Sections 24 and 26

Functions and Compositions

Class Objectives

- Define and classify functions as one-to-one, onto, or bijective.
- Combine functions by composition.
- Prove two functions are equal

Definition

domain / codomain

A relation $f \subseteq A \times B$ is called a **function** provided

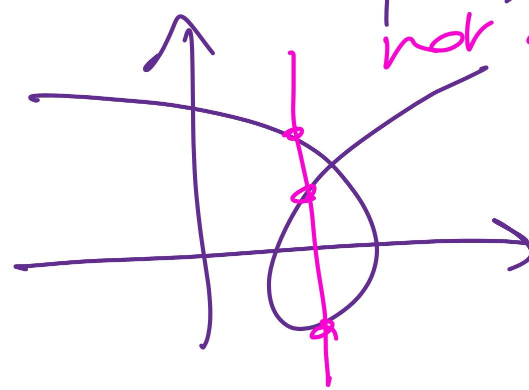
$$(a, b_1) \in f \text{ and } (a, b_2) \in f \text{ imply } b_1 = b_2.$$

Recall: the vertical line test for graphs of functions

(every input has, at most, one output)

✓ function

not a function



Notation: $(a,b) \in f \subseteq A \times B \iff f(a) = b \quad \exists a \in A, b \in B$

(Poll Everywhere)

Definition: Domain and Image

$f: A \to B$

<u>Domain</u> of f = set of all inputs (or, first element in (a,b))

$= \{ x \in A \mid f(x) = y \ \exists y \in B \}$

$= \{ x \in A \mid (x,y) \in f \}$

$= \mathrm{dom}(f) = \mathrm{dom}\, f = A$

<u>Image</u> of f = set of all outputs (or, second element in (a,b))

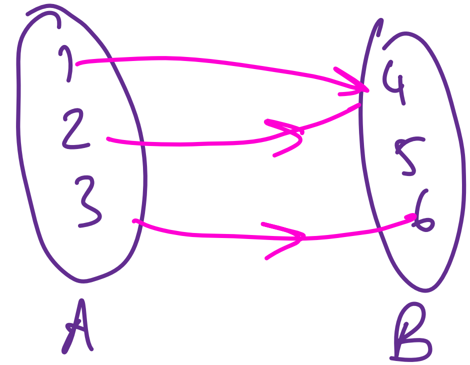
$= \{ y \in B \mid f(x) = y \ \exists x \in A \}$

$= \{ y \in B \mid (x,y) \in f \}$

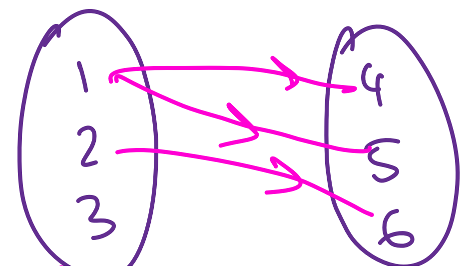
$= \mathrm{im}(f) = \mathrm{im}\, f \subseteq B$ ← codomain

$A = \{1,2,3\}$ $\quad$ $B = \{4,5,6\}$

$f = \{(1,4),(2,4),(3,6)\} \subseteq A \times B$



$g = \{(1,4),(2,6),(1,5)\} \subseteq A \times B$



Examples

ex: $f: \mathbb{Z} \to \mathbb{Z}$

$n \mapsto n^2$

(or $f(n) = n^2$)

here: $\text{dom } f = \mathbb{Z}$

$\text{codom } f = \mathbb{Z}$

$\text{im } f = \mathbb{P} \subseteq \mathbb{Z}$

perfect squares

$f: \mathbb{Z} \to \mathbb{Z}$

$n \mapsto n-3$

(or $f(n) = n-3$)

$g: \mathbb{Z} \to \mathbb{Z}$

$n \mapsto 2n$

$g: \mathbb{E} \to \mathbb{Z}$

$n \mapsto n/2$

Explain why the functions $f(x) = \dfrac{9-x^2}{x+3}$ and $g(x) = 3-x$ are not equal.

Proposition

$f : A \to B$ (at this point f is a relation. Goal: f is a function)

Proof Template.

To prove that f is a function from a set A to a set B:

- Prove that f is a function. (every $x \in A$ has, at most, one output)
- Prove that $\mathrm{dom}(f) = A$.
- Prove that $\mathrm{im}(f) \subseteq B$.

