Geometric Constraints and Variational Approaches to Image Analysis

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Outline

1. Motivation
    - Image analysis and geometric priors
    - Elastica model and completion property
    - State-of-the-art
2. Contribution
    - Digital sets and convergent estimators
    - A combinatorial model for elastica
    - A quadratic non-submodular formulation for elastica
    - Elastica minimization via graph-cuts
3. Conclusion and perspectives

Motivation

Image analysis

The problems we are interested in come from *image analysis*.

Segmentation **Denoising** **Inpainting**

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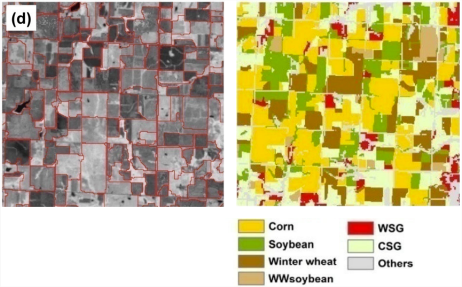
X. Li, Zhao, Han, Tong, and Yang 2019



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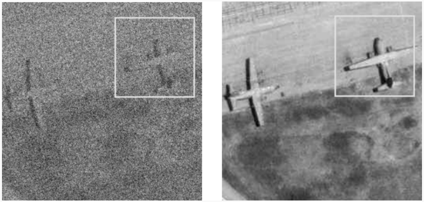
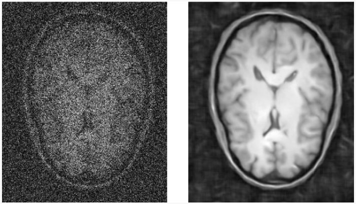


Xu et al. 2018



Jiang et al. 2018

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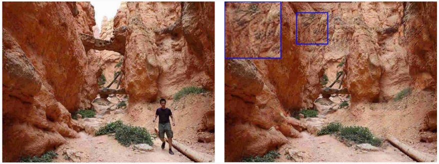
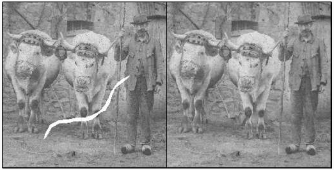


Yu et al. 2018



Masnou and Morel 1998

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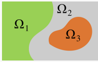
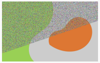


Denoising: $f_{\widehat{\boldsymbol{I}}} = \arg\min_f E_{den}(f, f_{\widetilde{\boldsymbol{I}}})$.



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We focused on *variational approaches* to solve these problems.

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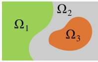

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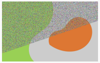

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We focused on *variational approaches* to solve these problems.
Energies are defined by terms that guide the optimization towards the solution of interest, e.g.,

- *Data fidelity*. The solution should not differ much from the input.
- *Spatial coherence*. Images are composed of regions with low variability in color.

Motivation

Geometric priors

The *Mumford Shah* (Mumford and Shah 1989) is a model for segmentation and denoising.

$$\min_{f,\mathcal{K}} \alpha \int_{\Omega} \|f_{\boldsymbol{I}} - f\|^2 dx + \beta \int_{\Omega\setminus\mathcal{K}} \|\nabla f\|^2 dx + \lambda Per(\mathcal{K}).$$

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In this thesis, we are interested in the combined use of *perimeter* and *squared curvature* as geometric priors.

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Completion property

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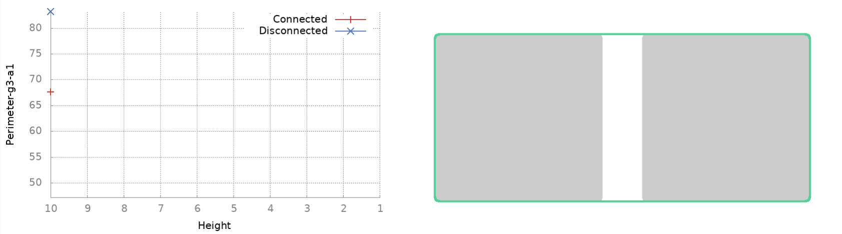

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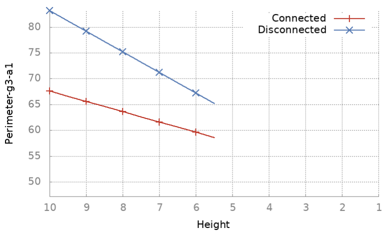

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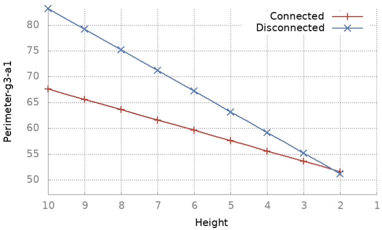

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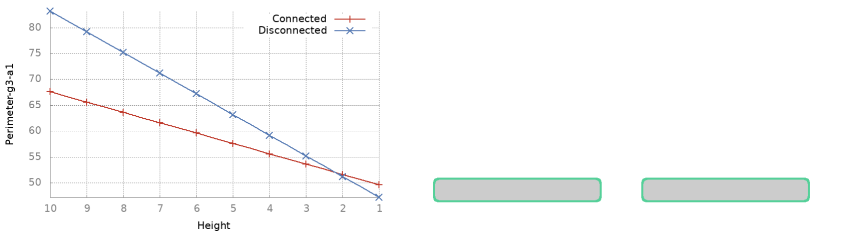

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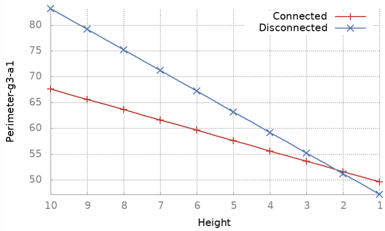

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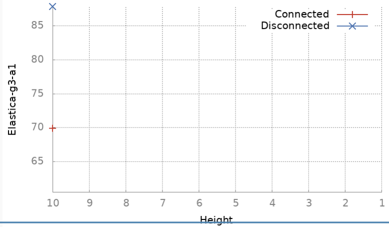

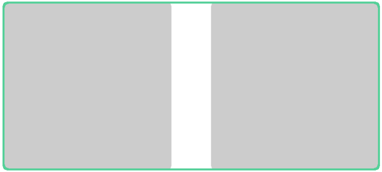

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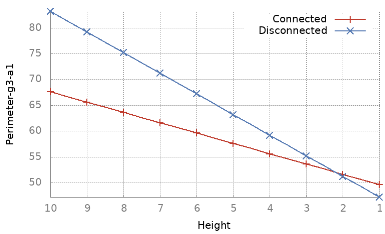

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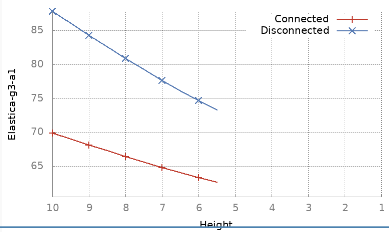

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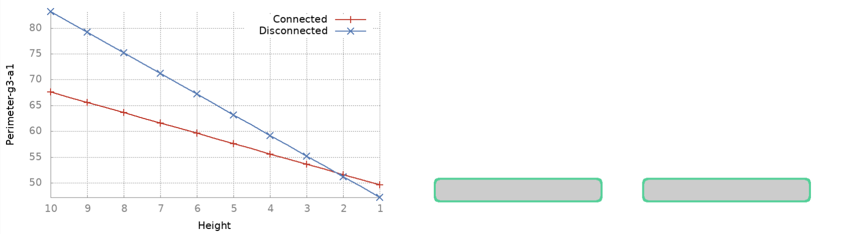

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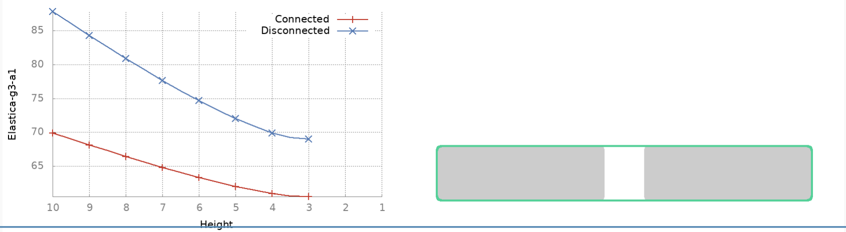

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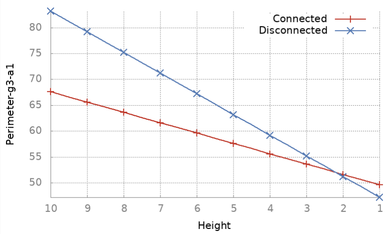

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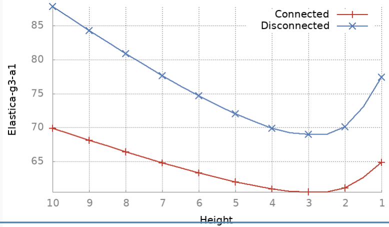

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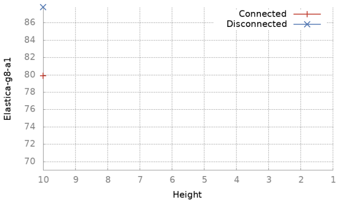

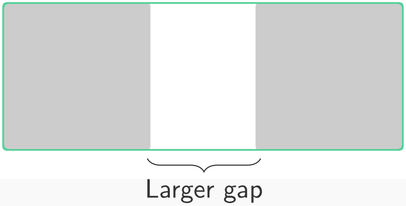

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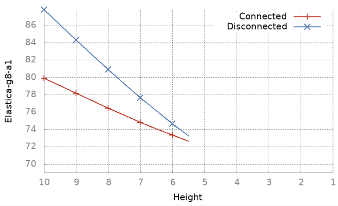

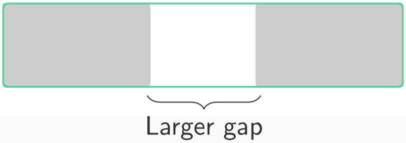

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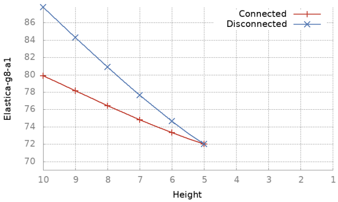

