

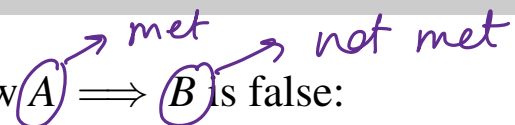
# Sections 6, 7

Today:

- ▶ §6: Counterexamples
  - ▶ Disprove statements that are false by providing a counterexample
- ▶ §7: Boolean Algebra
  - ▶ Define Boolean algebra and its basic operations
  - ▶ Show two statements are logically equivalent
  - ▶ Determine if Boolean expressions are tautologies, contingencies or contradictions

## Counterexamples for If-Then Statements

To show  $A \implies B$  is false:



- ▶ Find an instance where  $A$  is true, but  $B$  is false.
- ▶ Just one counterexample disproves a statement!

Note: Proof by example is NOT valid.



A "proof" is for a correct statement

## Example

Let  $n$  be an integer. If  $5 \mid (n^2 - 1)$  then  $5 \mid (n - 1)$ .

$n=1$ :  $\underbrace{5 \mid 0} \Rightarrow 5 \mid 0$  (This is true!)  
 $T \Rightarrow T$

$a \mid b$  means  $b = ca$  for some  $c \in \mathbb{Z}$   
(here:  $0 = c \cdot 5$  for some  $c \in \mathbb{Z}$ )

So  $n=1$  is not a counterexample.

$n=4$ :  $5 \mid (4^2 - 1)$  but  $5 \nmid (4 - 1)$   
 $5 \mid 15$  ✓ since  $5 \nmid 3$   
(condition is verified)

**(Poll Everywhere)**

## Counterexamples for If and Only If Statements

To show  $A \iff B$  is false:  $(A \iff B \text{ means } A \Rightarrow B \text{ and } B \Rightarrow A)$

- Find an instance where  $A$  is true, but  $B$  is false.

OR

$A \Rightarrow B$  is false

- Find an instance where  $B$  is true, but  $A$  is false.

$B \Rightarrow A$  is false

- One direction is enough
- One direction might be true while the other is false (that still makes  $A \iff B$  false)
- Clearly state which direction you're disproving

## Exercises

The following statements are false. Explain why.

- ❶ If  $a, b$ , and  $c$  are positive integers with  $a|(bc)$ , then  $a|b$  or  $a|c$ .
- ❷  $a, b$ , and  $c$  are positive integers.  $a^{(b^c)} = (a^b)^c$ .
- ❸  $p$  is prime if and only if  $2^p - 1$  is also prime.

# Algebra v.s. Boolean Algebra

	Algebra	Boolean Algebra
Expression	$3x - 2y$	$\neg x \vee y$
Variables	Numbers	True or False
Basic Operations	Addition $+$ Multiplication $\times$	And $\wedge$ Or $\vee$ Not $\neg$ Implication $\rightarrow$ Equivalence $\leftrightarrow$
Properties	Commutative Associative Distributive	Commutative Associative Distributive

core

shortcuts

$$\begin{aligned}
 x + y &= y + x \\
 &\equiv \\
 x \vee y &= y \vee x
 \end{aligned}$$

$$x \vee (y \vee z) = (x \vee y) \vee z$$

$$\begin{aligned}
 x \cdot (y + z) &= x \cdot y + x \cdot z \quad \equiv \quad x \wedge (y \vee z) \\
 &= (x \wedge y) \vee (x \wedge z)
 \end{aligned}$$

## Truth Table for Basic Operations

Let  $x$  and  $y$  be Boolean variables.

$x$	$y$	$x \wedge y$	$x \vee y$	$\neg x$	$x \rightarrow y$	$x \leftrightarrow y$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

$\wedge$  and

$\vee$  or

$\neg$  neg

$\rightarrow$  implies " $\Rightarrow$ "

iff " $\Leftrightarrow$ "  
 $\leftrightarrow$  left right arrow

**Remark:** Order of operation:  $\neg$  first, then read left to right. Expressions inside parentheses must be evaluated first.

## Examples

1.  $\neg \text{False} \vee \text{False} \wedge \text{True} =$

2. Suppose that  $x$  is True,  $y$  is False. Evaluate

$$\neg(x \wedge y) \vee ((\neg x) \wedge y)$$

**(Poll Everywhere)**



## Definition

When two Boolean expressions have the same output value for all possible values of their variables, we call these expressions **logically equivalent**.

**Example:** Show that  $\neg(x \wedge y)$  and  $(\neg x) \vee (\neg y)$  are logically equivalent.

$x$	$y$	$x \wedge y$	$\neg(x \wedge y)$	$\neg x$	$\neg y$	$(\neg x) \vee (\neg y)$
T	T					
T	F					
F	T					
F	F					

Prove logical equivalence > Method 1: TRUTH TABLE

## Example

Show that  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ .

