Case Study # 4: Linear 1D Transport Equation

Background: The transport of various scalar quantities (e.g species mass fraction, temperature) in flows can be modeled using a linear convection-diffusion equation (presented here in a 1-D form),

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \mathcal{D}\frac{\partial^2 \phi}{\partial x^2}$$

ϕ is the transported scalar, u and $\mathcal{D}$ are known parameters (flow velocity and diffusion coefficient resp.). The purpose of this case study is to investigate the behavior of various numerical solutions of this equation.

Investigation: This case study focuses on the solution of the 1-D linear transport equation (above) for $x \in [0, L]$ and $t \in [0, \tau]$ (where $\tau = 1/k^2\mathcal{D}$) subject to periodic boundary conditions and the following initial condition

$$\phi(x, 0) = \sin(kx),$$

with $k = 2\,\pi/L$ and L=1 m.

NOTE—This problem has an analytical solution [1],

$$\Phi(x, t) = \exp\left(-k^2\mathcal{D}t\right)\ \sin\left[k(x - ut)\right]$$

Implement a numerical solution of this problem using each of the following schemes,

- **Trapezoidal**—(AKA Crank-Nicholson) with central differencing for both the convective flux and the diffusive flux.
- **Central Differencing**—Use an appropriate ODE solver of your choice for time integration together with central differencing for both the convective flux and the diffusive flux.
- **Upwind**—Finite Volume method: Use an appropriate ODE solver of your choice for time integration together with the convective flux treated using the basic upwind method and the diffusive flux treated using central differencing.
- **QUICK**—Finite Volume method: Use an appropriate ODE solver of your choice for time integration together with the convective flux treated using the QUICK method and the diffusive flux treated using central differencing.

The convection velocity is $u = 0.2$ m/s, and the diffusion coefficient is $\mathcal{D} = 0.005$ m^2/s. Use a uniform mesh. Consider the following cases:

$$(C, s) \in \{(0.1, 0.25), (0.5, 0.25), (2, 0.25), (0.5, 0.5), (0.5, 1)\}$$

where $C = u\Delta t/\Delta x$ and $s = \mathcal{D}\Delta t/\Delta x^2$. Comment on the stability and accuracy of your solutions.

Report: Prepare a report (3-6 pages max in ASME's two-colum article format, templates are available on-line) describing your work and including:

1. Short description of the problem

2. Numerical Solution Approach (including boundary condition implementation)

3. Results and Discussion – Comment on the stability, accuracy, and effective order (w.r.t. Δt and Δx) of your solutions and include the necessary plots to support the discussion.

4. Conclusion.

The Report (single file, PDF format only) and your source code(s) (`.py` uploaded as a single separate file (please zip source files if there are more than one) are due on **November 18, 2015 at 5pm** and must be submitted electronically using the class SmartSite.

References

[1] J. C. Tannehill, D. A. Anderson, and R. C. Pletcher. *Computational Fluid Mechanics and Heat Transfer*. Computational and Physical Processes in Mechanics and Thermal Sciences. Taylor & Francis, 2nd edition, 1997.