Problem 103

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) $\neq$ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) $>$ S(C).

If S(A) is minimised for a given *n*, we shall call it an optimum special sum set. The first five optimum special sum sets are given below.

n = 1: {1}
n = 2: {1, 2}
n = 3: {2, 3, 4}
n = 4: {3, 5, 6, 7}
n = 5: {6, 9, 11, 12, 13}

It *seems* that for a given optimum set, A = $\{a_1, a_2, \ldots, a_n\}$, the next optimum set is of the form B = $\{b, a_1+b, a_2+b, \ldots, a_n+b\}$, where *b* is the "middle" element on the previous row.

By applying this "rule" we would expect the optimum set for *n* = 6 to be A = {11, 17, 20, 22, 23, 24}, with S(A) = 117. However, this is not the optimum set, as we have merely applied an algorithm to provide a near optimum set. The optimum set for *n* = 6 is A = {11, 18, 19, 20, 22, 25}, with S(A) = 115 and corresponding set string: 111819202225.

Given that A is an optimum special sum set for *n* = 7, find its set string.

NOTE: This problem is related to problems 105 and 106.

Problem 105

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) $\neq$ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) $>$ S(C).

For example, {81, 88, 75, 42, 87, 84, 86, 65} is not a special sum set because 65 + 87 + 88 = 75 + 81 + 84, whereas {157, 150, 164, 119, 79, 159, 161, 139, 158} satisfies both rules for all possible subset pair combinations and S(A) = 1286.

Using sets.txt (right click and "Save Link/Target As..."), a 4K text file with one-hundred sets containing seven to twelve elements (the two examples given above are the first two sets in the file), identify all the special sum sets, A_1, A_2, ..., A_k, and find the value of $S(A_1) + S(A_2) + ... + S(A_k)$.

NOTE: This problem is related to problems 103 and 106.

Problem 106

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. $S(B) \neq S(C)$; that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then $S(B) > S(C)$.

For this problem we shall assume that a given set contains *n* strictly increasing elements and it already satisfies the second rule.

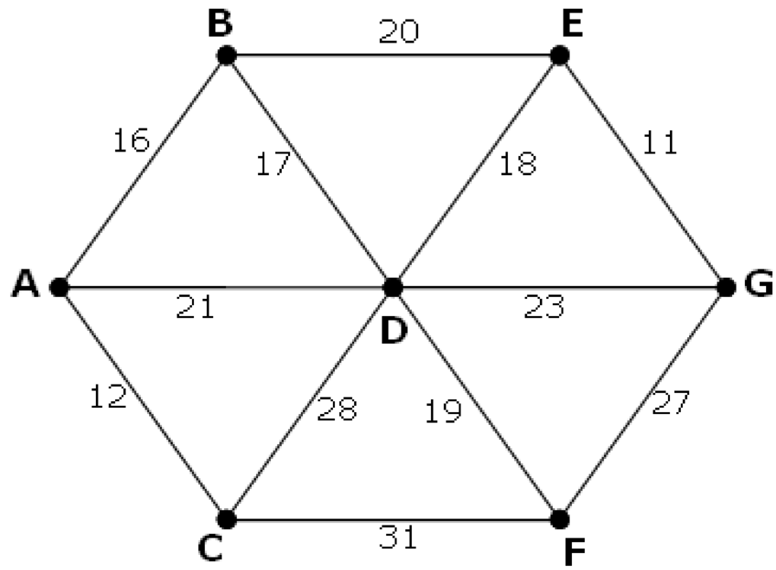
Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which $n = 4$, only 1 of these pairs need to be tested for equality (first rule). Similarly, when $n = 7$, only 70 out of the 966 subset pairs need to be tested.

For $n = 12$, how many of the 261625 subset pairs that can be obtained need to be tested for equality?

NOTE: This problem is related to problems 103 and 105.

Problem 107

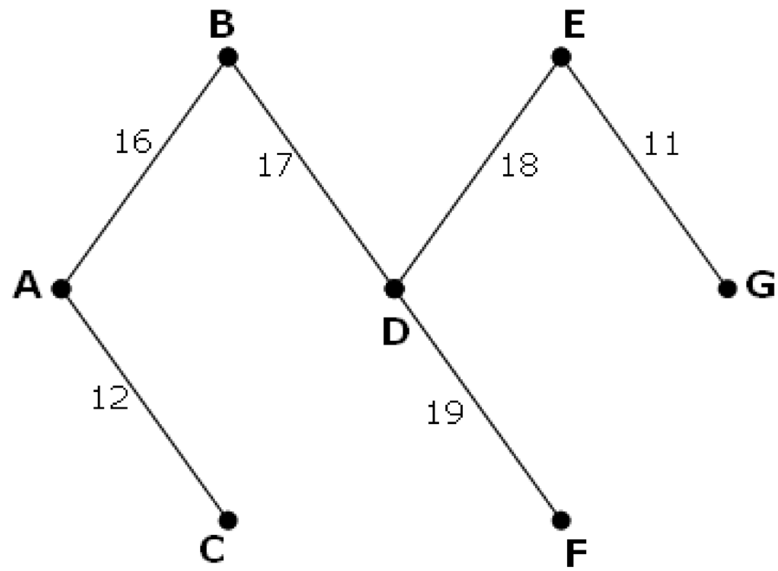
The following undirected network consists of seven vertices and twelve edges with a total weight of 243.



The same network can be represented by the matrix below.

	A	B	C	D	E	F	G
A	-	16	12	21	-	-	-
B	16	-	-	17	20	-	-
C	12	-	-	28	-	31	-
D	21	17	28	-	18	19	23
E	-	20	-	18	-	-	11
F	-	-	31	19	-	-	27
G	-	-	-	23	11	27	-

However, it is possible to optimise the network by removing some edges and still ensure that all points on the network remain connected. The network which achieves the maximum saving is shown below. It has a weight of 93, representing a saving of 243 − 93 = 150 from the original network.



Using network.txt (right click and 'Save Link/Target As...'), a 6K text file containing a network with forty vertices, and given in matrix form, find the maximum saving which can be achieved by removing redundant edges whilst ensuring that the network remains connected.

Problem 108

In the following equation x, y, and n are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

For $n = 4$ there are exactly three distinct solutions:

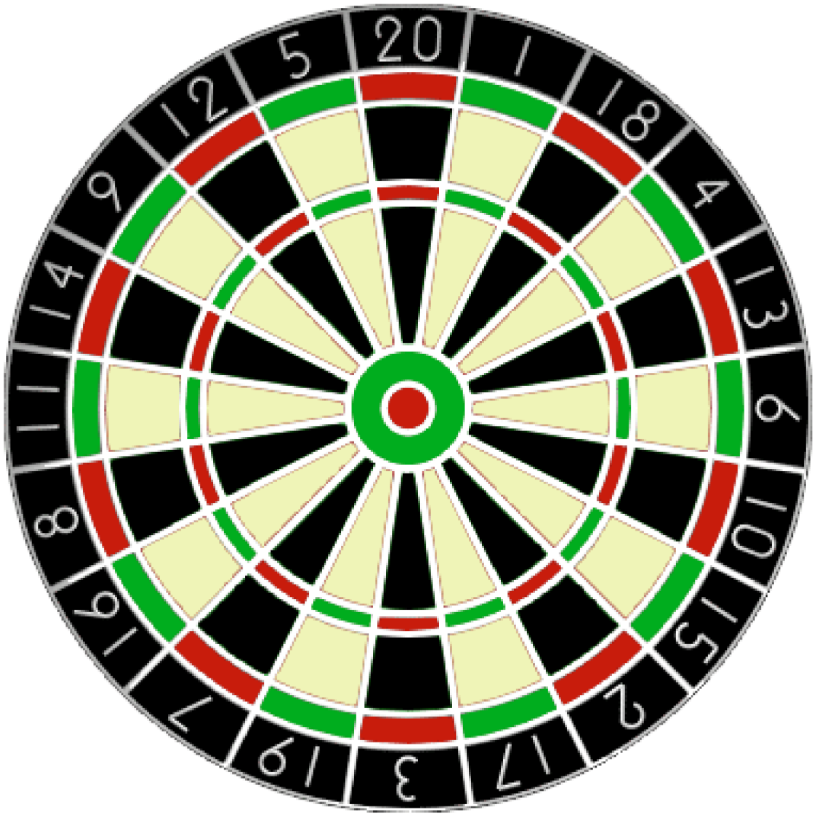
$$\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$$
$$\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$
$$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

What is the least value of n for which the number of distinct solutions exceeds one-thousand?

NOTE: This problem is an easier version of problem 110; it is strongly advised that you solve this one first.

Problem 109

In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty.



The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero. The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively.

At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points.

There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system, which means that the player must land a double (including the double bulls-eye at the centre of the board) on their final dart to win; any other dart that would reduce their running total to one or lower means the score for that set of three darts is "bust".

When a player is able to finish on their current score it is called a "checkout" and the highest checkout is 170: T20 T20 D25 (two treble 20s and double bull).

There are exactly eleven distinct ways to checkout on a score of 6:

D3		
D1	D2	
S2	D2	
D2	D1	
S4	D1	

S1	S1	D2
S1	T1	D1
S1	S3	D1
D1	D1	D1
D1	S2	D1
S2	S2	D1

Note that D1 D2 is considered **different** to D2 D1 as they finish on different doubles. However, the combination S1 T1 D1 is considered the **same** as T1 S1 D1.

In addition we shall not include misses in considering combinations; for example, D3 is the **same** as 0 D3 and 0 0 D3.

Incredibly there are 42336 distinct ways of checking out in total.

How many distinct ways can a player checkout with a score less than 100?

Problem 110

In the following equation x, y, and n are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

It can be verified that when $n = 1260$ there are 113 distinct solutions and this is the least value of n for which the total number of distinct solutions exceeds one hundred.

What is the least value of n for which the number of distinct solutions exceeds four million?

NOTE: This problem is a much more difficult version of problem 108 and as it is well beyond the limitations of a brute force approach it requires a clever implementation.

Problem 111

Considering 4-digit primes containing repeated digits it is clear that they cannot all be the same: 1111 is divisible by 11, 2222 is divisible by 22, and so on. But there are nine 4-digit primes containing three ones:

1117, 1151, 1171, 1181, 1511, 1811, 2111, 4111, 8111

We shall say that M(n, d) represents the maximum number of repeated digits for an n-digit prime where d is the repeated digit, N(n, d) represents the number of such primes, and S(n, d) represents the sum of these primes.

So M(4, 1) = 3 is the maximum number of repeated digits for a 4-digit prime where one is the repeated digit, there are N(4, 1) = 9 such primes, and the sum of these primes is S(4, 1) = 22275. It turns out that for *d* = 0, it is only possible to have M(4, 0) = 2 repeated digits, but there are N(4, 0) = 13 such cases.

In the same way we obtain the following results for 4-digit primes.

Digit, *d*	M(4, *d*)	N(4, *d*)	S(4, *d*)
0	2	13	67061
1	3	9	22275
2	3	1	2221
3	3	12	46214
4	3	2	8888
5	3	1	5557
6	3	1	6661
7	3	9	57863
8	3	1	8887
9	3	7	48073

For *d* = 0 to 9, the sum of all S(4, *d*) is 273700.

Find the sum of all S(10, *d*).

Problem 113

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

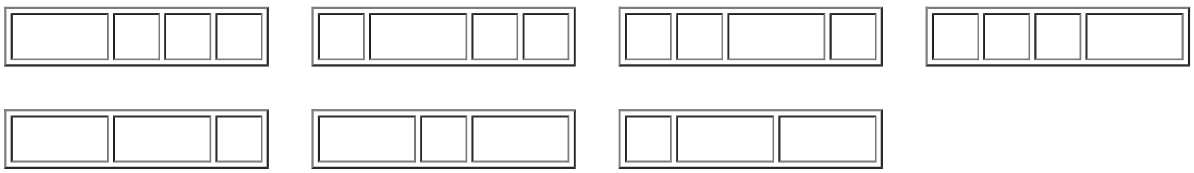
As *n* increases, the proportion of bouncy numbers below *n* increases such that there are only 12951 numbers below one-million that are not bouncy and only 277032 non-bouncy numbers below 10^{10}.

How many numbers below a googol (10^{100}) are not bouncy?

Problem 116

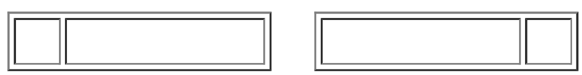
A row of five black square tiles is to have a number of its tiles replaced with coloured oblong tiles chosen from red (length two), green (length three), or blue (length four).

If red tiles are chosen there are exactly seven ways this can be done.



If green tiles are chosen there are three ways.

And if blue tiles are chosen there are two ways.



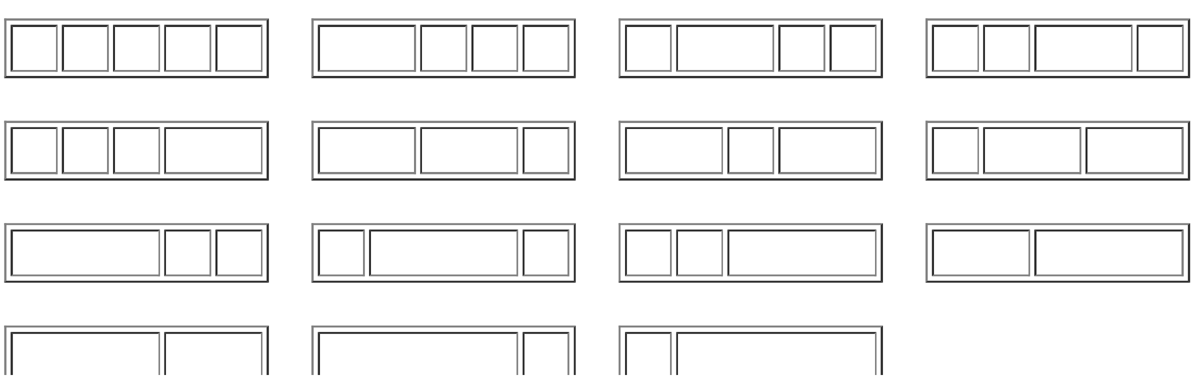
Assuming that colours cannot be mixed there are 7 + 3 + 2 = 12 ways of replacing the black tiles in a row measuring five units in length.

How many different ways can the black tiles in a row measuring fifty units in length be replaced if colours cannot be mixed and at least one coloured tile must be used?

NOTE: This is related to problem 117.

Problem 117

Using a combination of black square tiles and oblong tiles chosen from: red tiles measuring two units, green tiles measuring three units, and blue tiles measuring four units, it is possible to tile a row measuring five units in length in exactly fifteen different ways.



How many ways can a row measuring fifty units in length be tiled?

NOTE: This is related to problem 116.

Problem 118

Using all of the digits 1 through 9 and concatenating them freely to form decimal integers, different sets can be formed. Interestingly with the set {2,5,47,89,631}, all of the elements belonging to it are prime.

How many distinct sets containing each of the digits one through nine exactly once contain only prime elements?

Problem 122

The most naive way of computing n^{15} requires fourteen multiplications:

$$n \times n \times \ldots \times n = n^{15}$$

But using a "binary" method you can compute it in six multiplications:

$$n \times n = n^2$$
$$n^2 \times n^2 = n^4$$
$$n^4 \times n^4 = n^8$$
$$n^8 \times n^4 = n^{12}$$
$$n^{12} \times n^2 = n^{14}$$
$$n^{14} \times n = n^{15}$$

However it is yet possible to compute it in only five multiplications:

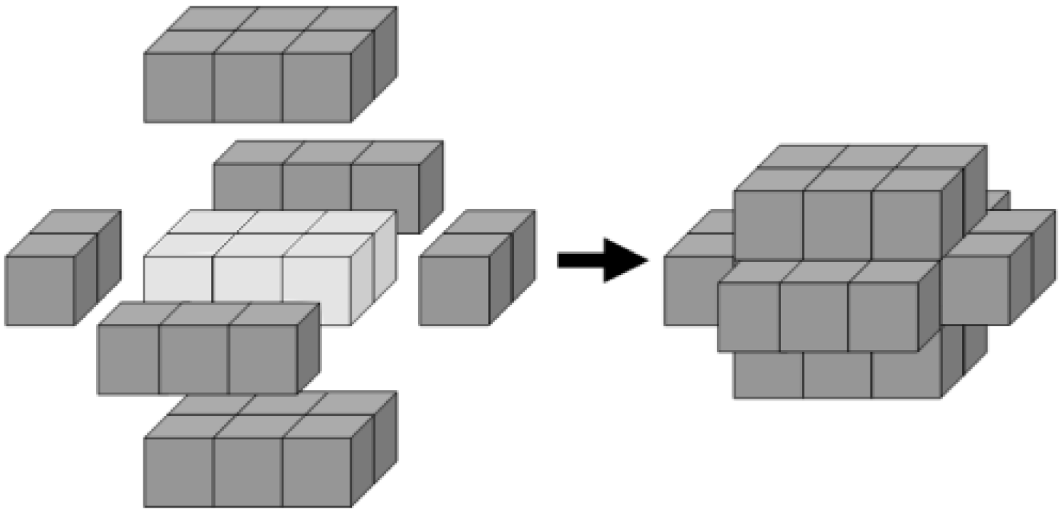
$$n \times n = n^2$$
$$n^2 \times n = n^3$$
$$n^3 \times n^3 = n^6$$
$$n^6 \times n^6 = n^{12}$$
$$n^{12} \times n^3 = n^{15}$$

We shall define m(k) to be the minimum number of multiplications to compute n^k; for example m(15) = 5.

For $1 \leq k \leq 200$, find $\sum$ m(k).

Problem 126

The minimum number of cubes to cover every visible face on a cuboid measuring 3 x 2 x 1 is twenty-two.



If we then add a second layer to this solid it would require forty-six cubes to cover every visible face, the third layer would require seventy-eight cubes, and the fourth layer would require one-hundred and eighteen cubes to cover every visible face.

However, the first layer on a cuboid measuring 5 x 1 x 1 also requires twenty-two cubes; similarly the first layer on cuboids measuring 5 x 3 x 1, 7 x 2 x 1, and 11 x 1 x 1 all contain forty-six cubes.

We shall define C(*n*) to represent the number of cuboids that contain *n* cubes in one of its layers. So C(22) = 2, C(46) = 4, C(78) = 5, and C(118) = 8.

It turns out that 154 is the least value of *n* for which C(*n*) = 10.

Find the least value of *n* for which C(*n*) = 1000.

Problem 127

The radical of *n*, rad(*n*), is the product of distinct prime factors of *n*. For example, $504 = 2^3 \times 3^2 \times 7$, so rad(504) = $2 \times 3 \times 7 = 42$.

We shall define the triplet of positive integers (*a*, *b*, *c*) to be an abc-hit if:

1. GCD(*a*, *b*) = GCD(*a*, *c*) = GCD(*b*, *c*) = 1
2. $a < b$
3. $a + b = c$
4. rad(*abc*) $< c$

For example, (5, 27, 32) is an abc-hit, because:

1. GCD(5, 27) = GCD(5, 32) = GCD(27, 32) = 1
2. $5 < 27$
3. $5 + 27 = 32$
4. rad(4320) = $30 < 32$

It turns out that abc-hits are quite rare and there are only thirty-one abc-hits for $c < 1000$, with $\sum c = 12523$.

