Econ 712 Macroeconomics I

Fall 2020, University of Wisconsin-Madison

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Problem Set[1] #2 - Due 9/18/19

Problem 1: Two-dimensional non-linear system

Consider the Ramsey model of consumption c_t and capital k_t:

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \tag{1a}$$

$$\beta u'(c_{t+1}) = \frac{u'(c_t)}{1-\delta+f'(k_{t+1})} \tag{1b}$$

parametrized by: $f(k) = zk^\alpha$,$z = 1$, $\alpha = 0.3$, $\delta = 0.1$, $\beta = 0.97$, $u(c) = \log(c)$.

1. Solve for steady state $(\bar{k}, \bar{c})$.

   Substitute $k_{t+1} = k_t = \bar{k}$ and $c_{t+1} = c_t = \bar{c}$ into (1) and solve for $\bar{k}$ and $\bar{c}$. From (1b):

$$\frac{\beta}{c_{t+1}} = \frac{1}{c_t(1-\delta+z\alpha k_{t+1}^{\alpha-1})}$$

$$1-\delta+z\alpha\bar{k}^{\alpha-1} = 1/\beta$$

$$\bar{k} = \left(\frac{z\alpha}{1/\beta-1+\delta}\right)^{1/(1-\alpha)} = 3.2690$$

   From (1a):

$$\bar{c} = z\bar{k}^\alpha - \delta\bar{k} = 1.0998$$

2. Linearize the system around it's steady state.

   (a) Rewrite equations (1) as

$$k_{t+1} = g(k_t, c_t)$$

$$c_{t+1} = h(k_t, c_t)$$

[1] **Based on previous problem sets by Anton Babkin, Fu Tan and Eirik Brandsås**

$$
\begin{aligned}
k_{t+1} &= zk_t^\alpha + (1-\delta)k_t - c_t \\
c_{t+1} &= \beta c_t(1-\delta+f'(k_{t+1})) \\
&= \beta c_t\left(1-\delta+z\alpha(zk_t^\alpha+(1-\delta)k_t-c_t)^{\alpha-1}\right)
\end{aligned}
$$

(b) Analytically calculate Jacobian $J = \begin{pmatrix} dk_{t+1}/dk_t & dk_{t+1}/dc_t \\ dc_{t+1}/dk_t & dc_{t+1}/dc_t \end{pmatrix}$ (use provided functional forms, but don't plug in parameters yet).

$$
\begin{aligned}
\frac{\partial k_{t+1}}{\partial k_t} &= f'(k_t) + 1 - \delta \\
\frac{\partial k_{t+1}}{\partial c_t} &= -1 \\
\frac{\partial c_{t+1}}{\partial k_t} &= \beta c_t f''(k_{t+1})\frac{\partial k_{t+1}}{\partial k_t} \\
\frac{\partial c_{t+1}}{\partial c_t} &= \beta\left[1-\delta+f'(k_{t+1})+c_t f''(k_{t+1})\frac{\partial k_{t+1}}{\partial c_t}\right]
\end{aligned}
$$

(c) Using Taylor expansion (first-order approximation here), system can be written in terms of deviations from steady state $\tilde{k}_t = k_t - \bar{k}$ and $\tilde{c}_t = c_t - \bar{c}$:

$$
\begin{pmatrix} \tilde{k}_{t+1} \\ \tilde{c}_{t+1} \end{pmatrix} = J \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix}
$$

3. Compute numerically eigenvalues and eigenvectors of the Jacobian at the steady state. Verify that the system has a saddle path. What is the slope of the saddle path at the steady state?

From Matlab: $\Lambda = \begin{pmatrix} 1.2060 & 0 \\ 0 & 0.8548 \end{pmatrix}, E = \begin{pmatrix} 0.9850 & 0.9848 \\ -0.1725 & 0.1734 \end{pmatrix}$

$\lambda_1 > 1, \lambda_2 < 1$, so yes, the system has a saddle path solution. Eigenvector $(e_{12}, e_{22}) = (0.9848, 0.1734$ shows direction of the saddle path, it's slope is $e_{22}/e_{12} = 0.1761$.

4. On a phase diagram in (k_t, c_t) show how the system evolves after an unexpected positive productivity shock at t_0, $z' > z$. (You don't need to plot lines precisely, but pay attention to

vector field (arrows), relative position of old and new steady states, directions of saddle paths and system trajectory after the shock.)

Phase diagram is shown in Figure 1. It can be shown analytically that $\bar{k}' > \bar{k}$ and $\bar{c}' > \bar{c}$.

