Proposition

Recall $|S| = \text{card}(S) \in \mathbb{N}$ (size of set S)

Let A and B be finite sets and let $f : A \to B$. Then:

1. If $|A| > |B|$, then f is not one-to-one and
2. if $|A| < |B|$, then f is not onto.

Contrapositive

(1) if f is one-to-one, then $|A| \le |B|$

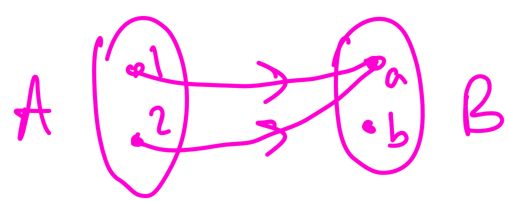
(2) if f is onto, then $|A| \ge |B|$

Thus:

if f is bijective then $|A| = |B|$

⚠ [if $|A| = |B|$, then f is bijective] is not true

ex:



$|A| = |B| = 2$, f is not bijective

How do we compare the size of two collections of objects?

Without counting, determine which of the two collections below has more elements/members.

No

```
Collection 1:  @ @ @ @ @ @ @ @ @ @ @ @ @ @
Collection 2:  & & & & & & & & & & & & & & & &
```

? ?

attempt to pair 1-1

Yes

```
Collection 1:  @ @ @ @ @ @ @ @ @ @ @ @ @ @
Collection 2:  # # # # # # # # # # # # # #
```

all elements can be paired, no leftovers one-to-one function vs. one-to-one correspondence = bijection

Recall: A and B are finite sets and there is a bijection f from A to B, then $|A| = |B|$.
We ask: can we compare "the size/cardinality" of infinite sets in a similar way?

Definition

Two sets A and B **have the same cardinality** provided that there is a bijection $f : A \to B$, or a *one-to-one correspondence* between the elements in sets A and B.

Consider the set of natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

and the set of natural number 1 and greater

$$\mathbb{N} - \{0\} = \mathbb{N}^* = \{1, 2, 3, 4, 5, 6, \ldots\}$$

Which of the above sets is "larger" (i.e., has more elements)?

(Poll Everywhere)

Two possible arguments

$$
\begin{array}{ccccccc}
\color{blue}{0} & \color{blue}{1} & \color{blue}{2} & \color{blue}{3} & \color{blue}{4} & \color{blue}{5} & \color{blue}{\dots} \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
 & \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} & \color{red}{\dots}
\end{array}
$$

not a one-to-one correspondence.

Instead:

$$
\begin{array}{ccccccccc}
\color{blue}{0} & & \color{blue}{1} & & \color{blue}{2} & & \color{blue}{3} & & \color{blue}{4} & & \color{blue}{5} & & \color{blue}{\dots} & & \color{blue}{n} \\
 & \nwarrow\!\!\searrow & & \nwarrow\!\!\searrow & & \nwarrow\!\!\searrow & & \nwarrow\!\!\searrow & & \nwarrow\!\!\searrow & & \nwarrow\!\!\searrow & & \nwarrow\!\!\searrow & \updownarrow \\
 & & \color{red}{1} & & \color{red}{2} & & \color{red}{3} & & \color{red}{4} & & \color{red}{5} & & \color{red}{\dots} & & \color{red}{n+1}
\end{array}
$$

for $n = 0, 1, 2....$

Formally: There exists a bijection between those two sets.

let $f : \mathbb{N} \to \mathbb{N} - \{0\}$
$n \longmapsto n+1$

or let $g : \mathbb{N} - \{0\} \to \mathbb{N}$
$n \longmapsto n-1$

Observe that $f^{-1} = g$.
EFY: prove that f (or g) is bijective

Example

Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

Rewrite $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$

This enumeration is generalizable to an expression.

$f : \mathbb{N} \longrightarrow \mathbb{Z}$

$$f(n) = \begin{cases} -\frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We propose a function $f : \mathbb{N} \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} -\dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

That is,

n	$f(n)$
0	0
1	1
2	-1
3	2
4	-2
5	3
6	-3
$\vdots$	$\vdots$

Note that f is indeed a function from $\mathbb{N}$ to $\mathbb{Z}$. We only need to show that f is one-to-one and onto.

First, we show that f is one-to-one. Suppose that $m, n \in \mathbb{N}$ and $f(m) = f(n)$. Since even numbers and mapped to negative numbers and odd numbers mapped to nonnegative numbers, then it must be the case that both m and n are even, or both m and n are odd.

- If both are even, then
$$-\frac{m}{2} = -\frac{n}{2}.$$
Multiply both sides by -2; then, $m = n$.
- If both are odd, then
$$\frac{m+1}{2} = \frac{n+1}{2}.$$
Multiply both sides by 2 and subtract 1; then, $m = n$.

So, f is one-to-one.

Next, we show that f is onto. Consider an arbitrary $y \in \mathbb{Z}$.

