Problem Set #1

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Discussed and/or compared answers with Sarah Bass, Emily Case, Katherine Kwok, Michael Nattinger, and Alex Von Hafften

Question 1

Since both u and w are increasing, strinctly concave, and twice differentiable, the solution to the consumer problem comes from the first-order conditions of the Lagrangian function:

$$\begin{aligned}
\mathcal{L} &= \theta u(c^1) + (1-\theta)w(c^2) - \lambda(c^1 + c^2 - c) \\
\frac{\partial \mathcal{L}}{\partial c^1} &= \theta u'(c^1) - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial c^2} &= (1-\theta)w'(c^2) - \lambda = 0 \\
\Rightarrow \theta u'(c^1) &= (1-\theta)w'(c^2)
\end{aligned}$$

Using the envelope condition, we can show the change in total utility given a change in c:

$$\begin{aligned}
v_\theta'(c) &= \theta u'(c^1)\frac{\partial c^1}{\partial c} + (1-\theta)w'(c^2)\frac{\partial c^2}{\partial c} \\
&= \theta u'(c^1)\frac{\partial c^1}{\partial c} + \theta u'(c^1)\frac{\partial c^2}{\partial c} \\
&= \theta u'(c^1)\frac{\partial (c^1 + c^2)}{\partial c} = \theta u'(c^1) = (1-\theta)w'(c^2)
\end{aligned}$$

To show that $v_\theta(c)$ is concave, we must prove that, for any $c < c'$ and $\lambda \in (0,1)$,

$$v_\theta(\lambda c + (1-\lambda)c') \geq \lambda v_\theta(c) + (1-\lambda)v_\theta(c')$$

Let $f(c) = c^1$ and $g(c) = c^2$ determine the value of c^1 and c^2 that maximize v_θ such that $f(c) + g(c) = c$. Then,

$$\begin{aligned}
v_\theta(\lambda c + (1-\lambda)c') &= \theta u(f(\lambda c + (1-\lambda)c')) + (1-\theta)w(g(\lambda c + (1-\lambda)c')) \\
\lambda v_\theta(c) + (1-\lambda)v_\theta(c') &= \theta\left[\lambda u(f(c)) + (1-\lambda)u(f(c'))\right] + (1-\theta)\left[\lambda w(g(c)) + (1-\lambda)w(g(c'))\right]
\end{aligned}$$

Where, since u and w are concave,

$$u(f(\lambda c + (1-\lambda)c')) \geq \lambda u(f(c)) + (1-\lambda)u(f(c'))$$
$$w(g(\lambda c + (1-\lambda)c')) \geq \lambda w(g(c)) + (1-\lambda)w(g(c'))$$

Thus, $v_\theta(\lambda c + (1-\lambda)c') - (\lambda v_\theta(c) + (1-\lambda)v_\theta(c')) \geq 0$, so v_θ is concave.

Question 2

Exercise 8.3 from Ljungqvist and Sargent

a. A competitive equilibrium in this economy is a price system, $\{q_t\}_{t=0}^{\infty}$, and an allocation, $\left\{c_t^1, c_t^2\right\}_{t=0}^{\infty}$, that solves each consumer's problem and clears both the goods and claims markets in every period, t:

$$c_t^1 + c_t^2 = y_t^1 + y_t^2 \qquad\qquad q_t^1 + q_t^2 = 0$$

b. Each agent solves their utility maximization problem in period zero, which is represented by the following Lagrangian function and features a single budget constraint:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t^i) - \lambda^i \left(\sum_{t=0}^{\infty} q_t c_t^i - q_t y_t^i \right)$$

Then optimal consumption in each period, for each agent, has the following first order condition:

$$\frac{\partial \mathcal{L}}{\partial c_t^i} = \beta^t u'(c_t^i) - \lambda^i \left(q_t\right) = 0$$

Thus, we can determine that relative consumption across consumers is constant in every state and time period:

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\lambda^1}{\lambda^2}$$

Furthermore, since markets are complete with full commitment, we know that consumers with perfectly insure, such that $c_t^i(s^t) = c^i$ for all t and s^t. Thus, using each consumer's first order condition, we can solve:

$$\frac{\beta^t u'(c_t^i)}{u'(c_0^i)} = \frac{q_t}{q_0} \Rightarrow q_t = \beta^t q_0$$

Then each consumer's budget constraint implies:

$$\sum_{t=0}^{\infty} q_t(s^t) c_t^i = \sum_{t=0}^{\infty} q_t y_t^i$$
$$c^i q_0 \sum_{t=0}^{\infty} \beta^t = q_0 \sum_{t=0}^{\infty} \beta^t y_t^i$$

So the allocation is independent of date 0 prices and the right-hand side of this equation depends on each consumer's endowment series:

$$\sum_{t=0}^{\infty} \beta^t y_t^1 = 1 + \beta^3 + \beta^6 + ... = \sum_{t=0}^{\infty} (\beta^3)^t = \frac{1}{1-\beta^3}$$

$$\sum_{t=0}^{\infty} \beta^t y_t^2 = \beta + \beta^2 + \beta^4 + \beta^5 + ... = \beta \sum_{t=0}^{\infty} (\beta^3)^t + \beta^2 \sum_{t=0}^{\infty} (\beta^3)^t = \frac{\beta + \beta^2}{1-\beta^3}$$

Thus, the competitive equilibrium allocation is:

$$c_t^1 = \frac{1-\beta}{1-\beta^3} \qquad\qquad c_t^2 = \frac{\beta-\beta^3}{1-\beta^3}$$

c. The present discounted value of this asset is

$$p = \sum_{t=0}^{\infty} 0.05\beta^t = \frac{1}{20(1-\beta)}$$

So this would also be the price of the asset.

Question 3

Exercise 8.4 from Ljungqvist and Sargent