# Instructor Solutions

For Version 2.0 (HTML)

## Section 1.4

**Exercise Solution 1.4.1.** *General solution* $u(t) = t^2/2 + C$*, particular solution* $u(t) = t^2/2 + 3$.

**Exercise Solution 1.4.2.** *General solution* $u(t) = \sin(t) + C$*, particular solution* $u(t) = \sin(t)$.

**Exercise Solution 1.4.3.** *General solution* $u(t) = e^t + C$*, particular solution* $u(t) = e^t + 3$.

**Exercise Solution 1.4.4.** *General solution* $u(t) = \ln|t| + C$*, particular solution* $u(t) = \ln|t| - \ln(2)$.

**Exercise Solution 1.4.5.** *General solution* $u(t) = \sin(t) + C$*, particular solution* $u(t) = \sin(t) + 1$.

**Exercise Solution 1.4.6.** *General solution* $u(t) = t\sin(t) + \cos(t) + C$*, particular solution* $u(t) = t\sin(t) + \cos(t)$.

**Exercise Solution 1.4.7.** *General solution* $v(t) = gt$*, particular solution* $v(t) = gt + v_0$.

**Exercise Solution 1.4.8.** *General solution* $h(t) = t^{n+1}/(n+1) + C$*, particular solution* $v(t) = t^{n+1}/(n+1)$.

**Exercise Solution 1.4.9.** *General solution* $u(t) = t^3/6 + C_1 t + C_2$*, particular solution* $u(t) = t^3/6 + 3t + 1$.

**Exercise Solution 1.4.10.** *General solution* $u(t) = -\sin(t) + C_1 t + C_2$*, particular solution* $u(t) = -\sin(t) + t + 1$.

**Exercise Solution 1.4.11.** *General solution* $y(t) = -gt^2/2 + C_1 t + C_2$*, particular solution* $y(t) = -gt^2/2 + 10$.

**Exercise Solution 1.4.12.** *General solution* $x(t) = 5t^2/2 - e^{-2t}/4 + C_1 t + C_2$*, particular solution* $x(t) = 5t^2/2 - e^{-2t}/4 - t/2 + 1/4$.

**Exercise Solution 1.4.13.** *The input salt rate to the tank is* $5\frac{\text{liter}}{\text{min}} \times 50\frac{\text{grams}}{\text{liter}} = 250\frac{\text{grams}}{\text{minute}}$. *The outflow rate of salt is* $5\frac{\text{liter}}{\text{min}} \times \frac{u(t)}{100}\frac{\text{grams}}{\text{liter}} = \frac{u(t)}{20}\frac{\text{grams}}{\text{minute}}$. *The ODE is*

$$u'(t) = 250 - \frac{u(t)}{20}$$

*with initial condition* $u(0) = 0$. *The solution is* $u(t) = 5000 - 5000e^{-t/20}$ *grams. The solution rises from* $u(0) = 0$ *and asymptotically approaches* $u = 5000$ *grams of salt in the tank. The limiting concentration is* $5000/100 = 50$ *grams per liter, the same as the incoming salt solution.*

## Section 1.5

**Exercise Solution 1.5.1.**

(a) *Momentum is mass times velocity, so has dimension* $MLT^{-1}$.

(b) *Angular velocity is measured in radians per unit time, so has dimension* $T^{-1}$.

(c) *From force times distance we have* $[Fd] = [F][d] = MLT^{-2}L = ML^2T^{-2}$.

(d) *Pressure is force per area, so has dimension* $MLT^{-2}L^{-2} = ML^{-1}T^{-2}$.

**Exercise Solution 1.5.2.** *We have* $[E] = ML^2T^{-2}$ *(from the text) and* $[c] = LT^{-1}$, *while* $[\lambda] = L$. *Then from* $h = E\lambda/c$ *we find* $[h] = (ML^2T^{-2})(L)/(LT^{-1}) = ML^2T^{-1}$.

**Exercise Solution 1.5.3.** *From* $v' = P - kv$ *we see that we need* $[v'] = [kv]$, *or* $LT^{-2} = [k]LT^{-1}$, *so* $[k] = T^{-1}$.

**Exercise Solution 1.5.4.** *Consider the difference quotient* $(f(x + \Delta x) - f(x))/\Delta x$. *The dimension of the numerator is* $M^aL^bT^c$ *and the dimension of the denominator is* $M^\alpha L^\beta T^\gamma$. *The difference quotient has dimension* $(M^aL^bT^c)(M^\alpha L^\beta T^\gamma) = M^{a-\alpha}L^{b-\beta}T^{c-\gamma}$. *Then* $f'(x)$ *should also have this dimension.*

**Exercise Solution 1.5.5.** *The function* $u(t)$ *has dimension* $M$ *(mass), so* $[u'(t)] = MT^{-1}$. *Also,* $[r] = L^3T^{-1}$ *(volume per time) and* $[c_1] = ML^{-3}$ *(mass per volume). Also* $[V] = L^3$. *Then* $[rc_1] = L^3T^{-1}ML^{-3} = MT^{-1}$ *and* $[ru/V] = L^3T^{-1}ML^{-3} = MT^{-1}$. *Thus each of* $u', rc_1,$ *and* $ru/V$ *has dimension* $MT^{-1}$ *and the ODE is dimensionally consistent.*

*In the solution* $u(t) = c_1V(1-e^{-rt/V})$ *we find that* $[-rt/V] = L^3T^{-1}TL^{-3} = 1$, *so the argument to the exponential is dimensionless, and hence so is the quantity* $(1 - e^{-rt/V})$. *The quantity* $[c_1V] = ML^{-3}L^3 = M$ *has dimension mass, and this is consistent with* $[u] = M$.

**Exercise Solution 1.5.6.** *If* $[u(t)] = N$ *then we must have* $[u/K] = 1$ *in order to compute* $(1 - u/K)$. *Then* $[u'] = NT^{-1}$ *and then we must have* $[ru] = NT^{-1}$ *if* $ru(1 - u/K)$ *is to have dimension* $NT^{-1}$. *this means that* $[r] = T^{-1}$, *typical for any "rate" constant.*

**Exercise Solution 1.5.7.** *We have* $[P] = T$, $[2\pi] = 1$, $[r] = L$, $[G] = M^{-1}L^3T^{-2}$, *and* $[m] = M$. *Then*

$$[2\pi\sqrt{r^3/(Gm)}] = (1)L^{3/2}M^{1/2}L^{-3/2}T^1M^{-1/2} = T$$

*which is* $[P]$, *so this is dimensionally consistent.*

**Exercise Solution 1.5.8.** *With* $[v] = LT^{-1}$, $[G] = M^{-1}L^3T^{-2}$, $[m] = M$, *and* $[r] = L$ *this requires* $LT^{-1} = (M^{-a}L^{3a}T^{-2a})M^bL^c = M^{-a+b}L^{3a+c}T^{-2a}$ *or*

$$-a + b = 0, \quad 3a + c = 1, \quad -2a = -1$$

*which leads to* $a = 1/2, b = 1/2$, *and* $c = -1/2$. *Then* $v = K\sqrt{Gm/r}$ *for some dimensionless constant* $K$. *This is in fact correct with* $K = \sqrt{2}$.

**Exercise Solution 1.5.9.** *We have* $[P] = T$, $[\ell] = L$, $[m] = M$, *and* $[g] = LT^{-2}$. *A formula of the form* $P = \ell^a m^b g^c$ *requires* $T = L^aM^bL^cT^{-2c}$, *which leads to* $b = 0, a + c = 0, -2c = 1$, *so* $a = 1/2, b = 0, c = -1/2$, *and then*

$$P = K\sqrt{\ell/g}$$

*for some dimensionless constant* $K$ *For the "linearized pendulum" this is correct, with* $K = 2\pi$; *for the general nonlinear pendulum this is also correct, but* $K$ *depends on the initial angle of the pendulum.*

**Exercise Solution 1.5.10.** *With* $[c] = LT^{-1}$, $[P] = ML^{-1}T^{-2}$, *and* $[\rho] = ML^{-3}$ *and a formula of the form* $c = P^a\rho^b$ *we need* $LT^{-1} = M^aL^{-a}T^{-2a}M^bL^{-3b}$, *which leads to* $a + b = 0, -a - 3b = 1$, *and* $-2a = -1$, *three equations in two unknowns, but they are consistent. The solution is* $a = 1/2$ *and* $b = -1/2$. *Then*

$$c = K\sqrt{P/\rho}$$

*for some dimensionless constant* $K$.

**Exercise Solution 1.5.11.** *We have* $[f] = T^{-1}$, $[\lambda] = ML^{-1}$, $[\tau] = MLT^{-2}$, *and* $[\ell] = L$. *Then* $f = \lambda^a\tau^b\ell^c$ *forces* $T^{-1} = M^aL^{-a}M^bL^bT^{-2b}L^c$ *or*

$$a + b = 0, \quad , -a + b + c = 0, \quad -2b = -1$$

*with solution* $a = -1/2, b = 1/2,$ *and* $c = -1$. *Then*

$$f = \frac{K}{\ell}\sqrt{\tau/\lambda}$$

*for some dimensionless constant* $K$ *(which turns out as* $K = 1/2$ *in ideal situations.)*

**Exercise Solution 1.5.12.** *Consider a dimensionless product of the form*

$$e^a k_e^\beta \hbar^\gamma c^\delta.$$

*Then*

$$[e^a k_e^\beta \hbar^\gamma c^\delta] = M^{\beta+\gamma} L^{3\beta+2\gamma+\delta} T^{-2\beta-\gamma-\delta} Q^{a-2\beta} = 1.$$

*This leads to equations*

$$\beta + \gamma = 0, \quad 3\beta + 2\gamma + \delta = 0, \quad -2\beta - \gamma - \delta = 0, \quad a - 2\beta = 0.$$

*The equations are dependent; we can choose (for example)* $\beta$ *arbitrarily and then* $a = 2\beta, \gamma = -\beta,$ *and* $\delta = -\beta$. *Then* $\beta = 1$ *leads to* $a = 2, \gamma = \delta = -1,$ *and dimensionless product*

$$\alpha = \frac{e^2 k_e}{\hbar c}.$$

*From the given data we have* $a \approx 0.00729735257 \approx 1/137.036$.

## Section 2.1

**Exercise Solution 2.1.1.** *Integrating factor* $e^{-t}$*, general solution* $u(t) = Ce^t - 3$*, specific solution is* $u(t) = 6e^t - 3$.

**Exercise Solution 2.1.2.** *Integrating factor* $e^{-2t}$*, general solution* $u(t) = Ce^{2t} - 2$*, specific solution is* $u(t) = 2e^{2t} - 2$.

**Exercise Solution 2.1.3.** *Integrating factor* $e^{3t}$*, general solution* $u(t) = Ce^{-3t} + 1$*, specific solution is* $u(t) = 4e^{-3t} + 1$.

**Exercise Solution 2.1.4.** *Integrating factor* $e^{3t}$*, general solution* $u(t) = Ce^{-3t} + 3t - 1$*, specific solution is* $u(t) = 6e^{-3t} + 3t - 1$.

**Exercise Solution 2.1.5.** *Integrating factor* $e^{-t}$*, general solution* $u(t) = Ce^t - \sin(t) - \cos(t)$*, specific solution is* $u(t) = 2e^t - \sin(t) - \cos(t)$.

**Exercise Solution 2.1.6.** *Integrating factor* $e^{4t}$*, general solution* $u(t) = Ce^{-4t} + e^t/5$*, specific solution is* $u(t) = \frac{9}{5}e^{-4t} + \frac{e^t}{5}$.

**Exercise Solution 2.1.7.** *Integrating factor* $e^{-t^2/2}$*, general solution* $u(t) = Ce^{t^2/2} - 1$*, specific solution is* $u(t) = 3e^{t^2/2} - 1$.

**Exercise Solution 2.1.8.** *Integrating factor* $1/t$*, general solution* $u(t) = Ct + 2t\ln(t)$*, specific solution is* $u(t) = 3t + 2t\ln(t)$.

**Exercise Solution 2.1.9.** *Integrating factor* $e^{-\cos(t)}$*, general solution* $u(t) = Ce^{-\cos(t)} - 1$*, specific solution is* $u(t) = 5e^1e^{-\cos(t)} - 1 = 5e^{1-\cos(t)} - 1$.

**Exercise Solution 2.1.10.** *Integrating factor* $e^{-at}$*, general solution* $u(t) = Ce^{at} - b/a$*, specific solution is* $u(t) = (u_0 + b/a)e^{at} - b/a$.

**Exercise Solution 2.1.11.** *Rearrange as* $u'(t)+ru(t)/V = rc_1$ *and multiply through by integrating factor* $e^{rt/V}$ *to obtain*

$$\frac{d}{dt}(e^{rt/V}u(t)) = rc_1e^{rt/V}.$$

*Integrate and obtain* $e^{rt/V}u(t) = Vc_1e^{rt/V} + C$ *so* $u(t) = Vc_1 + Ce^{-rt/V}$ *is the general solution. Then* $u(0) = 0$ *forces* $Vc_1 + C = 0$*, so* $C = -Vc_1$ *and* $u(t) = Vc_1(1 - e^{-rt/V})$.

**Exercise Solution 2.1.12.**

*(a)* $[k] = T^{-1}$.

(b) *Write the ODE as* $u'(t) + ku(t) = 0$ *and use integrating factor* $e^{kt}$ *to find* $u(t) = Ce^{-kt}$*, Then* $u(0) = u_0$ *implies* $C = u_0$*, so* $u(t) = u_0e^{-kt}$*. Since* $k$ *is positive the exponential decays to zero as* $t$ *increases to infinity.*

(c) *The equation* $u(t + \Delta t) = u(t)/2$ *becomes* $u_0e^{-k(t+\Delta t)} = u_0e^{-kt}/2$*, which simplifies to* $e^{-k\Delta t} = 1/2$*. Solve for* $\Delta t = \ln(2)/k$*. This does not depend on the variable* $t$ *itself.*

**Exercise Solution 2.1.13.** *Write the ODE as* $u'(t) + ku(t) = kA$ *and use integrating factor* $e^{kt}$ *to find* $d(e^{kt}u(t))/dt = kAe^{kt}$*. Integrate to find* $e^{kt}u(t) = Ae^{kt} + C$ *and so* $u(t) = A + Ce^{-kt}$ *is the general solution. Then* $u(0) = u_0$ *yields* $A + C = u_0$*, so* $C = u_0 - A$ *and* $u(t) = A + (u_0 - A)e^{-kt}$*.*

**Exercise Solution 2.1.14.** *Write the ODE as* $x'(t) + x(t)/100 = 0.2$ *and use integrating factor* $e^{t/100}$ *to find* $d(e^{t/100}x(t))/dt = 0.2e^{t/100}$*. Integrate to find* $e^{t/100}x(t) = 20e^{t/100} + C$ *and so* $x(t) = 20 + Ce^{-t/100}$ *is the general solution. Then* $x(0) = 3$ *yields* $20 + C = 3$*, so* $C = -17$ *and* $x(t) = 20 - 17e^{-t/100}$*.*

**Exercise Solution 2.1.15.** *The solution to* $u'(t) = -k(u(t) - 70)$ *with* $u(0) = 92.6$ *is* $u(t) = 70 + 22.6e^{-kt}$*. If* $u(2) = 90$ *then* $70 + 22.6e^{-2k} = 90$ *and so* $k \approx 0.0611$*. Then solve* $u(t) = 70 + 22.6e^{-0.0611t} = 102.4$ *for* $t$ *to find* $t \approx -5.895$ *hours.*

**Exercise Solution 2.1.16.** *The rate in is* $(0.2)(4) = 0.8$ *kg per minute, and the rate out is* $(x(t)/400)(4) = x(t)/100$ *kg per minute. The ODE is* $x'(t) = 0.8 - x(t)/100$ *with* $x(0) = 0$*. The solution is* $x(t) = 80 - 80e^{-t/100}$*. The amount of salt limits to* $80$ *kg.*

**Exercise Solution 2.1.17.**

(a) *The volume in the tank at time* $t$ *is* $V(t) = 400 - t$*.*

(b) *The rate in is* $(0.2)(4) = 0.8$ *kg per minute, and the rate out is* $(x(t)/V(t))(5) = 5x(t)/(400 - t)$ *kg per minute. The appropriate ODE is thus*

$$x'(t) = 0.8 - 5x(t)/(400 - t)$$

*with initial condition* $x(0) = 0$*.*

(c) *To solve write the ODE as* $x'(t) + 5x(t)/(400 - t) = 0.8$*. The integrating factor is* $e^{-5\ln(400-t)} = 1/(t - 400)^5$ *(drop absolute values by using the* *fact that* $0 \le t \le 400$*, since the tank will be empty after* $t = 400$*.) Multiply both sides of the ODE by this integrating factor and find*

$$\frac{d}{dt}\left(\frac{x(t)}{(t-400)^5}\right) = \frac{0.8}{(t-400)^5}.$$

*Integrate and find*

$$\frac{x(t)}{(t-400)^5} = \frac{-0.2}{(t-400)^4} + C.$$

*The general solution is then*

$$x(t) = 0.2(400-t) + C(t-400)^5.$$

*The condition* $x(0) = 0$ *forces* $80 - C(400)^5 = 0$*, so* $C = 80/400^5$*. The solution is*

$$x(t) = 80 - t/5 + 80(t/400 - 1)^5$$

*after simplifying.*

(d) *The function* $x(t)$ *rises from* $x(0) = 0$*, peaks at about* $x(132.5) \approx 42.8$*, then drops to* $x(400) = 0$*, when the tank is empty.*

**Exercise Solution 2.1.18.**

(a) *Let us take a coordinate system in which up is the positive direction, down is negative. As the rock falls through the air the ODE that governs its motion is (from* $ma = F$*)* $5v'(t) = 5g - 0.8v(t)$ *or* $v'(t) = g - 0.16v(t)$ *with* $g = -9.8$*, and initial condition* $v(0) = 0$*, assuming* $t = 0$ *is the time the rock is dropped. Note that when* $v < 0$ *the air resistance term is* $-0.8v > 0$*, upward. The solution is*

$$v(t) = 61.25(e^{-0.16t} - 1).$$

*With* $x = 0$ *as the air/water interface, the altitude of the rock at time* $t$ *is*

$$x(t) = \int_{z=0}^{t} v(z)\,dz + 200 = 582.8125 - 61.25t - 382.8125e^{-0.16t}.$$

*The rock hits the water when* $x(t) = 0$*, which leads to* $t \approx 7.689$ *seconds. At this instant* $v(7.689) \approx -43.35$ *meters per second.*

(b) *For convenience, now set* $t = 0$ *as the moment that the rock hits the water. The ODE that now governs the rock's motion is* $5v'(t) = 5g - 5v(t)$ *or just* $v'(t) = g - v(t)$ *with initial condition* $v(0) = -43.35$ *meters per second. The solution is* $v(t) = -9.8 - 33.65e^{-t}$*. The position of the rock, with* $x = 0$ *as the air/water interface, is*

$$x(t) = \int_{z=0}^{t} v(z)\, dz = -33.65 - 9.8t + 33.65e^{-t}.$$

*The rock hits the bottom when* $x(t) = -10$*, which occurs at* $t \approx 0.253$ *seconds. The rock thus hits the bottom about* $7.689 + 0.253 = 7.942$ *seconds after being dropped.*

(c) *The velocity of the rock is* $v(0.253) \approx -35.85$ *meters per second when it hits the bottom.*

**Exercise Solution 2.1.19.**

(a) *Write the ODE as* $q'(t) + q(t)/RC = V_0/R$ *and use integrating factor* $e^{t/RC}$ *to obtain*

$$\frac{d}{dt}(q(t)e^{t/RC}) = (V_0/R)e^{t/RC}.$$

*Integrate to find*

$$e^{t/RC}q(t) = V_0Ce^{t/RC} + A$$

*for some arbitrary constant of integration* $A$*. The general solution is then* $q(t) = V_0C + Ae^{-t/RC}$*. If* $q(0) = 0$ *then* $A = -V_0C$ *and the solution is* $q(t) = V_0C(1 - e^{-t/RC})$*.*

(b) *As* $t \to \infty$ *we find* $q(t) \to V_0C$*.*

(c) *With* $[C] = [q]/[V] = M^{-1}L^{-2}T^2Q^2$ *and* $[R] = ML^2T^{-1}Q^{-2}$ *we find* $[RC] = [R][C] = T$*.*

(d) *This occurs when* $e{-}t/RC = 1/100$*, which leads to* $t = RC\ln(100) \approx 4.6RC$*.*

**Exercise Solution 2.1.20.**

(a) *Write the ODE as* $q'(t) + q(t)/RC = V_0\sin(\omega t)/R$ *and use integrating factor* $e^{t/RC}$ *to obtain*

$$\frac{d}{dt}(q(t)e^{t/RC}) = V_0e^{t/RC}\sin(\omega t)/R.$$

*Integrate to find*

$$e^{t/RC}q(t) = e^{t/RC}(A\cos(\omega t) + B\sin(\omega t)) + D$$

*where* $A = -V_0\omega RC^2/(1+(RC\omega)^2)$ *and* $B = V_0C/(1+(RC\omega)^2)$ *and* $D$ *is an arbitrary constant. The general solution is then*

$$q(t) = A\cos(\omega t) + B\sin(\omega t) + De^{-t/RC}.$$

*With* $R$ *and* $C$ *as specified (note* $RC = 1$*) this is as asserted.*

*(b) We find (using* $R = 10^6, C = 10^{-6}$*)*

$$V_C(t) = e^{-t}(A_2\cos(\omega t) + B_2\sin(\omega t)) + De^{-t}.$$

*with* $A_2 = -\frac{V_0\omega}{\omega^2+1}$ *and* $B_2 = \frac{V_0}{\omega^2+1}$*. As* $t$ *increases the exponential term decays to zero.*

*(c) The amplitude of* $V_C(t)$ *is* $\sqrt{A_2^2+B_2^2}$ *which simplifies to* $V_0/\sqrt{\omega^2+1}$*.*

*(d) When* $\omega \approx 0$ *we find the amplitude of* $V_c$ *is about equal to* $V_0$*, but as* $\omega$ *increases the amplitude of* $V_c$ *tapers to* $0$*.*

**Exercise Solution 2.1.21.** *Write* $-w(t)h(t)u(t) = w'(t)u(t)$ *as* $w'(t)u(t) + w(t)h(t)u(t) = 0$*, divide by* $u(t)$ *to obtain*

$$w'(t) + w(t)h(t) = 0,$$

*which we can consider as an ODE for* $w(t)$*. Take integrating factor* $e^{H(t)}$ *with* $H'(t) = h(t)$ *and find* $d(w(t)e^{H(t)})/dt = 0$*, so* $w(t)e^{H(t)} = C$ *or* $w(t) = Ce^{-H(t)}$ *as desired.*

## Section 2.2

**Exercise Solution 2.2.1.** *General solution* $u(t) = Ce^t - 3$*, specific solution is* $u(t) = 6e^t - 3$*.*

**Exercise Solution 2.2.2.** *General solution* $u(t) = Ce^{2t} - 2$*, specific solution is* $u(t) = 2e^{2t} - 2$*.*

**Exercise Solution 2.2.3.** *General solution* $u(t) = Ce^{-3t} + 1$*, specific solution is* $u(t) = 4e^{-3t} + 1$*.*

**Exercise Solution 2.2.4.** *General solution* $u(t) = Ce^{t^2/2} - 1$*, specific solution is* $u(t) = 3e^{t^2/2} - 1$*.*

**Exercise Solution 2.2.5.** *General solution* $u(t) = Ce^{-\cos(t)} - 1$*, specific solution is* $u(t) = 5e^1e^{-\cos(t)} - 1 = 5e^{1-\cos(t)} - 1$*.*

**Exercise Solution 2.2.6.** *General solution* $u(t) = Ce^{at} - b/a$*, specific solution is* $u(t) = (u_0 + b/a)e^{at} - b/a$*.*

**Exercise Solution 2.2.7.** *General solution* $u(t) = Ce^{-\cos(t)}$*, specific solution is* $u(t) = e^1e^{-\cos(t)} = e^{1-\cos(t)}$*.*

**Exercise Solution 2.2.8.** *General solution* $u(t) = Ce^{t^3/3}$*, specific solution is* $u(t) = 2e^{(t^3-1)/3}$*.*

**Exercise Solution 2.2.9.** *General solution* $u(t) = e^{e^t}$*, specific solution is* $u(t) = 3e^{e^t-1}$*.*

**Exercise Solution 2.2.10.** *Separate variables as* $du/(-k(u - A)) = dt$ *and integrate to find* $-\frac{1}{k}\ln|A - u| = t + C$*. Then* $\ln|A - u| = -kt + C$ *and so* $A - u = Ce^{-kt}$ *where* $C \neq 0$ *(though* $C = 0$ *is permissible, since it corresponds to* $u(t) = A$*). Solve to find general solution* $u(t) = A - Ce^{-kt}$*. Then* $u(0) = u_0$ *implies* $C = A - u_0$*, so* $u(t) = A + (u_0 - A)e^{-kt}$*.*

**Exercise Solution 2.2.11.** *Separate variables as* $dv/(P - kv) = dt$ *and integrate to find* $-\frac{1}{k}\ln|P - kv| = t + C$*. Then* $\ln|P - kv| = -kt + C$ *and so* $P - kv = Ce^{-kt}$ *(*$C \neq 0$*, but again,* $C = 0$ *is permissible, it corresponds to* $v(t) = P/k$*). Solve for* $v = P/k + Ce^{-kt}$ *and then* $v(0) = 0$ *implies* $C = -P/k$*, so* $v(t) = \frac{P}{k}(1 - e^{-kt})$*.*

**Exercise Solution 2.2.12.**

(a) *Start with* $F = kr^a\mu^b v^c$. *With* $[F] = MLT^{-2}$, $[r] = L$, $\mu = ML^{-1}T^{-1}$, *and* $[v] = LT^{-1}$ *this means that* $MLT^{-2} = L^a(ML^{-1}T^{-1})^b(LT^{-1})^c = M^bL^{a-b+c}T^{-b-c}$. *This dictates* $b = 1, a-b+c = 1, -b-c = -2$, *with unique solution* $a = b = c = 1$. *So* $F = kr\mu v$ *is the only dimensionally consistent formula, where* $k$ *is a dimensionless constant.*

(b) *The net force on the object is* $mg - kr\mu v$, *and from Newton's Second Law we have* $mv' = mg - kr\mu v$, *so* $v' = g - kr\mu v/m$.

(c) *This can be done with an integrating factor or with separation of variables. For separation of variables, separate (Leibnitz notation) as* $dv/(g - kr\mu v/m) = dt$ *and integrate to find*

$$-\frac{m\ln|g - kr\mu v/m|}{kr\mu} = t + C.$$

*Then* $\ln|g - kr\mu v/m| = -kr\mu t/m + C$ *(redefining* $C$*) and then* $g - kr\mu v/m = Ce^{-kr\mu t/m}$ *or*

$$v(t) = \frac{mg}{kr\mu} + Ce^{-kr\mu t/m},$$

*again redefining* $C$ *on the fly. For any* $C$ *this solution decays to terminal velocity* $\frac{mg}{kr\mu}$.

**Exercise Solution 2.2.13.** *It's much easier to take the hint. With* $\tilde{r} = r-h$ *and* $\tilde{K} = ((1 - h/r)K$ *we find that*

$$u' = \tilde{r}u(1-u/\tilde{K}) = (r-h)u(1-ru/K(r-h)) = (r-h)u-ru/K = ru(1-u/K)-hu$$

*which is the harvested logistic equation. The solution to the "standard" logistic equation* $u' = \tilde{r}u(1 - u/\tilde{K})$ *is*

$$\begin{aligned} u(t) &= \frac{\tilde{K}}{1 + e^{-\tilde{r}t}(\tilde{K}/u_0 - 1)} \\ &= \frac{(1 - h/r)K}{1 + e^{-(r-h)t}(\frac{K}{u_0}(1 - h/r) - 1)}. \end{aligned}$$

**Exercise Solution 2.2.14.** *As suggested, use* $a = \sqrt{mg/k}$ *to write*

$$\int \frac{dv}{kv^2/m - g} = \frac{m}{k}\int \frac{dv}{v^2 - a^2}$$

*and from*

$$\frac{1}{v^2-a^2} = \frac{1}{2a}\frac{1}{v-a} - \frac{1}{2a}\frac{1}{v+a}$$

*we find*

$$\frac{m}{k}\frac{1}{v^2-a^2} = \frac{m}{2ak}(\ln|v-a| - \ln|v+a|).$$

*Using the definition of* $a$ *and combining the logarithms shows that*

$$\int \frac{dv}{kv^2/m-g} = \frac{1}{2}\sqrt{\frac{m}{gk}}\ln\left|\frac{v-\sqrt{mg/k}}{v+\sqrt{mg/k}}\right|.$$

**Exercise Solution 2.2.15.** *Separate as* $dx/(0.2-x/100) = dt$ *and integrate to find* $-100\ln|0.2-x/100| = t+C$*. Solve for* $x$ *as* $x = 20 - Ce^{-t/100}$*. Then* $x=3$ *when* $t=0$ *yields* $C=17$*, so* $x(t) = 20 - 17e^{-t/100}$*.*

**Exercise Solution 2.2.16.** *Figure 2.1 shows the solution with* $u_0 = 9.6, K = 675,$ *and* $r = 0.53$.

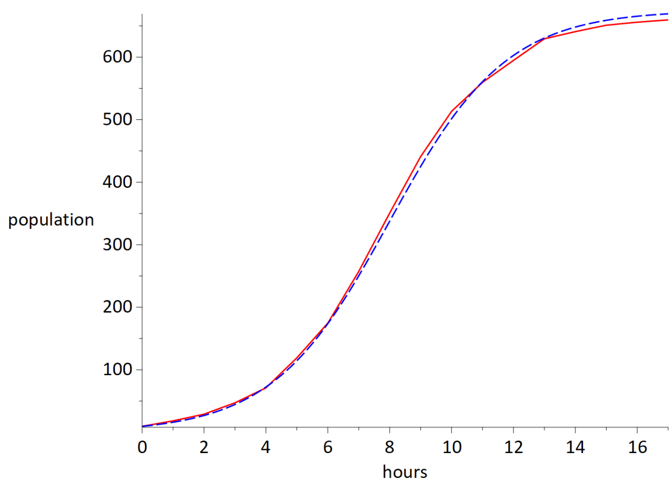


Figure 2.1: Yeast population (red, solid) and model fit (blue, dashed).

**Exercise Solution 2.2.17.**

*(a) Either of the solutions*

$$v(t) = \sqrt{\frac{g}{\tilde{k}}}\left(\frac{1-e^{-2t\sqrt{g\tilde{k}}}}{1+e^{-2t\sqrt{g\tilde{k}}}}\right). \tag{2.1}$$

*or equivalently,*

$$v(t) = \sqrt{\frac{g}{\tilde{k}}}\tanh(t\sqrt{g\tilde{k}}). \tag{2.2}$$

*follows easily by substituting in $\tilde{k} = k/m$ into the solution obtained in the text (or they can be checked directly in the ODE itself).*

(b) *Use the hint to find the distance $d(t)$ fallen by the shuttlecock with $d(0) = 0$ is*

$$d(t) = \int_0^t \sqrt{\frac{g}{\tilde{k}}}\tanh(z\sqrt{g\tilde{k}})\,dz = \frac{\ln(\cosh(t\sqrt{\tilde{k}g}))}{\tilde{k}}.$$

(c) *With $\tilde{k} = 0.21$ the plot is shown in Figure 2.2.*

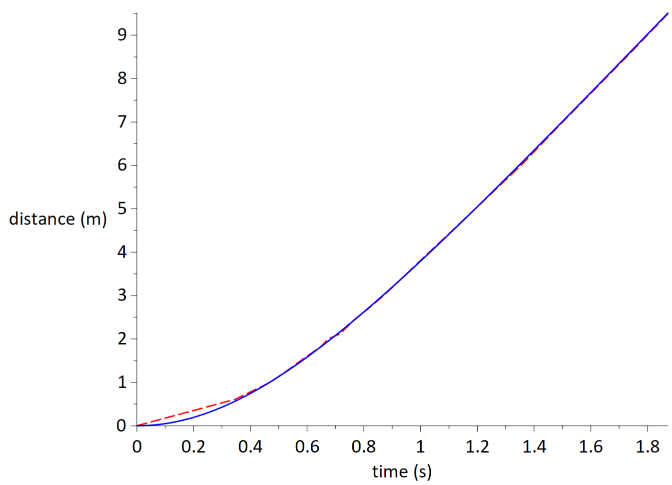


Figure 2.2: Shuttlecock data (red,dashed) and best fit quadratic resistance model (blue,solid).

(d) *It looks pretty good. The official least-squares estimate is $\tilde{k} \approx 0.2117$ and yields residual* 0.01008880649.

(e) *The linear model follows from net force $F = mg - kv(t)$ and Newton's Second Law $mv' = F$.*

*(f) The linear model has solution* $v(t) = \frac{g}{\tilde{k}}(1 - e^{-\tilde{k}t})$. *The optimal choice for* $\tilde{k}$ *is about* 1.06 *(least squares estimate is* $\tilde{k} \approx 1.0612$, *residual* 0.522, *quite a bit larger than the quadratic model.) A plot of the model versus the data is shown in Figure 2.3.*

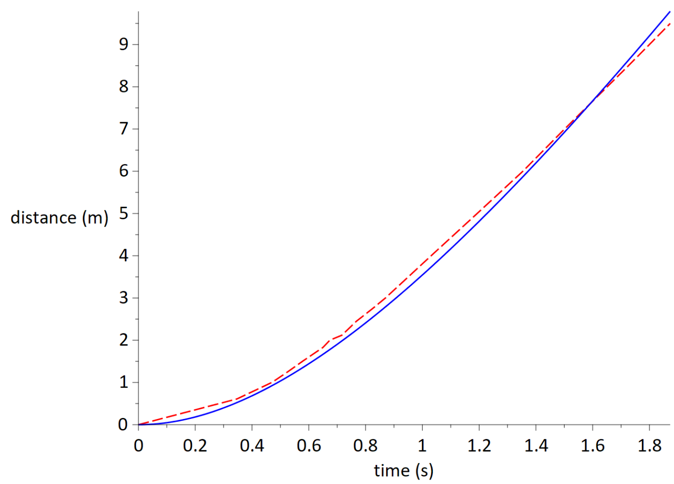


Figure 2.3: Shuttlecock data (red,dashed) and best fit linear resistance model (blue,solid).

## Section 2.3

**Exercise Solution 2.3.1.** *The ODE is* $u' = f(t,u)$ *with* $f(t,u) = u - 2t$. *Then* $f(0,0) = 0, f(0,1) = 1, f(1,0) = -2, f(1,1) = -1$. *Crude slope field shown in left panel of Figure 2.4.*

**Exercise Solution 2.3.2.** *The ODE is* $u' = f(t,u)$ *with* $f(t,u) = u^2+t+1$. *Then* $f(0,0) = 1, f(0,1) = 2, f(1,0) = 2, f(1,1) = 3$. *Crude slope field shown in right panel of Figure 2.4.*

**Exercise Solution 2.3.3.** *The ODE is* $u' = f(t,u)$ *with* $f(t,u) = -u$. *Then* $f(0,1) = -1, f(0,2) = -2, f(1,1) = -1, f(1,3) = -3$. *Crude slope field shown in left panel of Figure 2.5.*

**Exercise Solution 2.3.4.** *The ODE is* $u' = f(t,u)$ *with* $f(t,u) = -1/u$. *Then* $f(0,1) = -1, f(0,2) = -1/2, f(1,1) = -1, f(1,3) = -1/3$. *Crude slope field shown in right panel of Figure 2.5.*

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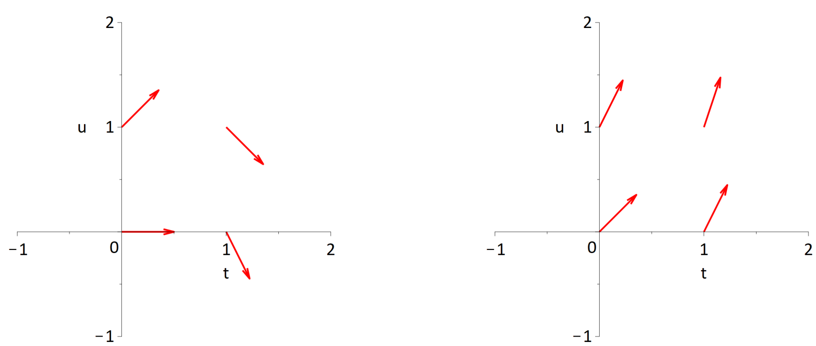


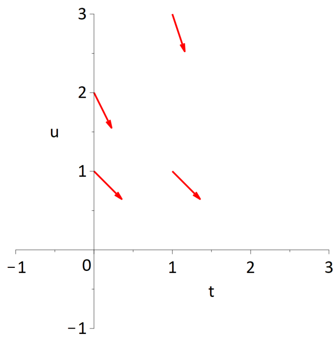

Figure 2.4: Slope fields for Exercises 2.3.1 (left) and 2.3.2 (right).

**Exercise Solution 2.3.5.** *Slope field shown in left panel of Figure 2.6.*

**Exercise Solution 2.3.6.** *Slope field shown in right panel of Figure 2.6.*

**Exercise Solution 2.3.7.** *Slope field shown in left panel of Figure 2.7. In this case* $u = 0$ *is an equilibrium solution.*

**Exercise Solution 2.3.8.** *Slope field shown in right panel of Figure 2.7. It sort of looks like* $u = 0$ *is an equilibrium, but the right side of the ODE is undefined there.*



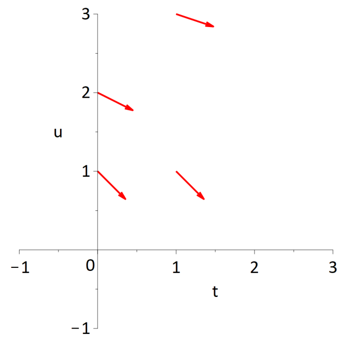



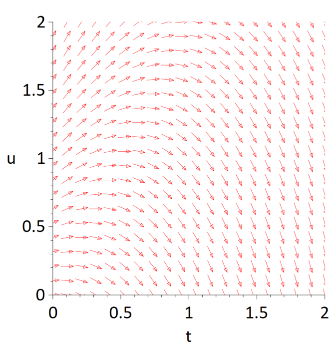

Figure 2.5: Slope fields for Exercises 2.3.3 (left) and 2.3.4 (right).

**Exercise Solution 2.3.9.** *Slope field shown in left panel of Figure 2.8. In this case* $u = 0$ *and* $u = 3$ *are equilibrium solutions.*

**Exercise Solution 2.3.10.** *Slope field shown in right panel of Figure 2.8. In this case* $u = -1$ *and* $u = 1$ *are equilibrium solutions.*

**Exercise Solution 2.3.11.** *Slope field shown in left panel of Figure 2.9. In this case* $u = 0$ *and* $u = 3$ *are equilibrium solutions.*

**Exercise Solution 2.3.12.** *Slope field shown in right panel of Figure 2.9. In this case* $u = -1$ *and* $u = 1$ *are equilibrium solutions.*



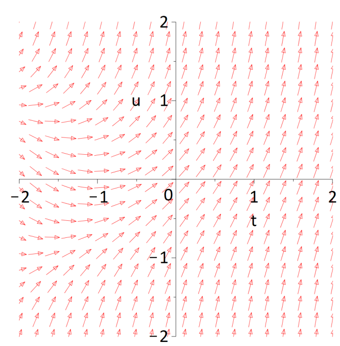




Figure 2.6: Slope fields for Exercises 2.3.5 (left) and 2.3.6 (right).

**Exercise Solution 2.3.13.** *The phase portrait is in Figure 2.10, solutions with* $u(0) = 2$ *and* $u(0) = -2$ *in the right panel.*

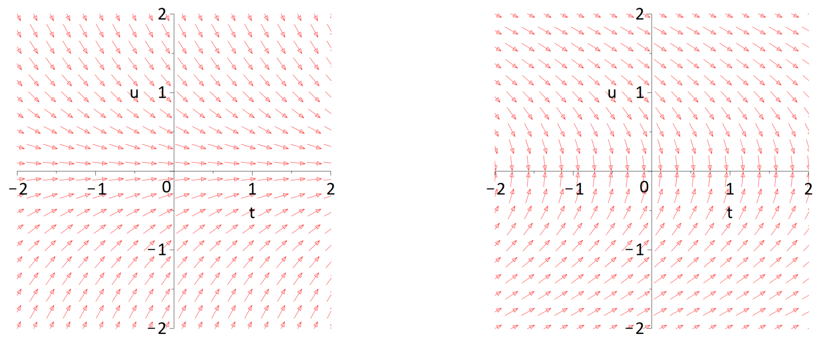



Figure 2.7: Slope fields for Exercises 2.3.7 (left) and 2.3.8 (right).

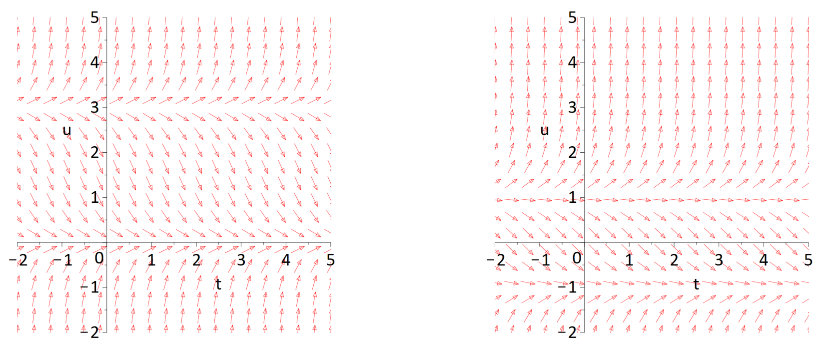



Figure 2.8: Slope fields for Exercises 2.3.9 (left) and 2.3.10 (right).

**Exercise Solution 2.3.14.** *The phase portrait is in Figure 2.11, solutions with* $v(0) = 0$ *and* $v(0) = 15$ *in the right panel.*

**Exercise Solution 2.3.15.** *The phase portrait is in Figure 2.12, solutions with* $v(0) = 0$ *and* $v(0) = 15/k$ *in the right panel.*

**Exercise Solution 2.3.16.** *The phase portrait is in Figure 2.13, solutions with* $u(0) = 1/2$, $u(0) = 2$, $u(0) = 4$ *in the right panel.*

**Exercise Solution 2.3.17.** *The phase portrait is in Figure 2.14, solutions with* $u(0) = 1/2$, $u(0) = 3/2$ *in the right panel.*

**Exercise Solution 2.3.18.** *The phase portrait is in Figure 2.15, solutions with* $u(0) = 1/2$, $u(0) = 3/2$ *in the right panel.*

**Exercise Solution 2.3.19.** *See Figure 2.16. Solution with* $u(0) = 0$ *increases asymptotically to equilibrium at* $u = c_1 V$*, solution with* $u(0) = 2c_1 V$ *decreases asymptotically to equilibrium at* $u = c_1 V$*.*

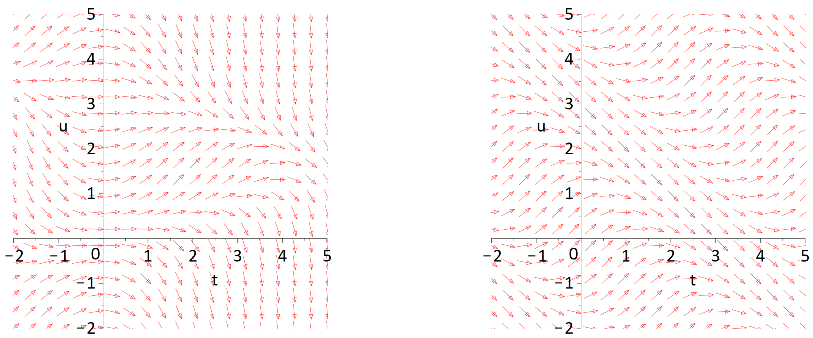



Figure 2.9: Slope fields for Exercises 2.3.11 (left) and 2.3.12 (right).

**Exercise Solution 2.3.20.** *See Figure 2.17. Solution with* $v(0) = 0$ *increases asymptotically to equilibrium at* $v = \sqrt{mg/k}$*, solution with* $u(0) = \sqrt{mg/k}$ *decreases asymptotically to equilibrium at* $v = \sqrt{mg/k}$*.*

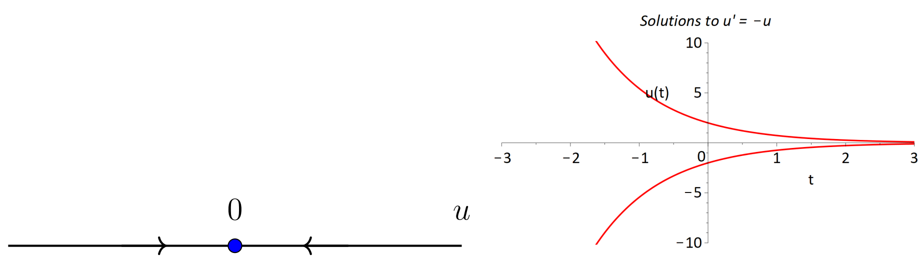



Figure 2.10: Phase portrait for $u' = -u$ (left) and some solutions (right).

**Exercise Solution 2.3.21.** *Take* $u' = (u-1)(u-3)$ *(the right side can be multiplied by any positive constant).*

**Exercise Solution 2.3.22.** *Take* $u' = (u+3)u(u-4)(u-5)$ *(the right side can be multiplied by any positive constant).*

**Exercise Solution 2.3.23.** *Take* $u' = -(u-1)^2(u-3)$ *(the right side can be multiplied by any positive constant).*

**Exercise Solution 2.3.24.** *Take* $u' = \sin(u)$ *(the right side can be multiplied by any positive constant).*

**Exercise Solution 2.3.25.** *The ODE is* $u' = f(u)$ *with* $f(u) = hu - u^2$*. Here* $u = 0$ *and* $u = h$ *are always the only fixed points. We have* $f'(u) =$

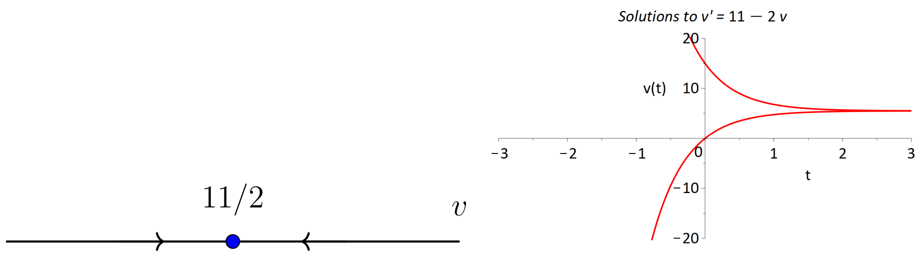



Figure 2.11: Phase portrait for $v' = 11 - 2v$ (left) and some solutions (right).

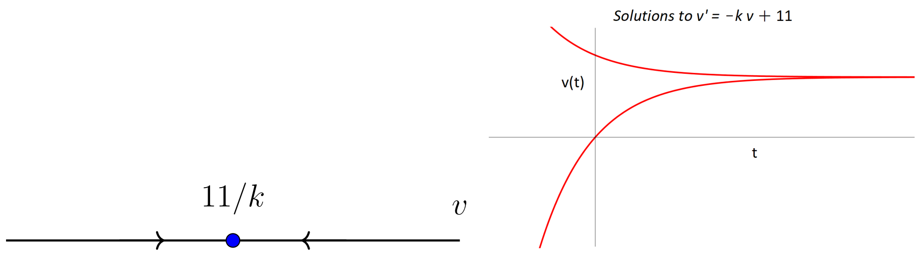



Figure 2.12: Phase portrait for $v' = 11 - kv$ (left) and some solutions (right).

*$h - 2u$. For $h > 0$ the fixed point at 0 is unstable ($f'(0) = h$) and the fixed point at $u = h$ is stable ($f'(h) = -h$). For $h < 0$ the stability is reversed. A bifurcation occurs at $h = 0$. See Figure 2.18 for the bifurcation diagram.*

**Exercise Solution 2.3.26.** *The ODE is $u' = f(u)$ with $f(u) = hu - u^3$. Here $u = 0$ is always a fixed point. For $h > 0$ we also have fixed points $u = \sqrt{h}$ and $u = -\sqrt{h}$. Compute $f'(u) = h - 3u^2$. When $h < 0$ the sole fixed point at $u = 0$ is stable. When $h > 0$ the fixed point at $u = 0$ is unstable, but $-\sqrt{h}$ and $\sqrt{h}$ are stable. At $h = 0$ we have a bifurcation, where the fixed point 0 is stable. See Figure 2.19 for a bifurcation diagram. This is an example of a pitchfork bifurcation.*

**Exercise Solution 2.3.27.** *The ODE is $u' = f(u)$ with $f(u) = u(1-u) - h$. For $0 < h < 1/4$ there are two fixed points, $u_1^* = \frac{1-\sqrt{1-4h}}{2}$ and $u_2^* = \frac{1+\sqrt{1-4h}}{2}$. For $h > 1/4$ there are no fixed points, and fact completing the square shows that $f(u) = u(1-u) - h = -(u-1/2)^2 + (1/4 - h)$, so we can see that in this case $f'(u) < 0$ for all $u$. When $h = 1/4$ there is a single fixed point at $u = 1/2$. We have $f'(u) = 1 - 2u$. For $h < 1/4$ we have $f'(u_1^*) = \sqrt{1-4h} > 0$, so this fixed point is unstable. Also, $f'(u_2^*) = -\sqrt{1-4h} < 0$, so this fixed point is stable. A bifurcation occurs as $h = 1/4$, where the single fixed point $u = 1/2$ is semi-stable. See Figure 2.20.*

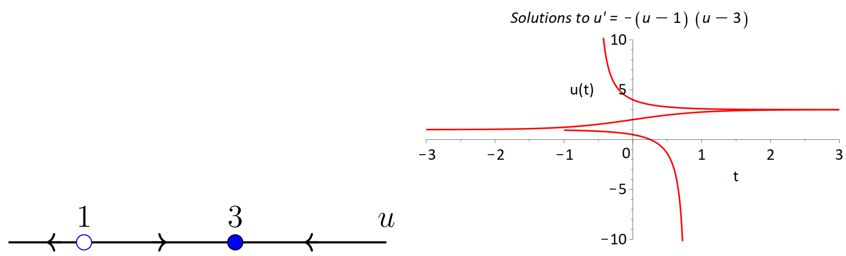



Figure 2.13: Phase portrait for $u' = -(u-1)(u-3)$ (left) and some solutions (right).

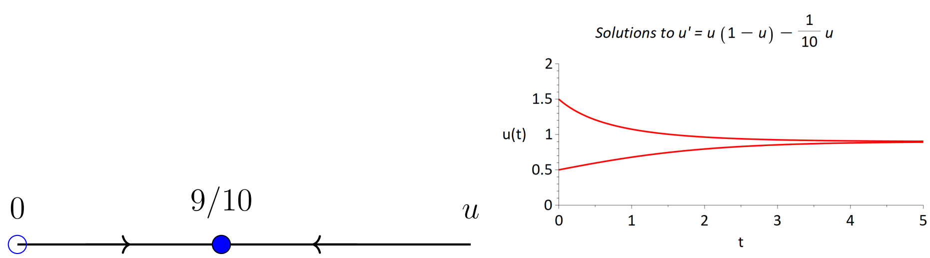



Figure 2.14: Phase portrait for $u'(t) = u(t)(1-u(t)) - u(t)/10$ (left) and some solutions (right).

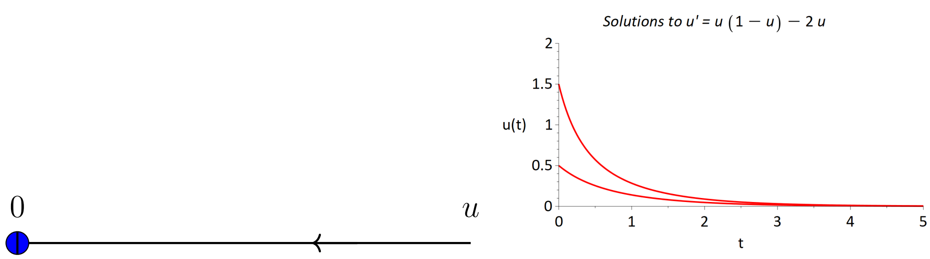



Figure 2.15: Phase portrait for $u'(t) = u(t)(1-u(t)) - 2u(t)$ (left) and some solutions (right).

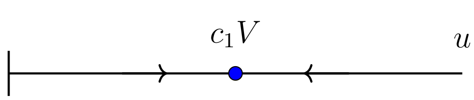



Figure 2.16: Phase portrait for $u'(t) = rc_1 - ru(t)/V$.

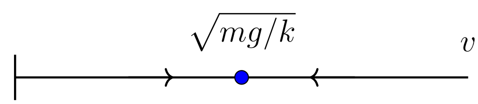



Figure 2.17: Phase portrait for $v'(t) = g - kv^2(t)/m$.

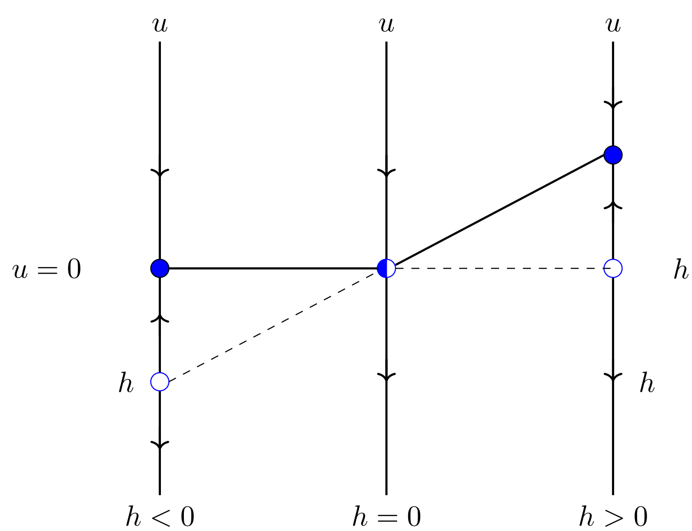



Figure 2.18: Bifurcation diagram for $u' = hu - u^2$.

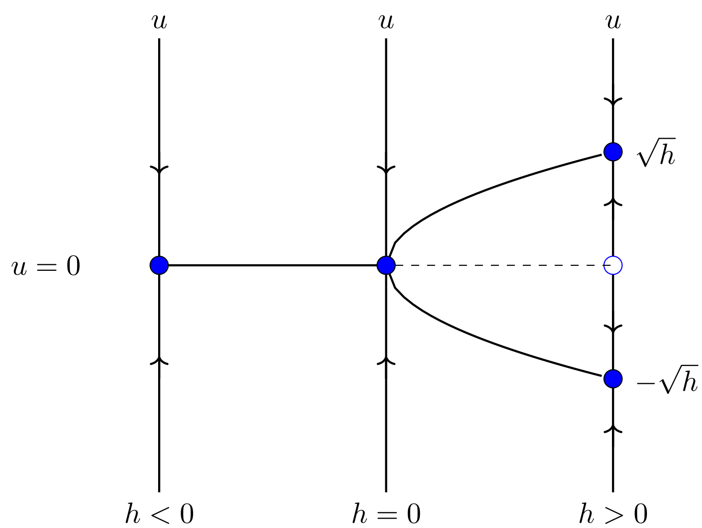



Figure 2.19: Bifurcation diagram for $u' = hu - u^3$, an example of a pitchfork bifurcation.

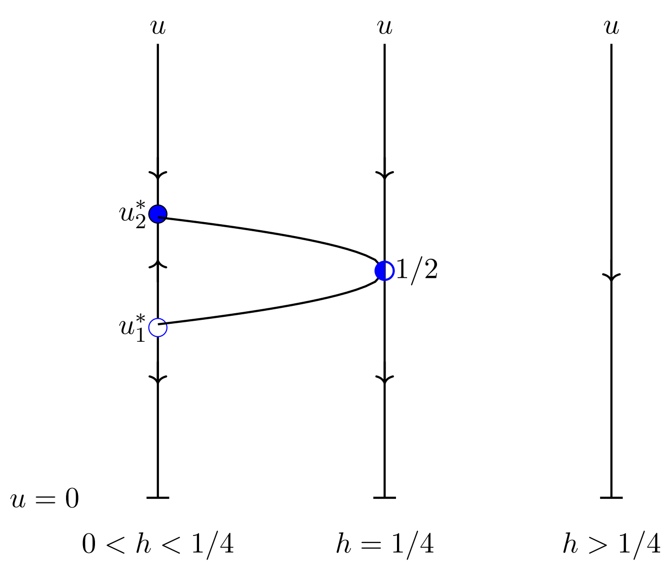



Figure 2.20: Bifurcation diagram for $u' = u(1-u) - h$.

## Section 2.4

**Exercise Solution 2.4.1.** *Here $f(t,u) = u+3$, which is continuous for all $u$ and $t$. Also $\frac{\partial f}{\partial u} = 1$, also continuous everywhere.*

**Exercise Solution 2.4.2.** *Here $f(t,u) = -u^2+\sin(t)$, which is continuous for all $u$ and $t$. Also $\frac{\partial f}{\partial u} = -2u$, also continuous everywhere.*

**Exercise Solution 2.4.3.** *Here $f(t,u) = 1/u$, which is continuous near $u = 2$ (everywhere except $u = 0$). Also $\frac{\partial f}{\partial u} = 1/u^2$, which is continuous near $u = 2$.*

**Exercise Solution 2.4.4.** *Here $f(t,u) = ru(1-u/K)$, which is continuous for all $u$ and $t$, as is $\frac{\partial f}{\partial u} = r - 2ru/K$.*

**Exercise Solution 2.4.5.**

(a) *The ODE is $u' = f(t,u)$ with $f(t,u) = h(u - A)$ (so this ODE is in fact autonomous if $A$ is constant). Then $\frac{\partial f}{\partial u} = h'(u - A)$, and since $h$ is continuously-differentiable, $\frac{\partial f}{\partial u}$ is continuous. Then $u' = h(u - A)$ has a unique solution for any $u(t_0) = u_0$.*

(b) *If $u(t_0) = A$ then the object is already at ambient temperature, so it should be the case that $u(t) = A$ for all $t$, so $u(t) = A$ should satisfy $u' = h(u - A)$, which means $h(0) = 0$.*

(c) *If $u_1(t)$ and $u_2(t)$ are solutions to $u' = h(u-A)$ with $u_1(0) > u_2(0) \geq A$ (object 1 is hotter than object 2) then object 1 should be cooling faster than object 2, that is, $u_1'(0) < u_2'(0) < 0$. This means that $h(u_1(0) - A) < h(u_2(0) - A)$ for any choice $u_1(0) > u_2(0) \geq A$, or equivalently $h(x) < h(y) < 0$ when $x > y$. The same argument works for $A \leq u_1(0) < u_2(0)$.*

**Exercise Solution 2.4.6.** *Solution is $u(t) = 2$, maximum domain $-\infty < t < \infty$.*

**Exercise Solution 2.4.7.** *Solution is $u(t) = \tan(t)$, maximum domain $-\pi/2 < t < \pi/2$.*

**Exercise Solution 2.4.8.** *Solution is $u(t) = -\ln(1-t)$, maximum domain $-\infty < t < 1$.*

**Exercise Solution 2.4.9.** *Solution is $u(t) = \sqrt{2t+9}$, maximum domain $-9/2 < t < \infty$.*

## Section 3.1

**Exercise Solution 3.1.1.** *Find* $u_2 = 6.0$*, true solution is* $u(t) = 4e^t - 3$ *with* $u(1) \approx 7.873$*.*

**Exercise Solution 3.1.2.** *Find* $u_4 = 1.582$*, true solution is* $u(t) = 5e^{-t} + 3t - 3$ *with* $u(1) \approx 1.839$*.*

**Exercise Solution 3.1.3.** *Find* $u_4 = 2.460$*, true solution is* $u(t) = \sqrt{2t+4}$ *with* $u(1) \approx 2.449$*.*

**Exercise Solution 3.1.4.** *Find* $u_5 = 5.285$*, true solution is* $u(t) = 3e^{(t^2-1)/2}$ *with* $u(1.5) \approx 5.605$*.*

**Exercise Solution 3.1.5.** *True solution is* $u(t) = 3 - e^{-t/3}$ *and* $u(5) \approx 2.811124397$*. With* $h = 1, 0.1, 0.01$ *Euler estimates are* $2.8683, 2.8164, 2.8116$*, errors*
$0.0572, 0.005291, 0.000525$*, roughly. This is consistent with first order accuracy.*

**Exercise Solution 3.1.6.** *True solution is* $u(t) = \ln(t^2/2 + e)$ *with* $u(3) \approx 1.976616951$*. With* $h = 1, 0.1, 0.01$ *Euler estimates are* $1.8772, 1.9678, 1.9758$*, errors*
$0.09944, 0.00877, 0.000863$*, roughly. This is consistent with first order accuracy.*

**Exercise Solution 3.1.7.** *True solution is* $u(t) = 2/(1-2t)$*, which has an asymptote at* $t = 1/2$*. With* $h = 0.5, 0.1, 0.01, 0.001$ *the Euler estimates are* $4, 8.2182, 36.257, 217.64$*. It's clear the Euler's method is reproducing the asymptotic blow-up.*

**Exercise Solution 3.1.8.** *True solution is* $u(t) = \sqrt{2t+4}$ *with* $u(4) = \sqrt{12} \approx 3.464101615$*. With* $h = 1, 0.01, 0.001$ *Euler estimates are* $3.553, 3.4721, 3.4649, 3.4642$*, errors* $8.89 \times 10^{-2}, 8.01 \times 10^{-3}, 7.937 \times 10^{-4}, 7.929 \times 10^{-5}$*, roughly. This is consistent with first order accuracy.*

**Exercise Solution 3.1.9.** *The true solution is* $u(t) = t$ *for all* $t$*, a straight line. Euler's method reproduces this solution for any step size* $h$*, because it is based on linear extrapolation—it tracks the solution perfectly (aside from any round-off error.)*

**Exercise Solution 3.1.10.**

(a) *From* $F = ma$ *with* $F_p = mP$ *and* $F_r = -kmv^r$ *we have* $mv' = mP - kmv^r$ *which yields* $v' = P - kv^r$*.*

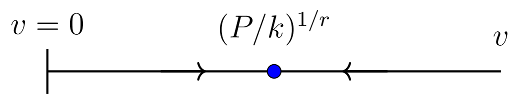



Figure 3.21: Phase portrait $v' = P - kv^r$.

*(b) Phase portrait shown in Figure 3.21 All solutions with $v(0) \geq 0$ should approach the fixed point $v = (P/k)^{1/r}$.*

*(c) For $v \geq 0$ the function $f(t,v) = P - kv^r$ is continuous, as is $\frac{\partial f}{\partial v} = -rkv^{r-1}$. The existence-uniqueness theorem applies.*

*(d) With $P = 11$, $r = 3/2$, and $k = 0.258$ the fixed point is at $v = (22/0.258)^{2/3} \approx 12.2$ (12.204, more accurately).*

*(e) The plots are as shown in Figure 3.22, $h = 1$ on the left, $h = 0.1$ on the right.*

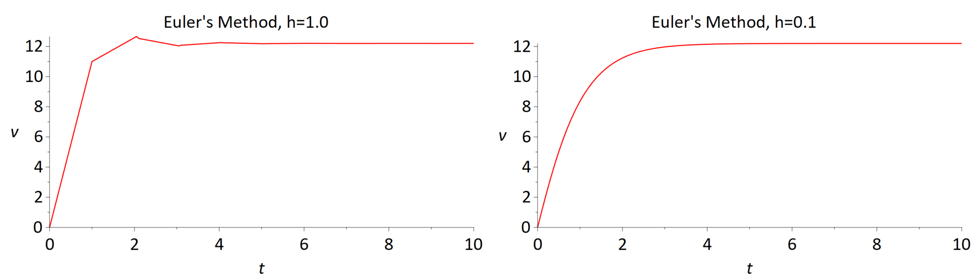



Figure 3.22: Euler's Method for $v' = 11 - 0.258v^{3/2}$ with $v(0) = 0$, step size $h = 1$ (left panel) and $h = 0.1$ (right panel).

*(f) With $h = 2$ and $t_0 = 0, v_0 = 0$ we find $v_1 = v_0 + (2)f(t_0, v_0) = 22$ and then, with $t_1 = 2$, $v_2 = v_1 + (2)f(t_1, v_1) = -9.25$ (an estimate of $v(0.4)$, true value about 12.128. With $h = 5$ and $t_0 = 0, v_0 = 0$ we find $v_1 = v_0 + (5)f(t_0, v_0) = 55$ and then, with $t_1 = 5$, $v_2 = v_1 + (5)f(t_1, v_1) = -416.2$, an estimate of $v(10) \approx 12.204$.*

**Exercise Solution 3.1.11.** *The true solution is $u(t) = 1/(1-t)$, but the maximum domain of this solution is $(-\infty, 1)$ (given that we started at $t = 0$). Euler's Method with step sizes $h = 1, 0.1, 0.01, 0.001$ produces estimates for $u(1)$ equal to $2, 6.13, 30.39$, and $193.1$. For $u(2)$ we obtain $6, 5.65 \times 10^{103}, \infty, \infty$ (the last two are really floating point overflow.) All*

*Euler estimates are nonsense, since we are trying to push the solution out of its maximal domain.*

**Exercise Solution 3.1.12.**

(a) *A general solution is* $u(t) = \sin(t) + Ce^t$.

(b) *The solution with* $u(0) = 0$ *is* $u(t) = \sin(t)$.

(c) *The direction field is shown in Figure 3.23.*

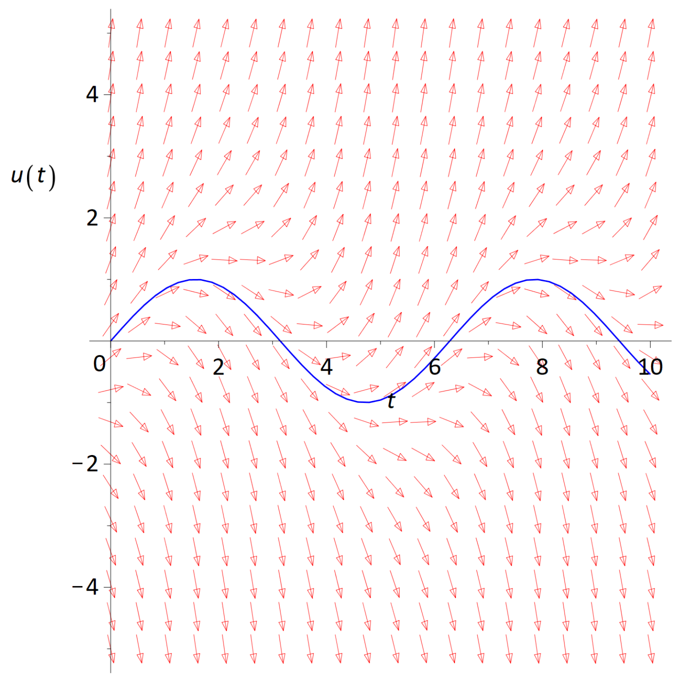



Figure 3.23: Direction field for $u'(t) = u(t) - \sin(t) + \cos(t)$, solution with $u(0) = 0$ in blue.

(d) *Euler's method with step sizes* $h = 1, 0.1, 0.01, 0.001$ *produces estimates* $222.75, 338.41, 51.770, 4.9346$, *compared to the true value* $u(10) = \sin(10) \approx -0.544$. *When Euler's Method invariably steps off the true solution curve (on the first step) the* $C$ *in the general solution lights up to a nonzero value, and we begin to track* $u(t) = \sin(t) + Ce^t$ *for some* $C \neq 0$.

**Exercise Solution 3.1.13.**

(a) $u(t) = e^{-10t}$, $\lim_{t\to 0} u(t) = 0$.

(b) *The first iterate is* $u_1 = 0$, *as are all following iterates!*

(c) *We find* $u_1 = -1, u_2 = 1$, *and so on, so* $u_{25} = -1$. *The iterates oscillate.*

(d) *The iterates are* $u_1 = -9, u_2 = 81, u_3 = -729, u_4 = 6561, u_5 = -59049$.

(e) *From Euler's method we have* $u_{k+1} = u_k + hf(t_k, u_k) = (1-10h)u_k$, *or* $u_k = (1-10h)u_{k-1}$ *after shifting indices. Then* $u_k = (1-10h)u_{k-1} = (1-10h)^2 u_{k-2} = \cdots (1-10h)^k u_0 = (1-10h)^k$.

(f) *We need* $|1 - 10h| < 1$, *meaning* $0 < h < 1/5$.

## Section 3.2

**Exercise Solution 3.2.1.** *Find* $u_1 = 3.5, u_2 = 7.5625$. *True solution is* $u(t) = 4e^t - 3$ *with* $u(1) \approx 7.873$.

**Exercise Solution 3.2.2.** *Find* $u_1 = 1.65625, u_2 = 1.55176, u_3 = 1.63419, u_4 = 1.86265$. *True solution is* $u(t) = 5e^{-t} + 3t - 3$ *with* $u(1) = 5/e \approx 1.839$.

**Exercise Solution 3.2.3.** *Find* $u_1 = 2.12132, u_2 = 2.23607, u_3 = 2.34521, u_4 = 2.44950$. *True solution is* $u(t) = \sqrt{2t+4}$ *with* $u(1) = \sqrt{6} \approx 2.44950$.

**Exercise Solution 3.2.4.** *Here* $u_1 = 3.3315, u_2 = 3.73661, u_3 = 4.23283, u_4 = 4.84278, u_5 = 5.59584$. *True solution is* $u(t) = 3e^{(t^2-1)/2}$ *with* $u(0.5) = 3e^{5/8} \approx 5.60474$.

**Exercise Solution 3.2.5.** *For* $h = 1$ *we find approximation* $2.8035$*; for* $h = 0.1$, $2.81106$*; for* $h = 0.01$, $2.81112$. *True solution is* $u(t) = 3 - e^{-t/3}$ *and* $u(5) = 3e^{-5/3} \approx 2.81112$.

**Exercise Solution 3.2.6.** *For* $h = 1$ *we find approximation* $1.97876$*; for* $h = 0.1$, $1.97654$*; for* $h = 0.01$, $1.97662$. *True solution is* $u(t) = \ln(t^2/2 + e)$ *and* $u(3) = \ln(9/2 + e) \approx 1.97662$.

**Exercise Solution 3.2.7.** *For* $h = 0.5$ *we find approximation* $7.0$*; for* $h = 0.1$, $23.76$*; for* $h = 0.01$, $211.2$*; for* $h = 0.001$, $2086$. *True solution is* $u(t) = \frac{1}{1/2-t}$ *and* $u(0.5)$ *is undefined (u limits to* $\infty$ *as* $t \to 1/2$ *from the left). Clearly the improved Euler iterates try to track this.*

**Exercise Solution 3.2.8.** *For* $h = 1.0$ *we find approximation* $3.46467$*; for* $h = 0.1$, $3.46410$*; for* $h = 0.01$, $3.46410$. *True solution is* $u(t) = \sqrt{2t+4}$ *and* $u(4) = \sqrt{12} \approx 3.46410$.

**Exercise Solution 3.2.9.** *The true solution is* $u(t) = t$ *for all* $t$*, a straight line. Like Euler's method, the improved Euler Method reproduces this solution for any step size* $h$*, because it is based on linear extrapolation—it tracks the solution perfectly (aside from any round-off error.)*

**Exercise Solution 3.2.10.** *The true solution is* $u(t) = 1/(1-t)$*, but the maximum domain of this solution is* $(-\infty, 1)$ *(given that we started at* $t = 0$*). The improved Euler method with step sizes* $h = 1, 0.1, 0.01, 0.001$ *produces estimates for* $u(2)$ *equal to* $133.65, \infty, \infty, \infty$ *(the last three are really floating point overflow.) All improved Euler estimates are nonsense, since we are trying to push the solution out of its maximal domain.*

**Exercise Solution 3.2.11.**

*(a)* *A general solution is* $u(t) = \sin(t) + Ce^t$.

*(b)* *The solution with* $u(0) = 0$ *is* $u(t) = \sin(t)$.

*(c)* *The direction field is shown in Figure 3.24.*

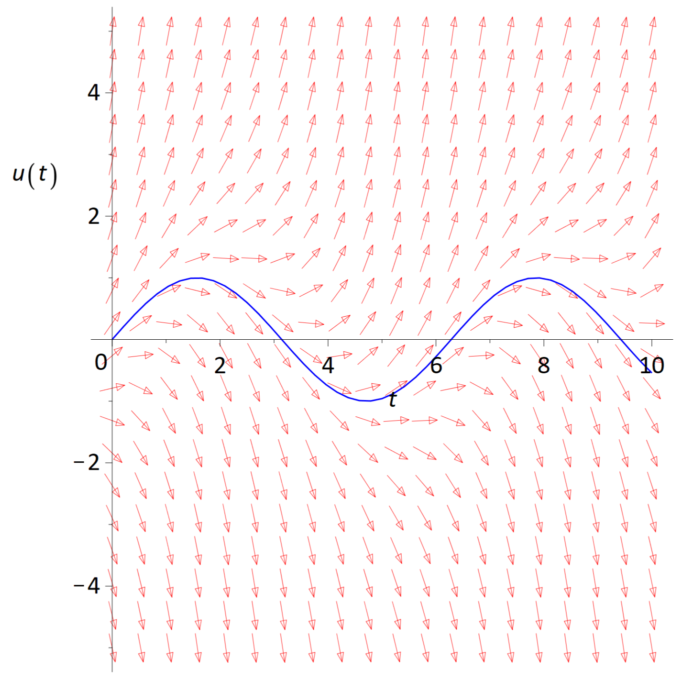



Figure 3.24: Direction field for $u'(t) = u(t) - \sin(t) + \cos(t)$, solution with $u(0) = 0$ in blue.

*(d)* *Improved Euler with step sizes* $h = 1, 0.1, 0.01, 0.001$ *produces estimates* $506.47, 16.282, -0.3618, -0.5422$, *compared to the true value* $u(10) = \sin(10) \approx -0.5440$. *When the improved Euler method invariably steps off the true solution curve (on the first step) the* $C$ *in the general solution lights up to a nonzero value, and we begin to track* $u(t) = \sin(t) + Ce^t$ *for some* $C \neq 0$.

**Exercise Solution 3.2.12.**

*(a)* $u(t) = e^{-10t}$, $\lim_{t \to 0} u(t) = 0$.