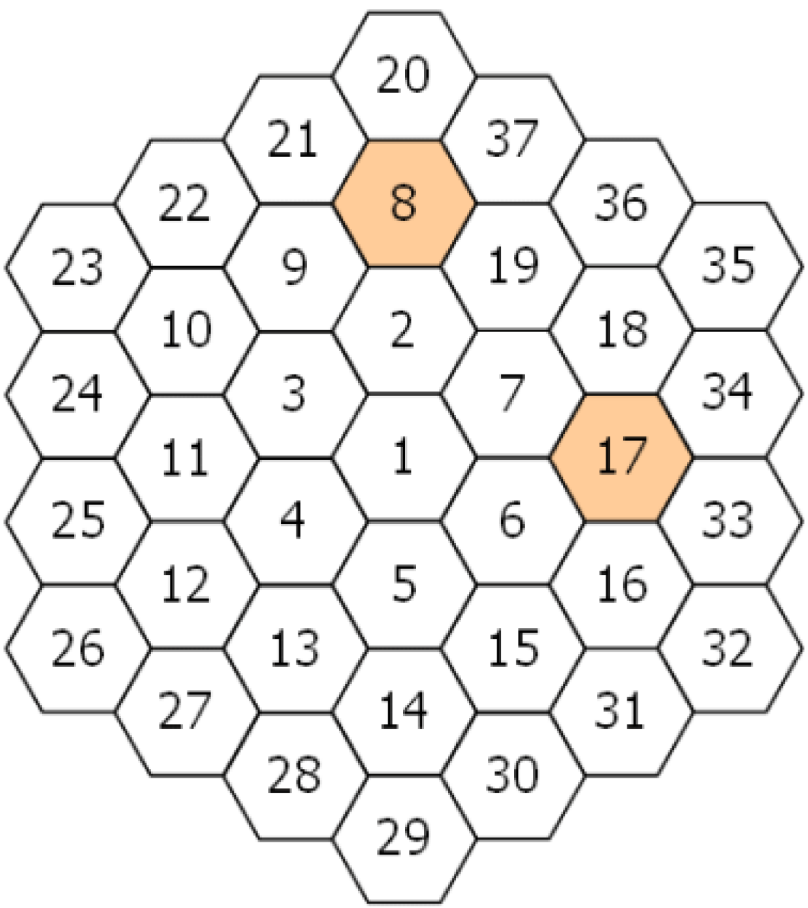
Find $\sum c$ for $c < 120000$.

Note: This problem has been changed recently, please check that you are using the right parameters.

Problem 128

A hexagonal tile with number 1 is surrounded by a ring of six hexagonal tiles, starting at "12 o'clock" and numbering the tiles 2 to 7 in an anti-clockwise direction.

New rings are added in the same fashion, with the next rings being numbered 8 to 19, 20 to 37, 38 to 61, and so on. The diagram below shows the first three rings.



By finding the difference between tile n and each its six neighbours we shall define PD(n) to be the number of those differences which are prime.

For example, working clockwise around tile 8 the differences are 12, 29, 11, 6, 1, and 13. So PD(8) = 3.

In the same way, the differences around tile 17 are 1, 17, 16, 1, 11, and 10, hence PD(17) = 2.

It can be shown that the maximum value of PD(n) is 3.

If all of the tiles for which PD(n) = 3 are listed in ascending order to form a sequence, the 10th tile would be 271.

Find the 2000th tile in this sequence.

Problem 129

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Given that n is a positive integer and GCD(n, 10) = 1, it can be shown that there always exists a value, k, for which R(k) is divisible by n, and let A(n) be the least such value of k; for example, A(7) = 6 and A(41) = 5.

The least value of n for which A(n) first exceeds ten is 17.

Find the least value of n for which A(n) first exceeds one-million.

Problem 130

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Given that n is a positive integer and GCD(n, 10) = 1, it can be shown that there always exists a value, k, for which R(k) is divisible by n, and let A(n) be the least such value of k; for example, A(7) = 6 and A(41) = 5.

You are given that for all primes, $p > 5$, that $p - 1$ is divisible by A(p). For example, when p = 41, A(41) = 5, and 40 is divisible by 5.

However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703.

Find the sum of the first twenty-five composite values of n for which GCD(n, 10) = 1 and $n - 1$ is divisible by A(n).

Problem 131

There are some prime values, p, for which there exists a positive integer, n, such that the expression $n^3 + n^2p$ is a perfect cube.

For example, when $p = 19$, $8^3 + 8^2 \times 19 = 12^3$.

What is perhaps most surprising is that for each prime with this property the value of n is unique, and there are only four such primes below one-hundred.

How many primes below one million have this remarkable property?

Problem 132

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k.

For example, R(10) = 1111111111 = 11×41×271×9091, and the sum of these prime factors is 9414.

Find the sum of the first forty prime factors of R(10^9).

Problem 133

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Let us consider repunits of the form R(10^n).

Although R(10), R(100), or R(1000) are not divisible by 17, R(10000) is divisible by 17. Yet there is no value of n for which R(10^n) will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of R(10^n).

Find the sum of all the primes below one-hundred thousand that will never be a factor of R(10^n).

Problem 134

Consider the consecutive primes $p_1 = 19$ and $p_2 = 23$. It can be verified that 1219 is the smallest number such that the last digits are formed by p_1 whilst also being divisible by p_2.

In fact, with the exception of $p_1 = 3$ and $p_2 = 5$, for every pair of consecutive primes, $p_2 > p_1$, there exist values of n for which the last digits are formed by p_1 and n is divisible by p_2. Let S be the smallest of these values of n.

Find $\sum S$ for every pair of consecutive primes with $5 \leq p_1 \leq 1000000$.

Problem 135

Given the positive integers, x, y, and z, are consecutive terms of an arithmetic progression, the least value of the positive integer, n, for which the equation, $x^2 - y^2 - z^2 = n$, has exactly two solutions is $n = 27$:

$$34^2 - 27^2 - 20^2 = 12^2 - 9^2 - 6^2 = 27$$

It turns out that $n = 1155$ is the least value which has exactly ten solutions.

How many values of n less than one million have exactly ten distinct solutions?

Problem 136

The positive integers, x, y, and z, are consecutive terms of an arithmetic progression. Given that n is a positive integer, the equation, $x^2 - y^2 - z^2 = n$, has exactly one solution when $n = 20$:

$$13^2 - 10^2 - 7^2 = 20$$

In fact there are twenty-five values of n below one hundred for which the equation has a unique solution.

How many values of n less than fifty million have exactly one solution?

Problem 137

Consider the infinite polynomial series $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + \ldots$, where F_k is the kth term in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, ... ; that is, $F_k = F_{k-1} + F_{k-2}$, $F_1 = 1$ and $F_2 = 1$.

For this problem we shall be interested in values of x for which $A_F(x)$ is a positive integer.

Surprisingly $A_F(1/2) = (1/2).1 + (1/2)^2.1 + (1/2)^3.2 + (1/2)^4.3 + (1/2)^5.5 + \ldots$
$= 1/2 + 1/4 + 2/8 + 3/16 + 5/32 + \ldots$
$= 2$

The corresponding values of x for the first five natural numbers are shown below.

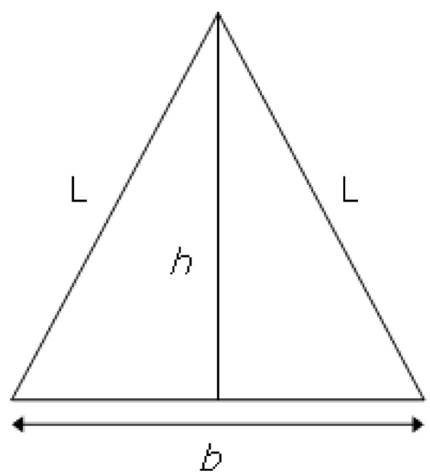
x	$A_F(x)$
$\sqrt{2}-1$	1
1/2	2
$(\sqrt{13}-2)/3$	3
$(\sqrt{89}-5)/8$	4
$(\sqrt{34}-3)/5$	5

We shall call $A_F(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 10th golden nugget is 74049690.

Find the 15th golden nugget.

Problem 138

Consider the isosceles triangle with base length, b = 16, and legs, L = 17.



By using the Pythagorean theorem it can be seen that the height of the triangle, $h = \sqrt{(17^2 - 8^2)} = 15$, which is one less than the base length.

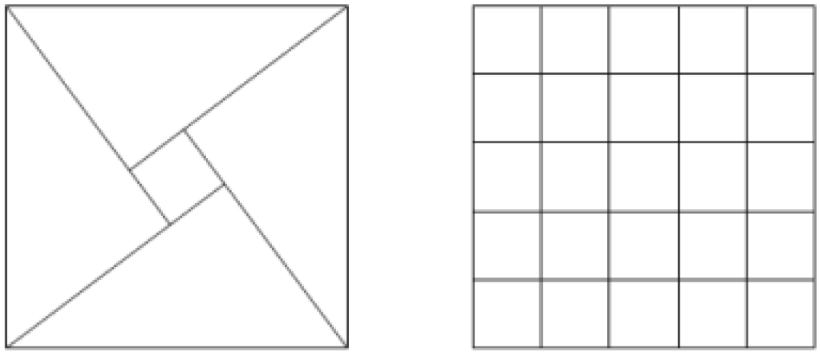
With b = 272 and L = 305, we get h = 273, which is one more than the base length, and this is the second smallest isosceles triangle with the property that $h = b \pm 1$.

Find $\sum$ L for the twelve smallest isosceles triangles for which $h = b \pm 1$ and b, L are positive integers.

Problem 139

Let (a, b, c) represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length *c*.

For example, (3, 4, 5) triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares.



However, if (5, 12, 13) triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square.

Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?

Problem 140

Consider the infinite polynomial series $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + \ldots$, where G_k is the kth term of the second order recurrence relation $G_k = G_{k-1} + G_{k-2}$, $G_1 = 1$ and $G_2 = 4$; that is, 1, 4, 5, 9, 14, 23, ...

For this problem we shall be concerned with values of x for which $A_G(x)$ is a positive integer.

The corresponding values of x for the first five natural numbers are shown below.

x	$A_G(x)$
$(\sqrt{5}-1)/4$	1
$2/5$	2
$(\sqrt{22}-2)/6$	3
$(\sqrt{137}-5)/14$	4
$1/2$	5

We shall call $A_G(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365.

Find the sum of the first thirty golden nuggets.

Problem 141

A positive integer, *n*, is divided by *d* and the quotient and remainder are *q* and *r* respectively. In addition *d*, *q*, and *r* are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio 3/2).
We will call such numbers, *n*, progressive.

Some progressive numbers, such as 9 and 10404 = 102^2, happen to also be perfect squares.
The sum of all progressive perfect squares below one hundred thousand is 124657.

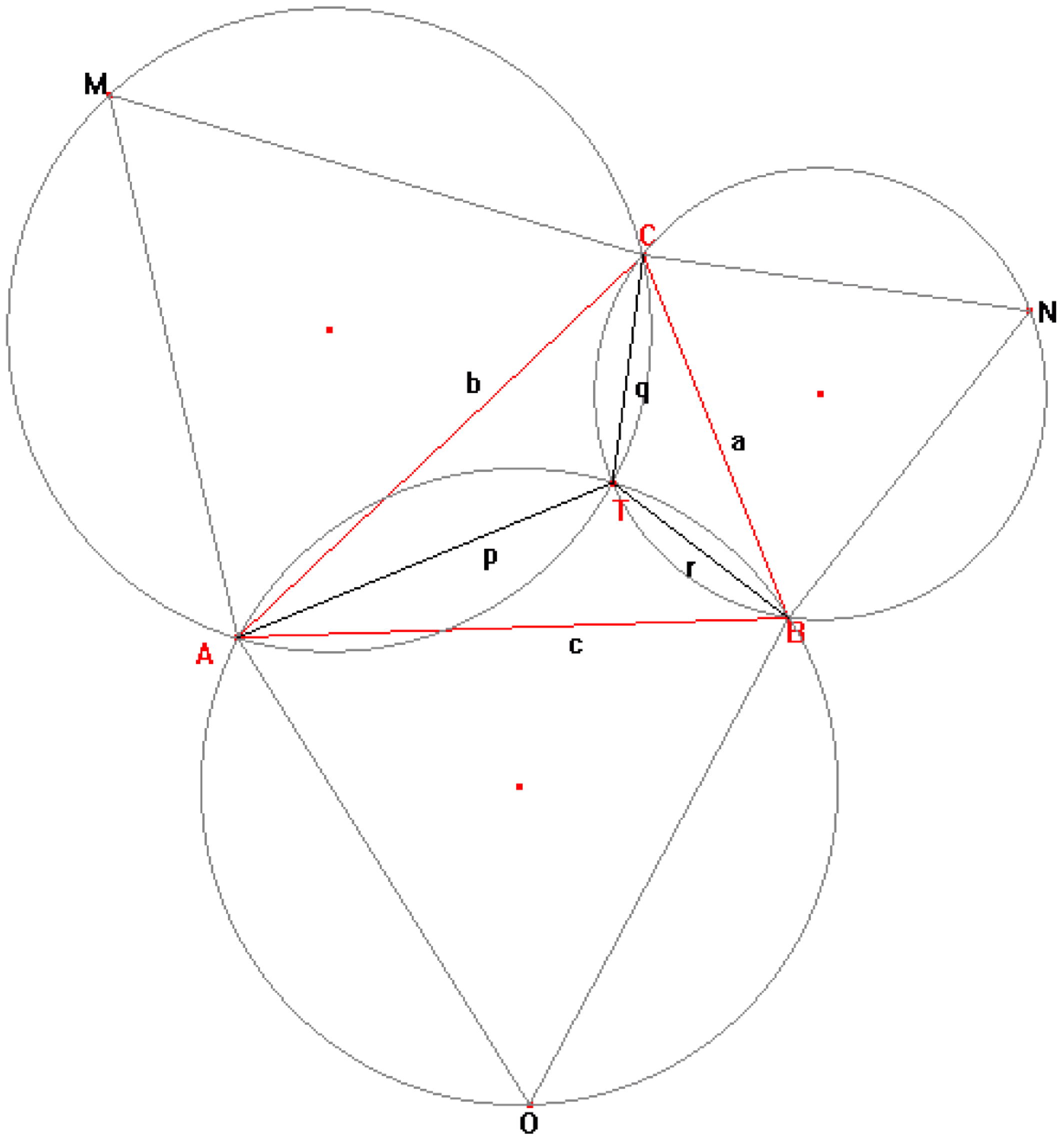
Find the sum of all progressive perfect squares below one trillion (10^{12}).

Problem 143

Let ABC be a triangle with all interior angles being less than 120 degrees. Let X be any point inside the triangle and let XA = p, XB = q, and XC = r.

Fermat challenged Torricelli to find the position of X such that p + q + r was minimised.

Torricelli was able to prove that if equilateral triangles AOB, BNC and AMC are constructed on each side of triangle ABC, the circumscribed circles of AOB, BNC, and AMC will intersect at a single point, T, inside the triangle. Moreover he proved that T, called the Torricelli/Fermat point, minimises p + q + r. Even more remarkable, it can be shown that when the sum is minimised, AN = BM = CO = p + q + r and that AN, BM and CO also intersect at T.



If the sum is minimised and a, b, c, p, q and r are all positive integers we shall call triangle ABC a Torricelli triangle. For example, a = 399, b = 455, c = 511 is an example of a Torricelli triangle, with p + q + r = 784.

Find the sum of all distinct values of p + q + r ≤ 120000 for Torricelli triangles.

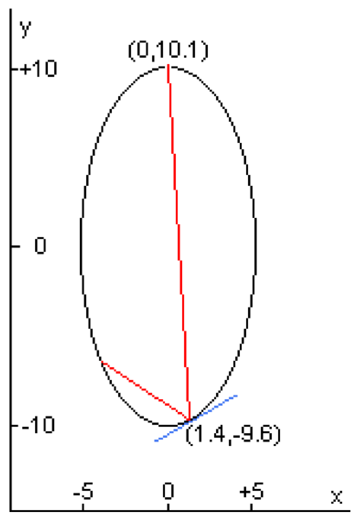
Note: This problem has been changed recently, please check that you are using the right parameters.

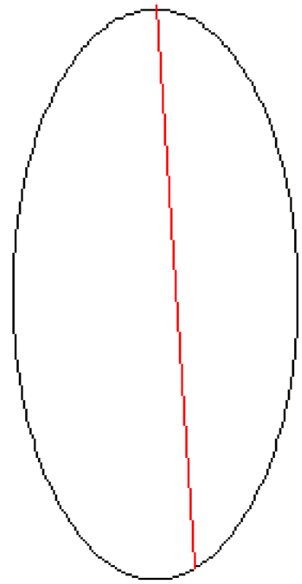
Problem 144

In laser physics, a "white cell" is a mirror system that acts as a delay line for the laser beam. The beam enters the cell, bounces around on the mirrors, and eventually works its way back out.

The specific white cell we will be considering is an ellipse with the equation $4x^2 + y^2 = 100$

The section corresponding to $-0.01 \leq x \leq +0.01$ at the top is missing, allowing the light to enter and exit through the hole.





The light beam in this problem starts at the point (0.0,10.1) just outside the white cell, and the beam first impacts the mirror at (1.4,-9.6).

Each time the laser beam hits the surface of the ellipse, it follows the usual law of reflection "angle of incidence equals angle of reflection." That is, both the incident and reflected beams make the same angle with the normal line at the point of incidence.

In the figure on the left, the red line shows the first two points of contact between the laser beam and the wall of the white cell; the blue line shows the line tangent to the ellipse at the point of incidence of the first bounce.

The slope m of the tangent line at any point (x,y) of the given ellipse is: $m = -4x/y$

The normal line is perpendicular to this tangent line at the point of incidence.

The animation on the right shows the first 10 reflections of the beam.

How many times does the beam hit the internal surface of the white cell before exiting?

Problem 145

Some positive integers n have the property that the sum [n + reverse(n)] consists entirely of odd (decimal) digits. For instance, 36 + 63 = 99 and 409 + 904 = 1313. We will call such numbers *reversible*; so 36, 63, 409, and 904 are reversible. Leading zeroes are not allowed in either n or reverse(n).

There are 120 reversible numbers below one-thousand.

How many reversible numbers are there below one-billion (10^9)?

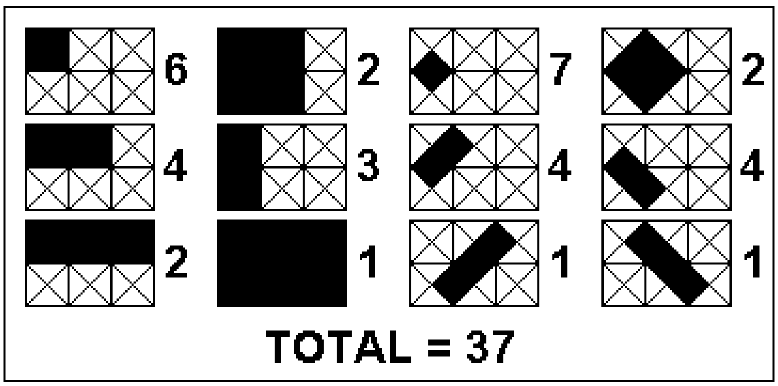
Problem 146

The smallest positive integer n for which the numbers n^2+1, n^2+3, n^2+7, n^2+9, n^2+13, and n^2+27 are consecutive primes is 10. The sum of all such integers n below one-million is 1242490.

What is the sum of all such integers n below 150 million?

Problem 147

In a 3x2 cross-hatched grid, a total of 37 different rectangles could be situated within that grid as indicated in the sketch.



There are 5 grids smaller than 3x2, vertical and horizontal dimensions being important, i.e. 1x1, 2x1, 3x1, 1x2 and 2x2. If each of them is cross-hatched, the following number of different rectangles could be situated within those smaller grids:

1x1: 1
2x1: 4
3x1: 8
1x2: 4
2x2: 18

Adding those to the 37 of the 3x2 grid, a total of 72 different rectangles could be situated within 3x2 and smaller grids.

How many different rectangles could be situated within 47x43 and smaller grids?

Problem 149

Looking at the table below, it is easy to verify that the maximum possible sum of adjacent numbers in any direction (horizontal, vertical, diagonal or anti-diagonal) is 16 (= 8 + 7 + 1).

−2	5	3	2
9	−6	5	1
3	2	7	3
−1	8	−4	8

Now, let us repeat the search, but on a much larger scale:

First, generate four million pseudo-random numbers using a specific form of what is known as a "Lagged Fibonacci Generator":

For $1 \leq k \leq 55$, $s_k = [100003 - 200003k + 300007k^3] \pmod{1000000} - 500000$.
For $56 \leq k \leq 4000000$, $s_k = [s_{k-24} + s_{k-55} + 1000000] \pmod{1000000} - 500000$.

