Fast low-rank metric learning

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Abstract

This doctoral thesis will present the results of my work into the reanimation of lifeless human tissues.

Acknowledgements

Many thanks to my mummy for the numerous packed lunches; and of course to Igor, my faithful lab assistant. DP IM.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

(Dan-Theodor Oneaţă)

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Chapter 1

Introduction

k Nearest Neighbours (kNN) is one of the oldest and simplest classification methods. It has its origins in an unpublished technical report by **?** and since then it became standard textbook material (Russell et al., 1996; Mitchell, 1997; Bishop, 2006). In the last 50 years, kNN was present in most of the machine learning related fields (pattern recognition, statistical classification, data mining, information retrieval, data compression) and it plays a role in many attractive applications (e.g., face recognition, plagiarism detection, vector quantization).

The idea behind kNN is intuitive and straightforward: classify a given point according to a majority vote of its neighbours; the selected class is the one that is the most represented amongst the k nearest neighbours. This is easy to implement and it usually represents a good way to approach new problems or data sets. Despite its simplicity kNN is still a powerful tool, performing surprisingly well in practice, as most of the simple classifiers (Holte, 1993).

Yet there are also other characteristics that make kNN an interesting method. First of all, kNN makes no assumptions about the underlying structure of the data, but lets the data "speak for themselves". This means that no a priori knowledge is needed beforehand and, more importantly, the accuracy increases with the number of points in the data set (in fact, it approaches Bayes optimality as the cardinality of the training set approaches infinity and k is sufficiently large; Cover and Hart, 1967). Secondly, kNN is able to represent complex functions with non-linear decision boundaries by using only simple local approximations (this behaviour is hardly captured by any parametric method). Lastly, kNN operates in a "lazy" fashion: the training data is just stored and their use is delayed until the testing. The quasi-inexistent training allows to easily add new

training examples.

On the other hand, kNN has some drawbacks that influence both the computational performance and also the quality of its predictions. Firstly, because it has to memorise all the exemplars, the storage requirements are directly proportional with the number of instances in the training set. Furthermore, a serious practical disadvantage is represented by the fact that all the computations are done at testing. The cost for this is also linear in the cardinality of the training set and it is often prohibitive for large data sets. However, more important are the issues related to the accuracy. On one hand, there is not clear how we should choose a dissimilarity metric and how notions as "close"/"far" are defined for our data set. This is non-trivial and usually the standard Euclidean metric is not satisfactory (see Subsection ??, for a more detailed discussion). On the other hand, there have been raised concerns with the usefulness of Nearest Neighbours (NN) methods for high dimensional data (Beyer et al., 1999; Hinneburg et al., 2000). The curse of dimensionality arises for kNN, because the distances become indiscernible for many dimensions; alternatively formulated, for a given distribution the maximum and the minimum distance between points become equal in the limit of dimensions.

All these problems are even more acute nowadays when we have to operate on huge sets of data with many attributes (e.g., images, videos, DNA sequences, etc.). There is an entire literature that tries to come up with possible solutions (some of the most prominent papers are discussed in Section ??). One elegant answer is provided by Goldberger et al. (2004). They proposed a new method, Neighbourhood Component Analysis (NCA), that copes with the drawbacks in a unified manner by learning a low-rank metric. This reduces both the storage and the computational cost, because the algorithm uses the data set projected into a lower subspace, with fewer attributes needed. Also the accuracy is improved, because the label information is used for constructing a proper metric that selects the relevant attributes. However, NCA introduces a consistent training time, because we now have an objective function whose gradient is quadratic in the number of data points that is optimized using iterative methods.

The main aim of this project is to see whether NCA's training and testing time can be reduced without significant loses in accuracy. We will try to achieve this by making use of k-d trees (a space partitioning structure; Bentley 1975). These have been successfully applied for speeding up kNN at query time (Friedman et al.,

1977) and we will use them in a similar manner for NCA's testing operations. For training, we will apply k-d trees in a slightly different way (as in Deng and Moore, 1995); as k-d trees organize the space, they can be used to group together near points and quickly compute approximations of the gradient and objective function at each iteration. This idea was also followed by Weinberger and Saul (2009) in accelerating a different distance metric technique, Large Margin Nearest Neighbour (LMNN). As an alternative approach, we could interpret NCA as a class-conditional kernel-density estimate and then use different types of kernels instead of the Gaussian ones; we will focus especially on kernels with compact support, because they do not introduce approximation errors and computations can be done more efficiently than in the previous case. If time permits, it would be interesting to experiment with different tree structures, such as ball trees (Omohundro, 1989; Moore, 2000) (which can perform well in higher dimensional spaces) or dual-trees (Gray and Moore, 2003) (a faster representation of the typcal k-d tree). Also we could try this approach on other distance metric learning methods, with different objective functions (e.g., Xing et al., 2003).

Chapter 2

Background

This chapter sets the theoretical basis for following discussion on low-rank metrics. We start with a brief mathematical introduction into distance metrics. Then we outline the characteristics of the Mahalanobis-like metric. At the end of the chapter we review some of the most prominent work in the area. The method we study, Neighbourhood Component Analysis (NCA), is presented in chapter 3.

2.1 Theoretical background

A distance metric is a mapping from a pair of inputs to a scalar. Let $d(x, y)$ denote the distance function between the arguments x and y. We say that d is a metric if it satisfies the following properties for any x, y, z:

- Non-negativity: $d(x, y) \geq 0$.

- Distinguishability: $d(x, y) = 0 \Leftrightarrow x \equiv y$.

- Symmetry: $d(x, y) = d(y, x)$.

- Triangle Inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

Essentially, a metric can be defined on any set of objects. The distance metric is valid as long as the above properties hold for any elements in the set. However, for the scope of our applications we limit ourselves to points in D dimensional real space. The inputs will be represented by column vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$. In this case the distance d is a function from $\mathbb{R}^D \times \mathbb{R}^D$ to $\mathbb{R}$. A well known example of such a function is the Euclidean metric. This is defined by:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(\mathbf{x} - \mathbf{y})^{\mathrm{T}}(\mathbf{x} - \mathbf{y})}, \quad \text{with } \mathbf{x}, \mathbf{y} \in \mathbb{R}^D. \tag{2.1}$$

Albeit useful in many scenarios, the Euclidean metric has two properties that are problematic in machine learning applications:[1]

- **Sensitivity to variable scaling**. In geometrical situations, the values across all D dimensions are measured in the same unit of length. In practical situations however, each dimension encodes a different feature of the data set. The Euclidean metric is sensitive to scaling and it returns different results if we change the measurement units for one of the D variables. We compensate this differences by dividing the values on dimension i by the standard deviation s_i on that direction:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\left(\frac{x_1 - y_1}{s_1}\right)^2 + \cdots + \left(\frac{x_D - y_D}{s_D}\right)^2} =$$
$$= \sqrt{(\mathbf{x} - \mathbf{y})^{\mathrm{T}}\mathbf{S}^{-1}(\mathbf{x} - \mathbf{y})}, \tag{2.2}$$

  where $\mathbf{S} = \mathrm{diag}(s_1^2, \cdots, s_D^2)$ and $s_i = \mathrm{var}[x_i], \forall i = 1, \cdots, D$.

- **Invariance to correlated variables**. Euclidean distance treats equally the discrepancy across any of the D dimensions. Often many features of a data set are redundant and, because of their number, they significantly influence the distance. Take the example of images and face recognition. The pixels from the image background are highly correlated and they reflect the same information: the colour of the background. If the two pictures of the same subject have different backgrounds, we get a high distance between the images. This effect is alleviated if we replace $\mathbf{S}$ in equation (2.2) with the covariance matrix of the data:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(\mathbf{x} - \mathbf{y})^{\mathrm{T}}\mathbf{S}^{-1}(\mathbf{x} - \mathbf{y})}, \tag{2.3}$$

  where $\mathbf{S} = \mathrm{cov}[\mathcal{D}]$, $\mathcal{D}$ is the dataset. The metric defined in equation 2.3 is known as the Mahalanobis distance.

The Mahalanobis metric extends the Euclidean distance by incorporating data specific information. But a good metric should be even more flexible than that. Apart for the data set, we are also given a task to perform. We want our metric to extract and discriminate only between information that is relevant to our given task. Again, in the face recognition example we are not interested at all in the

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[1] http://matlabdatamining.blogspot.com/2006/11/mahalanobis-distance.html
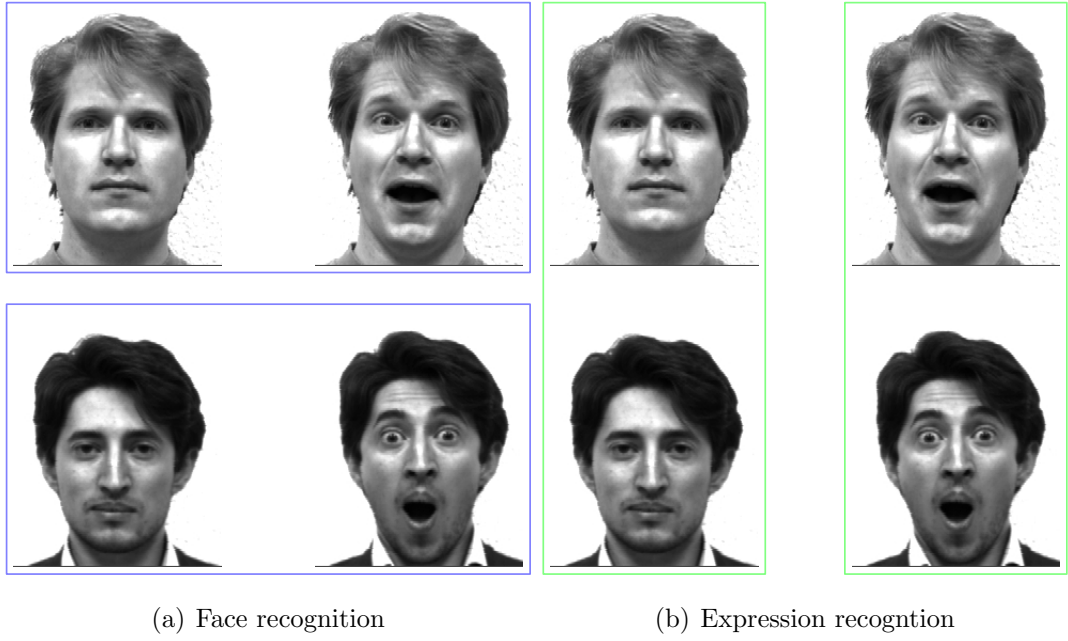
(a) Face recognition (b) Expression recogntion

Figure 2.1: On a given data set we might need to perform multiple tasks; in this example: face recognition and expression recognition. Euclidean metric is not optimal since we want different pairs of equivalence for the two tasks. A good distance metric should be task dependent. These images are an extract from ORL database.

background colour so the importance of that feature should be down-weighted. In figure 2.1, we illustrate that for a given data set it is not optimal to use a fixed metric.

We can further generalize the result from equation 2.3. Instead of setting $\mathbf{S}$ as the covariance of the data, we let $\mathbf{S}$ be a different matrix. The problem of determining a suitable matrix $\mathbf{S}^{-1}$ is called Mahalanobis (or quadratic) distance metric learning. We note that $\mathbf{S}^{-1}$ ought to be positive semi-definite $\mathbf{x}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{x} \geq 0, \forall\, \mathbf{x}$. This ensures that the norm is a real value: $d(\mathbf{x}, \mathbf{0}) \equiv \|\mathbf{x}\| = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{S}^{-1}\mathbf{x}} \in \mathbb{R}$.

There is an interesting equivalence between a Mahalanobis-like metric and a linear transformation. Using Cholesky decomposition we can write $\mathbf{S}^{-1} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$, which gives $\|\mathbf{x}\| = \sqrt{(\mathbf{A}\mathbf{x})^{\mathrm{T}}(\mathbf{A}\mathbf{x})}$, where $\mathbf{A}$ represents a linear transformation of the data. Hence, the problem of finding a suitable distance metric $\mathbf{S}$ is equivalent to the problem of finding a good linear transformation $\mathbf{A}$ and then applying Euclidean distance in the projected space. We will discuss these two variants interchangeably and we will often parametrize our models in terms of $\mathbf{A}$.

2.2   Related methods

The first attempts in metric learning were simple alterations of the Euclidean metric to improve kNN performance. For example, Mitchell (1997) suggests to weight each attribute of the data points such that the cross validation error is minimized. This technique is equivalent to stretching the axis of the relevant attributes and compressing the axis of the attributes that are less informative. The approach is similar to learning a diagonal metric, equation 2.2. Another method (Moore and Lee, 1994) is to eliminate the least relevant attributes: they are given a weight of zero. Albeit useful, these techniques are restrictive as the feature scaling does not reflect how the data covary.

Hastie and Tibshirani (1996) proposed Discriminant Adaptive Nearest Neighbours (DANN), a method that constructs a local metric for each particular query. The metric is chosen such that it provides the best discrimination amongst classes in the neighbourhood of the query point. This increases the already costly testing time of kNN. However, DANN method provides a non-linear metric if we take into account all the possible resulted local linear distance metrics. Even though non-linear transformations represent a very interesting domain, we will concentrate our attention on linear metrics as they are less prone to over-fitting and easier to learn and represent.

Xing et al. (2003) proposed learning a Mahalanobis-like distance metric for clustering in a semi-supervised context. There are specified pairs of similar data points and the algorithm finds the linear projection that minimizes the distance between these pairs without collapsing the whole data set to a single point. This approach is formulated as a convex optimization problem with constraints. The solution is attained using an iterative procedure, such as Newton-Raphson. This method assumes that the data set is unimodal and leverages the assumption-free kNN. Another drawback is the training time, because the iterative process is costly for large data sets.