*(b) The improved Euler iteration turns out to be $u_k = 1/2^k$, and the estimate for $u(5)$ is $u_{50} \approx 8.88 \times 10^{-16}$.*

*(c) All iterates $u_k = 1$.*

*(d) The iterates are $41, 1681, 68921, 2.826 \times 10^6$, and $1.16 \times 10^8$. Clearly unstable.*

*(e) We have $f(t,u) = -10u$, so with $t_k = kh$ the improved Euler iteration becomes (using intermediate result $w = u_k + hf(t_k, u_k) = u_k - 10hu_k = (1-10h)u_k$)*

$$\begin{aligned} u_{k+1} &= u_k + h\left(\frac{f(t_k,u_k) + f(t_k+h,w)}{2}\right) \\ &= u_k + h\left(\frac{-10u_k - 10w}{2}\right) \\ &= u_k + h\left(\frac{-10u_k - 10(1-10h)u_k}{2}\right) \\ &= (50h^2 - 10h + 1)u_k \end{aligned}$$

*after simplifying. Since the multiplicative factor $(50h^2-10h+1)$ doesn't depend on $k$ and $u_0 = 1$ this makes it easy to see that $u_1 = (50h^2 - 10h + 1), u_2 = (50h^2 - 10h + 1)^2, u_3 = (50h^2 - 10h + 1)^3$, and so on.*

*(f) We need $|50h^2 - 10h + 1| < 1$ in order to obtain $u_k \to 0$. A plot of $50h^2 - 10h + 1$ for $h > 0$ reveals that this is satisfied when $0 < h < 1/5$.*

## Section 3.3

**Exercise Solution 3.3.1.** *Find* $u_2 = 7.8694$*, true solution is* $u(t) = 4e^t - 3$ *with* $u(1) = 4e - 3 \approx 7.8731$*.*

**Exercise Solution 3.3.2.** *Find* $u_2 = 1.84086$*, true solution is* $u(t) = 5e^{-t} + 3t - 3$ *with* $u(1) = 5/e \approx 1.83940$*.*

**Exercise Solution 3.3.3.** *Find* $u_4 = 2.44949$*, true solution is* $u(t) = \sqrt{2t+4}$ *with* $u(1) = \sqrt{6} \approx 2.44949$*.*

**Exercise Solution 3.3.4.** *Here* $u_2 = 5.60447$*, true solution is* $u(t) = 3e^{(t^2-1)/2}$ *with* $u(1.5) \approx 5.60474$*.*

**Exercise Solution 3.3.5.** *For* $h = 1$ *we find approximation* $2.81108$*; for* $h = 0.1$*,* $2.81112$*; for* $h = 0.01$*,* $2.81112$*. True solution is* $u(t) = 3 - e^{-t/3}$ *and* $u(5) = 3e^{-5/3} \approx 2.81112$*.*

**Exercise Solution 3.3.6.** *For* $h = 1$ *we find approximation* $1.97683$*; for* $h = 0.1$*,* $1.97662$*; for* $h = 0.01$*,* $1.97662$*. True solution is* $u(t) = \ln(t^2/2 + e)$ *and* $u(3) = \ln(9/2 + e) \approx 1.97662$*.*

**Exercise Solution 3.3.7.** *For* $h = 0.5$ *we find approximation* $16.98$*; for* $h = 0.1$*,* $82.03$*; for* $h = 0.01$*,* $819.9$*; for* $h = 0.001$*,* $8199.1$*. True solution is* $u(t) = \frac{1}{1/2 - t}$ *and* $u(0.5)$ *is undefined (*$u$ *limits to* $\infty$ *as* $t \to 1/2$ *from the left). Clearly RK4 tries to track this.*

**Exercise Solution 3.3.8.** *For* $h = 1.0$ *we find approximation* $3.46412$*; for* $h = 0.1$*,* $3.46410$*; for* $h = 0.01$*,* $3.46410$*. True solution is* $u(t) = \sqrt{2t+4}$ *and* $u(4) = \sqrt{12} \approx 3.46410$*.*

**Exercise Solution 3.3.9.**

(a) *Euler's method gives estimate* $u(1) \approx 10.76871524$*, error* $0.79737$*.*

(b) *The improved Euler method gives estimate* $u(1) \approx 8.817775300$*, error* $1.1536$*.*

(c) *The RK4 method gives estimate* $u(1) \approx 9.974349526$*, error* $3.003 \times 10^{-3}$*.*

(d) *Maple's* `dsolve` *command with the "numeric" option gives estimate* $u(1) \approx 1.86 \times 10^{-6}$*. Matlab's* `ode45` *command gives estimate* $u(1) \approx 9.971559281$*, error* $2.14 \times 10^{-4}$*.*

**Exercise Solution 3.3.10.** *The true solution is $u(t) = 1/(1-t)$, but the maximum domain of this solution is $(-\infty, 1)$ (given that we started at $t = 0$). The RK4 method with step sizes $h = 1, 0.1, 0.01, 0.001$ produces estimates for $u(2)$ equal to $1.67 \times 10^{11}, \infty, \infty, \infty$ (the last three are really floating point overflow.) All RK4 estimates are nonsense, since we are trying to push the solution out of its maximal domain.*

**Exercise Solution 3.3.11.**

*(a)* *A general solution is $u(t) = \sin(t) + Ce^t$.*

*(b)* *The solution with $u(0) = 0$ is $u(t) = \sin(t)$.*

*(c)* *The direction field is shown in Figure 3.25.*

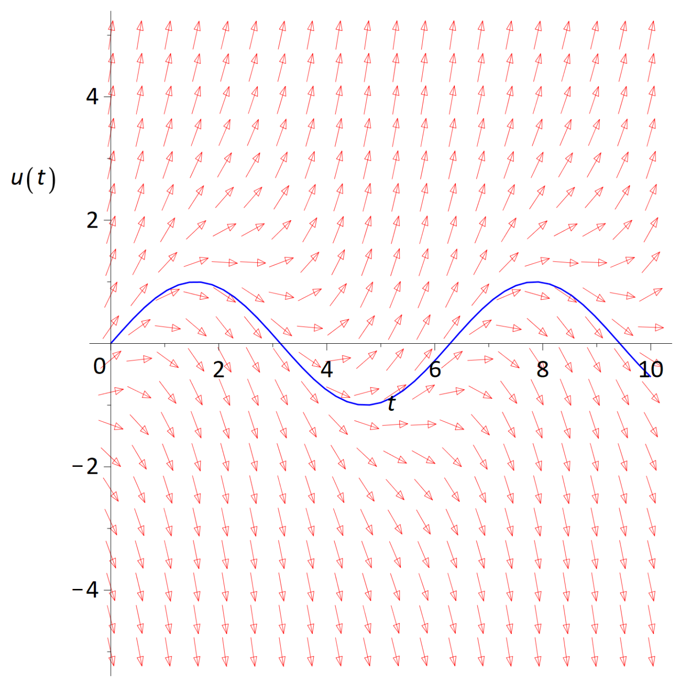



Figure 3.25: Direction field for $u'(t) = u(t) - \sin(t) + \cos(t)$, solution with $u(0) = 0$ in blue.

*(d)* *RK4 with step sizes $h = 1, 0.1, 0.01, 0.001$ produces estimates* $13.229, -0.53863, -0.54402, -0.54402$*, compared to the true value* $u(10) = \sin(10) \approx -0.54402$*. When the RK4 method invariably steps off the true solution curve (on the first step) the $C$ in the general solution*

*lights up to a nonzero value, and we begin to track* $u(t) = \sin(t) + Ce^t$ *for some* $C \neq 0$*. But RK4 does well here. The* $h = 0.001$ *solution is accurate to ten significant figures! However, by the time we get out to* $t = 50$ *(true value* $\sin(50) \approx -0.2624$*) even RK4 with step size* $0.001$ *estimates* $u(50) \approx -2.52 \times 10^7$*.*

**Exercise Solution 3.3.12.**

*(a)* $u(t) = e^{-10t}$, $\lim_{t\to 0} u(t) = 0$.

*(b) The RK4 iteration yields* $u(5) \approx 5.03 \times 10^{-22}$*, compare to the true value* $u(5) = e^{-50} \approx 1.929 \times 10^{-22}$*. Both are effectively zero.*

*(c) RK4 now estimates* $u(5) \approx 1.18 \times 10^{-12}$*, about* $10^{10}$ *times too big, but still effectively zero.*

*(d) RK4 now estimates* $u(5) \approx 2.087 \times 10^{12}$*, and the iterates keep growing. Clearly nonsense.*

*(e) A little experimentation shows that around* $h \approx 0.28$ *the estimated solution grows rather than decays.*

**Exercise Solution 3.3.13.**

*(a) A single Euler step produces* $u_1 = 1 + (0.1)f(0.5, 1.0) \approx 1.097942554$*.*

*(b) The first Euler step of size* $0.05$ *produces* $u_{k+1/2} = 1 + (0.05)f(0.5, 1.0) \approx 1.048971277$*. The second step of size* $0.05$ *produces* $\tilde{u}_1 = 1.048971277 + (0.05)f(0.55, 1.048971277) \approx 1.105365009$*.*

*(c) The local truncation error is estimated as* $|LTE| = 2|1.105365009 - 1.097942554| \approx 1.484 \times 10^{-2}$*. This does not meet the error tolerance. But cutting* $h$ *in half to* $0.05$ *and then taking a step of size* $0.05$ *(in conjunction with two steps of size* $0.025$*) yields* $|LTE| \approx 3.64 \times 10^{-3}$*, which is acceptable.*

# Section 3.4

**Exercise Solution 3.4.1.**

(a) *The sum of squares function is*

$$S(a) = (0.1a - 0.11)^2 + (0.6a - 0.5)^2 + (1.1a - 0.6)^2 + (1.4a - 0.5)^2.$$

*Setting* $S'(a) = 0$ *yields minimizer* $a \approx 0.472$*, easily confirmed with a graph of* $S(a)$*. The residual is* $0.0833$*. The fit to the data is shown in Figure 3.26, left panel.*

(b) *The sum of squares function is*

$$S(a, b) = (0.1a+b-0.11)^2+(0.6a+b-0.5)^2+(1.1a+b-0.6)^2+(1.4a+b-0.5)^2.$$

*Setting* $\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0$ *and solving for* $a$ *and* $b$ *yields minimizer* $a \approx 0.309, b \approx 0.180$*, easily confirmed with a graph of* $S(a, b)$*. The residual is* $0.0474$*. Of course this residual is smaller since throwing* $b$ *into the computation gives us "more to work with" when fitting the data (informally). The fit to the data is shown in Figure 3.26, right panel.*

(c) *The sum of squares function is*

$$S(a) = (0.01a-0.11)^2+(0.36a-0.5)^2+(1.21a-0.6)^2+(1.96a-0.5)^2.$$

*Setting* $S'(a) = 0$ *yields minimizer* $a \approx 0.347$*, easily confirmed with a graph of* $S(a)$*. The residual is* $0.217$*. The fit to the data is shown in Figure 3.27, left panel.*

(d) *The sum of squares function is*

$$\begin{aligned} S(a, b, c) &= (0.01a + 0.1b + c - 0.11)^2 + (0.36a + 0.6b + c - 0.5)^2 \\ &\quad + (1.21a + 1.1b + c - 0.6)^2 + (1.96a + 1.4b + c - 0.5)^2. \end{aligned}$$

*Setting* $\frac{\partial S}{\partial a} = 0, \frac{\partial S}{\partial b} = 0, \frac{\partial S}{\partial c} = 0$ *yields minimizer* $a \approx -0.61, b \approx 1.217, c \approx -0.0069$*, residual* $5.4 \times 10^{-5}$*. The fit to the data is shown in Figure 3.27, right panel.*

**Exercise Solution 3.4.2.** *Minimizing* $S(k)$ *as given yields* $k \approx 8.4 \times 10^{-4}$*, with residual* $2.34 \times 10^{-4}$*. A plot of* $\ln(y_0) - k^* \ln(t)$ *and the data pairs (time,* $\ln([H_2O_2])$*) is shown in Figure 3.28.*

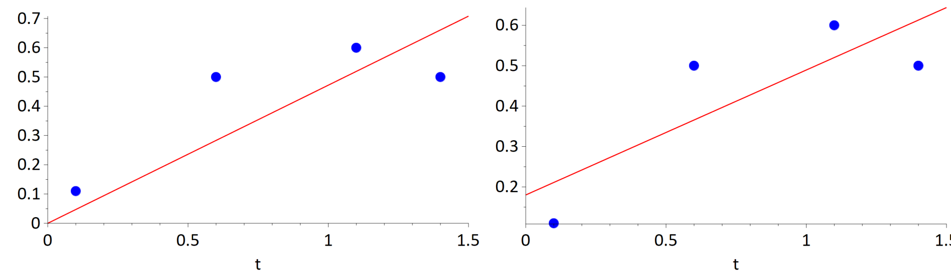



Figure 3.26: Best fit to data for Exercise 3.4.1, $u(t) = at$ (left panel) and $u(a, b, t) = at + b$ (right panel).

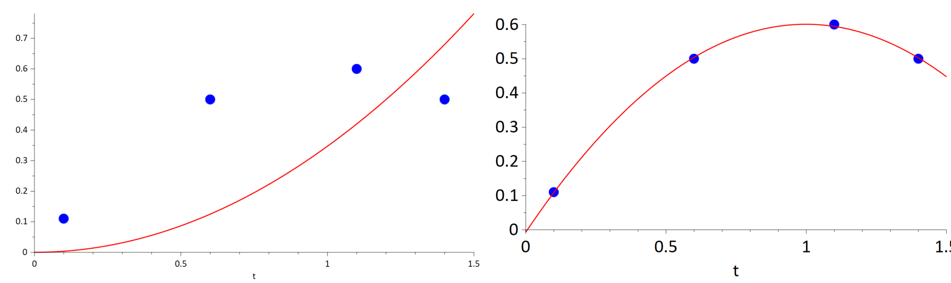



Figure 3.27: Best fit to data for Exercise 3.4.1, $u(t) = at^2$ (left panel) and $u(a, b, c, t) = at^2 + bt + c$ (right panel).

**Exercise Solution 3.4.3.** *Forming an appropriate sum of squares* $S(k, P)$ *and minimizing by solving* $\frac{\partial S}{\partial k} = 0, \frac{\partial S}{\partial P} = 0$ *yields minimizer* $P \approx 8.5997, k \approx 0.8072$. *A plot of the Hill-Keller solution with these parameters and the data is shown in Figure 3.29.*

**Exercise Solution 3.4.4.** *The predicted position for Bolt in the* 100 *meter race was given by*

$$x(t) = 12t + 16.001(1 - e^{-0.865t}).$$

*To use this to estimate Bolt's time for* 200 *meters solve* $x(t) = 200$ *for* $t$. *This yields* $t \approx 18.00$ *seconds, compare to the world record time of* 19.19 *seconds (as of 2020, set in 2009 by Bolt). One possible source for the discrepancy is that the* 200 *is usually run around a turn, which slows the runners a bit.*

*The same technique for the mile yields estimated time* $t \approx 135.45$ *seconds (*2 *minutes,* 15.45 *seconds), ridiculously faster than the record as of 2020,* 3

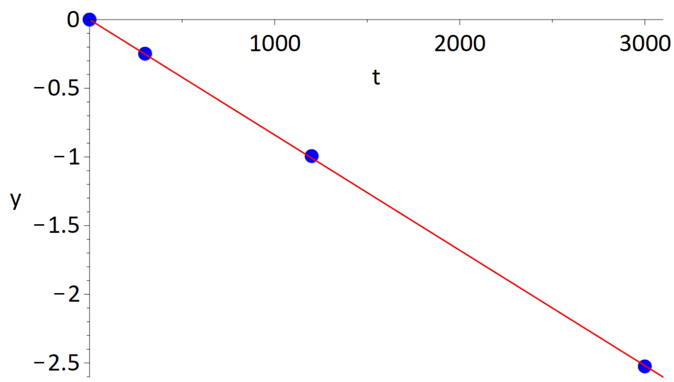



Figure 3.28: Reaction data (time, $\ln([H_2O_2])$) and best fit line $\ln(y(t)) = -kt$.

*minutes* 43.13 *seconds. Of course, the explosive effort in a sprint cannot be maintained for a mile! The situation for the marathon is even worse, with predicted time* 3517.6 *seconds, or* 58 *minutes,* 37.6 *seconds, compared to the record (as of 2020) of* 2 *hours,* 1 *minutes,* 39 *seconds.*

**Exercise Solution 3.4.5.** *From the hint it's easy to see that*

$$S''(m) = 2\sum_{j=1}^{n} x_j^2.$$

*If any $x_j$ is nonzero then this quantity is positive. Also, given that $S(m)$ is of the form $Am^2 + Bm + C$ where $A > 0$, it's clear that $S(m)$ limits to infinity as $m \to \pm\infty$.*

**Exercise Solution 3.4.6.** *In this case the sum of squares is*

$$S(m) = \sum_{j=1}^{n} (y_j - mx_j - b)^2$$

*if $b$ is considered known. This is entirely equivalent to the previous case if we define $\tilde{y}_j = y_j - b$.*

**Exercise Solution 3.4.7.** *A plot of*

$$S_1(k) = |\ln(0.78) + 300k| + |\ln(0.37) + 1200k| + |\ln(0.08) + 3000k|$$

*on the range $0 \le k \le 0.002$ is shown in Figure 3.30, left panel. In the right panel is a zoomed in graph near $k = 0.00084$, which is about the minimum.*

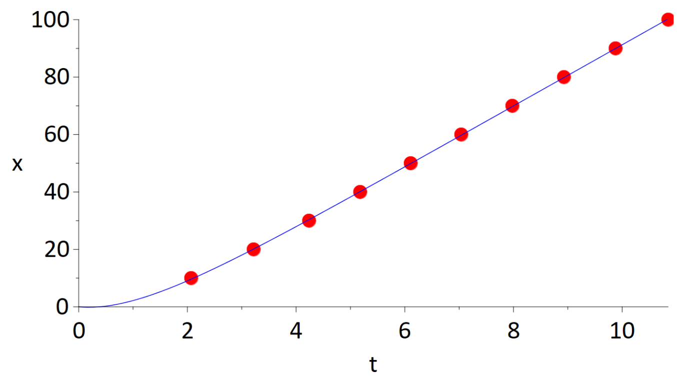



Figure 3.29: Position $x(t)$ from Hill-Keller solution with $P = 8.5997, k = 0.8072$ (blue) and data from Tori Bowie's 2017 race (red).

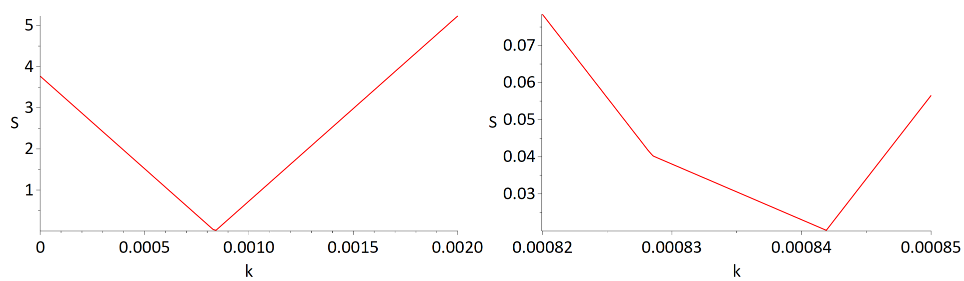



Figure 3.30: Plot of $S_1(k)$ and close-up near $k = 0.00084$.

*In this case $L^1$ minimization yields essentially the same estimate as least-squares.*

**Exercise Solution 3.4.8.**

(a) *The data is shown in the left panel of Figure 3.31. It looks very logistic.*

(b) *The sum of squares is a rather complicated mess.*

(c) *The plot indicates that $K \approx 650$ is a good guess, and playing around with $r$ yields $r \approx 0.5$ as reasonable. With these as an initial guess in a solver for $\frac{\partial S}{\partial r} = 0$ and $\frac{\partial S}{\partial K} = 0$ we find $K \approx 663.9$ and $r \approx 0.541$.*

(d) *This plot is shown in the right panel of Figure 3.31. It is an excellent fit.*

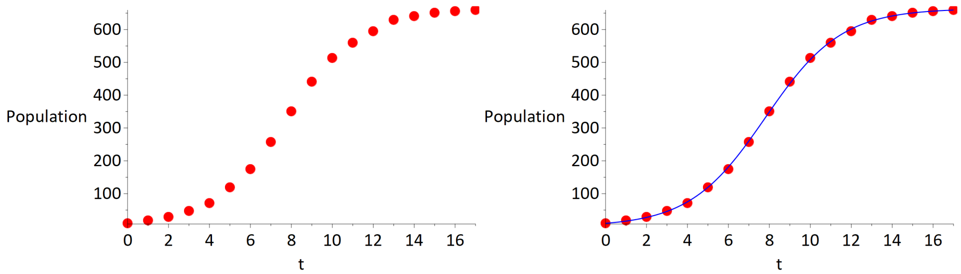



Figure 3.31: Yeast data (left) and best fit logistic curve (right) superimposed.

**Exercise Solution 3.4.9.**

*(a) A graph of $S(k)$ on the range $0.01 \leq k \leq 0.05$ is shown in Figure 3.32, left panel. Obvious minimum near $k = 0.03$.*

*(b) Set $S'(k) = 0$ and solve to find $k^* \approx 0.03156$. The residual sum of squares is $173.97$.*

*(c) A plot of $u(t)$ is shown superimposed on the data in the right panel of Figure 3.32. A decent fit, not perfect.*

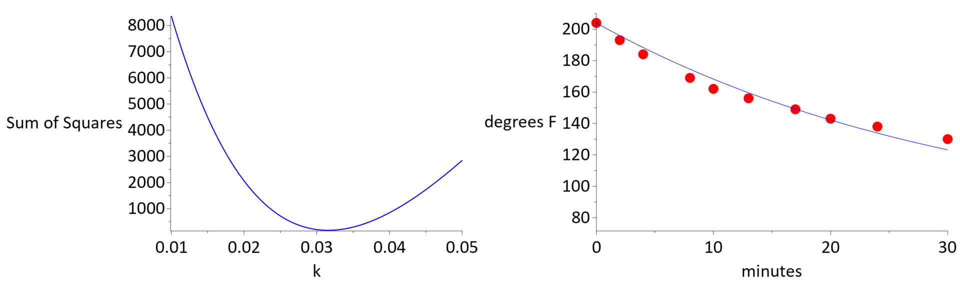



Figure 3.32: Sum of squares for potato data (left panel), best fit solution for Newton's Law of Cooling (right panel).

**Exercise Solution 3.4.10.**

*(a) A graph of $\tilde{S}(k)$ on the range $0.01 \leq k \leq 0.05$ is shown in Figure 3.33, left panel. Obvious minimum near $k = 0.03$.*

*(b) Set $S'(k) = 0$ and solve to find $k^* \approx 0.03013$.*

(c) *A plot of $u(t)$ is shown superimposed on the data in the right panel of Figure 3.33. A decent fit, not perfect. Almost the same as the previous exercise.*

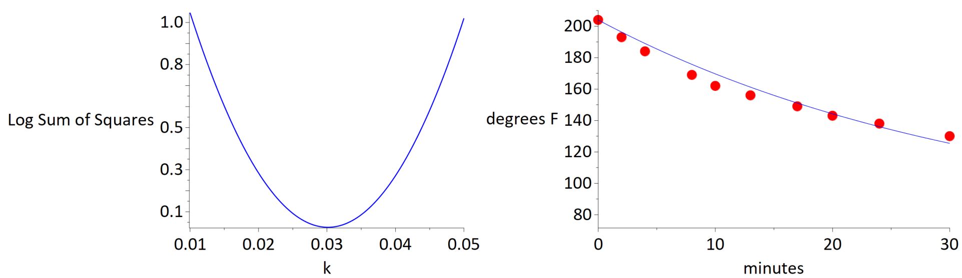



Figure 3.33: Log sum of squares for potato data (left panel), best fit solution for Newton's Law of Cooling (right panel).

**Exercise Solution 3.4.11.**

(a) *To take $F(0) = 0$ means that if the object starts at ambient temperature it stays at ambient, while if $F$ is strictly increasing it means that if object 1 is hotter than object 2 (say both hotter than ambient) then object 1 is cooling faster than object 2. Under this condition the ODE has a unique fixed point at $u = A$, stable.*

(b) *If $v > 0$ then $F(v) = kv^r$ and $F'(v) = krv^r > 0$, so $F$ is strictly increasing for $v > 0$. If $v < 0$ then $F(v) = -k(-v)^r$ and $F'(v) = kr(-v)^r > 0$ and again $F$ is strictly increasing. It's easy to see that $F(v) \neq 0$ unless $v = 0$.*

(c) *If we define*

$$u(t) = A + \left((u_0 - A)^{1-r} + k(r-1)t\right)^{1/(1-r)}$$

*then $u(0) = A + ((u_0 - A)^{1-r})^{1/(r-1)} = A + (u_0 - A) = u_0$, making use of $r > 1$ and $u_0 > A$. Also*

$$u'(t) = -k\left((u_0 - A)^{1-r} + k(r-1)t\right)^{r/(1-r)}$$

*and*

$$k(u(t) - A)^r = k\left((u_0 - A)^{1-r} + k(r-1)t\right)^{r/(1-r)}$$

*which makes it easy to see that $u$ satisfies $u'(t) = -k(u(t) - A)^r$ with the required initial condition.*

(d) *A plot of* $\ln(S(k,r))$ *on the range* $0 \le k \le 0.0004, 2 \le r \le 2.5$ *is shown in Figure 3.34 in the left panel. Quite difficult to see where a global minimum might occur, but rotating the figure around helps to see that around* $r = 2.2, k = 0.00015$ *looks promising.*

(e) *Setting* $\frac{\partial S}{\partial k} = 0$ *and* $\frac{\partial S}{\partial r} = 0$ *and solving for* $k$ *and* $r$ *in the range* $0 \le k \le 0.0004, 2 \le r \le 2.5$ *yields (at least one) solution* $k \approx 1.03 \times 10^{-4}$ *and* $r \approx 2.253$*, residual* $3.782$*. A plot of resulting* $u(t)$ *superimposed on the data is shown in the right panel of Figure 3.34.*

*The fit is much superior to the standard Newton's Law of Cooling. The residual here,* $3.782$*, is much smaller than the residual of* $173.97$ *obtained from the usual Newton's Law of Cooling.*

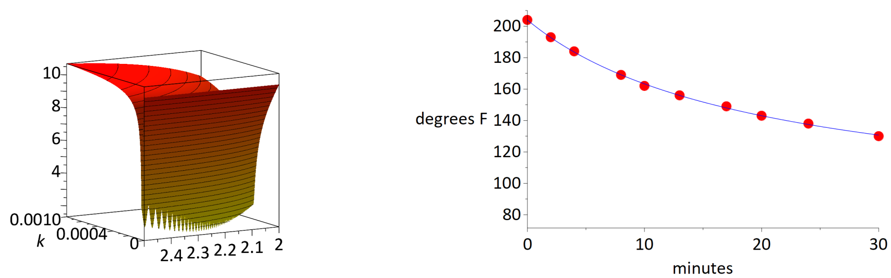



Figure 3.34: Plot of $\ln(S(k,r))$ (left panel) and graph of best fit $u(t)$ (right panel).

# Section 4.1

**Exercise Solution 4.1.1.** *Suppose the mass is at position $u(t)$ at time $t$. In this position the spring on the left exerts force $-k_1 u$ (pulling the mass back to the left if $u > 0$, pushing it right if $u < 0$) and the spring on the right exerts a similar force $-k_2 u$. If $u' > 0$ (mass moving to the right) then the dashpot on the left exerts force $-c_1 u'$, and the dashpot on the right exerts force $-c_2 u'$. The total force on the mass is thus $-(k_1 + k_2)u - (c_1 + c_2)u'$, and Newton's Second Law yields $mu'' = -(k_1 + k_2)u - (c_1 + c_2)u'$ or*

$$mu'' + (c_1 + c_2)u' + (k_1 + k_2)u = 0.$$

**Exercise Solution 4.1.2.**

(a) *Suppose $F_{friction} = -c\ \mathrm{sgn}(u'(t))(u'(t))^2$ and $u' > 0$. Then $(u')^2 > 0$ and $\mathrm{sgn}(u') > 0$, so $F_{friction} < 0$ and $|F_{friction}| = c|u'(t)|^2$, as asserted. If $u' < 0$ then $(u')^2 > 0$ and $\mathrm{sgn}(u') < 0$, so $F_{friction} > 0$ and $|F_{friction}| = c|u'(t)|^2$, as asserted. In this model the forces on the mass are $F_{friction} + F_{spring} = -c\ \mathrm{sgn}(u'(t))(u'(t))^2 - ku$, so Newton's Second Law gives $mu'' + c\ \mathrm{sgn}(u'(t))(u'(t))^2 + ku = 0$.*

(b) *In this case $F_{friction} = -F\mathrm{sgn}(u')$ and the equation governing the mass motion is $mu'' + F\mathrm{sgn}(u') + ku = 0$.*

**Exercise Solution 4.1.3.**

(a) *The ODE is*

$$5000u''(t) + (2 \times 10^4)u'(t) + (5 \times 10^5)u = 0.$$

(b) *Compute*

$$\begin{aligned} u(t) &= \frac{\sqrt{6}e^{-2t}}{1200}\sin(4\sqrt{6}t) + \frac{e^{-2t}}{100}\cos(4\sqrt{6}t) \\ u'(t) &= -\frac{\sqrt{6}}{24}e^{-2t}\sin(4\sqrt{6}t) \\ u''(t) &= \frac{\sqrt{6}e^{-2t}}{12}\sin(4\sqrt{6}t) - e^{-2t}\cos(4\sqrt{6}t). \end{aligned}$$

*Simple algebra shows that the ODE is satisfied (write the ODE as $5000(u''(t) + 4u'(t) + 100u(t)) = 0$). A plot of the solution is shown in the left panel of Figure 4.35.*

(c) *The building goes through a full oscillation in $P$ seconds where $4\sqrt{6}P = 2\pi$, so $P = \pi/(2\sqrt{6}) \approx 0.64$ seconds.*

(d) *The acceleration $u''(t)$ is graphed in the middle panel of Figure 4.35. Maximum occurs initially, 1 meter per second squared, about $1/9.8 \approx 0.102$ g's.*

(e) *The ODE is now*

$$5000u''(t) + (5 \times 10^5)u = 0.$$

*A solution of the form $u(t) = u_0 \cos(\omega)$ exists if $\omega = 10$, and taking $u_0 = 0.01$ yields the initial data. The solution is graphed in the right panel of Figure 4.35.*

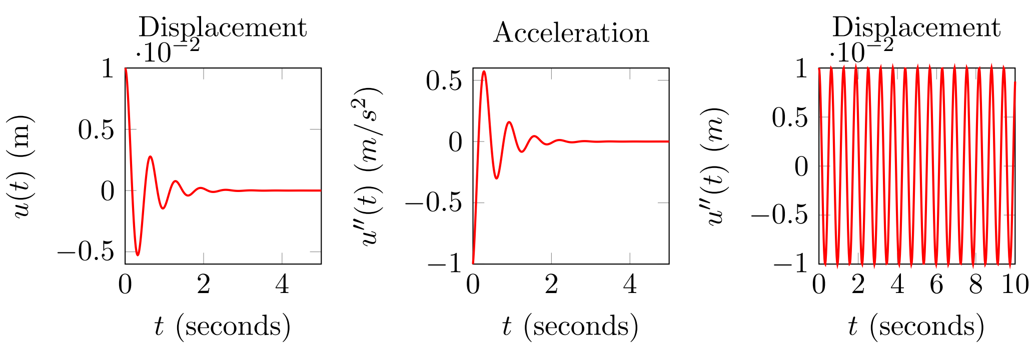



Figure 4.35: Solution $u(t) = \frac{\sqrt{6}e^{-2t}}{1200}\sin(4\sqrt{6}t) + \frac{e^{-2t}}{100}\cos(4\sqrt{6}t)$ (left panel) and $u''(t)$ (middle panel), undamped displacement (right panel).

**Exercise Solution 4.1.4.**

(a) *The volume of the submerged portion of the buoy is $-Ay$ (note $y < 0$, so this volume is positive). The mass of water displaced is thus $-A\rho y$ and the weight of water displaced is $-A\rho g y$, where we agreed to use $g > 0$. The buoyant force on the buoy is thus $F_{buoyancy} = -Ag\rho y$.*

(b) *$F_{gravity} = -mg$, accounting for the fact that it points downward.*

(c) *From Newton's Second Law of Motion $my'' = F_{buoyancy} + F_{gravity} = -mg - Ag\rho y$, which simplifies to*

$$y'' + \frac{A\rho g}{m}y = -g.$$

**Exercise Solution 4.1.5.** *The ODE is*

$$10^{-3}q''(t) + 10q'(t) + 10^4 q(t) = 3.$$

*And equilibrium solution* $q(t) = q^*$ *occurs when* $10^4 q^* = 3$ *(since* $q'' = q' = 0$*) and so* $q^* = 3\times 10^{-4}$ *coulombs. The current in the circuit is* $I(t) = q'(t) = 0$.

**Exercise Solution 4.1.6.**

(a) *If the object is at distance* $r(t)$ *from the center of the earth the gravitational force is* $\frac{GMm}{r^2(t)}$*, and so Newton's second law (assuming pure radial motion) given* $mr''(t) = \frac{GMm}{r^2(t)}$ *or*

$$r''(t) = -\frac{GM}{r^2(t)}.$$

*From the given information the initial data is* $r(0) = R$ *and* $r'(0) = v_0$.

(b) *For somewhere around* $v_0 < 11000$ *meter per second the object falls back to earth and if* $v_0 > 11000$ *the object never returns.*

(c) *Multiply both sides of* $r''(t) = -\frac{GM}{r^2(t)}$ *by* $r'(t)$ *and integrate from* $t = 0$ *to* $t = T$ *to obtain*

$$\int_0^T r'(t)r''(t)\,dt = -GM\int_0^T \frac{r'(t)}{r^2(t)}\,dt.$$

*The quantity* $r'(t)r''(t)$ *has antiderivative* $(r'(t))^2/2$ *and* $r'(t)/(r^2(t))$ *has antiderivative* $-1/r(t)$*. Making use of these antiderivatives leads to*

$$\frac{1}{2}((r'(T))^2 - v_0^2) = GM\left(\frac{1}{r(T)} - \frac{1}{R}\right)$$

*making use of* $r(0) = R$ *and* $r'(0) = v_0$*. This can be rearranged to (multiply by the object mass* $m$*)*

$$\frac{m(r'(T))^2}{2} + GMm\left(\frac{1}{R} - \frac{1}{r(T)}\right) = \frac{mv_0^2}{2}.$$

*As noted, the left side is the object's total energy, kinetic* $(mv_0^2/2)$ *plus potential (0) when it is launched. The term* $m(r'(T))^2/2$ *is the object's kinetic energy at time* $T$*. The remaining terms on the left are* *the object's potential energy when it is at altitude $r(T)$. To see this note that the work needed to lift the object from $r = R$ to $r = r(T)$ against gravity is*

$$\int_R^{r(T)} \frac{GMm}{r^2}\,dr = GMm\left(\frac{1}{R} - \frac{1}{r(T)}\right).$$

*(d) If $r(T) \to \infty$ and $r'(T) \to 0$ then from part (d) or (4.17) as $T \to \infty$ we find*

$$\frac{GMm}{R} = \frac{mv_0^2}{2}.$$

*Divide by $m$ and solve for $v_0 = \sqrt{2GM/R} \approx 11186.6$ meters per second. This is the escape velocity of the earth, about $6.95$ miles per second.*

*From Reading Exercise 29 we have $[G] = M^{-1}L^3T^{-2}$. Also $[M] = M$ (here $M$ is doing double-duty as the earth's mass and the dimension $M$!), and $[R] = L$ so that $[2GM/R] = L^2T^{-2}$ and $[v_0] = LT^{-1}$, a velocity.*

## Section 4.2

**Exercise Solution 4.2.1.** *ODE is* $3u''(t) + 24u'(t) + 60u(t) = 0$*, characteristic equation* $3r^2 + 24r + 60 = 0$*, roots* $-4 \pm 2i$*, underdamped.*

**Exercise Solution 4.2.2.** *ODE is* $u''(t) + 20u(t) = 0$*, characteristic equation* $r^2 + 20 = 0$*, roots* $\pm 2i\sqrt{5}$*, undamped.*

**Exercise Solution 4.2.3.** *ODE is* $2u''(t) + 12u'(t) + 10u(t) = 0$*, characteristic equation* $2r^2 + 12r + 10 = 0$*, roots* $-1, -5$*, overdamped.*

**Exercise Solution 4.2.4.** *ODE is* $2u''(t) + 16u'(t) + 64u(t) = 0$*, characteristic equation* $2r^2 + 16r + 64 = 0$*, roots* $-4 \pm 4i$*, underdamped.*

**Exercise Solution 4.2.5.** *ODE is* $2u''(t) + 4u'(t) + 10u(t) = 0$*, characteristic equation* $2r^2 + 4r + 10 = 0$*, roots* $-1 \pm 2i$*, underdamped.*

**Exercise Solution 4.2.6.** *ODE is* $3u''(t) + 21u'(t) + 36u(t) = 0$*, characteristic equation* $3r^2 + 21r + 36 = 0$*, roots* $-3, -4$*, overdamped.*

**Exercise Solution 4.2.7.** *ODE is* $2u''(t) + 12u'(t) + 18u(t) = 0$*, characteristic equation* $2r^2 + 12r + 18 = 0$*, double root* $-3$*, critically damped.*

**Exercise Solution 4.2.8.** *ODE is* $3u''(t) + 18u'(t) + 75u(t) = 0$*, characteristic equation* $3r^2 + 18r + 75 = 0$*, roots* $-3 \pm 4i$*, underdamped.*

**Exercise Solution 4.2.9.** *ODE is* $2u''(t)+8u'(t)+6u(t) = 0$*, characteristic equation* $2r^2 + 8r + 6 = 0$*, roots* $-1, -3$*, overdamped.*

**Exercise Solution 4.2.10.** *ODE is* $5u''(t) + 10u'(t) + 5u(t) = 0$*, characteristic equation* $5r^2 + 10r + 5 = 0$*, double root* $-1$*, critically damped.*

**Exercise Solution 4.2.11.** *ODE is* $u''(t)+6u'(t)+8u(t) = 0$*, characteristic equation* $r^2+6r+8 = 0$*, roots* $-2, -4$*, general solution* $u(t) = c_1e^{-2t}+c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-2t}/2 - 7e^{-4t}/2$*.*

**Exercise Solution 4.2.12.** *ODE is* $3u''(t) + 9u'(t) + 6u(t) = 0$*, characteristic equation* $3r^2 + 9r + 6 = 0$*, roots* $-1, -2$*, general solution* $u(t) = c_1e^{-t} + c_2e^{-2t}$*. Specific solution is* $u(t) = 7e^{-t} - 5e^{-2t}$*.*

**Exercise Solution 4.2.13.** *ODE is* $2u''(t) + 10u'(t) + 12u(t) = 0$*, characteristic equation* $2r^2 + 10r + 12 = 0$*, roots* $-2, -3$*, general solution* $u(t) = c_1e^{-2t} + c_2e^{-3t}$*. Specific solution is* $u(t) = 9e^{-2t} - 7e^{-3t}$*.*

**Exercise Solution 4.2.14.** *ODE is* $3u''(t) + 21u'(t) + 36u(t) = 0$*, characteristic equation* $3r^2 + 21r + 36 = 0$*, roots* $-3, -4$*, general solution* $u(t) = c_1e^{-3t} + c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-3t} - 9e^{-4t}$*.*

**Exercise Solution 4.2.15.** *ODE is* $2u''(t) + 10u'(t) + 8u(t) = 0$*, characteristic equation* $2r^2 + 10r + 86 = 0$*, roots* $-1, -4$*, general solution* $u(t) = c_1e^{-t} + c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-t}/3 - 5e^{-4t}/3$*.*

**Exercise Solution 4.2.16.** *ODE is* $u''(t) + 7u'(t) + 12u(t) = 0$*, characteristic equation* $r^2 + 7r + 12 = 0$*, roots* $-3, -4$*, general solution* $u(t) = c_1e^{-3t} + c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-3t} - 9e^{-4t}$*.*

**Exercise Solution 4.2.17.** *ODE is* $3u''(t) + 18u'(t) + 24u(t) = 0$*, characteristic equation* $3r^2 + 18r + 24 = 0$*, roots* $-2, -4$*, general solution* $u(t) = c_1e^{-2t} + c_2e^{-4t}$*. Specific solution is* $u(t) = 11e^{-2t}/2 - 7e^{-4t}/2$*.*

**Exercise Solution 4.2.18.** *ODE is* $u''(t)+4u'(t)+3u(t) = 0$*, characteristic equation* $r^2+4r+3 = 0$*, roots* $-1, -3$*, general solution* $u(t) = c_1e^{-t}+c_2e^{-3t}$*. Specific solution is* $u(t) = 9e^{-t}/2 - 5e^{-3t}/2$*.*

**Exercise Solution 4.2.19.** *ODE is* $u''(t)+4u'(t)+5u(t) = 0$*, characteristic equation* $r^2 + 4r + 5 = 0$*, roots* $-2 \pm i$*, general solution* $u(t) = c_1e^{(-2+i)t} + c_2e^{(-2-i)t}$*. Specific solution is* $u(t) = (1-4i)e^{(-2+i)t} + (1+4i)e^{(-2-i)t}$*. The real-valued general solution is* $u(t) = d_1e^{-2t}\cos(t) + d_2e^{-2t}\sin(t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-2t}\cos(t)+8e^{-2t}\sin(t)$*.*

**Exercise Solution 4.2.20.** *ODE is* $2u''(t) + 4u'(t) + 20u(t) = 0$*, characteristic equation* $2r^2 + 4r + 20 = 0$*, roots* $-1 \pm 3i$*, general solution* $u(t) = c_1e^{(-1+3i)t} + c_2e^{(-1-3i)t}$*. Specific solution is* $u(t) = (1-i)e^{(-1+3i)t} + (1+i)e^{(-1-3i)t}$*. The real-valued general solution is* $u(t) = d_1e^{-t}\cos(3t)+d_2e^{-t}\sin(3t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-t}\cos(3t) + 2e^{-t}\sin(3t)$*.*

**Exercise Solution 4.2.21.** *ODE is* $2u''(t) + 16u'(t) + 64u(t) = 0$*, characteristic equation* $2r^2 + 16r + 64 = 0$*, roots* $-4 \pm 4i$*, general solution* $u(t) = c_1e^{(-4+4i)t} + c_2e^{(-4-4i)t}$*. Specific solution is* $u(t) = (1-3i/2)e^{(-4+4i)t} + (1+3i/2)e^{(-4-4i)t}$*. The real-valued general solution is* $u(t) = d_1e^{-4t}\cos(4t) + d_2e^{-4t}\sin(4t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-4t}\cos(4t) + 3e^{-4t}\sin(4t)$*.*

**Exercise Solution 4.2.22.** *ODE is* $u''(t) + 6u'(t) + 18u(t) = 0$*, characteristic equation* $r^2 + 6r + 18 = 0$*, roots* $-3 \pm 3i$*, general solution* $u(t) = c_1e^{(-3+3i)t} + c_2e^{(-3-3i)t}$*. Specific solution is* $u(t) = (1-5i/3)e^{(-3+3i)t} + (1+$

$5i/3)e^{(-3-3i)t}$. *The real-valued general solution is* $u(t) = d_1 e^{-3t}\cos(3t) + d_2 e^{-3t}\sin(3t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-3t}\cos(3t) + 10e^{-3t}\sin(3t)/3$.

**Exercise Solution 4.2.23.** *ODE is* $2u''(t) + 8u'(t) + 10u(t) = 0$*, characteristic equation* $2r^2 + 8r + 10 = 0$*, roots* $-2 \pm i$*, general solution* $u(t) = c_1 e^{(-2+i)t} + c_2 e^{(-2-i)t}$. *Specific solution is* $u(t) = (1-4i)e^{(-2+i)t} + (1+4i)e^{(-2-i)t}$. *The real-valued general solution is* $u(t) = d_1 e^{-2t}\cos(t) + d_2 e^{-2t}\sin(t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-2t}\cos(t) + 8e^{-2t}\sin(t)$.

**Exercise Solution 4.2.24.** *ODE is* $3u''(t) + 12u'(t) + 60u(t) = 0$*, characteristic equation* $3r^2 + 12r + 60 = 0$*, roots* $-2 \pm 4i$*, general solution* $u(t) = c_1 e^{(-2+4i)t} + c_2 e^{(-2-4i)t}$. *Specific solution is* $u(t) = (1-i)e^{(-2+4i)t} + (1+i)e^{(-2-4i)t}$. *The real-valued general solution is* $u(t) = d_1 e^{-2t}\cos(4t) + d_2 e^{-2t}\sin(4t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-2t}\cos(4t) + 2e^{-2t}\sin(4t)$.

**Exercise Solution 4.2.25.** *ODE is* $2u''(t) + 16u'(t) + 50u(t) = 0$*, characteristic equation* $2r^2 + 16r + 50 = 0$*, roots* $-4 \pm 3i$*, general solution* $u(t) = c_1 e^{(-4+3i)t} + c_2 e^{(-4-3i)t}$. *Specific solution is* $u(t) = (1-2i)e^{(-4+3i)t} + (1+2i)e^{(-4-3i)t}$. *The real-valued general solution is* $u(t) = d_1 e^{-4t}\cos(3t) + d_2 e^{-4t}\sin(3t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-4t}\cos(3t) + 4e^{-4t}\sin(3t)$.

**Exercise Solution 4.2.26.** *ODE is* $3u''(t) + 12u'(t) + 39u(t) = 0$*, characteristic equation* $3r^2 + 12r + 39 = 0$*, roots* $-2 \pm 3i$*, general solution* $u(t) = c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t}$. *Specific solution is* $u(t) = (1-4i/3)e^{(-2+3i)t} + (1+4i/3)e^{(-2-3i)t}$. *The real-valued general solution is* $u(t) = d_1 e^{-2t}\cos(3t) + d_2 e^{-2t}\sin(3t)$ *and with the initial conditions yields specific solution* $u(t) = 2e^{-2t}\cos(3t) + 8e^{-2t}\sin(3t)/3$.

**Exercise Solution 4.2.27.** *ODE is* $u''(t) + 4u'(t) + 4u(t) = 0$*, characteristic equation* $r^2 + 4r + 4 = 0$*, double root* $-2$*, general solution* $u(t) = c_1 e^{-2t} + c_2 t e^{-2t}$. *Specific solution is* $u(t) = 2e^{-2t} + 8te^{-2t}$.

**Exercise Solution 4.2.28.** *ODE is* $3u''(t) + 6u'(t) + 3u(t) = 0$*, characteristic equation* $3r^2 + 6r + 3 = 0$*, double root* $-12$*, general solution* $u(t) = c_1 e^{-t} + c_2 t e^{-t}$. *Specific solution is* $u(t) = 2e^{-t} + 6te^{-t}$.

**Exercise Solution 4.2.29.** *ODE is* $2u''(t) + 8u'(t) + 8u(t) = 0$*, characteristic equation* $2r^2 + 8r + 8 = 0$*, double root* $-2$*, general solution* $u(t) = c_1 e^{-2t} + c_2 t e^{-2t}$. *Specific solution is* $u(t) = 2e^{-2t} + 8te^{-2t}$.

**Exercise Solution 4.2.30.** *ODE is* $5u''(t) + 40u'(t) + 80u(t) = 0$*, characteristic equation* $5r^2 + 40r + 80 = 0$*, double root* $-4$*, general solution* $u(t) = c_1e^{-4t} + c_2te^{-4t}$*. Specific solution is* $u(t) = 2e^{-4t} + 12te^{-4t}$*.*

**Exercise Solution 4.2.31.**

(a) *The ODE is* $20000u''(t) + 80000u'(t) + 60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 4r + 3) = 20000(r+1)(r+3) = 0$*, roots* $r = -1, -3$*. The general solution to the ODE is* $u(t) = c_1e^{-t} + c_2e^{-3t}$ *and the initial data requires* $c_1 + c_2 = 0, -c_1 - 3c_2 = 0.1$*, solution* $c_1 = 0.05, c_2 = -0.05$*. The solution is thus* $u(t) = 0.05e^{-t} - 0.05e^{-3t}$*. This system is overdamped. A plot of* $u(t)$ *is shown in the left panel of Figure 4.36.*

(b) *The ODE is* $20000u''(t)+40000u'(t)+60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 2r + 3) = 0$*, roots* $r = -1 \pm i\sqrt{2}$*. The general solution to the ODE is* $u(t) = c_1e^{(-1+i\sqrt{2})t} + c_2e^{(-1-i\sqrt{2})t}$ *and the initial data requires* $c_1 + c_2 = 0, (-1 + i\sqrt{2})c_1 + (-1 - i\sqrt{2})c_2 = 0.1$*, solution* $c_1 = -i\sqrt{2}/40 \approx -0.0353i, c_2 = i\sqrt{2}/40 \approx 0.0353i$*. The real-valued version of the solution is* $u(t) = \sqrt{2}e^{-t}\sin(t\sqrt{2})/20$*. This system is underdamped. A plot of* $u(t)$ *is shown in the right panel of Figure 4.36.*

(c) *The ODE is* $20000u''(t)+60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 3) = 0$*, roots* $r = \pm i\sqrt{3}$*. The general solution to the ODE is* $u(t) = c_1e^{it\sqrt{3}} + c_2e^{-it\sqrt{3}}$ *and the initial data requires* $c_1 + c_2 = 0, i\sqrt{3}c_1 - i\sqrt{3}c_2 = 0.1$*, solution* $c_1 = -i\sqrt{3}/60 \approx -0.0289i, c_2 = i\sqrt{6}/60 \approx 0.0289i$*. The real-valued version of the solution is* $u(t) = \sqrt{3}\sin(t\sqrt{3})/30$*. This system is underdamped. A plot of* $u(t)$ *is shown in the left panel of Figure 4.37.*

(d) *The choice* $c = 40000\sqrt{3} \approx 69282$ *yields a critically damped system. The ODE is* $20000u''(t)+40000\sqrt{3}u'(t)+60000u(t) = 0$*, with* $u(0) = 0$ *and* $u'(0) = 0.1$*. The characteristic equations is* $20000(r^2 + 2\sqrt{3}r + 3) = 0$*, double root* $r = -\sqrt{3}$*. The general solution to the ODE is* $u(t) = c_1e^{-t\sqrt{3}} + c_2te^{-t\sqrt{3}}$ *and the initial data requires* $c_1 = 0$ *and* $c_2 = 1/10$*. The solution is* $u(t) = te^{-t\sqrt{3}}/10$*. A plot of* $u(t)$ *is shown in the right panel of Figure 4.37.*

**Exercise Solution 4.2.32.**

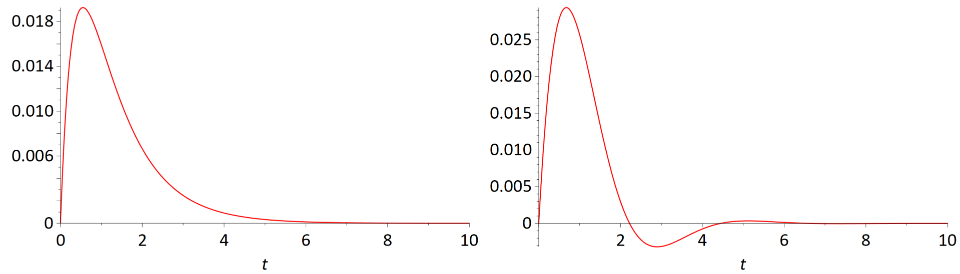



Figure 4.36: Solution to $20000u''(t) + 80000u'(t) + 60000u(t) = 0$ (left) and $20000u''(t) + 40000u'(t) + 60000u(t) = 0$ (right), both with $u(0) = 0, u'(0) = 0.1$.

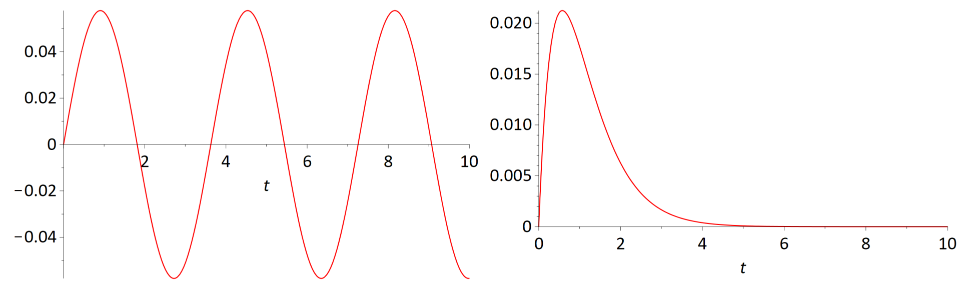



Figure 4.37: Solution to $20000u''(t) + 60000u(t) = 0$ (left) and $20000u''(t) + 40000\sqrt{3}u'(t) + 60000u(t) = 0$ (right), both with $u(0) = 0, u'(0) = 0.1$.

(a) *This follows from substituting* $d(t) = d'(t) = d''(t) = 0$ *into the vibration table ODE.*

(b) *The equilibrium position comes from substituting* $y(t) = y_{eq}$ *(so* $y'(t) = y''(t) = 0$*) into the ODE to obtain* $ky_{eq} = -mg$*, so* $y_{eq} = -mg/k$*.*

(c) *From* $y(t) = y_{eq} + w(t)$ *it follows that* $y'(t) = w'(t), y''(t) = w''(t)$*, and substituting these into the ODE* $my''(t) + cy'(t) + ky(t) = -mg$ *yields* $mw''(t) + cw'(t) + kw(t) = 0$*.*

(d) *The governing ODE is* $100w''(t) + 2000w'(t) + 10^4 w(t) = 0$ *with* $g = 9.8$*. The system is critically damped since* $(2000)^2 - 4(100)(10^4) = 0$*.*

(e) $w(t) = 0.01te^{-10t}$*. The table top returns to within* $0.0001$ *of equilibrium when* $w(t) = 0.0001$*, which leads to* $t \approx 0.3577$ *seconds.*

**Exercise Solution 4.2.33.**

(a) *This system is an undamped spring-mass system.*

(b) *The characteristic equation is* $r^2 + gr/L = 0$ *with roots* $r = \pm i\sqrt{g/L}$. *The general solution will be of the form*

$$\theta(t) = c_1 \cos(t\sqrt{g/L}) + c_2 \sin(t\sqrt{g/L}).$$

(c) *The period is* $P = 2\pi/\sqrt{g/L} = 2\pi\sqrt{L/g}$. *This makes perfect sense: period increases as* $L$ *increases, decreases as* $g$ *decreases. Moreover,* $[g] = LT^{-2}$, $[L] = L$, *and so* $[P] = T$.

**Exercise Solution 4.2.34.** *The ODE is*

$$(1.0 \times 10^{-4})q''(t) + 0.1q'(t) + 10^4 q(t) = 0.$$

*This system is underdamped, since* $(0.1)^2 - 4(1.0e-4)((1.0e4) = -3.99 < 0$.

*The characteristic equation is* $(1.0 \times 10^{-4})r^2 + 0.1r + 10^4 = 0$ *with roots* $r \approx -500 \pm 9987.5i$. *The solution with the given initial data is*

$$q(t) \approx (2.5 \times 10^{-5})e^{-500t} \sin(9987.5t) + (5.0 \times 10^{-4})e^{-500t} \cos(9987.5t).$$

*A plot is shown in Figure 4.38.*

**Exercise Solution 4.2.35.**

(a) *The identity* $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ *with* $x = \omega t$ *and* $y = \phi$ *becomes (after multiplying by* $C$*)*

$$C\sin(\omega t + \phi) = C\sin(\omega t)\cos(\phi) + C\cos(\omega t)\sin(\phi).$$

*Comparison of the right side above to* $A\cos(\omega t) + B\sin(\omega t)$ *shows they will be identical as functions of* $t$ *is* $C\sin(\phi) = A$ *and* $C\cos(\phi) = B$.

(b) *Squaring each side of each of* $C\sin(\phi) = A$ *and* $C\cos(\phi) = B$ *and adding yields* $C^2 = A^2 + B^2$, *so* $C = \sqrt{A^2 + B^2}$.

(c) *Take the quotient of the left and right sides of* $C\sin(\phi) = A$ *and* $C\cos(\phi) = B$ *to obtain* $\tan(\phi) = A/B$ *or* $\phi = \arctan(A/B)$ *if* $B > 0$. *If* $B < 0, A > 0$ *then* $\phi = \arctan(A/B) + \pi$, *while if* $B < 0, A < 0$ *then* $\phi = \arctan(A/B) - \pi$.

**Exercise Solution 4.2.36.** *Start with*

$$u(t) = -\frac{r_1}{r_1 - r_2}e^{-r_1 t} + \frac{r_2}{r_1 - r_2}e^{-r_2 t}$$

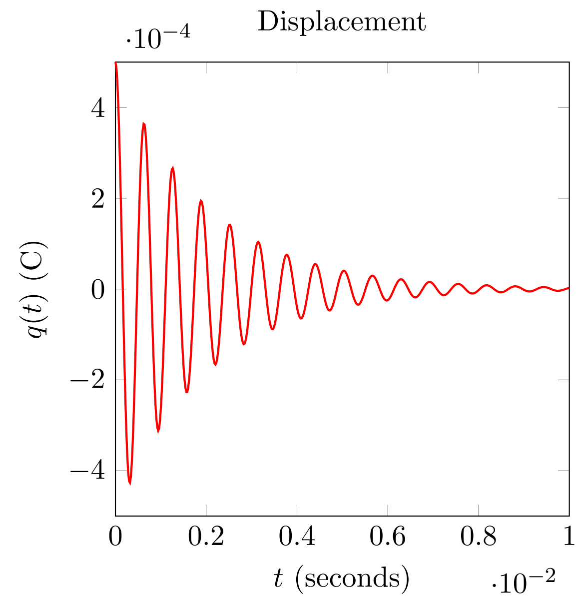



Figure 4.38: Charge on capacitor for Exercise 4.2.34.

*A bit of experimentation/plotting of this function shows that* $r_1 \approx 0.35$ *and* $r_2 >> r_1$ *yields reasonable agreement to the data. An actual least-squares fit yields* $r_1 \approx 0.39$ *and* $r_2 \approx 2.11$*. Then the characteristic equation must be approximately* $3.3(r+r_1)(r+r_2) = 3.3r^2 + 8.25r + 2.72 = 0$*, so* $c \approx 8.25$ *and* $k \approx 2.72$*.*

**Exercise Solution 4.2.37.** *As suggested, start with*

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

*and note that* $r_1 \neq r_2$ *with* $r_1$ *and* $r_2$ *both negative. We may as well assume that* $r_2 < r_1 < 0$*. If* $u(t^*) = 0$ *and* $u(t^{**}) = 0$ *for two times* $t = t^*$ *and* $t = t^{**}$*, with* $t^* \neq t^{**}$ *then this means*

$$\begin{aligned} c_1 e^{r_1 t^*} + c_2 e^{r_2 t^*} &= 0 \\ c_1 e^{r_1 t^{**}} + c_2 e^{r_2 t^{**}} &= 0. \end{aligned}$$

*Divide the first equation by* $e^{r_1 t^*}$ *to obtain* $c_1 + c_2 e^{(r_2 - r_1)t^*} = 0$*. Divide the second equation by* $e^{r_1 t^{**}}$ *to obtain* $c_1 + c_2 e^{(r_2 - r_1)t^{**}} = 0$*. Combine these two equations to find*

$$(e^{(r_2 - r_1)t^{**}} - e^{(r_2 - r_1)t^*})c_2 = 0. \tag{4.3}$$

*But the quantity* $(e^{(r_2-r_1)t^{**}} - e^{(r_2-r_1)t^*})$ *is nonzero, for if* $(e^{(r_2-r_1)t^{**}} = e^{(r_2-r_1)t^*})$ *we can take the logarithm of both sides and divide by* $r_2 - r_1$ *to find* $t^* = t^{**}$*, a contradiction. Thus the quantity multiplying* $c_2$ *on the left in (4.3) is nonzero and this forces* $c_2 = 0$*. Then* $c_1 + c_2 e^{(r_2-r_1)t^*} = 0$ *from above forces* $c_1 = 0$*.*

**Exercise Solution 4.2.38.** *As suggested, start with*

$$u(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}$$

*so that* $u(t^*) = 0$ *and* $u(t^{**}) = 0$ *become*

$$\begin{aligned} c_1 e^{-\alpha t^*} + c_2 t^* e^{-\alpha t^*} &= 0 \\ c_1 e^{-\alpha t^{**}} + c_2 t^{**} e^{-\alpha t^{**}} &= 0. \end{aligned}$$

*Divide the first equation by* $e^{-\alpha t^*}$ *to obtain* $c_1 + c_2 t^* = 0$*. The second equation similar yields* $c_1 + c_2 t^{**} = 0$*. Then* $c_2(t^{**} - t^*) = 0$*, which implies* $c_2 = 0$ *since* $t^{**} \neq t^*$*. Then* $c_1 = -c_2 t^*$ *forces* $c_1 = 0$*.*

**Exercise Solution 4.2.39.**

(a) *The roots of the characteristic equation* $mr^2 + cr + k = 0$ *are (using* $\sqrt{c^2 - 4mk} = i\sqrt{4mk - c^2}$*)*

$$r_1 = -\frac{c}{2m} + i\frac{\sqrt{4mk - c^2}}{2m} \text{ and } r_2 = -\frac{c}{2m} - i\frac{\sqrt{4mk - c^2}}{2m}.$$

(b) *As in the text, let* $r_1 = -\alpha + i\omega$ *and* $r_2 = -\alpha - i\omega$ *where* $\alpha = c/(2m)$ *and* $\omega = \sqrt{4mk - c^2}/(2m)$ *(both real). Use Euler's identity to compute*

$$\begin{aligned} e^{r_1 t} &= e^{(-\alpha + i\omega)t} = e^{-\alpha t} e^{i\omega t} = e^{-\alpha t}\cos(\omega t) + ie^{-\alpha t}\sin(\omega t) \\ e^{r_2 t} &= e^{(-\alpha - i\omega)t} = e^{-\alpha t} e^{-i\omega t} = e^{-\alpha t}\cos(\omega t) - ie^{-\alpha t}\sin(\omega t) \end{aligned}$$

*using the facts that* $\cos(-\omega t) = \cos(\omega t)$ *and* $\sin(-\omega t) = -\sin(\omega t)$*. This shows that* $e^{r_1 t}$ *and* $e^{r_2 t}$ *are conjugate.*

(c) *Take the hint and write the conditions that* $c_1 + c_2 = u_0$ *and* $r_1 c_1 + r_2 c_2 = v_0$ *as* $c_1 + c_2 = u_0$ *and* $(-\alpha + i\omega)c_1 + (-\alpha - i\omega)c_2 = v_0$*. Solve for*

$$c_1 = \frac{u_0}{2} - i(\alpha u_0 + v_0)/(2m) \text{ and } c_2 = \frac{u_0}{2} + i(\alpha u_0 + v_0)/(2m).$$

*Since* $u_0, v_0, \alpha, \omega,$ *and* $m$ *are real,* $c_1$ *and* $c_2$ *are conjugate.*

(d) *Since* $\overline{c_2} = c_1$ *and* $\overline{e^{r_2 t}} = e^{r_1 t}$ *we have* $\overline{c_2 e^{r_2 t}} = c_1 e^{r_1 t}$ *and so*

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{r_1 t} + \overline{c_1 e^{r_1 t}}$$

*which is real.*

**Exercise Solution 4.2.40.**

(a) *The downward force of gravity on the piston of mass* $m$ *is* $-mg$*. The upward force of the gas on the piston is* $PA$*, but since* $A = V/y$ *this is* $PV/y$*. Finally, from the gas law* $PV = nRT$*, so the upward force on the piston is* $nRT/y$*. The net force on the piston is then*

$$F = \frac{nRT}{y} - mg.$$

*From Newton's Second Law of Motion we have* $ma = F$*. With* $a = y''$ *and* $F$ *as above this yields*

$$my''(t) = \frac{nRT}{y(t)} - mg.$$

(b) *At equilibrium* $y(t) = y_{eq}$ *we have* $F = 0$*, so* $nRT/y_{eq} - mg = 0$*. Solve for* $y_{eq} = nRT/mg$*.*

(c) *With* $u(t) = y(t) - y_{eq}$ *we have* $y(t) = u(t) + y_{eq}$*, so* $y''(t) = u''(t)$*. Insert this information into* $my''(t) - \frac{nRT}{y(t)} = -mg$ *and find*

$$mu''(t) = -mg + \frac{nRT}{u(t) + y_{eq}}$$

*as asserted. This is a nonlinear second order ODE for* $u(t)$*.*

(d) *The motion is plotted in the left panel of Figure 4.39. The period is about 3 or so.*

(e) *With* $f(u) = 1/(u + a)$ *the approximation*

$$f(u) \approx 1/a - u/a^2 + O(u^2)$$

*with* $a = y_{eq} = nRT/mg$ *yields*

$$-mg + \frac{nRT}{u + y_{eq}} \approx -mg + nRT\left(\frac{mg}{nRT} - \frac{m^2 g^2}{n^2 R^2 T^2} u\right) = -\frac{m^2 g^2}{nRT} u.$$

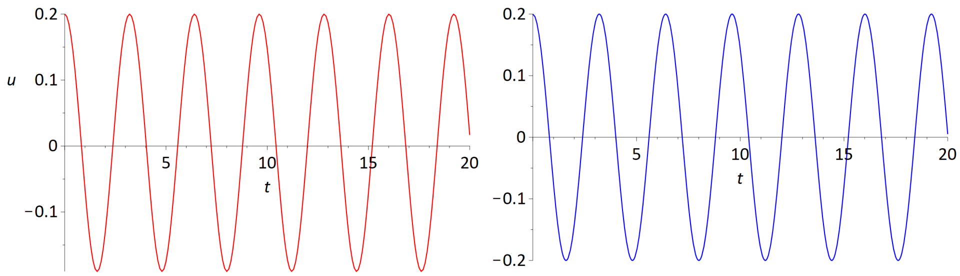



Figure 4.39: Solution to $mu''(t) = -mg + \frac{nRT}{u(t)+y_{eq}}$ (left panel) and linearized equation $u''(t) + mg^2u(t)/(nRT) = 0$ (right panel), initial data $u(0) = 0.2, u'(0) = 0$.

*In the nonlinear ODE this gives* $mu'' = -\frac{m^2g^2}{nRT}u$ *or*

$$u''(t) + \frac{mg^2}{nRT}u(t) = 0.$$

*With the given numbers this is* $u''(t) + 3.8503u(t) = 0$. *The solution with* $u(0) = 0.2, u'(0) = 0$ *is*

$$u(t) \approx 0.2\cos(1.9622t).$$

*The frequency of oscillation is* $1.9622/(2\pi) \approx 0.312$ *Hz, period* $2\pi/1.9622 \approx 3.2$ *seconds.*

*(f) The solutions to the nonlinear and linearized equations are shown in Figure 4.40.*

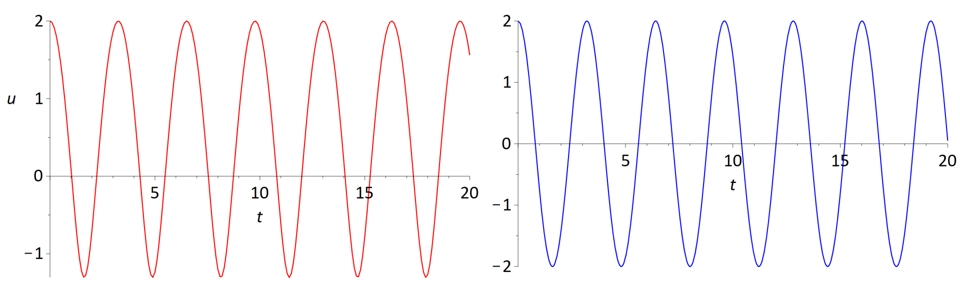



Figure 4.40: Solution to $mu''(t) = -mg + \frac{nRT}{u(t)+y_{eq}}$ (left panel) and linearized equation $u''(t) + mg^2u(t)/(nRT) = 0$ (right panel), initial data $u(0) = 2, u'(0) = 0$.

## Section 4.3

**Exercise Solution 4.3.1.** $u_h(t) = c_1e^{-4t} + c_2e^{-5t}$, $u_p(t) = e^{-3t}$. *General solution* $u(t) = e^{-3t}+c_1e^{-4t}+c_2e^{-5t}$, *specific solution* $u(t) = e^{-3t}+11e^{-4t}-10e^{-5t}$.

**Exercise Solution 4.3.2.** $u_h(t) = c_1e^{-t} + c_2e^{-3t}$, $u_p(t) = -e^{t}$. *General solution* $u(t) = -e^{t}+c_1e^{-t}+c_2e^{-3t}$, *specific solution* $u(t) = -e^{t}+13e^{-t}/2-7e^{-3t}/2$.

**Exercise Solution 4.3.3.** $u_h(t) = c_1e^{-4t}\cos(4t)+c_2e^{-4t}\sin(4t)$, $u_p(t) = 1$. *General solution* $u(t) = 1 + c_1e^{-4t}\cos(4t) + c_2e^{-4t}\sin(4t)$, *specific solution* $u(t) = 1 + e^{-4t}\cos(4t) + 7e^{-4t}\sin(4t)/4$.

**Exercise Solution 4.3.4.** $u_h(t) = c_1e^{-t} + c_2e^{-2t}$, $u_p(t) = 1$. *General solution* $u(t) = 1 + c_1e^{-t} + c_2e^{-2t}$, *specific solution* $u(t) = 1 + 5e^{-t} - 4e^{-2t}$.

**Exercise Solution 4.3.5.** $u_h(t) = c_1e^{-t} + c_2e^{-3t}$, $u_p(t) = 3t - 4$. *General solution* $u(t) = 3t-4+c_1e^{-t}+c_2e^{-3t}$, *specific solution* $u(t) = 3t-4+9e^{-t}-3e^{-3t}$.

**Exercise Solution 4.3.6.** $u_h(t) = c_1e^{-2t}\cos(2t) + c_2e^{-2t}\sin(2t)$, $u_p(t) = \sin(2t)/2-\cos(2t)$. *General solution* $u(t) = \sin(2t)/2-\cos(2t)+c_1e^{-2t}\cos(2t)+c_2e^{-2t}\sin(2t)$, *specific solution* $u(t) = \sin(2t)/2 - \cos(2t) + 3e^{-2t}\cos(2t) + 4e^{-2t}\sin(2t)$.

**Exercise Solution 4.3.7.** $u_h(t) = c_1e^{-t} + c_2e^{-4t}$, $u_p(t) = -\cos(3t)/5 - \sin(3t)/15$. *General solution* $u(t) = c_1e^{-t} + c_2e^{-4t} - \cos(3t)/5 - \sin(3t)/15$, *specific solution* $u(t) = 4e^{-t} - 9e^{-4t}/5 - \cos(3t)/5 - \sin(3t)/15$.

**Exercise Solution 4.3.8.** $u_h(t) = c_1e^{-2t}\cos(3t) + c_2e^{-2t}\sin(3t)$, $u_p(t) = -e^{-2t}\cos(4t)$, *general solution* $u(t) = -e^{-2t}\cos(4t)+c_1e^{-2t}\cos(3t)+c_2e^{-2t}\sin(3t)$, *specific solution* $u(t) = -e^{-2t}\cos(4t) + 3e^{-2t}\cos(3t) + 7e^{-2t}\sin(3t)/3$.

**Exercise Solution 4.3.9.** $u_h(t) = c_1e^{-3t/2} + c_2te^{-3t/2}$, $u_p(t) = t^2/9 - 5t/27 + 4/27$. *General solution* $u(t) = c_1e^{-3t/2} + c_2te^{-3t/2} + t^2/9 - 5t/27 + 4/27$, *specific solution* $u(t) = 50e^{-3t/2}/27 + 161te^{-3t/2}/27 + t^2/9 - 5t/27 + 4/27$.

**Exercise Solution 4.3.10.** $u_h(t) = c_1e^{-t} + c_2e^{-4t}$, $u_p(t) = -(2t + 1)e^{-2t}$. *General solution* $u(t) = -(2t + 1)e^{-2t} + c_1e^{-t} + c_2e^{-4t}$, *specific solution* $u(t) = -(2t + 1)e^{-2t} + 5e^{-t} - 2e^{-4t}$.

**Exercise Solution 4.3.11.** $u_h(t) = c_1 e^{-2t} + c_2 e^{-5t}$, $u_p(t) = -e^{-3t}(2t^2 + 2t+3)$. *General solution* $u(t) = -e^{-3t}(2t^2+2t+3)+c_1e^{-2t}+c_2e^{-5t}$, *specific solution* $u(t) = -e^{-3t}(2t^2 + 2t + 3) + 7e^{-2t} - 2e^{-5t}$.

**Exercise Solution 4.3.12.** $u_h(t) = c_1 \cos(t)+c_2 \sin(t)$, $u_p(t) = -t^2 \cos(t)/4+ t\sin(t)/4$. *General solution* $u(t) = -t^2 \cos(t)/4 + t\sin(t)/4 + c_1 \cos(t) + c_2 \sin(t)$, *specific solution* $u(t) = -t^2 \cos(t)/4+t\sin(t)/4+2\cos(t)+3\sin(t)$.

**Exercise Solution 4.3.13.** $u_h(t) = c_1e^{-t} \cos(3t) + c_2e^{-t} \sin(3t)$, $u_p(t) = e^{-2t}$. *General solution* $u(t) = e^{-2t} + c_1e^{-t} \cos(3t) + c_2e^{-t} \sin(3t)$, *specific solution* $u(t) = e^{-2t} + e^{-t} \cos(3t) + 2e^{-t} \sin(3t)$.

**Exercise Solution 4.3.14.** $u_h(t) = c_1e^{-t} \cos(t) + c_2e^{-t} \sin(t)$, $u_p(t) = te^{-t} \sin(t)$. *General solution* $u(t) = te^{-t} \sin(t) + c_1e^{-t} \cos(t) + c_2e^{-t} \sin(t)$, *specific solution* $u(t) = te^{-t} \sin(t) + 2e^{-t} \cos(t) + 5e^{-t} \sin(t)$.

**Exercise Solution 4.3.15.** $u_h(t) = c_1e^{-2t} \cos(3t) + c_2e^{-2t} \sin(3t)$, $u_p(t) = te^{-2t}$. *General solution* $u(t) = te^{-2t} + c_1e^{-2t} \cos(3t) + c_2e^{-2t} \sin(3t)$, *specific solution* $u(t) = te^{-2t} + 2e^{-2t} \cos(3t) + 2e^{-2t} \sin(3t)$.

**Exercise Solution 4.3.16.** $u_h(t) = c_1e^{-4t} \cos(4t) + c_2e^{-4t} \sin(4t)$, $u_p(t) = 1$. *General solution* $u(t) = 1+c_1e^{-4t} \cos(4t)+c_2e^{-4t} \sin(4t)$, *specific solution* $u(t) = 1 + e^{-4t} \cos(4t) + 7e^{-4t} \sin(4t)/4$.

**Exercise Solution 4.3.17.** $u_h(t) = c_1e^{-t} + c_2e^{-4t}$, $u_p(t) = -\cos(2t)$. *General solution* $u(t) = -\cos(2t) + c_1e^{-t} + c_2e^{-4t}$, *specific solution* $u(t) = -\cos(2t) + 5e^{-t} - 2e^{-4t}$.

**Exercise Solution 4.3.18.** $u_h(t) = c_1e^{-t} + c_2e^{-5t}$, $u_p(t) = -2e^{-2t}/3$. *General solution* $u(t) = -2e^{-2t}/3 + c_1e^{-t} + c_2e^{-5t}$, *specific solution* $u(t) = -2e^{-2t}/3 + 15e^{-t}/4 - 13e^{-5t}/12$.

**Exercise Solution 4.3.19.** $u_h(t) = c_1e^{-2t} + c_2e^{-5t}$, $u_p(t) = 5t/2 - 1/4$. *General solution* $u(t) = 5t/2-1/4+c_1e^{-2t}+c_2e^{-5t}$, *specific solution* $u(t) = 5t/2 - 1/4 + 47e^{-2t}/12 - 5e^{-5t}/3$.

**Exercise Solution 4.3.20.** $u_h(t) = c_1e^{-t} \cos(2t) + c_2e^{-t} \sin(2t)$, $u_p(t) = e^{-t} \cos(t)$. *General solution* $u(t) = e^{-t} \cos(t) + c_1e^{-t} \cos(2t) + c_2e^{-t} \sin(2t)$, *specific solution* $u(t) = e^{-t} \cos(t) + e^{-t} \cos(2t) + 5e^{-t} \sin(2t)/2$.

**Exercise Solution 4.3.21.** $u_h(t) = c_1e^{-t} \cos(t)+c_2e^{-t} \sin(t)$, $u_p(t) = (5t-2) \cos(t) + (10t - 14) \sin(t)$. *General solution* $u(t) = (5t - 2) \cos(t) + (10t - 14) \sin(t)+c_1e^{-t} \cos(t)+c_2e^{-t} \sin(t)$, *specific solution* $u(t) = (5t-2) \cos(t)+ (10t - 14) \sin(t) + 4e^{-t} \cos(t) + 16e^{-t} \sin(t)$.

**Exercise Solution 4.3.22.** $u_h(t) = c_1e^{-2t}\cos(t) + c_2e^{-2t}\sin(t)$, $u_p(t) = -\cos(5t)-\sin(5t)$. *General solution* $u(t) = -\cos(5t)-\sin(5t)+c_1e^{-2t}\cos(t)+c_2e^{-2t}\sin(t)$, *specific solution* $u(t) = -\cos(5t) - \sin(5t) + 3e^{-2t}\cos(t) + 14e^{-2t}\sin(t)$.

**Exercise Solution 4.3.23.** $u_h(t) = c_1\cos(t) + c_2\sin(t)$, $u_p(t) = t$, *general solution* $u(t) = t + c_1\cos(t) + c_2\sin(t)$, *specific solution* $u(t) = t + 2\cos(t) + 2\sin(t)$.

**Exercise Solution 4.3.24.** $u_h(t) = c_1e^{-4t}+c_2e^{-5t}$, $u_p(t) = 2te^{-4t}$, *general solution* $u(t) = 2te^{-4t} + c_1e^{-4t} + c_2e^{-5t}$, *specific solution* $u(t) = 2te^{-4t} + 11e^{-4t} - 9e^{-5t}$.

**Exercise Solution 4.3.25.** $u_h(t) = c_1e^{-t} + c_2e^{-5t}$, $u_p(t) = te^{-t}/2$, *general solution* $u(t) = te^{-t}/2 + c_1e^{-t} + c_2e^{-5t}$, *specific solution* $u(t) = te^{-t}/2 + 25e^{-t}/8 - 9e^{-5t}/8$.

