Problem 103: Special subset sums: optimum

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) ≠ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) > S(C).

If S(A) is minimised for a given *n*, we shall call it an optimum special sum set. The first five optimum special sum sets are given below.

n = 1: {1}
n = 2: {1, 2}
n = 3: {2, 3, 4}
n = 4: {3, 5, 6, 7}
n = 5: {6, 9, 11, 12, 13}

It *seems* that for a given optimum set, A = $\{a_1, a_2, \ldots, a_n\}$, the next optimum set is of the form B = $\{b, a_1+b, a_2+b, \ldots, a_n+b\}$, where *b* is the "middle" element on the previous row.

By applying this "rule" we would expect the optimum set for *n* = 6 to be A = {11, 17, 20, 22, 23, 24}, with S(A) = 117. However, this is not the optimum set, as we have merely applied an algorithm to provide a near optimum set. The optimum set for *n* = 6 is A = {11, 18, 19, 20, 22, 25}, with S(A) = 115 and corresponding set string: 111819202225.

Given that A is an optimum special sum set for *n* = 7, find its set string.

NOTE: This problem is related to problems 105 and 106.

Problem 106: Special subset sums: meta-testing

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) ≠ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) > S(C).

For this problem we shall assume that a given set contains *n* strictly increasing elements and it already satisfies the second rule.

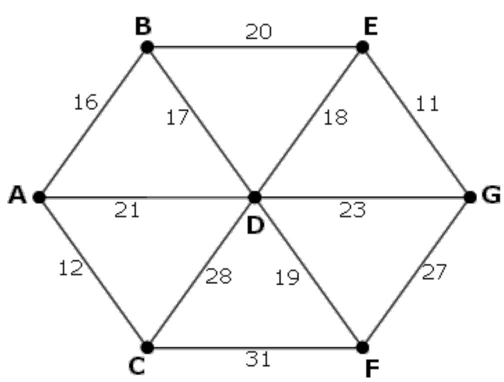
Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which *n* = 4, only 1 of these pairs need to be tested for equality (first rule). Similarly, when *n* = 7, only 70 out of the 966 subset pairs need to be tested.

For *n* = 12, how many of the 261625 subset pairs that can be obtained need to be tested for equality?

NOTE: This problem is related to problems 103 and 105.

Problem 107: Minimal network

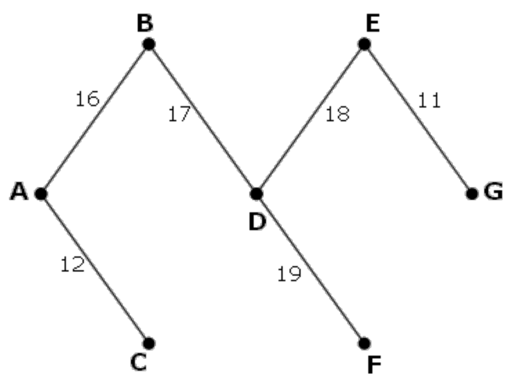
The following undirected network consists of seven vertices and twelve edges with a total weight of 243.



The same network can be represented by the matrix below.

	A	B	C	D	E	F	G
A	-	16	12	21	-	-	-
B	16	-	-	17	20	-	-
C	12	-	-	28	-	31	-
D	21	17	28	-	18	19	23
E	-	20	-	18	-	-	11
F	-	-	31	19	-	-	27
G	-	-	-	23	11	27	-

However, it is possible to optimise the network by removing some edges and still ensure that all points on the network remain connected. The network which achieves the maximum saving is shown below. It has a weight of 93, representing a saving of 243 − 93 = 150 from the original network.



Using network.txt (right click and 'Save Link/Target As...'), a 6K text file containing a network with forty vertices, and given in matrix form, find the maximum saving which can be achieved by removing redundant edges whilst ensuring that the network remains connected.

Problem 109: Darts

In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty.

The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero.

The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively.

At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points.

There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system, which means that the player must land a double (including the double bulls-eye at the centre of the board) on their final dart to win; any other dart that would reduce their running total to one or lower means the score for that set of three darts is "bust".

When a player is able to finish on their current score it is called a "checkout" and the highest checkout is 170: T20 T20 D25 (two treble 20s and double bull).

There are exactly eleven distinct ways to checkout on a score of 6:

D3		
D1	D2	
S2	D2	
D2	D1	
S4	D1	
S1	S1	D2
S1	T1	D1
S1	S3	D1
D1	D1	D1
D1	S2	D1
S2	S2	D1

Note that D1 D2 is considered **different** to D2 D1 as they finish on different doubles. However, the combination S1 T1 D1 is considered the **same** as T1 S1 D1.

In addition we shall not include misses in considering combinations; for example, D3 is the **same** as 0 D3 and 0 0 D3.

Incredibly there are 42336 distinct ways of checking out in total.

How many distinct ways can a player checkout with a score less than 100?

Problem 111: Primes with runs

Considering 4-digit primes containing repeated digits it is clear that they cannot all be the same: 1111 is divisible by 11, 2222 is divisible by 22, and so on. But there are nine 4-digit primes containing three ones:

1117, 1151, 1171, 1181, 1511, 1811, 2111, 4111, 8111

We shall say that M(n, d) represents the maximum number of repeated digits for an n-digit prime where d is the repeated digit, N(n, d) represents the number of such primes, and S(n, d) represents the sum of these primes.

So M(4, 1) = 3 is the maximum number of repeated digits for a 4-digit prime where one is the repeated digit, there are N(4, 1) = 9 such primes, and the sum of these primes is S(4, 1) = 22275. It turns out that for d = 0, it is only possible to have M(4, 0) = 2 repeated digits, but there are N(4, 0) = 13 such cases.

In the same way we obtain the following results for 4-digit primes.

Digit, d	M(4, d)	N(4, d)	S(4, d)
0	2	13	67061
1	3	9	22275
2	3	1	2221
3	3	12	46214
4	3	2	8888

5	3	1	5557
6	3	1	6661
7	3	9	57863
8	3	1	8887
9	3	7	48073

For d = 0 to 9, the sum of all S(4, d) is 273700.

Find the sum of all S(10, d).

Problem 122: Efficient exponentiation

The most naive way of computing n^{15} requires fourteen multiplications:

$n \times n \times \ldots \times n = n^{15}$

But using a "binary" method you can compute it in six multiplications:

$n \times n = n^2$
$n^2 \times n^2 = n^4$
$n^4 \times n^4 = n^8$
$n^8 \times n^4 = n^{12}$
$n^{12} \times n^2 = n^{14}$
$n^{14} \times n = n^{15}$

However it is yet possible to compute it in only five multiplications:

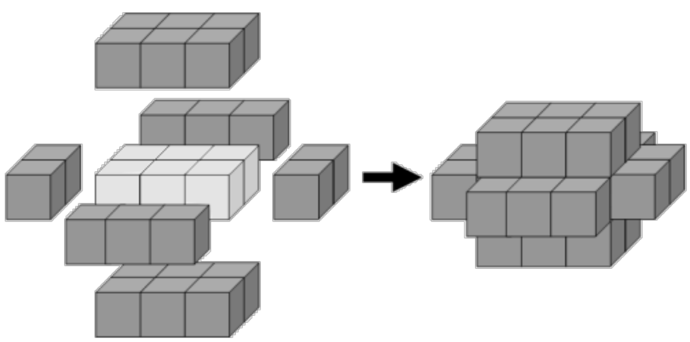
$n \times n = n^2$
$n^2 \times n = n^3$
$n^3 \times n^3 = n^6$
$n^6 \times n^6 = n^{12}$
$n^{12} \times n^3 = n^{15}$

We shall define m(k) to be the minimum number of multiplications to compute n^k; for example m(15) = 5.

For $1 \le k \le 200$, find $\sum$ m(k).

Problem 126: Cuboid layers

The minimum number of cubes to cover every visible face on a cuboid measuring 3 x 2 x 1 is twenty-two.



If we then add a second layer to this solid it would require forty-six cubes to cover every visible face, the third layer would require seventy-eight cubes, and the fourth layer would require one-hundred and eighteen cubes to cover every visible face.

However, the first layer on a cuboid measuring 5 x 1 x 1 also requires twenty-two cubes; similarly the first layer on cuboids measuring 5 x 3 x 1, 7 x 2 x 1, and 11 x 1 x 1 all contain forty-six cubes.

We shall define C(n) to represent the number of cuboids that contain n cubes in one of its layers. So C(22) = 2, C(46) = 4, C(78) = 5, and C(118) = 8.

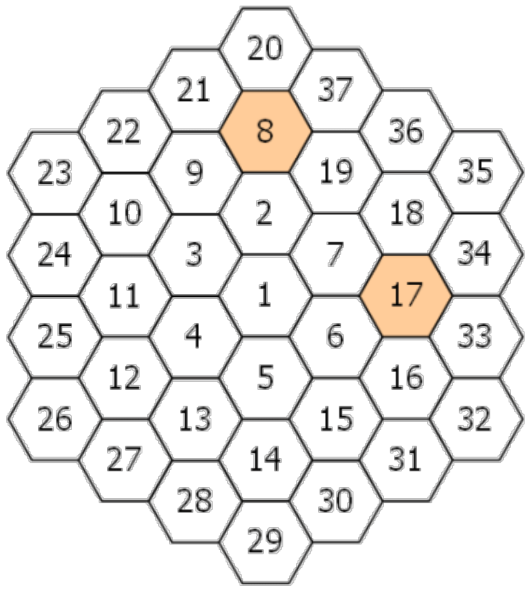
It turns out that 154 is the least value of n for which C(n) = 10.

Find the least value of n for which C(n) = 1000.

Problem 128: Hexagonal tile differences

A hexagonal tile with number 1 is surrounded by a ring of six hexagonal tiles, starting at "12 o'clock" and numbering the tiles 2 to 7 in an anti-clockwise direction.

New rings are added in the same fashion, with the next rings being numbered 8 to 19, 20 to 37, 38 to 61, and so on. The diagram below shows the first three rings.



By finding the difference between tile n and each its six neighbours we shall define PD(n) to be the number of those differences which are prime.

For example, working clockwise around tile 8 the differences are 12, 29, 11, 6, 1, and 13. So PD(8) = 3.

In the same way, the differences around tile 17 are 1, 17, 16, 1, 11, and 10, hence PD(17) = 2.

It can be shown that the maximum value of PD(n) is 3.

If all of the tiles for which PD(n) = 3 are listed in ascending order to form a sequence, the 10th tile would be 271.

Find the 2000th tile in this sequence.

Problem 130: Composites with prime repunit property

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Given that n is a positive integer and GCD(n, 10) = 1, it can be shown that there always exists a value, k, for which R(k) is divisible by n, and let A(n) be the least such value of k; for example, A(7) = 6 and A(41) = 5.

You are given that for all primes, $p > 5$, that $p - 1$ is divisible by A(p). For example, when p = 41, A(41) = 5, and 40 is divisible by 5.

However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703.

Find the sum of the first twenty-five composite values of n for which
GCD(n, 10) = 1 and $n - 1$ is divisible by A(n).

Problem 131: Prime cube partnership

There are some prime values, p, for which there exists a positive integer, n, such that the expression $n^3 + n^2p$ is a perfect cube.

For example, when $p = 19$, $8^3 + 8^2 \times 19 = 12^3$.

What is perhaps most surprising is that for each prime with this property the value of n is unique, and there are only four such primes below one-hundred.

How many primes below one million have this remarkable property?

Problem 133: Repunit nonfactors

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Let us consider repunits of the form R(10^n).

Although R(10), R(100), or R(1000) are not divisible by 17, R(10000) is divisible by 17. Yet there is no value of n for which R(10^n) will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of R(10^n).

Find the sum of all the primes below one-hundred thousand that will never be a factor of R(10^n).

Problem 141: Investigating progressive numbers, *n*, which are also square.

A positive integer, n, is divided by d and the quotient and remainder are q and r respectively. In addition d, q, and r are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio 3/2).
We will call such numbers, n, progressive.

Some progressive numbers, such as 9 and $10404 = 102^2$, happen to also be perfect squares.
The sum of all progressive perfect squares below one hundred thousand is 124657.

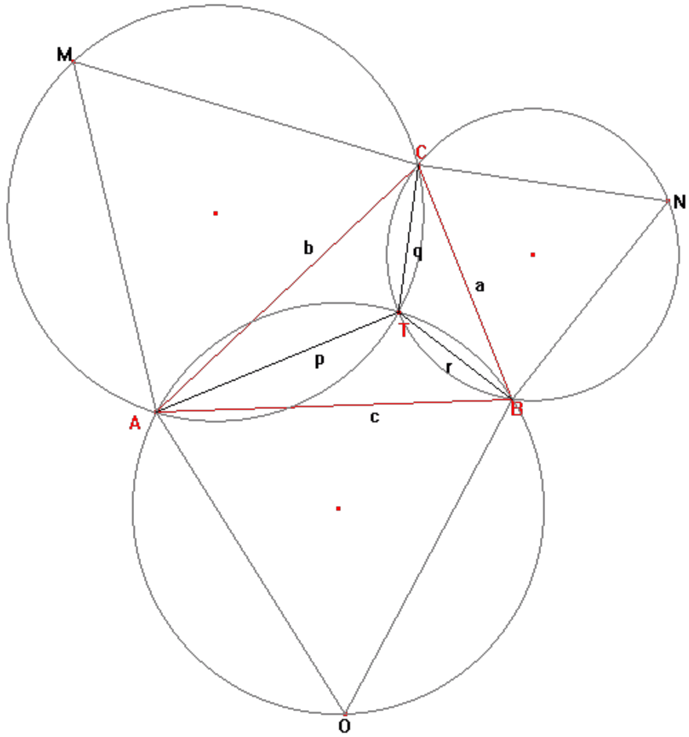
Find the sum of all progressive perfect squares below one trillion (10^{12}).

Problem 143: Investigating the Torricelli point of a triangle

Let ABC be a triangle with all interior angles being less than 120 degrees. Let X be any point inside the triangle and let XA = p, XC = q, and XB = r.

Fermat challenged Torricelli to find the position of X such that p + q + r was minimised.

Torricelli was able to prove that if equilateral triangles AOB, BNC and AMC are constructed on each side of triangle ABC, the circumscribed circles of AOB, BNC, and AMC will intersect at a single point, T, inside the triangle. Moreover he proved that T, called the Torricelli/Fermat point, minimises p + q + r. Even more remarkable, it can be shown that when the sum is minimised, AN = BM = CO = p + q + r and that AN, BM and CO also intersect at T.



If the sum is minimised and a, b, c, p, q and r are all positive integers we shall call triangle ABC a Torricelli triangle. For example, a = 399, b = 455, c = 511 is an example of a Torricelli triangle, with p + q + r = 784.

Find the sum of all distinct values of p + q + r ≤ 120000 for Torricelli triangles.

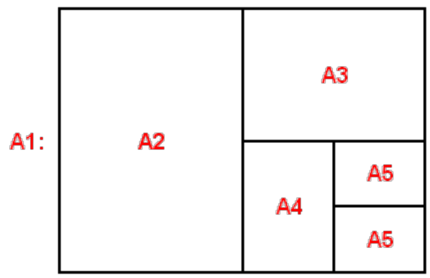
Problem 151: Paper sheets of standard sizes: an expected-value problem.

A printing shop runs 16 batches (jobs) every week and each batch requires a sheet of special colour-proofing paper of size A5.

Every Monday morning, the foreman opens a new envelope, containing a large sheet of the special paper with size A1.

He proceeds to cut it in half, thus getting two sheets of size A2. Then he cuts one of them in half to get two sheets of size A3 and so on until he obtains the A5-size sheet needed for the first batch of the week.

All the unused sheets are placed back in the envelope.



At the beginning of each subsequent batch, he takes from the envelope one sheet of paper at random. If it is of size A5, he uses it. If it is larger, he repeats the 'cut-in-half' procedure until he has what he needs and any remaining sheets are always placed back in the envelope.

Excluding the first and last batch of the week, find the expected number of times (during each week) that the foreman finds a single sheet of paper in the envelope.

Give your answer rounded to six decimal places using the format x.xxxxxx .

Problem 152: Writing 1/2 as a sum of inverse squares

There are several ways to write the number 1/2 as a sum of inverse squares using *distinct* integers.

For instance, the numbers {2,3,4,5,7,12,15,20,28,35} can be used:

$$\frac{1}{2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{28^2} + \frac{1}{35^2}$$

In fact, only using integers between 2 and 45 inclusive, there are exactly three ways to do it, the remaining two being: {2,3,4,6,7,9,10,20,28,35,36,45} and {2,3,4,6,7,9,12,15,28,30,35,36,45}.

How many ways are there to write the number 1/2 as a sum of inverse squares using distinct integers between 2 and 80 inclusive?

Problem 153: Investigating Gaussian Integers

As we all know the equation x^2=-1 has no solutions for real x.
If we however introduce the imaginary number i this equation has two solutions: $x=i$ and $x=-i$.
If we go a step further the equation $(x-3)^2$=-4 has two complex solutions: x=3+2i and x=3-2i.
x=3+2i and x=3-2i are called each others' complex conjugate.
Numbers of the form $a+bi$ are called complex numbers.
In general $a+bi$ and $a-bi$ are each other's complex conjugate.

A Gaussian Integer is a complex number $a+bi$ such that both a and b are integers.
The regular integers are also Gaussian integers (with b=0).
To distinguish them from Gaussian integers with $b \neq 0$ we call such integers "rational integers."
A Gaussian integer is called a divisor of a rational integer n if the result is also a Gaussian integer.
If for example we divide 5 by 1+2i we can simplify $\frac{5}{1+2i}$ in the following manner:
Multiply numerator and denominator by the complex conjugate of 1+2i: 1−2i.
The result is $\frac{5}{1+2i} = \frac{5}{1+2i}\frac{1-2i}{1-2i} = \frac{5(1-2i)}{1-(2i)^2} = \frac{5(1-2i)}{1-(-4)} = \frac{5(1-2i)}{5} = 1-2i$.
So 1+2i is a divisor of 5.

Note that 1+i is not a divisor of 5 because $\frac{5}{1+i} = \frac{5}{2} - \frac{5}{2}i$.
Note also that if the Gaussian Integer ($a+bi$) is a divisor of a rational integer n, then its complex conjugate ($a-bi$) is also a divisor of n.

In fact, 5 has six divisors such that the real part is positive: {1, 1 + 2i, 1 − 2i, 2 + i, 2 − i, 5}.
The following is a table of all of the divisors for the first five positive rational integers:

n	Gaussian integer divisors with positive real part	Sum s(n) of these divisors
1	1	1
2	1, 1+i, 1-i, 2	5
3	1, 3	4
4	1, 1+i, 1-i, 2, 2+2i, 2-2i,4	13
5	1, 1+2i, 1-2i, 2+i, 2-i, 5	12

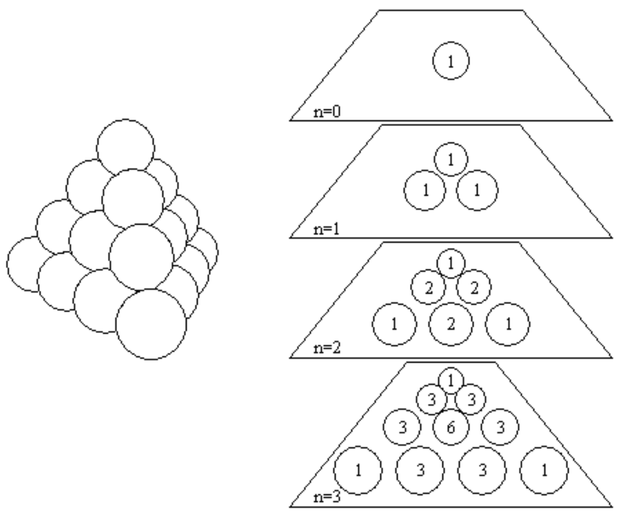
For divisors with positive real parts, then, we have: $\sum_{n=1}^{5} s(n) = 35$.

For $1 \le n \le 10^5$, $\sum$ s(n)=17924657155.

What is $\sum$ s(n) for $1 \le n \le 10^8$?

Problem 154: Exploring Pascal's pyramid.

A triangular pyramid is constructed using spherical balls so that each ball rests on exactly three balls of the next lower level.



Then, we calculate the number of paths leading from the apex to each position:

A path starts at the apex and progresses downwards to any of the three spheres directly below the current position.

Consequently, the number of paths to reach a certain position is the sum of the numbers immediately above it (depending on the position, there are up to three numbers above it).

The result is *Pascal's pyramid* and the numbers at each level n are the coefficients of the trinomial expansion $(x + y + z)^n$.

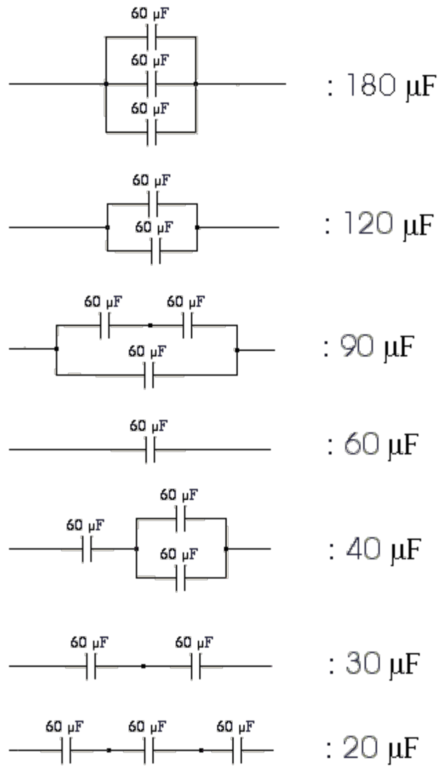
How many coefficients in the expansion of $(x + y + z)^{200000}$ are multiples of 10^{12}?

Problem 155: Counting Capacitor Circuits.

An electric circuit uses exclusively identical capacitors of the same value C.
The capacitors can be connected in series or in parallel to form sub-units, which can then be connected in series or in parallel with other

capacitors or other sub-units to form larger sub-units, and so on up to a final circuit.

Using this simple procedure and up to n identical capacitors, we can make circuits having a range of different total capacitances. For example, using up to n=3 capacitors of 60 μF each, we can obtain the following 7 distinct total capacitance values:



If we denote by $D(n)$ the number of distinct total capacitance values we can obtain when using up to n equal-valued capacitors and the simple procedure described above, we have: $D(1)$=1, $D(2)$=3, $D(3)$=7 ...

Find $D(18)$.

Reminder : When connecting capacitors C_1, C_2 etc in parallel, the total capacitance is $C_T = C_1 + C_2 + \ldots$,
whereas when connecting them in series, the overall capacitance is given by: $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots$

Problem 156: Counting Digits

Starting from zero the natural numbers are written down in base 10 like this:
0 1 2 3 4 5 6 7 8 9 10 11 12....

Consider the digit d=1. After we write down each number n, we will update the number of ones that have occurred and call this number $f(n,1)$. The first values for $f(n,1)$, then, are as follows:

n	$f(n,1)$
0	0
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1

10	2
11	4
12	5

Note that $f(n,1)$ never equals 3.
So the first two solutions of the equation $f(n,1)=n$ are n=0 and n=1. The next solution is n=199981.

In the same manner the function $f(n,d)$ gives the total number of digits d that have been written down after the number n has been written.
In fact, for every digit $d \neq 0$, 0 is the first solution of the equation $f(n,d)=n$.

Let $s(d)$ be the sum of all the solutions for which $f(n,d)=n$.
You are given that $s(1)$=22786974071.

Find $\sum s(d)$ for $1 \leq d \leq 9$.

Note: if, for some n, $f(n,d)=n$ for more than one value of d this value of n is counted again for every value of d for which $f(n,d)=n$.

Problem 157: Solving the diophantine equation $^1/_a+^1/_b= {^p}/_{10^n}$

Consider the diophantine equation $^1/_a+^1/_b= {^p}/_{10^n}$ with a, b, p, n positive integers and $a \leq b$.
For n=1 this equation has 20 solutions that are listed below:

$^1/_1+^1/_1=^{20}/_{10}$	$^1/_1+^1/_2=^{15}/_{10}$	$^1/_1+^1/_5=^{12}/_{10}$	$^1/_1+^1/_{10}=^{11}/_{10}$	$^1/_2+^1/_2=^{10}/_{10}$
$^1/_2+^1/_5=^7/_{10}$	$^1/_2+^1/_{10}=^6/_{10}$	$^1/_3+^1/_6=^5/_{10}$	$^1/_3+^1/_{15}=^4/_{10}$	$^1/_4+^1/_4=^5/_{10}$
$^1/_4+^1/_{20}=^3/_{10}$	$^1/_5+^1/_5=^4/_{10}$	$^1/_5+^1/_{10}=^3/_{10}$	$^1/_6+^1/_{30}=^2/_{10}$	$^1/_{10}+^1/_{10}=^2/_{10}$
$^1/_{11}+^1/_{110}=^1/_{10}$	$^1/_{12}+^1/_{60}=^1/_{10}$	$^1/_{14}+^1/_{35}=^1/_{10}$	$^1/_{15}+^1/_{30}=^1/_{10}$	$^1/_{20}+^1/_{20}=^1/_{10}$

How many solutions has this equation for $1 \leq n \leq 9$?

Problem 158: Exploring strings for which only one character comes lexicographically after its neighbour to the left.

Taking three different letters from the 26 letters of the alphabet, character strings of length three can be formed.
Examples are 'abc', 'hat' and 'zyx'.
When we study these three examples we see that for 'abc' two characters come lexicographically after its neighbour to the left.
For 'hat' there is exactly one character that comes lexicographically after its neighbour to the left. For 'zyx' there are zero characters that come lexicographically after its neighbour to the left.
In all there are 10400 strings of length 3 for which exactly one character comes lexicographically after its neighbour to the left.

We now consider strings of $n \leq 26$ different characters from the alphabet.
For every n, p(n) is the number of strings of length n for which exactly one character comes lexicographically after its neighbour to the left.

What is the maximum value of p(n)?

Problem 159: Digital root sums of factorisations.

A composite number can be factored many different ways. For instance, not including multiplication by one, 24 can be factored in 7 distinct ways:

24 = 2x2x2x3
24 = 2x3x4
24 = 2x2x6
24 = 4x6
24 = 3x8
24 = 2x12
24 = 24

Recall that the digital root of a number, in base 10, is found by adding together the digits of that number, and repeating that process until a number is arrived at that is less than 10. Thus the digital root of 467 is 8.

We shall call a Digital Root Sum (DRS) the sum of the digital roots of the individual factors of our number.

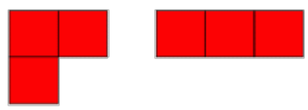
The chart below demonstrates all of the DRS values for 24.

Factorisation	Digital Root Sum
2x2x2x3	9
2x3x4	9
2x2x6	10
4x6	10
3x8	11
2x12	5
24	6

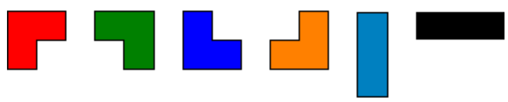
The maximum Digital Root Sum of 24 is 11.
The function mdrs(n) gives the maximum Digital Root Sum of n. So mdrs(24)=11.
Find $\sum$mdrs(n) for $1 < n < 1,000,000$.

Problem 161: Triominoes

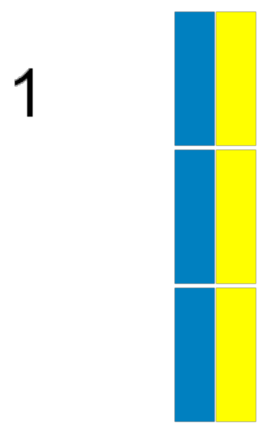
A triomino is a shape consisting of three squares joined via the edges. There are two basic forms:



If all possible orientations are taken into account there are six:



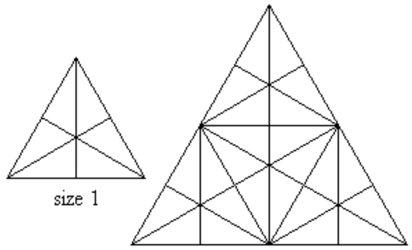
Any n by m grid for which nxm is divisible by 3 can be tiled with triominoes.
If we consider tilings that can be obtained by reflection or rotation from another tiling as different there are 41 ways a 2 by 9 grid can be tiled with triominoes:



In how many ways can a 9 by 12 grid be tiled in this way by triominoes?

Problem 163: Cross-hatched triangles

Consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side, such as in the *size 1* triangle in the sketch below.



Sixteen triangles of either different shape or size or orientation or location can now be observed in that triangle. Using *size 1* triangles as building blocks, larger triangles can be formed, such as the *size 2* triangle in the above sketch. One-hundred and four triangles of either different shape or size or orientation or location can now be observed in that *size 2* triangle.

It can be observed that the *size 2* triangle contains 4 *size 1* triangle building blocks. A *size 3* triangle would contain 9 *size 1* triangle building blocks and a *size n* triangle would thus contain n^2 *size 1* triangle building blocks.

If we denote T(n) as the number of triangles present in a triangle of *size* n, then

T(1) = 16
T(2) = 104

Find T(36).

Problem 167: Investigating Ulam sequences

For two positive integers a and b, the Ulam sequence U(a,b) is defined by $U(a,b)_1 = a$, $U(a,b)_2 = b$ and for k > 2, $U(a,b)_k$ is the smallest integer greater than $U(a,b)_{(k-1)}$ which can be written in exactly one way as the sum of two distinct previous members of U(a,b).

For example, the sequence U(1,2) begins with
1, 2, 3 = 1 + 2, 4 = 1 + 3, 6 = 2 + 4, 8 = 2 + 6, 11 = 3 + 8;
5 does not belong to it because 5 = 1 + 4 = 2 + 3 has two representations as the sum of two previous members, likewise 7 = 1 + 6 = 3 + 4.

Find $\sum U(2,2n+1)_k$ for $2 \le n \le 10$, where $k = 10^{11}$.

Problem 168: Number Rotations

Consider the number 142857. We can right-rotate this number by moving the last digit (7) to the front of it, giving us 714285.
It can be verified that 714285=5×142857.
This demonstrates an unusual property of 142857: it is a divisor of its right-rotation.

Find the last 5 digits of the sum of all integers n, $10 < n < 10^{100}$, that have this property.

Problem 169: Exploring the number of different ways a number can be expressed as a sum of powers of 2.

Define f(0)=1 and f(n) to be the number of different ways n can be expressed as a sum of integer powers of 2 using each power no more than twice.

For example, f(10)=5 since there are five different ways to express 10:

1 + 1 + 8
1 + 1 + 4 + 4
1 + 1 + 2 + 2 + 4
2 + 4 + 4
2 + 8

What is $f(10^{25})$?

Problem 170: Find the largest 0 to 9 pandigital that can be formed by concatenating products.

Take the number 6 and multiply it by each of 1273 and 9854:

$6 \times 1273 = 7638$
$6 \times 9854 = 59124$

By concatenating these products we get the 1 to 9 pandigital 763859124. We will call 763859124 the "concatenated product of 6 and (1273,9854)". Notice too, that the concatenation of the input numbers, 612739854, is also 1 to 9 pandigital.

