Asymptotic Analysis Examples

Determine the asymptotic relationship between f and g.

- $f(n) = \frac{1}{2}n^2 - 3n$
  $g(n) = n^2$
  We want to find a $c > 0$ and $n_0 > 0$ such that
  $$\frac{1}{2}n^2 - 3n \leq cn^2$$
  for all $n \geq n_0$. Dividing through by n^2, we see
  $$\frac{1}{2} - \frac{3}{n} \leq c$$
  so we can choose $n_0 = 1$ and then
  $$\Big(0 <\Big) \quad c = 1 \quad \left(\geq \frac{1}{2} - \frac{3}{n_0} = -\frac{5}{2}\right)$$
  to satisify the definition of
  $$f(n) = O\big(g(n)\big).$$
  Then we try to find a $c > 0$ and $n_0 > 0$ such that
  $$n^2 \leq c\left(\frac{1}{2}n^2 - 3n\right)$$
  for all $n \geq n_0$. Dividing through by n^2, we see
  $$1 \leq c\left(\frac{1}{2} - \frac{3}{n}\right).$$
  If we choose something like $n_0 = 12$, then
  $$1 \leq c\left(\frac{1}{2} - \frac{3}{n_0}\right) \Longrightarrow 1 \leq \frac{1}{4}c \Longrightarrow c \geq 4$$
  so choosing $c = 4$ satisfies the definition of
  $$g(n) = O\big(f(n)\big).$$
  We've shown both $f = O(g)$ and $g = O(f)$ so that means $f = \Theta(g)$.

- $f(n) = n^3 - 5n$

  $g(n) = 2n^2 + 25n$

  We suspect $f \neq O(g)$, so we need to show, for any given $c > 0$, how to find n large enough to violate the inequality

  $$n^3 - 5n \leq c\left(2n^2 + 25n\right).$$

  To that end, consider

  $$\frac{n^3 - 5n}{2n^2 + 25n} \leq c. \tag{1}$$

  Note that

  $$\frac{n^3 - 5n}{2n^2 + 25n} > \frac{n^3 - \frac{1}{2}n^3}{27n^2} = \frac{1}{54}n$$

  for $n \geq 4$. So, for any c, choosing n so that

  $$\frac{1}{54}n > c \quad \text{(i.e., } n > 54c\text{)}$$

  means

  $$\frac{n^3 - 5n}{2n^2 + 25n} > \frac{1}{54}n > c$$

  which violates inequality (1) and, in turn, the original inequality. Having shown that, no matter how large we choose c, it is always possible for n to get large enough to violate the definition of $f = O(g)$, we conclude that in fact $f \neq O(g)$. Hence, the asymptotic relationship between f and g is $f = \omega(g)$.

- $f(n) = 6n^3$, $g(n) = n^2$

  $$6n^3 \leq cn^2 \implies 6n \leq c \implies \nexists c, n_0 \text{ s.t. } 6n^3 \leq cn^2 \text{ for } n > n_0$$

  so $f(n) \neq O(g(n))$.

  $$6n^3 \geq cn^2 \implies 6n > c \implies n > \frac{c}{6}$$

  so for any c, choosing $n > \frac{c}{6}$ makes $f(n) > g(n)$. That means $f = \omega(g)$ or $g = o(f)$.

- $f(n) = n^{1/2}$, $g(n) = n^{2/3}$

  $$\lim_{n\to\infty} \frac{n^{1/2}}{n^{2/3}} = \lim n^{1/2-2/3} = \lim n^{-1/6} = 0$$

  so $f(n) = o(g(n))$.

Note that, correspondingly,

$$\lim_{n\to\infty} \frac{n^{2/3}}{n^{1/2}} \Longrightarrow \lim n^{1/6} = \infty.$$

- $f(n) = 100n + \log(n)$, $g(n) = n + (\log(n))^2$

$$\lim_{n\to\infty} \frac{100n + \log(n)}{n + (\log(n))^2} = \lim_{n\to\infty} \frac{100 + \frac{1}{n}}{1 + 2(\log(n))\frac{1}{n}} = \lim_{n\to\infty} \frac{100n + 1}{n + 2\log(n)} = \lim_{n\to\infty} \frac{100}{1 + \frac{2}{n}} = 100$$

so $f(n) = O(g(n))$.

$$\lim_{n\to\infty} \frac{n + (\log(n))^2}{100n + \log(n)} = \frac{1}{100}$$

so $f(n) = \Omega(g(n))$.

Together, that means $f(n) = \Theta(g(n))$.