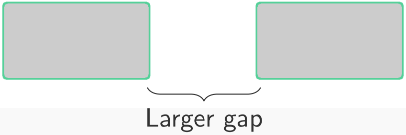

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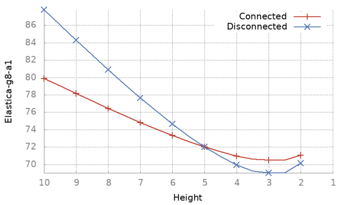

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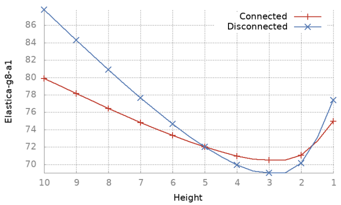

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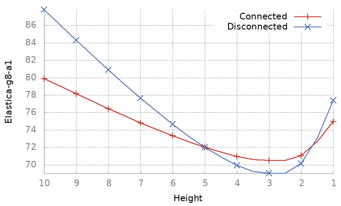

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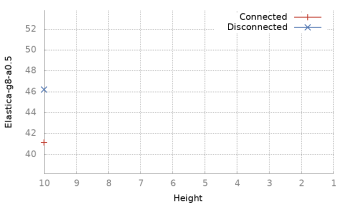

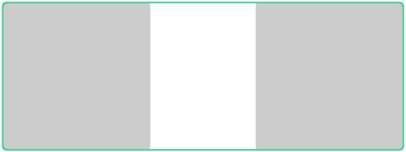

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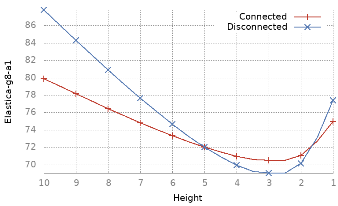

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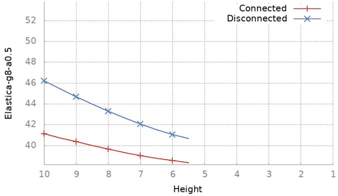

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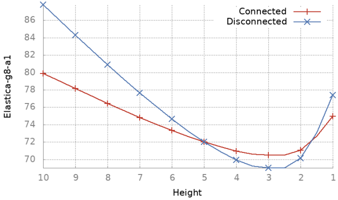

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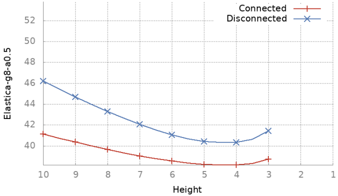

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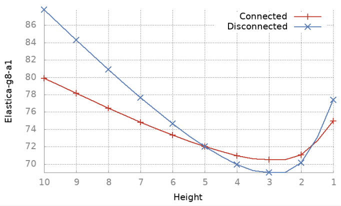

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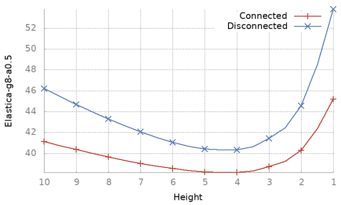

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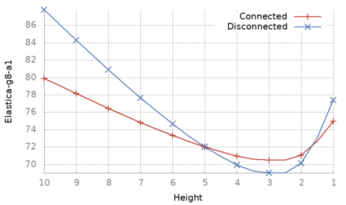

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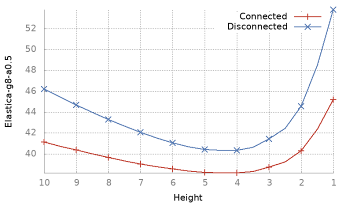
$$\min_{X \in \Omega} \int_{\partial X} \alpha + \beta\kappa^2 ds. \quad - \quad \text{The elastica energy}$$



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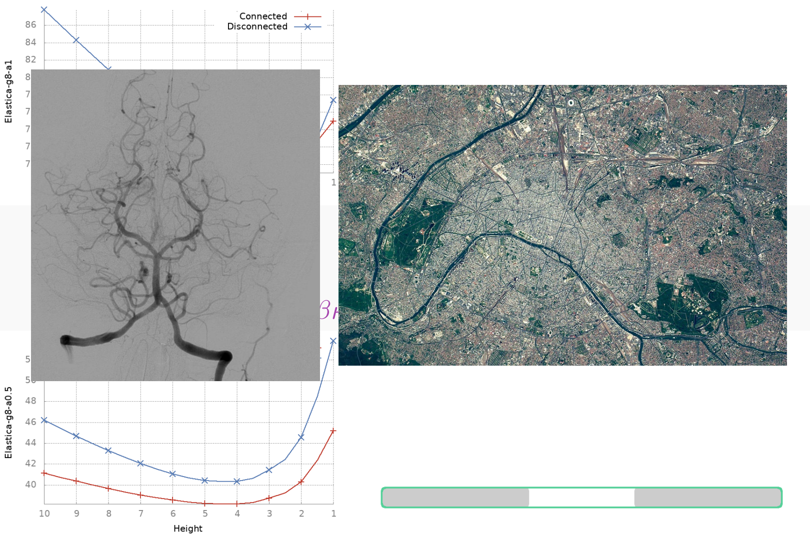

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State-of-the-art

Continuous setting: Define the energy over the whole domain and minimize the elastica with respect the level-curves (Chan, S. H. Kang, Kang, and Shen 2002).

$$\int_{\Omega} \left(\alpha + \beta \nabla \cdot \left(\frac{\nabla f_I}{\|\nabla f_I\|} \right)^2 \right) \|\nabla f_I\| d\Omega.$$

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Masnou and Morel 1998

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Triple cliques Nieuwenhuis, Toeppe, Gorelick, Veksler, and Boykov 2014	Global formulation, quadratic non-submodular energy. Limited precision due combinatorial explosion.

Motivation

Goals

Models based on the minimization of the elastica energy

	Continuous	Discrete	**Digital**
Numerical instability	Yes	No	No
Suitable for digital sets	No	No	Yes
Rounding issues	Yes	No	No
Contour completion	Partial	Partial	Extended
Global optimum (Free elastica)	-	-	Yes

Outline

1. Motivation
   - Image analysis and geometric priors
   - Elastica model and completion property
   - State-of-the-art
2. Contribution
   - Digital sets and convergent estimators
   - A combinatorial model for elastica
   - A quadratic non-submodular formulation for elastica
   - Elastica minimization via graph-cuts
3. Conclusion and perspectives

Digital sets and convergent estimators

- Digital grid particularities and restrictions.
- Multigrid convergence of geometric estimators.

Digital sets and convergent estimators

Digital set peculiarities

Where can we do better?

- Most of models neglect the digital character of digital images and ignore the fact that geometric measurements (mainly those local as tangent and curvature) in such objects should be done with a definition of *convergence* that is specific for digital shapes.

Digital sets and convergent estimators

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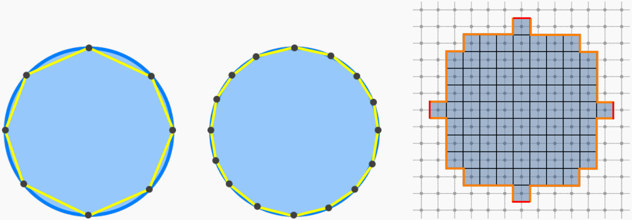
Exact sampling x digitization



Digital sets and convergent estimators

Digital set peculiarities

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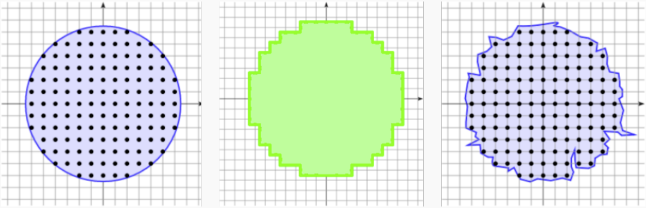
Digitization ambiguity



Digital sets and convergent estimators

Multigrid convergent estimators

Definition (Multigrid convergence)

Let $\mathcal{X}$ be a family of shapes in $\mathbb{R}^n$ and u a geometric quantity that is defined for every shape $X \in \mathcal{X}$. Further, let $D_h(X)$ denote the digitization of X with grid step h.

The estimator $\hat{u}$ is multigrid convergent for $\mathcal{X}$ if and only if, for any $X \in \mathcal{X}$ there exists $h_X > 0$ such that for every $0 < h < h_X$

$$|\hat{u}(D_h(X)) - u(X)| \leq \tau(h), \quad \text{with } \lim_{h \to 0} \tau(h) = 0.$$

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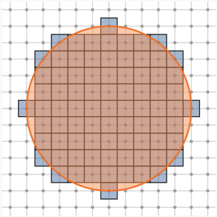
Multigrid convergent estimator of area

$$\widehat{Area(X)} = h^2 |D_h(X)|.$$

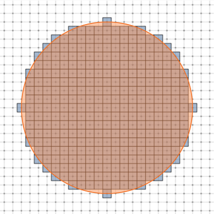
Motivation

Multigrid convergent estimators

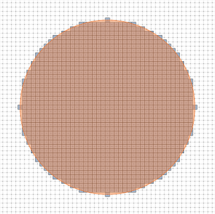
Disk of radius $5 (Area \approx 78.54)$.



