Proof of the Heine-Borel Theorem

Definitely not Danni Shi's Original Idea Though

February 2019

Heine-Borel Theorem:
A subset S of $\mathbb{R}$ is a compact set if and only if S is closed and bounded.

Proof: "$\underline{\Rightarrow}$": Suppose S is compact. I want to first prove the <u>bounded</u> part, then the <u>closed</u> part.
<u>Bounded</u>: Let $\{U_\lambda : \lambda \in \Lambda\}$ be a collection of open sets that covers $\mathbb{R}$, with $U_{\lambda i} = B(0, \lambda_i)$. By compactness, S can be covered by $\bigcup_{i=1}^{k} U_{\lambda i}$ with some finite $k \in \mathbb{N}$.
Now choose $m = \max\{\lambda_1, \cdots, \lambda_k\}$. Then for all s in $S, |s| \leqslant m$, so S is bounded.
<u>Closed</u>: I want to show that $S^c = \mathbb{R}\backslash S$ is open. Let $x \in \mathbb{R}\backslash S$, and $s \in S$. Let $ds < \frac{1}{2}d(x,s)$. So $\{B(s, ds)\}$ is an open cover of S. Since S is compact, S can be covered by $\bigcup_{i=1}^{m} B(s_i, ds_i)$ for some finite $m \in \mathbb{N}$.
Since $ds < \frac{1}{2}d(x,s)$, so $\bigcup_{i=1}^{k} (B(x, ds_i) \cap B(s_i, ds_i)) = \varnothing$ for any finite k.
Since x is arbitrary, this means for any x, its neighborhood is a subset of $\mathbb{R}\backslash S$, and thus x is its interior point. Hence $\mathbb{R}\backslash S$ is open and S is closed.

"$\underline{\Rightarrow}$": Suppose S is closed and bounded, and I want to prove that S is compact. Before I go on, I want to bring out this **lemma** first: a closed and bounded set contains its maximum and minimum. I will skip the proof and just use it because I am very lazy.
Let $\{V_\lambda, \lambda \in \Lambda\}$ be an open cover of S. Let $\ell = \inf S$ and $u = \sup S$. $\ell, u \in S$. I want to show that V has finite subcover for S.
Define $S_x = S \cap (-\infty, x]\, \forall x \in \mathbb{R}$. Let B be a collection of x such that S_x can be covered by a finite subcover of V. So B keeps track of how much of S can be finitely covered by V. (B is nonempty. Consider $S_\ell = \{\ell\}$, which obviously can be finitely covered by V and $\ell \in B$.)
Assume S is true, then all of S can be finitely covered by V, and for all $x > u, x \in B$, as $S_x = \varnothing$ now. Then if my assumption is true, S is compact and B should not be bounded above. So first let me assume that $b = \sup B$ exists.
Case 1, $b \in S$: we can assume $b \in V_{\lambda 0}$ for some $\lambda 0 \in \Lambda$. Since for all $j, V_{\lambda j}$ is open, there exists an interval $[x_1, x_2]$ in $V_{\lambda 0}$ s.t. $x_1 < b < x_2$. Then there exists some $V_{\lambda k}$ (or a union of some) that covers S_{x_1}, which may also cover S_{x_2}, so $x_2 \in B$ also and this contradicts the supremum assumption of b.
Case 2, $b \notin S$: Since S is closed, there exists some $\epsilon > 0$ s.t. $B(b, \epsilon) \cap S = \varnothing$. So $S_{b-\epsilon} = \varnothing = S_{b+\epsilon}$, and both $b - \epsilon, b + \epsilon$ belongs to B, which again contradicts the supremum assumption of b.
Hence, B is not bounded above and S is compact.