The same can be done for 0 to 9 pandigital numbers.

What is the largest 0 to 9 pandigital 10-digit concatenated product of an integer with two or more other integers, such that the concatenation of the input numbers is also a 0 to 9 pandigital 10-digit number?

Problem 171: Finding numbers for which the sum of the squares of the digits is a square.

For a positive integer n, let f(n) be the sum of the squares of the digits (in base 10) of n, e.g.

$f(3) = 3^2 = 9$,
$f(25) = 2^2 + 5^2 = 4 + 25 = 29$,
$f(442) = 4^2 + 4^2 + 2^2 = 16 + 16 + 4 = 36$

Find the last nine digits of the sum of all n, $0 < n < 10^{20}$, such that f(n) is a perfect square.

Problem 172: Investigating numbers with few repeated digits.

How many 18-digit numbers n (without leading zeros) are there such that no digit occurs more than three times in n?

Problem 175: Fractions involving the number of different ways a number can be expressed as a sum of powers of 2.

Define f(0)=1 and f(n) to be the number of ways to write n as a sum of powers of 2 where no power occurs more than twice.

For example, f(10)=5 since there are five different ways to express 10:
10 = 8+2 = 8+1+1 = 4+4+2 = 4+2+2+1+1 = 4+4+1+1

It can be shown that for every fraction p/q (p>0, q>0) there exists at least one integer n such that
f(n)/f(n-1)=p/q.

For instance, the smallest n for which f(n)/f(n-1)=13/17 is 241.
The binary expansion of 241 is 11110001.
Reading this binary number from the most significant bit to the least significant bit there are 4 one's, 3 zeroes and 1 one. We shall call the string 4,3,1 the *Shortened Binary Expansion* of 241.

Find the Shortened Binary Expansion of the smallest n for which
f(n)/f(n-1)=123456789/987654321.

Give your answer as comma separated integers, without any whitespaces.

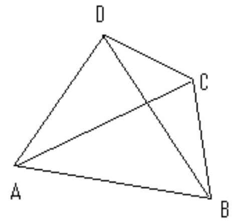
Problem 176: Right-angled triangles that share a cathetus.

The four right-angled triangles with sides (9,12,15), (12,16,20), (5,12,13) and (12,35,37) all have one of the shorter sides (catheti) equal to 12. It can be shown that no other integer sided right-angled triangle exists with one of the catheti equal to 12.

Find the smallest integer that can be the length of a cathetus of exactly 47547 different integer sided right-angled triangles.

Problem 177: Integer angled Quadrilaterals.

Let ABCD be a convex quadrilateral, with diagonals AC and BD. At each vertex the diagonal makes an angle with each of the two sides, creating eight corner angles.



For example, at vertex A, the two angles are CAD, CAB.

We call such a quadrilateral for which all eight corner angles have integer values when measured in degrees an "integer angled quadrilateral". An example of an integer angled quadrilateral is a square, where all eight corner angles are 45°. Another example is given by DAC = 20°, BAC = 60°, ABD = 50°, CBD = 30°, BCA = 40°, DCA = 30°, CDB = 80°, ADB = 50°.

What is the total number of non-similar integer angled quadrilaterals?

Note: In your calculations you may assume that a calculated angle is integral if it is within a tolerance of 10^{-9} of an integer value.

Problem 178: Step Numbers

Consider the number 45656.
It can be seen that each pair of consecutive digits of 45656 has a difference of one.
A number for which every pair of consecutive digits has a difference of one is called a step number.
A pandigital number contains every decimal digit from 0 to 9 at least once.
How many pandigital step numbers less than 10^{40} are there?

Problem 180: Rational zeros of a function of three variables.

For any integer n, consider the three functions

$$f_{1,n}(x,y,z) = x^{n+1} + y^{n+1} - z^{n+1}$$
$$f_{2,n}(x,y,z) = (xy + yz + zx)*(x^{n-1} + y^{n-1} - z^{n-1})$$
$$f_{3,n}(x,y,z) = xyz*(x^{n-2} + y^{n-2} - z^{n-2})$$

and their combination

$$f_n(x,y,z) = f_{1,n}(x,y,z) + f_{2,n}(x,y,z) - f_{3,n}(x,y,z)$$

We call (x,y,z) a golden triple of order k if x, y, and z are all rational numbers of the form a / b with $0 < a < b \le k$ and there is (at least) one integer n, so that $f_n(x,y,z) = 0$.

Let $s(x,y,z) = x + y + z$.
Let $t = u / v$ be the sum of all distinct $s(x,y,z)$ for all golden triples (x,y,z) of order 35.
All the $s(x,y,z)$ and t must be in reduced form.

Find $u + v$.

Problem 181: Investigating in how many ways objects of two different colours can be grouped.

Having three black objects B and one white object W they can be grouped in 7 ways like this:

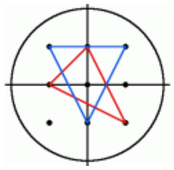
(BBBW) (B,BBW) (B,B,BW) (B,B,B,W) (B,BB,W) (BBB,W) (BB,BW)

In how many ways can sixty black objects B and forty white objects W be thus grouped?

Problem 184: Triangles containing the origin.

Consider the set I_r of points (x,y) with integer co-ordinates in the interior of the circle with radius r, centered at the origin, i.e. $x^2 + y^2 < r^2$.

For a radius of 2, I_2 contains the nine points (0,0), (1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1) and (1,-1). There are eight triangles having all three vertices in I_2 which contain the origin in the interior. Two of them are shown below, the others are obtained from these by rotation.



For a radius of 3, there are 360 triangles containing the origin in the interior and having all vertices in I_3 and for I_5 the number is 10600.

How many triangles are there containing the origin in the interior and having all three vertices in I_{105}?

Problem 185: Number Mind

The game Number Mind is a variant of the well known game Master Mind.

Instead of coloured pegs, you have to guess a secret sequence of digits. After each guess you're only told in how many places you've guessed the correct digit. So, if the sequence was 1234 and you guessed 2036, you'd be told that you have one correct digit; however, you would NOT be told that you also have another digit in the wrong place.

For instance, given the following guesses for a 5-digit secret sequence,

90342 ;2 correct
70794 ;0 correct
39458 ;2 correct
34109 ;1 correct
51545 ;2 correct
12531 ;1 correct

The correct sequence 39542 is unique.

Based on the following guesses,

5616185650518293 ;2 correct
3847439647293047 ;1 correct
5855462940810587 ;3 correct
9742855507068353 ;3 correct
4296849643607543 ;3 correct
3174248439465858 ;1 correct
4513559094146117 ;2 correct
7890971548908067 ;3 correct
8157356344118483 ;1 correct
2615250744386899 ;2 correct
8690095851526254 ;3 correct
6375711915077050 ;1 correct
6913859173121360 ;1 correct
6442889055042768 ;2 correct
2321386104303845 ;0 correct
2326509471271448 ;2 correct
5251583379644322 ;2 correct
1748270476758276 ;3 correct
4895722652190306 ;1 correct
3041631117224635 ;3 correct
1841236454324589 ;3 correct
2659862637316867 ;2 correct

Find the unique 16-digit secret sequence.

Problem 186: Connectedness of a network.

Here are the records from a busy telephone system with one million users:

RecNr	Caller	Called

1	200007	100053
2	600183	500439
3	600863	701497
…	…	…

The telephone number of the caller and the called number in record n are Caller(n) = S_{2n-1} and Called(n) = S_{2n} where $S_{1,2,3,\ldots}$ come from the "Lagged Fibonacci Generator":

For $1 \le k \le 55$, $S_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000)
For $56 \le k$, $S_k = [S_{k-24} + S_{k-55}]$ (modulo 1000000)

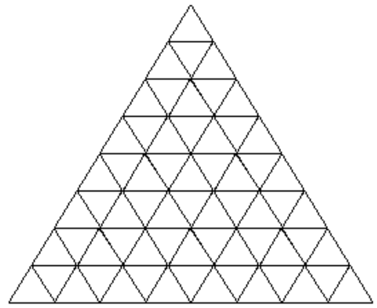
If Caller(n) = Called(n) then the user is assumed to have misdialled and the call fails; otherwise the call is successful.

From the start of the records, we say that any pair of users X and Y are friends if X calls Y or vice-versa. Similarly, X is a friend of a friend of Z if X is a friend of Y and Y is a friend of Z; and so on for longer chains.

The Prime Minister's phone number is 524287. After how many successful calls, not counting misdials, will 99% of the users (including the PM) be a friend, or a friend of a friend etc., of the Prime Minister?

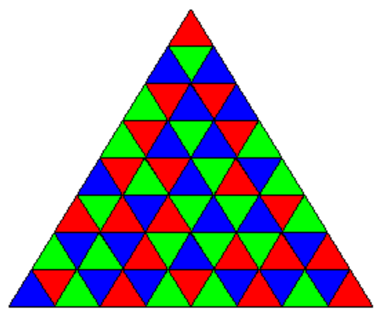
Problem 189: Tri-colouring a triangular grid

Consider the following configuration of 64 triangles:



We wish to colour the interior of each triangle with one of three colours: red, green or blue, so that no two neighbouring triangles have the same colour. Such a colouring shall be called valid. Here, two triangles are said to be neighbouring if they share an edge.
Note: if they only share a vertex, then they are not neighbours.

For example, here is a valid colouring of the above grid:



A colouring C' which is obtained from a colouring C by rotation or reflection is considered *distinct* from C unless the two are identical.

How many distinct valid colourings are there for the above configuration?

Problem 190: Maximising a weighted product

Let $S_m = (x_1, x_2, \ldots, x_m)$ be the m-tuple of positive real numbers with $x_1 + x_2 + \ldots + x_m = m$ for which $P_m = x_1 * x_2^2 * \ldots * x_m^m$ is maximised.

For example, it can be verified that $[P_{10}] = 4112$ ([ ] is the integer part function).

Find $\Sigma[P_m]$ for $2 \leq m \leq 15$.

Problem 192: Best Approximations

Let x be a real number.
A *best approximation* to x for the *denominator bound* d is a rational number r/s *in reduced form*, with $s \leq d$, such that any rational number which is closer to x than r/s has a denominator larger than d:

$$|p/q-x| < |r/s-x| \Rightarrow q > d$$

For example, the best approximation to $\sqrt{13}$ for the denominator bound 20 is 18/5 and the best approximation to $\sqrt{13}$ for the denominator bound 30 is 101/28.

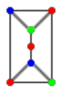
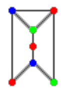
Find the sum of all denominators of the best approximations to $\sqrt{n}$ for the denominator bound 10^{12}, where n is not a perfect square and $1 < n \leq 100000$.

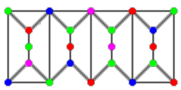
Problem 193: Squarefree Numbers

A positive integer n is called squarefree, if no square of a prime divides n, thus 1, 2, 3, 5, 6, 7, 10, 11 are squarefree, but not 4, 8, 9, 12.

How many squarefree numbers are there below 2^{50}?

Problem 194: Coloured Configurations

Consider graphs built with the units A:  and B: , where the units are glued along the vertical edges as in the graph

.

A configuration of type (a,b,c) is a graph thus built of a units A and b units B, where the graph's vertices are coloured using up to c colours, so that no two adjacent vertices have the same colour.
The compound graph above is an example of a configuration of type (2,2,6), in fact of type (2,2,c) for all $c \geq 4$.

Let N(a,b,c) be the number of configurations of type (a,b,c).
For example, N(1,0,3) = 24, N(0,2,4) = 92928 and N(2,2,3) = 20736.

Find the last 8 digits of N(25,75,1984).

Problem 195: Inscribed circles of triangles with one angle of 60 degrees

Let's call an integer sided triangle with exactly one angle of 60 degrees a 60-degree triangle.
Let r be the radius of the inscribed circle of such a 60-degree triangle.

There are 1234 60-degree triangles for which $r \leq 100$.
Let T(n) be the number of 60-degree triangles for which $r \leq n$, so
T(100) = 1234, T(1000) = 22767, and T(10000) = 359912.

Find T(1053779).

Problem 196: Prime triplets

Build a triangle from all positive integers in the following way:

```
  1
  2  3
  4  5  6
  7  8  9 10
 11 12 13 14 15
 16 17 18 19 20 21
 22 23 24 25 26 27 28
 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45
 46 47 48 49 50 51 52 53 54 55
 56 57 58 59 60 61 62 63 64 65 66
 . . .
```

Each positive integer has up to eight neighbours in the triangle.

A set of three primes is called a *prime triplet* if one of the three primes has the other two as neighbours in the triangle.

For example, in the second row, the prime numbers 2 and 3 are elements of some prime triplet.

If row 8 is considered, it contains two primes which are elements of some prime triplet, i.e. 29 and 31.

If row 9 is considered, it contains only one prime which is an element of some prime triplet: 37.

Define S(n) as the sum of the primes in row n which are elements of any prime triplet.
Then S(8)=60 and S(9)=37.

You are given that S(10000)=950007619.

Find S(5678027) + S(7208785).

Problem 198: Ambiguous Numbers

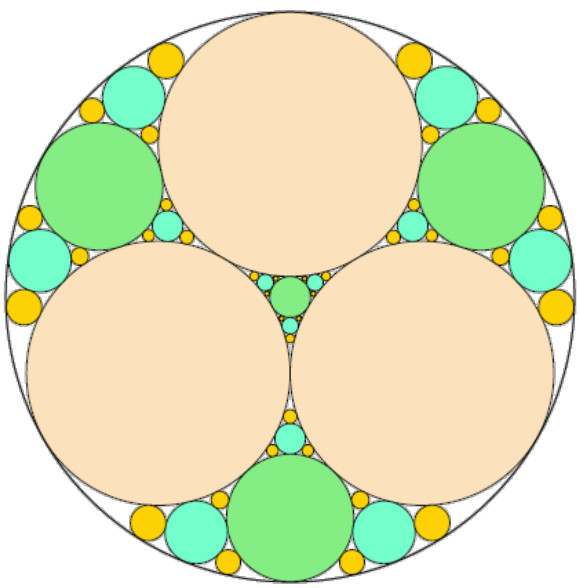
A best approximation to a real number x for the denominator bound d is a rational number r/s (in reduced form) with $s \le d$, so that any rational number p/q which is closer to x than r/s has $q > d$.

Usually the best approximation to a real number is uniquely determined for all denominator bounds. However, there are some exceptions, e.g. 9/40 has the two best approximations 1/4 and 1/5 for the denominator bound 6. We shall call a real number x *ambiguous*, if there is at least one denominator bound for which x possesses two best approximations. Clearly, an ambiguous number is necessarily rational.

How many ambiguous numbers $x = p/q$, $0 < x < 1/100$, are there whose denominator q does not exceed 10^8?

Problem 199: Iterative Circle Packing

Three circles of equal radius are placed inside a larger circle such that each pair of circles is tangent to one another and the inner circles do not overlap. There are four uncovered "gaps" which are to be filled iteratively with more tangent circles.



At each iteration, a maximally sized circle is placed in each gap, which creates more gaps for the next iteration. After 3 iterations (pictured), there are 108 gaps and the fraction of the area which is not covered by circles is 0.06790342, rounded to eight decimal places.

What fraction of the area is not covered by circles after 10 iterations?
Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

Problem 200: Find the 200th prime-proof sqube containing the contiguous sub-string "200"

We shall define a sqube to be a number of the form, p^2q^3, where p and q are distinct primes.
For example, 200 = $5^2 2^3$ or 120072949 = $23^2 61^3$.

The first five squbes are 72, 108, 200, 392, and 500.

Interestingly, 200 is also the first number for which you cannot change any single digit to make a prime; we shall call such numbers, prime-proof.
The next prime-proof sqube which contains the contiguous sub-string "200" is 1992008.

Find the 200th prime-proof sqube containing the contiguous sub-string "200".

Problem 201: Subsets with a unique sum

For any set A of numbers, let sum(A) be the sum of the elements of A.
Consider the set B = {1,3,6,8,10,11}.
There are 20 subsets of B containing three elements, and their sums are:

sum({1,3,6}) = 10,
sum({1,3,8}) = 12,
sum({1,3,10}) = 14,
sum({1,3,11}) = 15,
sum({1,6,8}) = 15,
sum({1,6,10}) = 17,
sum({1,6,11}) = 18,
sum({1,8,10}) = 19,
sum({1,8,11}) = 20,
sum({1,10,11}) = 22,
sum({3,6,8}) = 17,
sum({3,6,10}) = 19,
sum({3,6,11}) = 20,
sum({3,8,10}) = 21,
sum({3,8,11}) = 22,
sum({3,10,11}) = 24,
sum({6,8,10}) = 24,
sum({6,8,11}) = 25,
sum({6,10,11}) = 27,
sum({8,10,11}) = 29.

Some of these sums occur more than once, others are unique.
For a set A, let U(A,k) be the set of unique sums of k-element subsets of A, in our example we find U(B,3) = {10,12,14,18,21,25,27,29} and sum(U(B,3)) = 156.

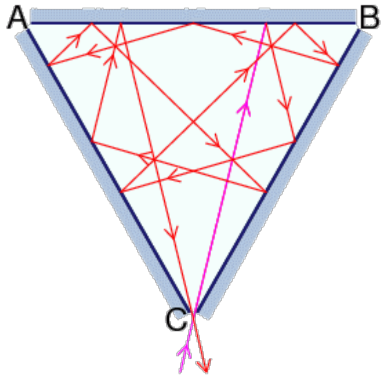
Now consider the 100-element set S = $\{1^2, 2^2, \ldots , 100^2\}$.
S has 100891344545564193334812497256 50-element subsets.

Determine the sum of all integers which are the sum of exactly one of the 50-element subsets of S, i.e. find sum(U(S,50)).

Problem 202: Laserbeam

Three mirrors are arranged in the shape of an equilateral triangle, with their reflective surfaces pointing inwards. There is an infinitesimal gap at each vertex of the triangle through which a laser beam may pass.

Label the vertices A, B and C. There are 2 ways in which a laser beam may enter vertex C, bounce off 11 surfaces, then exit through the same vertex: one way is shown below; the other is the reverse of that.



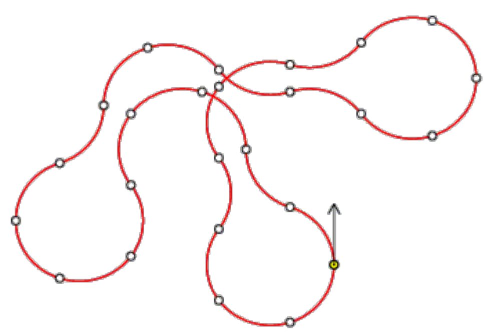
There are 80840 ways in which a laser beam may enter vertex C, bounce off 1000001 surfaces, then exit through the same vertex.

In how many ways can a laser beam enter at vertex C, bounce off 12017639147 surfaces, then exit through the same vertex?

Problem 208: Robot Walks

A robot moves in a series of one-fifth circular arcs (72°), with a free choice of a clockwise or an anticlockwise arc for each step, but no turning on the spot.

One of 70932 possible closed paths of 25 arcs starting northward is



Given that the robot starts facing North, how many journeys of 70 arcs in length can it take that return it, after the final arc, to its starting position?
(Any arc may be traversed multiple times.)

Problem 209: Circular Logic

A k-input *binary truth table* is a map from k input bits (binary digits, 0 [false] or 1 [true]) to 1 output bit. For example, the 2-input binary truth tables for the logical AND and XOR functions are:

x	y	x AND y
0	0	0
0	1	0
1	0	0
1	1	1

x	y	x XOR y
0	0	0
0	1	1
1	0	1
1	1	0

How many 6-input binary truth tables, τ, satisfy the formula

$$\tau(a, b, c, d, e, f) \text{ AND } \tau(b, c, d, e, f, a \text{ XOR } (b \text{ AND } c)) = 0$$

for all 6-bit inputs (a, b, c, d, e, f)?

Problem 210: Obtuse Angled Triangles

Consider the set S(r) of points (x,y) with integer coordinates satisfying $|x| + |y| \le r$.
Let O be the point (0,0) and C the point (r/4,r/4).
Let N(r) be the number of points B in S(r), so that the triangle OBC has an obtuse angle, i.e. the largest angle α satisfies 90°<α<180°.
So, for example, N(4)=24 and N(8)=100.

What is N(1,000,000,000)?

Problem 212: Combined Volume of Cuboids

An *axis-aligned cuboid*, specified by parameters { (x_0,y_0,z_0), (dx,dy,dz) }, consists of all points (X,Y,Z) such that $x_0 \le X \le x_0+dx$, $y_0 \le Y \le y_0+dy$ and $z_0 \le Z \le z_0+dz$. The volume of the cuboid is the product, dx × dy × dz. The *combined volume* of a collection of cuboids is the volume of their union and will be less than the sum of the individual volumes if any cuboids overlap.

Let $C_1,\ldots,C_{50000}$ be a collection of 50000 axis-aligned cuboids such that C_n has parameters

$x_0 = S_{6n-5}$ modulo 10000
$y_0 = S_{6n-4}$ modulo 10000
$z_0 = S_{6n-3}$ modulo 10000
$dx = 1 + (S_{6n-2}$ modulo 399)
$dy = 1 + (S_{6n-1}$ modulo 399)
$dz = 1 + (S_{6n}$ modulo 399)

where $S_1,\ldots,S_{300000}$ come from the "Lagged Fibonacci Generator":

For $1 \le k \le 55$, $S_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000)
For $56 \le k$, $S_k = [S_{k-24} + S_{k-55}]$ (modulo 1000000)

Thus, C_1 has parameters {(7,53,183),(94,369,56)}, C_2 has parameters {(2383,3563,5079),(42,212,344)}, and so on.

The combined volume of the first 100 cuboids, $C_1,\ldots,C_{100}$, is 723581599.

What is the combined volume of all 50000 cuboids, $C_1,\ldots,C_{50000}$?

Problem 213: Flea Circus

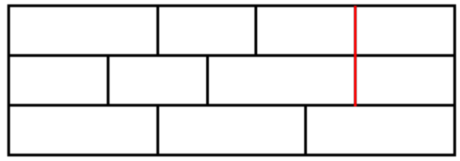
A 30×30 grid of squares contains 900 fleas, initially one flea per square.
When a bell is rung, each flea jumps to an adjacent square at random (usually 4 possibilities, except for fleas on the edge of the grid or at the corners).

What is the expected number of unoccupied squares after 50 rings of the bell? Give your answer rounded to six decimal places.

Problem 215: Crack-free Walls

Consider the problem of building a wall out of 2×1 and 3×1 bricks (horizontal×vertical dimensions) such that, for extra strength, the gaps between horizontally-adjacent bricks never line up in consecutive layers, i.e. never form a "running crack".

For example, the following 9×3 wall is not acceptable due to the running crack shown in red:



There are eight ways of forming a crack-free 9×3 wall, written W(9,3) = 8.

Calculate W(32,10).

Problem 217: Balanced Numbers

A positive integer with k (decimal) digits is called balanced if its first $\lceil k/2 \rceil$ digits sum to the same value as its last $\lceil k/2 \rceil$ digits, where $\lceil x \rceil$, pronounced *ceiling* of x, is the smallest integer $\geq x$, thus $\lceil \pi \rceil = 4$ and $\lceil 5 \rceil = 5$.

So, for example, all palindromes are balanced, as is 13722.

Let T(n) be the sum of all balanced numbers less than 10^n.
Thus: T(1) = 45, T(2) = 540 and T(5) = 334795890.

Find T(47) mod 3^{15}

Problem 218: Perfect right-angled triangles

Consider the right angled triangle with sides a=7, b=24 and c=25. The area of this triangle is 84, which is divisible by the perfect numbers 6 and 28.
Moreover it is a primitive right angled triangle as gcd(a,b)=1 and gcd(b,c)=1.
Also c is a perfect square.

We will call a right angled triangle perfect if
-it is a primitive right angled triangle
-its hypotenuse is a perfect square

We will call a right angled triangle super-perfect if
-it is a perfect right angled triangle and
-its area is a multiple of the perfect numbers 6 and 28.

How many perfect right-angled triangles with $c \leq 10^{16}$ exist that are not super-perfect?

Problem 219: Skew-cost coding

Let **A** and **B** be bit strings (sequences of 0's and 1's).
If **A** is equal to the leftmost length(**A**) bits of **B**, then **A** is said to be a *prefix* of **B**.
For example, 00110 is a prefix of 001101001, but not of 00111 or 100110.

A *prefix-free code of size* n is a collection of n distinct bit strings such that no string is a prefix of any other. For example, this is a prefix-free code of size 6:

0000, 0001, 001, 01, 10, 11

Now suppose that it costs one penny to transmit a '0' bit, but four pence to transmit a '1'.
Then the total cost of the prefix-free code shown above is 35 pence, which happens to be the cheapest possible for the skewed pricing scheme in question.
In short, we write Cost(6) = 35.

What is Cost(10^9) ?

Problem 220: Heighway Dragon

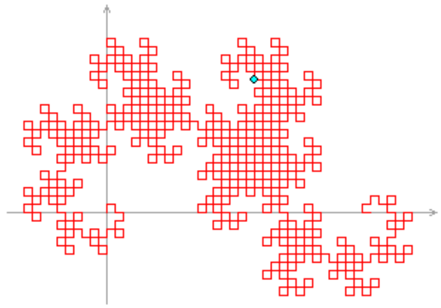
Let D_0 be the two-letter string "Fa". For n≥1, derive D_n from D_{n-1} by the string-rewriting rules:

"a" → "aRbFR"
"b" → "LFaLb"

Thus, D_0 = "Fa", D_1 = "FaRbFR", D_2 = "FaRbFRRLFaLbFR", and so on.

These strings can be interpreted as instructions to a computer graphics program, with "F" meaning "draw forward one unit", "L" meaning "turn left 90 degrees", "R" meaning "turn right 90 degrees", and "a" and "b" being ignored. The initial position of the computer cursor is (0,0), pointing up towards (0,1).

Then D_n is an exotic drawing known as the *Heighway Dragon* of order *n*. For example, D_{10} is shown below; counting each "F" as one step, the highlighted spot at (18,16) is the position reached after 500 steps.



What is the position of the cursor after 10^{12} steps in D_{50} ?
Give your answer in the form ***x***,***y*** with no spaces.

Problem 221: Alexandrian Integers

We shall call a positive integer A an "Alexandrian integer", if there exist integers p, q, r such that:

$$A = p \cdot q \cdot r \quad \text{and} \quad \frac{1}{A} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

For example, 630 is an Alexandrian integer (p = 5, q = −7, r = −18). In fact, 630 is the 6^{th} Alexandrian integer, the first 6 Alexandrian integers being: 6, 42, 120, 156, 420 and 630.

Find the 150000^{th} Alexandrian integer.

Problem 222: Sphere Packing

What is the length of the shortest pipe, of internal radius 50mm, that can fully contain 21 balls of radii 30mm, 31mm, ..., 50mm?

Give your answer in micrometres (10^{-6} m) rounded to the nearest integer.

Problem 223: Almost right-angled triangles I

Let us call an integer sided triangle with sides $a \le b \le c$ *barely acute* if the sides satisfy
$a^2 + b^2 = c^2 + 1$.

How many barely acute triangles are there with perimeter ≤ 25,000,000?

Problem 224: Almost right-angled triangles II

Let us call an integer sided triangle with sides $a \le b \le c$ *barely obtuse* if the sides satisfy
$a^2 + b^2 = c^2 - 1$.

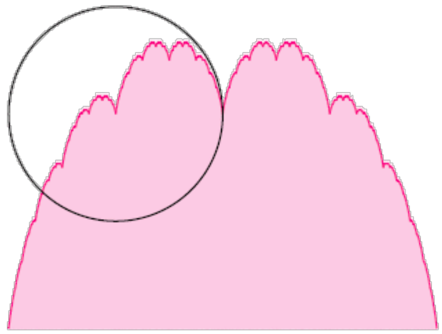
How many barely obtuse triangles are there with perimeter ≤ 75,000,000?

Problem 226: A Scoop of Blancmange

The *blancmange curve* is the set of points (x,y) such that $0 \le x \le 1$ and $y = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$,

where $s(x)$ = the distance from x to the nearest integer.

The area under the blancmange curve is equal to ½, shown in pink in the diagram below.



Let C be the circle with centre (¼,½) and radius ¼, shown in black in the diagram.

What area under the blancmange curve is enclosed by C?
Give your answer rounded to eight decimal places in the form *0.abcdefgh*

Problem 227: The Chase

"*The Chase*" is a game played with two dice and an even number of players.

The players sit around a table; the game begins with two opposite players having one die each. On each turn, the two players with a die roll it. If a player rolls a 1, he passes the die to his neighbour on the left; if he rolls a 6, he passes the die to his neighbour on the right; otherwise, he keeps the die for the next turn.
The game ends when one player has both dice after they have been rolled and passed; that player has then lost.

In a game with 100 players, what is the expected number of turns the game lasts?

Give your answer rounded to ten significant digits.

Problem 228: Minkowski Sums

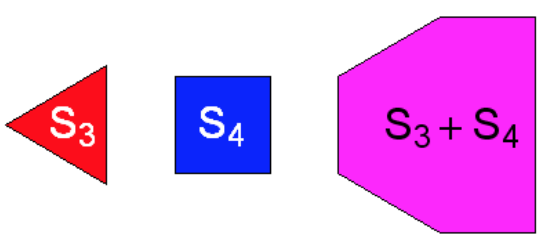
Let S_n be the regular n-sided polygon – or *shape* – whose vertices v_k ($k = 1,2,\ldots,n$) have coordinates:

$$x_k = \cos\left({}^{2k-1}/_{n} \times 180^\circ \right)$$
$$y_k = \sin\left({}^{2k-1}/_{n} \times 180^\circ \right)$$

Each S_n is to be interpreted as a filled shape consisting of all points on the perimeter and in the interior.

The *Minkowski sum*, $S+T$, of two shapes S and T is the result of adding every point in S to every point in T, where point addition is performed coordinate-wise: $(u, v) + (x, y) = (u+x, v+y)$.

For example, the sum of S_3 and S_4 is the six-sided shape shown in pink below:



How many sides does $S_{1864} + S_{1865} + \ldots + S_{1909}$ have?

Problem 229: Four Representations using Squares

Consider the number 3600. It is very special, because

$$3600 = 48^2 + \quad 36^2$$

$$3600 = 20^2 + 2\times40^2$$

$$3600 = 30^2 + 3\times30^2$$

$$3600 = 45^2 + 7\times15^2$$

Similarly, we find that $88201 = 99^2 + 280^2 = 287^2 + 2\times54^2 = 283^2 + 3\times52^2 = 197^2 + 7\times84^2$.

In 1747, Euler proved which numbers are representable as a sum of two squares. We are interested in the numbers n which admit representations of all of the following four types:

$$n = a_1^2 + \quad b_1^2$$

$$n = a_2^2 + 2\ b_2^2$$

$$n = a_3^2 + 3\ b_3^2$$

$$n = a_7^2 + 7\ b_7^2,$$

where the a_k and b_k are positive integers.