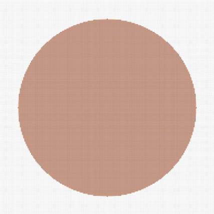
$h = 1.0, \ \widehat{A} = 81.$



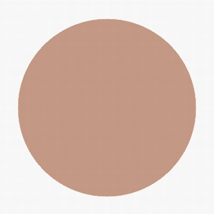
$h = \frac{1}{2}, \ \hat{A} = 79.25.$



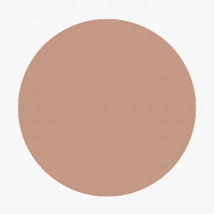
$h = \frac{1}{4}, \ \hat{A} = 78.56.$



$h = \frac{1}{16}, \ \hat{A} = 78.44.$



$h = \frac{1}{32}, \ \hat{A} = 78.5.$



$h = \frac{1}{64}, \ \hat{A} = 78.53.$

Digital sets and convergent estimators

Multigrid convergent estimators

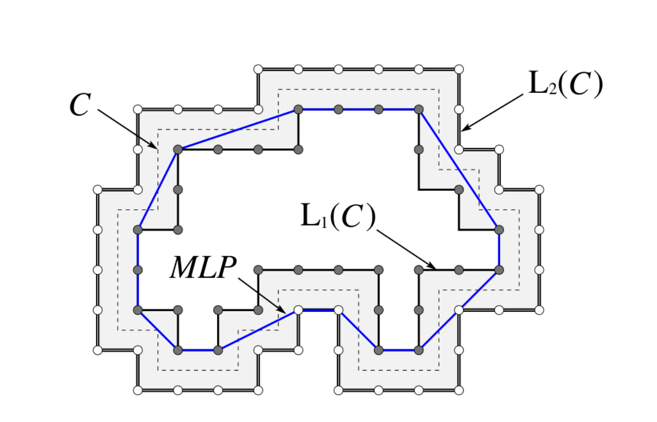
- Minimum Length Polygon (MLP) Sloboda 1998
  - Proved multigrid convergent for piecewise 3-smooth convex shapes.

Digital sets and convergent estimators

Multigrid convergent estimators

- Minimum Length Polygon (MLP) Sloboda 1998

shapes.



Digital sets and convergent estimators

Multigrid convergent estimators

- Minimum Length Polygon (MLP) Sloboda 1998
  - Proved multigrid convergent for piecewise 3-smooth convex shapes.

- Integral Invariant (II) Coeurjolly, Lachaud, and Levallois 2013
  - Proved multigrid convergent for C^2 convex shapes with bounded curvature.

Digital sets and convergent estimators

Multigrid convergent estimators

- Minimum Length Polygon (MLP) Sloboda 1998
  - hapes.
- Ir
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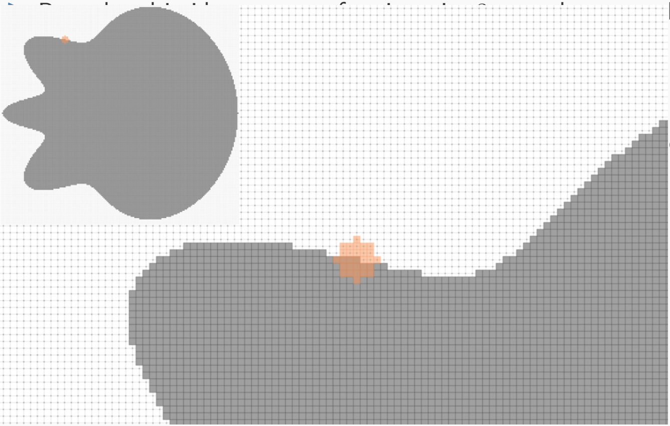

$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$

Digital sets and convergent estimators

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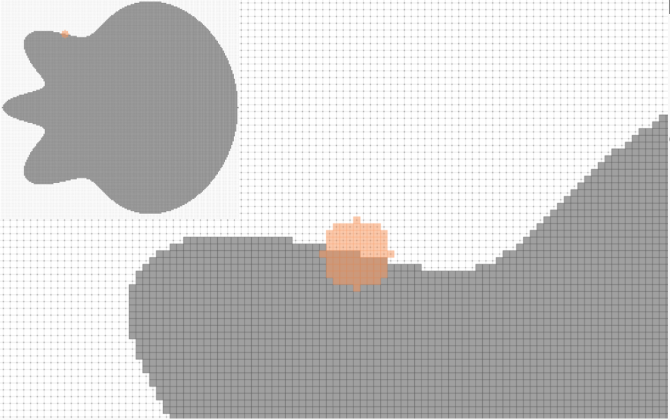

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Digital sets and convergent estimators

Conclusion

- Digital sets are ambiguous and are constrained to the digital grid.
- The multigrid convergence is an adapted definition of convergence for geometric estimation on digital sets.

Digital sets and convergent estimators

Conclusion

- Digital sets are ambiguous and are constrained to the digital grid.
- The multigrid convergence is an adapted definition of convergence for geometric estimation on digital sets.

Can we construct optimization models using multigrid convergent estimators?

A combinatorial model for elastica

- Validate that multigrid convergent estimators can be used in optimization models.
- LocalSearch algorithm.
- Global optimum for the free elastica.

Combinatorial Elastica

Digital elastica

Continuous elastica: $\displaystyle\int_{\partial S} \alpha + \beta\kappa^2 ds.$

Definition (Digital elastica energy)

Let $\hat{\kappa}$ and $\hat{s}$ multigrid convergent estimators of curvature and local length. The digital elastica energy of a digital shape $D \subset \Omega \subset \mathbb{Z}^2$ of parameters $\boldsymbol{\theta} = (\alpha \geq 0, \beta \geq 0)$ is defined as

$$\hat{E}_{\boldsymbol{\theta}}(D) = \sum_{\dot{e} \in \partial_h(D)} \hat{s}(\dot{e}) \Big(\alpha + \beta\hat{\kappa}^2(\dot{e}) \Big).$$

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- The digital elastica energy converges (multigrid) to the continuous elastica.

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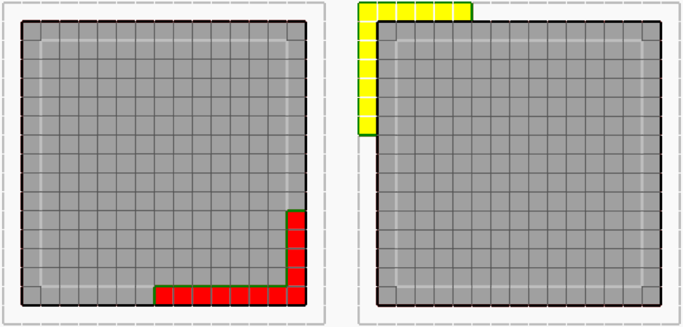
- The digital elastica energy converges (multigrid) to the continuous elastica.
- *Local search*: set a local neighborhood $\mathcal{W}(D)$ of D and pick the shape $X^\star \in \mathcal{W}(D)$ among those of minimum digital elastica value.

Combinatorial Elastica

Neighborhood of shapes

- Members of $\mathcal{W}(D)$ are constructed by removing or adding a set of connected pixels to D.



Combinatorial Elastica

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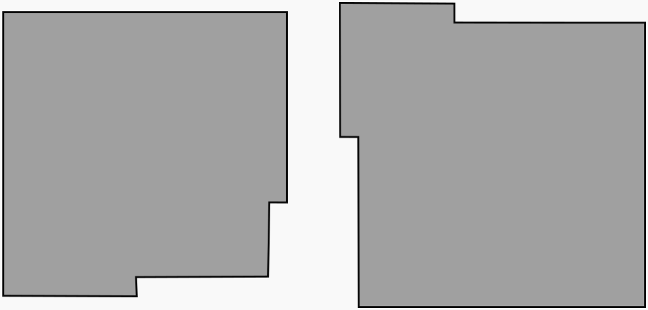

Combinatorial Elastica
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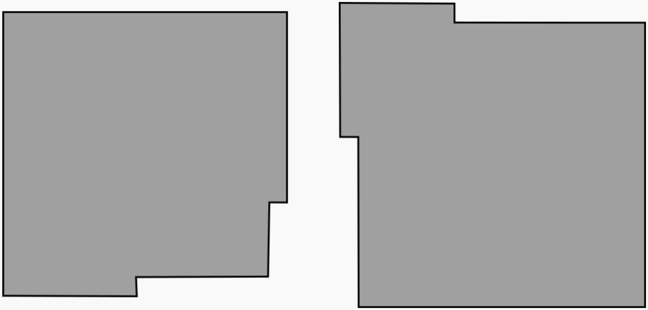


$$D^{(k+1)} \longleftarrow \underset{X \in \mathcal{W}(D^{(k)})}{\arg\min} \hat{E}_{\boldsymbol{\theta}}(X).$$

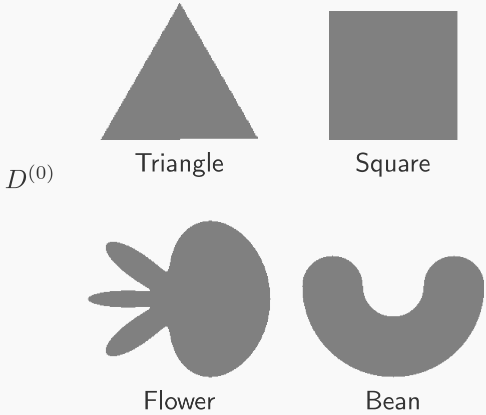
- We use the integral invariant estimator (II-r) to estimate the curvature.

