Geometric Constraints and Variational Approaches to Image Analysis

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Motivation

Image analysis

The problems we are interested in come from *Image Analysis*.

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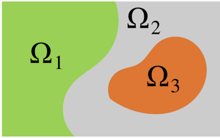
Segmentation: Given input image $f_I : \Omega \to [0,1]$, find natural number n and partition $\mathcal{I}^\star = \{\Omega_i \subset \Omega \mid i \leq n\}$ such that $f_{\widehat{\boldsymbol{I}}}$

$$\mathcal{I}^\star = \arg\min_{\mathcal{I}} E_{seg}(\mathcal{I}, f_{\boldsymbol{I}}) \quad \text{subject to} \quad \begin{array}{l} \forall i \neq j : \ \Omega_i \cap \Omega_j = \emptyset \\ \bigcup_i^n \Omega_i = \Omega \end{array}$$



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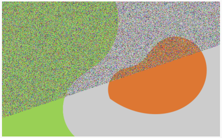
Denoising: Given a noisy image $f_{\widetilde{\boldsymbol{I}}}$ corrupted, find an estimation $f_{\widehat{\boldsymbol{I}}}$ of the original image such that

$$f_{\widehat{\boldsymbol{I}}} = \arg\min_{f} E_{den}(f, f_{\widetilde{\boldsymbol{I}}})$$



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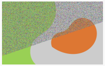

Inpainting: Given an image $f_{\widetilde{\boldsymbol{I}}}$ and a collection of missing patches $\mathcal{P}$, complete the patches such that

$$f_{\widehat{\boldsymbol{I}}} = \arg\min_f E_{inp}(f, f_{\widetilde{\boldsymbol{I}}}) \quad \text{subject to } f(\Omega \setminus \mathcal{P}) = f_{\widetilde{\boldsymbol{I}}}(\Omega \setminus \mathcal{P}).$$

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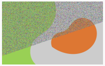

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Energies are defined according to assumptions made about the solution, e.g.,

- *Data fidelity*. The solution should not differ much from the input.
- *Spatial coherence*. Images are composed of regions with low variability in color.

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Geometric priors

The *Mumford Shah* is a model for segmentation and denoising.

$$\min_{f,\mathcal{K}} \alpha \int_{\Omega} \|f_{\boldsymbol{I}} - f\|^2 dx + \beta \int_{\Omega \setminus \mathcal{K}} \|\nabla f\|^2 dx + \lambda Per(\mathcal{K}).$$

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- A measure of length is present in both models.
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In this thesis, we are interested in the combined use of *length* and *squared curvature* as geometric priors.

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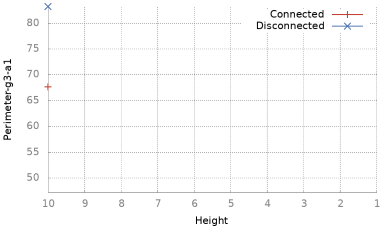
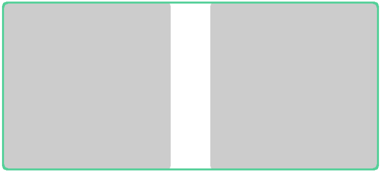
Completion property

$$\min_{\Omega \in \{\Omega_c, \Omega_d\}} \int_{\partial\Omega} ds.$$





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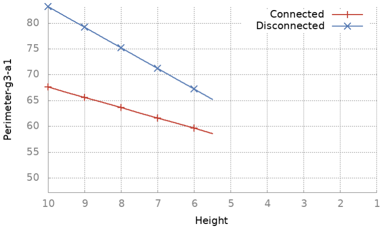

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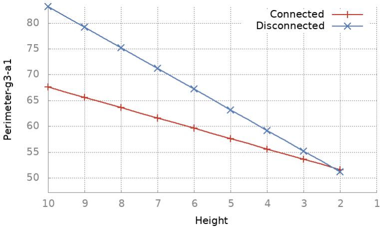

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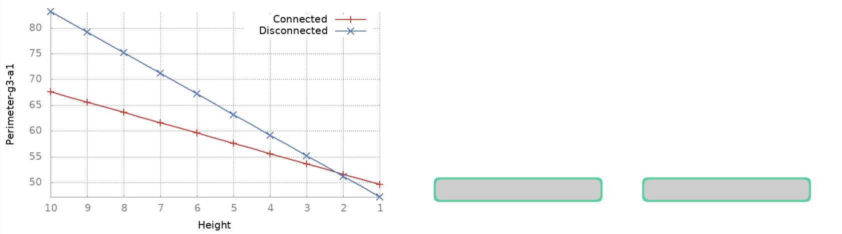

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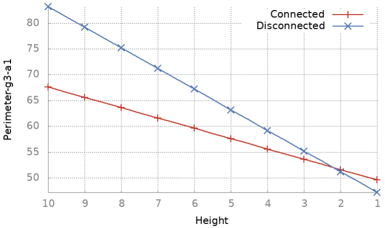

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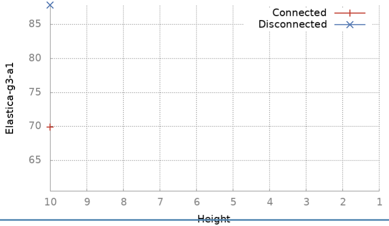

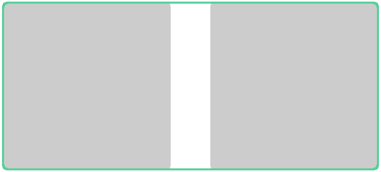

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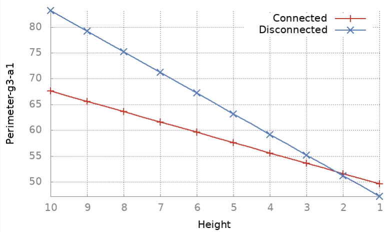

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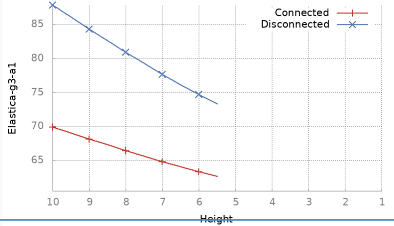

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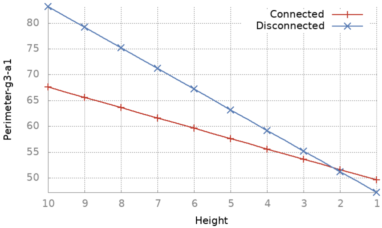

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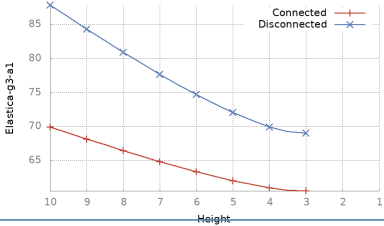

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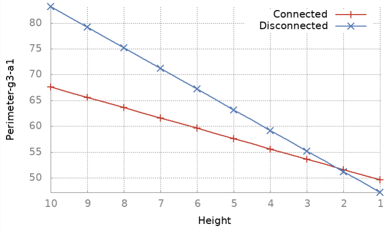

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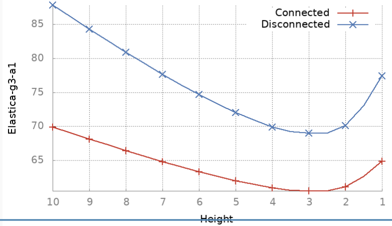

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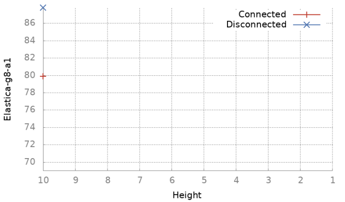