There are 75373 such numbers that do not exceed 10^7.
How many such numbers are there that do not exceed 2×10^9?

Problem 230: Fibonacci Words

For any two strings of digits, A and B, we define $F_{A,B}$ to be the sequence (A,B,AB,BAB,ABBAB,...) in which each term is the concatenation of the previous two.

Further, we define $D_{A,B}(n)$ to be the n^{th} digit in the first term of $F_{A,B}$ that contains at least n digits.

Example:

Let A=1415926535, B=8979323846. We wish to find $D_{A,B}(35)$, say.

The first few terms of $F_{A,B}$ are:
1415926535
8979323846
14159265358979323846
897932384614159265358979323846
14159265358979323846897932384614159265358979323846

Then $D_{A,B}(35)$ is the 35^{th} digit in the fifth term, which is 9.

Now we use for A the first 100 digits of π behind the decimal point:

14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679

and for B the next hundred digits:

82148086513282306647093844609550582231725359408128
48111745028410270193852110555964462294895493038196 .

Find $\sum_{n = 0,1,...,17} \ 10^n \times D_{A,B}((127+19n)\times 7^n)$.

Problem 232: The Race

Two players share an unbiased coin and take it in turns to play "The Race". On Player 1's turn, he tosses the coin once: if it comes up Heads, he scores one point; if it comes up Tails, he scores nothing. On Player 2's turn, she chooses a positive integer T and tosses the coin T times: if it comes up all Heads, she scores 2^{T-1} points; otherwise, she scores nothing. Player 1 goes first. The winner is the first to 100 or more points.

On each turn Player 2 selects the number, T, of coin tosses that maximises the probability of her winning.

What is the probability that Player 2 wins?

Give your answer rounded to eight decimal places in the form 0.*abcdefgh* .

Problem 233: Lattice points on a circle

Let $f(N)$ be the number of points with integer coordinates that are on a circle passing through (0,0), $(N,0)$,$(0,N)$, and (N,N).

It can be shown that $f(10000) = 36$.

What is the sum of all positive integers $N \le 10^{11}$ such that $f(N) = 420$?

Problem 236: Luxury Hampers

Suppliers 'A' and 'B' provided the following numbers of products for the luxury hamper market:

Product	'A'	'B'
Beluga Caviar	5248	640
Christmas Cake	1312	1888
Gammon Joint	2624	3776
Vintage Port	5760	3776
Champagne Truffles	3936	5664

Although the suppliers try very hard to ship their goods in perfect condition, there is inevitably some spoilage - *i.e.* products gone bad.

The suppliers compare their performance using two types of statistic:

- The five *per-product spoilage rates* for each supplier are equal to the number of products gone bad divided by the number of products supplied, for each of the five products in turn.
- The *overall spoilage rate* for each supplier is equal to the total number of products gone bad divided by the total number of products provided by that supplier.

To their surprise, the suppliers found that each of the five per-product spoilage rates was worse (higher) for 'B' than for 'A' by the same factor (ratio of spoilage rates), m>1; and yet, paradoxically, the overall spoilage rate was worse for 'A' than for 'B', also by a factor of m.

There are thirty-five m>1 for which this surprising result could have occurred, the smallest of which is 1476/1475.

What's the largest possible value of m?
Give your answer as a fraction reduced to its lowest terms, in the form u/v.

Problem 238: Infinite string tour

Create a sequence of numbers using the "Blum Blum Shub" pseudo-random number generator:

$$s_0 = 14025256$$
$$s_{n+1} = s_n^2 \text{ mod } 20300713$$

Concatenate these numbers $s_0s_1s_2$… to create a string w of infinite length.
Then, w = 14025256741014958470038053646…

For a positive integer k, if no substring of w exists with a sum of digits equal to k, $p(k)$ is defined to be zero. If at least one substring of w exists with a sum of digits equal to k, we define $p(k) = z$, where z is the starting position of the earliest such substring.

For instance:

The substrings 1, 14, 1402, …
with respective sums of digits equal to 1, 5, 7, …
start at position **1**, hence $p(1) = p(5) = p(7) = \ldots = \mathbf{1}$.

The substrings 4, 402, 4025, …
with respective sums of digits equal to 4, 6, 11, …
start at position **2**, hence $p(4) = p(6) = p(11) = \ldots = \mathbf{2}$.

The substrings 02, 0252, …
with respective sums of digits equal to 2, 9, …
start at position **3**, hence $p(2) = p(9) = \ldots = \mathbf{3}$.

Note that substring 025 starting at position **3**, has a sum of digits equal to 7, but there was an earlier substring (starting at position **1**) with a sum of digits equal to 7, so $p(7) = 1$, *not* 3.

We can verify that, for $0 < k \leq 10^3$, $\sum p(k)$ = 4742.

Find $\sum p(k)$, for $0 < k \leq 2 \cdot 10^{15}$.

Problem 239: Twenty-two Foolish Primes

A set of disks numbered 1 through 100 are placed in a line in random order.

What is the probability that we have a partial derangement such that exactly 22 prime number discs are found away from their natural positions? (Any number of non-prime disks may also be found in or out of their natural positions.)

Give your answer rounded to 12 places behind the decimal point in the form 0.abcdefghijkl.

Problem 240: Top Dice

There are 1111 ways in which five 6-sided dice (sides numbered 1 to 6) can be rolled so that the top three sum to 15. Some examples are:

D_1,D_2,D_3,D_4,D_5 = 4,3,6,3,5

D_1,D_2,D_3,D_4,D_5 = 4,3,3,5,6
D_1,D_2,D_3,D_4,D_5 = 3,3,3,6,6
D_1,D_2,D_3,D_4,D_5 = 6,6,3,3,3

In how many ways can twenty 12-sided dice (sides numbered 1 to 12) be rolled so that the top ten sum to 70?

Problem 241: Perfection Quotients

For a positive integer n, let $\sigma(n)$ be the sum of all divisors of n, so e.g. $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

A perfect number, as you probably know, is a number with $\sigma(n) = 2n$.

Let us define the **perfection quotient** of a positive integer as $p(n) = \dfrac{\sigma(n)}{n}$.

Find the sum of all positive integers $n \le 10^{18}$ for which $p(n)$ has the form $k + {}^1\!/_2$, where k is an integer.

Problem 242: Odd Triplets

Given the set {1,2,...,n}, we define $f(n,k)$ as the number of its k-element subsets with an odd sum of elements. For example, $f(5,3) = 4$, since the set {1,2,3,4,5} has four 3-element subsets having an odd sum of elements, i.e.: {1,2,4}, {1,3,5}, {2,3,4} and {2,4,5}.

When all three values n, k and $f(n,k)$ are odd, we say that they make an *odd-triplet* $[n,k,f(n,k)]$.

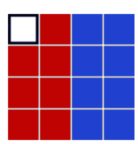
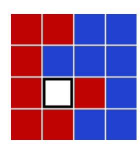
There are exactly five odd-triplets with $n \le 10$, namely:
$[1,1,f(1,1)=1]$, $[5,1,f(5,1)=3]$, $[5,5,f(5,5)=1]$, $[9,1,f(9,1)=5]$ and $[9,9,f(9,9)=1]$.

How many odd-triplets are there with $n \le 10^{12}$?

Problem 244: Sliders

You probably know the game *Fifteen Puzzle*. Here, instead of numbered tiles, we have seven red tiles and eight blue tiles.

A move is denoted by the uppercase initial of the direction (Left, Right, Up, Down) in which the tile is slid, e.g. starting from configuration (**S**), by the sequence **LULUR** we reach the configuration (**E**):

(**S**)  , (**E**) 

For each path, its checksum is calculated by (pseudocode):

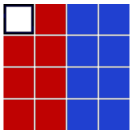
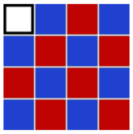
checksum = 0
checksum = (checksum × 243 + m_1) mod 100 000 007
checksum = (checksum × 243 + m_2) mod 100 000 007
…
checksum = (checksum × 243 + m_n) mod 100 000 007

where m_k is the ASCII value of the k^{th} letter in the move sequence and the ASCII values for the moves are:

L	76
R	82
U	85
D	68

For the sequence **LULUR** given above, the checksum would be 19761398.

Now, starting from configuration (**S**), find all shortest ways to reach configuration (**T**).

(S)  , (T) 

What is the sum of all checksums for the paths having the minimal length?

Problem 245: Coresilience

We shall call a fraction that cannot be cancelled down a resilient fraction.
Furthermore we shall define the resilience of a denominator, $R(d)$, to be the ratio of its proper fractions that are resilient; for example, $R(12) = {}^{4}/_{11}$.

The resilience of a number $d > 1$ is then $\dfrac{\varphi(d)}{d - 1}$, where φ is Euler's totient function.

We further define the **coresilience** of a number $n > 1$ as $C(n) = \dfrac{n - \varphi(n)}{n - 1}$.

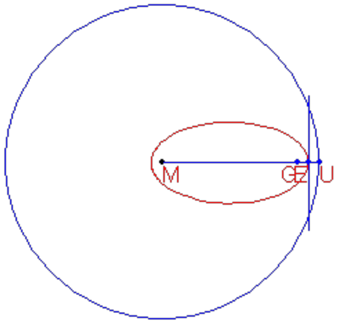
The coresilience of a prime p is $C(p) = \dfrac{1}{p - 1}$.

Find the sum of all **composite** integers $1 < n \le 2\times10^{11}$, for which $C(n)$ is a unit fraction.

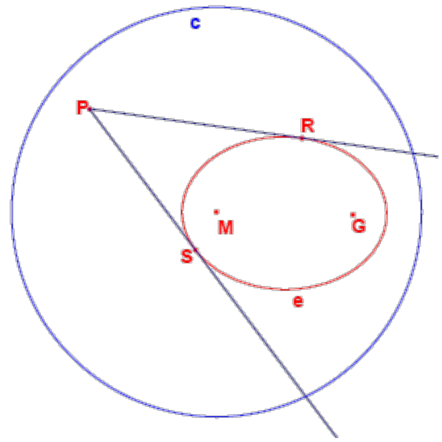
Problem 246: Tangents to an ellipse

A definition for an ellipse is:
Given a circle c with centre M and radius r and a point G such that d(G,M)<r, the locus of the points that are equidistant from c and G form an ellipse.

The construction of the points of the ellipse is shown below.



Given are the points M(-2000,1500) and G(8000,1500).
Given is also the circle c with centre M and radius 15000.
The locus of the points that are equidistant from G and c form an ellipse e.
From a point P outside e the two tangents t_1 and t_2 to the ellipse are drawn.
Let the points where t_1 and t_2 touch the ellipse be R and S.

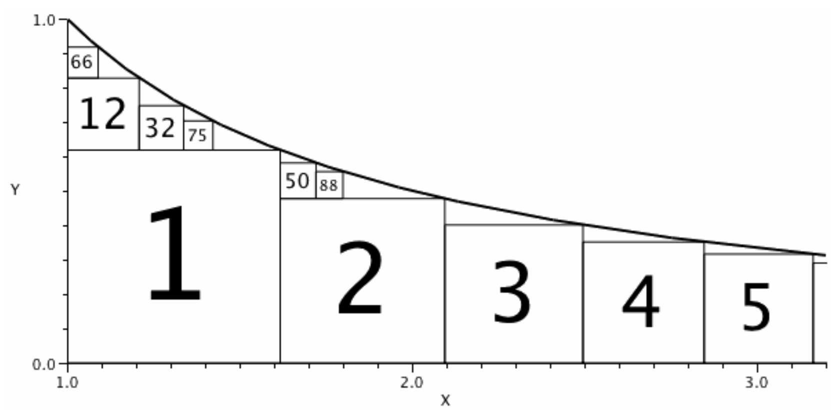


For how many lattice points P is angle RPS greater than 45 degrees?

Problem 247: Squares under a hyperbola

Consider the region constrained by $1 \le x$ and $0 \le y \le {}^1/_x$.

Let S_1 be the largest square that can fit under the curve.
Let S_2 be the largest square that fits in the remaining area, and so on.
Let the *index* of S_n be the pair (left, below) indicating the number of squares to the left of S_n and the number of squares below S_n.



The diagram shows some such squares labelled by number.
S_2 has one square to its left and none below, so the index of S_2 is (1,0).
It can be seen that the index of S_{32} is (1,1) as is the index of S_{50}.
50 is the largest n for which the index of S_n is (1,1).

What is the largest n for which the index of S_n is (3,3)?

Problem 248: Numbers for which Euler's totient function equals 13!

The first number n for which φ(n)=13! is 6227180929.

Find the 150,000th such number.

Problem 249: Prime Subset Sums

Let S = {2, 3, 5, ..., 4999} be the set of prime numbers less than 5000.

Find the number of subsets of S, the sum of whose elements is a prime number.
Enter the rightmost 16 digits as your answer.

Problem 250: 250250

Find the number of non-empty subsets of $\{1^1, 2^2, 3^3,..., 250250^{250250}\}$, the sum of whose elements is divisible by 250. Enter the rightmost 16 digits as your answer.

Problem 251: Cardano Triplets

A triplet of positive integers (a,b,c) is called a Cardano Triplet if it satisfies the condition:

$$\sqrt[3]{a+b\sqrt{c}}+\sqrt[3]{a-b\sqrt{c}}=1$$

For example, (2,1,5) is a Cardano Triplet.

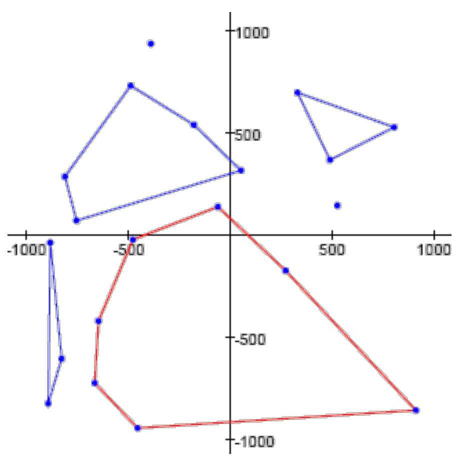
There exist 149 Cardano Triplets for which $a+b+c \leq 1000$.

Find how many Cardano Triplets exist such that $a+b+c \leq 110,000,000$.

Problem 252: Convex Holes

Given a set of points on a plane, we define a convex hole to be a convex polygon having as vertices any of the given points and not containing any of the given points in its interior (in addition to the vertices, other given points may lie on the perimeter of the polygon).

As an example, the image below shows a set of twenty points and a few such convex holes. The convex hole shown as a red heptagon has an area equal to 1049694.5 square units, which is the highest possible area for a convex hole on the given set of points.



For our example, we used the first 20 points (T_{2k-1}, T_{2k}), for $k=1,2,\ldots,20$, produced with the pseudo-random number generator:

$$\begin{aligned} S_0 &= 290797 \\ S_{n+1} &= S_n^2 \bmod 50515093 \\ T_n &= (S_n \bmod 2000) - 1000 \end{aligned}$$

i.e. $(527, 144)$, $(-488, 732)$, $(-454, -947)$, …

What is the maximum area for a convex hole on the set containing the first 500 points in the pseudo-random sequence?
Specify your answer including one digit after the decimal point.

Problem 253: Tidying up

A small child has a “number caterpillar” consisting of forty jigsaw pieces, each with one number on it, which, when connected together in a line, reveal the numbers 1 to 40 in order.

Every night, the child's father has to pick up the pieces of the caterpillar that have been scattered across the play room. He picks up the pieces at random and places them in the correct order.
As the caterpillar is built up in this way, it forms distinct segments that gradually merge together.
The number of segments starts at zero (no pieces placed), generally increases up to about eleven or twelve, then tends to drop again before finishing at a single segment (all pieces placed).

For example:

Piece Placed	Segments So Far
12	1
4	2
29	3
6	4
34	5
5	4
35	4
…	…

Let M be the maximum number of segments encountered during a random tidy-up of the caterpillar.
For a caterpillar of ten pieces, the number of possibilities for each M is

M	Possibilities
1	512
2	250912
3	1815264
4	1418112
5	144000

so the most likely value of M is 3 and the average value is $^{385643}/_{113400}$ = 3.400732, rounded to six decimal places.

The most likely value of M for a forty-piece caterpillar is 11; but what is the average value of M?

Give your answer rounded to six decimal places.

Problem 254: Sums of Digit Factorials

Define f(n) as the sum of the factorials of the digits of n. For example, f(342) = 3! + 4! + 2! = 32.

Define sf(n) as the sum of the digits of f(n). So sf(342) = 3 + 2 = 5.

Define g(i) to be the smallest positive integer n such that sf(n) = i. Though sf(342) is 5, sf(25) is also 5, and it can be verified that g(5) is 25.

Define sg(i) as the sum of the digits of g(i). So sg(5) = 2 + 5 = 7.

Further, it can be verified that g(20) is 267 and $\sum$sg(i) for $1 \le i \le 20$ is 156.

What is $\sum$sg(i) for $1 \le i \le 150$?

Problem 255: Rounded Square Roots

We define the *rounded-square-root* of a positive integer n as the square root of n rounded to the nearest integer.

The following procedure (essentially Heron's method adapted to integer arithmetic) finds the rounded-square-root of n:

Let d be the number of digits of the number n.
If d is odd, set $x_0 = 2\times10^{(d-1)/2}$.
If d is even, set $x_0 = 7\times10^{(d-2)/2}$.

Repeat:

$$x_{k+1} = \left\lfloor \frac{x_k + \lceil n/x_k \rceil}{2} \right\rfloor$$

until $x_{k+1} = x_k$.

As an example, let us find the rounded-square-root of n = 4321.
n has 4 digits, so $x_0 = 7\times10^{(4-2)/2} = 70$.

$$x_1 = \left\lfloor \frac{70 + \lceil 4321/70 \rceil}{2} \right\rfloor = 66.$$

$$x_2 = \left\lfloor \frac{66 + \lceil 4321/66 \rceil}{2} \right\rfloor = 66.$$

Since $x_2 = x_1$, we stop here.
So, after just two iterations, we have found that the rounded-square-root of 4321 is 66 (the actual square root is 65.7343137…).

The number of iterations required when using this method is surprisingly low.
For example, we can find the rounded-square-root of a 5-digit integer ($10{,}000 \le n \le 99{,}999$) with an average of 3.2102888889 iterations (the average value was rounded to 10 decimal places).

Using the procedure described above, what is the average number of iterations required to find the rounded-square-root of a 14-digit number ($10^{13} \le n < 10^{14}$)?
Give your answer rounded to 10 decimal places.

Note: The symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ represent the floor function and ceiling function respectively.

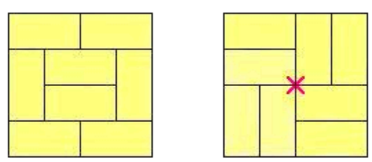
Problem 256: Tatami-Free Rooms

Tatami are rectangular mats, used to completely cover the floor of a room, without overlap.

Assuming that the only type of available tatami has dimensions 1×2, there are obviously some limitations for the shape and size of the rooms that can be covered.

For this problem, we consider only rectangular rooms with integer dimensions a, b and even size $s = a\cdot b$.
We use the term 'size' to denote the floor surface area of the room, and — without loss of generality — we add the condition $a \le b$.

There is one rule to follow when laying out tatami: there must be no points where corners of four different mats meet.
For example, consider the two arrangements below for a 4×4 room:



The arrangement on the left is acceptable, whereas the one on the right is not: a red "X" in the middle, marks the point where four tatami meet.

Because of this rule, certain even-sized rooms cannot be covered with tatami: we call them tatami-free rooms.
Further, we define $T(s)$ as the number of tatami-free rooms of size s.

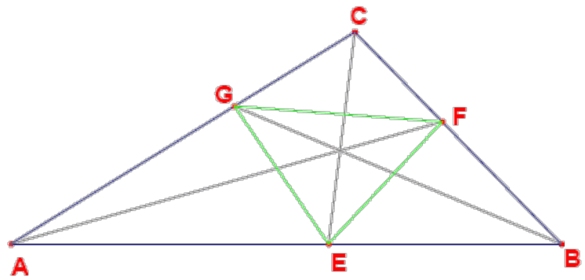
The smallest tatami-free room has size s = 70 and dimensions 7×10.
All the other rooms of size s = 70 can be covered with tatami; they are: 1×70, 2×35 and 5×14.
Hence, $T(70) = 1$.

Similarly, we can verify that $T(1320) = 5$ because there are exactly 5 tatami-free rooms of size s = 1320:
20×66, 22×60, 24×55, 30×44 and 33×40.
In fact, s = 1320 is the smallest room-size s for which $T(s) = 5$.

Find the smallest room-size s for which $T(s) = 200$.

Problem 257: Angular Bisectors

Given is an integer sided triangle ABC with sides $a \leq b \leq c$. (AB = c, BC = a and AC = b).
The angular bisectors of the triangle intersect the sides at points E, F and G (see picture below).



The segments EF, EG and FG partition the triangle ABC into four smaller triangles: AEG, BFE, CGF and EFG.
It can be proven that for each of these four triangles the ratio area(ABC)/area(subtriangle) is rational.
However, there exist triangles for which some or all of these ratios are integral.

How many triangles ABC with perimeter≤100,000,000 exist so that the ratio area(ABC)/area(AEG) is integral?

Problem 258: A lagged Fibonacci sequence

A sequence is defined as:

- $g_k = 1$, for $0 \leq k \leq 1999$
- $g_k = g_{k-2000} + g_{k-1999}$, for $k \geq 2000$.

Find g_k mod 20092010 for $k = 10^{18}$.

Problem 260: Stone Game

A game is played with three piles of stones and two players.
At her turn, a player removes one or more stones from the piles. However, if she takes stones from more than one pile, she must remove the same number of stones from each of the selected piles.

In other words, the player chooses some N>0 and removes:

- N stones from any single pile; or
- N stones from each of any two piles (2N total); or
- N stones from each of the three piles (3N total).

The player taking the last stone(s) wins the game.

A *winning configuration* is one where the first player can force a win.
For example, (0,0,13), (0,11,11) and (5,5,5) are winning configurations because the first player can immediately remove all stones.

A *losing configuration* is one where the second player can force a win, no matter what the first player does.
For example, (0,1,2) and (1,3,3) are losing configurations: any legal move leaves a winning configuration for the second player.

Consider all losing configurations (x_i,y_i,z_i) where $x_i \leq y_i \leq z_i \leq 100$.
We can verify that $\Sigma(x_i+y_i+z_i) = 173895$ for these.

Find $\Sigma(x_i+y_i+z_i)$ where (x_i,y_i,z_i) ranges over the losing configurations
with $x_i \leq y_i \leq z_i \leq 1000$.

Problem 261: Pivotal Square Sums

Let us call a positive integer k a *square-pivot*, if there is a pair of integers $m > 0$ and $n \geq k$, such that the sum of the $(m+1)$ consecutive squares up to k equals the sum of the m consecutive squares from $(n+1)$ on:

$$(k-m)^2 + \ldots + k^2 = (n+1)^2 + \ldots + (n+m)^2.$$

Some small square-pivots are

- **4**: 3^2 + $\mathbf{4}^2$ = 5^2
- **21**: 20^2 + $\mathbf{21}^2$ = 29^2
- **24**: 21^2 + 22^2 + 23^2 + $\mathbf{24}^2$ = 25^2 + 26^2 + 27^2
- **110**: 108^2 + 109^2 + $\mathbf{110}^2$ = 133^2 + 134^2

Find the sum of all **distinct** square-pivots $\leq 10^{10}$.

Problem 262: Mountain Range.

The following equation represents the *continuous* topography of a mountainous region, giving the elevation h at any point (x,y):

$$h = \left(5000 - \frac{x^2 + y^2 + xy}{200} + \frac{25(x+y)}{2}\right) \cdot e^{-\left|\frac{x^2+y^2}{1000000} - \frac{3(x+y)}{2000} + \frac{7}{10}\right|}$$

A mosquito intends to fly from A(200,200) to B(1400,1400), without leaving the area given by $0 \leq x, y \leq 1600$.

Because of the intervening mountains, it first rises straight up to a point A', having elevation f. Then, while remaining at the same elevation f, it flies around any obstacles until it arrives at a point B' directly above B.

First, determine f_{min} which is the minimum constant elevation allowing such a trip from A to B, while remaining in the specified area.
Then, find the length of the shortest path between A' and B', while flying at that constant elevation f_{min}.

Give that length as your answer, rounded to three decimal places.

Note: For convenience, the elevation function shown above is repeated below, in a form suitable for most programming languages:
h=(5000-0.005*(x*x+y*y+x*y)+12.5*(x+y)) * exp(-abs(0.000001*(x*x+y*y)-0.0015*(x+y)+0.7))

Problem 263: An engineers' dream come true

Consider the number 6. The divisors of 6 are: 1,2,3 and 6.
Every number from 1 up to and including 6 can be written as a sum of distinct divisors of 6:
1=1, 2=2, 3=1+2, 4=1+3, 5=2+3, 6=6.
A number n is called a practical number if every number from 1 up to and including n can be expressed as a sum of distinct divisors of n.

A pair of consecutive prime numbers with a difference of six is called a sexy pair (since "sex" is the Latin word for "six"). The first sexy pair is (23, 29).

We may occasionally find a triple-pair, which means three consecutive sexy prime pairs, such that the second member of each pair is the first member of the next pair.

We shall call a number n such that :

- (n-9, n-3), (n-3,n+3), (n+3, n+9) form a triple-pair, and
- the numbers n-8, n-4, n, n+4 and n+8 are all practical,

an engineers' paradise.

Find the sum of the first four engineers' paradises.

Problem 264: Triangle Centres

Consider all the triangles having:

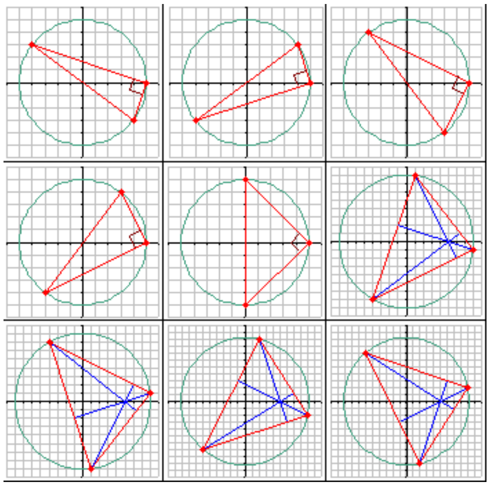
- All their vertices on lattice points.
- Circumcentre at the origin O.
- Orthocentre at the point H(5, 0).

There are nine such triangles having a perimeter ≤ 50.
Listed and shown in ascending order of their perimeter, they are:

A(-4, 3), B(5, 0), C(4, -3)
A(4, 3), B(5, 0), C(-4, -3)
A(-3, 4), B(5, 0), C(3, -4)

A(3, 4), B(5, 0), C(-3, -4)
A(0, 5), B(5, 0), C(0, -5)
A(1, 8), B(8, -1), C(-4, -7)

A(8, 1), B(1, -8), C(-4, 7)
A(2, 9), B(9, -2), C(-6, -7)
A(9, 2), B(2, -9), C(-6, 7)



The sum of their perimeters, rounded to four decimal places, is 291.0089.

Find all such triangles with a perimeter $\leq 10^5$.
Enter as your answer the sum of their perimeters rounded to four decimal places.

Problem 269: Polynomials with at least one integer root

A root or zero of a polynomial P(x) is a solution to the equation P(x) = 0.
Define P_n as the polynomial whose coefficients are the digits of n.
For example, $P_{5703}(x) = 5x^3 + 7x^2 + 3$.

We can see that:

- $P_n(0)$ is the last digit of n,
- $P_n(1)$ is the sum of the digits of n,
- $P_n(10)$ is n itself.

Define Z(k) as the number of positive integers, n, not exceeding k for which the polynomial P_n has at least one integer root.

It can be verified that Z(100 000) is 14696.

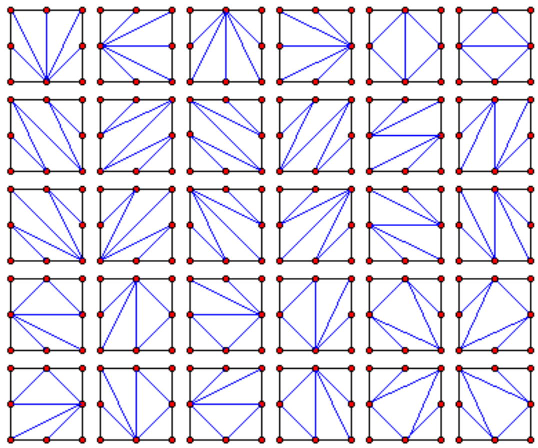
What is Z(10^{16})?

Problem 270: Cutting Squares

A square piece of paper with integer dimensions N×N is placed with a corner at the origin and two of its sides along the x- and y-axes. Then, we cut it up respecting the following rules:

- We only make straight cuts between two points lying on different sides of the square, and having integer coordinates.
- Two cuts cannot cross, but several cuts can meet at the same border point.
- Proceed until no more legal cuts can be made.

Counting any reflections or rotations as distinct, we call C(N) the number of ways to cut an N×N square. For example, C(1) = 2 and C(2) = 30 (shown below).



What is $C(30) \bmod 10^8$?

Problem 271: Modular Cubes, part 1

For a positive number n, define $S(n)$ as the sum of the integers x, for which $1<x<n$ and $x^3 \equiv 1 \bmod n$.

When $n=91$, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81.
Thus, $S(91)=9+16+22+29+53+74+79+81=363$.

Find S(13082761331670030).

Problem 272: Modular Cubes, part 2

For a positive number n, define $C(n)$ as the number of the integers x, for which $1<x<n$ and $x^3 \equiv 1 \bmod n$.

When $n=91$, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81.
Thus, $C(91)=8$.

Find the sum of the positive numbers $n \leq 10^{11}$ for which $C(n)=242$.

Problem 273: Sum of Squares

Consider equations of the form: $a^2 + b^2 = N$, $0 \leq a \leq b$, a, b and N integer.

For $N=65$ there are two solutions:

$a=1$, $b=8$ and $a=4$, $b=7$.

We call $S(N)$ the sum of the values of a of all solutions of $a^2 + b^2 = N$, $0 \leq a \leq b$, a, b and N integer.

Thus $S(65) = 1 + 4 = 5$.

Find $\sum S(N)$, for all squarefree N only divisible by primes of the form $4k+1$ with $4k+1 < 150$.

Problem 274: Divisibility Multipliers

For each integer $p > 1$ coprime to 10 there is a positive *divisibility multiplier* $m < p$ which preserves divisibility by p for the following function on any positive integer, n:

$f(n)$ = (all but the last digit of n) + (the last digit of n) * m

That is, if m is the divisibility multiplier for p, then $f(n)$ is divisible by p if and only if n is divisible by p.

(When n is much larger than p, $f(n)$ will be less than n and repeated application of f provides a multiplicative divisibility test for p.)

For example, the divisibility multiplier for 113 is 34.

