# Kepler's laws and calculation

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**Abstract**

A brief explanation as to how to code a simulation of our solar system.

## 1 Introduction

Exploration has always been part of what makes us human. The captivating to space endeavor inspires all scientists and storytellers. Central to this is celestial dynamics - how planets and objects travel through our solar system. Johannes Kepler found insights into how to predict these movements formulating them into 3 laws which we will discuss today.

Our primary objective in this paper is to traverse through a web app here https://orbits-2aa96.web.app/ and go through the maths of why I did some calculations I did.

## 2 Kepler's first law

### 2.1 What is it?

Kepler's first law states

> Each planet's orbit about the Sun is an ellipse. The Sun's center is always located at one focus of the orbital ellipse. The Sun is at one focus. [kep]

We could think of this much like the procedure to draw a mathematically accurate ellipse. If you have ever seen the method of drawing, it you can skip this explanation.

If we look at this image

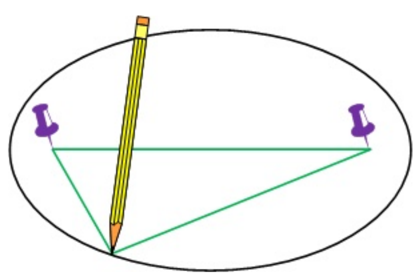

Figure 1: How to draw an ellipse.

we can get an idea as to how to draw an ellipse. The two pins represent the two foci of the ellipse and the string forms a triangle between the two and a pencil. With this information, we can start to formulate an equation for how to draw an ellipse.

## 2.2 Calculations

First we can label some diagrams and initialise some variables.

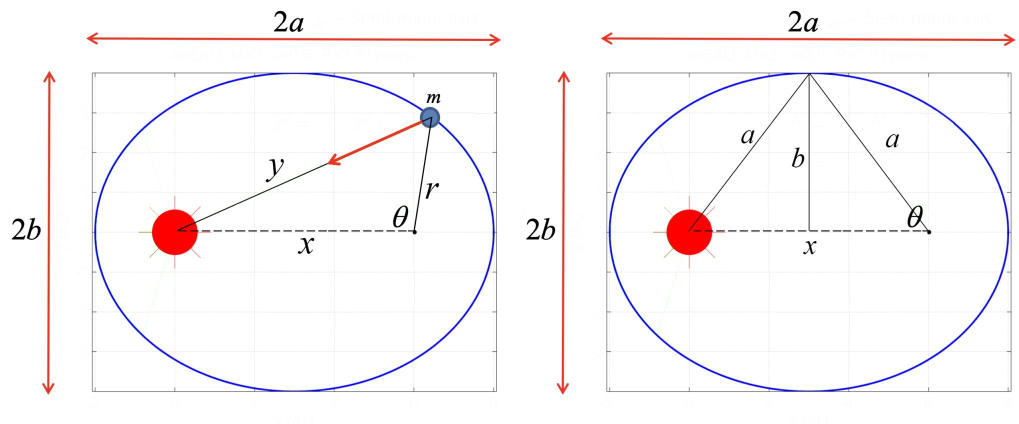


Figure 2: Diagrams labeled with letters.

From these diagrams we can immediately link some of the variables together. We can immediately link $y$, $r$ and $a$ together

$$y + r = 2a \tag{1}$$

where we can get

$$y = 2a - r \tag{2}$$

From the second diagram we can see using pythagoras' theorem

$$b^2 = a^2 - \left(\frac{x}{2}\right)^2 \tag{3}$$

We will use these equation later on.
Now we can introduce the eccentricity, $\sigma$ where

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}} \tag{4}$$

if we square both sides we get,

$$b^2 = a^2 - \epsilon^2 a^2 \tag{5}$$

Now if we compare equation 3 with equation 5 we get

$$\left(\frac{x}{2}\right)^2 = (\epsilon a)^2 \tag{6}$$

Now if we know that $a$, $\epsilon$ and $x$ are positive so

$$x = 2\epsilon a \tag{7}$$

If we look at the first diagram again we can also formulate another equation from it using the cosine rule.

$$y^2 = x^2 + r^2 - 2xrcos\theta \tag{8}$$

Now we substitute this into equation 2 to get

$$(2a - r)^2 = x^2 + r^2 - 2xrcos\theta \tag{9}$$

,

which we can do some algebraic manipulation to get

$$4a^2 - 4ar = x^2 - 2xr\cos\theta \tag{10}$$

Now we can substitute this into equation 7 to get

$$4a^2 - 4ar = (2\epsilon a)^2 - 2(2\epsilon a)r\cos\theta \tag{11}$$

If we cancel the 4s and rearrange we obtain the equation for an ellipse

$$r = \frac{a(1-\epsilon^2)}{1-\epsilon\cos\theta} \tag{12}$$

Now we have the equation for an ellipse we can use it to form parametric equations which we can use to display in a Cartesian plane or in 3 dimensional space.