Relevant Component Analysis (RCA; Bar-Hillel et al., 2003; Shental et al., 2002) is another method that makes use of the labels of the classes in a semi-supervised way to provide a distance metric. More precisely, RCA uses the so-called chunklet information. A chunklet contains points from the same class, but the class label is not known; data points that belong to the same chunklet have the same label, while data points from different chunklets do not necessarily belong

to different classes. The main idea behind RCA is to find a linear transformation which "whitens" the data with respect to the averaged within-chunklet covariance matrix (Weinberger and Saul, 2009). Compared to the method of Xing et al. (2003), RCA has the advantage of presenting a closed form solution, but it has even stricter assumptions about the data: it considers that the data points in each class are normally distributed so they can be described using only second order statistics (Goldberger et al., 2004).

Large Margin Nearest Neighbour (LMNN; Weinberger and Saul, 2009) was designed especially for kNN classification. And it can be regarded as the support vector machine (SVM) counter part of metric learning techniques. LMNN also separates the classes by a certain margin and it inherits convexity property. Also it falls into the category of semi-definite programming. The authors also introduces various approaches to speed up the computations and to learn local metrics.

Metric learning is an ongoing research area in machine learning with an abundance of different techniques. For a good report that covers many related methods, the reader is referred to (Yang and Jin, 2006).

Chapter 3

Neighbourhood component analysis

Neighbourhood Component Analysis (NCA; Goldberger et al., 2004) learns a Mahalanobis metric that improves the performance of k nearest neighbours (kNN). From a practical point of view, NCA is regarded as a desirable additional step before doing classification with kNN. It improves the classification accuracy and it is also provides good low-dimensional representation of the data.

3.1 General presentation

Because the goal is to enhance the kNN performance, the first idea Goldberger et al. (2004) had was to maximize the leave one out cross validation performance with respect to a linear projection $\mathbf{A}$. The procedure can be described as follows: apply the linear transformation $\mathbf{A}$ to the whole data set, then take each point $\mathbf{A}\mathbf{x}_i$ and classify it using kNN on the transformed data set $\{\mathbf{A}\mathbf{x}_j\}_{j=1}^N$. The matrix $\mathbf{A}$ that achieves the highest number of correct classifications will be used for testing. Unfortunately, any objective function based on the k nearest neighbours is piecewise constant and discontinuous and, hence, hard to optimize. The reason is that there does not exist an exact correlation between $\mathbf{A}$ and the neighbours of a given point: a small perturbation in $\mathbf{A}$ might cause strong changes or, conversely, it might leave the neighbours unchanged.

The authors' solution lies in the concept of *stochastic* nearest neighbours. Remember that in the classical scenario, a query point gets the label of the closest point. In the stochastic nearest neighbour case, the query point inherits the label of a neighbour with a probability that is inverse proportional with the distance. The stochastic function is reminiscent of the softmax activation used

for neural networks or the generalized logistic function. So p_{ij} is the probability that the point $\mathbf{x}_j$ is selected as the nearest neighbour of the point $\mathbf{x}_i$ and it is given by:

$$p_{ij} = \frac{\exp(-d_{ij}^2)}{\sum_{\substack{k=1 \\ k \neq i}}^{N} \exp(-d_{ik}^2)}, \tag{3.1}$$

where $d_{ij} = d(\mathbf{A}\mathbf{x}_i; \mathbf{A}\mathbf{x}_j) = (\mathbf{x}_i - \mathbf{x}_j)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}(\mathbf{x}_i - \mathbf{x}_j)$; also we set $p_{ii} = 0$: point $\mathbf{x}_i$ cannot pick itself as the nearest neighbour. Now we can construct a continuous objective function using the stochastic assignments p_{ij} which are differentiable with respect to $\mathbf{A}$.

A suitable quantity to maximize is the probability of each point of getting correctly classified. A point $\mathbf{x}_i$ is correctly classified when it is selected by a point $\mathbf{x}_j$ that has the same label as $\mathbf{x}_i$:

$$p_i = \sum_{j \in c_i} p_{ij}. \tag{3.2}$$

The objective function considers each point in the data set and incorporates their probability of belonging to the true class:

$$
\begin{aligned}
f(\mathbf{A}) &= \sum_{i=1}^{N} p_i \\
&= \sum_{i=1}^{N} \sum_{j \in c_i} \frac{\exp(-d_{ij}^2)}{\sum_{k \neq i} \exp(-d_{ik}^2)}.
\end{aligned} \tag{3.3}
$$

The score given by the objective function can be interpreted as the expected number of the correctly classified points.

We maximise $f(\mathbf{A})$ using an iterative gradient based solver such as gradient ascent, conjugate gradients or delta-bar-delta, see Subsection 3.3.1. For the optimisation, we need the numerical expression of the gradient. If we differentiate with respect to $\mathbf{A}$, we obtain:

$$\frac{\partial f}{\partial \mathbf{A}} = 2\mathbf{A} \sum_{i=1}^{N} \left(p_i \sum_{k=1}^{N} p_{ik}\mathbf{x}_{ik}\mathbf{x}_{ik}^{\mathrm{T}} - \sum_{j \in c_i} p_{ij}\mathbf{x}_{ij}\mathbf{x}_{ij}^{\mathrm{T}} \right), \tag{3.4}$$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$. The interested reader can find the derivation of the gradient in the appendix.

It is useful to note that NCA's objective function is not convex. So care must be taken to avoid poor local optima. The choice of initialization and of the

optimization method affect the quality of the final solution. Section 3.3 discusses these and other practical issues. On the other hand, the non-convexity property has its advantage. It allows NCA to get good results on more complicated data sets that are not non convex.

Another important aspect is the computational cost of the method. For each point, we need to compute all the pairwise distances d_{ij} which has the cost of $\mathcal{O}(dN^2)$. But prior to that, we have to compute the point projections $\{\mathbf{A}\mathbf{x}_i\}_{i=1}^N$; this operation is done in $\mathcal{O}(dDN)$ flops. So the total cost of evaluating the objective function $f(\mathbf{A})$ is $\mathcal{O}(dDN + dN^2)$. The cost of the gradient given in the original paper, equation 3.4, scales in D^2N^2. We can improve this by "pushing" the matrix $\mathbf{A}$ into the sum:

This has a cost in $\mathcal{O}(dDN^2)$. However, automatic differentiation (AD) lets us calculate the gradient in the same time as the original objective function (**?**). AD uses the chain rule to derivative the objective function; the gradient will consist of elementary functions. Instead of evaluating the complicated analytical gradient, we work on a series of cheap operations. This method is very useful in practice and there exist packages for it in most programming languages. In our implementation we used the analytical differentiation, because we were not aware of AD.

Chapter 4 treats NCA's computational drawback; there are presented different ideas of speeding up the computations.

The general scenario in which NCA is applied can be described by the training and testing steps: describe algorithm.

3.2 Class-conditional kernel density estimation interpretation

In this section we will present NCA into a class-conditional kernel density estimation framework. This interpretation will allow us to understand what are the assumptions behind NCA. Moreover, this also offers the possibility of altering the model in a suitable way that is efficient for computations. We will see this in the sections 4.4 and 4.5. Similar ideas were previously presented by and , but the following were derived independently and they offer different insights. The following interpretation was inspired by the probabilistic kNN presented by Barber
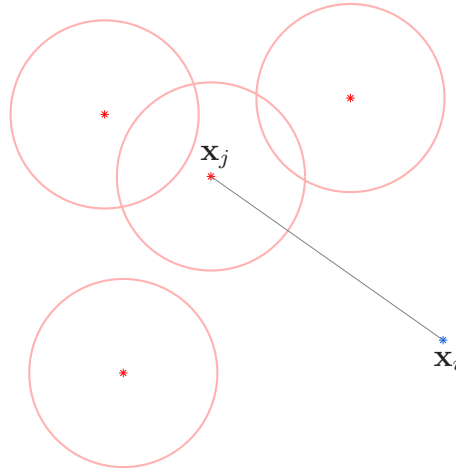
(a) Illustration of the class as a mixture of Gaussians.

Figure 3.1: Formulating NCA as a class-conditional kernel density estimation framework.

(2011).

We start with the basic assumption that each class can be modelled by a mixture of Gaussians. For each of the N_c data points in class c we consider a Gaussian "bump" centred around it. From a generative perspective, we can view that each point $\mathbf{x}_j$ can generate a point $\mathbf{x}_i$ with a probability given by an isotropic normal distribution with variance σ^2:

$$p(\mathbf{x}_i|\mathbf{x}_j) = \mathcal{N}(\mathbf{x}_i|\mathbf{x}_j, \sigma^2 \mathrm{I}_D) \tag{3.5}$$

$$= \frac{1}{(2\pi)^{D/2}} \exp\left\{-\frac{1}{2\sigma^2}(\mathbf{x}_i - \mathbf{x}_j)^{\mathrm{T}}(\mathbf{x}_i - \mathbf{x}_j)\right\}. \tag{3.6}$$

By changing the position of the points through a linear transformation $\mathbf{A}$, the probability changes as follows:

$$p(\mathbf{A}\mathbf{x}_i|\mathbf{A}\mathbf{x}_j) \propto \exp\left\{-\frac{1}{2\sigma^2}(\mathbf{x}_i - \mathbf{x}_j)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}(\mathbf{x}_i - \mathbf{x}_j)\right\}. \tag{3.7}$$

We can note that this is similar to the p_{ij} from NCA. Both $p(\mathbf{A}\mathbf{x}_i|\mathbf{A}\mathbf{x}_j)$ and p_{ij} are directly proportional with the same quantity.

Using the mixture of Gaussians assumption, we have that the probability of a point of being generated by class c is equal to the sum of all Gaussians in class

c:

$$p(\mathbf{x}_i|c) = \frac{1}{N_c} \sum_{\mathbf{x}_j \in c} p(\mathbf{x}_i|\mathbf{x}_j) \tag{3.8}$$

$$= \frac{1}{N_c} \sum_{\mathbf{x}_j \in c} \mathcal{N}(\mathbf{x}_i|\mathbf{x}_j, \mathbf{I}_D). \tag{3.9}$$

However, we are interested on the inverse probability, given a point $\mathbf{x}_i$ what is the probability of $\mathbf{x}_i$ belonging to class c. We can obtain an expression for $p(c|\mathbf{x}_i)$ using Bayes' theorem:

$$p(c|\mathbf{x}_i) = \frac{p(\mathbf{x}_i|c)p(c)}{p(c|\mathbf{x}_i)} = \frac{p(\mathbf{x}_i|c)p(c)}{\sum_c p(\mathbf{x}_i|c)p(c)}. \tag{3.10}$$

Now if further consider the classes to be equal probable (which might a reasonable assumption if we have no a priori information) we arrive at result that resembles the expression of p_i (see equation):

$$p(c|\mathbf{A}\mathbf{x}_i) = \frac{\frac{1}{N_c} \sum_{\mathbf{x}_j \in c} \exp\left\{-\frac{1}{2\sigma^2}(\mathbf{x}_i - \mathbf{x}_j)^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \mathbf{A}(\mathbf{x}_i - \mathbf{x}_j)\right\}}{\frac{1}{N_{c'}} \sum_{c'} \sum_{\mathbf{x}_k \in c'} \exp\left\{-\frac{1}{2\sigma^2}(\mathbf{x}_i - \mathbf{x}_k)^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \mathbf{A}(\mathbf{x}_i - \mathbf{x}_k)\right\}} \tag{3.11}$$

We are interested in predicting the correct class of the point $\mathbf{x}_i$. So we try to find that linear transformation $\mathbf{A}$ that maximises the class conditional probability of this point $p(c_i|\mathbf{x}_i)$ to its true class c_i.

$$f(\mathbf{A}) = \sum_i p(c_i|\mathbf{A}\mathbf{x}_i). \tag{3.12}$$

.

The gradient of the objective function is the following:

$$\frac{\partial f}{\partial \mathbf{A}} = \sum_i \left\{ \frac{\frac{\partial p(\mathbf{A}\mathbf{x}_i|c)}{\partial \mathbf{A}}p(c)}{\sum_c p(\mathbf{x}_i|c)p(c)} - \underbrace{\frac{p(\mathbf{A}\mathbf{x}_i|c)p(c)}{\sum_c p(\mathbf{A}\mathbf{x}_i|c)p(c)}}_{p(c|\mathbf{A}\mathbf{x}_i)} \frac{\sum_c \frac{\partial p(\mathbf{A}\mathbf{x}_i|c)}{\partial \mathbf{A}}p(c)}{\sum_c p(\mathbf{A}\mathbf{x}_i|c)p(c)} \right\}. \tag{3.13}$$

3.3 Practical notes

This section provides some practical advice for the questions that can be raised while implementing NCA. While NCA is not that hard to implement, there is needed certain care in order to achieve good solutions.

3.3.1 Optimization methods

The function $f(\mathbf{A})$, equation 3.3, can be maximised using any gradient based optimizer. We considered two popular approaches for our implementation: gradient ascent and conjugate gradients. These are only briefly presented here. The interested reader is pointed to Bishop (1995).

Gradient ascent

Gradient ascent is one of the simplest optimisation methods. It is an iterative algorithm that aims to find the maximum of a function by following the gradient direction at each step. The entire procedure is summarized by algorithm 3.1.

Algorithm 3.1 Gradient ascent (batch version)

Require: Data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$.

1: Get initial $\mathbf{A}_0$

2: **repeat**

3:     Update parameter: $\mathbf{A}_{t+1} \leftarrow \mathbf{A}_t + \eta \frac{\partial f(\mathbf{A}_t, \mathcal{D})}{\partial \mathbf{A}}$

4:     Update learning rate η

5:     $t \leftarrow t + 1$

6: **until** convergence

7: **return** $\mathbf{A}_t$.

There are three aspects we need to carefully consider:

1. The algorithm starts the search in the parameter space from the an initial parameter $\mathbf{A}_0$. Different values for $\mathbf{A}_0$ will give different final solutions. In the NCA context, initialization is related to finding a good initial linear transformation; this is discussed separately in subsection 3.3.2. For the moment, we can assume the values of $\mathbf{A}_0$ are randomly chosen.

2. The parameter η is called *step size* or, in neural networks related literature, it also known as *learning rate*. As name suggests, η controls how much we move the parameter in the gradient direction.