$f(76275) = 7627 + 5 * 34 = 7797$: 76275 and 7797 are both divisible by 113
$f(12345) = 1234 + 5 * 34 = 1404$: 12345 and 1404 are both not divisible by 113

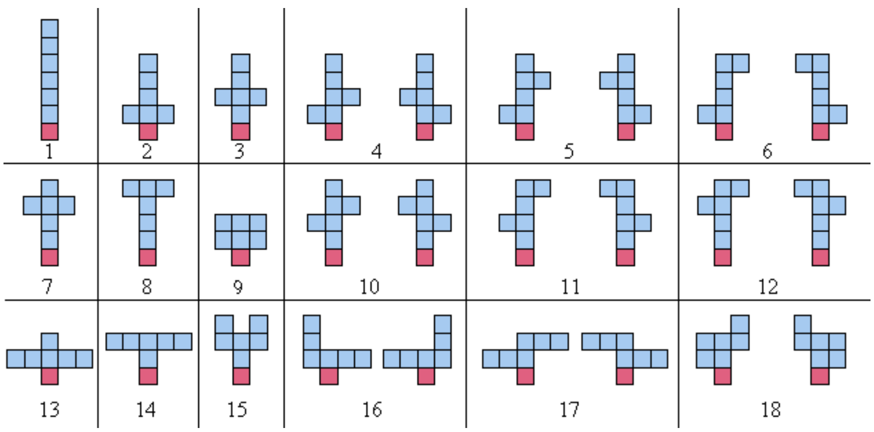
The sum of the divisibility multipliers for the primes that are coprime to 10 and less than 1000 is 39517. What is the sum of the divisibility multipliers for the primes that are coprime to 10 and less than 10^7?

Problem 275: Balanced Sculptures

Let us define a *balanced sculpture* of order n as follows:

- A polyomino made up of n+1 tiles known as the *blocks* (n tiles) and the *plinth* (remaining tile);
- the plinth has its centre at position (x = 0, y = 0);
- the blocks have y-coordinates greater than zero (so the plinth is the unique lowest tile);
- the centre of mass of all the blocks, combined, has x-coordinate equal to zero.

When counting the sculptures, any arrangements which are simply reflections about the y-axis, are not counted as distinct. For example, the 18 balanced sculptures of order 6 are shown below; note that each pair of mirror images (about the y-axis) is counted as one sculpture:



There are 964 balanced sculptures of order 10 and 360505 of order 15.
How many balanced sculptures are there of order 18?

Problem 276: Primitive Triangles

Consider the triangles with integer sides a, b and c with a ≤ b ≤ c.
An integer sided triangle (a,b,c) is called primitive if gcd(a,b,c)=1.
How many primitive integer sided triangles exist with a perimeter not exceeding 10 000 000?

Problem 278: Linear Combinations of Semiprimes

Given the values of integers $1 < a_1 < a_2 <\ldots < a_n$, consider the linear combination
$q_1a_1 + q_2a_2 + \ldots + q_na_n = b$, using only integer values $q_k \geq 0$.

Note that for a given set of a_k, it may be that not all values of b are possible.
For instance, if $a_1 = 5$ and $a_2 = 7$, there are no $q_1 \geq 0$ and $q_2 \geq 0$ such that b could be
1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18 or 23.
In fact, 23 is the largest impossible value of b for $a_1 = 5$ and $a_2 = 7$.
We therefore call $f(5, 7) = 23$.
Similarly, it can be shown that $f(6, 10, 15)=29$ and $f(14, 22, 77) = 195$.

Find $\sum f(p*q,p*r,q*r)$, where p, q and r are prime numbers and $p < q < r < 5000$.

Problem 279: Triangles with integral sides and an integral angle

How many triangles are there with integral sides, at least one integral angle (measured in degrees), and a perimeter that does not exceed 10^8?

Problem 280: Ant and seeds

A laborious ant walks randomly on a 5x5 grid. The walk starts from the central square. At each step, the ant moves to an adjacent square at random, without leaving the grid; thus there are 2, 3 or 4 possible moves at each step depending on the ant's position.

At the start of the walk, a seed is placed on each square of the lower row. When the ant isn't carrying a seed and reaches a square of the lower row containing a seed, it will start to carry the seed. The ant will drop the seed on the first empty square of the upper row it eventually reaches.

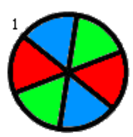
What's the expected number of steps until all seeds have been dropped in the top row?
Give your answer rounded to 6 decimal places.

Problem 281: Pizza Toppings

You are given a pizza (perfect circle) that has been cut into $m \cdot n$ equal pieces and you want to have exactly one topping on each slice.

Let $f(m,n)$ denote the number of ways you can have toppings on the pizza with m different toppings ($m \geq 2$), using each topping on exactly n slices ($n \geq 1$).
Reflections are considered distinct, rotations are not.

Thus, for instance, $f(2,1) = 1$, $f(2,2) = f(3,1) = 2$ and $f(3,2) = 16$.
$f(3,2)$ is shown below:



Find the sum of all $f(m,n)$ such that $f(m,n) \leq 10^{15}$.

Problem 282: The Ackermann function

For non-negative integers m, n, the Ackermann function $A(m, n)$ is defined as follows:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m,n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

For example $A(1, 0) = 2$, $A(2, 2) = 7$ and $A(3, 4) = 125$.

Find $\sum_{n=0}^{6} A(n, n)$ and give your answer mod 14^8.

Problem 283: Integer sided triangles for which the area/perimeter ratio is integral.

Consider the triangle with sides 6, 8 and 10. It can be seen that the perimeter and the area are both equal to 24. So the area/perimeter ratio is equal to 1.
Consider also the triangle with sides 13, 14 and 15. The perimeter equals 42 while the area is equal to 84. So for this triangle the area/perimeter ratio is equal to 2.

Find the sum of the perimeters of all integer sided triangles for which the area/perimeter ratios are equal to positive integers not exceeding 1000.

Problem 284: Steady Squares

The 3-digit number 376 in the decimal numbering system is an example of numbers with the special property that its square ends with the same digits: 376^2 = 141376. Let's call a number with this property a steady square.

Steady squares can also be observed in other numbering systems. In the base 14 numbering system, the 3-digit number c37 is also a steady square: $c37^2$ = aa0c37, and the sum of its digits is c+3+7=18 in the same numbering system. The letters a, b, c and d are used for the 10, 11, 12 and 13 digits respectively, in a manner similar to the hexadecimal numbering system.

For $1 \le n \le 9$, the sum of the digits of all the n-digit steady squares in the base 14 numbering system is 2d8 (582 decimal). Steady squares with leading 0's are not allowed.

Find the sum of the digits of all the n-digit steady squares in the base 14 numbering system for
$1 \le n \le 10000$ (decimal) and give your answer in the base 14 system using lower case letters where necessary.

Problem 285: Pythagorean odds

Albert chooses a positive integer k, then two real numbers a, b are randomly chosen in the interval [0,1] with uniform distribution.
The square root of the sum $(k \cdot a+1)^2 + (k \cdot b+1)^2$ is then computed and rounded to the nearest integer. If the result is equal to k, he scores k points; otherwise he scores nothing.

For example, if $k=6$, $a=0.2$ and $b=0.85$, then $(k \cdot a+1)^2 + (k \cdot b+1)^2 = 42.05$.
The square root of 42.05 is 6.484... and when rounded to the nearest integer, it becomes 6.
This is equal to k, so he scores 6 points.

It can be shown that if he plays 10 turns with $k=1$, $k=2$, ..., $k=10$, the expected value of his total score, rounded to five decimal places, is 10.20914.

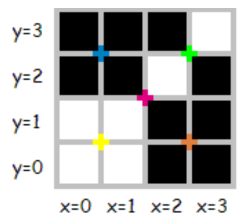
If he plays 10^5 turns with $k=1$, $k=2$, $k=3$, ..., $k=10^5$, what is the expected value of his total score, rounded to five decimal places?

Problem 287: Quadtree encoding (a simple compression algorithm)

The quadtree encoding allows us to describe a $2^N \times 2^N$ black and white image as a sequence of bits (0 and 1). Those sequences are to be read from left to right like this:

- the first bit deals with the complete $2^N \times 2^N$ region;
- "0" denotes a split:
  the current $2^n \times 2^n$ region is divided into 4 sub-regions of dimension $2^{n-1} \times 2^{n-1}$,
  the next bits contains the description of the top left, top right, bottom left and bottom right sub-regions - in that order;
- "10" indicates that the current region contains only black pixels;
- "11" indicates that the current region contains only white pixels.

Consider the following 4×4 image (colored marks denote places where a split can occur):



This image can be described by several sequences, for example : "001010101001011111011010101010", of length 30, or "0100101111101110", of length 16, which is the minimal sequence for this image.

For a positive integer N, define D_N as the $2^N \times 2^N$ image with the following coloring scheme:

- the pixel with coordinates $x = 0$, $y = 0$ corresponds to the bottom left pixel,
- if $(x - 2^{N-1})^2 + (y - 2^{N-1})^2 \leq 2^{2N-2}$ then the pixel is black,
- otherwise the pixel is white.

What is the length of the minimal sequence describing D_{24}?

Problem 288: An enormous factorial

For any prime p the number $N(p,q)$ is defined by $N(p,q) = \sum_{n=0 \text{ to } q} T_n * p^n$
with T_n generated by the following random number generator:

$S_0 = 290797$
$S_{n+1} = S_n^2 \bmod 50515093$
$T_n = S_n \bmod p$

Let Nfac(p,q) be the factorial of N(p,q).
Let NF(p,q) be the number of factors p in Nfac(p,q).

You are given that NF(3,10000) mod 3^{20}=624955285.

Find NF(61,10^7) mod 61^{10}

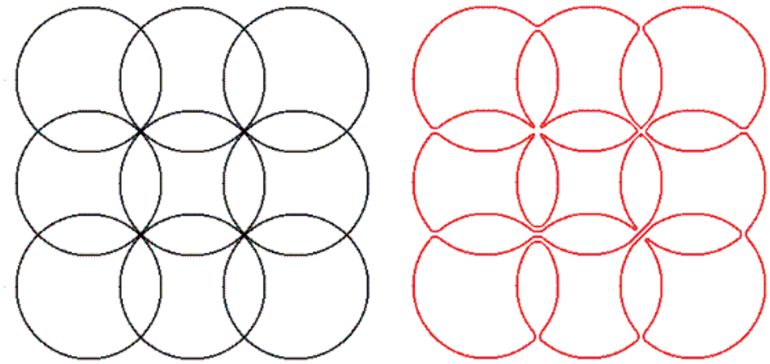
Problem 289: Eulerian Cycles

Let C(x,y) be a circle passing through the points (x, y), (x, y+1), (x+1, y) and (x+1, y+1).

For positive integers m and n, let E(m,n) be a configuration which consists of the $m \cdot n$ circles:
{ C(x,y): $0 \leq x < m$, $0 \leq y < n$, x and y are integers }

An Eulerian cycle on E(m,n) is a closed path that passes through each arc exactly once.
Many such paths are possible on E(m,n), but we are only interested in those which are not self-crossing: A non-crossing path just touches itself at lattice points, but it never crosses itself.

The image below shows E(3,3) and an example of an Eulerian non-crossing path.



Let L(m,n) be the number of Eulerian non-crossing paths on E(m,n).
For example, L(1,2) = 2, L(2,2) = 37 and L(3,3) = 104290.

Find L(6,10) mod 10^{10}.

Problem 290: Digital Signature

How many integers $0 \le n < 10^{18}$ have the property that the sum of the digits of n equals the sum of digits of $137n$?

Problem 291: Panaitopol Primes

A prime number p is called a Panaitopol prime if $p = \dfrac{x^4 - y^4}{x^3 + y^3}$ for some positive integers x and y.

Find how many Panaitopol primes are less than 5×10^{15}.

Problem 292: Pythagorean Polygons

We shall define a *pythagorean polygon* to be a **convex polygon** with the following properties:

- there are at least three vertices,
- no three vertices are aligned,
- each vertex has **integer coordinates**,
- each edge has **integer length**.

For a given integer n, define P(n) as the number of distinct pythagorean polygons for which the perimeter is $\le n$.
Pythagorean polygons should be considered distinct as long as none is a translation of another.

You are given that P(4) = 1, P(30) = 3655 and P(60) = 891045.
Find P(120).

Problem 294: Sum of digits - experience #23

For a positive integer k, define d(k) as the sum of the digits of k in its usual decimal representation. Thus d(42) = 4+2 = 6.

For a positive integer n, define S(n) as the number of positive integers $k < 10^n$ with the following properties :

- k is divisible by 23 and
- d(k) = 23.

You are given that S(9) = 263626 and S(42) = 6377168878570056.

Find $S(11^{12})$ and give your answer mod 10^9.

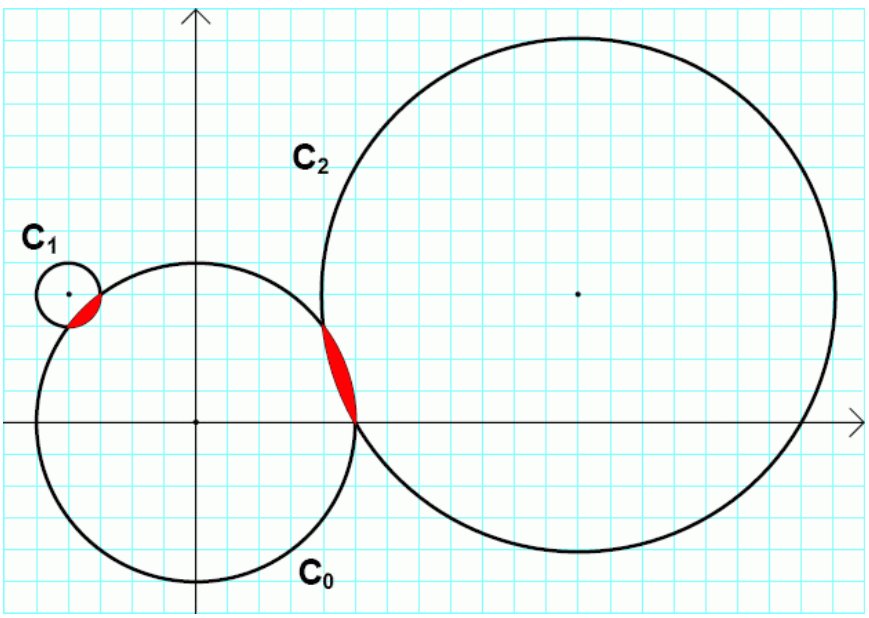
Problem 295: Lenticular holes

We call the convex area enclosed by two circles a *lenticular hole* if:

- The centres of both circles are on lattice points.
- The two circles intersect at two distinct lattice points.
- The interior of the convex area enclosed by both circles does not contain any lattice points.

Consider the circles:
C_0: $x^2+y^2=25$
C_1: $(x+4)^2+(y-4)^2=1$
C_2: $(x-12)^2+(y-4)^2=65$

The circles C_0, C_1 and C_2 are drawn in the picture below.



C_0 and C_1 form a lenticular hole, as well as C_0 and C_2.

We call an ordered pair of positive real numbers (r_1, r_2) a *lenticular pair* if there exist two circles with radii r_1 and r_2 that form a lenticular hole. We can verify that (1, 5) and (5, $\sqrt{65}$) are the lenticular pairs of the example above.

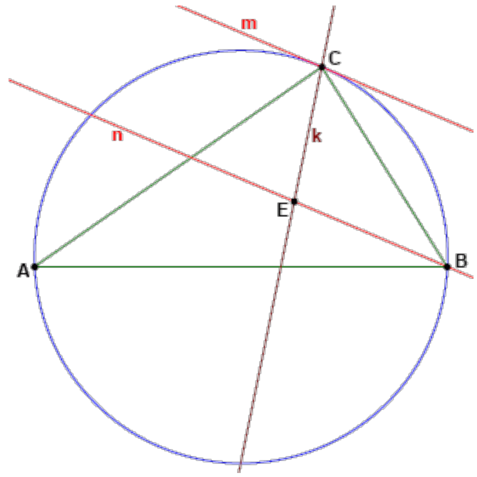
Let L(N) be the number of **distinct** lenticular pairs (r_1, r_2) for which $0 < r_1 \leq r_2 \leq N$.
We can verify that L(10) = 30 and L(100) = 3442.

Find L(100 000).

Problem 296: Angular Bisector and Tangent

Given is an integer sided triangle ABC with $BC \leq AC \leq AB$.
k is the angular bisector of angle ACB.

m is the tangent at C to the circumscribed circle of ABC.
n is a line parallel to m through B.
The intersection of n and k is called E.



How many triangles ABC with a perimeter not exceeding 100 000 exist such that BE has integral length?

Problem 297: Zeckendorf Representation

Each new term in the Fibonacci sequence is generated by adding the previous two terms.
Starting with 1 and 2, the first 10 terms will be: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

Every positive integer can be uniquely written as a sum of nonconsecutive terms of the Fibonacci sequence. For example, 100 = 3 + 8 + 89.
Such a sum is called the **Zeckendorf representation** of the number.

For any integer $n>0$, let $z(n)$ be the number of terms in the Zeckendorf representation of n.
Thus, $z(5) = 1$, $z(14) = 2$, $z(100) = 3$ etc.
Also, for $0<n<10^6$, $\sum z(n) = 7894453$.

Find $\sum z(n)$ for $0<n<10^{17}$.

Problem 298: Selective Amnesia

Larry and Robin play a memory game involving of a sequence of random numbers between 1 and 10, inclusive, that are called out one at a time. Each player can remember up to 5 previous numbers. When the called number is in a player's memory, that player is awarded a point. If it's not, the player adds the called number to his memory, removing another number if his memory is full.

Both players start with empty memories. Both players always add new missed numbers to their memory but use a different strategy in deciding which number to remove:
Larry's strategy is to remove the number that hasn't been called in the longest time.
Robin's strategy is to remove the number that's been in the memory the longest time.

Example game:

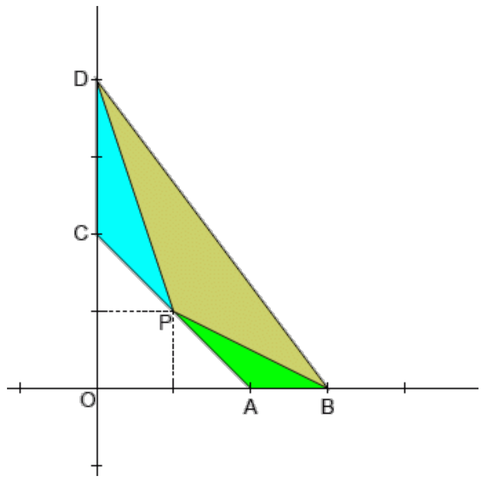
Turn	Called number	Larry's memory	Larry's score	Robin's memory	Robin's score
1	1	1	0	1	0
2	2	1,2	0	1,2	0
3	4	1,2,4	0	1,2,4	0
4	6	1,2,4,6	0	1,2,4,6	0
5	1	1,2,4,6	1	1,2,4,6	1
6	8	1,2,4,6,8	1	1,2,4,6,8	1
7	10	1,4,6,8,10	1	2,4,6,8,10	1
8	2	1,2,6,8,10	1	2,4,6,8,10	2

9	4	1,2,4,8,10	1	2,4,6,8,10	3
10	1	1,2,4,8,10	2	1,4,6,8,10	3

Denoting Larry's score by L and Robin's score by R, what is the expected value of $|L-R|$ after 50 turns? Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

Problem 299: Three similar triangles

Four points with integer coordinates are selected:
A(a, 0), B(b, 0), C(0, c) and D(0, d), with $0 < a < b$ and $0 < c < d$.
Point P, also with integer coordinates, is chosen on the line AC so that the three triangles ABP, CDP and BDP are all similar.



It is easy to prove that the three triangles can be similar, only if $a=c$.

So, given that $a=c$, we are looking for triplets (a,b,d) such that at least one point P (with integer coordinates) exists on AC, making the three triangles ABP, CDP and BDP all similar.

For example, if (a,b,d)=(2,3,4), it can be easily verified that point P(1,1) satisfies the above condition. Note that the triplets (2,3,4) and (2,4,3) are considered as distinct, although point P(1,1) is common for both.

If $b+d < 100$, there are 92 distinct triplets (a,b,d) such that point P exists.
If $b+d < 100\,000$, there are 320471 distinct triplets (a,b,d) such that point P exists.

If $b+d < 100\,000\,000$, how many distinct triplets (a,b,d) are there such that point P exists?

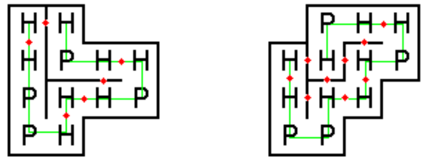
Problem 300: Protein folding

In a very simplified form, we can consider proteins as strings consisting of hydrophobic (H) and polar (P) elements, e.g. HHPPHHHPHHPH.

For this problem, the orientation of a protein is important; e.g. HPP is considered distinct from PPH. Thus, there are 2^n distinct proteins consisting of n elements.

When one encounters these strings in nature, they are always folded in such a way that the number of H-H contact points is as large as possible, since this is energetically advantageous.
As a result, the H-elements tend to accumulate in the inner part, with the P-elements on the outside.
Natural proteins are folded in three dimensions of course, but we will only consider protein folding in two dimensions.

The figure below shows two possible ways that our example protein could be folded (H-H contact points are shown with red dots).



The folding on the left has only six H-H contact points, thus it would never occur naturally.
On the other hand, the folding on the right has nine H-H contact points, which is optimal for this string.

Assuming that H and P elements are equally likely to occur in any position along the string, the average number of H-H contact points in an optimal folding of a random protein string of length 8 turns out to be 850 / 2^8=3.3203125.

What is the average number of H-H contact points in an optimal folding of a random protein string of length 15?
Give your answer using as many decimal places as necessary for an exact result.

Problem 301: Nim

Nim is a game played with heaps of stones, where two players take it in turn to remove any number of stones from any heap until no stones remain.

We'll consider the three-heap normal-play version of Nim, which works as follows:
- At the start of the game there are three heaps of stones.
- On his turn the player removes any positive number of stones from any single heap.
- The first player unable to move (because no stones remain) loses.

If (n_1,n_2,n_3) indicates a Nim position consisting of heaps of size n_1, n_2 and n_3 then there is a simple function $X(n_1,n_2,n_3)$ — that you may look up or attempt to deduce for yourself — that returns:

- zero if, with perfect strategy, the player about to move will eventually lose; or
- non-zero if, with perfect strategy, the player about to move will eventually win.

For example $X(1,2,3)$ = 0 because, no matter what the current player does, his opponent can respond with a move that leaves two heaps of equal size, at which point every move by the current player can be mirrored by his opponent until no stones remain; so the current player loses. To illustrate:
- current player moves to (1,2,1)
- opponent moves to (1,0,1)
- current player moves to (0,0,1)
- opponent moves to (0,0,0), and so wins.

For how many positive integers $n \le 2^{30}$ does $X(n,2n,3n)$ = 0 ?

Problem 302: Strong Achilles Numbers

A positive integer n is **powerful** if p^2 is a divisor of n for every prime factor p in n.

A positive integer n is a **perfect power** if n can be expressed as a power of another positive integer.

A positive integer n is an **Achilles number** if n is powerful but not a perfect power. For example, 864 and 1800 are Achilles numbers: 864 = $2^5 \cdot 3^3$ and 1800 = $2^3 \cdot 3^2 \cdot 5^2$.

We shall call a positive integer S a *Strong Achilles number* if both S and φ(S) are Achilles numbers.[1]
For example, 864 is a Strong Achilles number: φ(864) = 288 = $2^5 \cdot 3^2$. However, 1800 isn't a Strong Achilles number because: φ(1800) = 480 = $2^5 \cdot 3^1 \cdot 5^1$.

There are 7 Strong Achilles numbers below 10^4 and 656 below 10^8.

How many Strong Achilles numbers are there below 10^{18}?

[1] φ denotes **Euler's totient function**.

Problem 303: Multiples with small digits

For a positive integer n, define $f(n)$ as the least positive multiple of n that, written in base 10, uses only digits ≤ 2.

Thus $f(2)=2$, $f(3)=12$, $f(7)=21$, $f(42)=210$, $f(89)=1121222$.

Also, $\sum_{n=1}^{100} \frac{f(n)}{n} = 11363107$.

Find $\sum_{n=1}^{10000} \frac{f(n)}{n}$.

Problem 304: Primonacci

For any positive integer n the function next_prime(n) returns the smallest prime p
such that p>n.

The sequence a(n) is defined by:
a(1)=next_prime(10^{14}) and a(n)=next_prime(a(n-1)) for n>1.

The fibonacci sequence f(n) is defined by: f(0)=0, f(1)=1 and f(n)=f(n-1)+f(n-2) for n>1.

The sequence b(n) is defined as f(a(n)).

Find ∑b(n) for 1≤n≤100 000. Give your answer mod 1234567891011.

Problem 305: Reflexive Position

Let's call S the (infinite) string that is made by concatenating the consecutive positive integers (starting from 1) written down in base 10.
Thus, S = 1234567891011121314151617181920212223242...

It's easy to see that any number will show up an infinite number of times in S.

Let's call f(n) the starting position of the n^{th} occurrence of n in S.
For example, f(1)=1, f(5)=81, f(12)=271 and f(7780)=111111365.

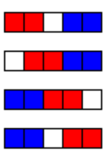
Find ∑f(3^k) for 1≤k≤13.

Problem 306: Paper-strip Game

The following game is a classic example of Combinatorial Game Theory:

Two players start with a strip of n white squares and they take alternate turns.
On each turn, a player picks two contiguous white squares and paints them black.
The first player who cannot make a move loses.

- If n = 1, there are no valid moves, so the first player loses automatically.
- If n = 2, there is only one valid move, after which the second player loses.
- If n = 3, there are two valid moves, but both leave a situation where the second player loses.
- If n = 4, there are three valid moves for the first player; she can win the game by painting the two middle squares.
- If n = 5, there are four valid moves for the first player (shown below in red); but no matter what she does, the second player (blue) wins.



So, for 1 ≤ n ≤ 5, there are 3 values of n for which the first player can force a win.
Similarly, for 1 ≤ n ≤ 50, there are 40 values of n for which the first player can force a win.

For $1 \le n \le 1\,000\,000$, how many values of n are there for which the first player can force a win?

Problem 308: An amazing Prime-generating Automaton

A program written in the programming language Fractran consists of a list of fractions.

The internal state of the Fractran Virtual Machine is a positive integer, which is initially set to a seed value. Each iteration of a Fractran program multiplies the state integer by the first fraction in the list which will leave it an integer.

For example, one of the Fractran programs that John Horton Conway wrote for prime-generation consists of the following 14 fractions:

$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}.$$

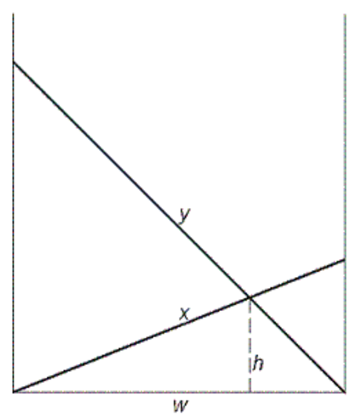
Starting with the seed integer 2, successive iterations of the program produce the sequence:
15, 825, 725, 1925, 2275, 425, ..., 68, **4**, 30, ..., 136, **8**, 60, ..., 544, **32**, 240, ...

The powers of 2 that appear in this sequence are 2^2, 2^3, 2^5, ...
It can be shown that *all* the powers of 2 in this sequence have prime exponents and that *all* the primes appear as exponents of powers of 2, in proper order!

If someone uses the above Fractran program to solve Project Euler Problem 7 (find the 10001st prime), how many iterations would be needed until the program produces $2^{10001\text{st prime}}$?

Problem 309: Integer Ladders

In the classic "Crossing Ladders" problem, we are given the lengths x and y of two ladders resting on the opposite walls of a narrow, level street. We are also given the height h above the street where the two ladders cross and we are asked to find the width of the street (w).



Here, we are only concerned with instances where all four variables are positive integers.
For example, if $x = 70$, $y = 119$ and $h = 30$, we can calculate that $w = 56$.

In fact, for integer values x, y, h and $0 < x < y < 200$, there are only five triplets (x,y,h) producing integer solutions for w:
(70, 119, 30), (74, 182, 21), (87, 105, 35), (100, 116, 35) and (119, 175, 40).

For integer values x, y, h and $0 < x < y < 1\,000\,000$, how many triplets (x,y,h) produce integer solutions for w?

Problem 310: Nim Square

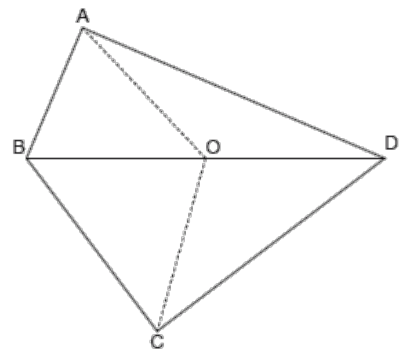
Alice and Bob play the game Nim Square.
Nim Square is just like ordinary three-heap normal play Nim, but the players may only remove a square number of stones from a heap.
The number of stones in the three heaps is represented by the ordered triple (a,b,c).
If $0 \le a \le b \le c \le 29$ then the number of losing positions for the next player is 1160.

Find the number of losing positions for the next player if $0 \le a \le b \le c \le 100\,000$.

Problem 311: Biclinic Integral Quadrilaterals

ABCD is a convex, integer sided quadrilateral with $1 \le AB < BC < CD < AD$.
BD has integer length. O is the midpoint of BD. AO has integer length.
We'll call ABCD a *biclinic integral quadrilateral* if $AO = CO \le BO = DO$.

For example, the following quadrilateral is a biclinic integral quadrilateral:
AB = 19, BC = 29, CD = 37, AD = 43, BD = 48 and AO = CO = 23.

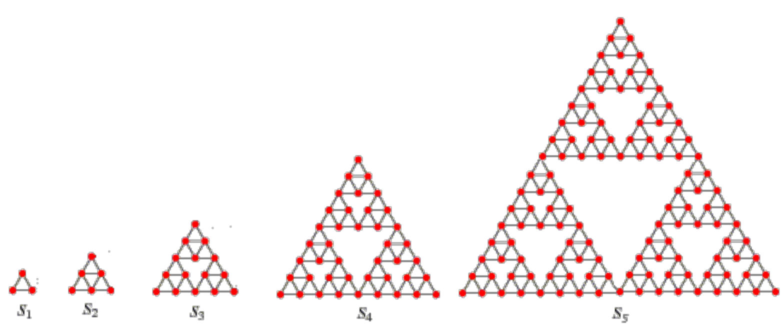


Let $B(N)$ be the number of distinct biclinic integral quadrilaterals ABCD that satisfy $AB^2+BC^2+CD^2+AD^2 \le N$.
We can verify that B(10 000) = 49 and B(1 000 000) = 38239.