□

Definition

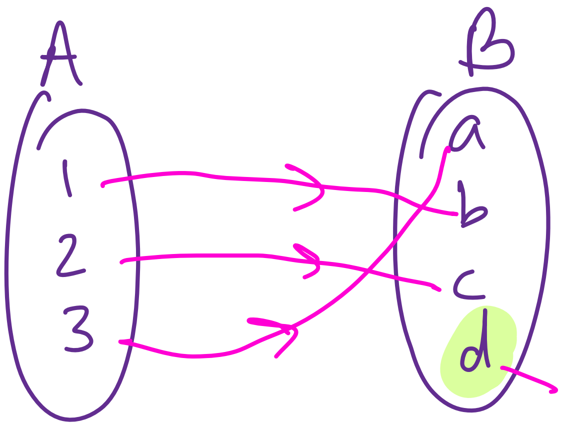
A function $f : A \to B$ is called **one-to-one** (or *injective*) provided that for all $a, b \in A$:

$$\textit{if } f(a) = f(b), \textit{ then } a = b.$$

Contrapositive: if $a \neq b$ then $f(a) \neq f(b)$

or every element in $\mathrm{im}(f) \subseteq B$ is mapped to exactly once in the function

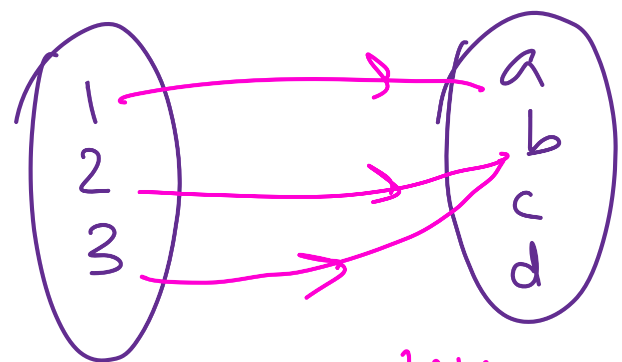
one-to-one



it's ok to have $\exists d \in B$ that does not have a preimage

(Poll Everywhere)

not one-to-one



still a function, but not injective

How to prove $f : A \to B$ is one-to-one

Template: let $a, b \in A$.

1. Direct method: ← More common

   Suppose $f(a) = f(b)$

   ⋮

   … Thus $a = b$

or

2. Contrapositive:

   Suppose $a \neq b$

   ⋮

   … Thus $f(a) \neq f(b)$

Prove that the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ defined by $f(x) = \dfrac{5x+1}{x-2}$ is one-on-one

$\mathbb{R} - \{2\}$ $\mathbb{R} - \{5\}$

f is one-to-one (injective)

tip: use the direct method

Suppose $f(a) = f(b)$ $\exists\, a, b \in A$. Then $a \neq 2$ and $b \neq 2$, and

$$\frac{5a+1}{a-2} = \frac{5b+1}{b-2} \iff (5a+1)(b-2) = (5b+1)(a-2)$$

$$\iff 5ab - 10a + b - 2 = 5ab - 10b + a - 2$$

$$\iff 11a = 11b \,;$$ Thus $a = b$

Note: This won't work for any function

ex: $f : \mathbb{Z} \to \mathbb{N}$; $f(x) = x^2 + 3$

$$f(a) = f(b) \Rightarrow a^2 + 3 = b^2 + 3 \Rightarrow a^2 = b^2$$

$$\Rightarrow a = b \text{ or } a = -b$$

(not one-to-one)

Definition

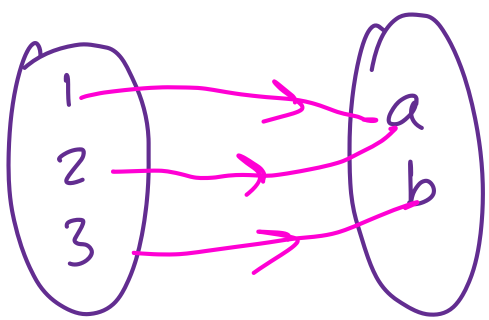
A function $f : A \to B$ is called **onto** B (or *surjective*) provided that for all $y \in B$:

there exists $x \in A$ *such that* $f(x) = y$.

