Problem Set #2

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Question 1

Recall that $\hat{\beta}_{OLS} = (X'X)^{-1}X'y$. Then, if $Z = XC$,

$$\begin{aligned}
\hat{\beta}_Z &= (Z'Z)^{-1}Zy = \left[(XC)'XC\right]^{-1}(XC)'y \\
&= \left(C'X'XC\right)^{-1}C'X'y = C^{-1}(X'X)^{-1}C'^{-1}C'X'y \\
&= C^{-1}(X'X)^{-1}X'y = C^{-1}\hat{\beta}_{OLS}
\end{aligned}$$

Also recall that $\hat{\varepsilon}_{OLS} = Y - X\hat{\beta}_{OLS}$. Then,

$$\begin{aligned}
\hat{\varepsilon}_Z &= Y - Z\hat{\beta}_Z = y - Z(Z'Z)^{-1}Zy \\
&= \left(I - XC((XC)'XC)^{-1}XC\right)y = \left(I - XCC^{-1}(X'X)^{-1}C'^{-1}C'X\right)y \\
&= \left(I - X(X'X)^{-1}X\right)y = y - X(X'X)^{-1}Xy \\
&= y - X\hat{\beta}_{OLS} = \hat{\varepsilon}_{OLS}
\end{aligned}$$

Question 2

3.5) Recall from question 1 that $\hat{\varepsilon}_{OLS} = (I - X(X'X)^{-1}X')Y$. Then,

$$\begin{aligned}
\hat{\beta}_e &= (X'X)^{-1}X'\hat{\varepsilon}_{OLS} = (X'X)^{-1}X'(I - X(X'X)^{-1}X')Y \\
&= ((X'X)^{-1}X' - (X'X)^{-1}X'X(X'X)^{-1}X')Y = ((X'X)^{-1}X' - (X'X)^{-1}X')Y \\
&= 0
\end{aligned}$$

3.6) Let $\hat{Y} = X(X'X)^{-1}X'Y$ and $\hat{\beta}_Y$ represent the OLS coefficient from a regression of $\hat{Y}$ on X. Then,

$$\begin{aligned}
\hat{\beta}_Y &= (X'X)^{-1}X'\hat{Y} = (X'X)^{-1}X'X(X'X)^{-1}X'Y \\
&= (X'X)^{-1}X'Y = \hat{\beta}_{OLS}
\end{aligned}$$

3.7) Let $X = [X_1 \ X_2]$ be an m by n matrix and recall that $P = X(X'X)^{-1}X'$ and $M = I - P$. Let $n = n_1 + n_2$, where X_1 is an m by n_1 matrix and X_2 is m by n_2. Then, we can define $\Gamma = \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix}$ such that $X_1 = X\Gamma$. Thus,

$$\begin{aligned} PX_1 &= PX\Gamma = X(X'X)^{-1}X'X\Gamma = X\Gamma = X_1 \\ MX_1 &= (I-P)X_1 = X_1 - PX_1 = X_1 - X_1 = 0 \end{aligned}$$

Question 3

3.11) Let X contain only a non-zero constant, $c \in \mathbb{R}$, such that $X = c\mathbb{1}_n$, where n is the number of elements in Y and $\mathbb{1}_n$ is an $n \times 1$ vector of ones. Then,

$$\begin{aligned} \hat{Y} &= X(X'X)^{-1}X'Y = (c\mathbb{1}_n)\left[(c\mathbb{1}_n)'(c\mathbb{1}_n)\right]^{-1}(c\mathbb{1}_n)'Y \\ &= c\mathbb{1}_n\left(c^2(\mathbb{1}_n'\mathbb{1}_n)\right)^{-1}c(\mathbb{1}_n'Y) = c^2\mathbb{1}_n\left(c^2 n\right)^{-1}n\bar{Y} \\ &= \frac{c^2 n}{c^2 n}\bar{Y}\mathbb{1}_n = \bar{Y}\mathbb{1}_n \end{aligned}$$

Thus, $\hat{Y}$ is a column vector where every entry is $\overline{Y}$

3.12) Equation (3.53) cannot be estimated by OLS. Equation (3.53) can be rewritten as $Y = X\beta + \varepsilon$, where $X = [\mathbb{1}_n \ D_1 \ D_2]$ and $\beta = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \end{pmatrix}$. $D_1 + D_2 = \mathbb{1}_n$, so $\text{rank}(X) \neq k$, violating the first Gauss-Markov asusumption.