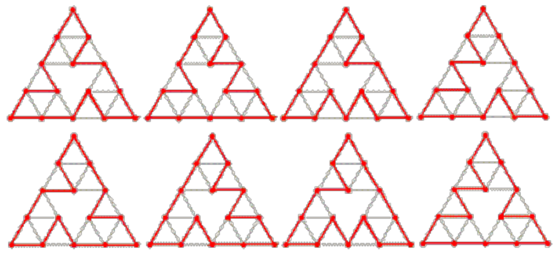
Find B(10 000 000 000).

Problem 312: Cyclic paths on Sierpiński graphs

- A **Sierpiński graph** of order-1 (S_1) is an equilateral triangle.
- S_{n+1} is obtained from S_n by positioning three copies of S_n so that every pair of copies has one common corner.



Let $C(n)$ be the number of cycles that pass exactly once through all the vertices of S_n.
For example, C(3) = 8 because eight such cycles can be drawn on S_3, as shown below:

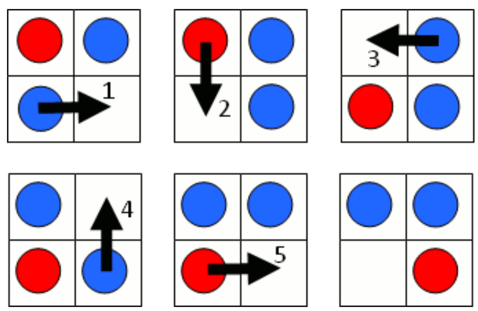


It can also be verified that :
C(1) = C(2) = 1
C(5) = 71328803586048
C(10 000) mod 10^8 = 37652224
C(10 000) mod 13^8 = 617720485

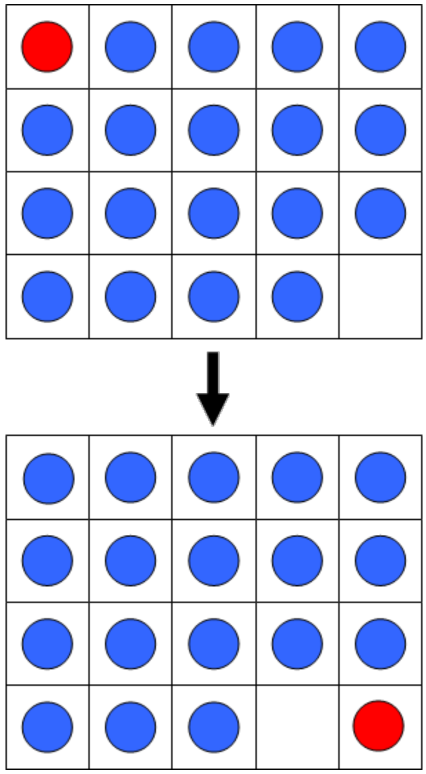
Find C(C(C(10 000))) mod 13^8.

Problem 313: Sliding game

In a sliding game a counter may slide horizontally or vertically into an empty space. The objective of the game is to move the red counter from the top left corner of a grid to the bottom right corner; the space always starts in the bottom right corner. For example, the following sequence of pictures show how the game can be completed in five moves on a 2 by 2 grid.



Let S(m,n) represent the minimum number of moves to complete the game on an m by n grid. For example, it can be verified that S(5,4) = 25.



There are exactly 5482 grids for which $S(m,n) = p^2$, where $p < 100$ is prime.

How many grids does $S(m,n) = p^2$, where $p < 10^6$ is prime?

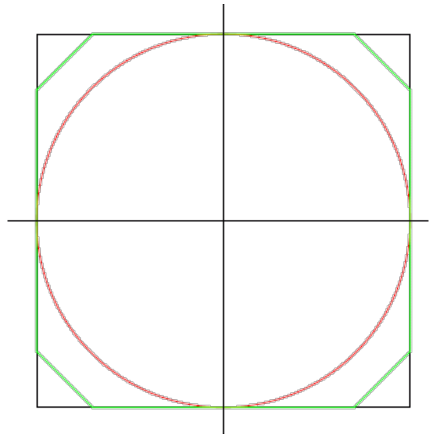
Problem 314: The Mouse on the Moon

The moon has been opened up, and land can be obtained for free, but there is a catch. You have to build a wall around the land that you stake out, and building a wall on the moon is expensive. Every country has been allotted a 500 m by 500 m square area, but they will possess only that area which they wall in. 251001 posts have been placed in a rectangular grid with 1 meter spacing. The wall must be a closed series of straight lines, each line running from post to post.

The bigger countries of course have built a 2000 m wall enclosing the entire 250 000 m^2 area. The Duchy of Grand Fenwick, has a tighter budget, and has asked you (their Royal Programmer) to compute what shape would get best maximum enclosed-area/wall-length ratio.

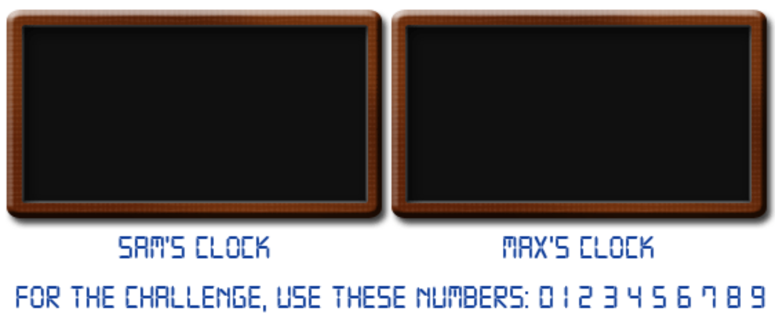
You have done some preliminary calculations on a sheet of paper. For a 2000 meter wall enclosing the 250 000 m^2 area the enclosed-area/wall-length ratio is 125.
Although not allowed , but to get an idea if this is anything better: if you place a circle inside the square area touching the four sides the area will be equal to $\pi*250^2$ m^2 and the perimeter will be $\pi*500$ m, so the enclosed-area/wall-length ratio will also be 125.

However, if you cut off from the square four triangles with sides 75 m, 75 m and $75\sqrt{2}$ m the total area becomes 238750 m^2 and the perimeter becomes $1400+300\sqrt{2}$ m. So this gives an enclosed-area/wall-length ratio of 130.87, which is significantly better.



Find the maximum enclosed-area/wall-length ratio.
Give your answer rounded to 8 places behind the decimal point in the form abc.defghijk.

Problem 315: Digital root clocks



Sam and Max are asked to transform two digital clocks into two "digital root" clocks.
A digital root clock is a digital clock that calculates digital roots step by step.

When a clock is fed a number, it will show it and then it will start the calculation, showing all the intermediate values until it gets to the result.
For example, if the clock is fed the number 137, it will show: "**137**" → "**11**" → "**2**" and then it will go black, waiting for the next number.

Every digital number consists of some light segments: three horizontal (top, middle, bottom) and four vertical (top-left, top-right, bottom-left, bottom-right).
Number "**1**" is made of vertical top-right and bottom-right, number "**4**" is made by middle horizontal and vertical top-left, top-right and bottom-right. Number "**8**" lights them all.

The clocks consume energy only when segments are turned on/off.
To turn on a "**2**" will cost 5 transitions, while a "**7**" will cost only 4 transitions.

Sam and Max built two different clocks.

Sam's clock is fed e.g. number 137: the clock shows "**137**", then the panel is turned off, then the next number ("**11**") is turned on, then the panel is turned off again and finally the last number ("**2**") is turned on and, after some time, off.
For the example, with number 137, Sam's clock requires:

"**137**" : $(2 + 5 + 4) \times 2 = 22$ transitions ("**137**" on/off).
"**11**" : $(2 + 2) \times 2 = 8$ transitions ("**11**" on/off).
"**2**" : $(5) \times 2 = 10$ transitions ("**2**" on/off).
For a grand total of 40 transitions.

Max's clock works differently. Instead of turning off the whole panel, it is smart enough to turn off only those segments that won't be needed for the next number.
For number 137, Max's clock requires:

"**137**" : $2 + 5 + 4 = 11$ transitions ("**137**" on)
7 transitions (to turn off the segments that are not needed for number "**11**").

"**11**" : 0 transitions (number "**11**" is already turned on correctly)
3 transitions (to turn off the first "**1**" and the bottom part of the second "**1**"; the top part is common with number "**2**").

"**2**" : 4 tansitions (to turn on the remaining segments in order to get a "**2**")
5 transitions (to turn off number "**2**").

For a grand total of 30 transitions.

Of course, Max's clock consumes less power than Sam's one.
The two clocks are fed all the prime numbers between A = 10^7 and B = 2×10^7.
Find the difference between the total number of transitions needed by Sam's clock and that needed by Max's one.

Problem 316: Numbers in decimal expansions

Let $p = p_1\, p_2\, p_3 \ldots$ be an infinite sequence of random digits, selected from {0,1,2,3,4,5,6,7,8,9} with equal probability.
It can be seen that p corresponds to the real number $0.p_1\, p_2\, p_3 \ldots$.
It can also be seen that choosing a random real number from the interval [0,1) is equivalent to choosing an infinite sequence of random digits selected from {0,1,2,3,4,5,6,7,8,9} with equal probability.

For any positive integer n with d decimal digits, let k be the smallest index such that
$p_k, p_{k+1}, \ldots p_{k+d-1}$ are the decimal digits of n, in the same order.
Also, let $g(n)$ be the expected value of k; it can be proven that $g(n)$ is always finite and, interestingly, always an integer number.

For example, if n = 535, then
for p = 31415926**535**897...., we get k = 9
for p = 355287143650049560000490848764084685**35**4..., we get k = 36
etc and we find that $g(535)$ = 1008.

Given that $\sum_{n=2}^{999} g\left(\left\lfloor \frac{10^6}{n} \right\rfloor\right) = 27280188$, find $\sum_{n=2}^{999999} g\left(\left\lfloor \frac{10^{16}}{n} \right\rfloor\right)$

Note: $\lfloor x \rfloor$ represents the floor function.

Problem 317: Firecracker

A firecracker explodes at a height of 100 m above level ground. It breaks into a large number of very small fragments, which move in every direction; all of them have the same initial velocity of 20 m/s.

We assume that the fragments move without air resistance, in a uniform gravitational field with g=9.81 m/s^2.

Find the volume (in m^3) of the region through which the fragments move before reaching the ground. Give your answer rounded to four decimal places.

Problem 318: 2011 nines

Consider the real number $\sqrt{2}+\sqrt{3}$.
When we calculate the even powers of $\sqrt{2}+\sqrt{3}$ we get:
$(\sqrt{2}+\sqrt{3})^2 = 9.898979485566356\ldots$
$(\sqrt{2}+\sqrt{3})^4 = 97.98979485566356\ldots$
$(\sqrt{2}+\sqrt{3})^6 = 969.998969071069263\ldots$
$(\sqrt{2}+\sqrt{3})^8 = 9601.99989585502907\ldots$
$(\sqrt{2}+\sqrt{3})^{10} = 95049.999989479221\ldots$
$(\sqrt{2}+\sqrt{3})^{12} = 940897.9999989371855\ldots$
$(\sqrt{2}+\sqrt{3})^{14} = 9313929.99999989263\ldots$
$(\sqrt{2}+\sqrt{3})^{16} = 92198401.99999998915\ldots$

It looks like that the number of consecutive nines at the beginning of the fractional part of these powers is non-decreasing.
In fact it can be proven that the fractional part of $(\sqrt{2}+\sqrt{3})^{2n}$ approaches 1 for large n.

Consider all real numbers of the form $\sqrt{p}+\sqrt{q}$ with p and q positive integers and p<q, such that the fractional part of $(\sqrt{p}+\sqrt{q})^{2n}$ approaches 1

for large n.

Let C(p,q,n) be the number of consecutive nines at the beginning of the fractional part of $(\sqrt{p}+\sqrt{q})^{2n}$.

Let N(p,q) be the minimal value of n such that C(p,q,n) $\geq$ 2011.

Find $\sum$N(p,q) for p+q $\leq$ 2011.

Problem 319: Bounded Sequences

Let $x_1, x_2,..., x_n$ be a sequence of length n such that:

- $x_1 = 2$
- for all $1 < i \leq n : x_{i-1} < x_i$
- for all i and j with $1 \leq i, j \leq n : (x_i)^j < (x_j + 1)^i$

There are only five such sequences of length 2, namely: {2,4}, {2,5}, {2,6}, {2,7} and {2,8}.
There are 293 such sequences of length 5; three examples are given below:
{2,5,11,25,55}, {2,6,14,36,88}, {2,8,22,64,181}.

Let $t(n)$ denote the number of such sequences of length n.
You are given that $t(10) = 86195$ and $t(20) = 5227991891$.

Find $t(10^{10})$ and give your answer modulo 10^9.

Problem 320: Factorials divisible by a huge integer

Let N(i) be the smallest integer n such that $n!$ is divisible by $(i!)^{1234567890}$

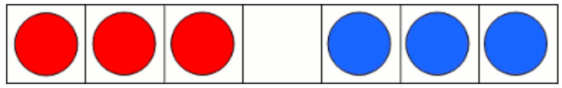
Let S(u)=$\sum$N(i) for $10 \leq i \leq u$.

S(1000)=614538266565663.

Find S(1 000 000) mod 10^{18}.

Problem 321: Swapping Counters

A horizontal row comprising of $2n + 1$ squares has n red counters placed at one end and n blue counters at the other end, being separated by a single empty square in the centre. For example, when $n = 3$.



A counter can move from one square to the next (slide) or can jump over another counter (hop) as long as the square next to that counter is unoccupied.

Let M(n) represent the minimum number of moves/actions to completely reverse the positions of the coloured counters; that is, move all the red counters to the right and all the blue counters to the left.

It can be verified M(3) = 15, which also happens to be a triangle number.

If we create a sequence based on the values of n for which M(n) is a triangle number then the first five terms would be:
1, 3, 10, 22, and 63, and their sum would be 99.

Find the sum of the first forty terms of this sequence.

Problem 322: Binomial coefficients divisible by 10

Let T(m, n) be the number of the binomial coefficients ${}^iC_n$ that are divisible by 10 for $n \le i < m$ (i, m and n are positive integers).
You are given that T(10^9, 10^7-10) = 989697000.

Find T(10^{18}, 10^{12}-10).

Problem 323: Bitwise-OR operations on random integers

Let y_0, y_1, y_2,... be a sequence of random unsigned 32 bit integers
(i.e. $0 \le y_i < 2^{32}$, every value equally likely).

For the sequence x_i the following recursion is given:

- $x_0 = 0$ and
- $x_i = x_{i-1} \mid y_{i-1}$, for $i > 0$. (| is the bitwise-OR operator)

It can be seen that eventually there will be an index N such that $x_i = 2^{32} - 1$ (a bit-pattern of all ones) for all $i \ge$ N.

Find the expected value of N.
Give your answer rounded to 10 digits after the decimal point.

Problem 324: Building a tower

Let $f(n)$ represent the number of ways one can fill a $3\times3\times n$ tower with blocks of $2\times1\times1$.
You're allowed to rotate the blocks in any way you like; however, rotations, reflections etc of the tower itself are counted as distinct.

For example (with q = 100000007) :
$f(2)$ = 229,
$f(4)$ = 117805,
$f(10)$ mod q = 96149360,
$f(10^3)$ mod q = 24806056,
$f(10^6)$ mod q = 30808124.

Find $f(10^{10000})$ mod 100000007.

Problem 325: Stone Game II

A game is played with two piles of stones and two players. At her turn, a player removes a number of stones from the larger pile. The number of stones she removes must be a positive multiple of the number of stones in the smaller pile.

E.g., let the ordered pair(6,14) describe a configuration with 6 stones in the smaller pile and 14 stones in the larger pile, then the first player can remove 6 or 12 stones from the larger pile.

The player taking all the stones from a pile wins the game.

A *winning configuration* is one where the first player can force a win. For example, (1,5), (2,6) and (3,12) are winning configurations because the first player can immediately remove all stones in the second pile.

A *losing configuration* is one where the second player can force a win, no matter what the first player does. For example, (2,3) and (3,4) are losing configurations: any legal move leaves a winning configuration for the second player.

Define S(N) as the sum of (x_i+y_i) for all losing configurations (x_i,y_i), $0 < x_i < y_i \le N$. We can verify that S(10) = 211 and S(10^4) = 230312207313.

Find S(10^{16}) mod 7^{10}.

Problem 326: Modulo Summations

Let a_n be a sequence recursively defined by: $a_1 = 1, \quad a_n = \left(\sum_{k=1}^{n-1} k \cdot a_k\right) \bmod n$.

So the first 10 elements of a_n are: 1,1,0,3,0,3,5,4,1,9.

Let $f(N,M)$ represent the number of pairs (p,q) such that:

$$1 \le p \le q \le N \quad \text{and} \quad \left(\sum_{i=p}^{q} a_i\right) \bmod M = 0$$

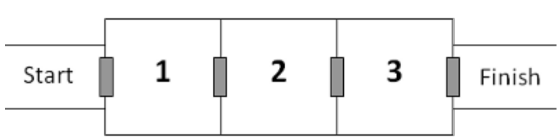
It can be seen that $f(10,10)$=4 with the pairs (3,3), (5,5), (7,9) and (9,10).

You are also given that $f(10^4,10^3)$=97158.

Find $f(10^{12},10^6)$.

Problem 327: Rooms of Doom

A series of three rooms are connected to each other by automatic doors.



Each door is operated by a security card. Once you enter a room the door automatically closes and that security card cannot be used again. A machine at the start will dispense an unlimited number of cards, but each room (including the starting room) contains scanners and if they detect that you are holding more than three security cards or if they detect an unattended security card on the floor, then all the doors will become permanently locked. However, each room contains a box where you may safely store any number of security cards for use at a later stage.

If you simply tried to travel through the rooms one at a time then as you entered room 3 you would have used all three cards and would be trapped in that room forever!

However, if you make use of the storage boxes, then escape is possible. For example, you could enter room 1 using your first card, place one card in the storage box, and use your third card to exit the room back to the start. Then after collecting three more cards from the dispensing machine you could use one to enter room 1 and collect the card you placed in the box a moment ago. You now have three cards again and will be able to travel through the remaining three doors. This method allows you to travel through all three rooms using six security cards in total.

It is possible to travel through six rooms using a total of 123 security cards while carrying a maximum of 3 cards.

Let C be the maximum number of cards which can be carried at any time.

Let R be the number of rooms to travel through.

Let M(C,R) be the minimum number of cards required from the dispensing machine to travel through R rooms carrying up to a maximum of C cards at any time.

For example, M(3,6)=123 and M(4,6)=23.
And, ΣM(C,6)=146 for $3 \leq C \leq 4$.

You are given that ΣM(C,10)=10382 for $3 \leq C \leq 10$.

Find ΣM(C,30) for $3 \leq C \leq 40$.

Problem 328: Lowest-cost Search

We are trying to find a hidden number selected from the set of integers {1, 2, ..., n} by asking questions. Each number (question) we ask, has a cost equal to the number asked and we get one of three possible answers:

- "Your guess is lower than the hidden number", or
- "Yes, that's it!", or
- "Your guess is higher than the hidden number".

Given the value of n, an *optimal strategy* minimizes the total cost (i.e. the sum of all the questions asked) for the worst possible case. E.g.

If n=3, the best we can do is obviously to ask the number "**2**". The answer will immediately lead us to find the hidden number (at a total cost = 2).

If n=8, we might decide to use a "binary search" type of strategy: Our first question would be "**4**" and if the hidden number is higher than 4 we will need one or two additional questions.
Let our second question be "**6**". If the hidden number is still higher than 6, we will need a third question in order to discriminate between 7 and 8.
Thus, our third question will be "**7**" and the total cost for this worst-case scenario will be 4+6+7=**17**.

We can improve considerably the worst-case cost for n=8, by asking "**5**" as our first question.
If we are told that the hidden number is higher than 5, our second question will be "**7**", then we'll know for certain what the hidden number is (for a total cost of 5+7=**12**).
If we are told that the hidden number is lower than 5, our second question will be "**3**" and if the hidden number is lower than 3 our third question will be "**1**", giving a total cost of 5+3+1=**9**.
Since **12**>**9**, the worst-case cost for this strategy is **12**. That's better than what we achieved previously with the "binary search" strategy; it is also better than or equal to any other strategy.
So, in fact, we have just described an optimal strategy for n=8.

Let C(n) be the worst-case cost achieved by an optimal strategy for n, as described above.
Thus C(1) = 0, C(2) = 1, C(3) = 2 and C(8) = 12.
Similarly, C(100) = 400 and $\sum_{n=1}^{100}$ C(n) = 17575.

Find $\sum_{n=1}^{200000}$ C(n).

Problem 329: Prime Frog

Susan has a prime frog.
Her frog is jumping around over 500 squares numbered 1 to 500. He can only jump one square to the left or to the right, with equal probability, and he cannot jump outside the range [1;500].
(if it lands at either end, it automatically jumps to the only available square on the next move.)

When he is on a square with a prime number on it, he croaks 'P' (PRIME) with probability 2/3 or 'N' (NOT PRIME) with probability 1/3 just before jumping to the next square.
When he is on a square with a number on it that is not a prime he croaks 'P' with probability 1/3 or 'N' with probability 2/3 just before jumping to the next square.

Given that the frog's starting position is random with the same probability for every square, and given that she listens to his first 15 croaks, what is the probability that she hears the sequence PPPPNNPPPNPPNPN?

Give your answer as a fraction p/q in reduced form.

Problem 330: Euler's Number

An infinite sequence of real numbers $a(n)$ is defined for all integers n as follows:

$$a(n) = \begin{cases} 1 & n < 0 \\ \displaystyle\sum_{i=1}^{\infty} \frac{a(n-i)}{i!} & n \geq 0 \end{cases}$$

For example,

$$a(0) = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e - 1$$

$$a(1) = \frac{e-1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2e - 3$$

$$a(2) = \frac{2e-3}{1!} + \frac{e-1}{2!} + \frac{1}{3!} + \ldots = \frac{7}{2} e - 6$$

with e = 2.7182818... being Euler's constant.

It can be shown that $a(n)$ is of the form $\dfrac{A(n)\, e + B(n)}{n!}$ for integers $A(n)$ and $B(n)$.

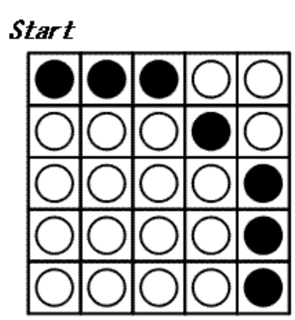
For example $a(10) = \dfrac{328161643\, e - 652694486}{10!}$.

Find $A(10^9) + B(10^9)$ and give your answer mod 77 777 777.

Problem 331: Cross flips

$N \times N$ disks are placed on a square game board. Each disk has a black side and white side.

At each turn, you may choose a disk and flip all the disks in the same row and the same column as this disk: thus $2 \times N$-1 disks are flipped. The game ends when all disks show their white side. The following example shows a game on a 5×5 board.



It can be proven that 3 is the minimal number of turns to finish this game.

The bottom left disk on the $N \times N$ board has coordinates (0,0);
the bottom right disk has coordinates (N-1,0) and the top left disk has coordinates (0,N-1).

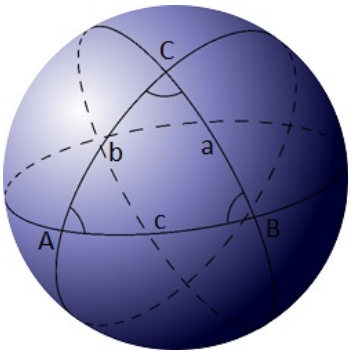
Let C_N be the following configuration of a board with $N \times N$ disks:
A disk at (x,y) satisfying $N-1 \le \sqrt{x^2+y^2} < N$, shows its black side; otherwise, it shows its white side. C_5 is shown above.

Let T(N) be the minimal number of turns to finish a game starting from configuration C_N or 0 if configuration C_N is unsolvable.
We have shown that T(5)=3. You are also given that T(10)=29 and T(1 000)=395253.

Find $\sum_{i=3}^{31} T(2^i - i)$.

Problem 332: Spherical triangles

A **spherical triangle** is a figure formed on the surface of a sphere by three **great circular arcs** intersecting pairwise in three vertices.



Let C(r) be the sphere with the centre (0,0,0) and radius r.
Let Z(r) be the set of points on the surface of C(r) with integer coordinates.
Let T(r) be the set of spherical triangles with vertices in Z(r). Degenerate spherical triangles, formed by three points on the same great arc, are not included in T(r).
Let A(r) be the area of the smallest spherical triangle in T(r).

For example A(14) is 3.294040 rounded to six decimal places.

Find $\sum_{r=1}^{50}$ A(r). Give your answer rounded to six decimal places.

Problem 333: Special partitions

All positive integers can be partitioned in such a way that each and every term of the partition can be expressed as 2^ix3^j, where i,j $\ge$ 0.

Let's consider only those such partitions where none of the terms can divide any of the other terms.
For example, the partition of 17 = 2 + 6 + 9 = (2^1x3^0 + 2^1x3^1 + 2^0x3^2) would not be valid since 2 can divide 6. Neither would the partition 17 = 16 + 1 = (2^4x3^0 + 2^0x3^0) since 1 can divide 16. The only valid partition of 17 would be 8 + 9 = (2^3x3^0 + 2^0x3^2).

Many integers have more than one valid partition, the first being 11 having the following two partitions.
11 = 2 + 9 = (2^1x3^0 + 2^0x3^2)
11 = 8 + 3 = (2^3x3^0 + 2^0x3^1)

Let's define P(n) as the number of valid partitions of n. For example, P(11) = 2.

Let's consider only the prime integers q which would have a single valid partition such as P(17).

The sum of the primes q <100 such that P(q)=1 equals 233.

Find the sum of the primes q <1000000 such that P(q)=1.

Problem 334: Spilling the beans

In Plato's heaven, there exist an infinite number of bowls in a straight line.
Each bowl either contains some or none of a finite number of beans.
A child plays a game, which allows only one kind of move: removing two beans from any bowl, and putting one in each of the two adjacent bowls.
The game ends when each bowl contains either one or no beans.

For example, consider two adjacent bowls containing 2 and 3 beans respectively, all other bowls being empty. The following eight moves will finish the game:

You are given the following sequences:

$t_0 = 123456.$

$$t_i = \begin{cases} \dfrac{t_{i-1}}{2}, & \text{if } t_{i-1} \text{ is even} \\ \left\lfloor \dfrac{t_{i-1}}{2} \right\rfloor \oplus 926252, & \text{if } t_{i-1} \text{ is odd} \end{cases}$$

where $\lfloor x \rfloor$ is the floor function
and $\oplus$ is the bitwise XOR operator.

$b_i = (t_i \bmod 2^{11}) + 1.$

The first two terms of the last sequence are $b_1 = 289$ and $b_2 = 145$.
If we start with b_1 and b_2 beans in two adjacent bowls, 3419100 moves would be required to finish the game.

Consider now 1500 adjacent bowls containing $b_1, b_2, \ldots, b_{1500}$ beans respectively, all other bowls being empty. Find how many moves it takes before the game ends.

Problem 335: Gathering the beans

Whenever Peter feels bored, he places some bowls, containing one bean each, in a circle. After this, he takes all the beans out of a certain bowl and drops them one by one in the bowls going clockwise. He repeats this, starting from the bowl he dropped the last bean in, until the initial situation appears again. For example with 5 bowls he acts as follows:

So with 5 bowls it takes Peter 15 moves to return to the initial situation.

Let $M(x)$ represent the number of moves required to return to the initial situation, starting with x bowls. Thus, $M(5) = 15$. It can also be verified that $M(100) = 10920$.

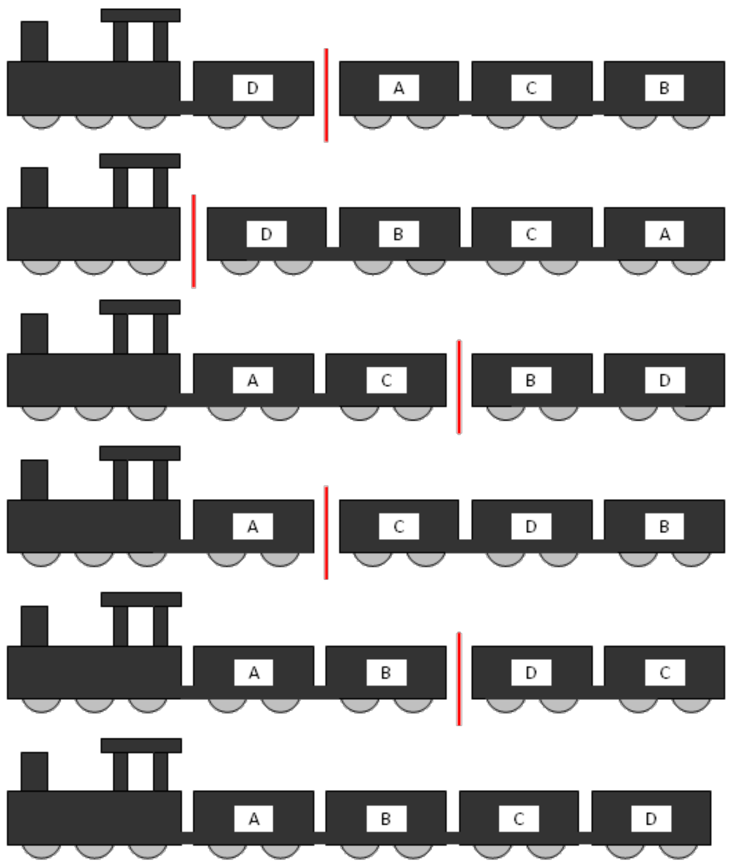
Find $\sum_{k=0}^{10^{18}} M(2^k+1)$. Give your answer modulo 7^9.

Problem 336: Maximix Arrangements

A train is used to transport four carriages in the order: ABCD. However, sometimes when the train arrives to collect the carriages they are not in the correct order.
To rearrange the carriages they are all shunted on to a large rotating turntable. After the carriages are uncoupled at a specific point the train moves off the turntable pulling the carriages still attached with it. The remaining carriages are rotated 180 degrees. All of the carriages are then rejoined and this process is repeated as often as necessary in order to obtain the least number of uses of the turntable.
Some arrangements, such as ADCB, can be solved easily: the carriages are separated between A and D, and after DCB are rotated the correct order has been achieved.

However, Simple Simon, the train driver, is not known for his efficiency, so he always solves the problem by initially getting carriage A in the correct place, then carriage B, and so on.

Using four carriages, the worst possible arrangements for Simon, which we shall call *maximix arrangements*, are DACB and DBAC; each requiring him five rotations (although, using the most efficient approach, they could be solved using just three rotations). The process he uses for DACB is shown below.



It can be verified that there are 24 maximix arrangements for six carriages, of which the tenth lexicographic maximix arrangement is DFAECB.

Find the 2011^{th} lexicographic maximix arrangement for eleven carriages.