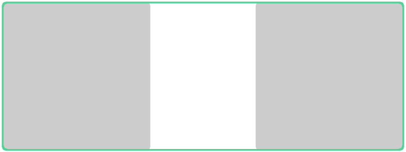

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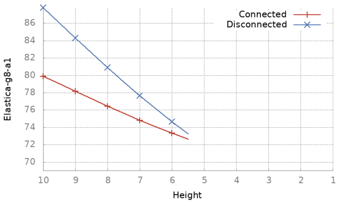

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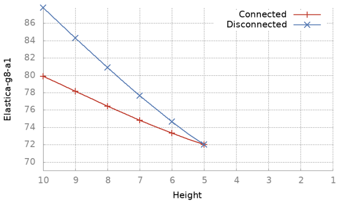

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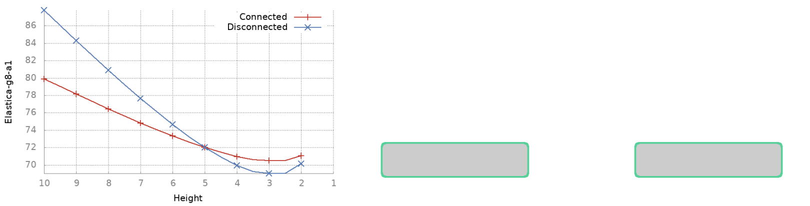

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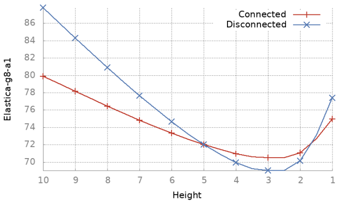

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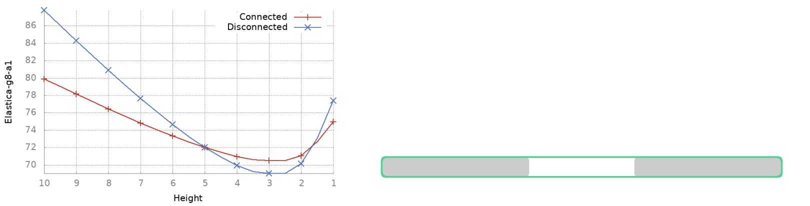

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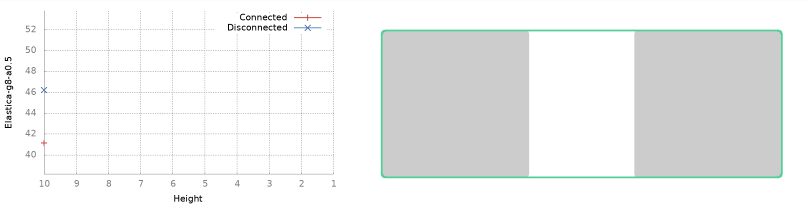

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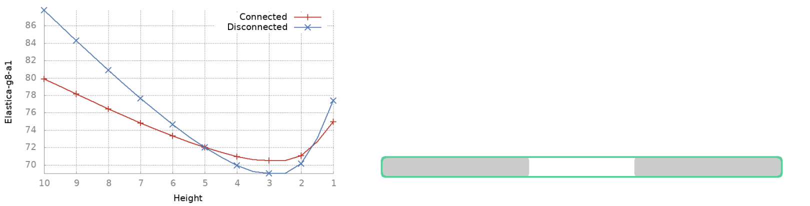

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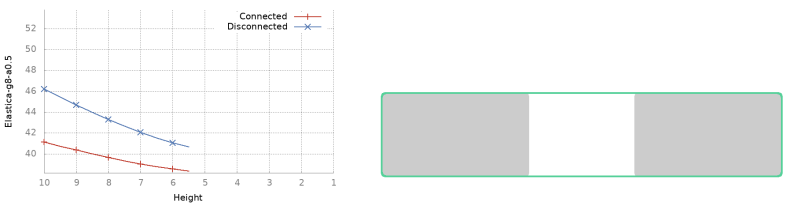

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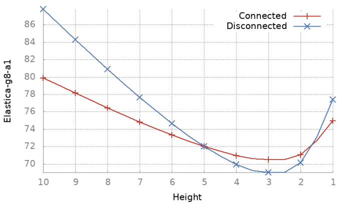

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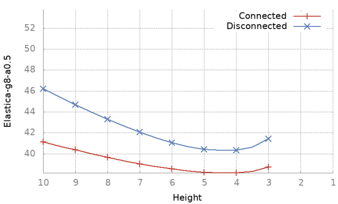

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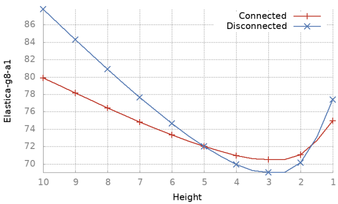

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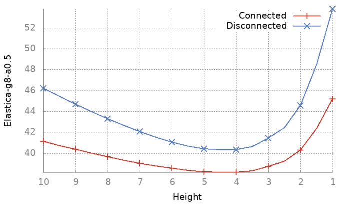

Motivation

State-of-art

Continuous world: Define the energy over the whole domain and minimize the elastica with respect the level-curves.

$$\int_{\Omega} \left(\alpha + \beta \nabla \cdot \left(\frac{\nabla f_{\boldsymbol{I}}}{\|\nabla f_{\boldsymbol{I}}\|} \right)^2 \right) \|\nabla f_{\boldsymbol{I}}\| d\Omega.$$

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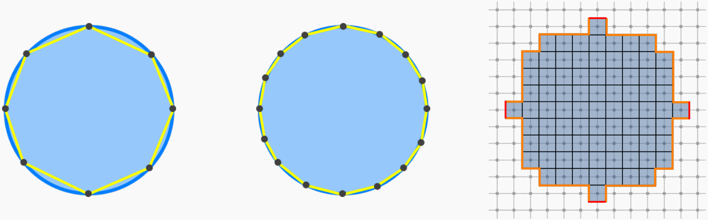
- Triple cliques: global optimization, non-submodular energy. Limited precision.

Motivation
Digital set peculiarities

Exact sampling x digitization



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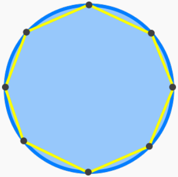
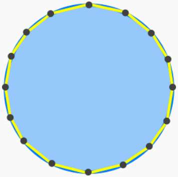
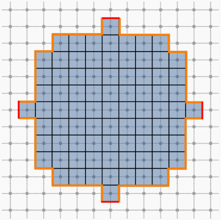
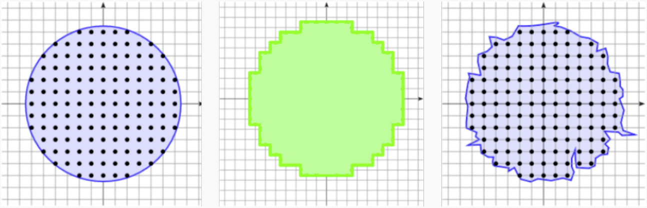






Digitization ambiguity



Motivation

Multigrid convergent estimators

Definition (Multigrid convergence)

Let $\mathcal{X}$ be a family of shapes in $\mathbb{R}^n$ and u a geometric quantity that is defined for every shape $X \in \mathcal{X}$. Further, let $D_h(X)$ denote the digitization of X with grid step h. The estimator $\hat{u}$ is multigrid convergent

for $\mathcal{X}$ if and only if, for any $X \in \mathcal{X}$ there exists $h_X > 0$ such that for every $0 < h < h_X$

$$|\hat{u}(D_h(X)) - u(X)| \leq \tau(h), \quad \text{with } \lim_{h \to 0} \tau(h) = 0$$

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Multigrid convergent estimator of area

$$\widehat{Area(X)} = h^2 |D_h(X)|$$

Motivation

Multigrid convergent estimators

Examples of multigrid convergent estimators of perimeter (for piecewise 3-smooth convex shapes).

