Sections 4, 5

Today:

- §4: Theorem
- §5: Proof
- Objectives:
  - Distinguish theorems from other mathematical statements.
  - Reword statements using if-then constructions.
  - Give the mathematical meanings of *true*, *and*, *or*, and *not*.

A **theorem** is a declarative statement about mathematics for which there is a proof.

A proof is an “essay” that shows that a statement is true.

Examples

1 Pythagorean Theorem

2 There are infinitely many primes (Euclid)

3 ~~All prime numbers are odd~~
not a theorem.

All prime numbers greater than 2 are odd ✓

Theorem = Mathematical Truth

- Absolute
- Unconditional
- Without exception
- Always true

Statements that are not absolutely true in this strict sense are called *false*.

Theorem (The Pythagorean Theorem)

If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

Theorem

If a function $f(x)$ is differentiable at a point, then it is continuous at that point.

Assumption / condition / what we know is true:

$f(x)$ is differentiable at $x=a$

what we can conclude: $f(x)$ is continuous at $x=a$

But: we cannot claim (here) that if $f(x)$ is continuous at $x=a$ then $f(x)$ is differentiable at $x=a$.

If-Then Statements: in Math vs. in daily use

“If you don’t eat your broccoli, (then) you won’t get dessert.”

This means (generally)

If (you don't eat your broccoli) Then (you won't get dessert)

But this could be understood (in everyday language) as:

~~If (you eat your broccoli) Then (you will get dessert)~~

not true mathematically!

(for now): the only thing we are guaranteed is:

If (you don't eat your broccoli) Then (you won't get dessert)

If-Then Statements

Suppose A and B are two statements.
We can form implications in the following ways:

\Rightarrow

$A \Rightarrow B$

If A, then B.

A is sufficient for B.

A only if B.

B is necessary for A.

This is also read as "A implies B"

A: hypothesis/condition

B: conclusion/consequence

Suppose A and B are two statements.
We can form implications in the following ways:

$A \Rightarrow B$	$B \Rightarrow A$
If A, then B.	If B, then A.
A is sufficient for B.	B is sufficient for A.
A only if B.	B only if A.
B is necessary for A.	A is necessary for B.

This is read as "B implies A"

or $A \Leftarrow B$ "A is implied by B"

\leftarrow

$A \Rightarrow B$
$B \Rightarrow A$
These are known as Converse statements.

If-Then Statements

Suppose A and B are two statements.
We can form implications in the following ways:

\leftrightarrow

$A \Rightarrow B$	$B \Rightarrow A$	$A \Leftrightarrow B$
If A, then B.	If B, then A.	A if and only if B.
A is sufficient for B.	B is sufficient for A.	A is necessary and sufficient.
		B is necessary and sufficient.
A only if B.	B only if A.	
B is necessary for A.	A is necessary for B.	A iff B.

A and B are equivalent

$A \Leftrightarrow B$ means $A \Rightarrow B$ and $B \Rightarrow A$

ex: Let $x \in \mathbb{Z}$. $(x \text{ is even}) \Leftrightarrow (x+1 \text{ is odd})$

If-Then Statements: Examples

So far, we've explored if-then statements of the form $A \Rightarrow B$ where A and B do not have a fixed truth value.

e.g.: let $x \in \mathbb{Z}$

A: x is even

B: x^2 is even

here: $A \Rightarrow B$ (we know x is even, so x^2 is even)

and $B \Rightarrow A$ (we know x^2 is even, so x is even)

However: sometimes A and B have truth values that affect the truth value of the implication.

e.g.: "$2+4=6$" (This is always true)

"$2+4=5$" (This is always false)

"pigs can fly" (This is always false)

If-Then Statements: Examples

ex1 If $\underbrace{(2+4=6)}_{\text{True}}$ Then $\underbrace{(\text{NY is a state of the USA})}_{\text{True}}$

$T \Rightarrow T$
Overall, the statement is true

ex2 If $\underbrace{(2+4=6)}_{\text{True}}$ Then $\underbrace{(\text{pigs can fly})}_{\text{False}}$

$T \Rightarrow F$
Overall, the statement is False

ex3 if $\underbrace{(\text{pigs can fly})}_{\text{False}}$ Then $\underbrace{(2+4=6)}_{\text{True}}$

$F \Rightarrow T$
Overall, this statement is True

A is impossible > VACUOUS TRUTH
It cannot be proven false (no exceptions).

ex4 If $\underbrace{(\text{pigs can fly})}_{\text{False}}$ Then $\underbrace{(2+4=7)}_{\text{False}}$

$F \Rightarrow F$
Overall, this statement is True

Summary:

Let A and B be two mathematical statements.

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$A \Leftrightarrow B$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Converse Statements

The **converse** of the statement "If *A*, then *B*" is

"If *B*, then *A*".

Examples (In general, converse statements do not have the same truth value)

[If (I took the train at W 4th st in NYC) Then (I went underground)] True

[If (I went underground) Then (I took the train at W 4th st in NYC)] False

 WTS : Want To Show

1. **A and B**

   The statement "A and B" means that both statements are true.