Problem 337: Totient Stairstep Sequences

Let $\{a_1, a_2, \ldots, a_n\}$ be an integer sequence of length n such that:

- $a_1 = 6$
- for all $1 \le i < n$: $\varphi(a_i) < \varphi(a_{i+1}) < a_i < a_{i+1}$ [1]

Let $S(N)$ be the number of such sequences with $a_n \le N$.
For example, S(10) = 4: {6}, {6, 8}, {6, 8, 9} and {6, 10}.
We can verify that S(100) = 482073668 and S(10 000) mod 10^8 = 73808307.

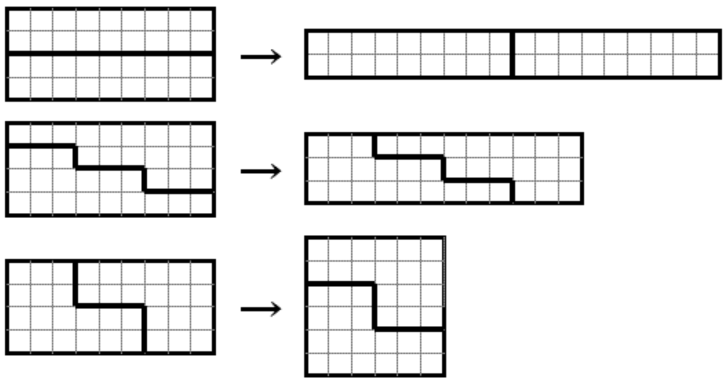
Find S(20 000 000) mod 10^8.

[1] φ denotes **Euler's totient function**.

Problem 338: Cutting Rectangular Grid Paper

A rectangular sheet of grid paper with integer dimensions $w \times h$ is given. Its grid spacing is 1.
When we cut the sheet along the grid lines into two pieces and rearrange those pieces without overlap, we can make new rectangles with different dimensions.

For example, from a sheet with dimensions 9×4 , we can make rectangles with dimensions 18×2, 12×3 and 6×6 by cutting and rearranging as below:



Similarly, from a sheet with dimensions 9×8 , we can make rectangles with dimensions 18×4 and 12×6 .

For a pair w and h, let F(w,h) be the number of distinct rectangles that can be made from a sheet with dimensions $w \times h$.
For example, F(2,1) = 0, F(2,2) = 1, F(9,4) = 3 and F(9,8) = 2.
Note that rectangles congruent to the initial one are not counted in F(w,h).
Note also that rectangles with dimensions $w \times h$ and dimensions $h \times w$ are not considered distinct.

For an integer N, let G(N) be the sum of F(w,h) for all pairs w and h which satisfy $0 < h \le w \le N$.
We can verify that G(10) = 55, G(10^3) = 971745 and G(10^5) = 9992617687.

Find G(10^{12}). Give your answer modulo 10^8.

Problem 339: Peredur fab Efrawg

"And he came towards a valley, through which ran a river; and the borders of the valley were wooded, and on each side of the river were level meadows. And on one side of the river he saw a flock of white sheep, and on the other a flock of black sheep. And whenever one of the white sheep bleated, one of the black sheep would cross over and become white; and when one of the black sheep bleated, one of the white sheep would cross over and become black."
en.wikisource.org

Initially each flock consists of n sheep. Each sheep (regardless of colour) is equally likely to be the next sheep to bleat. After a sheep has bleated and a sheep from the other flock has crossed over, Peredur may remove a number of white sheep in order to maximize the expected final number of black sheep. Let E(n) be the expected final number of black sheep if Peredur uses an optimal strategy.

You are given that E(5) = 6.871346 rounded to 6 places behind the decimal point.
Find E(10 000) and give your answer rounded to 6 places behind the decimal point.

Problem 340: Crazy Function

For fixed integers a, b, c, define the *crazy function* F(n) as follows:
F(n) = n - c for all n > b
F(n) = F(a + F(a + F(a + F(a + n)))) for all $n \le$ b.

Also, define S(a, b, c) = $\sum_{n=0}^{b} F(n)$.

For example, if a = 50, b = 2000 and c = 40, then F(0) = 3240 and F(2000) = 2040.
Also, S(50, 2000, 40) = 5204240.

Find the last 9 digits of S(21^7, 7^{21}, 12^7).

Problem 341: Golomb's self-describing sequence

The **Golomb's self-describing sequence** {G(n)} is the only nondecreasing sequence of natural numbers such that n appears exactly G(n) times in the sequence. The values of G(n) for the first few n are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	…
G(n)	1	2	2	3	3	4	4	4	5	5	5	6	6	6	6	…

You are given that G(10^3) = 86, G(10^6) = 6137.
You are also given that ΣG(n^3) = 153506976 for $1 \le n < 10^3$.

Find ΣG(n^3) for $1 \le n < 10^6$.

Problem 342: The totient of a square is a cube

Consider the number 50.
$50^2 = 2500 = 2^2 \times 5^4$, so $\varphi(2500) = 2 \times 4 \times 5^3 = 8 \times 5^3 = 2^3 \times 5^3$. [1]
So 2500 is a square and φ(2500) is a cube.

Find the sum of all numbers n, $1 < n < 10^{10}$ such that $\varphi(n^2)$ is a cube.

[1] φ denotes **Euler's totient function**.

Problem 343: Fractional Sequences

For any positive integer k, a finite sequence a_i of fractions x_i/y_i is defined by:
$a_1 = 1/k$ and
$a_i = (x_{i-1}+1)/(y_{i-1}-1)$ reduced to lowest terms for i>1.
When a_i reaches some integer n, the sequence stops. (That is, when y_i=1.)
Define f(k) = n.
For example, for k = 20:

$1/20 \to 2/19 \to 3/18 = 1/6 \to 2/5 \to 3/4 \to 4/3 \to 5/2 \to 6/1 = 6$

So f(20) = 6.

Also f(1) = 1, f(2) = 2, f(3) = 1 and Σf(k^3) = 118937 for $1 \le k \le 100$.

Find Σf(k^3) for $1 \le k \le 2\times10^6$.

Problem 344: Silver dollar game

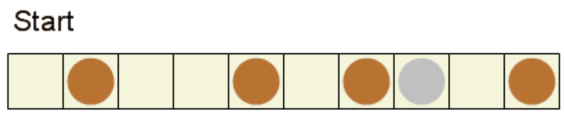
One variant of N.G. de Bruijn's **silver dollar** game can be described as follows:

On a strip of squares a number of coins are placed, at most one coin per square. Only one coin, called the **silver dollar**, has any value. Two players take turns making moves. At each turn a player must make either a *regular* or a *special* move.

A *regular* move consists of selecting one coin and moving it one or more squares to the left. The coin cannot move out of the strip or jump on or over another coin.

Alternatively, the player can choose to make the *special* move of pocketing the leftmost coin rather than making a regular move. If no regular moves are possible, the player is forced to pocket the leftmost coin.

The winner is the player who pockets the silver dollar.



A *winning configuration* is an arrangement of coins on the strip where the first player can force a win no matter what the second player does.

Let W(n,c) be the number of winning configurations for a strip of n squares, c worthless coins and one silver dollar.

You are given that W(10,2) = 324 and W(100,10) = 1514704946113500.

Find W(1 000 000, 100) modulo the semiprime 1000 036 000 099 (= 1 000 003 · 1 000 033).

Problem 350: Constraining the least greatest and the greatest least

A *list of size n* is a sequence of n natural numbers.
Examples are (2,4,6), (2,6,4), (10,6,15,6), and (11).

The **greatest common divisor**, or gcd, of a list is the largest natural number that divides all entries of the list.
Examples: gcd(2,6,4) = 2, gcd(10,6,15,6) = 1 and gcd(11) = 11.

The **least common multiple**, or lcm, of a list is the smallest natural number divisible by each entry of the list.
Examples: lcm(2,6,4) = 12, lcm(10,6,15,6) = 30 and lcm(11) = 11.

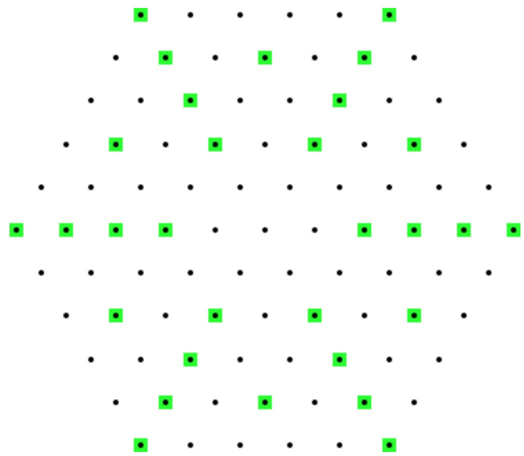
Let f(G, L, N) be the number of lists of size N with gcd $\geq G$ and lcm $\leq L$. For example:

f(10, 100, 1) = 91.
f(10, 100, 2) = 327.
f(10, 100, 3) = 1135.
f(10, 100, 1000) mod 101^4 = 3286053.

Find f(10^6, 10^{12}, 10^{18}) mod 101^4.

Problem 351: Hexagonal orchards

A *hexagonal orchard* of order n is a triangular lattice made up of points within a regular hexagon with side n. The following is an example of a hexagonal orchard of order 5:



Highlighted in green are the points which are hidden from the center by a point closer to it. It can be seen that for a hexagonal orchard of order 5, 30 points are hidden from the center.

Let H(n) be the number of points hidden from the center in a hexagonal orchard of order n.

H(5) = 30. H(10) = 138. H(1 000) = 1177848.

Find H(100 000 000).

Problem 352: Blood tests

Each one of the 25 sheep in a flock must be tested for a rare virus, known to affect 2% of the sheep population. An accurate and extremely sensitive PCR test exists for blood samples, producing a clear positive / negative result, but it is very time-consuming and expensive.

Because of the high cost, the vet-in-charge suggests that instead of performing 25 separate tests, the following procedure can be used instead:

The sheep are split into 5 groups of 5 sheep in each group. For each group, the 5 samples are mixed together and a single test is performed. Then,

- If the result is negative, all the sheep in that group are deemed to be virus-free.
- If the result is positive, 5 additional tests will be performed (a separate test for each animal) to determine the affected individual(s).

Since the probability of infection for any specific animal is only 0.02, the first test (on the pooled samples) for each group will be:

- Negative (and no more tests needed) with probability 0.98^5 = 0.9039207968.
- Positive (5 additional tests needed) with probability 1 - 0.9039207968 = 0.0960792032.

Thus, the expected number of tests for each group is 1 + 0.0960792032 × 5 = 1.480396016.
Consequently, all 5 groups can be screened using an average of only 1.480396016 × 5 = **7.40198008** tests, which represents a huge saving of more than 70% !

Although the scheme we have just described seems to be very efficient, it can still be improved considerably (always assuming that the test is sufficiently sensitive and that there are no adverse effects caused by mixing different samples). E.g.:

- We may start by running a test on a mixture of all the 25 samples. It can be verified that in about 60.35% of the cases this test will be negative, thus no more tests will be needed. Further testing will only be required for the remaining 39.65% of the cases.
- If we know that at least one animal in a group of 5 is infected and the first 4 individual tests come out negative, there is no need to run a test on the fifth animal (we know that it must be infected).
- We can try a different number of groups / different number of animals in each group, adjusting those numbers at each level so that the total expected number of tests will be minimised.

To simplify the very wide range of possibilities, there is one restriction we place when devising the most cost-efficient testing scheme: whenever we start with a mixed sample, all the sheep contributing to that sample must be fully screened (i.e. a verdict of infected / virus-free must be reached for all of them) before we start examining any other animals.

For the current example, it turns out that the most cost-efficient testing scheme (we'll call it the *optimal strategy*) requires an average of just **4.155452** tests!

Using the optimal strategy, let T(s,p) represent the average number of tests needed to screen a flock of s sheep for a virus having probability p to be present in any individual.
Thus, rounded to six decimal places, T(25, 0.02) = 4.155452 and T(25, 0.10) = 12.702124.

Find ΣT(10000, p) for p=0.01, 0.02, 0.03, ... 0.50.
Give your answer rounded to six decimal places.

Problem 353: Risky moon

A moon could be described by the sphere C(r) with centre (0,0,0) and radius r.

There are stations on the moon at the points on the surface of C(r) with integer coordinates. The station at (0,0,r) is called North Pole station, the station at (0,0,-r) is called South Pole station.

All stations are connected with each other via the shortest road on the great arc through the stations. A journey between two stations is risky. If d is the length of the road between two stations, $(d/(\pi r))^2$ is a measure for the risk of the journey (let us call it the risk of the road). If the journey includes more than two stations, the risk of the journey is the sum of risks of the used roads.

A direct journey from the North Pole station to the South Pole station has the length πr and risk 1. The journey from the North Pole station to the South Pole station via (0,r,0) has the same length, but a smaller risk: $(½\pi r/(\pi r))^2+(½\pi r/(\pi r))^2=0.5$.

The minimal risk of a journey from the North Pole station to the South Pole station on C(r) is M(r).

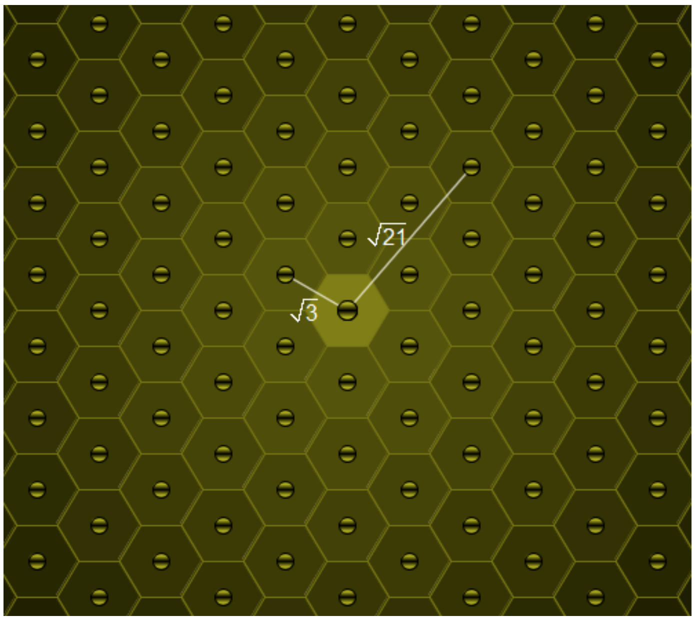
You are given that M(7)=0.1784943998 rounded to 10 digits behind the decimal point.

Find $\sum M(2^n-1)$ for $1 \le n \le 15$.

Give your answer rounded to 10 digits behind the decimal point in the form a.bcdefghijk.

Problem 354: Distances in a bee's honeycomb

Consider a honey bee's honeycomb where each cell is a perfect regular hexagon with side length 1.



One particular cell is occupied by the queen bee.
For a positive real number L, let B(L) count the cells with distance L from the queen bee cell (all distances are measured from centre to centre); you may assume that the honeycomb is large enough to accommodate for any distance we wish to consider.
For example, B($\sqrt{3}$) = 6, B($\sqrt{21}$) = 12 and B(111 111 111) = 54.

Find the number of $L \le 5\cdot10^{11}$ such that B(L) = 450.

Problem 355: Maximal coprime subset

Define Co(n) to be the maximal possible sum of a set of mutually co-prime elements from {1, 2, ..., n}.
For example Co(10) is 30 and hits that maximum on the subset {1, 5, 7, 8, 9}.

You are given that Co(30) = 193 and Co(100) = 1356.

Find Co(200000).

Problem 356: Largest roots of cubic polynomials

Let a_n be the largest real root of a polynomial $g(x) = x^3 - 2^n \cdot x^2 + n$.
For example, $a_2 = 3.86619826...$

Find the last eight digits of $\sum_{i=1}^{30} \lfloor a_i^{987654321} \rfloor$.

Note: $\lfloor a \rfloor$ represents the floor function.

Problem 357: Prime generating integers

Consider the divisors of 30: 1,2,3,5,6,10,15,30.
It can be seen that for every divisor d of 30, $d+30/d$ is prime.

Find the sum of all positive integers n not exceeding 100 000 000
such that for every divisor d of n, $d+n/d$ is prime.

Problem 358: Cyclic numbers

A **cyclic number** with n digits has a very interesting property:
When it is multiplied by 1, 2, 3, 4, ... n, all the products have exactly the same digits, in the same order, but rotated in a circular fashion!

The smallest cyclic number is the 6-digit number 142857 :
142857 × 1 = 142857
142857 × 2 = 285714
142857 × 3 = 428571
142857 × 4 = 571428
142857 × 5 = 714285
142857 × 6 = 857142

The next cyclic number is 0588235294117647 with 16 digits :
0588235294117647 × 1 = 0588235294117647
0588235294117647 × 2 = 1176470588235294
0588235294117647 × 3 = 1764705882352941
...
0588235294117647 × 16 = 9411764705882352

Note that for cyclic numbers, leading zeros are important.

There is only one cyclic number for which, the eleven leftmost digits are 00000000137 and the five rightmost digits are 56789 (i.e., it has the form 00000000137...56789 with an unknown number of digits in the middle). Find the sum of all its digits.

Problem 359: Hilbert's New Hotel

An infinite number of people (numbered 1, 2, 3, etc.) are lined up to get a room at Hilbert's newest infinite hotel. The hotel contains an infinite number of floors (numbered 1, 2, 3, etc.), and each floor contains an infinite number of rooms (numbered 1, 2, 3, etc.).

Initially the hotel is empty. Hilbert declares a rule on how the n^{th} person is assigned a room: person n gets the first vacant room in the lowest numbered floor satisfying either of the following:

- the floor is empty
- the floor is not empty, and if the latest person taking a room in that floor is person m, then $m + n$ is a perfect square

Person 1 gets room 1 in floor 1 since floor 1 is empty.
Person 2 does not get room 2 in floor 1 since 1 + 2 = 3 is not a perfect square.
Person 2 instead gets room 1 in floor 2 since floor 2 is empty.
Person 3 gets room 2 in floor 1 since 1 + 3 = 4 is a perfect square.

Eventually, every person in the line gets a room in the hotel.

Define P(f, r) to be n if person n occupies room r in floor f, and 0 if no person occupies the room. Here are a few examples:
P(1, 1) = 1
P(1, 2) = 3
P(2, 1) = 2
P(10, 20) = 440
P(25, 75) = 4863
P(99, 100) = 19454

Find the sum of all P(f, r) for all positive f and r such that $f \times r$ = 71328803586048 and give the last 8 digits as your answer.

Problem 360: Scary Sphere

Given two points (x_1,y_1,z_1) and (x_2,y_2,z_2) in three dimensional space, the **Manhattan distance** between those points is defined as $|x_1-x_2|+|y_1-y_2|+|z_1-z_2|$.

Let C(r) be a sphere with radius r and center in the origin O(0,0,0).
Let I(r) be the set of all points with integer coordinates on the surface of C(r).
Let S(r) be the sum of the Manhattan distances of all elements of I(r) to the origin O.

E.g. S(45)=34518.

Find S(10^{10}).

Problem 361: Subsequence of Thue-Morse sequence

The **Thue-Morse sequence** {T_n} is a binary sequence satisfying:

- $T_0 = 0$
- $T_{2n} = T_n$
- $T_{2n+1} = 1 - T_n$

The first several terms of {T_n} are given as follows:
01101001100101101001011001101001....

We define {A_n} as the sorted sequence of integers such that the binary expression of each element appears as a subsequence in {T_n}.
For example, the decimal number 18 is expressed as 10010 in binary. 10010 appears in {T_n} (T_8 to T_{12}), so 18 is an element of {A_n}.
The decimal number 14 is expressed as 1110 in binary. 1110 never appears in {T_n}, so 14 is not an element of {A_n}.

The first several terms of A_n are given as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	…
A_n	0	1	2	3	4	5	6	9	10	11	12	13	18	…

We can also verify that A_{100} = 3251 and A_{1000} = 80852364498.

Find the last 9 digits of $\sum_{k=1}^{18} A_{10^k}$.

Problem 362: Squarefree factors

Consider the number 54.

54 can be factored in 7 distinct ways into one or more factors larger than 1:
54, 2×27, 3×18, 6×9, 3×3×6, 2×3×9 and 2×3×3×3.
If we require that the factors are all squarefree only two ways remain: 3×3×6 and 2×3×3×3.

Let's call Fsf(n) the number of ways n can be factored into one or more squarefree factors larger than 1, so Fsf(54)=2.

Let S(n) be $\sum$Fsf(k) for k=2 to n.

S(100)=193.

Find S(10 000 000 000).

Problem 363: Bézier Curves

A cubic Bézier curve is defined by four points: P_0, P_1, P_2 and P_3.

The curve is constructed as follows:
On the segments P_0P_1, P_1P_2 and P_2P_3 the points Q_0,Q_1 and Q_2 are drawn such that $P_0Q_0/P_0P_1=P_1Q_1/P_1P_2=P_2Q_2/P_2P_3$=t (t in [0,1]).
On the segments Q_0Q_1 and Q_1Q_2 the points R_0 and R_1 are drawn such that $Q_0R_0/Q_0Q_1=Q_1R_1/Q_1Q_2$=t for the same value of t.
On the segment R_0R_1 the point B is drawn such that R_0B/R_0R_1=t for the same value of t.
The Bézier curve defined by the points P_0, P_1, P_2, P_3 is the locus of B as Q_0 takes all possible positions on the segment P_0P_1. (Please note that for all points the value of t is the same.)

In the applet to the right you can drag the points P_0, P_1, P_2 and P_3 to see what the Bézier curve (green curve) defined by those points looks like. You can also drag the point Q_0 along the segment P_0P_1.

From the construction it is clear that the Bézier curve will be tangent to the segments P_0P_1 in P_0 and P_2P_3 in P_3.

A cubic Bézier curve with P_0=(1,0), P_1=(1,v), P_2=(v,1) and P_3=(0,1) is used to approximate a quarter circle.
The value v>0 is chosen such that the area enclosed by the lines OP_0, OP_3 and the curve is equal to $^{\pi}/_4$ (the area of the quarter circle).

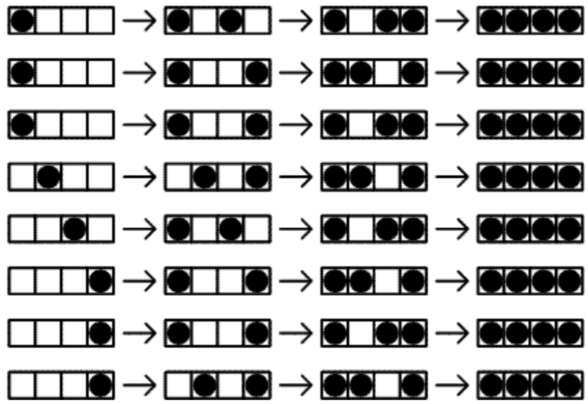
By how many percent does the length of the curve differ from the length of the quarter circle?
That is, if L is the length of the curve, calculate $100*^{(L-\pi/2)}/_{(\pi/2)}$.
Give your answer rounded to 10 digits behind the decimal point.

Problem 364: Comfortable distance

There are N seats in a row. N people come after each other to fill the seats according to the following rules:

1. If there is any seat whose adjacent seat(s) are not occupied take such a seat.
2. If there is no such seat and there is any seat for which only one adjacent seat is occupied take such a seat.
3. Otherwise take one of the remaining available seats.

Let T(N) be the number of possibilities that N seats are occupied by N people with the given rules.
The following figure shows T(4)=8.



We can verify that T(10) = 61632 and T(1 000) mod 100 000 007 = 47255094.

Find T(1 000 000) mod 100 000 007.

Problem 366: Stone Game III

Two players, Anton and Bernhard, are playing the following game.
There is one pile of n stones.
The first player may remove any positive number of stones, but not the whole pile.
Thereafter, each player may remove at most twice the number of stones his opponent took on the previous move.
The player who removes the last stone wins.

E.g. n=5
If the first player takes anything more than one stone the next player will be able to take all remaining stones.
If the first player takes one stone, leaving four, his opponent will take also one stone, leaving three stones.
The first player cannot take all three because he may take at most 2x1=2 stones. So let's say he takes also one stone, leaving 2. The second player can take the two remaining stones and wins.
So 5 is a losing position for the first player.
For some winning positions there is more than one possible move for the first player.
E.g. when n=17 the first player can remove one or four stones.

Let M(n) be the maximum number of stones the first player can take from a winning position *at his first turn* and M(n)=0 for any other position.

$\sum$M(n) for n$\leq$100 is 728.

Find $\sum$M(n) for n$\leq 10^{18}$. Give your answer modulo 10^8.

Problem 367: bozo sort

Bozo sort, not to be confused with the slightly less efficient **bogo sort**, consists out of checking if the input sequence is sorted and if not swapping randomly two elements. This is repeated until eventually the sequence is sorted.

If we consider all permutations of the first 4 natural numbers as input the expectation value of the number of swaps, averaged over all 4! input sequences is 24.75.
The already sorted sequence takes 0 steps.

In this problem we consider the following variant on bozo sort.
If the sequence is not in order we pick three elements at random and shuffle these three elements randomly.
All 3!=6 permutations of those three elements are equally likely.
The already sorted sequence will take 0 steps.
If we consider all permutations of the first 4 natural numbers as input the expectation value of the number of shuffles, averaged over all 4! input sequences is 27.5.
Consider as input sequences the permutations of the first 11 natural numbers.
Averaged over all 11! input sequences, what is the expected number of shuffles this sorting algorithm will perform?

Give your answer rounded to the nearest integer.

Problem 368: A Kempner-like series

The **harmonic series** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is well known to be divergent.

If we however omit from this series every term where the denominator has a 9 in it, the series remarkably enough converges to approximately 22.9206766193.
This modified harmonic series is called the **Kempner** series.

Let us now consider another modified harmonic series by omitting from the harmonic series every term where the denominator has 3 or more equal consecutive digits. One can verify that out of the first 1200 terms of the harmonic series, only 20 terms will be omitted.
These 20 omitted terms are:

$\frac{1}{111}, \frac{1}{222}, \frac{1}{333}, \frac{1}{444}, \frac{1}{555}, \frac{1}{666}, \frac{1}{777}, \frac{1}{888}, \frac{1}{999}, \frac{1}{1000}, \frac{1}{1110},$

$\frac{1}{1111}, \frac{1}{1112}, \frac{1}{1113}, \frac{1}{1114}, \frac{1}{1115}, \frac{1}{1116}, \frac{1}{1117}, \frac{1}{1118}$ and $\frac{1}{1119}$.

This series converges as well.

Find the value the series converges to.
Give your answer rounded to 10 digits behind the decimal point.

Problem 369: Badugi

In a standard 52 card deck of playing cards, a set of 4 cards is a **Badugi** if it contains 4 cards with no pairs and no two cards of the same suit.

Let f(n) be the number of ways to choose n cards with a 4 card subset that is a Badugi. For example, there are 2598960 ways to choose five cards from a standard 52 card deck, of which 514800 contain a 4 card subset that is a Badugi, so f(5) = 514800.

Find $\sum$f(n) for $4 \le n \le 13$.

Problem 370: Geometric triangles

Let us define a *geometric triangle* as an integer sided triangle with sides $a \le b \le c$ so that its sides form a **geometric progression**, i.e. $b^2 = a \cdot c$.

An example of such a geometric triangle is the triangle with sides a = 144, b = 156 and c = 169.

There are 861805 geometric triangles with perimeter $\le 10^6$.

How many geometric triangles exist with perimeter $\le 2.5 \cdot 10^{13}$?

Problem 371: Licence plates

Oregon licence plates consist of three letters followed by a three digit number (each digit can be from [0..9]).
While driving to work Seth plays the following game:
Whenever the numbers of two licence plates seen on his trip add to 1000 that's a win.

E.g. MIC-012 and HAN-988 is a win and RYU-500 and SET-500 too. (as long as he sees them in the same trip).

Find the expected number of plates he needs to see for a win.
Give your answer rounded to 8 decimal places behind the decimal point.

Note: We assume that each licence plate seen is equally likely to have any three digit number on it.

Problem 372: Pencils of rays

Let R(M, N) be the number of lattice points (x, y) which satisfy $M<x\le N$, $M<y\le N$ and $\left\lfloor \frac{y^2}{x^2} \right\rfloor$ is odd.

We can verify that R(0, 100) = 3019 and R(100, 10000) = 29750422.
Find R(2·10^6, 10^9).

Note: $\lfloor x \rfloor$ represents the floor function.

Problem 373: Circumscribed Circles

Every triangle has a circumscribed circle that goes through the three vertices. Consider all integer sided triangles for which the radius of the circumscribed circle is integral as well.

Let S(n) be the sum of the radii of the circumscribed circles of all such triangles for which the radius does not exceed n.

S(100)=4950 and S(1200)=1653605.

Find S(10^7).

Problem 374: Maximum Integer Partition Product

An integer partition of a number n is a way of writing n as a sum of positive integers.

Partitions that differ only in the order of their summands are considered the same. A partition of n into **distinct parts** is a partition of n in which every part occurs at most once.

The partitions of 5 into distinct parts are:
5, 4+1 and 3+2.

Let f(n) be the maximum product of the parts of any such partition of n into distinct parts and let m(n) be the number of elements of any such partition of n with that product.

So f(5)=6 and m(5)=2.

For n=10 the partition with the largest product is 10=2+3+5, which gives f(10)=30 and m(10)=3.
And their product, f(10)·m(10) = 30·3 = 90

It can be verified that
$\sum$f(n)·m(n) for 1 ≤ n ≤ 100 = 1683550844462.

Find $\sum$f(n)·m(n) for 1 ≤ n ≤ 10^{14}.
Give your answer modulo 982451653, the 50 millionth prime.

Problem 375: Minimum of subsequences

Let S_n be an integer sequence produced with the following pseudo-random number generator:

$$S_0 = 290797$$
$$S_{n+1} = S_n^2 \bmod 50515093$$

Let A(i, j) be the minimum of the numbers $S_i, S_{i+1}, \ldots, S_j$ for $i \le j$.
Let M(N) = ΣA(i, j) for $1 \le i \le j \le N$.
We can verify that M(10) = 432256955 and M(10 000) = 3264567774119.

Find M(2 000 000 000).

Problem 376: Nontransitive sets of dice

Consider the following set of dice with nonstandard pips:

Die A: 1 4 4 4 4 4
Die B: 2 2 2 5 5 5
Die C: 3 3 3 3 3 6

A game is played by two players picking a die in turn and rolling it. The player who rolls the highest value wins.

If the first player picks die A and the second player picks die B we get
P(second player wins) = $^7/_{12} > {}^1/_2$

If the first player picks die B and the second player picks die C we get
P(second player wins) = $^7/_{12} > {}^1/_2$

If the first player picks die C and the second player picks die A we get
P(second player wins) = $^{25}/_{36} > {}^1/_2$

So whatever die the first player picks, the second player can pick another die and have a larger than 50% chance of winning.
A set of dice having this property is called a **nontransitive set of dice**.

We wish to investigate how many sets of nontransitive dice exist. We will assume the following conditions:

- There are three six-sided dice with each side having between 1 and N pips, inclusive.
- Dice with the same set of pips are equal, regardless of which side on the die the pips are located.
- The same pip value may appear on multiple dice; if both players roll the same value neither player wins.
- The sets of dice {A,B,C}, {B,C,A} and {C,A,B} are the same set.