Combinatorial Elastica

Free elastica

Free elastica:

$$D^{(k+1)} \longleftarrow \underset{X \in \mathcal{W}(D^{(k)})}{\arg\min} \; \hat{E}_{\boldsymbol{\theta}}(X).$$

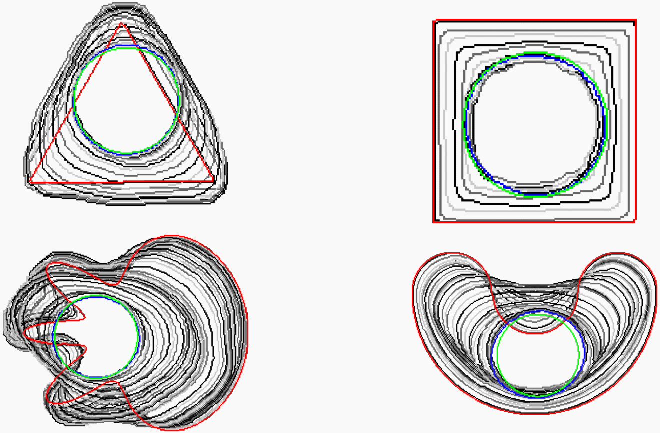


Combinatorial Elastica

Free elastica evolution

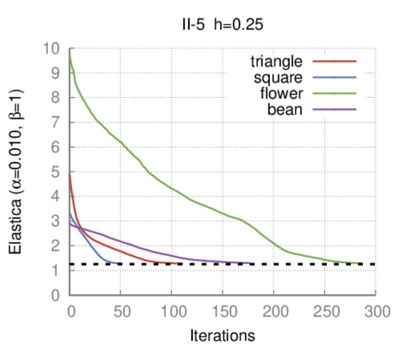
$$\hat{E}_{\boldsymbol{\theta}}(D) = \sum_{\dot{e} \in \partial_h(D)} \hat{s}(\dot{e}) \Big(\alpha + \beta \hat{\kappa}^2(\dot{e}) \Big).$$

$$\text{II-5},\ \alpha = 0.01, \beta = 1.$$



Combinatorial Elastica

Energy evolution



$$\min E(X) = \int_{\partial X} \alpha + \beta\kappa^2 ds$$

$$= 4\pi\beta\frac{1}{r} = 4\pi\beta\left(\frac{\alpha}{\beta}\right)^{1/2},$$

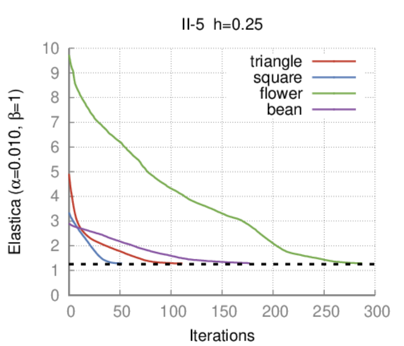
where $\frac{\partial}{\partial r}2\pi(\alpha r + \frac{\beta}{r}) = 0.$

For $\alpha = 0.01,\ \beta = 1,$

$$\min E(X) \approx 1.2566.$$

Combinatorial Elastica

Energy evolution



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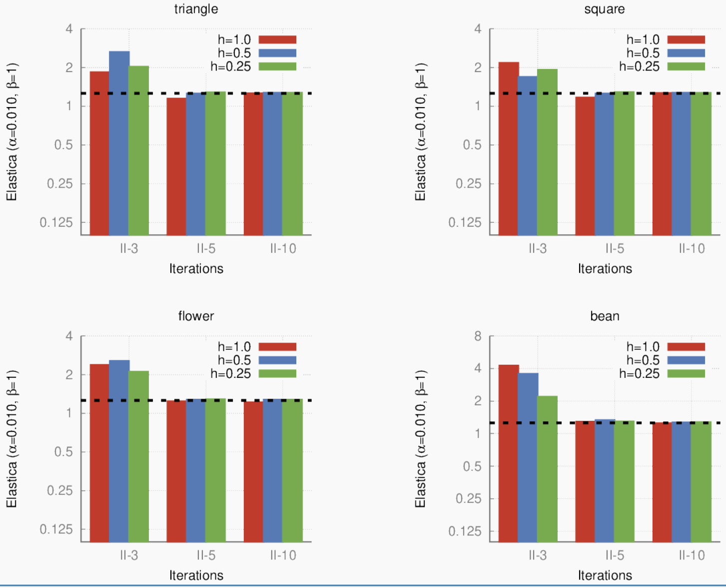
- What is the influence of the radius of the estimation disk?

Combinatorial Elastica

Radius and grid resolution



Combinatorial Elastica

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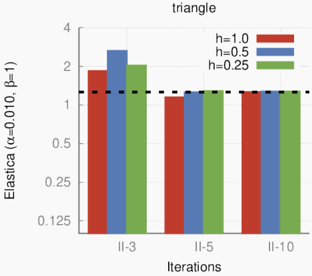
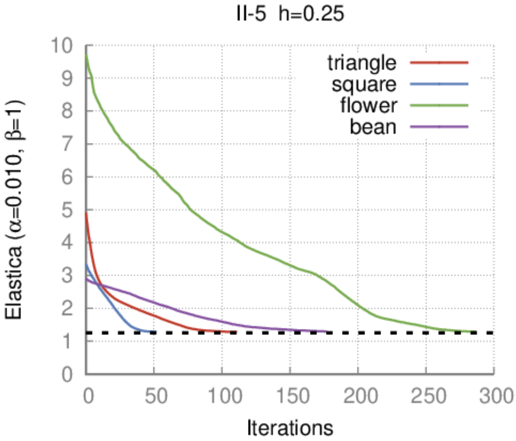
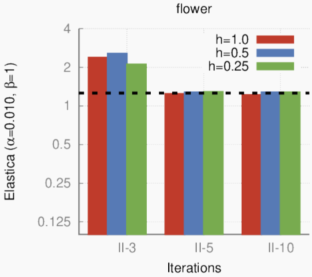
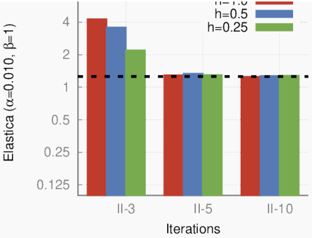






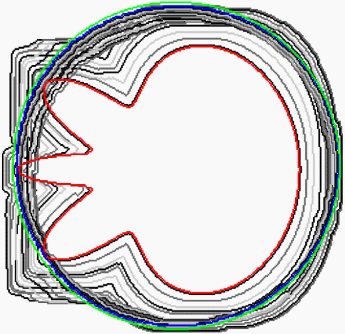


Combinatorial Elastica

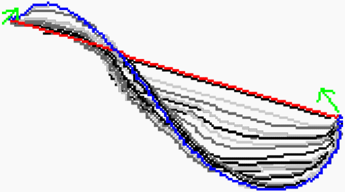
Other experiments

$$\hat{E}_{\boldsymbol{\theta}}(D) = \sum_{\dot{e} \in \partial_h(D)} \hat{s}(\dot{e})\Big(\alpha + \beta\hat{\kappa}^2(\dot{e}) \Big).$$

II-10, $\alpha = 0.001, \beta = 1.$



Free elastica.



Constrained elastica.

Combinatorial Elastica

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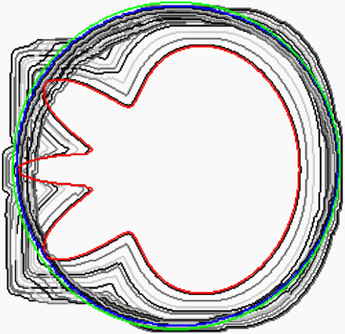
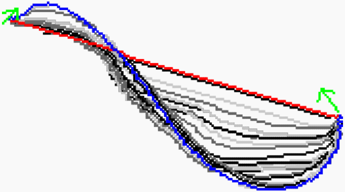


Free elastica.



Constrained elastica.

- What about running time?

Combinatorial Elastica

Running time

	$h = 1.0$		$h = 0.5$		$h = 0.25$	
	Pixels	Time	Pixels	Time	Pixels	Time
Triangle	521	2s (0.07s/it)	2080	43s (0.81s/it)	8315	532s(4.8s/it)
Square	841	0.9s (0.09s/it)	3249	8s (0.3s/it)	12769	102s (2s/it)
Flower	1641	13s (0.24s/it)	6577	209s (1.68s/it)	26321	3534s (12.3s/it)
Bean	1574	7s (0.16s/it)	6278	88s (1.08s/it)	25130	1131s (6.4s/it)
Ellipse	626	1s (0.14s/it)	2506	16s (0.44s/it)	10038	286s (3.1s/it)

Table: **Running time for the free elastica problem.** Quite high running times. The geometry of the shape influences in the total running time.

Combinatorial Elastica

Conclusion

- Multigrid convergent estimators are suitable for elastica minimization
- A simple neighborhood is sufficient to escape bad local minimum. Some solutions are very close to global optimum.
- Too slow. It cannot be used in practice.

A quadratic non-submodular formulation for elastica

- Global formulation attempt.
- Fall back on a local formulation.
- FlipFlow algorithm. Up to 10x faster than LocalSearch.

Non-submodular elastica

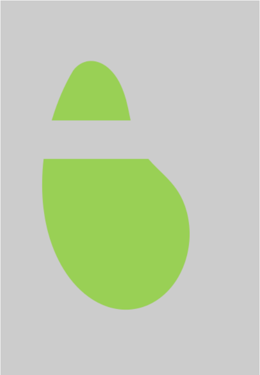
Local models and completion effect

The completion effect can be difficult to recover in local formulations.

Non-submodular elastica

Local models and completion effect

The completion effect can be difficult to recover in local formulations.



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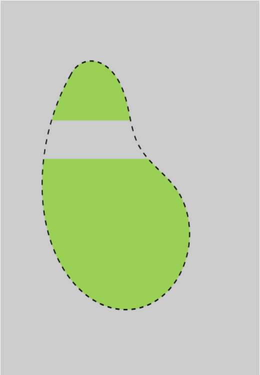

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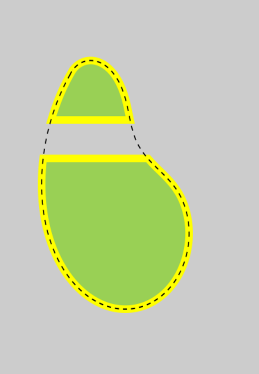

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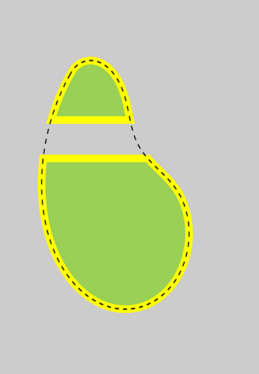
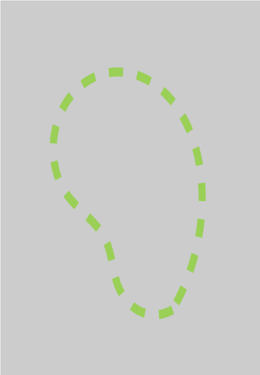


Let's try a global formulation.

Non-submodular elastica

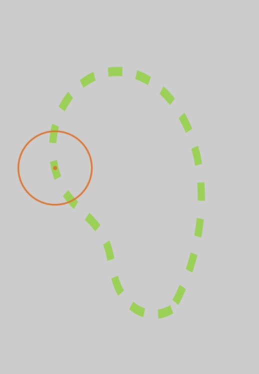
Difficulties with a global formulation



- m pixels and n edges.