**Exercise Solution 4.3.26.** $u_h(t) = c_1e^{-t}+c_2e^{-3t}$, $u_p(t) = -te^{-3t}$, *general solution* $u(t) = -te^{-3t} + c_1e^{-t} + c_2e^{-3t}$, *specific solution* $u(t) = -te^{-3t} + 5e^{-t} - 3e^{-3t}$.

**Exercise Solution 4.3.27.** $u_h(t) = c_1\cos(t) + c_2\sin(t)$, $u_p(t) = t\sin(t)/2$, *general solution* $u(t) = t\sin(t)/2+c_1\cos(t)+c_2\sin(t)$, *specific solution* $u(t) = t\sin(t)/2 + 2\cos(t) + 3\sin(t)$

**Exercise Solution 4.3.28.** $u_h(t) = c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t)$, $u_p(t) = -te^{-t}\cos(t)$, *general solution* $u(t) = -te^{-t}\cos(t)+c_1e^{-t}\cos(t)+c_2e^{-t}\sin(t)$, *specific solution* $u(t) = -te^{-t}\cos(t) + 2e^{-t}\cos(t) + 6e^{-t}\sin(t)$.

**Exercise Solution 4.3.29.** $u_h(t) = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$, $u_p(t) = -te^{-t}\cos(3t)/6$, *general solution* $u(t) = -te^{-t}\cos(3t)/6 + c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$, *specific solution* $u(t) = -te^{-t}\cos(3t)/6+2e^{-t}\cos(3t)+31e^{-t}\sin(3t)/18$.

**Exercise Solution 4.3.30.** $u_h(t) = c_1e^{-2t}\cos(2t) + c_2e^{-2t}\sin(2t)$, $u_p(t) = 4te^{-2t}\sin(2t)$, *general solution* $u(t) = 4te^{-2t}\sin(2t)+c_1e^{-2t}\cos(2t)+c_2e^{-2t}\sin(2t)$, *specific solution* $u(t) = 4te^{-2t}\sin(2t) + 2e^{-2t}\cos(2t) + 7e^{-2t}\sin(2t)/2$.

**Exercise Solution 4.3.31.** $u_h(t) = c_1e^{-2t} + c_2te^{-2t}$, $u_p(t) = t^3e^{-2t}/6$, *general solution* $u(t) = t^3e^{-2t}/6 + c_1e^{-2t} + c_2te^{-2t}$, *specific solution* $u(t) = t^3e^{-2t}/6 + 2e^{-2t} + 7te^{-2t}$.

**Exercise Solution 4.3.32.** $u_h(t) = c_1\cos(t)+c_2\sin(t)$, $u_p(t) = -t\cos(t)/2$, *general solution* $u(t) = -t\cos(t)/2 + c_1\cos(t) + c_2\sin(t)$, *specific solution* $u(t) = -t\cos(t)/2 + 2\cos(t) + 7\sin(t)/2$.

**Exercise Solution 4.3.33.** *Substituting $u_p(t) = Ae^{at}$ into $mu''(t)+cu'(t)+ku(t) = e^{at}$ produces $A(ma^2+ca+k)e^{at} = e^{at}$, so that $A(ma^2+ca+k) = 1$. Since $a$ is not a root of the characteristic equation, $ma^2+ca+k \neq 0$ and so we can solve uniquely for $A$ as $A = 1/(ma^2+ca+k)$.*

**Exercise Solution 4.3.34.**

*(a)* *Substitute $u_p(t) = A_0$ and $f(t) = a_0$ into the ODE and find $kA_0 = a_0$, so $A_0 = a_0/k$.*

*(b)* *Substitute $u_p(t) = A_0 + A_1 t$ and $f(t) = a_0 + a_1 t$ into the ODE and find*

$$(cA_1 + kA_0) + kA_1 t = a_0 + a_1 t.$$

*Then match $t$ coefficients on both sides to find $kA_1 = a_1$, so $A_1 = a_1/k$. Then match constant terms to find $A_0 = (a_0 - cA_1)/k = (a_0 - ca_1/k)/k$.*

*(c)* *Substitute $u_p(t) = \sum_{j=0}^{n} A_j t^j$ and $f(t) = \sum_{j=0}^{n} a_j t^j$ into the ODE and find*

$$kA_n t^n + \sum_{j=0}^{n-1} B_j t^j = \sum_{j=0}^{n} a_j t^j$$

*where $B_j$ involves $A_j, A_{j+1}, \ldots, A_n$ and contains the term $kA_j$. Then match $t$ coefficients on both sides above to find $kA_n = a_n$, so $A_n = a_n/k$. From this we can solve for $B_{n-1} = a_{n-1}$ for $A_{n_1}$, then $B_{n-2} = a_{n-2}$ for $A_{n-2}$, and so on.*

**Exercise Solution 4.3.35.**

*(a)* *The solution is now*

$$u(t) \approx -0.03 + 0.005e^{-1.51t} + 0.0251e^{-215.9t}.$$

*The graph is shown in the left panel of Figure 4.41. The maximum deflection is now $-0.03$, but the solution is much more "abrupt" near $t = 0$, e.g., subjects the rider to a much higher acceleration.*

*(b)* *The solution is now*

$$u(t) \approx -0.03 - 0.403e^{-13.04t}\sin(12.49t) + 0.03e^{-13.04t}\cos(12.49t).$$

*The graph is shown in the right panel of Figure 4.41. The maximum deflection is now $-0.146$ (which would actually bottom out the shock at a 140mm travel). A significantly underdamped system would feel unpleasantly "bouncy."*

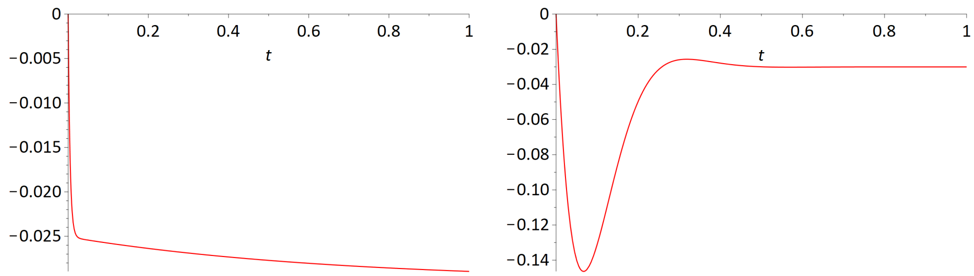



Figure 4.41: Solution to shock absorber ODE with $c = 10^4$ (left) and $c = 1000$ (right).

**Exercise Solution 4.3.36.**

*(a)* *With the ODE in the form $x''(t) + kx'(t) = P$. The characteristic equation for the homogeneous ODE $x''(t) + kx'(t) = 0$ is $r^2 + kr = 0$, with roots $r = 0$ and $r = -k$.*

*(b)* *The general solution to the homogeneous equation is therefore $x(t) = c_1 + c_2 e^{-kt}$.*

*(c)* *A particular solution to $x''(t) + kx'(t) = P$ can be found with a guess $x(t) = at$, which leads to $ka = P$ and so $a = P/k$. Thus the solution is $x_p(t) = Pt/k$.*

*(d)* *The general solution to the nonhomogeneous ODE is*

$$x(t) = Pt/k + c_1 + c_2 e^{-kt}.$$

*To obtain $x(t_0) = 0, x'(t_0) = 0$ we need (use $x'(t) = P/k - c_2 e^{-kt}$)*

$$Pt_0/k + c_1 + c_2 e^{-kt_0}, \quad P/k - c_2 e^{-kt} = 0.$$

*This yields $c_2 = Pe^{kt_0}/k$ and $c_1 = -Pt_0/k - P/k^2$. With these in the general solution we obtain*

$$x(t) = \frac{P}{k^2}(e^{-k(t-t_0)} - 1 + k(t - t_0)$$

*as before.*

**Exercise Solution 4.3.37.**

*(a)* *Substitute $y(t) = y_{eq}$ into the ODE with $d(t) = 0$ and obtain $ky_{eq} = kL_0 - mg$, so $y_{eq} = L_0 - mg/k \approx 0.902$ meters. That is, the table compresses about 10 cm due to gravity.*

(b) *The solution is*

$$y(t) \approx 1.88\times10^{-6}(\cos(125.66t)-e^{-10t})-0.002te^{-10t}+1.57\times10^{-5}\sin(125.66t)+0.902.$$

*A plot is shown in the left panel of Figure 4.42. The amplitude for* $t > 1$ *is about* $1.58 \times 10^{-5}$ *meters, about* 16 *percent that of* $d(t)$.

(c) *In this case the solution is*

$$y(t) = 0.902 - te^{-10t}/10.$$

*A graph is shown in the right panel of Figure 4.42.*

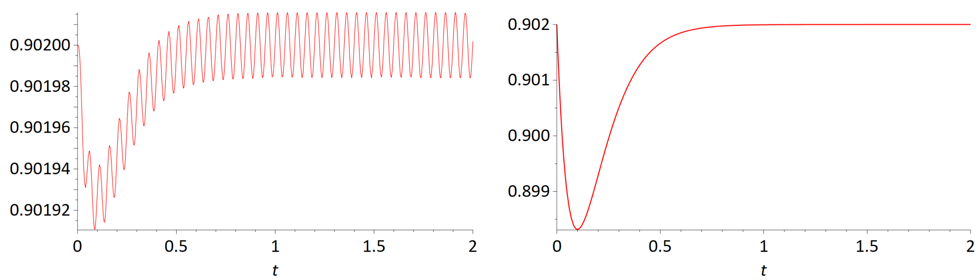



Figure 4.42: Motion of vibration table with ground shaking (left) or initial velocity $y'(0) = -0.1$ (right).

**Exercise Solution 4.3.38.**

(a) *The ODE is*

$$0.1q''(t) + 20q(t) + 10^4 q(t) = 5$$

*with* $q(0) = q'(0) = 0$.

(b) *The homogeneous general solution is* $q_h(t) = c_1e^{-100t}\cos(300t)+c_2e^{-100t}\sin(300t)$. *This system is underdamped.*

(c) *Guess* $q_p(t) = A$, *substitute into the ODE, and find* $A = 1/2000$.

(d) *The general solution is* $q(t) = 1/2000+c_1e^{-100t}\cos(300t)+c_2e^{-100t}\sin(300t)$. *The solution with the required initial conditions is*

$$q(t) = 1/2000 - e^{-100t}\cos(300t)/2000 - e^{-100t}\sin(300t)/6000.$$

(e) *A graph of* $q(t)$ *is shown in the left panel of Figure 4.43 and the current* $q'(t)$ *in the right panel.*

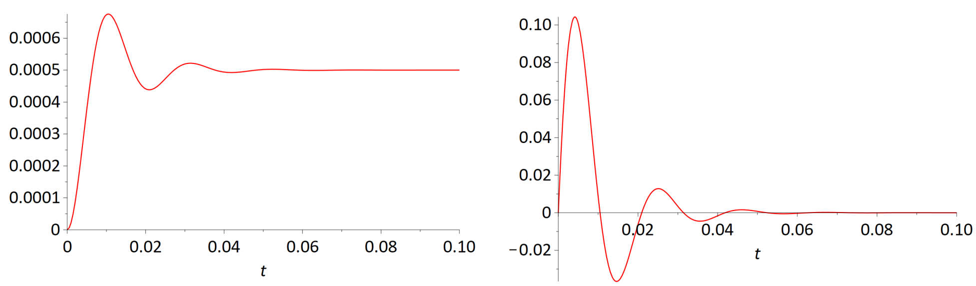



Figure 4.43: Capacitor charge (left) and current (right).

## Section 4.4

**Exercise Solution 4.4.1.** $G(\omega) = 1/\sqrt{(2\omega^2 - 8)^2 + \omega^2}$. *Resonance occurs at* $\omega = \sqrt{62}/4 \approx 1.969$. *A plot is shown in the left panel of Figure 4.44. Periodic response is* $-\frac{9\,\sin(4\,t)}{74} - \frac{3\,\cos(4\,t)}{148}$ *with amplitude* $3\sqrt{37}/148 \approx 0.123$.

**Exercise Solution 4.4.2.** $G(\omega) = 1/2\sqrt{\omega^4 + \omega^2 + 16}$. *Resonance does not occur here. A plot is shown in the right panel of Figure 4.44. Periodic response is* $-1/16\,\sin(4\,t) - 1/16\,\cos(4\,t)$, *with amplitude* $\sqrt{2}/16 \approx 0.0883$.

**Exercise Solution 4.4.3.** $G(\omega) = 1/2\sqrt{\omega^4 - 16\omega^2 + 100}$. *Resonance occurs at* $\omega = 2\sqrt{2} \approx 2.828$. *A plot is shown in the left panel of Figure 4.45. Periodic response is* $\frac{5\,\sin(2\,t)}{26} + \frac{15\,\cos(2\,t)}{52}$ *with amplitude* $5\sqrt{13}/52 \approx 0.347$.

**Exercise Solution 4.4.4.** $G(\omega) = 1/2\sqrt{100\omega^4 - 999\omega^2 + 2500}$. *Resonance occurs at* $\omega = 3\sqrt{222}/20 \approx 2.235$. *A plot is shown in the right panel of Figure 4.45. Periodic response is* $-\frac{20\,\cos(9/4\,t)}{349} + \frac{72\,\sin(9/4\,t)}{349}$ *with amplitude* $0.214$.

**Exercise Solution 4.4.5.** *The gain is the same as part (d),* $G(\omega) = 1/2\sqrt{100\omega^4 - 999\omega^2 + 2500}$, *and again resonance occurs at* $\omega = 3\sqrt{222}/20 \approx 2.235$. *A plot is shown the left panel of Figure 4.46. Periodic response is* $-(5.26\times10^{-4})\sin(10t) - (5.54\times10^{-6})\cos(10t)$, *amplitude* $5.26\times10^{-4}$. *Much smaller than (d), even though the amplitude of the driving force is the same.*

**Exercise Solution 4.4.6.** $G(\omega) = 1/\sqrt{(\omega^2 - 10000)^2 + \omega^2}$. *Resonance occurs at* $\omega = \sqrt{39998}/2 \approx 99.9975$. *A plot is shown in the right panel of Figure 4.46. Periodic response is* $-0.01\cos(100t) + 0.05\sin(100t)$ *with amplitude about* $0.51$.

**Exercise Solution 4.4.7.** $G(\omega) = 1/\sqrt{(\omega^2 - 1)^2 + 100\omega^2}$. *Resonance does not occur here. A plot is shown in the left panel of Figure 4.47. Periodic response is* $-\frac{6\,\cos(2\,t)}{409} + \frac{40\,\sin(2\,t)}{409} \approx (-0.0147\,\cos(2.0\,t) + 0.0978\,\sin(2.0\,t))$ *with amplitude* $2/\sqrt{409} \approx 0.0989$.

**Exercise Solution 4.4.8.** $G(\omega) = 1/|1 - \omega^2|$. *Resonance occurs at* $\omega = 1$ *(where $G$ is undefined). A plot is shown in the right panel of Figure 4.47. Periodic response is* $-\cos(2t)/3$, *although the full general solution is* $-\cos(2t)/3 + c_1\cos(t) + c_2\sin(t)$ *(which might also be considered a "periodic" response.) Amplitude is* $1/3$ *for the* $-\cos(2t)/3$ *piece.*

**Exercise Solution 4.4.9.** *The relevant ODE is*

$$10^{-4}q''(t) + 2q'(t) + 10^6 q(t) = \sin(\omega t).$$

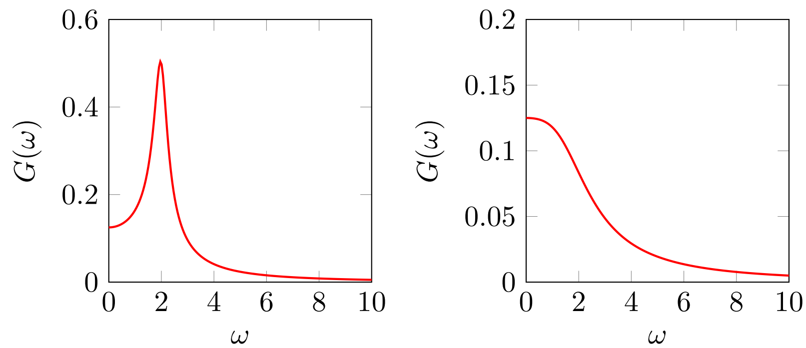



Figure 4.44: Gain functions for Exercises 4.4.1 and 4.4.2.

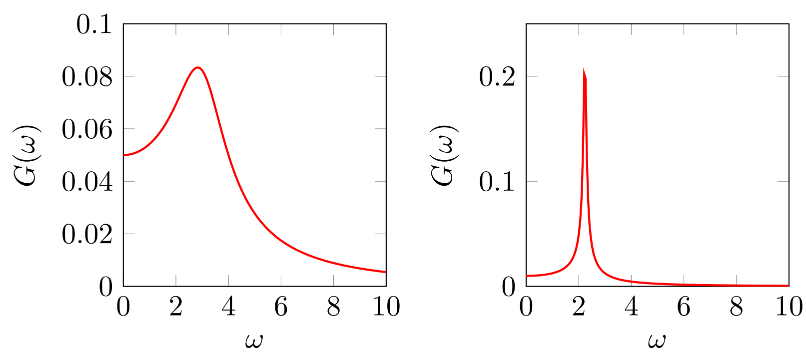



Figure 4.45: Gain functions for Exercises 4.4.3 and 4.4.4.

*The gain function is*

$$G(\omega) = \frac{1}{\sqrt{(\omega^2/10000 - 10^6)^2 + 4\omega^2}}$$

*and is shown in Figure 4.48. The resonant frequency is about* $98,995$ *radians per second, or* $17,756$ *hz.*

**Exercise Solution 4.4.10.** *The gain function is*

$$G(\omega) = \frac{1}{\sqrt{(L\omega^2 - 1/C)^2 + R^2\omega^2}}.$$

*If resonance occurs for* $\omega > 0$ *then* $G'(\omega) = 0$ *at that frequency, which leads to*

$$G'(\omega) = -\frac{\omega(2CL^2\omega^2 + CR^2 - 2L)}{C((L\omega^2 - 1/C)^2 + R^2\omega^2)^{3/2}} = 0.$$

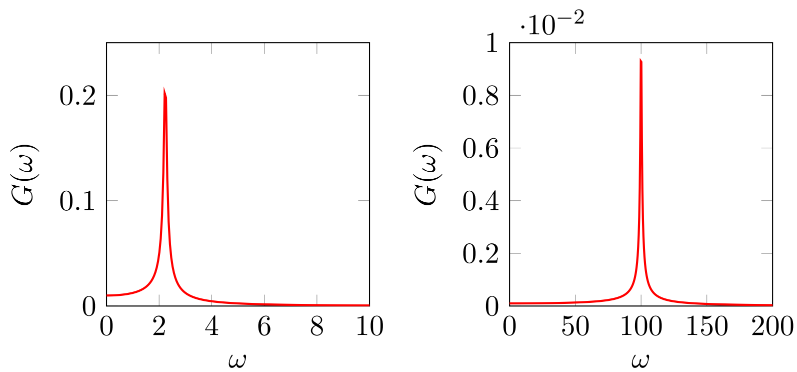



Figure 4.46: Gain functions for Exercises 4.4.5 and 4.4.6.

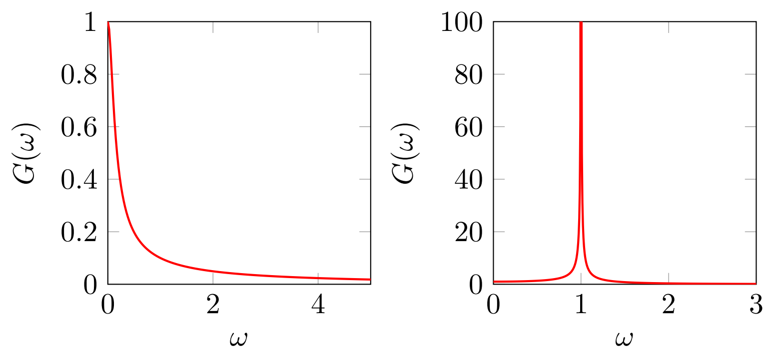



Figure 4.47: Gain functions for Exercises 4.4.7 and 4.4.8.

*The numerator is zero for* $\omega > 0$ *when* $2CL^2\omega^2 + R^2C - 2L = 0$*, which yields*

$$\omega = \frac{\sqrt{4L/C - 2R^2}}{2L}.$$

**Exercise Solution 4.4.11.**

*(a) See Figure 4.49.*

*(b) Resonance occurs at* $\omega = 2\sqrt{6}/5 \approx 0.98$*.*

*(c) See Figure 4.50. The transient is gone (to visual approximation) after time* $t = 25$ *or so.*

*(d) The radial frequency of the response after* $t = 25$ *is* $\omega = 1$ *(period* $2\pi$*), to good approximation. Even though* $f(t)$ *contains terms* $\cos(t)$

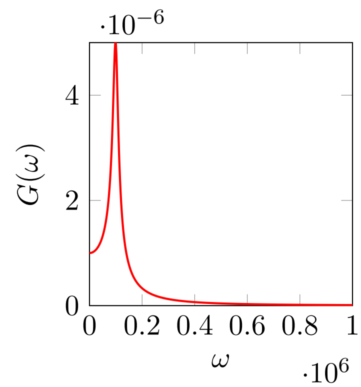



Figure 4.48: Gain function for RLC circuit.

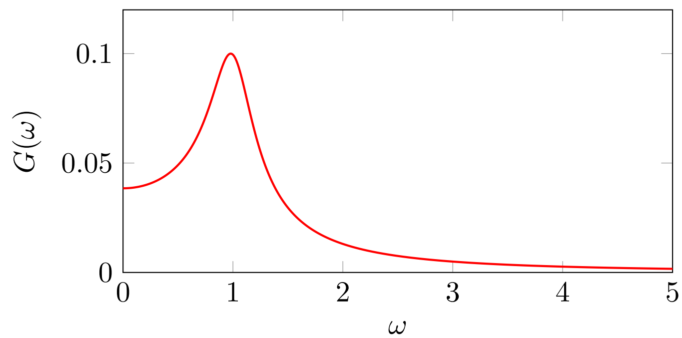



Figure 4.49: Gain function.

*and* $\cos(5t)$*, the* $\cos(5t)$ *terms have little impact since* $G(5) \approx 0.00166$ *while* $G(1) \approx 0.0995$*. The* $\cos(5t)$ *forcing is mostly filtered out.*

**Exercise Solution 4.4.12.** *The gain function is*

$$G(\omega) = \frac{1}{(m\omega^2 - k)^2 + c^2\omega^2}.$$

*Resonance occurs at* $\omega_{res} = \sqrt{k/m - (c/m)^2/2}$*. Then* $(m\omega_{res}^2 - k)^2 = c^4/4m^2$ *while* $c^2\omega_{res}^2 = c^4/2m^2 + kc^2/m$*. Then*

$$(m\omega_{res}^2 - k)^2 + c^2\omega_{res}^2 = kc^2/m - c^4/4m^2 = c^2(k/m - c^2/4m^2).$$

*Then* $\sqrt{(m\omega_{res}^2 - k)^2 + c^2\omega_{res}^2} = c\sqrt{k/m - c^2/4m^2} = c\omega_{nat}$ *so that the peak gain at resonance is*

$$G(\omega_{res}) = \frac{1}{c\omega_{nat}}.$$

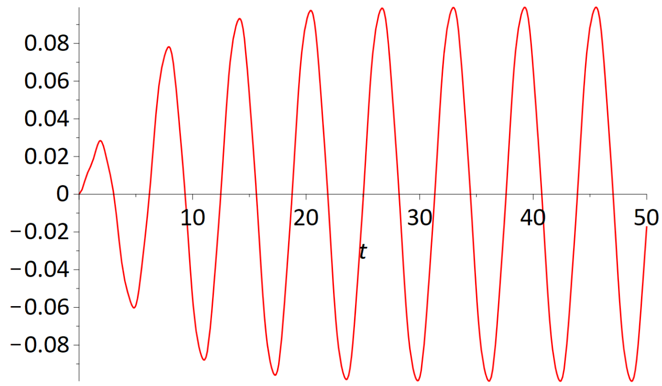



Figure 4.50: Time domain response.

**Exercise Solution 4.4.13.**

*(a)* *We find that* $u_p'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$ *has amplitude*

$$\omega\sqrt{A^2+B^2} = \frac{C\omega}{\sqrt{(m\omega^2-k)^2+c^2\omega^2}} = C\omega G(\omega).$$

*(b)* *Note that* $\omega G(\omega)$ *is nonnegative for* $\omega \geq 0$*, equals* $0$ *at* $\omega = 0$ *and limits to* $0$ *as* $\omega \to \infty$*. The maximum occurs when*

$$\frac{d(\omega G(\omega))}{d\omega} = \frac{k^2 - m^2\omega^4}{((m\omega^2-k)^2+c^2\omega^2)^{3/2}} = 0$$

*which occurs at* $\omega_{res}' = \sqrt{k/m}$ *(the unique nonnegative critical point.) When* $c$ *is close to zero,* $\omega_{res} \approx \omega_{res}'$.

*(c)* *In this case the amplitude of* $u_p''(t)$ *is* $C\omega^2 G(\omega)$*. The maximum occurs when*

$$\frac{d(\omega^2 G(\omega))}{d\omega} = \frac{\omega(\omega^2(c^2-2mk)+2k^2)}{((m\omega^2-k)^2+c^2\omega^2)^{3/2}} = 0.$$

*The only critical point with* $\omega > 0$ *is* $\omega_{res}'' = 2k/\sqrt{4mk-2c^2} = 1/\sqrt{m/k-(c/k)^2/2}$. *When* $c \approx 0$, $\omega_{res}'' \approx \omega_{res}' \approx \omega_{res}$.

**Exercise Solution 4.4.14.**

*(a)* *Here* $\omega_{res} \approx 0.98, \omega_- \approx 0.748, \omega_+ \approx 1.166,$ *and* $Q \approx 2.345$.

(b) *Here* $\omega_{res} \approx 3.142, \omega_- \approx 2.881, \omega_+ \approx 3.834,$ *and* $Q \approx 6.245$.

(c) *Here* $\omega_{res} \approx 3.162, \omega_- \approx 3.137, \omega_+ \approx 3.187,$ *and* $Q \approx 63.24$.

(d) *Here* $\omega_{res} \approx 3.1623, \omega_- \approx 3.15977, \omega_+ \approx 3.16477,$ *and* $Q \approx 632.5$.

(e) *In this case no real computation is needed—it's clear the we should take "*$Q = \infty$*".*

*Note that in (b)-(d) the quantity* $Q$ *scales in proportion to* $1/c$.

**Exercise Solution 4.4.15.**

(a) *A plot of* $G(\omega)$ *is shown in the left panel of Figure 4.51. Resonance occurs where* $G'(\omega) = 0$*, yielding* $\omega_{res} = 7\sqrt{2} \approx 9.90$*, or* $1.576$ *Hz. The maximum amplitude of the building's periodic motion is* $G(9.90) \cdot 10^4 \approx 0.1$ *meter.*

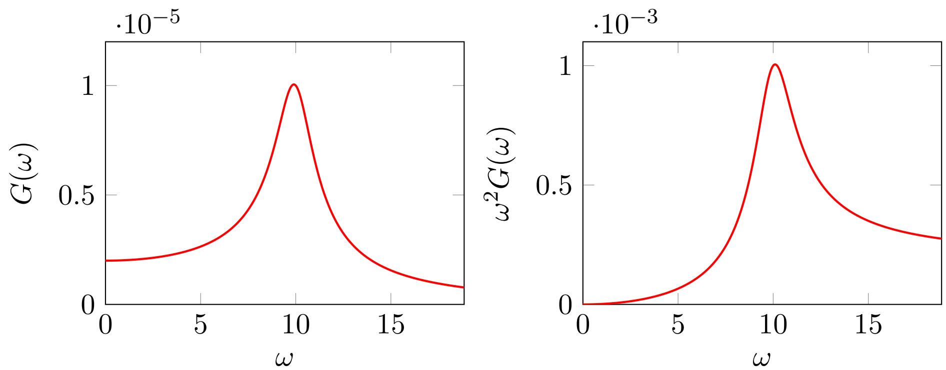



Figure 4.51: Gain function $G(\omega)$ (left) and $\omega^2 G(\omega)$ (right).

(b) *A plot of the amplitude of* $u_p''(t)$*, the acceleration, is shown in the right panel of Figure 4.51. The maximum value is at* $\omega^* \approx 10.1$*, with peak value* $0.001$*. The maximum acceleration is then about* $(0.001)(10^4) = 10$ *meters per second squared.*

(c) *A little experimentation shows that* $c \approx 16910$ *is the smallest value of* $c$ *that works.*

**Exercise Solution 4.4.16.**

(a) *Here the solution is* $u(t) \approx -5.263\cos(t) + 5.263\cos(0.9t)$ *with* $\omega_0 = 1$, $\omega = 0.9$*, and* $\delta = 0.1$*. The period of the beats is* $20\pi \approx 62.8$*. See the left panel in Figure 4.52* *(b)* *Here the solution is* $u(t) \approx 2.273\cos(t) - 2.273\cos(1.2t)$ *with* $\omega_0 = 1$, $\omega = 1.2$, *and* $\delta = -0.2$. *The period of the beats is* $10\pi \approx 31.4$. *See the right panel in Figure 4.52*

*(c)* *Here the solution is* $u(t) \approx -2.564\cos(2t) + 2.564\cos(1.9t)$ *with* $\omega_0 = 2$, $\omega = 1.9$, *and* $\delta = 0.1$. *The period of the beats is* $20\pi \approx 62.8$. *See the left panel in Figure 4.53*

*(d)* *Here the solution is* $u(t) \approx -25.06\cos(2t) + 25.06\cos(1.99t)$ *with* $\omega_0 = 2$, $\omega = 1.99$, *and* $\delta = 0.01$. *The period of the beats is* $200\pi \approx 628$. *See the right panel in Figure 4.53*

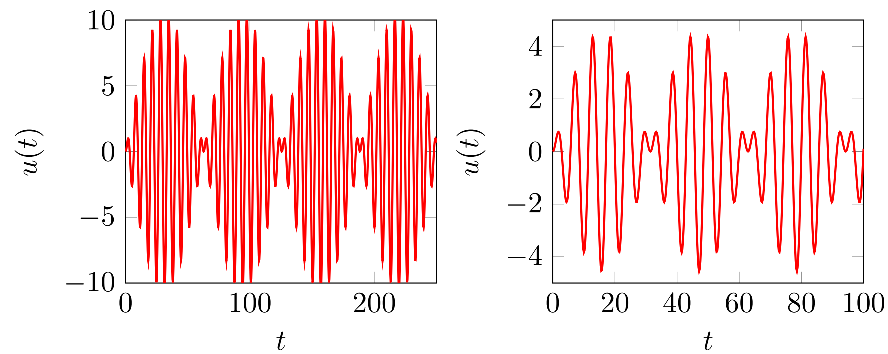



Figure 4.52: Solution $u(t)$ for part (a) (left) and part (b) (right).

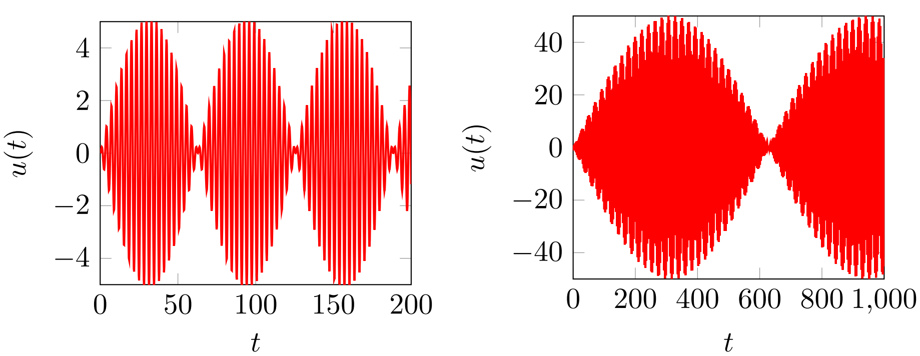



Figure 4.53: Solution $u(t)$ for part (c) (left) and part (d) (right).

**Exercise Solution 4.4.17.**

*(a)* *From*

$$u(t) = -\frac{C\omega}{m\omega_0(\omega_0^2 - \omega^2)}\sin(\omega_0 t) + \frac{C}{m(\omega_0^2 - \omega^2)}\sin(\omega t).$$

*one easily verifies that*

$$\begin{aligned} u''(t) &= \frac{\omega\omega_0^2}{m\omega_0(\omega_0^2 - \omega^2)}\sin(\omega_0 t) - \frac{\omega_0^2}{m(\omega_0^2 - \omega^2)}\sin(\omega t) \\ &= \frac{k\omega}{m^2\omega_0(\omega_0^2 - \omega^2)}\sin(\omega_0 t) - \frac{k}{m^2(\omega_0^2 - \omega^2)}\sin(\omega t) \end{aligned}$$

*from which* $mu''(t) + ku(t) = C\sin(\omega t)$ *follows. That* $u(0) = 0$ *is obvious. We also compute*

$$u'(t) = -\frac{C\omega}{m(\omega_0^2 - \omega^2)}\cos(\omega_0 t) + \frac{C\omega}{m(\omega_0^2 - \omega^2)}\cos(\omega t).$$

*Then*

$$u'(0) = -\frac{C\omega}{m(\omega_0^2 - \omega^2)} + \frac{C\omega}{m(\omega_0^2 - \omega^2)} = 0.$$

*(b)* *Since* $|\sin(z)| \le 1$ *and* $|\cos(z)| \le 1$ *for all* $z$ *it follows that* $|A\sin(\omega_0 t)| \le |A|$ *and* $|B\sin(\omega t)| \le |B|$. *From the triangle inequality then*

$$|A\sin(\omega_0 t) + B\sin(\omega t)| \le |A\sin(\omega_0 t)| + |B\sin(\omega t)| \le |A| + |B|.$$

*From part (a) with* $A = -\frac{C\omega}{m\omega_0(\omega_0^2-\omega^2)}$ *and* $B = \frac{C}{m(\omega_0^2-\omega^2)}$ *we find that for* $u(t)$

$$\begin{aligned} |u(t)| &= \left| -\frac{C\omega}{m\omega_0(\omega_0^2 - \omega^2)}\sin(\omega_0 t) + \frac{C}{m(\omega_0^2 - \omega^2)}\sin(\omega t) \right| \\ &\le \left| \frac{C\omega}{m\omega_0(\omega_0^2 - \omega^2)} \right| + \left| \frac{C}{m(\omega_0^2 - \omega^2)} \right| \\ &= \frac{C}{m|\omega_0^2 - \omega^2|}(\omega/\omega_0 + 1) \\ &= \frac{C}{m\omega_0|\omega_0^2 - \omega^2|}(\omega + \omega_0) \\ &= \frac{C}{m\omega_0|\omega_0 - \omega|} \end{aligned}$$

*using* $C, \omega, \omega_0, m > 0$.

(c) *In Example 4.27 we have* $m = 2$, $k = 5$, *and* $\omega = 3.2$, *with amplitude* $C = 5$*). In this case* $\omega_0 = 3$ *and the bound from part (b) yields* $|u(t)| \leq 4.17$, *which is in accord with the figure from that example.*

**Exercise Solution 4.4.18.** *The solution is approximately*

$$u(t) \approx e^{-t/80}(0.516\sin(t) + 0.061\cos(t)) - 0.061\cos(1.1t) - 0.468\sin(1.1t).$$

*This function is graphed in Figure 4.54. The predicted period of the beats is* $20\pi \approx 62.83$, *which is (visually) approximately correct here.*

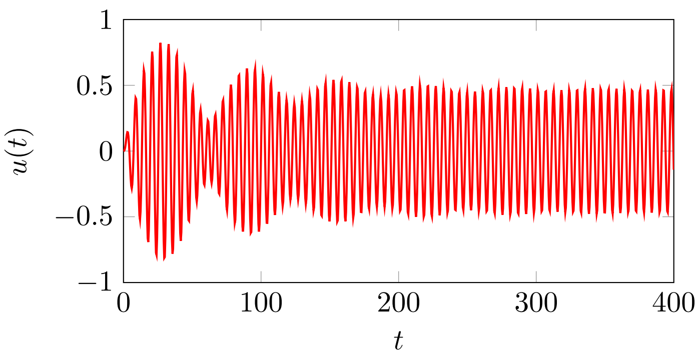



Figure 4.54: Solution to $10u''(t) + 0.25u'(t) + 10u(t) = \sin(1.1t)$ with $u(0) = 0, u'(0) = 0$.

## Section 4.5

**Exercise Solution 4.5.1.** *We find $[k] = T^{-1}$. If $t_c = k^\alpha u_0^\beta$ then taking the dimension of each side yields $T = T^{-\alpha}M^\beta$ which forces $\alpha = -1, \beta = 0$, and so $t_c = k^{-1}$. Since $[u_0] = M$, any characteristic mass scale of the form $u_c = k^\alpha u_0^\beta$ has $M = T^{-\alpha}M^\beta$, so $\alpha = 0, \beta = 1$, and $u_c = u_0$. With $\tau = t/t_c = kt$ or $t = \tau/k$ and $u(t) = u_c\bar{u}(\tau) = u_0\bar{u}(kt)$ we find $du/dt = ku_0\frac{d\bar{u}}{d\tau}$ and the ODE $du/dt = -ku$ becomes $ku_0\frac{d\bar{u}}{d\tau} = -ku_0\bar{u}$ or $d\bar{u}/dt = -\bar{u}$ with initial data $\bar{u}(0) = u_0/u_0 = 1$.*

**Exercise Solution 4.5.2.** *Since $[u] = M$ and $[u'] = MT^{-1}$ we find that $MT^{-1} = [k]M^2$ and so $[k] = M^{-1}T^{-1}$. If $t_c = k^\alpha u_0^\beta$ then taking the dimension of each side yields $T = T^{-\alpha}M^{-\alpha+\beta}$ which forces $\alpha = -1, \beta = -1$, and so $t_c = k^{-1}u_0^{-1}$. Since $[u_0] = M$, any characteristic mass scale of the form $u_c == k^\alpha u_0^\beta$ has $M = T^{-\alpha}M^{-\alpha+\beta}$, so $\alpha = 0, \beta = 1$, and $u_c = u_0$. With $\tau = t/t_c = ku_0t$ or $t = \tau/ku_0$ and $u(t) = u_c\bar{u}(\tau) = u_0\bar{u}(ku_0t)$ we find $du/dt = ku_0^2\frac{d\bar{u}}{d\tau}$. The ODE $du/dt = -ku^2$ becomes $ku_0^2\frac{d\bar{u}}{d\tau} = -ku_0^2\bar{u}^2$ or $d\bar{u}/dt = -\bar{u}^2$ with initial data $\bar{u}(0) = u_0/u_0 = 1$.*

*The solution to $u'(t) = -ku^2(t)$ with initial condition $u(0) = u_0$ is $u(t) = u_0/(1 + ku_0t)$ and $u(t) = u_0/2$ when $t = 1/(ku_0)$, which is exactly $t_c$.*

**Exercise Solution 4.5.3.** *We find $[u'] = \Theta T^{-1}$, and since $[u] = [A] = \Theta$ we must have $k = T^{-1}$. We try a characteristic time scale of the form*

$$t_c = k^\alpha A^\beta.$$

*This leads to $M^0L^0T^1\Theta^0 = M^0T^{-\alpha}L^0\Theta^\beta$ with solution $\alpha = -1$, $\beta = 0$. The only characteristic scale of this form is $t_c = 1/k$. Similarly consider a characteristic scale for $u$ of the form*

$$u_c = k^\alpha A^\beta.$$

*This leads to $M^0L^0T^0\Theta^1 = M^0T^{-\alpha}L^0\Theta^\beta$ with solution $\alpha = 0$, $\beta = 1$. The only characteristic scale of this form is $u_c = A$.*

*Take $\tau = t/t_c = kt$ (so $t = \tau/k$) and $\bar{u} = u/u_c = u/A$ (so $u(t) = A\bar{u}(\tau)$). Then $du/dt = \frac{A}{t_c}d\bar{u}/d\tau = kAd\bar{u}/\tau$. The Newton cooling ODE $du/dt = -k(u - A)$ becomes $kAd\bar{u}/d\tau = -k(A\bar{u} - A)$ or*

$$\frac{d\bar{u}}{d\tau} = -(\bar{u} - 1).$$

*The initial condition $u(0) = u_0$ becomes $\bar{u}(0) = u_0/A$. The characteristic scale $u_c = A$ is exactly the ambient temperature to which all solutions decay.*

**Exercise Solution 4.5.4.** *We find $[v] = LT^{-1}$ (velocity), $[v'] = LT^{-2}$ (acceleration), and so $[P] = L^1T^{-2}$ and $[k] = T^{-1}$. We try a characteristic time scale of the form*

$$t_c = k^\alpha P^\beta.$$

*This leads to $M^0L^0T^1 = L^\beta T^{-\alpha-2\beta}$ with solution $\alpha = -1$, $\beta = 0$. The only characteristic scale of this form is $t_c = 1/k$. Similarly consider a characteristic scale for $v$ of the form*

$$v_c = k^\alpha P^\beta.$$

*This leads to $M^0L^1T^{-1} = L^\beta T^{-\alpha-2\beta}$ with solution $\alpha = -1$, $\beta = 1$. The only characteristic scale of this form is $v_c = P/k$, which is the top speed of the runner.*

*Then $\tau = t/t_c = kt$ (or $t = \tau/k$) and $\bar{v}(\tau) = v(t)/v_c$, or $v(t) = v_c\bar{v}(\tau) = \frac{P}{k}\bar{v}(\tau)$. This yields $dv/dt = Pd\bar{v}/d\tau$ and the Hill-Keller ODE $dv/dt = P - kv(t)$ becomes*

$$\frac{d\bar{v}}{d\tau} = 1 - \bar{v}(\tau).$$

*The initial condition becomes $\bar{v}(kt_0) = 0$.*

**Exercise Solution 4.5.5.** *We have $[u] = M$ and so $[u'] = MT^{-1}$. Also $[V] = L^3$, $[r] = L^3T^{-1}$ and $[c_1] = ML^{-3}$. A characteristic time scale is of the form*

$$t_c = V^\alpha r^\beta c_1^\gamma$$

*which leads to $M^0L^0T^1 = M^\gamma L^{3\alpha+3\beta-3\gamma}T^{-\beta}$. We conclude that $\gamma = 0, 3(\alpha+\beta-\gamma) = 0, -\beta = 1$, with solution $\alpha = 1, \beta = -1, \gamma = 0$. That is, $t_c = V/r$.*

*A characteristic mass scale $u_c$ for $u$ is of the form*

$$u_c = V^\alpha r^\beta c_1^\gamma$$

*which leads to $M^1L^0T^0 = M^\gamma L^{3\alpha+3\beta-3\gamma}T^{-\beta}$. We conclude that $\gamma = 1, 3(\alpha+\beta-\gamma) = 0, -\beta = 0$, with solution $\alpha = 1, \beta = 0, \gamma = 1$. That is, $u_c = c_1V$.*

*We then have $\tau = t/t_c = rt/V$ or $t = V\tau/r$. Also, $\bar{u}(\tau) = u(t)/u_c = u(t)/(c_1V)$ or $u(t) = c_1V\bar{u}(\tau)$. Then $du/dt = c_1V\frac{d\bar{u}}{d\tau}\frac{d\tau}{dt} = rc_1d\bar{u}d\tau$. The original ODE $du/dt = rc_1 - ru/V$ becomes, after cancellations,*

$$\frac{d\bar{u}}{d\tau} = 1 - \bar{u}(\tau).$$

**Exercise Solution 4.5.6.**

(a) *We have* $[R] = L, [g] = LT^{-2}$, *and* $[m] = M$, *so* $t_c = R^\alpha g^\beta m^\gamma$ *leads to* $T = M^\gamma L^{\alpha+\beta} T^{-2\beta}$ *or* $\gamma = 0, \alpha + \beta = 0, -2\beta = 1$. *The unique solution is* $\alpha = 1/2, \beta = -1/2, \gamma = 0$ *and the characteristic time scale is* $t_c = R^{1/2} g^{-1/2} = \sqrt{R/g}$, *independent of* $m$.

(b) *We have* $\tau = t/t_c = t\sqrt{g/R}$ *or* $t = \tau\sqrt{R/g}$. *Also define* $\bar{\theta}(\tau) = \theta(t)/\theta_c = \theta(t)$. *Then* $d\theta/dt = \frac{1}{t_c} d\bar{\theta}/d\tau = \sqrt{g/R} d\bar{\theta}/d\tau$ *and* $d^2\theta/dt^2 = \frac{g}{R} d^2\bar{\theta}/d\tau^2$. *The ODE* $\frac{d^2\theta}{dt^2} + \frac{g}{R}\theta(t) = 0$ *becomes*

$$\frac{d^2\bar{\theta}}{d\tau^2} + \bar{\theta}(\tau) = 0.$$

**Exercise Solution 4.5.7.** *If we take* $t_c = R^\alpha C^\beta q_0^\gamma$ *then we need*

$$T = M^{\alpha-\beta} L^{2\alpha-2\beta} T^{-\alpha+2\beta} Q^{-2\alpha+2\beta+\gamma},$$

*which leads to* $\alpha - \beta = 0$, $2\alpha - 2\beta = 0$, $-\alpha + 2\beta = 1$, *and* $-2\alpha + 2\beta + \gamma = 0$. *The solution is* $\alpha = 1, \beta = 1, \gamma = 0$ *(despite the fact that there are four equations in three unknowns, the system is consistent.) Then* $t_c = RC$.

*For a characteristic charge scale* $q_c = R^\alpha C^\beta q_0^\gamma$ *we need*

$$Q = M^{\alpha-\beta} L^{2\alpha-2\beta} T^{-\alpha+2\beta} Q^{-2\alpha+2\beta+\gamma},$$

*which leads to* $\alpha - \beta = 0$, $2\alpha - 2\beta = 0$, $-\alpha + 2\beta = 0$, *and* $-2\alpha + 2\beta + \gamma = 1$. *The solution is* $\alpha = 0, \beta = 0, \gamma = 1$ *(again despite the fact that there are four equations in three unknowns, the system is consistent.) Then* $q_c = q_0$.

*Define* $\tau = t/t_c = t/(RC)$, *or* $t = \tau RC$. *Also define* $\bar{q}(\tau) = q(t)/q_0$ *so* $q(t) = q_0\bar{q}(\tau)$, *so* $dq/dt = \frac{q_0}{RC} d\bar{q}/d\tau$. *The ODE* $R\frac{dq}{dt} + \frac{q(t)}{C} = 0$ *becomes* $(q_0)/C d\bar{q}/d\tau + (q_0/C\bar{q}(\tau) = 0$ *or*

$$\frac{d\bar{q}}{d\tau} + \bar{q}(\tau) = 0$$

*with* $\bar{q}(0) = q(0)/q_0 = 1$.

**Exercise Solution 4.5.8.**

(a) *If we start with* $t_c = L^\alpha R^\beta C^\gamma q_0^\delta$ *and use* $[R] = ML^2T^{-1}Q^{-2}$, $[C] = M^{-1}L^{-2}T^2Q^2$ *and* $[q_0] = Q$. *If*

$$t_c = L^\alpha R^\beta C^\gamma q_0^\delta$$

*then we require*

$$T = M^{\alpha+\beta-\gamma} L^{2\alpha+2\beta-2\gamma} T^{-\beta+2\gamma} Q^{-2\alpha-2\beta+2\gamma+\delta}.$$

*This yields three equations in four unknowns,* $\alpha+\beta-\gamma = 0, 2\alpha+2\beta-2\gamma = 0, -\beta+2\gamma = 1, -2\alpha-2\beta+2\gamma+\delta = 0$, *and the system is in fact dependent. We can solve for* $\alpha = 1/2-\beta/2, \gamma = 1/2+\beta/2, \delta = 0$ *treating* $\beta$ *as a free variable. This means*

$$t_c = \sqrt{LC}(CR^2/L)^{\beta/2}$$

*where* $\beta$ *is any real number.*

*The same computation starting with* $q_c = L^\alpha R^\beta C^\gamma q_0^\delta$ *leads to equations* $\alpha+\beta-\gamma = 0, 2\alpha+2\beta-2\gamma = 0, -\beta+2\gamma = 0, -2\alpha-2\beta+2\gamma+\delta = 1$, *and again the system is in fact dependent. We can solve for* $\alpha = -\beta/2, \gamma = \beta/2, \delta = 1$ *treating* $\beta$ *as a free variable. This means*

$$q_c = q_0 \left(\frac{CR^2}{L}\right)^{\beta/2}.$$

(b) *Define* $\tau = t/t_c = t/\sqrt{LC}$, *or* $t = \tau\sqrt{LC}$. *Also define* $\bar{q}(\tau) = q(t)/q_0$ *so* $q(t) = q_0\bar{q}(\tau)$. *Then* $dq/dt = q_0/sqrtLC d\bar{q}/d\tau$ *and* $d^2q/dt^2 = (q_0/(LC))d^2\bar{q}/d\tau^2$. *Substitute this into the ODE* $Lq'' + Rq' + q/C = 0$ *to find*

$$\frac{Lq_0}{LC}\frac{d^2\bar{q}}{d\tau^2} + \frac{Rq_0}{\sqrt{LC}}\frac{d\bar{q}}{d\tau} + \frac{q_0}{C}\bar{q} = 0.$$

*Multiply both sides above by* $C/q_0$ *to arrive at*

$$\frac{d^2\bar{q}}{d\tau^2} + \gamma\frac{d\bar{q}}{d\tau} + \bar{q}(\tau) = 0$$

*with* $\gamma = R\sqrt{C/L}$. *The initial condition* $q(0) = q_0$ *becomes* $\bar{q}(0) = 1$ *and* $dq/dt(0) = 0$ *becomes* $\frac{d\bar{q}}{d\tau}(0) = 0$.

**Exercise Solution 4.5.9.**

(a) *The fixed points for this ODE are* $u = 0, P$, *and* $K$. *The phase portrait is shown in Figure 4.55.*

(b) *Here* $[r] = T^{-1}$.

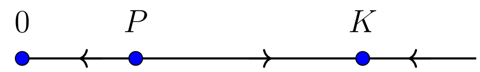



Figure 4.55: Phase portrait for $u' = -ru(1 - u/P)(1 - u/K)$.

*(c) If $t_c = r^{\alpha} K^{\beta} P^{\gamma}$ then we must have*

$$T = T^{-\alpha} N^{\beta+\gamma}$$

*which yields $\alpha = -1$ and $\beta + \gamma = 0$, so $\gamma = -\beta$. The characteristic time scale is of the form $t_c = r^{-1}(K/P)^{\beta}$. One such time scale is $t_c = 1/r$. That is, the system changes in time on a scale comparable to $1/r$, so that dictates the scale on which data should be collected.*

*(d) If $u_c = r^{\alpha} K^{\beta} P^{\gamma}$ then we must have*

$$N = T^{-\alpha} N^{\beta+\gamma}$$

*which yields $\alpha = 0$ and $\beta + \gamma = 1$ or $\gamma = 1 - \beta$. The characteristic population scale is of the form*

$$u_c = K^{1-\gamma} P^{\gamma} = K \left( \frac{P}{K} \right)^{\gamma}.$$

*(e) Define $\tau = t/t_c = rt$, or $t = \tau/r$, and $\bar{u}(\tau) = u(t)/u_c = u(t)/K$. Equivalently $u(t) = K\bar{u}(\tau)$. Then $du/dt = rK d\bar{u}/d\tau$ and the ODE $u' = u(1 - u/K)(1 - u/P)$ becomes*

$$\frac{d\bar{u}}{d\tau} = -\bar{u}(1 - \bar{u})(1 - \alpha\bar{u})$$

*where $\alpha = K/P$. The initial data $u(0) = u_0$ becomes $\bar{u}(0) = u_0/K$.*

*(f) Define $\tau = t/t_c = rt$, or $t = \tau/r$, and $\bar{u}(\tau) = u(t)/u_c = u(t)/P$. Equivalently $u(t) = P\bar{u}(\tau)$. Then $du/dt = rP d\bar{u}/d\tau$ and the ODE $u' = u(1 - u/K)(1 - u/P)$ becomes*

$$\frac{d\bar{u}}{d\tau} = -\bar{u}(1 - \bar{u})(1 - \alpha\bar{u})$$

*where $\alpha = P/K$. The initial data $u(0) = u_0$ becomes $\bar{u}(0) = u_0/P$. The role of $P$ and $K$ here is somewhat symmetric.*

(g) *Define* $\tau = t/t_c = rt$, *or* $t = \tau/r$, *and* $\bar{u}(\tau) = u(t)/u_c = u(t)/K$. *Equivalently* $u(t) = K\bar{u}(\tau)$. *Then* $du/dt = rKd\bar{u}/d\tau$ *and the ODE* $u' = u(1-u/K)(1-u/P) - hu$ *becomes*

$$\frac{d\bar{u}}{d\tau} = -\bar{u}\,(1-\bar{u})\left(1 - \frac{K}{P}\bar{u}\right) - \epsilon\bar{u}$$

*where* $\epsilon = h/r$.

(h) *In this case* $K/P = 10$ *and the nondimensionalized ODE becomes*

$$\frac{d\bar{u}}{d\tau} = -\bar{u}\,(1-\bar{u})\,(1-10\bar{u}) - \epsilon\bar{u}$$

*The fixed points for this ODE are the solutions to* $-\bar{u}(1-\bar{u})(1-10\bar{u}) - \epsilon\bar{u} = 0$, *which are* $\bar{u} = 0$ *and* $\bar{u} = 11/20 \pm \sqrt{81-40\epsilon}/20$. *Note that when* $\epsilon = 0$ *these roots are at* $\bar{u} = 0, 1/10$, *and* 1. *A typical phase portrait for a small positive value of* $\epsilon$ *is shown in Figure 4.56, with* $u_1^* = 11/20 - \sqrt{81-40\epsilon}/20$ *and* $u_2^* = 11/20 + \sqrt{81-40\epsilon}/20$. *As* $\epsilon$ *increases from* 0 *to* 2.025 *(where the discriminant* $81 - 40\epsilon = 0$*) the positive fixed points draw closer and closer together (toward* $\bar{u} = 11/20$*). At* $\epsilon = 2.025$ *a bifurcation occurs and these fixed points cease to exist. All solutions then decrease to* $\bar{u} = 0$.

*In the original ODE this occurs at* $h/r = 2.025$, *so the condition* $h > 2.025r$ *assures the extinction of the species for any initial condition.*

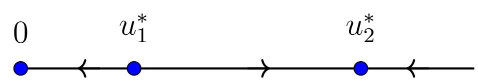



Figure 4.56: Phase portrait for $\bar{u}' = \bar{u}(1-\bar{u})(1-10\bar{u}) - \epsilon\bar{u}$, $\epsilon$ small and positive; $u_1^* = 11/20 - \sqrt{81-40\epsilon}/20$ and $u_2^* = 11/20 + \sqrt{81-40\epsilon}/20$.

**Exercise Solution 4.5.10.** *From* $t_c = m^\alpha k_1^\beta k_2^\gamma u_0^\delta$ *and* $[m] = M, [k_1] = MT^{-2}, [k_2] = MT^{-2}L^{-2}, [u_0] = L$ *we obtain the condition* $T = M^{\alpha+\beta+\gamma}L^{-2\gamma+\delta}T^{-2\beta-2\gamma}$. *This requires*

$$\alpha + \beta + \gamma = 0, \quad -2\gamma + \delta = 0, \quad -2\beta - 2\gamma = 1,$$

*three equations in four unknowns* $\alpha, \beta, \gamma, \delta$. *If we take* $\gamma$ *as a free variable then* $\alpha = 1/2, \beta = -1/2 - \gamma$, *and* $\delta = 2g$, *so that*

$$t_c = m^{1/2}k_1^{-1/2}(k_2u_0^2/k_1)^\gamma u_0^\gamma = \sqrt{\frac{m}{k}}\left(\frac{k_2u_0^2}{k_1}\right)^\gamma.$$

*Similarly if* $u_c = m^\alpha k_1^\beta k_2^\gamma u_0^\delta$ *we are led to*

$$\alpha + \beta + \gamma = 0, \quad -2\gamma + \delta = 1, \quad -2\beta - 2\gamma = 0,$$

*three equations in four unknowns* $\alpha, \beta, \gamma, \delta$. *If we take* $\gamma$ *as a free variable then* $\alpha = 0, \beta = -\gamma$, *and* $\delta = 1 + 2\gamma$, *so that*

$$u_c = u_0(u_0^2 k_2/k_1)^\gamma = u_0 \left(\frac{k_2 u_0^2}{k_1}\right)^\gamma.$$

**Exercise Solution 4.5.11.**

(a) *Since force (which is the dimension of the* $mdv/dt$ *and* $mg$ *terms) has dimension* $MLT^{-2}$ *we find that* $[k_1] = MLT^{-2}(LT^{-1})^{-1} = MT^{-1}$ *and* $[k_2] = MLT^{-2}(LT^{-1})^{-2} = ML^{-1}$.

(b) *Given the dimensions in part (a),* $t_c = m^\alpha g^\beta k_1^\gamma k_2^\delta$ *leads to*

$$T = M^{\alpha+\gamma+\delta} L^{\beta-\delta} T^{-2\beta-\gamma}$$

*so that* $\alpha + \gamma + \delta = 0, \beta - \delta = 0, -2\beta - \gamma = 1$. *If we treat* $\delta = \delta_t$ *as a free variable then* $\alpha = 1 + \delta, \beta = \delta, \gamma = -1 - 2\delta$. *Then*

$$t_c = \frac{m}{k_1}\left(\frac{mgk_2}{k_1^2}\right)^{\delta_t}$$

(c) *Given the dimensions in part (a),* $v_c = m^\alpha g^\beta k_1^\gamma k_2^\delta$ *leads to*

$$LT^{-1} = M^{\alpha+\gamma+\delta} L^{\beta-\delta} T^{-2\beta-\gamma}$$

*so that* $\alpha + \gamma + \delta = 0, \beta - \delta = 1, -2\beta - \gamma = -1$. *If we treat* $\delta = \delta_v$ *as a free variable then* $\alpha = 1 + \delta, \beta = 1 + \delta, \gamma = -1 - 2\delta$. *Then*

$$v_c = \frac{mg}{k_1}\left(\frac{mgk_2}{k_1^2}\right)^{\delta_v}.$$

(d) *Define* $\tau = t/t_c = k_1 t/m$, *or* $t = m\tau/k_1$. *Also define* $\bar{v}(\tau) = v(t)/v_c = k_1 v(t)/(mg)$. *Then* $v(t) = mg\bar{v}(\tau)/k_1$ *and* $dv/dt = gd\bar{v}/d\tau$. *The ODE* $mdv/dt = mg - k_1 v - k_2 v^2$ *becomes*

$$mg\frac{d\bar{v}}{d\tau} = mg - mg\bar{v}(\tau) - k_2 m^2 g^2/k_1^2 \bar{v}^2(\tau)$$

*or, if we divide through by* $mg$,

$$\frac{d\bar{v}}{d\tau} = 1 - \bar{v} - \epsilon\bar{v}^2$$

*where* $\epsilon = mgk_2/k_1^2$ *(dimensionless).*

*The solution to this ODE with* $\bar{v}(0) = 0$ *and* $\epsilon = 0$ *is the function* $\bar{v}_0(\tau) = 1 - e^{-\tau}$. *The solutions with* $\epsilon = 1, 0.1,$ *and* $0.01$ *are shown in Figure 4.57, along with* $\bar{v}_0(\tau)$. *Depending on what is needed, some-*

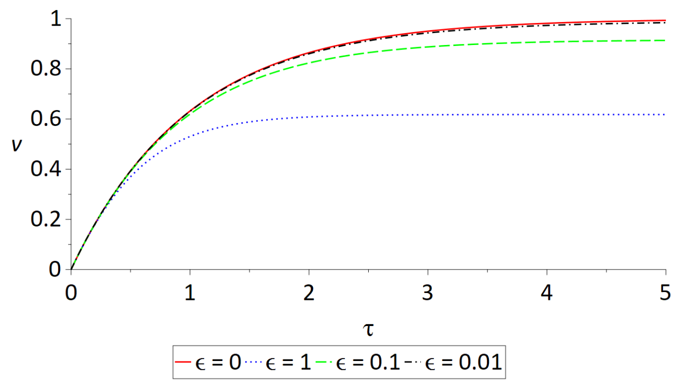



Figure 4.57: Solution to $\frac{d\bar{v}}{d\tau} = 1 - \bar{v} - \epsilon\bar{v}^2$ with $\bar{v}(0) = 0$ for $\epsilon = 0, 0.01, 0.1, 1$.

*thing in the range* $\epsilon \le \epsilon_0$ *with* $\epsilon_0 = 0.01 to 0.1$ *seems reasonable. This corresponds to the criteria* $mgk_2/k_1^2 \le \epsilon_0$ *or* $k_2 \le \epsilon_0 k_1^2/(mg)$ *in the original ODE.*

(e) *Taking* $\delta_t = -1/2$ *for* $t_c$ *leads to* $t_c = \sqrt{\frac{m}{gk_2}}$, *while* $\delta_v = -1/2$ *for* $v_c$ *yields* $v_c = \sqrt{mg/k_2}$. *With* $v(t) = v_c\bar{v}(\tau) = \sqrt{mg/k_2}\bar{v}$ *and* $\tau = t/t_c$ *we find* $dv/dt = g d\bar{v}/d\tau$. *Then* $m\frac{dv}{dt} = mg - k_1 v(t) - k_2 v^2(t)$ *becomes (after dividing by* $mg$*)*

$$\frac{d\bar{v}}{d\tau} = 1 - \epsilon\bar{v} - \bar{v}^2$$

*with* $\epsilon = k_1/\sqrt{mgk_2}$.

*The solution to this ODE with* $\bar{v}(0) = 0$ *and* $\epsilon = 0$ *is the function* $\bar{v}_0(\tau) = \tanh(\tau)$. *The solutions with* $\epsilon = 1, 0.1,$ *and* $0.01$ *are shown in Figure 4.58, along with* $\bar{v}_0(\tau)$. *As in part (d), depending on what is needed, something in the range* $\epsilon \le \epsilon_0$ *with* $\epsilon_0 = 0.01 to 0.1$ *seems reasonable. This corresponds to the criteria* $k_1/\sqrt{mgk_2} \le \epsilon_0$ *or* $k_1 \le \epsilon_0\sqrt{mgk_2}$ *in the original ODE.*

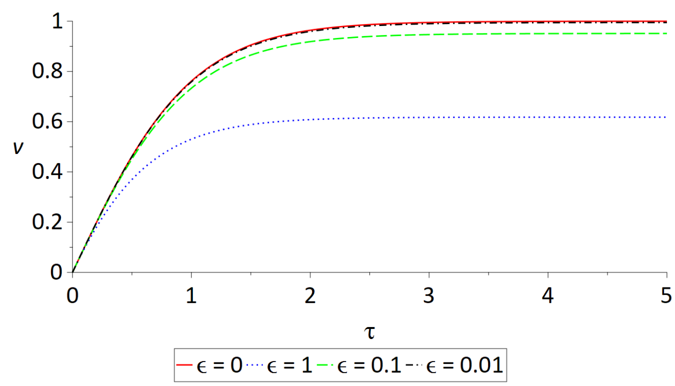



Figure 4.58: Solution to $\frac{d\bar{v}}{d\tau} = 1 - \epsilon\bar{v} - \bar{v}^2$ with $\bar{v}(0) = 0$ for $\epsilon = 0, 0.01, 0.1, 1$.

# Section 5.1

**Exercise Solution 5.1.1.**

*(a) The solution is* $u_1(t) \approx 5.78 - 0.78e^{-kt}$ *for* $0 < t < 12$.

*(b) The initial data for* $u_2(t)$ *is* $u_2(12) = u_1(12) \approx 5.683$ *mg. Then* $u_2(t) \approx 8.67 - 2.99e^{-k(t-12)}$. *This can also be expressed as* $u_2(t) \approx 8.67 - 23.82e^{-kt}$.

*(c) The function* $u_3(t)$ *will satisfy* $u_3(18) = u_2(18) + 5 \approx 7.61$ *mg, with* $u_3' = -ku_3 + 1$ *for* $t > 18$. *The solution is* $u_3(t) \approx 5.78 + 6.83e^{-k(t-18)}$ *or alternatively, as* $u_3(t) \approx 5.78 + 153.79e^{-kt}$.

*(d) The solution is plotted in Figure 5.59.*

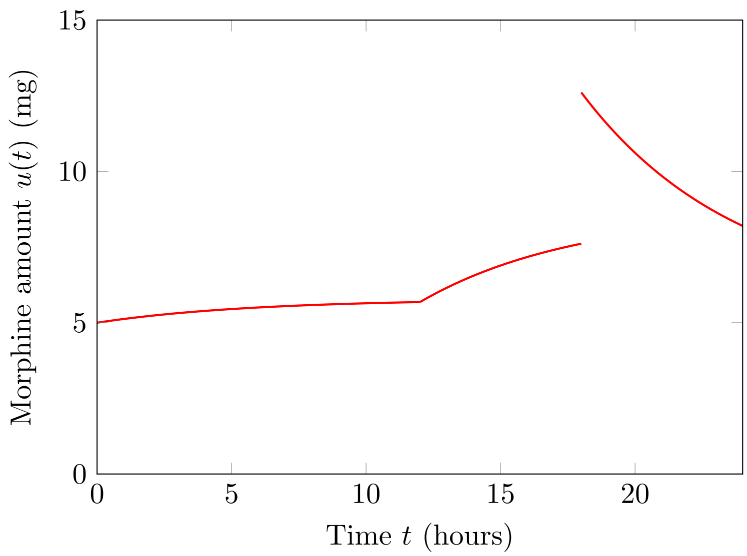



Figure 5.59: Amount of morphine (mg) in patient's system.

**Exercise Solution 5.1.2.**

*(a) The solution is* $p(t) = 1000e^{0.02t}$, *and* $p(5) \approx \$1105.17$.

*(b) The statement of the problem pretty much makes the argument. The solution is* $p(t) = 27000e^{0.02t} - 26000$, *and* $p(5) \approx \$3839.61$.

(c) *For time* $0 < t < 2$ *we still have* $p_1(t) = 27000e^{0.02t} - 26000$*, and* $p_1(2) \approx \$2101.89$*. For* $t > 2$ *we have* $p_2'(t) = 0.02p_2(t) + 200$ *with* $p_2(2) = p_1(2) = 2101.89$*. The solution is* $p_2(t) = 12101.89e^{0.02(t-2)} - 10000$*. The solution may also be expressed as* $p(t) = 11627.37e^{0.02t} - 10000$*.*

(d) *With the information from part (c) the balance just prior to* $t = 5$ *is* $p_2(5) \approx \$2850.23$*. When the lump sum deposit of \$1000 is made the balance jumps "instantaneously" to \$3850.23. After that the account grows according to* $p'(t) = 0.02p(t)$ *with* $p(5) = 3850.23$*, and the balance is* $p(t) = 3850.23e^{0.02(t-5)}$*, or* $p(t) = 3483.83e^{0.02t}$*.*

**Exercise Solution 5.1.3.** *For* $0 < t < 50$ *the appropriate ODE is* $u'(t) = -k(u(t) - 80)$ *with initial condition* $u(0) = 120$*. The solution is* $u = u_1$ *with* $u_1(t) = 80 + 40e^{-0.05t}$*. Then* $u_1(50) \approx 83.28$*. For* $t > 50$ *the temperature obeys* $u'(t) = -k(u(t) - 90)$ *with initial condition* $u(50) = 83.28$*. The solution is* $u = u_2$ *where* $u_2(t) = 90 - 6.717e^{-k(t-50)} \approx 90 - 81.825e^{-kt}$*. At* $t = 70$ *the temperature is* $u_2(70) \approx 87.53$*.*

**Exercise Solution 5.1.4.** *For* $0 < t < 10$ *the governing ODE is* $2u''(t) + 8u(t) = 0$ *with initial data* $u(0) = u'(0) = 0$*, and the solution is simply* $u = u_1$ *where* $u_1(t) = 0$*. For* $10 < t < 15$ *the ODE is* $2u''(t) + 8u(t) = 40$ *with initial data* $u(0) = u'(0) = 0$ *and the solution is* $u = u_2$ *with* $u_2(t) = 5 - 5\cos(20)\cos(2t) - 5\sin(20)\sin(2t)$ *or* $u_2(t) = 5 - 5\cos(2(t-10))$*. At* $t = 15$ *we find* $u_2(15) \approx 9.20$ *and* $u_2'(15) \approx -5.44$*. At this time the ODE becomes* $2u''(t) + 8u(t) = 0$ *again, but with data* $u(15) = 9.20, u_2'(15) = -5.44$*. The solution is* $u = u_3$ *where* $u_3(t) \approx 9.20\cos(2(t-15)) - 2.72\sin(2(t-15))$*.*

**Exercise Solution 5.1.5.** *The relevant ODE for* $0 < t < 0.003$ *is* $10q'(t) + 10^4 q(t) = 2$ *with initial condition* $q(0) = 0$*. The solution is* $q = q_1$ *where* $q_1(t) = (1 - e^{-1000t})/5000$*. For* $t > 0.003$ *the ODE becomes* $10q'(t) + 10^4 q(t) = 5$ *with initial condition* $q(0.003) = q_1(0.003) \approx 0.00019$*. The solution to this ODE is* $q = q_2$ *with* $q_2(t) \approx 5 \times 10^{-4} - (6.226 \times 10^{-3})e^{-1000t} \approx 5 \times 10^{-4} - (3.1 \times 10^{-4})e^{-1000(t-0.003)}$*. At* $t = 0.005$ *the charge is* $q_2(0.005) \approx 4.58 \times 10^{-4}$*.*

## Section 5.2

**Exercise Solution 5.2.1.** $F(s) = 6/s^3$.

**Exercise Solution 5.2.2.** $G(s) = 4/(s^2+16) + 7/s^2 - 1/(s-2)$

**Exercise Solution 5.2.3.** $P(s) = (s+3)/((s+3)^2+49)$

**Exercise Solution 5.2.4.** $F(s) = 2/s^3 - 2/s^2 + 1/s$

**Exercise Solution 5.2.5.** $Q(s) = 6/(s-5)^4$

**Exercise Solution 5.2.6.** *Use linearity.* $f(t) = t - 2$

**Exercise Solution 5.2.7.** $q(t) = \sin(2t)/2$.

**Exercise Solution 5.2.8.** *Write* $G(s) = 2\frac{s}{s^2+4} + \frac{2}{s^2+4}$ *so* $g(t) = 2\cos(2t) + \sin(2t)$.

**Exercise Solution 5.2.9.** *Complete the square to find* $F(s) = \frac{4s}{(s+2)^2+2^2}$. *Write this as* $F(s) = \frac{4(s+2)}{(s+2)^2+2^2} - \frac{8}{(s+2)^2+2^2}$. *Since* $\mathcal{L}^{-1}(s/(s^2+2^2)) = \cos(2t)$ *and* $\mathcal{L}^{-1}(1/(s^2+2^2)) = \sin(2t)/2$ *the first shifting theorem and linearity yield* $f(t) = 4e^{-2t}\cos(2t) - 4e^{-2t}\sin(2t)$.

**Exercise Solution 5.2.10.** *From* $\mathcal{L}^{-1}(2/s^3) = t^2$ *it follows that* $f(t) = t^2e^{-3t}$.

**Exercise Solution 5.2.11.** *The poles of* $F(s)$ *are at* $s = -1$ *and* $s = -2$ *(both multiplicity* 1*), so* $f(t)$ *is a linear combination of* $e^{-t}$ *and* $e^{-2t}$.

**Exercise Solution 5.2.12.** *The poles of* $F(s)$ *are at* $s = -1$ *(multiplicity* 2*) and* $s = -2$ *(multiplicity* 1*), so* $f(t)$ *is a linear combination of* $e^{-t}, te^{-t}$ *and* $e^{-2t}$.

**Exercise Solution 5.2.13.** *The poles of* $F(s)$ *are at* $s = i$ *and* $s = -i$*, both of multiplicity* 1*, so* $f(t)$ *is a linear combination of* $e^{it}$ *and* $e^{-it}$*, or* $\sin(t)$ *and* $\cos(t)$.

**Exercise Solution 5.2.14.** *The poles of* $F(s)$ *are at* $s = i, s = -i, s = 3i$ *and* $s = -3i$*, all of multiplicity* 1*, so* $f(t)$ *is a linear combination of* $e^{it}, e^{-it}, e^{3it}$ *and* $e^{-3it}$*, or* $\sin(t), \cos(t), \sin(3t)$ *and* $\cos(3t)$.

**Exercise Solution 5.2.15.** $F(s)$ *has a pole at* $s = 1$ *of multiplicity* 3 *and poles at* $s = -1 \pm i$ *of multiplicity* 1*, so* $f(t)$ *will contain terms* $e^t, te^t$, $t^2e^t$, *and* $e^{(-1+i)t}$, $e^{(-1-i)t}$. *These last two terms are equivalent to* $e^{-t}\sin(t)$ *and* $e^{-t}\cos(t)$.