   The learning rate can be either fixed or adaptive. For the first case, choosing the correct value for η is critical. If η is set too large, the algorithm diverges. On the other hand, a small η results in slow convergence.

Usually, a variable learning rate η is preferred since it is more flexible. The "bold driver" is a heuristic for automatically modifying η during training. The idea is to gradually increase the learning rate after each iteration as long as the objective function keeps improving. Then we go back to the previous position and start decreasing the learning rate until a new increase in the function is found. This procedure is described in algorithm. The parameters indicate how much the learning rate is adjusted at each step. Usually, $\rho = 1.1$ and $\sigma = 0.5$.

Another way is to change the learning rate η using a decreasing rule proportional to the number of current iteration: $\eta = \frac{\eta_0}{t+t_0}$. Now we have two parameters instead of one. Intuitively, η_0 can be regarded as the initial learning rate and t_0 is the number of iterations after which η_0 starts to drop off. Using a learning rate of such form is especially motivated in the stochastic gradient case. If $\eta \propto 1/t$, the algorithm converges to the true minimum as t goes to infinity. However, in practice the converges is very slow; so setting the constant η_0 and t_0 is important for reaching a good solution in a short time. Leon Bottou recommends to set η_0 to be regularization value and select t_0 such that the update will result in a sensible new value for $\mathbf{A}$. Since we did not use regularization, we followed some of the tricks presented in (LeCun et al., 1998). We fixed the value for η_0 and we did an exponential search for t_0 using a cross validation set.

3. Convergence. We set a maximum number of iterations. Also we use "early stopping": we monitor the performance of the new $\mathbf{A}$ on a cross validation set. If the objective function did not increase for more than a preset number of iterations we stop and return to the previous best solution. This avoids overfitting.

Conjugate gradients

3.3.2 Initialization

Initialization is important because the function $f(\mathbf{A})$ is not convex. The quality of the final solution relies on the starting point of the optimisation algorithm. A general rule of thumb is to try multiple initial seeds and select that final $\mathbf{A}$ that gives the highest score.

Score: 48.88

(a) Initial random projection

Score: 97.75

(b) NCA projection after random initialization

Score: 94.94

(c) Initial projection using PCA

Score: 100.00

(d) NCA projection after PCA initialization

Score: 98.88

(e) Initial projection using LDA

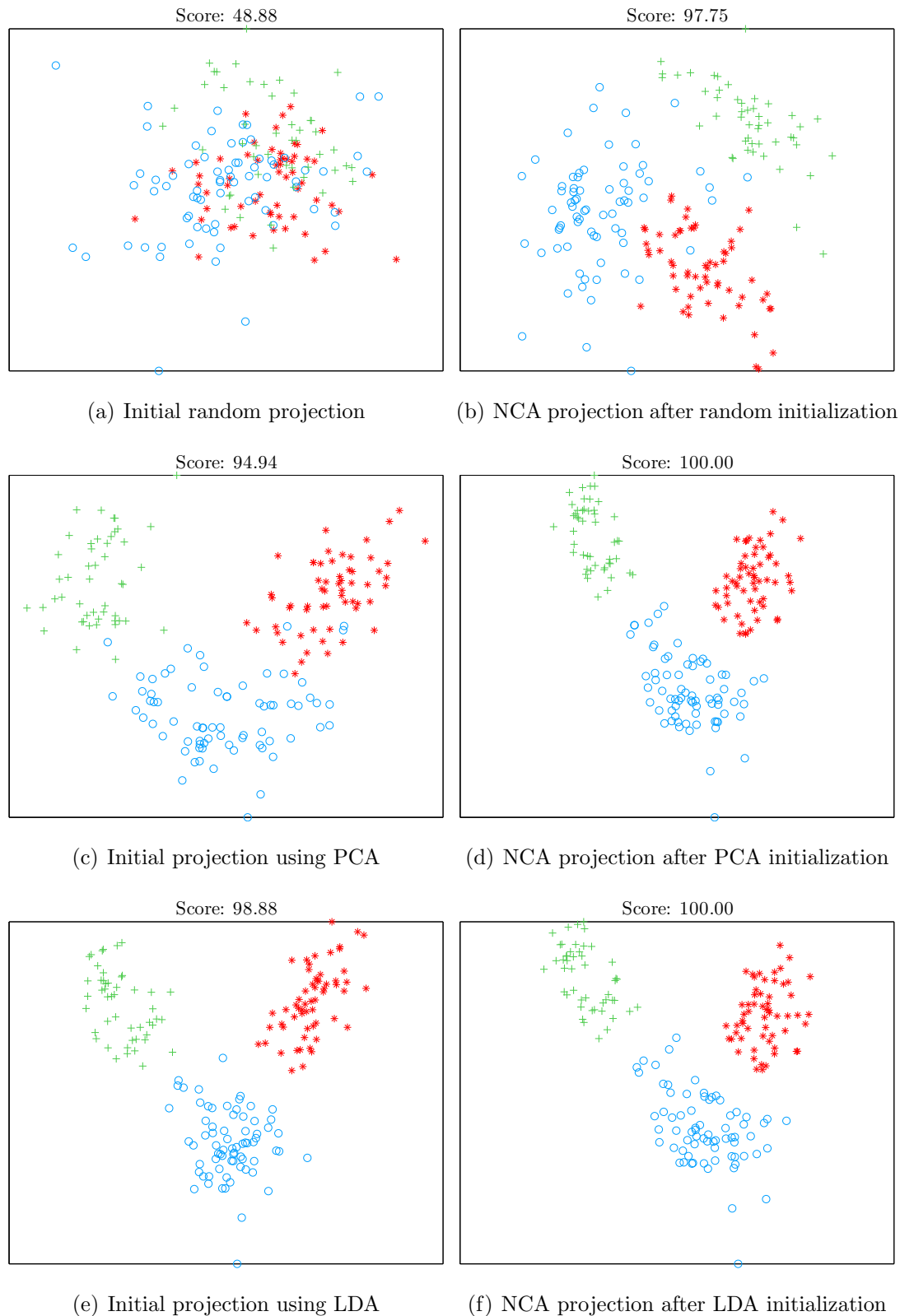Score: 100.00

(f) NCA projection after LDA initialization

Figure 3.2: Results for different initializations (random, PCA and LDA) on `wine` data set. The images on the left side represent the projections of the data set using the initial **A**. The images on the right side are data set projection with the final **A**. Above each figure there is presented the LOOCV score which is normalized to 100.

We have already mentioned random initialization in subsection 3.3.1. Beside this, we can linear transformations that are cheap to compute. Such examples include principal component analysis (PCA; Pearson, 1901), linear discriminant analysis (LDA; Fisher and Others, 1936) and relevant component analysis (RCA; Bar-Hillel et al., 2003). For completeness, we give here the equations and also include further notes:

- PCA finds an orthogonal linear transformation of the data. This is obtained by computing the eigendecomposition of the outer covariance matrix:

$$\mathbf{S} = \frac{1}{N} \sum_{i=1}^{N} (\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}}. \tag{3.14}$$

- LDA finds a linear transformation $\mathbf{A}$ by maximizing the variance between classes $\mathbf{S}_B$ relative to the amount of within-class variance $\mathbf{S}_W$:

$$\mathbf{S}_B = \frac{1}{C} \sum_{c=1}^{C} \boldsymbol{\mu}_c \boldsymbol{\mu}_c^{\mathrm{T}} \tag{3.15}$$

$$\mathbf{S}_W = \frac{1}{N} \sum_{c=1}^{C} \sum_{i \in c} (\mathbf{x}_i - \boldsymbol{\mu}_c)(\mathbf{x}_i - \boldsymbol{\mu}_c)^{\mathrm{T}}. \tag{3.16}$$

  The projection matrix $\mathbf{A}$ that achieves this maximization consists of the eigenvectors of $\mathbf{S}_W^{-1}\mathbf{S}_B$.

  Unlike PCA, LDA makes use of the class labels and this usually guarantees a better initial projection.

- RCA finds a linear transformation $\mathbf{A}$ that "whitens" the data with respect to the within-chunklet covariance matrix. Because for NCA we restrict ourselves to fully labelled data, the within-chunklet covariance is the within-class covariance $\mathbf{S}_W$, Equation 3.16. The whitening transformation is then $\mathbf{A} = \mathbf{S}_W^{-1/2}$.

If the projection is full-rank, $\mathbf{A} \in \mathbb{R}^{D \times D}$, other obvious initializations are the identity matrix $\mathbf{A} = \mathrm{I}_D$ and the Mahalanobis linear transformation $\mathbf{A} = \mathbf{S}^{-1/2}$.

If we want to learn a low-rank projection, $\mathbf{A} \in \mathbb{R}^{d \times D}, d < D$, then we can still use the eigendecomposition based methods. We construct $\mathbf{A}$ using only the top d most discriminative eigenvectors, *i.e.*, those eigenvectors that have the highest eigenvalues associated.

From our experiments, we conclude that a good initialization reflects in a good solution and a better convergence; this benefits are more evident large data sets. As advertised in , we found RCA to work the best. Figure depicts initialization effects on a small data set.

3.3.3 Numerical issues

Numerical problems can easily appear when computing the soft assignments p_i. If a point $\mathbf{x}_i$ is far away from the rest of the points, the stochastic probabilities $p_{ij}, \forall j$, are all 0 in numerical precision. Consequently, the result p_i is undetermined: $\frac{0}{0}$. To make an idea of how far $\mathbf{x}_i$ has to be for this to happen, let us consider an example in MATLAB. The answer to `exp(-d^2)` is 0 whenever `d` exceeds 30 units. This is problematic in practice, since distances larger than 30 often appear. Some common cases include data sets that contain outliers or data sets that present a large scale.

The large scale effect can be partially alleviated if we initialize $\mathbf{A}$ with small values. This is idea was used by Laurens van der Maaten in his implementation: `A = 0.01*randn(d,D)`. However, this does not guarantee to compensate the scale variation for any data set.

A more robust solution is to normalize the data, *i.e.*, centre and making it unit variance:

$$x_{ij} \leftarrow \frac{x_{ij} - \mu_j}{\sigma_j}, i = \{1, \cdots, N\}, j = \{1, \cdots, D\}. \tag{3.17}$$

In this case, we have to store the initial mean and variance of the data and, at test time, transform the test data accordingly: subtract the mean and scale it using the variance coefficients. The variance scaling can be regarded as a linear transformation. We can combine this transformation with the learnt transformation $\mathbf{A}$ and get:

$$\mathbf{A}_{\text{total}} \leftarrow \mathbf{A} \cdot \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_D \end{pmatrix}. \tag{3.18}$$

In general, data normalization avoids numerical issues for the first iterations. But during training, the scale of $\mathbf{A}$ increases and data points can be "thrown" away. We adopt a rather simplistic approach to avoid any numerical problems: replace p_i with a very small value whenever we are in the $\frac{0}{0}$ case, as in Laurens

van der Maaten implementation. In MATLAB, this is done using the following command `max(p_i,eps)`.

A more robust more rigorous way of dealing with this by multiplying both the numerator and denominator of p_i with a certain quantity $\exp(L)$:

$$p_i = \frac{\sum_{j \in c_i} \exp(L - d_{ij}^2)}{\sum_{k \neq i} \exp(L - d_{ik}^2)}, \tag{3.19}$$

where $L = \min_{k \neq i} d_{ik}^2$. This value of L ensures that at least one term in the denominator does not undeflow.

In our implementation, we preferred the previous trick because of its simplicity and speed.

3.3.4   Regularization

Although the original paper (Goldberger et al., 2004) claimed that there were no problems with overfitting, we observed that NCA objective function has the tendency to increase the scale of the linear projection $\mathbf{A}$. Butman and Goldberger (2008) pointed out that this is especially problematic for data sets whose size N is significantly smaller than the dimensionality D. In such a case, the linear transformation $\mathbf{A}$ can be chosen to project each point $\mathbf{x}_i$ to a pre-specified a location $\mathbf{y}_i$. The values of $\mathbf{A}$ are obtained by solving the linear system:

$$\mathbf{A}\mathbf{x}_i = \mathbf{y}_i, \quad i = \{1, \cdots, N\}. \tag{3.20}$$

Because the system has more equations dN than unknowns dD, it has exact solutions. If we set the same positions $\mathbf{y}_i$ for points in the same class, we can virtually increase the scale of the projection to infinity and get an error-free classification. This degeneracy can be corrected with the help of regularization. We alter the objective function by subtracting a regularization term proportional with the magnitude of $\mathbf{A}$. This gives:

$$g(\mathbf{A}) = f(\mathbf{A}) - \lambda \sum_{i=1}^{d} \sum_{j=1}^{D} A_{ij}^2. \tag{3.21}$$

$$\frac{\partial g}{\partial \mathbf{A}} = \frac{\partial f}{\partial \mathbf{A}} - 2\lambda \mathbf{A}, \tag{3.22}$$

where λ is a positive constant that is tuned via cross-validation.

Another effect of a large scale linear projection $\mathbf{A}$ is the fact that only the nearest neighbour is considered for each point. This is not usually the optimal thing to do, so regularization is also used to prevent this (Singh-Miller, 2010).

In our implementation, we use "early stopping" technique which has a similar effect to regularization.

3.3.5 Doing classification

For testing, Goldberger et al. (2004) considered only kNN decision rule. Since we are optimizing a particular function $f(\mathbf{A})$, another sensible tool for classification is an NCA based function. If we are using the probabilistic interpretation, section 3.2, a query point $\mathbf{x}_i$ is labelled with most probable class c:

$$c = \text{argmax}_c \, p(c|\mathbf{x}_i). \tag{3.23}$$

This can be re-written in NCA specific notation:

$$c = \text{argmax}_c \frac{\sum_{j \in c} p_{ij}}{\sum_k p_{ik}}. \tag{3.24}$$

In our experiments, 1NN and the NCA classification rule described by equation (3.24) give very similar results if we train $\mathbf{A}$ using the un-regularized function $f(\mathbf{A})$. On the cases where we used early stopping, the NCA classification rule yielded better performances, see chapter 5. In this case, kNN with $k > 1$ should also perform better than simple 1NN.