For $N = 7$ we find there are 9780 such sets.
How many are there for $N = 30$?

Problem 377: Sum of digits, experience 13

There are 16 positive integers that do not have a zero in their digits and that have a digital sum equal to 5, namely:
5, 14, 23, 32, 41, 113, 122, 131, 212, 221, 311, 1112, 1121, 1211, 2111 and 11111.
Their sum is 17891.

Let $f(n)$ be the sum of all positive integers that do not have a zero in their digits and have a digital sum equal to n.

Find $\sum_{i=1}^{17} f(13^i)$.

Give the last 9 digits as your answer.

Problem 378: Triangle Triples

Let T(n) be the n^{th} triangle number, so T(n) = $\frac{n(n+1)}{2}$.

Let dT(n) be the number of divisors of T(n).
E.g.: T(7) = 28 and dT(7) = 6.

Let Tr(n) be the number of triples (i, j, k) such that $1 \le i < j < k \le n$ and dT(i) > dT(j) > dT(k).
Tr(20) = 14, Tr(100) = 5772 and Tr(1000) = 11174776.

Find Tr(60 000 000).
Give the last 18 digits of your answer.

Problem 379: Least common multiple count

Let $f(n)$ be the number of couples (x,y) with x and y positive integers, $x \le y$ and the least common multiple of x and y equal to n.

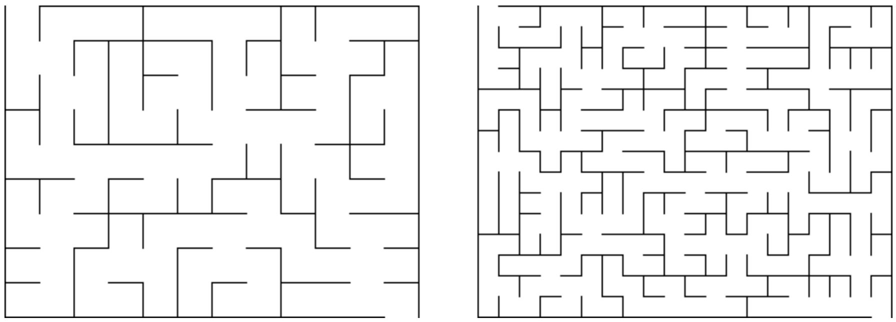
Let g be the **summatory function** of f, i.e.: $g(n) = \sum f(i)$ for $1 \le i \le n$.

You are given that $g(10^6) = 37429395$.

Find $g(10^{12})$.

Problem 380: Amazing Mazes!

An m×n maze is an m×n rectangular grid with walls placed between grid cells such that there is exactly one path from the top-left square to any other square.
The following are examples of a 9×12 maze and a 15×20 maze:



Let C(m,n) be the number of distinct m×n mazes. Mazes which can be formed by rotation and reflection from another maze are considered distinct.

It can be verified that C(1,1) = 1, C(2,2) = 4, C(3,4) = 2415, and C(9,12) = 2.5720e46 (in scientific notation rounded to 5 significant digits).
Find C(100,500) and write your answer in scientific notation rounded to 5 significant digits.

When giving your answer, use a lowercase e to separate mantissa and exponent. E.g. if the answer is 1234567891011 then the answer format would be 1.2346e12.

Problem 381: (prime-k) factorial

For a prime p let S(p) = ($\sum$(p-k)!) mod(p) for $1 \le k \le 5$.

For example, if p=7,
(7-1)! + (7-2)! + (7-3)! + (7-4)! + (7-5)! = 6! + 5! + 4! + 3! + 2! = 720+120+24+6+2 = 872.
As 872 mod(7) = 4, S(7) = 4.

It can be verified that $\sum$S(p) = 480 for $5 \le p < 100$.

Find $\sum$S(p) for $5 \le p < 10^8$.

Problem 382: Generating polygons

A **polygon** is a flat shape consisting of straight line segments that are joined to form a closed chain or circuit. A polygon consists of at least three sides and does not self-intersect.

A set S of positive numbers is said to *generate a polygon* P if:

- no two sides of P are the same length,
- the length of every side of P is in S, and
- S contains no other value.

For example:
The set {3, 4, 5} generates a polygon with sides 3, 4, and 5 (a triangle).
The set {6, 9, 11, 24} generates a polygon with sides 6, 9, 11, and 24 (a quadrilateral).
The sets {1, 2, 3} and {2, 3, 4, 9} do not generate any polygon at all.

Consider the sequence s, defined as follows:

- $s_1 = 1$, $s_2 = 2$, $s_3 = 3$
- $s_n = s_{n-1} + s_{n-3}$ for $n > 3$.

Let U_n be the set $\{s_1, s_2, ..., s_n\}$. For example, U_{10} = {1, 2, 3, 4, 6, 9, 13, 19, 28, 41}.
Let f(n) be the number of subsets of U_n which generate at least one polygon.
For example, f(5) = 7, f(10) = 501 and f(25) = 18635853.

Find the last 9 digits of f(10^{18}).

Problem 383: Divisibility comparison between factorials

Let $f_5(n)$ be the largest integer x for which 5^x divides n.
For example, $f_5(625000) = 7$.

Let $T_5(n)$ be the number of integers i which satisfy $f_5((2\cdot i-1)!) < 2\cdot f_5(i!)$ and $1 \leq i \leq n$.
It can be verified that $T_5(10^3) = 68$ and $T_5(10^9) = 2408210$.

Find $T_5(10^{18})$.

Problem 384: Rudin-Shapiro sequence

Define the sequence a(n) as the number of adjacent pairs of ones in the binary expansion of n (possibly overlapping).
E.g.: $a(5) = a(101_2) = 0$, $a(6) = a(110_2) = 1$, $a(7) = a(111_2) = 2$

Define the sequence $b(n) = (-1)^{a(n)}$.
This sequence is called the **Rudin-Shapiro** sequence.

Also consider the summatory sequence of b(n): $s(n)=\sum_{i=0}^{n} b(i)$.

The first couple of values of these sequences are:

n	0	1	2	3	4	5	6	7
a(n)	0	0	0	1	0	0	1	2
b(n)	1	1	1	-1	1	1	-1	1
s(n)	1	2	3	2	3	4	3	4

The sequence s(n) has the remarkable property that all elements are positive and every positive integer k occurs exactly k times.

Define g(t,c), with $1 \leq c \leq t$, as the index in s(n) for which t occurs for the c'th time in s(n).
E.g.: g(3,3) = 6, g(4,2) = 7 and g(54321,12345) = 1220847710.

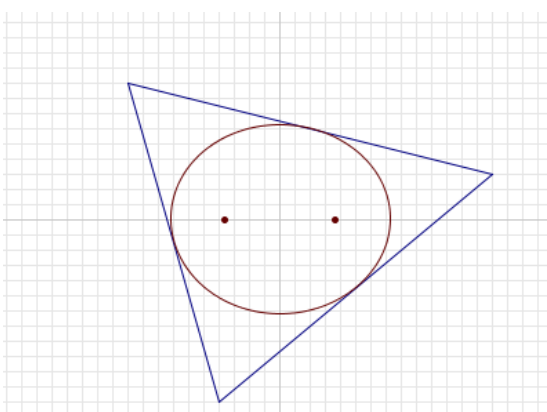
Let F(n) be the fibonacci sequence defined by:
F(0)=F(1)=1 and
F(n)=F(n-1)+F(n-2) for n>1.

Define GF(t)=g(F(t),F(t-1)).

Find ΣGF(t) for $2 \leq t \leq 45$.

Problem 385: Ellipses inside triangles

For any triangle T in the plane, it can be shown that there is a unique ellipse with largest area that is completely inside T.



For a given n, consider triangles T such that:
- the vertices of T have integer coordinates with absolute value $\leq$ n, and
- the **foci**[1] of the largest-area ellipse inside T are ($\sqrt{13}$,0) and ($-\sqrt{13}$,0).

Let A(n) be the sum of the areas of all such triangles.

For example, if n = 8, there are two such triangles. Their vertices are (-4,-3),(-4,3),(8,0) and (4,3),(4,-3),(-8,0), and the area of each triangle is 36. Thus A(8) = 36 + 36 = 72.

It can be verified that A(10) = 252, A(100) = 34632 and A(1000) = 3529008.

Find A(1 000 000 000).

[1]The **foci** (plural of **focus**) of an ellipse are two points A and B such that for every point P on the boundary of the ellipse, $AP + PB$ is constant.

Problem 386: Maximum length of an antichain

Let n be an integer and $S(n)$ be the set of factors of n.

A subset A of $S(n)$ is called an **antichain** of $S(n)$ if A contains only one element or if none of the elements of A divides any of the other elements of A.

For example: $S(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$
$\{2, 5, 6\}$ is not an antichain of $S(30)$.
$\{2, 3, 5\}$ is an antichain of $S(30)$.

Let $N(n)$ be the maximum length of an antichain of $S(n)$.

Find $\Sigma N(n)$ for $1 \le n \le 10^8$

Problem 387: Harshad Numbers

A **Harshad or Niven number** is a number that is divisible by the sum of its digits.
201 is a Harshad number because it is divisible by 3 (the sum of its digits.)
When we truncate the last digit from 201, we get 20, which is a Harshad number.
When we truncate the last digit from 20, we get 2, which is also a Harshad number.
Let's call a Harshad number that, while recursively truncating the last digit, always results in a Harshad number a *right truncatable Harshad number*.

Also:
201/3=67 which is prime.
Let's call a Harshad number that, when divided by the sum of its digits, results in a prime a *strong Harshad number*.

Now take the number 2011 which is prime.
When we truncate the last digit from it we get 201, a strong Harshad number that is also right truncatable.
Let's call such primes *strong, right truncatable Harshad primes*.

You are given that the sum of the strong, right truncatable Harshad primes less than 10000 is 90619.

Find the sum of the strong, right truncatable Harshad primes less than 10^{14}.

Problem 388: Distinct Lines

Consider all lattice points (a,b,c) with $0 \le a,b,c \le N$.

From the origin O(0,0,0) all lines are drawn to the other lattice points.
Let D(N) be the number of *distinct* such lines.

You are given that D(1 000 000) = 831909254469114121.

Find D(10^{10}). Give as your answer the first nine digits followed by the last nine digits.

Problem 389: Platonic Dice

An unbiased single 4-sided die is thrown and its value, T, is noted.
T unbiased 6-sided dice are thrown and their scores are added together. The sum, C, is noted.

C unbiased 8-sided dice are thrown and their scores are added together. The sum, O, is noted.
O unbiased 12-sided dice are thrown and their scores are added together. The sum, D, is noted.
D unbiased 20-sided dice are thrown and their scores are added together. The sum, I, is noted.
Find the variance of I, and give your answer rounded to 4 decimal places.

Problem 390: Triangles with non rational sides and integral area

Consider the triangle with sides $\sqrt{5}$, $\sqrt{65}$ and $\sqrt{68}$. It can be shown that this triangle has area 9.

S(n) is the sum of the areas of all triangles with sides $\sqrt{(1+b^2)}$, $\sqrt{(1+c^2)}$ and $\sqrt{(b^2+c^2)}$ (for positive integers b and c) that have an integral area not exceeding n.

The example triangle has b=2 and c=8.

$S(10^6)=18018206$.

Find $S(10^{10})$.

Problem 391: Hopping Game

Let s_k be the number of 1's when writing the numbers from 0 to k in binary.
For example, writing 0 to 5 in binary, we have 0, 1, 10, 11, 100, 101. There are seven 1's, so $s_5 = 7$.
The sequence $S = \{s_k : k \geq 0\}$ starts $\{0, 1, 2, 4, 5, 7, 9, 12, \ldots\}$.

A game is played by two players. Before the game starts, a number n is chosen. A counter c starts at 0. At each turn, the player chooses a number from 1 to n (inclusive) and increases c by that number. The resulting value of c must be a member of S. If there are no more valid moves, the player loses.

For example:
Let $n = 5$. c starts at 0.
Player 1 chooses 4, so c becomes $0 + 4 = 4$.
Player 2 chooses 5, so c becomes $4 + 5 = 9$.
Player 1 chooses 3, so c becomes $9 + 3 = 12$.
etc.
Note that c must always belong to S, and each player can increase c by at most n.

Let $M(n)$ be the highest number the first player can choose at her first turn to force a win, and $M(n) = 0$ if there is no such move. For example, $M(2) = 2$, $M(7) = 1$ and $M(20) = 4$.

Given $\Sigma(M(n))^3 = 8150$ for $1 \leq n \leq 20$.

Find $\Sigma(M(n))^3$ for $1 \leq n \leq 1000$.

Problem 392: Enmeshed unit circle

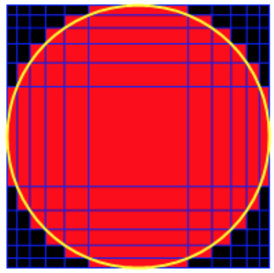
A rectilinear grid is an orthogonal grid where the spacing between the gridlines does not have to be equidistant.
An example of such grid is logarithmic graph paper.

Consider rectilinear grids in the Cartesian coordinate system with the following properties:

- The gridlines are parallel to the axes of the Cartesian coordinate system.
- There are N+2 vertical and N+2 horizontal gridlines. Hence there are (N+1) x (N+1) rectangular cells.
- The equations of the two outer vertical gridlines are x = -1 and x = 1.
- The equations of the two outer horizontal gridlines are y = -1 and y = 1.
- The grid cells are colored red if they overlap with the unit circle, black otherwise.

For this problem we would like you to find the postions of the remaining N inner horizontal and N inner vertical gridlines so that the area occupied by the red cells is minimized.

E.g. here is a picture of the solution for N = 10:



The area occupied by the red cells for N = 10 rounded to 10 digits behind the decimal point is 3.3469640797.

Find the positions for N = 400.
Give as your answer the area occupied by the red cells rounded to 10 digits behind the decimal point.

Problem 393: Migrating ants

An $n \times n$ grid of squares contains n^2 ants, one ant per square.
All ants decide to move simultaneously to an adjacent square (usually 4 possibilities, except for ants on the edge of the grid or at the corners).
We define $f(n)$ to be the number of ways this can happen without any ants ending on the same square and without any two ants crossing the same edge between two squares.

You are given that $f(4) = 88$.
Find $f(10)$.

Problem 394: Eating pie

Jeff eats a pie in an unusual way.
The pie is circular. He starts with slicing an initial cut in the pie along a radius.
While there is at least a given fraction F of pie left, he performs the following procedure:
- He makes two slices from the pie centre to any point of what is remaining of the pie border, any point on the remaining pie border equally likely. This will divide the remaining pie into three pieces.
- Going counterclockwise from the initial cut, he takes the first two pie pieces and eats them.
When less than a fraction F of pie remains, he does not repeat this procedure. Instead, he eats all of the remaining pie.

For $x \geq 1$, let $E(x)$ be the expected number of times Jeff repeats the procedure above with $F = {}^1/_x$.
It can be verified that $E(1) = 1$, $E(2) \approx 1.2676536759$, and $E(7.5) \approx 2.1215732071$.

Find E(40) rounded to 10 decimal places behind the decimal point.

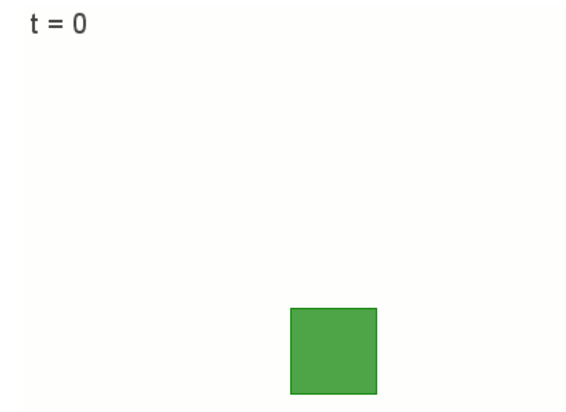
Problem 395: Pythagorean tree

The **Pythagorean tree** is a fractal generated by the following procedure:

Start with a unit square. Then, calling one of the sides its base (in the animation, the bottom side is the base):

1. Attach a right triangle to the side opposite the base, with the hypotenuse coinciding with that side and with the sides in a 3-4-5 ratio. Note that the smaller side of the triangle must be on the 'right' side with respect to the base (see animation).
2. Attach a square to each leg of the right triangle, with one of its sides coinciding with that leg.
3. Repeat this procedure for both squares, considering as their bases the sides touching the triangle.

The resulting figure, after an infinite number of iterations, is the Pythagorean tree.



It can be shown that there exists at least one rectangle, whose sides are parallel to the largest square of the Pythagorean tree, which encloses the Pythagorean tree completely.

Find the smallest area possible for such a bounding rectangle, and give your answer rounded to 10 decimal places.

Problem 396: Weak Goodstein sequence

For any positive integer n, the **nth weak Goodstein sequence** $\{g_1, g_2, g_3, ...\}$ is defined as:

- $g_1 = n$
- for $k > 1$, g_k is obtained by writing g_{k-1} in base k, interpreting it as a base $k + 1$ number, and subtracting 1.

The sequence terminates when g_k becomes 0.

For example, the 6th weak Goodstein sequence is {6, 11, 17, 25, ...}:

- $g_1 = 6$.
- $g_2 = 11$ since $6 = 110_2$, $110_3 = 12$, and $12 - 1 = 11$.
- $g_3 = 17$ since $11 = 102_3$, $102_4 = 18$, and $18 - 1 = 17$.
- $g_4 = 25$ since $17 = 101_4$, $101_5 = 26$, and $26 - 1 = 25$.

and so on.

It can be shown that every weak Goodstein sequence terminates.

Let $G(n)$ be the number of nonzero elements in the nth weak Goodstein sequence.
It can be verified that $G(2) = 3$, $G(4) = 21$ and $G(6) = 381$.
It can also be verified that $\Sigma G(n) = 2517$ for $1 \le n < 8$.

Find the last 9 digits of $\Sigma G(n)$ for $1 \le n < 16$.

Problem 397: Triangle on parabola

On the parabola $y = x^2/k$, three points $A(a, a^2/k)$, $B(b, b^2/k)$ and $C(c, c^2/k)$ are chosen.

Let $F(K, X)$ be the number of the integer quadruplets (k, a, b, c) such that at least one angle of the triangle ABC is 45-degree, with $1 \le k \le K$ and $-X \le a < b < c \le X$.

For example, F(1, 10) = 41 and F(10, 100) = 12492.
Find F(10^6, 10^9).

Problem 398: Cutting rope

Inside a rope of length n, n-1 points are placed with distance 1 from each other and from the endpoints. Among these points, we choose m-1 points at random and cut the rope at these points to create m segments.

Let E(n, m) be the expected length of the second-shortest segment.
For example, E(3, 2) = 2 and E(8, 3) = 16/7.
Note that if multiple segments have the same shortest length the length of the second-shortest segment is defined as the same as the shortest length.

Find E(10^7, 100).
Give your answer rounded to 5 decimal places behind the decimal point.

Problem 399: Squarefree Fibonacci Numbers

The first 15 fibonacci numbers are:
1,1,2,3,5,8,13,21,34,55,89,144,233,377,610.
It can be seen that 8 and 144 are not squarefree: 8 is divisible by 4 and 144 is divisible by 4 and by 9.
So the first 13 squarefree fibonacci numbers are:
1,1,2,3,5,13,21,34,55,89,233,377 and 610.

The 200th squarefree fibonacci number is: 971183874599339129547649988289594072811608739584170445.
The last sixteen digits of this number are: 1608739584170445 and in scientific notation this number can be written as 9.7e53.

Find the 100 000 000th squarefree fibonacci number.
Give as your answer its last sixteen digits followed by a comma followed by the number in scientific notation (rounded to one digit after the decimal point).
For the 200th squarefree number the answer would have been: 1608739584170445,9.7e53

Note:
For this problem, assume that for every prime p, the first fibonacci number divisible by p is not divisible by p^2 (this is part of **Wall's conjecture**). This has been verified for primes $\leq 3 \cdot 10^{15}$, but has not been proven in general.
If it happens that the conjecture is false, then the accepted answer to this problem isn't guaranteed to be the 100 000 000th squarefree fibonacci number, rather it represents only a lower bound for that number.

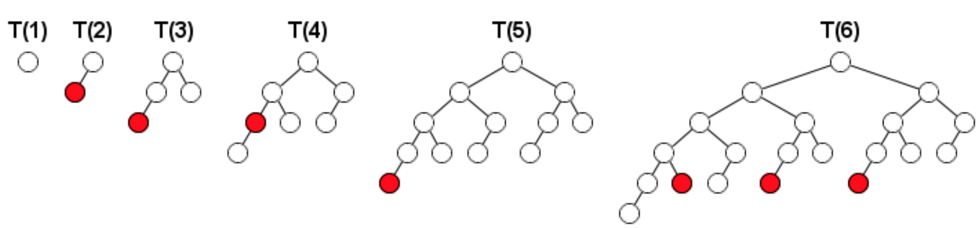
Problem 400: Fibonacci tree game

A **Fibonacci tree** is a binary tree recursively defined as:

- T(0) is the empty tree.
- T(1) is the binary tree with only one node.
- T(k) consists of a root node that has T(k-1) and T(k-2) as children.

On such a tree two players play a take-away game. On each turn a player selects a node and removes that node along with the subtree rooted at that node.
The player who is forced to take the root node of the entire tree loses.

Here are the winning moves of the first player on the first turn for T(k) from k=1 to k=6.



Let $f(k)$ be the number of winning moves of the first player (i.e. the moves for which the second player has no winning strategy) on the first turn of the game when this game is played on T(k).

For example, $f(5) = 1$ and $f(10) = 17$.

Find $f(10000)$. Give the last 18 digits of your answer.

Problem 401: Sum of squares of divisors

The divisors of 6 are 1,2,3 and 6.
The sum of the squares of these numbers is 1+4+9+36=50.

Let sigma2(n) represent the sum of the squares of the divisors of n. Thus sigma2(6)=50.

Let SIGMA2 represent the summatory function of sigma2, that is SIGMA2(n)=∑sigma2(i) for i=1 to n.
The first 6 values of SIGMA2 are: 1,6,16,37,63 and 113.

Find SIGMA2(10^{15}) modulo 10^9.

Problem 402: Integer-valued polynomials

It can be shown that the polynomial $n^4 + 4n^3 + 2n^2 + 5n$ is a multiple of 6 for every integer n. It can also be shown that 6 is the largest integer satisfying this property.

Define M(a, b, c) as the maximum m such that $n^4 + an^3 + bn^2 + cn$ is a multiple of m for all integers n. For example, M(4, 2, 5) = 6.

Also, define S(N) as the sum of M(a, b, c) for all $0 < a, b, c \leq N$.

We can verify that S(10) = 1972 and S(10000) = 2024258331114.

Let F_k be the Fibonacci sequence:
$F_0 = 0$, $F_1 = 1$ and
$F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$.

Find the last 9 digits of Σ S(F_k) for $2 \leq k \leq 1234567890123$.

Problem 403: Lattice points enclosed by parabola and line

For integers a and b, we define $D(a, b)$ as the domain enclosed by the parabola $y = x^2$ and the line $y = a \cdot x + b$:
$D(a, b) = \{ (x, y) \mid x^2 \leq y \leq a \cdot x + b \}$.

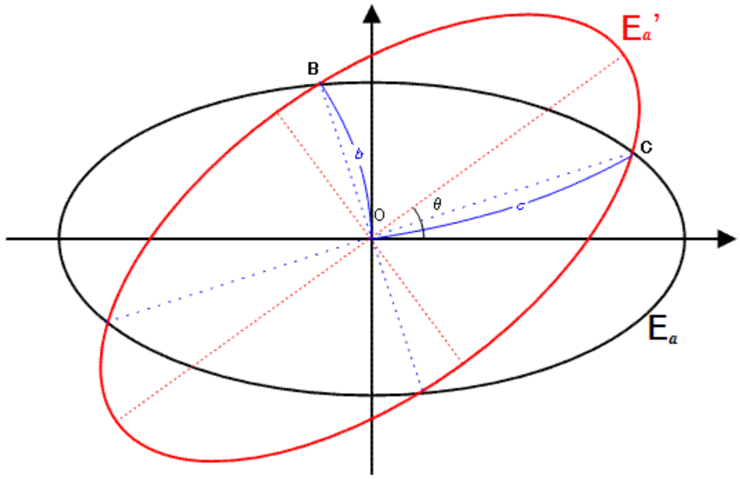
L(a, b) is defined as the number of lattice points contained in $D(a, b)$.
For example, L(1, 2) = 8 and L(2, -1) = 1.

We also define S(N) as the sum of L(a, b) for all the pairs (a, b) such that the area of $D(a, b)$ is a rational number and $|a|,|b| \leq N$.
We can verify that S(5) = 344 and S(100) = 26709528.

Find S(10^{12}). Give your answer mod 10^8.

Problem 404: Crisscross Ellipses

E_a is an ellipse with an equation of the form $x^2 + 4y^2 = 4a^2$.
E_a' is the rotated image of E_a by θ degrees counterclockwise around the origin O(0, 0) for 0° < θ < 90°.



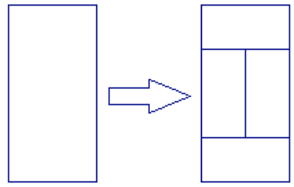
b is the distance to the origin of the two intersection points closest to the origin and c is the distance of the two other intersection points.
We call an ordered triplet (a, b, c) a *canonical ellipsoidal triplet* if a, b and c are positive integers.
For example, (209, 247, 286) is a canonical ellipsoidal triplet.

Let C(N) be the number of distinct canonical ellipsoidal triplets (a, b, c) for $a \leq N$.
It can be verified that C(10^3) = 7, C(10^4) = 106 and C(10^6) = 11845.

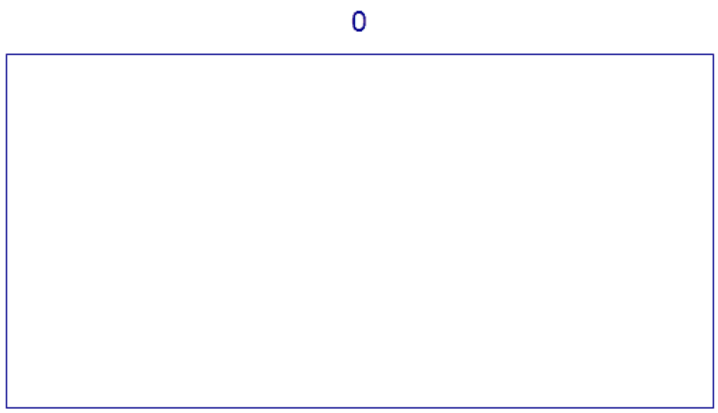
Find C(10^{17}).

Problem 405: A rectangular tiling

We wish to tile a rectangle whose length is twice its width.
Let $T(0)$ be the tiling consisting of a single rectangle.
For $n > 0$, let $T(n)$ be obtained from $T(n-1)$ by replacing all tiles in the following manner:



The following animation demonstrates the tilings $T(n)$ for n from 0 to 5:



Let $f(n)$ be the number of points where four tiles meet in $T(n)$.
For example, $f(1) = 0$, $f(4) = 82$ and $f(10^9) \bmod 17^7 = 126897180$.

Find $f(10^k)$ for $k = 10^{18}$, give your answer modulo 17^7.

Problem 406: Guessing Game

We are trying to find a hidden number selected from the set of integers {1, 2, ..., n} by asking questions. Each number (question) we ask, we get one of three possible answers:

- "Your guess is lower than the hidden number" (and you incur a cost of a), or
- "Your guess is higher than the hidden number" (and you incur a cost of b), or
- "Yes, that's it!" (and the game ends).

Given the value of n, a, and b, an *optimal strategy* minimizes the total cost for the worst possible case.

For example, if $n = 5$, $a = 2$, and $b = 3$, then we may begin by asking "**2**" as our first question.

If we are told that 2 is higher than the hidden number (for a cost of b=3), then we are sure that "**1**" is the hidden number (for a total cost of **3**).
If we are told that 2 is lower than the hidden number (for a cost of a=2), then our next question will be "**4**".
If we are told that 4 is higher than the hidden number (for a cost of b=3), then we are sure that "**3**" is the hidden number (for a total cost of 2+3=**5**).
If we are told that 4 is lower than the hidden number (for a cost of a=2), then we are sure that "**5**" is the hidden number (for a total cost of 2+2=**4**).
Thus, the worst-case cost achieved by this strategy is **5**. It can also be shown that this is the lowest worst-case cost that can be achieved. So, in fact, we have just described an optimal strategy for the given values of n, a, and b.

Let C(n, a, b) be the worst-case cost achieved by an optimal strategy for the given values of n, a, and b.

Here are a few examples:
C(5, 2, 3) = 5
C(500, $\sqrt{2}$, $\sqrt{3}$) = 13.22073197...
C(20000, 5, 7) = 82
C(2000000, $\sqrt{5}$, $\sqrt{7}$) = 49.63755955...

Let F_k be the Fibonacci numbers: $F_k = F_{k-1} + F_{k-2}$ with base cases $F_1 = F_2 = 1$.
Find $\sum_{1 \le k \le 30} C(10^{12}, \sqrt{k}, \sqrt{F_k})$, and give your answer rounded to 8 decimal places behind the decimal point.

Problem 407: Idempotents

If we calculate a^2 mod 6 for $0 \le a \le 5$ we get: 0,1,4,3,4,1.

The largest value of a such that $a^2 \equiv a$ mod 6 is 4.
Let's call M(n) the largest value of $a < n$ such that $a^2 \equiv a \pmod{n}$.
So M(6) = 4.

Find $\sum$M(n) for $1 \le n \le 10^7$.

Problem 408: Admissible paths through a grid

Let's call a lattice point (x, y) *inadmissible* if x, y and $x + y$ are all positive perfect squares.
For example, (9, 16) is inadmissible, while (0, 4), (3, 1) and (9, 4) are not.

Consider a path from point (x_1, y_1) to point (x_2, y_2) using only unit steps north or east.
Let's call such a path *admissible* if none of its intermediate points are inadmissible.

Let P(n) be the number of admissible paths from (0, 0) to (n, n).
It can be verified that P(5) = 252, P(16) = 596994440 and P(1000) mod 1 000 000 007 = 341920854.

Find P(10 000 000) mod 1 000 000 007.

Problem 409: Nim Extreme

Let n be a positive integer. Consider **nim** positions where:

- There are n non-empty piles.
- Each pile has size less than 2^n.
- No two piles have the same size.

Let W(n) be the number of winning nim positions satisfying the above conditions (a position is winning if the first player has a winning strategy).
For example, W(1) = 1, W(2) = 6, W(3) = 168, W(5) = 19764360 and W(100) mod 1 000 000 007 = 384777056.

Find W(10 000 000) mod 1 000 000 007.

Problem 410: Circle and tangent line

Let C be the circle with radius r, $x^2 + y^2 = r^2$. We choose two points P(a, b) and Q($-a$, c) so that the line passing through P and Q is tangent to C.

For example, the quadruplet (r, a, b, c) = (2, 6, 2, -7) satisfies this property.

