



## In this chapter

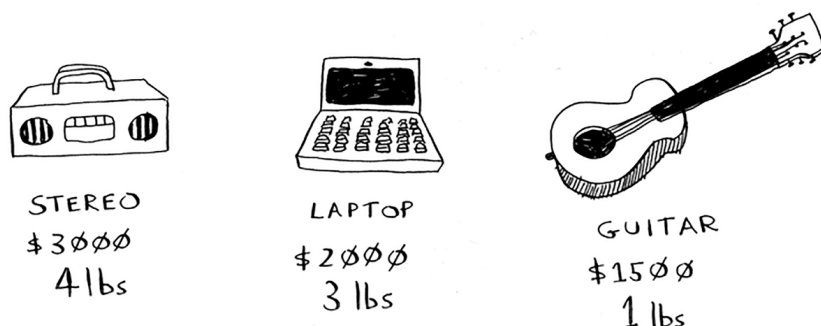
- You learn dynamic programming, a technique to solve a hard problem by breaking it up into subproblems and solving those subproblems first.
- Using examples, you learn to how to come up with a dynamic programming solution to a new problem.

## The knapsack problem

Let's revisit the knapsack problem from chapter 8. You're a thief with a knapsack that can carry 4 lb of goods.



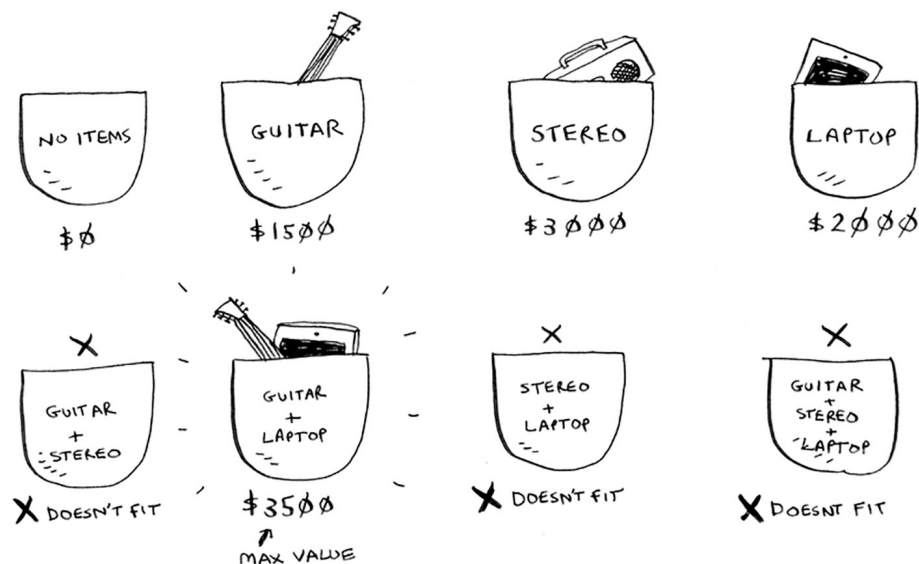
You have three items that you can put into the knapsack.



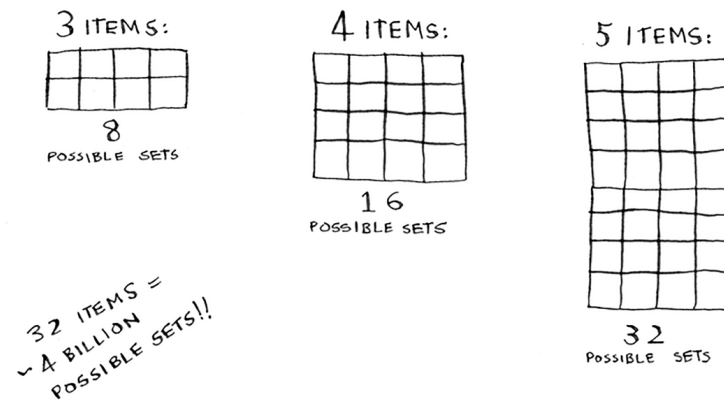
What items should you steal so that you steal the maximum money's worth of goods?

### The simple solution

The simplest algorithm is this: you try every possible set of goods and find the set that gives you the most value.