Thus, $s_{10} = -393027$ and $s_{100} = 86613$.

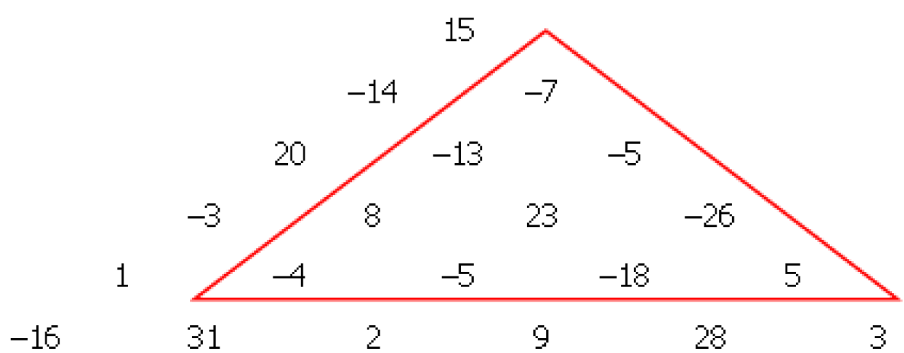
The terms of s are then arranged in a 2000×2000 table, using the first 2000 numbers to fill the first row (sequentially), the next 2000 numbers to fill the second row, and so on.

Finally, find the greatest sum of (any number of) adjacent entries in any direction (horizontal, vertical, diagonal or anti-diagonal).

Problem 150

In a triangular array of positive and negative integers, we wish to find a sub-triangle such that the sum of the numbers it contains is the smallest possible.

In the example below, it can be easily verified that the marked triangle satisfies this condition having a sum of −42.



We wish to make such a triangular array with one thousand rows, so we generate 500500 pseudo-random numbers s_k in the range $\pm 2^{19}$, using a type of random number generator (known as a Linear Congruential Generator) as follows:

$t := 0$

for k = 1 up to k = 500500:

t := (615949*t + 797807) modulo 2^{20}

s_k := $t-2^{19}$

Thus: s_1 = 273519, s_2 = −153582, s_3 = 450905 etc

Our triangular array is then formed using the pseudo-random numbers thus:

$$\begin{array}{c} s_1 \\ s_2 \ \ s_3 \\ s_4 \ \ s_5 \ \ s_6 \\ s_7 \ \ s_8 \ \ s_9 \ \ s_{10} \\ \cdots \end{array}$$

Sub-triangles can start at any element of the array and extend down as far as we like (taking-in the two elements directly below it from the next row, the three elements directly below from the row after that, and so on).
The "sum of a sub-triangle" is defined as the sum of all the elements it contains.
Find the smallest possible sub-triangle sum.

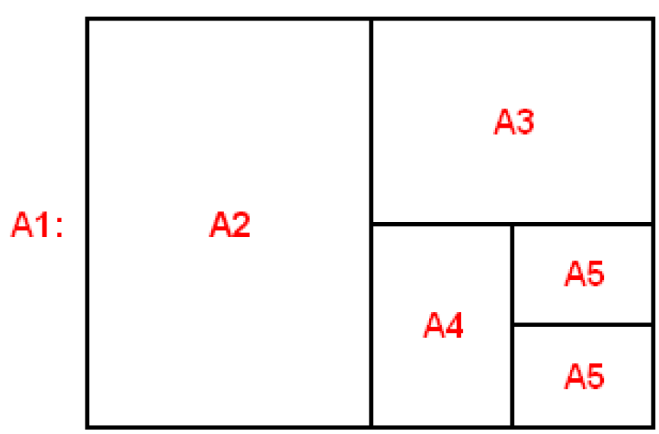
Problem 151

A printing shop runs 16 batches (jobs) every week and each batch requires a sheet of special colour-proofing paper of size A5.

Every Monday morning, the foreman opens a new envelope, containing a large sheet of the special paper with size A1.

He proceeds to cut it in half, thus getting two sheets of size A2. Then he cuts one of them in half to get two sheets of size A3 and so on until he obtains the A5-size sheet needed for the first batch of the week.

All the unused sheets are placed back in the envelope.



At the beginning of each subsequent batch, he takes from the envelope one sheet of paper at random. If it is of size A5, he uses it. If it is larger, he repeats the 'cut-in-half' procedure until he has what he needs and any remaining sheets are always placed back in the envelope.

Excluding the first and last batch of the week, find the expected number of times (during each week) that the foreman finds a single sheet of paper in the envelope.

Give your answer rounded to six decimal places using the format x.xxxxxx .

Problem 152

There are several ways to write the number 1/2 as a sum of inverse squares using *distinct* integers.

For instance, the numbers {2,3,4,5,7,12,15,20,28,35} can be used:

$$\frac{1}{2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{28^2} + \frac{1}{35^2}$$

In fact, only using integers between 2 and 45 inclusive, there are exactly three ways to do it, the remaining two being: {2,3,4,6,7,9,10,20,28,35,36,45} and {2,3,4,6,7,9,12,15,28,30,35,36,45}.

How many ways are there to write the number 1/2 as a sum of inverse squares using distinct integers between 2 and 80 inclusive?

Problem 153

As we all know the equation x^2=-1 has no solutions for real x.
If we however introduce the imaginary number i this equation has two solutions: $x=i$ and $x=-i$.
If we go a step further the equation $(x-3)^2=-4$ has two complex solutions: $x=3+2i$ and $x=3-2i$.
$x=3+2i$ and $x=3-2i$ are called each others' complex conjugate.
Numbers of the form $a+bi$ are called complex numbers.
In general $a+bi$ and $a-bi$ are each other's complex conjugate.

A Gaussian Integer is a complex number $a+bi$ such that both a and b are integers.
The regular integers are also Gaussian integers (with b=0).
To distinguish them from Gaussian integers with $b \neq 0$ we call such integers "rational integers."
A Gaussian integer is called a divisor of a rational integer n if the result is also a Gaussian integer.
If for example we divide 5 by 1+2i we can simplify $\frac{5}{1+2i}$ in the following manner:
Multiply numerator and denominator by the complex conjugate of 1+2i: 1−2i.

The result is $\frac{5}{1+2i} = \frac{5}{1+2i}\frac{1-2i}{1-2i} = \frac{5(1-2i)}{1-(2i)^2} = \frac{5(1-2i)}{1-(-4)} = \frac{5(1-2i)}{5} = 1-2i$.

So $1+2i$ is a divisor of 5.

Note that $1+i$ is not a divisor of 5 because $\frac{5}{1+i} = \frac{5}{2} - \frac{5}{2}i$.

Note also that if the Gaussian Integer $(a+bi)$ is a divisor of a rational integer n, then its complex conjugate $(a-bi)$ is also a divisor of n.

In fact, 5 has six divisors such that the real part is positive: $\{1, 1 + 2i, 1 - 2i, 2 + i, 2 - i, 5\}$.
The following is a table of all of the divisors for the first five positive rational integers:

n	Gaussian integer divisors with positive real part	Sum s(n) of these divisors
1	1	1
2	1, $1+i$, $1-i$, 2	5
3	1, 3	4
4	1, $1+i$, $1-i$, 2, $2+2i$, $2-2i$, 4	13
5	1, $1+2i$, $1-2i$, $2+i$, $2-i$, 5	12

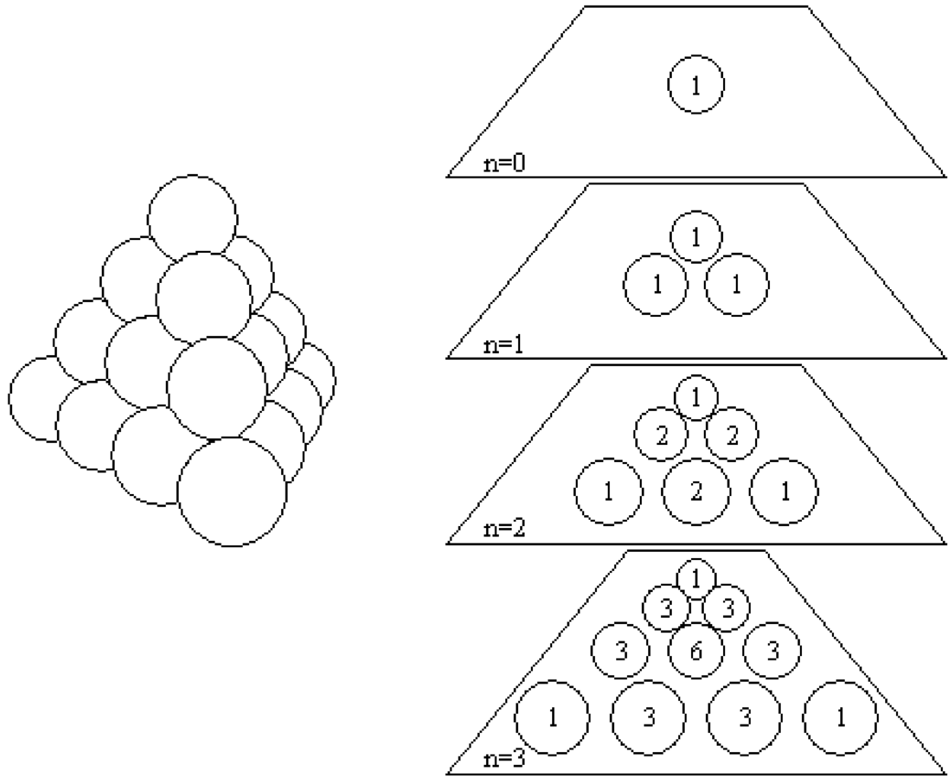
For divisors with positive real parts, then, we have: $\sum_{n=1}^{5} s(n) = 35$.

For $1 \le n \le 10^5$, $\sum$ s(n)=17924657155.

What is $\sum$ s(n) for $1 \le n \le 10^8$?

Problem 154

A triangular pyramid is constructed using spherical balls so that each ball rests on exactly three balls of the next lower level.



Then, we calculate the number of paths leading from the apex to each position:

A path starts at the apex and progresses downwards to any of the three spheres directly below the current position.

Consequently, the number of paths to reach a certain position is the sum of the numbers immediately above it (depending on the position, there are up to three numbers above it).

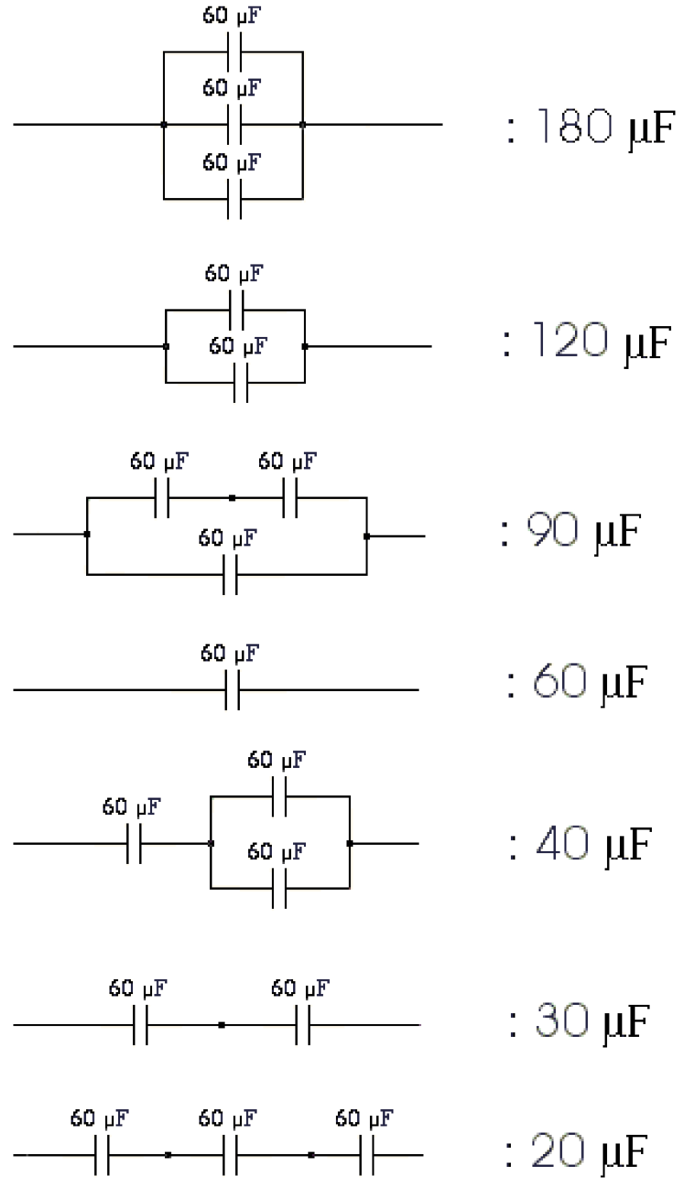
The result is *Pascal's pyramid* and the numbers at each level n are the coefficients of the trinomial expansion $(x + y + z)^n$.

How many coefficients in the expansion of $(x + y + z)^{200000}$ are multiples of 10^{12}?

Problem 155

An electric circuit uses exclusively identical capacitors of the same value C.
The capacitors can be connected in series or in parallel to form sub-units, which can then be connected in series or in parallel with other capacitors or other sub-units to form larger sub-units, and so on up to a final circuit.

Using this simple procedure and up to n identical capacitors, we can make circuits having a range of different total capacitances. For example, using up to n=3 capacitors of 60 μF each, we can obtain the following 7 distinct total capacitance values:



If we denote by $D(n)$ the number of distinct total capacitance values we can obtain when using up to n equal-valued capacitors and the simple procedure described above, we have: $D(1)$=1, $D(2)$=3, $D(3)$=7 …

Find $D(18)$.

Reminder : When connecting capacitors C_1, C_2 etc in parallel, the total capacitance is
$C_T = C_1 + C_2 + \ldots$,
whereas when connecting them in series, the overall capacitance is given by:

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots$$

Problem 156

Starting from zero the natural numbers are written down in base 10 like this:

0 1 2 3 4 5 6 7 8 9 10 11 12....

Consider the digit d=1. After we write down each number n, we will update the number of ones that have occurred and call this number $f(n,1)$. The first values for $f(n,1)$, then, are as follows:

n	$f(n,1)$
0	0
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	2
11	4
12	5

Note that $f(n,1)$ never equals 3.
So the first two solutions of the equation $f(n,1)=n$ are n=0 and n=1. The next solution is n=199981.

In the same manner the function $f(n,d)$ gives the total number of digits d that have been written down after the number n has been written.
In fact, for every digit $d \neq 0$, 0 is the first solution of the equation $f(n,d)=n$.

Let $s(d)$ be the sum of all the solutions for which $f(n,d)=n$.
You are given that $s(1)$=22786974071.

Find $\sum s(d)$ for $1 \leq d \leq 9$.

Note: if, for some n, $f(n,d)=n$ for more than one value of d this value of n is counted again for every value of d for which $f(n,d)=n$.

Problem 157

Consider the diophantine equation $^1/_a+^1/_b=^p/_{10^n}$ with a, b, p, n positive integers and $a \leq b$.
For n=1 this equation has 20 solutions that are listed below:

$^1/_1+^1/_1=^{20}/_{10}$	$^1/_1+^1/_2=^{15}/_{10}$	$^1/_1+^1/_5=^{12}/_{10}$	$^1/_1+^1/_{10}=^{11}/_{10}$	$^1/_2+^1/_2=^{10}/_{10}$
$^1/_2+^1/_5=^7/_{10}$	$^1/_2+^1/_{10}=^6/_{10}$	$^1/_3+^1/_6=^5/_{10}$	$^1/_3+^1/_{15}=^4/_{10}$	$^1/_4+^1/_4=^5/_{10}$

$^1/_4+^1/_{20}=^3/_{10}$	$^1/_5+^1/_5=^4/_{10}$	$^1/_5+^1/_{10}=^3/_{10}$	$^1/_6+^1/_{30}=^2/_{10}$	$^1/_{10}+^1/_{10}=^2/_{10}$
$^1/_{11}+^1/_{110}=^1/_{10}$	$^1/_{12}+^1/_{60}=^1/_{10}$	$^1/_{14}+^1/_{35}=^1/_{10}$	$^1/_{15}+^1/_{30}=^1/_{10}$	$^1/_{20}+^1/_{20}=^1/_{10}$

How many solutions has this equation for $1 \leq n \leq 9$?

Problem 158

Taking three different letters from the 26 letters of the alphabet, character strings of length three can be formed.
Examples are 'abc', 'hat' and 'zyx'.
When we study these three examples we see that for 'abc' two characters come lexicographically after its neighbour to the left.
For 'hat' there is exactly one character that comes lexicographically after its neighbour to the left. For 'zyx' there are zero characters that come lexicographically after its neighbour to the left.
In all there are 10400 strings of length 3 for which exactly one character comes lexicographically after its neighbour to the left.

We now consider strings of $n \leq 26$ different characters from the alphabet.
For every n, p(n) is the number of strings of length n for which exactly one character comes lexicographically after its neighbour to the left.

What is the maximum value of p(n)?

Problem 159

A composite number can be factored many different ways. For instance, not including multiplication by one, 24 can be factored in 7 distinct ways:

24 = 2x2x2x3
24 = 2x3x4
24 = 2x2x6
24 = 4x6
24 = 3x8
24 = 2x12
24 = 24

Recall that the digital root of a number, in base 10, is found by adding together the digits of that number, and repeating that process until a number is arrived at that is less than 10. Thus the digital root of 467 is 8.

We shall call a Digital Root Sum (DRS) the sum of the digital roots of the individual factors of our number.
The chart below demonstrates all of the DRS values for 24.

Factorisation	Digital Root Sum
2x2x2x3	9
2x3x4	9
2x2x6	10
4x6	10
3x8	11
2x12	5
24	6

The maximum Digital Root Sum of 24 is 11.
The function mdrs(n) gives the maximum Digital Root Sum of n. So mdrs(24)=11.
Find $\sum$mdrs(n) for 1 $<n<$ 1,000,000.

Problem 160

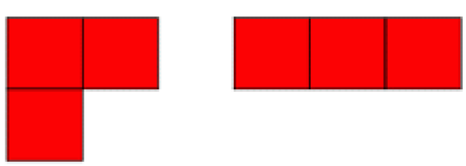
For any N, let f(N) be the last five digits before the trailing zeroes in N!.
For example,

9! = 362880 so f(9)=36288
10! = 3628800 so f(10)=36288
20! = 2432902008176640000 so f(20)=17664

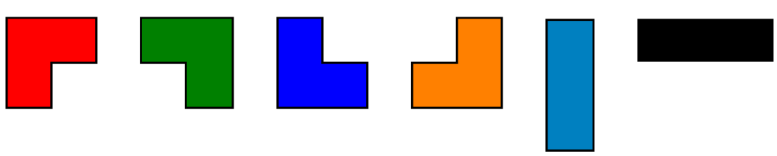
Find f(1,000,000,000,000)

Problem 161

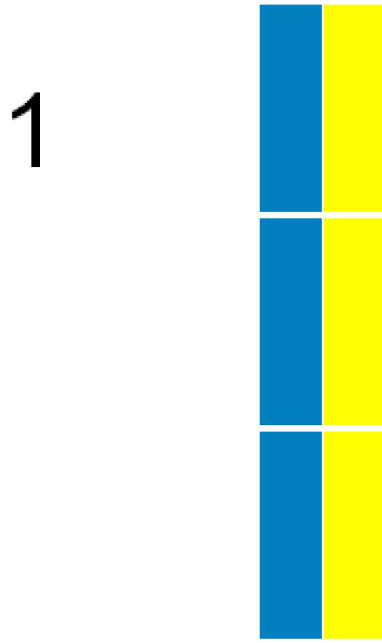
A triomino is a shape consisting of three squares joined via the edges. There are two basic forms:



If all possible orientations are taken into account there are six:



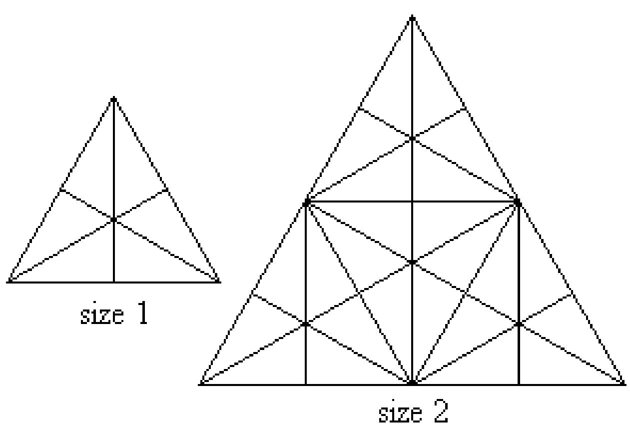
Any n by m grid for which nxm is divisible by 3 can be tiled with triominoes.
If we consider tilings that can be obtained by reflection or rotation from another tiling as different there are 41 ways a 2 by 9 grid can be tiled with triominoes:



In how many ways can a 9 by 12 grid be tiled in this way by triominoes?