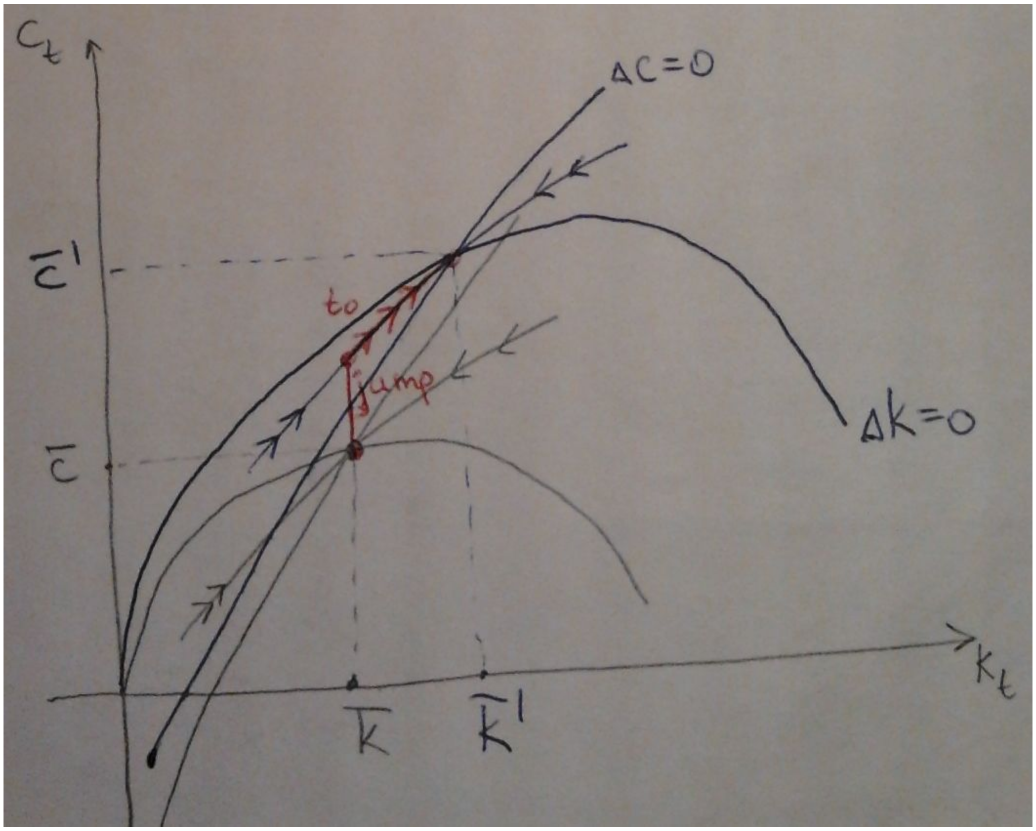

Figure 1: Ramsey phase diagram: dynamics after shock $z \to z'$.

5. Compute numerically and plot trajectories of k_t and c_t for $t = 1, 2, ..., 20$ if $t_0 = 5$ and $z' = z + 0.1$. For this question, we will be looking at the linearized version of the nonlinear system around the new steady state.

   (a) Compute the new steady state $(\bar{k}', \bar{c}')$. and Jacobian matrix at that point.

   $\bar{k}' = 3.7458 > \bar{k}$, $\bar{c}' = 1.2602 > \bar{c}$.

(b) Diagonalize the system using eigenvectors and rewrite it in terms of $\hat{k}_t$ and $\hat{c}_t$:

$$\begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = E^{-1} \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix}$$

$$E^{-1} J E = \Lambda$$

$$E^{-1} \begin{pmatrix} \tilde{k}_{t+1} \\ \tilde{c}_{t+1} \end{pmatrix} = \Lambda E^{-1} \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix}$$

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \Lambda \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}$$

(c) Write down non-explosive solution for $(\hat{k}_t, \hat{c}_t)$, rewrite in terms of original variables (k_t, c_t).