   Example

   If x and y are even integers, then xy is divisible by 4.

$$\underbrace{\begin{pmatrix} x \text{ is even} \\ \text{and} \\ y \text{ is even} \end{pmatrix}}_{\text{both hold}}$$

. Let's prove this (first proof!)

Condition / Assumption / know is true: x is even
y is even

x is even means: $x = 2a$, $a \in \mathbb{Z}$. y is even means $y = 2b$, $b \in \mathbb{Z}$

Conclusion : what we need to show : xy is divisible by 4

WTS: $xy = 4c$, $c \in \mathbb{Z}$

1. ***A* and *B***
   The statement "*A* and *B*" means that both statements are true.
   Example
   If x and y are even integers, then xy is divisible by 4.

Pf: Let x and y be even integers.
Then $x = 2a$ and $y = 2b$ where $a, b \in \mathbb{Z}$.
Multiplying x and y, we get $xy = (2a)(2b)$ which is equivalent to $xy = 4(ab)$. Let $c = ab$. Then $c \in \mathbb{Z}$ since both a and b are integers.
This means $xy = 4c$, $c \in \mathbb{Z}$.
Thus, xy is divisible by 4.

And, Or, Not

1. ***A* and *B***
   The statement "*A* and *B*" means that both statements are true.
   Example
   If x and y are even integers, then xy is divisible by 4.
2. ***A* or *B***
   The statement "*A* or *B*" means that at least one of the statement is true.
   Example
   If x is even or y is even, then xy is even.

one condition is enough, but both could hold.

So "x is even or y is even" means:

- x is even and y is odd
- or x is odd and y is even
- or x is even and y is even

All are possible conditions

"or" does not mean "mutually exclusive"

And, Or, Not

1. **A and B**
   The statement "A and B" means that both statements are true.
   Example
   If x and y are even integers, then xy is divisible by 4.
2. **A or B**
   The statement "A or B" means that at least one of the statement is true.
   Example
   If x is even or y is even, then xy is even.
3. **Not A**
   The statement "not A" is a true statement if and only if A is false.
   Example
   "If $f(x)$ is not continuous at a, then $f(x)$ is not differentiable at a.

   A: $f(x)$ is continuous at a
   B: $f(x)$ is differentiable at a

   "If (not A), then (not B)."

Contrapositive of "If A, then B"

$A \Rightarrow B$

	Condition A	Condition B		
Case 1	True	True	possible	(True)
Case 2	True	False	impossible	(False)
Case 3	False	True	possible	(True)
Case 4	False	False	possible	(True)

The **contrapositive** of the statement "If A, then B": $(\text{not } B) \Rightarrow (\text{not } A)$

	not B Condition ~~B~~	not A Condition ~~A~~		
Case 1	~~True~~ F	~~True~~ F	possible	(True)
Case 2	~~False~~ T	~~True~~ F	impossible	(False)
Case 3	~~True~~ F	~~False~~ T	possible	(True)
Case 4	~~False~~ T	~~False~~ T	possible	(True)

Conclusion: $(A \Rightarrow B)$ is equivalent to $((\text{not } B) \Rightarrow (\text{not } A))$

Vacuous Truth

Statements of the form "If A, then B" in which condition A is impossible are called **vacuous truths**.

Mathematicians consider such statements true because they have no exceptions.

ex: 1, 4, 9, 16, 25 ... are perfect squares

Example:

- If $\underbrace{\text{an integer is both a perfect square and prime}}_{\text{This condition never happens}}$, then it is negative.

This is a vacuously true statement

- $\underbrace{k \in \mathbb{Z} \text{ is odd and divisible by } 2}_{\substack{\text{Absurd} \\ \text{(always false)}}} \Rightarrow k$ is a perfect square.

(True)

Theorems & Friends

Theorem	A theorem is a declarative statement about mathematics for which there is a proof.
Proposition	A minor theorem.
Lemma	A theorem whose main purpose to help prove another, more important theorem.
Corollary	A theorem with a short proof, whose main step is the use of another theorem.
Result	A modest, generic word for theorem.
Conjecture	A mathematical statement that has not been proven.

(page 12-13 in textbook)

Conjecture Examples

What can we say about the sum of consecutive perfect cubes?

$1^3 = 1$

$1^3 + 2^3 \quad (1+2)^2 = 3^2 = 9$

$1^3 + 2^3 + 3^3 \quad (1+2+3)^2 = 6^2 = 36$

$\vdots$

It seems like:

$$1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2$$

(proof: section 22)

Tips for Writing If-Then Proofs

1. Rewrite statement as an if-then statement.
   - Restate the hypothesis of the result
   - Invent suitable notation
   - Assign letters to represent variables

Exercises

Write as an ‘If A, then B’ statement:

1. ‘All dogs go to heaven.’

2. ‘Every perfect square is positive.’

Tips for Writing If-Then Proofs

1. Rewrite statement as an if-then statement.
   - Restate the hypothesis of the result
   - Invent suitable notation
   - Assign letters to represent variables
2. Write the last statement of the proof by restating the conclusion of the result.