- Minimum Length Polygon (MLP)

  Sloboda, Zatko, and Stoer (1998)

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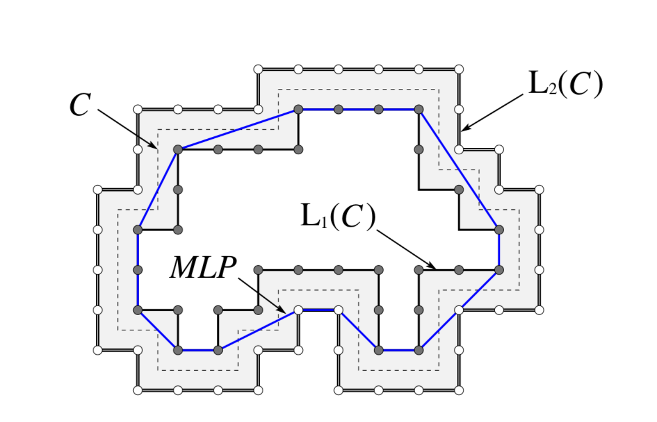

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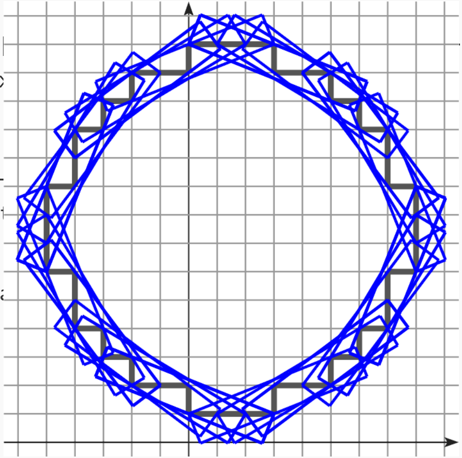

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Examples of multigrid convergent estimators of curvature:

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  Roussillon and Lachaud (2011)
  Schindele, Massopust, and Forster (2017)
  - Proved multigrid convergent for convex shapes with continuous curvature.

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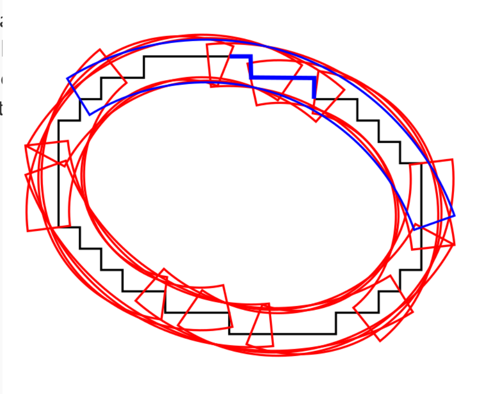

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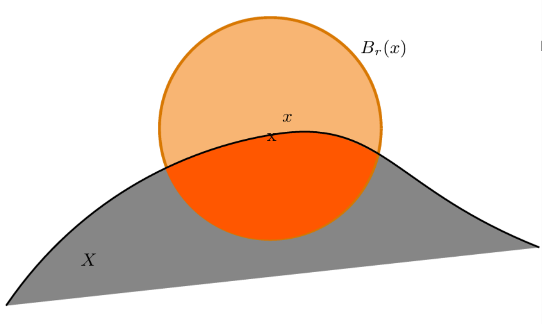
  Schindele, Massopust, and Forster (2017)
  - Proved multigrid convergent for convex shapes with continuous curvature.
- Integral Invariant (II)
  Coeurjolly, Lachaud, and Levallois (2013)
  - Proved multigrid convergent for C^2 convex shapes with bounded curvature.

Motivation
Digital geometric estimator

Examples of multigrid convergent estimators of curvature:

- λ-Maximal Digital Circular Arcs (λ-MDCA)
  Roussillon and Lachaud (2011)
  Schindel
  - Pro ... ntinuous
    cur
- Integral
  Coeurjol
  - Pro ... bounded
    cur



$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$

Motivation

Goals

- Can we define an elastica-based model for image analysis using multigrid convergent estimators?

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- Can we recover the completion property of elastica?

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- Can we recover the completion property of elastica?
- Can we escape bad local minima?

Motivation

Goals

- Can we define an elastica-based model for image analysis using multigrid convergent estimators? Yes!
- Can we recover the completion property of elastica? Yes!
- Can we escape bad local minima? Yes!

Presentation plan

1. Motivation
   - Image analysis and Geometric priors
   - Elastica and completion property
   - State-of-art
   - Multigrid convergent estimators

2. Contribution
   - A combinatorial model for elastica
   - A quadratic non-submodular formulation for elastica
   - Elastica minimization via graph-cuts

3. Conclusion and perspectives

A combinatorial model for elastica

Combinatorial Elastica

Digital elastica

Definition (Digital elastica energy)

Let $\hat{s}$ and $\hat{\kappa}$ multigrid convergent estimators of local length and curvature. The digital elastica energy of a digital shape $S \subset \Omega \subset \mathbb{Z}^2$ is defined as

$$\hat{E}(S) = \sum_{e \in \partial_h(S)} \hat{s}(e)\left(\alpha + \beta \hat{\kappa}^2(e)\right).$$

Combinatorial Elastica

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$$\hat{E}(S) = \sum_{e \in \partial_h(S)} \hat{s}(e)\Big(\alpha + \beta \hat{\kappa}^2(e) \Big).$$

- *Local search*: set a local neighborhood $\mathcal{N}(S)$ of S and pick the shape $X^\star \in \mathcal{N}(S)$ among those of minimum digital elastica value.

Combinatorial Elastica

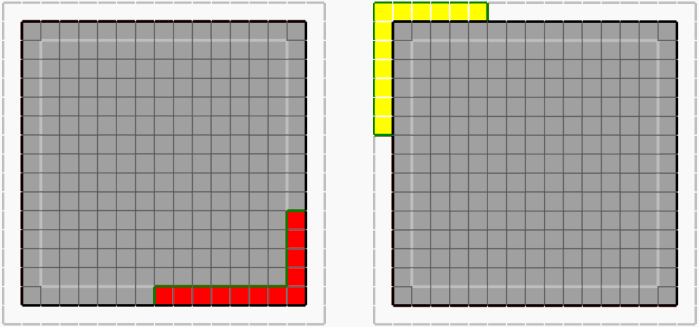
Local search algorithm

input : A digital set S; coefficient α, β; the maximum number of iterations maxIt; and a stop condition tolerance

delta $\longleftarrow$ tolerance $+1$;
$i \longleftarrow 0$;
$S^{(0)} \longleftarrow S$;
$X^{\star} \longleftarrow S$;
while i $<$ maxIt **and** delta $>$ tolerance **do**
    for $X \in \mathcal{N}(S^{(i)})$ **do**
        if $\hat{E}(X) < \hat{E}(X^{\star})$ **then**
            $X^{\star} \longleftarrow X$
    i $\longleftarrow$ i $+1$;
    $S^{(i)} \longleftarrow X^{\star}$;
    delta $\longleftarrow \hat{E}(S^{(i-1)}) - \hat{E}(S^{(i)})$;