This works, but it's really slow. For 3 items, you have to calculate 8 possible sets. For 4 items, you have to calculate 16 sets. With every item you add, the number of sets you have to calculate doubles! This algorithm takes  $O(2^n)$  time, which is very, very slow.



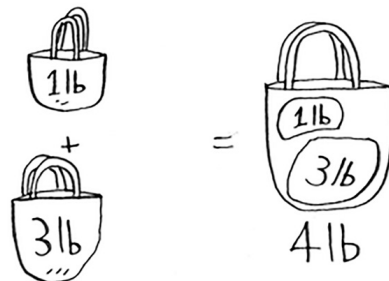
That's impractical for any reasonable number of goods. In chapter 8, you saw how to calculate an *approximate* solution. That solution will be close to the optimal solution, but it may not be the optimal solution.

So how do you calculate the optimal solution?

## Dynamic programming

Answer: With dynamic programming! Let's see how the dynamic-programming algorithm works here. Dynamic programming starts by solving subproblems and builds up to solving the big problem.

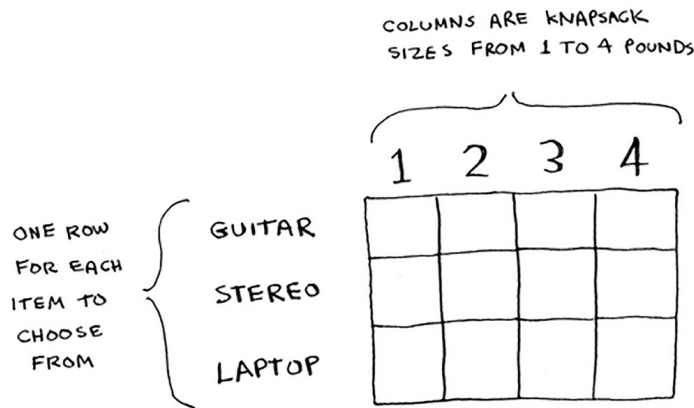
For the knapsack problem, you'll start by solving the problem for smaller knapsacks (or "sub-knapsacks") and then work up to solving the original problem.



*Dynamic programming is a hard concept, so don't worry if you don't get it right away. We're going to look at a lot of examples.*

I'll start by showing you the algorithm in action first. After you've seen it in action once, you'll have a lot of questions! I'll do my best to address every question.

Every dynamic-programming algorithm starts with a grid. Here's a grid for the knapsack problem.



The rows of the grid are the items, and the columns are knapsack weights from 1 lb to 4 lb. You need all of those columns because they will help you calculate the values of the sub-knapsacks.

The grid starts out empty. You're going to fill in each cell of the grid. Once the grid is filled in, you'll have your answer to this problem! Please follow along. Make your own grid, and we'll fill it out together.

### The guitar row

I'll show you the exact formula for calculating this grid later. Let's do a walkthrough first. Start with the first row.

	1	2	3	4
GUITAR				
STEREO				
LAPTOP				

This is the *guitar* row, which means you're trying to fit the guitar into the knapsack. At each cell, there's a simple decision: do you steal the guitar or not? Remember, you're trying to find the set of items to steal that will give you the most value.

The first cell has a knapsack of capacity 1 lb. The guitar is also 1 lb, which means it fits into the knapsack! So the value of this cell is \$1,500, and it contains a guitar.

Let's start filling in the grid.

	1	2	3	4
GUITAR	\$1500 G			
STEREO				
LAPTOP				

Like this, each cell in the grid will contain a list of all the items that fit into the knapsack at that point.

Let's look at the next cell. Here you have a knapsack of capacity 2 lb. Well, the guitar will definitely fit in there!

	1	2	3	4
GUITAR	\$1500 G	\$1500 G		
STEREO				
LAPTOP				

The same for the rest of the cells in this row. Remember, this is the first row, so you have *only* the guitar to choose from. You're pretending that the other two items aren't available to steal right now.

	1	2	3	4
GUITAR	\$1500 G	\$1500 G	\$1500 G	\$1500 G
STEREO				
LAPTOP				

At this point, you're probably confused. *Why* are you doing this for knapsacks with a capacity of 1 lb, 2 lb, and so on, when the problem talks about a 4 lb knapsack? Remember how I told you that dynamic programming starts with a small problem and builds up to the big problem? You're solving subproblems here that will help you to solve the big problem. Read on, and things will become clearer.

At this point, your grid should look like this.