**Exercise Solution 5.2.16.** *The function $s^2 + 5s + 4$ has poles at $s = -1$ and $s = -4$, both of multiplicity 1, so $F(s)$ has poles here as well but of multiplicity 2. Thus $f(t)$ will contain terms $e^{-t}, te^{-t}, e^{-4t}$, and $te^{-4t}$.*

**Exercise Solution 5.2.17.** *The function $s^2 + 4s + 13$ has poles at $s = -2 \pm 3i$, so $(s^2 + 4s + 13)^3$ has these same poles but of multiplicity 3. The factor $(s+3)^7$ has a pole at $s = -3$ of multiplicity 7. As such we can expect $f(t)$ to contain terms*

$$\begin{aligned}&e^{(-2+3i)t}, te^{(-2+3i)t}, t^2e^{(-2+3i)t}, e^{(-2-3i)t}, te^{(-2-3i)t}, t^2e^{(-2-3i)t},\\ &e^{-3t}, te^{-3t}, \ldots, t^3e^{-3t}.\end{aligned}$$

*Alternatively, the pair $t^ne^{(-2+3i)t}, t^ne^{(-2-3i)t}$ can be replaced by $t^ne^{-2t}\cos(3t)$ and $t^ne^{-2t}\sin(3t)$.*

**Exercise Solution 5.2.18.** *Laplace transform both sides of the ODE and fill in the initial data to find $sU(s) - 6 = 2U(s)$, so $U(s) = 6/(s-2)$ and $u(t) = 6e^{2t}$.*

**Exercise Solution 5.2.19.** *Laplace transform both sides of the ODE and fill in the initial data to find $sU(s) + 4 = -5U(s)$, so $U(s) = -4/(s+5)$ and $u(t) = -4e^{-5t}$.*

**Exercise Solution 5.2.20.** *Laplace transform both sides of the ODE and fill in the initial data to find $sU(s) - u_0 = aU(s)$, so $U(s) = u_0/(s-a)$ and $u(t) = u_0e^{at}$.*

**Exercise Solution 5.2.21.** *Laplace transform both sides of the ODE, fill in the initial data, and collect the $U(s)$ terms on the left, all other terms on the right to find $(s^2 + 3s + 2)U(s) = 6s + 22$. Then*

$$U(s) = \frac{6s + 22}{s^2 + 3s + 2} = \frac{16}{s+1} - \frac{10}{s+2}$$

*after a partial fraction decomposition. Then $u(t) = 16e^{-t} - 10e^{-2t}$.*

**Exercise Solution 5.2.22.** *Laplace transform both sides of the ODE, fill in the initial data, and collect the $U(s)$ terms on the left, all other terms on the right to find $(4s^2 + 8s + 4)U(s) = 20s + 52$. Then*

$$U(s) = \frac{5s + 13}{s^2 + 2s + 1} = \frac{5}{s+1} + \frac{8}{(s+1)^2}$$

*after a partial fraction decomposition. Then $u(t) = 5e^{-t} + 8te^{-t}$.*

**Exercise Solution 5.2.23.** *Laplace transform both sides of the ODE, fill in the initial data, and collect the $U(s)$ terms on the left, all other terms on the right to find $(s^2 + 2s + 10)U(s) = s + 4$. Then*

$$U(s) = \frac{s + 4}{s^2 + 2s + 10} = \frac{s + 4}{(s + 1)^2 + 3^2}$$

*after completing the square in the denominator. This can also be written*

$$U(s) = \frac{3}{(s + 1)^2 + 3^2} + \frac{s + 1}{(s + 1)^2 + 3^2}$$

*which has inverse transform $u(t) = e^{-t} \sin(3t) + e^{-t} \cos(3t)$.*

**Exercise Solution 5.2.24.** *Laplace transform both sides of the ODE, fill in the initial data, and collect the $U(s)$ terms on the left, all other terms on the right to find $(2s^2 + 22s + 36)U(s) = 2s + 46$. Then*

$$U(s) = \frac{s + 23}{s^2 + 11s + 18} = \frac{3}{s + 2} - \frac{2}{s + 9}.$$

*Then $u(t) = 3e^{-2t} - 2e^{-9t}$.*

**Exercise Solution 5.2.25.** *Laplace transform both sides of the ODE, fill in the initial data, and collect the $U(s)$ terms on the left, all other terms on the right to find $(3s^2 + 6s + 6)U(s) = 3s$. Then*

$$U(s) = \frac{s}{s^2 + 2s + 2} = \frac{s}{(s + 1)^2 + 1}$$

*after completing the square in the denominator. This can also be written*

$$U(s) = \frac{s + 1}{(s + 1)^2 + 1} - \frac{1}{(s + 1)^2 + 1}$$

*which has inverse transform $u(t) = e^{-t} \cos(t) - e^{-t} \sin(t)$.*

**Exercise Solution 5.2.26.** *Laplace transform both sides of the ODE, fill in the initial data, and collect the $U(s)$ terms on the left, all other terms on the right to find $(3s^2 + 18s + 27)U(s) = 3s + 12$. Then*

$$U(s) = \frac{s + 4}{s^2 + 6s + 9} = \frac{1}{s + 3} + \frac{1}{(s + 3)^2}$$

*after a partial fraction decomposition. Then $u(t) = e^{-3t} + te^{-3t}$.*

**Exercise Solution 5.2.27.** *We find* $\mathcal{L}(\sinh(t)) = (1/(s-1)-1/(s+1))/2 = 1/(s^2-1)$ *and* $\mathcal{L}(\cosh(t)) = (1/(s-1)+1/(s+1))/2 = s/(s^2-1)$. *Quite similar to the standard sine and cosine transforms.*

**Exercise Solution 5.2.28.** *Using the table with* $a = i$ *yields* $\mathcal{L}(e^{it}) = 1/(s-i)$. *The transform of* $\cos(t) + i\sin(t)$ *is* $s/(s^2+1) + i/(s^2+1)$ *or* $(s+i)/(s^2+1)$. *But*

$$\frac{1}{s-i} = \frac{s+i}{(s+i)(s-i)} = \frac{s+i}{s^2+1},$$

*exactly the same thing!*

**Exercise Solution 5.2.29.** *The function* $H(t-c)$ *is identically zero for* $t < c$ *and* 1 *for* $t \geq c$. *Then for* $s > 0$

$$\begin{aligned}
\mathcal{L}(H(t-c)) &= \int_0^\infty H(t-c)e^{-st}\,dt \\
&= \int_0^c 0e^{-st}\,dt + \int_c^\infty 1e^{-st}\,dt \\
&= \int_c^\infty e^{-st}\,dt \\
&= -\frac{e^{-st}}{s}\Big|_{t=c}^{t\to\infty} \\
&= \frac{e^{-cs}}{s}.
\end{aligned}$$

**Exercise Solution 5.2.30.** *Compute*

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\
&= \int_0^5 3e^{-st}\,dt + \int_5^1 07e^{-st}\,dt + \int_7^\infty 0e^{-st}\,dt \\
&= -\frac{3e^{-st}}{s}\Big|_{t=0}^{t=5} - \frac{7e^{-st}}{s}\Big|_{t=5}^{t=10} \\
&= \frac{3}{s} + 4\frac{e^{-5s}}{s} - 7\frac{e^{-10s}}{s}
\end{aligned}$$

*after collecting terms. The argument* $s$ *can assume any value.*

**Exercise Solution 5.2.31.** *We begin with the very definition of the Laplace transform*

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

*Differentiate both sides with respect to $s$ and assume we can pass the $d/ds$ derivative under the integral, noting that $d(e^{-st})/ds = -te^{-st}$. Then*

$$\frac{dF}{ds} = \int_0^\infty -tf(t)e^{-st}\,dt.$$

*But the right side above is exactly the definition of $\mathcal{L}(-tf(t))$.*

**Exercise Solution 5.2.32.** *Let $f(t) = e^{-2t}\sin(3t)$ so $F(s) = 3/((s+2)^2 + 9)$. Then from the previous exercise $\mathcal{L}(tf(t)) = -dF/ds = (6s+12)/(s^2 + 4s+13)^2$.*

**Exercise Solution 5.2.33.**

(a) *If $f(t) = 1$ then $F(s) = 1/s$. Also, $\lim_{t\to 0^+} f(t) = 1$ and $\lim_{s\to\infty} sF(s) = 1$.*

(b) *If $f(t) = t$ then $F(s) = 1/s^2$. Also, $\lim_{t\to 0^+} f(t) = 0$ and $\lim_{s\to\infty} sF(s) = 0$.*

(c) *If $f(t) = e^t$ then $F(s) = 1/(s-1)$. Also, $\lim_{t\to 0^+} f(t) = 1$ and $\lim_{s\to\infty} sF(s) = 1$.*

(d) *If $f(t) = \cos(t)$ then $F(s) = s/(s^2+1)$. Also, $\lim_{t\to 0^+} f(t) = 1$ and $\lim_{s\to\infty} sF(s) = 1$.*

(e) *If $f(t) = \sin(t)/t$ then $F(s) = \arctan(1/s)$. Also, $\lim_{t\to 0^+} f(t) = 1$ (use L'Hopital's Rule) and $\lim_{s\to\infty} sF(s) = 1$ (write the limits as $\lim_{s\to\infty} F(s)/(1/s)$ and apply L'Hopital's rule).*

**Exercise Solution 5.2.34.**

(a) *If $f(t) = 4$ then $F(s) = 4/s$. Here $F$ has a pole at $s = 0$ of multiplicity 1, so the theorem is applicable. Also, $\lim_{t\to\infty} f(t) = 4$ and $\lim_{s\to 0^+} sF(s) = 4$.*

(b) *If $f(t) = e^{-t}$ then $F(s) = 1/(s+1)$. Here $F$ has a pole at $s = -1$ so the theorem is applicable. Also, $\lim_{t\to\infty} f(t) = 0$ and $\lim_{s\to 0^+} sF(s) = 0$.*

(c) *If $f(t) = t^4e^{-t}$ then $F(s) = 24/(s+1)^5$. Here $F$ has a pole at $s = -1$ so the theorem is applicable. Also, $\lim_{t\to\infty} f(t) = 0$ and $\lim_{s\to 0^+} sF(s) = 0$.*

(d) *If $f(t) = 2 + e^{-3t}\cos(t)$ then $F(s) = 2/s + (s+3)/((s+3)^2+1) = (3s^2+15s+20)/(s((s+3)^2+1))$. Here $F$ has poles at $s = 0, -3 \pm i$ all of multiplicity 1, so the theorem is applicable. Also, $\lim_{t\to\infty} f(t) = 2$ and $\lim_{s\to 0^+} sF(s) = 2$.*

**Exercise Solution 5.2.35.** *The ODE is* $2x''(t) + 8x'(t) + 40x(t) = 0$. *Laplace transforming both sides and filling in the initial data yields* $(2s^2 + 8s + 40)X(s) = (s + 4)$, *so* $X(s) = (s + 4)/(2s^2 + 8s + 20)$. *Complete the square to write*

$$X(s) = \frac{s+4}{(2(s^2+4s+20)} = \frac{s+4}{2((s+2)^2+4^2)}.$$

*Write this as*

$$X(s) = \frac{s+4}{2((s+2)^2+4^2)} = \frac{1}{2}\frac{s+2}{(s+2)^2+4^2} + \frac{1}{(s+2)^2+4^2}.$$

*The inverse transform gives* $x(t) = e^{-2t}\cos(4t)/2 + e^{-2t}\sin(4t)/4$.

**Exercise Solution 5.2.36.** *The ODE is* $(2\times10^{-4})q''(t)+2q'(t)+(10^4)q(t) = 0$. *Laplace transform and fill in initial data to find* $(s^2/5000+2s+10000)Q(s)-1/5000 = 0$ *and solve to find*

$$Q(s) = \frac{1}{s^2+10^4s+5\times10^7} = \frac{1}{(s+5000)^2+5000^2}.$$

*The inverse transform yields* $q(t) = e^{-5000t}\sin(5000t)/5000$.

**Exercise Solution 5.2.37.** *This equation is nonlinear. There is no simple way to relate the transform* $\mathcal{L}(u^2(t))$ *to* $\mathcal{L}(u(t))$.

**Exercise Solution 5.2.38.**

(a) *From the rule for first derivatives we have*

$$\mathcal{L}(f''') = \mathcal{L}((f'')') = s\mathcal{L}(f'') - f''(0).$$

*Using the rule for* $\mathcal{L}(f'') = s^2F(s) - sf(0) - f'(0)$ *yields* $\mathcal{L}(f''') = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$.

(b) *From the rule for first derivatives we have*

$$\mathcal{L}(f^{(4)}) = \mathcal{L}((f''')') = s\mathcal{L}(f''') - f'''(0).$$

*Using the rule for* $\mathcal{L}(f''') = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$ *yields* $\mathcal{L}(f^{(4)}) = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$.

(c) *An induction works. Assume the result holds for $f^{(n-1)}$, that is,*

$$\mathcal{L}(f^{(n-1)}) = s^{n-1}F(s) - s^{n-2}f(0) - s^{n-3}f'(0) - \cdots - sf^{(n-3)}(0) - f^{(n-2)}(0).$$

*From the rule for first derivatives and noting that $f^{(n)} = (f^{n-1})'$ we have*

$$\begin{aligned}\mathcal{L}(f^{(n)}) &= s\mathcal{L}(f^{(n-1)}) - f^{n-1}(0)\\ &= s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0)\\ &\quad - \cdots - s^2 f^{(n-3)}(0) - sf^{(n-2)}(0) - f^{(n-1)}(0).\end{aligned}$$

**Exercise Solution 5.2.39.** *Note that the by the fundamental theorem of calculus $g'(t) = f(t)$. Also, $g(0) = 0$. Laplace transform both sides of $g' = f$ and use $g(0) = 0$ to find $sG(s) = F(s)$, so $G(s) = F(s)/s$.*

**Exercise Solution 5.2.40.**

(a) *When $k = 1$ the expression is $(-1)(1/t)^2 F'(1/t) = 1/(1+t)^2$ (use $F'(s) = -1/(s+1)^2$.) A plot of $1/(1+t)^2$ and $e^{-t}$ is shown in the left panel of Figure 5.60.*

(b) *When $k = 2$ the expression is $((-1)^2/2)(2/t)^3 F''(2/t) = 1/(1+t/2)^3$ (use $F''(s) = 2/(s+1)^3$.) A plot of $1/(1+t/2)^3$ and $e^{-t}$ is shown in the right panel of Figure 5.60.*

(c) *When $k = 5$ the expression is $((-1)^5/120)(5/t)^6 F^{(5)}(5/t) = 1/(1+t/5)^6$ (use $F^{(5)}(s) = -120/(s+1)^6$.) A plot of $1/(1+t/5)^6$ and $e^{-t}$ is shown in Figure 5.61.*

(d) *The conjecture is that for a general $k$ we have*

$$\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right) = \frac{1}{(1+t/k)^{k+1}}.$$

*This is not hard to prove by using $F^{(k)}(s) = (-1)^k(k+1)/(s+1)^{k+1}$.*

*To examine the limit of $h(k) = \frac{1}{(1+t/k)^{k+1}}$ for fixed $t > 0$ compute $\ln(h(k))$ and write this as $\ln(h(k)) = \ln(k/(k+t))/(1/(k+1))$, and apply L'Hopital's Rule (consider $k$ as a real variable) to find $\ln(h(k)) \to -t$, so $h(k) \to e^{-t}$.*

(e) *It's easy to compute that* $F^{(k)}(s) = (-1)^k(k+1)!/s^{k+2}$ *and then simplify to find*

$$\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right) = (1+1/k)t.$$

*As* $k \to \infty$ *this obviously limits to* $t$*.*

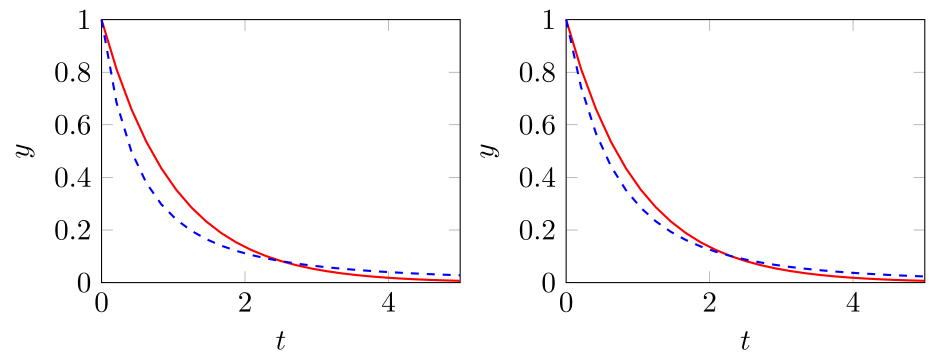



Figure 5.60: Left panel: Graph of $e^{-t}$ (red,solid) and $1/(1+t)^2$ (blue, dashed). Right panel: Graph of $e^{-t}$ (red,solid) and $1/(1+t/2)^3$ (blue, dashed).

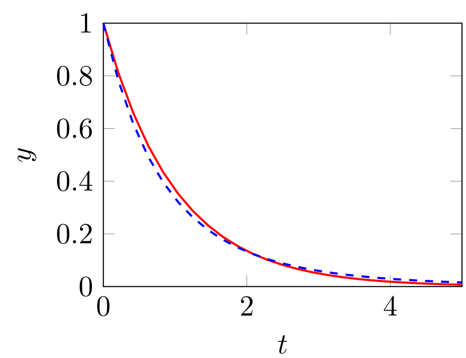



Figure 5.61: Graph of $e^{-t}$ (red,solid) and $1/(1+t/5)^6$ (blue, dashed).

## Section 5.3

**Exercise Solution 5.3.1.** $f(t) = 7H(t-5)$.

**Exercise Solution 5.3.2.** $f(t) = 7-7H(t-5)$, *or* $f(t) = 7H(t)-7H(t-5)$.

**Exercise Solution 5.3.3.** $f(t) = 2(1-H(t-3))+5(H(t-3)-H(t-6))-3H(t-6) = 2+3H(t-3)-8H(t-6)$.

**Exercise Solution 5.3.4.** $f(t) = 1-H(t-3)+t(H(t-3)-H(t-6))+e^{-t}H(t-6) = 1+(t-1)H(t-3)+(e^{-t}-t)H(t-6)$.

**Exercise Solution 5.3.5.** $f(t) = e^t(H(t-3)-H(t-6))+e^{2t}(H(t-6)-H(t-10))+4H(t-10) = e^tH(t-3)+(e^{2t}-e^t)H(t-6)+(4-e^{2t})H(t-10)$.

**Exercise Solution 5.3.6.** *We find*

- $F(s) = 7e^{-5s}/s$.
- $F(s) = 7/s - 7e^{-5s}/s$.
- $F(s) = 2/s + 3e^{-3s}/s - 8e^{-6s}/s$.
- *Use the second shifting theorem (Version 2) to find* $F(s) = 1/s + 2e^{-3s}/s + e^{-3s}/s^2 + e^{-6s-6}/(s+1) - 6e^{-6s}/s - e^{-6s}/s^2$.
- *Use the second shifting theorem (Version 2) to find* $F(s) = e^{-3(s-1)}/(s-1) - e^{-6(s-1)}/(s-1) + e^{-6(s-2)}/(s-2) - e^{-10(s-2)}/(s-2) + 4e^{-10s}/s$.

**Exercise Solution 5.3.7.** *The inverse transform of* $2/s^2$ *is* $2t$*, so by the second shifting theorem* $f(t) = 2H(t-3)(t-3)$.

**Exercise Solution 5.3.8.** *The inverse transform of* $1/(s^2+16)$ *is* $\sin(4t)/4$*, so* $q(t) = H(t-1)\sin(4(t-1))/4$.

**Exercise Solution 5.3.9.** *The inverse transform of* $(3s+2)/(s^2+4) = 3s/(s^2+4) + 2/(s^2+4)$ *is* $3\cos(2t)+\sin(2t)$ *so* $g(t) = H(t-5)(3\cos(2(t-5)) + \sin(2(t-5)))$.

**Exercise Solution 5.3.10.** *First,*

$$\frac{s}{s^2+6s+25} = \frac{s}{(s+3)^2+4^2} = \frac{s+3}{(s+3)^2+4^2} - \frac{3}{(s+3)^2+4^2}$$

*has inverse Laplace transform* $\phi(t) = e^{-3t}\cos(4t) - 3e^{-3t}\sin(4t)/4$. *Then* $f(t) = H(t-2\pi)\phi(t-2\pi) = H(t-2\pi)e^{-3(t-2\pi)}(\cos(4(t-2\pi)) - 3\sin(4(t-2\pi))/4)$. *This can also be written as* $f(t) = H(t-2\pi)e^{-3(t-2\pi)}(\cos(4t) - 3\sin(4t)/4)$ *since* $\sin(4(t-2\pi)) = \sin(4t)$ *and* $\cos(4(t-2\pi)) = \cos(4t)$.

**Exercise Solution 5.3.11.** *The inverse transform of* $12/(s+2)^4$ *is* $2t^3e^{-2t}$*, so* $f(t) = 2H(t-3)(t-3)^3e^{-2(t-3)}$*.*

**Exercise Solution 5.3.12.** *Transform both sides of the ODE and use the initial data to find* $sU(s) - 1 = -2U(s) + 4e^{-5s}/s$*. Then* $U(s) = 1/(s+2) + 4e^{-5s}/(s(s+2))$*. The inverse transform of* $1/(s+2)$ *is* $e^{-2t}$*. The inverse transform of* $1/(s(s+2)) = 1/(2s) - 1/(2(s+2))$ *is* $1/2 - e^{-2t}/2$ *so the inverse transform of* $4e^{-5s}/(s(s+2))$ *is* $4H(t-5)(1-e^{-2(t-5)})/2$*. All in all* $u(t) = e^{-2t} + 2H(t-5)(1-e^{-2(t-5)})$*. Graph shown in the left panel of Figure 5.62.*

**Exercise Solution 5.3.13.** *Transform both sides of the ODE and use the initial data to find* $sU(s) - 1 = -3U(s) + 3e^{-3s}/s - 6e^{-5s}/s$*. Then* $U(s) = 1/(s+3) + 3e^{-3s}/(s(s+3)) - 6e^{-5s}/(s(s+3))$*. The inverse transform of* $1/(s+3)$ *is* $e^{-3t}$*. The inverse transform of* $3/(s(s+3)) = 1/s - 1/(s+3)$ *is* $1 - e^{-3t}$ *so the inverse transform of* $3e^{-3s}/(s(s+3))$ *is* $H(t-3)(1-e^{-3(t-3)}$*. The inverse transform of* $6e^{-5s}/(s(s+3))$ *is* $2H(t-5)(1-e^{-3(t-5)})$*. All in all* $u(t) = e^{-3t} + H(t-3)(1-e^{-3(t-3)}) - 2H(t-5)(1-e^{-3(t-5)})$*. Graph shown in the middle panel of Figure 5.62.*

**Exercise Solution 5.3.14.** *Transform both sides of the ODE and use the initial data to find* $sU(s) - 2 = -U(s) + e^{-s}/s + e^{-s}/s^2 - e^{-2s}/s$*. Then* $U(s) = 2/(s+1) + e^{-s}/(s(s+1)) + e^{-s}/(s^2(s+1)) - e^{-2s}/(s(s+1))$*. The inverse transform of* $2/(s+1)$ *is* $2e^{-t}$*. The inverse transform of* $e^{-s}/(s(s+1))$ *is* $H(t-1)(1-e^{-(t-1)})$*, the inverse transform of* $e^{-s}/(s^2(s+1))$ *is* $H(t-1)(t-2+e^{-(t-1)})$*, and the inverse transform of* $-e^{-2s}/(s(s+1))$ *is* $-H(t-2)(1-e^{-(t-2)})$*.) All in all*

$$\begin{aligned} u(t) &= 2e^{-t} + H(t-1)(1-e^{-(t-1)}) + H(t-1)(t-2+e^{-(t-1)}) \\ &\quad - H(t-2)(1-e^{-(t-2)}) \\ &= 2e^{-t} + H(t-1)(t-1) - H(t-2)(1-e^{-(t-2)}). \end{aligned}$$

*Graph shown in the right panel of Figure 5.62.*

**Exercise Solution 5.3.15.** *Transforming both sides and using the initial data yields* $s^2U(s) + 4sU(s) + 3U(s) = e^{-s}/s$ *so that* $U(s) = \frac{e^{-s}}{s(s^2+4s+3)} = \frac{e^{-s}}{s(s+1)(s+3)}$*. Then*

$$U(s) = e^{-s}\left(\frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}\right).$$

*An inverse transform yields* $u(t) = H(t-1)(1/3 - e^{-(t-1)}/2 + e^{-3(t-1)}/6)$*. Graph shown in the left panel of Figure 5.63.*

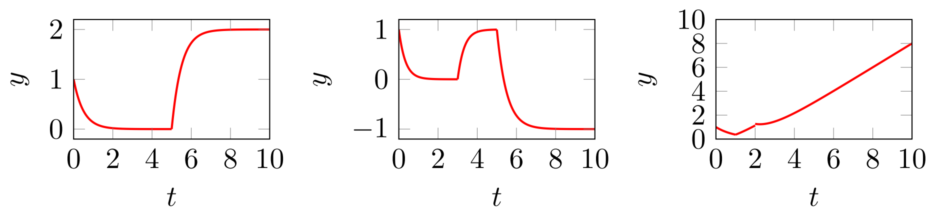



Figure 5.62: Graph of solutions for Exercises 5.3.12 (left), 5.3.13 (middle), and 5.3.14 (right).

**Exercise Solution 5.3.16.** *Transforming both sides and using the initial data yields* $s^2U(s) + 16U(s) = e^{-3s}/s$ *so that* $U(s) = \frac{e^{-3s}}{s(s^2+16)}$. *Then*

$$U(s) = e^{-3s}\left(\frac{1}{16s} - \frac{s}{16(s^2+16)}\right).$$

*An inverse transform yields* $u(t) = H(t-3)(1-\cos(4(t-3)))/16$. *Graph shown in the right panel of Figure 5.63.*

**Exercise Solution 5.3.17.** *Laplace transform and fill in the initial data to find* $(s^2+4s+4)U(s) - s - 6 = 4/s + 8e^{-3s}/s$. *Then*

$$U(s) = \frac{s+6}{(s+2)^2} + \frac{4}{s(s+2)^2} + \frac{8e^{-3s}}{s(s+2)^2}.$$

*A partial fraction decomposition shows*

$$\frac{s+6}{(s+2)^2} = \frac{1}{s+2} + \frac{4}{(s+2)^2}.$$

*and*

$$\frac{4}{s(s+2)^2} = \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}.$$

*Use this to find*

$$\begin{aligned} u(t) &= e^{-2t} + 4te^{-2t} + 1 - e^{-2t} - 2te^{-2t} \\ &\quad + 2H(t-3)(1 - e^{-2(t-3)} - 2(t-3)e^{-2(t-3)}) \\ &= 1 + 2te^{-2t} + 2H(t-3)(1 - e^{-2(t-3)} - 2(t-3)e^{-2(t-3)}). \end{aligned}$$

*Graph shown in the left panel of Figure 5.64.*

**Exercise Solution 5.3.18.** *Transform both sides of the ODE and fill in initial data to find* $(s^2+16)U(s)+1 = e^{-s\pi}s/(s^2+16)$*. Then*

$$U(s) = -\frac{1}{s^2+16} + \frac{e^{-s\pi}s}{(s^2+16)^2}.$$

*The inverse transform of the first term above is* $-\sin(4t)/4$*. For the second term take the hint and note that* $s/(s^2+16)^2 = -\frac{1}{2}\frac{dF}{ds}$ *where* $F(s) = 1/(s^2+16)$*. Since* $F(s)$ *inverse transforms to* $f(t) = \sin(4t)/4$*,* $s/(s^2+16)^2$ *corresponds to* $t\sin(4t)/8$*. Then*

$$\begin{aligned} u(t) &= -\sin(4t)/4 + H(t-\pi)(t-\pi)\sin(4(t-\pi))/8 \\ &= -\sin(4t)/4 + H(t-\pi)(t-\pi)\sin(4t)/8 \end{aligned}$$

*(since* $\sin(4(t-\pi)) = \sin(4t)$*.) Graph shown in the right panel of Figure 5.64.*

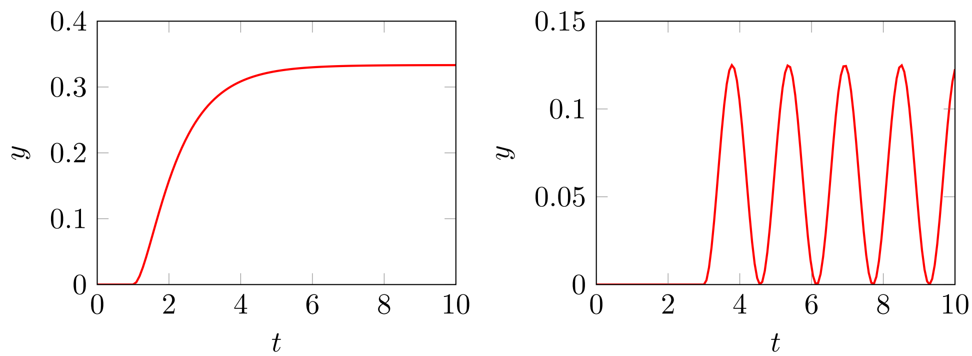



Figure 5.63: Graph of solutions to Exercise 5.3.15 (left) and 5.3.16 (right).

**Exercise Solution 5.3.19.** *The ODE is* $u'(t) = -ku(t) + 1 + 0.5H(t-12)$ *(recall* $k = 0.173$*) with initial condition* $u(0) = 5$*. Laplace transforming, using the initial data, and then solving for* $U(s)$ *yields*

$$U(s) = \frac{5}{s+k} + \frac{1}{s(s+k)} + \frac{e^{-12s}}{2s(s+k)}$$

*Inverse transforming yields*

$$u(t) = 5e^{-kt} + \frac{1-e^{-kt}}{k} + H(t-12)\frac{1-e^{-k(t-12)}}{2k}.$$

*A graph is shown in Figure 5.65.*

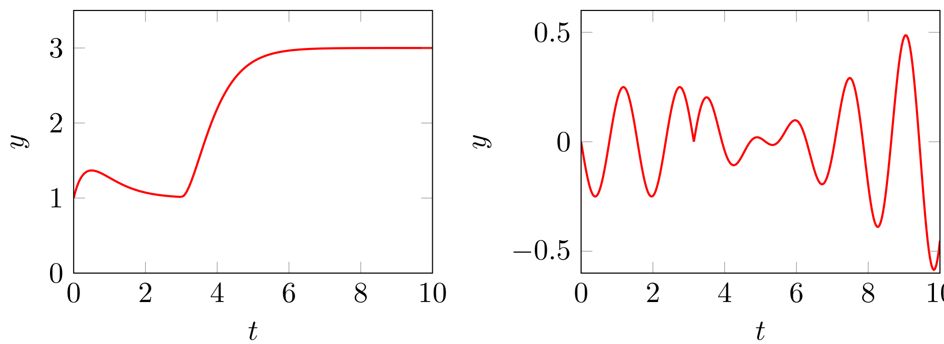



Figure 5.64: Graph of solutions to Exercise 5.3.17 (left) and 5.3.18 (right).

**Exercise Solution 5.3.20.** *The ODE is* $p'(t) = 0.02p(t) + 520 - 320H(t-2)$ *with* $p(0) = 1000$. *Laplace transform to find* $sP(s) - 1000 = 0.02P(s) + 520/s - 320e^{-2s}/s$. *Then*

$$P(s) = \frac{1000}{s-0.02} + \frac{520}{s(s-0.02)} - \frac{320e^{-2s}}{s(s-0.02)}.$$

*An inverse transform shows that*

$$p(t) = 1000e^{0.02t} + 26000(e^{0.02t} - 1) - 16000H(t-2)(e^{0.02(t-2)} - 1).$$

*A graph is shown in Figure 5.66.*

**Exercise Solution 5.3.21.** *The ODE is* $u'(t) = -0.05(u(t) - A)$ *with* $A = 80 + 10H(t-50)$, *so* $u'(t) = -0.05u(t) + 4 + 0.5H(t-50)$, *with* $u(0) = 120$. *Laplace transform, fill in the initial data, and solve for*

$$U(s) = \frac{120}{s+0.05} + \frac{4}{s(s+0.05)} + \frac{0.5e^{-50s}}{s(s+0.05)}.$$

*An inverse transform yields*

$$u(t) = 40e^{-0.05t} + 80 + 10H(t-50)(1 - e^{-0.05(t-50)})$$

*A graph is shown in Figure 5.67.*

**Exercise Solution 5.3.22.** *The ODE is* $2u''(t) + 8u(t) = 40H(t-10) - 40H(t-15)$ *with initial data* $u(0) = u'(0) = 0$. *Laplace transform and fill*

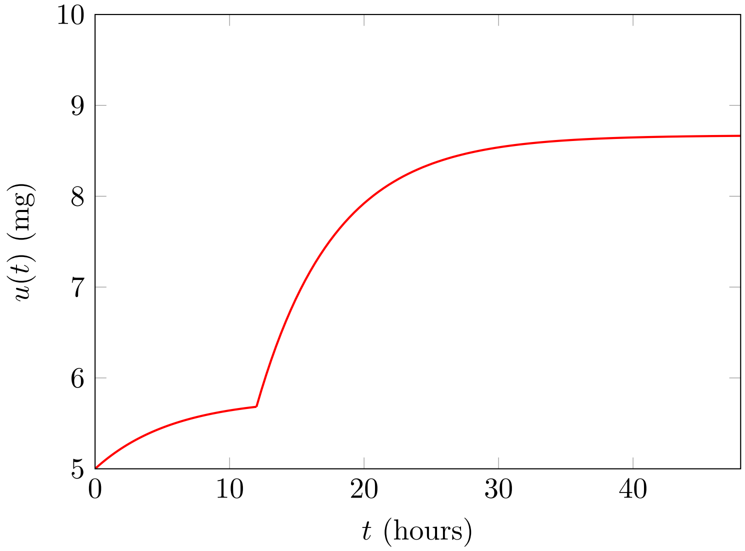



Figure 5.65: Plot of morphine level (mg).

*in the initial data to find* $2(s^2+4)U(s) = 40e^{-10s}/s - 40e^{-15s}/s$. *Solve for* $U(s)$ *to find*

$$U(s) = \frac{20e^{-10s}}{s(2^2+4)} - \frac{20e^{-15s}}{s(2^2+4)}.$$

*From a partial fraction decomposition we have* $\frac{1}{s(s^2+4)} = \frac{1}{4s} - \frac{s}{4(s^2+4)}$. *The inverse transform of* $\frac{1}{s(s^2+4)}$ *is then* $1/4 - \cos(2t)/4$. *It follows that*

$$u(t) = 5H(t-10)(1-\cos(2(t-10))) - 5H(t-15)(1-\cos(2(t-15))).$$

*A graph is shown in Figure 5.68.*

**Exercise Solution 5.3.23.** *The ODE is* $10q'(t)+10^4q(t) = 2+3H(t-0.003)$ *with initial condition* $q(0) = 0$. *Laplace transforming, substituting in the initial condition, and solving for* $Q(s)$ *yields*

$$Q(s) = \frac{1}{5s(s+10^3)} + \frac{3e^{-0.003s}}{10s(s+10^3)}.$$

*Write* $\frac{1}{s(s+10^3)} = \frac{1}{1000s} - \frac{1}{1000(s+1000)}$ *to see that* $\frac{1}{s(s+10^3)}$ *has inverse transform* $(1-e^{-1000t})/1000$. *From this it follows that*

$$q(t) = \frac{1-e^{-1000t}}{5000} + \frac{3H(t-0.003)(1-e^{-1000(t-0.003)})}{10000}.$$

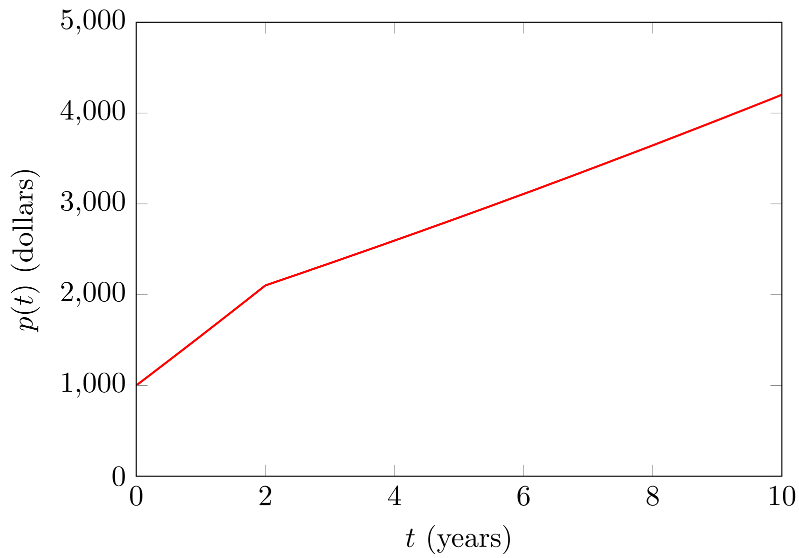



Figure 5.66: Account balance.

*A graph is shown in Figure 5.69.*

**Exercise Solution 5.3.24.**

(a) *From the definition of the Laplace transform*

$$\mathcal{L}(H(t-c)f(t-c)) = \int_0^\infty e^{-st}H(t-c)f(t-c)\,dt.$$

*Since $H(t-c) = 0$ for $0 < t < c$ we can begin the integration with lower limit $t = c$. For $t > c$ the function $H(t-c) = 1$, so the integrand is just $e^{-st}f(t-c)$ and then*

$$\mathcal{L}(H(t-c)f(t-c)) = \int_c^\infty e^{-st}f(t-c)\,dt.$$

(b) *Make a substitution $w = t-c$ (so $t = w+c$ and $dt = dw$) in the integral on the right above. The lower limit $t = c$ becomes $w = c-c = 0$, while $t = \infty$ becomes $w = \infty$; more accurately, if $t = T$ then $w = T-c$ and as $T \to \infty$ so does $T-c$. Using $e^{-st} = e^{-s(w+c)} = e^{-sc}e^{-st}$ leads to*

$$\mathcal{L}(H(t-c)f(t-c)) = e^{-cs}\int_0^\infty e^{-ws}f(w)\,dw$$

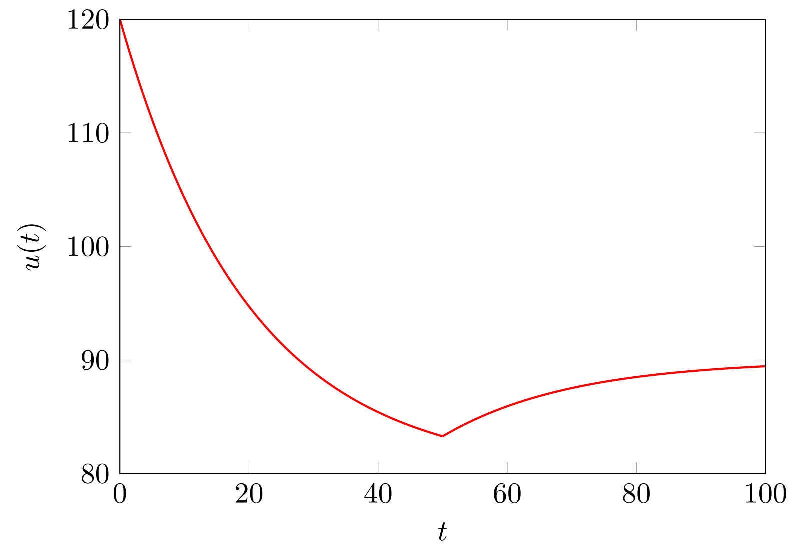



Figure 5.67: Newton cooling solution.

*since $e^{-sc}$ is constant with respect to the variable of integration $w$. But the above integral is the very definition of $F(s) = \mathcal{L}(f)$, so we've shown that $\mathcal{L}(H(t-c)f(t-c)) = e^{-sc}F(s)$.*

**Exercise Solution 5.3.25.** *Consider $\mathcal{L}(H(t-c)g(t))$. Define the function $f(t) = g(t+c)$ so that $g(t) = f(t-c)$. Note that if $g(t)$ is defined for $t > c$ then $f(t)$ is defined for $t > 0$. Also, $f$ is piecewise continuous and of exponential order if $g$ is. Then*

$$\begin{aligned}\mathcal{L}(H(t-c)g(t)) &= \mathcal{L}(H(t-c)f(t-c))\\ &= \mathcal{L}(H(t-c)f(t-c))\\ &= e^{-sc}F(s)\end{aligned}$$

*where the last step follows from the second shifting theorem Version 1. This completes the proof.*

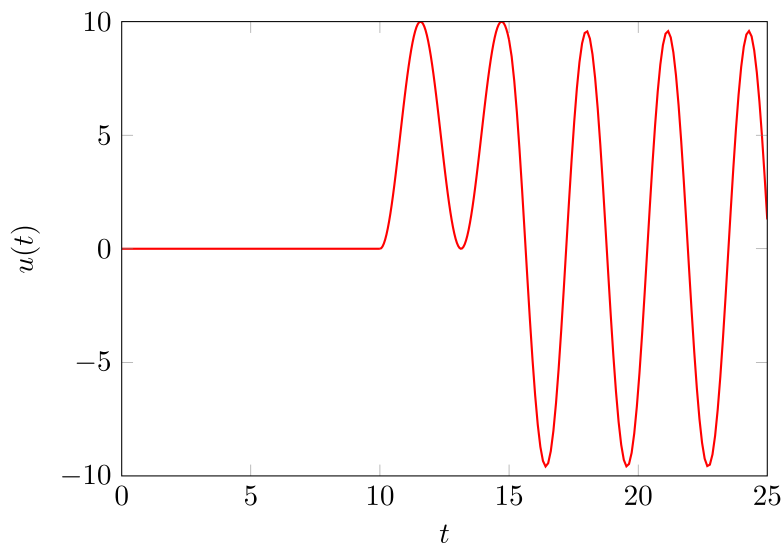



Figure 5.68: Motion of mass.

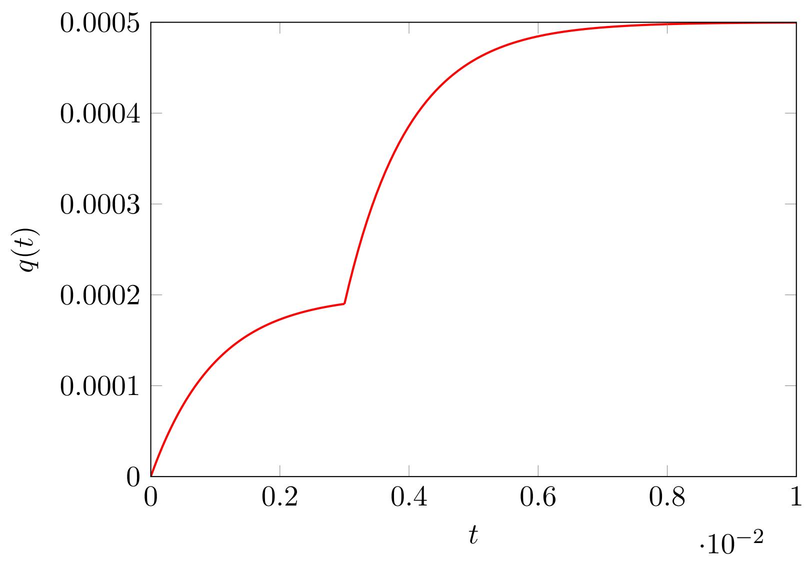



Figure 5.69: Charge on capacitor.

## Section 5.4

**Exercise Solution 5.4.1.** *Laplace transform to find* $sU(s)-1=-2U(s)+4e^{-5s}$*, so* $U(s)=1/(s+2)+4e^{-5s}/(s+2)$ *and* $u(t)=e^{-2t}+4H(t-5)e^{-2(t-5)}$*. Graph shown in the left panel of Figure 5.70.*

**Exercise Solution 5.4.2.** *Transform to find* $sU(s)-1=-3U(s)+3e^{-3s}-6e^{-5s}/s$ *so* $U(s)=1/(s+3)+3e^{-3s}/(s+3)-6e^{-5s}/(s(s+3))$ *with inverse transform* $u(t)=e^{-3t}+3H(t-3)e^{-3(t-3)}-2H(t-5)(1-e^{-3(t-5)})$*. Graph shown in the middle panel of Figure 5.70.*

**Exercise Solution 5.4.3.** *Transform to find* $sU(s)-2=-U(s)+e^{-s}/s+e^{-s}/s^2-3e^{-2s}$ *so* $U(s)=2/(s+1)+e^{-s}/(s(s+1))+e^{-s}/(s^2(s+1))-3e^{-2s}/(s+1)$ *with inverse transform* $u(t)=2e^{-t}+(t-1)H(t-1)-3H(t-2)e^{-(t-2)}$*. Graph shown in the right panel of Figure 5.70.*

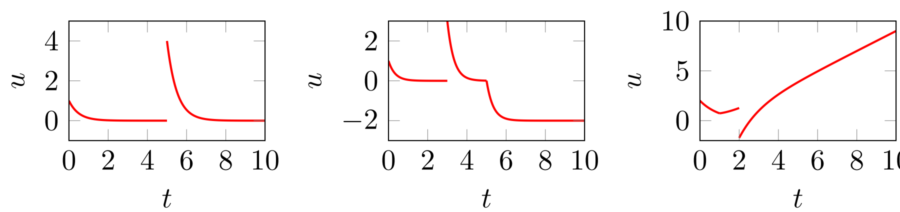



Figure 5.70: Graph of solutions to Exercises 5.4.1 (left), 5.4.2 (middle), and 5.4.3 (right).

**Exercise Solution 5.4.4.** *Transform to find* $(s^2+4s+3)U(s)=e^{-s}$*, so* $U(s)=e^{-s}/(s^2+4s+3)$ *and* $u(t)=H(t-1)(e^{-(t-1)}-e^{-3(t-1)})/2$*. Graph in left panel of Figure 5.71.*

**Exercise Solution 5.4.5.** *Transform to find* $(s^2+1)U(s)=e^{-3s}$ *so* $U(s)=e^{-3s}/(s^2+1)$ *and* $u(t)=H(t-3)\sin(t-3)$*. Graph in right panel of Figure 5.71.*

**Exercise Solution 5.4.6.** *Transform to find* $(s^2+4s+4)U(s)-s-6=1/s+5e^{-2s}$*, so* $U(s)=(s+6)/(s^2+4s+4)+1/(s(s^2+4s+4))+5e^{-2s}/(s^2+4s+4)$*. An inverse transform yields* $u(t)=1/4+e^{-2t}(14t+3)/4+5H(t-2)(t-2)e^{-2(t-2)}$*. Graph in left panel of Figure 5.72.*

**Exercise Solution 5.4.7.** *Transform to find* $(s^2+4)U(s)-s=s/(s^2+4)-20e^{-3s}$*, so* $U(s)=s/(s^2+4)-20e^{-3s}/(s^2+4)+s/(s^2+4)^2$*. An inverse transform yields* $u(t)=\cos(2t)+t\sin(2t)/4-10H(t-3)\sin(2(t-3))$*. Graph in right panel of Figure 5.72.*

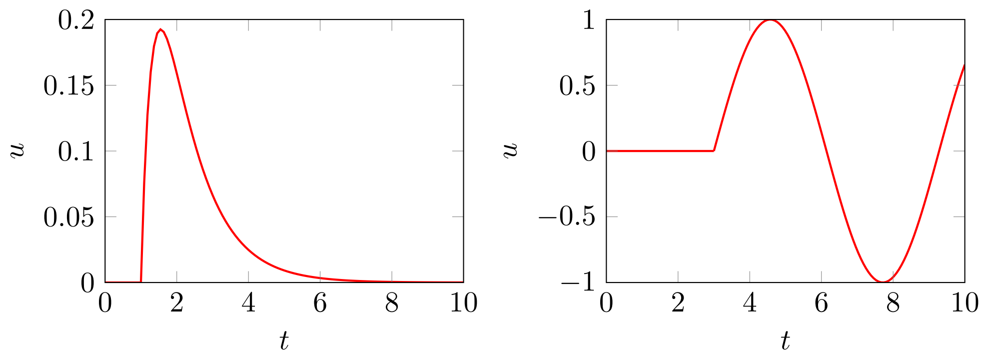



Figure 5.71: Graph of solutions to Exercises 5.4.4 (left) and 5.4.5 (right).

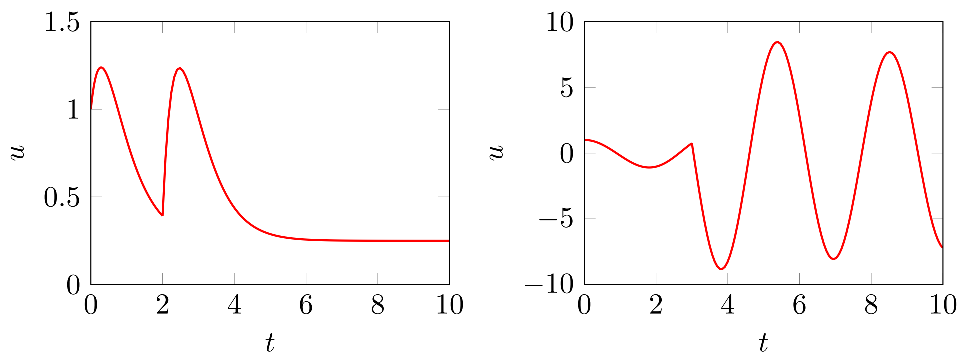



Figure 5.72: Graph of solutions to Exercises 5.4.6 (left) and 5.4.7 (right).

**Exercise Solution 5.4.8.** *If we define*

$$\phi(t) = \int_{-\infty}^{t} \delta(z)\, dz$$

*then* $\phi(t) = 0$ *for* $t < 0$*, since the mass of the Dirac function is not in* $(-\infty, t)$*. For* $t > 0$ *we find* $\phi(t) = 1$*, while* $\phi(0)$ *is ambiguous. That is,* $\phi(t) = H(t)$ *at all points* $t \neq 0$*, which means we can consider* $H(t)$ *as an antiderivative for* $\delta(t)$ *in some sense. This is consistent with the statement that* $H'(t) = \delta(t)$*.*

**Exercise Solution 5.4.9.**

(a) *The ODE is* $4u''(t)+16u'(t)+116u(t) = 20\delta(t-5)$ *with* $u(0) = u'(0) = 0$*, if* $u(t)$ *denotes the mass position.*

(b) *Transform both sides to find* $(4s^2+16s+116)U(s) = 20e^{-5s}$*, so* $U(s) = 5e^{-5s}/(s^2 + 4s + 29)$*. An inverse transform shows that* $u(t) = H(t-5)e^{-2(t-5)}\sin(5(t-5))$*. The mass remains motionless up until time* $t = 5$*, at which time the blow sets the mass in motion; it oscillates and decays back to position* $u = 0$*.*

**Exercise Solution 5.4.10.**

(a) *Salt enters at a rate of* $0.2$ *kg per minute and exits at a rate of* $-2x/100$ *or* $-x/50$ *kg per minute. The ODE is* $x'(t) = -x(t)/50 + 0.2$ *with* $x(0) = 0$*. Laplace transforming yields* $sX(s) = -X(s)/50 + 0.2/s$ *and so* $X(s) = 0.2/(s(s + 1/50))$*. Inverse transforming yields* $x(t) = 10 - 10e^{-t/50}$*.*

(b) *This can be modeled with the ODE* $x'(t) = -x(t)/50 + 0.2 + 5\delta(t-20)$ *with* $x(0) = 0$*. A Laplace transform shows* $sX(s) = -X(s)/50 + 0.2/s + 5e^{-20s}$*, and then* $X(s) = 0.2/(s(s+1/50)) + 5e^{-20s}/(s+1/50)$*. Inverse transforming shows*

$$x(t) = 10 - 10e^{-t/50} + 5H(t-20)e^{-(t-20)/50}.$$

*The solution is graphed in Figure 5.73. We can see the* $5$ *kg jump in* $x(t)$ *at* $t = 20$*.*

**Exercise Solution 5.4.11.** *If* $u(t)$ *denotes the amount of morphine present then*

$$u'(t) = -ku(t) + 5\delta(t-4) + 5\delta(t-8) + 5\delta(t-12)$$

*with* $u(0) = 10$*. (Alternatively,* $u'(t) = -ku(t) + 10\delta(t) + 5\delta(t-4) + 5\delta(t-8) + 5\delta(t-12)$ *with* $u(0) = 0$*, to model the initial 10 mg bolus "just after"* $t = 0$*.) To solve, Laplace transform and find* $sU(s) - 10 = -kU(s) + 5e^{-4s} + 5e^{-8s} + 5e^{-12s}$ *and so*

$$U(s) = \frac{10}{s+k} + \frac{5e^{-4s}}{s+k} + \frac{5e^{-8s}}{s+k} + \frac{5e^{-12s}}{s+k}.$$

*An inverse transform shows*

$$u(t) = 10e^{-kt} + 5H(t-4)e^{-k(t-4)} + 5H(t-8)e^{-k(t-8)} + 5H(t-12)e^{-k(t-12)}.$$

*The solution is graphed in Figure 5.74.*

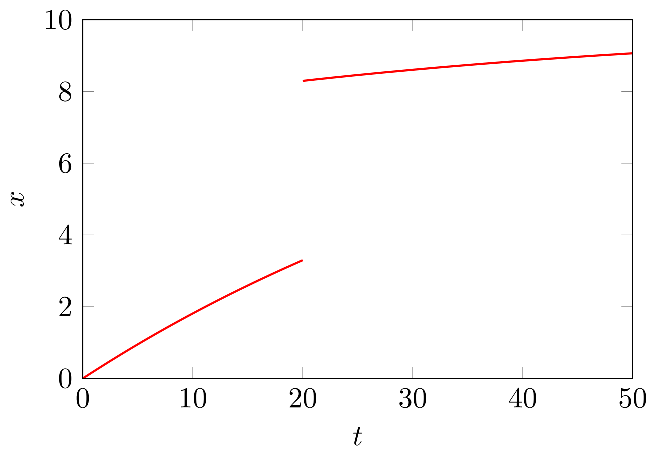



Figure 5.73: Solution to $x' = -x/50 + 0.2 + 5\delta(t-20)$, $x(0) = 0$.

**Exercise Solution 5.4.12.**

(a) *The ODE is* $p'(t) = 0.05p(t) + 500$ *with* $p(0) = 1000$. *The solution is* $p(t) = 11000e^{t/20} - 10000$.

(b) *The ODE is* $p'(t) = 0.05p(t) + 500 + 1000\delta(t-2)$ *with* $p(0) = 1000$. *A Laplace transform yields* $sP(s) - 1000 = P(s)/20 + 500/s + 1000e^{-2s}$ *and* $P(s) = 1000/(s+1/20) + 500/(s(s+1/20)) + 1000e^{-2s}/(s+1/20)$. *Inverse transforming shows that* $p(t) = 11000e^{t/20} - 10000 + 1000H(t-2)e^{(t-2)/20}$. *Then* $p(10) \approx 9627.76$.

(c) *The ODE is* $p'(t) = 0.05p(t) + 500 + 1000\delta(t-8)$ *with* $p(0) = 1000$. *A Laplace transform yields* $sP(s) - 1000 = P(s)/20 + 500/s + 1000e^{-9s}$ *and* $P(s) = 1000/(s+1/20) + 500/(s(s+1/20)) + 1000e^{-8s}/(s+1/20)$. *Inverse transforming shows that* $p(t) = 11000e^{t/20} - 10000 + 1000H(t-8)e^{(t-8)/20}$. *Then* $p(10) \approx 9241.10$. *In (b) the balance is higher because the $1,000 deposit has more time to accrue interest.*

**Exercise Solution 5.4.13.**

(a) *The solution is* $u(t) = e^{-t}\cos(5t)$.

(b) *This occurs at time* $t = t_2 = 3\pi/10 \approx 0.9425$.

(c) *With* $m = 2$ *it's easy to compute that the momentum of the mass as* $t = t_2$ *is* $mu'(t_2) = 10e^{-3\pi/10} \approx 3.8966$. *So we should take* $A = -10e^{-3\pi/10}$.

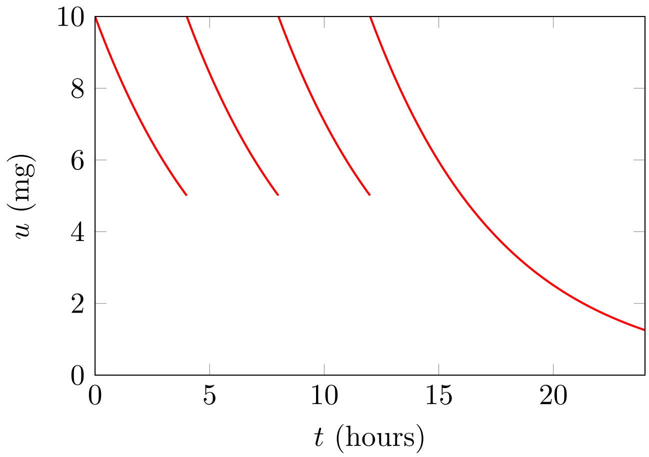



Figure 5.74: Solution to $u'(t) = -ku(t) + 5\delta(t-4) + 5\delta(t-8) + 5\delta(t-12)$, $u(0) = 10$.

*(d) Laplace transform the ODE* $2u''(t) + 4u'(t) + 52u(t) = A\delta(t-t_2)$ *to obtain* $(2s^2 + 4s + 52)U(s) - 2s - 2 = Ae^{-t_2 s}$. *Solve for*

$$U(s) = \frac{s+1}{s^2+2s+26} + \frac{Ae^{-t_2 s}}{2s^2+4s+52}$$

*and inverse transform to find*

$$u(t) = e^{-t}\cos(5t) + \frac{A}{10}H(t-t_2)e^{-(t-t_2)}\sin(5(t-t_2))$$

*But if we fill in the computed values for* $A$ *and* $t_2$ *this becomes*

$$u(t) = e^{-t}\cos(5t)(1 - H(t - 3\pi/10)).$$

*The mass oscillates naturally for* $0 < t < 3\pi/10$, *but the hammer blow at time* $t_2 = 3\pi/10$ *with impulse* $A = -10e^{-3\pi/10}$ *brings the mass to a dead stop.*

**Exercise Solution 5.4.14.**

*(a) The ODE is* $p'(t) = 0.04p(t) + A\delta(t-2) + A\delta(t-4)$ *with* $p(0) = 2000$.

*(b) The solution is* $p(t) = 2000e^{0.04t} + AH(t-2)e^{0.04(t-2)} + AH(t-4)e^{0.04(t-4)}$.

(c) *Compute* $p(10) = 2983.65 + 2.6484A$*, so solve* $2983.65 + 2.6484A = 10000$ *for* $A$ *to find* $A \approx 2649.30$.

**Exercise Solution 5.4.15.**

(a) *Laplace transform both sides of* $mv'(t) = A\delta(t - t_0)$ *and use* $v(0) = 0$ *to find* $msV(s) = Ae^{-st_0}$*. Then* $V(s) = Ae^{-st_0}/(ms)$ *and an inverse transform shows that*

$$v(t) = AH(t - t_0)/m.$$

(b) *Given that* $H(t - t_0) = 1$ *for* $t > t_0$*, it's quite trivial to see that*

$$\lim_{t \to t_0^+} mv(t) = \lim_{t \to t_0^+} AH(t - t_0) = A,$$

*the impulse of the blow.*

## Section 5.5

**Exercise Solution 5.5.1.** *$F_1(s) = 1/s^2$, $F_2(s) = 1/s$, $p(t) = t^2/2$, and $P(s) = 1/s^3$.*

**Exercise Solution 5.5.2.** *$F_1(s) = 1/s^2$, $F_2(s) = 1/(s-1)$, $p(t) = e^t - t - 1$, and $P(s) = 1/(s^2(s-1))$.*

**Exercise Solution 5.5.3.** *$F_1(s) = 1/(s-a)$, $F_2(s) = 1/(s-b)$, $p(t) = (e^{at} - e^{bt})/(a-b)$, and $P(s) = 1/((s-a)(s-b))$.*

**Exercise Solution 5.5.4.** *$F_1(s) = F_2(s) = 1/(s^2+1)$, $p(t) = (\sin(t) - t\cos(t))/2$, and $P(s) = 1/(s^2+1)^2$.*

**Exercise Solution 5.5.5.** *$F_1(s) = 2/s^3$, $F_2(s) = 1$, $p(t) = t^2$, and $P(s) = 2/s^3$.*

**Exercise Solution 5.5.6.** *$F_1(s) = 1/s^2 + 3/s$, $F_2(s) = e^{-2s}$, $p(t) = H(t-2)(t+1)$, and $P(s) = e^{-2s}/s^2 + 3e^{-2s}/s$.*

**Exercise Solution 5.5.7.** *Unit impulse response is $\mathcal{L}^{-1}(1/(s+4)) = e^{-4t}$.*

**Exercise Solution 5.5.8.** *Unit impulse response is $\mathcal{L}^{-1}(1/(s-2)) = e^{2t}$.*

**Exercise Solution 5.5.9.** *Unit impulse response is $\mathcal{L}^{-1}(1/s) = H(t)$ or $1$.*

**Exercise Solution 5.5.10.** *Unit impulse response is $\mathcal{L}^{-1}(1/(s^2+3s+2)) = e^{-t} - e^{-2t}$.*

**Exercise Solution 5.5.11.** *Unit impulse response is $\mathcal{L}^{-1}(1/(s^2+1)) = \sin(t)$.*

**Exercise Solution 5.5.12.** *Unit impulse response is $\mathcal{L}^{-1}(1/(2s^2+4s+10)) = e^{-t}\sin(2t)/4$.*

**Exercise Solution 5.5.13.** *Unit impulse response is $\mathcal{L}^{-1}(1/(s^2+4s+4)) = te^{-2t}$.*

**Exercise Solution 5.5.14.** *Unit impulse response is $\mathcal{L}^{-1}(1/s^2) = t$.*

**Exercise Solution 5.5.15.** *Given the convolution equation $f * g = p$ we can Laplace transform to find that $F(s)G(s) = P(s)$ or*

$$F(s)\frac{1}{s^2+1} = \frac{4}{(s+1)^2(s^2+1)}$$

*from which we find $F(s) = 4/(s+1)^2$. Inverse transform to find $f(t) = 4te^{-t}$.*

**Exercise Solution 5.5.16.** *Laplace transform the ODE and use the initial data to find* $(as+b)U(s) = F(s)$. *We can compute* $U(s) = 1/(s(s+5))$ *and* $F(s) = 1/s$, *from which it follows that* $(as+b)/(s(s+5)) = 1/s$ *or* $(as+b)/(s+5) = 1$. *We conclude that* $a = 1$ *and* $b = 5$.

**Exercise Solution 5.5.17.** *Laplace transform the ODE and use the initial data to find* $(as+b)U(s) = F(s)$. *We can compute* $U(s) = e^{-3s}/(s+2)$ *and* $F(s) = e^{-3s}$, *from which it follows that* $(as+b)e^{-3s}/(s+2) = e^{-3s}$ *or* $(as+b)/(s+2) = 1$. *We conclude that* $a = 1$ *and* $b = 2$.

**Exercise Solution 5.5.18.** *From* $U(s) = G(s)F(s) = F(s)/(ms^2+cs+k)$ *along with* $U(s) = 4e^{-s}((s+1)(s+5))$ *and* $F(s) = 4e^{-5s}$ *we find* $G(s) = 1/(ms^2+cs+k) = 1/(s^2+6s+5)$. *Then* $m = 1, c = 6,$ *and* $k = 5$.

**Exercise Solution 5.5.19.** *Laplace transforming* $mu''+cu'+ku = 3\delta(t-1)$ *and using the initial data yields* $U(s) = G(s)F(s)$ *where* $U(s) \approx (1.56)(1.61e^{-s}/((s+0.092)^2+1.61^2))$, $F(s) = 3e^{-s}$, *and* $G(s) = 1/(ms^2+cs+k)$. *We can deduce that*

$$\begin{aligned} G(s) &\approx \frac{(1.56)(1.61)}{3((s+0.092)^2+1.61^2)} \\ &\approx \frac{1}{1.19s^2+0.220s+3.11}. \end{aligned}$$

*Reasonable estimates are* $m = 1.19, c = 0.220, k = 3.11$.

**Exercise Solution 5.5.20.** *Laplace transform both sides of the ODE* $Lq''(t)+Rq'(t)+q(t)/C = V(t)$ *to find*

$$(Ls^2+8s+1/C)Q(s) = 5/s$$

*where* $Q(s) = \mathcal{L}(q)$. *It follows that*

$$Q(s) = \frac{5}{s(Ls^2+8s+1/C)}. \tag{5.4}$$

*To make use of the given data, note that the current through the resistor is* $q'(t)$ *and the voltage* $v_r(t)$ *across the resistor is therefore* $v_r(t) = 8q'(t)$ *(Ohm's law). We know* $v_r(t)$ *and so can compute* $\mathcal{L}(v_r) = (2.383)(1199)/((s+285.7)^2+1199^2)$. *Since* $\mathcal{L}(q') = sQ(s)-q(0) = sQ(s)$ *we can determine from* $v_r(t) = 8q'(t)$ *that* $8sQ(s) = (2.383)(1199)/((s+285.7)^2+1199^2)$ *or*

$$Q(s) = \frac{(2.383)(1199)}{8s((s+285.7)^2+1199^2)}. \tag{5.5}$$

*If we normalize each expression for $Q(s)$ in (5.4) and (5.5) to have numerator* 1 *we obtain*

$$\frac{1}{s((L/5)s^2 + 1.6s + 1/(5C))} = \frac{1}{s(0.0028s^2 + 1.6s + 4250)}.$$

*From this it follows that* $L/5 \approx 0.0028$ *so* $L \approx 0.014$ *h and* $1/(5C) = 4250$ *so that* $C \approx 4.7 \times 10^{-5}$ *f.*

**Exercise Solution 5.5.21.** *The appropriate ODE model here is* $x'(t) = 0.2 - x(t)/50 + A\delta(t - t_0)$ *with* $t_0 = 111$ *and* $x(0) = 0$*, where* $x(t)$ *is the amount of salt (kg) in the tank at time* $t$*. The solution to the ODE can be obtained by Laplace transforming to find* $sX(s) = 0.2/s - X(s)/50 + Ae^{-st_0}$*.*

$$X(s) = \frac{0.2}{s(s + 1/50)} + \frac{Ae^{-st_0}}{s + 1/50}.$$

*An inverse transform shows that*

$$x(t) = 10 - 10e^{-t/50} + AH(t - t_0)e^{-(t-t_0)/50}.$$

*For* $t > t_0$ *this becomes*

$$x(t) = 10 - 10e^{-t/50} + Ae^{-(t-t_0)/50} = 10 + (-10 + Ae^{t_0/50})e^{-t/50}.$$

*The concentration of salt would be given by* $c(t) = x(t)/100$ *or*

$$c(t) = 0.1 + (-0.1 + Ae^{t_0/50}/100)e^{-t/50}.$$

*Match this to the measured value* $0.1 + 0.915e^{-0.02t}$ *and deduce that* $-0.1 + Ae^{t_0/50}/100 = 0.195$*. Given that* $t_0 = 111$ *this leads to* $A = (100)(0.295/e^{t_0/50}) \approx 3.2$ *kg.*

**Exercise Solution 5.5.22.** *Laplace transform both sides of* $ay''(t)+by'(t)+cy(t) = f$ *to find*

$$(as^2 + bs + c)Y(s) = F(s).$$

*From this information we can determine that*

$$\frac{F(s)}{as^2 + bs + c} = Y(s) = \frac{4e^{-2s}}{(s + 2)(s + 3)(s + 4)}.$$

*There are many choices that lead to consistency with the data, for example,* $F(s) = 4e^{-2s}/(s + 2)$ *and then choosing* $as^2 + bs + c = (s + 3)(s + 4) = s^2 + 7s + 12$*, so* $a = 1, b = 7, c = 12$*. And of course, multiplying each of* $F(s), a, b, c$ *by a common scalar leaves the quotient on the right above unchanged.*

**Exercise Solution 5.5.23.**

(a) *Laplace transform both sides of the ODE and fill in the initial data to find* $(ms^2 + cs + k)U(s) = mv_0$ *so that*

$$U(s) = \frac{v_0}{ms^2 + cs + k} = mv_0 G(s).$$

(b) *Inverse transform and use the definition of the impulse response to find* $u(t) = mv_0 u_\delta(t)$.

**Exercise Solution 5.5.24.** *In each case let's use the convolution theorem (though they can be done directly from the definition of convolution).*

- **Commutativity:** *This is equivalent to the s-domain statement* $F_1(s)G(s) = G(s)F_1(s)$, *which is clearly true.*

- **Distributivity:** *This is equivalent to the s-domain statement* $(aF_1(s)+ bF_2(s))G(s) = aF_1(s)G(s) + bF_2(s)G(s)$, *also clearly true.*

- **Associativity:** *This is equivalent to the s-domain statement* $(F_1(s)F_2(s))G(s) = F_1(s)(F_2(s)G(s))$, *also true.*

**Exercise Solution 5.5.25.** (a) *The definition of convolution is*

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

*Take the absolute value of both sides above and make use of* $\left|\int_a^b \phi(t)\,dt\right| \le \int_a^b |\phi(t)|\,dt$ *to find that*

$$\begin{aligned} |(f * g)(t)| &= \left|\int_0^t f(\tau)g(t-\tau)\,d\tau\right| \\ &\le \int_0^t |f(\tau)||g(t-\tau)|\,d\tau. \end{aligned}$$

(b) *With the assumptions that* $|f(t)| \le M_1 e^{at}$ *and* $g(t) \le M_2 e^{bt}$ *as well as* $a < c$, $b < c$ *it follows that for* $t \ge 0$ *we have* $e^{at} \le e^{ct}$ *and* $e^{bt} \le e^{ct}$ *so that* $|f(t)| \le M_1 e^{ct}$ *and* $g(t) \le M_2 e^{ct}$.

(c) *Using the bounds in part (b) we find that*

$$\begin{aligned}
|(f * g)(t)| &\leq \int_0^t |f(\tau)g(t-\tau)|\, d\tau \\
&\leq \int_0^t (M_1 e^{c\tau})(M_2 e^{c(t-\tau)})\, d\tau \\
&= M_1 M_2 e^{ct} \int_0^t \, d\tau \\
&= M_1 M_2 t e^{ct}.
\end{aligned}$$

(d) *For any* $d > c$ *we can choose a constant* $K$ *so that* $M_1 M_2 t e^{ct} \leq K e^{dt}$ *for all* $t \geq 0$. *To see this rewrite this last inequality as the equivalent statement* $t e^{(c-d)t} \leq K/(M_1 M_2)$. *The function* $t e^{(c-d)t}$ *has a maximum value of* $1/(e(d-c))$ *for* $t \geq 0$, *so choose* $K = M_1 M_2/(e(d-c))$.

## Section 5.6

**Exercise Solution 5.6.1.** *Substitute* $u(t) = \frac{r'(t)+kr(t)}{K}$ *into* $y'(t) = -ky(t) + Ku(t)$ *to find ODE*

$$y'(t) = -ky(t) + r'(t) + kr(t).$$

*With* $y(0) = r(0)$ *it is easy to check that* $y(t) = r(t)$ *is the unique solution to this ODE. If we Laplace transform both sides of* $u(t) = \frac{r'(t)+kr(t)}{K}$ *we obtain* $U(s) = (sR(s) + kR(s))/K = G_c(s)R(s)$. *This corresponds to the s-domain computation.*

**Exercise Solution 5.6.2.**

*(a) With* $u(t) = r'(t)$ *the ODE for* $y(t)$ *is* $y'(t) = r'(t)$, *and with* $y(0) = r(0) = 0$, *and the unique solution is* $y(t) = r(t)$.

*(b) In the s-domain we have* $sY(s) = U(s)$ *so that* $G_p(s) = 1/s$. *With control* $u(t) = r'(t)$ *we have* $U(s) = sR(s)$, *so that* $G_c(s) = s$. *Then* $G_p(s)G_c(s) = 1$ *and* $Y(s) = G_p(s)U(s) = G_p(s)C_c(s)R(s) = R(s)$. *This of course implies that* $y(t) = r(t)$.

**Exercise Solution 5.6.3.**

*(a) We find* $G_c(s) = K_p$. *With* $G_p(s) = 1/s$ *we then have* $G(s) = G_p(s)G_c(s)/(1 + G_p(s)G_c(s)) = K_p/(s + K_p)$.

*(b) We find* $R(s) = \frac{10}{s(s+2)}$, *and with* $K_p = 1$, $Y(s) = R(s)/(s+1) = \frac{10}{s(s+1)(s+2)}$. *Then* $y(t) = 5 + 5e^{2t} - 10e^{-t}$. *The solution approaches* $r(t)$, *which itself approaches* 5. *A plot is shown in Figure 5.75. Larger values of* $K_p$ *cause* $y(t)$ *to approach* $r(t)$ *more rapidly.*

*(c) Suppose that* $\lim_{s\to 0^+} G(s) = 1$. *The Final Value Theorem shows that if* $\lim_{t\to r(t)} = r_0$ *then* $\lim_{s\to 0^+} sR(s) = r_0$ *and then it follows that* $\lim_{s\to 0^+} sY(s) = \lim_{s\to 0^+} sG(s)R(s) = r_0$. *By the final value theorem* $\lim_{t\to\infty} y(t) = r_0$ *too.*

**Exercise Solution 5.6.4.**

*(a) We have* $G_c(s) = K_p + K_i/s + K_d s$. *Given* $G_p(s) = 1/s$ *we find*

$$G(s) = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)} = \frac{K_d s^2 + K_p s + K_i}{(K_d + 1)s^2 + K_p s + K_i}.$$

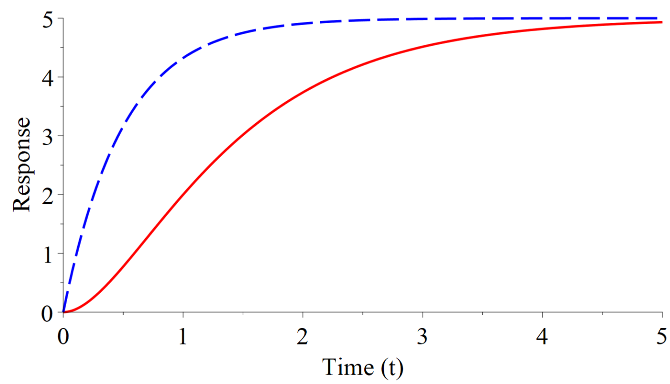



Figure 5.75: Setpoint $r(t)$ (dashed, blue) and process variable/response (solid, red).

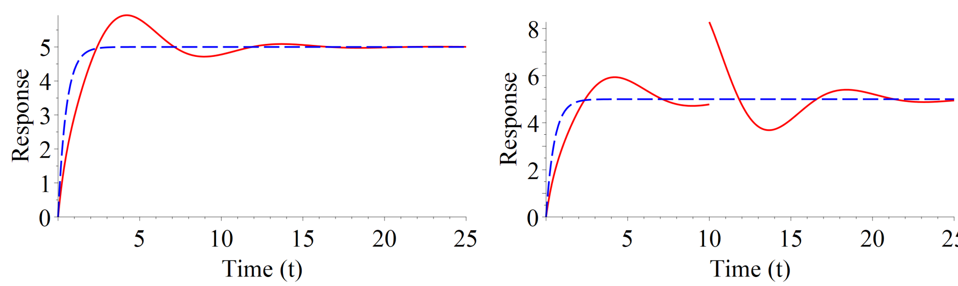



Figure 5.76: Left panel: Setpoint $r(t)$ (dashed, blue) and process variable/response (solid, red) for PID control. Right panel: Same with disturbance at time $t = 10$.

(b) *With $r(t) = 5 - 5e^{-2t}$ we have $R(s) = 10/(s(s+2))$ and $Y(s) = \frac{10(s^2+s+1)}{s(s+2)(2s^2+s+1)}$. Then $y(t) = 5 - 15e^{-2t}/7 - 20e^{-t/4}\cos(t\sqrt{7}/4)/7$. The solution $y(t)$ approaches 5 (as does $r(t)$) but oscillates a bit in doing so. This is shown in the left panel of Figure 5.76.*

(c) *As shown in the right panel of Figure 5.76, there is an obvious spike in the solution at $t = 10$, but the solution quickly settles back to 5.*

**Exercise Solution 5.6.5.**

(a) *A Laplace transform shows that $(ms^2 + cs + k)X(s) = U(s)$ or $X(s) = U(s)/(ms^2 + cs + k)$, so $G_p(s) = 1/(ms^2 + cs + k)$.*

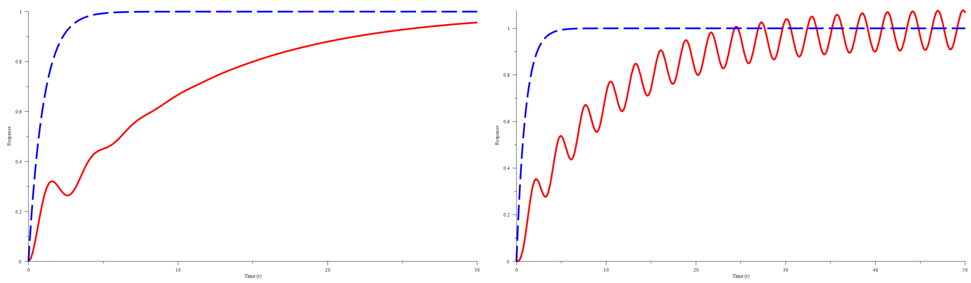


Figure 5.77: Left panel: Setpoint $r(t)$ (dashed, blue) and process variable/response (solid, red) for $K_p = 1, K_i = 0.5, K_d = 1$. Right panel: Same, but with $K_p = 1, K_i = 0.5, K_d = 0$.

*(b)* *Routine algebra shows that* $G(s) = G_c(s)G_p(s)/(1 + G_c(s)G_p(s))$ *leads to*

$$G(s) = \frac{K_d s^2 + K_p s + K_i}{ms^3 + (c + K_d)s^2 + (k + K_p)s + K_i}.$$

*(c)* *We find that* $G(s) = \frac{s^2+s+0.5}{s^3+1.1s^2+1.4s+0.5}$ *with* $R(s) = \frac{1}{s(s+1)}$. *Then* $Y(s) = \frac{s^2+s+0.5}{s(s+1)(s^3+1.1s^2+5s+0.5)}$. *The solution* $y(t)$ *tracks* $r(t)$ *as* $r(t)$ *approaches* 1, *although* $y(t)$ *oscillates a bit in doing so.*

*(d)* *Here* $u(0) = 1$ *and* $u(t)$ *gradually increases to a limiting value of* 4 *(since it requires* 4 *newtons to maintain the mass in position* $x = 1$, *which is what* $r(t)$ *limits to).*

*(e)* *The function* $y(t)$ *oscillates and never settles down, as shown in the right panel of Figure 5.77. Here the process variable contains terms like* $\cos(2.236t)$ *and* $\sin(2.2.36t)$. *Not surprisingly, the transfer function* $G(s)$ *has poles at approximately* $\pm 2.236i$ *(and also* $-0.1$*).*

## Section 6.1

**Exercise Solution 6.1.1.** *Nonlinear (has $x_1x_2$).*

**Exercise Solution 6.1.2.** *Linear, constant coefficient, and homogenous.*

**Exercise Solution 6.1.3.** *Nonlinear.*

**Exercise Solution 6.1.4.** *Linear, variable coefficient, homogeneous.*

**Exercise Solution 6.1.5.** *Nonlinear ($x_1/x_2$).*

**Exercise Solution 6.1.6.** *Nonlinear ($e^{x_1+2x_2}$).*

**Exercise Solution 6.1.7.** *Linear, variable coefficient, homogeneous.*

**Exercise Solution 6.1.8.** *Nonlinear ($x_2x_3$ product).*

**Exercise Solution 6.1.9.** *Linear, constant coefficient, nonhomogeneous.*

**Exercise Solution 6.1.10.** *Linear, variable coefficient, homogeneous.*

**Exercise Solution 6.1.11.** *Linear, variable coefficient, nonhomogeneous.*

**Exercise Solution 6.1.12.** *With $x_1 = u$ and $x_2 = u'$*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -4x_1/3 - 5x_2/3\end{aligned}$$

*with $x_1(0) = 7$ and $x_2(0) = 5$.*

**Exercise Solution 6.1.13.** *With $x_1 = u$ and $x_2 = u'$*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - x_2\end{aligned}$$

*with $x_1(0) = 1$ and $x_2(0) = 0$.*

**Exercise Solution 6.1.14.** *With $x_1 = u$ and $x_2 = u'$*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1/2 - \cos(x_2)\end{aligned}$$

*with $x_1(0) = 3$ and $x_2(0) = -1$.*

**Exercise Solution 6.1.15.** *With* $x_1 = u$ *and* $x_2 = u'$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 x_2 + 7\end{aligned}$$

*with* $x_1(0) = 2$ *and* $x_2(0) = 4$.

