Linear Algebra Reference Sheet

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1 Basic Knowledge

A vector space is an Abelian group under vector addition with defined scalar multiplication over $\mathbb{F}$
$v_1, v_2, ..., v_n$ linearly independent when $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ has only the trivial solution
If E is an elementary matrix, then EA performs a row operation and AE a column operation
$A \sim B$ if they represent the same linear operator under possibly different bases, written $B = PAP^{-1}$

1.1 Positive Definiteness

A symmetric matrix A is positive semi-definite if $x^TAx \geq 0\ \forall x \neq \mathbf{0} \iff Av = \lambda v \implies \lambda \geq 0$
Positive definite matrices $x^TAx > 0$ are nonsingular and have positive diagonal elements
$f : \mathbb{R}^n \to \mathbb{R}$ is convex $\iff \boldsymbol{H_f}$ is positive semi-definite $\iff f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x})\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
Taylor expansion of f at $\boldsymbol{z} \in [\boldsymbol{x}, \boldsymbol{y}]$ is $f(\boldsymbol{y}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{x})^T H_f(\boldsymbol{z})(\boldsymbol{y} - \boldsymbol{x})$

1.2 Vectorization

$\nabla_x x^TAx = \nabla_x(\sum_{i,j} x_ix_jA_{ij}) = x^T(A + A^T)$, $\nabla_x Ax = A$, $\nabla_x x^Ty = y^T$, $\nabla_x x^Tx = 2x^T$
$\sum_i(\sigma(\theta^Tx^{(i)}) - y^{(i)})x^{(i)} = X^T(\sigma(X\theta) - y)$, $\nabla_\theta\sigma(X\theta) = \text{diag}(\sigma'(X\theta))X$
$\nabla_x f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, $\boldsymbol{J_f} = \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial x_1} \cdots \frac{\partial \boldsymbol{f}}{\partial x_n} \end{bmatrix}$, $\boldsymbol{H_f} = \boldsymbol{J}(\nabla_x f)^T$

2 Change of Basis

Let $B = \{b_1, b_2, ..., b_n\}$ a basis for $\mathcal{V}$ over $\mathbb{F}$ and $T \in \mathcal{L}(\mathcal{V})$. Then
$\mathcal{P}_B = \begin{bmatrix}[e_1]_B & [e_2]_B \ldots [e_n]_B\end{bmatrix}$ where $\mathcal{P}_B(v) = [v]_B$ and $\mathcal{P}_B^{-1} = \begin{bmatrix}b_1 & b_2 \ldots b_n\end{bmatrix}$
$\mathcal{M}(T) = \begin{bmatrix}Te_1 & Te_2 \ldots Te_n\end{bmatrix}$ and $\mathcal{M}(T)^{-1} = \begin{bmatrix}T^{-1}e_1 & T^{-1}e_2 \ldots T^{-1}e_n\end{bmatrix}$
$[\mathcal{M}(T)]_B = \mathcal{P}_B\mathcal{M}(T)\mathcal{P}_B^{-1} = \mathcal{P}_{BE}[\mathcal{M}(T)]_{EE}\mathcal{P}_{EB} = \begin{bmatrix}[Te_1]_B & [Te_2]_B \ldots [Te_n]_B\end{bmatrix}$

3 Orthonormal Bases

A list of vectors $(v_1, \ldots, v_m)$ is orthonormal if $\langle v_i, v_j\rangle = 0$ for $i \neq j$ and $\|v_i\| = 1$
If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathcal{V}$ and $v \in \mathcal{V}$, then $v = \langle v, e_1\rangle e_1 + \ldots + \langle v, e_n\rangle e_n$
A matrix Q is orthogonal if its columns and rows form orthonormal bases or if $Q^{-1} = Q^T$
An orthogonal matrix can be thought of as a change of basis or a rotation to a new coordinate axes
Orthogonal operators preserve inner product and norm. $\langle Tx, Ty\rangle = \langle x, y\rangle$ and $\|Tv\| = \|v\|$