- Suppose $y \leq 0$. Let $n = -2y$. Note that n is even and $n \in \mathbb{N}$. So, $n \in \operatorname{dom} f$. Also note that

$$f(n) = -\frac{n}{2} = -\frac{-2y}{2} = y.$$

- Suppose that $y > 0$. Let $n = 2y - 1$. Note that n is odd, and $n \in \mathbb{N}$. So, $n \in \operatorname{dom} f$. Also note that

$$f(n) = \frac{n+1}{2} = \frac{2y-1+1}{2} = y.$$

So, f is onto.
Therefore, $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

Examples

Using the notion of one-to-one correspondence, compare the cardinality of the given pairs of sets.

- The set of all natural numbers and the set of all even natural numbers.

If this sounds counter-intuitive to you...

Keep in mind:

1. Our intuition about sizes is largely confined by our experience, which is mostly with finite sets!

2. When cardinality of infinite sets are involved, rely on one-to-one correspondence.

Note: just because we can't find an expression for the bijection, it does not mean that one does not exist. For example, consider the set S:

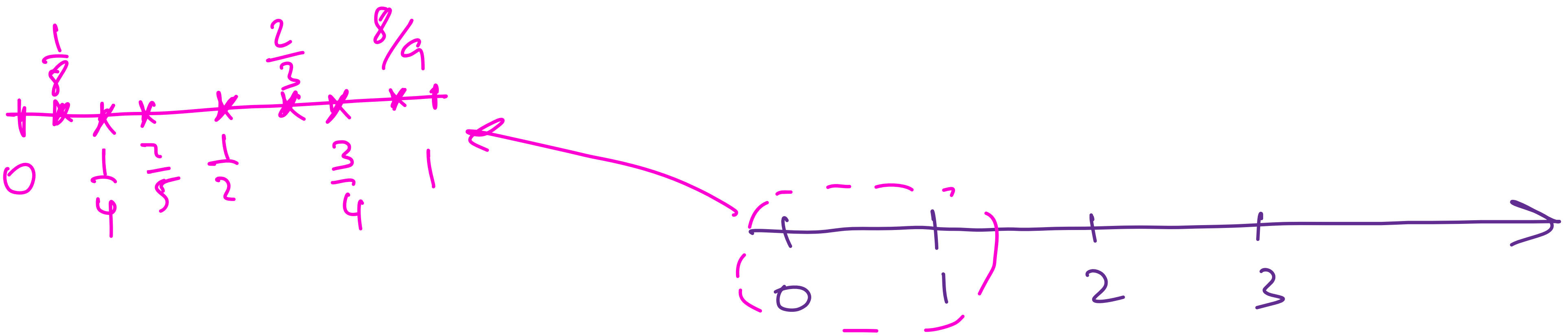
$$S = \{x \in \mathbb{N} \mid 3 | x \text{ or } x \text{ is prime}\}$$

$$S = \{0, 2, 3, 5, 6, 9, 11, 12, 13, 15, 17, \dots\}$$

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots, 205, \dots\}$$

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{l} a \in \mathbb{Z} \\ b \in \mathbb{Z}^{*} \end{array} \right\}$$

True or False: there are more rational numbers than natural numbers.



What about $\mathbb{Q}$?

start with $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x \geq 0\}$

The set of all natural numbers and the set of all (nonnegative) rational numbers

$$
\begin{array}{cccccc}
0 & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \;\; \frac{5}{1} \\
 & \frac{1}{2} & \cancel{\frac{2}{2}} & \frac{3}{2} & \cancel{\frac{4}{2}} & \cdots \\
 & \frac{1}{3} & \frac{2}{3} & \cancel{\frac{3}{3}} & \frac{4}{3} & \cdots \\
 & \frac{1}{4} & \cancel{\frac{2}{4}} & \frac{3}{4} & \frac{4}{4} & \cdots \\
 & \vdots \;\; \frac{1}{5} & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

start

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots\} = \mathbb{N}$

$\{0, \frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \ldots\} = \mathbb{Q}^+$

$|\mathbb{N}| = |\mathbb{Q}^+|$

Can you come up with a similar argument for the set of all natural numbers and the set of all rational numbers?

$$\begin{array}{ccccccccccc}
 & & & & & \vdots & \vdots & \vdots & \vdots & \cdots \\
 & & & & & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
 & & & & & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
 & & & & & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
 & & & & 0 & & & & & \\
\cdots & -\frac{3}{1} & -\frac{2}{1} & -\frac{1}{1} & & & & & & \\
\cdots & -\frac{3}{2} & -\frac{2}{2} & -\frac{1}{2} & & & & & & \\
\cdots & -\frac{3}{3} & -\frac{2}{3} & -\frac{1}{3} & & & & & & \\
\cdots & \vdots & \vdots & \vdots & & & & & &
\end{array}$$

Ok so do **ALL** infinite sets have the same cardinality?

Cardinality of the set of natural numbers and the set of real numbers

The set of natural numbers:

$$\{0, 1, 2, 3, 4, 5, 6, \ldots\}.$$

The set of real numbers:

$\{0, 0.0, 0.00 \ldots\ldots ? \ldots\}$

(how to list **all** real numbers systematically?)

Furthermore, do you think that we will be able to find a one-to-one correspondence between the natural numbers and the real numbers?