**Exercise Solution 6.1.16.** *With* $x_1 = u$, $x_2 = u'$, *and* $x_3 = u''$,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -5x_1 - x_2 - 2x_3\end{aligned}$$

*with* $x_1(0) = 1$, $x_2(0) = 0$, *and* $x_3(0) = -1$.

**Exercise Solution 6.1.17.** *With* $x_1 = u$, $x_2 = u'$, $x_3 = u''$, *and* $x_4 = u'''$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -3x_1 - 5x_2 - 4x_3 - x_4\end{aligned}$$

*with* $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = -1$, *and* $x_4(0) = 4$.

**Exercise Solution 6.1.18.** *Let* $x_1 = u_1, x_2 = u_1'$, *and* $x_3 = u_2$. *Then*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + x_3 + \sin(t) \\ \dot{x}_3 &= -3x_1 + x_3\end{aligned}$$

*with* $x_1(0) = 1, x_2(0) = 3$, *and* $x_3(0) = -2$.

**Exercise Solution 6.1.19.** *Let* $x_1 = u_1$, $x_2 = u_1'$, $x_3 = u_2$, *and* $x_4 = u_2'$. *Then*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3/3 - \sin(x_4)/3 + t \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_1 + 2x_3\end{aligned}$$

*with* $x_1(0) = 1, x_2(0) = 3$, $x_3(0) = -2$, *and* $x_4(0) = 5$.

**Exercise Solution 6.1.20.** *With*

$$u_P(t) = \frac{e^{-2t}}{2} + \frac{e^{-6t}}{2}$$
$$u_T(t) = \frac{e^{-2t}}{4} - \frac{e^{-6t}}{4}$$

*with find that*

$$\dot{u}_P(t) = -e^{-2t} - 3e^{-6t}$$
$$\dot{u}_T(t) = -\frac{e^{-2t}}{2} + \frac{3e^{-6t}}{2}.$$

*Inserting these into each of* $\dot{u}_P(t) = -4u_P(t) + 4u_T(t)$ *and* $\dot{u}_T(t) = u_P(t) - 4u_T(t)$ *shows that they work. On* $0 \leq t \leq 5$ *the graphs are shown in Figure 6.78. This makes sense:* $u_P(t)$ *starts high and falls monotonically, while*

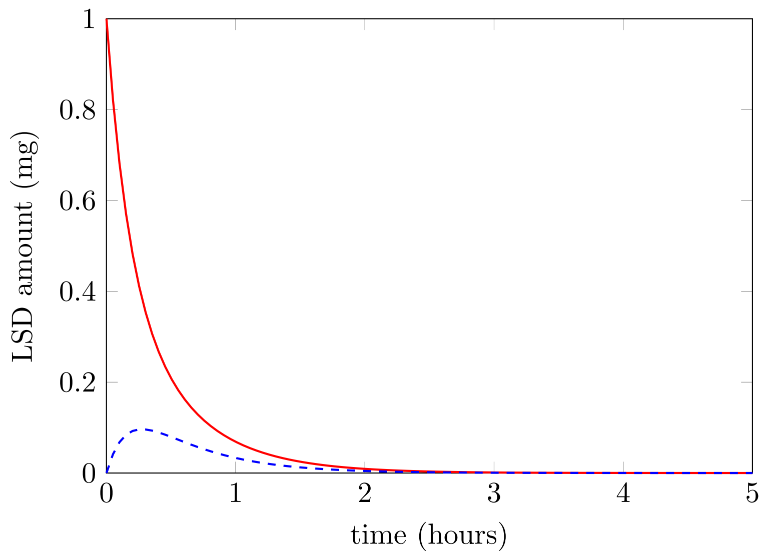



Figure 6.78: Plot of $u_P(t)$ (solid, red) and $u_T(t)$ (dashed, blue).

$u_T(t)$ *(tissue) starts with no LSD, acquires some from plasma, and then as the LSD is cleared from the plasma falls back to zero.*

**Exercise Solution 6.1.21.**

*(a) The transform of the* $\dot{x}_1$ *equation yields* $sX_1(s) - x_1(0) = X_2(s)$ *or* $sX_1(s) = X_2(s)$*. The transform of the* $\dot{x}_2$ *equation yields* $sX_2(s) - x_2(0) = -X_1(s)$ *or* $sX_2(s) - 1 = -X_1(s)$*.*

(b) *We have* $X_2(s) = sX_1(s)$*, so substitute this into* $sX_2(s) - 1 = -X_1(s)$ *to obtain* $s^2X_1(s) - 1 = -X_1(s)$*, so that* $X_1(s) = 1/(s^2+1)$*. Then* $X_2(s) = s/(s^2+1)$*.*

(c) *Straight out of the Laplace transform table we find* $x_1(t) = \sin(t)$ *and* $x_2(t) = \cos(t)$*, which are easily seen to satisfy the original ODEs.*

(d) *Transforming each ODE and filling in the initial data yields*

$$sX_1(s)+8 = 2X_1(s)+3X_2(s)+12/(s+1) \qquad \text{and} \qquad sX_2(s)-2 = X_1(s)+4X_2(s).$$

*Solve for* $X_1(s)$ *and* $X_2(s)$ *to find*

$$X_1(s) = \frac{-8s+2}{s^2-1} \qquad \text{and} \qquad X_2(s) = \frac{2s}{s^2-1}.$$

*Inverse transforming (after a partial fraction expansion) shows that* $x_1(t) = -3e^t - 5e^{-t}$ *and* $x_2(t) = e^t + e^{-t}$*.*

**Exercise Solution 6.1.22.**

(a) *Tank 1 has two inflow pipes, each five liters per minute, and one outflow of ten liters per minute, so a net rate of volume change of zero. Tank 2 has two outflow pipes, each five liters per minute, and one inflow of ten liters per minute, so also a net rate of volume change of zero.*

(b) *Standard conservation modeling: the inflow at the upper left on tank 1 has five liters per minute times* 0.1 *kg per liter, so* 0.5 *kg per minute. The amount of salt in tank 1 at time* $t$ *is* $x_1(t)$ *and is divided evenly amount* 1200 *liters, so each liter has* $x_1/1200$ *kg of salt and* 10 *liters leave per minute, hence the* $-x_1/120$ *term in the* $\dot{x}_1$ *equation, which is* $x_1/120$ *in the* $\dot{x}_2$ *equation. Similarly tank 2 has* $x_2/500$ *kg of salt in each liter, and* 5 *liters enter tank* 1 *per minute, hence the* $x_2/100$ *term in the* $\dot{x}_1$ *equation, and a corresponding negative in the* $\dot{x}_2$ *equation, double since another* 5 *liters leave on the lower right.*

(c) *From above*

$$\dot{x}_1 = \frac{1}{2} - \frac{x_1}{120} + \frac{x_2}{100}.$$

(d) *From above*

$$\dot{x}_2 = \frac{x_1}{120} - \frac{x_2}{50}.$$

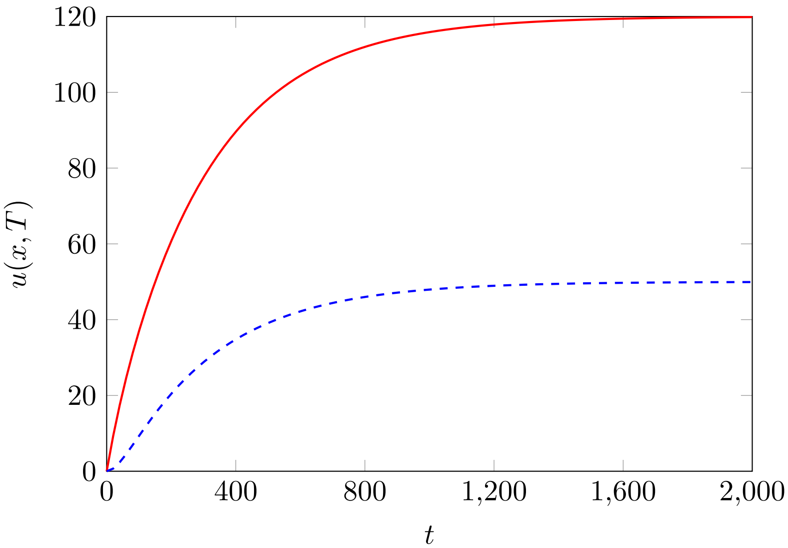



Figure 6.79: Plot of $x_1(t)$ (solid, red) and $x_2(t)$ (dashed, blue).

(e) *Verifying that the given functions satisfy the ODEs is routine. A graph of* $x_1(t)$ *and* $x_2(t)$ *is shown in Figure 6.79. In each case the limiting concentration is* $120/1200 = 50/500 = 0.1$ *kg per liter, the same as the incoming fluid.*

## Section 6.2

**Exercise Solution 6.2.1.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} 7 & -4 \\ 20 & -11 \end{bmatrix}$$

*with* $\lambda_1 = -1, \lambda_2 = -3$*, and*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

*A general solution is*

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

*The initial data is obtained with* $c_1 = -1, c_2 = 2$*.*

**Exercise Solution 6.2.2.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 6 & -5 \end{bmatrix}$$

*with* $\lambda_1 = -2, \lambda_2 = -3$*, and*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

*A general solution is*

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

*The initial data is obtained with* $c_1 = 1, c_2 = 1$*.*

**Exercise Solution 6.2.3.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$$

*with* $\lambda_1 = -1 + i$*,* $\lambda_2 = -1 - i$*, and*

$$\mathbf{v}_1 = \begin{bmatrix} 2 + i \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 - i \\ 5 \end{bmatrix}.$$

*A complex-valued general solution is*

$$\mathbf{x}(t) = c_1 e^{(-1+i)t} \begin{bmatrix} 2+i \\ 5 \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} 2-i \\ 5 \end{bmatrix}.$$

*A real-valued general solution is*

$$\mathbf{x}(t) = d_1 e^{-t} \begin{bmatrix} 2\cos(t) - \sin(t) \\ 5\cos(t) \end{bmatrix} + d_2 e^{-t} \begin{bmatrix} 2\sin(t) + \cos(t) \\ 5\sin(t) \end{bmatrix}.$$

*The initial data is obtained with* $d_1 = 2/5, d_2 = -4/5$.

**Exercise Solution 6.2.4.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$$

*with* $\lambda_1 = -2+3i$, $\lambda_2 = -2-3i$, *and*

$$\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

*A complex-valued general solution is*

$$\mathbf{x}(t) = c_1 e^{(-2+3i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 e^{(-2-3i)t} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}.$$

*A real-valued general solution is*

$$\mathbf{x}(t) = d_1 e^{-2t} \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix} + d_2 e^{-2t} \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix}.$$

*The initial data is obtained with* $d_1 = -2, d_2 = 2$.

**Exercise Solution 6.2.5.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} -6 & 9 & -4 \\ -6 & 11 & -6 \\ -10 & 21 & -12 \end{bmatrix}$$

*with* $\lambda_1 = -4, \lambda_2 = -2$, $\lambda_3 = -1$, *and*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

*A general solution is*

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

*The initial data is obtained with* $c_1 = 1, c_2 = 0, c_3 = -2$.

**Exercise Solution 6.2.6.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} -7 & 2 & 6 \\ -6 & -1 & 4 \\ -9 & 2 & 8 \end{bmatrix}$$

*with* $\lambda_1 = 2, \lambda_2 = -1 + 2i,\ \lambda_3 = -1 - 2i,$ *and*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}.$$

*A complex-valued general solution is*

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + c_2 e^{(-1+2i)t} \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} + c_3 e^{(-1-2i)t} \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}.$$

*A real-valued general solution is*

$$\mathbf{x}(t) = d_1 e^{2t} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + d_2 e^{-t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \\ \cos(2t) \end{bmatrix} + d_3 e^{-t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \\ \sin(2t) \end{bmatrix}.$$

*The initial data is obtained with* $d_1 = -2, d_2 = 2, d_3 = 2$.

**Exercise Solution 6.2.7.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} -4 & -1 & 2 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -2 \end{bmatrix}$$

*with* $\lambda_1 = -1, \lambda_2 = -1,\ \lambda_3 = -3,\ \lambda_4 = -2,$ *and*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 3 \end{bmatrix}.$$

*A general solution is*

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix}1\\1\\2\\0\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} + c_3 e^{-3t}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} + c_4 e^{-2t}\begin{bmatrix}-1\\-1\\0\\3\end{bmatrix}.$$

*The initial data is obtained with* $c_1 = 2, c_2 = 1, c_3 = 0, c_4 = 0.$

**Exercise Solution 6.2.8.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix}3 & -1\\4 & -1\end{bmatrix}$$

*with double eigenvalue* $\lambda = 1$, *and eigenvector*

$$\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}.$$

*By solving* $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}$ *we obtain* $\mathbf{v}_1 = \langle 0, -1\rangle$ *(or more generally,* $\mathbf{v}_1 = \langle t_1, 2t_1 - 1\rangle$ *for a free variable* $t_1$*). We can construct a general solution*

$$\mathbf{x}(t) = c_1 e^{t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{t}\begin{bmatrix}t\\2t-1\end{bmatrix}.$$

*The initial data is obtained with* $c_1 = 1, c_2 = -1.$

**Exercise Solution 6.2.9.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix}7 & -3\\12 & -5\end{bmatrix}$$

*with double eigenvalue* $\lambda = 1$, *and eigenvector*

$$\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}.$$

*By solving* $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}$ *we obtain* $\mathbf{v}_1 = \langle 0, -1/3\rangle$ *(or more generally,* $\mathbf{v}_1 = \langle t_1, 2t_1 - 1/3\rangle$ *for a free variable* $t_1$*).. We can construct a general solution*

$$\mathbf{x}(t) = c_1 e^{t}\begin{bmatrix}1\\2\end{bmatrix} + c_2 e^{t}\begin{bmatrix}t\\2t-1/3\end{bmatrix}.$$

*The initial data is obtained with* $c_1 = 1, c_2 = 3.$

**Exercise Solution 6.2.10.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} -10 & -8 \\ 8 & 6 \end{bmatrix}$$

*with double eigenvalue* $\lambda = -2$*, and eigenvector*

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

*By solving* $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}$ *we obtain* $\mathbf{v}_1 = \langle 1/8, 0\rangle$ *(or more generally,* $\mathbf{v}_1 = \langle 1/8 - t_1, t_1\rangle$ *for a free variable* $t_1$*). We can construct a general solution*

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -t + 1/8 \\ t \end{bmatrix}.$$

*The initial data is obtained with* $c_1 = 0, c_2 = 16$*.*

**Exercise Solution 6.2.11.** *Matrix is*

$$\mathbf{A} = \begin{bmatrix} 6 & 5 & 4 \\ -2 & -1 & -1 \\ -6 & -6 & -5 \end{bmatrix}$$

*with double eigenvalue* $\lambda_1 = \lambda_2 = \lambda = 1$ *and another eigenvalue* $\lambda_3 = -2$*. The only eigenvector for* $\lambda$ *is*

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

*and the eigenvector for* $\lambda_3$ *is*

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

*By solving* $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}$ *we obtain* $\mathbf{v}_1 = \langle -1, 0, 1\rangle$ *(more generally,* $\mathbf{v}_1 = \langle -1 - t_1, t_1, 1\rangle$ *for a free variable* $t_1$*). We can construct a general solution*

$$\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -t - 1 \\ t \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

*The initial data is obtained with* $c_1 = 2, c_2 = -5, c_3 = 2$*.*

**Exercise Solution 6.2.12.**

(a) *The characteristic equation is* $r^2+3r+2=0$*, roots* $r_1=-1, r_2=-2$*. A general solution is*

$$x(t)=c_1e^{-t}+c_2e^{-2t}.$$

(b) *The equivalent system is* $\dot{x}_1=x_2$ *and* $\dot{x}_2=-2x_1-3x_2$*. The relevant matrix is*

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$$

(c) *The eigenvalues are* $\lambda_1=-1$ *and* $\lambda_2=-2$*, with eigenvectors*

$$\mathbf{v}_1=\begin{bmatrix}1\\ -1\end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}_2=\begin{bmatrix}1\\ -2\end{bmatrix}.$$

*The general solution is then*

$$\mathbf{x}(t)=c_1e{-t}\begin{bmatrix}1\\ -1\end{bmatrix}+c_2e^{-2t}\begin{bmatrix}1\\ -2\end{bmatrix}.$$

*Then* $x_1(t)$ *is of precisely the same form as* $x(t)$ *in part (a).*

(d) *The equivalent system is* $\dot{x}_1=x_2$ *and* $\dot{x}_2=-kx_1/m-cx_2/m$*. The relevant matrix is*

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -k/m & -c/m\end{bmatrix}$$

*The eigenvalues are* $\lambda_1=\frac{-c+\sqrt{c^2-4mk}}{2m}$ *and* $\lambda_2=\frac{-c-\sqrt{c^2-4mk}}{2m}$*. These are precisely the roots of the characteristic equation* $mr^2+cr+k=0$*. The eigenvectors have the asserted form, namely*

$$\mathbf{v}_1=\begin{bmatrix}1\\ \lambda_1\end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}_2=\begin{bmatrix}1\\ \lambda_2\end{bmatrix}.$$

*Then general system has a general solution*

$$\mathbf{x}(t)=c_1e^{\lambda_1 t}\begin{bmatrix}1\\ \lambda_1\end{bmatrix}+c_2e^{\lambda_2 t}\begin{bmatrix}1\\ \lambda_2\end{bmatrix}.$$

*Since* $r_1=\lambda_1$ *and* $r_2=\lambda_2$*,* $x_1(t)$ *is of exactly the same form as* $x(t)=c_1e^{r_1t}+c_2e^{r_2t}$*.*

**Exercise Solution 6.2.13.**

(a) *The matrix is*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b) *There is a double eigenvalue* $\lambda_1 = \lambda_2 = \lambda = 0$ *and another eigenvalue* $\lambda_3 = -1$*. The only eigenvector for* $\lambda$ *is*

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

*and the eigenvector for* $\lambda_3$ *is*

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) *By solving* $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{v}$ *we obtain* $\mathbf{v}_1 = \langle 1, 0, 0\rangle$*, or more generally,* $\mathbf{v}_1 = \langle 1, 0, t_1\rangle$ *for a free variable* $t_1$*. We can construct a general solution*

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

*When* $c_2 = c_3 = 0$ *the solutions are constant, but* $c_2 \neq 0$ *yields solutions that grow without bound. If* $c_1 = c_2 = 0$ *we obtain solutions that decay to zero.*

**Exercise Solution 6.2.14.**

(a) *The matrix is*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1 + k_2)/m_1 & -c_1/m_1 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -k_2/m_2 & -c_2/m_2 \end{bmatrix}.$$

*(b) The matrix becomes*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -1/2 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -4 & -1/2 \end{bmatrix}.$$

*The eigenvalues are complex conjugates, $-0.25 \pm 3.01i$ and $-0.25 \pm 0.90i$, approximately. This indicates oscillatory behavior that decays to zero, as one might expect from a lightly damped system.*

*(c) The matrix becomes*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -10 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -4 & -10 \end{bmatrix}.$$

*The eigenvalues are real and equal to $-8.98$, $-1.02$, $-9.91$, and $-0.09$, approximately. This indicates decay to zero without oscillatory behavior—just what one would expect from a heavily damped system.*

**Exercise Solution 6.2.15.**

*(a) The system is*

$$\begin{aligned} \dot{x}_1 &= -\frac{9x_1}{100} + \frac{x_2}{40} \\ \dot{x}_2 &= \frac{9x_1}{100} - \frac{9x_2}{200} \end{aligned}$$

*with $x_1(0) = 10$ and $x_2(0) = 5$. The relevant matrix is*

$$\mathbf{A} = \begin{bmatrix} -9/100 & 1/40 \\ 9/100 & -9/200 \end{bmatrix}.$$

*(b) Laplace transforming yields*

$$\begin{aligned} sX_1(s) - 10 &= -\frac{9X_1(s)}{100} + \frac{X_2(s)}{40} \\ sX_2(s) - 5 &= \frac{9X_1(s)}{100} - \frac{9X_2(s)}{200} \end{aligned}$$

*The solution is*

$$X_1(s) = \frac{10s + 23/40}{s^2 + 27s/200 + 9/5000}$$
$$X_2(s) = \frac{5s + 27/20}{s^2 + 27s/200 + 9/5000}.$$

*The denominators factor as* $(s+3/200)(s+3/25)$. *Inverse transforming yields*

$$x_1(t) = \frac{85}{21}e^{-3t/200} + \frac{125}{21}e^{-3t/25}$$
$$x_2(t) = \frac{85}{7}e^{-3t/200} - \frac{50}{7}e^{-3t/25}$$

(c) *The eigenvalues of* $\mathbf{A}$ *are* $\lambda_1 = -3/200$ *and* $\lambda_2 = -3/25$ *with eigenvectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

*A general solution is thus*

$$\mathbf{x}(t) = c_1 e^{-3t/200} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-3t/25} \begin{bmatrix} -5 \\ 6 \end{bmatrix}.$$

*The initial data leads to* $c_1 - 5c_2 = 10$ *and* $3c_1 + 6c_2 = 5$ *with solution* $c_1 = 85/21$ *and* $c_2 = -25/21$. *In the general solution this gives exactly the same result as part (b).*

**Exercise Solution 6.2.16.** *First, if* $\lambda_1 = a + bi$ *and* $\mathbf{v}_1 = \mathbf{v}_r + i\mathbf{v}_i$ *then*

$$\begin{aligned} e^{\lambda_1 t}\mathbf{v}_1 &= e^{at}(\cos(bt) + i\sin(bt))(\mathbf{v}_r + i\mathbf{v}_i) \\ &= e^{at}(\cos(bt)\mathbf{v}_r - \sin(bt)\mathbf{v}_i) + ie^{at}(\sin(bt)\mathbf{v}_r + \cos(bt)\mathbf{v}_i). \end{aligned}$$

*Then*

$$\begin{aligned} \mathbf{y}_r(t) &= e^{at}(\cos(bt)\mathbf{v}_r - \sin(bt)\mathbf{v}_i) \\ \mathbf{y}_i(t) &= e^{at}(\sin(bt)\mathbf{v}_r + \cos(bt)\mathbf{v}_i). \end{aligned}$$

*Then*

$$\begin{aligned} \mathbf{x}(t) &= d_1\mathbf{y}_r(t + d_2\mathbf{y}_i(t) \\ &= d_1 e^{at}(\cos(bt)\mathbf{v}_r - \sin(bt)\mathbf{v}_i) + d_2 e^{at}(\sin(bt)\mathbf{v}_r + \cos(bt)\mathbf{v}_i). \end{aligned}$$

# Section 6.3

**Exercise Solution 6.3.1.** *Laplace transforming and solving for* $X_1(s), X_2(s)$ *yields*

$$\begin{aligned} X_1(s) &= \frac{3s+1}{s^2+4s+3} \\ X_2(s) &= \frac{8s+4}{s^2+4s+3}. \end{aligned}$$

*An inverse transform shows that* $x_1(t) = 4e^{-3t} - e^{-t}$ *and* $x_2(t) = 10e^{-3t} - 2e^{-t}$.

**Exercise Solution 6.3.2.** *Laplace transforming and solving for* $X_1(s), X_2(s)$ *yields*

$$\begin{aligned} X_1(s) &= \frac{2s^2+17s+25}{s^3+6s^2+11s+6} \\ X_2(s) &= \frac{3s^2+32s+49}{s^3+6s^2+11s+6}. \end{aligned}$$

*The denominators factor as* $(s+1)(s+2)(s+3)$*. An inverse transform shows that* $x_1(t) = 5e^{-t} - 4e^{-3t} + e^{-2t}$ *and* $x_2(t) = 10e^{-t} - 10e^{-3t} + 3e^{-2t}$.

**Exercise Solution 6.3.3.** *Laplace transforming and solving for* $X_1(s), X_2(s)$ *yields*

$$\begin{aligned} X_1(s) &= \frac{s^2-s-6}{s(s+1)(s+3)} \\ X_2(s) &= \frac{2(s^2-3s-9)}{s(s+1)(s+3)}. \end{aligned}$$

*An inverse transform shows that* $x_1(t) = -2 + 2e^{-t} + e^{-3t}$ *and* $x_2(t) = -6 + 5e^{-t} + 3e^{-3t}$.

**Exercise Solution 6.3.4.** *Laplace transforming and solving for* $X_1(s), X_2(s)$ *yields*

$$\begin{aligned} X_1(s) &= \frac{s^2+s+1}{s^2(s+1)} \\ X_2(s) &= \frac{2s^2-s-1}{s^2(s+1)}. \end{aligned}$$

*An inverse transform shows that* $x_1(t) = t + e^{-t}$ *and* $x_2(t) = -t + 2e^{-t}$.

**Exercise Solution 6.3.5.** *Laplace transforming and solving for* $X_1(s), X_2(s)$ *yields*

$$X_1(s) = \frac{s(s-3)}{(s+1)(s^2+1)}$$
$$X_2(s) = \frac{s(3s-5)}{(s+1)(s^2+1)}.$$

*An inverse transform shows that* $x_1(t) = 2e^{-t} - \cos(t) - 2\sin(t)$ *and* $x_2(t) = 4e^{-t} - \cos(t) - 4\sin(t)$.

**Exercise Solution 6.3.6.** *Laplace transforming and solving for* $X_1(s), X_2(s)$ *yields*

$$X_1(s) = -\frac{s(s-3)}{(s^2+1)(s+1)(s+2)}$$
$$X_2(s) = \frac{3s^3+3s^2+5s-3}{(s^2+1)(s+1)(s+2)}.$$

*An inverse transform shows that* $x_1(t) = 2e^{-2t} - 2e^{-t} + \sin(t)$ *and* $x_2(t) = 5e^{-2t} - 4e^{-t} + 2\cos(t)$.

**Exercise Solution 6.3.7.** *Laplace transforming and solving for* $X_1(s), X_2(s), X_3(s)$ *yields*

$$X_1(s) = \frac{s^3+2s^2+s+6}{s(s+1)(s+2)(s+3)}$$
$$X_2(s) = \frac{s+4}{(s+2)(s+3)}$$
$$X_3(s) = -\frac{s^2+10s+3}{s(s+1)(s+3)}.$$

*An inverse transform shows that* $x_1(t) = 1 + e^{-3t} + 2e^{-2t} - 3e^{-t}$, $x_2(t) = 2e^{-2t} - e^{-3t}$, *and* $x_3(t) = -1 - 3e^{-t} + 3e^{-3t}$.

**Exercise Solution 6.3.8.** *Laplace transforming and solving for* $X_1(s), X_2(s), X_3(s)$ *yields*

$$X_1(s) = \frac{s^4-2s^3-9s^2-2s-20}{(s+1)(s+2)(s+3)(s^2+1)}$$
$$X_2(s) = \frac{2(s+4)}{(s+2)(s+3)}$$
$$X_3(s) = \frac{s^3-8s^2+3s-12}{(s+1)(s+3)(s^2+1)}.$$

*An inverse transform shows that* $x_1(t) = 2e^{-3t} + 4e^{-2t} - 6e^{-t} + \cos(t)$, $x_2(t) = 4e^{-2t} - 2e^{-3t}$, *and* $x_3(t) = 6e^{-3t} - 6e^{-t} + \cos(t)$.

**Exercise Solution 6.3.9.**

$$\mathbf{A} = \begin{bmatrix} 7 & -4 \\ 20 & -11 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = e^{-2t}\begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

*A guess of the form* $\mathbf{x}_p(t) = e^{-2t}\mathbf{v}$ *with* $\mathbf{f}(t) = e^{-2t}\mathbf{w}$ *where* $\mathbf{w} = \langle 3, 7\rangle$ *leads to* $(\mathbf{A} + 2\mathbf{I})\mathbf{v} = -\mathbf{w}$ *and then* $\mathbf{v} = (\mathbf{A} + 2\mathbf{I})^{-1}\mathbf{w} = \langle 1, 3\rangle$. *So*

$$\mathbf{x}_p(t) = e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

*A homogeneous general solution is*

$$\mathbf{x}_h(t) = c_1 e^{-3t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

*and the general solution to the nonhomogeneous system is*

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

*The initial data yields* $c_1 = -2$, $c_2 = 5$.

**Exercise Solution 6.3.10.**

$$\mathbf{A} = \begin{bmatrix} -6 & 2 \\ -15 & 5 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = e^{-3t}\begin{bmatrix} -3 \\ -9 \end{bmatrix}.$$

*A guess of the form* $\mathbf{x}_p(t) = e^{-3t}\mathbf{v}$ *with* $\mathbf{f}(t) = e^{-3t}\mathbf{w}$ *where* $\mathbf{w} = \langle -3, -9\rangle$ *leads to* $(\mathbf{A} + 3\mathbf{I})\mathbf{v} = -\mathbf{w}$ *and then* $\mathbf{v} = (\mathbf{A} + 3\mathbf{I})^{-1}\mathbf{w} = \langle 1, 3\rangle$. *So*

$$\mathbf{x}_p(t) = e^{-3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

*A homogeneous general solution is*

$$\mathbf{x}_h(t) = c_1 e^{-t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

*and the general solution to the nonhomogeneous system is*

$$\mathbf{x}(t) = c_1 e^{-t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

*The initial data yields* $c_1 = 1$, $c_2 = -2$.

**Exercise Solution 6.3.11.**

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}(t) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

*A guess of the form* $\mathbf{x}_p(t) = \mathbf{v}$ *with* $\mathbf{f}(t) = \mathbf{w}$ *where* $\mathbf{w} = \langle 2, -2 \rangle$ *leads to* $\mathbf{A}\mathbf{v} = -\mathbf{w}$ *and then* $\mathbf{v} = (\mathbf{A})^{-1}\mathbf{w} = \langle 8, 13 \rangle$*. So*

$$\mathbf{x}_p(t) = \begin{bmatrix} 8 \\ 13 \end{bmatrix}.$$

*A homogeneous general solution is*

$$\mathbf{x}_h(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

*and the general solution to the nonhomogeneous system is*

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 13 \end{bmatrix}.$$

*The initial data yields* $c_1 = 3$, $c_2 = -13$.

**Exercise Solution 6.3.12.**

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}(t) = \cos(t) \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} -1 \\ -3 \end{bmatrix}.$$

*Follow the hints: take a guess of the form* $\mathbf{x}_p(t) = \cos(t)\mathbf{v}_1 + \sin(t)\mathbf{v}_2$ *with* $\mathbf{f}(t) = \cos(t)\mathbf{w}_1 + \sin(t)\mathbf{w}_2$ *where* $\mathbf{w}_1 = \langle -1, 0 \rangle$ *and* $\mathbf{w}_2 = \langle -1, -3 \rangle$*. Then solving the linear system* $(\mathbf{A}^2 + \mathbf{I})\mathbf{v}_1 = -(\mathbf{A}\mathbf{w}_1 + \mathbf{w}_2)$ *yields* $\mathbf{v}_1 = \langle -1, -1 \rangle$ *and then* $\mathbf{v}_2 = \mathbf{A}\mathbf{v}_1 + \mathbf{w}_1 = \langle -2, -4 \rangle$*. A particular solution is*

$$\mathbf{x}_p(t) = \cos(t) \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \sin(t) \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

*A homogeneous general solution is*

$$\mathbf{x}_h(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

*and the general solution to the nonhomogeneous system is*

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \cos(t) \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \sin(t) \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

*The initial data yields* $c_1 = 0$, $c_2 = 2$.

**Exercise Solution 6.3.13.**

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = \cos(t) \begin{bmatrix} 5 \\ 12 \end{bmatrix} + \sin(t) \begin{bmatrix} -3 \\ -12 \end{bmatrix}.$$

*Again follow the hints: take a guess of the form* $\mathbf{x}_p(t) = \cos(t)\mathbf{v}_1 + \sin(t)\mathbf{v}_2$ *with* $\mathbf{f}(t) = \cos(t)\mathbf{w}_1 + \sin(t)\mathbf{w}_2$ *where* $\mathbf{w}_1 = \langle 5, 12 \rangle$ *and* $\mathbf{w}_2 = \langle -3, -12 \rangle$. *Then solving the linear system* $(\mathbf{A}^2 + \mathbf{I})\mathbf{v}_1 = -(\mathbf{A}\mathbf{w}_1 + \mathbf{w}_2)$ *yields* $\mathbf{v}_1 = \langle 0, 2 \rangle$ *and then* $\mathbf{v}_2 = \mathbf{A}\mathbf{v}_1 + \mathbf{w}_1 = \langle 1, 0 \rangle$. *A particular solution is*

$$\mathbf{x}_p(t) = \cos(t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

*A homogeneous general solution is*

$$\mathbf{x}_h(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

*and the general solution to the nonhomogeneous system is*

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \cos(t) \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

*The initial data yields* $c_1 = 1$, $c_2 = -2$.

**Exercise Solution 6.3.14.**

$$\mathbf{A} = \begin{bmatrix} -6 & 9 & -4 \\ -6 & 11 & -6 \\ -10 & 21 & -12 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = \begin{bmatrix} -4 \\ -4 \\ -8 \end{bmatrix}.$$

*Guess a particular solution* $\mathbf{x}_p(t) = \mathbf{v}$, *constant. Then* $\mathbf{0} = \mathbf{A}\mathbf{v} + \mathbf{w}$ *where* $\mathbf{w} = \langle -4, -4, -8 \rangle$ *and so solve for* $\mathbf{v} = \langle 1, 2, 3 \rangle$. *So*

$$\mathbf{x}_p(t) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

*A homogeneous general solution is*

$$\mathbf{x}_h(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-4t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

*and the general solution to the nonhomogeneous system is*

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-4t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

*The initial data yields* $c_1 = 1$, $c_2 = 2$, $c_3 = -1$.

**Exercise Solution 6.3.15.**

$$\mathbf{A} = \begin{bmatrix} -6 & 9 & -4 \\ -6 & 11 & -6 \\ -10 & 21 & -12 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = -e^t \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}.$$

*Guess a particular solution* $\mathbf{x}_p(t) = e^t\mathbf{v}$. *In the ODE this leads to* $e^t\mathbf{v} = e^t\mathbf{A}\mathbf{v} + e^t\mathbf{w}$ *with* $\mathbf{w} = -\langle 4, 4, 8\rangle$, *or* $\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{w}$ *after dividing by* $e^t$. *Write this as* $(\mathbf{A} - \mathbf{I})\mathbf{v} = -\mathbf{w}$ *so* $\mathbf{v} = -(\mathbf{A} - \mathbf{I})^{-1}\mathbf{w} = \langle -2/15, 2/5, 2/15\rangle$. *A particular solution is then*

$$\mathbf{x}_p(t) = e^t \begin{bmatrix} -2/15 \\ 2/5 \\ 2/15 \end{bmatrix}.$$

*A homogeneous general solution is*

$$\mathbf{x}_h(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-4t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

*and the general solution to the nonhomogeneous system is*

$$\mathbf{x}(t) = e^t \begin{bmatrix} -2/15 \\ 2/5 \\ 2/15 \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-4t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

*The initial data yields* $c_1 = 3$, $c_2 = -1/3$, $c_3 = -6/5$.

**Exercise Solution 6.3.16.**

(a) *The system is*

$$\dot{x}_1 = -\frac{9x_1}{100} + \frac{x_2}{40} + 0.4$$
$$\dot{x}_2 = \frac{9x_1}{100} - \frac{9x_2}{200}$$

*with* $x_1(0) = 10$ *and* $x_2(0) = 5$. *The relevant matrix and right side function is*

$$\mathbf{A} = \begin{bmatrix} -9/100 & 1/40 \\ 9/100 & -9/200 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}.$$

(b) *First, guess a particular solution* $\mathbf{x}_p = \mathbf{v}$, *a constant vector. This leads to* $\mathbf{Av} + \mathbf{f} = \mathbf{0}$ ($\mathbf{f}(t)$ *is constant) with solution* $\mathbf{v} = \langle 10, 20 \rangle$. *Thus*

$$\mathbf{x}_p(t) = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

*The eigenvalues of* $\mathbf{A}$ *are* $\lambda_1 = -3/200$ *and* $\lambda_2 = -3/25$ *with eigenvectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

*A general solution to* $\mathbf{x} = \mathbf{Ax}$ *is thus*

$$\mathbf{x}_h(t) = c_1 e^{-3t/200} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-3t/25} \begin{bmatrix} -5 \\ 6 \end{bmatrix}.$$

*The general solution to the nonhomogeneous problem* $\mathbf{x} = \mathbf{Ax} + \mathbf{f}$ *is then*

$$\mathbf{x}(t) = \begin{bmatrix} 10 \\ 20 \end{bmatrix} + c_1 e^{-3t/200} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-3t/25} \begin{bmatrix} -5 \\ 6 \end{bmatrix}.$$

*The initial data leads to* $c_1 = -25/7$ *and* $c_2 = -5/7$.

(c) *Laplace transforming both sides and using the initial data produces*

$$\begin{aligned} sX_1(s) - 10 &= -\frac{9}{100} X_1(s) + \frac{1}{40} X_2(s) + 0.4/s \\ sX_2(s) - 5 &= \frac{9}{100} X_1(s) - \frac{9}{200} X_2(s). \end{aligned}$$

*Solve to find*

$$\begin{aligned} X_1(s) &= \frac{50000s^2 + 4875s + 90}{s(5000s^2 + 675s + 9)} \\ X_2(s) &= \frac{25000s^2 + 6750s + 180}{s(5000s^2 + 675s + 9)}. \end{aligned}$$

*(The denominators factor as $s(25s+3)(200s+3)$.) An inverse transform yields*

$$x_1(t) = 10 - \frac{25}{7}e^{-3t/200} + \frac{25}{7}e^{-3t/25}$$
$$x_2(t) = 20 - \frac{75}{7}e^{-3t/200} - \frac{30}{7}e^{-3t/25}.$$

*This is the same as part (b).*

**Exercise Solution 6.3.17.**

(a) *If we emulate the computation of that example we find* $\mathbf{a} = \langle 0, 1/7, 0, -5/7\rangle$ *and* $\mathbf{b} = \langle 1/14, 0, -5/14, 0\rangle$. *Then*

$$\mathbf{w}_p(t) = \cos(2t)\begin{bmatrix} 0 \\ 1/7 \\ 0 \\ -5/7 \end{bmatrix} + \sin(2t)\begin{bmatrix} 1/14 \\ 0 \\ -5/14 \\ 0 \end{bmatrix}.$$

(b) *In this case the matrix* $\mathbf{A}^2 + \omega^2\mathbf{I}$ *is singular and there is no solution of the form desired. This is one of the resonant frequencies of the system.*

(c) *The other frequency is* $\omega = 1/\sqrt{2}$, *the other resonant frequency of the system. Again,* $\mathbf{A}^2 + \omega^2\mathbf{I}$ *is singular.*

**Exercise Solution 6.3.18.**

(a) *Starting on the minus side of the voltage source and moving clockwise yields* $V(t) - R_1 I_1 - q/C = 0$.

(b) *Start in the minus side of the capacitor move clockwise around the right side loop to conclude that* $q/C - L\dot{I}_2 - R_2 I_2 = 0$.

(c) *We have current* $I_1$ *entering* $N$, $I$ *and* $I_2$ *exiting, so* $I_1 - I - I_2 = 0$ *or* $I = I_1 - I_2$.

(d) *Use* $\dot{q} = I = I_1 - I_2$ *from part (c) along with* $I_1 = V(t)/R - q/(R_1 C)$ *from part (a) to obtain*

$$\dot{q} = \frac{V(t)}{R_1} - \frac{1}{R_1 C}q - I_2$$

*A rearrangement of the equation in part (b) yields*

$$\dot{I}_2 = \frac{1}{LC}q - \frac{R_2}{L}I_2.$$

*(e) Here*

$$\mathbf{A} = \begin{bmatrix} -1/(R_1C) & -1 \\ 1/(LC) & -R_2/L \end{bmatrix} \qquad \textit{and} \qquad \mathbf{f}(t) = \begin{bmatrix} V(t)/R_1 \\ 0 \end{bmatrix}.$$

*(f) The charge $q(t)$ and current $I_2(t)$ are shown in Figure 6.80.*

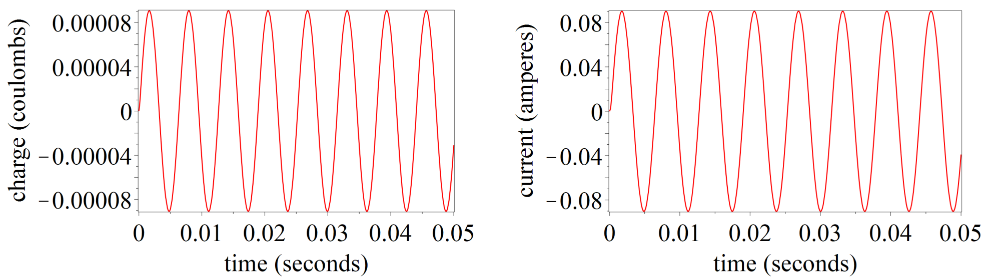



Figure 6.80: Charge $q(t)$ on capacitor (left) and current $I_2(t)$ (right).

# Section 6.4

**Exercise Solution 6.4.1.** *The eigenvalues and eigenvectors lead to*

$$\mathbf{D} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{P} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

*Then*

$$e^{t\mathbf{A}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1} = \begin{bmatrix} -3e^{-2t} + 4e^{-t} & 6e^{-2t} - 6e^{-t} \\ -2e^{-2t} + 2e^{-t} & 4e^{-2t} - 3e^{-t} \end{bmatrix}.$$

*For Putzer's algorithm (with $\lambda_1 = -2, \lambda_2 = -1$) we find*

$$\begin{aligned} \mathbf{P}_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{P}_1 &= \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \\ r_1(t) &= e^{-2t} \\ \mathbf{P}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ r_2(t) &= e^{-t} - e^{-2t}. \end{aligned}$$

*Putzer's algorithm yields the same result as diagonalization.*

*The solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \langle 1, 2 \rangle$ is*

$$\mathbf{x}(t) = \begin{bmatrix} -8\,\mathrm{e}^{-t} + 9\,\mathrm{e}^{-2\,t} \\ -4\,\mathrm{e}^{-t} + 6\,\mathrm{e}^{-2\,t} \end{bmatrix}.$$

**Exercise Solution 6.4.2.** *The eigenvalues and eigenvectors lead to*

$$\mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

*Then*

$$e^{t\mathbf{A}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{bmatrix}.$$

*For Putzer's algorithm (with $\lambda_1 = 4, \lambda_2 = -2$) we find*

$$\begin{aligned}
\mathbf{P}_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{P}_1 &= \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \\
r_1(t) &= e^{4t} \\
\mathbf{P}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
r_2(t) &= e^{4t}/6 - e^{-2t}/6.
\end{aligned}$$

*Putzer's algorithm yields the same result as diagonalization.*

*The solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \langle 4, 2 \rangle$ is*

$$\mathbf{x}(t) = \begin{bmatrix} 3\,\mathrm{e}^{4\,t} + \mathrm{e}^{-2\,t} \\ 3\,\mathrm{e}^{4\,t} - \mathrm{e}^{-2\,t} \end{bmatrix}.$$

**Exercise Solution 6.4.3.** *The eigenvalues and eigenvectors lead to*

$$\mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

*Then*

$$e^{t\mathbf{A}} = \mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1} = \begin{bmatrix} -2\,\mathrm{e}^{2\,t} + 3\,\mathrm{e}^{-t} & \mathrm{e}^{2\,t} - \mathrm{e}^{-t} \\ -6\,\mathrm{e}^{2\,t} + 6\,\mathrm{e}^{-t} & 3\,\mathrm{e}^{2\,t} - 2\,\mathrm{e}^{-t} \end{bmatrix}.$$

*For Putzer's algorithm (with $\lambda_1 = -1, \lambda_2 = 2$) we find*

$$\begin{aligned}
\mathbf{P}_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{P}_1 &= \begin{bmatrix} -6 & 3 \\ -18 & 9 \end{bmatrix} \\
r_1(t) &= e^{-t} \\
\mathbf{P}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
r_2(t) &= e^{2t}/3 - e^{-t}/3.
\end{aligned}$$

*Putzer's algorithm yields the same result as diagonalization.*

*The solution to* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *with* $\mathbf{x}(0) = \langle 0, -2 \rangle$ *is*

$$\mathbf{x}(t) = \left[ \begin{array}{c} -2\,\mathrm{e}^{2\,t} + 2\,\mathrm{e}^{-t} \\ -6\,\mathrm{e}^{2\,t} + 4\,\mathrm{e}^{-t} \end{array} \right].$$

**Exercise Solution 6.4.4.** *This matrix has a double eigenvalue* $\lambda = 2$*, defective. Using Putzer's algorithm we find*

$$\begin{aligned} \mathbf{P}_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{P}_1 &= \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \\ r_1(t) &= e^{2t} \\ \mathbf{P}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ r_2(t) &= te^{2t}. \end{aligned}$$

*Then*

$$e^{t\mathbf{A}t} = r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 = \left[ \begin{array}{cc} -\mathrm{e}^{2\,t}\,(t-1) & \mathrm{e}^{2\,t}t \\ -\mathrm{e}^{2\,t}t & \mathrm{e}^{2\,t}\,(t+1) \end{array} \right].$$

*The solution to* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *with* $\mathbf{x}(0) = \langle 1, 2 \rangle$ *is*

$$\mathbf{x}(t) = \left[ \begin{array}{c} \mathrm{e}^{2\,t}\,(t+1) \\ \mathrm{e}^{2\,t}\,(t+2) \end{array} \right].$$

**Exercise Solution 6.4.5.** *This matrix has one eigenvalue of* $-2$ *and a double eigenvalue* $\lambda = -1$*, defective. With eigenvalues in the order* $-2, -1, -1$

*and Putzer's algorithm we find*

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{P}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$
$$r_1(t) = e^{-2t}$$
$$\mathbf{P}_2 = \begin{bmatrix} 2 & -2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$
$$r_2(t) = e^{-2t} + e^{-t}$$
$$\mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$r_3(t) = (t-1)e^{-t} + e^{-2t}.$$

*Putzer's algorithm yields*

$$\begin{aligned} e^{t\mathbf{A}t} &= r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 + r_3(t)\mathbf{P}_2 \\ &= \begin{bmatrix} (2\,t-1)\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2\,t} & (-2\,t+3)\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2\,t} & (2\,t-1)\,\mathrm{e}^{-t} + \mathrm{e}^{-2\,t} \\ \mathrm{e}^{-t}t & -(t-1)\,\mathrm{e}^{-t} & \mathrm{e}^{-t}t \\ (-t+2)\,\mathrm{e}^{-t} - 2\,\mathrm{e}^{-2\,t} & (t-3)\,\mathrm{e}^{-t} + 3\,\mathrm{e}^{-2\,t} & (-t+2)\,\mathrm{e}^{-t} - \mathrm{e}^{-2\,t} \end{bmatrix}. \end{aligned}$$

*The solution to* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *with* $\mathbf{x}(0) = \langle 1, 0, -1 \rangle$ *is*

$$\mathbf{x}(t) = \begin{bmatrix} \mathrm{e}^{-2\,t} \\ 0 \\ -\mathrm{e}^{-2\,t} \end{bmatrix}.$$

**Exercise Solution 6.4.6.** *The matrix is*

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}.$$

*This matrix has eigenvalues* $2, 3, -2$ *and is diagonalizable, but we'll use Putzer's algorithm, with eigenvalues in the order* $2, 3, -2$. *We find*

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P}_1 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix}$$

$$r_1(t) = e^{2t}$$

$$\mathbf{P}_2 = \begin{bmatrix} 4 & 0 & 4 \\ -4 & 0 & -4 \\ 16 & 0 & 16 \end{bmatrix}$$

$$r_2(t) = e^{3t} - e^{2t}$$

$$\mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$r_3(t) = e^{3t}/5 - e^{2t}/4 + e^{-2t}/20.$$

*Putzer's algorithm yields*

$$\begin{aligned} e^{t\mathbf{A}t} &= r_1(t)\mathbf{P}_0 + r_2(t)\mathbf{P}_1 + r_3(t)\mathbf{P}_2 \\ &= \begin{bmatrix} \mathrm{e}^{2\,t} - 1/5\,\mathrm{e}^{3\,t} + 1/5\,\mathrm{e}^{-2\,t} & -\mathrm{e}^{3\,t} + \mathrm{e}^{2\,t} & -1/5\,\mathrm{e}^{3\,t} + 1/5\,\mathrm{e}^{-2\,t} \\ 1/5\,\mathrm{e}^{3\,t} - 1/5\,\mathrm{e}^{-2\,t} & \mathrm{e}^{3\,t} & 1/5\,\mathrm{e}^{3\,t} - 1/5\,\mathrm{e}^{-2\,t} \\ 1/5\,\mathrm{e}^{3\,t} - \mathrm{e}^{2\,t} + 4/5\,\mathrm{e}^{-2\,t} & \mathrm{e}^{3\,t} - \mathrm{e}^{2\,t} & 1/5\,\mathrm{e}^{3\,t} + 4/5\,\mathrm{e}^{-2\,t} \end{bmatrix} \end{aligned}$$

*The solution to* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *with* $\mathbf{x}(0) = \langle 1, 0, -1 \rangle$ *is*

$$\mathbf{x}(t) = \begin{bmatrix} \mathrm{e}^{2\,t} \\ 0 \\ -\mathrm{e}^{2\,t} \end{bmatrix}.$$

**Exercise Solution 6.4.7.** *If we take a cue from the scalar case, we start with*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$$

*and then write this as*

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t).$$

*Multiply both sides by $e^{-t\mathbf{A}}$ to find*

$$e^{-t\mathbf{A}}(\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)) = e^{-t\mathbf{A}}\mathbf{b}(t).$$

*The left side is $\frac{d}{dt}(e^{-t\mathbf{A}}\mathbf{x}(t))$, so we can write the above as*

$$\frac{d}{dt}(e^{-t\mathbf{A}}\mathbf{x}(t)) = e^{-t\mathbf{A}}\mathbf{b}(t).$$

*Rename the variable $t$ to $z$ above and integrate both sides from $z = 0$ to $z = t$ to find*

$$e^{-t\mathbf{A}}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-z\mathbf{A}}\mathbf{b}(z)\,dz.$$

*Then we can solve for $\mathbf{x}(t)$ as*

$$\mathbf{x}(t) = e^{t\mathbf{A}} \int_0^t e^{-z\mathbf{A}}\mathbf{b}(z)\,dz + e^{t\mathbf{A}}\mathbf{x}(0).$$

*Note that $e^{t\mathbf{A}}$ multiplies $\mathbf{x}(0)$ on the left (in fact, all computations above have matrices multiplying vectors on the left, as is required).*

*For the given system we have*

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & -5 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{b}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}.$$

*We also compute*

$$e^{t\mathbf{A}} = \left[\begin{array}{cc} 2\,\mathrm{e}^{-t} - \mathrm{e}^{-3\,t} & -\mathrm{e}^{-t} + \mathrm{e}^{-3\,t} \\ 2\,\mathrm{e}^{-t} - 2\,\mathrm{e}^{-3\,t} & -\mathrm{e}^{-t} + 2\,\mathrm{e}^{-3\,t} \end{array}\right].$$

*Carrying out the computation for the solution yields*

$$\mathbf{x}(t) = \begin{bmatrix} -3e^{-t} + \dfrac{13}{9}e^{-3t} - \dfrac{2t}{3} + \dfrac{23}{9} \\ -3e^{-t} + \dfrac{26}{9}e^{-3t} - \dfrac{t}{3} + \dfrac{19}{9} \end{bmatrix}.$$

## Section 7.1

**Exercise Solution 7.1.1.** *The vectors are shown in the left panel of Figure 7.81.*

**Exercise Solution 7.1.2.** *The vectors are shown in the right panel of Figure 7.81.*

**Exercise Solution 7.1.3.** *The vectors are shown in the left panel of Figure 7.82.*

**Exercise Solution 7.1.4.** *The vectors are shown in the right panel of Figure 7.82.*

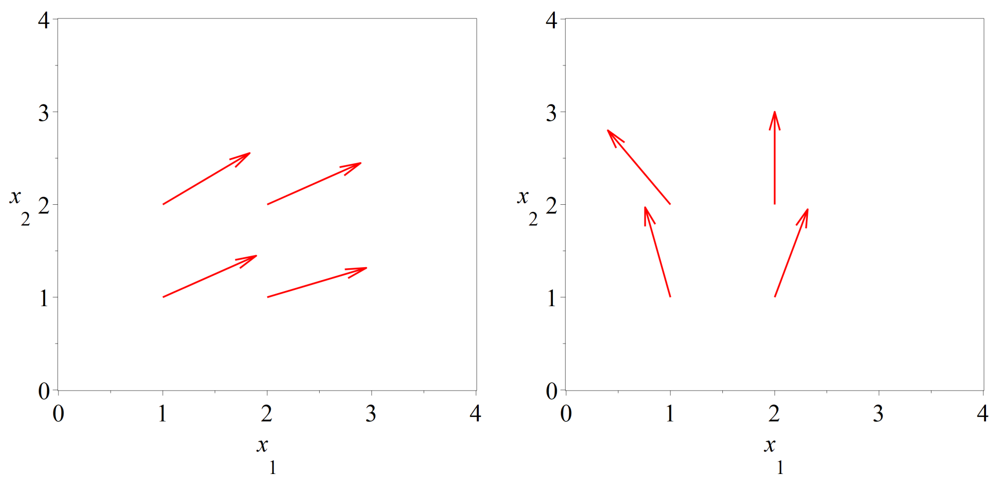



Figure 7.81: Vectors for Exercises 7.1.1 (left panel) and 7.1.2 (right panel).

**Exercise Solution 7.1.5.** *A direction field and a few solutions are shown in Figure 7.83. Solution converge to either* $(3, 0)$ *or* $(0, 3)$*. It appears that one species must go extinct, the other limits to its carrying capacity.*

**Exercise Solution 7.1.6.** *A direction field and a few solutions are shown in Figure 7.84. Solutions all approach* $(3, 3)$*, since each species behaves independently and each has carrying capacity* $3$*.*

**Exercise Solution 7.1.7.** *A direction field and a few solutions are shown in Figure 7.85. Solutions with initial conditions in this range all approach* $(0, 0)$*, since the pendulum is damped. This corresponds to the pendulum hanging motionless, straight down.*

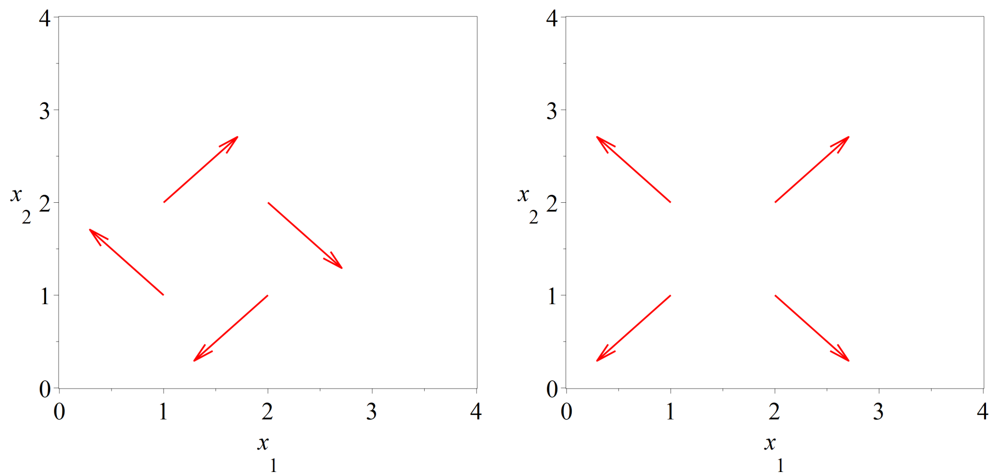



Figure 7.82: Vectors for Exercises 7.1.3 (left panel) and 7.1.4 (right panel).

**Exercise Solution 7.1.8.** *A direction field and a few solutions are shown in Figure 7.86. Solutions form closed orbits, indicating that the pendulum never stops moving. This makes perfect sense (no friction).*

**Exercise Solution 7.1.9.** *A direction field and a few solutions are shown in Figure 7.87. It looks like solutions approach the $S$ axis ($I = 0$). Each solution converges to some point with $I = 0$ and $S = S_1$ on this axis, for some $S_1 > 0$. That is, the number of infected individuals approaches zero, the number of susceptible individuals approaches $S_1$, and the number of recovered approaches $N - S_1$. Everyone ends up either recovered or having never had the disease.*

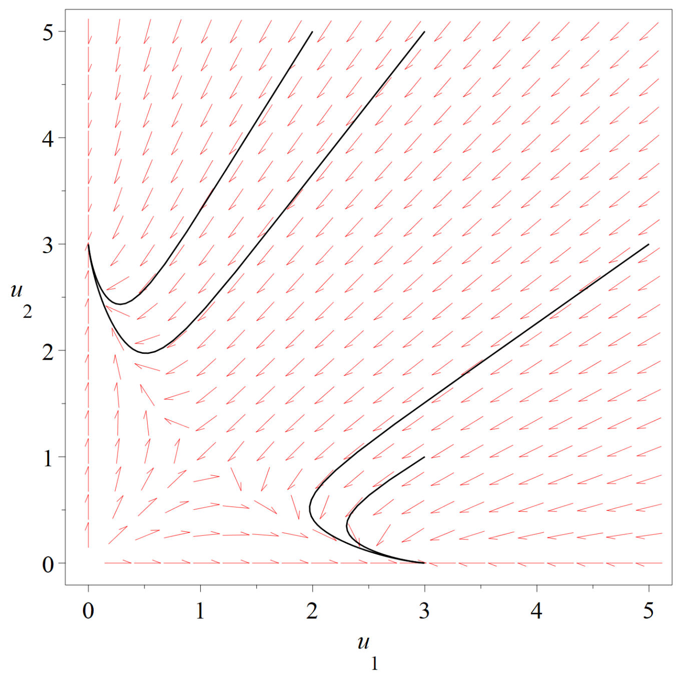



Figure 7.83: Direction field for competing species with $r_1 = 1$, $r_2 = 1$, $K_1 = 3$, $K_2 = 3$, $a = 2$, and $b = 2$, and a few solution trajectories.

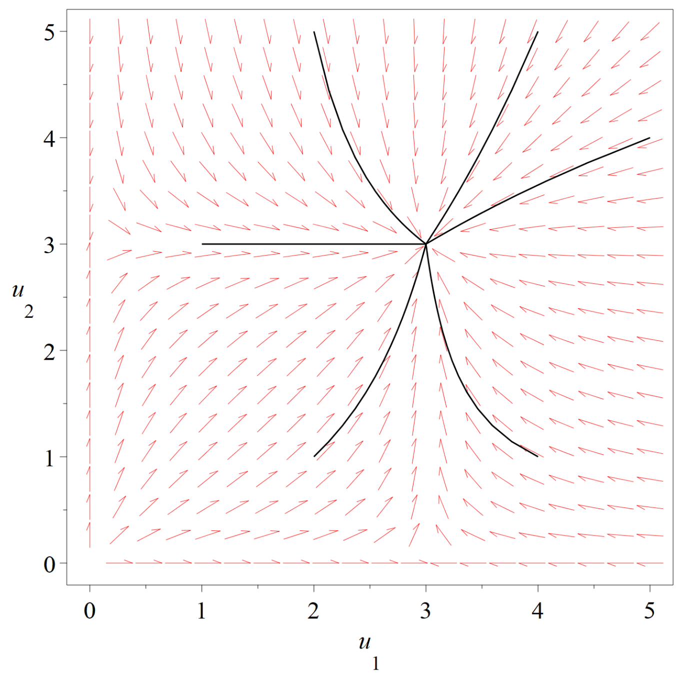



Figure 7.84: Direction field for competing species with $r_1 = 1$, $r_2 = 1$, $K_1 = 3$, $K_2 = 3$, $a = 0$, and $b = 0$, and a few solution trajectories.

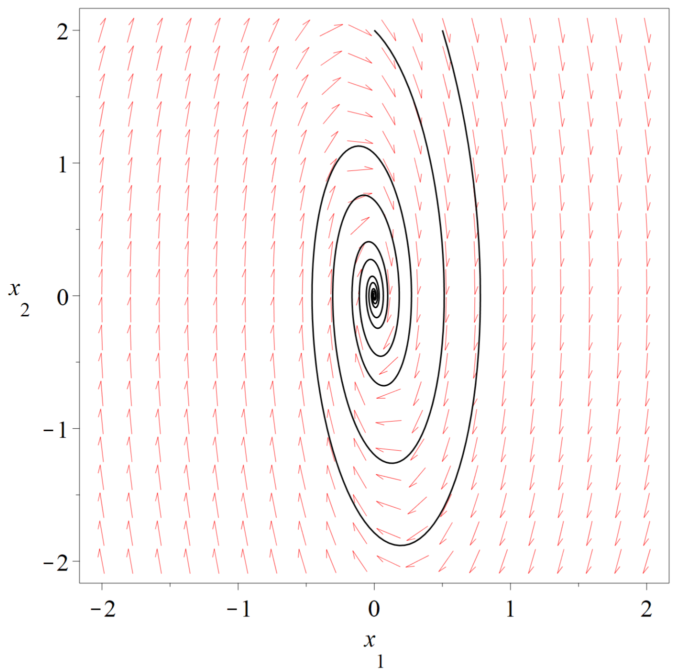



Figure 7.85: Direction field for damped pendulum equation (as a first order system), with a few solution trajectories.

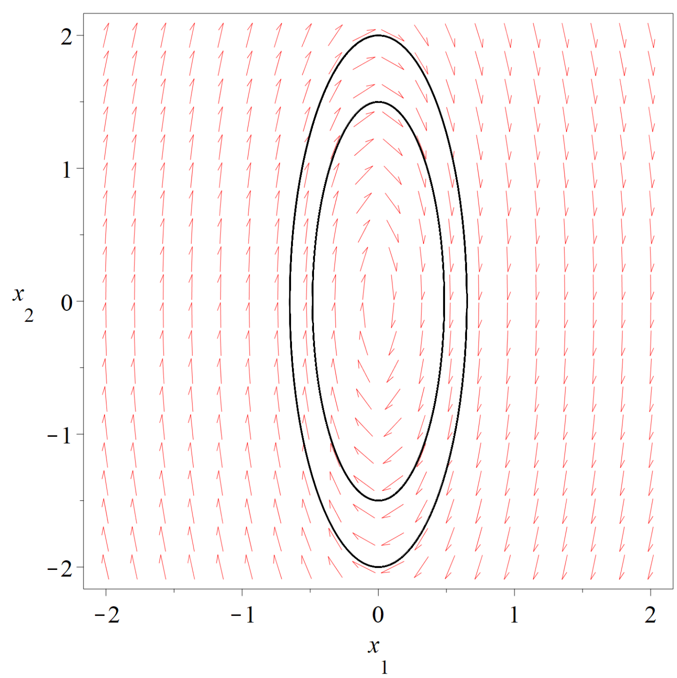



Figure 7.86: Direction field for undamped pendulum equation (as a first order system), with a few solution trajectories.

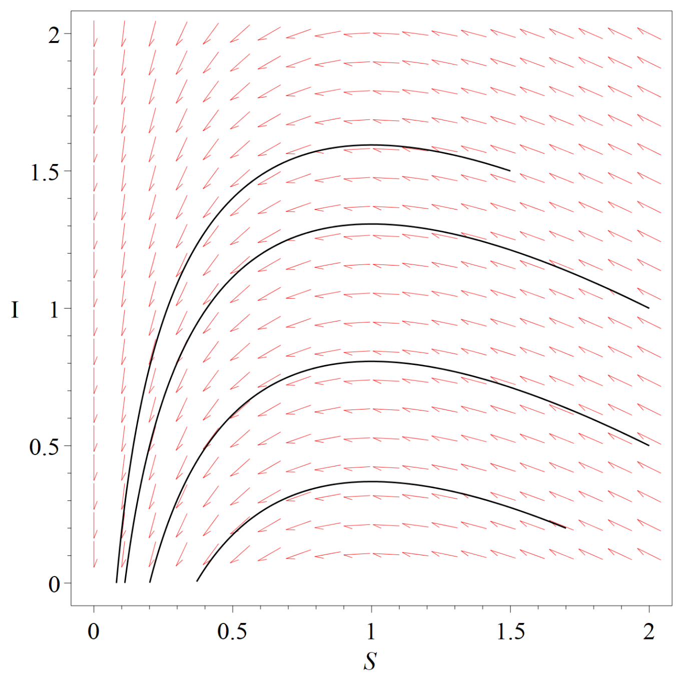



Figure 7.87: Direction field for SIR model and several solutions trajectories.

## Section 7.2

**Exercise Solution 7.2.1.** *See left panel in Figure 7.88. Eigenvalues are real,* $-2$ *and* $-4$*.*

**Exercise Solution 7.2.2.** *See right panel in Figure 7.88. Eigenvalues are real, mixed sign* $(1 \pm \sqrt{41})/2$*.*

**Exercise Solution 7.2.3.** *See left panel in Figure 7.89. Eigenvalues are real,* $2$ *and* $4$*.*

**Exercise Solution 7.2.4.** *See right panel in Figure 7.89. Eigenvalues are pure imaginary,* $\pm i$*.*

**Exercise Solution 7.2.5.** *See left panel in Figure 7.90. Eigenvalues are complex,* $-1 \pm 2i$*.*

**Exercise Solution 7.2.6.** *See right panel in Figure 7.90. Eigenvalues are complex,* $1 \pm 2i$*.*

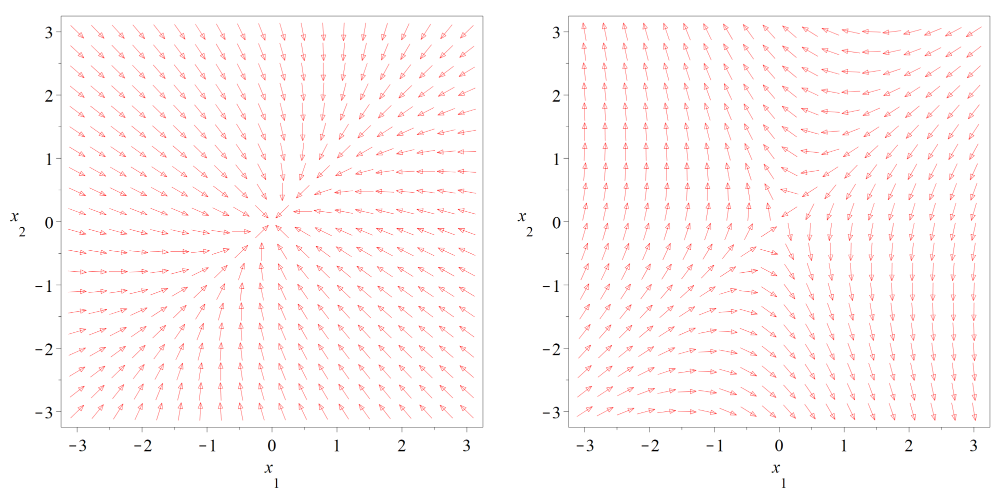



Figure 7.88: Direction fields for Exercises 7.2.1 (left) and 7.2.2 (right).

**Exercise Solution 7.2.7.** *See the left panel in Figure 7.91.*

**Exercise Solution 7.2.8.** *See the right panel in Figure 7.91.*

**Exercise Solution 7.2.9.** *See the left panel in Figure 7.92.*

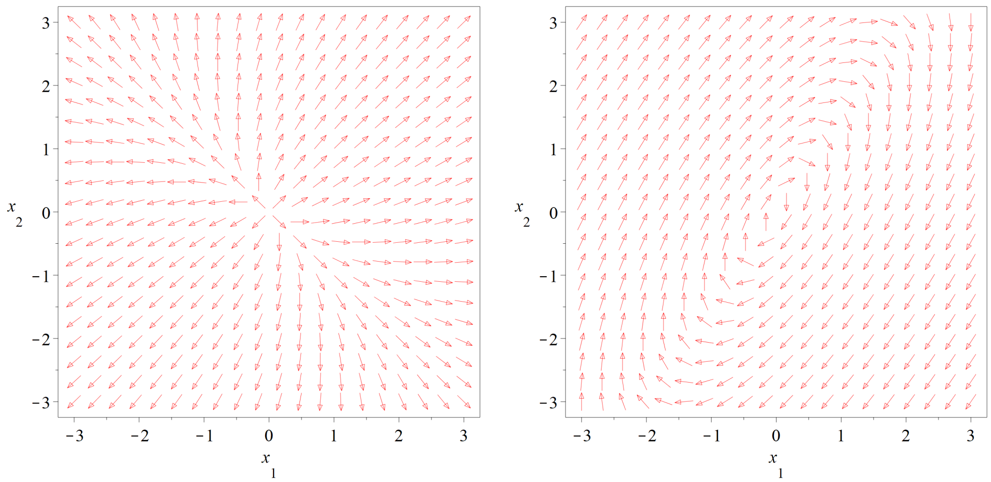



Figure 7.89: Direction fields for Exercises 7.2.3 (left) and 7.2.4 (right).

**Exercise Solution 7.2.10.** *See the right panel in Figure 7.92.*

**Exercise Solution 7.2.11.** *See Figure 7.93.*

**Exercise Solution 7.2.12.**

(a) *If we let "$x = x$" and $y = \dot{x}$ then clearly $\dot{x} = y$ and from $m\ddot{x}+c\dot{x}+kx = 0$ we obtain $m\dot{y} + cy + kx = 0$ or equivalently, $\dot{y} = -kx/m - cy/m$.*

(b) *The $x$ nullcline is $\dot{x} = 0$, which is just $y = 0$. The $y$ nullcline is $-kx/m - cy/m = 0$, which is equivalent to $y = -kx/c$.*

(c) *See Figure 7.94.*

(d) *See Figure 7.94. Solutions should spiral into the origin.*

(e) *This is similar to Figure 7.94, but the $y$ nullcline is now almost vertical. Solutions should still spiral into the origin, but more slowly.*

(f) *This is similar to Figure 7.94, but in this case the $y$ nullcline has a very shallow slope, almost horizontal. Solutions spiral rapidly into the origin.*

(g) *In this extreme case the $y$ nullcline is the vertical axis. Solutions spiral in closed orbits around the origin, and the system motion never damps out.*

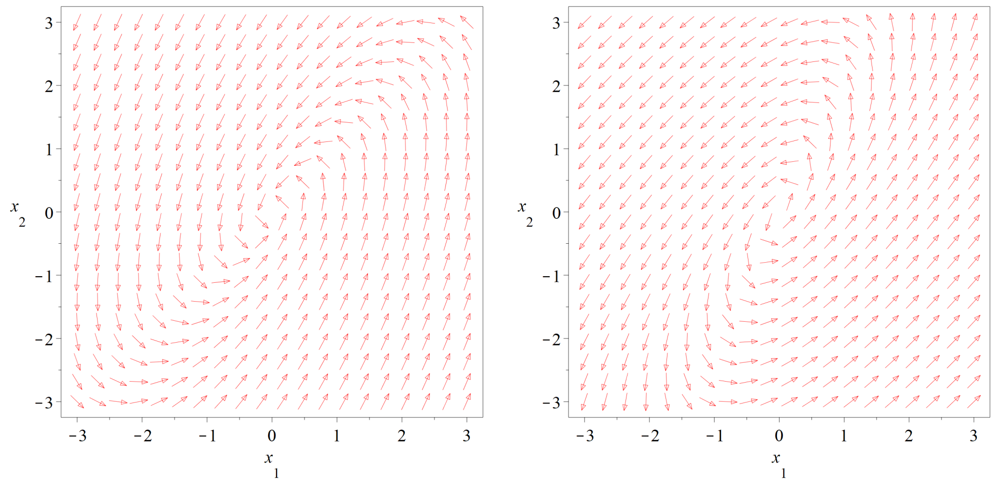


Figure 7.90: Direction fields for Exercises 7.2.5 (left) and 7.2.6 (right).

**Exercise Solution 7.2.13.**

(a) *Solving* $\frac{1}{2} - \frac{x_1}{120} + \frac{x_2}{100}$ *for* $x_2$ *yields* $x_2 = -50 + 5x_1/6$. *Solving* $\frac{x_1}{120} - \frac{x_2}{50}$ *yields* $x_2 = 5x_1/12$.

(b) *The intersection is at* $x_1 = 120, x_2 = 50$. *Here* $\dot{x}_1 = \dot{x}_2 = 0$, *so it is a fixed point.*

(c) *Typical solutions shown Figure 7.95. They approach the fixed point.*

(d) *The general solution is* $x_1(t) = 120 + c_1 e^{-t/300} + c_2 e^{-t/40}$ *and* $x_2(t) = 50 + c_1 e^{-t/300}/2 - 5c_2 e^{-t/40}/3$. *This shows analytically that solutions all approach the fixed point at* $(120, 50)$.

**Exercise Solution 7.2.14.**

(a) *It is straightforward to compute that* $D = \det(\mathbf{A}) = k_a k_e$ *and* $T = \operatorname{tr}(\mathbf{A}) = -(k_a + k_b + k_e)$. *Clearly* $D > 0$ *and* $T < 0$, *which already shows that the eigenvalues are either real and negative or complex with negative real part. The verification that*

$$T^2/4 - D = \frac{(k_a - k_e)^2}{4} + \frac{k_b(2k_a + k_b + 2k_e)}{4}$$

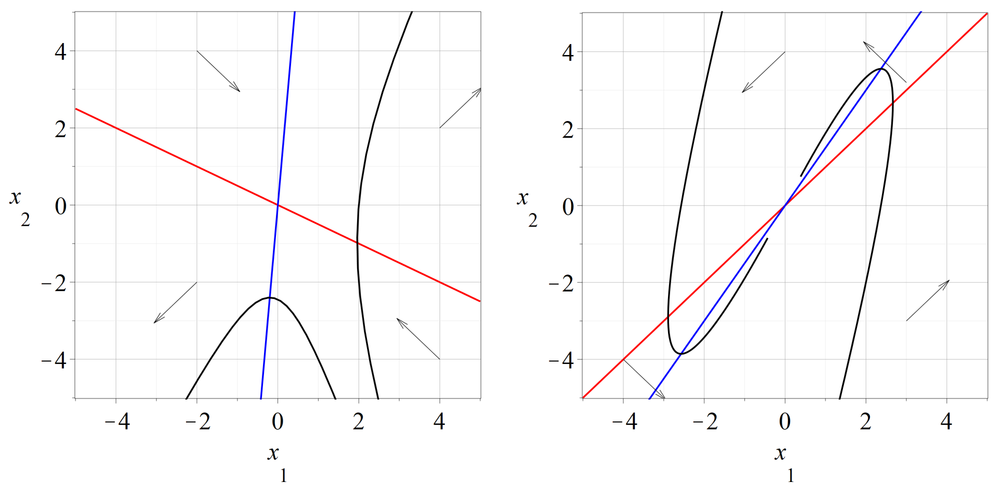



Figure 7.91: Phase portraits and solution curves for Exercises 7.2.7 (left) and 7.2.8 (right).

*is straightforward. It is clear the that the right side above is always positive since it is a sum of a square and a positive term. Thus* $T^2/4 - D > 0$.

*(b) Because* $0 < D < T^2/4$ *we conclude that both eigenvalues are real, and since* $T < 0$ *both are negative.*

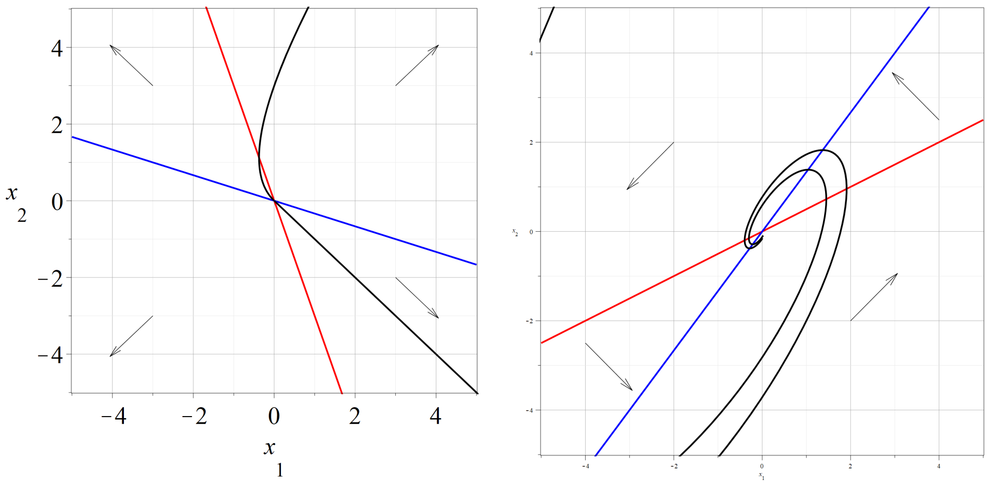



Figure 7.92: Phase portraits and solution curves for Exercises 7.2.9 (left) and 7.2.10 (right).