3.3.6 Dimensionality annealing

Dimensionality annealing is an approach to learning a low-rank metric in an way that avoids local optima (Hinton and Murray, oral communication). We start with a full rank projection $\mathbf{A}$ and gradually reduce the rank by introducing regularization terms on the lines of $\mathbf{A}$. Compared with the classical regularization procedure, subsection 3.3.4, here we use a regularization term on each of the dimension d. The new objective function and its gradient are given by the following equations:

$$g(\mathbf{A}) = f(\mathbf{A}) - \sum_{i=1}^{d} \lambda_i \sum_{j=1}^{D} A_{ij}^2 \tag{3.25}$$

$$\frac{\partial g}{\partial \mathbf{A}} = \frac{\partial f}{\partial \mathbf{A}} - 2 \begin{pmatrix} \lambda_1 A_{11} & \cdots & \lambda_1 A_{1D} \\ \vdots & \ddots & \vdots \\ \lambda_d A_{d1} & \cdots & \lambda_d A_{dD} \end{pmatrix}. \tag{3.26}$$

A large value of λ_d will impose a small magnitude of $\mathbf{A}$ on dimension d. We increase λ_d slowly; this permits for the rest of the values of $\mathbf{A}$ to readjust. We clarify these steps in algorithm 3.2.

Algorithm 3.2 Dimensionality annealing

Require: Data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$, initial linear transformation $\mathbf{A} \in \mathbb{R}^{D \times D}$
  and low dimension d.
1: **for** $i = 1, \cdots, D - d$ **do**
2:   Select dimension d to anneal
3:   **for** $j = 1, \ldots, P$ **do**
4:     Increase regularization coefficient $\lambda_d \leftarrow \lambda_d + \Delta\lambda$
5:     Optimize function g: $\mathbf{A} = \mathbf{A} + \eta \frac{\partial g(\mathbf{A}, \mathcal{D}, \boldsymbol{\lambda})}{\partial \mathbf{A}}$
6:   **end for**
7: **end for**

There are several notes to be made. There are alternatives for selecting the dimension d we want to anneal, step **??** from algorithm 3.2. We can choose d to be the direction that has:

- minimum variance in our data set $\{\mathbf{A}\mathbf{x}_i\}_{i=1}^N$

- minimum variance in the projected data set $\{\mathbf{A}\mathbf{x}_i\}_{i=1}^N$

- smallest magnitude in $\mathbf{A}$.

The function optimization step can be done either by gradient ascent or conjugate gradients. It is possible to do conjugate gradients for a small number of iterations, typically 2 or 3.

After a dimension is annealed, we reach the end of the inner for loop, we can remove that particular dimension; for example set the elements on the d row equal to 0.

A further idea would be to run conjugate gradients until convergence initialized with low dimensional $\mathbf{A}$ returned by algorithm 3.2.

Chapter 4

Reducing the computational cost

As emphasized in Section 3.1, Neighbourhood Component Analysis (NCA) is a computationally expensive method. The evaluation of its objective function is quadratic in the size of the data set. Given that the optimization is done iteratively, NCA becomes prohibitively slow when applied on large data sets. There is only little previous work that uses NCA for large scaled applications. One example is Singh-Miller (2010), who parallelizes the computations across multiple computers and adopts various heuristics to prune terms of the objective function and the gradient.

This chapter aims to complete the existing work. We present a series of methods that can improve NCA's speed. Most of these ideas are new in the context of NCA. They are also quite versatile. Each method proposes a new objective function and can be regarded as an independent model on its own.

4.1 Sub-sampling

Sub-sampling is the simplest idea that can help speeding up the computations. For the training procedure we use a randomly selected sub-set $\mathcal{D}_n$ of the original data set $\mathcal{D}$:

$$\mathcal{D}_n = \{\mathbf{x}_{i_1}, \cdots, \mathbf{x}_{i_n}\} \subseteq \mathcal{D}.$$

If n is the size of the sub-set then the cost of the gradient is reduced to $\mathcal{O}(dDn^2)$. After the projection matrix $\mathbf{A}$ is learnt, we apply it to the whole data set $\{\mathbf{x}_i\}_{i=1}^N$. Then all the new data points $\{\mathbf{A}\mathbf{x}_i\}_{i=1}^N$ are used for classification. The cost of the classification is $\mathcal{O}(dN)$; it is only linear in the total number of points N.
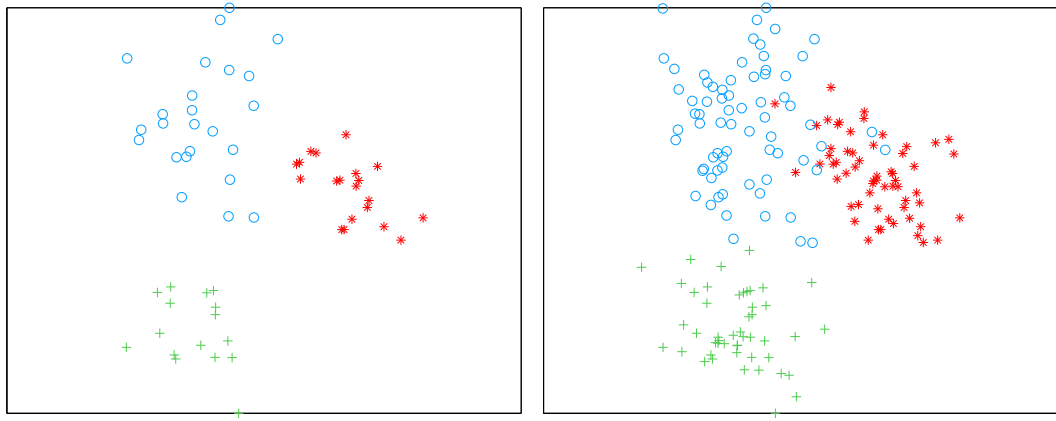
(a) Learnt projection **A** on the sub-sampled data set $\mathcal{D}_n$.

(b) The projection **A** applied to the whole data set $\mathcal{D}$.

Figure 4.1: Result of sub-sampling method on `wine`. There were used one third of the original data set for training, *i.e.*, $n = N/3$. We note that the points that belong to the sub-set $\mathcal{D}_n$ are perfectly separated. But after applying the metric to the whole data there appear different misclassification errors. The effects are even more acute if we use smaller sub-sets.

While easy to implement, this method discards a lot of information available. Also it is affected by the fact the sub-sampled data has a thinner density than the real data. The distances between the randomly selected points are larger than they are in the full data set. This causes the scale of the projection matrix **A** not to be large enough, Figure 4.1.

4.2 Mini-batches

The next obvious idea is to use sub-sets in an iterative manner, similarly to the stochastic gradient descent method: split the data into mini-batches and train on them successively. Again the cost for one evaluation of the gradient will be $\mathcal{O}(dDn^2)$ if the mini-batch consists of n points.

There are different possibilities for splitting the data-set:

1. Random selection. In this case the points are assigned randomly to each mini-batch and after one pass through the whole data set another random allocation is done. As in section 4.1, this suffers from the thin distribution problem. In order to alleviate this and achieve convergence, large-sized mini-batches should be used (similar to Laurens van der Maaten's imple-

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Algorithm 4.1 Training algorithm using mini-batches formed by clustering

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Require: Data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$ and initial linear transformation $\mathbf{A}$.

1: **repeat**

2:     Project each data point using $\mathbf{A}$: $\mathcal{D}_{\mathbf{A}} = \{\mathbf{A}\mathbf{x}_1, \cdots, \mathbf{A}\mathbf{x}_N\}$.

3:     Use either algorithm 4.2 or 4.3 on $\mathcal{D}_{\mathbf{A}}$ to split $\mathcal{D}$ into K mini-batches $\mathcal{M}_1, \cdots, \mathcal{M}_K$.

4:     **for all** $\mathcal{M}_i$ **do**

5:         Update parameter: $\mathbf{A} \leftarrow \mathbf{A} + \eta \frac{\partial f(\mathbf{A}, \mathcal{M}_i)}{\partial \mathbf{A}}$.

6:         Update learning rate η.

7:     **end for**

8: **until** convergence.

---

mentation). The algorithm is similar to Algorithm 4.1, but lines 2 and 3 will be replaced with a simple random selection.

2. Clustering. Constructing mini-batches by clustering ensures that the point density in each mini-batch is conserved. In order to maintain a low computational cost, we consider cheap clustering methods, *e.g.*, farthest point clustering (FPC; Gonzalez, 1985) and recursive projection clustering (RPC; Chalupka, 2011).

FPC gradually selects cluster centres until it reaches the desired number of clusters K. The point which is the farthest away from all the current centres is selected as new centre. The precise algorithm is presented below.

Algorithm 4.2 Farthest point clustering

Require: Data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$ and K number of clusters.

1: Randomly pick a point that will be the first centre $\mathbf{c}_1$.

2: Allocate all the points in the first cluster $\mathcal{M}_1 \leftarrow \mathcal{D}$.

3: **for** $i = 1$ to K **do**

4:     Select the i-th cluster centre $\mathbf{c}_i$ as the point that is farthest away from any cluster centre $\mathbf{c}_1, \cdots, \mathbf{c}_{i-1}$.

5:     Move to the cluster $\mathcal{M}_i$ those points that are closer to its centre than to any other cluster centre: $\mathcal{M}_i = \{\mathbf{x} \in \mathcal{D} \mid d(\mathbf{x}; \mathbf{c}_i) < d(\mathbf{x}; \mathbf{c}_j), \forall j \neq i\}$

6: **end for**

The computational cost of this method is $\mathcal{O}(NK)$. However, we do not have any control on the number of points in each cluster, so we might end up

with very unbalanced clusters. A very uneven split has a couple of obvious drawbacks: too large mini-batches will maintain high cost, while on too small clusters there is not too much to learn.

An alternative is RPC which was especially designed to mitigate this problem. It constructs the clusters similarly to how the k-d trees are build, Subsection 4.4.1. However instead of splitting the data set across axis aligned direction it chooses the splitting directions randomly, Algorithm 4.3. So, because it uses the median value it will result in similar sized clusters and we can easily control the dimension of each cluster.

Algorithm 4.3 Recursive projection clustering

Require: Data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$ and n size of clusters.

1: **if** $N < n$ **then**

2:     New cluster: $i \leftarrow i + 1$.

3:     **return** current points as a cluster: $\mathcal{M}_i \leftarrow \mathcal{D}$.

4: **else**

5:     Randomly select two points $\mathbf{x}_j$ and $\mathbf{x}_k$ from $\mathcal{D}$.

6:     Project all data points onto the line defined by $\mathbf{x}_j$ and $\mathbf{x}_k$. (Give equation?)

7:     Select the median value $\tilde{\mathbf{x}}$ from the projected points.

8:     Recurs on the data points above and below $\tilde{\mathbf{x}}$: $\text{RPC}(\mathcal{D}_{>\tilde{\mathbf{x}}})$ and $\text{RPC}(\mathcal{D}_{\leq\tilde{\mathbf{x}}})$.

9: **end if**

Note that we are re-clustering in the transformed space after one sweep through the whole data set. There are also other alternatives. For example, we could cluster in the original space. This can be done either only once or periodically. However the proposed variant has the advantage of a good behaviour for a low-rank projection matrix $\mathbf{A}$. Not only that is cheaper, but the clusters resulted in low dimensions by using RPC are closer to the real clusters then applying the same method in a high dimensional space.

4.3 Stochastic learning

The following technique is theoretically justified by stochastic approximation arguments. The main idea is to get an unbiased estimator of the gradient by looking only at a few points and how they relate to the entire data set.

More precisely, in the classical learning setting, we update our parameter $\mathbf{A}$ after we have considered each point $\mathbf{x}_i$ in the data set. In the stochastic learning procedure, we update $\mathbf{A}$ more frequently by considering only n randomly selected points. As in the previous case, we still need to compute the soft assignments $\{p_i\}_{i=1}^n$ using *all* the N points. To stress this further, this solution differs from the mini-batch approach. For the previous method, the contributions $\{p_i\}_{i=1}^n$ are calculated only between the n points that belong to the mini-batch.

The objective function that we need to optimize at each iteration and its gradient are given by the next equations:

$$f_{\text{sNCA}}(\mathbf{A}) = \sum_{l=1}^n p_{i_l} \tag{4.1}$$

$$\frac{\partial \hat{f}}{\partial \mathbf{A}} = \frac{\partial f_{\text{sNCA}}}{\partial \mathbf{A}} = \sum_{l=1}^n \frac{\partial p_{l_i}}{\partial \mathbf{A}}, \tag{4.2}$$

$$\text{with } p_i = \sum_{\substack{j=1 \\ j \in c_i}}^N p_{ij}. \tag{4.3}$$

This means that the theoretical cost of the stochastic learning method scales with nN.

Algorithm 4.4 Stochastic learning for NCA (sNCA)

Require: Data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$, n number of points to consider for the gradient estimation, $\mathbf{A}$ initial linear transformation.

1: **repeat**
2:     Split data $\mathcal{D}$ into groups $\mathcal{M}_i$ of size n.
3:     **for all** $\mathcal{M}_i$ **do**
4:         Update parameter using gradient given by Equation 4.2:
5:         $\mathbf{A} \leftarrow \mathbf{A} + \eta \frac{\partial f_{\text{sNCA}}(\mathbf{A}, \mathcal{M}_i)}{\partial \mathbf{A}}$.
6:         Update learning rate η.
7:     **end for**
8: **until** convergence.