Combinatorial Elastica

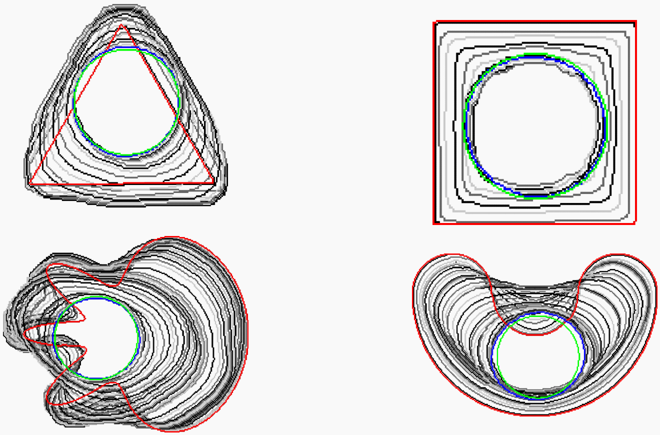
Neighborhood of shapes



Combinatorial Elastica

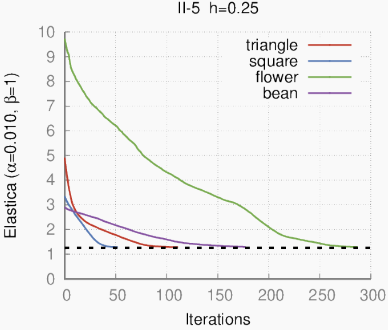
Shape evolution

$$\alpha = 0.01, \beta = 1$$



Combinatorial Elastica

Energy evolution



$$\min E(X) = \int_{\partial X} \alpha + \beta\kappa^2 ds$$

$$= 4\pi\beta\frac{1}{r} = 4\pi\beta\left(\frac{\alpha}{\beta}\right)^{1/2}$$

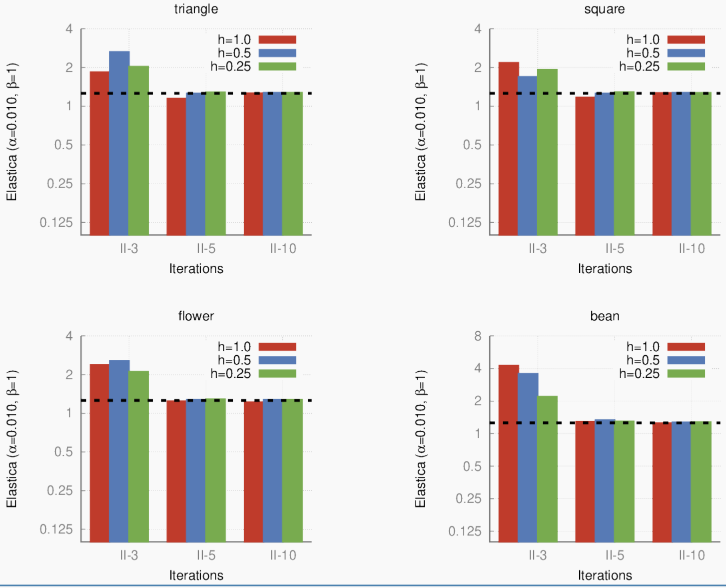
, where $\frac{\partial}{\partial r}2\pi(\alpha r + \frac{\beta}{r}) = 0$

For $\alpha = 0.01,\ \beta = 1$

$$\min E(X) \approx 1.2566$$

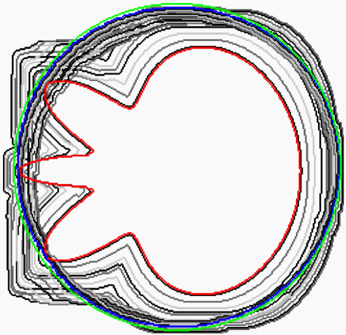
Combinatorial Elastica

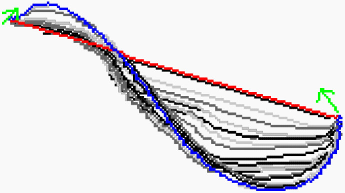
Radius and grid resolution



Combinatorial Elastica

Other experiments





Combinatorial Elastica

Running time

	$h = 1.0$		$h = 0.5$		$h = 0.25$	
	Pixels	Time	Pixels	Time	Pixels	Time
Triangle	521	2s (0.07s/it)	2080	43s (0.81s/it)	8315	532s(4.8s/it)
Square	841	0.9s (0.09s/it)	3249	8s (0.3s/it)	12769	102s (2s/it)
Flower	1641	13s (0.24s/it)	6577	209s (1.68s/it)	26321	3534s (12.3s/it)
Bean	1574	7s (0.16s/it)	6278	88s (1.08s/it)	25130	1131s (6.4s/it)
Ellipse	626	1s (0.14s/it)	2506	16s (0.44s/it)	10038	286s (3.1s/it)

Table: **Running time of LocalSearch.** The running times for the free elastica problem are displayed. Notice that even having a similar number of pixels, the square (bean) shape evolves much faster than the triangle (flower).

Combinatorial Elastica

Conclusion

- Multigrid convergent estimators are suitable for elastica minimization

Combinatorial Elastica

Conclusion

- Multigrid convergent estimators are suitable for elastica minimization
- A simple neighborhood is sufficient to escape bad local minima. Some solutions very close to global optimum.

Combinatorial Elastica

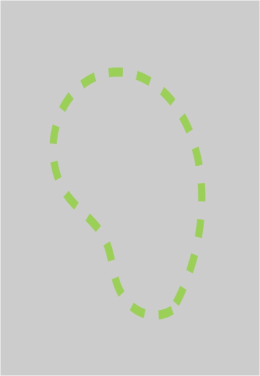
Conclusion

- Multigrid convergent estimators are suitable for elastica minimization
- A simple neighborhood is sufficient to escape bad local minima. Some solutions very close to global optimum.
- Too slow. It cannot be used in practice.

A quadratic non-submodular formulation for elastica

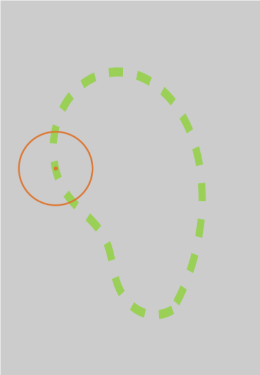
Non-submodular elastica

Difficulties with a global model



Non-submodular elastica

Difficulties with a global model

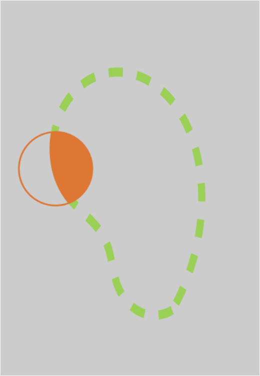


- Center of the estimation disk

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\, \alpha + \beta \hat{\kappa}_r^2(D, \ell_i) \, \right)$$

Non-submodular elastica

Difficulties with a global model



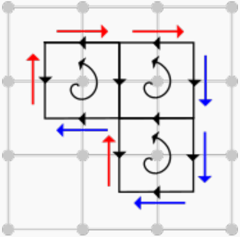
- Center of the estimation disk
- Pixel counting and estimation of curvature squared

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6} \beta \left(c^2 - 2c\boldsymbol{A}_i^T \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{A}_i^T \boldsymbol{x} \right) \right)$$

subject to $\quad \boldsymbol{x} \in \{0,1\}^m, \boldsymbol{y} \in \{0,1\}^n.$

Non-submodular elastica

Difficulties with a global model



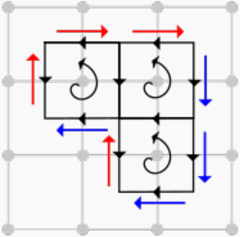
- Center of the estimation disk
- Pixel counting and estimation of curvature squared
- Topological constraints

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6} \beta \left(c^2 - 2c\boldsymbol{A}_i^T \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{A}_i^T \boldsymbol{x} \right) \right)$$

subject to $\quad \boldsymbol{x} \in \{0,1\}^m, \boldsymbol{y} \in \{0,1\}^n, T(\boldsymbol{x}, \boldsymbol{y}).$