(a) Neither (3.54) nor (3.55) is more general. The two specifications have the same explanatory power. In (3.54), the average of Y for men is given by α_1, and the average for women is given by α_2. In (3.55), the averages are $\mu + \phi$ and μ, respectively. Thus, given the parameters for one specification, you could calculate the parameters of the other with:

$$\begin{aligned} \mu + \phi &= \alpha_1 & \qquad \phi &= \alpha_2 - \alpha_1 \\ \mu &= \alpha_2 & \alpha_2 &= \mu \end{aligned}$$

(b) $\mathbb{1}_n'D_1 = n_1$, $\mathbb{1}_n'D_2 = n_2$

3.13) (a) Letting $X = [D_1 \ D_2]$ and $\hat{\beta} = \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix}$, we can solve:

$$\begin{aligned} \hat{\beta} &= (X'X)^{-1}X'Y \\ &= \begin{pmatrix} \mathbb{1}_n'D_1 & 0 \\ 0 & \mathbb{1}_n D_2 \end{pmatrix}^{-1} \begin{pmatrix} D_1'Y \\ D_2'Y \end{pmatrix} = \frac{1}{n_1 n_2}\begin{pmatrix} n_2 & 0 \\ 0 & n_1 \end{pmatrix}\begin{pmatrix} \sum_{i=1}^n D_{1i}Y_i \\ \sum_{i=1}^n D_{2i}Y_i \end{pmatrix} \\ &= \frac{1}{n_1 n_2}\begin{pmatrix} n_2\sum_{i=1}^n D_{1i}Y_i \\ n_1\sum_{i=1}^n D_{2i}Y_i \end{pmatrix} = \begin{pmatrix} \frac{1}{n_1}\sum_{i=1}^n D_{1i}Y_i \\ \frac{1}{n_2}\sum_{i=1}^n D_{2i}Y_i \end{pmatrix} = \begin{pmatrix} \overline{Y}_1 \\ \overline{Y}_2 \end{pmatrix} \end{aligned}$$

(b) In plain English, Y^* is the demeaned value of Y, using the means for men and women separately. Econometrically, as shown below, Y^* is the residualized value of Y, or, rather, the value of Y that cannot be explained by gender alone:

$$\begin{aligned} Y^* &= Y - D_1\overline{Y}_1 - D_2\overline{Y}_2 = Y - \big(D_1\overline{Y}_1 + D_2\overline{Y}_2\big) \\ &= Y - [D_1\ D_2]\begin{pmatrix}\hat{\gamma}_1 \\ \hat{\gamma}_2\end{pmatrix} = \hat{\mu} \end{aligned}$$

It logically follows, then, that X^* is the residualized value of X, from a regression of X on D_1 and D_2.

(c) Let $D = [D_1\ D_2]$, $\hat{\alpha} = \begin{pmatrix}\hat{\alpha}_1 \\ \hat{\alpha}_2\end{pmatrix}$, and $\hat{\gamma} = \begin{pmatrix}\hat{\gamma}_1 \\ \hat{\gamma}_2\end{pmatrix}$. From part (b), we can rewrite:

$$Y^* = Y - D_1\hat{\gamma}_1 - D_2\hat{\gamma}_2 = Y - D\hat{\gamma} = (I_n - D(D'D)^{-1}D')Y = M_D Y$$

Where M_D is the orthogonal projection matrix for D. Similarly, $X^* = M_D X$. Thus, we can derive:

$$\begin{aligned} Y^* &= X^*\widetilde{\beta} \\ M_D Y &= M_D X\widetilde{\beta} \\ \widetilde{\beta} &= (X'M_D X)^{-1}M_D X'Y \end{aligned}$$

Since The second recgression is a partition of D and X, then by Theorem 3.4,

$$\hat{\beta} = (X'M_D X)^{-1}X'M_D Y$$

Thus, $\hat{\beta} = \widetilde{\beta}$.

Question 4

Let $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$ and $X = [X_1 \ X_2]$. By the definition of R^2,

$$\begin{aligned} R_1^2 &= 1 - \frac{\hat{\varepsilon}'\hat{\varepsilon}}{(Y-\overline{Y})'(Y-\overline{Y})} = 1 - \frac{(Y-X\hat{\beta})'(Y-X\hat{\beta})}{(Y-\overline{Y})'(Y-\overline{Y})} \\ &= 1 - \frac{(Y-X_1\hat{\beta}_1 - X_2\hat{\beta}_2)'(Y-X_1\hat{\beta}_1 - X_2\hat{\beta}_2)}{(Y-\overline{Y})'(Y-\overline{Y})} \end{aligned}$$