This method comes with an additional facility. It can be used for on-line learning. Given a new point $\mathbf{x}_{N+1}$ we update $\mathbf{A}$ using the derivative $\frac{\partial p_{N+1}}{\partial \mathbf{A}}$.
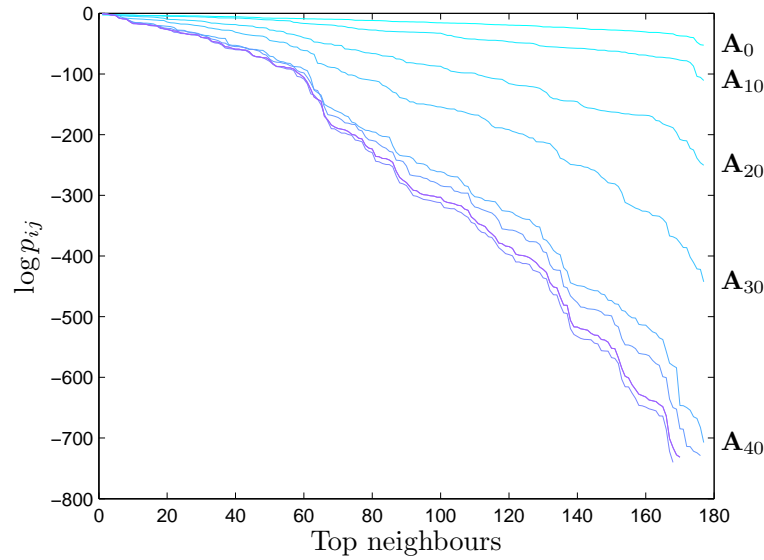
Figure 4.2: Evolution of the stochastic assignments p_{ij} during training for a given point $\mathbf{x}_i$.

4.4 Approximate computations

A straightforward way of speeding up the computations was previously mentioned in the original paper (Goldberger et al., 2004) and in some NCA related work (Weinberger and Tesauro, 2007; Singh-Miller, 2010). The observations involve pruning small terms in the original objective function. We then use an approximated objective function and its corresponding gradient for the optimization process.

The motivation lies in the fact that the contributions p_{ij} decay very quickly with distance:

$$p_{ij} \propto \exp\{-d(\mathbf{A}\mathbf{x}_i; \mathbf{A}\mathbf{x}_j)^2\}.$$

The evolution of the contributions during the training period is depicted in Figure 4.2. We notice that most of the values p_{ij} are insignificant compared to the largest contribution. This suggests that we might be able to preserve the accuracy of our estimations even if we discard a large part of the neighbours.

So, Weinberger and Tesauro (2007) choose to use only the top $m = 1000$ neighbours for each point $\mathbf{x}_i$. Also they disregard those points that are farther away than $d_{\max} = 34$ units from the query point: $p_{ij} = 0, \forall \mathbf{x}_j$ such that $d(\mathbf{A}\mathbf{x}_i; \mathbf{A}\mathbf{x}_j) > d_{\max}$. While useful in practical situations, these suggestions lack of a principled description: how can we optimally choose m and $d_{\max}$ in a general

setting? We would also like to be able to estimate the error introduced by the approximations.

We correct those drawbacks by making use of the KDE formulation of NCA (see Section 3.2) and adapting existing ideas for fast KDE (Deng and Moore, 1995; Gray and Moore, 2003) to our particular application. We will use a class of accelerated methods that are based on data partitioning structures (*e.g.*, k-d trees, ball trees). As we shall shortly see, these provide us with means to quickly find only the neighbours $\mathbf{x}_j$ that give significant values p_{ij} for any query point $\mathbf{x}_i$.

4.4.1 k-d trees

The k dimensional tree structure (k-d tree; Bentley, 1975) organises the data in a binary tree using axis-aligned splitting planes. The k-d tree has the property to place close in the tree those points that live nearby in the original geometrical space. This makes such structures efficient mechanisms for nearest neighbour searches (Friedman et al., 1977) or range searches (Moore, 1991).

There are different flavours of k-d trees. We choose for our application a variant of k-d tree that uses bounding boxes to describe the position of the points. Intuitively, we can imagine each node of the tree as a bounding hyper-rectangle in the D dimensional space of our data. The root node will represent the whole data set and it can be viewed as an hyper-rectangle that contains all the data points, see Figure 4.3(a). In the two-dimensional presented exampled, the points are enclosed by rectangles. From Figure 4.3(a) to 4.3(d), there are presented the existing bounding boxes at different levels of the binary tree. To understand how these are obtained, we discuss the k-d tree construction.

The building of the tree starts from the root node and is done recursively. At each node we select which of the points from the current node will be allocated to each of the two children. Because these are also described by hyper-rectangles, we just have to select a splitting plane. Then the two successors will consist of the points from the two sides of the hyper-plane.

A splitting hyper-plane can be fully defined by two parameters: a direction $\vec{d}$ on which the plane is perpendicular and a point P that is in the plane. Given that the splits are axis aligned, there are D possible directions $\vec{d}$. We can either choose this randomly or we can use each of the directions from 1 to D in a successive manner. A more common approach is to choose $\vec{d}$ to be the dimension

(a) Root node

(b) First level
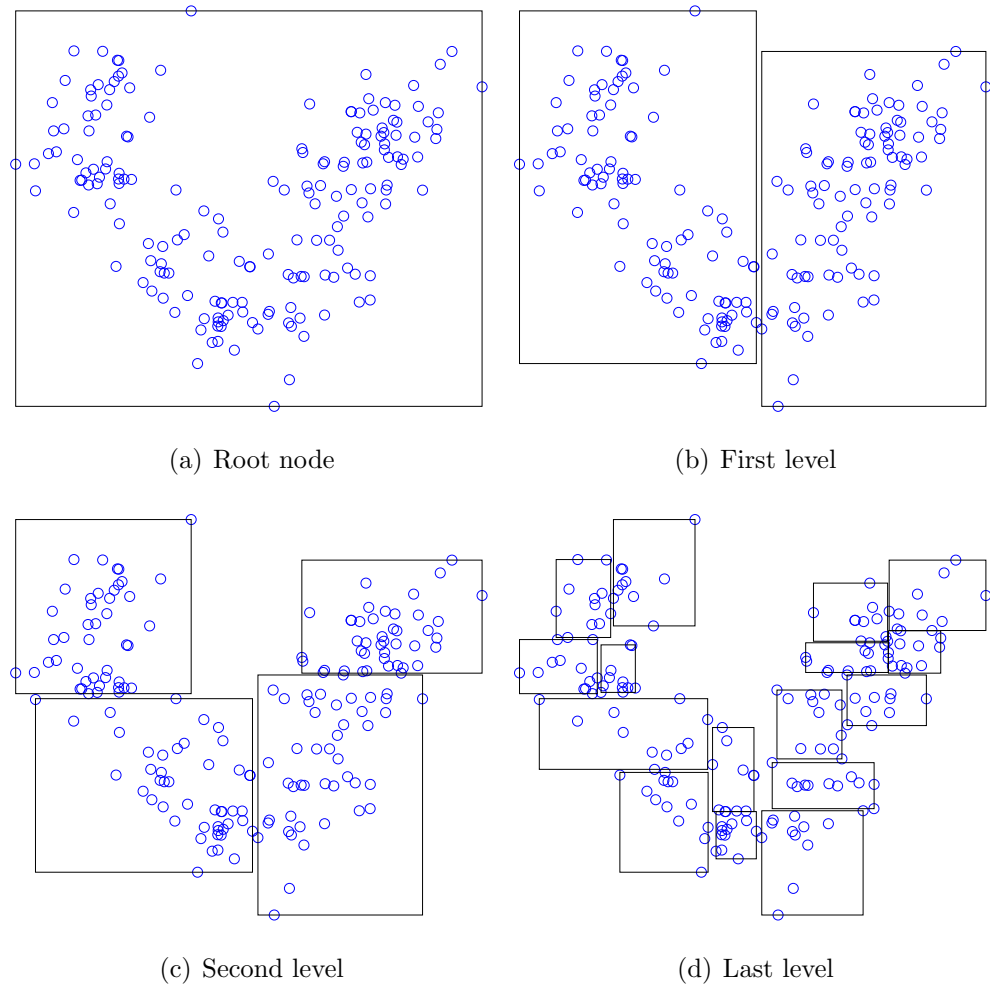
(c) Second level

(d) Last level

Figure 4.3: Illustration of the k-d tree with bounding boxes at different levels of depths. This figure also outlines the building phases of the tree.

that presents the largest variance:

$$\vec{d} \leftarrow \text{argmax}_d(\max_i x_{id} - \min_i x_{id}). \tag{4.4}$$

This results in a better clustering of the points and the shape of the bounding boxes will be closer to the shape of a square. Otherwise it might happen that points situated in the same node can still be further away.

Regarding the splitting value P, a usual choice is the median value $\tilde{x}_d$ on the previously selected direction $\vec{d}$. This choice of P guarantees a balanced tree which offers several advantages. We can allocate static memory for the entire data structure. This is faster to access than dynamical allocation. Also a balanced tree has a better worst case complexity than an unbalanced one. Other useful implementation tricks that can be applied to balanced k-d trees are suggested by Lang (2009).

After the splitting plane is chosen, the left child will contain the points that are on the left of the hyper-plane:

$$\mathcal{D}_{\leq \tilde{x}_d} = \{\mathbf{x} \in \mathcal{D}_{\mathbf{x}_i} | x_d \leq \tilde{x}_d\}, \tag{4.5}$$

where $\mathcal{D}_{\mathbf{x}_i}$ denotes the data points bounded by the current node $\mathbf{x}_i$. Similarly, the right child will contain the points that are placed on the right of the hyper-plane:

$$\mathcal{D}_{> \tilde{x}_d} = \{\mathbf{x} \in \mathcal{D}_{\mathbf{x}_i} | x_d > \tilde{x}_d\}. \tag{4.6}$$

This process is repeated until the number of points bounded by the current node goes below a threshold m. These nodes are the leaves of the tree and they store the effective points. The other non-leaf nodes store information regarding the bounding box and the splitting plane. Note that a hyper-rectangle is completely defined by only two D-dimensional points, one for the "top-right" corner and the other for the "bottom-left" corner.

The most used operation on a k-d tree is the nearest neighbour (NN) search. While we will not apply pure NN for the next method, we will use similar concepts. However, we can do NN retrieval with k-d trees after we applied NCA, as suggested by Goldberger et al. (2004). The search in the k-d tree is done in a depth-first search manner: start from the root and traverse the whole tree by selecting the closest node to the query point. In the leaf, we find the nearest neighbour from the m points and store it and the corresponding distance $d_{\min}$. Then we recurs up the tree and look at the farther node. If this is situated at

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Algorithm 4.5 k-d tree building algorithm

Require: Data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$, i position in tree and m number of points in leaves.

1: **if** $N < m$ **then**
2:    Mark node i as leaf: `splitting_direction(i)=-1`.
3:    Add points to leaf: `points(i)` $\leftarrow \mathcal{D}$.
4:    **return**
5: **end if**
6: Choose direction $\vec{d}$ using Equation 4.4: `splitting_direction(i)=d`.
7: Find the median value $\tilde{x}_d$ on the direction $\vec{d}$: `splitting_value(i)=`$\tilde{x}_d$.
8: Determine the subsets of points that are separated by the resulting splitting plane: $\mathcal{D}_{\leq \tilde{\mathbf{x}}}$ and $\mathcal{D}_{> \tilde{\mathbf{x}}}$, see Equations 4.5 and 4.6.
9: Build left child `build_kdtree(`$\mathcal{D}_{\leq \tilde{\mathbf{x}}}$`,2*i)`.
10: Build right child `build_kdtree(`$\mathcal{D}_{> \tilde{\mathbf{x}}}$`,2*i+1)`.

---

a minimum distance that is smaller than $d_{\min}$, we have to investigate also that node. In the other case, we can ignore the node and all the points it contains. Usually, a large fraction of the points can be omitted, especially when the data has structure. It is important to stress that the performance of k-d trees quickly degrades with the number of dimensions the data lives.

4.4.2 Approximate kernel density estimation

The following ideas are mostly inspired by previous work on fast kernel density estimators with k-d trees (Gray and Moore, 2001, 2003; Alexander Gray, 2003) and fast Gaussian Processes regression with k-d trees (Shen et al., 2006).

In kernel density estimation (KDE), we are given a data set $\mathcal{D}$ and the goal is to compute the probability denisty at a given point using a sum of the contributions from all the points:

$$p(\mathbf{x}_i) = \frac{1}{N} \sum_{j=1}^{N} k(\mathbf{x}_i | \mathbf{x}_j). \tag{4.7}$$

A common class of KDE problems are the N-body problems (Gray and Moore, 2001). This imposes to estimate $p(\mathbf{x}_i)$ for each data point $\{\mathbf{x}_i\}_{i=1}^{N}$ in the data set $\mathcal{D}$. Note that this operation is quadratic in the number of points, as it is the case of NCA.

If the number of samples N in $\mathcal{D}$ is sufficiently large, we expect to accurately approximate $p(\mathbf{x}_i)$ by using only nearby neighbours of $\mathbf{x}_i$.

To illustrate how this works, let us assume we are given a query point $\mathbf{x}_i$ and a group of points G. We try to reduce computations by replacing each individual contribution $k(\mathbf{x}_i|\mathbf{x}_j), \mathbf{x}_j \in G$, with a fixed quantity $k(\mathbf{x}_i|\mathbf{x}_g)$. The value of $k(\mathbf{x}_i|\mathbf{x}_g)$ is group specific and since it is used for all points in G, it is chosen such that it does not introduce a large error. A reasonable value for $k(\mathbf{x}_i|\mathbf{x}_g)$ is obtained by approximating each point $\mathbf{x}_i$ with a fixed $\mathbf{x}_g$, for example, the mean of the points in G. Then we compute the kernel value $k(\mathbf{x}_i|\mathbf{x}_g)$ using the estimated $\mathbf{x}_g$. A second possibility is to directly approximate $k(\mathbf{x}_i|\mathbf{x}_g)$. For example:

$$k(\mathbf{x}_i|\mathbf{x}_g) = \frac{1}{2}\left(\min_j k(\mathbf{x}_i|\mathbf{x}_j) + \max_j k(\mathbf{x}_i|\mathbf{x}_j)\right). \tag{4.8}$$

We prefer the this option, because it does not introduce any further computational expense. Both the minimum and the maximum contributions are previously calculated to decide whether to prune or not. Also it does not need storing any additional statistic, such as the mean.