Non-submodular elastica

Difficulties with a global model



- Center of the estimation disk
- Pixel counting and estimation of curvature squared
- Topological constraints
- Third order non-convex binary

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6} \beta \left(c^2 - 2c\boldsymbol{A}_i^T \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{A}_i^T \boldsymbol{x} \right) \right)$$

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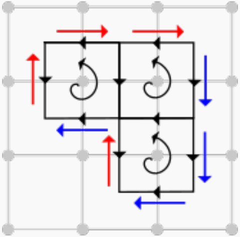
Non-submodular elastica

Difficulties with a global model



- Center of the estimation disk
- Pixel counting and estimation of curvature squared
- Topological constraints
- Third order non-convex binary
- Level 1 linearization: non semi-definite positive matrix

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6}\beta\left(c^2 - 2c\boldsymbol{A}_i^T\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{A}_i^T\boldsymbol{x}\right)\right)$$

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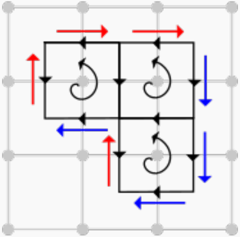
Non-submodular elastica

Difficulties with a global model



- Center of the estimation disk
- Pixel counting and estimation of curvature squared
- Topological constraints
- Third order non-convex binary
- Level 1 linearization: non semi-definite positive matrix
- Level 2 linearization: $O(m^3)$ variables

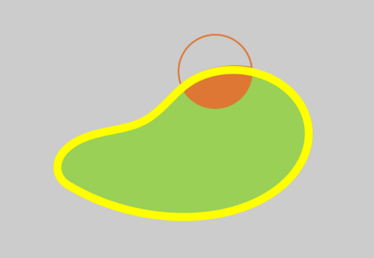
$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6} \beta \big(c^2 - 2c\boldsymbol{A}_i^T \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{A}_i^T \boldsymbol{x} \big) \right)$$

subject to $\quad \boldsymbol{x} \in \{0,1\}^m, \boldsymbol{y} \in \{0,1\}^n, T(\boldsymbol{x}, \boldsymbol{y}).$

Non-submodular elastica

Simplification

$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$



- Define an optimization band (yellow) as the inner contour of the shape, denoted I.
- Set pixels such that the curvature estimation is reduced

Non-submodular elastica

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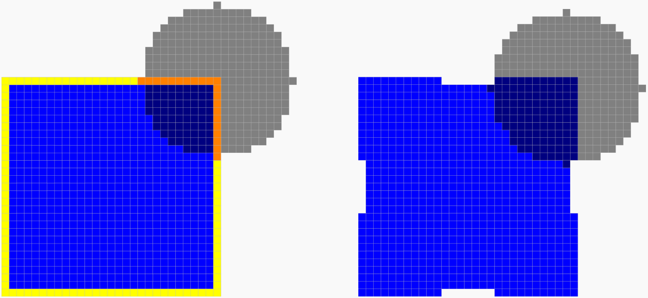

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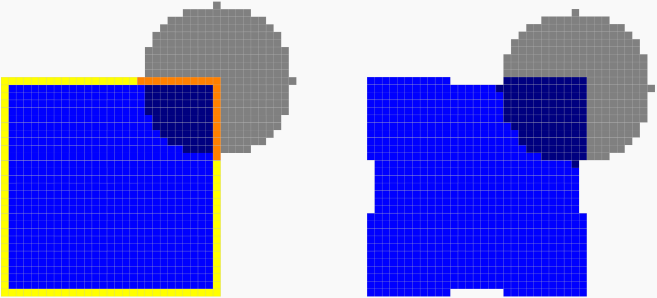
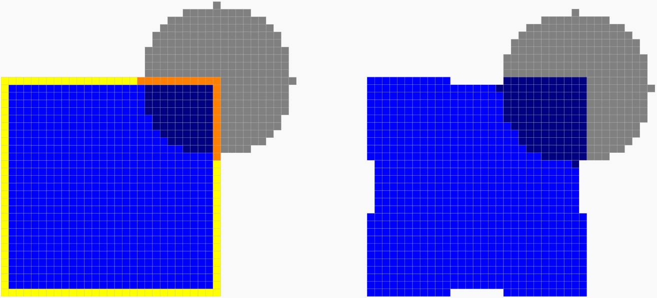


- Optimization identifies zones of shortage (convex) or abundance (concave) of pixels.

Non-submodular elastica

Simplification

$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$



- Optimization identifies zones of shortage (convex) or abundance (concave) of pixels.
- $x = 1 \rightarrow$ Zone of shortage of pixels (convex) $\rightarrow$ Estimator disk should be shifted towards the interior $\rightarrow$ This pixel does not belong to the next contour.

Non-submodular elastica

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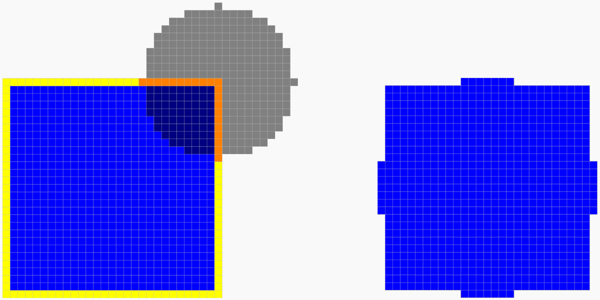
$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$



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- $x = 1 \rightarrow$ Zone of shortage of pixels (convex) $\rightarrow$ Estimator disk should be shifted towards the interior $\rightarrow$ This pixel does not belong to the next contour.

Non-submodular elastica

FlipFlow

$$E^{flip}_{(\boldsymbol{\theta},m)}(D^{(k)},X^{(k)}) = \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in R_m(D^{(k)})} \beta \hat{\kappa}(p)^2$$

Non-submodular elastica

FlipFlow

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$$D \subset \Omega \subset \mathbb{Z}^2, \quad X^{(k)} := \{\, x_i \in \{0,1\} \mid p_i \in I^{(k)} \,\}$$

Non-submodular elastica

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$$D \subset \Omega \subset \mathbb{Z}^2, \quad X^{(k)} := \{\, x_i \in \{0,1\} \mid p_i \in I^{(k)} \,\}$$

$$R_m(D) := \{p \mid m-1 < d_D(p) \leq m\} \cup \{p \mid -m+1 > d_D(p) \geq -m\}$$

Non-submodular elastica

FlipFlow

$$
\begin{aligned}
E^{flip}_{(\boldsymbol{\theta},m)}(D^{(k)},X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in R_m(D^{(k)})} \beta \hat{\kappa}(p)^2 \\
&= \sum_{x_j \in X^{(k)}} \alpha s(x_j) \\
&+ \sum_{\substack{p \in \\ R_m(D^{(k)})}} 2c_1\beta \Bigg((1/2 + |F_r^{(k)}(p)| - c_2) \cdot \sum_{\substack{x_j \in \\ X_r^{(k)}(p)}} x_j + \sum_{\substack{j<l, \\ x_j, x_l \in \\ X_r^{(k)}(p)}} x_j x_l \Bigg)
\end{aligned}
$$

Non-submodular elastica

FlipFlow

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\begin{aligned}
E^{flip}_{(\boldsymbol{\theta},m)}(D^{(k)},X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in R_m(D^{(k)})} \beta \hat{\kappa}(p)^2 \\
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\end{aligned}
$$

$$
s(x_j) = \sum_{q_i \in \mathcal{N}_4(p_j)} t(q_i), \quad \text{where } t(q_i) = \begin{cases} (x_j - x_i)^2, & \text{if } q_i \in I^{(k)} \\ (x_j - 1)^2, & \text{if } q_i \in F^{(k)} \\ (x_j - 0)^2, & \text{otherwise.} \end{cases}
$$