Let F(R, X) be the number of the integer quadruplets (r, a, b, c) with this property, and with $0 < r \le R$ and $0 < a \le X$.

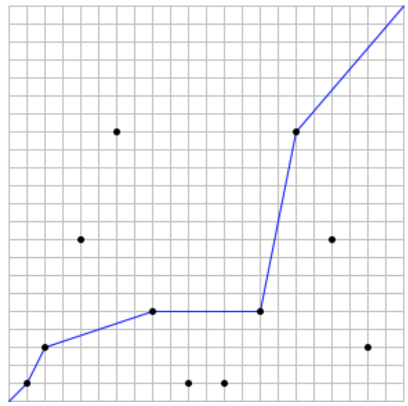
We can verify that F(1, 5) = 10, F(2, 10) = 52 and F(10, 100) = 3384.
Find F(10^8, 10^9) + F(10^9, 10^8).

Problem 411: Uphill paths

Let n be a positive integer. Suppose there are stations at the coordinates $(x, y) = (2^i \bmod n, 3^i \bmod n)$ for $0 \le i \le 2n$. We will consider stations with the same coordinates as the same station.

We wish to form a path from (0, 0) to (n, n) such that the x and y coordinates never decrease.
Let S(n) be the maximum number of stations such a path can pass through.

For example, if n = 22, there are 11 distinct stations, and a valid path can pass through at most 5 stations. Therefore, S(22) = 5. The case is illustrated below, with an example of an optimal path:



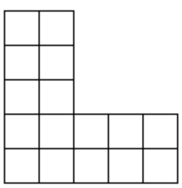
It can also be verified that S(123) = 14 and S(10000) = 48.

Find $\sum$ S(k^5) for $1 \le k \le 30$.

Problem 412: Gnomon numbering

For integers m, n ($0 \le n < m$), let L(m, n) be an $m \times m$ grid with the top-right $n \times n$ grid removed.

For example, L(5, 3) looks like this:



We want to number each cell of L(m, n) with consecutive integers 1, 2, 3, ... such that the number in every cell is smaller than the number below it and to the left of it.

For example, here are two valid numberings of L(5, 3):

6	4			
7	5			
14	9			
15	11	8	3	1
16	13	12	10	2

8	1			
11	2			
13	7			
14	12	9	5	3
16	15	10	6	4

Let LC(m, n) be the number of valid numberings of L(m, n).
It can be verified that LC(3, 0) = 42, LC(5, 3) = 250250, LC(6, 3) = 406029023400 and LC(10, 5) mod 76543217 = 61251715.

Find LC(10000, 5000) mod 76543217.

Problem 413: One-child Numbers

We say that a d-digit positive number (no leading zeros) is a one-child number if exactly one of its sub-strings is divisible by d.

For example, 5671 is a 4-digit one-child number. Among all its sub-strings 5, 6, 7, 1, 56, 67, 71, 567, 671 and 5671, only 56 is divisible by 4.
Similarly, 104 is a 3-digit one-child number because only 0 is divisible by 3.
1132451 is a 7-digit one-child number because only 245 is divisible by 7.

Let F(N) be the number of the one-child numbers less than N.

We can verify that F(10) = 9, F(10^3) = 389 and F(10^7) = 277674.

Find F(10^{19}).

Problem 414: Kaprekar constant

6174 is a remarkable number; if we sort its digits in increasing order and subtract that number from the number you get when you sort the digits in decreasing order, we get 7641-1467=6174.
Even more remarkable is that if we start from any 4 digit number and repeat this process of sorting and subtracting, we'll eventually end up with 6174 or immediately with 0 if all digits are equal.
This also works with numbers that have less than 4 digits if we pad the number with leading zeroes until we have 4 digits.
E.g. let's start with the number 0837:
8730-0378=8352
8532-2358=6174

6174 is called the **Kaprekar constant**. The process of sorting and subtracting and repeating this until either 0 or the Kaprekar constant is reached is called the **Kaprekar routine**.

We can consider the Kaprekar routine for other bases and number of digits.
Unfortunately, it is not guaranteed a Kaprekar constant exists in all cases; either the routine can end up in a cycle for some input numbers or the constant the routine arrives at can be different for different input numbers.
However, it can be shown that for 5 digits and a base b = 6t+3≠9, a Kaprekar constant exists.
E.g. base 15: $(10,4,14,9,5)_{15}$
base 21: $(14,6,20,13,7)_{21}$

Define C_b to be the Kaprekar constant in base b for 5 digits. Define the function $sb(i)$ to be

- 0 if i = C_b or if i written in base b consists of 5 identical digits
- the number of iterations it takes the Kaprekar routine in base b to arrive at C_b, otherwise

Note that we can define $sb(i)$ for all integers $i < b^5$. If i written in base b takes less than 5 digits, the number is padded with leading zero digits until we have 5 digits before applying the Kaprekar routine.

Define $S(b)$ as the sum of $sb(i)$ for $0 < i < b^5$.
E.g. S(15) = 5274369
S(111) = 400668930299

Find the sum of S(6k+3) for 2 ≤ k ≤ 300.
Give the last 18 digits as your answer.

Problem 415: Titanic sets

A set of lattice points S is called a *titanic set* if there exists a line passing through exactly two points in S.

An example of a titanic set is S = {(0, 0), (0, 1), (0, 2), (1, 1), (2, 0), (1, 0)}, where the line passing through (0, 1) and (2, 0) does not pass through any other point in S.

On the other hand, the set {(0, 0), (1, 1), (2, 2), (4, 4)} is not a titanic set since the line passing through any two points in the set also passes through the other two.

For any positive integer N, let $T(N)$ be the number of titanic sets S whose every point (x, y) satisfies $0 \le x, y \le N$. It can be verified that $T(1)$ = 11, $T(2)$ = 494, $T(4)$ = 33554178, $T(111)$ mod 10^8 = 13500401 and $T(10^5)$ mod 10^8 = 63259062.

Find $T(10^{11})$ mod 10^8.

Problem 416: A frog's trip

A row of n squares contains a frog in the leftmost square. By successive jumps the frog goes to the rightmost square and then back to the leftmost square. On the outward trip he jumps one, two or three squares to the right, and on the homeward trip he jumps to the left in a similar manner. He cannot jump outside the squares. He repeats the round-trip travel m times.

Let F(m, n) be the number of the ways the frog can travel so that at most one square remains unvisited.
For example, F(1, 3) = 4, F(1, 4) = 15, F(1, 5) = 46, F(2, 3) = 16 and F(2, 100) mod 10^9 = 429619151.

Find the last 9 digits of F(10, 10^{12}).

Problem 417: Reciprocal cycles II

A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given:

$^1/_2$ = 0.5
$^1/_3$ = 0.(3)
$^1/_4$ = 0.25
$^1/_5$ = 0.2
$^1/_6$ = 0.1(6)
$^1/_7$ = 0.(142857)
$^1/_8$ = 0.125
$^1/_9$ = 0.(1)
$^1/_{10}$ = 0.1

Where 0.1(6) means 0.166666..., and has a 1-digit recurring cycle. It can be seen that $^1/_7$ has a 6-digit recurring cycle.

Unit fractions whose denominator has no other prime factors than 2 and/or 5 are not considered to have a recurring cycle.
We define the length of the recurring cycle of those unit fractions as 0.

Let L(n) denote the length of the recurring cycle of 1/n. You are given that ∑L(n) for 3 ≤ n ≤ 1 000 000 equals 55535191115.

Find ∑L(n) for 3 ≤ n ≤ 100 000 000

Problem 418: Factorisation triples

Let n be a positive integer. An integer triple (a, b, c) is called a *factorisation triple* of n if:

- $1 \le a \le b \le c$
- $a \cdot b \cdot c = n$.

Define $f(n)$ to be $a + b + c$ for the factorisation triple (a, b, c) of n which minimises c / a. One can show that this triple is unique.

For example, $f(165) = 19$, $f(100100) = 142$ and $f(20!) = 4034872$.

Find $f(43!)$.

Problem 419: Look and say sequence

The **look and say** sequence goes 1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ...
The sequence starts with 1 and all other members are obtained by describing the previous member in terms of consecutive digits.
It helps to do this out loud:
1 is 'one one' ⟶ 11
11 is 'two ones' ⟶ 21
21 is 'one two and one one' ⟶ 1211
1211 is 'one one, one two and two ones' ⟶ 111221
111221 is 'three ones, two twos and one one' ⟶ 312211
...

Define A(n), B(n) and C(n) as the number of ones, twos and threes in the n'th element of the sequence respectively.
One can verify that A(40) = 31254, B(40) = 20259 and C(40) = 11625.

Find A(n), B(n) and C(n) for n = 10^{12}.
Give your answer modulo 2^{30} and separate your values for A, B and C by a comma.
E.g. for n = 40 the answer would be 31254,20259,11625

Problem 420: 2x2 positive integer matrix

A *positive integer matrix* is a matrix whose elements are all positive integers.
Some positive integer matrices can be expressed as a square of a positive integer matrix in two different ways. Here is an example:

$$\begin{pmatrix} 40 & 12 \\ 48 & 40 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 12 & 2 \end{pmatrix}^2 = \begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}^2$$

We define F(N) as the number of the 2x2 positive integer matrices which have a trace less than N and which can be expressed as a square of a positive integer matrix in two different ways.
We can verify that F(50) = 7 and F(1000) = 1019.

Find F(10^7).

Problem 421: Prime factors of $n^{15}+1$

Numbers of the form $n^{15}+1$ are composite for every integer $n > 1$.
For positive integers n and m let $s(n,m)$ be defined as the sum of the *distinct* prime factors of $n^{15}+1$ not exceeding m.

E.g. $2^{15}+1 = 3\times3\times11\times331$.
So $s(2,10) = 3$ and $s(2,1000) = 3+11+331 = 345$.

Also $10^{15}+1 = 7\times11\times13\times211\times241\times2161\times9091$.
So $s(10,100) = 31$ and $s(10,1000) = 483$.

Find $\sum s(n,10^8)$ for $1 \le n \le 10^{11}$.

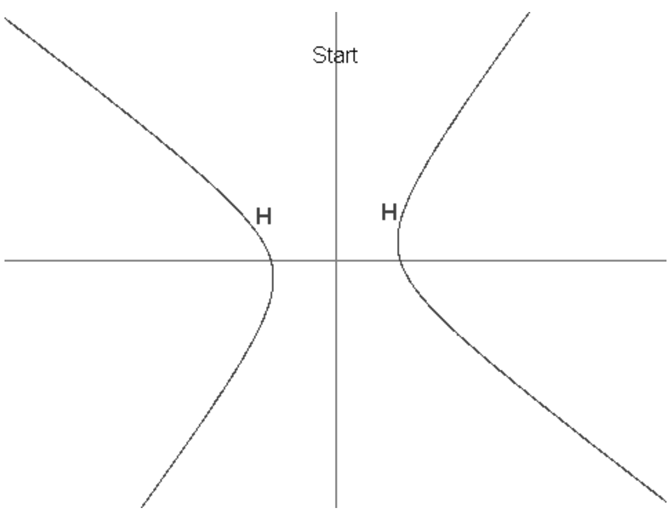
Problem 422: Sequence of points on a hyperbola

Let H be the hyperbola defined by the equation $12x^2 + 7xy - 12y^2 = 625$.

Next, define X as the point (7, 1). It can be seen that X is in H.

Now we define a sequence of points in H, $\{P_i : i \ge 1\}$, as:

- $P_1 = (13, 61/4)$.
- $P_2 = (-43/6, -4)$.
- For $i > 2$, P_i is the unique point in H that is different from P_{i-1} and such that line P_iP_{i-1} is parallel to line $P_{i-2}X$. It can be shown that P_i is well-defined, and that its coordinates are always rational.



You are given that $P_3 = (-19/2, -229/24)$, $P_4 = (1267/144, -37/12)$ and $P_7 = (17194218091/143327232, 274748766781/1719926784)$.

Find P_n for $n = 11^{14}$ in the following format:

If $P_n = (a/b,\ c/d)$ where the fractions are in lowest terms and the denominators are positive, then the answer is $(a + b + c + d)$ mod 1 000 000 007.

For $n = 7$, the answer would have been: 806236837.

Problem 423: Consecutive die throws

Let n be a positive integer.
A 6-sided die is thrown n times. Let c be the number of pairs of consecutive throws that give the same value.

For example, if $n = 7$ and the values of the die throws are (1,1,5,6,6,6,3), then the following pairs of consecutive throws give the same value:
(1,1,5,6,6,6,3)
(1,1,5,6,6,6,3)
(1,1,5,6,6,6,3)
Therefore, $c = 3$ for (1,1,5,6,6,6,3).

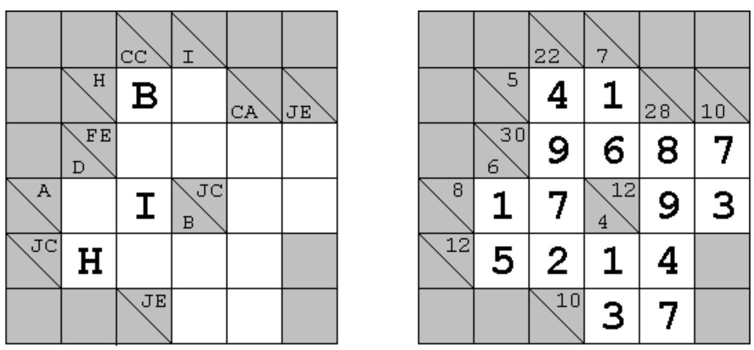
Define $C(n)$ as the number of outcomes of throwing a 6-sided die n times such that c does not exceed $\pi(n)$.[1]
For example, $C(3) = 216$, $C(4) = 1290$, $C(11) = 361912500$ and $C(24) = 4727547363281250000$.

Define $S(L)$ as $\sum C(n)$ for $1 \leq n \leq L$.
For example, $S(50)$ mod 1 000 000 007 = 832833871.

Find $S(50\ 000\ 000)$ mod 1 000 000 007.

[1] π denotes the **prime-counting function**, i.e. $\pi(n)$ is the number of primes $\leq n$.

Problem 424: Kakuro



A	B	C	D	E	F	G	H	I	J
8	4	2	6	0	3	9	5	7	1

The above is an example of a cryptic kakuro (also known as cross sums, or even sums cross) puzzle, with its final solution on the right. (The common rules of kakuro puzzles can be found easily on numerous internet sites. Other related information can also be currently found at krazydad.com whose author has provided the puzzle data for this challenge.)

The downloadable text file (kakuro200.txt) contains the description of 200 such puzzles, a mix of 5x5 and 6x6 types. The first puzzle in the file is the above example which is coded as follows:

6,X,X,(vCC),(vI),X,X,X,(hH),B,O,(vCA),(vJE),X,(hFE,vD),O,O,O,O,(hA),O,I,(hJC,vB),O,O,(hJC),H,O,O,O,X,X,X,(hJE),O,O,X

The first character is a numerical digit indicating the size of the information grid. It would be either a 6 (for a 5x5 kakuro puzzle) or a 7 (for a 6x6 puzzle) followed by a comma (,). The extra top line and left column are needed to insert information.

The content of each cell is then described and followed by a comma, going left to right and starting with the top line.
X = Gray cell, not required to be filled by a digit.
O (upper case letter)= White empty cell to be filled by a digit.
A = Or any one of the upper case letters from A to J to be replaced by its equivalent digit in the solved puzzle.
( ) = Location of the encrypted sums. Horizontal sums are preceded by a lower case "h" and vertical sums are preceded by a lower case "v". Those are followed by one or two upper case letters depending if the sum is a single digit or double digit one. For double digit sums, the first letter would be for the "tens" and the second one for the "units". When the cell must contain information for both a horizontal and a vertical sum, the

first one is always for the horizontal sum and the two are separated by a comma within the same set of brackets, ex.: (hFE,vD). Each set of brackets is also immediately followed by a comma.

The description of the last cell is followed by a Carriage Return/Line Feed (CRLF) instead of a comma.

The required answer to each puzzle is based on the value of each letter necessary to arrive at the solution and according to the alphabetical order. As indicated under the example puzzle, its answer would be 8426039571. At least 9 out of the 10 encrypting letters are always part of the problem description. When only 9 are given, the missing one must be assigned the remaining digit.

You are given that the sum of the answers for the first 10 puzzles in the file is 64414157580.

Find the sum of the answers for the 200 puzzles.

Problem 425: Prime connection

Two positive numbers A and B are said to be *connected* (denoted by "A ↔ B") if one of these conditions holds:
(1) A and B have the same length and differ in exactly one digit; for example, 123 ↔ 173.
(2) Adding one digit to the left of A (or B) makes B (or A); for example, 23 ↔ 223 and 123 ↔ 23.

We call a prime P a *2's relative* if there exists a chain of connected primes between 2 and P and no prime in the chain exceeds P.

For example, 127 is a 2's relative. One of the possible chains is shown below:
2 ↔ 3 ↔ 13 ↔ 113 ↔ 103 ↔ 107 ↔ 127
However, 11 and 103 are not 2's relatives.

Let F(N) be the sum of the primes ≤ N which are not 2's relatives.
We can verify that $F(10^3) = 431$ and $F(10^4) = 78728$.

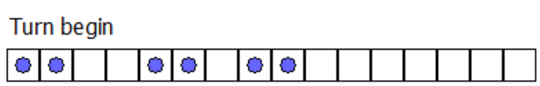
Find $F(10^7)$.

Problem 426: Box-ball system

Consider an infinite row of boxes. Some of the boxes contain a ball. For example, an initial configuration of 2 consecutive occupied boxes followed by 2 empty boxes, 2 occupied boxes, 1 empty box, and 2 occupied boxes can be denoted by the sequence (2, 2, 2, 1, 2), in which the number of consecutive occupied and empty boxes appear alternately.

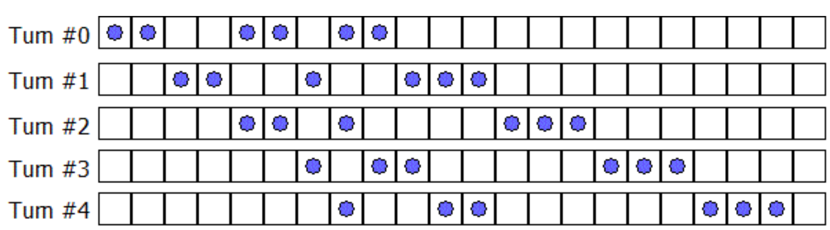
A turn consists of moving each ball exactly once according to the following rule: Transfer the leftmost ball which has not been moved to the nearest empty box to its right.

After one turn the sequence (2, 2, 2, 1, 2) becomes (2, 2, 1, 2, 3) as can be seen below; note that we begin the new sequence starting at the first occupied box.



A system like this is called a **Box-Ball System** or **BBS** for short.

It can be shown that after a sufficient number of turns, the system evolves to a state where the consecutive numbers of occupied boxes is invariant. In the example below, the consecutive numbers of **occupied boxes** evolves to [1, 2, 3]; we shall call this the final state.



We define the sequence $\{t_i\}$:

- $s_0 = 290797$
- $s_{k+1} = s_k^2 \bmod 50515093$
- $t_k = (s_k \bmod 64) + 1$

Starting from the initial configuration (t_0, t_1, …, t_{10}), the final state becomes [1, 3, 10, 24, 51, 75].
Starting from the initial configuration (t_0, t_1, …, $t_{10\,000\,000}$), find the final state.
Give as your answer the sum of the squares of the elements of the final state. For example, if the final state is [1, 2, 3] then 14 (= $1^2 + 2^2 + 3^2$) is your answer.

Problem 427: n-sequences

A sequence of integers S = {s_i} is called an *n-sequence* if it has n elements and each element s_i satisfies $1 \le s_i \le n$. Thus there are n^n distinct n-sequences in total. For example, the sequence S = {1, 5, 5, 10, 7, 7, 7, 2, 3, 7} is a 10-sequence.

For any sequence S, let L(S) be the length of the longest contiguous subsequence of S with the same value. For example, for the given sequence S above, L(S) = 3, because of the three consecutive 7's.

Let $f(n) = \sum$ L(S) for all n-sequences S.

For example, $f(3)$ = 45, $f(7)$ = 1403689 and $f(11)$ = 481496895121.

Find f(7 500 000) mod 1 000 000 009.

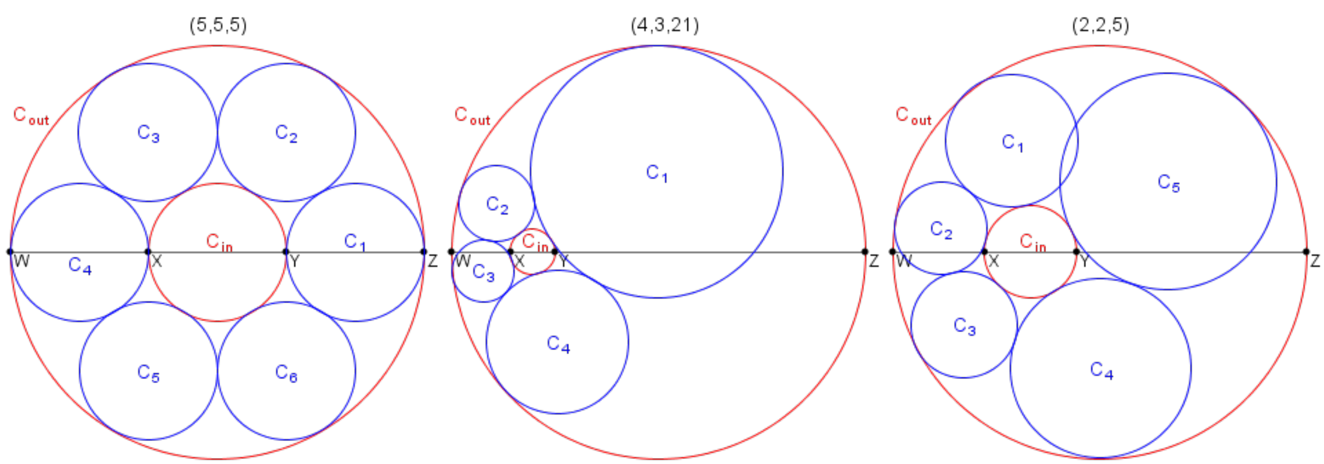
Problem 428: Necklace of circles

Let a, b and c be positive numbers.
Let W, X, Y, Z be four collinear points where |WX| = a, |XY| = b, |YZ| = c and |WZ| = $a + b + c$.
Let C_{in} be the circle having the diameter XY.
Let C_{out} be the circle having the diameter WZ.

The triplet (a, b, c) is called a *necklace triplet* if you can place $k \ge 3$ distinct circles C_1, C_2, ..., C_k such that:

- C_i has no common interior points with any C_j for $1 \le i, j \le k$ and $i \ne j$,
- C_i is tangent to both C_{in} and C_{out} for $1 \le i \le k$,
- C_i is tangent to C_{i+1} for $1 \le i < k$, and
- C_k is tangent to C_1.

For example, (5, 5, 5) and (4, 3, 21) are necklace triplets, while it can be shown that (2, 2, 5) is not.



Let T(n) be the number of necklace triplets (a, b, c) such that a, b and c are positive integers, and $b \le n$. For example, T(1) = 9, T(20) = 732 and T(3000) = 438106.

Find T(1 000 000 000).

Problem 429: Sum of squares of unitary divisors

A unitary divisor d of a number n is a divisor of n that has the property gcd(d, n/d) = 1.

The unitary divisors of 4! = 24 are 1, 3, 8 and 24.
The sum of their squares is $1^2 + 3^2 + 8^2 + 24^2 = 650$.

Let S(n) represent the sum of the squares of the unitary divisors of n. Thus S(4!)=650.

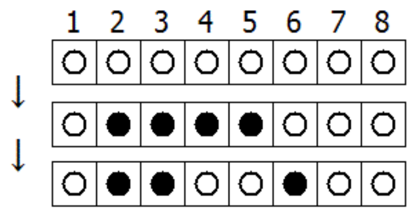
Find S(100 000 000!) modulo 1 000 000 009.

Problem 430: Range flips

N disks are placed in a row, indexed 1 to N from left to right.
Each disk has a black side and white side. Initially all disks show their white side.

At each turn, two, not necessarily distinct, integers A and B between 1 and N (inclusive) are chosen uniformly at random.
All disks with an index from A to B (inclusive) are flipped.

The following example shows the case N = 8. At the first turn A = 5 and B = 2, and at the second turn A = 4 and B = 6.



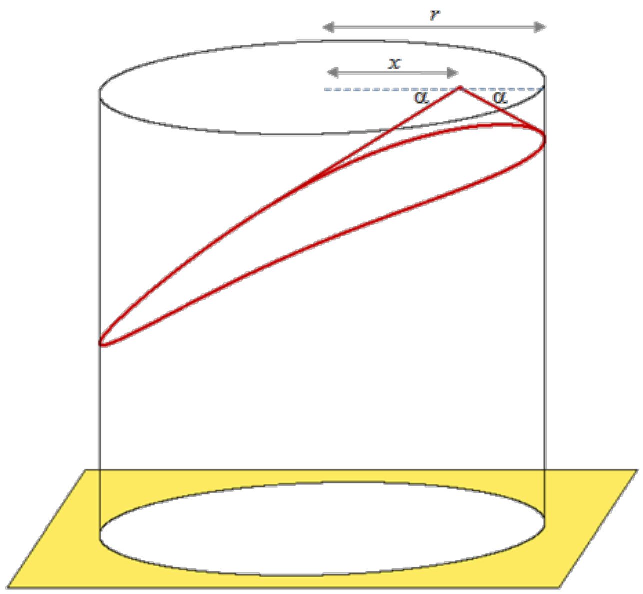
Let E(N, M) be the expected number of disks that show their white side after M turns.
We can verify that E(3, 1) = 10/9, E(3, 2) = 5/3, E(10, 4) ≈ 5.157 and E(100, 10) ≈ 51.893.

Find E(10^{10}, 4000).
Give your answer rounded to 2 decimal places behind the decimal point.

Problem 431: Square Space Silo

Fred the farmer arranges to have a new storage silo installed on his farm and having an obsession for all things square he is absolutely devastated when he discovers that it is circular. Quentin, the representative from the company that installed the silo, explains that they only manufacture cylindrical silos, but he points out that it is resting on a square base. Fred is not amused and insists that it is removed from his property.

Quick thinking Quentin explains that when granular materials are delivered from above a conical slope is formed and the natural angle made with the horizontal is called the angle of repose. For example if the angle of repose, $\alpha = 30$ degrees, and grain is delivered at the centre of the silo then a perfect cone will form towards the top of the cylinder. In the case of this silo, which has a diameter of 6m, the amount of space wasted would be approximately 32.648388556 m^3. However, if grain is delivered at a point on the top which has a horizontal distance of x metres from the centre then a cone with a strangely curved and sloping base is formed. He shows Fred a picture.



We shall let the amount of space wasted in cubic metres be given by $V(x)$. If $x = 1.114785284$, which happens to have three squared decimal places, then the amount of space wasted, $V(1.114785284) \approx 36$. Given the range of possible solutions to this problem there is exactly one other option: $V(2.511167869) \approx 49$. It would be like knowing that the square is king of the silo, sitting in splendid glory on top of your grain.

Fred's eyes light up with delight at this elegant resolution, but on closer inspection of Quentin's drawings and calculations his happiness turns to despondency once more. Fred points out to Quentin that it's the radius of the silo that is 6 metres, not the diameter, and the angle of repose for his grain is 40 degrees. However, if Quentin can find a set of solutions for this particular silo then he will be more than happy to keep it.

If Quick thinking Quentin is to satisfy frustratingly fussy Fred the farmer's appetite for all things square then determine the values of x for all possible square space wastage options and calculate $\sum x$ correct to 9 decimal places.

Problem 432: Totient sum

Let S(n,m) = $\sum\varphi(n \times i)$ for $1 \le i \le m$. (φ is Euler's totient function)
You are given that S(510510,10^6)= 45480596821125120.

Find S(510510,10^{11}).
Give the last 9 digits of your answer.

Problem 433: Steps in Euclid's algorithm

Let E(x_0, y_0) be the number of steps it takes to determine the greatest common divisor of x_0 and y_0 with **Euclid's algorithm**. More formally:
$x_1 = y_0$, $y_1 = x_0 \bmod y_0$
$x_n = y_{n-1}$, $y_n = x_{n-1} \bmod y_{n-1}$
E(x_0, y_0) is the smallest n such that $y_n = 0$.

We have E(1,1) = 1, E(10,6) = 3 and E(6,10) = 4.

Define S(N) as the sum of E(x,y) for $1 \le x,y \le$ N.
We have S(1) = 1, S(10) = 221 and S(100) = 39826.

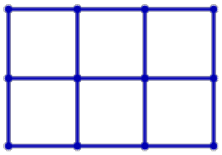
Find S($5 \cdot 10^6$).

Problem 434: Rigid graphs

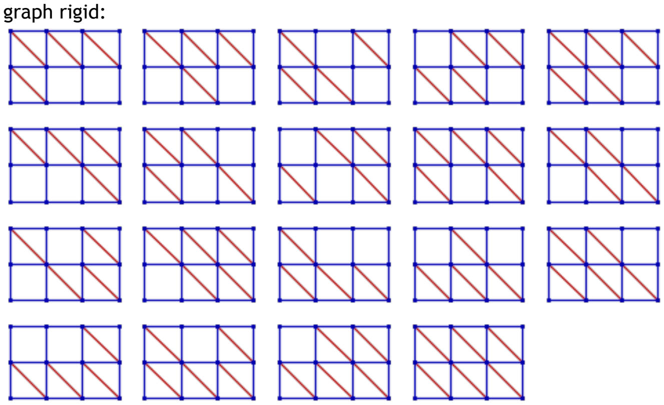
Recall that a graph is a collection of vertices and edges connecting the vertices, and that two vertices connected by an edge are called adjacent.
Graphs can be embedded in Euclidean space by associating each vertex with a point in the Euclidean space.
A **flexible** graph is an embedding of a graph where it is possible to move one or more vertices continuously so that the distance between at least

two nonadjacent vertices is altered while the distances between each pair of adjacent vertices is kept constant.
A **rigid** graph is an embedding of a graph which is not flexible.
Informally, a graph is rigid if by replacing the vertices with fully rotating hinges and the edges with rods that are unbending and inelastic, no parts of the graph can be moved independently from the rest of the graph.

The **grid graphs** embedded in the Euclidean plane are not rigid, as the following animation demonstrates:



However, one can make them rigid by adding diagonal edges to the cells. For example, for the 2x3 grid graph, there are 19 ways to make the graph rigid:



Note that for the purposes of this problem, we do not consider changing the orientation of a diagonal edge or adding both diagonal edges to a cell as a different way of making a grid graph rigid.

Let $R(m,n)$ be the number of ways to make the $m \times n$ grid graph rigid.
E.g. $R(2,3) = 19$ and $R(5,5) = 23679901$

Define $S(N)$ as $\sum R(i,j)$ for $1 \leq i, j \leq N$.
E.g. $S(5) = 25021721$.
Find $S(100)$, give your answer modulo 1000000033