3.1 Gram-Schmidt

If v_i are independent, then there exist orthonormal e_i such that $\text{span}(v_1, \ldots, v_m) = \text{span}(e_1, \ldots, e_m)$
$e_1 = \frac{v_1}{\|v_1\|}$, $e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|}$, $e_3 = \frac{v_3 - \langle v_3, e_2\rangle e_2 - \langle v_3, e_1\rangle e_1}{\|v_3 - \langle v_3, e_2\rangle e_2 - \langle v_3, e_1\rangle e_1\|} \ldots$

4 Invertible Matrix Theorem

Let A be an $n \times n$ matrix that represents $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Note this implies $\dim \mathcal{V} = \dim \mathcal{W}$. Then
A is invertible $\iff$ A^T is invertible $\iff$ $\det(A) \neq 0$ $\iff$ $\operatorname{rank}(A) = n$ $\iff$ 0 is not an eigenvalue of A
$\iff$ $\operatorname{range} T = \mathcal{W}$ $\iff$ $\ker T = \{\mathbf{0}\}$ $\iff$ T is injective $\iff$ T is surjective

5 Determinant and Trace

The determinant is a function from $n \times n$ matrices to $\mathbb{F}$ defined recursively by $\det A = a$ if $A = [a]$
and $\det A = \sum_j (-1)^{1+j} a_{1j} \det A_{1j}$ otherwise where A_{ij} denotes deleting the ith row and jth column
$|\det A|$ scales volume of vector transformed by A and is 0 when it is collapsed to a lower dimension
If A is triangular, then $\det A = \prod_i a_{ii}$ where a_{ii} are the diagonal elements
Interchanging rows or columns, $\det A = -\det A$. Adding rows or columns $\det A = \det A$
$\det A = \det A^T$, $\det A^{-1} = \frac{1}{\det A}$, $\det(cA) = c^n \det A$, $\det(AB) = \det A \det B$, $\det A = \prod_i^n \lambda_i$

5.1 Characteristic Polynomial

$\det(\lambda I - T) = p(\lambda) = (\lambda - \lambda_1)^{d_1} \ldots (\lambda - \lambda_m)^{d_m}$ where $d_i = \dim G(\lambda_i, T) = \dim(\operatorname{null}(T - \lambda I)^{\dim \mathcal{V}})$
If $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$, then $c_{n-1} = -\operatorname{tr} A$ and $c_0 = (-1)^n \det A$
$p(T) = O$ and the zeros of $p(\lambda)$ are the eigenvalues of T

5.2 Trace

The trace is a function from $n \times n$ matrices to $\mathbb{F}$ defined by $\operatorname{tr} A = \sum_i a_{ii}$
Trace and determinant are both invariant under similarity, so $\operatorname{tr} A = \operatorname{tr}(PJP^{-1}) = \sum_i^n \lambda_i$
$\operatorname{tr}(X^T Y) = \sum_{i,j} X_{ij} Y_{ij} = \operatorname{vec}(X)^T \operatorname{vec}(Y) \approx \langle X, Y \rangle$, $\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$

6 Fundamental Theorem of Linear Algebra

Each $m \times n$ real matrix A contains four fundamental subspaces described by $A = U\Sigma V^T$

1. $\operatorname{Im} A = \operatorname{col} A = \operatorname{range} A$ has $\dim r$ with basis of first r columns of U
2. $\ker A^T$ has $\dim(m - r)$ with basis of last $m - r$ columns of U
3. $\operatorname{Im} A^T = \operatorname{row} A$ has $\dim r$ with basis of first r columns of V
4. $\ker A$ has $\dim(n - r)$ with basis of first $n - r$ columns of V

$\ker A = (\operatorname{Im} A^T)^\perp$, $\ker A^T = (\operatorname{Im} A)^\perp$ and $x \mapsto Ax$ is one-to-one $\iff$ $\ker A = \{\mathbf{0}\}$
This implies if $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, then $\mathcal{W} = \operatorname{row} T^* \oplus \ker T^*$ and $\mathcal{V} = \operatorname{row} T \oplus \ker T$
A takes $v \in \operatorname{row} A \mapsto \operatorname{col} A$ and $v \in \ker A \mapsto \{\mathbf{0}\}$. Hence $Av_i = \sigma_i u_i \implies AV = \Sigma U$

7 Rank-Nullity Theorem

Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, then $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$. Equivalently, $\dim(\operatorname{Im} T) + \dim(\ker T) = \dim V$.

