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Math143 - Homework 3

Problem 1: Show that $\mathbb{R}P^N$ is a differentiable manifold by definition.

Let $(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1}$. Recall that $\mathbb{R}P^n$ is the set of lines in $\mathbb{R}^{n+1}$ which pass through the origin, so $\mathbb{R}P^n$ can be identified as the quotient space of $\mathbb{R}^{n+1} - \{0\}$ by an equivalence relation. $(x_1, x_2, \dots, x_{n+1}) \sim (\lambda x_1, \dots, \lambda x_{n+1})$ for $\lambda \in \mathbb{R}, \lambda \neq 0$. The points of $\mathbb{RP}^\ltimes$ will be denoted by $[x_1,\dots,x_{n+1}]$.

To prove property (1) of differentiable manifolds, define subsets $V_1,\dots,V_{n+1}$ of $\mathbb{R}P^n$ by $V_i = \{[x_1,\dots,x_{n+1}\}|x_i \neq 0\}$. We claim that $\mathbb{R}P^n$ can be covered by $V_1,\dots,V_{n+1}$.

To prove property (2), we define $X_i : \mathbb{R}^n \to V_i$ so that $X_i(y_1,\dots,y_n) - [y_1,\dots,y_{i-1},1,y_i,\dots,y_n]$ where $y_i = \frac{x_i+1}{x_i}$. Easy to check $V_1 \cap V_j \neq \emptyset$ and $X^{-1}(V_i \cap V_j)$ are open.
In addition, $X_j^{-1}X_i(y_1,\dots,y_n) = X_j^{-1}[y_1,\dots,y_{i-1},1,y_i,\dots,y_n]$
$= X^{j-1}[\frac{y_1}{y_j},\dots,1,\frac{y_{j+1}}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_n}{y_j}] = (\frac{y_1}{y_j},\frac{y_{j-1}}{y_j},\frac{y_{j+1}}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_n}{y_j})$.
This mapping is differentiable, thus done.

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Problem 2: Show why the set of tangent vectors which is tangent to all the curves starting from a point on a manifold M form a linear space (called a tangent space at p of M, denoted by T_pM).

A tangent vector at p of M is the tangent vector at $t = 0$ of some curve $(-\epsilon, \epsilon) \to M$ with $\alpha'(0) = p$.
T_pM is a vector space. Moreover if we chose a parametrization $\underline{X} : U \to M$, then T_pM has a basis $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\}$ and we define its dual basis as $\{dx_1,\dots,dx_n\}$.
Recall for a basis $\{e_1,\dots,e_n\}$ of V^n, then $\{e_1^*,\dots,e_n^*\}$ is the dual basis of $(V^n)^*$. Key idea is to mimic the orthonormal basis of $\mathbb{R}^n$. Define $\{e_1^*,\dots,e_n^*\}$ such that $e_i^*(e_j) = \delta_{ij}$.
We see that we can eventually derive $\alpha'(0) = \sum_i x_i'(0)(\frac{\partial}{\partial x_i})_0$, which implies that every vector on T_pM is a linear combination of the basis vectors, so it is a vector space.

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Problem 3: Show we can put a differentiable structure on a tangent bundle of a differentiable manifold.

Recall that if M is a differentiable manifold, then the set $TM = \{(p,v) : p \in M, v \in T_pM\}$ with a differential structure is called the tangent bundle of M. We prove that such a differential structure always exists.

Let $\{(U_\alpha, X_\alpha)\}$ be a maximal differential structure on M. Denote the coordinates of U_α by $(x_1^\alpha,\dots,x_n^\alpha)$ and the associated bases of the tangent spaces $X_\alpha(U_\alpha)$ by $\{\frac{\partial}{\partial x_1^\alpha},\dots,\frac{\partial}{\partial x_n^\alpha}\}$. Then for every α, define $Y_\alpha : U \times \mathbb{R}^n \to TM$ by $Y_\alpha(x_1^\alpha,\dots,x_n^\alpha,\mu_1,\dots,\mu_n) = (\underline{X}_\alpha(x_1^\alpha,\dots,x_n^\alpha), \sum_{i=1}^n u_i \frac{\partial}{\partial x_1^\alpha})$

where $\mu \in \mathbb{R}^n$. Then $(U_\alpha \times \mathbb{R}^n, Y_\alpha)$ is a differential structure on TM. The proof for this is similar enough to problem 1 that it is left as an exercise to the reader.

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Problem 4: If M is a manifold and G is a group that acts discontinuously on M, Show M/G is a manifold. (See theorem on page 23, Do Carmo Riemanninan Geometry).

Recall that we say a group G acts on a differentiable manifold M if there exists a mapping $\phi : G \times M \to M$ such that
(1) for each $g \in G$, the mapping $\phi_i : M \to M$ given by $\phi_g(p) = \phi(g, p)$ for $p \in M$ is a diffeomorphism and ϕ_e is the identity
(2) if $g_1, g_2 \in G$, then $\phi_{g_1 g_2} = \phi_{g_1}\phi_{g_2}$
We also say that an action is properly discontinuous if every $p \in M$ has a neighborhood $U \subset M$ such that $U \cup g(U) = \emptyset$ for all $g \neq e$. Note that each such group action determines an equivalence relation on M. We can form a quotient space M/G using this equivalence relation. In fact, M/G has a differential structure with respect to which the projection $M \to M/G$ is a local diffeomorphism. Hence M/G is also a differentiable manifold.
An example is the Torus viewed as $\mathbb{R}^k/G$ where $G = \mathbb{Z}^k$.

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