In other words, $\mathrm{im}(f) = B$. (originally, for any function, $\mathrm{im}(f) \subseteq B$)

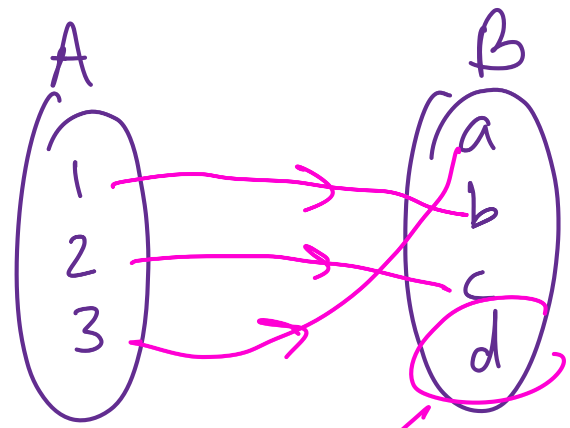
Every element in B is mapped to by some element in A.

onto



(onto, but not one to one)

not onto



$d \in B$, but no $x \in A$ satisfies $f(x) = d$

Template:

expression

1. Direct Method: (common when given $f(x) = \dots$)

   let $y \in B$ be arbitrary

   ⋮ "Construct" some $x \in A$ such that $f(x) = y$

   Show that $y = f(x)$ $\exists x \in A$

2. Set Method: (common in composition proofs)

   Proof by double inclusion

   - $\mathrm{im}(f) \subseteq B$ (always true)
   - $B \subseteq \mathrm{im}(f)$

Prove that the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ defined by $f(x) = \dfrac{5x+1}{x-2}$ is onto R\{5}

$\mathbb{R}-\{2\}$ $\mathbb{R}-\{5\}$

f is onto

Need to show that $\forall b \in B\ (\mathbb{R}-\{5\}),\ \exists a \in A\ (\mathbb{R}-\{2\}) \mid f(a) = b$

scratch: "construct" a s.t. $f(a) = b$

$f(a) = b = \dfrac{5a+1}{a-2}$, solve for a:

$b(a-2) = 5a+1 \Rightarrow ab - 2b = 5a+1 \Rightarrow a(b-5) = 2b+1$

so $a = \dfrac{2b+1}{b-5}$

let $a = \dfrac{2b+1}{b-5}$ for $b \in \mathbb{R}-\{5\}$ (codomain of f). Then

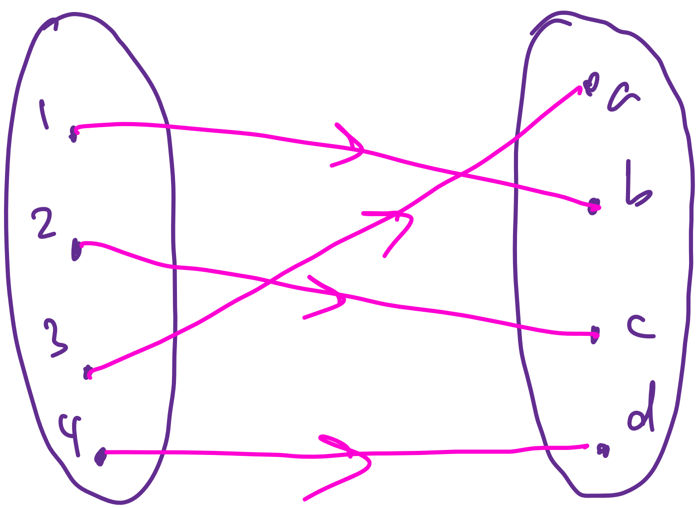
$$f(a) = \frac{5\left(\frac{2b+1}{b-5}\right)+1}{\frac{2b+1}{b-5} - 2} = \frac{10b+5+b-5}{2b+1-2(b-5)} = \frac{11b}{11} = b.$$

Definition

Let $f : A \to B$. We call f a **bijection** (or a bijective function) from A to B provided that f is one-to-one and onto B.

i.e. $\forall y \in B, \exists! x \in A \mid f(x) = y$

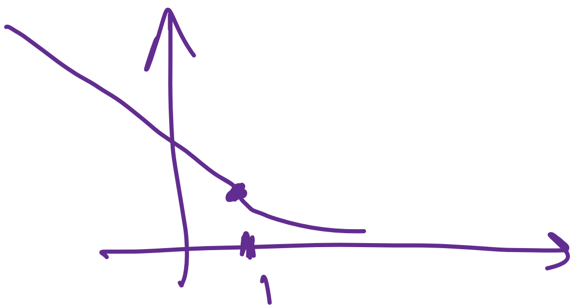
or every element in A maps, uniquely, to some element in B
and every element in B is mapped to by some element in A



Let $A = \{3, 17, 29, 45\}$ and $B = \{4, 6, 22, 60\}$. A relation R from A to B is defined by aRb if $a + b$ is a prime number. Is R a function from A to B? Is it a bijection? (no formal proof needed, a graph will do)

Prove that the piecewise function

$$f(x) = \begin{cases} 2-x, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$



is one-to-one but not onto over $\mathbb{R}$.