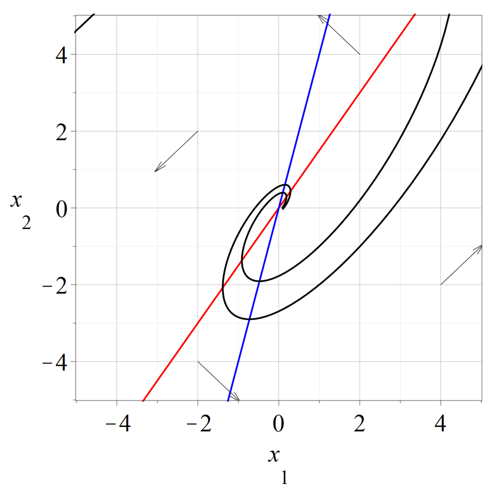



Figure 7.93: Phase portrait and solution curves for Exercise 7.2.11.

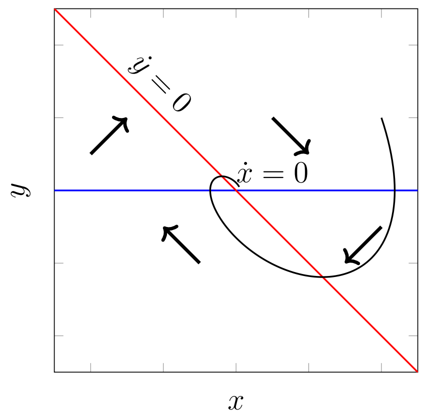



Figure 7.94: Nullclines ($x$ nullcline is $y = 0$ shown in blue, $y$ nullcline is $y = -x$ shown in red) and direction arrows for system $\dot{x} = y$ and $\dot{y} = -kx/m - cy/m$ with $k/c = 1$, and typical solution.

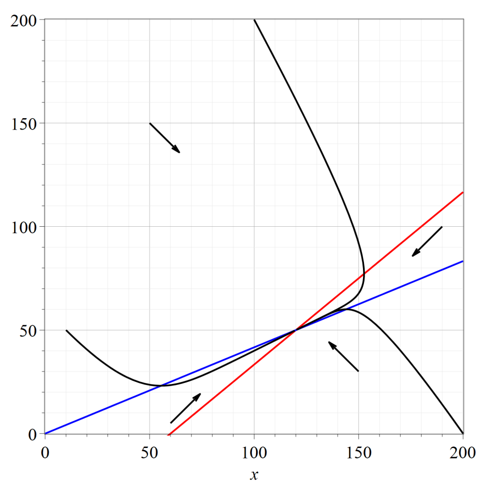



Figure 7.95: Phase portrait for Exercise 7.2.13.

## Section 7.3

**Exercise Solution 7.3.1.** *See Figure 7.96 for the phase portrait, Figure 7.97 for solution sketches with the given initial conditions. The solution with initial conditions* $x_1(0) = -1, x_2(0) = 3$ *does not extended past about* $t \approx 1.2$. *The fixed points are* $(-2, -2)$ *and* $(1, 1)$. *The Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} -2x_1 & -1 \\ 1 & -1 \end{bmatrix}.$$

*Then*

$$\mathbf{J}(-2, -2) = \begin{bmatrix} 4 & -1 \\ 1 & -1 \end{bmatrix}.$$

*has approximate eigenvalues* $3.79$ *and* $-0.79$, *so this is a saddle point. Also*

$$\mathbf{J}(1, 1) = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}.$$

*has approximate eigenvalues* $-1.5 \pm 0.866i$, *so this is an asymptotically stable spiral point.*

**Exercise Solution 7.3.2.** *See Figure 7.98 for the phase portrait, Figure 7.99 for solution sketches with the given initial conditions. The fixed points are* $(0, -2)$ *and* $(-1, 0)$. *The Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} -2 & -1 \\ -x_2 & -x_1 \end{bmatrix}.$$

*Then*

$$\mathbf{J}(0, -2) = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}.$$

*has approximate eigenvalues* $-1 \pm i$, *so this is an asymptotically stable spiral point. Also*

$$\mathbf{J}(-1, 0) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}.$$

*has approximate eigenvalues* $1$ *and* $-2$, *so this is a saddle point.*

**Exercise Solution 7.3.3.** *See Figure 7.100 for the phase portrait, Figure 7.101 for solution sketches with the given initial conditions. The fixed points are* $(-3,0)$ *and* $(-1,1)$. *The Jacobian is*

$$\mathbf{J}(x_1,x_2) = \begin{bmatrix} x_2 & x_1+2x_2 \\ 1 & -2 \end{bmatrix}.$$

*Then*

$$\mathbf{J}(-3,0) = \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix}.$$

*has eigenvalues* $-1 \pm i\sqrt{2}$, *so this is an asymptotically stable spiral point. Also*

$$\mathbf{J}(-1,1) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

*has approximate eigenvalues* $1.3$ *and* $-2.3$, *so this is a saddle point.*

**Exercise Solution 7.3.4.** *See Figure 7.102 for the phase portrait, Figure 7.103 for solution sketches with the given initial conditions. The fixed points are* $(0,0), (2,2)$, *and* $(-2,-2)$. *The Jacobian is*

$$\mathbf{J}(x_1,x_2) = \begin{bmatrix} 3x_1^2-3 & -1 \\ 1 & -1 \end{bmatrix}.$$

*Then*

$$\mathbf{J}(0,0) = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}.$$

*has double eigenvalue* $-2$, *so this is an asymptotically stable node. Also*

$$\mathbf{J}(2,2) = \begin{bmatrix} 9 & -1 \\ 1 & -1 \end{bmatrix}.$$

*has approximate eigenvalues* $8.9$ *and* $-0.9$, *so this is a saddle point. Also*

$$\mathbf{J}(-2,-2) = \begin{bmatrix} 9 & -1 \\ 1 & -1 \end{bmatrix}.$$

*has approximate eigenvalues* $8.9$ *and* $-0.9$, *so this is also a saddle point.*

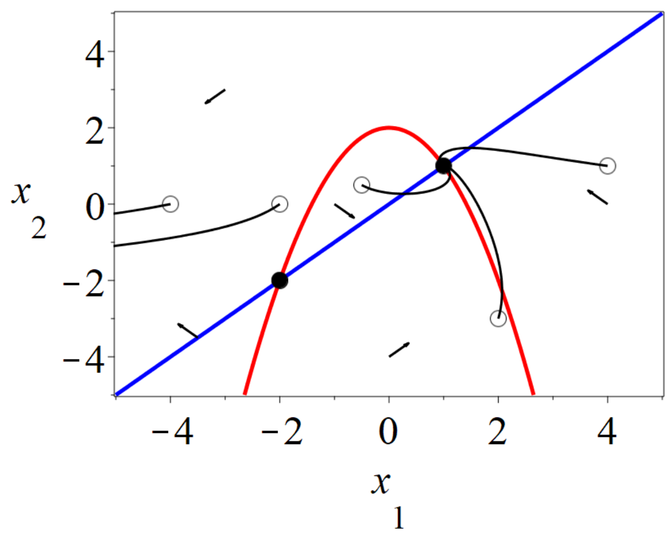



Figure 7.96: Phase portrait for Exercise 7.3.1.

**Exercise Solution 7.3.5.** *We find* $f_1(u_1, u_2) = u_1(2 - u_1 - 3u_2)/2$ *and* $f_2(u_1, u_2) = 2u_2(3 - u_2 - 3u_1)/3$. *Solving* $u_1(2 - u_1 - 3u_2)/2 = 0$ *and* $2u_2(3 - u_2 - 3u_1)/3 = 0$ *simultaneously yields fixed points* $(0, 0)$, $(2, 0)$, $(0, 3)$, *and* $(7/8, 3/8)$. *The Jacobian matrix is*

$$\mathbf{J}(u_1, u_2) = \begin{bmatrix} 1 - u_1 - 3u_2/2 & -3u_1/2 \\ -2u_2 & 2 - 2u_1 - 4u_2/3 \end{bmatrix}.$$

*Then*

$$\mathbf{J}(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

*has eigenvalues* $1$ *and* $2$, *so* $(0, 0)$ *is an unstable node. Also*

$$\mathbf{J}(2, 0) = \begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix}.$$

*has eigenvalues* $-1$ *and* $-2$, *so* $(2, 0)$ *is an asymptotically stable node. Also*

$$\mathbf{J}(0, 3) = \begin{bmatrix} -7/2 & 0 \\ -6 & -2 \end{bmatrix}.$$

*has eigenvalues* $-7/2$ *and* $-2$, *so* $(0, 3)$ *is an asymptotically stable node. Finally,*

$$\mathbf{J}(7/8, 3/8) = \begin{bmatrix} -7/16 & -21/16 \\ -3/4 & -1/4 \end{bmatrix}.$$

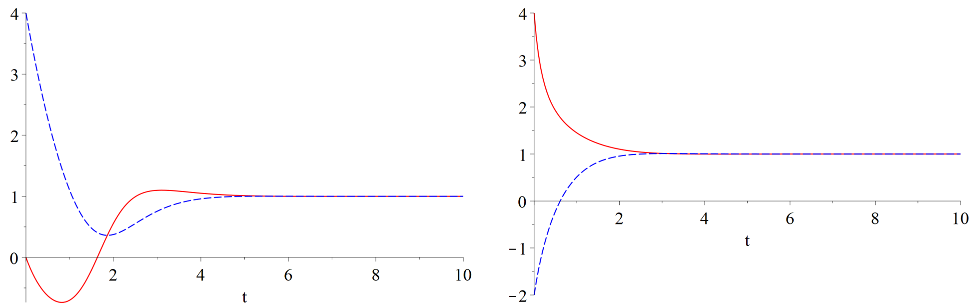



Figure 7.97: Individual solutions components for Exercise 7.3.1, $x_1(t)$ (red, solid) and $x_2(t)$ (blue, dashed) for $x_1(0) = 0, x_2(0) = 4$ (left panel) and $x_1(0) = 4, x_2(0) = -2$ (right panel).

*has approximate eigenvalues* $0.65$ *and* $-1.34$*, so* $(7/8, 3/8)$ *is a saddle point.*

**Exercise Solution 7.3.6.**

(a) *The $S$ nullcline consists of points such that $SI = 0$, which is precisely the axes $S = 0$ and $I = 0$. The $I$ nullcline consists of points such that $SI - 2I = 0$ or $I(S-2) = 0$, which is $I = 0$ and $S = 2$. See Figure 7.104.*

(b) *As noted, the $S$ nullcline consists of the coordinate axes; all solutions in the first quadrant move in the negative $S$ direction (left). See Figure 7.104.*

(c) *The $I$ nullcline consists of the horizontal axis $I = 0$ and the vertical line $S = 2$. Solutions with $0 < S < 2$ have $\dot{I} = I(S-2) < 0$ and so $I$ is decreasing (the solution is moving downward). Solutions with $S > 2$ have $\dot{I} = I(S-2) > 0$ and so $I$ is increasing (the solution is moving upward). See Figure 7.104.*

(d) *The fixed points are simultaneous solutions to $SI = 0$ and $I(S-2) = 0$. This is exactly all points for which $I = 0$, that is, the horizontal $S$ axis.*

(e) *The Jacobian matrix is*

$$\mathbf{J}(S, I) = \begin{bmatrix} -I & -S \\ I & S-2 \end{bmatrix}.$$

*At any fixed point $(S, I) = (S, 0)$ we find*

$$\mathbf{J}(S, 0) = \begin{bmatrix} 0 & -S \\ 0 & S-2 \end{bmatrix}$$

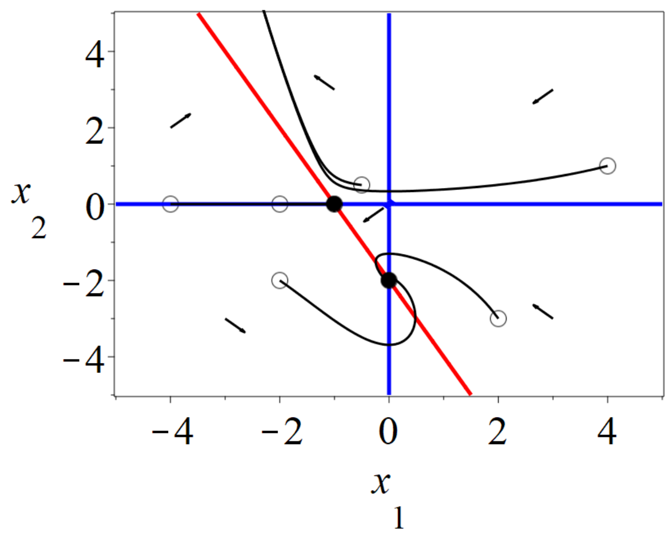



Figure 7.98: Phase portrait for Exercise 7.3.2.

*has eigenvalues* $0$ *and* $S-2$*. These fixed points are not hyperbolic, so the Hartman-Grobman Theorem does not apply.*

(f) *See Figure 7.104. This is compelling evidence that the number of infected will decrease to* $0$ *in time, while* $S$ *will decrease to some fixed positive number* $S^*$*. That is, a certain number of people will never get infected, and this number* $S^*$ *is less than* $2$ *(in whatever units we are measured population in). The number of recovered people will be* $R^* = N - S^*$ *if the population size is* $N$*.*

**Exercise Solution 7.3.7.**

(a) *The* $x_1$ *nullcline is all points with* $x_2 = 0$*, the horizontal axis. Solutions above this axis have* $x_1$ *increasing (they move to the right), solutions below this axis have* $x_1$ *decreasing (they move to the left). See Figure 7.105.*

(b) *The* $x_2$ *nullcline is obtained by setting* $-\frac{g}{L}\sin(x_1) - cx_2 = 0$*, which yields* $x_2 = -\frac{g}{cL}\sin(x_1)$ *or* $x_2 = -9.81\sin(x_1)$ *with the specified parameters. This nullcline is an inverted sine curve. See Figure 7.105. Solutions above this nullcline have* $\dot{x}_2 < 0$*, so they move downward. Solutions below this nullcline have* $\dot{x}_2 > 0$*, so they move upward.*

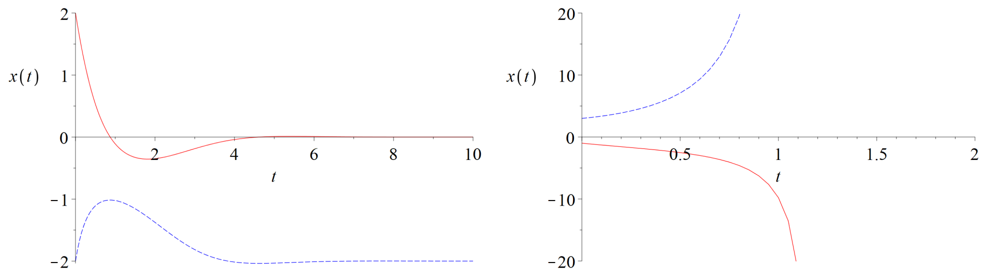



Figure 7.99: Individual solutions components for Exercise 7.3.2, $x_1(t)$ (red, solid) and $x_2(t)$ (blue, dashed) for $x_1(0) = 2, x_2(0) = -2$ (left panel) and $x_1(0) = -1, x_2(0) = 3$ (right panel).

(c) *Solving $x_2 = 0$ and $-\frac{g}{L}\sin(x_1) - cx_2 = 0$ simultaneously yields solutions $x_1 = k\pi$ and $x_2 = 0$ for any integer $k$. The case in which $k = 2j$ is even corresponds to $x_1 = \theta = 2j\pi$, so the pendulum is hanging straight down, and $x_2 = \dot{\theta} = 0$ means the pendulum is motionless. The case in which $k = 2j+1$ is odd corresponds to $x_1 = \theta = (2j+1)\pi$, so the pendulum is perfectly balanced upside down, and $x_2 = \dot{\theta} = 0$ means the pendulum is motionless.*

(d) *The Jacobian matrix is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -g\cos(x_1)/L & -c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9.81\cos(x_1) & -1 \end{bmatrix}.$$

*At any equilibrium solution of the form $x_1 = 2j\pi$ (an even multiple of $\pi$, corresponding to the pendulum hanging motionless straight down) the Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -9.81 & -1 \end{bmatrix}$$

*and has approximate eigenvalues $-0.5 \pm 3.09i$. These equilibrium points are asymptotically stable spiral points. Near such a point solutions spiral to the fixed point, corresponding to the pendulum oscillating back and forth to rest at that point.*

*At any equilibrium solution of the form $x_1 = (2j+1)\pi$ (an odd multiple of $\pi$, corresponding to the pendulum motionless upside down) the Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 9.81 & -1 \end{bmatrix}$$

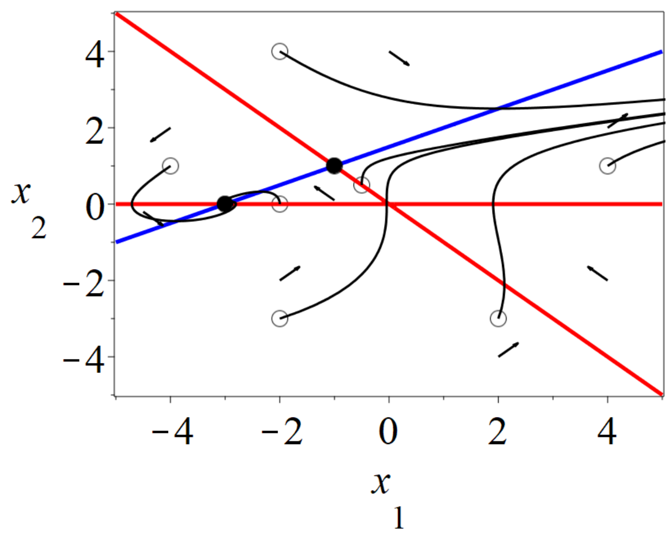



Figure 7.100: Phase portrait for Exercise 7.3.3.

*and has approximate eigenvalues* 2.67 *and* −3.67*. These equilibrium points are saddle points. Near such a point typical solutions move away from the fixed point. Physically, the pendulum swings away from vertical.*

(e) *See Figure 7.105. Solutions all spiral into stable fixed points as friction robs them of energy. The greater the initial angular velocity (the larger the value of* $|x_2|$*) the longer it takes to settle into a fixed point.*

**Exercise Solution 7.3.8.** *The fixed points are* $(-1,-2,1)$ *and* $(1,0,-1)$*. The Jacobian matrix is*

$$\mathbf{J}(x,y,z) = \begin{bmatrix} -1 & 1 & 0 \\ z & 0 & x \\ -1 & 0 & -1 \end{bmatrix}.$$

*Then*

$$\mathbf{J}(-1,-2,1) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

*has approximate eigenvalues* 0.84 *and* $1.42 \pm 0.61i$*. This is a hyperbolic*

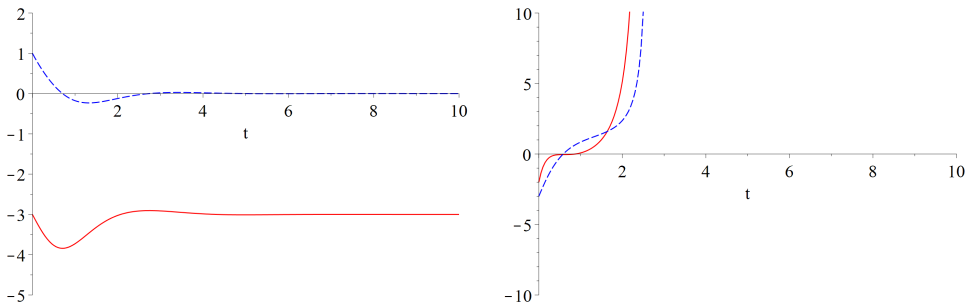



Figure 7.101: Individual solutions components for Exercise 7.3.3, $x_1(t)$ (red, solid) and $x_2(t)$ (blue, dashed) for $x_1(0) = -3, x_2(0) = 1$ (left panel) and $x_1(0) = -2, x_2(0) = -3$ (right panel).

*equilibrium point, and unstable (due to the positive real eigenvalue). Also*

$$\mathbf{J}(1, 0, -1) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

*has approximate eigenvalues* $-1.54$ *and* $-0.23 \pm 1.12i$. *This is a hyperbolic equilibrium point, and is stable.*

*A bit of numerical solving confirms these results. Solutions that start close to* $(1, 0, -1)$ *approach that fixed point, but solutions do not approach* $(-1, -2, 1)$ *no matter how close they start. Many solutions also blow up to infinity.*

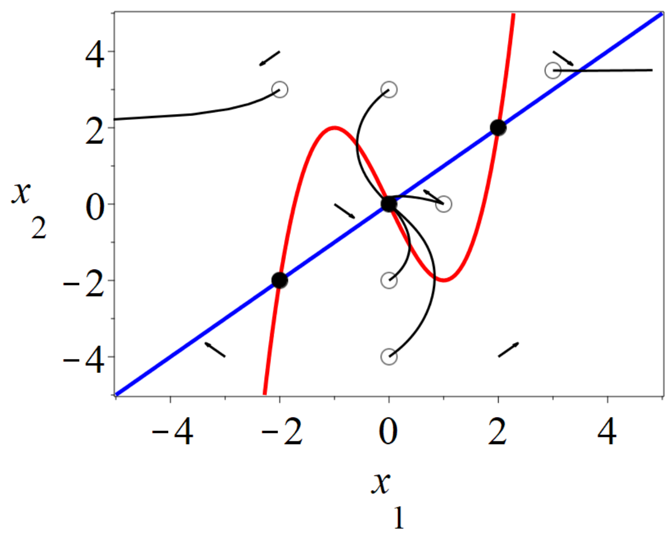



Figure 7.102: Phase portrait for Exercise 7.3.4.

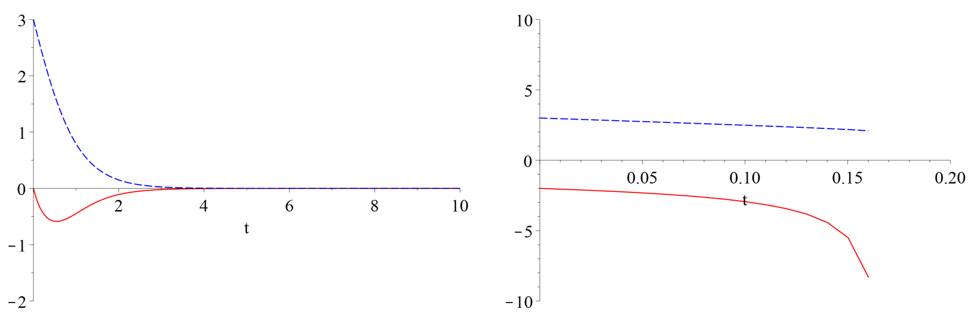



Figure 7.103: Individual solutions components for Exercise 7.3.4, $x_1(t)$ (red, solid) and $x_2(t)$ (blue, dashed) for $x_1(0) = 0, x_2(0) = 3$ (left panel) and $x_1(0) = -2, x_2(0) = 3$ (right panel).

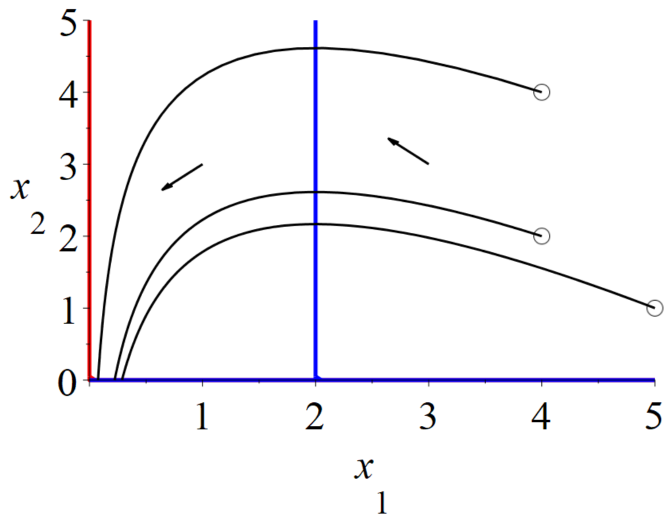



Figure 7.104: Phase portrait for SIR (SI only) model with $a = 1, b = 2$.

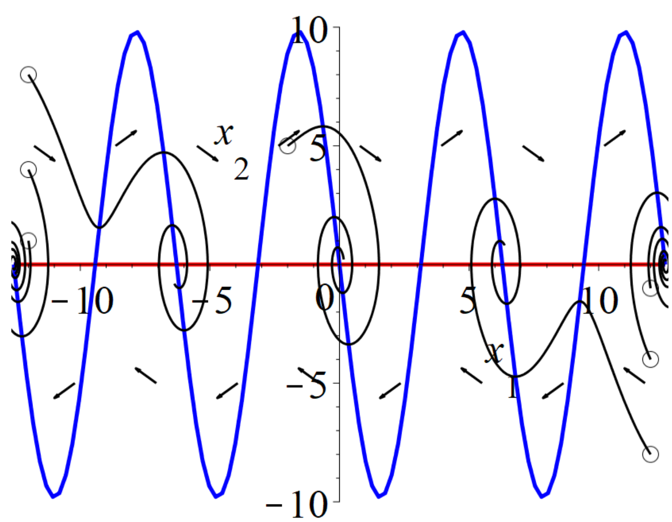



Figure 7.105: Phase portrait for nonlinear pendulum.

# Section 7.4

**Exercise Solution 7.4.1.**

(a) *The equation* $-ax_2+x_2^2 = 0$ *forces* $x_2 = 0$ *or* $x_2 = a$ *and then* $x_1-x_2 = 0$ *yields* $x_1 = 0$ *or* $x_1 = a$. *The fixed points are* $(0, 0)$ *and* $(a, a)$.

(b) *The* $x_1$ *nullcline consists of the horizontal lines* $x_2 = 0$ *and* $x_2 = a$. *For* $x_2 < 0$ *we find* $\dot{x}_1 > 0$ *so solutions move in the direction of increasing* $x_1$ *(to the right). For* $0 < x_2 < a$ *solutions move to the left, and for* $x_2 > a$ *solutions move to the right. This nullcline is shown in the left panel of Figure 7.106.*

(c) *The* $x_2$ *nullcline consists of the diagonal line* $x_2 = x_1$. *For* $x_2 < x_1$ *we find* $\dot{x}_2 < 0$ *so solutions move in the direction of decreasing* $x_2$ *(down). For* $x_2 > x_1$ *solutions upward. This nullcline is shown in the right panel of Figure 7.106.*

(d) *The Jacobian is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} =0 & -a+2x_2 \\ 1 & -1 \end{bmatrix}.$$

*At the fixed point* $(0, 0)$ *we find*

$$\mathbf{J}(0, 0) = \begin{bmatrix} =0 & -a \\ 1 & -1 \end{bmatrix}.$$

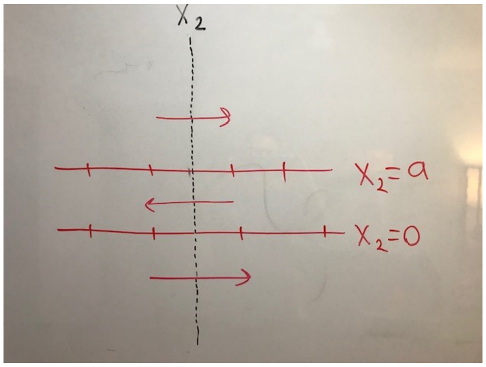
*The determinant* $D$ *of this matrix equals* $a$, *which is positive by assumption, so* $(0, 0)$ *is always stable. The trace* $T$ *of this matrix is* $-1$. *If* $0 < a < 1/4$ *(so* $0 < D < T^2/4$*) then* $(0, 0)$ *is an asymptotically stable node and if* $a > 1/4$ *then* $(0, 0)$ *is an asymptotically stable spiral point.*

*At* $(a, a)$ *the Jacobian is*

$$\mathbf{J}(a, a) = \begin{bmatrix} 0 & a \\ 1 & -1 \end{bmatrix}.$$

*The determinant here is* $D = -a$, *so if* $a > 0$ *this is a saddle.*

(e) *See Figure 7.107 for the case* $a > 1/4$ *and Figure 7.108 for the case* $a < 1/4$. *The solutions have the same general behavior, except when* $a < 1/4$ *they do not spiral as they approach the fixed point* $(0, 0)$.



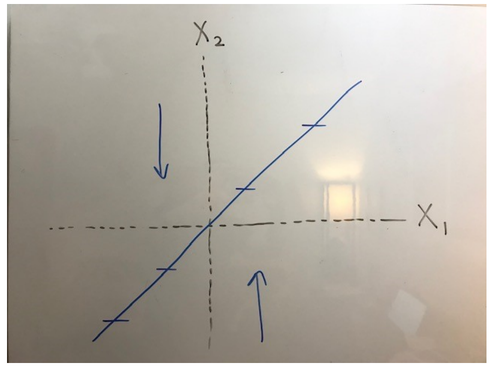




Figure 7.106: Nullclines for $x_1$ (left) and $x_2$ (right) for Problem 7.4.1.

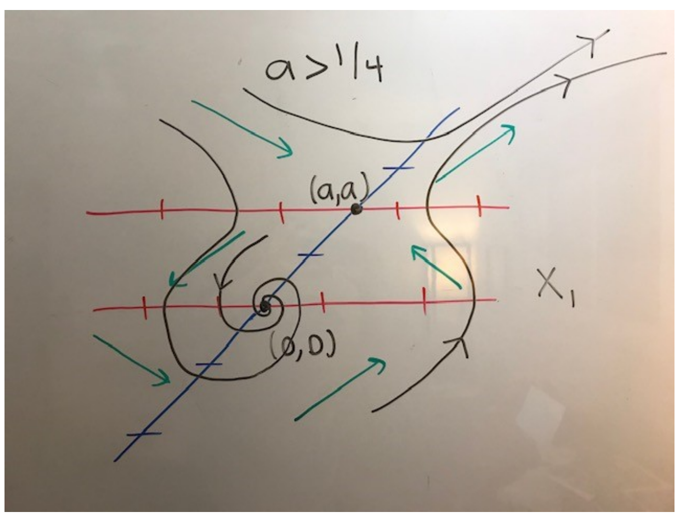



Figure 7.107: Phase portrait for system in Problem 7.4.1, $a > 1/4$.

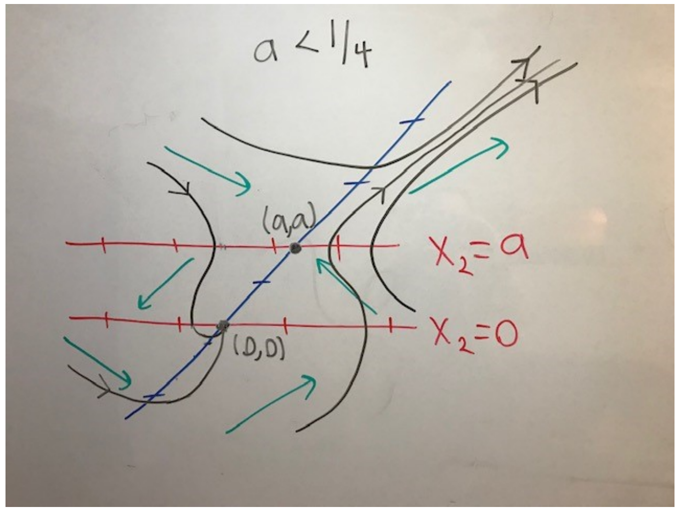



Figure 7.108: Phase portrait for system in Problem 7.4.1, $a < 1/4$.

**Exercise Solution 7.4.2.**

(a) *See Figure 7.109.*

(b) *It's easy to see that the fourth fixed point* $((\bar{a}-1)/(\bar{a}\bar{b}-1), (\bar{b}-1)/(\bar{a}\bar{b}-1))$ *has a negative component, since* $\bar{a}-1 < 0$ *and* $\bar{b}-1 > 0$*; whatever the sign of* $\bar{a}\bar{b}-1$*, one component of the fixed point must be negative. The other fixed points* $(0,0), (1,0)$*, and* $(0,1)$ *are always in the first quadrant for any choice of* $\bar{a}$ *and* $\bar{b}$*.*

(c) *The Jacobian is*

$$\mathbf{J}(v_1, v_2) = \begin{bmatrix} r_1(1-2v_1-\bar{a}v_2) & -r_1 a v_1 \\ -r_2 b v_2 & r_2(1-2v_2-\bar{b}v_1) \end{bmatrix}.$$

*Then*

$$\mathbf{J}(0,0) = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

*which has eigenvalues* $r_1$ *and* $r_2$*. This is an unstable node. Also*

$$\mathbf{J}(1,0) = \begin{bmatrix} -r_1 & -r_1\bar{a} \\ 0 & r_2(1-\bar{b}) \end{bmatrix}.$$

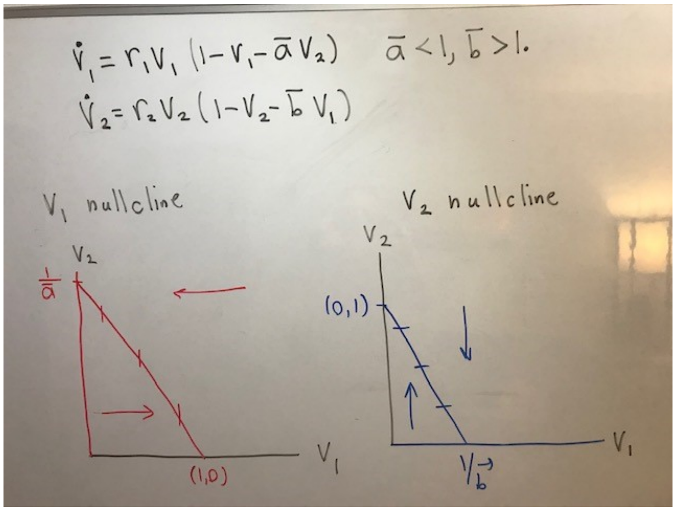



Figure 7.109: Nullclines for $v_1$ (left) and $v_2$ (right) for Problem 7.4.2.

*This is upper triangular with eigenvalues* $-r_1 < 0$ *and* $r_2(1 - \bar{b}) < 0$, *so this is a stable node. Finally*

$$\mathbf{J}(0,1) = \begin{bmatrix} r_1(1-\bar{a}) & 0 \\ -r_2\bar{b} & -r_2 \end{bmatrix}.$$

*This is lower triangular with eigenvalues* $r_1(1 - \bar{a}) > 0$ *and* $-r_2 < 0$, *so this is a saddle point (unstable).*

(d) *See Figure 7.110.*

(e) *This is pretty apparent from the phase portrait in Figure 7.110. All solutions clearly go to* $(1,0)$.

(f) *This precisely reverses the role of the species and all solutions go to* $(0,1)$. *The first species will go extinct in this model.*

**Exercise Solution 7.4.3.** *In each case the Jacobian matrix is*

$$\mathbf{J}(v_1, v_2) = \begin{bmatrix} r_1(1 - 2v_1 - \bar{a}v_2) & -r_1 a v_1 \\ -r_2 b v_2 & r_2(1 - 2v_2 - \bar{b}v_1) \end{bmatrix}.$$

*The eigenvalues of* $\mathbf{J}(0,0)$ *in every case are* $r_1$ *and* $r_2$, *both positive, so the origin is always an unstable node.*

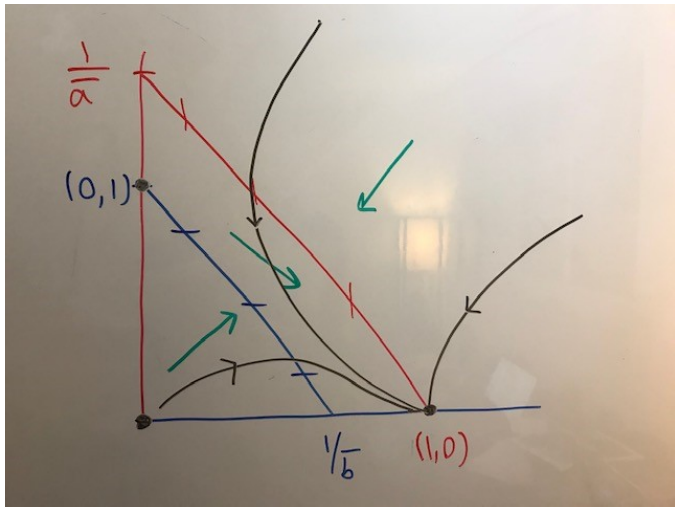



Figure 7.110: Phase portrait for Problem 7.4.2.

(a) *See Figure 7.111. The fixed points here are* $(0,0), (0,1)$, *and* $(1,0)$. *At* $(0,1)$ *the eigenvalues are* $0$ *and* $-r_2$, *so this is not a hyperbolic equilibrium point. At* $(1,0)$ *the eigenvalues are* $-r_1 < 0$ *and* $r_2(1-\bar{b}) > 0$, *so this is a saddle. Although we can't use the Hartman-Grobman Theorem at* $(0,1)$, *it certainly looks stable.*

(b) *See Figure 7.112. The fixed points here are* $(0,0), (0,1)$, *and* $(1,0)$. *At* $(1,0)$ *the eigenvalues are* $0$ *and* $-r_1$, *so this is not a hyperbolic equilibrium point. At* $(0,1)$ *the eigenvalues are* $-r_2 < 0$ *and* $r_1(1-\bar{a}) > 0$, *so this is a saddle. Although we can't use the Hartman-Grobman Theorem at* $(1,0)$, *it certainly looks stable.*

(c) *See Figure 7.113. Here every point on the line* $v_1 + v_2 = 1$ *in the first quadrant is a fixed point. At any such fixed point the Jacobian can be written as*

$$\mathbf{J}(v_1, v_2) = \begin{bmatrix} -r_1 v_1 & -r_1 v_1 \\ -r_2(1-v_1) & -r_2(1-v_1) \end{bmatrix}.$$

*This has eigenvalues* $0$ *and* $-r_1 v_1 - r_2(1-v_1)$ *and is not hyperbolic. Still, it appears that solutions that start off the line* $v_1+v_2 = 1$ *converge to a point on that line.*

**Exercise Solution 7.4.4.**

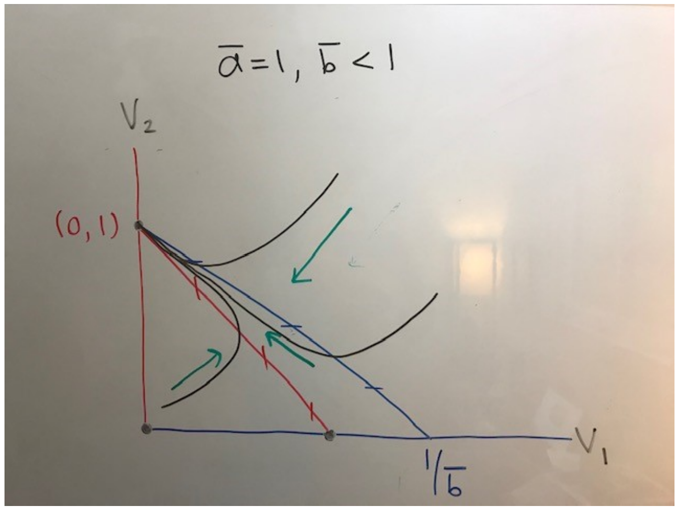



Figure 7.111: Phase portrait for Problem 7.4.3 part (a).

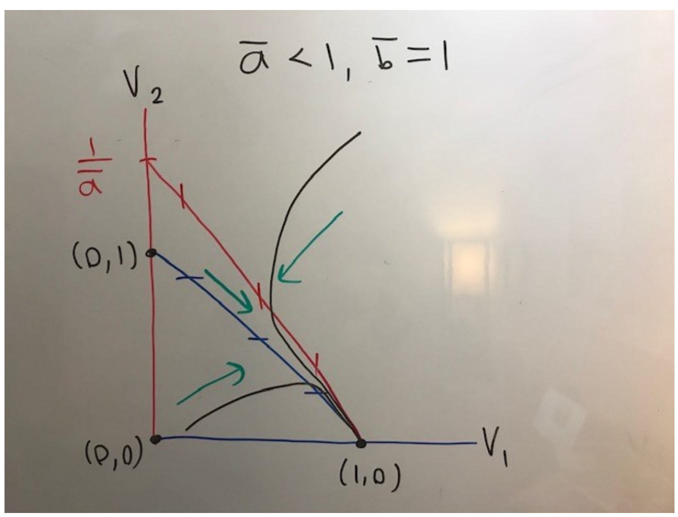



Figure 7.112: Phase portrait for Problem 7.4.3 part (b).

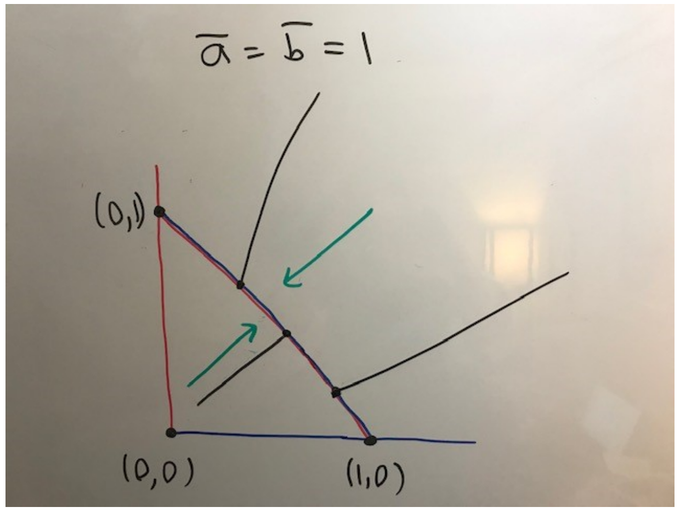



Figure 7.113: Phase portrait for Problem 7.4.3 part (c).

(a) *The $S$ nullcline consists of points such that $SI = 0$, which is precisely the axes $S = 0$ and $I = 0$. The $I$ nullcline consists of points such that $aSI - bI = 0$ or $I(aS - b) = 0$, which is $I = 0$ and $S = b/a$.*

(b) *See phase portrait in Figure 7.114.*

(c) *See phase portrait in Figure 7.114.*

(d) *The fixed points are simultaneous solutions to $SI = 0$ and $I(aS-b) = 0$. This is exactly all points for which $I = 0$, that is, the horizontal $S$ axis.*

(e) *The Jacobian matrix is*

$$\mathbf{J}(S, I) = \begin{bmatrix} -aI & -aS \\ aI & aS - b \end{bmatrix}.$$

*At any fixed point $(S, I) = (S, 0)$ we find*

$$\mathbf{J}(S, 0) = \begin{bmatrix} 0 & -aS \\ 0 & aS - b \end{bmatrix}$$

*has eigenvalues $0$ and $aS - b$. These fixed points are not hyperbolic, so the Hartman-Grobman Theorem does not apply.*


Figure 7.114: Phase portrait for Problem 7.4.4.

(f) *See Figure 7.114. This is compelling evidence that the number of infected will decrease to 0 in time, while $S$ will decrease to some fixed positive number $S^*$. That is, a certain number of people will never get infected, and this number $S^*$ is less than $b/a$ (in whatever units we are measured population in). The number of recovered people will be $R^* = N - S^*$ if the population size is $N$.*

**Exercise Solution 7.4.5.**

(a) *The $x_1$ nullcline is all points with $x_2 = 0$, the horizontal axis. Solutions above this axis have $x_1$ increasing (they move to the right), solutions below this axis have $x_1$ decreasing (they move to the left). See Figure 7.115.*

(b) *The $x_2$ nullcline is obtained by setting $-\frac{g}{L}\sin(x_1) - cx_2 = 0$, which yields $x_2 = -\frac{g}{cL}\sin(x_1)$. This nullcline is an inverted sine curve. See Figure 7.116. Solutions above this nullcline have $\dot{x}_2 < 0$, so they move downward. Solutions below this nullcline have $\dot{x}_2 > 0$, so they move upward.*

(c) *Solving $x_2 = 0$ and $-\frac{g}{L}\sin(x_1) - cx_2 = 0$ simultaneously yields solutions $x_1 = k\pi$ and $x_2 = 0$ for any integer $k$. The case in which $k = 2j$ is even corresponds to $x_1 = \theta = 2j\pi$, so the pendulum is hanging straight down, and $x_2 = \dot{\theta} = 0$ means the pendulum is motionless.*

*The case in which* $k = 2j+1$ *is odd corresponds to* $x_1 = \theta = (2j+1)\pi$, *so the pendulum is perfectly balanced upside down, and* $x_2 = \dot{\theta} = 0$ *means the pendulum is motionless.*

(d) *The Jacobian matrix is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -g\cos(x_1)/L & -c \end{bmatrix}.$$

*At any equilibrium solution of the form* $x_1 = 2j\pi$ *(an even multiple of* $\pi$*, corresponding to the pendulum hanging motionless straight down) the Jacobian is*

$$\mathbf{J}(2j\pi, 0) = \begin{bmatrix} 0 & 1 \\ -g/L & -c \end{bmatrix}$$

*The determinant is* $D = g/L$ *and the trace is* $T = -c$. *We conclude that since* $D > 0$ *and* $T < 0$ *this fixed point is stable. If* $0 < D < T^2/4$ *or equivalently,* $c^2/4 - g/L > 0$, *this is an asymptotically stable node. If* $c^2/4 - g/L < 0$ *this is an asymptotically stable spiral point.*

(e) *The Jacobian matrix at a point* $x_1 = (2j+1)\pi$, $x_2 = 0$ *is*

$$\mathbf{J}((2j+1)\pi, 0) = \begin{bmatrix} 0 & 1 \\ g/L & -c \end{bmatrix}$$

*The determinant is* $D = -g/L$, *so this will always be a saddle point.*

(f) *See Figure 7.117 for the case* $(c/2)^2 - g/L < 0$. *When* $(c/2)^2 - g/L > 0$ *the picture is similar, but the solutions trajectories do not spiral as they converge to the fixed points.*

**Exercise Solution 7.4.6.**

(a) *Straightforward algebra. Use* $\dot{z} = 0$ *to substitute out* $z = -x$ *in the first equation and find* $-(x+a)x = 0$ *so* $x = 0$ *or* $x = -a$. *Then* $z = 0$ *or* $z = a$ *and* $y = 1$ *or* $y = a+1$ *from the second equation.*

(b) *The Jacobian is*

$$\mathbf{J}(x, y, z) = \begin{bmatrix} z & 0 & x+a \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}.$$

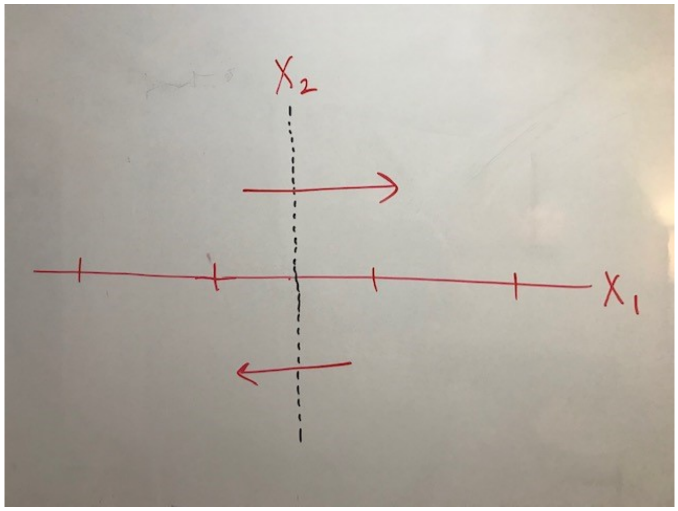



Figure 7.115: $x_1$ nullcline for damped pendulum.

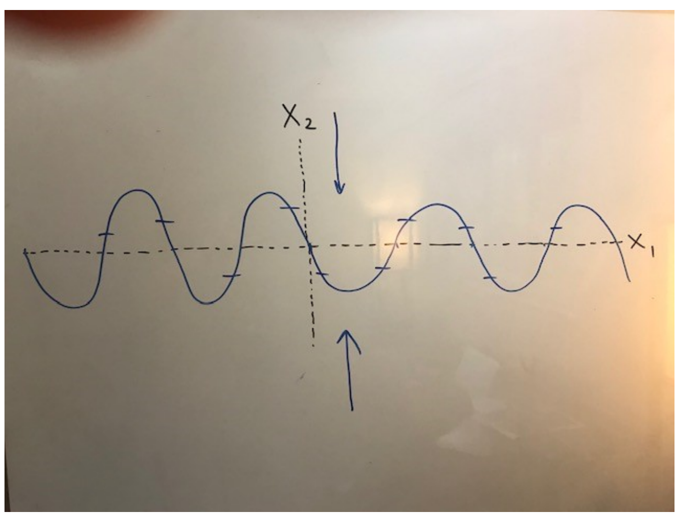



Figure 7.116: $x_2$ nullcline for damped pendulum.

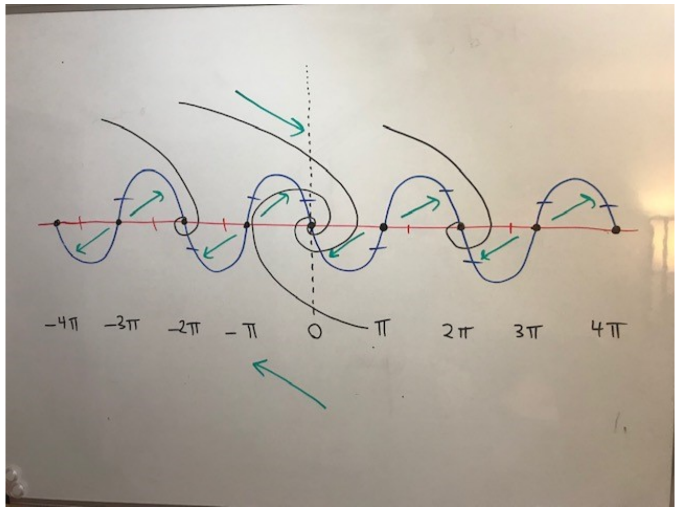



Figure 7.117: Phase portrait for damped pendulum, $(c/2)^2 - g/L < 0$.

*At* $(0, 1, 0)$ *we find*

$$\mathbf{J}(0,1,0) = \begin{bmatrix} 0 & 0 & a \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

*with eigenvalues* $\lambda_1 = -1, \lambda_2 = (-1 + \sqrt{1-4a})/2$, *and* $\lambda_3 = (-1 - \sqrt{1-4a})/2$. *If* $a < 0$ *then* $\lambda_2 > 0$, *so this fixed point is unstable. If* $a > 0$ *then all eigenvalues are negative or have negative real part. The eigenvalues are real when* $a < 1/4$ *and complex with negative real part if* $a > 1/4$.

*(c) At* $(-a, a+1, a)$ *we find*

$$\mathbf{J}(-a,a+1,a) = \begin{bmatrix} a & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

*with eigenvalues* $-1, -1$, *and* $a$. *This fixed point is asymptotically stable for* $a < 0$ *and unstable for* $a > 0$.

*(d) The numerical results support this analysis.*

# Section 7.5

**Exercise Solution 7.5.1.**

(a) *We set* $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $\mathbf{x}^0 = \langle 1, 2\rangle$*. Then with* $\mathbf{f}(t, \mathbf{x}) = \langle x_1 - x_2, x_1 + x_2\rangle$ *we have true solution* $\mathbf{x}(t) = \langle e^t(\cos(t) - 2\sin(t)), e^t(2\cos(t) + \sin(t))\rangle$ *with* $\mathbf{x}(1.0) \approx \langle -3.11, 5.22\rangle$*.*

$$\mathbf{x}^1 = \mathbf{x}^0 + (0.5)\mathbf{f}(0, \langle 1, 2\rangle) = \langle 0.5, 3.5\rangle$$

*and*

$$\mathbf{x}^2 = \mathbf{x}^1 + (0.5)\mathbf{f}(0.5, \langle 0.5, 3.5\rangle) = \langle -1, 5.5\rangle.$$

(b) *We set* $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $\mathbf{x}^0 = \langle 1, 2\rangle$*. Then with* $\mathbf{f}(t, \mathbf{x}) = \langle x_1 + x_2, x_1 + x_2\rangle$ *we have true solution* $\mathbf{x}(t) = \langle -1/2 + 3e^{2t}/2, 1/2 + 3e^{2t}/2\rangle$ *with* $\mathbf{x}(1.0) \approx \langle 10.58, 11.58\rangle$*. Also*

$$\mathbf{x}^1 = \mathbf{x}^0 + (0.5)\mathbf{f}(0, \langle 1, 2\rangle) = \langle 2.5, 3.5\rangle$$

*and*

$$\mathbf{x}^2 = \mathbf{x}^1 + (0.5)\mathbf{f}(0.5, \langle 2.5, 3.5\rangle) = \langle 5.5, 6.5\rangle.$$

(c) *We set* $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $\mathbf{x}^0 = \langle 0, 0, 1\rangle$*. Define* $\mathbf{f}(t, \mathbf{x}) = \langle x_1 x_2 + 1 - t^3, x_1 + x_2 + t - t^2, x_2 x_3 - 1 - t^2 + t^3\rangle$*. Compute*

$$\mathbf{x}^1 = \mathbf{x}^0 + (0.5)\mathbf{f}(0, \langle 0, 0, 1\rangle) = \langle 0.5, 0, 0.5\rangle$$

*and*

$$\mathbf{x}^2 = \mathbf{x}^1 + (0.5)\mathbf{f}(0.5, \langle 0.5, 0, 0.5\rangle) = \langle 0.9375, 0.375, -0.0625\rangle.$$

(d) *The error for each step size is* $0.567, 0.0604,$ *and* $0.00607$*, approximately proportional to* $h$*.*

(e) *The error for each step size is* $0.175$*,* $0.0196$*, and* $0.00199$*, approximately proportional to* $h$*.*

**Exercise Solution 7.5.2.**

(a) *We obviously have* $\dot{x}_1 = x_2$*. From the ODE* $\ddot{\theta}(t) + c\dot{\theta}(t) + \frac{g}{L}\sin(\theta(t)) = 0$ *we find* $\dot{x}_2 + cx_2 + g\sin(x_1)/L = 0$*, from which the second ODE follows.*

(b) *It is easily verified that the analytical solution is* $x_1(t) = \pi, x_2(t) = 0$ *for all* $t$. *Equivalently,* $\theta(t) = \pi, \dot{\theta}(t) = 0$ *for all* $t$. *The pendulum remains perfectly balanced.*

(c) *First, the initial condition* $x_1(0) = \pi$ *cannot be captured perfectly in floating point arithmetic. As a result the pendulum starts off slightly off vertical, and eventually tips over, around* $t = 8$ *in Maple (with* $\pi$ *to ten significant figures). The pendulum then swings back and forth and settles to a stable downward equilibrium.*

(d) *The results here are quite similar. Taking* $h$ *smaller, say* $h = 0.0001$ *with RK4, makes little difference, but representing* $\pi$ *to more significant figures helps.*

(e) *In this case the pendulum eventually tips over and swings around almost perfectly upright again, remains there for some time, then tips over again. The process repeats.*

**Exercise Solution 7.5.3.**

(a) *In this case we have* $x^n = (1 - \lambda h)^n$ *and with* $h = T/n$ *this becomes*

$$x^n = (1 - \lambda T/n)^n.$$

*If we make use of* $\lim_{n\to\infty}(1 - A/n)^n = e^{-A}$ *with* $A = \lambda T$ *we find that* $\lim_{n\to\infty} x^n = e^{-\lambda T} = x(T)$.

(b) *In this case we have* $x^n = 1/(1 + \lambda h)^n$ *and with* $h = T/n$ *this becomes*

$$x^n = \frac{1}{(1 + \lambda T/n)^n}.$$

*If we make use of* $\lim_{n\to\infty}(1 - A/n)^n = e^{-A}$ *and apply this to* $1/x_n$ *with* $A = \lambda T$ *we find that* $\lim_{n\to\infty} 1/x^n = e^{\lambda T}$ *(which is never zero) and so* $\lim_{n\to\infty} x^n = e^{-\lambda T} = x(T)$.

**Exercise Solution 7.5.4.**

(a) *First, the analytical solution is* $x(t) = e^{-0.25t}$.

*Set* $t_0 = 0, t_1 = 0.5, t_2 = 1.0$ *and* $x_0 = 1$. *Then* $x_1$ *satisfies* $x^1 = (0.5)(-0.25x^1) + 1$, *which leads to* $x^1 \approx 0.889$. *Then* $x_2$ *satisfies* $x^2 = (0.5)(-0.25x^2) + 0.889$, *which leads to* $x^1 \approx 0.790$. *The true solution value is* $x(1) = e^{-0.25} \approx 0.779$.

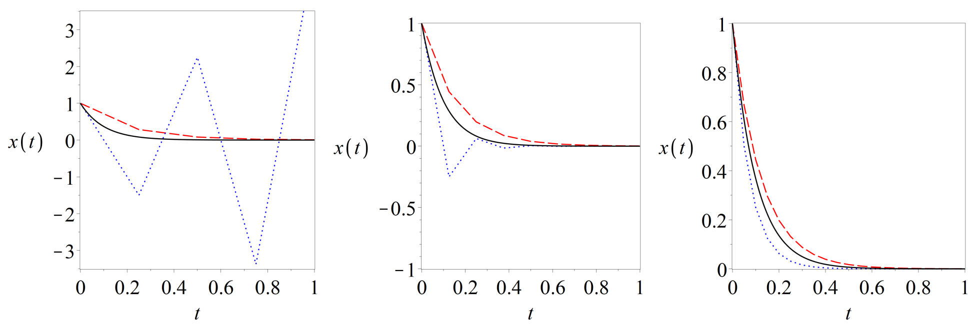



Figure 7.118: Left panel: step size $h = 0.25$ for ODE $x' = -10x$ with $x(0) = 1$ for Euler's method (dotted/blue), the implicit Euler method (dashed/red) and true solution $x(t) = e^{-10t}$ (solid/black). Middle panel: same, step size $h = 0.125$. Right panel: same, step size $h = 0.05$.

*(b)* *Set $t_0 = 0, t_1 = 1, t_2 = 2$ and $x_0 = 1$. Then $x_1$ satisfies $x^1 = 0.5x^1(2 - x^1) + 1$, which leads to $x^1 = \sqrt{2} \approx 1.4142$. Then $x_2$ satisfies $x^2 = 0.5x^2(2 - x^2) + 1.4142$, which leads to $x^2 \approx 1.682$. The true solution is $x(t) = 2/(1 + e^{-t})$ so $x(2) = 2/(1 + e^{-2}) \approx 1.762$.*

*(c)* *We have $t_1 = 1, t_2 = 2, t_3 = 3$ and $\mathbf{x}^0 = \langle 1, 3\rangle$. Then $\mathbf{x}^1 \approx \langle -0.167, 0.167\rangle$, $\mathbf{x}^2 \approx \langle -0.194, -0.139\rangle$, and $\mathbf{x}^3 \approx \langle -0.116, -0.106\rangle$. The true solution is $\mathbf{x}(t) = \langle 2e^{-5t} - e^{-t}, 4e^{-5t} - e^{-t}\rangle$ and $\mathbf{x}(3) \approx \langle -0.0498, -0.0498\rangle$.*

*(d)* *With $t_0 = 0, t_1 = 0.2, t_2 = 0.4, t_3 = 0.6, t_4 = 0.8, t_5 = 1.0$ and $\mathbf{x}^0 = \langle 1, 3\rangle$ we find iterates*

$$\mathbf{x}^1 \approx \langle 0.589, 2.402\rangle, \mathbf{x}^2 \approx \langle 0.204, 1.968\rangle, \mathbf{x}^3 \approx \langle -0.125, 1.660\rangle,$$
$$\mathbf{x}^4 \approx \langle -0.385, 1.448\rangle, \mathbf{x}^5 \approx \langle -0.579, 1.303\rangle.$$

**Exercise Solution 7.5.5.**

*(a)* *See the left panel of Figure 7.118 for step size $h = 0.25$, the middle panel for $h = 0.15$, and the right panel for $h = 0.05$. According to (7.48) (with $\lambda = 10$) the iterates here converge to zero when $h < 0.2$, which is in accordance with the figure. From Reading Exercise 7.5.4 the iterates should remain positive when $h < 0.1$, which again seems correct.*

*(b)* *The analytical solution is $x_1(t) = 3e^{-t} - 2e^{-5t}$, $x_2(t) = 3e^{-t} - 4e^{-5t}$. See Figure 7.119 for parametric plots. When $h = 1.0$ the solution goes well outside the view range.*

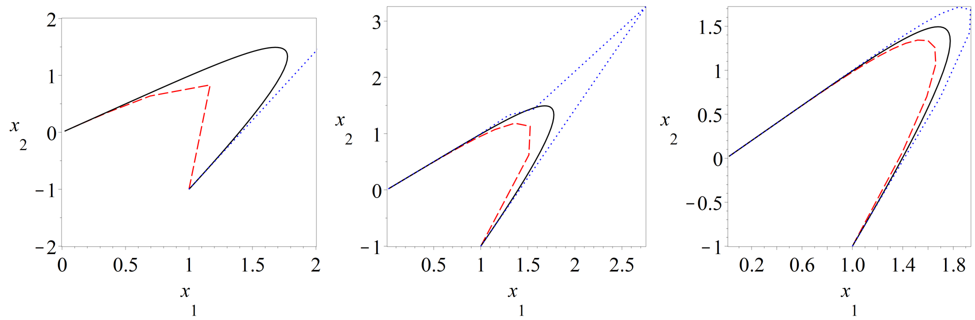



Figure 7.119: Left panel: parametric plot $x_1(t)$ vs $x_2(t)$ for step size $h = 1.0$ for Euler's method (dotted/blue), the implicit Euler method (dashed/red) and true solution $x_1(t) = 3e^{-t} - 2e^{-5t}$, $x_2(t) = 3e^{-t} - 4e^{-5t}$ (solid/black). Middle panel: same, step size $h = 0.25$. Right panel: same, step size $h = 0.1$.

**Exercise Solution 7.5.6.**

*(a)* *The true solution is* $x(t) = t - 1 + 2e^{-t}$ *and* $x(1) = 2/e$. *The errors for implicit Euler with step sizes* $h = 0.1, 0.01, 0.001,$ *and* $0.0001$ *are* $0.0353276965, 0.0036635421, 0.0003677258, 0.0000367826$*, respectively.*

*(b)* *The analytical solution is* $x_1(t) = 6e^{-t} - 52e^{-5t}/25 + 13t/5 - 73/25$, $x_2(t) = 6e^{-t} - 104e^{-5t}/25 + 11t/5 - 71/25$. *The errors for* $h = 0.1, 0.01,$ *and* $0.001$ *are* $0.104268966843117747, 0.0117885987798727332,$ *and* $0.00119297073597383397$.

**Exercise Solution 7.5.7.**

*(a)* *The errors for the trapezoidal method are* $0.0006137976, 6.13 \times 10^{-6}$, *and* $6.2 \times 10^{-8}$.

*(b)* *The errors for the trapezoidal method are* $0.0011104183, 0.000011436$, *and* $1.145 \times 10^{-7}$.

**Exercise Solution 7.5.8.**

*(a)* *This is simply the fundamental theorem of calculus in the form*

$$x(T) - x(t_0) = \int_{t_0}^{T} \dot{x}(t)\, dt = \int_{t_0}^{T} f(t)\, dt$$

*along with the observation that* $x(t_0) = 0$.

(b) *From* $x_{k+1} = x_k + hf(t_k)$ *we have* $x_1 = x_0 + hf(t_0) = f(t_0)$ *(since* $x_0 = x(t_0) = 0$*). Then* $x_2 = x_1 + hf(t_1) = h(f(t_0) + f(t_1))$, $x_3 = x_2 + hf(t_2) = h(f(t_0) + f(t_1) + f(t_2)$, *and so on. We find*

$$x_n = h(f(t_0)+f(t_1)+\cdots+f(t_{n-1})) = \frac{T-t_0}{n}(f(t_0)+f(t_1)+\cdots+f(t_{n-1})).$$

*The right side is just the familiar left Riemann sum approximation to* $\int_{t_0}^{T} f(t)\,dt$ *from Calculus 2.*

(c) *The implicit Euler method takes the form* $x^k = x^{k-1} + hf(t_k)$*. Then* $x^1 = x^0 + hf(t_1) = hf(t_1)$, $x^2 = x^1 + hf(t_2) = h(f(t_1) + f(t_2))$, *and more generally*

$$x^n = h(f(t_1)+f(t_2)+\cdots+f(t_n)) = \frac{T-t_0}{n}(f(t_1)+f(t_2)+\cdots+f(t_n)).$$

*The resulting approximation* $x^n$ *to* $x(T)$ *is the right Riemann sum approximation to* $\int_{t_0}^{T} f(t)\,dt$.

**Exercise Solution 7.5.9.**

(a) *The system is* $\dot{x}_1 = x_2, \dot{x}_2 = -101x_1 - 2x_2$ *with* $\mathbf{x}(0) = \langle 1, 0\rangle$.

(b) *The eigenvalues and eigenvectors of* $\mathbf{A}$ *are* $-1\pm 10i$ *and* $\langle -1-10i, 101\rangle$ *and* $\langle -1+10i, 101\rangle$*, respectively. A real-valued general solution is*

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{-t}\sin(10t) \\ e^{-t}(10\cos(10t) - \sin(10t)) \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\cos(10t) \\ -e^{-t}(\cos(10t) + 10\sin(10t)) \end{bmatrix}.$$

*With the given initial data the solution is*

$$\mathbf{x}(t) = e^{-t} \begin{bmatrix} \cos(10t) + \sin(10t)/10 \\ -101\sin(10t)/10 \end{bmatrix}.$$

*The solution spirals toward the asymptotically stable fixed point at* $\langle 0, 0\rangle$.

(c) *The true solution value is* $\mathbf{x}(5) \approx \langle 0.0063, 0.0173\rangle$*. Implicit Euler gives estimate* $\langle 1.52\times 10^{-9}, 1.79\times 10^{-9}\rangle$*. Standard Euler's method explodes. A plot is shown in the left panel of Figure 7.120.*

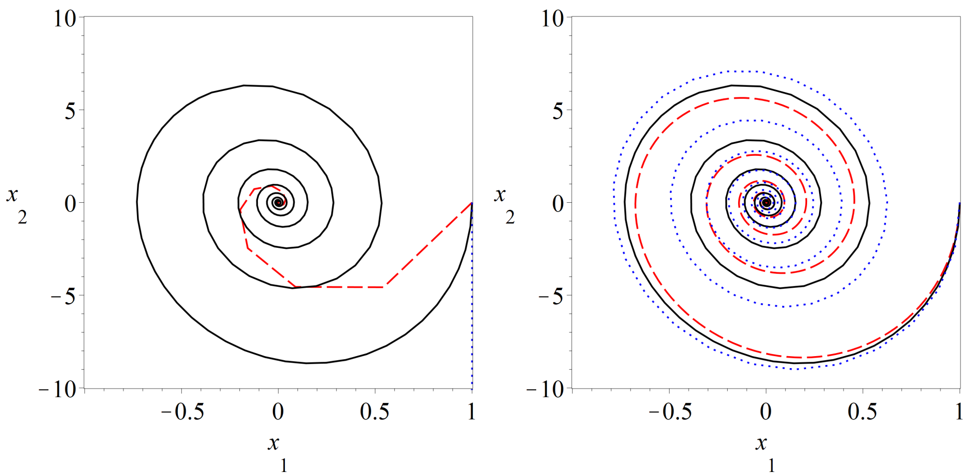



Figure 7.120: Left panel: True solution (solid/black), Euler estimate (dotted/blue) and implicit Euler (dashed/red), step size $h = 0.1$. Right panel: Same, but with $h = 0.005$.

(d) *A step size $h \leq 0.005$ tames Euler's method. With $h = 0.005$ implicit Euler gives estimate $\langle 0.00158, 0.0106 \rangle$. Standard Euler's method gives $\langle 0.0233, 0.0134 \rangle$. A plot is shown in the right panel of Figure 7.120.*

**Exercise Solution 7.5.10.**

(a) *The matrix* $\mathbf{A}$ *is*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10001 & 1/10 & 10000 & 0 \\ 0 & 0 & 0 & 1 \\ 10^7 & 0 & -10^7 & -100 \end{bmatrix}$$

*with approximate eigenvalues $-0.1 \pm 994i$ and $-49.95 \pm 3163i$. This means any solution contains products of the form $e^{\alpha t}\sin(\beta t)$ and $e^{\alpha t}\cos(\beta t)$ with $\alpha = -0.1, \beta = 0.994$ or $\alpha = -49.95, \beta = 3163$. Terms involving the former decay on a scale of about 50 seconds and oscillate with a period of about $2\pi/0.994 \approx 6.32$ seconds, while terms involving the latter values decay on a scale of $0.1$ seconds and oscillate with a period of about $2\pi/3163 \approx 0.002$ seconds. Corresponding frequencies are 0.158 Hz and 503 Hz.*

(b) *See Figure 7.121. The left panel is $w_1(t)$ on $0 \leq t \leq 0.5$, the right*

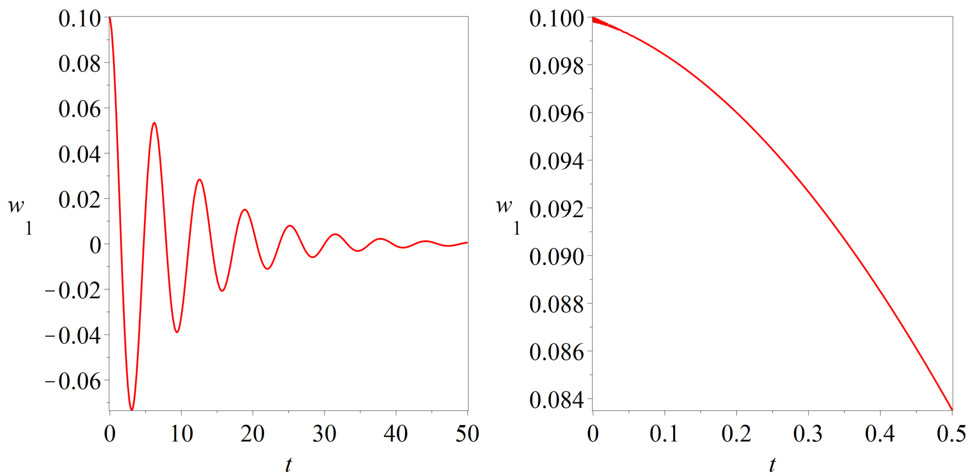



Figure 7.121: Left panel: Function $w_1(t)$ on the interval $0 \le t \le 5$. Right panel: Function $w_1(t)$ on the interval $0 \le t \le 0.5$.

*panel is $w_1(t)$ for $0 \le t \le 0.5$. In the right panel some of the initial fast oscillations can be seen for $t < 0.01$.*

(c) *The numerical solution grows rapidly and is meaningless.*

(d) *See Figure 7.122 in which the graph of the analytical solution $w_1(t)$ is shown as a solid red curve and the approximate solution obtained from the implicit Euler method as the dashed blue curve. The curves are difficult to distinguish. The true value for $w_1(5)$ is approximately* 0.01558*, while the implicit Euler estimate is* 0.0149*.*

**Exercise Solution 7.5.11.**

(a) *Refer to Figure 7.125. From Kirchhoff's voltage law applied to the loop involving $R_1$ and $C_1$ we have*

$$R_1 I_1(t) + q_1(t)/C_1 = 0. \tag{7.6}$$

*For the loop involving $C_1, C_2,$ and $R_2$ we have*

$$R_2 I_3(t) + q_2(t)/C_2 - q_1(t)/C_1 = 0. \tag{7.7}$$

*At the node labeled "N" between $R_1, R_2,$ and $C_1$ we must have*

$$I_1 - I_2 - I_3 = 0. \tag{7.8}$$

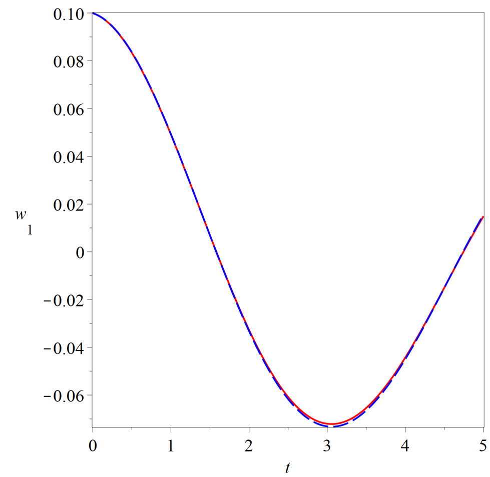



Figure 7.122: Analytical solution $w_1(t)$ shown as solid red graph, approximation from the implicit Euler method as the dashed blue curve.

*Finally, we also have*

$$\dot{q}_1 = I_2 \tag{7.9}$$
$$\dot{q}_2 = I_3. \tag{7.10}$$

*Using (7.8) to eliminate* $I_1$ *in (7.6)-(7.7), then (7.9)-(7.10) to replace* $I_2, I_3$*, and finally solving for* $\dot{q}_1$ *and* $\dot{q}_2$ *leads to*

$$\dot{q}_1(t) = -\left(\frac{1}{R_2C_1} + \frac{1}{R_1C_1}\right) q_1(t) + \frac{1}{R_2C_2} q_2(t) \tag{7.11}$$
$$\dot{q}_2(t) = \frac{1}{R_2C_1} q_1(t) - \frac{1}{R_2C_2} q_2(t) \tag{7.12}$$

*a linear system of ODE's for functions* $q_1(t), q_2(t)$*.*

*(b) A straightforward substitution of* $C_1 = C_2 = 0.001$ *farad,* $R_1 = 1000$ *ohms, and* $R_2 = 0.1$ *ohm, yields*

$$\dot{q}_1(t) = -10001 q_1(t) + 10000 q_2(t) \tag{7.13}$$
$$\dot{q}_2(t) = 10000 q_1(t) - 10000 q_2(t). \tag{7.14}$$

*The solution with* $q_1(0) = 0.000001$ *and* $q_2(0) = 0.000001$ *is easily*

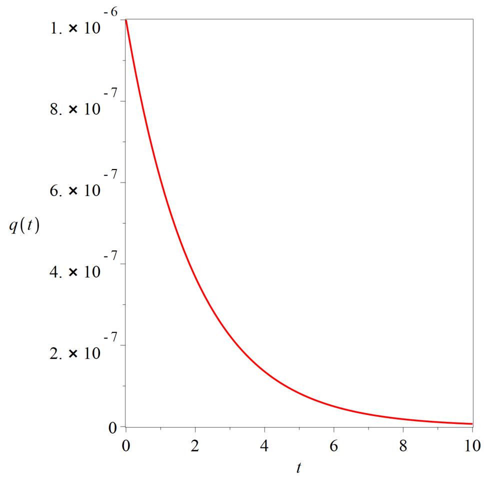



Figure 7.123: Charge $q_1(t)$ on capacitor $C_1$.

*obtained via eigenvalue analysis and is*

$$q_1(t) = (9.99975 \times 10^{-7})e^{-0.49999t} + (2.5 \times 10^{-11})e^{-20000.5t} \tag{7.15}$$

$$q_2(t) = (1.0 \times 10^{-6})e^{-0.49999t} - (2.5 \times 10^{-11})e^{-20000.5t}. \tag{7.16}$$

(c) *The function $q_1(t)$ is shown in Figure 7.123. The graph of $q_2(t)$ is virtually identical.*

(d) *The Euler algorithm blows up with $h = 0.1$, and we have to decrease $h$ to less than about $0.001$ to obtain good results.*

(e) *The function $q_1(t)$ is shown in Figure 7.124 as the solid black graph and the implicit Euler estimate with step size $h = 0.1$ as the dashed red graph. The estimate is very good.*

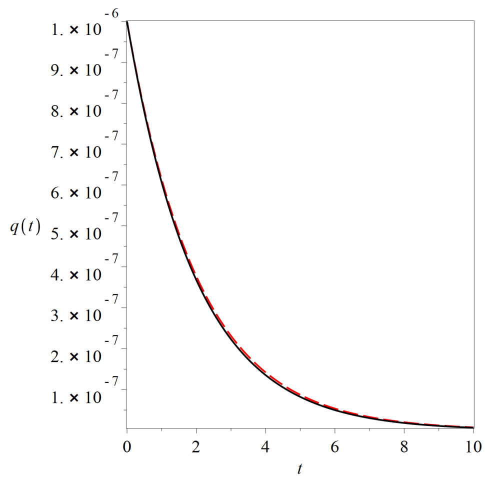



Figure 7.124: Charge $q_1(t)$ on capacitor $C_1$ (solid black) and implicit Euler estimate (dashed red).