Problem 163

Consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side, such as in the *size 1* triangle in the sketch below.



Sixteen triangles of either different shape or size or orientation or location can now be observed in that triangle. Using *size 1* triangles as building blocks, larger triangles can be formed, such as the *size 2* triangle in the above sketch. One-hundred and four triangles of either different shape or size or orientation or location can now be observed in that *size 2* triangle.

It can be observed that the *size 2* triangle contains 4 *size 1* triangle building blocks. A *size 3* triangle would contain 9 *size 1* triangle building blocks and a *size n* triangle would thus contain n^2 *size 1* triangle building blocks.

If we denote T(n) as the number of triangles present in a triangle of *size* n, then

T(1) = 16
T(2) = 104

Find T(36).

Problem 164

How many 20 digit numbers n (without any leading zero) exist such that no three consecutive digits of n have a sum greater than 9?

Problem 165

A segment is uniquely defined by its two endpoints.
By considering two line segments in plane geometry there are three possibilities:
the segments have zero points, one point, or infinitely many points in common.

Moreover when two segments have exactly one point in common it might be the case that that common point is an endpoint of either one of the segments or of both. If a common point of two segments is not an endpoint of either of the segments it is an interior point of both segments.
We will call a common point T of two segments L_1 and L_2 a true intersection point of L_1 and L_2 if T is the only common point of L_1 and L_2 and T is an interior point of both segments.

Consider the three segments L_1, L_2, and L_3:

L_1: (27, 44) to (12, 32)
L_2: (46, 53) to (17, 62)
L_3: (46, 70) to (22, 40)

It can be verified that line segments L_2 and L_3 have a true intersection point. We note that as the one of the end points of L_3: (22,40) lies on L_1 this is not considered to be a true point of intersection. L_1 and L_2 have no common point. So among the three line segments, we find one true intersection point.

Now let us do the same for 5000 line segments. To this end, we generate 20000 numbers using the so-called "Blum Blum Shub" pseudo-random number generator.

$s_0 = 290797$

$s_{n+1} = s_n \times s_n$ (modulo 50515093)

$t_n = s_n$ (modulo 500)

To create each line segment, we use four consecutive numbers t_n. That is, the first line segment is given by:

(t_1, t_2) to (t_3, t_4)

The first four numbers computed according to the above generator should be: 27, 144, 12 and 232. The first segment would thus be (27,144) to (12,232).

How many distinct true intersection points are found among the 5000 line segments?

Problem 167

For two positive integers a and b, the Ulam sequence U(a,b) is defined by $U(a,b)_1 = a$, $U(a,b)_2 = b$ and for k > 2, $U(a,b)_k$ is the smallest integer greater than $U(a,b)_{(k-1)}$ which can be written in exactly one way as the sum of two distinct previous members of U(a,b).

For example, the sequence U(1,2) begins with
1, 2, 3 = 1 + 2, 4 = 1 + 3, 6 = 2 + 4, 8 = 2 + 6, 11 = 3 + 8;
5 does not belong to it because 5 = 1 + 4 = 2 + 3 has two representations as the sum of two previous members, likewise 7 = 1 + 6 = 3 + 4.

Find $\sum U(2,2n+1)_k$ for $2 \leq n \leq 10$, where $k = 10^{11}$.

Problem 168

Consider the number 142857. We can right-rotate this number by moving the last digit (7) to the front of it, giving us 714285.
It can be verified that $714285 = 5 \times 142857$.
This demonstrates an unusual property of 142857: it is a divisor of its right-rotation.

Find the last 5 digits of the sum of all integers n, $10 < n < 10^{100}$, that have this property.

Problem 169

Define f(0)=1 and f(n) to be the number of different ways n can be expressed as a sum of integer powers of 2 using each power no more than twice.

For example, f(10)=5 since there are five different ways to express 10:

1 + 1 + 8
1 + 1 + 4 + 4
1 + 1 + 2 + 2 + 4
2 + 4 + 4
2 + 8

What is f(10^{25})?

Problem 170

Take the number 6 and multiply it by each of 1273 and 9854:

$6 \times 1273 = 7638$
$6 \times 9854 = 59124$

By concatenating these products we get the 1 to 9 pandigital 763859124. We will call 763859124 the "concatenated product of 6 and (1273,9854)". Notice too, that the concatenation of the input numbers, 612739854, is also 1 to 9 pandigital.

The same can be done for 0 to 9 pandigital numbers.

What is the largest 0 to 9 pandigital 10-digit concatenated product of an integer with two or more other integers, such that the concatenation of the input numbers is also a 0 to 9 pandigital 10-digit number?

Problem 171

For a positive integer n, let f(n) be the sum of the squares of the digits (in base 10) of n, e.g.

f(3) = 3^2 = 9,
f(25) = $2^2 + 5^2$ = 4 + 25 = 29,
f(442) = $4^2 + 4^2 + 2^2$ = 16 + 16 + 4 = 36

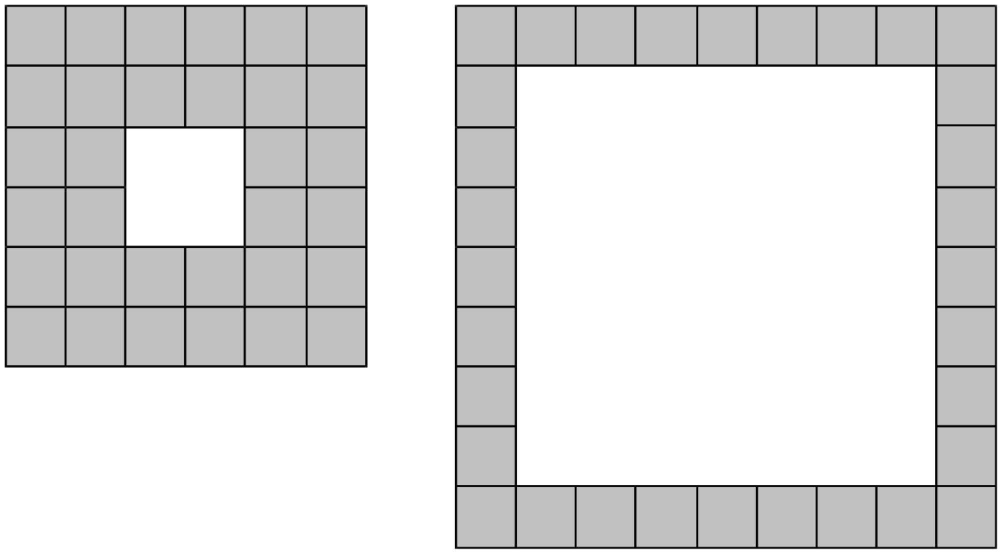
Find the last nine digits of the sum of all n, $0 < n < 10^{20}$, such that f(n) is a perfect square.

Problem 172

How many 18-digit numbers n (without leading zeros) are there such that no digit occurs more than three times in n?

Problem 173

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry. For example, using exactly thirty-two square tiles we can form two different square laminae:



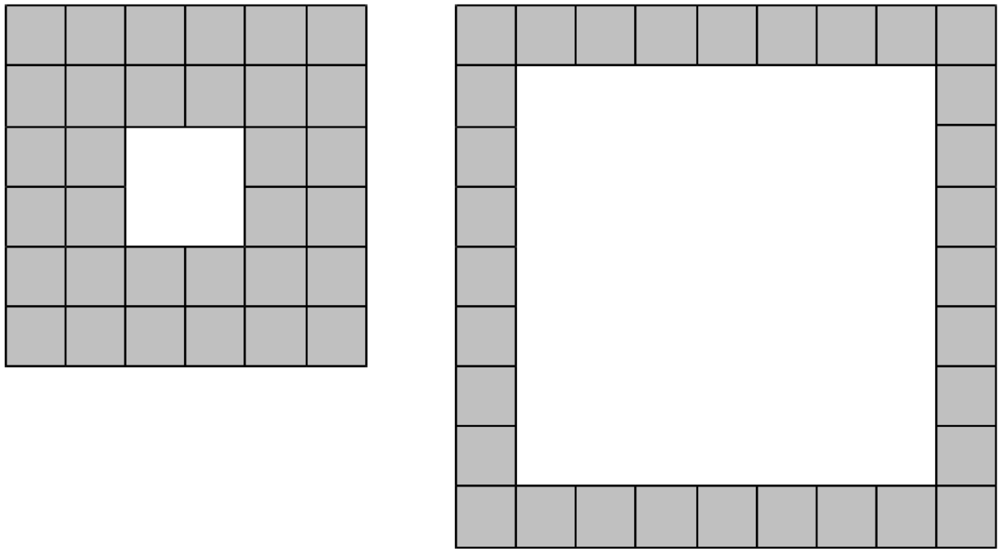
With one-hundred tiles, and not necessarily using all of the tiles at one time, it is possible to form forty-one different square laminae.

Using up to one million tiles how many different square laminae can be formed?

Problem 174

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry.

Given eight tiles it is possible to form a lamina in only one way: 3x3 square with a 1x1 hole in the middle. However, using thirty-two tiles it is possible to form two distinct laminae.



If t represents the number of tiles used, we shall say that t = 8 is type L(1) and t = 32 is type L(2).

Let N(n) be the number of $t \leq$ 1000000 such that t is type L(n); for example, N(15) = 832.

What is $\sum$ N(n) for 1 $\leq n \leq$ 10?

Problem 175

Define f(0)=1 and f(n) to be the number of ways to write n as a sum of powers of 2 where no power occurs more than twice.

For example, f(10)=5 since there are five different ways to express 10:
10 = 8+2 = 8+1+1 = 4+4+2 = 4+2+2+1+1 = 4+4+1+1

It can be shown that for every fraction p/q (p>0, q>0) there exists at least one integer n such that
f(n)/f(n-1)=p/q.

For instance, the smallest n for which f(n)/f(n-1)=13/17 is 241.
The binary expansion of 241 is 11110001.
Reading this binary number from the most significant bit to the least significant bit there are 4 one's, 3 zeroes and 1 one. We shall call the string 4,3,1 the *Shortened Binary Expansion* of 241.

Find the Shortened Binary Expansion of the smallest n for which
f(n)/f(n-1)=123456789/987654321.

Give your answer as comma separated integers, without any whitespaces.

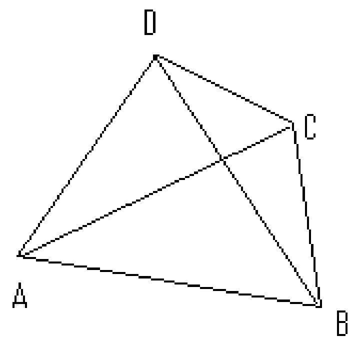
Problem 176

The four rectangular triangles with sides (9,12,15), (12,16,20), (5,12,13) and (12,35,37) all have one of the shorter sides (catheti) equal to 12. It can be shown that no other integer sided rectangular triangle exists with one of the catheti equal to 12.

Find the smallest integer that can be the length of a cathetus of exactly 47547 different integer sided rectangular triangles.

Problem 177

Let ABCD be a convex quadrilateral, with diagonals AC and BD. At each vertex the diagonal makes an angle with each of the two sides, creating eight corner angles.



For example, at vertex A, the two angles are CAD, CAB.

We call such a quadrilateral for which all eight corner angles have integer values when measured in degrees an "integer angled quadrilateral". An example of an integer angled quadrilateral is a square, where all eight corner angles are 45°. Another example is given by DAC = 20°, BAC = 60°, ABD = 50°, CBD = 30°, BCA = 40°, DCA = 30°, CDB = 80°, ADB = 50°.

What is the total number of non-similar integer angled quadrilaterals?

Note: In your calculations you may assume that a calculated angle is integral if it is within a tolerance of 10^{-9} of an integer value.

Problem 178

Consider the number 45656.
It can be seen that each pair of consecutive digits of 45656 has a difference of one.
A number for which every pair of consecutive digits has a difference of one is called a step number.
A pandigital number contains every decimal digit from 0 to 9 at least once.
How many pandigital step numbers less than 10^{40} are there?

Problem 180

For any integer n, consider the three functions

$$f_{1,n}(x,y,z) = x^{n+1} + y^{n+1} - z^{n+1}$$
$$f_{2,n}(x,y,z) = (xy + yz + zx)*(x^{n-1} + y^{n-1} - z^{n-1})$$
$$f_{3,n}(x,y,z) = xyz*(x^{n-2} + y^{n-2} - z^{n-2})$$

and their combination

$$f_n(x,y,z) = f_{1,n}(x,y,z) + f_{2,n}(x,y,z) - f_{3,n}(x,y,z)$$

We call (x,y,z) a golden triple of order k if x, y, and z are all rational numbers of the form a / b with
$0 < a < b \leq k$ and there is (at least) one integer n, so that $f_n(x,y,z) = 0$.

Let $s(x,y,z) = x + y + z$.
Let $t = u$ / v be the sum of all distinct $s(x,y,z)$ for all golden triples (x,y,z) of order 35.
All the $s(x,y,z)$ and t must be in reduced form.

Find $u + v$.

Problem 181

Having three black objects B and one white object W they can be grouped in 7 ways like this:

(BBBW) (B,BBW) (B,B,BW) (B,B,B,W) (B,BB,W) (BBB,W) (BB,BW)

In how many ways can sixty black objects B and forty white objects W be thus grouped?

Problem 182

The RSA encryption is based on the following procedure:

Generate two distinct primes p and q.
Compute $n=pq$ and $\varphi=(p-1)(q-1)$.
Find an integer e, $1<e<\varphi$, such that $\gcd(e,\varphi)=1$.

A message in this system is a number in the interval $[0,n-1]$.
A text to be encrypted is then somehow converted to messages (numbers in the interval $[0,n-$

1]).
To encrypt the text, for each message, m, $c=m^e$ mod n is calculated.

To decrypt the text, the following procedure is needed: calculate d such that ed=1 mod φ, then for each encrypted message, c, calculate $m=c^d$ mod n.

There exist values of e and m such that m^e mod $n=m$.
We call messages m for which m^e mod $n=m$ unconcealed messages.

An issue when choosing e is that there should not be too many unconcealed messages.
For instance, let p=19 and q=37.
Then n=19*37=703 and φ=18*36=648.
If we choose e=181, then, although gcd(181,648)=1 it turns out that all possible messages m (0≤m≤n-1) are unconcealed when calculating m^e mod n.
For any valid choice of e there exist some unconcealed messages.
It's important that the number of unconcealed messages is at a minimum.

Choose p=1009 and q=3643.
Find the sum of all values of e, 1<e<φ(1009,3643) and gcd(e,φ)=1, so that the number of unconcealed messages for this value of e is at a minimum.

Problem 183

Let N be a positive integer and let N be split into k equal parts, r = N/k, so that N = r + r + ... + r.
Let P be the product of these parts, P = $r \times r \times \ldots \times r = r^k$.

For example, if 11 is split into five equal parts, 11 = 2.2 + 2.2 + 2.2 + 2.2 + 2.2, then P = 2.2^5 = 51.53632.

Let M(N) = P_{max} for a given value of N.

It turns out that the maximum for N = 11 is found by splitting eleven into four equal parts which leads to P_{max} = $(11/4)^4$; that is, M(11) = 14641/256 = 57.19140625, which is a terminating decimal.

However, for N = 8 the maximum is achieved by splitting it into three equal parts, so M(8) = 512/27, which is a non-terminating decimal.

Let D(N) = N if M(N) is a non-terminating decimal and D(N) = -N if M(N) is a terminating decimal.

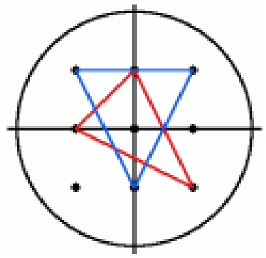
For example, ΣD(N) for 5 ≤ N ≤ 100 is 2438.

Find ΣD(N) for 5 ≤ N ≤ 10000.

Problem 184

Consider the set I_r of points (x,y) with integer co-ordinates in the interior of the circle with radius r, centered at the origin, i.e. $x^2 + y^2 < r^2$.

For a radius of 2, I_2 contains the nine points (0,0), (1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1) and (1,-1). There are eight triangles having all three vertices in I_2 which contain the origin in the interior. Two of them are shown below, the others are obtained from these by rotation.



For a radius of 3, there are 360 triangles containing the origin in the interior and having all vertices in I_3 and for I_5 the number is 10600.

How many triangles are there containing the origin in the interior and having all three vertices in I_{105}?

Problem 185

The game Number Mind is a variant of the well known game Master Mind.

Instead of coloured pegs, you have to guess a secret sequence of digits. After each guess you're only told in how many places you've guessed the correct digit. So, if the sequence was 1234 and you guessed 2036, you'd be told that you have one correct digit; however, you would NOT be told that you also have another digit in the wrong place.

For instance, given the following guesses for a 5-digit secret sequence,

90342 ;2 correct
70794 ;0 correct
39458 ;2 correct
34109 ;1 correct
51545 ;2 correct
12531 ;1 correct

The correct sequence 39542 is unique.

Based on the following guesses,

5616185650518293 ;2 correct
3847439647293047 ;1 correct
5855462940810587 ;3 correct

9742855507068353 ;3 correct
4296849643607543 ;3 correct
3174248439465858 ;1 correct
4513559094146117 ;2 correct
7890971548908067 ;3 correct
8157356344118483 ;1 correct
2615250744386899 ;2 correct
8690095851526254 ;3 correct
6375711915077050 ;1 correct
6913859173121360 ;1 correct
6442889055042768 ;2 correct
2321386104303845 ;0 correct
2326509471271448 ;2 correct
5251583379644322 ;2 correct
1748270476758276 ;3 correct
4895722652190306 ;1 correct
3041631117224635 ;3 correct
1841236454324589 ;3 correct
2659862637316867 ;2 correct

Find the unique 16-digit secret sequence.

Problem 186

Here are the records from a busy telephone system with one million users:

RecNr	Caller	Called
1	200007	100053
2	600183	500439
3	600863	701497
...	...	...

The telephone number of the caller and the called number in record n are Caller(n) = S_{2n-1} and Called(n) = S_{2n} where $S_{1,2,3,...}$ come from the "Lagged Fibonacci Generator":

For $1 \leq k \leq 55$, $S_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000)
For $56 \leq k$, $S_k = [S_{k-24} + S_{k-55}]$ (modulo 1000000)

If Caller(n) = Called(n) then the user is assumed to have misdialled and the call fails; otherwise the call is successful.

From the start of the records, we say that any pair of users X and Y are friends if X calls Y or vice-versa. Similarly, X is a friend of a friend of Z if X is a friend of Y and Y is a friend of Z; and so on for longer chains.

The Prime Minister's phone number is 524287. After how many successful calls, not counting misdials, will 99% of the users (including the PM) be a friend, or a friend of a friend etc., of the Prime Minister?