Non-submodular elastica

Difficulties with a global formulation

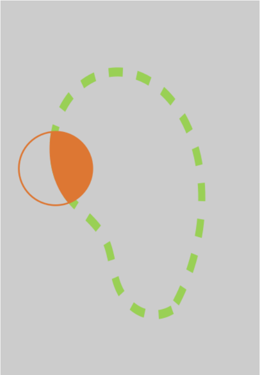


- m pixels and n edges.
- Center of the estimation disk.

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \beta \hat{\kappa}_r^2(D, \ell_i) \right)$$

Non-submodular elastica

Difficulties with a global formulation



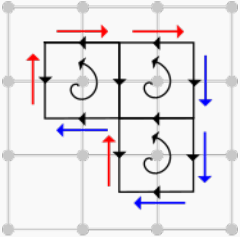
- m pixels and n edges.
- Center of the estimation disk.
- Pixel counting and estimation of curvature squared.

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6}\beta\left(c^2 - 2c\boldsymbol{A}_i^T\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{A}_i^T\boldsymbol{x}\right)\right)$$

subject to $\quad \boldsymbol{x} \in \{0,1\}^m, \boldsymbol{y} \in \{0,1\}^n.$

Non-submodular elastica

Difficulties with a global formulation



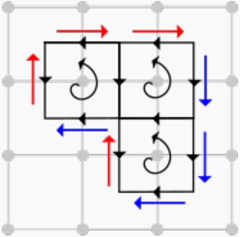
- m pixels and n edges.
- Center of the estimation disk.
- Pixel counting and estimation of curvature squared.
- Linear topological constraints.

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6}\beta\big(c^2 - 2c\boldsymbol{A}_i^T\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{A}_i^T\boldsymbol{x}\big)\right)$$

subject to $\quad \boldsymbol{x} \in \{0,1\}^m, \boldsymbol{y} \in \{0,1\}^n, T(\boldsymbol{x}, \boldsymbol{y}).$

Non-submodular elastica

Difficulties with a global formulation



- m pixels and n edges.
- Center of the estimation disk.
- Pixel counting and estimation of curvature squared.
- Linear topological constraints.
- Third order constrained non-convex binary problem.

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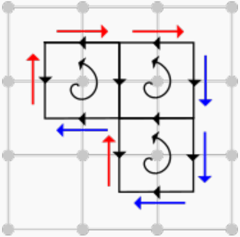


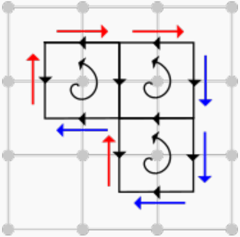
- m pixels and n edges.
- Center of the estimation disk.
- Pixel counting and estimation of curvature squared.
- Linear topological constraints.
- Third order constrained non-convex binary problem.
- Level 1 linearization: non semi-definite positive quadratic problem.

$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6} \beta \left(c^2 - 2c\boldsymbol{A}_i^T \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{A}_i^T \boldsymbol{x} \right) \right)$$

subject to $\quad \boldsymbol{x} \in \{0,1\}^m, \boldsymbol{y} \in \{0,1\}^n, T(\boldsymbol{x},\boldsymbol{y}).$

Non-submodular elastica

Difficulties with a global formulation



- m pixels and n edges.
- Center of the estimation disk.
- Pixel counting and estimation of curvature squared.
- Linear topological constraints.
- Third order constrained non-convex binary problem.
- Level 1 linearization: non semi-definite positive quadratic problem.
- Level 2 linearization: $O(m^3)$ variables.

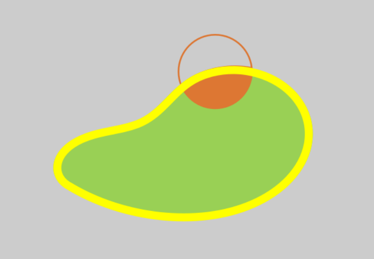
$$\sum_{\ell_i \in \mathcal{L}} \boldsymbol{y}_i \left(\alpha + \frac{9}{r^6}\beta\left(c^2 - 2c\boldsymbol{A}_i^T\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{A}_i^T\boldsymbol{x}\right)\right)$$

subject to $\quad \boldsymbol{x} \in \{0,1\}^m, \boldsymbol{y} \in \{0,1\}^n, T(\boldsymbol{x},\boldsymbol{y}).$

Non-submodular elastica

Simplification

$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$

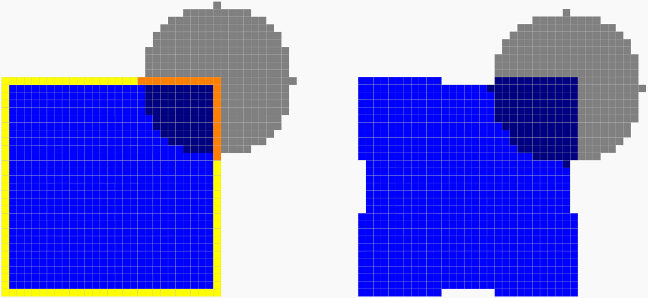


- Define the optimization region (yellow) as the inner contour of the shape, denoted I.
- Evolve the estimation disks in the current contour.
- Set pixels such that the curvature estimation is reduced.

Non-submodular elastica

Simplification

$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$



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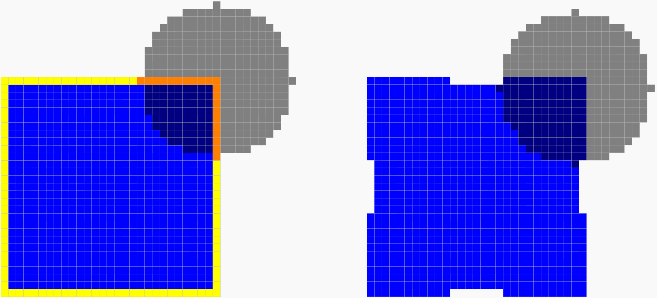

- Optimization identifies zones of shortage (convex) or abundance (concave) of pixels.

Non-submodular elastica

Simplification

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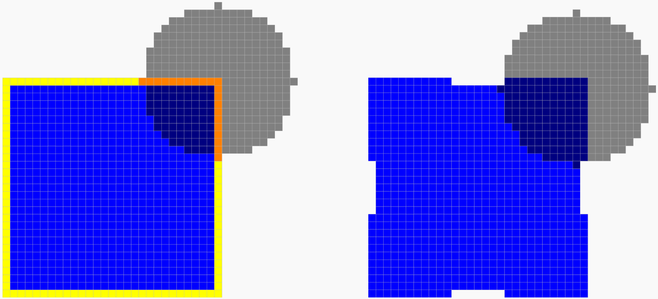
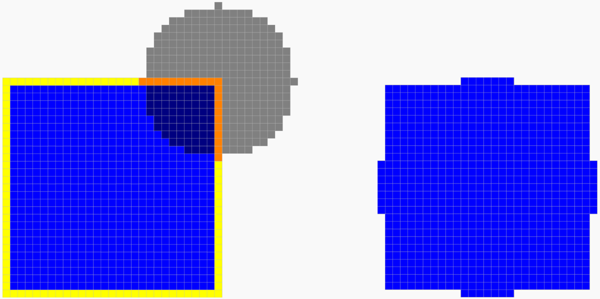


- Optimization identifies zones of shortage (convex) or abundance (concave) of pixels.
- $x = 1 \rightarrow$ Zone of shortage of pixels (convex) $\rightarrow$ Estimator disk should be shifted towards the interior $\rightarrow$ This pixel does not belong to the next contour.

Non-submodular elastica

Simplification

$$\hat{\kappa}(p) = \frac{3}{r^3}\left(\frac{\pi r^2}{2} - |B_r(p) \cap X|\right)$$



- Optimization identifies zones of shortage (convex) or abundance (concave) of pixels.
- $x = 1 \rightarrow$ Zone of shortage of pixels (convex) $\rightarrow$ Estimator disk should be shifted towards the interior $\rightarrow$ This pixel does not belong to the next contour.
- Therefore, we invert the optimal labeling.

Non-submodular elastica

FlipFlow

$$D \subset \Omega \subset \mathbb{Z}^2, \quad X^{(k)} := \{ \, x_i \in \{0,1\} \mid p_i \in \underbrace{I^{(k)}}_{\text{Inner contour}} \}$$

$$E_{\boldsymbol{\theta}}^{flip}(D^{(k)}, X^{(k)}) = \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in I^{(k)}} \beta \hat{\kappa}(p)^2$$

Non-submodular elastica

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$$D \subset \Omega \subset \mathbb{Z}^2, \quad X^{(k)} := \{\, x_i \in \{0,1\} \mid p_i \in \underbrace{I^{(k)}}_{\text{Inner contour}} \,\}$$

$$\begin{aligned} E^{flip}_{\boldsymbol{\theta}}(D^{(k)}, X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in I^{(k)}} \beta \hat{\kappa}(p)^2 \\ &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) \\ &+ \sum_{\substack{p \in \\ I^{(k)}}} 2c_1\beta \Bigg((1/2 + |F_r^{(k)}(p)| - c_2) \cdot \sum_{\substack{x_j \in \\ X_r^{(k)}(p)}} x_j + \sum_{\substack{j<l, \\ x_j, x_l \in \\ X_r^{(k)}(p)}} x_j x_l \Bigg) \end{aligned}$$

Non-submodular elastica

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\end{aligned}$$

$$s(x_j) = \sum_{q_i \in \mathcal{N}_4(p_j)} t(q_i), \quad \text{where } t(q_i) = \begin{cases} (x_j - x_i)^2, & \text{if } q_i \in I^{(k)} \\ (x_j - 1)^2, & \text{if } q_i \in F^{(k)} \\ (x_j - 0)^2, & \text{otherwise.} \end{cases}$$