General solution:

$$\begin{cases} \hat{k}_t = c_1 \lambda_1^t \\ \hat{c}_t = c_2 \lambda_2^t \end{cases}$$

Non-explosive solution $(\lambda_1 > 1, \lambda_2 < 1)$:

$$\begin{cases} \hat{k}_t = 0 \\ \hat{c}_t = c_2 \lambda_2^t \end{cases}$$

$$\begin{pmatrix} k_1 - \bar{k}' \\ c_t - \bar{c}' \end{pmatrix} = E \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = \begin{pmatrix} e_{11}\hat{k}_t + e_{12}\hat{c}_t \\ e_{21}\hat{k}_t + e_{22}\hat{c}_t \end{pmatrix}$$

$$\begin{pmatrix} k_t \\ c_t \end{pmatrix} = \begin{pmatrix} \bar{k}' + e_{12} c_2 \lambda_2^t \\ \bar{c}' + e_{22} c_2 \lambda_2^t \end{pmatrix}$$

(d) Pin down a particular saddle path trajectory using a boundary condition $k_{t_0} = \bar{k}$ (capital can't jump from the old steady state at the time of the shock, so pick suitable c_{t_0}).

$$k_{t_0} = \bar{k} = \bar{k}' + e_{12}c_2\lambda_2^{t_0}$$
$$c_2 = \frac{\bar{k} - \bar{k}'}{e_{12}\lambda_2^{t_0}}$$

(e) Use the particular solution to compute and graph k_t and c_t after the shock.

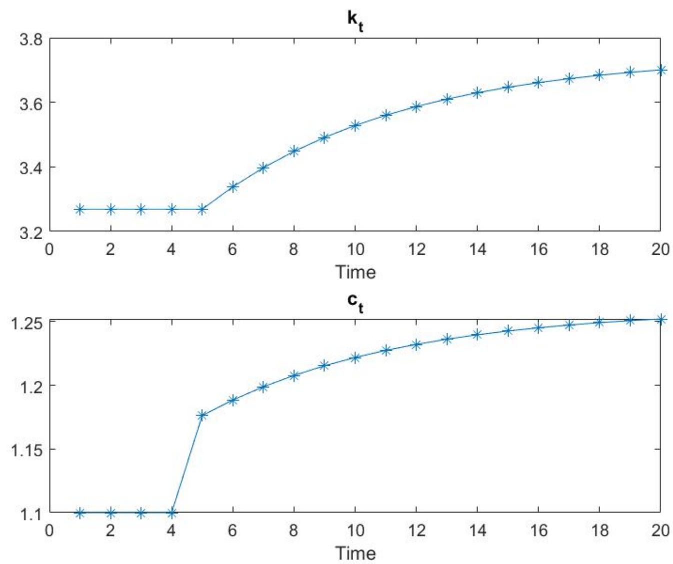

Figure 2: Transition on linear saddle path

6. For this question, we explore the nonlinear nature of the system and numerically solve the actual transition path using the “shooting method".

   (a) In the previous question, you solve c_{t_0} under the linear system. Put (k_{t_0}, c_{t_0}) into the nonlinear system (1a) and (1b). Compute and graph how the system evolves. Does it converge to a steady state?

   As shown in figure 3, if the c_{t_0} that is computed under the linear system is used, the actual nonlinear system will not converge.

   (b) Use “shooting method" to find the actual c_{t_0} needed. The method is to try different values of c_{t_0} such that after long enough time, the system will converge to the new steady state.

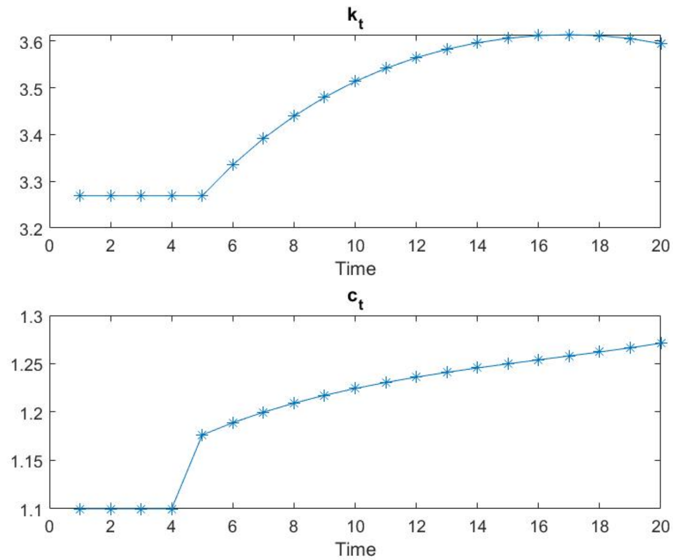

Figure 3: Actual transition if c_{t_0} in the previous question is used

Figure 4 shows the actual transition path of k_t and c_t using the shooting algorithm.