Problem 188

The *hyperexponentiation* or *tetration* of a number a by a positive integer b, denoted by a↑↑b or ^{b}a, is recursively defined by:

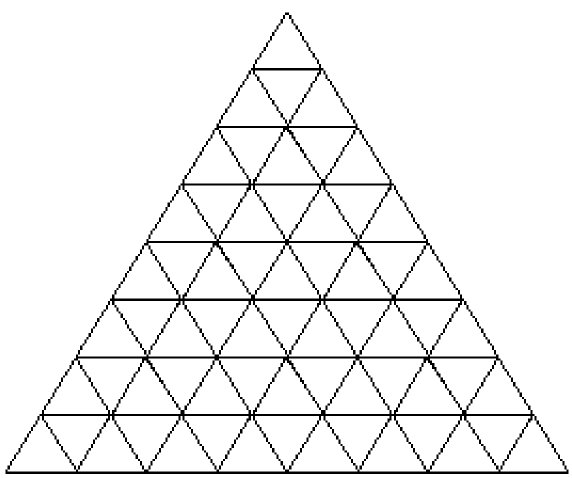
a↑↑1 = a,
a↑↑(k+1) = $a^{(a↑↑k)}$.

Thus we have e.g. 3↑↑2 = 3^3 = 27, hence 3↑↑3 = 3^{27} = 7625597484987 and 3↑↑4 is roughly $10^{3.6383346400240996*10^{12}}$.

Find the last 8 digits of 1777↑↑1855.

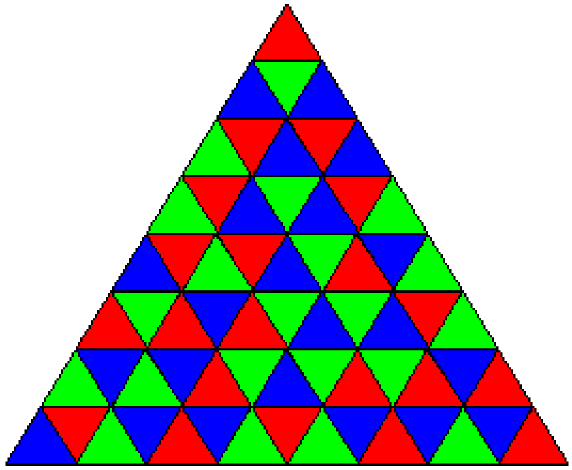
Problem 189

Consider the following configuration of 64 triangles:



We wish to colour the interior of each triangle with one of three colours: red, green or blue, so that no two neighbouring triangles have the same colour. Such a colouring shall be called valid. Here, two triangles are said to be neighbouring if they share an edge.
Note: if they only share a vertex, then they are not neighbours.

For example, here is a valid colouring of the above grid:



A colouring C' which is obtained from a colouring C by rotation or reflection is considered *distinct* from C unless the two are identical.

How many distinct valid colourings are there for the above configuration?

Problem 190

Let $S_m = (x_1, x_2, \ldots , x_m)$ be the m-tuple of positive real numbers with $x_1 + x_2 + \ldots + x_m = m$ for which $P_m = x_1 * x_2^2 * \ldots * x_m^m$ is maximised.

For example, it can be verified that $[P_{10}] = 4112$ ([ ] is the integer part function).

Find $\Sigma[P_m]$ for $2 \leq m \leq 15$.

Problem 192

Let x be a real number.
A *best approximation* to x for the *denominator bound* d is a rational number r/s *in reduced form*, with $s \leq d$, such that any rational number which is closer to x than r/s has a denominator larger than d:

$$|p/q-x| < |r/s-x| \Rightarrow q > d$$

For example, the best approximation to $\sqrt{13}$ for the denominator bound 20 is 18/5 and the best approximation to $\sqrt{13}$ for the denominator bound 30 is 101/28.

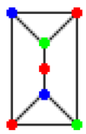
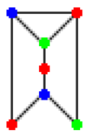
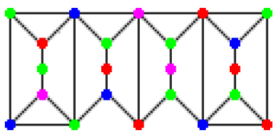
Find the sum of all denominators of the best approximations to $\sqrt{n}$ for the denominator bound 10^{12}, where n is not a perfect square and $1 < n \leq 100000$.

Problem 193

A positive integer n is called squarefree, if no square of a prime divides n, thus 1, 2, 3, 5, 6, 7, 10, 11 are squarefree, but not 4, 8, 9, 12.

How many squarefree numbers are there below 2^{50}?

Problem 194

Consider graphs built with the units A:  and B: , where the units are glued along the vertical edges as in the graph .

A configuration of type (a,b,c) is a graph thus built of a units A and b units B, where the graph's vertices are coloured using up to c colours, so that no two adjacent vertices have the same colour.
The compound graph above is an example of a configuration of type (2,2,6), in fact of type $(2,2,c)$ for all $c \geq 4$.

Let N(a,b,c) be the number of configurations of type (a,b,c).
For example, N(1,0,3) = 24, N(0,2,4) = 92928 and N(2,2,3) = 20736.

Find the last 8 digits of N(25,75,1984).

Problem 195

Let's call an integer sided triangle with exactly one angle of 60 degrees a 60-degree triangle.
Let r be the radius of the inscribed circle of such a 60-degree triangle.

There are 1234 60-degree triangles for which $r \leq 100$.
Let T(n) be the number of 60-degree triangles for which $r \leq n$, so
T(100) = 1234, T(1000) = 22767, and T(10000) = 359912.

Find T(1053779).

Problem 196

Build a triangle from all positive integers in the following way:

```
 1
 2  3
 4  5  6
 7  8  9 10
11 12 13 14 15
16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31 32 33 34 35 36
37 38 39 40 41 42 43 44 45
46 47 48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63 64 65 66
. . .
```

Each positive integer has up to eight neighbours in the triangle.

A set of three primes is called a *prime triplet* if one of the three primes has the other two as neighbours in the triangle.

For example, in the second row, the prime numbers 2 and 3 are elements of some prime triplet.

If row 8 is considered, it contains two primes which are elements of some prime triplet, i.e. 29 and 31.
If row 9 is considered, it contains only one prime which is an element of some prime triplet: 37.

Define S(n) as the sum of the primes in row n which are elements of any prime triplet.
Then S(8)=60 and S(9)=37.

You are given that S(10000)=950007619.

Find S(5678027) + S(7208785).

Problem 197

Given is the function $f(x) = \lfloor 2^{30.403243784-x^2} \rfloor \times 10^{-9}$ ($\lfloor \ \rfloor$ is the floor-function),
the sequence u_n is defined by $u_0 = -1$ and $u_{n+1} = f(u_n)$.

Find $u_n + u_{n+1}$ for $n = 10^{12}$.
Give your answer with 9 digits after the decimal point.

Problem 198

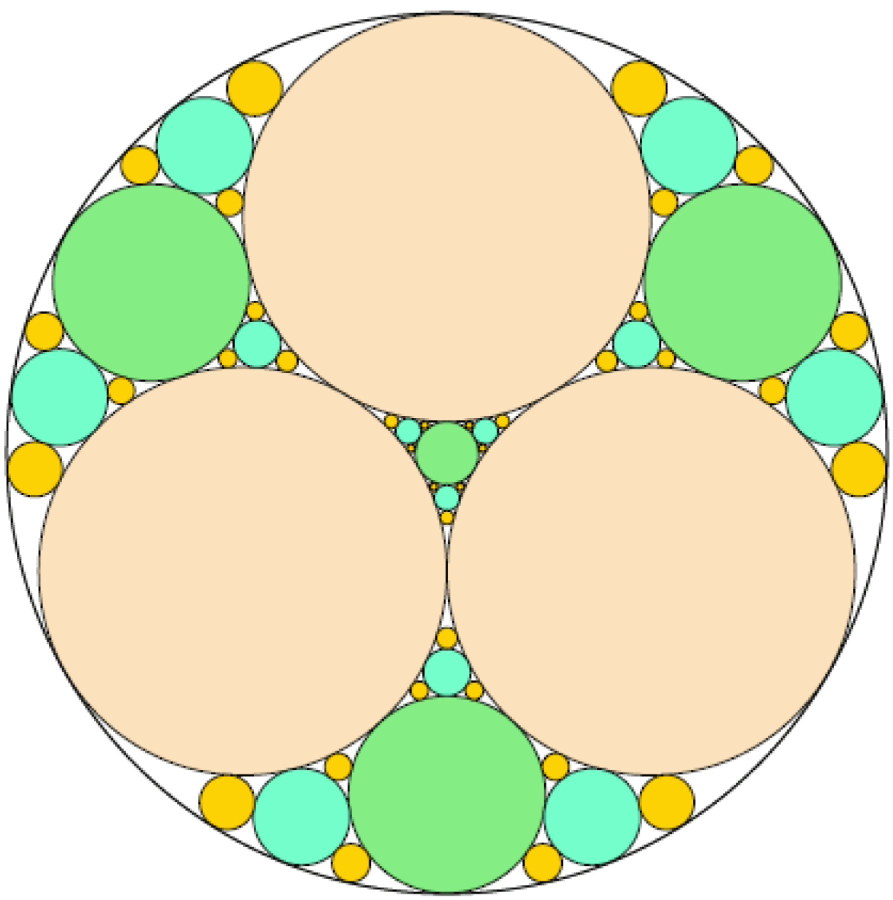
A best approximation to a real number x for the denominator bound d is a rational number r/s (in reduced form) with $s \le d$, so that any rational number p/q which is closer to x than r/s has $q > d$.

Usually the best approximation to a real number is uniquely determined for all denominator bounds. However, there are some exceptions, e.g. 9/40 has the two best approximations 1/4 and 1/5 for the denominator bound 6. We shall call a real number x *ambiguous*, if there is at least one denominator bound for which x possesses two best approximations. Clearly, an ambiguous number is necessarily rational.

How many ambiguous numbers $x = p/q$, $0 < x < 1/100$, are there whose denominator q does not exceed 10^8?

Problem 199

Three circles of equal radius are placed inside a larger circle such that each pair of circles is tangent to one another and the inner circles do not overlap. There are four uncovered "gaps" which are to be filled iteratively with more tangent circles.



At each iteration, a maximally sized circle is placed in each gap, which creates more gaps for the next iteration. After 3 iterations (pictured), there are 108 gaps and the fraction of the area which is not covered by circles is 0.06790342, rounded to eight decimal places.

What fraction of the area is not covered by circles after 10 iterations?
Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

Problem 200

We shall define a sqube to be a number of the form, p^2q^3, where p and q are distinct primes.
For example, 200 = $5^2 2^3$ or 120072949 = $23^2 61^3$.

The first five squbes are 72, 108, 200, 392, and 500.

Interestingly, 200 is also the first number for which you cannot change any single digit to make a prime; we shall call such numbers, prime-proof. The next prime-proof sqube which contains the contiguous sub-string "200" is 1992008.

Find the 200th prime-proof sqube containing the contiguous sub-string "200".

Problem 201

For any set A of numbers, let sum(A) be the sum of the elements of A.
Consider the set B = {1,3,6,8,10,11}.
There are 20 subsets of B containing three elements, and their sums are:

sum({1,3,6}) = 10,
sum({1,3,8}) = 12,
sum({1,3,10}) = 14,
sum({1,3,11}) = 15,
sum({1,6,8}) = 15,
sum({1,6,10}) = 17,
sum({1,6,11}) = 18,
sum({1,8,10}) = 19,
sum({1,8,11}) = 20,
sum({1,10,11}) = 22,
sum({3,6,8}) = 17,
sum({3,6,10}) = 19,
sum({3,6,11}) = 20,
sum({3,8,10}) = 21,
sum({3,8,11}) = 22,
sum({3,10,11}) = 24,
sum({6,8,10}) = 24,
sum({6,8,11}) = 25,
sum({6,10,11}) = 27,
sum({8,10,11}) = 29.

Some of these sums occur more than once, others are unique.
For a set A, let U(A,k) be the set of unique sums of k-element subsets of A, in our example we find U(B,3) = {10,12,14,18,21,25,27,29} and sum(U(B,3)) = 156.

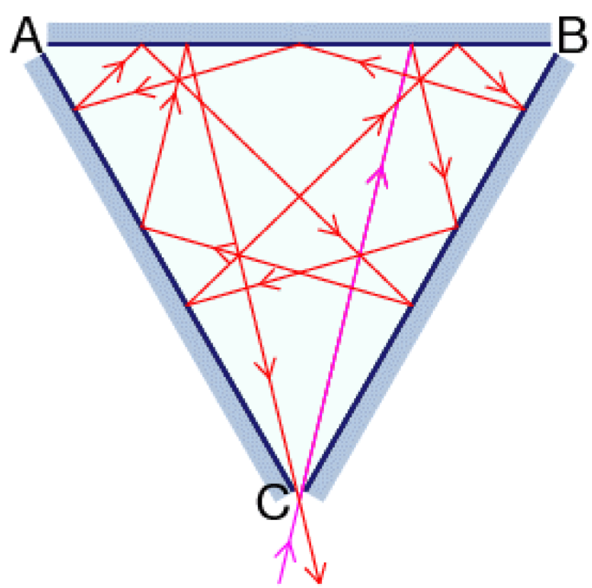
Now consider the 100-element set S = $\{1^2, 2^2, ... , 100^2\}$.
S has 100891344545564193334812497256 50-element subsets.

Determine the sum of all integers which are the sum of exactly one of the 50-element subsets of S, i.e. find sum(U(S,50)).

Problem 202

Three mirrors are arranged in the shape of an equilateral triangle, with their reflective surfaces pointing inwards. There is an infinitesimal gap at each vertex of the triangle through which a laser beam may pass.

Label the vertices A, B and C. There are 2 ways in which a laser beam may enter vertex C, bounce off 11 surfaces, then exit through the same vertex: one way is shown below; the other is the reverse of that.



There are 80840 ways in which a laser beam may enter vertex C, bounce off 1000001 surfaces, then exit through the same vertex.

In how many ways can a laser beam enter at vertex C, bounce off 12017639147 surfaces, then exit through the same vertex?

Problem 204

A Hamming number is a positive number which has no prime factor larger than 5.
So the first few Hamming numbers are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15.
There are 1105 Hamming numbers not exceeding 10^8.

We will call a positive number a generalised Hamming number of type n, if it has no prime factor larger than n.
Hence the Hamming numbers are the generalised Hamming numbers of type 5.

How many generalised Hamming numbers of type 100 are there which don't exceed 10^9?

Problem 205

Peter has nine four-sided (pyramidal) dice, each with faces numbered 1, 2, 3, 4.
Colin has six six-sided (cubic) dice, each with faces numbered 1, 2, 3, 4, 5, 6.

Peter and Colin roll their dice and compare totals: the highest total wins. The result is a draw if the totals are equal.

What is the probability that Pyramidal Pete beats Cubic Colin? Give your answer rounded to seven decimal places in the form 0.abcdefg

---

Problem 207

For some positive integers k, there exists an integer partition of the form $4^t = 2^t + k$,
where 4^t, 2^t, and k are all positive integers and t is a real number.

The first two such partitions are $4^1 = 2^1 + 2$ and $4^{1.5849625...} = 2^{1.5849625...} + 6$.

Partitions where t is also an integer are called *perfect*.
For any $m \geq 1$ let P(m) be the proportion of such partitions that are perfect with $k \leq m$.
Thus P(6) = 1/2.

In the following table are listed some values of P(m)

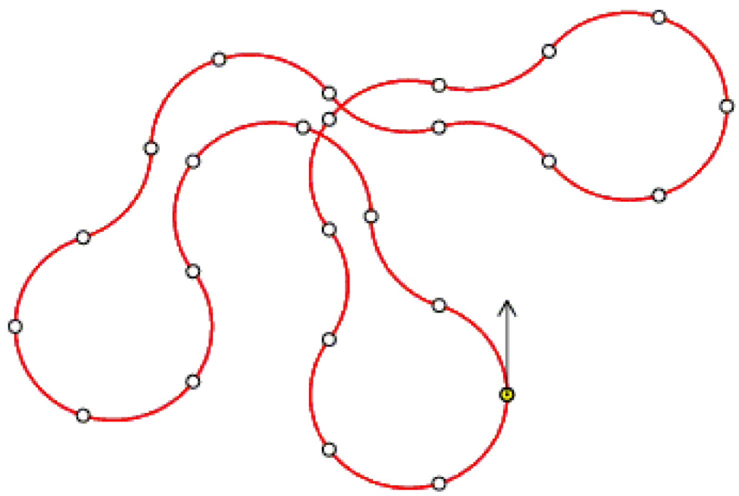
P(5) = 1/1
P(10) = 1/2
P(15) = 2/3
P(20) = 1/2
P(25) = 1/2
P(30) = 2/5
...
P(180) = 1/4
P(185) = 3/13

Find the smallest m for which P(m) < 1/12345

Problem 208

A robot moves in a series of one-fifth circular arcs (72°), with a free choice of a clockwise or an anticlockwise arc for each step, but no turning on the spot.

One of 70932 possible closed paths of 25 arcs starting northward is



Given that the robot starts facing North, how many journeys of 70 arcs in length can it take that return it, after the final arc, to its starting position?
(Any arc may be traversed multiple times.)

Problem 209

A k-input *binary truth table* is a map from k input bits (binary digits, 0 [false] or 1 [true]) to 1 output bit. For example, the 2-input binary truth tables for the logical AND and XOR functions are:

x	y	x AND y
0	0	0
0	1	0
1	0	0
1	1	1

x	y	x XOR y
0	0	0
0	1	1
1	0	1
1	1	0

How many 6-input binary truth tables, τ, satisfy the formula

$$\tau(a, b, c, d, e, f) \text{ AND } \tau(b, c, d, e, f, a \text{ XOR } (b \text{ AND } c)) = 0$$

for all 6-bit inputs (a, b, c, d, e, f)?

Problem 210

Consider the set S(r) of points (x,y) with integer coordinates satisfying $|x| + |y| \leq r$.
Let O be the point (0,0) and C the point (r/4,r/4).
Let N(r) be the number of points B in S(r), so that the triangle OBC has an obtuse angle, i.e. the largest angle α satisfies 90°<α<180°.
So, for example, N(4)=24 and N(8)=100.

What is N(1,000,000,000)?

Problem 212

An *axis-aligned cuboid*, specified by parameters { (x_0,y_0,z_0), (dx,dy,dz) }, consists of all points (X,Y,Z) such that $x_0 \leq X \leq x_0+dx$, $y_0 \leq Y \leq y_0+dy$ and $z_0 \leq Z \leq z_0+dz$. The volume of the cuboid is the product, dx × dy × dz. The *combined volume* of a collection of cuboids is the volume of their union and will be less than the sum of the individual volumes if any cuboids overlap.

Let $C_1,...,C_{50000}$ be a collection of 50000 axis-aligned cuboids such that C_n has parameters

$x_0 = S_{6n-5}$ modulo 10000
$y_0 = S_{6n-4}$ modulo 10000
$z_0 = S_{6n-3}$ modulo 10000
dx = 1 + (S_{6n-2} modulo 399)
dy = 1 + (S_{6n-1} modulo 399)
dz = 1 + (S_{6n} modulo 399)

where $S_1,...,S_{300000}$ come from the "Lagged Fibonacci Generator":

For $1 \leq k \leq 55$, $S_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000)
For $56 \leq k$, $S_k = [S_{k-24} + S_{k-55}]$ (modulo 1000000)

Thus, C_1 has parameters {(7,53,183),(94,369,56)}, C_2 has parameters {(2383,3563,5079),(42,212,344)}, and so on.

The combined volume of the first 100 cuboids, $C_1,...,C_{100}$, is 723581599.

What is the combined volume of all 50000 cuboids, $C_1,...,C_{50000}$?

Problem 213

A 30×30 grid of squares contains 900 fleas, initially one flea per square.
When a bell is rung, each flea jumps to an adjacent square at random (usually 4 possibilities, except for fleas on the edge of the grid or at the corners).

What is the expected number of unoccupied squares after 50 rings of the bell? Give your answer rounded to six decimal places.

Problem 214

Let φ be Euler's totient function, i.e. for a natural number n, $\varphi(n)$ is the number of k, $1 \le k \le n$, for which $\gcd(k,n) = 1$.