Non-submodular elastica

FlipFlow

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$$\begin{aligned}
E_{\boldsymbol{\theta}}^{flip}(D^{(k)}, 1 - X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in I^{(k)}} \beta \hat{\kappa}(p)^2 \\
&= \sum_{x_j \in X^{(k)}} \alpha s(x_j) \\
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&\quad + \sum_{p \in I^{(k)}} 2c_1\beta \Big((1/2 + |F_r^{(k)}(p)| - c_2) \cdot \sum_{\substack{x_j \in \\ X_r^{(k)}(p)}} x_j + \sum_{\substack{j<l, \\ x_j, x_l \in \\ X_r^{(k)}(p)}} x_j x_l \Big)
\end{aligned}$$

Shrink mode (convexities)

$$\begin{aligned}
a^{(k)} &\leftarrow \arg\min_{X^{(k)}} E_{\boldsymbol{\theta}}^{flip}(D^{(k)}, 1 - X^{(k)}); \\
D^{(k+1)} &\leftarrow F^{(k)} + a^{(k)}.
\end{aligned}$$

Non-submodular elastica

FlipFlow

$$D \subset \Omega \subset \mathbb{Z}^2, \quad X^{(k)} := \{ x_i \in \{0,1\} \mid p_i \in \underbrace{I^{(k)}}_{\text{Inner contour}} \}$$

$$\begin{aligned} E_{\boldsymbol{\theta}}^{flip}(D^{(k)}, 1 - X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in I^{(k)}} \beta \hat{\kappa}(p)^2 \\ &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) \\ &+ \sum_{p \in I^{(k)}} 2c_1\beta \Bigg((1/2 + |F_r^{(k)}(p)| - c_2) \cdot \sum_{\substack{x_j \in \\ X_r^{(k)}(p)}} x_j + \sum_{\substack{j<l, \\ x_j, x_l \in \\ X_r^{(k)}(p)}} x_j x_l \Bigg) \end{aligned}$$

Shrink mode (convexities)

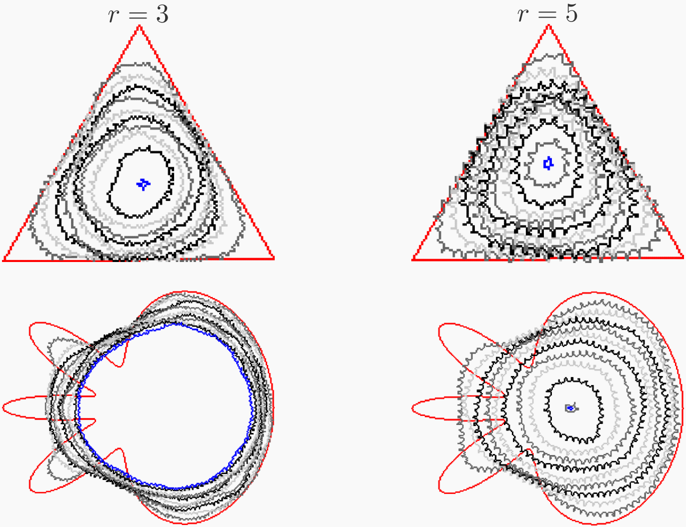
$$\begin{aligned} a^{(k)} &\leftarrow \arg\min_{X^{(k)}} E_{\boldsymbol{\theta}}^{flip}(D^{(k)}, 1 - X^{(k)}); \\ D^{(k+1)} &\leftarrow F^{(k)} + a^{(k)}. \end{aligned}$$

Expansion mode (concavities)

$$\begin{aligned} a^{(k)} &\leftarrow \arg\min_{\overline{X}^{(k)}} E_{\boldsymbol{\theta}}^{flip}(\overline{D}^{(k)}, 1 - \overline{X}^{(k)}); \\ D^{(k+1)} &\leftarrow \overline{\overline{F}^{(k)} + a^{(k)}}. \end{aligned}$$

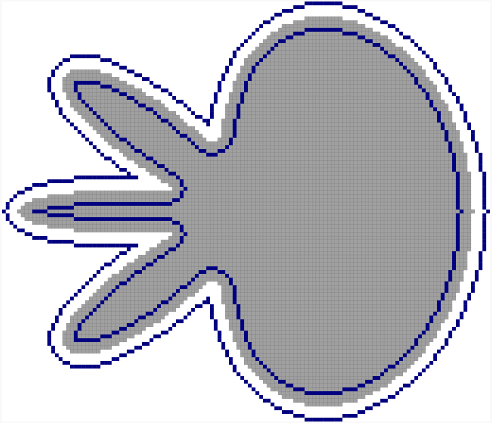
Non-submodular elastica

FlipFlow



Non-submodular elastica

Evaluation on farther rings



Non-submodular elastica

Evaluation on farther rings

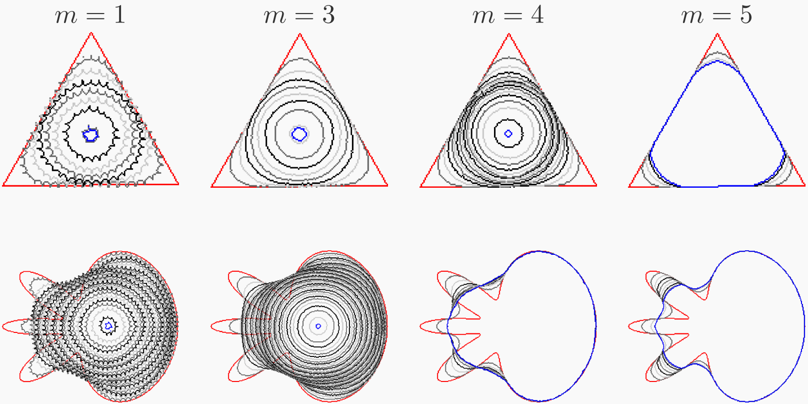
$$
\begin{aligned}
E^{flip}_{(\boldsymbol{\theta},m)}(D^{(k)}, 1 - X^{(k)}) &= \sum_{x_j \in X^{(k)}} \alpha s(x_j) + \sum_{p \in R_m(D^{(k)})} \beta \hat{\kappa}(p)^2 \\
&= \sum_{x_j \in X^{(k)}} \alpha s(x_j) \\
&+ \sum_{\substack{p \in \\ R_m(D^{(k)})}} 2c_1\beta \Big((1/2 + |F_r^{(k)}(p)| - c_2) \cdot \sum_{\substack{x_j \in \\ X_r^{(k)}(p)}} x_j + \sum_{\substack{j<l, \\ x_j, x_l \in \\ X_r^{(k)}(p)}} x_j x_l \Big)
\end{aligned}
$$

$$
R_m(D) := \{p \mid m - 1 < d_D(p) \leq m\} \cup \{p \mid -m + 1 > d_D(p) \geq -m\}
$$

Non-submodular elastica

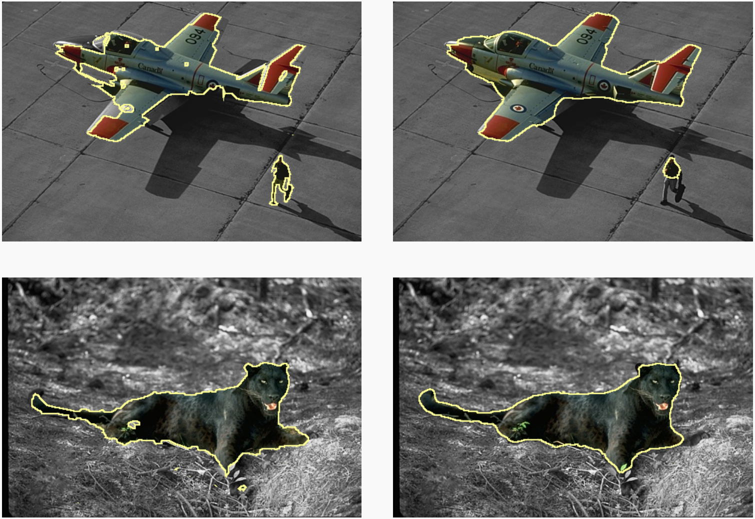
Evaluation on farther rings

$r = 5$

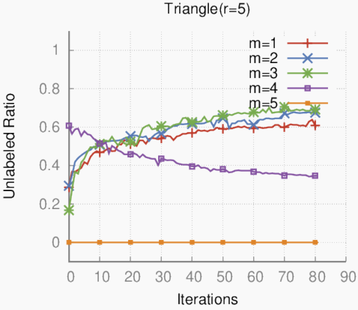


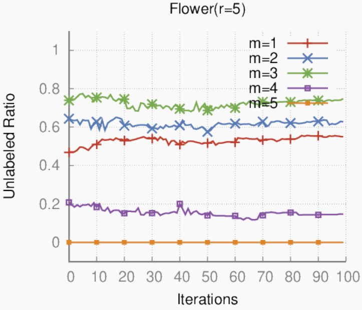
Non-submodular elastica

Contour correction



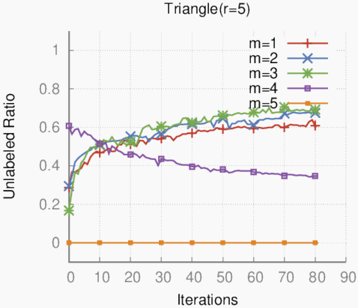
Non-submodular elastica
Unlabeled ratio

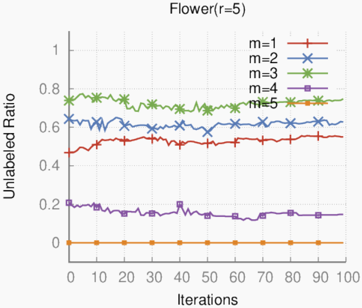




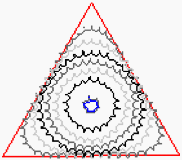
Non-submodular elastica

Unlabeled ratio





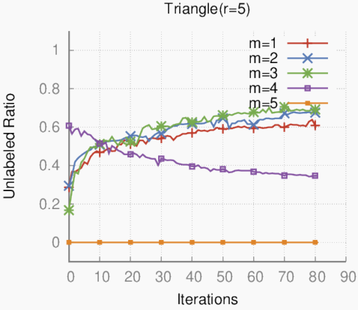
$m = 1$

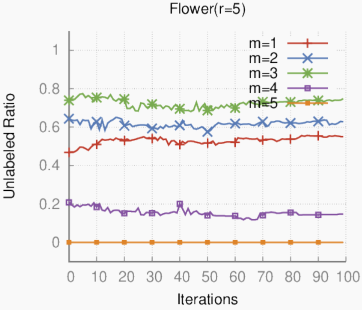


$m = 3$

Non-submodular elastica

Unlabeled ratio





- Unlabeled ratio is not sufficient to explain the smoothness at farther rings.
- We are more confident to use values of m closer to the estimation disk radius value.
- Conjecture: For $m = r$ the energy is submodular.