8 Diagonalization

Let $T \in \mathcal{L}(\mathcal{V})$ and $\{\lambda_1, \dots, \lambda_m\}$ denote the distinct eigenvalues of T. Then
T is diagonalizable $\iff$ $\mathcal{V}$ has a basis consisting of eigenvectors of T $\iff$ $\dim G(\lambda_i, T) = \dim E(\lambda_i, T)$
$\iff$ $\mathcal{V} = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$ $\iff$ $\mathcal{V} = U_1 \oplus \cdots \oplus U_n$ where each U_i is invariant under T
$A = PDP^{-1}$ where $P = \begin{bmatrix} v_1 & v_2 \dots v_n \end{bmatrix}$ and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$
$D = [A]_B$ where B is the basis consisting of eigenvectors of A and $P^{-1} = P_{BE}$

8.1 Spectral Theorem

If $\mathcal{V}$ is a real inner-product space and $T \in \mathcal{L}(\mathcal{V})$, then $\mathcal{V}$ has an orthonormal basis consisting of eigenvectors of T if and only if T is self-adjoint (symmetric).
If A is orthogonally diagonalizable, then $A^T = (PDP^{-1})^T = (P^{-1})^T DP^T = PDP^{-1} = A$
If A is symmetric, then $(A-\lambda I)^2 v = 0 \implies v^T(A-\lambda I)^2 v = 0 \implies ((A-\lambda I)v)^2 = 0 \implies Av = \lambda v$

9 Singular Value Decomposition

Any real $m \times n$ matrix A can be factored into $A = U\Sigma V^T$ where U, V are orthogonal matrices whose columns are the orthonormal eigenvectors of AA^T and $A^T A$ respecitvely and Σ is the $m \times n$ diagonal matrix of the square roots of the nonzero eigenvalues values of $A^T A$.

9.1 Key Insights

$A^T A$ and AA^T are symmetric positive semi-definite, the condition where $U\Sigma V^T = PDP^{-1}$
$A^T A = (U\Sigma V^T)^T U\Sigma V^T = V\Sigma^2 V^T,$ from which V and Σ can be implied by spectral decomposition
When A is square, the transformation can be viewed as a change of basis (rotation 1) V^T, a scaling in that intermediate basis Σ and then another change of basis (rotation 2) U
$A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$ viewed as inverse rotation (2), same scaling and inverse rotation (1)
$A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$. Inverse scaling, assuming that A^{-1} exists.

10 QR Decomposition

Any $m \times n$ matrix A with linearly independent columns can be factored into $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns and R is a nonsingular upper triangular matrix.

$$A = \begin{bmatrix} u_1 & u_2 \dots u_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \dots q_n \end{bmatrix} \begin{bmatrix} \langle q_1, u_1 \rangle & \langle q_1, u_2 \rangle & \cdots \\ 0 & \langle q_2, u_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

If A does not have linearly independent columns, then R will be singular

11 Triangular Matrices

The inverse, product and sum of an upper (lower) triangular matrix is upper (lower) triangular
A triangular matrix is invertible if and only if the entires on its main diagonal are nonzero
$\mathcal{M}(T - \lambda I)$ is not invertible $\iff$ $\lambda = d_i$ for some diagonal element d_i $\implies$ eigenvalues on main diagonal
If $Ax = b$ where A is lower triangular and nonsingular, then $x_1 = {}^{b_1}/_{a_{11}}$, ..., $x_n = \frac{b_n - \sum_{k=1}^{n-1} a_{nk} x_k}{a_{nn}}$
Inverse can be computed by solving $A \begin{bmatrix} x_1 & x_2 \dots x_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \dots e_n \end{bmatrix}$ column by column
If $A = LU,$ then let $Ux = y$ and solve $Ly = b$ via forward-substitution and $Ux = y$ by back-substitution

11.1 LU Decomposition

If A is square and nonsingular, then $A = LU$ for unit lower triangular L and upper triangular U
Defining E_{ij} to remove the ith row in the jth column, we have that $U = E_{n,n-1} \dots E_{32}E_{n,1} \dots E_{21}A$
$A = E_{21}^{-1} \dots E_{n,1}^{-1}E_{32}^{-1} \dots E_{n,n-1}^{-1}U = LU$, where E_{ij} has $-\frac{a_{ij}}{a_{jj}}$ at index (i, j)
If A is singular, then there exists P such that the algorithm $PA = LU$ avoids dividing by zero