By iterating φ, each positive integer generates a decreasing chain of numbers ending in 1.
E.g. if we start with 5 the sequence 5,4,2,1 is generated.
Here is a listing of all chains with length 4:

5,4,2,1
7,6,2,1
8,4,2,1
9,6,2,1
10,4,2,1
12,4,2,1
14,6,2,1
18,6,2,1

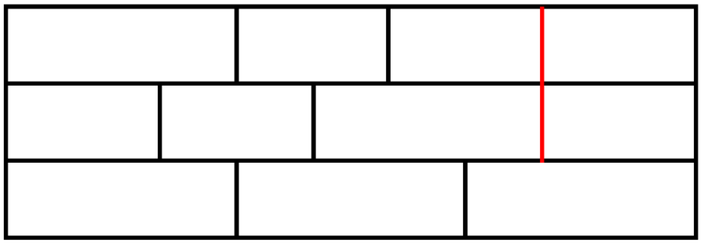
Only two of these chains start with a prime, their sum is 12.

What is the sum of all primes less than 40000000 which generate a chain of length 25?

Problem 215

Consider the problem of building a wall out of 2×1 and 3×1 bricks (horizontal×vertical dimensions) such that, for extra strength, the gaps between horizontally-adjacent bricks never line up in consecutive layers, i.e. never form a "running crack".

For example, the following 9×3 wall is not acceptable due to the running crack shown in red:



There are eight ways of forming a crack-free 9×3 wall, written W(9,3) = 8.

Calculate W(32,10).

Problem 216

Consider numbers $t(n)$ of the form $t(n) = 2n^2-1$ with $n > 1$.
The first such numbers are 7, 17, 31, 49, 71, 97, 127 and 161.
It turns out that only 49 = 7*7 and 161 = 7*23 are not prime.
For $n \leq$ 10000 there are 2202 numbers $t(n)$ that are prime.

How many numbers $t(n)$ are prime for $n \leq$ 50,000,000 ?

Problem 217

A positive integer with k (decimal) digits is called balanced if its first $\lceil k/2 \rceil$ digits sum to the same value as its last $\lceil k/2 \rceil$ digits, where $\lceil x \rceil$, pronounced *ceiling* of x, is the smallest integer $\geq x$, thus $\lceil \pi \rceil = 4$ and $\lceil 5 \rceil = 5$.

So, for example, all palindromes are balanced, as is 13722.

Let T(n) be the sum of all balanced numbers less than 10^n.
Thus: T(1) = 45, T(2) = 540 and T(5) = 334795890.

Find T(47) mod 3^{15}

Problem 218

Consider the right angled triangle with sides a=7, b=24 and c=25. The area of this triangle is 84, which is divisible by the perfect numbers 6 and 28.
Moreover it is a primitive right angled triangle as gcd(a,b)=1 and gcd(b,c)=1.
Also c is a perfect square.

We will call a right angled triangle perfect if
-it is a primitive right angled triangle
-its hypotenuse is a perfect square

We will call a right angled triangle super-perfect if
-it is a perfect right angled triangle and
-its area is a multiple of the perfect numbers 6 and 28.

How many perfect right-angled triangles with $c \le 10^{16}$ exist that are not super-perfect?

Problem 219

Let **A** and **B** be bit strings (sequences of 0's and 1's).
If **A** is equal to the leftmost length(**A**) bits of **B**, then **A** is said to be a *prefix* of **B**.
For example, 00110 is a prefix of 001101001, but not of 00111 or 100110.

A *prefix-free code of size* n is a collection of n distinct bit strings such that no string is a prefix of any other. For example, this is a prefix-free code of size 6:

0000, 0001, 001, 01, 10, 11

Now suppose that it costs one penny to transmit a '0' bit, but four pence to transmit a '1'.
Then the total cost of the prefix-free code shown above is 35 pence, which happens to be the cheapest possible for the skewed pricing scheme in question.
In short, we write Cost(6) = 35.

What is Cost(10^9) ?

Problem 220

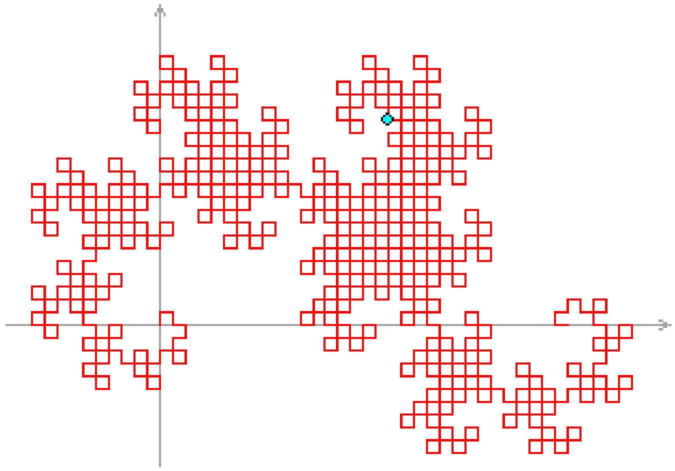
Let D_0 be the two-letter string "Fa". For $n \ge 1$, derive D_n from D_{n-1} by the string-rewriting rules:

"a" → "aRbFR"
"b" → "LFaLb"

Thus, D_0 = "Fa", D_1 = "FaRbFR", D_2 = "FaRbFRRLFaLbFR", and so on.

These strings can be interpreted as instructions to a computer graphics program, with "F" meaning "draw forward one unit", "L" meaning "turn left 90 degrees", "R" meaning "turn right 90 degrees", and "a" and "b" being ignored. The initial position of the computer cursor is (0,0), pointing up towards (0,1).

Then D_n is an exotic drawing known as the *Heighway Dragon* of order *n*. For example, D_{10} is shown below; counting each "F" as one step, the highlighted spot at (18,16) is the position reached after 500 steps.



What is the position of the cursor after 10^{12} steps in D_{50} ?
Give your answer in the form *x*,*y* with no spaces.

Problem 221

We shall call a positive integer A an "Alexandrian integer", if there exist integers p, q, r such that:

$$A = p \cdot q \cdot r \quad \text{and} \quad \frac{1}{A} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

For example, 630 is an Alexandrian integer ($p = 5$, $q = -7$, $r = -18$). In fact, 630 is the 6^{th} Alexandrian integer, the first 6 Alexandrian integers being: 6, 42, 120, 156, 420 and 630.

Find the 150000^{th} Alexandrian integer.

Problem 222

What is the length of the shortest pipe, of internal radius 50mm, that can fully contain 21 balls of radii 30mm, 31mm, ..., 50mm?

Give your answer in micrometres (10^{-6} m) rounded to the nearest integer.

Problem 223

Let us call an integer sided triangle with sides $a \leq b \leq c$ *barely acute* if the sides satisfy

$a^2 + b^2 = c^2 + 1$.

How many barely acute triangles are there with perimeter $\leq$ 25,000,000?

Problem 224

Let us call an integer sided triangle with sides $a \leq b \leq c$ *barely obtuse* if the sides satisfy $a^2 + b^2 = c^2 - 1$.

How many barely obtuse triangles are there with perimeter $\leq$ 75,000,000?

Problem 225

The sequence 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201 ...
is defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$.

It can be shown that 27 does not divide any terms of this sequence.
In fact, 27 is the first odd number with this property.

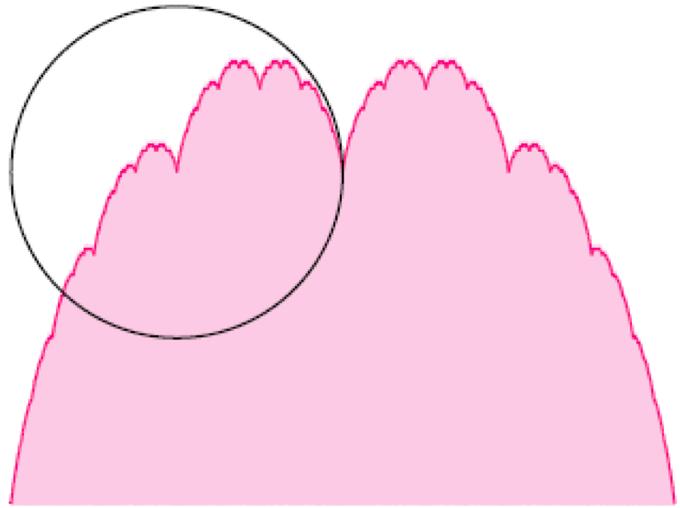
Find the 124^{th} odd number that does not divide any terms of the above sequence.

Problem 226

The *blancmange curve* is the set of points (x,y) such that $0 \leq x \leq 1$ and $y = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$,

where $s(x)$ = the distance from x to the nearest integer.

The area under the blancmange curve is equal to ½, shown in pink in the diagram below.



Let C be the circle with centre (¼,½) and radius ¼, shown in black in the diagram.

What area under the blancmange curve is enclosed by C?
Give your answer rounded to eight decimal places in the form *0.abcdefgh*

Problem 227

"*The Chase*" is a game played with two dice and an even number of players.

The players sit around a table; the game begins with two opposite players having one die each. On each turn, the two players with a die roll it.
If a player rolls a 1, he passes the die to his neighbour on the left; if he rolls a 6, he passes the die to his neighbour on the right; otherwise, he keeps the die for the next turn.
The game ends when one player has both dice after they have been rolled and passed, that player has then lost.

In a game with 100 players, what is the expected number of turns the game lasts?

Give your answer rounded to ten significant digits.

Problem 228

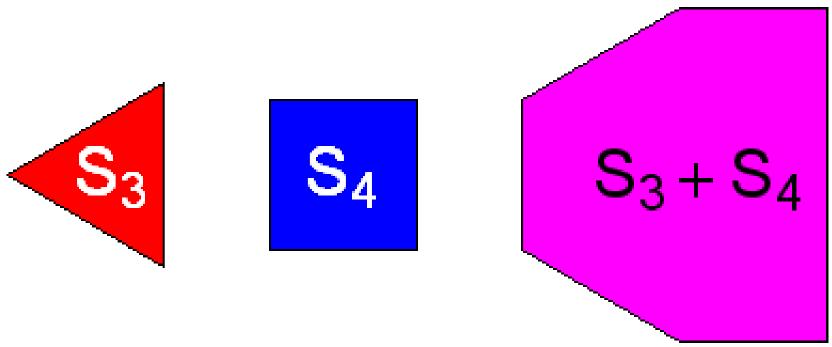
Let S_n be the regular n-sided polygon – or *shape* – whose vertices v_k (k = 1,2,…,n) have coordinates:

$$x_k = \cos\left({}^{2k-1}/_{n} \times 180^\circ \right)$$
$$y_k = \sin\left({}^{2k-1}/_{n} \times 180^\circ \right)$$

Each S_n is to be interpreted as a filled shape consisting of all points on the perimeter and in the interior.

The *Minkowski sum*, $S+T$, of two shapes S and T is the result of adding every point in S to every point in T, where point addition is performed coordinate-wise: $(u, v) + (x, y) = (u+x, v+y)$.

For example, the sum of S_3 and S_4 is the six-sided shape shown in pink below:



How many sides does $S_{1864} + S_{1865} + \ldots + S_{1909}$ have?

Problem 229

Consider the number 3600. It is very special, because

$$3600 = 48^2 + 36^2$$

$$3600 = 20^2 + 2\times 40^2$$

$$3600 = 30^2 + 3\times 30^2$$

$$3600 = 45^2 + 7\times 15^2$$

Similarly, we find that $88201 = 99^2 + 280^2 = 287^2 + 2\times 54^2 = 283^2 + 3\times 52^2 = 197^2 + 7\times 84^2$.

In 1747, Euler proved which numbers are representable as a sum of two squares. We are interested in the numbers n which admit representations of all of the following four types:

$$n = a_1^2 + b_1^2$$

$$n = a_2^2 + 2\, b_2^2$$

$$n = a_3^2 + 3\, b_3^2$$

$$n = a_7^2 + 7\, b_7^2,$$

where the a_k and b_k are positive integers.

There are 75373 such numbers that do not exceed 10^7.
How many such numbers are there that do not exceed 2×10^9?

Problem 230

For any two strings of digits, A and B, we define $F_{A,B}$ to be the sequence (A,B,AB,BAB,ABBAB,...) in which each term is the concatenation of the previous two.

Further, we define $D_{A,B}(n)$ to be the n^{th} digit in the first term of $F_{A,B}$ that contains at least n digits.

Example:

Let A=1415926535, B=8979323846. We wish to find $D_{A,B}(35)$, say.

The first few terms of $F_{A,B}$ are:
1415926535
8979323846
14159265358979323846
897932384614159265358979323846
14159265358979323846897932384614159265358979323846

Then $D_{A,B}(35)$ is the 35^{th} digit in the fifth term, which is 9.

Now we use for A the first 100 digits of π behind the decimal point:

14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679

and for B the next hundred digits:

82148086513282306647093844609550582231725359408128
48111745028410270193852110555964462294895493038196 .

Find $\sum_{n=0,1,\ldots,17} \ 10^n\times D_{A,B}((127+19n)\times7^n)$.

Problem 231

The binomial coefficient ${}^{10}C_3 = 120$.

$120 = 2^3 \times 3 \times 5 = 2 \times 2 \times 2 \times 3 \times 5$, and $2 + 2 + 2 + 3 + 5 = 14$.
So the sum of the terms in the prime factorisation of ${}^{10}C_3$ is 14.

Find the sum of the terms in the prime factorisation of ${}^{20000000}C_{15000000}$.

Problem 232

Two players share an unbiased coin and take it in turns to play "The Race". On Player 1's turn, he tosses the coin once: if it comes up Heads, he scores one point; if it comes up Tails, he scores nothing. On Player 2's turn, she chooses a positive integer T and tosses the coin T times: if it comes up all Heads, she scores 2^{T-1} points; otherwise, she scores nothing. Player 1 goes first. The winner is the first to 100 or more points.

On each turn Player 2 selects the number, T, of coin tosses that maximises the probability of her winning.

What is the probability that Player 2 wins?

Give your answer rounded to eight decimal places in the form 0.*abcdefgh* .

Problem 233

Let $f(N)$ be the number of points with integer coordinates that are on a circle passing through (0,0), $(N,0)$,$(0,N)$, and (N,N).

It can be shown that $f(10000) = 36$.

What is the sum of all positive integers $N \leq 10^{11}$ such that $f(N) = 420$?

Problem 234

For an integer $n \geq 4$, we define the *lower prime square root* of n, denoted by lps(n), as the largest prime $\leq \sqrt{n}$ and the *upper prime square root* of n, ups(n), as the smallest prime $\geq \sqrt{n}$.

So, for example, lps(4) = 2 = ups(4), lps(1000) = 31, ups(1000) = 37.
Let us call an integer $n \geq 4$ *semidivisible*, if one of lps(n) and ups(n) divides n, but not both.

The sum of the semidivisible numbers not exceeding 15 is 30, the numbers are 8, 10 and 12.
15 is not semidivisible because it is a multiple of both lps(15) = 3 and ups(15) = 5.
As a further example, the sum of the 92 semidivisible numbers up to 1000 is 34825.

What is the sum of all semidivisible numbers not exceeding 999966663333 ?

Problem 235

Given is the arithmetic-geometric sequence u(k) = (900-3k)r^{k-1}.
Let s(n) = $\Sigma_{k=1...n}$u(k).

Find the value of r for which s(5000) = -600,000,000,000.

Give your answer rounded to 12 places behind the decimal point.

Problem 236

Suppliers 'A' and 'B' provided the following numbers of products for the luxury hamper market:

Product	'A'	'B'
Beluga Caviar	5248	640
Christmas Cake	1312	1888
Gammon Joint	2624	3776
Vintage Port	5760	3776
Champagne Truffles	3936	5664

Although the suppliers try very hard to ship their goods in perfect condition, there is inevitably some spoilage - *i.e.* products gone bad.

The suppliers compare their performance using two types of statistic:

- The five *per-product spoilage rates* for each supplier are equal to the number of products gone bad divided by the number of products supplied, for each of the five products in turn.
- The *overall spoilage rate* for each supplier is equal to the total number of products gone bad divided by the total number of products provided by that supplier.

To their surprise, the suppliers found that each of the five per-product spoilage rates was worse (higher) for 'B' than for 'A' by the same factor (ratio of spoilage rates), m>1; and yet, paradoxically, the overall spoilage rate was worse for 'A' than for 'B', also by a factor of m.

There are thirty-five m>1 for which this surprising result could have occurred, the smallest of which is 1476/1475.

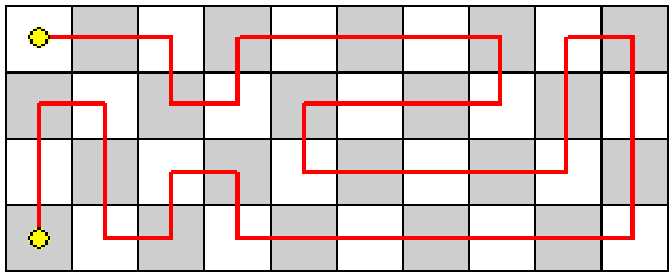
What's the largest possible value of m?
Give your answer as a fraction reduced to its lowest terms, in the form u/v.

Problem 237

Let T(*n*) be the number of tours over a 4 × *n* playing board such that:

- The tour starts in the top left corner.
- The tour consists of moves that are up, down, left, or right one square.
- The tour visits each square exactly once.
- The tour ends in the bottom left corner.

The diagram shows one tour over a 4 × 10 board:



T(10) is 2329. What is T(10^{12}) modulo 10^8?

Problem 238

Create a sequence of numbers using the "Blum Blum Shub" pseudo-random number generator:

$$s_0 = 14025256$$
$$s_{n+1} = s_n^2 \bmod 20300713$$

Concatenate these numbers $s_0s_1s_2\ldots$ to create a string w of infinite length.
Then, w = 14025256741014958470038053646…

For a positive integer k, if no substring of w exists with a sum of digits equal to k, $p(k)$ is defined to be zero. If at least one substring of w exists with a sum of digits equal to k, we define $p(k) = z$, where z is the starting position of the earliest such substring.

For instance:

The substrings 1, 14, 1402, …
with respective sums of digits equal to 1, 5, 7, …
start at position **1**, hence $p(1) = p(5) = p(7) = \ldots =$ **1**.

The substrings 4, 402, 4025, …
with respective sums of digits equal to 4, 6, 11, …
start at position **2**, hence $p(4) = p(6) = p(11) = \ldots =$ **2**.

The substrings 02, 0252, …

with respective sums of digits equal to 2, 9, ...
start at position **3**, hence $p(2) = p(9) = ... = 3$.

Note that substring 025 starting at position **3**, has a sum of digits equal to 7, but there was an earlier substring (starting at position **1**) with a sum of digits equal to 7, so $p(7) = 1$, *not* 3.

We can verify that, for $0 < k \leq 10^3$, $\sum p(k) = 4742$.

Find $\sum p(k)$, for $0 < k \leq 2 \cdot 10^{15}$.

Problem 239

A set of disks numbered 1 through 100 are placed in a line in random order.

What is the probability that we have a partial derangement such that exactly 22 prime number discs are found away from their natural positions?
(Any number of non-prime disks may also be found in or out of their natural positions.)

Give your answer rounded to 12 places behind the decimal point in the form 0.abcdefghijkl.

Problem 240

There are 1111 ways in which five 6-sided dice (sides numbered 1 to 6) can be rolled so that the top three sum to 15. Some examples are:

$D_1,D_2,D_3,D_4,D_5 = 4,3,6,3,5$
$D_1,D_2,D_3,D_4,D_5 = 4,3,3,5,6$
$D_1,D_2,D_3,D_4,D_5 = 3,3,3,6,6$
$D_1,D_2,D_3,D_4,D_5 = 6,6,3,3,3$

In how many ways can twenty 12-sided dice (sides numbered 1 to 12) be rolled so that the top ten sum to 70?

Problem 241

For a positive integer n, let $\sigma(n)$ be the sum of all divisors of n, so e.g. $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

A perfect number, as you probably know, is a number with $\sigma(n) = 2n$.

Let us define the **perfection quotient** of a positive integer as $p(n) = \frac{\sigma(n)}{}$

Let us define the **perfection quotient** of a positive integer as $p(n) = \frac{\quad}{n}$.

Find the sum of all positive integers $n \le 10^{18}$ for which $p(n)$ has the form $k + {}^1\!/_2$, where k is an integer.