Tips for Writing If-Then Proofs

1. Rewrite statement as an if-then statement.
   - Restate the hypothesis of the result
   - Invent suitable notation
   - Assign letters to represent variables
2. Write the last statement of the proof by restating the conclusion of the result.
3. Use definitions
   - Work forward from the beginning of the proof
   - Work backwards from the end of the proof

Tips for Writing If-Then Proofs

1. Rewrite statement as an if-then statement.
    - Restate the hypothesis of the result
    - Invent suitable notation
    - Assign letters to represent variables
2. Write the last statement of the proof by restating the conclusion of the result.
3. Use definitions
    - Work forward from the beginning of the proof
    - Work backwards from the end of the proof
4. Find the “aha!” moment that links the two halves of your argument.

Example of Proof: Proposition

The sum of an even integer and an odd integer is odd.

WTS: Want To Show

Equivalently: "If $\underbrace{x \text{ is even and } y \text{ is odd}}_{\text{Given, Can use}}$, then $\underbrace{x+y}_{\text{WTS}}$ is odd."

Given,

Can use

WTS

Draft:

x is even : $x = 2k$

y is odd : $y = 2k' + 1$

$k, k' \in \mathbb{Z}$

Thus: $x+y$ is odd $(x+y = 2m+1)$

Equivalently: "If x is even and y is odd, then $x+y$ is odd."

Proof.

- Suppose that x is even and y is odd.
- Since x is even, $2|x$, by the definition of even.
  So, there is an integer a for which $x = 2a$, by the definition of divisible.
- Since y is odd, there is some integer b for which $y = 2b+1$, by the definition of odd.
- Therefore,

$$\begin{aligned} x+y &= 2a+2b+1 \\ &= 2(a+b)+1. \end{aligned}$$

- Let $c = a+b$, which is an integer.
- There is an integer c such that $x+y = 2c+1$, then $x+y$ is odd.

$\square$

To prove a statement of the form "A iff B":

($\Longrightarrow$) Prove "If A, then B."

($\Longleftarrow$) Prove "If B, then A."

EFY: Exercise For You

Example: An integer n is odd if and only if n^2 is odd.

<u>Proof outline:</u>

First, prove $A \Rightarrow B$

$\Rightarrow$) Suppose $n \in \mathbb{Z}$ is odd

Thus, n^2 is odd

EFY

Then, prove $B \Rightarrow A$

$\Leftarrow$) Suppose n^2 is odd

Thus, n is odd

EFY (after section 20)

Exercises

For the following statements $A\&B$, which of the following is true where $n \in \mathbb{Z}$?

A: n is divisible by 3.

B: $n+1$ is even.

- $A \implies B$ only.
- $B \implies A$ only.
- $A \iff B$.
- Neither $A \implies B$ nor $B \implies A$.

For the following statements $A\&B$, which of the following is true where $n \in \mathbb{Z}$?

A: n is prime

B: $2n$ is not prime.

- $A \implies B$ only.
- $B \implies A$ only.
- $A \iff B$
- Neither $A \implies B$ nor $B \implies A$.

Find the error in the following proof.

Theorem

Let $a, b, c \in \mathbb{Z}$. If $a|bc$, then $a|b$ or $a|c$.

Proof

Let $a, b, c \in \mathbb{Z}$. We will show that if $a|bc$, then $a|b$ or $a|c$. Let $a = 5$, $b = 3$ and $c = 10$. We observe that $a|bc$ because $5|30$. Also, since $5|10$ it is true that $a|c$. Therefore, $a|b$ or $a|c$. $\square$

Find the error in the following proof

Theorem

The sum of two odd integers is even.

Proof

We show that if x and y are odd integers, then $x + y$ is an even integer. Let x and y be odd integers. By definition of odd, we know there exists an integer a such that $x = 2a + 1$ and $y = 2a + 1$. Observe that $x + y = (2a + 1) + (2a + 1) = 2(2a + 1)$. Therefore there is an integer $c = 2a + 1$ such that $x + y = 2c$. Hence $2|(x + y)$, and therefore $x + y$ is even. $\square$

Exercises

1. Prove that the sum of two odd integers is even.
2. Suppose a, b, and c are integers. Prove that if $a|b$ and $a|c$, then $a|(b+c)$.

Exercises

1. Prove that the sum of two odd integers is even.
2. Suppose a, b, and c are integers. Prove that if $a|b$ and $a|c$, then $a|(b+c)$.
3. Let x be an integer. Prove that x is odd if and only if there is an integer b such that $x = 2b - 1$.

Summary of 4

- This section introduced the notion of a theorem: a declarative statement about mathematics that has a proof.
- We discussed the absolute nature of the word true in mathematics.
- We examined the if-then and if-and-only-if forms of theorems, as well as alternative language to express such results.
- We clarified the way in which mathematicians use the words and, or, and not.
- We presented a number of synonyms for theorem and explained their connotations.
- Finally, we discussed vacuous if-then statements and noted that mathematicians regard such statements as true.

Summary of 5

- We introduced the concept of proof and presented the basic technique of writing a direct proof for an if-then statement.
- For if-and-only-if statements, we apply this basic technique to both the forward $\Longrightarrow$ and the backward $\Longleftarrow$ implications.