$\mathbb{R} \to \mathbb{R}$

one-to-one:

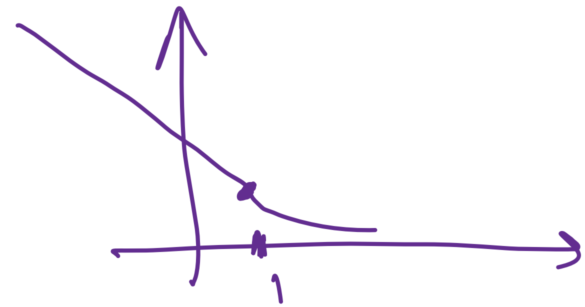
$x \leq 1$:

$\begin{pmatrix} a \leq 1 \\ b \leq 1 \end{pmatrix}$

Prove that the piecewise function

$$f(x) = \begin{cases} 2 - x, & x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

is one-to-one but not onto over $\mathbb{R}$.



$\mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$

f is not onto because $\exists\, y \in \mathbb{R} \mid f(x) \neq y \;\forall x \in \mathbb{R}$

$f^{-1} \neq \frac{1}{f}$

Definition

Let $f : A \to B$. The **inverse** relation, denoted f^{-1} is the relation f^{-1} defined by:

$$f^{-1} = \{(y,x) \ : \ (x,y) \in f\} \subseteq B \times A.$$

That is, $f(x) = y$ if and only if $f^{-1}(y) = x$. Also, $\text{dom} f = \text{im} f^{-1}$ and $\text{im} f = \text{dom} f^{-1}$

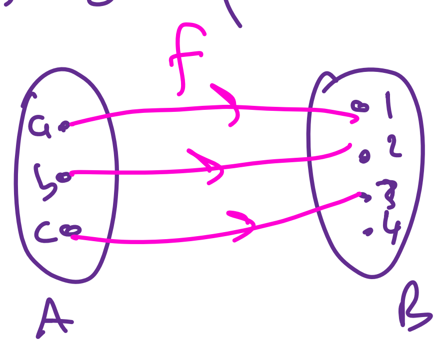
If f is a function, f^{-1} is not necessarily a function.

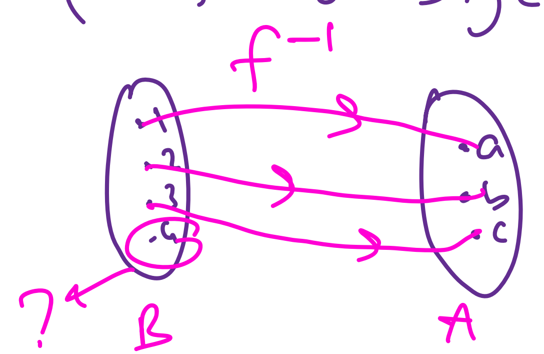
ex $A = \{-1, 0, 1, 2\}$ $f = \{(-1,1), (0,0), (1,1), (2,4)\}$ is a function

$B = \{0, 1, 2, 4\}$

but $f^{-1} = \{(1,-1), (0,0), (1,1), (4,2)\}$ is not a function

f^{-1} is a function iff f is a bijection.





$f^{-1} : B \to A$ is not a function ($\text{dom} f^{-1} \neq B$)

Proposition

Let $f : A \to B$. f^{-1} is a function from B to A (i.e., $f^{-1} : B \to A$) if and only if f is a bijection.

$\Rightarrow$) $f^{-1} : B \to A$ is a function $\Rightarrow$ f is a bijection

Suppose $f^{-1} : B \to A$ is a function. WTS that f is bijection.