We see that the error introduced by each approximation is bounded by the following quantity:

$$\epsilon_{\max} \leq \frac{1}{2}\left(\max_j k(\mathbf{x}_i|\mathbf{x}_j) - \min_j k(\mathbf{x}_i|\mathbf{x}_j)\right). \tag{4.9}$$

This can be controlled to be small if we approximate only for those groups that are far away from the query point or when the variation of the kernel value is small within the group. It is better still to consider the error relative to the total quantity we want to estimate. Of course we do not know the total sum we want to estimate in advance, but we can use a lower bound $p_{\text{SoFar}}(\mathbf{x}_i) + \max k(\mathbf{x}_i|\mathbf{x}_j)$. Hence, a possible cut-off rule is:

Note that in this case it is important the order in which we accumulate. A large $p_{\text{SoFar}}(\mathbf{x}_i)$ in the early stage will allow more compuational savings.

We use k-d trees to form groups of points G that will be described as hyperrectangles. To compute the probability density function $p(\mathbf{x}_i)$, we start at the root, the largest group, and traverse the tree going through the nearest node each time until we reach the leaf. This guarantees that we add large contributions at the begining. Then we recurs up the tree and visit other nodes only if necessary, when the cut-off condition is not satisfied.

4.4.3 Approximate KDE for NCA

We recall that NCA was formulated as a class-conditional kernel density estimation problem, Section 3.2. By combining ideas from the previous two subsections, we can develop an NCA specific approximation algorithm.

There are some differences from the classical KDE approximation. In this case, we deal with class-conditional probabilities $p(\mathbf{Ax}_i|c), \forall c$. So each class c needs to be treated separately: we build a k-d tree with the projected data points $\{\mathbf{Ax}_j\}_{j \in c}$ and calculate the estimated probability $\hat{p}(\mathbf{Ax}_i|c)$ for each class. Another distictintion is that for NCA our final interest are the objective function and its gradient. We can easily obtain an approximated version of the objective function by replacing $p(\mathbf{Ax}_i|c)$ with the approximated $\hat{p}(\mathbf{Ax}_i|c)$ in Equation 3.12. To obtain the gradient of this new objective function we can use Equation 3.13. The derivative of $\frac{\partial}{\partial \mathbf{A}} \hat{p}(\mathbf{Ax}|c)$ will be different only for those groups where we do approximations. So, for such a group we obtain:

$$\frac{\partial}{\partial \mathbf{A}} \sum_{j \in G} k(\mathbf{Ax}|\mathbf{Ax}_j) \approx \frac{\partial}{\partial \mathbf{A}} \frac{1}{2} \left\{ \min_{j \in G} k(\mathbf{Ax}|\mathbf{Ax}_j) + \max_{j \in G} k(\mathbf{Ax}|\mathbf{Ax}_j) \right\}$$

$$= \frac{1}{2} \left\{ \frac{\partial}{\partial \mathbf{A}} k(\mathbf{Ax}|\mathbf{Ax}_c) + \frac{\partial}{\partial \mathbf{A}} k(\mathbf{Ax}|\mathbf{Ax}_f) \right\}, \qquad (4.10)$$

where $\mathbf{Ax}_c$ denotes the closest point in G to the query point $\mathbf{Ax}$ and $\mathbf{Ax}_f$ is the farthest point in G to $\mathbf{Ax}$. Here we made use of the fact the kernel function is a monotonic function of the distance. This means that the closest point gives the maximum contribution, while the farthest point the minimum.

4.5 Exact computations

Exact methods are the counterpart of approximate methods. We can have both efficient and exact computations just by modifying the NCA model. Again, the idea is motivated by the rapid decay of the exponential function. Instead of operating on very small values, we will make them exactly zero. This is achieved by replacing the squared exponential kernel with a compact support function. So, the points that lie outside the support of the kernel are ignored and just a fraction of the total number of points is used for computing the contributions p_{ij}. Further gains in speed are obtained if the search for those points is done with k-d trees (the range search algorithm is suitable for this task; Moore, 1991).

Algorithm 4.6 Approximate NCA objective function and gradient computation

Require: Projection matrix $\mathbf{A}$, data set $\mathcal{D} = \{\mathbf{x}_1, \cdots, \mathbf{x}_N\}$ and error ϵ.

1: **for all** classes c **do**

2:     Build k-d tree for the points in class c.

3: **end for**

4: **for all** data points $\mathbf{x}_i$ **do**

5:     **for all** classes c **do**

6:         Compute estimated probability $\hat{p}(\mathbf{A}\mathbf{x}_i|c)$ and the corresponding deriva-
        tives $\frac{\partial}{\partial \mathbf{A}}\hat{p}(\mathbf{A}\mathbf{x}_i|c)$ using approximated KDE Algorithm:

7:     **end for**

8:     Compute soft probability $\hat{p}_i \equiv \hat{p}(c|\mathbf{A}\mathbf{x}_i) = \frac{\hat{p}(\mathbf{A}\mathbf{x}_i|c_i)}{\sum_c \hat{p}(\mathbf{A}\mathbf{x}_i|c)}$.

9:     Compute gradient $\frac{\partial}{\partial \mathbf{A}}\hat{p}_i$ using Equation 3.13.

10:    Update function value and gradient value.

11: **end for**

The choice of the compact support kernel is restricted by a single requirement: differentiability. We will use the simplest polynomial function that has this property. This is given by the following expression:

$$k_{\mathrm{CS}}(u) = \begin{cases} c\,(a^2 - u^2)^2 & \text{if } u \in [-a; +a] \\ 0 & \text{otherwise,} \end{cases} \tag{4.11}$$

where c is a constant that controls the height of the kernel and a is a constant that controls the width of the kernel. In the given context, the kernel will be a function of the distance between two points: $k_{\mathrm{CS}}(u) = k_{\mathrm{CS}}(d_{ij})$, where $d_{ij} = d(\mathbf{A}\mathbf{x}_i; \mathbf{A}\mathbf{x}_j)$. Note that the constant a can be absorbed by the linear projection $\mathbf{A}$. This means that the scale of the learnt metric will compensate for the kernel's width. Also the value for c is not important: from Equation 4.13 we see that this reduces. For convenience, we set both $a = 1$ and $c = 1$. So, we obtain the following simplified version of the kernel:

$$k_{\mathrm{CS}}(d_{ij}) = (1 - d_{ij}^2)^2\, \mathrm{I}(|d_{ij}| \leq 1), \tag{4.12}$$

where $\mathrm{I}(\cdot)$ denotes the indicator function: $\mathrm{I}(\cdot)$ return 1 when its argument is true and 0 when its argument is false.

Now we reiterate the steps of the NCA algorithm (presented in Section 3.1), and replace $\exp(\cdot)$ with $k_{\mathrm{CS}}(\cdot)$. We obtain the following new stochastic neighbour

assignments:

$$q_{ij} = \frac{k_{\mathrm{CS}}(d_{ij})}{\sum_{k \neq i} k_{\mathrm{CS}}(d_{ik})}. \tag{4.13}$$

These can be compared to the classical soft assignments given by Equation 3.1. Next we do not need to change the general form of the objective function:

$$f_{\mathrm{CS}}(\mathbf{A}) = \sum_i \sum_{j \in c_i} q_{ij}. \tag{4.14}$$

In order to derive the gradient of the function f_{CS}, we start by computing the gradient of the kernel:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{A}} k_{\mathrm{CS}}(d_{ij}) &= \frac{\partial}{\partial \mathbf{A}} \left[(1 - d_{ij}^2)^2 \cdot \mathrm{I}(|d_{ij}| \leq 1) \right] \\ &= -4\mathbf{A}(1 - d_{ij}^2)\mathbf{x}_{ij}\mathbf{x}_{ij}^{\mathrm{T}} \cdot \mathrm{I}(|d_{ij}| \leq 1), \end{aligned} \tag{4.15}$$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$.

The gradient of the new objective function is:

$$\frac{\partial f_{\mathrm{CS}}}{\partial \mathbf{A}} = 4\mathbf{A} \sum_{i=1}^{N} \left(q_i \sum_{k=1}^{N} \frac{q_{ik}}{1 - d_{ik}^2} \mathbf{x}_{ik}\mathbf{x}_{ik}^{\mathrm{T}} - \sum_{j \in c_i} \frac{q_{ij}}{1 - d_{ij}^2} \mathbf{x}_{ij}\mathbf{x}_{ij}^{\mathrm{T}} \right). \tag{4.16}$$

This method can be applied in same way as classic NCA: learn a metric $\mathbf{A}$ that maximizes the objective function $f_{\mathrm{CS}}(\mathbf{A})$. Since the function is differentiable, any gradient based method is suitable for optimization and can be used on Equation 4.16.

There is one concern with the compact support version of NCA. There are situations when a point $\mathbf{x}_i$ is placed outside the support of any other point in the data set. Intuitively, this means that the point $\mathbf{x}_i$ is not selected by any point, hence it is not assigned any class label. Also this causes mathematical problems: as in Subsection 3.3.3, the contributions p_{ij} will have an indeterminate value $\frac{0}{0}$. Except of the log-sum-exp trick, the advice from Subsection 3.3.3 can be applied here as well. A more robust way of dealing with this is discussed in the next Section.

4.6 NCA with compact support kernels and background distribution

We extend the previous model to handle cases where points fall outside the support of any other neighbours. The idea is to use for each class a background
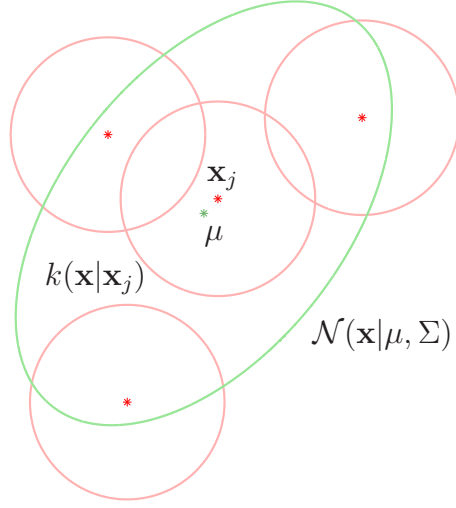
Figure 4.4: Neighbourhood component analysis with compact support kernels and background distribution. The main assumption is that each class is a mixture of compact support distributions $k(\mathbf{x}|\mathbf{x}_j)$ plus a normal background distribution $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

distribution that explains the unallocated points. The background distribution should have an infinite support and an obvious example is the normal distribution.

To introduce a background distribution in a principled manner, we return to the class conditional kernel density estimation (CC-KDE) formulation of NCA, Section 3.2. First, we recast the compact support NCA in the probabilistic framework and consider each class as mixture of compact support distributions:

$$p(\mathbf{x}_i|c) = \frac{1}{N} \sum_{j \in c} k_{\text{CS}}(\mathbf{x}_i|\mathbf{x}_j), \tag{4.17}$$

where $k_{\text{CS}}(\mathbf{x}_i|\mathbf{x}_j) = k_{\text{CS}}(d_{ij})$ and is defined by Equation 4.11. Because $k_{\text{CS}}(\mathbf{x}_i|\mathbf{x}_j)$ denotes a distribution, it ought to integrate to 1. For $c = \frac{15}{16}$ and $a = 1$ the requirement is satisfied.

We further change the model and incorporate an additional distribution in the class-conditional probability $p(\mathbf{x}_i|c)$. From a generative perspective this can be interpreted as follows: a point $\mathbf{x}_i$ is generated by either the compact support distribution from each point $k_{\text{CS}}(\mathbf{x}_i|\mathbf{x}_j)$ or by a class-specific normal distribution $\mathcal{N}(\mathbf{x}_i|\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c)$. So, the distribution $p(\mathbf{x}_i|c)$ can be written as the sum of these components:

$$p(\mathbf{x}_i|c) = \beta \mathcal{N}(\mathbf{x}_i|\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c) + (1 - \beta)\frac{1}{N_c} \sum_{j \in c} k_{\text{CS}}(\mathbf{x}_i|\mathbf{x}_j), \tag{4.18}$$

where β is the mixing coefficient between the background distribution and the compact support model, $\boldsymbol{\mu}_c$ is the sample mean of the class c and $\boldsymbol{\Sigma}_c$ is the sample

covariance of the class c. The constant β can be set to $\frac{1}{N_c+1}$. This will give equal weights to the background distribution and to each compact support distribution. It might be better to treat β as a parameter and fit it during training. We expect β to adapt to the data set: for example, β should increase for data sets with convex classes.

To finalize this method, we just need to plug Equation 4.18 into the set of Equations 3.10, 3.12 and 3.13. The only difficulty is the gradient computation. We give here only derivatives for each individual component (the full derivations and equations can be found in the Appendix):

- The gradient of the compact support distribution $k_{\mathrm{CS}}(\mathbf{x}_i|\mathbf{x}_j)$ with respect to $\mathbf{A}$ is very similar to what is given in Equation 4.15. The only difference is that in this case we have everything multiplied by the constant $c = \frac{15}{16}$.

- For the gradient of the background distribution it is useful to note that projecting the points $\{\mathbf{x}_i\}_{i=1}^N$ into a new space $\{\mathbf{A}\mathbf{x}_i\}_{i=1}^N$ will change the sample mean $\boldsymbol{\mu}_c$ to $\mathbf{A}\boldsymbol{\mu}_c$ and the sample covariance $\boldsymbol{\Sigma}_c$ to $\mathbf{A}\boldsymbol{\Sigma}_c\mathbf{A}^{\mathrm{T}}$. Hence, we have:

$$\frac{\partial}{\partial \mathbf{A}}\mathcal{N}(\mathbf{A}\mathbf{x}_i|\mathbf{A}\boldsymbol{\mu}_c, \mathbf{A}\boldsymbol{\Sigma}_c\mathbf{A}^{\mathrm{T}}) = \mathcal{N}(\mathbf{A}\mathbf{x}_i|\mathbf{A}\boldsymbol{\mu}_c, \mathbf{A}\boldsymbol{\Sigma}_c\mathbf{A}^{\mathrm{T}})$$
$$\times \{-(\mathbf{A}\boldsymbol{\Sigma}_c\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}\boldsymbol{\Sigma}_c + \mathbf{v}\mathbf{v}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Sigma}_c - \mathbf{v}(\mathbf{x} - \boldsymbol{\mu}_c)^{\mathrm{T}}\}, \quad (4.19)$$

where $\mathbf{v} = (\mathbf{A}\boldsymbol{\Sigma}_c\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}(\mathbf{x} - \boldsymbol{\mu}_c)$.