For the Cartesian plane we can get the pair of equations

$$x = \cos\theta\frac{a(1-\epsilon^2)}{1-\epsilon\cos\theta}$$

$$y = \sin\theta\frac{a(1-\epsilon^2)}{1-\epsilon\cos\theta}$$

where theta for a full ellipse will vary between 0 and $2\pi$ and $\epsilon$ is constant.

# 3 Kepler's second law

## 3.1 What is it?

Well the definition states that

> The imaginary line joining a planet and the Sun sweeps equal areas of space during equal time intervals as the planet orbits. [kep]

It is truly amazing how Kepler was able to obtain this conclusion from the resources which he had during his time.

Obtaining this law involves a lot of calculus.

Deriving the equation from first principles is quite tedious and it is a bit beyond the scope of this paper. Perhaps a better use of time is see how it applies in the simulation.

## 3.2 Caluclations

We start off with the equation which can be derived from considering limiting the distance travelled in a certain amount of time to 0 and calculating the limit of area between the two positions in time.

$$\frac{dA}{dt} = \frac{1}{2}\sqrt{G(m+M)(1-\epsilon^2)a} \tag{13}$$

where $\frac{dA}{dt}$ is the rate of change of area swept, G is Newton's gravitational constant and m and M are the masses of the two objects.

Now if we consider the relationship between rate of change of angle and area we can obtain

$$\frac{d\theta}{dA} = \frac{2}{r^2} \tag{14}$$

Evaluating these two equations we can obtain

$$r^2\frac{d\theta}{dt} = \sqrt{G(m+M)(1-\epsilon^2)a} \tag{15}$$

Solving the differential equation and substituting equation 12

$$t = \frac{a^2(1-\epsilon^2)^2}{\sqrt{G(m+M)(1-\epsilon^2)a}} \int_{\theta_0}^{\theta} \frac{1}{(1-\epsilon\cos\theta)^2}\, d\theta \tag{16}$$

we can now obtain by squaring the numerator and canceling

$$t = \sqrt{\frac{a^3(1-\epsilon^2)^3}{G(m+M)}} \int_{\theta_0}^{\theta} \frac{1}{(1-\epsilon\cos\theta)^2}\, d\theta \tag{17}$$

If we are a bit sneaky and take an equation from Kepler's Third Law which we will derive later

$$P^2 = \frac{4\pi^2}{G(m+M)} a^3 \tag{18}$$

and substitute it into a slightly rearrange equation 17

$$t = \sqrt{\frac{4\pi^2}{G(m+M)} a^3 (1-\epsilon^2)^3} \frac{1}{2\pi} \int_{\theta_0}^{\theta} \frac{1}{(1-\epsilon\cos\theta)^2}\, d\theta \tag{19}$$

we get a much simpler looking

$$t = P(1-\epsilon)^{\frac{3}{2}} \frac{1}{2\pi} \int_{\theta_0}^{\theta} \frac{1}{(1-\epsilon\cos\theta)^2}\, d\theta \tag{20}$$

Now instead of evaluating the integral directly, as it is quite tough, we can use Simpson's rule where

$$y = \frac{1}{(1-\epsilon\cos\theta)^2} \tag{21}$$

$$\int_{\theta_0}^{\theta} y\, d\theta \approx \frac{1}{3} h[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n] \tag{22}$$

.

In my calculations in my simulation I had a slightly different set up where the the starting coefficient is multiplied out and for obtaining the areas cumulatively, I had an accumulator which calculated the first term and then used the result to calculate the second term etc.