First, f is one-to-one:

Next, f is onto:

`$ g\circ f$`

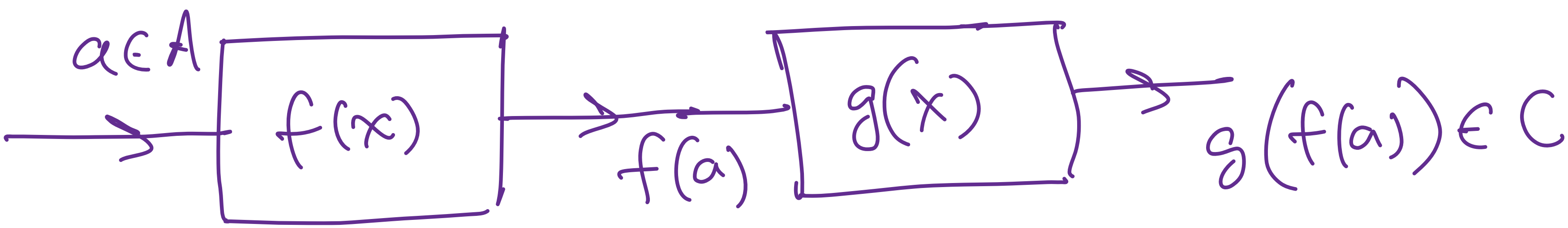
Definition

Let A, B, C be sets. Suppose $f : A \to B$ and $g : B \to C$. Define the **composition** of g and f, denoted $g \circ f$, to be a function from A to C where $\forall a \in A$,

$$(g \circ f)(a) = g(f(a)).$$

(read as "g of f of a")

In particular, $g \circ f : A \to C$; that is, $\operatorname{dom}(g \circ f) = \operatorname{dom} f = A$.



Note: $g \circ f \neq f \circ g$.

ex: $g : \mathbb{Z} \to \mathbb{Z}$
$x \mapsto x^2$
$f : \mathbb{Z} \to \mathbb{Z}$
$x \mapsto x+1$

$g(f(2)) = g(3) = 9$

$f(g(2)) = f(4) = 5$

(Poll Everywhere, 1-2/6)

Example

Let $f = \{(1,4),(2,2),(3,1),(4,2)\}$ and $g = \{(1,3),(2,3),(3,4),(4,1)\}$.

1. Find $g \circ f$
2. Find $f \circ g$
3. Find $f \circ f$

Theorem

If $f : A \to B$ and $g : B \to C$ are each one-to-one, then $g \circ f : A \to C$ is one-to-one.

Note: here, $f \circ g$ is not defined. For both $f \circ g$ and $g \circ f$ to be defined, $f : A \to A$ and $g : A \to A$

<u>proof:</u> Let $f : A \to B$ and $g : B \to C$ be one-to-one functions.

WTS $g \circ f$ is one-to-one, so $g \circ f(a_1) = g \circ f(a_2) \Rightarrow a_1 = a_2$

Assume $a_1, a_2 \in A$ and $g \circ f(a_1) = g \circ f(a_2)$

Then $g(\underline{f(a_1)}) = g(\underline{f(a_2)})$ by def of composition

Thus $\underline{f(a_1)} = \underline{f(a_2)}$ since g is one-to-one,

and $a_1 = a_2$ since f is one-to-one

Proposition

Let f and g be functions. Then $f = g$.

Proof Template.

Suppose that f and g are functions. To prove that $f = g$:

1. Show that $\operatorname{dom} f = \operatorname{dom} g$
2. Show that for every $x \in \operatorname{dom} f$, $f(x) = g(x)$.

$\square$

(Poll Everywhere, 5/6)

Definition

Let A be a set. The identity function on A, denoted Id_A, is a function whose domain is A, and for all $a \in A$, $\mathrm{Id}_A(a) = a$. That is,

$$\mathrm{Id}_A = \{(a,a) \,:\, a \in A\}.$$

(Poll Everywhere, 6/6)

Proposition

Suppose $f : A \to B$ is one-to-one and onto. Then:

1. $f \circ f^{-1} = \mathrm{Id}_B$ and
2. $f^{-1} \circ f = \mathrm{Id}_A$.

1. Let $f(x) = \dfrac{x}{x+1}$. Make a conjecture about $\overbrace{f \circ f \circ \cdots \circ f(x)}^{n \text{ times}}$ in terms of n and prove it.

2. Let E and F be two sets and $f : E \to F$. Let $A, B \subseteq E$.
   1. Prove that $A \subset B \implies f(A) \subset f(B)$. Is the reverse implication true?
   2. Prove that $f(A \cap B) \subset f(A) \cap f(B)$. Is the reverse inclusion true?

3. Let E and F be two sets and $f : E \to F$. Let $A, B \subseteq F$.
   1. Prove that $A \subset B \implies f^{-1}(A) \subset f^{-1}(B)$. Is the reverse implication true?