	1	2	3	4
GUITAR	\$1500 G	\$1500 G	\$1500 G	\$1500 G
STEREO				
LAPTOP				

Remember, you're trying to maximize the value of the knapsack. *This row represents the current best guess for this max.* So right now, according to this row, if you had a knapsack of capacity 4 lb, the max value you could put in there would be \$1,500.

	1	2	3	4
GUITAR	\$1500 G	\$1500 G	\$1500 G	\$1500 G
STEREO				
LAPTOP				

← OUR CURRENT BEST GUESS FOR WHAT THE THIEF SHOULD STEAL: THE GUITAR FOR \$1500

You know that's not the final solution. As we go through the algorithm, you'll refine your estimate.

### The stereo row

Let's do the next row. This one is for the stereo. Now that you're on the second row, you can steal the stereo or the guitar. At every row, you can steal the item at that row or the items in the rows above it. So you can't choose to steal the laptop right now, but you can steal the stereo and/or the guitar. Let's start with the first cell, a knapsack of capacity 1 lb. The current max value you can fit into a knapsack of 1 lb is \$1,500.

	1	2	3	4
GUITAR	\$1500 G	\$1500 G	\$1500 G	\$1500 G
STEREO				
LAPTOP				

CURRENT MAX FOR A 1lb KNAPSACK →

← NEW MAX FOR A 1lb KNAPSACK

Should you steal the stereo or not?

You have a knapsack of capacity 1 lb. Will the stereo fit in there? Nope, it's too heavy! Because you can't fit the stereo, \$1,500 *remains* the max guess for a 1 lb knapsack.

	1	2	3	4
GUITAR	\$1500 ↓ G	\$1500 G	\$1500 G	\$1500 G
STEREO	\$1500 G			
LAPTOP				

Same thing for the next two cells. These knapsacks have a capacity of 2 lb and 3 lb. The old max value for both was \$1,500.

	1	2	3	4
GUITAR	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$1500 G
STEREO	\$1500 G	\$1500 G	\$1500 G	
LAPTOP				

The stereo still doesn't fit, so your guesses remain unchanged.

What if you have a knapsack of capacity 4 lb? Aha: the stereo finally fits! The old max value was \$1,500, but if you put the stereo in there instead, the value is \$3,000! Let's take the stereo.

	1	2	3	4
GUITAR	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$1500 G
STEREO	\$1500 G	\$1500 G	\$1500 G	\$3000 S
LAPTOP				

You just updated your estimate! If you have a 4 lb knapsack, you can fit at least \$3,000 worth of goods in it. You can see from the grid that you're incrementally updating your estimate.

	1	2	3	4	
GUITAR	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	← OLD ESTIMATE
STEREO	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$3000 ↓ S	← NEW ESTIMATE
LAPTOP					← FINAL SOLUTION

### The laptop row

Let's do the same thing with the laptop! The laptop weighs 3 lb, so it won't fit into a 1 lb or a 2 lb knapsack. The estimate for the first two cells stays at \$1,500.

	1	2	3	4
GUITAR	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G
STEREO	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$3000 ↓ S
LAPTOP	\$1500 ↓ G	\$1500 ↓ G		

At 3 lb, the old estimate was \$1,500. But you can choose the laptop instead, and that's worth \$2,000. So the new max estimate is \$2,000!

	1	2	3	4
GUITAR	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G
STEREO	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$3000 ↓ S
LAPTOP	\$1500 ↓ G	\$1500 ↓ G	\$2000 ↓ L	

At 4 lb, things get really interesting. This is an important part. The current estimate is \$3,000. You can put the laptop in the knapsack, but it's only worth \$2,000.



$$\begin{array}{ccc} \$3000 & \text{VS} & \$2000 \\ \text{STEREO} & & \text{LAPTOP} \end{array}$$

Hmm, that's not as good as the old estimate. But wait! The laptop weighs only 3 lb, so you have 1 lb free! You could put something in this 1 lb.

$$\begin{array}{ccc} \$3000 & \text{VS} & \left( \$2000 + \frac{???}{1 \text{ LB OF FREE SPACE}} \right) \\ \text{STEREO} & & \text{LAPTOP} \end{array}$$

What's the maximum value you can fit into 1 lb of space? Well, you've been calculating it all along.