- If we also consider β a parameter, we also need the derivative of the objective function with respect to β. This can be easily obtained, if we use the derivative of the class conditional distribution with respect to β:

$$\frac{\partial}{\partial \beta}p(\mathbf{x}_i|c) = \mathcal{N}(\mathbf{x}_i|\boldsymbol{\mu}_c, \boldsymbol{\Sigma}_c) - \frac{1}{N_c}\sum_{j\in c} k_{\mathrm{CS}}(\mathbf{x}_i|\mathbf{x}_j). \quad (4.20)$$

Chapter 5

Evaluation

In this chapter we present the evaluation of the methods proposed in the previous part, chapter 4. These models were tested in terms of speed and accuracy. The data sets selected for evaluation vary both as number of samples N and as dimensionality D, table 5.1. The diversity of the data sets made assessing the performance of the algorithms difficult. It is known that there is no free lunch in accurately solving widely different problems.

For evaluation we used 70% of the data set for training and the rest of 30% was kept for testing. We made exceptions for two data sets: `landsat` and `mnist`. These are commonly already split in training and testing sets.

We started testing NCA on small data sets. This allowed us to have a baseline to compare the new models against, table 5.2. This can also be viewed as a replication of the work in the original article Goldberger et al. (2004). The authors did not provide us with information about their implementation. For this series of experiments, we used random initialization and we optimized the function using conjugate gradients. The results are averaged over 40 runs. We also considered the score obtained in the original space. We computed this using leave one out cross validation using the Euclidean metric. The results are quite similar to those presented in the original paper.

However, in order to achieve a robust implementation of NCA we had to carry out additional experiments. The learnt lessons were summarized in section 3.3. We will see the influence of the implementation tricks in the following results, section 5.1. Additional results are attached at the end of the thesis, appendix B:

- Tables B.3 and B.4 presents a compares two possible optimization methods for NCA: conjugate gradients and gradient ascent with "bold driver"

Data set name	Abbrevation	N	D	C
Balance scale	`balance`	625	4	3
Ecoli	`ecoli`	336	7	8
Glass identification	`glass`	214	9	6
Ionosphere	`ionosphere`	351	33	2
Iris	`iris`	150	4	3
Landsat satellite	`landsat`	6435	36	6
MAGIC Gamma telescope	`magic`	19020	10	2
MNIST digits	`mnist`	70000	784	10
Pima Indians diabetes	`pima`	768	8	2
Image segmentation	`segment`	2310	18	7
SPECTF heart	`spectf`	267	44	2
Blood transfusion	`transfusion`	748	4	2
USPS digits	`usps`	11000	256	10
Wine	`wine`	178	13	3
Yeast	`yeast`	1484	8	10

Table 5.1: This table presents the characteristics of the data sets used: number of samples N, dimensionality of the data D and number of classes C. The two digits data sets `mnist` and `usps` were downloaded from the following URL `http://cs.nyu.edu/~roweis/data.html`. All the others data sets are available in the UCI repository `http://archive.ics.uci.edu/ml/datasets.html`.

heuristic.

- Figures B.1, B.2 and B.3 illustrate the initialization effect.

5.1   Mini-batches methods comparison

First we present individual results for each of the methods. We also give details about the methodology and the parameter choices. The final comparison is illustrated at the end of this subsection using an error *vs.* time representation.

The methods were compared in this section on the top 3 largest data sets as number of points N.

Data set		Train score	Test scores		Baseline
	d	$f(A)$	1-NN	NCA	Eucl.
`balance`	2	92.86 ± 0.47	90.78 ± 0.53	90.61 ± 0.55	
	$D = 4$	95.36 ± 0.38	93.40 ± 0.47	93.04 ± 0.51	76.18
`ionosphere`	2	98.31 ± 0.14	79.86 ± 0.75	79.74 ± 0.78	
	$D = 33$	72.07 ± 0.71	86.22 ± 0.64	72.87 ± 0.71	85.38
`iris`	2	99.38 ± 0.11	94.94 ± 0.39	94.72 ± 0.39	
	$D = 4$	99.48 ± 0.10	95.10 ± 0.44	95.15 ± 0.44	95.53
`wine`	2	99.15 ± 0.14	92.4 ± 1.0	92.4 ± 1.0	
	$D = 13$	98.95 ± 0.15	95.36 ± 0.51	95.36 ± 0.51	74.53

Table 5.2: Accuracy of standard NCA on four small data sets. Scores are averaged over 40 runs. The second column presents the dimensionality d the data set is reduced to. The last column shows the leave one out cross validation performance on the data set using Euclidean metric.

Data set	Train	1NN	NCA
`usps`	99.112 ± 0.099	89.30 ± 0.18	89.41 ± 0.16
`magic`	82.71 ± 0.41	79.25 ± 0.45	80.22 ± 0.72
`mnist`	96.867	82.130	82.470

Subsampling. For this method we trained NCA on a subset of $n = 3000$ samples. We used conjugate gradients for optimization and for initialization RCA linear transformation. We observe that the training scores are significantly better than the train performance, especially on `usps` data set. As previously mentioned, the thinning effect is a cause for this.

Mini-batches. To get significant gradients we used large mini-batches $n = 2000$. The points in a batch are selected via recursive projection clustering (RPC) algorithm. We trained the metric using gradient ascent with early stopping. The learning rate had a the form $\frac{\eta}{t+t_0}$. We fixed $\eta = 1$ and tuned parameter t_0 using cross validation across an exponential scale; also finer tuning is done across a linear scale. We used 5% of the training set for cross validation and we monitor the accuracy on the cross validation set at each iteration.

Data set	Train	1NN	NCA
usps	92.00 ± 0.43	91.26 ± 0.17	92.37 ± 0.17
magic	80.04 ± 0.48	79.14 ± 0.52	79.8 ± 1.1
mnist	87.47 ± 0.49	87.52 ± 0.36	89.88 ± 0.25

		Train score	Test scores	
Data set	d	$f(A)$	1-NN	NCA
balance	2	88.35 ± 0.83	87.37 ± 0.49	90.45 ± 0.38
	$D = 4$	94.70 ± 0.87	95.32 ± 0.34	96.14 ± 0.29
ionosphere	2	89.0 ± 1.7	85.71 ± 0.94	87.08 ± 0.95
	$D = 33$	92.6 ± 1.5	84.72 ± 0.57	84.34 ± 0.59
iris	2	96.41 ± 0.94	96.33 ± 0.57	97.00 ± 0.46
	$D = 4$	97.5 ± 1.3	95.67 ± 0.71	96.11 ± 0.66
wine	2	98.80 ± 0.70	97.22 ± 0.65	97.50 ± 0.49
	$D = 13$	99.25 ± 0.62	96.85 ± 0.41	96.85 ± 0.41

Table 5.3: Stochastic NCA results on small data sets.

If the performance does not increase for 25 iterations, we stop the learning process.

Stochastic learning. We considered $n = 50$ neighbours to look at for each iteration. The learning procedure is similar to the one applied for the mini-batches method: gradient ascent with early stopping. The same observation as before apply here.
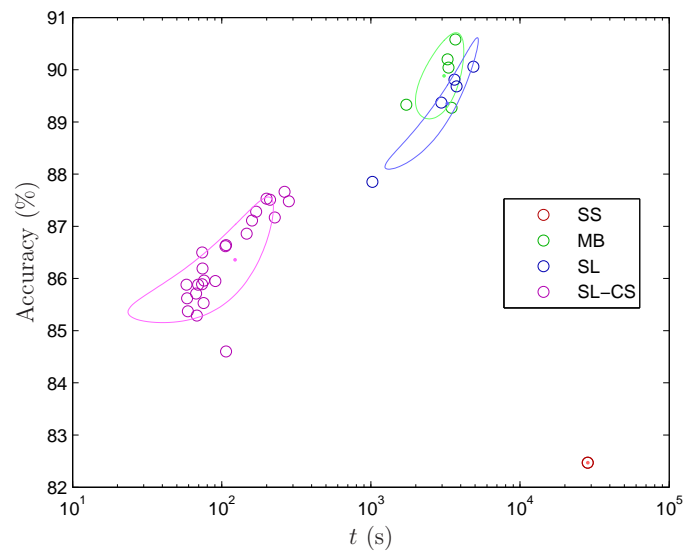
Besides the results on the large data sets, we also present the performance of this method when used for small data sets. We mostly observe similar results as the baseline NCA. The most significant difference is that the classification done using NCA objective function is better than 1NN classification.
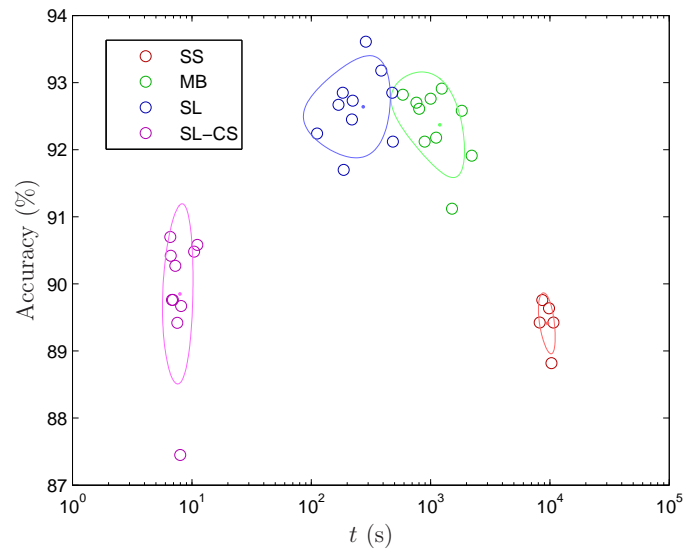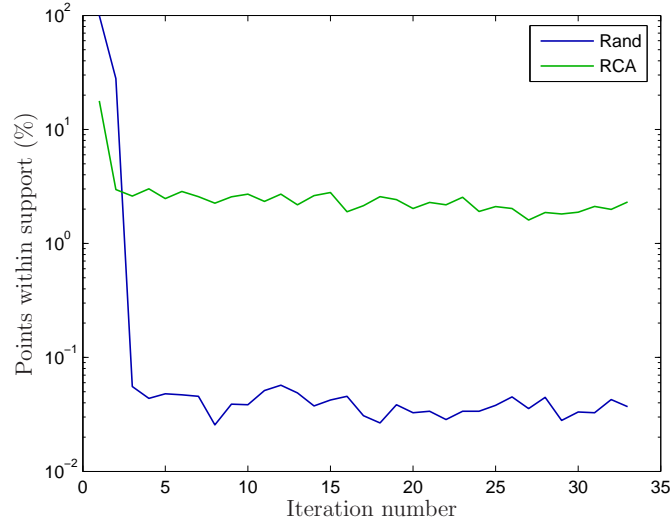
(a) `usps`
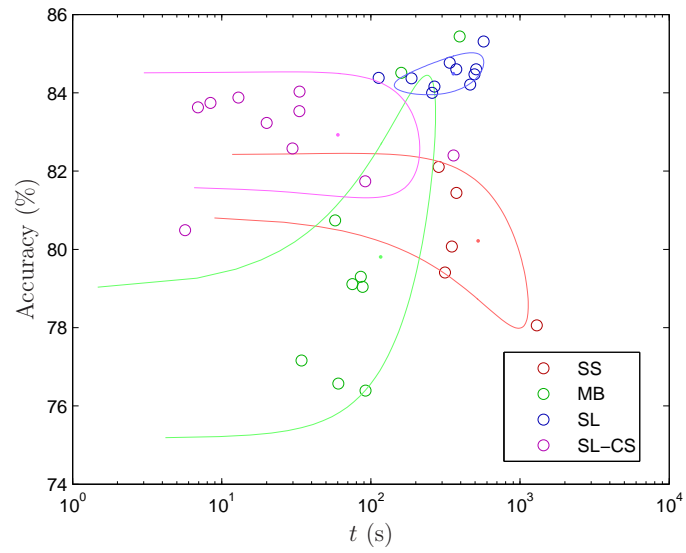
(b) `magic`

(c) `mnist`

Figure 5.1: Time *vs.* accuracy plots on larger data sets.

Data set	Train	1NN	NCA
usps	90.23 ± 0.50	90.68 ± 0.22	92.64 ± 0.17
magic	78.39 ± 0.25	79.76 ± 0.13	84.49 ± 0.12
mnist	85.97 ± 0.37	86.07 ± 0.43	89.35 ± 0.39

5.2 Approximate computations

For KDE, demonstrate how k-d trees work better in low dimensions. Also show that if data has structure than the fraction of points visited for NN is lower. Contrast this with the case when we randomly initialize the projection matrix. We can arrive in such a scenario by initalizing with RCA.

Present fraction of points pruned.

Present results of KDE in terms accuracy. Argue that a reliable speeding can be achieved via a C implementation. We have investigated this using C code and found that is KDE is done fast enough.

NCA with compact support kernels

Demonstrate performance in terms of accuracy. Present also average fraction of points that lie outside the kernel. Test it on large data sets.

Point to Appendix for further experimentations.

