Sections 6, 7

Today:

- §6: Counterexamples
  - Disprove statements that are false by providing a counterexample
- §7: Boolean Algebra
  - Define Boolean algebra and its basic operations
  - Show two statements are logically equivalent
  - Determine if Boolean expressions are tautologies, contingencies or contradictions

Counterexamples for If-Then Statements

met   not met

To show $A \implies B$ is false:

- Find an instance where A is true, but B is false.
- Just one counterexample disproves a statement!

Note: Proof by example is NOT valid.

A "proof" is for a correct statement

Example

Let n be an integer. If $\underbrace{5 \mid (n^2 - 1)}$ then $5 \mid (n-1)$.

$n=1$: $5 \mid 0 \Rightarrow 5 \mid 0$ (This is true! $T \Rightarrow T$)

$a \mid b$ means $b = ca$ for some $c \in \mathbb{Z}$

(here: $0 = c \cdot 5$ for some $c \in \mathbb{Z}$)

So $n=1$ is not a counterexample.

$n=4$: $5 \mid (4^2-1)$ but $5 \nmid (4-1)$

$5 \mid 15$ ✓ since $5 \nmid 3$

(condition is verified)

(Poll Everywhere)

Counterexamples for If and Only If Statements

To show $A \iff B$ is false: ($A \Leftrightarrow B$ means $A \Rightarrow B$ and $B \Rightarrow A$)

- Find an instance where A is true, but B is false.

$\underbrace{\text{where } A \text{ is true, but } B \text{ is false}}_{A \Rightarrow B \text{ is false}}$

OR

- Find an instance where B is true, but A is false.

$\underbrace{\text{where } B \text{ is true, but } A \text{ is false}}_{B \Rightarrow A \text{ is false}}$

- One direction is enough
- One direction might be true while the other is false (that still makes $A \Leftrightarrow B$ false)
- Clearly state which direction you're disproving

Exercises

The following statements are false. Explain why.

1. If a, b, and c are positive integers with $a|(bc)$, then $a|b$ or $a|c$.
2. a, b, and c are positive integers. $a^{(b^c)} = (a^b)^c$.
3. p is prime if and only if $2^p - 1$ is also prime.

Algebra v.s. Boolean Algebra

	Algebra	Boolean Algebra
Expression	$3x - 2y$	$\neg x \vee y$
Variables	Numbers	True or False
Basic Operations	Addition $+$ Multiplication $\times$	And $\wedge$ Or $\vee$ Not $\neg$ Implication $\rightarrow$ Equivalence $\leftrightarrow$
Properties	Commutative Associative Distributive	Commutative Associative Distributive

core

shortcuts

$$x + y = y + x$$
$$\equiv$$
$$x \vee y = y \vee x$$

$$x \vee (y \vee z) = (x \vee y) \vee z$$

$$x \cdot (y + z) = x \cdot y + x \cdot z \quad // \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Truth Table for Basic Operations

\land and   \vee or   \neg neg   \rightarrow implies "$\Rightarrow$"   iff "$\Leftrightarrow$" \leftrightarrow

Let x and y be Boolean variables.

x	y	$x \wedge y$	$x \vee y$	$\neg x$	$x \rightarrow y$	$x \leftrightarrow y$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

Remark: Order of operation: $\neg$ first, then read left to right. Expressions inside parentheses must be evaluated first.

Examples

1. $\neg$ False $\vee$ False $\wedge$ True $=$

2. Suppose that x is True, $y =$is False. Evaluate

$$\neg(x \wedge y) \vee ((\neg x) \wedge y)$$

(Poll Everywhere)

Definition

When two Boolean expressions have the same output value for all possible values of their variables, we call these expressions **logically equivalent**.
Example: Show that $\neg(x \wedge y)$ and $(\neg x) \vee (\neg y)$ are logically equivalent.

x	y	$x \wedge y$	$\neg(x \wedge y)$	$\neg x$	$\neg y$	$(\neg x) \vee (\neg y)$
T	T					
T	F					
F	T					
F	F					

Prove logical equivalence > Method 1: TRUTH TABLE

Example

Show that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

x	y	z	$y \vee z$	$x \wedge (y \vee z)$	$x \wedge y$	$x \wedge z$	$(x \wedge y) \vee (x \wedge z)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Proposition (Proposition 7.3)

The expressions $x \to y$ and $(\neg x) \vee y$ are logically equivalent.