MAX  
VALUE FOR  
1 lb →

	1	2	3	4
	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G
	\$1500 ↓ G	\$1500 ↓ G	\$1500 ↓ G	\$3000 ↓ 5
	\$1500 ↓ G	\$1500 ↓ G	\$2000 ↓ L	

According to the last best estimate, you can fit the guitar into that 1 lb space, and that's worth \$1,500. So the real comparison is as follows.

$$\begin{array}{ccc} \$3000 & \text{VS} & \left( \$2000 + \$1500 \right) \\ \text{STEREO} & & \text{LAPTOP} \quad \text{GUITAR} \end{array}$$

You might have been wondering why you were calculating max values for smaller knapsacks. I hope now it makes sense! When you have space left over, you can use the answers to those subproblems to figure out what will fit in that space. It's better to take the laptop + the guitar for \$3,500.

The final grid looks like this.

	1	2	3	4
GUITAR	\$1500	\$1500	\$1500	\$1500
STEREO	\$1500	\$1500	\$1500	\$3000
LAPTOP	\$1500	\$1500	\$2000	\$3500

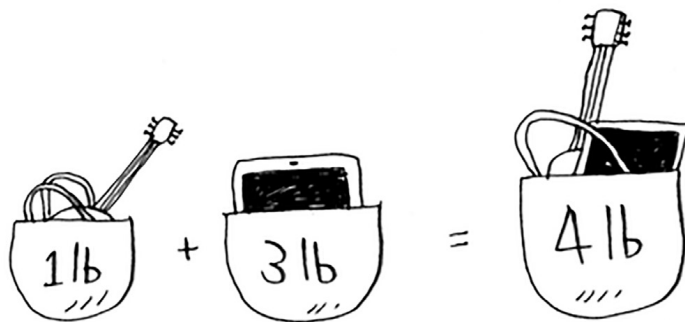
↑  
THE ANSWER!

There's the answer: the maximum value that will fit in the knapsack is \$3,500, made up of a guitar and a laptop!

Maybe you think that I used a different formula to calculate the value of that last cell. That's because I skipped some unnecessary complexity when filling in the values of the earlier cells. Each cell's value gets calculated with the same formula. Here it is.

$$\begin{array}{c} \text{ROW} \quad \text{COLUMN} \\ \downarrow \quad \downarrow \\ \text{CELL}[i][j] \end{array} = \text{MAX OF} \left\{ \begin{array}{l} 1. \text{ THE PREVIOUS MAX (VALUE AT CELL } [i-1][j]) \\ \text{VS} \\ 2. \text{ VALUE OF CURRENT ITEM + VALUE OF THE REMAINING SPACE} \\ \quad \quad \quad \uparrow \\ \quad \quad \quad \text{CELL}[i-1][j - \text{ITEM'S WEIGHT}] \end{array} \right.$$

You can use this formula with every cell in this grid, and you should end up with the same grid I did. Remember how I talked about solving subproblems? You combined the solutions to two subproblems to solve the bigger problem.



## Knapsack problem FAQ

Maybe this still feels like magic. This section answers some common questions.

### What happens if you add an item?

Suppose you realize there's a fourth item you can steal that you didn't notice before! You can also steal an iPhone.

Do you have to recalculate everything to account for this new item? Nope. Remember, dynamic programming keeps progressively building on your estimate. So far, these are the max values.



IPHONE  
\$2000  
1lb

	1	2	3	4
GUITAR	\$1500 G	\$1500 G	\$1500 G	\$1500 G
STEREO	\$1500 G	\$1500 G	\$1500 G	\$3000 S
LAPTOP	\$1500 G	\$1500 G	\$2000 L	\$3500 LG

That means for a 4 lb knapsack, you can steal \$3,500 worth of goods. You thought that was the final max value. But let's add a row for the iPhone.

	1	2	3	4
GUITAR	\$1500 G	\$1500 G	\$1500 G	\$1500 G
STEREO	\$1500 G	\$1500 G	\$1500 G	\$3000 S
LAPTOP	\$1500 G	\$1500 G	\$2000 L	\$3500 LG
IPHONE				

↑  
NEW ANSWER

Turns out you have a new max value! Try to fill in this new row before reading on.

Let's start with the first cell. The iPhone fits into the 1 lb knapsack. The old max was \$1,500, but the iPhone is worth \$2,000. Let's take the iPhone instead.