Non-submodular elastica
Conclusion

- Global formulation not computable in practice.

Non-submodular elastica

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- Global formulation not computable in practice.
- Local formulation 10x faster than combinatorial model.

Non-submodular elastica

Conclusion

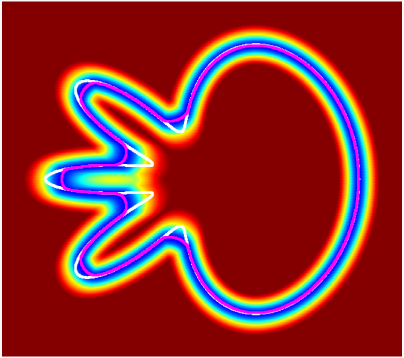
- Global formulation not computable in practice.
- Local formulation 10x faster than combinatorial model.
- Useful as a post-processing procedure: contour correction.

Elastica minimization via graph-cuts

- Balance coefficient to estabilize curvature estimation.
- Set up a graph whose minimum cut approximates the zero level set of the balance coefficient.
- GraphFlow algorithm. Up to 10x faster than FlipFlow.

Non-submodular elastica

Balance coefficient



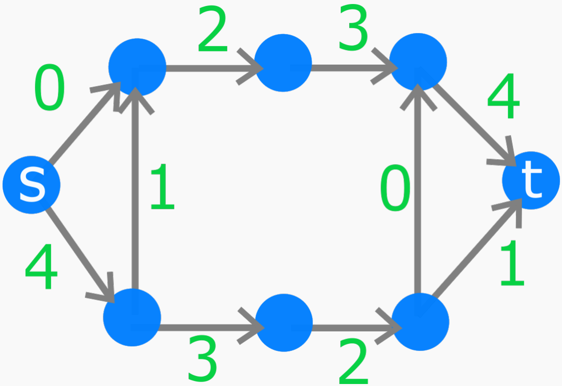
- Balance coefficient

$$u_r(D,p) = \left(\frac{\pi r^2}{2} - |B_r(p) \cap D|\right)^2$$

- White contour: contour of the shape
- Pink contour: ϵ-level set of the balance coefficient

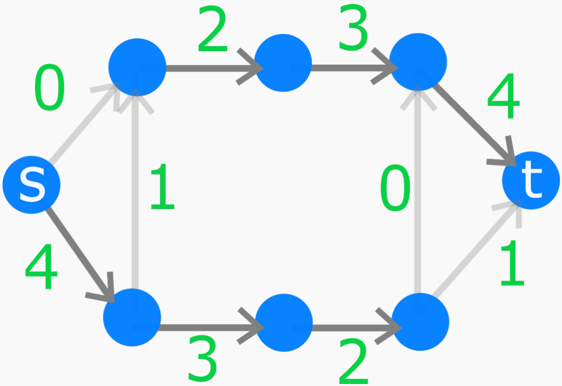
Elastica minimization via graph-cuts

Graph cut

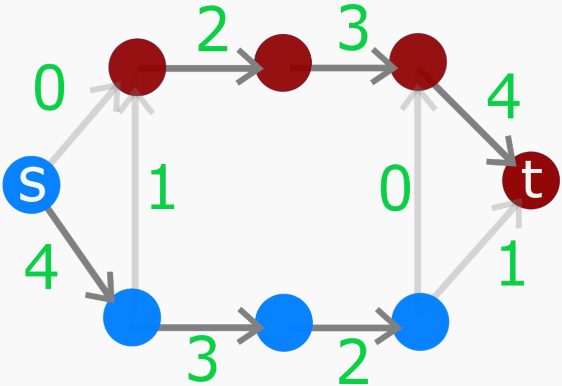


Elastica minimization via graph-cuts

Graph cut

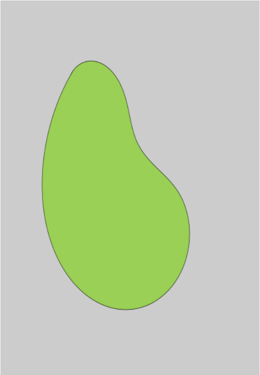


Elastica minimization via graph-cuts
Graph cut



Elastica minimization via graph-cuts

Building the graph

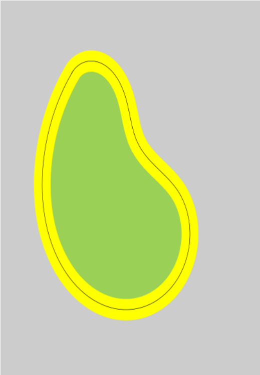


Elastica minimization via graph-cuts

Building the graph

- Optimization band

$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$

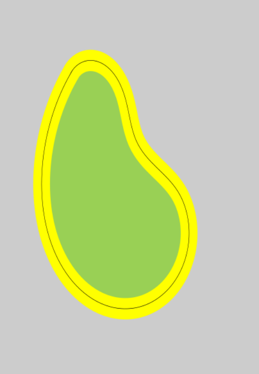


Elastica minimization via graph-cuts

Building the graph

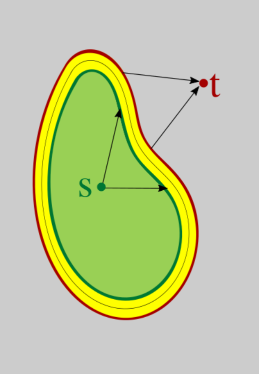
- Optimization band

$$
\begin{aligned}
O(D) :=& \{p \in D \mid -n \leq d_D(p) \leq n\} \\
F(D) :=& D \setminus O(D)
\end{aligned}
$$



Elastica minimization via graph-cuts

Building the graph



- Optimization band

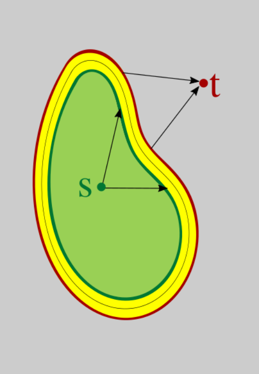
$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$
$$F(D) := D \setminus O(D)$$

- Graph $\mathcal{G}_D(\mathcal{V}, \mathcal{E}, c)$

$$\mathcal{V} = \{v_p \mid p \in O(D)\} \cup \{s, t\}$$
$$\mathcal{E} = \{\{v_p, v_q\} \mid p, q \in O(D) \text{ and } q \in \mathcal{N}_4(p)\} \cup \mathcal{E}_{st}$$
$$\mathcal{E}_{st} = \{(s, v_p), (v_p, t) \mid p \in O(D)\}$$

Elastica minimization via graph-cuts

Building the graph



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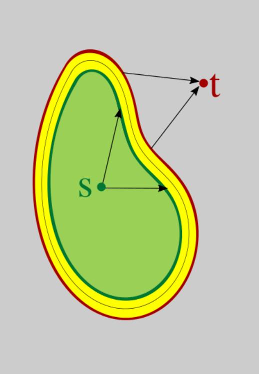
Elastica minimization via graph-cuts

Building the graph



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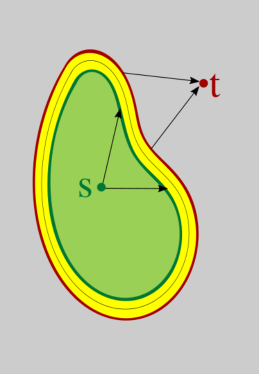
Elastica minimization via graph-cuts

Building the graph



- Optimization band

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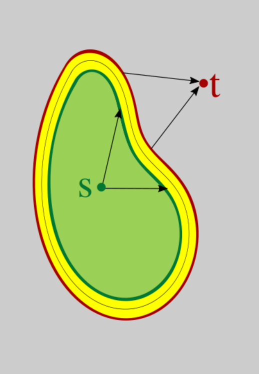
$$\mathcal{E}_{st} = \{(s, v_p), (v_p, t) \mid p \in O(D)\}$$

- Edge's weight

edge e	$\mathbf{c(e)}$
$\{v_p, v_q\}$	$\frac{1}{2}\left(u_r(D,p) + u_r(D,q)\right)$
(s, v_p)	M
(v_p, t)	M

Elastica minimization via graph-cuts

Building the graph



- Optimization band

$$O(D) := \{p \in D \mid -n \leq d_D(p) \leq n\}$$
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- Graph $\mathcal{G}_D(\mathcal{V}, \mathcal{E}, c)$

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$$\mathcal{E}_{st} = \{(s, v_p), (v_p, t) \mid p \in O(D)\}$$

- Edge's weight

edge e	$\mathbf{c(e)}$
$\{v_p, v_q\}$	$\frac{1}{2}\left(u_r(D,p) + u_r(D,q)\right)$
(s, v_p)	M
(v_p, t)	M