Theorem *

$\mathbb{N}$ and $\mathbb{R}$ have different cardinality.

countably infinite

uncountably infinite

Notation: $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = \aleph_1$

(\$\aleph_0\$) (\$\aleph_1\$)

so $|\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{N}| = |\mathbb{N}^*| = |\mathbb{E}| = |\mathbb{P}| = |\mathbb{N}_3| = \aleph_0$ (and so many more)

*Pf: We will show that $\mathbb{N}$ and $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ have different cardinalities.

How? We will show that there is no bijection between $\mathbb{N}$ and $(0,1)$

Cardinality of the set of natural numbers and the set of real numbers

Theorem

$\mathbb{N}$ and $\mathbb{R}$ have different cardinality.

Proof (idea): BWOC, suppose that there exists a bijection between $\mathbb{N}$ and $(0,1)$. Then we can rearrange the elements in that enumeration in a particular way, WLOG assume the following table:

Cantor's diagonal argument

$\mathbb{N}$	$x \in (0,1)$
0	$0.\boxed{3}34178\ldots$
1	$0.4\boxed{1}5670\ldots$
2	$0.00\boxed{0}721\ldots$
3	$0.123\boxed{4}567\ldots$
4	$0.21780\boxed{5}9\ldots$
5	$0.07432\boxed{1}98\ldots$
$\vdots$	$\vdots$

Let ~~$y = 0.\boxed{3}\boxed{1}\boxed{0}\boxed{4}\boxed{5}\boxed{1}\ldots$~~

$y = 0.\boxed{4}\boxed{2}\boxed{1}\boxed{5}\boxed{6}\boxed{2}\ldots$

(take the ith decimal place from the ith element in the enumeration and modify it by adding 1 to it. if it's 9, change it to zero)

Then $y \notin$ in the enumeration.

Conclusion

- For **any** pairing of natural numbers with real numbers, we can always point out a real number that has not been listed.
- Any such pairing is **not** a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{R}$.
- Therefore, the real numbers are "more numerous" than the natural numbers.

The cardinality of the real numbers is larger than the cardinality of the natural numbers!

- This argument is called "Cantor's Diagonalization Argument"

Thus: $\underbrace{|(0,1)|}_{\aleph_1} > \underbrace{|\mathbb{N}|}_{\aleph_0}$. $\Rightarrow \underbrace{|\mathbb{R}|}_{\aleph_1} > \underbrace{|\mathbb{N}|}_{\aleph_0}$

- There are infinitely-many natural numbers and infinitely-many real numbers

  but "the infinity of the real numbers is larger than the infinity of the natural numbers" } countability

  $\aleph_0$

- We refer to the cardinality of the natural numbers as **countable infinity** and the cardinality of the real numbers as **uncountable infinity**.

Continuum Hypothesis: let $\overset{\aleph_1}{S}$ be an infinite sized set.

$$(\nexists S \mid \aleph_0 < |S| < \aleph_1)$$

Cantor Diagonalization Argument
To prove Cantor's Power Set Theorem

if A is a finite set; $|A| = n$.

Then $|\underbrace{P(A)}_{\text{power set}}| = |2^A| = 2^{|A|} = 2^n > n$.

Cantor's Power Set Theorem
For any set S (finite or infinite), the cardinality of the power set of S is strictly greater than the cardinality of S.

ex: Consider $\mathbb{N}$. $|\mathbb{N}| = \aleph_0$

and $P(\mathbb{N})$: $|P(\mathbb{N})| > \aleph_0$ (continuum hypothesis)

$= \{\{1\}, \{2\}, \ldots, \{0,1\}, \ldots, \{1,4,5\}, \ldots\}$ $\Rightarrow |P(\mathbb{N})| = \aleph_1$

Theorem (Cantor's Theorem)

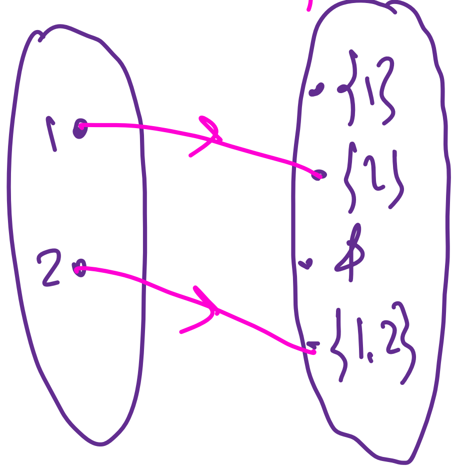
Let A be a set. If $f : A \to 2^A$, then f is not onto.

Note that if A is a finite set, then the proof is simple: Suppose that $n = |A|$. Since $|A| = n < 2^n = |2^A|$, then by the pigeonhole principle, f is not onto.
However, the theorem applies to both finite and infinite set. Therefore, the proof needs to be more general.

$n = 2$: $A = \{1, 2\}$ $P(A) = \{\{1\}, \{2\}, \emptyset, \{1,2\}\}$

$f: A \to P(A)$

WLOG



not onto

This is always true since $|2^A| > |A|$ $\forall$ set A (finite or infinite)