	1	2	3	4
GUITAR	\$1500 G	\$1500 G	\$1500 G	\$1500 G
STEREO	\$1500 G	\$1500 G	\$1500 G	\$3000 S
LAPTOP	\$1500 G	\$1500 G	\$2000 L	\$3500 LG
IPHONE	\$2000 I			

In the next cell, you can fit the iPhone *and* the guitar.

\$1500 G	\$1500 G	\$1500 G	\$1500 G
\$1500 G	\$1500 G	\$1500 G	\$3000 S
\$1500 G	\$1500 G	\$2000 L	\$3500 LG
\$2000 I	\$3500 IG		

For cell 3, you can't do better than take the iPhone and the guitar again, so leave it as is.

For the last cell, things get interesting. The current max is \$3,500. You can steal the iPhone instead, and you have 3 lb of space left over.

$$\begin{array}{c} \$3500 \\ \text{LAPTOP + GUITAR} \end{array} \quad \text{vs} \quad \left( \begin{array}{c} \$2000 \\ \text{IPHONE} \end{array} + \frac{???}{3 \text{ LBS FREE}} \right)$$

Those 3 lb are worth \$2,000! \$2,000 from the iPhone + \$2,000 from the old subproblem: that's \$4,000. A new max!

Here's the new final grid.

\$1500 G	\$1500 G	\$1500 G	\$1500 G
\$1500 G	\$1500 G	\$1500 G	\$3000 S
\$1500 G	\$1500 G	\$2000 L	\$3500 LG
\$3500 I	\$3500 IG	\$3500 IG	\$4000 IL

↑  
NEW  
ANSWER

Question: Would the value of a column ever go *down*? Is this possible?

	1	2	3	4
	\$1500	\$1500	\$1500	\$1500
	∅	∅	∅	\$3000

MAX VALUE  
DECREASED  
AS WE  
WENT ON ↓

Think of an answer before reading on.

Answer: No. At every iteration, you're storing the current max estimate. The estimate can never get worse than it was before!

## EXERCISE

**9.1** Suppose you can steal another item: an MP3 player. It weighs 1 lb and is worth \$1,000. Should you steal it?

## What happens if you change the order of the rows?

Does the answer change? Suppose you fill the rows in this order: stereo, laptop, guitar. What does the grid look like? Fill out the grid for yourself before moving on.

Here's what the grid looks like.

	1	2	3	4
STEREO	∅	∅	∅	\$3000 S
LAPTOP	∅	∅	\$2000 L	\$3000 S
GUITAR	\$1500 G	\$1500 G	\$2000 L	\$3500 LG

The answer doesn't change. The order of the rows doesn't matter.

## Can you fill in the grid column-wise instead of row-wise?

Try it for yourself! For this problem, it doesn't make a difference. It could make a difference for other problems.

## What happens if you add a smaller item?

Suppose you can steal a necklace. It weighs 0.5 lb, and it's worth \$1,000. So far, your grid assumes that all weights are integers. Now you decide to steal the necklace. You have 3.5 lb left over. What's the max value you can fit in 3.5 lb? You don't know! You only calculated values for 1 lb, 2 lb, 3 lb, and 4 lb knapsacks. You need to know the value of a 3.5 lb knapsack.

*Because of the necklace, you have to account for finer granularity, so the grid has to change.*

	0.5	1	1.5	2	2.5	3	3.5	4
GUITAR								
STEREO								
LAPTOP								
JEWELRY								

## Can you steal fractions of an item?

Suppose you're a thief in a grocery store. You can steal bags of lentils and rice. If a whole bag doesn't fit, you can open it and take as much as you can carry. So now it's not all or nothing—you can take a fraction of an item. How do you handle this using dynamic programming?

Answer: You can't. With the dynamic-programming solution, you either take the item or not. There's no way for it to figure out that you should take half an item.

But this case is also easily solved using a greedy algorithm! First, take as much as you can of the most valuable item. When that runs out, take as much as you can of the next most valuable item, and so on.

For example, suppose you have these items to choose from.



Quinoa is more expensive per pound than anything else. So, take all the quinoa you can carry! If that fills your knapsack, that's the best you can do.

If the quinoa runs out and you still have space in your knapsack, take the next most valuable item, and so on.