If we plot all of equation 17 we get a graph which looks like this

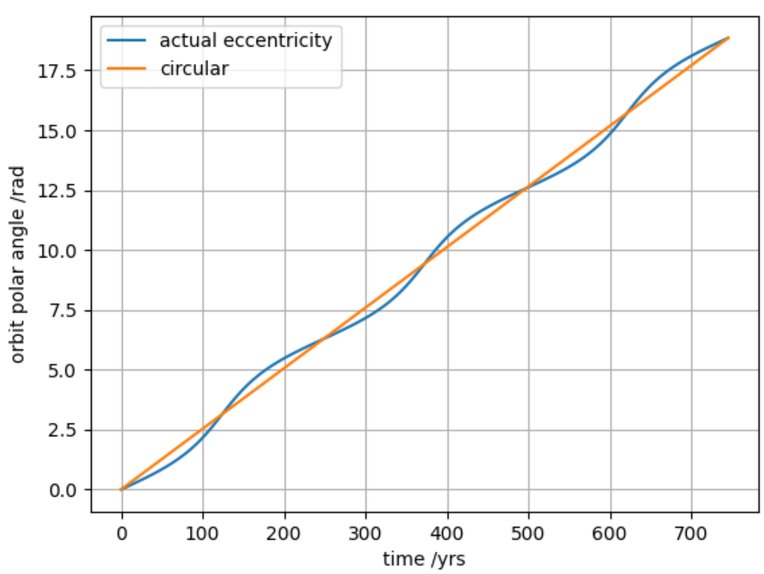


Figure 3: Plot of orbit polar angle vs time

From this we can interpolate values from time to obtain the angle as in our program time is constant and we want the angle to vary with different times.

# 4 Kepler's third law

## 4.1 What is it?

> The squares of the orbital periods of the planets are directly proportional to the cubes of the semi-major axes of their orbits. [kep]

Or in other words this equation

$$T^2 \propto a^3 \tag{23}$$

We can use Newton's law of gravitation [Kos20] to arrive at this conclusion.

$$F = -\frac{GMm}{r^2} \tag{24}$$

and also thinking about centripetal force we can arrive at the conclusion that the centripetal force and gravitation force due to the star must be equal

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \tag{25}$$

rearranging if $M >> m$ gives us

$$v = \sqrt{\frac{GM}{r}} \tag{26}$$

then using the equation from circular motion

$$P = \frac{2\pi r}{v} \tag{27}$$

we get

$$P^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \tag{28}$$

.

If we consider both masses then it would be

$$P^2 = \left(\frac{4\pi^2}{G(M+m)}\right) r^3 \tag{29}$$

.

which we saw in the previous section. However, we can verify this by plotting some actual data [BPh] from our solar system from a slightly adjusted equation

$$Yr = \left(\frac{4\pi^2}{GM}\right)^{\frac{1}{2}} AU^{\frac{3}{2}} \tag{30}$$

We will get a line which looks like this

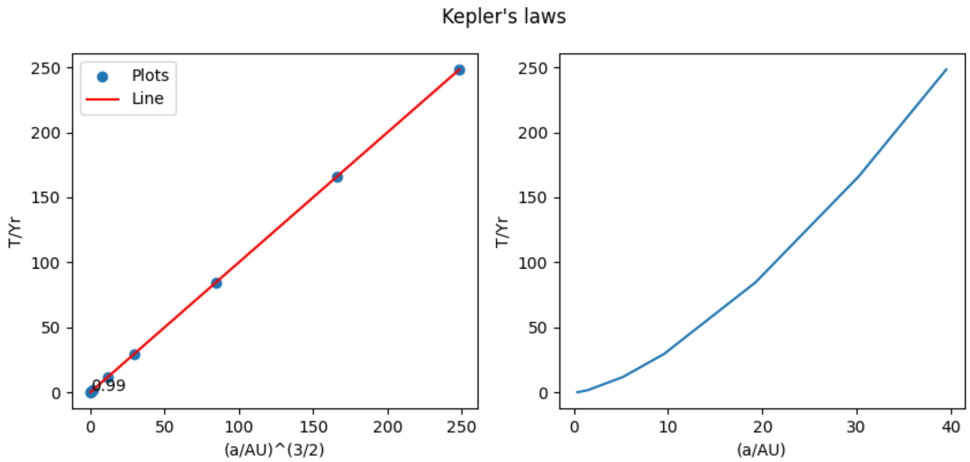


Figure 4: Plot of time period vs semi-major axis

Which we can see from the r squared value, is a very strong correlation.

# 5 Conclusion

Hopefully after reading some of this paper, you have a better grasp on Kepler's laws and how they might be derived. Our exploration of the mathematics is quite fascinating and it still surprises me how Kepler was able to calculate and arrive at these conclusions without having technology like we have today.

# References

[BPh] Bpho computational physics project and competition. https://www.bpho.org.uk/bpho/computational-challenge/BPhO_CompPhys_Challenge_2023_briefing.pdf. Accessed: 2023-08-12.

[kep] Orbits and kepler's laws — nasa solar system exploration. https://solarsystem.nasa.gov/resources/310/orbits-and-keplers-laws/#:~:text=Kepler's%20First%20Law%3A%20each%20planet's,Sun%20is%20at%20one%20focus. Accessed: 2023-08-12.

[Kos20] Alex Ely Kossovsky. *Newton's Law of Universal Gravitation*, pages 171–172. Springer International Publishing, Cham, 2020.