Problem 242

Given the set $\{1,2,...,n\}$, we define $f(n,k)$ as the number of its k-element subsets with an odd sum of elements. For example, $f(5,3) = 4$, since the set $\{1,2,3,4,5\}$ has four 3-element subsets having an odd sum of elements, i.e.: $\{1,2,4\}$, $\{1,3,5\}$, $\{2,3,4\}$ and $\{2,4,5\}$.

When all three values n, k and $f(n,k)$ are odd, we say that they make an *odd-triplet* $[n,k,f(n,k)]$.

There are exactly five odd-triplets with $n \le 10$, namely:
$[1,1,f(1,1) = 1]$, $[5,1,f(5,1) = 3]$, $[5,5,f(5,5) = 1]$, $[9,1,f(9,1) = 5]$ and $[9,9,f(9,9) = 1]$.

How many odd-triplets are there with $n \le 10^{12}$?

Problem 243

A positive fraction whose numerator is less than its denominator is called a proper fraction.
For any denominator, d, there will be $d-1$ proper fractions; for example, with $d = 12$:
${}^1/_{12}$, ${}^2/_{12}$, ${}^3/_{12}$, ${}^4/_{12}$, ${}^5/_{12}$, ${}^6/_{12}$, ${}^7/_{12}$, ${}^8/_{12}$, ${}^9/_{12}$, ${}^{10}/_{12}$, ${}^{11}/_{12}$.

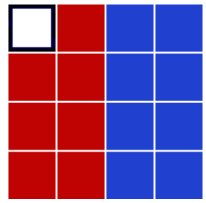
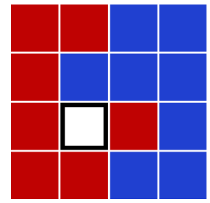
We shall call a fraction that cannot be cancelled down a *resilient fraction*.
Furthermore we shall define the *resilience* of a denominator, $R(d)$, to be the ratio of its proper fractions that are resilient; for example, $R(12) = {}^4/_{11}$.
In fact, $d = 12$ is the smallest denominator having a resilience $R(d) < {}^4/_{10}$.

Find the smallest denominator d, having a resilience $R(d) < {}^{15499}/_{94744}$.

Problem 244

You probably know the game *Fifteen Puzzle*. Here, instead of numbered tiles, we have seven red tiles and eight blue tiles.

A move is denoted by the uppercase initial of the direction (Left, Right, Up, Down) in which the tile is slid, e.g. starting from configuration (**S**), by the sequence **LULUR** we reach the configuration (**E**):

(**S**)  , (**E**) 

For each path, its checksum is calculated by (pseudocode):

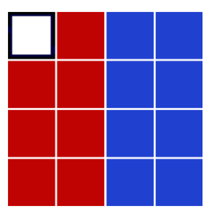
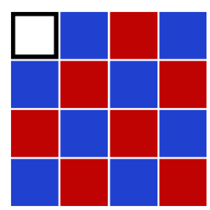
checksum = 0
checksum = (checksum $\times$ 243 + m_1) mod 100 000 007
checksum = (checksum $\times$ 243 + m_2) mod 100 000 007
…
checksum = (checksum $\times$ 243 + m_n) mod 100 000 007

where m_k is the ASCII value of the k^{th} letter in the move sequence and the ASCII values for the moves are:

L	76
R	82
U	85
D	68

For the sequence **LULUR** given above, the checksum would be 19761398.

Now, starting from configuration (**S**), find all shortest ways to reach configuration (**T**).

(**S**)  , (**T**) 

What is the sum of all checksums for the paths having the minimal length?

Problem 245

We shall call a fraction that cannot be cancelled down a resilient fraction.
Furthermore we shall define the resilience of a denominator, $R(d)$, to be the ratio of its proper fractions that are resilient; for example, $R(12) = {}^4/_{11}$.

The resilience of a number $d > 1$ is then $\dfrac{\varphi(d)}{d - 1}$, where φ is Euler's totient function.

We further define the **coresilience** of a number $n > 1$ as $C(n) = \dfrac{n - \varphi(n)}{n - 1}$.

The coresilience of a prime p is $C(p) = \frac{1}{p - 1}$.

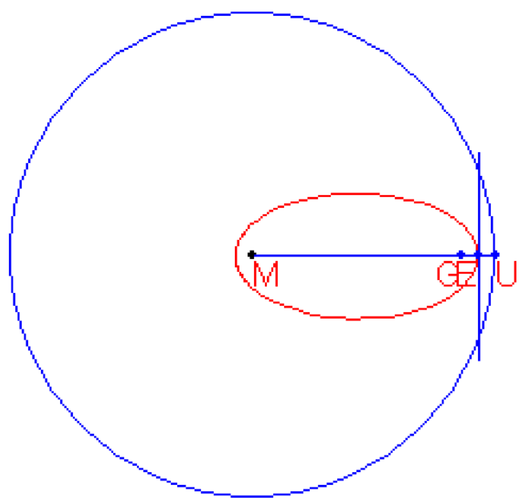
Find the sum of all **composite** integers $1 < n \leq 2\times10^{11}$, for which $C(n)$ is a unit fraction.

Note: the upper limit has been changed recently. Check out that you use the right upper limit.

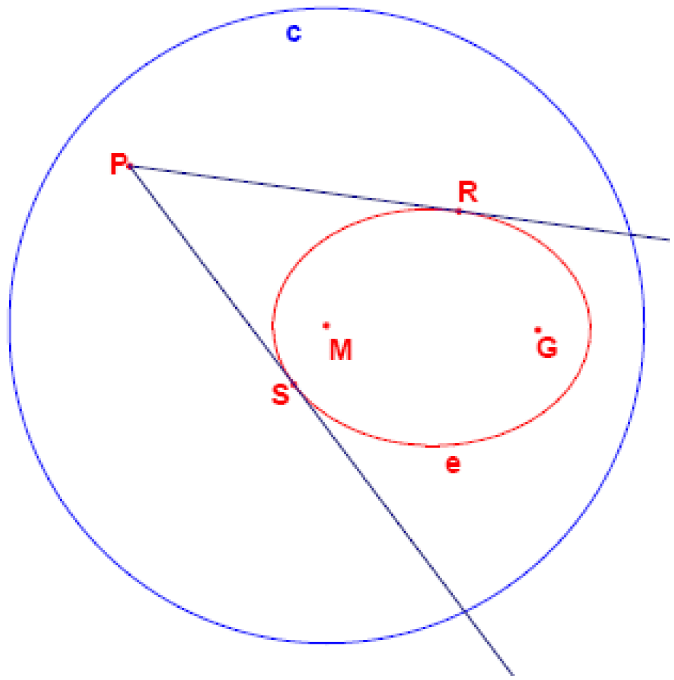
Problem 246

A definition for an ellipse is:
Given a circle c with centre M and radius r and a point G such that d(G,M)<r, the locus of the points that are equidistant from c and G form an ellipse.

The construction of the points of the ellipse is shown below.



Given are the points M(-2000,1500) and G(8000,1500).
Given is also the circle c with centre M and radius 15000.
The locus of the points that are equidistant from G and c form an ellipse e.
From a point P outside e the two tangents t_1 and t_2 to the ellipse are drawn.
Let the points where t_1 and t_2 touch the ellipse be R and S.



For how many lattice points P is angle RPS greater than 45 degrees?

Problem 247

Consider the region constrained by $1 \le x$ and $0 \le y \le {}^1/_x$.

Let S_1 be the largest square that can fit under the curve.
Let S_2 be the largest square that fits in the remaining area, and so on.
Let the *index* of S_n be the pair (left, below) indicating the number of squares to the left of S_n and the number of squares below S_n.

The diagram shows some such squares labelled by number.
S_2 has one square to its left and none below, so the index of S_2 is (1,0).
It can be seen that the index of S_{32} is (1,1) as is the index of S_{50}.
50 is the largest n for which the index of S_n is (1,1).

What is the largest n for which the index of S_n is (3,3)?

Problem 248

The first number n for which φ(n)=13! is 6227180929.

Find the 150,000th such number.

Note: The number to find has been changed recently. Check out that you have computed the right number.

Problem 249

Let S = {2, 3, 5, ..., 4999} be the set of prime numbers less than 5000.

Find the number of subsets of S, the sum of whose elements is a prime number.
Enter the rightmost 16 digits as your answer.

Problem 250

Find the number of non-empty subsets of $\{1^1, 2^2, 3^3, \ldots, 250250^{250250}\}$, the sum of whose elements is divisible by 250. Enter the rightmost 16 digits as your answer.

Problem 251

A triplet of positive integers (a,b,c) is called a Cardano Triplet if it satisfies the condition:

For example, (2,1,5) is a Cardano Triplet.

There exist 149 Cardano Triplets for which $a+b+c \leq 1000$.

Find how many Cardano Triplets exist such that $a+b+c \leq 110,000,000$.

Note: This problem has been changed recently, please check that you are using the right parameters.

Problem 252

Given a set of points on a plane, we define a convex hole to be a convex polygon having as vertices any of the given points and not containing any of the given points in its interior (in addition to the vertices, other given points may lie on the perimeter of the polygon).

As an example, the image below shows a set of twenty points and a few such convex holes. The convex hole shown as a red heptagon has an area equal to 1049694.5 square units, which is the highest possible area for a convex hole on the given set of points.

For our example, we used the first 20 points (T_{2k-1}, T_{2k}), for k = 1,2,…,20, produced with the pseudo-random number generator:

$$S_0 = 290797$$
$$S_{n+1} = S_n^2 \bmod 50515093$$
$$T_n = (S_n \bmod 2000) - 1000$$

i.e. (527, 144), (–488, 732), (–454, –947), …

What is the maximum area for a convex hole on the set containing the first 500 points in the pseudo-random sequence?
Specify your answer including one digit after the decimal point.

Problem 253

A small child has a "number caterpillar" consisting of forty jigsaw pieces, each with one number on it, which, when connected together in a line, reveal the numbers 1 to 40 in order.

Every night, the child's father has to pick up the pieces of the caterpillar that have been scattered across the play room. He picks up the pieces at random and places them in the correct order.
As the caterpillar is built up in this way, it forms distinct segments that gradually merge together.
The number of segments starts at zero (no pieces placed), generally increases up to about eleven or twelve, then tends to drop again before finishing at a single segment (all pieces placed).

For example:

Piece Placed	Segments So Far
12	1
4	2
29	3
6	4
34	5
5	4
35	4
…	…

Let M be the maximum number of segments encountered during a random tidy-up of the caterpillar.
For a caterpillar of ten pieces, the number of possibilities for each M is

M	Possibilities
1	512
2	250912
3	1815264
4	1418112
5	144000

so the most likely value of M is 3 and the average value is $^{385643}/_{113400}$ = 3.400732, rounded to six decimal places.

The most likely value of M for a forty-piece caterpillar is 11; but what is the average value of M?

Give your answer rounded to six decimal places.

Problem 254

Define f(n) as the sum of the factorials of the digits of n. For example, f(342) = 3! + 4! + 2! = 32.

Define sf(n) as the sum of the digits of f(n). So sf(342) = 3 + 2 = 5.

Define g(i) to be the smallest positive integer n such that sf(n) = i. Though sf(342) is 5, sf(25) is also 5, and it can be verified that g(5) is 25.

Define sg(i) as the sum of the digits of g(i). So sg(5) = 2 + 5 = 7.

Further, it can be verified that g(20) is 267 and $\sum$ sg(i) for 1 $\leq i \leq$ 20 is 156.

What is $\sum$ sg(i) for 1 $\leq i \leq$ 150?

Problem 255

We define the *rounded-square-root* of a positive integer n as the square root of n rounded to the nearest integer.

The following procedure (essentially Heron's method adapted to integer arithmetic) finds the rounded-square-root of n:

Let d be the number of digits of the number n.
If d is odd, set $x_0 = 2\times10^{(d-1)/2}$.
If d is even, set $x_0 = 7\times10^{(d-2)/2}$.
Repeat:

until $x_{k+1} = x_k$.

As an example, let us find the rounded-square-root of $n = 4321$.
n has 4 digits, so $x_0 = 7\times10^{(4-2)/2} = 70$.

Since $x_2 = x_1$, we stop here.
So, after just two iterations, we have found that the rounded-square-root of 4321 is 66 (the actual square root is 65.7343137…).

The number of iterations required when using this method is surprisingly low.
For example, we can find the rounded-square-root of a 5-digit integer ($10,000 \leq n \leq 99,999$) with an average of 3.2102888889 iterations (the average value was rounded to 10 decimal places).

Using the procedure described above, what is the average number of iterations required to find the rounded-square-root of a 14-digit number ($10^{13} \leq n < 10^{14}$)?
Give your answer rounded to 10 decimal places.

Note: The symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ represent the floor function and ceiling function respectively.

Problem 256

Tatami are rectangular mats, used to completely cover the floor of a room, without overlap.

Assuming that the only type of available tatami has dimensions 1×2, there are obviously some limitations for the shape and size of the rooms that can be covered.

For this problem, we consider only rectangular rooms with integer dimensions a, b and even size $s = a\cdot b$.
We use the term 'size' to denote the floor surface area of the room, and — without loss of generality — we add the condition $a \leq b$.

There is one rule to follow when laying out tatami: there must be no points where corners of four different mats meet.
For example, consider the two arrangements below for a 4×4 room:

The arrangement on the left is acceptable, whereas the one on the right is not: a red "X" in the middle, marks the point where four tatami meet.

Because of this rule, certain even-sized rooms cannot be covered with tatami: we call them tatami-free rooms.
Further, we define $T(s)$ as the number of tatami-free rooms of size s.

The smallest tatami-free room has size $s = 70$ and dimensions 7×10.
All the other rooms of size $s = 70$ can be covered with tatami; they are: 1×70, 2×35 and 5×14.
Hence, $T(70) = 1$.

Similarly, we can verify that $T(1320) = 5$ because there are exactly 5 tatami-free rooms of size s = 1320:
20×66, 22×60, 24×55, 30×44 and 33×40.
In fact, s = 1320 is the smallest room-size s for which $T(s) = 5$.

Find the smallest room-size s for which $T(s) = 200$.

Problem 257

Given is an integer sided triangle ABC with sides a $\leq$ b $\leq$ c. (AB = c, BC = a and AC = b).
The angular bisectors of the triangle intersect the sides at points E, F and G (see picture below).

The segments EF, EG and FG partition the triangle ABC into four smaller triangles: AEG, BFE, CGF and EFG.
It can be proven that for each of these four triangles the ratio area(ABC)/area(subtriangle) is rational.
However, there exist triangles for which some or all of these ratios are integral.

How many triangles ABC with perimeter$\leq$100,000,000 exist so that the ratio area(ABC)/area(AEG) is integral?

Problem 258

A sequence is defined as:

- $g_k = 1$, for $0 \leq k \leq 1999$
- $g_k = g_{k\text{-}2000} + g_{k\text{-}1999}$, for $k \geq 2000$.

Find g_k mod 20092010 for $k = 10^{18}$.

Problem 259

A positive integer will be called *reachable* if it can result from an arithmetic expression obeying the following rules:

- Uses the digits 1 through 9, in that order and exactly once each.
- Any successive digits can be concatenated (for example, using the digits 2, 3 and 4 we obtain the number 234).

- Only the four usual binary arithmetic operations (addition, subtraction, multiplication and division) are allowed.
- Each operation can be used any number of times, or not at all.
- Unary minus is not allowed.
- Any number of (possibly nested) parentheses may be used to define the order of operations.

For example, 42 is reachable, since (1/23) * ((4*5)-6) * (78-9) = 42.

What is the sum of all positive reachable integers?

Problem 260

A game is played with three piles of stones and two players.
At her turn, a player removes one or more stones from the piles. However, if she takes stones from more than one pile, she must remove the same number of stones from each of the selected piles.

In other words, the player chooses some N>0 and removes:

- N stones from any single pile; or
- N stones from each of any two piles (2N total); or
- N stones from each of the three piles (3N total).

The player taking the last stone(s) wins the game.

A *winning configuration* is one where the first player can force a win.
For example, (0,0,13), (0,11,11) and (5,5,5) are winning configurations because the first player can immediately remove all stones.

A *losing configuration* is one where the second player can force a win, no matter what the first player does.
For example, (0,1,2) and (1,3,3) are losing configurations: any legal move leaves a winning configuration for the second player.

Consider all losing configurations (x_i,y_i,z_i) where $x_i \le y_i \le z_i \le 100$.
We can verify that $\Sigma(x_i+y_i+z_i) = 173895$ for these.

Find $\Sigma(x_i+y_i+z_i)$ where (x_i,y_i,z_i) ranges over the losing configurations
with $x_i \le y_i \le z_i \le 1000$.

Problem 261

Let us call a positive integer k a $square\text{-}pivot$, if there is a pair of integers $m > 0$ and $n \ge k$, such that the sum of the $(m+1)$ consecutive squares up to k equals the sum of the m

consecutive squares from (n+1) on:

$$(k-m)^2 + \ldots + k^2 = (n+1)^2 + \ldots + (n+m)^2.$$

Some small square-pivots are

- **4**: 3^2 + $\mathbf{4}^2$ = 5^2
- **21**: 20^2 + $\mathbf{21}^2$ = 29^2
- **24**: 21^2 + 22^2 + 23^2 + $\mathbf{24}^2$ = 25^2 + 26^2 + 27^2
- **110**: 108^2 + 109^2 + $\mathbf{110}^2$ = 133^2 + 134^2

Find the sum of all **distinct** square-pivots $\le 10^{10}$.

Problem 262

The following equation represents the *continuous* topography of a mountainous region, giving the elevation h at any point (x,y):

A mosquito intends to fly from A(200,200) to B(1400,1400), without leaving the area given by $0 \le x, y \le 1600$.

Because of the intervening mountains, it first rises straight up to a point A', having elevation f. Then, while remaining at the same elevation f, it flies around any obstacles until it arrives at a point B' directly above B.

First, determine f_{min} which is the minimum constant elevation allowing such a trip from A to B, while remaining in the specified area.
Then, find the length of the shortest path between A' and B', while flying at that constant elevation f_{min}.

Give that length as your answer, rounded to three decimal places.

Note: For convenience, the elevation function shown above is repeated below, in a form suitable for most programming languages:
h=(5000-0.005*(x*x+y*y+x*y)+12.5*(x+y)) * exp(-abs(0.000001*(x*x+y*y)-0.0015*(x+y)+0.7))

Problem 263

Consider the number 6. The divisors of 6 are: 1,2,3 and 6.
Every number from 1 up to and including 6 can be written as a sum of distinct divisors of 6:
1=1, 2=2, 3=1+2, 4=1+3, 5=2+3, 6=6.
A number n is called a practical number if every number from 1 up to and including n can be expressed as a sum of distinct divisors of n.

A pair of consecutive prime numbers with a difference of six is called a sexy pair (since "sex" is the Latin word for "six"). The first sexy pair is (23, 29).

We may occasionally find a triple-pair, which means three consecutive sexy prime pairs, such that the second member of each pair is the first member of the next pair.

We shall call a number n such that :

- (n-9, n-3), (n-3,n+3), (n+3, n+9) form a triple-pair, and
- the numbers n-8, n-4, n, n+4 and n+8 are all practical,

an engineers’ paradise.

Find the sum of the first four engineers’ paradises.

Problem 264

Consider all the triangles having:

- All their vertices on lattice points.
- Circumcentre at the origin O.
- Orthocentre at the point H(5, 0).

There are nine such triangles having a perimeter $\leq$ 50.
Listed and shown in ascending order of their perimeter, they are:

A(-4, 3), B(5, 0), C(4, -3)
A(4, 3), B(5, 0), C(-4, -3)
A(-3, 4), B(5, 0), C(3, -4)

A(3, 4), B(5, 0), C(-3, -4)
A(0, 5), B(5, 0), C(0, -5)
A(1, 8), B(8, -1), C(-4, -7)

A(8, 1), B(1, -8), C(-4, 7)
A(2, 9), B(9, -2), C(-6, -7)
A(9, 2), B(2, -9), C(-6, 7)

The sum of their perimeters, rounded to four decimal places, is 291.0089.

