Econ 711 – Fall 2020 – Problem Set 5

Due online Monday night October 12 at midnight.

Please feel free to work together on these problems (and all homeworks), but each student needs to write up his/her own answers at the end, rather than directly copying from one master solution.

Question 1. The Consumer Problem

Solve the Consumer Problem and state the Marshallian demand $x(p, w)$ and indirect utility $v(p, w)$ for the following utility functions:

(a) $u(x) = x_1^\alpha + x_2^\alpha$ for $\alpha < 1$

(b) $u(x) = x_1 + x_2$

(c) $u(x) = x_1^\alpha + x_2^\alpha$ for $\alpha > 1$

(d) $u(x) = \min\{x_1, x_2\}$ (Leontief utility)

(e) $u(x) = \min\{x_1 + x_2, x_3 + x_4\}$

(f) $u(x) = \min\{x_1, x_2\} + \min\{x_3, x_4\}$

(For parts (e) and (f), you may describe the Marshallian demand in words rather than giving mathematical formulas if you prefer, and you can ignore the "knife-edge" cases where two prices or sums of prices are exactly equal, but you should still give formulas for the indirect utility function.)

Question 2. CES Utility

Throughout this problem, let $X = \mathbb{R}^k_+$, and let $(a_1, a_2, \ldots, a_k)$ be a set of strictly positive coefficients which sum to 1. You may assume prices and wealth are strictly positive, and ignore cases where two or more prices are identical.

(a) For each of the following utility functions, solve the consumer problem and state $x(p, w)$:

i. linear utility $u(x) = x_1 + x_2 + \ldots + x_k$

ii. Cobb-Douglas utility $u(x) = x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$

iii. Leontief utility $u(x) = \min\left\{\frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_k}{a_k}\right\}$

(b) Consider the Constant Elasticity of Substitution (CES) utility function

$$u(x) \quad = \quad \left(\sum_{i=1}^{k} a_i^{\frac{1}{s}} x_i^{\frac{s-1}{s}}\right)^{\frac{s}{s-1}}$$

with $s \in (0, 1) \cup (1, +\infty)$. Solve the consumer problem and state $x(p, w)$. (Recall that maximizing a function $(f(x))^{\frac{s}{s-1}}$ is the same as maximizing $f(x)$ when $s > 1$, and the same as minimizing $f(x)$ when $s < 1$.)

(c) Show that CES utility gives the same demand as linear utility in the limit $s \to +\infty$,

as Cobb-Douglas utility in the limit $s \to 1$,

and as Leontief utility in the limit $s \to 0$.

(d) The Elasticity of Substitution between goods 1 and 2 is defined as

$$\xi_{1,2} \quad = \quad -\frac{\partial \log\left(\frac{x_1(p,w)}{x_2(p,w)}\right)}{\partial \log\left(\frac{p_1}{p_2}\right)} \quad = \quad -\frac{\partial\left(\frac{x_1(p,w)}{x_2(p,w)}\right)}{\partial\left(\frac{p_1}{p_2}\right)} \frac{\frac{p_1}{p_2}}{\frac{x_1(p,w)}{x_2(p,w)}}$$

While this looks complicated, in the case of CES demand, we can write the ratio $\frac{x_1}{x_2}$ as a relatively simple function of the price ratio $\frac{p_1}{p_2}$, and calculate this elasticity without much difficulty. Calculate the elasticity of substitution for CES demand, and note its value as $s \to +\infty$, $s \to 1$, and $s \to 0$.

Question 3. Exchange Economies

We've been considering the problem facing a consumer with wealth w at prices p. An "exchange economy" is a different model where instead of money, each consumer is endowed with an initial bundle of goods $e \in \mathbb{R}^k_+$, and can either buy or sell any quantity of the goods at market prices p. The consumer's problem is then

$$\max_{x \in \mathbb{R}^k_+} u(x) \qquad \text{subject to} \qquad p \cdot x \leq p \cdot e$$

i.e., the consumer's "budget" is the market value of the goods they start with.

Assume preferences are locally non-satiated and the consumer's problem has a unique solution $x(p,e)$. We'll say the consumer is a *net buyer* of good i if $x_i(p,e) > e_i$, and a net seller if $x_i(p,e) < e_i$.

(a) Show that if p_i increases, the consumer cannot switch from being a net seller to a net buyer.

(b) Suppose u is differentiable and concave. Use the Lagrangian and the envelope theorem to show that $\frac{\partial v}{\partial p_i}$ is negative if the consumer is a net buyer of good i, and positive if the consumer is a net seller.

(c) Consider the following statement. "If the consumer is a net buyer of good i and its price goes up, the consumer must be worse off." True or false? Explain.