Non-submodular elastica

FlipFlow

$$
\begin{aligned}
E^{flip}_{(\boldsymbol{\theta},m)}(D^{(k)}, 1 - X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in R_m(D^{(k)})} \beta \hat{\kappa}(p)^2 \\
&= \sum_{x_j \in X^{(k)}} \alpha s(x_j) \\
&+ \sum_{\substack{p \in \\ R_m(D^{(k)})}} 2c_1\beta \Bigg((1/2 + |F_r^{(k)}(p)| - c_2) \cdot \sum_{\substack{x_j \in \\ X_r^{(k)}(p)}} x_j + \sum_{\substack{j<l, \\ x_j, x_l \in \\ X_r^{(k)}(p)}} x_j x_l \Bigg)
\end{aligned}
$$

$$
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$$

Non-submodular elastica

FlipFlow

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\begin{aligned}
E^{flip}_{(\boldsymbol{\theta},m)}(D^{(k)},1-X^{(k)}) &= \sum_{x_j\in X^{(k)}} \alpha s(x_j) + \sum_{p\in R_m(D^{(k)})} \beta\hat{\kappa}(p)^2\\
&= \sum_{x_j\in X^{(k)}} \alpha s(x_j)\\
&+ \sum_{\substack{p\in\\ R_m(D^{(k)})}} 2c_1\beta\left((1/2+|F^{(k)}_r(p)|-c_2)\cdot \sum_{\substack{x_j\in\\ X^{(k)}_r(p)}} x_j + \sum_{\substack{j<l,\\ x_j,x_l\in\\ X^{(k)}_r(p)}} x_j x_l\right)
\end{aligned}
$$

Shrink mode (convexities)

$$
\begin{aligned}
a^{(k)} &\leftarrow \arg\min_{X^{(k)}} E^{flip}_{\boldsymbol{\Theta},m}(D^{(k)},1-X^{(k)});\\
D^{(k+1)} &\leftarrow F^{(k)} + a^{(k)}.
\end{aligned}
$$

Non-submodular elastica

FlipFlow

$$
\begin{aligned}
E^{flip}_{(\boldsymbol{\theta},m)}(D^{(k)},1-X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in R_m(D^{(k)})} \beta \hat{\kappa}(p)^2 \\
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&+ \sum_{\substack{p \in \\ R_m(D^{(k)})}} 2c_1\beta \Bigg((1/2 + |F^{(k)}_r(p)| - c_2) \cdot \sum_{\substack{x_j \in \\ X^{(k)}_r(p)}} x_j + \sum_{\substack{j<l, \\ x_j,x_l \in \\ X^{(k)}_r(p)}} x_j x_l \Bigg)
\end{aligned}
$$

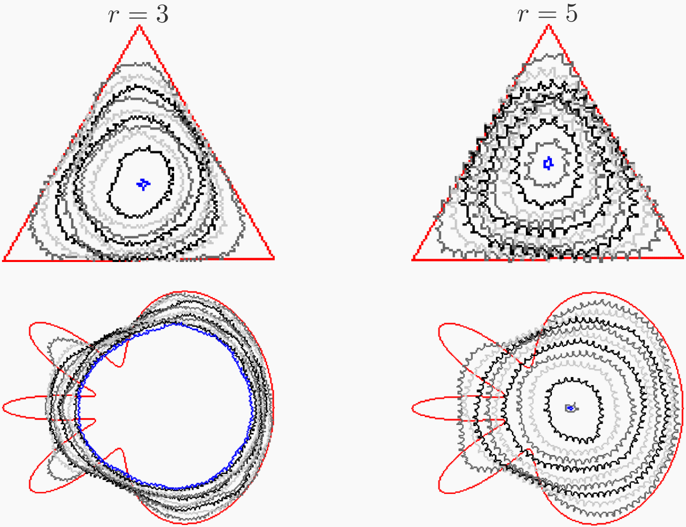
Shrink mode (convexities)

$$
\begin{aligned}
a^{(k)} &\leftarrow \arg\min_{X^{(k)}} E^{flip}_{\boldsymbol{\Theta},m}(D^{(k)},1-X^{(k)}); \\
D^{(k+1)} &\leftarrow F^{(k)} + a^{(k)}.
\end{aligned}
$$

Expansion mode (concavities)

$$
\begin{aligned}
a^{(k)} &\leftarrow \arg\min_{\overline{X}^{(k)}} E^{flip}_{\boldsymbol{\Theta},m}(\overline{D}^{(k)},1-\overline{X}^{(k)}); \\
D^{(k+1)} &\leftarrow \overline{\overline{F}^{(k)} + a^{(k)}}.
\end{aligned}
$$

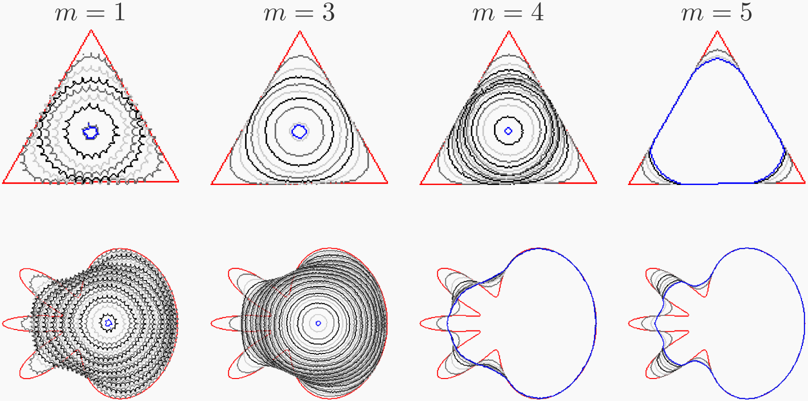
Non-submodular elastica
Evaluation on farther rings