11.2 LDU Decomposition

If A admits an LU decomposition, then $A = LDU$ for unit triangular L, U and diagonal D
Let $D = \operatorname{diag}(u_{11}, \dots, u_{nn})$ and $U_1 = \frac{U}{\operatorname{vec} D}$, then $A = LDU_1$ as L is already unit triangular

11.3 Cholesky Decomposition

If A is real positive definite, then $A = LL^T$ for a lower triangular L with positive diagonal entries.
$A \succ 0 \implies A = LDU = A^T = (LDU)^T = U^T DL^T = LDL^T = (LD^{\frac{1}{2}})(D^{\frac{1}{2}}L^T) = L_1L_1^T$
Since L is nonsingular, let $y = L^T x$. Then $y^T Dy = x^T LDLx = x^T Ax > 0 \implies D \succ 0$
Once existence proved, find L by $l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}$ and $l_{ij} = \frac{1}{l_{jj}}(a_{ij} - \sum_{k=1}^{j-1} l_{ik}l_{jk})$ for $i > j$

12 Inner Products

$z = a + bi$, $|z|^2 = z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$, $\langle u, v\rangle = \overline{\langle v, u\rangle}$, $\|v\|^2 = \langle v, v\rangle$
Euclidean inner product over $\mathbb{C}$ becomes $\langle(u_1, \dots, u_n), (v_1, \dots, v_n)\rangle = u_1\overline{v}_1 + \dots + u_n\overline{v}_n$
If $\langle u, v\rangle = 0$, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ (Pythagorean)
$|\langle u, v\rangle| \leq \|u\|\|v\|$ (Cauchy-Schwartz) and $\|u + v\| \leq \|u\| + \|v\|$ (Triangle)

12.1 Adjoints

The adjoint of $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ is $T^* \in \mathcal{L}(\mathcal{W}, \mathcal{V})$ such that $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for all $v \in \mathcal{V}, w \in \mathcal{W}$
If B is an orthonormal basis for $\mathcal{V}$, then $[T^*]_B = [T]_B^*$ where $[T]_B^*$ is the conjugate transpose of $[T]_B$

12.2 Orthogonal Projections

$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle u,v\rangle}{\langle v,v\rangle}v = \frac{\|u\|\|v\|\cos\theta}{\|v\|^2}v = \frac{(u_1v_2+\dots u_nv_n)\hat{v}}{\|v\|} = \|u\|\cos\theta\hat{v}$
The orthogonal complement of $\mathcal{U} \subseteq \mathcal{V}$ is $\mathcal{U}^\perp = \{v \in \mathcal{V} \mid \langle v, u\rangle = 0 \ \forall u \in \mathcal{U}\}$
$\mathcal{V}^\perp = \{\mathbf{0}\}$ and $\mathcal{U}^\perp$ is always a subspace of $\mathcal{V}$. If $\mathcal{U} \subseteq \mathcal{V}$ is a subsapce, then $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}^\perp$
Let $\mathcal{U} \subseteq \mathcal{V}$ be a subspace. The orthogonal projection of $\mathcal{V}$ onto $\mathcal{U}$ is the operator $P_{\mathcal{U}}$ where if $v = u + w$ where $u \in \mathcal{U}$ and $w \in \mathcal{U}^\perp$, then $P_{\mathcal{U}}(v) = u$
If $T \in \mathcal{L}(\mathcal{V})$, then U is invariant under $T \iff P_UTP_U = TP_U \iff U^\perp$ is invariant under T^*
Given a subspace $\mathcal{U}$ of $\mathcal{V}$ and a vector $v \in V$, then $P_{\mathcal{U}}(v) := \underset{u \in U}{\operatorname{argmin}}\|u - v\|$

12.3 Riesz Representation Theorem

If $\varphi : \mathcal{V} \to \mathbb{F}$ is a linear form, then there exists a unique $u \in \mathcal{V}$ such that $\varphi(v) = \langle v, u\rangle$ for all $v \in \mathcal{V}$