i=1}^n x_j^i(p, w_i) \end{aligned}$$

Question 3: Homothetic Proferences

Complete, transitive preferences, $\succsim$, are homothetic if, $\forall x, y \in \mathbb{R}^k_+, t > 0$,

$$x \succsim y \iff tx \succsim ty$$

(a) Let $x^* \in x(p, w)$ and define $Y = \{y \in \mathbb{R}^k_+ | y \notin x(p, w)\}$.

1. Suppose, for some $t > 0$, that $tx^*(p, w) \notin x(p, tw)$
   a. Since preferences are complete, there must exist some $y^* \in Y$ such that $ty^* \in x(p, tw)$
   b. $x^* \in x(p, w) \land y^* \notin x(p, w) \Rightarrow x^* \succsim y^*$. By homothetic preferences, this implies that $tx^* \succsim ty^*$ $\forall t > 0$
   c. $tx^* \succsim ty^* \Rightarrow u(tx) \geq u(ty)$. Since $x^* \in x(p, w)$, then $p \cdot x^* \leq w$. This implies also that $p \cdot (tx^*) \leq tw$
   d. Since $ty^* \in x(p, tw)$,

   $$ty^* = \underset{x}{\operatorname{argmax}} u(x) \text{ s.t. } p \cdot x \leq tw$$

   And by c., $u(tx^*) \geq u(ty^*)$, where $p \cdot (tx^*) \leq tw$. Thus, $tx^* \in x(p, tw)$

   $\therefore$ by contradiction, $x^* \in x(p, w) \Rightarrow tx^* \in x(p, tw)$ ■
2. Suppose $\exists y^* \in Y$ such that $ty^* \in x(p, tw)$
   a. By definition, $ty \succsim z$ $\forall z \in \mathbb{R}^k_+$ such that $p \cdot z \leq tw$
   b. $p \cdot (tx^*) = t(p \cdot x^*)$ where, by definition, $p \cdot x^* \leq w$. Then $p \cdot (tx^*) \leq tw$. Thus, $ty^* \succsim tx^*$
   c. Since preferences are homothetic, $ty^* \succsim tx^* \Rightarrow y^* \succsim x^*$. Thus, $y^* \in x(p, w)$

   $\therefore$ by contradiction, $tx^* \in x(p, tw) \Rightarrow x^* \in x(p, w)$ ■

$\therefore$ for any $t > 0$, $x(p, tw) = tx(p, w)$ ■