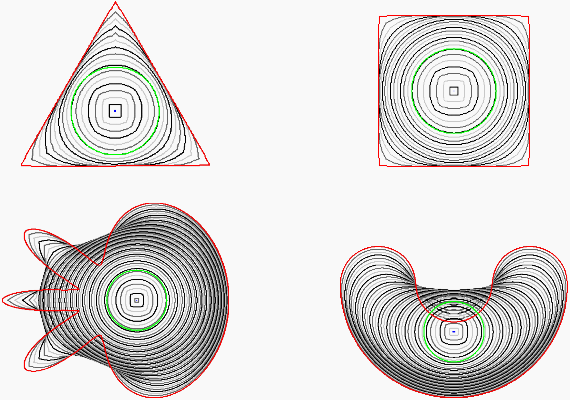
- Digital shape update

$$D^{(k+1)} = F(D^{(k)}) + S^{(k)}$$

Elastica minimization via graph-cuts

Shape evolution

$$\alpha = 1/8^2, \beta = 1.$$



Elastica minimization via graph-cuts

Shape evolution

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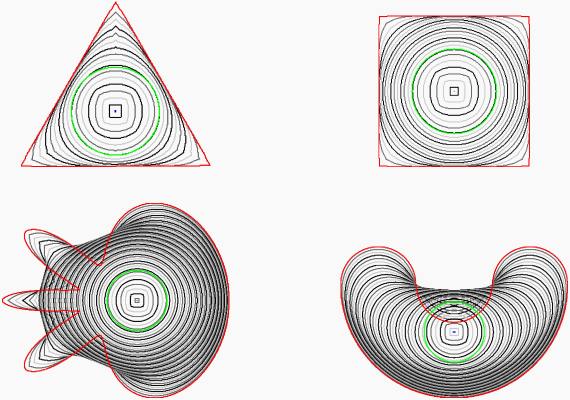
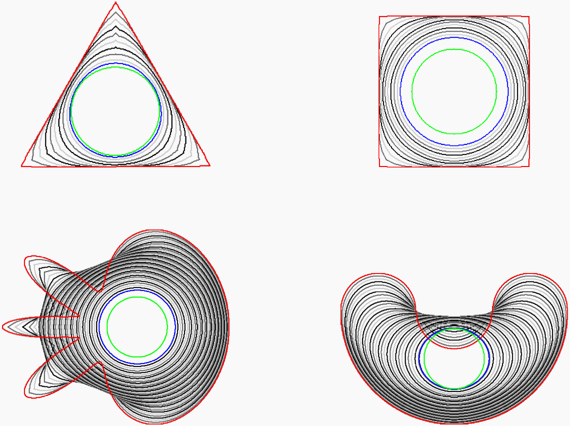


- What if we stop the evolution when elastica increases?

Elastica minimization via graph-cuts

Shape evolution

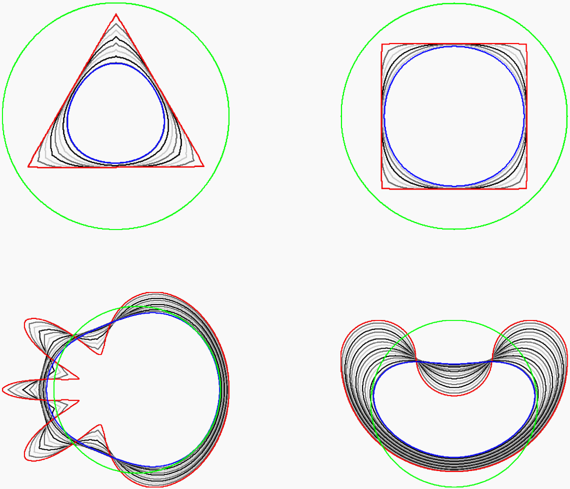
Stop if elastica increases $(\alpha = 1/8^2, \beta = 1)$



Elastica minimization via graph-cuts

Shape evolution

Stop if elastica increases $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

The a-probe set

Definition (a-probe set)

Let $D \subset \Omega \subset \mathbb{Z}^2$ a digital set and a a natural number. The a-probe set of D is defined as

$$\mathcal{P}_a(D) = D \cup \bigcup_{a'<a} D^{+a'} \cup D^{-a'},$$

where $D^{+a}(D^{-a})$ denotes a dilation(erosion) by a disk of radius a.

Candidate selection

$$sol(D^{(k)}) \longleftarrow \bigcup_{D' \in \mathcal{P}_a(D^{(k)})} \left\{ F^{(k)} + S \mid mincut(S, \mathcal{G}_{D'}) \right\}$$

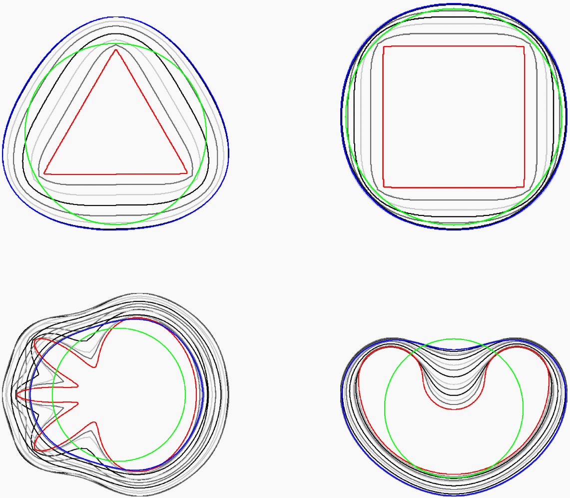
Candidate validation

$$D^{(k+1)} \longleftarrow \underset{D' \in sol(D^{(k)})}{\arg\min} \; \hat{E}_{\boldsymbol{\theta}}(D')$$

Elastica minimization via graph-cuts

Shape evolution with α-probe set

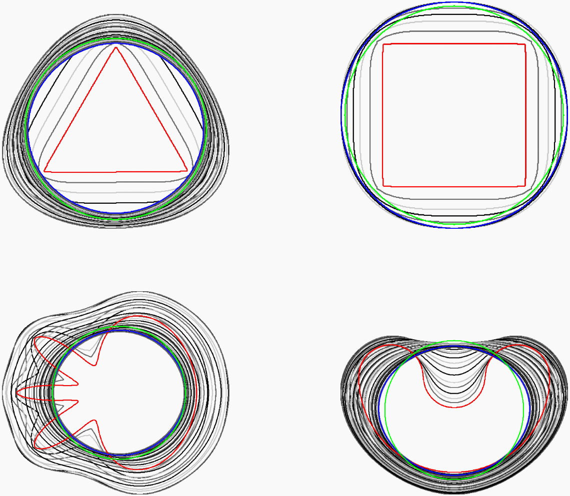
Stop if elastica increases $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

Shape evolution with α-probe set

Always update $(\alpha = 1/22^2, \beta = 1)$



Elastica minimization via graph-cuts

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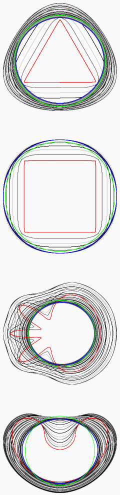
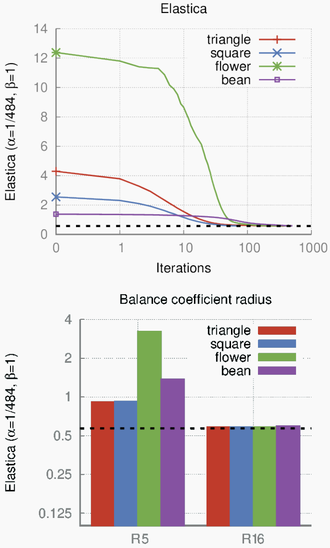




Elastica minimization via graph-cuts

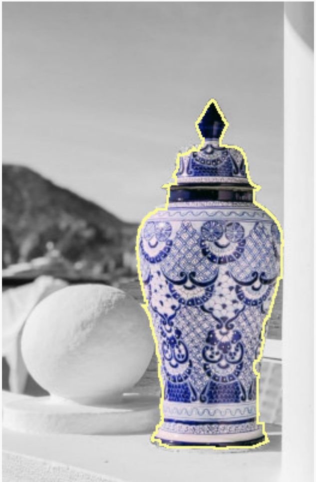
Contour correction

Initial segmentation

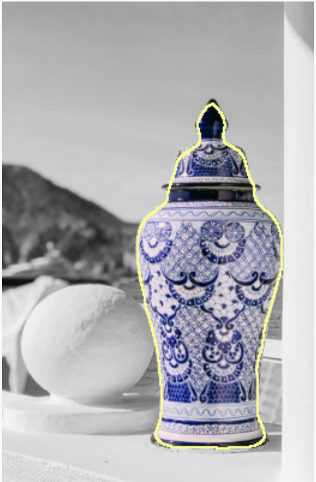
$0.825s$ (3 it)

Elastica minimization via graph-cuts

Contour correction



Initial segmentation



$0.746s$ (3 it)

Elastica minimization via graph-cuts

Contour correction

Initial segmentation

$1.1s$ (3 it)

Elastica minimization via graph-cuts

Contour correction

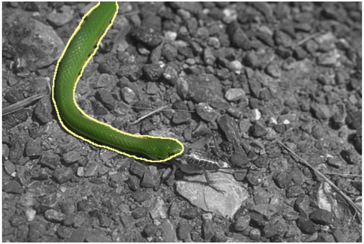
Initial segmentation

$10s$ (30 it)

Elastica minimization via graph-cuts

Contour completion

Initial segmentation



$17s$ (62 it)

Conclusion

Summary of models

Model	Implementation	Running time	Free elastica	Constrained elastica	Image term
LocalSearch	medium	slow	yes(**opt**)	yes	no
FlipFlow	hard	acceptable	yes	no	yes
(BalanceFlow)	medium	acceptable	yes	no	yes
GraphFlow	easy	fast	yes(**opt**)	no	yes

Table: Models summary. The qualitative attributes are relative, e.g., the GraphFlow presents the lowest running time while LocalSearch presents the highest.

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Table: **Models summary.** The qualitative attributes are relative, e.g., the GraphFlow presents the lowest running time while LocalSearch presents the highest.

	Pixels	LocalSearch	FlipFlow	BalanceFlow	GraphFlow
Triangle	8315	4.8s/it	0.4s/it	0.38s/it	0.14s/it
Square	12769	2s/it	0.51s/it	0.47s/it	0.12s/it
Ellipse	10038	3.1s/it	0.64s/it	0.57s/it	0.1s/it
Flower	26321	12.3s/it	1.23s/it	0.94s/it	0.14s/it
Bean	25130	6.4s/it	1.2s/it	1.17s/it	0.16s/it

Table: **Free elastica running times.** Running time and input size for the free elastica experiment.

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Cons

- Susceptible to bad local minimum (we can escape it with a proper definition of the neighborhood).

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- **Global formulation and multigrid convergent estimators**: Do there exist a practicable model for elastica?

Thank you!

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