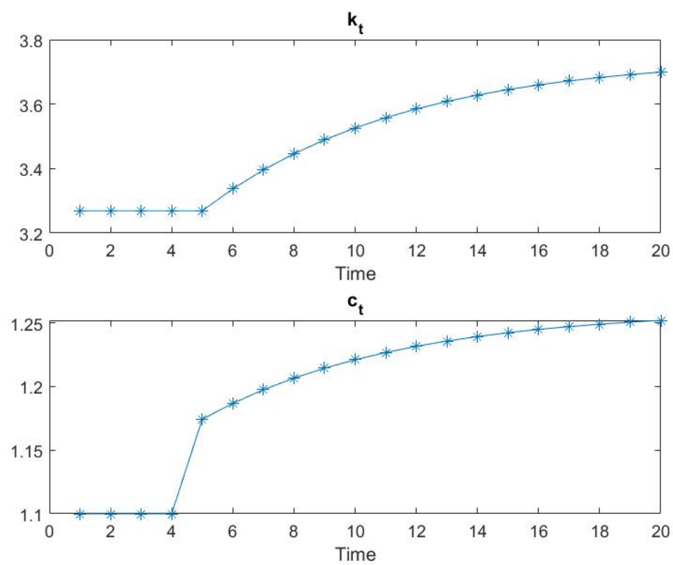

Figure 4: Transition using “shooting method"

Problem 2: Setting up a model

- For the problems below, state the Social Planner Problem (SPP), the Consumer Problem (CP), and define the Competitive Equilibrium (CE) *(Don't solve)*.

1. Consider an overlapping generations economy of 2-period-lived agents. There is a constant measure of N agents in each generation. New young agents enter the economy at each date $t \geq 1$. Half of the young agents are endowed with w_1 when young and 0 when old. The other half are endowed with 0 when young and w_2 when old. There is no savings technology. Agents order their consumption stream by $U(c_t^t, c_{t+1}^t) = \ln c_t^t + \ln c_{t+1}^t$. There is a measure N of inital old agents. Half of them are endowed with w_2 and the other half endowed with 0. Each old agent order their consumption by c_1^0. Each old agent is endowed with M units of fiat currency. No other generation is endowed with fiat currency, and the stock of fiat currency is fixed over time.

   SPP: $\max_{\{c_t^t, c_t^{t-1}\}} N * c_1^0 + \sum_{t=1}^{\infty} N * (\ln c_t^t + \ln c_{t+1}^t)$

   s.t. $N * c_t^t + N * c_t^{t-1} \leq N/2 * w_1 + N/2 * w_2 \ \ \forall t$; $(c_t^t, c_t^{t-1}) \geq (0, 0) \ \ \forall t$

   CP: Initial old: $\max_{c_1^0} c_1^0$ s.t. $c_1^0 \leq w_{o,i} + M/P_1$, with $w_{o,1} = 0, w_{o,2} = w_2$

   Other generations: Agents can (and will) trade within there generations. Hence we can include a 1-period bond b_{t+1}^t with price Q_t.

   $\max_{c_t^t, c_{t+1}^t, M_{t+1}^t, b_{t+1}^t} \ln c_t^t + \ln c_{t+1}^t$ s.t.

   $c_t^t + M_{t+1}^t/P_t + Q_t b_{t+1}^t \leq w_{y,i}$; $c_{t+1}^t \leq M_{t+1}^t/P_{t+1} + b_{t+1}^t + w_{o,i}$; $(c_t^t, c_{t+1}^t) \geq 0$; $M_{t+1}^t \geq 0$; $w_{y,i}/Q_t \geq b_{t+1}^t \geq -(M_{t+1}^t/P_{t+1} + w_{o,i})$

   with $(w_{y,1}, w_{o,1}) = (w_1, 0)$, $(w_{y,2}, w_{o,2}) = (0, w_2)$

   CE: Allocation $\{c_{t,i}^{t-1}, c_{t,i}^t, M_{t+1,i}^t, b_{t+1,i}^t\}$ and prices $\{P_t, Q_t\}$ s.t. given prices, allocation solves consumer problem (agents optimize), and markets clear:

   $N/2 * (c_{t,1}^t + c_{t,2}^t) + N/2 * (c_{t,1}^{t-1} + c_{t,2}^{t-1}) \leq N/2 * w_1 + N/2 * w_2$

   $N/2 * (M_{t+1,1}^t + M_{t+1,2}^t) = N * M$

   $N/2 * (b_{t+1,1}^t + b_{t+1,2}^t) = 0$

2. Consider an overlapping generations economy of 3-period-lived agents. Denote these periods as *young, mid, old.* At each date $t \geq 1$, N_t new young agents enter the economy, each endowed

with w_1 units of the consumption good when young, w_2 units when mid, and w_3 units when old. The consumption good is non-storable. The population is described by $N_{t+1} = n * N_t$, where $n > 0$. Consumption preference is described by $\ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t$. At time $t = 1$, there is a measure N_{-1} of old agents, each endowed with w_3 units of the consumption good, and a measure of N_0 mid agents, each endowed with w_2 units of the consumption good at $t = 1$ and w_3 units at $t = 2$. Additionally, each inital old agent is endowed with 1 unit a fiat currency.

