Vandermonde's Identity: an Algebraic Proof

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November 2018

An algebraic approach to prove the Vandermonde's Identity:

$$\sum_{x=0}^{z} \binom{n_1}{x}\binom{n_2}{z-x} = \binom{n_1+n_2}{z}$$

Recall the polynomial expansion

$$(k+1)^{n_1+n_2} = \sum_{z=0}^{n_1+n_2} \binom{n_1+n_2}{z} k^z (*)$$

Also note that $(k+1)^{n_1+n_2} = (k+1)^{n_1} \cdot (k+1)^{n_2}$, we can have

$$\begin{aligned}
&(k+1)^{n_1+n_2} = (k+1)^{n_1} \cdot (k+1)^{n_2} = \left(\sum_{i=0}^{n_1} \binom{n_1}{i} k^i\right) \cdot \left(\sum_{j=0}^{n_2} \binom{n_2}{j} k^j\right) \\
&= \left(\binom{n_1}{0} + \binom{n_1}{1}k + \binom{n_1}{2}k^2 + \cdots + \binom{n_1}{n_1}k^{n_1}\right) \cdot \left(\binom{n_2}{0} + \binom{n_2}{1}k + \binom{n_2}{2}k^2 + \cdots + \binom{n_2}{n_2}k^{n_2}\right) \\
&= \binom{n_1}{0}\binom{n_2}{0} + \left(\binom{n_1}{0}\binom{n_2}{1} + \binom{n_1}{1}\binom{n_2}{0}\right)k + \left(\binom{n_1}{0}\binom{n_2}{2} + \binom{n_1}{1}\binom{n_2}{1} + \binom{n_1}{2}\binom{n_2}{0}\right)k^2 + \cdots
\end{aligned}$$

Let's go back to (*), so now the coefficient for a k^z can also be expressed as

$$\binom{n_1}{0}\binom{n_2}{z} + \binom{n_1}{1}\binom{n_2}{z-1} + \cdots + \binom{n_1}{z}\binom{n_2}{0} = \sum_{x=0}^{z} \binom{n_1}{z}\binom{n_2}{z-x}$$

which implies that

$$\sum_{x=0}^{z} \binom{n_1}{x}\binom{n_2}{z-x} = \binom{n_1+n_2}{z}$$