Find all such triangles with a perimeter $\leq 10^5$.
Enter as your answer the sum of their perimeters rounded to four decimal places.

Problem 269

A root or zero of a polynomial P(x) is a solution to the equation P(x) = 0.
Define P_n as the polynomial whose coefficients are the digits of n.
For example, $P_{5703}(x) = 5x^3 + 7x^2 + 3$.

We can see that:

- $P_n(0)$ is the last digit of n,
- $P_n(1)$ is the sum of the digits of n,
- $P_n(10)$ is n itself.

Define Z(k) as the number of positive integers, n, not exceeding k for which the polynomial P_n has at least one integer root.

It can be verified that Z(100 000) is 14696.

What is Z(10^{16})?

Problem 270

A square piece of paper with integer dimensions N×N is placed with a corner at the origin and two of its sides along the x- and y-axes. Then, we cut it up respecting the following rules:

- We only make straight cuts between two points lying on different sides of the square, and having integer coordinates.
- Two cuts cannot cross, but several cuts can meet at the same border point.
- Proceed until no more legal cuts can be made.

Counting any reflections or rotations as distinct, we call C(N) the number of ways to cut an N×N square. For example, C(1) = 2 and C(2) = 30 (shown below).

What is C(30) mod 10^8 ?

Problem 271

For a positive number n, define S(n) as the sum of the integers x, for which $1<x<n$ and x^3 1 mod n.

When n=91, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81.
Thus, S(91)=9+16+22+29+53+74+79+81=363.

Find S(13082761331670030).

Problem 272

For a positive number n, define C(n) as the number of the integers x, for which $1<x<n$ and x^3 1 mod n.

When n=91, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81.
Thus, C(91)=8.

Find the sum of the positive numbers $n \leq 10^{11}$ for which C(n)=242.

Problem 273

Consider equations of the form: $a^2 + b^2 = N$, $0 \leq a \leq b$, a, b and N integer.

For N=65 there are two solutions:

a=1, b=8 and a=4, b=7.

We call S(N) the sum of the values of a of all solutions of $a^2 + b^2 = N$, $0 \leq a \leq b$, a, b and N integer.

Thus S(65)=1+4=5.

Find $\sum$S(N), for all squarefree N only divisible by primes of the form $4k+1$ with $4k+1 < 150$.

Problem 274

For each integer $p > 1$ coprime to 10 there is a positive *divisibility multiplier* $m < p$ which preserves divisibility by p for the following function on any positive integer, n:

$f(n)$ = (all but the last digit of n) + (the last digit of n) * m

That is, if m is the divisibility multiplier for p, then $f(n)$ is divisible by p if and only if n is divisible by p.

(When n is much larger than p, $f(n)$ will be less than n and repeated application of f provides a multiplicative divisibility test for p.)

For example, the divisibility multiplier for 113 is 34.

$f(76275) = 7627 + 5 * 34 = 7797$: 76275 and 7797 are both divisible by 113
$f(12345) = 1234 + 5 * 34 = 1404$: 12345 and 1404 are both not divisible by 113

The sum of the divisibility multipliers for the primes that are coprime to 10 and less than 1000 is 39517. What is the sum of the divisibility multipliers for the primes that are coprime to 10 and less than 10^7?

Problem 275

Let us define a *balanced sculpture* of order n as follows:

- A polyomino made up of n+1 tiles known as the *blocks* (n tiles) and the *plinth* (remaining tile);
- the plinth has its centre at position (x = 0, y = 0);
- the blocks have y-coordinates greater than zero (so the plinth is the unique lowest tile);
- the centre of mass of all the blocks, combined, has x-coordinate equal to zero.

When counting the sculptures, any arrangements which are simply reflections about the y-axis, are not counted as distinct. For example, the 18 balanced sculptures of order 6 are shown below; note that each pair of mirror images (about the y-axis) is counted as one sculpture:

There are 964 balanced sculptures of order 10 and 360505 of order 15.
How many balanced sculptures are there of order 18?

Problem 276

Consider the triangles with integer sides a, b and c with a ≤ b ≤ c.
An integer sided triangle (a,b,c) is called primitive if gcd(a,b,c)=1.
How many primitive integer sided triangles exist with a perimeter not exceeding 10 000 000?

Problem 278

Given the values of integers $1 < a_1 < a_2 <\dots < a_n$, consider the linear combination $q_1a_1 + q_2a_2 + \dots + q_na_n = b$, using only integer values $q_k \geq 0$.

Note that for a given set of a_k, it may be that not all values of b are possible.
For instance, if $a_1 = 5$ and $a_2 = 7$, there are no $q_1 \geq 0$ and $q_2 \geq 0$ such that b could be 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18 or 23.
In fact, 23 is the largest impossible value of b for $a_1 = 5$ and $a_2 = 7$.

We therefore call $f(5, 7) = 23$.
Similarly, it can be shown that $f(6, 10, 15)=29$ and $f(14, 22, 77) = 195$.

Find $\sum f(p*q,p*r,q*r)$, where p, q and r are prime numbers and $p < q < r < 5000$.

Problem 279

How many triangles are there with integral sides, at least one integral angle (measured in degrees), and a perimeter that does not exceed 10^8?

Problem 280

A laborious ant walks randomly on a 5x5 grid. The walk starts from the central square. At each step, the ant moves to an adjacent square at random, without leaving the grid; thus there are 2, 3 or 4 possible moves at each step depending on the ant's position.

At the start of the walk, a seed is placed on each square of the lower row. When the ant isn't carrying a seed and reaches a square of the lower row containing a seed, it will start to carry the seed. The ant will drop the seed on the first empty square of the upper row it eventually reaches.

What's the expected number of steps until all seeds have been dropped in the top row?
Give your answer rounded to 6 decimal places.

Problem 281

You are given a pizza (perfect circle) that has been cut into $m \cdot n$ equal pieces and you want to have exactly one topping on each slice.

Let $f(m,n)$ denote the number of ways you can have toppings on the pizza with m different toppings ($m \geq 2$), using each topping on exactly n slices ($n \geq 1$).
Reflections are considered distinct, rotations are not.

Thus, for instance, $f(2,1) = 1$, $f(2,2) = f(3,1) = 2$ and $f(3,2) = 16$.
$f(3,2)$ is shown below:

Find the sum of all $f(m,n)$ such that $f(m,n) \leq 10^{15}$.

Problem 282

For non-negative integers m, n, the Ackermann function $A(m, n)$ is defined as follows:

For example $A(1, 0) = 2$, $A(2, 2) = 7$ and $A(3, 4) = 125$.

Find $A(n, n)$ and give your answer mod 14^8.

Problem 283

Consider the triangle with sides 6, 8 and 10. It can be seen that the perimeter and the area are both equal to 24. So the area/perimeter ratio is equal to 1.
Consider also the triangle with sides 13, 14 and 15. The perimeter equals 42 while the area is equal to 84. So for this triangle the area/perimeter ratio is equal to 2.

Find the sum of the perimeters of all integer sided triangles for which the area/perimeter ratios are equal to positive integers not exceeding 1000.

Problem 284

The 3-digit number 376 in the decimal numbering system is an example of numbers with the special property that its square ends with the same digits: 376^2 = 141376. Let's call a number with this property a steady square.

Steady squares can also be observed in other numbering systems. In the base 14 numbering system, the 3-digit number c37 is also a steady square: $c37^2$ = aa0c37, and the sum of its digits is c+3+7=18 in the same numbering system. The letters a, b, c and d are used for the 10, 11, 12 and 13 digits respectively, in a manner similar to the hexadecimal numbering system.

For $1 \le n \le 9$, the sum of the digits of all the n-digit steady squares in the base 14 numbering system is 2d8 (582 decimal). Steady squares with leading 0's are not allowed.

Find the sum of the digits of all the n-digit steady squares in the base 14 numbering system for $1 \le n \le 10000$ (decimal) and give your answer in the base 14 system using lower case letters where necessary.

Problem 285

Albert chooses a positive integer k, then two real numbers a, b are randomly chosen in the interval [0,1] with uniform distribution.

The square root of the sum $(k \cdot a+1)^2 + (k \cdot b+1)^2$ is then computed and rounded to the nearest integer. If the result is equal to k, he scores k points; otherwise he scores nothing.

For example, if $k = 6$, $a = 0.2$ and $b = 0.85$, then $(k \cdot a+1)^2 + (k \cdot b+1)^2 = 42.05$.
The square root of 42.05 is 6.484... and when rounded to the nearest integer, it becomes 6.
This is equal to k, so he scores 6 points.

It can be shown that if he plays 10 turns with $k = 1$, $k = 2$, ..., $k = 10$, the expected value of his total score, rounded to five decimal places, is 10.20914.

If he plays 10^5 turns with $k = 1$, $k = 2$, $k = 3$, ..., $k = 10^5$, what is the expected value of his total score, rounded to five decimal places?

Problem 286

Barbara is a mathematician and a basketball player. She has found that the probability of scoring a point when shooting from a distance x is exactly $(1 - {}^x/_q)$, where q is a real constant greater than 50.

During each practice run, she takes shots from distances $x = 1$, $x = 2$, ..., $x = 50$ and, according to her records, she has precisely a 2 % chance to score a total of exactly 20 points.

Find q and give your answer rounded to 10 decimal places.

Problem 287

The quadtree encoding allows us to describe a $2^N \times 2^N$ black and white image as a sequence of bits (0 and 1). Those sequences are to be read from left to right like this:

- the first bit deals with the complete $2^N \times 2^N$ region;
- "0" denotes a split:
  the current $2^n \times 2^n$ region is divided into 4 sub-regions of dimension $2^{n-1} \times 2^{n-1}$,
  the next bits contains the description of the top left, top right, bottom left and bottom right sub-regions - in that order;
- "10" indicates that the current region contains only black pixels;
- "11" indicates that the current region contains only white pixels.

Consider the following 4×4 image (colored marks denote places where a split can occur):

This image can be described by several sequences, for example :
"001010101001011111011010101010", of length 30, or
"0100101111101110", of length 16, which is the minimal sequence for this image.

For a positive integer N, define D_N as the $2^N \times 2^N$ image with the following coloring scheme:

- the pixel with coordinates $x = 0$, $y = 0$ corresponds to the bottom left pixel,
- if $(x - 2^{N-1})^2 + (y - 2^{N-1})^2 \leq 2^{2N-2}$ then the pixel is black,
- otherwise the pixel is white.

What is the length of the minimal sequence describing D_{24}?

Problem 288

For any prime p the number N(p,q) is defined by N(p,q) = $\sum_{n=0 \text{ to } q} T_n * p^n$
with T_n generated by the following random number generator:

$S_0 = 290797$

$S_{n+1} = S_n^2 \bmod 50515093$

$T_n = S_n \bmod p$

Let Nfac(p,q) be the factorial of N(p,q).
Let NF(p,q) be the number of factors p in Nfac(p,q).

You are given that NF(3,10000) mod 3^{20}=624955285.

Find NF(61,10^7) mod 61^{10}

Problem 289

Let C(x,y) be a circle passing through the points (x, y), (x, y+1), (x+1, y) and (x+1, y+1).

For positive integers m and n, let E(m,n) be a configuration which consists of the $m \cdot n$ circles:
{ C(x,y): $0 \leq x < m$, $0 \leq y < n$, x and y are integers }

An Eulerian cycle on E(m,n) is a closed path that passes through each arc exactly once.
Many such paths are possible on E(m,n), but we are only interested in those which are not self-crossing: A non-crossing path just touches itself at lattice points, but it never crosses itself.

The image below shows E(3,3) and an example of an Eulerian non-crossing path.

Let L(m,n) be the number of Eulerian non-crossing paths on E(m,n).
For example, L(1,2) = 2, L(2,2) = 37 and L(3,3) = 104290.

Find L(6,10) mod 10^{10}.

Problem 290

How many integers $0 \leq n < 10^{18}$ have the property that the sum of the digits of n equals the sum of digits of $137n$?

Problem 291

A prime number p is called a Panaitopol prime if for some positive integers x and y.

Find how many Panaitopol primes are less than 5×10^{15}.

Problem 292

We shall define a *pythagorean polygon* to be a **convex polygon** with the following properties:

- there are at least three vertices,
- no three vertices are aligned,
- each vertex has **integer coordinates**,
- each edge has **integer length**.

For a given integer n, define P(n) as the number of distinct pythagorean polygons for which the perimeter is $\leq n$.
Pythagorean polygons should be considered distinct as long as none is a translation of another.

You are given that P(4) = 1, P(30) = 3655 and P(60) = 891045.
Find P(120).

Problem 293

An even positive integer N will be called admissible, if it is a power of 2 or its distinct prime factors are consecutive primes.
The first twelve admissible numbers are 2,4,6,8,12,16,18,24,30,32,36,48.

If N is admissible, the smallest integer M > 1 such that N+M is prime, will be called the pseudo-Fortunate number for N.

For example, N=630 is admissible since it is even and its distinct prime factors are the

consecutive primes 2,3,5 and 7.
The next prime number after 631 is 641; hence, the pseudo-Fortunate number for 630 is M=11.
It can also be seen that the pseudo-Fortunate number for 16 is 3.

Find the sum of all distinct pseudo-Fortunate numbers for admissible numbers N less than 10^9.

Problem 294

For a positive integer k, define d(k) as the sum of the digits of k in its usual decimal representation. Thus d(42) = 4+2 = 6.

For a positive integer n, define S(n) as the number of positive integers $k < 10^n$ with the following properties :

- k is divisible by 23 and
- d(k) = 23.

You are given that S(9) = 263626 and S(42) = 6377168878570056.

Find $S(11^{12})$ and give your answer mod 10^9.

Problem 295

We call the convex area enclosed by two circles a *lenticular hole* if:

- The centres of both circles are on lattice points.
- The two circles intersect at two distinct lattice points.
- The interior of the convex area enclosed by both circles does not contain any lattice points.

Consider the circles:
C_0: $x^2+y^2=25$
C_1: $(x+4)^2+(y-4)^2=1$
C_2: $(x-12)^2+(y-4)^2=65$

The circles C_0, C_1 and C_2 are drawn in the picture below.

C_0 and C_1 form a lenticular hole, as well as C_0 and C_2.

We call an ordered pair of positive real numbers (r_1, r_2) a *lenticular pair* if there exist two circles with radii r_1 and r_2 that form a lenticular hole. We can verify that (1, 5) and (5, $\sqrt{65}$) are the lenticular pairs of the example above.

Let L(N) be the number of **distinct** lenticular pairs (r_1, r_2) for which $0 < r_1 \leq r_2 \leq N$.
We can verify that L(10) = 30 and L(100) = 3442.

Find L(100 000).

Problem 296

Given is an integer sided triangle ABC with $BC \leq AC \leq AB$.
k is the angular bisector of angle ACB.
m is the tangent at C to the circumscribed circle of ABC.
n is a line parallel to m through B.
The intersection of n and k is called E.

How many triangles ABC with a perimeter not exceeding 100 000 exist such that BE has integral length?

Problem 297

Each new term in the Fibonacci sequence is generated by adding the previous two terms.
Starting with 1 and 2, the first 10 terms will be: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

Every positive integer can be uniquely written as a sum of nonconsecutive terms of the Fibonacci sequence. For example, 100 = 3 + 8 + 89.
Such a sum is called the **Zeckendorf representation** of the number.

For any integer n>0, let $z(n)$ be the number of terms in the Zeckendorf representation of n.
Thus, $z(5) = 1$, $z(14) = 2$, $z(100) = 3$ etc.
Also, for $0<n<10^6$, $\sum z(n) = 7894453$.

Find $\sum z(n)$ for $0<n<10^{17}$.

Problem 298

Larry and Robin play a memory game involving of a sequence of random numbers between 1 and 10, inclusive, that are called out one at a time. Each player can remember up to 5 previous numbers. When the called number is in a player's memory, that player is awarded a point. If it's not, the player adds the called number to his memory, removing another number if his memory is full.

Both players start with empty memories. Both players always add new missed numbers to their

memory but use a different strategy in deciding which number to remove:
Larry's strategy is to remove the number that hasn't been called in the longest time.
Robin's strategy is to remove the number that's been in the memory the longest time.

Example game:

Turn	Called number	Larry's memory	Larry's score	Robin's memory	Robin's score
1	1	1	0	1	0
2	2	1,2	0	1,2	0
3	4	1,2,4	0	1,2,4	0
4	6	1,2,4,6	0	1,2,4,6	0
5	1	1,2,4,6	1	1,2,4,6	1
6	8	1,2,4,6,8	1	1,2,4,6,8	1
7	10	1,4,6,8,10	1	2,4,6,8,10	1
8	2	1,2,6,8,10	1	2,4,6,8,10	2
9	4	1,2,4,8,10	1	2,4,6,8,10	3
10	1	1,2,4,8,10	2	1,4,6,8,10	3

Denoting Larry's score by L and Robin's score by R, what is the expected value of $|L-R|$ after 50 turns? Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

Problem 299

Four points with integer coordinates are selected:
A(a, 0), B(b, 0), C(0, c) and D(0, d), with $0 < a < b$ and $0 < c < d$.
Point P, also with integer coordinates, is chosen on the line AC so that the three triangles ABP, CDP and BDP are all similar.

It is easy to prove that the three triangles can be similar, only if a=c.

So, given that a=c, we are looking for triplets (a,b,d) such that at least one point P (with integer coordinates) exists on AC, making the three triangles ABP, CDP and BDP all similar.

For example, if (a,b,d)=(2,3,4), it can be easily verified that point P(1,1) satisfies the above condition. Note that the triplets (2,3,4) and (2,4,3) are considered as distinct, although point P(1,1) is common for both.

If b+d < 100, there are 92 distinct triplets (a,b,d) such that point P exists.
If b+d < 100 000, there are 320471 distinct triplets (a,b,d) such that point P exists.

If b+d < 100 000 000, how many distinct triplets (a,b,d) are there such that point P exists?

Problem 300

In a very simplified form, we can consider proteins as strings consisting of hydrophobic (H) and polar (P) elements, e.g. HHPPHHHPHHPH.
For this problem, the orientation of a protein is important; e.g. HPP is considered distinct from PPH. Thus, there are 2^n distinct proteins consisting of n elements.

When one encounters these strings in nature, they are always folded in such a way that the number of H-H contact points is as large as possible, since this is energetically advantageous.
As a result, the H-elements tend to accumulate in the inner part, with the P-elements on the outside.
Natural proteins are folded in three dimensions of course, but we will only consider protein folding in two dimensions.

The figure below shows two possible ways that our example protein could be folded (H-H contact points are shown with red dots).

The folding on the left has only six H-H contact points, thus it would never occur naturally.
On the other hand, the folding on the right has nine H-H contact points, which is optimal for this string.

Assuming that H and P elements are equally likely to occur in any position along the string, the average number of H-H contact points in an optimal folding of a random protein string of length 8 turns out to be 850 / 2^8=3.3203125.

What is the average number of H-H contact points in an optimal folding of a random protein string of length 15?
Give your answer using as many decimal places as necessary for an exact result.

Problem 301

Nim is a game played with heaps of stones, where two players take it in turn to remove any number of stones from any heap until no stones remain.

We'll consider the three-heap normal-play version of Nim, which works as follows:
- At the start of the game there are three heaps of stones.
- On his turn the player removes any positive number of stones from any single heap.
- The first player unable to move (because no stones remain) loses.

If (n_1,n_2,n_3) indicates a Nim position consisting of heaps of size n_1, n_2 and n_3 then there is a simple function $X(n_1,n_2,n_3)$ — that you may look up or attempt to deduce for yourself — that returns:

- zero if, with perfect strategy, the player about to move will eventually lose; or
- non-zero if, with perfect strategy, the player about to move will eventually win.

For example $X(1,2,3) = 0$ because, no matter what the current player does, his opponent can respond with a move that leaves two heaps of equal size, at which point every move by the current player can be mirrored by his opponent until no stones remain; so the current player loses. To illustrate:
- current player moves to (1,2,1)
- opponent moves to (1,0,1)
- current player moves to (0,0,1)
- opponent moves to (0,0,0), and so wins.

For how many positive integers $n \leq 2^{30}$ does $X(n,2n,3n) = 0$?