Non-submodular elastica

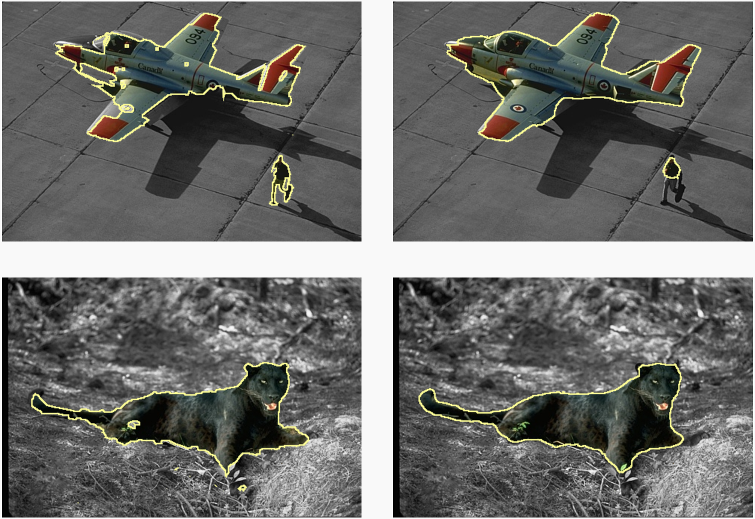
Evaluation on farther rings

$r = 5$



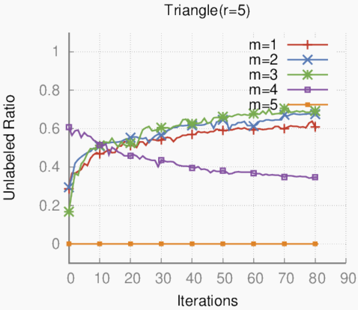
Non-submodular elastica

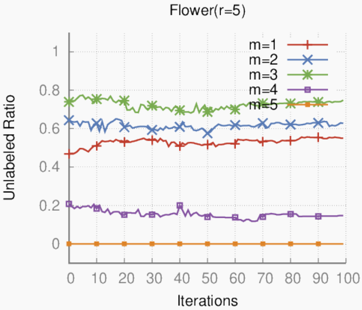
Contour correction



Non-submodular elastica

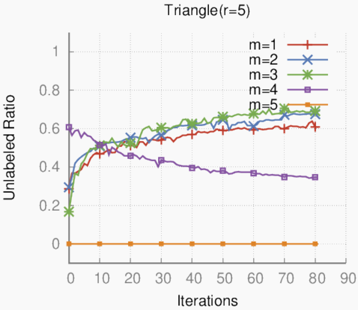
Unlabeled ratio

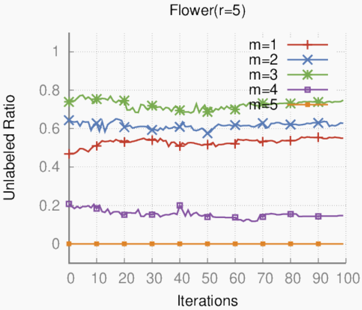




Non-submodular elastica

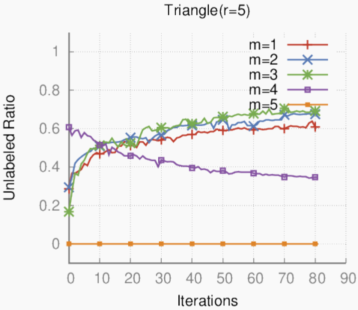
Unlabeled ratio

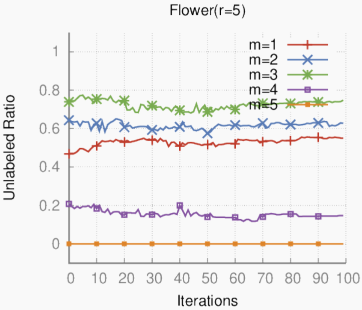




- Given that we evaluate the curvature estimator on the initial contour, how to change the pixels in this contour to reduce the difference of inner and outer pixels?

Non-submodular elastica
Unlabeled ratio

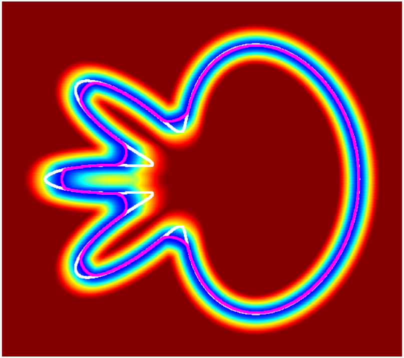




- Given that we evaluate the curvature estimator on the initial contour, how to change the pixels in this contour to reduce the difference of inner and outer pixels?
- Given that the shape does not change, where the estimation disks should be centered in order to reduce the difference of inner and outer pixels?

Non-submodular elastica

Balance coefficient



- Balance coefficient

$$u_r(D,p) = \left(\frac{\pi r^2}{2} - |B_r(p) \cap D| \right)^2$$

- White contour: contour of the shape
- Pink contour: zero level set of the balance coefficient

Non-submodular elastica

Conclusion

- Change pixels in the contour in order to reduce the difference between inner and outer pixels.
- Quadratic non-submodular energy.

Non-submodular elastica

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- Farther rings: better results when disks are evaluated farther from the contour.

Non-submodular elastica

Conclusion

- Change pixels in the contour in order to reduce the difference between inner and outer pixels.
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- Pos-processing procedure: contour correction.

Non-submodular elastica

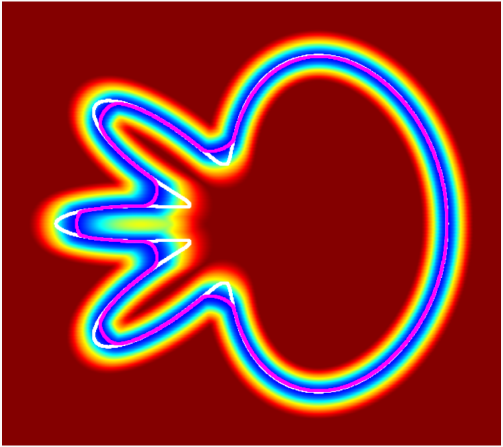
Conclusion

- Change pixels in the contour in order to reduce the difference between inner and outer pixels.
- Quadratic non-submodular energy.
- Farther rings: better results when disks are evaluated farther from the contour.
- Pos-processing procedure: contour correction.
- Unstable hypothesis: fixing the contour (dimension 1) is more sensitive than fixing the shape (dimension 2).
- Make the shape evolve to the zero level set of its balance coefficient.

Elastica minimization via graph-cuts

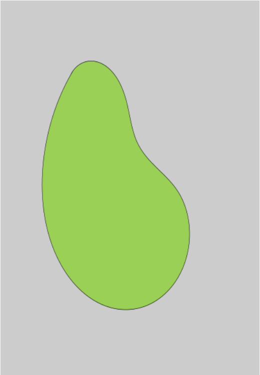
Elastica minimization via graph-cuts

Zero level set of balance coefficient



Elastica minimization via graph-cuts

Building the graph

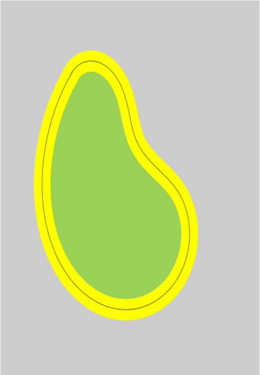


Elastica minimization via graph-cuts

Building the graph

- Optimization band

$$O_n(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$

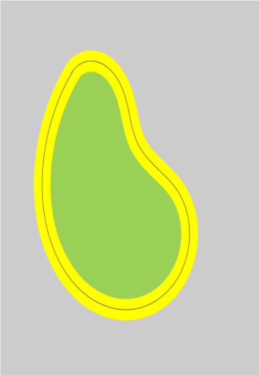


Elastica minimization via graph-cuts

Building the graph

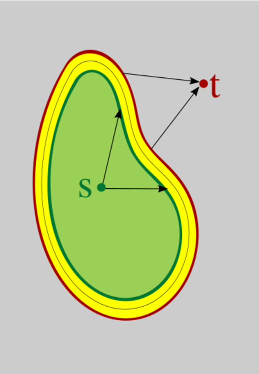
- Optimization band

$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$



Elastica minimization via graph-cuts

Building the graph

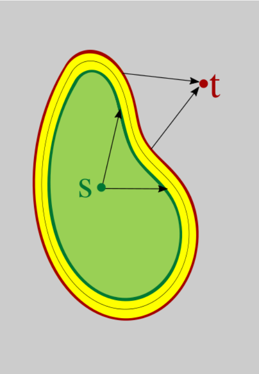


- Optimization band

$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$
$$F(D) := D \setminus O(D)$$

Elastica minimization via graph-cuts

Building the graph



- Optimization band

$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$
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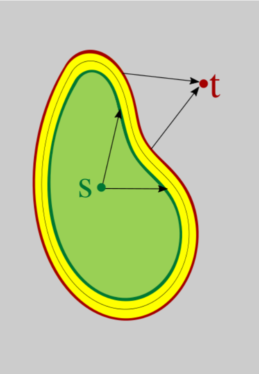
- Graph $\mathcal{G}_D(\mathcal{V}, \mathcal{E}, c)$

$$\mathcal{V} = \{v_p \mid p \in O(D)\} \cup \{s,t\}$$
$$\mathcal{E} = \{\{v_p, v_q\} \mid p,q \in O(D) \text{ and } q \in \mathcal{N}_4(p)\} \cup \mathcal{E}_{st}$$
$$\mathcal{E}_{st} = \{(s, v_p), (v_p, t) \mid p \in O(D)\}$$