Now let $\widetilde{\beta}_2 = 0^*\hat{\beta}_2$. Then, for $\widetilde{\beta} = \begin{pmatrix} \widetilde{\beta}_1 \\ \widetilde{\beta}_2 \end{pmatrix}$, $X\widetilde{\beta} = X_1\widetilde{\beta}_1$ and $\widetilde{\beta}_1 = \hat{\beta}_1 = (X_1'M_2X_1)^{-1}X_1'M_2Y$:

$$\begin{aligned} R_2^2 &= 1 - \frac{(Y-X\widetilde{\beta})'(Y-X\widetilde{\beta})}{(Y-\overline{Y})'(Y-\overline{Y})} = 1 - \frac{(Y-X_1\widetilde{\beta}_1 - X_2\widetilde{\beta}_2)'(Y-X_1\widetilde{\beta}_1 - X_2\widetilde{\beta}_2)}{(Y-\overline{Y})'(Y-\overline{Y})} \\ &= 1 - \frac{(Y-X_1\hat{\beta}_1)'(Y-X_1\hat{\beta}_1)}{(Y-\overline{Y})'(Y-\overline{Y})} \\ &\leq 1 - \frac{(Y-X_1\hat{\beta}_1 - X_2\hat{\beta}_2)'(Y-X_1\hat{\beta}_1 - X_2\hat{\beta}_2)}{(Y-\overline{Y})'(Y-\overline{Y})} = R_1^2 \end{aligned}$$

It is possible for $R_1^2 = R_2^2$, if $\hat{\beta}_2 = 0$, which occurs if X_2 is orthogonal to Y.

Question 5

3.21) As a standard OLS coefficient in a non-partitioned regression, $\widetilde{\beta}_1 = (X_1'X_1)^{-1}X_1'Y$. By Theorem 3.4, $\hat{\beta}_1 = (X_1'M_2X_1)^{-1}X_1'M_2Y$. Thus, $\widetilde{\beta}_1 = \hat{\beta}_1$ if $X_1'M_2 = X_1'$. This will be true if:

$$\begin{aligned} X_1'M_2 &= X_1' \\ X_1'(I - X_2(X_2'X_2)^{-1}X_2') &= X_1' \\ X_1' - X_1'X_2(X_2'X_2)^{-1}X_2') &= X_1' \end{aligned}$$

Which holds if $X_1'X_2 = 0$, in which case X_1 and X_2 are orthogonal. The same is true for $\widetilde{\beta}_2$ and $\hat{\beta}_2$, by the same mathematical logic. The coefficients will also be equal if either X_1 or X_2 (or both) are orthogonal to Y, since this will lead to OLS coefficients of zero.

3.22) Recall that, by Theorem 3.4, $\hat{\beta}_2 = (X_2'M_1X_2)^{-1}X_2'M_1Y$. Then,

$$\begin{aligned}
\widetilde{u} &= Y - X_1\widetilde{\beta}_1 = Y - X_1(X_1'X_1)^{-1}X_1'Y = (I - X_1(X_1'X_1)^{-1}X_1')Y = M_1Y \\
\widetilde{u} &= X_2\widetilde{\beta}_2 \\
M_1Y &= X_2\widetilde{\beta}_2 \\
M_1M_1Y &= M_1X_2\widetilde{\beta}_2 \\
X_2'M_1Y &= X_2'M_1X_2\widetilde{\beta}_2 \\
\widetilde{\beta}_2 &= (X_2'M_1X_2)^{-1}X_2'M_1Y = \hat{\beta}_2
\end{aligned}$$

3.23)

Question 6

(due w/ PS3)

3.24)

3.25)

Question 7

Given the $n \times 1$ vector y and the $n \times k$ matrix X, assume Prank$(X) = k$; $\mathbb{E}(y|X) = X\beta$; and $Var(y|X) = \sigma^2 I$.
Partiion X: $X = [X_1 \ X_2]$ where X_1 is $n \times k_1$, X_2 is $n \times k_2$, and $k_1 + k_2 = k$.
Similarly partition $\beta : \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$, where β_1 is $k_1 \times 1$ and β_2 is $k_2 \times 1$.

(a) **Consider the OLS regression of y on X that yields the OLS estimator $\hat{\beta}$. Whatis $\mathbb{E}(\hat{\beta}_1|X)$? Simplify your answer.**

(b) **Let $\hat{y} = X\hat{\beta}$. Now, consider the OLS regression of $\hat{y}$ on X_1 that yields the OLS estimator $\hat{\beta}_1$. What is $\mathbb{E}(\hat{\beta}_1|X)$? (Simplify your answer.) Is $\hat{\beta}_1$ an unbiasedestimator of β_1?**

(c) **Consider the OLS regression of y on X_1 that yields the OLS estimator β_1. Let $\widetilde{y} = X_1\widetilde{\beta}_1$. Now consider the OLS regression of $\widetilde{y}$ on X that yields the OLS estimator $\widetilde{\beta}$. How is $\widetilde{\beta}$ related to $\widetilde{\beta}_1$? (Provide a mapping between $\widetilde{\beta}$ and $\widetilde{\beta}_1$ that does not involve X.)**

(d) **What is the R^2 for the OLS regression of $\widetilde{y}$ on X (from part (c))? Simplify your answer.**

(e) **What is $Var(\widetilde{\beta}_1|X)$? Simplify your answer.**