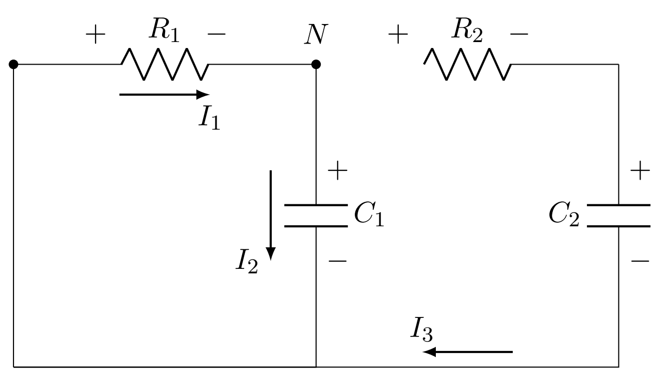



Figure 7.125: Double-loop RC circuit for Exercise 7.5.11.

# Section 7.6

**Exercise Solution 7.6.1.**

(a) *Compute* $D_1 = a_1$ *and* $D_2 = a_1 a_2$ *(recall that* $a_k = 0$ *for* $k \geq n$*, so* $a_3 = 0$ *here). The roots of* $p(z) = z^2 + a_1 z + a_2$ *thus have negative real part if and only if* $a_1 > 0$ *and* $a_1 a_2 > 0$*. But* $a_1 > 0$ *and* $a_1 a_2 > 0$ *implies* $a_2 > 0$*, and conversely* $a_1 > 0$ *and* $a_2 > 0$ *implies* $a_1 a_2 > 0$*.*

(b) *Compute* $D_1 = a_1, D_2 = a_1 a_2 - a_3$*, and* $D_3 = (a_1 a_2 - a_3) a_3$*. The roots of the polynomial* $p(z) = z^3 + a_1 z^2 + a_2 z + a_3$ *all have negative real part exactly when* $D_1, D_2$*, and* $D_3$ *are all positive, so* $a_1 > 0, a_1 a_2 - a_3 > 0$*, and* $a_3(a_1 a_2 - a_3) > 0$*. The last condition* $a_3(a_1 a_2 - a_3) > 0$ *can be replaced by* $a_1 a_2 - a_3 > 0$ *when* $a_3 > 0$*.*

(c) *Compute* $D_1 = a_1$*,* $D_2 = a_1 a_2 - a_3$*,* $D_3 = (a_1 a_2 - a_3) a_3 - a_1^2 a_4$*, and* $D_4 = (a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) a_4$*; note that* $D_4 = a_4 D_3$*. The simplest condition is to require that each of* $D_1, D_2, D_3$*, and* $D_4$ *be positive, although* $D_4$ *is positive if and only if* $a_4 > 0$ *and* $D_3 > 0$*.*

**Exercise Solution 7.6.2.** *If*

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

*then the characteristic polynomial of* $\mathbf{A}$ *is*

$$p(\lambda) = \lambda^2 - T\lambda + D$$

*where* $T = tr(\mathbf{A}) = a_{1,1} + a_{2,2}$ *is the trace of* $\mathbf{A}$ *and* $D = det(\mathbf{A}) = a_{2,2} a_{1,1} - a_{1,2} a_{2,1}$ *is the determinant of* $\mathbf{A}$*. For* $p$ *to have roots with negative real part (so* $\mathbf{A}$ *has eigenvalues with negative real part) we need, from Routh-Hurwitz,* $-T > 0$ *(which is equivalent to* $T < 0$*), and* $D > 0$*. These are in exact accordance with parts (2) and (3) of the trace-determinant theorem.*

**Exercise Solution 7.6.3.**

(a) *The system is* $\dot{x}_1 = x_2$*,* $m\dot{x}_2 = 0$ *(or just* $\dot{x}_2 = 0$*, since* $m > 0$*). Then* $\mathbf{f}(\mathbf{x}) = \langle x_2, 0 \rangle$*.*

(b) *We have* $\nabla P = \langle 0, m \rangle$ *and then* $\nabla P \cdot \mathbf{f} = 0$*, so* $P$ *is a first integral and represents a conserved quantity. The function* $P$ *is just the momentum* $m\dot{x}$ *of the particle, so this is conservation of momentum.*

(c) *We compute* $\nabla E = \langle 0, mu_2 \rangle$ *and so* $\nabla E \cdot \mathbf{f} = 0$*, so* $E$ *is a first integral and represents a conserved quantity. The function* $E$ *is just the kinetic energy* $m\dot{x}^2/2$ *of the particle, so this is conservation of energy in some form.*

*In this very simple setting, in both (b) and (c) here the essential fact is that* $\dot{x}$ *is constant.*

**Exercise Solution 7.6.4.**

(a) *The system is* $\dot{x}_1 = x_2$ *and* $\dot{x}_2 = -kx_1/m$*, or* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *where* $\mathbf{f}(\mathbf{x}) = \langle x_2, -kx_1/m \rangle$ *and* $\mathbf{x} = \langle x_1, x_2 \rangle$*.*

(b) *Compute* $\nabla E = \langle ku_1, mu_2 \rangle$ *and so* $\nabla E \cdot \mathbf{f} = 0$*. This expresses that the total energy* $E$ *of the frictionless system (kinetic energy* $m\dot{x}^2$ *plus potential energy* $kx^2/2$ *stored in the spring) remains constant in time. The system trajectory in the* $x_1 x_2$ *phase plane is a curve defined implicitly by* $E(x_1, x_2) = c$ *for some constant* $c > 0$*. This curve is an ellipse centered at the origin.*

(c) *The system is now* $\dot{x}_1 = x_2$ *and* $\dot{x}_2 = -kx_1/m - cx_2/m$*, so* $\mathbf{f}(\mathbf{x}) = \langle x_2, -kx_1/m - cx_2/m \rangle$*. With* $\nabla E = \langle ku_1, mu_2 \rangle$ *we find* $\nabla E(\mathbf{u}) \cdot \mathbf{f}(\mathbf{u}) = -cu_2^2 \leq 0$*. According to Theorem 7.6.1 (with* $V = E$ *there) the fixed point* $\mathbf{x}^* = \langle 0, 0 \rangle$ *is stable. We may also cite Theorem 7.6.3 by noting that the set of points* $(u_1, u_2)$ *with* $\nabla E \cdot \mathbf{f} = 0$ *consists exactly of those points of the form* $(u_1, 0)$*, and this set does not contain any solution trajectories except* $(0, 0)$ *(such a trajectory would have* $x_2(t) \equiv 0$ *for all* $t$*, but then* $\dot{x}_2 = -kx_1/m - cx_2/m$ *yields* $x_1(t) = 0$*.) We conclude that the origin* $\mathbf{x}^* = \langle 0, 0 \rangle$ *is asymptotically stable, which is the mass motionless at equilibrium.*

**Exercise Solution 7.6.5.** *It's easy to check that* $x_1 = x_2 = 0$ *is an isolated fixed point. A direction field is shown in the left panel of Figure 7.126, with a few solution curves and the level curves for the function* $V(x_1, x_2) = x_1^2 + x_2^2$*.*

*The linearized system at the origin has Jacobian matrix*

$$\mathbf{J}(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*with double eigenvalue* $0$*, which does not allow us to make any conclusion about stability. With* $V(x_1, x_2) = x_1^2 + x_2^2$ *with have* $\nabla V = \langle 2x_1, 2x_2 \rangle$ *and with* $\mathbf{f}(\mathbf{x}) = \langle -x_1^3, -x_2^3$ *we find* $\nabla V \cdot \mathbf{f} = -2(x_1^4 + x_2^4) < 0$ *for* $(x_1, x_2) \neq (0, 0)$*. We conclude that this fixed point is asymptotically stable.*

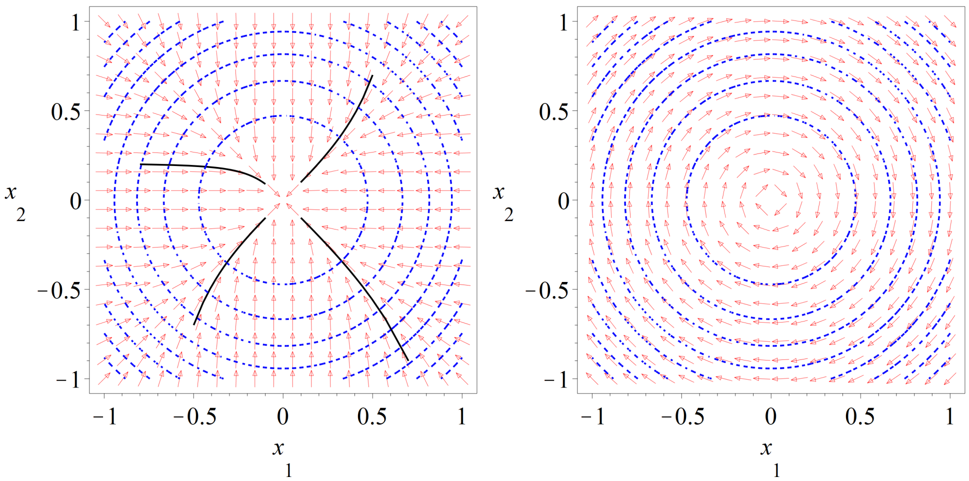



Figure 7.126: Left panel: Direction field and solution curves (solid black) for system $\dot{x}_1 = -x_1^3, \dot{x}_2 = -x_2^3$, with level curves for $V(x_1, x_2) = x_1^2 + x_2^2$ (dashed blue). Right panel: same, for system $\dot{x}_1 = x_2, \dot{x}_2 = -x_1$.

**Exercise Solution 7.6.6.** *An easy computation shows that $x_1 = x_2 = 0$ is an isolated fixed point. A direction field is shown in the right panel of Figure 7.126 with level curves for the function $V(x_1, x_2) = x_1^2+x_2^2$ (see computations below). The linearized system at the origin has Jacobian matrix*

$$\mathbf{J}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

*with eigenvalues $\pm i$. These both have zero real part, which does not allow us to make any conclusion about stability. With $V(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ with have $\nabla V = \langle 2ax_1 + bx_2, bx_1 + 2cx_2 \rangle$ and with $\mathbf{f}(\mathbf{x}) = \langle x_1, -x_2 \rangle$ we find*

$$\nabla V \cdot \mathbf{f} = b(x_2^2 - x_1^2) + 2x_1x_2(a - c).$$

*In order for this to be positive semidefinite we need $b = 0$, so that $V(x_1, x_2) = ax_1^2+bx_2^2$, which will be positive definite if both $a$ and $c$ are nonzero. However, for $\nabla V \cdot \mathbf{f} \geq 0$ we need $a = c$, in which case we obtain a valid Lyapunov function $V(x_1, x_2) = ax_1^2 + ax_2^2$ with $\nabla V \cdot \mathbf{f} = 0$, so this $V$ is actually a first integral. We conclude this fixed point is stable, but we cannot conclude it is asymptotically stable (which it is not, since trajectories are closed orbits that coincide with the level curves of $V$.)*

**Exercise Solution 7.6.7.** *This system has infinitely many fixed points, all along the diagonal line* $x_2 = -x_1/2$*; see the left panel in Figure 7.127, in which the direction field is plotted. The fixed points are shown along the dashed blue line, and a few solution trajectories are shown as solid black curves. The Jacobian at each fixed point is*

$$\mathbf{J} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}$$

*with eigenvalues* $0$ *and* $-5$*, which does not (by itself) allow us to make conclusions about the stability of any of these fixed points. For the Lyapunov approach, if we take* $V(x_1, x_2) = x_1^2 + x_2^2$ *as suggested, a straightforward computation shows that* $\nabla V \cdot \mathbf{f} = -2x_1^2 - 8x_1x_2 - 8x_2^2$*. This last expression factors as* $-2(x_1 - 2x_2)^2$*, which is non-positive for all* $x_1$ *and* $x_2$*. We can conclude that fixed point at* $(0, 0)$ *(and in fact, any of the fixed points) is stable. We cannot conclude that any given fixed point is asymptotically stable, since they are not isolated. In fact by solving the system analytically we can see that the solution trajectories that start at a point* $(a, b)$ *are straight lines that converge to the fixed point* $((4a - 2b)/5, (-2a + b)/5)$*.*

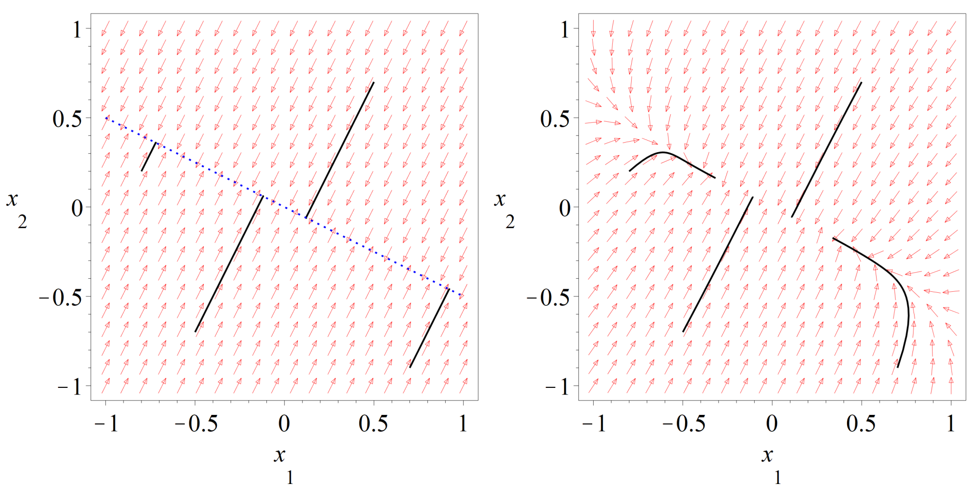



Figure 7.127: Left panel: Direction field and fixed points (dashed blue line) for system $\dot{x}_1 = -x_1 - 2x_2, \dot{x}_2 = -2x_1 - 4x_2$, with solution trajectories (solid black). Right panel: Direction field and solution trajectories (solid black) for system $\dot{x}_1 = -x_1 - 2x_2 - x_1^3, \dot{x}_2 = -2x_1 - 4x_2$, with solution trajectories (solid black).

**Exercise Solution 7.6.8.** *Straightforward algebra shows that this system has an isolated fixed point at* $x_1 = x_2 = 0$*; see the right panel in Figure 7.127, in which the direction field is plotted. The Jacobian at* $(0,0)$ *is*

$$\mathbf{J} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}$$

*with eigenvalues* $0$ *and* $-5$ *(same as part (c)), which does not by itself allow us to make conclusions about the stability of this fixed point. For the Lyapunov approach, if we take* $V(x_1, x_2) = x_1^2 + x_2^2$ *as suggested, a straightforward computation shows that*

$$\nabla V \cdot \mathbf{f} = -2x_1^4 - 2x_1^2 - 8x_1x_2 - 8x_2^2 = -2x_1^4 - 2(x_1 + 2x_2)^2.$$

*It's not hard to see that this expression is always negative if* $(x_1, x_2) \neq (0,0)$*. We can conclude that the origin is asymptotically stable for this system.*

**Exercise Solution 7.6.9.** *Straightforward algebra shows that this system has an isolated fixed point at* $x_1 = x_2 = 0$*. The Jacobian at* $(0,0)$ *is the zero matrix with double eigenvalue* $0$*, which does not allow us to make conclusions about the stability of this fixed point. For the Lyapunov approach, if we take* $V(x_1, x_2) = x_1^4 + x_2^4$ *as suggested, a straightforward computation shows that*

$$\nabla V \cdot \mathbf{f} = 0$$

*Thus this is a stable fixed point, but we cannot assert asymptotic stability. In fact, the solutions form closed orbits.*

**Exercise Solution 7.6.10.** *A bit of easy algebra shows that* $x_1 = x_2 = x_3 = 0$ *is the only fixed point for this system. With* $V(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2$ *we obtain*

$$\nabla V \cdot \mathbf{f} = -4\,a{x_1}^2{x_2}^4 - 8\,b{x_1}^2{x_2}^4 - 4\,a{x_1}^2{x_3}^2 - 4\,c{x_3}^4 - 4\,b{x_2}^2 - 4\,c{x_3}^2$$

*which is easily seen to be non-positive for any choice of* $a, b, c$ *all positive (which also makes* $V$ *itself positive definite). Thus the origin is stable, but no choice for* $a, b, c$ *works to prove asymptotic stability (if* $x_2 = x_3 = 0$ *we can take any value for* $x_1$*.) The Jacobian at the origin is*

$$\mathbf{J}(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

**Exercise Solution 7.6.11.** *A bit of easy algebra shows that* $x_1 = x_2 = x_3 = 0$ *is the only fixed point for this system. With* $V(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2$ *we obtain*

$$\nabla V \cdot \mathbf{f} = -8\,ax^4 - 4\,ax^2y^2 - 4\,ax^2z^2 - 4\,bx^2y^2 - 4\,cz^4 - 4\,by^2 - 4\,cz^2$$

*which is easily seen to be non-positive for any choice of* $a, b, c$ *all positive (which also makes* $V$ *itself positive definite). Thus the origin is stable. And (unlike the last problem, part (f)) this Lyapunov function yields* $\nabla V \cdot \mathbf{f} < 0$ *if we choose, for example,* $a = b = c = 1$*. Thus the origin here is asymptotically stable. The Jacobian at the origin is*

$$\mathbf{J}(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

*and yields no hard information about the stability.*

**Exercise Solution 7.6.12.** *Since* $k_a, k_b$*, and* $k_e$ *are all positive by assumption, it's clear that* $V(\mathbf{x}) = k_b x_1^2 + k_a x_2^2$ *satisfies* $V > 0$ *for* $\mathbf{x} \neq \mathbf{0}$*. It is also straightforward to verify what the hint says,*

$$\nabla V \cdot \mathbf{f} = -2(k_b x_1 - k_a x_2)^2 - 2k_b k_e x_1^2.$$

*Both terms are non-positive. If* $x_1 \neq 0$ *then the term* $-2k_b k_2 x_1^2$ *on the right side above is negative, and hence so is the right side. If* $x_1 = 0$ *and* $x_2 \neq 0$ *the first term equals* $-2k_a^2 x_2^2$ *and is negative unless* $x_2 = 0$*. Thus* $\nabla V \cdot \mathbf{f} < 0$ *if* $\mathbf{x}^* \neq \mathbf{0}$ *and this is a strict Lyapunov function. The origin is thus asymptotically stable (something we already knew from the eigenvalue solution procedure).*

**Exercise Solution 7.6.13.** *With* $V(x) = (x - K)^2$ *with have* $\nabla V = 2(x - K)$ *(of course* $\nabla V$ *is really just* $dV/dx$ *here). Then*

$$\nabla V \cdot f(x) = 2rx(1 - x/K)(x - K) = -2rx(x - K)^2/K$$

*which clearly satisfies* $\nabla V \cdot f < 0$ *for* $x$ *near* $K$*. We conclude that* $x^* = K$ *is an asymptotically stable fixed point. The basin of attraction is all positive* $x$*.*

**Exercise Solution 7.6.14.**

(a) *Since* $Q(1, 0) = a$ *this follows immediately.*

(b) *Since* $Q(-b/2a, 1) = c - b^2/4a$*, if* $Q < 0$ *then* $c - b^2/4a < 0$*. This inequality is equivalent to* $c < b^2/4a$*. Multiply by* $4a$ *(noting that* $a < 0$ *from part (a)) to find* $4ac > b^2$*, or* $b^2 - 4ac < 0$*.*

(c) *This is a straightforward algebraic verification.*

(d) *Given that*

$$Q(x, y) = a\left(\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right).$$

*it's easy to check that* $(4ac - b^2)/4a^2 > 0$*, so the quantity in the outer pair of parentheses above is always nonnegative. Any since either* $x \neq 0$ *or* $y \neq 0$ *it's easy to see that this quantity is positive. Since* $a < 0$ *it follows that* $Q < 0$*.*

**Exercise Solution 7.6.15.**

(a) *The* $ax_1$ *term in the first ODE embodies the exponential growth of the prey species, while* $-bx_1x_2$ *embodies the statement that the "presence of the predator negatively impacts the prey population"; this impact is assumed to be jointly proportional to* $x_1$ *and* $x_2$*. The* $-cx_2$ *term in the second ODE captures the exponential decay of the predator population, while the* $dx_1x_2$ *term captures "the presence of the prey species gives the predators sustenance," again in joint proportion to* $x_1$ *and* $x_2$*.*

(b) *Setting* $\dot{x_1} = 0$ *yields* $ax_1 - bx_1x_2 = x_1(a - bx_2) = 0$*, so either* $x_1 = 0$ *or* $x_2 = a/b$*. Setting* $\dot{x_2} = 0$ *yields* $-cx_2 + dx_1x_2 = x_2(-c + dx_2) = 0$*, so either* $x_2 = 0$ *or* $x_1 = c/d$*. If* $x_1 = 0$ *then we must have* $x_2 = 0$*, while if* $x_2 = a/b$ *then we must have* $x_1 = c/d$*.*

(c) *The Jacobian matrix is*

$$\mathbf{J}(x_1, x_2) = \begin{bmatrix} a - bx_2 & -bx_1 \\ dx_2 & -c + dx_1 \end{bmatrix}$$

*At the origin* $x_1 = x_2 = 0$ *(mutual extinction) we have*

$$\mathbf{J}(0, 0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

*with real eigenvalues* $a$ *and* $-c$*, of mixed sign (since by assumption* $a, b, c,$ *and* $d$ *are positive). The origin is always a saddle point.*

*At* $x_1 = c/d, x_2 = a/b$ *we find*

$$\mathbf{J}(c/d, a/b) = \begin{bmatrix} 0 & -bc/d \\ ad/b & 0 \end{bmatrix}$$

*with purely imaginary eigenvalues* $\pm i\sqrt{ac}$. *Since these have zero real part, the Hartmann-Grobman theorem does not allow us to make any claim about the stability of this fixed point.*

(d) *Given* $V(x_1, x_2) = dx_1 - c\ln(x_1) + bx_2 - a\ln(x_2)$ *we can compute*

$$\nabla V(x_1, x_2) = \langle d - c/x, b - a/y\rangle.$$

*The ODEs are of the form* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *where* $\mathbf{f}(\mathbf{x}) = \langle ax_1 - bx_1x_2, -cx_2 + dx_1x_2\rangle$ *and it is easy to check that* $\nabla V \cdot \mathbf{f} = 0$.

*To check that* $V$ *is a strict local minimum for* $V$ *we can use the second derivative test from Calculus 3. We find that* $V_{xx}V_{yy} - V_{xy}^2 = ca/(x^2y^2)$ *and at* $x = d/c, y = a/b$ *this evaluates to* $b^2d^2/(ac) > 0$. *Since* $V_{xx}(c/d, a/b) = d^2/c > 0$, *this point is a strict local minimum. In fact a plot of* $V$ *in the first quadrant (as in the next part) makes it clear that this fixed point is the only minimizer for* $V$ *in the first quadrant.*

(e) *A graph of* $V$ *with the choices* $a = 1, b = 2, c = 1, d = 3$ *is shown in Figure 7.128. The level curves for* $V$ *are the trajectories for the predator-prey ODE system, which makes it clear that the solutions are periodic, especially when we look at the contour plot in the right panel.*

(f) *The* $x_1$ *nullcline is the vertical axis* $x_1 = 0$ *(the* $x_2$ *axis) and the horizontal line* $x_2 = a/b$. *The* $x_2$ *nullcline is the horizontal axis* $x_2 = 0$ *(the* $x_1$ *axis) and the vertical line* $x_1 = c/d$. *These divide the first quadrant into four rectangular pieces (unbounded). Solutions spiral counterclockwise around the fixed point in closed loops.*

(g) *One significant critique is that the prey species can never go extinct, now matter how that population falls. The same observation holds for the predator species.*

**Exercise Solution 7.6.16.**

(a) *The first assumption that "The overall population* $N(t) = S(t) + I(t) + R(t)$ *grows according to* $\dot{N} = rN$ *for some positive growth rate* $r$*" and*

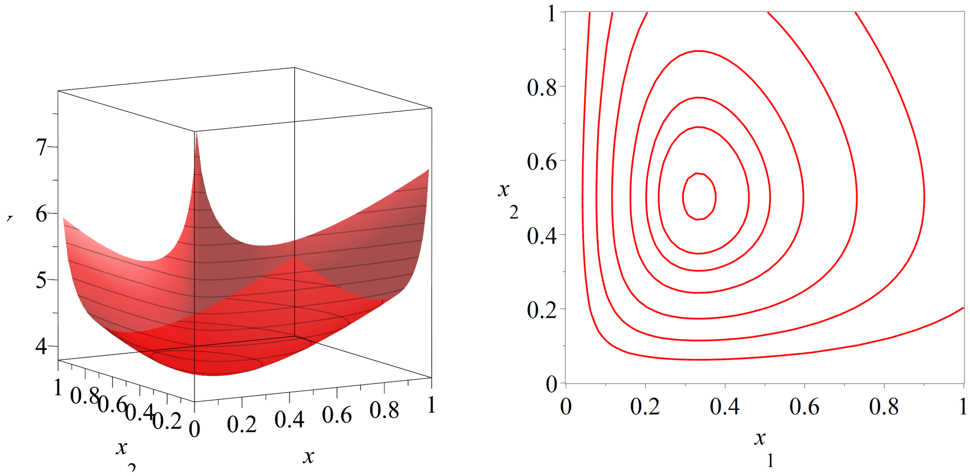



Figure 7.128: Left panel: Graph of $V(x_1, x_2) = dx_1 - c\ln(x_1) + bx_2 - a\ln(x_2)$ with $a = 1, b = 2, c = 1, d = 3$ on the region $0 < x_1, x_2 < 1$. Right panel: Contour plot of this same function.

*that "the new members of the population enter into the $S$ category" leads to the term $r(S+I+R)$ on the right in the $\dot{S}$ equation. The $-aSI$ term in the $\dot{S}$ equation indicates that the susceptible become infected at a rate jointly proportional to $S$ and $I$ (as in the standard SIR model) and the $dR$ term quantifies the fact that the recovered become susceptible again (lose immunity) at a rate proportional to the number of recovered individuals.*

*The term $-aSI$ in the $\dot{I}$ ODE captures the fact that susceptible individual leave the $S$ category and enter the $I$ category. The $-bI$ term captures rate at which infected individuals recover (leave the $I$ category, enter $R$) and the $-cI$ term captures the rate at which the infected die. Finally, in the $\dot{R}$ equation the term $bI$ reflects the rate at which those who leave the $I$ category enter the $R$ category, and $-dR$ the rate at which those who leave the $I$ category enter the $S$ category (lose immunity).*

(b) *Use a computer algebra system, but if you do it by hand, the $\dot{I} = 0$ equation yields $(aS - b - c)I = 0$, which means either $I = 0$ or $S = (b+c)/a$. If $I = 0$ the equation $\dot{R} = 0$ yields $R = 0$ and then $\dot{S} = 0$ yields $S = 0$. So $(0,0,0)$ is one fixed point. But (returning to $\dot{I} = 0$)*

*if* $S = (b+c)/a$ *then the* $\dot{S} = 0$ *equation becomes*

$$(r-b-c)aI + a(d+r)R = -r(b+c).$$

*In conjunction with* $bI - dR = 0$ *this yields (some messy algebra) the fixed point*

$$\left(\frac{b+c}{a}, \frac{rd(b+c)}{a((c-r)d-rb)}, \frac{rb(b+c)}{a((c-r)d-rb)}\right).$$

(c) *Routine differentiation shows that*

$$\mathbf{J}(S,I,R) = \begin{bmatrix} r-aI & r-aS & r+d \\ aI & aS-b-c & 0 \\ 0 & b & -d \end{bmatrix}.$$

*Then*

$$\mathbf{J}(0,0,0) = \begin{bmatrix} r & r & r+d \\ 0 & -b-c & 0 \\ 0 & b & -d \end{bmatrix}.$$

*The eigenvalues are* $-d, -b-c$, *and* $r$, *two positive, one negative. This fixed point is unstable (some kind of three-dimensional saddle).*

(d) *Simply substitute the values* $a = 1/2, b = 1/2, c = 1, d = 1,$ *and* $e = 1$ *into the expression for the second fixed point given in part (b). The point* $\mathbf{q} = \langle 3, 6r/(2-3r), 3r/(2-3r)\rangle$ *has all positive components precisely when* $0 < r < 2/3$.

(e) *We find that*

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \dfrac{r((b+d)r+bd)}{(b+d)r-cd} & r-b-c & r+d \\ -\dfrac{r(b+c)d}{(b+d)r-cd} & 0 & 0 \\ 0 & b & -d \end{bmatrix}.$$

*The characteristic polynomial of* $\mathbf{J}(\mathbf{q})$ *is* $p(\lambda) = det(\lambda\mathbf{I} - \mathbf{J}(\mathbf{q}))$ *and a bit of computation shows that* $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ *where*

$$a_1 = \frac{3r^2-2r+2}{2-3r}, \quad a_2 = \frac{11r}{6-4r}, \quad a_3 = \frac{3r}{2}.$$

*We can the use the Routh-Hurwitz theorem to compute*

$$D_1 = \frac{3r^2 - 2r + 2}{2 - 3r}$$

$$D_2 = \frac{r(3r^2 + 7r + 5)}{(2 - 3r)^2} D_3 \qquad = \frac{3r^2(3r^2 + 7r + 5)}{2(2 - 3r)^2}.$$

*It's easy to see that if $r > 0$ then $D_2$ and $D_3$ are always positive, and $D_1$ is positive precisely when $0 < r < 2/3$ (the numerator $3r^2 - 2r + 2$ is positive for $r > 0$, and the denominator $2 - 3r$ is positive for $0 < r < 2/3$).*

(f) *The eigenvalues of $\mathbf{J}(\mathbf{q})$ are $-0.15$ and $-1.675 \pm 1.48i$. A plot of the solution trajectory for the system with initial condition $x_1(0) = 3.3, x_2(0) = 5.6,$ and $x_3(0) = 3.1$ is shown in Figure 7.129.*

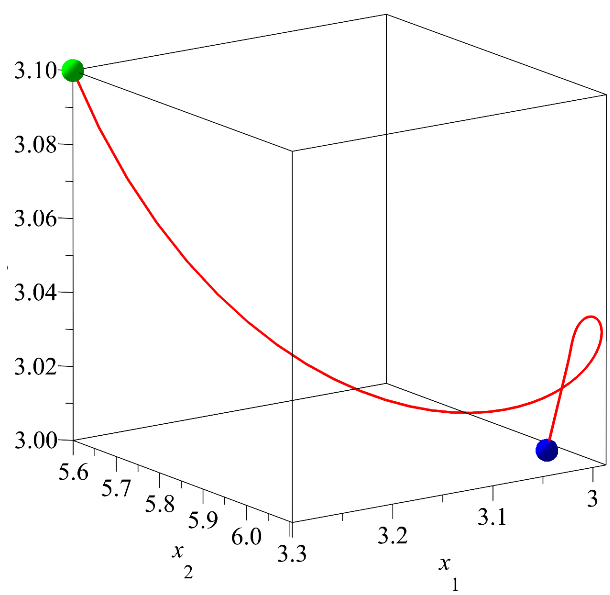



Figure 7.129: Trajectory of SIR system with $r = 0.5$, initial condition $\mathbf{x}(0) = \langle 3.3, 5.6, 3.1 \rangle$. The initial point is shown as a green dot, the fixed point $\langle 3, 6, 3 \rangle$ as a blue dot.

*A bit of experimentation indicates (but does not prove) that this fixed point is asymptotically globally stable.*

(g) *The solutions grow without bound.*

(h) *First, there is no hard limit, e.g., logistic, on the total population, and no cause of death other than the disease is modeled. Many other refinements are possible.*

**Exercise Solution 7.6.17.**

(a) *This follows immediately from* $dy/dx = (dy/dt)/(dx/dt)$ *along with* $dx/dt = y, dy/dt = -x$.

$$\frac{dy}{dx} = -\frac{x}{y}.$$

(b) *If we consider* $y = y(x)$ *as a function of* $x$ *we can separate the above ODE to find* $y\,dy = -x\,dx$. *Integrating shows that* $y^2/2 = -x^2/2 + C$ *for an arbitrary constant* $C$, *or that* $x^2 + y^2 = c$, *where* $c = 2C$ *is also arbitrary. That is to say, on curves of the form* $x^2 + y^2 = C$ *(circles centered at the origin) the solutions to this system of ODEs remains at a constant value. As a result we may take* $V(x, y) = x^2 + y^2$ *as a first integral for this system.*

(c) *Consider* $y = y(x)$ *as a description of a system trajectory, so that* $dy/dx = \dot{y}/\dot{x}$ *or*

$$\frac{dy}{dx} = \frac{(-c + dx)y}{(a - by)x}.$$

*This is a separable ODE. Separate as*

$$\frac{a - by}{y}\,dy = \frac{dx - c}{x}\,dx.$$

*Integrate both sides and find*

$$a\ln(y) - by = dx - c\ln(x) + C$$

*where* $C$ *is some arbitrary constant and we assume* $x$ *and* $y$ *are positive so that absolute values are not necessary when integrating. This last equation can be written as* $V(x, y) = C$ *where*

$$V(x, y) = c\ln(x) - dx + a\ln(y) - by.$$

*The function* $V$ *is a first integral for the system.*

**Exercise Solution 7.6.18.**

(a) *Simple algebra, obviously from* $x_1 = x$ *and* $x_2 = \dot{x}$ *we have* $\dot{x}_1 = x_2$. *The equation* $m\ddot{x}+k_1x+k_2x^3 = 0$ *is equivalent to* $m\dot{x}_2+k_1x_1+k_2x_1^3 = 0$ *or* $\dot{x}_2 = -k_1x_1/m - k_2x_1^3/m$.

(b) *Setting* $\dot{x}_1 = 0$ *yields* $x_2 = 0$ *and if the* $\dot{x}_2 = 0$ *we obtain* $k_1x_1+k_2x_1^3 = 0$. *Since* $k_1$ *and* $k_2$ *are both positive, the only solution is* $x_1 = 0$. *The Jacobian matrix at* $(0,0)$ *is*

$$\mathbf{J}(0,0) = \begin{bmatrix} 0 & 1 \\ -k_1/m & 0 \end{bmatrix}$$

*with purely imaginary eigenvalues* $\pm i\sqrt{k_1/m}$.

(c) *We have* $\nabla E(\mathbf{x}) = \langle k_1x_1+k_2x_1^3, mx_2\rangle$ *and then with* $\mathbf{f}(\mathbf{x}) = \langle x_2, -k_1x_1/m - k_2x_1^3/m\rangle$ *compute*

$$\nabla E \cdot \mathbf{f} = x_2(k_1x_1 + k_2x_1^3) + mx_2(-k_1x_1/m - k_2x_1^3/m) = 0.$$

(d) *A contour plot of* $E(x_1, x_2)$ *is shown in Figure 7.130. Given that the level curves are the trajectories of this system, it is clear that the mass oscillates periodically (but not truly sinusoidally) without any decay in amplitude.*

(e) *The system is now* $\dot{x}_1 = x_2$ *and* $\dot{x}_2 = -k_1x_1/m - k_2x_1^3/m - cx_2/m$. *The same algebra as the undamped case shows that* $(0,0)$ *is the unique fixed point. The Jacobian at* $(0,0)$ *is*

$$\mathbf{J}(0,0) = \begin{bmatrix} 0 & 1 \\ -k_1/m & -c/m \end{bmatrix}$$

*with eigenvalues* $-\frac{c}{2m} \pm i\frac{\sqrt{4k_1m-c^2}}{2m}$. *These eigenvalues are either real and both negative, or complex with negative real part.*

*With* $E(x_1, x_2)$ *are previously defined we find*

$$\nabla E \cdot \mathbf{f} = -cx_2^2.$$

*The same argument using LaSalle's principle given for the nonlinear pendulum (namely, that the set* $S = \{(x_1, x_2) : x_2 = 0\}$ *contains no solution trajectories except the fixed point* $(0,0)$*) shows that the origin is asymptotically stable.*

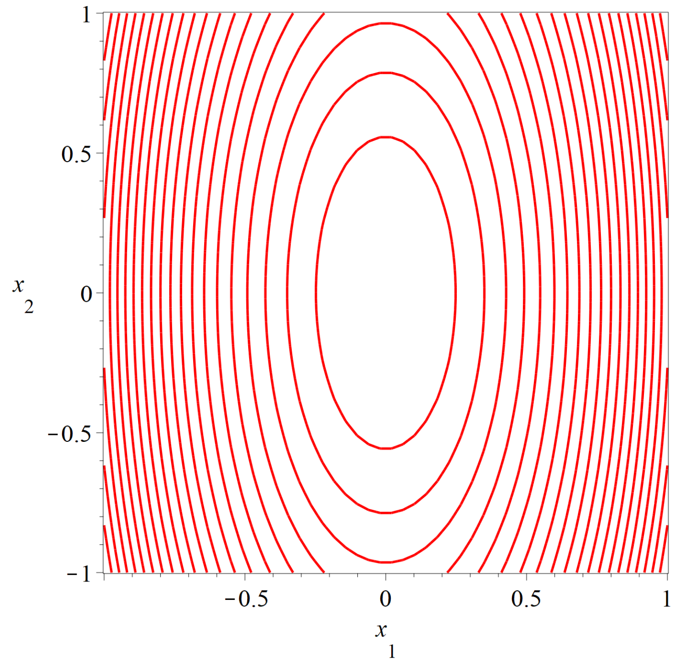



Figure 7.130: Contour plot for first integral $E(x_1, x_2) = k_1 u_1^2/2 + k_2 u_1^4/4 + m u_2^2/2$.

**Exercise Solution 7.6.19.**

(a) *The fixed point for (possible) mutual coexistence is*

$$\mathbf{x}^* = \langle K - af/e, f/e, (Kbe - fac - de)/(ec) \rangle.$$

*It's quite clear that the first component is positive exactly when $K - af/e > 0$, and the third component is positive precisely when $Kbe - fac - de > 0$.*

(b) *A routine differentiation shows that the Jacobian in general is*

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \dfrac{r(K - 2x_1 - ax_2)}{K} & -d\frac{rx_1 a}{K} & 0 \\ bx_2 & bx_1 - cx_3 - d & -cx_2 \\ 0 & ex_3 & ex_2 - f \end{bmatrix}.$$

*Substituting in the values of* $x_1, x_2,$ *and* $x_3$ *at* $\mathbf{x} = \mathbf{x}^*$ *yields*

$$\mathbf{J}(\mathbf{x}^*) = \begin{bmatrix} -r(1 - af/Ke) & -ra(1 - af/Ke) & 0 \\ bf/e & 0 & -cf/e \\ 0 & (Kbe - abf - de)/c & 0 \end{bmatrix}.$$

*By expanding* $\det(\lambda\mathbf{I} - \mathbf{J}(\mathbf{x}^*))$ *we obtain characteristic polynomial* $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ *where*

$$\begin{aligned} a_1 &= r\left(1 - \frac{af}{Ke}\right) \\ a_2 &= \frac{(bK^2e^2 - Ke(de + ab(f - r)) - a^2bfr)f}{Ke^2} \\ a_3 &= \frac{((Kb - d)e - abf)(Ke - af)rf}{Ke^2}. \end{aligned}$$

*(c) Expansion of the relevant determinants shows that*

$$\begin{aligned} D_1 &= r\left(1 - \frac{af}{Ke}\right) \\ D_2 &= \frac{r^2 fab}{e}\left(1 - \frac{af}{Ke}\right)^2 \\ D_3 &= \frac{r^3abf^2(Kbe - abf - de)}{e^2}\left(1 - \frac{af}{Ke}\right)^3. \end{aligned}$$

*If* $1 - af/(Ke) > 0$ *or equivalently, if* $K - af/e > 0$, *then* $D_1 > 0$ *and* $D_2 > 0$. *If* $Kbe - abf - de > 0$ *holds then* $D_3 > 0$. *We conclude from the Routh-Hurwitz theorem that* $\mathbf{J}(\mathbf{x}^*)$ *has eigenvalues with negative real part, and so the fixed point* $\mathbf{x}^*$ *is asymptotically stable, if it is physically relevant (has positive components).*

**Exercise Solution 7.6.20.** *Proceed by contradiction: suppose* $\mathbf{x}^*$ *is asymptotically stable. Then there is a point* $\mathbf{x}_0$ *such that the solution* $\mathbf{x}(t)$ *to* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *satisfies* $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$. *This means (if* $E$ *is a continuous function) that*

$$\lim_{t\to\infty} E(\mathbf{x}(t)) = E(\mathbf{x}^*).$$

*However, if* $E$ *is a first integral then* $E(\mathbf{x}(t)) = E(\mathbf{x}_0)$ *for all* $t$, *and so*

$$\lim_{t\to\infty} E(\mathbf{x}(t)) = E(x_0)$$

*since* $E(\mathbf{x}(t)) = E(\mathbf{x}_0)$ *for all* $t$*. Based on the last two above displayed equations we conclude that* $E(\mathbf{x}^*) = E(\mathbf{x}_0)$*, which is a contradiction since* $\mathbf{x}^*$ *was assumed to be a strict local minimum for* $E$ *(that is,* $E(\mathbf{x}^*) < E(\mathbf{x})$ *for all points* $\mathbf{x}$ *near* $\mathbf{x}^*$*.)*

## Section 8.1

**Exercise Solution 8.1.1.** *Start with the continuity equation* $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$ *and use the given fact that* $\frac{\partial \rho}{\partial t} = 0$ *to find* $\frac{\partial q}{\partial x} = 0$*. That is,* $q$ *is independent of* $x$*. Conversely if* $\frac{\partial q}{\partial x} = 0$ *it is immediate that* $\frac{\partial \rho}{\partial t} = 0$*, so* $\rho$ *does not depend on time.*

**Exercise Solution 8.1.2.** $u(x,t) = 3e^{-\pi^2 t^2}\sin(\pi x)$*. See top left panel in Figure 8.131.*

**Exercise Solution 8.1.3.** $u(x,t) = 3e^{-\pi^2 t^2}\sin(\pi x) + 5e^{-36\pi^2 t}\sin(6\pi x)$*. See top right panel in Figure 8.131.*

**Exercise Solution 8.1.4.** $u(x,t) = e^{-4\pi^2 t^2}\sin(2\pi x) - 3e^{-64\pi^2 t}\sin(8\pi x)$*. See bottom left panel in Figure 8.131.*

**Exercise Solution 8.1.5.** $u(x,t) = 4e^{-16\pi^2 t^2}\sin(4\pi x) + 2e^{-196\pi^2 t}\sin(14\pi x)$*. See bottom right panel in Figure 8.131.*

**Exercise Solution 8.1.6.** $u(x,t) = 3e^{-\pi^2 t}\cos(\pi x)$*. See top left panel in Figure 8.132.*

**Exercise Solution 8.1.7.** $u(x,t) = 4 + 3e^{-\pi^2 t}\cos(\pi x)$*. See top right panel in Figure 8.132.*

**Exercise Solution 8.1.8.** $u(x,t) = 2 + 5e^{-36\pi^2 t}\cos(6\pi x)$*. See bottom left panel in Figure 8.132.*

**Exercise Solution 8.1.9.** $u(x,t) = 4e^{-16\pi^2 t}\cos(4\pi x) + 2e^{-196\pi^2 t}\cos(14\pi x)$*. See bottom right panel in Figure 8.132.*

**Exercise Solution 8.1.10.**

(a) *Start with the integral*

$$I = \int_0^L u(x,t)\,dx$$

*and compute*

$$\frac{dI}{dt} = \frac{d}{dt}\left(\int_0^L u(x,t)\,dx\right) = \int_0^L \frac{\partial u}{\partial t}(x,t)\,dx.$$

*But if* $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ *then from above*

$$\begin{aligned}\frac{dI}{dt} &= \alpha \int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)\,dx \\ &= \alpha\left(\frac{\partial u}{\partial x}(L,t) - \frac{\partial u}{\partial x}(0,t)\right) \\ &= 0.\end{aligned}$$

*So* $I$ *is constant.*

(b) *We know that* $\rho = \rho_0 + c\lambda(u - u_0)$ *so that*

$$\int_0^L \rho(x,t)\,dx = \int_0^L (\rho_0 + c\lambda(u-u_0))\,dx = L(\rho_0 - cu_0) + c\lambda \int_0^L u(x,t)\,dx.$$

*But the integral on the right above is constant, so the integral of* $\rho$ *on the left is constant. The integral of* $\rho$ *over the bar* $0 \leq x \leq L$ *is exactly the total amount of thermal energy in the bar at any time. If the boundaries are insulated, no energy can enter or leave.*

**Exercise Solution 8.1.11.**

(a) *The statement "the bar loses heat at* $x = 0$ *in proportion to the difference between the temperature* $u(0,t)$ *and the ambient environment temperature" pretty much nails down the relation* $q(0,t) = -\mu(u(0,t) - A)$. *To see that this is reasonable if* $\mu \geq 0$, *suppose* $u(0,t) > A$ *at some time, that is, the end of the bar is hotter than the ambient temperature. Then clearly heat should be flowing to the left, out of the bar (or at least not flowing in), so* $q(0,t) \geq 0$. *The choice* $\mu \geq 0$ *is consistent with this.*

(b) *With* $q = -k\frac{\partial u}{\partial x}$ *this boundary condition becomes* $\frac{\partial u}{\partial x}(0,t) = \mu(u(0,t) - A)/k$.

(c) *We need* $q(L,t) = \mu(u(L,t) - A)$ *with* $\mu \geq 0$ *(if* $u(L,t) > A$ *then* $q(L,t) \geq 0$, *energy is flowing to the right out of the bar). In terms of* $u$ *this becomes* $\frac{\partial u}{\partial x}(L,t) = -\mu(u(L,t) - A)/k$.

**Exercise Solution 8.1.12.**

(a) *If* $U(x)$ *is a solution to the heat equation then* $\frac{\partial U}{\partial t} - \alpha\frac{\partial^2 U}{\partial x^2} = 0$, *but if* $U$ *does not depend on* $t$ *this immediately becomes* $d^2U/dx^2 = 0$. *Integrating twice shows that* $U(x) = c_1 x + c_2$, *where* $c_1$ *and* $c_2$ *are arbitrary constants of integration. Thus* $U(x)$ *is a linear function.*

(b) *If* $U(0) = 20$ *then* $c_2 = 20$. *If* $U(5) = 80$ *then* $5c_1 + c_2 = 80$, *so* $c_1 = 12$. *So* $U(x) = 12x + 20$.

(c) *Similar to part (b):* $U(0) = u_0$ *forces* $c_2 = u_0$. *Then* $U(L) = u_L$ *means that* $c_1 L + c_2 = u_L$, *so* $c_1 = (u_L - u_0)/L$. *There is no choice in the value for* $c_1$ *and* $c_2$, *so* $U(x) = (u_L - u_0)x/L + u_0$ *is the unique solution.*

(d) *From part (a) we have* $U(x) = c_1 x + c_2$, *and so* $U'(x) = c_1$. *Then* $-kU'(0) = g_0$ *forces* $-kc_1 = g_0$. *But* $kU'(L) = g_1$ *forces* $kc_1 = g_1$. *We must therefore have* $g_1 = -g_0$ *(and so* $c_1 = g_1/k$ *or* $c_1 = -g_0/k$*) in order for a solution to exist. The solution is* $U(x) = g_1/x/k + c_1$. *But* $c_1$ *can be anything at all—it doesn't affect the Neumann boundary data or the fact that* $U''(x) = 0$.

*Of course if* $g_1 \neq -g_0$ *this means we are pumping more heat energy into one end of the bar than we are withdrawing it at the other, or withdrawing more at one end then we are introducing at the other. In either case, the net flow of heat energy into the bar is not zero, and so the temperature can never stabilize.*

**Exercise Solution 8.1.13.**

(a) *If stuff can be created (or destroyed) in* $\Omega$ *then the net change in the amount of stuff in* $\Omega$ *over any given time period is the amount that entered/left via the endpoints at* $x = x_0$ *and* $x = x_0 + \Delta x$ *plus the amount created in* $\Omega$ *over that time period (these are the only avenues by which stuff enters or leaves* $\Omega$*). If we limit the time interval to an arbitrary short span of time the the rate at which the amount of stuff in* $\Omega$ *is changing at time* $t$ *is the net rate at which stuff enters* $\Omega$ *plus the net rate of stuff creation in* $\Omega$.

(b) *In the equation*

$$\int_{x_0}^{x_0+\Delta x} \frac{\partial \rho}{\partial t}(x,t)\,dx = q(x_0,t) - q(x_0+\Delta x,t) + \int_{x_0}^{x_0+\Delta x} r(x,t)\,dx$$

*the left side integral is the rate at which the amount of stuff in* $\Omega$ *is changing. The terms* $q(x_0,t)$ *and* $-q(x_0+\Delta x,t)$ *are the rates at which stuff is entering at* $x = x_0$ *and* $x = x_0 + \Delta x$, *respectively. The integral on the right is the rate at which stuff is being created in* $\Omega$. *This expresses the reasoning of part (a).*

*(c) Divide both sides of the equation in part (b) by $\Delta x$ to obtain*

$$\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}\frac{\partial \rho}{\partial t}(x,t)\,dx = \frac{q(x_0,t)-q(x_0+\Delta x,t)}{\Delta x}+\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} r(x,t)\,dx.$$

*As $\Delta x \to 0$ the left side above limits to $\frac{\partial \rho}{\partial t}(x_0,t)$. The quantity $\frac{q(x_0,t)-q(x_0+\Delta x,t)}{\Delta x}$ limits to $-\frac{\partial q}{\partial x}(x_0,t)$. The integral on the right approaches $r(x_0,t)$. All in all we obtain*

$$\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x}=r.$$

*with all functions evaluated at an arbitrary point $(x,t)$.*

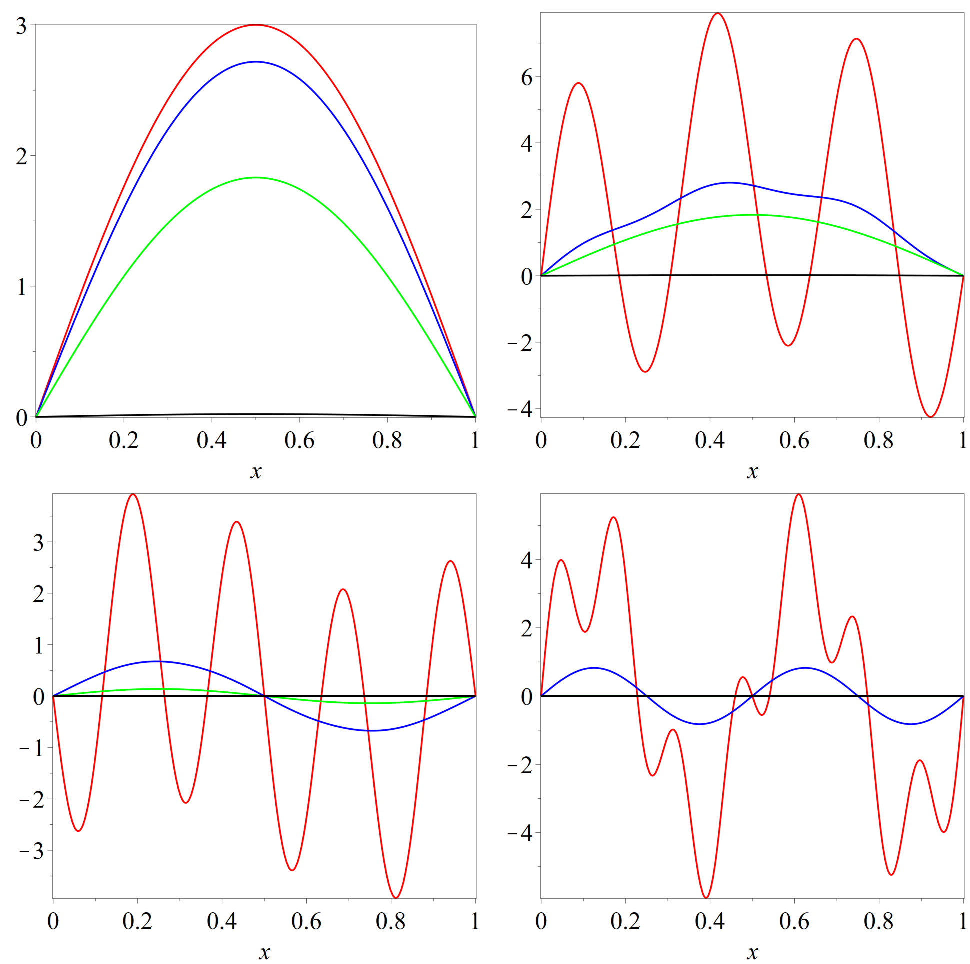



Figure 8.131: Figures for Exercises 8.1.2 (top left), 8.1.3 (top right), 8.1.4 (bottom left), and 8.1.5 (bottom right). In each case $t = 0$ is in red, $t = 0.01$ is blue, $t = 0.05$ is green, $t = 0.5$ is black. In each case the solution decays to 0 as $t$ increases, at all points.

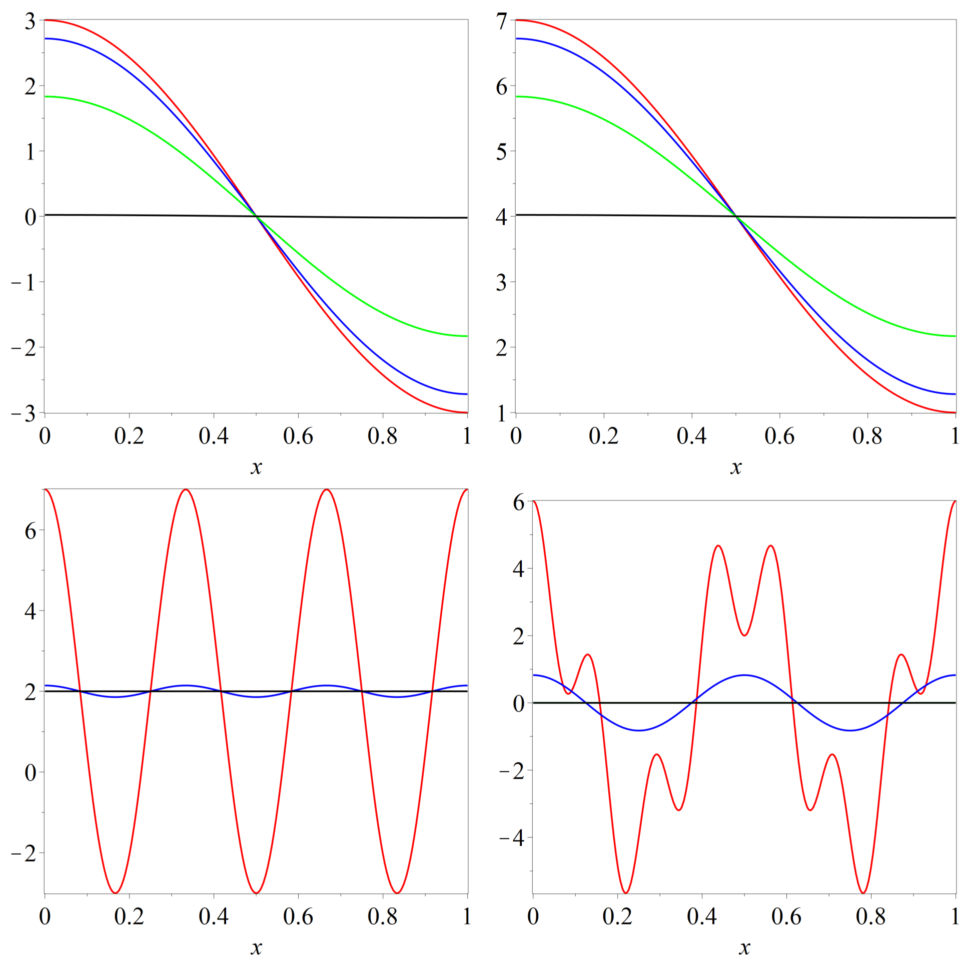



Figure 8.132: Figures for Exercises 8.1.6 (top left), 8.1.7 (top right), 8.1.8 (bottom left), and 8.1.9 (bottom right). In each case $t = 0$ is in red, $t = 0.01$ is blue, $t = 0.05$ is green, $t = 0.5$ is black. In each case the solution decays in time to a constant value (whatever the average value of $u(x, 0)$ is on the interval).

## Section 8.2

**Exercise Solution 8.2.1.** *For $n \leq 1$ we obtain $s_n(x) = 0$ and for $n \geq 2$ find $s_n(x) = f(x) = 3\cos(2\pi x)$. Then $\|f - s_n\|_2 \approx 0.212$ for $n = 0, 1$ and $\|f - s_n\|_2 = 0$ for $n \geq 2$. This graph is omitted.*

**Exercise Solution 8.2.2.** *For $n \leq 1$ we obtain $s_n(x) = 0$, for $n = 2, 3$ we find $s_n(x) = 3\cos(2\pi x)$, and for $n \geq 4$ find $s_n(x) = f(x) = 3\cos(2\pi x) - 4\cos(4\pi x)$. Then $\|f - s_n\|_2 \approx 3.54$ for $n = 0, 1$, $\|f - s_n\|_2 = 2.828$ for $n = 2, 3$ and $\|f - s_n\|_2 = 0$ for $n \geq 4$. This graph is omitted.*

**Exercise Solution 8.2.3.** *You should find that*

$$\begin{aligned}
s_0(x) &= 1 \\
s_1(x) &= 1 - \frac{8\cos(\pi x/2)}{\pi^2} \\
s_3(x) &= 1 - \frac{8\cos(\pi x/2)}{\pi^2} - \frac{8\cos(\pi 3x/2)}{9\pi^2} \\
s_5(x) &= 1 - \frac{8\cos(\pi x/2)}{\pi^2} - \frac{8\cos(\pi 3x/2)}{9\pi^2} - \frac{8\cos(5\pi x/2)}{25\pi^2}.
\end{aligned}$$

*Also, $s_2 = s_1$ and $s_4 = s_3$. Then $\|f - s_0\|_2 \approx 0.816, \|f - s_1\|_2 \approx 0.098, \|f - s_5\|_2 \approx 0.022$. A plot is shown in Figure 8.133, upper left panel.*

**Exercise Solution 8.2.4.** *The coefficients turn out to be $a_0 = 10/3$ and $a_k = -10\sin(2k\pi/3)/(k\pi)$ for $k \geq 1$, so for example*

$$s_{10}(x) = 5/3 - \frac{5\sqrt{3}}{\pi}\cos(\pi x/3) + \cdots - \frac{\sqrt{3}}{2\pi}\cos(10\pi x/3).$$

*Then $\|f - s_0\|_2 \approx 4.085, \|f - s_5\|_2 \approx 1.118, \|f - s_{10}\|_2 \approx 0.852$. A plot is shown in Figure 8.133, upper right panel.*

**Exercise Solution 8.2.5.** *The approximation $s_{10}$ is*

$$s_{10}(x) \approx -0.053\cos(\pi x/3) + 0.186\cos(2\pi x/3) + \cdots - 0.026\cos(10\pi x/3).$$

*The errors are $\|f - s_3\|_2 \approx 0.882, \|f - s_5\|_2 \approx 0.488, \|f - s_{10}\|_2 \approx 0.027$. A plot is shown in Figure 8.133, bottom left panel.*

**Exercise Solution 8.2.6.** *The coefficients $a_k = \cos(k\pi/2)$, or $a_0 = 1, a_1 = 0, a_2 = -1, a_3 = 0$, etc. Then*

$$s_n(x) = 1 - \cos(\pi x) + \cos(2\pi x) - \cos(3\pi x) + \cdots$$

*Plots of $s_n$ for $n = 5, 10, 20$ are shown in Figure 8.133, bottom right panel.*

**Exercise Solution 8.2.7.** *The coefficients here are* $b_k = 4\sin(k\pi x)/(k\pi)$ *when* $k$ *is odd,* $b_k = 0$ *for* $k$ *even. Then* $\|f - s_1\|_2 = 0.435$, $\|f - s_3\|_2 = 0.315$, $\|f - s_{10}\|_2 = 0.201$. *Plots of* $s_n$ *for* $n = 1, 3, 10$ *are shown in Figure 8.134, top left panel.*

**Exercise Solution 8.2.8.** *We find* $s_1(x) = 0$, $s_n(x) = 3\sin(2\pi x)$ *for* $n = 2, 3$, *and* $s_n(x) = f(x) = 3\sin(2\pi x) - 4\sin(4\pi x)$ *for* $n \geq 4$. *The errors are* $\|f - s_1\|_2 = 3.536$, $\|f - s_2\|_2 = 2.828$, $\|f - s_{10}\|_2 = 0$. *Graph here is omitted.*

**Exercise Solution 8.2.9.** *The coefficients are* $b_k = -4(-1)^k/(k\pi)$ *and*

$$s_n(x) = 4\sin(\pi x/2)/\pi - 4\sin(\pi x)/(2\pi) + 4\sin(3\pi x/2)/(3\pi) + \cdots$$

*Errors are* $\|f - s_1\|_2 = 1.023$, $\|f - s_5\|_2 = 0.542$, $\|f - s_{10}\|_2 = 0.393$. *Plots of* $s_n$ *for* $n = 1, 5, 10$ *are shown in Figure 8.134, top right panel.*

**Exercise Solution 8.2.10.** *The coefficients are* $b_k = 16(1 - (-1)^k)/(k^3\pi^3)$ *and*

$$s_n(x) = \frac{32}{\pi^3}\sin(\pi x/2) + \frac{32}{27\pi^3}\sin(3\pi x/2) + \frac{32}{125\pi^3}\sin(5\pi x/2) + \cdots$$

*Errors are* $\|f - s_1\|_2 = 0.0393$, $\|f - s_3\|_2 = 0.009$, $\|f - s_5\|_2 = 0.0035$. *Plots of* $s_n$ *for* $n = 1, 3, 5$ *are shown in Figure 8.134, bottom panel.*

**Exercise Solution 8.2.11.**

*(a) From* $\cos(-x) = \cos(x)$ *we have* $\cos(-k\pi x/L) = \cos(k\pi x/L)$ *for any choice of* $k$ *and* $L$, *and since the sum of even functions is even,* $s_n(-x) = s_n(x)$, *that is,* $s_n$ *is even. Also*

$$\cos(k\pi(x + 2L)/L) = \cos(k\pi x/L + 2k\pi) = \cos(k\pi x/L)$$

*so* $\cos(k\pi x/L)$ *is periodic with period* $2L$. *The sum of such periodic functions is periodic, i.e.,* $s_n(x) = s_n(x + 2L)$ *(though the period could actually be shorter, e.g., if* $a_1 = 0$*).*

*(b) See the left panel of Figure 8.135. The Fourier cosine expansion is periodic with period* $2L = 2$ *here, and an even function of* $x$.

*(c) From* $\sin(-x) = -\sin(x)$ *we have* $\sin(-k\pi x/L) = -\sin(k\pi x/L)$ *for any choice of* $k$ *and* $L$, *and since the sum of odd functions is odd,* $s_n(-x) = -s_n(x)$, *that is,* $s_n$ *is odd. Also*

$$\sin(k\pi(x + 2L)/L) = \sin(k\pi x/L + 2k\pi) = \sin(k\pi x/L)$$

*so* $\sin(k\pi x/L)$ *is periodic with period* $2L$. *The sum of such periodic functions is periodic, i.e.,* $s_n(x) = s_n(x+2L)$ *(though the period could actually be shorter, e.g., if* $b_1 = 0$*).*

(d) *See the right panel of Figure 8.135. The Fourier sine expansion is periodic with period* $2L = 2$ *here, and an odd function of* $x$.

**Exercise Solution 8.2.12.** *With* $\sin(x)\sin(y) = (\cos(x-y) - \cos(x+y))/2$ *we have*

$$\sin(j\pi x/L)\sin(k\pi x/L) = \frac{\cos((j-k)\pi x/L)}{2} - \frac{\cos((j+k)\pi x/L)}{2}.$$

*Integrate both sides above from* $x = 0$ *to* $x = L$*:*

$$\begin{aligned}
\int_0^L \sin(j\pi x/L)\sin(k\pi x/L)\,dx &= \int_0^L \left(\frac{\cos((j-k)\pi x/L)}{2} - \frac{\cos((j+k)\pi x/L)}{2}\right) dx \\
&= \left(\frac{\sin((j-k)\pi x/L)}{2\pi(j-k)} - \frac{\sin((j+k)\pi x/L)}{2\pi(j+k)}\right)\Bigg|_{x=0}^{x=L} \\
&= \left(\frac{\sin((j-k)\pi)}{2\pi(j-k)} - \frac{\sin((j+k)\pi)}{2\pi(j+k)}\right) \\
&= 0
\end{aligned}$$

*using the facts that* $\sin(0) = 0$ *and* $\sin(z) = 0$ *when* $z$ *is a multiple of* $\pi$. *Also note that* $j \neq \pm k$.

**Exercise Solution 8.2.13.** *A graphical proof is convincing, if not perfectly rigorous. Consider a function* $\phi(x)$ *that is nonnegative and continuous on* $a \leq x \leq b$. *The claim is that if*

$$\int_a^b \phi(x)\,dx = 0 \tag{8.17}$$

*then* $\phi(x) = 0$ *for all* $x$ *between* $a$ *and* $b$. *We can prove this by considering the contrapositive: if* $\phi(x_0) > 0$ *then the integral on the left in (8.17) must be positive. Thus suppose* $\phi(x_0) > 0$ *for some* $x_0$ *as illustrated in Figure 8.136. Since* $\phi$ *is continuous,* $\phi(x)$ *must remain positive in some interval around* $x_0$. *It is then clear that there is some positive area under the curve near* $x_0$*, and since* $\phi$ *can never assume negative values, the integral in (8.17) is positive. Equivalently, if the integral is zero then it must be that* $\phi(x) = 0$ *for all* $x$.

*Since $f^2 \geq 0$ if $f$ is real-valued (and $f^2$ is continuous), if $\|f\|_2 = 0$ we conclude that if* $\int_a^b f^2(x)\,dx = 0$ *and so $f$ is zero on the interval $a \leq x \leq b$. The same reasoning applied to $f = g - h$ shows that if $\|g-h\|_2 = 0$ for continuous functions $g$ and $h$ then $g(x) = h(x)$ for all $x$.*

**Exercise Solution 8.2.14.**

(a) *Here $s_0(x) = 0$, $s_1(x) = -\sin(\pi x)$, and $s_n(x) = f(x)$ for $n \geq 2$. See top left panel in Figure 8.137.*

(b) *Here $s_n(x)$ contains only cosine terms (all $b_k$ turn out to be zero) and is*

$$s_n(x) = \frac{2}{3} + \sum_{k=1}^{n} \frac{4(-1)^k}{k^2\pi^2}\cos(k\pi x).$$

*See top right panel in Figure 8.137.*

(c) *Here $s_n(x)$ contains only sine terms (all $a_k$ turn out to be zero) and is*

$$s_n(x) = \sum_{k=1}^{n} \frac{6(-1)^{k+1}}{k\pi}\sin(k\pi x/3).$$

*See bottom left panel in Figure 8.137.*

(d) *We find $a_k = 2(-1)^k(e^2 - 1/e^2)/(k^2\pi^2+4)$ and $b_k = (-1)^k k\pi(e^2 - 1/e^2)/(k^2\pi^2+4)$. See bottom right panel in Figure 8.137.*

**Exercise Solution 8.2.15.** *With rounding parameter $r = 0.1$ the approximate cosine expansion of $f(t)$ is*

$$\begin{aligned}\tilde{f}(t) = 0.35 - 1.1\cos(\pi x) + 0.4\cos(2\pi x) - 0.9\cos(3\pi x) + 0.1\cos(4\pi x) - 0.3\cos(5\pi x)\\ - 0.2\cos(7\pi x) - 0.1\cos(9\pi x) - 0.1\cos(11\pi x) - 0.1\cos(13\pi x).\end{aligned}$$

*The graph is shown in Figure 8.138 in the left panel.*

*For $r = 0.5$ the approximation becomes*

$$\tilde{f}(t) = 0.25 - 1.0\cos(\pi x) + 0.5\cos(2\pi x) - 1.0\cos(3\pi x) - 0.5\cos(5\pi x).$$

*The graph is shown in Figure 8.138 in the middle panel.*

*For $r = 0.01$ most of the first 20 coefficients survive the rounding process, and round to $\pm 0.01, \pm 0.02$, etc. The graph is shown in Figure 8.138 in the right panel.*

**Exercise Solution 8.2.16.**

(a) *We find that* $s_{0,0}(x, y) = 1/6$. *Both* $s_{2,2}$ *and* $s_{10,10}$ *have many terms, but the approximations are shown in Figure 8.139.*

(b) *The obvious (and correct) modification is to take*

$$s_{m,n}(x, y) = \sum_{j=0}^{m}\sum_{k=1}^{n} b_{j,k}\sin(j\pi x/A)\sin(k\pi y/B).$$

*with coefficients*

$$b_{j,k} = \frac{4}{AB}\int_0^A\int_0^B f(x, y)\sin(j\pi x/A)\sin(k\pi y/B)\,dy\,dx.$$

*We find* $s_{1,1}(x, y) \approx 0.503\sin(\pi x/2)\sin(\pi y)$. *Both* $s_{2,2}$ *and* $s_{10,10}$ *have many terms, but the approximations are shown in Figure 8.140.*

**Exercise Solution 8.2.17.**

(a) *These are routine integrations.*

(b) *If we write* $\phi(c_0, c_1, c_2) = \|f - s_2\|_2^2$ *and expand out* $\phi$ *then we find*

$$\phi(c_0, c_1, c_2) = \int_0^1 f^2(x)\,dx - 2\sum_{j=0}^{2} c_j\int_0^1 f(x)p_j(x)\,dx$$
$$+ 2\sum_{j,k=0,j\neq k}^{2} c_jc_k\int_0^1 p_j(x)p_k(x)\,dx + \sum_{j=0}^{2} c_j^2\int_0^1 p_j^2(x)\,dx.$$

*But since the* $p_j$ *are orthogonal this becomes*

$$\phi(c_0, c_1, c_2) = \int_0^1 f^2(x)\,dx - 2\sum_{0=1}^{2} c_j\int_0^1 f(x)p_j(x)\,dx + \sum_{0=1}^{2} c_j^2\int_0^1 p_j^2(x)\,dx.$$

*The* $c_jc_k$ *cross terms disappear, which makes things much easier. Differentiate with respect to each* $c_j$ *and set the result to zero, then solve for* $c_j$ *to find*

$$c_j = \frac{\displaystyle\int_{-1}^1 f(x)p_j(x)\,dx}{\displaystyle\int_{-1}^1 p_j^2(x)\,dx}.$$

*(This is in fact a minimum). Then the best approximation to a function $f(x)$ is given by*

$$s_2(x) = c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x).$$

(c) *With $f(x) = e^x$ we find $c_0 = (e - 1/e)/2 \approx 1.175, c_1 = 3/e \approx 1.104, c_2 = 15e/4 - 105/(4e) \approx 0.537$. The approximation is*

$$s_2(x) \approx 0.9963 + 1.1036x + 0.5367x^2.$$

*A graph of $s_2(x)$ and $f(x)$ on $-1 \le x \le 1$ is shown in the left panel of Figure 8.141. A graph of $f(x) - s_2(x)$ and $f(x) - T_2(x)$ is shown in the right panel. The approximation based on the orthogonal polynomials produces a lower maximum error on this interval. This is typical.*

(d) *This should lead to $p_3(x) = x^3 - 3x/5$. The optimal approximation here is*

$$s_3(x) \approx 0.9963 + 0.9980x + 0.5367x^2 + 0.1761x^3.$$

*A graph of $s_3(x)$ and $f(x)$ on $-1 \le x \le 1$ is shown in the left panel of Figure 8.142. A graph of $f(x) - s_3(x)$ and $f(x) - T_3(x)$ is shown in the right panel, where $T_3(x) = 1 + x + x^2/2 + x^3/6$ is the Taylor polynomial for $e^x$ at $x = 0$. The approximation based on the orthogonal polynomials again produces a lower maximum error on this interval.*

**Exercise Solution 8.2.18.**

(a) *Compute (use $\overline{e^{2\pi ikx/L}} = e^{-2\pi ikx/L}$)*

$$\int_0^L e^{2\pi ijx/L} e^{-2\pi ikx/L}\, dx = \int_0^L e^{2\pi i(j-k)x/L}\, dx = \left.\frac{e^{2\pi i(j-k)x/L}}{2\pi i(j-k)/L}\right|_{x=0}^{x=L}.$$

*But $e^{2\pi i(j-k)x/L} = 1$ when $x = 0$ and $x = L$, since $j - k$ is an integer. The right side in the displayed equation above is zero.*

*Similarly*

$$\int_0^L e^{2\pi ikx/L} e^{-2\pi ikx/L}\, dx = \int_0^L 1\, dx = L.$$

(b) *With* $f(x) = (x-1/2)^2$ *on* $0 \leq x \leq 2$ *we find that* $c_k = (2+i\pi k)/(k^2\pi^2)$ *for* $k \neq 0$ *(with* $c_0 = 7/12$*) and*

$$s_5(x) = \frac{(2-5\pi i}{25\pi^2}e^{-5\pi it} + \cdots + \frac{7}{12} + \cdots + \frac{(2+5\pi i}{25\pi^2}e^{5\pi it}.$$

*This is actually real-valued. Applying Euler's identity (group the* $e^{-k\pi it}$ *and* $e^{k\pi it}$ *terms) shows that*

$$s_5(t) = \frac{7}{12} + \frac{4}{\pi^2}\cos(\pi t) - \frac{2}{\pi}\sin(\pi t) + \cdots + \frac{4}{25\pi^2}\cos(5\pi t) - \frac{2}{5\pi}\sin(5\pi t).$$

*A graph is shown in Figure 8.143, including for* $s_{20}$*. The approximation clearly improves as* $n$ *increases.*

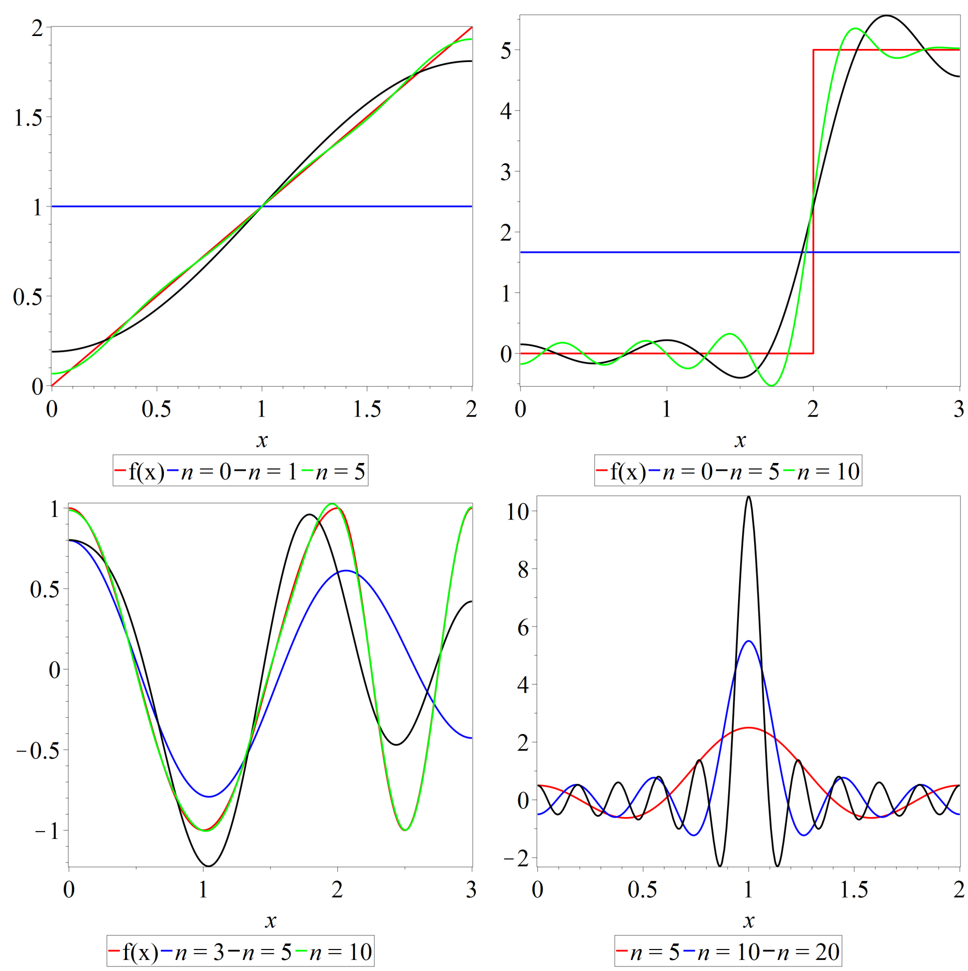



Figure 8.133: Graphs of $f(x)$ and $s_n(x)$ for various values of $n$ for Exercises 8.2.3 in top left, 8.2.4 top right, 8.2.5 bottom left, 8.2.6 bottom right.