$x$	$y$	$z$	$y \vee z$	$x \wedge (y \vee z)$	$x \wedge y$	$x \wedge z$	$(x \wedge y) \vee (x \wedge z)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

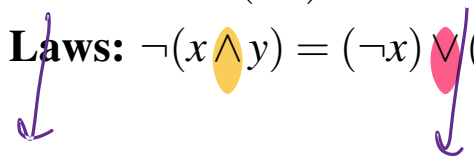
### Proposition (Proposition 7.3)

*The expressions  $x \rightarrow y$  and  $(\neg x) \vee y$  are logically equivalent.*

## Important Properties

pro-tip! prove these\* via truth table

[Theorem 7.2]

- ▶ **Commutative:**  $x \wedge y = y \wedge x$  and  $x \vee y = y \vee x$
  - \* ▶ **Associative:**  $(x \wedge y) \wedge z = x \wedge (y \wedge z)$  and  $(x \vee y) \vee z = x \vee (y \vee z)$
  - ▶ **Identity:**  $x \wedge \text{TRUE} = x$  and  $x \vee \text{FALSE} = x$
  - \* ▶ **Double negative:**  $\neg(\neg x) = x$
  - ▶ **Idempotency:**  $x \wedge x = x$  and  $x \vee x = x$
  - \* ▶ **Distributive:**  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  and  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
  - ▶  $x \wedge (\neg x) = \text{FALSE}$  and  $x \vee (\neg x) = \text{TRUE}$
  - \* ▶ **DeMorgan's Laws:**  $\neg(x \wedge y) = (\neg x) \vee (\neg y)$  and  $\neg(x \vee y) = (\neg x) \wedge (\neg y)$
-   
contradiction                      tautology

Prove logical equivalence > Method 2: PROPERTIES

## More Examples

1. Show that  $(x \wedge y) \vee (x \wedge \neg y) = x$ .

$$\begin{aligned}\underline{(x \wedge y)} \vee \underline{(x \wedge \neg y)} &= x \wedge (y \vee (\neg y)) \quad // \text{distributive} \\ &= x \wedge (\text{True}) \quad // \text{tautology} \\ &= x \quad // \text{identity}\end{aligned}$$

2. Show that  $x \rightarrow y = (\neg y) \rightarrow (\neg x)$ .

$$\begin{aligned}x \rightarrow y &= (\neg x) \vee y \quad // \text{prop. 7.3} \\ &= y \vee (\neg x) \quad // \text{commutative prop} \\ &= \neg(\underline{\neg y}) \vee \underline{(\neg x)} \quad // \text{double neg.} \\ &= \underline{(\neg y)} \rightarrow \underline{(\neg x)} \quad // \text{prop 7.3}\end{aligned}$$

(Poll everywhere)

## Exercises

Prove that  $x \leftrightarrow y$  is logically equivalent to  $(x \rightarrow y) \wedge ((\neg x) \rightarrow (\neg y))$

## Tautologies and Contradictions

- ▶ A **tautology** is a logical statement that is always true, for all truth values of its variables.

ex:  $p \rightarrow (p \vee q)$  (truth table E F Y)

- ▶ A **contradiction** is a logical statement that is always false, for all truth values of its variables.

ex:  $(p \vee q) \wedge (\neg p) \wedge (\neg q)$  (E F Y)

- ▶ A **contingency** is a logical statement that can be either true or false depending on the truth values of its variables.

see previous examples

## Classify the Boolean expression

Truth table

$$(x \vee y) \wedge (x \vee \neg y) \wedge (\neg x).$$

x	y	$\neg x$	$\neg y$	$x \vee y$	$x \vee \neg y$	Result
T	T	F	F	T	T	F
T	F	F	T	T	T	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

} contradiction

Logical algebra

$$\begin{aligned}
 (x \vee y) \wedge (x \vee \neg y) \wedge (\neg x) &= \underbrace{x \vee (y \wedge \neg y)}_{\text{False (Always)}} \wedge (\neg x) \quad (\text{distributive}) \\
 &= \underbrace{x \vee \text{False}}_{\text{(identity)}} \wedge (\neg x) \quad (\text{contradiction}) \\
 &= x \wedge (\neg x) = \text{False}
 \end{aligned}$$



## Examples of Classification

1.  $(\neg x) \vee y$

2.  $x \wedge (\neg x)$

(Poll Everywhere)

## Exercises

Without using a truth table, show that the following expression is a tautology

$$((x \rightarrow y) \wedge (x \rightarrow \neg y)) \rightarrow \neg x$$

## Summary of 6

- ▶ We cannot prove statements using examples.
- ▶ However, if there is one instance where the statement is false, we can *disproof* with an example.

## Summary of 7

- ▶ We use truth tables to show two Boolean expressions are *logically equivalent*.
- ▶ There are three classifications of Boolean expressions: *tautology* (all TRUE), *contradiction* (all FALSE), and *contingency* ( some TRUE and some FALSE).