SPP: $\max_{\{c_t^t, c_t^{t-1}, c_t^{t-2}\}} N_{-1} * \ln c_1^{-1} + N_0 * (\ln c_1^0 + \ln c_2^0) + \sum_{t=1}^{\infty} N_t * (\ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t)$

s.t. $N_t * c_t^t + N_{t-1} * c_t^{t-1} + N_{t-2} * c_t^{t-2} \leq N_t * w_1 + N_{t-1} * w_2 + N_{t-2} * w_3 \ \ \forall t$; $(c_t^t, c_t^{t-1}, c_t^{t-2}) \geq \mathbf{0} \ \ \forall t$

CP: Initial old: $\max_{c_1^{-1}} \ln c_1^{-1}$ s.t. $c_1^{-1} \leq w_3 + M/P_1$

Initial mid: $\max_{c_1^0, c_2^0} \ln c_1^0 + \ln c_2^0$ s.t.

$c_1^0 + M_2^0/P_1 \leq w_2$; $c_2^0 \leq M_2^0/P_2$

Other generations: Young agents can trade with mid agents. Hence we can include a 1-period bond b with price Q, available only when young and mid.

$\max_{c_t^t, c_{t+1}^t, c_{t+2}^t, b_{t+1}^t, b_{t+2}^t, M_{t+1}^t, M_{t+2}^t} \ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t$ s.t.

$c_t^t + M_{t+1}^t/P_t + Q_t b_{t+1}^t \leq w_1$; $c_{t+1}^t + M_{t+2}^t/P_{t+1} + Q_{t+1} b_{t+2}^t \leq M_{t+1}^t/P_{t+1} + b_{t+1}^t + w_2$; $c_{t+2}^t \leq M_{t+2}^t/P_{t+2} + b_{t+2}^t + w_3$;

$(c_t^t, c_{t+1}^t) \geq \mathbf{0}$; $(M_{t+1}^t, M_{t+2}^t) \geq \mathbf{0}$; $w_1/Q_t \geq b_{t+1}^t \geq -(M_{t+1}^t/P_{t+1} + w_2)$; $(M_{t+1}^t/P_{t+1} + b_{t+1}^t + w_2)/Q_{t+1} \geq b_{t+2}^t \geq -(M_{t+2}^t/P_{t+2} + w_3)$

CE: Allocation $\{c_t^{t-2}, c_t^{t-1}, c_t^t, M_{t+1}^t, M_{t+2}^t, b_{t+1}^t\}$ and prices $\{P_t, Q_t\}$ s.t. given prices, allocation solves consumer problem (agents optimize), and markets clear:

$N_t * c_t^t + N_{t-1} * c_t^{t-1} + N_{t-2} * c_t^{t-2} \leq N_t * w_1 + N_{t-1} * w_2 + N_{t-2} * w_3$

$N_t * M_{t+1,}^t + N_{t-1} M_{t+1}^{t-1} = N_{-1} * M$

$b_{t+1}^t + b_{t+1}^{t-1} = 0$

- (Cake eating problem) Consider a single infinitely lived agent with preference over their consumption stream $\boldsymbol{c} = \{c_t\}$ given by $U(\boldsymbol{c}) = \sum_{t=1}^{\infty} \beta^t u(c_t)$, where $\beta < 1$ and $u(.)$ is increasing and concave. Consumption cannot be negative in any period. The agent is endowed with k_1 units of the consumption good in period $t = 1$. There is a perfect storage technology, such that the consumption good is effectively infinitely durable. State the agent's problem *(Don't solve).*

$\max_{\{c_t\}} \sum_{t=1}^{\infty} \beta^t u(c_t)$ s.t $\sum_{t=1}^{\infty} c_t \leq k_1$

The problem can also be written sequentially as $\max_{\{c_t, k_{t+1}\}} \sum_{t=1}^{\infty} \beta^t u(c_t)$ s.t.

$c_t + k_{t+1} \leq (1-\delta)k_t$, where depreciation $\delta = 0$ (perfect storage)