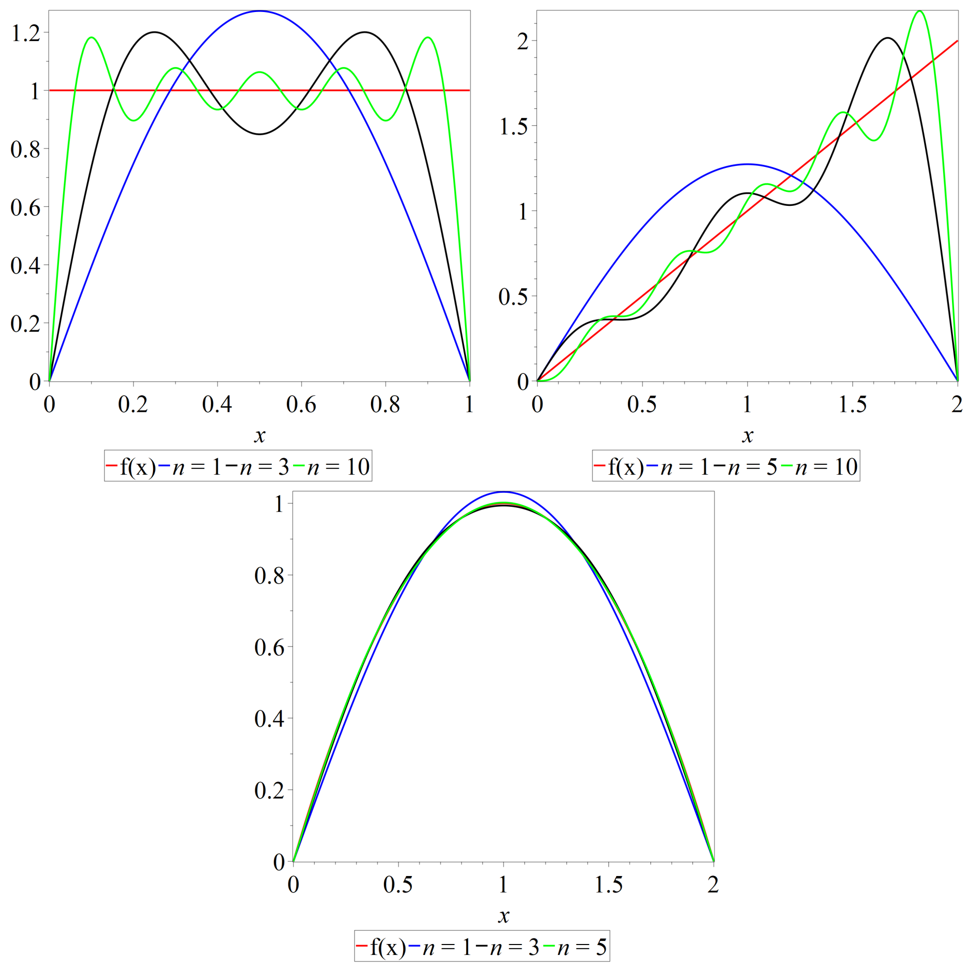



Figure 8.134: Graphs of $f(x)$ and $s_n(x)$ for various values of $n$ for Exercises 8.2.7 in top left, 8.2.9 top right, 8.2.10 bottom.

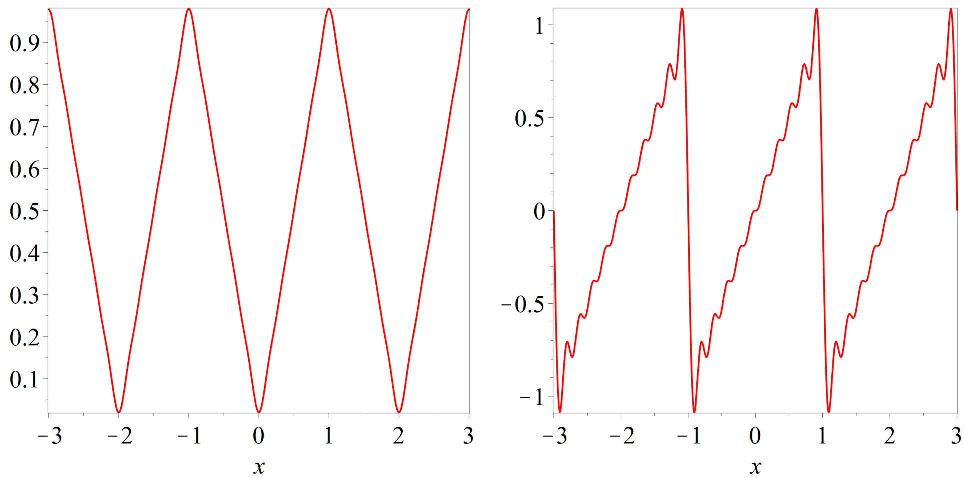



Figure 8.135: Left panel: Fourier cosine series for $f(x) = x$ on interval $-3 \leq x \leq 3$. Right panel: Fourier sine series for $f(x) = x$ on interval $-3 \leq x \leq 3$.

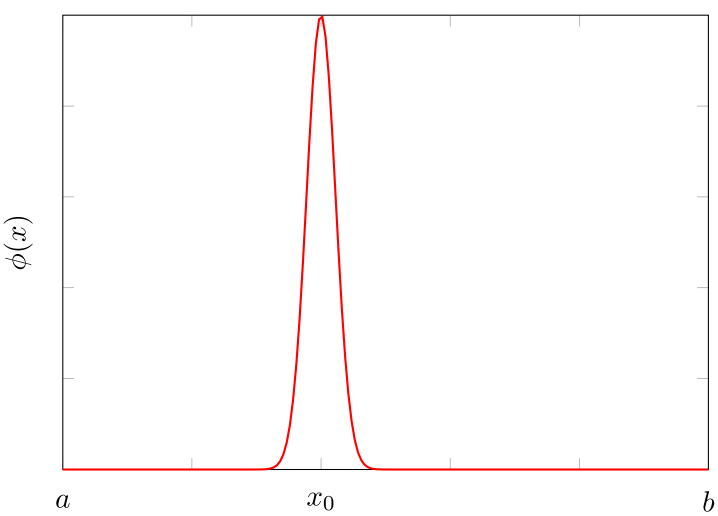



Figure 8.136: A nonnegative continuous function on some interval with $\phi(x_0) > 0$.

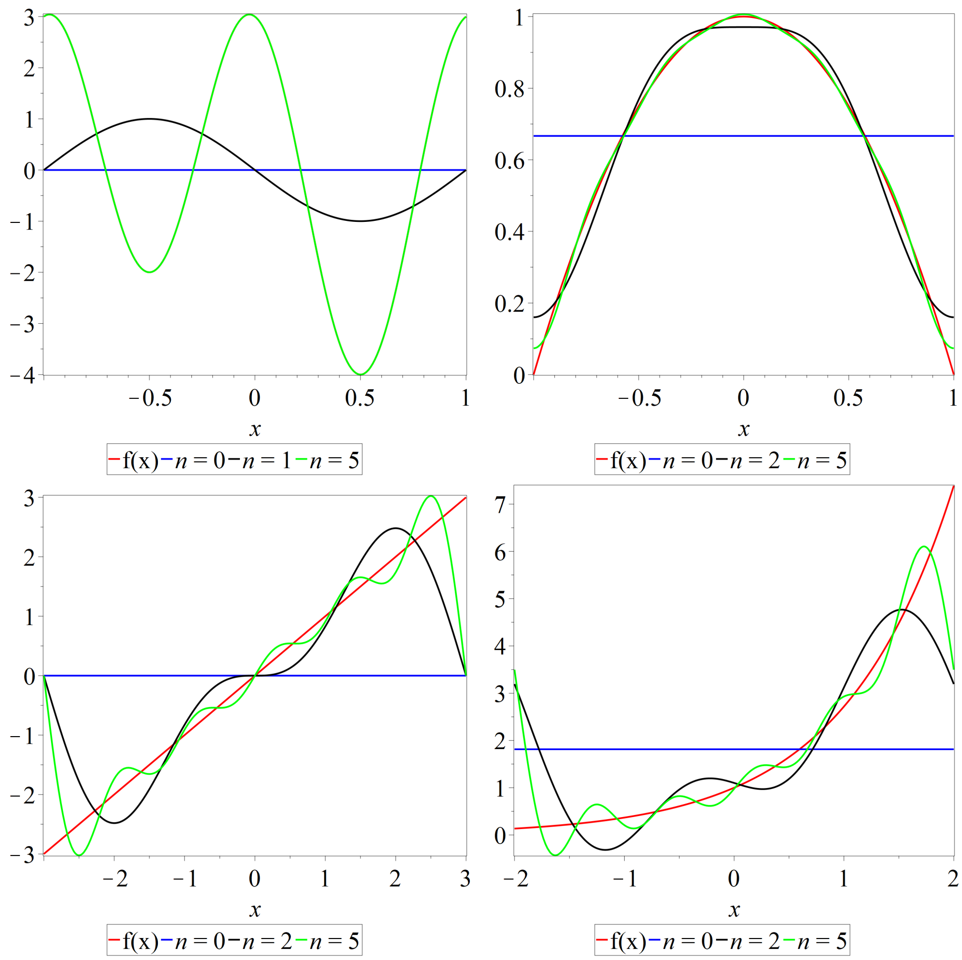



Figure 8.137: Top left panel: Fourier sine/cosine approximations for $f(x) = 3\cos(2\pi x) - \sin(\pi x)$ on interval $-1 \leq x \leq 1$. Top right panel: Fourier sine/cosine approximations for $f(x) = 1 - x^2$ on interval $-1 \leq x \leq 1$. Bottom left panel: Fourier sine/cosine approximations for $f(x) = x$ on interval $-3 \leq x \leq 3$. Bottom right panel: Fourier sine/cosine approximations for $f(x) = e^x$ on interval $-2 \leq x \leq 2$.

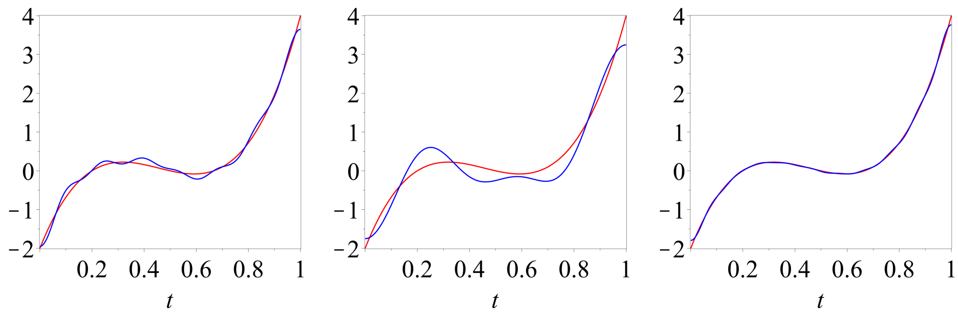



Figure 8.138: Left panel: Fourier cosine/JPEG approximation to $f(t)$, rounding parameter $r = 0.1$; $f$ shown in red, approximation in blue. Middle panel: same, with $r = 0.5$. Right panel: same, with $r = 0.01$.

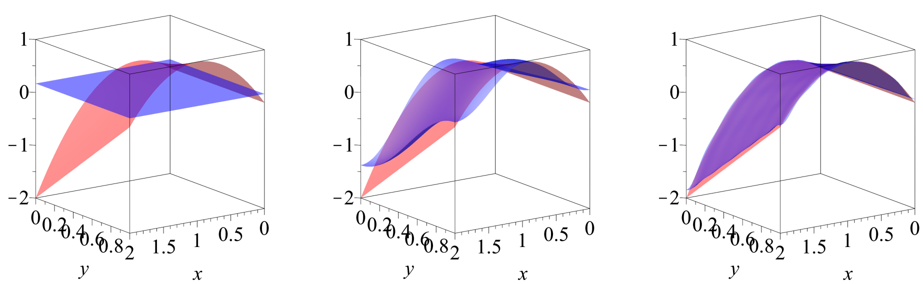



Figure 8.139: Left panel: Fourier double cosine approximation $s_{0,0}$ to $f$; graph of $f$ shown in red, approximation in blue. Middle panel: same, for $s_{2,2}$. Right panel: same, for $s_{10,10}$.

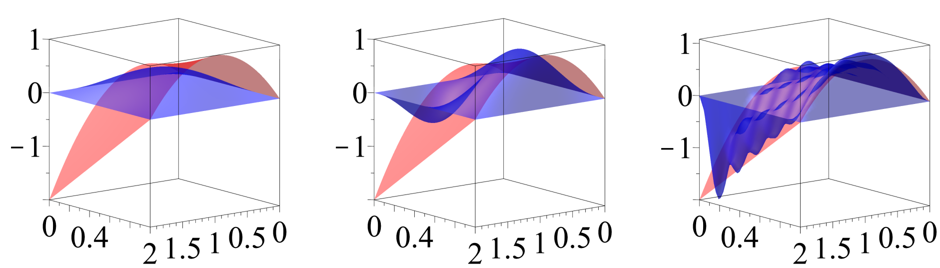



Figure 8.140: Left panel: Fourier double sine approximation $s_{1,1}$ to $f$; graph of $f$ shown in red, approximation in blue. Middle panel: same, for $s_{2,2}$. Right panel: same, for $s_{10,10}$.

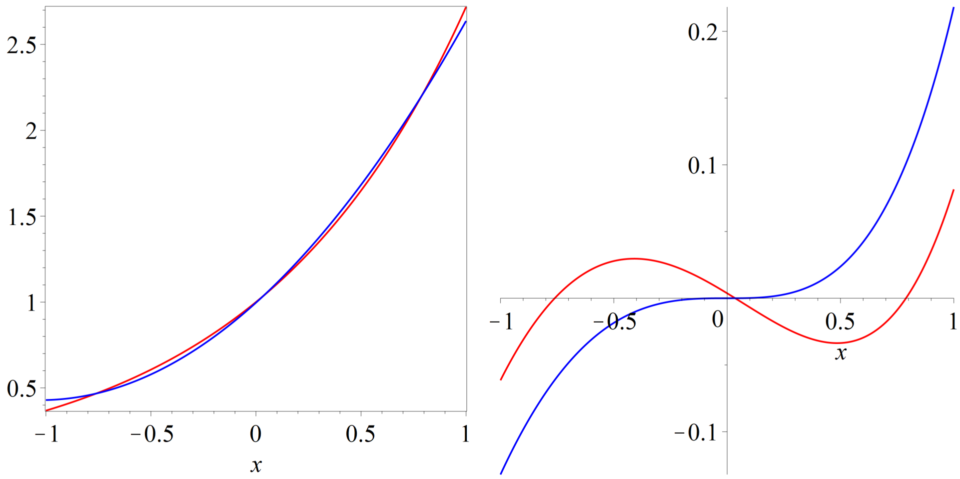



Figure 8.141: Left panel: Function $f(x) = e^x$ (red) and quadratic approximation $s_2(x)$ (blue). Right panel: error $f(x) - s_2(x)$ (red) and $f(x) - T_2(x)$ (blue) with Taylor polynomial $T_2(x) = 1 + x + x^2/2$.

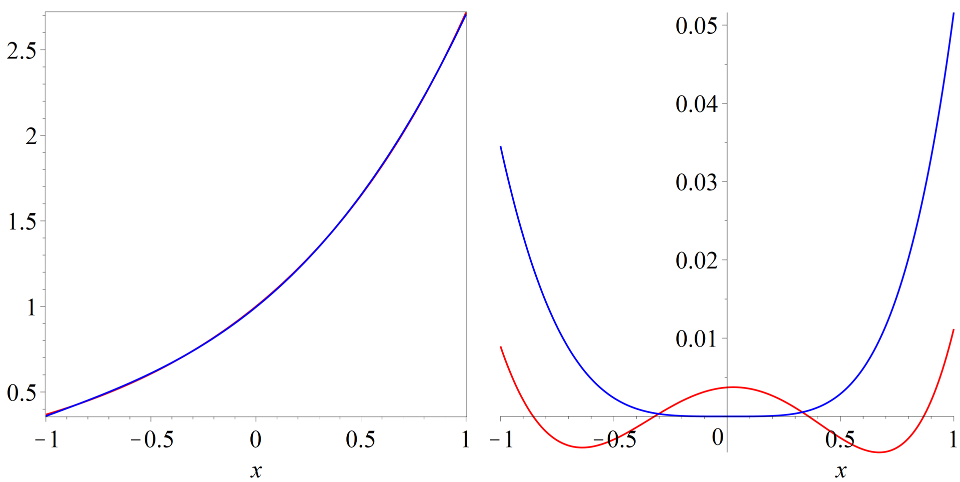



Figure 8.142: Left panel: Function $f(x) = e^x$ (red) and cubic approximation $s_3(x)$ (blue). Right panel: error $f(x) - s_3(x)$ (red) and $f(x) - T_3(x)$ (blue) with Taylor polynomial $T_3(x) = 1 + x + x^2/2 + x^3/6$.

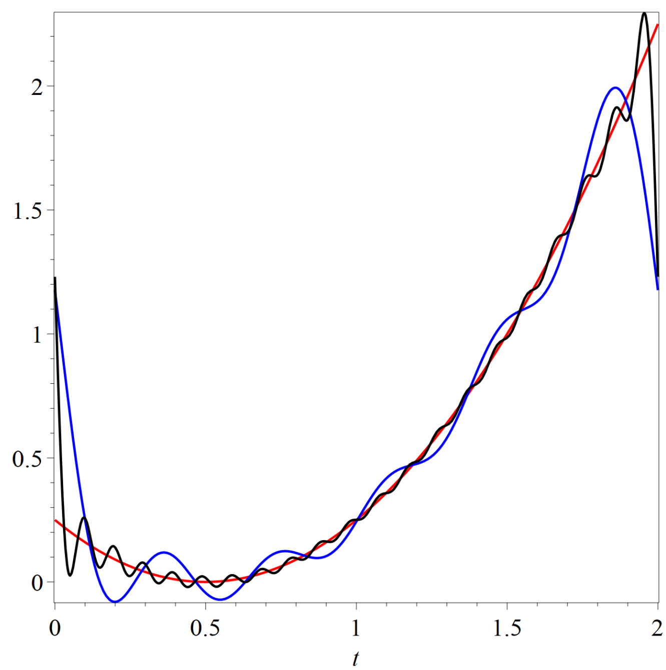



Figure 8.143: Function $f(x) = (x - 1/2)^2$ (red) and complex-exponential Fourier approximations $s_5(x)$ (blue) and $s_{20}(x)$ (black).

## Section 8.3

**Exercise Solution 8.3.1.** *The approximate solution is*

$$\begin{aligned} u(x,t) &\approx 0.918e^{-12.3t}\sin(1.57x) + 0e^{-49.3t}\sin(3.14x) \\ &+ 0.133e^{-110t}\sin(4.71x). \end{aligned}$$

*Note $b_2 = 0$ here. Graph shown in the top left panel of Figure 8.144.*

**Exercise Solution 8.3.2.** *The approximate solution is*

$$\begin{aligned} u(x,t) &\approx -0.360e^{-2.46t}\sin(1.57x) + e^{-9.86t}\sin(3.14x) \\ &- 0.388e^{-22.2t}\sin(4.71x). \end{aligned}$$

*Graph shown in the top right panel of Figure 8.144.*

**Exercise Solution 8.3.3.** *The approximate solution is*

$$\begin{aligned} u(x,t) &\approx -1.56e^{-1.85t}\sin(0.785x) + 1.59e^{-7.40t}\sin(1.57x) \\ &- 0.160e^{-16.7t}\sin(2.36x) + 0.051e^{-29.6t}\sin(3.14x). \end{aligned}$$

*Graph shown in the bottom left panel of Figure 8.144.*

**Exercise Solution 8.3.4.** *The approximate solution is*

$$\begin{aligned} u(x,t) &\approx 0.505e^{-9.86t}\sin(3.14x) - 0.137e^{-39.4t}\sin(6.28x) \\ &+ 0.0447e^{-88.7t}\sin(9.42x) - 0.0192e^{-158t}\sin(12.6x) \\ &+ 0.010e^{-246t}\sin(15.7x). \end{aligned}$$

*Graph shown in the bottom right panel of Figure 8.144. Changing $\alpha$ to 2 effectively doubles the decay rate; the $t$ variable in the solution is replaced by $2t$.*

**Exercise Solution 8.3.5.** *The approximate solution is*

$$\begin{aligned} u(x,t) &\approx \frac{1}{30} - \frac{3}{\pi^4}e^{-4\pi^2 t}\cos(2\pi x) \\ &\approx -0.033 - 0.031e^{-39.4t}\cos(6.28x). \end{aligned}$$

*(The coefficient $a_2 = 0$ here). Graph shown in the top left panel of Figure 8.145.*

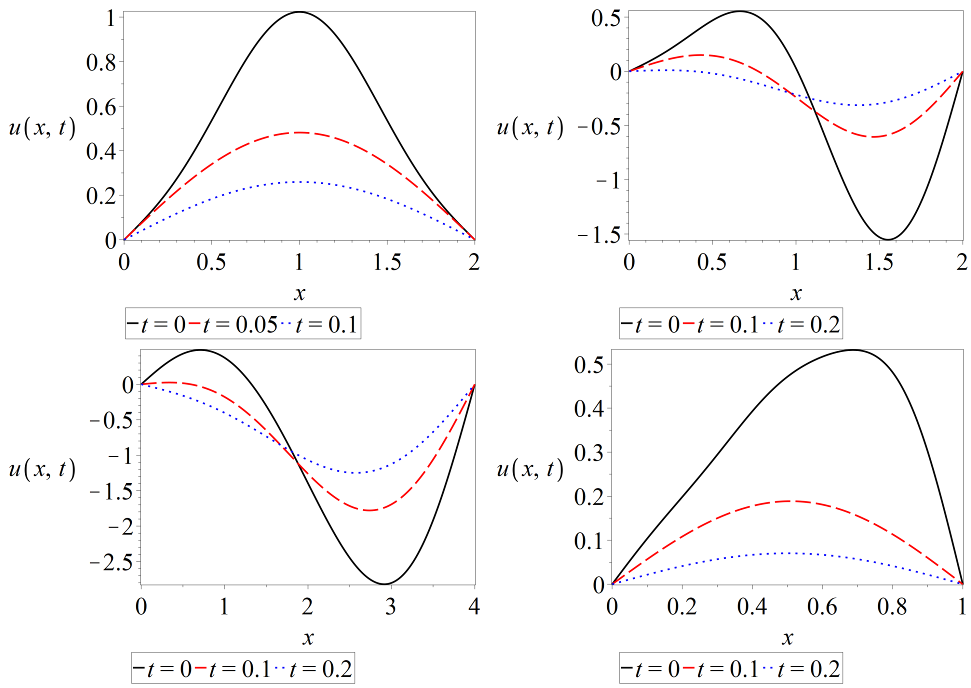



Figure 8.144: Solutions to Exercises 8.3.1 (top left), 8.3.2 (top right),8.3.3 (bottom left), 8.3.4 (bottom right).

**Exercise Solution 8.3.6.** *The approximate solution is*

$$\begin{aligned} u(x,t) \approx\ & 0.500 - 0.374e^{-2.46t}\cos(1.57x) \\ & + 0.162e^{-22.2t}\cos(4.71x) - 0.500e^{-39.4t}\cos(6.28x) \\ & + 0.188e^{-61.6t}\cos(7.85x). \end{aligned}$$

*Here* $a_2 = 0$. *Graph shown in the top right panel of Figure 8.145.*

**Exercise Solution 8.3.7.** *The approximate solution is*

$$\begin{aligned} u(x,t) \approx\ & 1 - 0.36e^{-1.85t}\cos(0.785x) \\ & - e^{-7.40t}\cos(1.57x) + 0.33e^{-16.7t}\cos(2.36x). \end{aligned}$$

*Graph shown in the bottom left panel of Figure 8.145.*

**Exercise Solution 8.3.8.** *The approximate solution is*

$$u(x,t) \approx 0.233 - 0.247e^{-9.86t}\cos(3.14x) + 0.0154e^{-39.4t}\cos(6.28x).$$

*Graph shown in the bottom right panel of Figure 8.145. Increasing the diffusivity $\alpha$ from $\alpha = 1$ to $\alpha = 3$ effectively speeds up time by a factor of 3—the solution diffuses to a constant value three time as fast.*

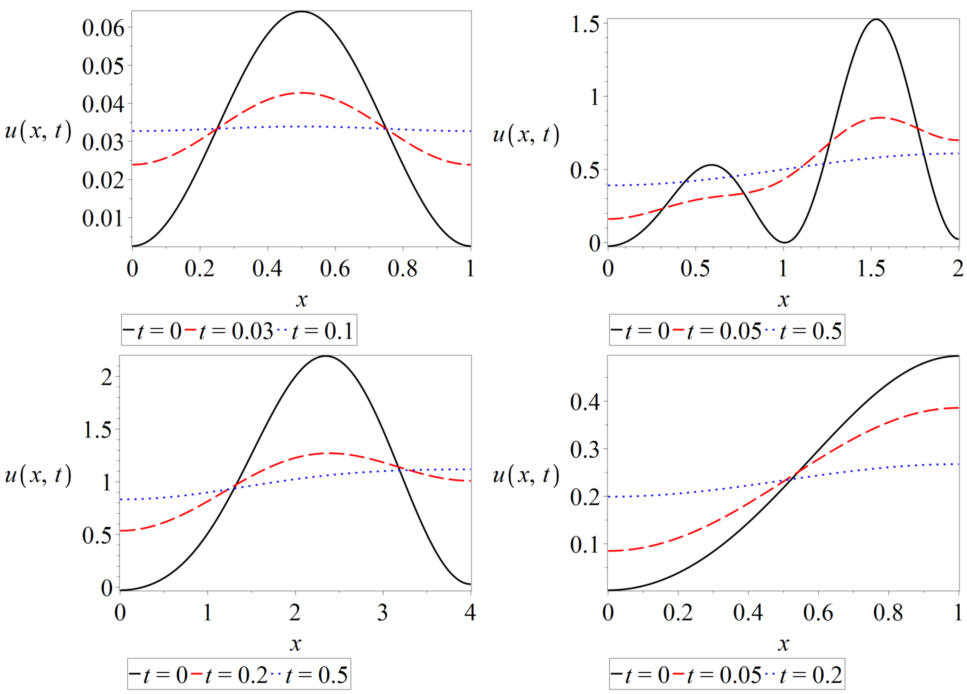



Figure 8.145: Solutions to Exercises 8.3.5 (top left), 8.3.6 (top right),8.3.7 (bottom left), 8.3.8 (bottom right).

**Exercise Solution 8.3.9.**

*(a) The approximate solution is*

$$u(x,t) \approx 1.01e^{-3.08t}\sin(0.785x) + 0.499e^{-27.7t}\sin(2.36x) \\ - 0.207e^{-77t}\sin(3.92x) - 0.0172e^{-151t}\sin(5.50x).$$

*Graph shown in the left panel of Figure 8.146.*

*(b) The approximate solution is*

$$u(x,t) \approx 1.49e^{-0.462t}\sin(0.392x) + 0.645e^{-4.16t}\sin(1.18x) \\ - 0.753e^{-11.5t}\sin(1.96x) + 0.0634e^{-22.7t}\sin(2.75x).$$

*Graph shown in the right panel of Figure 8.146.*

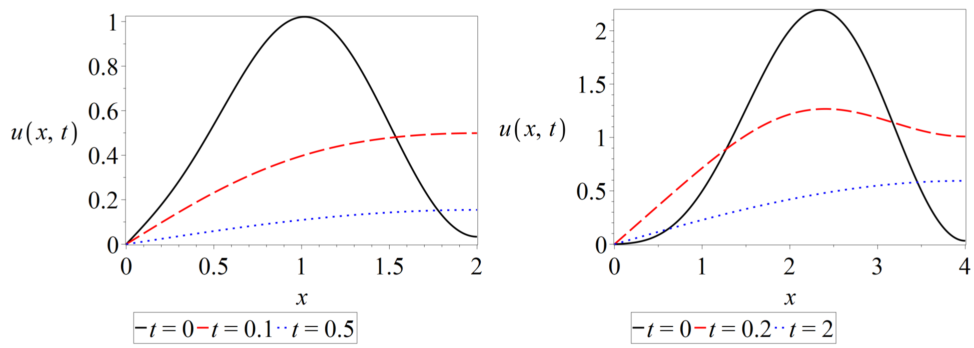



Figure 8.146: Solutions to Exercise 8.3.9. Top left panel (a), top right panel (b).

**Exercise Solution 8.3.10.** *Separating variables again leads to solutions*

$$u(x,t) = C_1 e^{-\alpha\lambda^2 t}\sin(\lambda x) + C_2 e^{-\alpha\lambda^2 t}\cos(\lambda x)$$

*where* $\lambda, C_1$, *and* $C_2$ *are constants. The boundary condition* $\dfrac{\partial u}{\partial x}(0,t) = 0$ *leads to*

$$C_1 \lambda e^{-\alpha\lambda^2 t} = 0$$

*which means that either* $\lambda = 0$ *(so* $u(x,t) = C_2$*, a constant, which is currently a valid choice) or* $C_1 = 0$*. When* $C_1 = 0$ *we find that*

$$u(x,t) = C_2 e^{-\alpha\lambda^2 t}\cos(\lambda x).$$

*Now* $u(L,t) = 0$ *means that either* $C_2 = 0$*, which is of no value, or* $\cos(\lambda L) = 0$*, which forces* $\lambda = (j + 1/2)\pi/L$ *for some integer* $j$*. The solution* $u(x,t) = C_2$ *is eliminated by the condition* $u(L,t) = 0$*.*

*All in all we are left with solutions that are multiples of*

$$u_j(x,t) = e^{-\alpha(j+1/2)^2\pi^2 t/L^2}\cos((j+1/2)\pi x/L).$$

*We can restrict our attention to* $j \geq 0$*. We construct a general solution to the heat equation with these boundary conditions as*

$$u(x,t) = \sum_{j=0}^{\infty} c_j e^{-\alpha(j+1/2)^2\pi^2 t/L^2}\cos((j+1/2)\pi x/L).$$

*To obtain initial data $u(x,0) = f(x)$ we need*

$$\sum_{j=0}^{\infty} c_j \cos((j+1/2)\pi x/L) = f(x).$$

*An argument very similar to that in the text shows that we should take*

$$c_j = \frac{\displaystyle\int_0^L f(x)\cos\left(\frac{\pi(j+1/2)x}{L}\right)\,dx}{\displaystyle\int_0^L \cos^2\left(\frac{\pi(j+1/2)x}{L}\right)\,dx}.$$

*With $\alpha = 1, L = 2$ and $f(x) = x^2(2-x)$ the approximate solution is*

$$\begin{aligned} u(x,t) &\approx 0.737e^{-0.616t}\cos(0.785x) - 0.809e^{-5.54t}\cos(2.36x) \\ &\quad + 0.107e^{-15.4t}\cos(3.92x) + 0.0547e^{-30.2t}\cos(5.50x). \end{aligned}$$

**Exercise Solution 8.3.11.**

(a) *Take the hint: replace $r(x,t)$ with $\frac{\partial\rho}{\partial t} - \alpha\frac{\partial^2\rho}{\partial x^2}$ in the specified integral to find*

$$\int_0^T\int_0^L r(x,t)\,dx\,dt = \int_0^T\int_0^L \frac{\partial\rho}{\partial t}\,dx\,dt - \alpha\int_0^T\int_0^L \frac{\partial^2\rho}{\partial x^2}\,dx\,dt. \tag{8.18}$$

*Evaluate the first double integral on the right in (8.18) in $t$ first and find*

$$\int_0^L\int_0^T \frac{\partial\rho}{\partial t}\,dt\,dx = \int_0^L (\rho(x,T) - f(x))\,dx \tag{8.19}$$

*and evaluate the second integral on the right in (8.18) to find*

$$\int_0^T\int_0^L \frac{\partial^2\rho}{\partial x^2}\,dx\,dt = \int_0^T \left(\frac{\partial\rho}{\partial x}(L,t) - \frac{\partial\rho}{\partial x}(0,t)\right)\,dt = 0 \tag{8.20}$$

*since $\frac{\partial\rho}{\partial x}(0,t) = \frac{\partial\rho}{\partial x}(L,t) = 0$. Combining (8.18)-(8.20) yields*

$$\int_0^L \rho(x,T)\,dx = \int_0^L f(x)\,dx + \int_0^T\int_0^L r(x,t)\,dx\,dt. \tag{8.21}$$

(b) *Since $\rho(x, T)$ is the density of stuff in the conduit at time $t = T$, the integral on the left in (8.21) is the total amount of stuff in the conduit at time $t = T$. Since $f(x) = \rho(x, 0)$ is the stuff density at time $t = 0$, the first integral on the right in (8.21) is the amount of stuff at time $t = 0$. Finally, if $r(x, t)$ is the rate of creation of stuff on a stuff per length per time basis, the second integral on the right in (8.21) tallies the total amount of stuff created (or destroyed) in the conduit from time $t = 0$ to $t = T$. Thus (8.21) is merely the statement that the amount of stuff at time $T$ is the amount present at time $t = 0$ plus all that was created from $t = 0$ to $t = T$.*

**Exercise Solution 8.3.12.** *The Fourier coefficients for $f(x)$ are all zero, of course. The Fourier cosine coefficients $a_0(t)$ to $a_3(t)$ for $r(x, t)$ with respect to $x$ are*

$$a_0(t) = 2e^{-t}, \quad a_1(t) = -8e^{-t}/\pi^2, \quad a_2(t) = 0, \quad a_3(t = -8e^{-t}/(9\pi^2).$$

*Solving for the $\phi_k(t)$ functions produces (rounded to three significant figures)*

$$\phi 0(t) = 2 - 2e^{-t}, \quad \phi_1(t) = 0.552(e^{-2.47t} - e^{-t}, \quad \phi_2(t) = 0,$$
$$\phi_3(t) = 0.00425(e^{-22.2} - e^{-t}).$$

*The approximation solution is*

$$u(x, t) \approx \phi_0(t)/2 + \phi_1(t)\cos(\pi x/2) + \phi_2(t)\cos(\pi x) + \phi_3(t)\cos(3\pi x/2)$$

*This is shown in the top left panel of Figure 8.147.*

**Exercise Solution 8.3.13.** *The Fourier coefficients for $f(x)$ are all zero. The Fourier cosine coefficients $a_0(t)$ to $a_2(t)$ for $r(x, t) = 1$ with respect to $x$ are $a_0(t) = 2, a_1(t) = a_2(t) = 0$. Solving for the $\phi_k(t)$ functions produces (rounded to three significant figures)*

$$\phi 0(t) = 2t, \quad \phi_1(t) = 0, \quad \phi_2(t) = 0.$$

*The approximate solution (which is exact here) is*

$$u(x, t) = t.$$

*This is shown in the top right panel of Figure 8.147. It makes perfect sense—the heat source $r$ is uniform in space and the entire bar's stuff density rises uniformly at a constant rate in time.*

**Exercise Solution 8.3.14.** *The Fourier coefficients for* $f(x)$ *are approximately* $f_0 = 2.0, f_1 = -0.360, f_2 = -1.0, f_3 = 0.330, f_4 = 0.0, f_5 = 0.0208$. *The Fourier cosine coefficients* $a_0(t)$ *to* $a_5(t)$ *for* $r(x,t) = x - 2$ *with respect to* $x$ *are independent of time (since* $r$ *is too) and given by* $a_0(t) = 0, a_1(t0 = -1.62, a_2(t) = 0, a_3(t) = -0.180, a_4(t) = 0, a_5(t) - 0.0646$. *More generally* $a_k(t) = 0$ *if* $k$ *is even and* $a_k(t) = -16/(k^2\pi^2)$ *if* $k$ *is odd.*

*Solving for the* $\phi_k(t)$ *functions produces (rounded to three significant figures)*

$$\begin{aligned}
&\phi 0(t) = 2, \quad \phi_1(t) = -0.876 + 0.516e^{-1.85t}, \quad \phi_2(t) = -e^{-7.40t},\\
&\phi_3(t) = -0.018 + 0.342e^{-16.7t}, \quad \phi_4(t) = 0,\\
&\phi_5(t) = -0.0014 + 0.0223e^{-46.3t}.
\end{aligned}$$

*The approximate solution is*

$$u(x,t) \approx 1 + \phi_1(t)\cos(\pi x/4) + \phi_2(t)\cos(\pi x/2) + \cdots + \phi_5(t)\cos(5\pi x/4).$$

*This is shown in the bottom left panel of Figure 8.147.*

**Exercise Solution 8.3.15.** *The Fourier cosine coefficients for* $f$ *are* $a_0 = 2$ *and* $a_k = 0$ *for* $k \geq 1$. *The Fourier cosine coefficients for* $r(x,t)$ *with respect to* $x$ *are*

$$a_0(t) = 4\sin(\pi t), \quad a_1(t) = -1.62\sin(\pi t), a_2(t) = 0.$$

*Solving for the* $\phi_k(t)$ *functions produces (rounded to three significant figures)*

$$\begin{aligned}
&\phi 0(t) = 3.27 - 1.27\cos(\pi t),\\
&\phi_1(t) = -0.383e^{-1.85t} - 0.226\sin(\pi t) + 0.383\cos(\pi t),\\
&\phi_2(t) = 0.
\end{aligned}$$

*The approximate solution is*

$$u(x,t) \approx \phi_0(t)/2 + \phi_1(t)\cos(\pi x/4).$$

*This is shown in the bottom right panel of Figure 8.147.*

**Exercise Solution 8.3.16.** *When* $x_0 = 1/4$ *the Fourier cosine coefficients are* $a_k = 2\cos(k\pi/4)$ *(so* $a_0 = 2, a_1 = \sqrt{2}, a_2 = 0$*, etc.) When* $x_0 = 1/3$ *the Fourier cosine coefficients are* $a_k = 2\cos(k\pi/3)$.

*Graphs of* $\rho(0,t)$ *for each case are shown in the left panel of Figure 8.148. The data at* $x = 0$ *results in a larger difference in the concentration over time, especially for about* $0.1 \leq t \leq 1$*—*$x = 0$ *is closer to where the sources are.*

**Exercise Solution 8.3.17.**

*(a) If $v(x,t) = u(x,t) - w(x,t)$ then*

$$\begin{aligned}\frac{\partial v}{\partial t} - \alpha \frac{\partial^2 v}{\partial x^2} &= \frac{\partial (u-w)}{\partial t} - \alpha \frac{\partial^2 (u-w)}{\partial x^2} \\ &= \left(\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2}\right) + \underbrace{- \left(\frac{\partial w}{\partial t} - \alpha \frac{\partial^2 w}{\partial x^2}\right)}_{r(x,t)} \\ &= 0 + r(x,t) = r(x,t)\end{aligned}$$

*where $r(x,t)$ is as indicated. Also $v(x,0) = u(x,0) - w(x,0) = f(x) - w(x,0) = h(x)$ as defined, and $v(0,t) = u(0,t) - w(0,t) = 0$ and $v(L,t) = u(L,t) - w(L,t) = 0$.*

*(b) With*

$$w(x,t) = u_0(t) + (u_L(t) - u_0(t))x/L$$

*it's simple to check that $w(0,t) = u_0(t)$ and $w(L,t) = u_L(t)$. In the specific case given with $L = 4$, $u_0(t) = \sin(t)$ and $u_L(t) = -\sin(2t)$ we find*

$$w(x,t) = \sin(t) + \frac{(-\sin(2t) - \sin(t))x}{4}.$$

*Also,*

$$r(x,t) = x\cos^2(t) + x\cos(t)/4 - \cos(t) - x/2.$$

*The initial data is $h(x) = x(4-x)$.*

*(c) First, the Fourier sine coefficients of $h(x)$ are $h_k = 128/(k^3\pi^3)$ for $k$ odd, $h_k = 0$ for $k$ even. The Fourier sine coefficients $a_k(t)$ of $r(x,t)$ are approximately*

$$\begin{aligned}a_1(t) &= 2.54\cos^2(t) - 0.636\cos(t) - 1.27, \\ a_2(t) &= -1.27\cos^2(t) - 0.318\cos(t) + 0.636, \\ a_3(t) &= 0.847\cos^2(t) - 0.212\cos(t) - 0.424, \\ a_4(t) &= -0.635\cos^2(t) - 0.159\cos(t) + 0.318 \\ a_5(t) &= 0.508\cos^2(t) - 0.127\cos(t) - 0.254.\end{aligned}$$

*The $\phi_k(t)$ are a bit of a mess but, for example,*

$$\phi_1(t) = 4.136e^{-1.23t} + 0.283943\cos(2t) - 0.31100\cos(t) + 0.46028\sin(2t) - 0.252221\sin(t).$$

*Then*

$$v(x,t) = \phi_1(t)\sin(\pi x/4) + \phi_2(t)\sin(\pi x/2) + \cdots + \phi_5(t)\sin(5\pi x/4).$$

*We may approximate $u(x,t)$ as*

$$u(x,t) = w(x,t) + v(x,t)$$

*where $w$ is from part (b) and $v$ is given by the series above.*

*For the specific parameters given, a graph of $u(x,t)$ is shown in Figure 8.149 at time $t = 0, \pi/4, \pi/2,$ and $t = \pi$. It's easy to see that the Dirichlet boundary data is satisfied at these times.*

(d) *The function $w(x,t) = xg_0(t) + (g_1(t) - g_0(t))x^2/(2L)$ satisfies the Neumann conditions $\frac{\partial u}{\partial x}(0,t) = g_0(t)$ and $\frac{\partial u}{\partial x}(L,t) = g_L(t)$. Let $v(x,t) = u(x,t) - w(x,t)$. Then $v(x,t)$ satisfies the nonhomogeneous version of the heat equation exactly as in part (b), with insulating boundary conditions and initial data $v(x,0) = u(x,0) - w(x,0)$. The function $v(x,t)$ can be obtained as in the text, and then $u(x,t) = v(x,t) + w(x,t)$.*

**Exercise Solution 8.3.18.**

(a) *That $u$ satisfies the heat equation is a simple consequence of linearity. Of course $u(0,t) = u_1(0,t) - u_2(0,t) = 0$ and $u(L,t) = u_1(L,t) - u_2(L,t) = 0$.*

(b) *If we multiply the heat equation through by $u$ we obtain*

$$u\frac{\partial u}{\partial t} - \alpha u\frac{\partial^2 u}{\partial x^2} = 0$$

*for $0 \le x \le L$ and $t \ge 0$. Integrating both sides above with respect to $x$ from $x = 0$ to $x = L$ and then in $t$ from $t = 0$ to $t = T$ (where $T$ is some fixed positive time) yields*

$$\int_0^T \int_0^L u(x,t)\frac{\partial u}{\partial t}(x,t)\,dx\,dt - \alpha\int_0^T \int_0^L u(x,t)\frac{\partial^2 u}{\partial x^2}(x,t)\,dx\,dt = 0.$$

(c) *The first double integral on the left in the last equation of part (b) can be partially evaluated by integrating first in* $t$ *and noting that an antiderivative for* $u\partial u/\partial t$ *with respect to* $t$ *is* $u^2/2$. *We find*

$$\int_0^T \int_0^L u(x,t)\frac{\partial u}{\partial t}(x,t)\,dx\,dt = \frac{1}{2}\int_0^L u^2(x,T)\,dx$$

*by taking into account* $u(x,0) = 0$.

(d) *Performing the* $x$ *integration in the second double integral on the left in part (b) using integration by parts* $\int_a^b fg'\,dx = fg|_{x=a}^{x=b} - \int_a^b f'g\,dx$ *with* $f = u$ *and* $g = \partial u/\partial x$ *shows that*

$$\begin{aligned}\int_0^L u(x,t)\frac{\partial^2 u}{\partial x^2}(x,t)\,dx &= u(L,t)\frac{\partial u}{\partial x}(L,t) - u(0,t)\frac{\partial u}{\partial x}(0,t) \\ &\quad - \int_0^L \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx \\ &= -\int_0^L \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx\end{aligned}$$

*since* $u(0,t) = u(L,t) = 0$.

(e) *Using the results of parts (c) and (d) in the last displayed equation of part (b) shows that*

$$\frac{1}{2}\int_0^L u^2(x,T)\,dx + \alpha\int_0^T \int_0^L \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx\,dt = 0.$$

(f) *Since the integrand of each integral in part (e) is nonnegative, each integral is nonnegative. Moreover, since the integrals sum to zero, each integral must itself be zero. From the cited Exercise this means that, for example,* $u^2(x,T) = 0$ *for all* $x$ *and each* $T$, *so* $u(x,T) = 0$ *for all* $x$ *and* $T$. *That is,* $u$ *is the zero function. Since* $u = u_1 - u_2$ *this means that* $u_1(x,t) = u_2(x,t)$ *for all* $x$ *and* $t$. *Thus if* $u_1$ *and* $u_2$ *both satisfy the heat equation with the same initial and boundary data, they are the same function. The solution is unique.*

(g) *This is exactly the same argument, noting now that* $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial t}(L,x) = 0$ *and using this in part (d).*

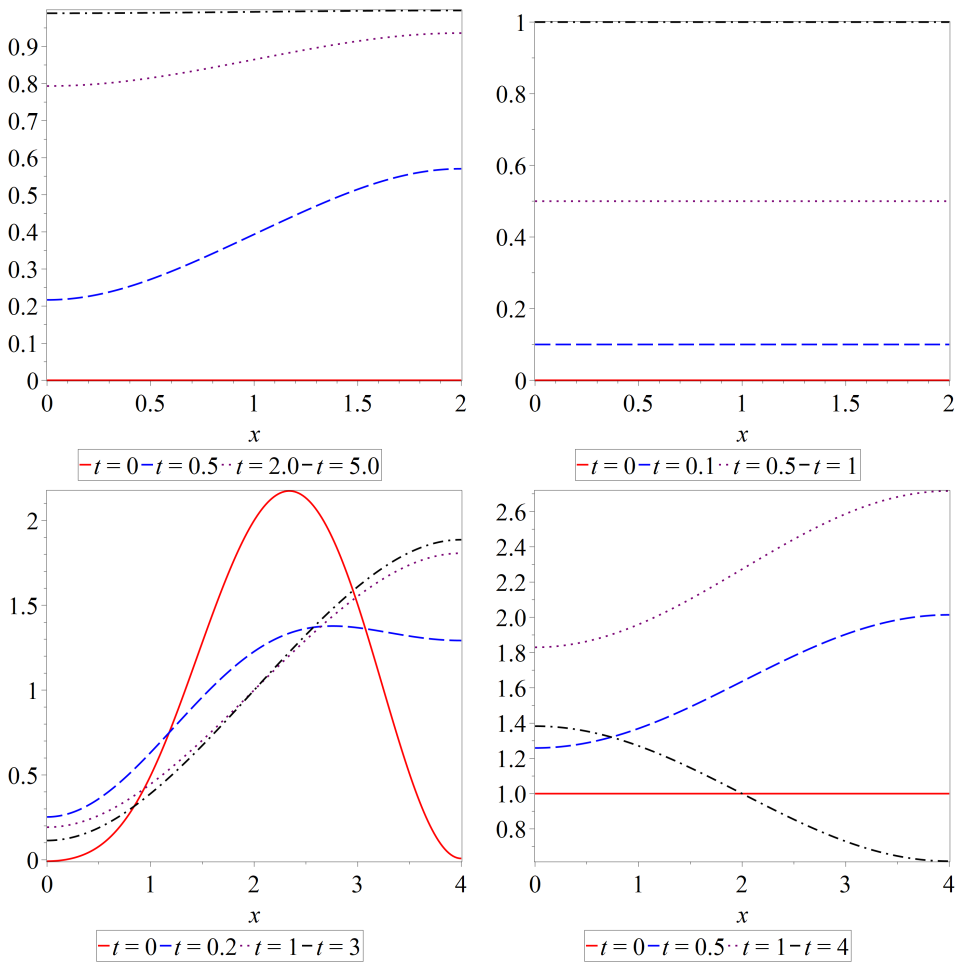



Figure 8.147: Solutions to Exercises 8.3.12 (top left), 8.3.13 (top right),8.3.14 (bottom left), 8.3.15 (bottom right).

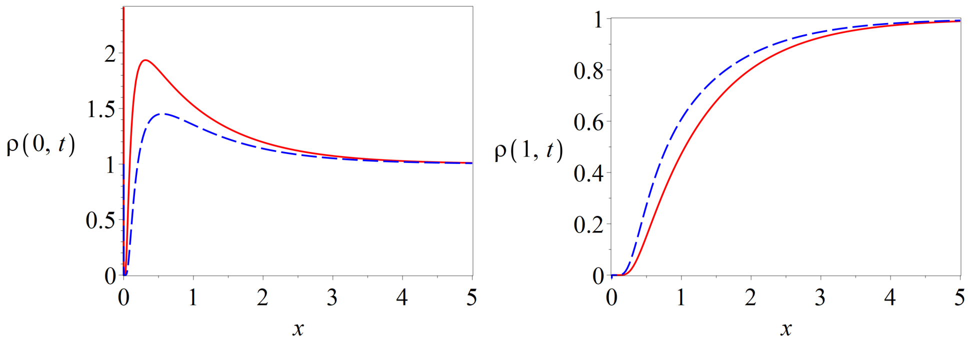



Figure 8.148: Left panel: concentration $\rho(0, t)$ for source at $x_0 = 1/4$ (solid red) and source at $x_0 = 1/3$ (dashed blue). Right panel: same, but data $\rho(1, t)$.

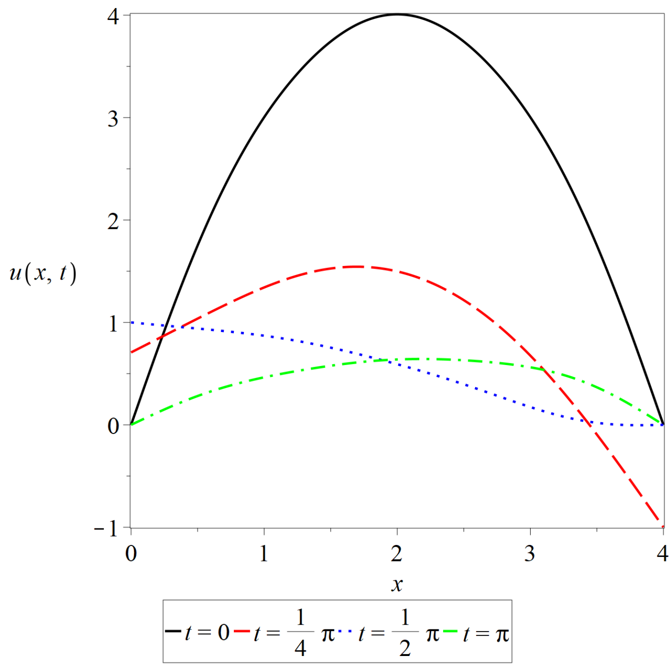



Figure 8.149: Solution to heat equation on $0 \leq x \leq 4$, diffusivity $\alpha = 2$, initial data $f(x) = x(4 - x)$, Dirichlet data $u(0, t) = \sin(t)$, $u(4, t) = -\sin(2t)$.

## Section 8.4

**Exercise Solution 8.4.1.** *The solution is* $\rho(x,t) = f(x - 2t) = (x - 2t)/((x-2t)^2+1)$*. See Figure 8.150, left panel.*

**Exercise Solution 8.4.2.** *The solution is* $\rho(x,t) = f(x + 2t) = (x + 2t)/((x+2t)^2+1)$*. See Figure 8.150, right panel.*

**Exercise Solution 8.4.3.** *The solution is* $\rho(x,t) = f(x - 0t) = f(x)$*. The situation is quite static, with nothing moving. This figure is not shown.*

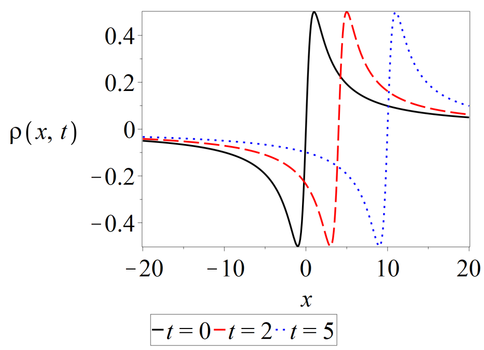



Figure 8.150: Left panel: solution to advection equation for Exercise 8.4.1. Right panel: solution to advection equation for Exercise 8.4.2.

**Exercise Solution 8.4.4.** *In this case the solution is* $u(x,t) = \cos(\pi t)\sin(\pi x)$ *and is exact (it is exact for any* $N \geq 2$*). Solution graphed in the top left panel of Figure 8.151.*

**Exercise Solution 8.4.5.** *In this case the solution is* $u(x,t) = \cos(\pi t)\sin(\pi x) + 3\sin(2\pi t)\sin(2\pi x)/(2\pi)$ *and is exact (it is exact for any* $N \geq 4$*). Solution graphed in the top right panel of Figure 8.151.*

**Exercise Solution 8.4.6.** *The solution is*

$$u(x,t) = \frac{32}{\pi^3}\sin(\pi t/2)\sin(\pi x/4) - \frac{32}{27\pi^3}\sin(3\pi t/2)\sin(3\pi x/4) + \frac{32}{125\pi^3}\sin(5\pi t/2)\sin(5\pi x/4).$$

*Solution graphed in the bottom left panel of Figure 8.151.*

**Exercise Solution 8.4.7.** *The solution is rather large, but involves the spatial components* $\cos(j\pi t/5)\sin(j\pi x/10)$ *and* $\sin(j\pi t/5)\sin(j\pi x/10)$. *The first and last terms are*

$$u(x,t) \approx 0.3459\cos(\pi t/5)\sin(\pi x/10) + \cdots - 0.03\sin(2\pi t)\sin(\pi x).$$

*The solution graphed in the bottom right panel of Figure 8.151 at times* $t = 0, 1, 2, \ldots, 10$. *Times* $t = 0, 1, 2$ *are in red and solution (a right-moving wave) is moving to the right at speed 2. The wave reflects off of the wall at* $x = 10$ *and reverses direction (and orientation); time* $t = 3$ *to* $t = 7$ *are shown in blue. At around* $t = 8$ *the wave reflects of the wall at* $x = 0$ *and reverse direction and orientation again, moving to the right; times* $t = 8, 9, 10$ *are shown in black.*

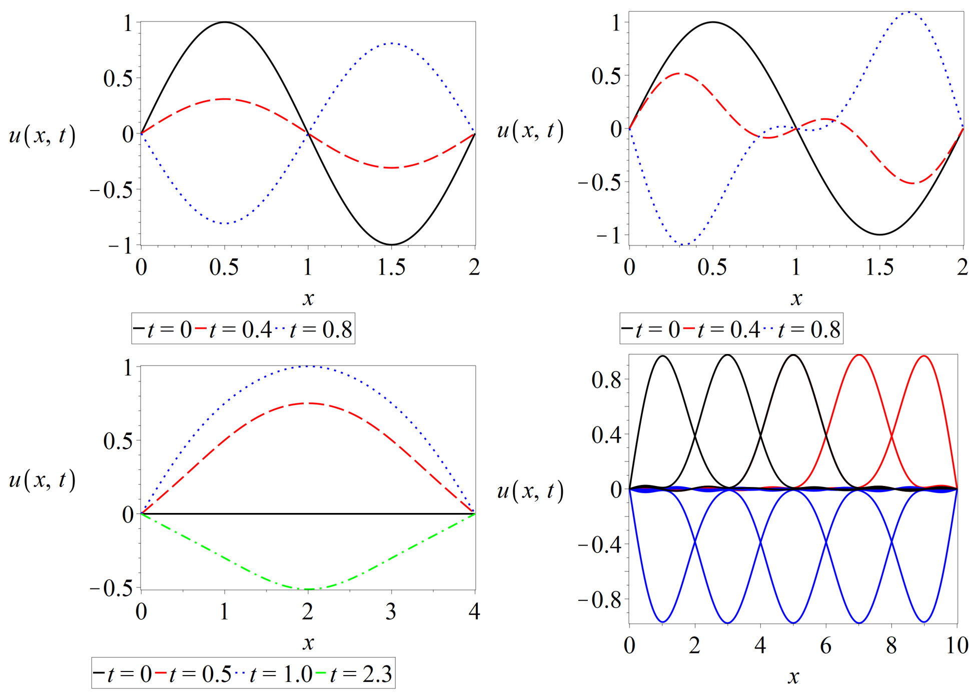



Figure 8.151: Solution to wave equation for Exercises 8.4.4 (top left), 8.4.5 (top right), 8.4.6 (bottom left), 8.4.7 (bottom right).

**Exercise Solution 8.4.8.** *We find* $D = P_1P_2$ *where* $P_1 = d/dt + I$ *and* $P_2 = d/dt + 8I$ *(or vice-versa). The solution or roots for* $P_1$ *and* $P_2$ *are* $c_1e^{-t}$ *and* $c_2e^{-8t}$ *for any constants* $c_1, c_2$.

**Exercise Solution 8.4.9.** *We find $D = P_1P_2$ where $P_1 = P_2 = d/dt + 2I$. The solution or roots for $P_1$ and $P_2$ are of the form are $ce^{-2t}$. This function is a solution to $D(u) = 0$, but this process does not find the solution $u(t) = te^{-2t}$.*

**Exercise Solution 8.4.10.** *Here $D = P_1P_2$ with $P_1 = d/dt + 3iI$ and $P_2 = d/dt - 3iI$ (or vice-versa). Then $c_1e^{-3it}$ and $c_2e^{3it}$ are the relevant solutions.*

**Exercise Solution 8.4.11.** *If $\rho(x,t)$ satisfies the wave equation with speed $c$ for $-\infty < x < \infty$ with initial condition $\rho(x,0) = f(x)$, and $\rho$ represents a right-moving wave then $\rho(x,t) = f(x-ct)$. Then $\frac{\partial \rho}{\partial t}(x,t) = -cf'(x-ct)$ and so $\frac{\partial \rho}{\partial t}(x,0) = -cf'(x)$.*

**Exercise Solution 8.4.12.** *If $P_1 = \frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}$ then the solutions to the advection equation $P_1(u) = 0$ are of the form $u(x,t) = f(x+2t)$ for some $f$; these are waves moving left at speed 2. If $P_2 = \frac{\partial}{\partial t} + 5\frac{\partial}{\partial x}$ then the solutions to the advection equation $P_2(u) = 0$ are of the form $u(x,t) = f(x-5t)$ for some $f$; these are waves moving right at speed 5. The composition of $P_1$ and $P_2$ is*

$$\begin{aligned} P_1P_2 &= \left(\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 5\frac{\partial}{\partial x}\right) \\ &= \frac{\partial^2}{\partial t^2} + 3\frac{\partial^2}{\partial t \partial x} - 10\frac{\partial^2}{\partial x^2}. \end{aligned}$$

*Then the partial differential equation*

$$\frac{\partial^2 u}{\partial t^2} + 3\frac{\partial^2 u}{\partial t \partial x} - 10\frac{\partial^2 u}{\partial x^2}$$

*should have solutions of the form $u(x,t) = f(x+2t) + g(x-5t)$ for arbitrary functions $f$ and $g$. This is easy to check, since then*

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= 4f''(x+2t) + 25g''(x-5t) \\ 3\frac{\partial^2 u}{\partial t \partial x} &= 6f''(x+2t) - 15g''(x-5t) \\ -10\frac{\partial^2 u}{\partial x^2} &= -10f''(x+2t) - 10g''(x-5t). \end{aligned}$$

*Adding up the right sides above shows that $u$ satisfies this PDE.*

**Exercise Solution 8.4.13.** *The terms that can appear in the solution to the wave equation contain* $\cos(k\pi ct/L)$ *and* $\sin(k\pi ct/L)$ *for* $k \geq 1$*, so the lowest radial frequency that can appear is* $\omega = \pi c/L$ *(when* $k = 1$*). This corresponds to a frequency of* $= \omega/(2\pi) = c/(2L)$ *hertz. But with* $c = \sqrt{T/\lambda}$ *this is* $f = \frac{\sqrt{T/\lambda}}{2L}$ *hertz.*

**Exercise Solution 8.4.14.**

(a) *The wave speed is* $c = \sqrt{58.69/(3.1 \times 10^{-4})} \approx 435.11$ *meters per second. Using the result of Exercise 8.4.13 the frequency here would be* $f = \frac{\sqrt{T/\lambda}}{2L} = 329.63$ *hz. This string is in tune.*

(b) *The approximate solution is*

$$\begin{aligned} u(x,t) &= 0.00790 \sin(4.76x)\cos(2070t) + 0.00197 \sin(9.52x)\cos(4140t) \\ &\quad - (4.67 \times 10^{-7}) \sin(14.3x)\cos(6210t) - 0.000493 \sin(19.0x)\cos(8280t) \\ &\quad - 0.000316 \sin(23.8x)\cos(10400t). \end{aligned}$$

*Plots shown in Figure 8.152.*

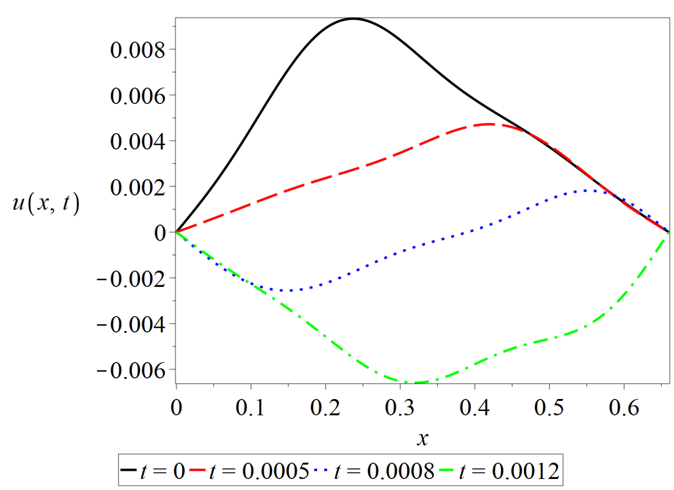



Figure 8.152: Solution to wave equation for string in Exercise 8.4.14.

**Exercise Solution 8.4.15.**

*(a) Given $w(x,t) = u(x,-t)$ compute*

$$\begin{aligned}
\frac{\partial w}{\partial t}(x,t) &= -\frac{\partial u}{\partial t}(x,-t) \\
\frac{\partial^2 w}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial t^2}(x,-t) \\
\frac{\partial w}{\partial x}(x,t) &= \frac{\partial u}{\partial x}(x,-t) \\
\frac{\partial^2 w}{\partial x^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,-t).
\end{aligned}$$

*Then* $\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0.$

*(b) The same computations as in part (a) lead to*

$$\frac{\partial w}{\partial t} + \alpha \frac{\partial^2 w}{\partial x^2} = -\left(\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2}\right) = 0.$$

**Exercise Solution 8.4.16.**

*(a) Given $u(x,t) = \phi(x - ct)$, compute*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -c\phi'(x - ct) \\
\frac{\partial u}{\partial x} &= \phi'(x - ct).
\end{aligned}$$

*Then it's easy to see that*

$$\left(\frac{\partial u}{\partial t}\right)^2 - c^2 \left(\frac{\partial u}{\partial x}\right)^2 = c^2(-\phi'(x - ct))^2 - c^2(\phi'(x - ct))^2 = 0$$

*so this nonlinear PDE is satisfied. Virtually the same computation works for $u(x,t) = \psi(x + ct)$.*

*(b) The PDE is nonlinear, so superposition will not hold. More specifically, suppose that $u(x,t) = \phi(x - ct) + \psi(x + ct)$. Then*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -c\phi'(x - ct) + c\psi'(x + ct) \\
\frac{\partial u}{\partial x} &= \phi'(x - ct) + \psi'(x + ct).
\end{aligned}$$

*Then*

$$\left(\frac{\partial u}{\partial t}\right)^2 - c^2\left(\frac{\partial u}{\partial x}\right)^2 = c^2(-\phi'(x-ct)+\psi'(x+ct))^2 - c^2(\phi'(x-ct)+\psi'(x+ct))^2$$
$$= -4c^2\phi'(x-ct)\psi'(x+ct).$$

*In general, unless $\phi'$ or $\psi'$ is zero (that is, at least one of them is constant), the superposition will not satisfy this PDE.*

**Exercise Solution 8.4.17.**

(a) *The continuity equation with stuff creation rate $r(x,t)$ requires that $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = r(x,t)$. In conjunction with the constitutive relation $q = c\rho$ this yields*

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = r(x,t).$$

(b) *In a control region $\Omega$ with short length $\Delta x$ of the conduit the amount of stuff present is approximately $m = \rho(x,t)\Delta x$. If we assume that the stuff decays in proportion to the amount present (like a classic radioactive decay) then the rate at which stuff is being created in $\Omega$ (really, destroyed) is $-km = -k\rho\Delta x$ where $k$ is a nonnegative constant with the dimension of reciprocal time. Given that $r$ quantifies the rate of stuff creation on a per length per time basis this means that in $\Omega$ the rate at which stuff is created also equals $r(x,t)\Delta x$. A comparison of $-k\rho\Delta x$ and $r(x,t)\Delta x$ shows that $r(x,t) = -k\rho(x,t)$ is an appropriate model. Note that if stuff is measured on a mass basis then $r$ has the dimension of mass per length per time; since $k$ has the dimension of reciprocal time and $\rho$ has dimension mass per length, $r$ and $-k\rho$ has the same physical dimension.*

(c) *This follows from $\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = r(x,t)$ in part (a) and $r(x,t) = -k\rho(x,t)$ from part (b).*

(d) *Let $phi(t) = \rho(x_0 + ct, t)$ be the value of the function $\rho$ on a characteristic curve $x = x_0 + ct$. Then from the chain rule*

$$\begin{aligned}\frac{d\phi(t)}{dt} &= c\frac{\partial \rho}{\partial x}(x_0+ct,t) + \frac{\partial \rho}{\partial t}(x_0+ct,t)\\ &= -k\rho(x_0+ct,t)\\ &= -k\phi(t).\end{aligned}$$

*This means that*

$$\phi(t) = Ce^{-kt}$$

*for some constant* $C$*, so that* $\rho(x_0+ct, t) = Ce^{-kt}$*. Given that* $\rho(x_0, 0) = f(x_0)$ *we have*

$$\rho(x_0 + ct, t) = f(x_0)e^{-kt}.$$

(e) *Use* $x = x_0 + ct$ *(so* $x_0 = x - ct$*)) above to find*

$$\rho(x, t) = f(x - ct)e^{-kt}.$$

*When* $k = 0$ *this is just advection of the initial data* $f$ *at velocity* $c$*, but when* $k > 0$ *the stuff decays in time. It's easy to compute that*

$$\frac{\partial \rho}{\partial t} = -cf'(x - ct)e^{-kt} - kf(x - ct)e^{-kt}$$
$$\frac{\partial \rho}{\partial x} = f'(x - ct)e^{-kt}.$$

*Then* $\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = -kf(x - ct)e^{-kt} = -k\rho.$

(f) *The solution is graphed in Figure 8.153. It makes perfect sense—the stuff propagates to the right at speed* 1 *and decays over time.*

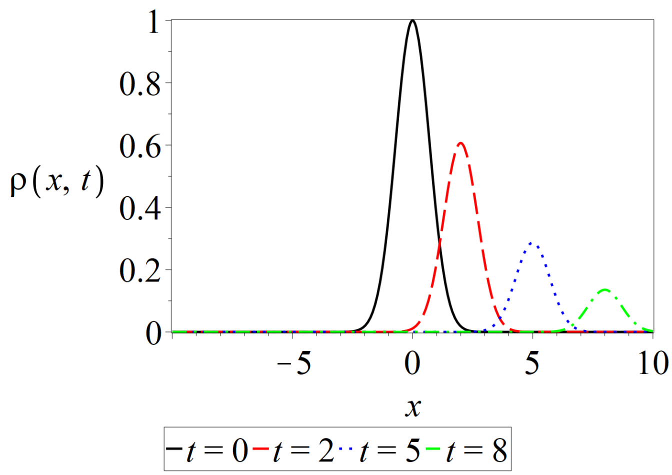



Figure 8.153: Solution to the advection equation with decay for Exercise 8.4.17.

# Appendix A

**Exercise Solution A.6.1.**

*(a)* $\mathrm{Re}(z) = 3$, $\mathrm{Im}(z) = 4$, $\mathrm{Re}(w) = 1$, *and* $\mathrm{Im}(w) = -1$. *Also* $z + w = 4+3i$, $z-w = 2+5i$, $zw = 7+i$, *and* $z/w = -1/2+7i/2$. *Also* $|z| = 5$, $|w| = \sqrt{2}$, *and* $|zw| = |z||w| = 5\sqrt{2}$. *Also* $\overline{z} = 3 - 4i$, $\overline{w} = 1 + i$, *and* $\overline{zw} = 7-i$. *Finally,* $e^z = e^3\cos(4)+ie^3\sin(4)$, $e^w = e\cos(1)-ie\sin(1)$,

$$e^z e^w = e^4(\cos(1)\cos(4)+\sin(1)\sin(4))+ie^4(\sin(4)\cos(1)-\sin(1)\cos(4)),$$

*and* $e^{z+w} = e^4\cos(3) + ie^4\sin(3)$. *That* $e^z e^w = e^{z+w}$ *follows by applying the given trigonometric identity.*

*(b)* $\mathrm{Re}(z) = 3$, $\mathrm{Im}(z) = 0$, $\mathrm{Re}(w) = 0$, *and* $\mathrm{Im}(w) = 1$. *Also* $z+w = 3+i$, $z - w = 3 - i$, $zw = 3i$, *and* $z/w = -3i$. *Also* $|z| = 3$, $|w| = 1$, *and* $|zw| = |z||w| = 3$. *Also* $\overline{z} = 3$, $\overline{w} = -i$, *and* $\overline{zw} = -3i$. *Finally,* $e^z = e^3$, $e^w = e^i = \cos(1) + i\sin(1)$,

$$e^z e^w = e^3\cos(1) + ie^3\sin(1)$$

*and* $e^{z+w} = e^{3+i} = e^3\cos(1) + ie^3\sin(1)$.

*(c)* $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) = \pi$, $\mathrm{Re}(w) = 1$, *and* $\mathrm{Im}(w) = \pi/2$. *Also* $z + w = 1 + 3i\pi/2$, $z - w = -1 + i\pi/2$, $zw = -\pi^2/2 + i\pi$, *and* $z/w = \frac{\pi^2}{2(1+\pi^2/4)} + i\frac{\pi}{1+\pi^2/4}$. *Also* $|z| = \pi$, $|w| = \sqrt{4+\pi^2}/2$, *and* $|zw| = |z||w| = \pi\sqrt{4+\pi^2}/2$. *Also* $\overline{z} = -i\pi$, $\overline{w} = 1-i\pi/2$, *and* $\overline{zw} = -\pi^2/2-i\pi$. *Finally,* $e^z = -1$, $e^w = ie$,

$$e^z e^w = -ie$$

*and* $e^{z+w} = e^{1+3i\pi/2} = -ie$.

**Exercise Solution A.6.2.** *Expand* $z^2 = (x + iy)^2 = x^2 + 2ixy - y^2$ *and set* $z^2 = i$ *to find* $x^2 - y^2 = 0$ *and* $2xy = 1$. *The solutions pairs are* $(x, y)$ *equals* $(\sqrt{2}/2, \sqrt{2}/2)$ *and* $(-\sqrt{2}/2, -\sqrt{2}/2)$, *so that* $z = \sqrt{2}/2 + i\sqrt{2}/2$ *and* $z = -\sqrt{2}/2 - i\sqrt{2}/2$ *are the solutions.*

**Exercise Solution A.6.3.**

*(a) Roots* $z = 2$ *with multiplicity* 3, $z = i$ *with multiplicity* 1, $z = -3$ *with multiplicity* 2, *and* $z = -i$ *with multiplicity* 1. *The roots do not appear in conjugate pairs, so* $p(z)$ *does not have real coefficients.*

(b) *Roots* $z = -1 - i$ *with multiplicity* $2$, $z = 0$ *with multiplicity* $7$, *and* $z = i$ *with multiplicity* $4$. *The roots do not appear in conjugate pairs, so* $p(z)$ *does not have real coefficients.*

(c) *Write* $z^2 + 1 = (z - i)(z + i)$ *so that* $p(z) = (z - i)^{14}(z + i)^{14}$. *The roots are then* $z = i$ *with multiplicity* $14$ *and* $z = -i$ *with multiplicity* $14$. *The roots are in conjugate pairs, so* $p(z)$ *has real coefficients (also clear if we just compute* $(z^2 + 1)^{14}$*).*

**Exercise Solution A.6.4.** *First, it's easy to see that* $z = 0$ *is a root, and we are given that* $z = i$ *is a root. Since* $p$ *has real coefficients* $z = -i$ *must be a root. Thus* $p(z) = z(z - i)(z + i)q(z) = (z^3 + z)q(z)$ *for some quadratic polynomial. A polynomial division shows that* $q(z) = p(z)/(z^3 + z) = z^2 - 2z + 2$. *The two roots of* $q$ *are* $z = 1 \pm i$, *and these are the two additional roots for* $p(z)$.

**Exercise Solution A.6.5.**

(a) *The zeros are* $z = 0$ *and* $z = 3$. *The poles are* $z = 1$ *and* $z = \pm 2i$. *The partial fraction decomposition is*

$$r(z) = \frac{-2/5}{z-1} + \frac{7/10 + 2i/5}{z - 2i} + \frac{7/10 - 2i/5}{z + 2i}.$$

(b) *The zeros are* $z = -1$ *and* $-1$ *(double root). The poles are* $z = 1$ *and* $z = -1 \pm i$. *The partial fraction decomposition is*

$$r(z) = \frac{4/5}{z-1} + \frac{1/10 + i/5}{z + 1 + i} + \frac{1/10 - i/5}{z + 1 - i}.$$

(c) *The only zero is* $z = 0$. *The poles are* $z = \pm i$ *and* $z = \pm 2i$. *The partial fraction decomposition is*

$$r(z) = \frac{1}{z-i} + \frac{1}{z+i} - \frac{1}{z-2i} - \frac{1}{z+2i}.$$

# Appendix B

**Exercise Solution B.6.1.**

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -4 & 1 \\ 3 & 1 \end{bmatrix}$$

**Exercise Solution B.6.2.**

$$\mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix}$$

**Exercise Solution B.6.3.**

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}$$

**Exercise Solution B.6.4.**

$$\mathbf{D} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -2 & 6 \\ 1 & 1 \end{bmatrix}$$

**Exercise Solution B.6.5.**

$$\mathbf{D} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

**Exercise Solution B.6.6.**

$$\mathbf{D} = \begin{bmatrix} 4+6i & 0 \\ 0 & 4-6i \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} -1-2i & -1+2i \\ 3 & 3 \end{bmatrix}$$

**Exercise Solution B.6.7.**

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 1 & -1 \\ 1 & 9 & 3 \end{bmatrix}$$

**Exercise Solution B.6.8.** *If we begin with* $\mathbf{Av} = \lambda\mathbf{v}$ *and conjugate both sides we obtain* $\overline{\mathbf{Av}} = \overline{\lambda\mathbf{v}}$. *But from the familiar properties of conjugation we have* $\overline{\mathbf{Av}} = \overline{\mathbf{A}}\overline{\mathbf{v}}$ *and* $\overline{\lambda\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$, *so that*

$$\overline{\mathbf{A}}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}.$$

*But since* $\mathbf{A}$ *has real entries we have* $\overline{\mathbf{A}} = \mathbf{A}$ *and so*

$$\mathbf{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}.$$

*This is precisely the statement that overline*$\mathbf{v}$ *is an eigenvector for* $\mathbf{A}$ *with eigenvalue* $\overline{\lambda}$.

*Thus if* $\lambda$ *is an eigenvalue for* $\mathbf{A}$ *so is* $\overline{\lambda}$. *This is an empty statement if* $\lambda$ *is real, but it means that complex eigenvalues must come in conjugate pairs.*