## Optimizing your travel itinerary

Suppose you're going to London for a nice vacation. You have two days there and a lot of things you want to do. You can't do everything, so you make a list.

ATTRACTION	TIME	RATING
WESTMINSTER ABBEY	1/2 DAY	7
GLOBE THEATER	1/2 DAY	6
NATIONAL GALLERY	1 DAY	9
BRITISH MUSEUM	2 DAYS	9
ST. PAUL'S CATHEDRAL	1/2 DAY	8

For each thing you want to see, you write down how long it will take and rate how much you want to see it. Can you figure out what you should see, based on this list?

It's the knapsack problem again! Instead of a knapsack, you have a limited amount of time. And instead of stereos and laptops, you have a list of places you want to go. Draw the dynamic-programming grid for this list before moving on.

Here's what the grid looks like.

	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
WESTMINSTER				
GLOBE THEATRE				
NATIONAL GALLERY				
BRITISH MUSEUM				
ST. PAUL'S				

Did you get it right? Fill in the grid. What places should you end up seeing? Here's the answer.

	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
WESTMINSTER	7 <sub>W</sub>	7 <sub>W</sub>	7 <sub>W</sub>	7 <sub>W</sub>
GLOBE THEATRE	7 <sub>W</sub>	13 <sub>WG</sub>	13 <sub>WG</sub>	13 <sub>WG</sub>
NATIONAL GALLERY	7 <sub>W</sub>	13 <sub>WG</sub>	16 <sub>WN</sub>	22 <sub>WGN</sub>
BRITISH MUSEUM	7 <sub>W</sub>	13 <sub>WG</sub>	16 <sub>WN</sub>	22 <sub>WGN</sub>
ST. PAUL'S	8 <sub>S</sub>	15 <sub>WS</sub>	21 <sub>WGS</sub>	24 <sub>WNS</sub>

↑  
FINAL ANSWER:  
WESTMINSTER ABBEY,  
NATIONAL GALLERY,  
ST. PAUL'S CATHEDRAL



## Handling items that depend on each other

Suppose you want to go to Paris, so you add a couple of items on the list.

EIFFEL TOWER	$1\frac{1}{2}$ DAY	8
THE LOUVRE	$1\frac{1}{2}$ DAY	9
NOTRE DAME	$1\frac{1}{2}$ DAY	7

These places take a lot of time, because first you have to travel from London to Paris. That takes half a day. If you want to do all three items, it will take four and a half days.

Wait, that's not right. You don't have to travel to Paris for each item. Once you're in Paris, each item should only take a day. So it should be one day per item + half a day of travel = 3.5 days, not 4.5 days.

If you put the Eiffel Tower in your knapsack, then the Louvre becomes "cheaper"—it will only cost you a day instead of 1.5 days. How do you model this in dynamic programming?

You can't. Dynamic programming is powerful because it can solve subproblems and use those answers to solve the big problem. *Dynamic programming only works when each subproblem is discrete—when it doesn't depend on other subproblems.* That means there's no way to account for Paris using the dynamic-programming algorithm.

## Is it possible that the solution will require more than two sub-knapsacks?

It's possible that the best solution involves stealing more than two items. The way the algorithm is set up, you're combining two knapsacks at most—you'll never have more than two sub-knapsacks. But it's possible for those sub-knapsacks to have their own sub-knapsacks.



## Is it possible that the best solution doesn't fill the knapsack completely?

Yes. Suppose you could also steal a diamond.

This is a big diamond: it weighs 3.5 pounds. It's worth a million dollars, way more than anything else. You should definitely steal it! But there's half a pound of space left, and nothing will fit in that space.



## EXERCISE

**9.2** Suppose you're going camping. You have a knapsack that will hold 6 lb, and you can take the following items. Each has a value, and the higher the value, the more important the item is:

- Water, 3 lb, 10
- Book, 1 lb, 3
- Food, 2 lb, 9
- Jacket, 2 lb, 5
- Camera, 1 lb, 6

What's the optimal set of items to take on your camping trip?

## Longest common substring

You've seen one dynamic programming problem so far. What are the takeaways?

- Dynamic programming is useful *when you're trying to optimize something given a constraint*. In the knapsack problem, you had to maximize the value of the goods you stole, constrained by the size of the knapsack.
- You can use dynamic programming when the problem can be broken into discrete subproblems, and they don't depend on each other.