Elastica minimization via graph-cuts

Building the graph



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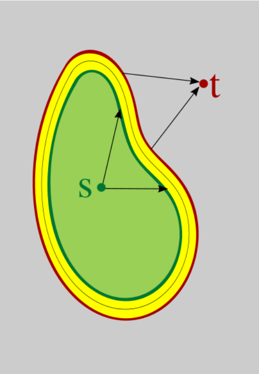
Elastica minimization via graph-cuts

Building the graph



- Optimization band

$$
\begin{aligned}
O(D) :=& \{p \in D \mid -n \leq d_D(p) \leq n\} \\
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\end{aligned}
$$

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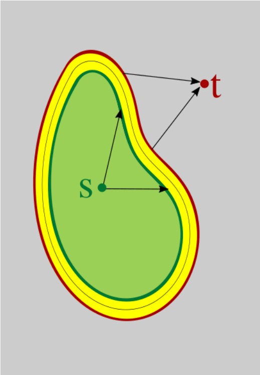
\end{aligned}
$$

Elastica minimization via graph-cuts

Building the graph



- Optimization band

$$
\begin{aligned}
O(D) &:= \{p \in D \mid -n \leq d_D(p) \leq n\} \\
F(D) &:= D \setminus O(D)
\end{aligned}
$$

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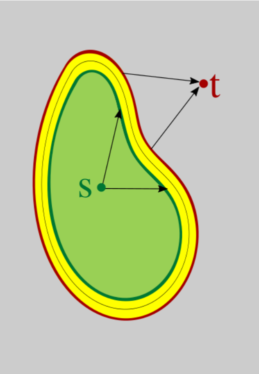
\mathcal{E}_{st} &= \{(s, v_p), (v_p, t) \mid p \in O(D)\}
\end{aligned}
$$

- Edge's weight

edge e	$\mathbf{c(e)}$
$\{v_p, v_q\}$	$u_r(D,p) + u_r(D,q)$
(s, v_p)	M
(v_p, t)	M

Elastica minimization via graph-cuts

Building the graph



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$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$
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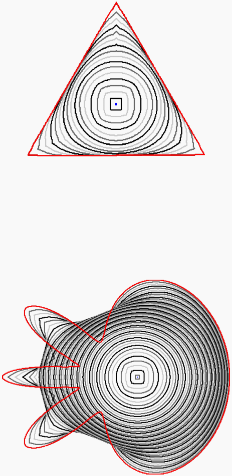
edge e	$\mathbf{c(e)}$
$\{v_p, v_q\}$	$u_r(D, p) + u_r(D, q)$
(s, v_p)	M
(v_p, t)	M

- Digital shape update

$$D^{(k+1)} = F(D^{(k)}) + S^{(k)}$$

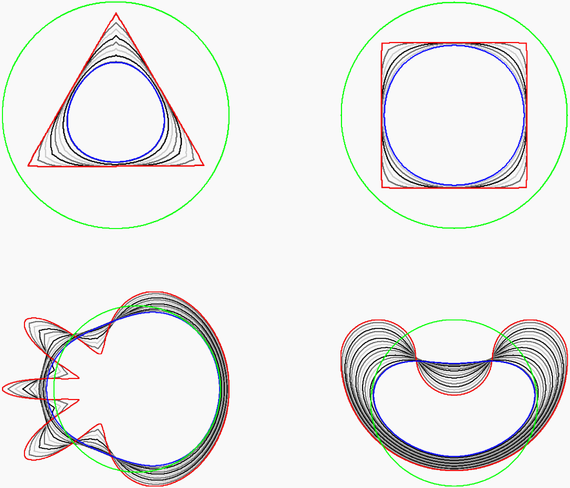
Elastica minimization via graph-cuts
Shape evolution



Elastica minimization via graph-cuts

Shape evolution

Stop if Elastica increases $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

The a-probe set

Definition (a-probe set)

Let $D \subset \Omega \subset \mathbb{Z}^2$ a digital set and a a natural number. The a-probe set of D is defined as

$$\mathcal{P}_a(D) = D \cup \bigcup_{a' < a} D^{+a'} \cup D^{-a'},$$

where $D^{+a}(D^{-a})$ denotes a dilation(erosion) by a disk of radius a.

Candidate selection

$$sol(D^{(k)}) \longleftarrow \bigcup_{D' \in \mathcal{P}_a(D^{(k)})} \left\{ F^{(k)} + S \mid mincut(S, \mathcal{G}_{D'}) \right\}$$

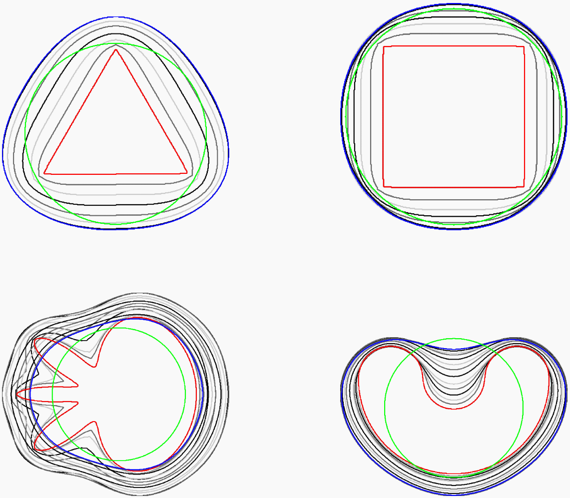
Candidate validation

$$D^{(k+1)} \longleftarrow \operatorname*{arg\,min}_{D' \in sol(D^{(k)})} \hat{E}_{\boldsymbol{\theta}}(D')$$

Elastica minimization via graph-cuts

Shape evolution with α-probe set

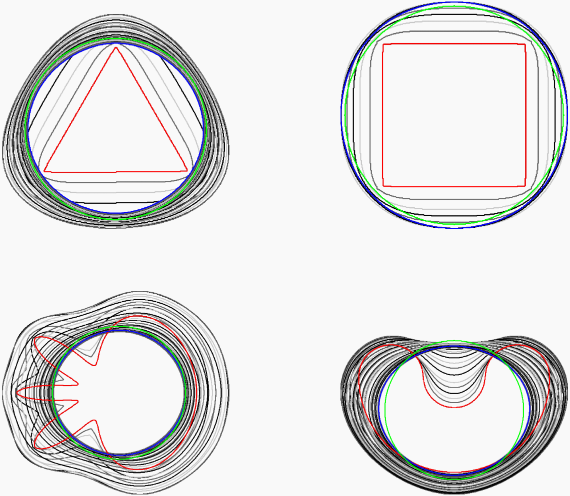
Stop if Elastica increases $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

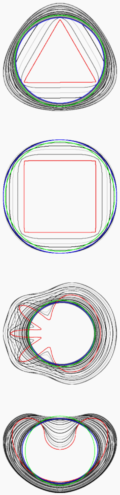
Shape evolution with α-probe set

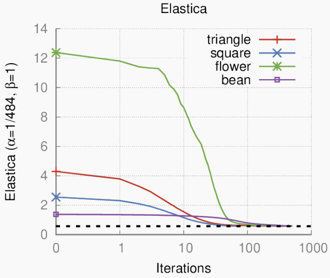
Always update $(\alpha = 1/22^2, \beta = 1)$

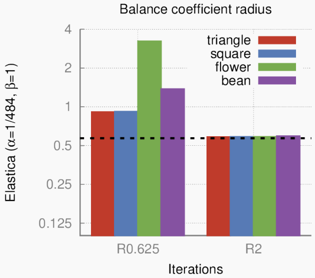


Elastica minimization via graph-cuts

Shape evolution with α-probe set







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