Important Properties

pro-tip: prove these* via truth table

[Theorem 7.2]

- **Commutative:** $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
- * **Associative:** $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$
- **Identity:** $x \wedge TRUE = x$ and $x \vee FALSE = x$
- * **Double negative:** $\neg(\neg x) = x$
- **Idempotency:** $x \wedge x = x$ and $x \vee x = x$
- * **Distributive:** $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (\neg x) = FALSE$ and $x \vee (\neg x) = TRUE$
- * **DeMorgan's Laws:** $\neg(x \wedge y) = (\neg x) \vee (\neg y)$ and $\neg(x \vee y) = (\neg x) \wedge (\neg y)$

contradiction tautology

Prove logical equivalence > Method 2: PROPERTIES

More Examples

1. Show that $(x \wedge y) \vee (x \wedge \neg y) = x$.

$$\begin{aligned}(x \wedge y) \vee (x \wedge \neg y) &= x \wedge (y \vee (\neg y)) && \text{// distributive}\\ &= x \wedge (\text{True}) && \text{// tautology}\\ &= x && \text{// identity}\end{aligned}$$

2. Show that $x \to y = (\neg y) \to (\neg x)$.

$$\begin{aligned}x \to y &= (\neg x) \vee y && \text{// prop. 7.3}\\ &= y \vee (\neg x) && \text{// commutative prop}\\ &= \neg(\neg y) \vee (\neg x) && \text{// double neg.}\\ &= (\neg y) \to (\neg x) && \text{// prop 7.3}\end{aligned}$$

(Poll everywhere)

Exercises

Prove that $x \leftrightarrow y$ is logically equivalent to $(x \to y) \wedge ((\neg x) \to (\neg y))$

Tautologies and Contradictions

- A **tautology** is a logical statement that is always true, for all truth values of its variables.

ex: $p \to (p \vee q)$ (truth table EFY)

- A **contradiction** is a logical statement that is always false, for all truth values of its variables.

ex: $(p \vee q) \wedge (\neg p) \wedge (\neg q)$ (EFY)

- A **contingency** is a logical statement that can be either true or false depending on the truth values of its variables.

See previous examples

Classify the Boolean expression

Truth table

$$(x \vee y) \wedge (x \vee \neg y) \wedge (\neg x).$$

x	y	$\neg x$	$\neg y$	$x \vee y$	$x \vee \neg y$	Result
T	T	F	F	T	T	F
T	F	F	T	T	T	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

contradiction

Logical algebra

$$\underbrace{(x \vee y) \wedge (x \vee \neg y)} \wedge (\neg x) = \underbrace{x \vee (\overset{\text{False (Always)}}{y \wedge \neg y})} \wedge (\neg x) \quad \text{(distributive)}$$

$$= \underbrace{x \vee \text{False}} \wedge (\neg x) \quad \text{(contradiction)}$$

$$\text{(identity)} \quad = x \wedge (\neg x) = \text{False}$$

Examples of Classification

1. $(\neg x) \vee y$
2. $x \wedge (\neg x)$

(Poll Everywhere)

Exercises

Without using a truth table, show that the following expression is a tautology

$$((x \to y) \wedge (x \to \neg y)) \to \neg x$$

Summary of 6

- We cannot prove statements using examples.
- However, if there is one instance where the statement is false, we can *disproof* with an example.

Summary of 7

- We use truth tables to show two Boolean expressions are *logically equivalent*.
- There are three classifications of Boolean expressions: *tautology* (all TRUE), *contradiction* (all FALSE), and *contingency* (some TRUE and some FALSE).