Appendix A

MATLAB code

A.1 NCA objective function

```matlab
function [f, df] = nca_obj(A, X, c)
%NCA_OBJ Neighbourhood Component Analysis objective function.
%Returns function value and the first order derivatives.
%
%       [f, df] = nca_obj(A, X, c)
%
% Inputs:
%         A dxD - projection matrix (d <= D).
%         X DxN - data.
%         c 1xN - class labels.
%
% Outputs:
%         f 1x1   - function value.
%        df 1xD*d - derivative values.

% Dan Oneata, June 2011

  [D N] = size(X);
  A = reshape(A,[],D);

  df = zeros(size(A,1),D);

  AX = A*X;
  dist_all = square_dist(AX,AX);
  kern_all = exp(-dist_all);
  kern_all(1:N+1:end) = 0;

  % Compute distance to the neighbours:
  row_dict = logical(eye(max(c)));
  neigh_mask = row_dict(c,c);
  kern_neigh = sum(kern_all.*neigh_mask, 1);
```

```matlab
  kern_sum = sum(kern_all, 1);
  p = kern_neigh ./ kern_sum;
  p = max(p, eps);

  % Compute function value:
  f = - sum(p);

  if nargout > 1,
    K = bsxfun(@rdivide, kern_all, kern_sum);
    K = max(K, eps);
    K = K';
    for i=1:N,
        x_ik = bsxfun(@minus,X(:,i),X);
        Ax_ik = bsxfun(@minus,AX(:,i),AX);
        x_ij = x_ik(:,c==c(i));
        Ax_ij = Ax_ik(:,c==c(i));

        % Update gradient value:
        df = df + p(i) * bsxfun(@times, K(i,:), Ax_ik) * x_ik' ...
            - bsxfun(@times, K(i,c==c(i)), Ax_ij) * x_ij';
    end
    df = -2*df;
    df = df(:);
  end

end
```

Notes:

1. `square_dist` computes the pairwise distances between two D-dimensional sets of points. The function was written by Iain Murray and it can be found in his MATLAB Toolbox, available at the following URL: `http://homepages.inf.ed.ac.uk/imurray2/code/imurray-matlab/square_dist.m`.

2. This implementation is suitable for rather small data sets whose pairwise distance matrix can fit in the RAM. If you are dealing with large data sets, it is better to iterate through the data set and compute the distances successively: from a point to the rest of the data set.

A.2 Objective function for stochastic learning

```matlab
function [f, df] = snca_obj(A, X, c, idxs)
%SNCA_OBJ Neighbourhood Component Analysis objective function for
```

```matlab
%stochastic learning.
%Computes function value and the first order derivatives of a few
%data-points with respect to rest of the whole data set.
%
%      [f, df] = snca_obj(A, X, c, idxs)
%
% Inputs:
%        A dxD - projection matrix (d <= D).
%        X DxN - data.
%        c 1xN - class labels.
%     idxs 1xn - indexes of the points that the function value
%                is computed for.
%
% Outputs:
%        f 1x1   - function value.
%       df 1xD*d - derivative values.

% Dan Oneata, June 2011

  [D N] = size(X);
  n = length(idxs);
  A = reshape(A,[],D);

  df = zeros(size(A,1),D);

  AX = A*X;
  dist_all = square_dist(AX,AX(:,idxs));
  kern_all = exp(-dist_all);
  inds = sub2ind([N n], idxs, 1:n);
  kern_all(inds) = 0;

  % Compute distance to the neighbours:
  row_dict = logical(eye(max(c)));
  neigh_mask = row_dict(c,c(idxs));
  kern_neigh = sum(kern_all.*neigh_mask, 1);

  kern_sum = sum(kern_all, 1);
  p = kern_neigh ./ kern_sum;
  p = max(p, eps);

  % Update function value:
  f = - sum(p);

  if nargout > 1,
    K = bsxfun(@rdivide, kern_all, kern_sum);
    K = max(K, eps);
    K = K';
    for i=1:n,
      ii = idxs(i);
```

```matlab
        x_ik = bsxfun(@minus,X(:,ii),X);
        Ax_ik = bsxfun(@minus,AX(:,ii),AX);
        x_ij = x_ik(:,c==c(ii));
        Ax_ij = Ax_ik(:,c==c(ii));

        % Update gradient value:
          df = df + p(i) * bsxfun(@times, K(i,:), Ax_ik) * x_ik' ...
              - bsxfun(@times, K(i,c==c(ii)), Ax_ij) * x_ij';
    end
    df = -2*df;
    df = df(:);
  end

end
```
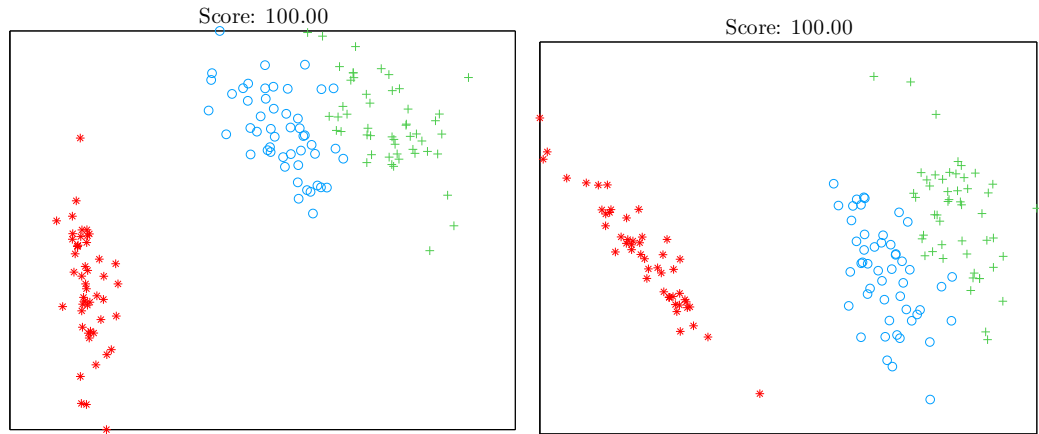
# Appendix B

Further results

B.1 Initialization

This section provides examples of NCA projections on various data sets using different initializations. This is particularly useful to contrast the importance of initializations. Also it shows how NCA typically performs on small sized data sets.

B.2 NCA with compact support kernels

Score: 100.00

(a) NCA projection after random initialization

Score: 100.00

(b) NCA projection after PCA initialization

Score: 99.33

(c) NCA projection after LDA initialization

Score: 100.00
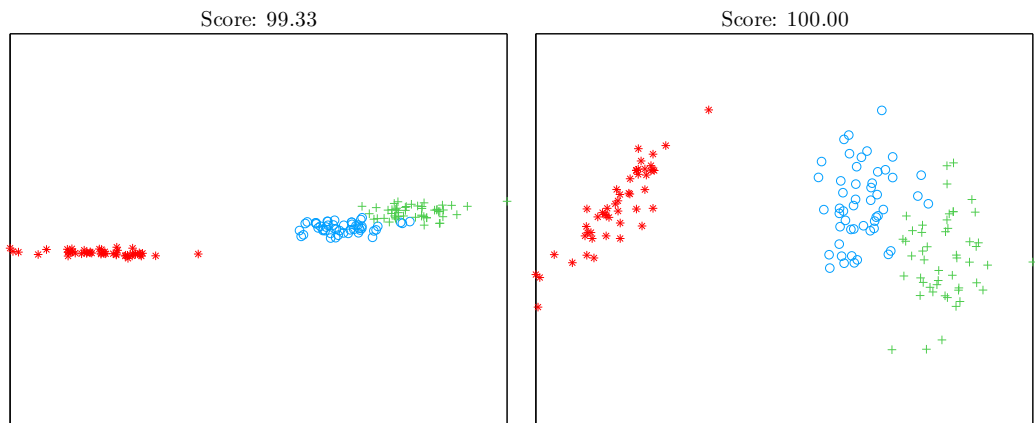
(d) NCA projection after RCA initialization

Figure B.1: Results on `iris` data set.

(a) NCA projection after random initialization

(b) NCA projection after PCA initialization

(c) NCA projection after LDA initialization

(d) NCA projection after RCA initialization

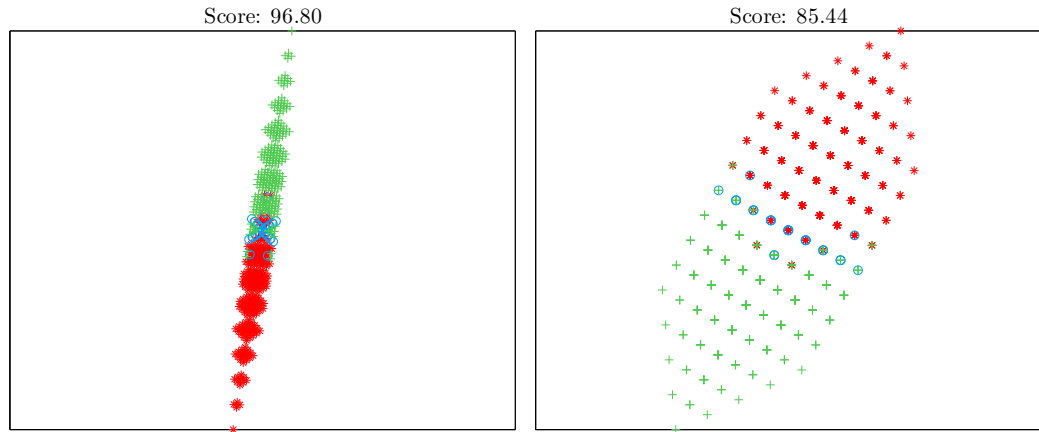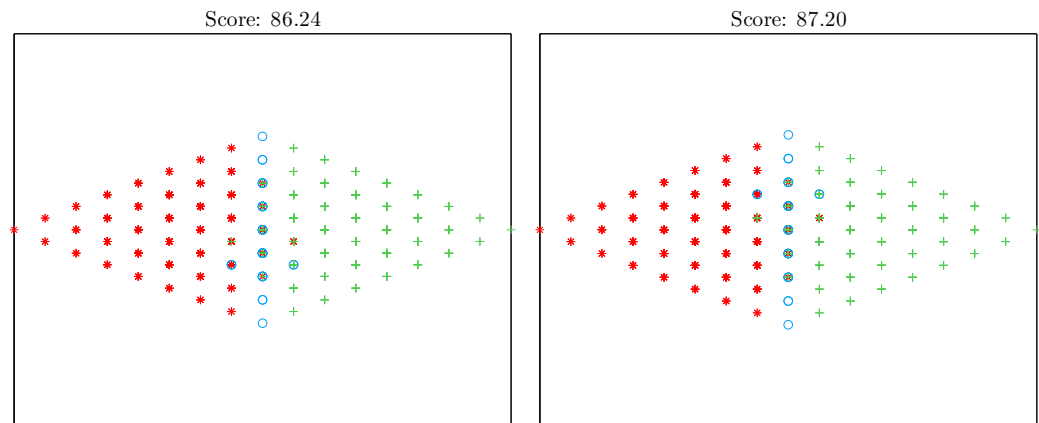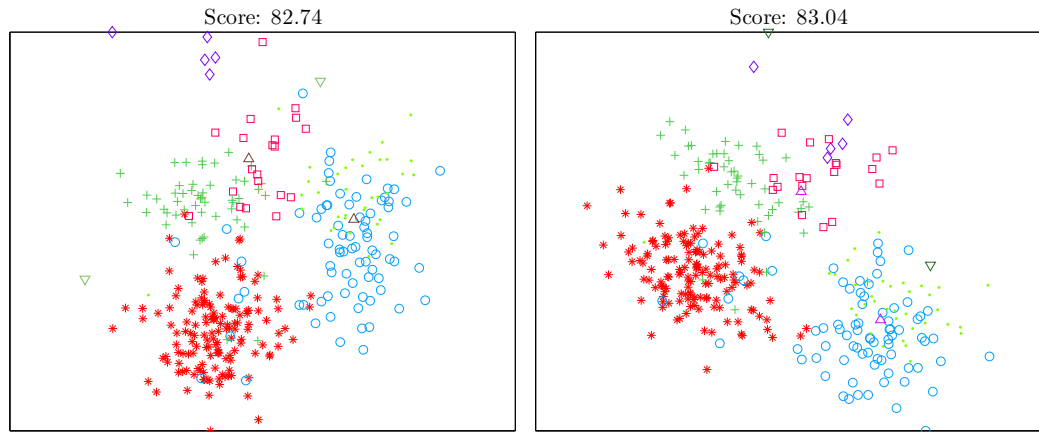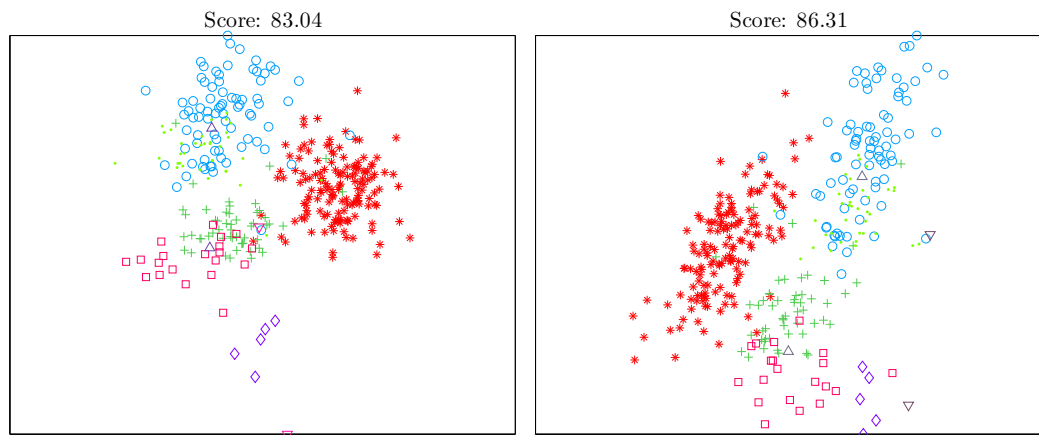Figure B.2: Results on `balance` data set.

(a) NCA projection after random initialization

(b) NCA projection after PCA initialization

(c) NCA projection after LDA initialization

(d) NCA projection after RCA initialization

Figure B.3: Results on `ecoli` data set.

Data set	d	NCA		NCA CS		NCA CS BACK	
		1-NN	NCA	1-NN	NCA	1-NN	NCA
balance	2	92.74 ± 0.57	92.29 ± 0.56	87.90 ± 0.68	89.04 ± 0.52	89.9 ± 1.3	91.04 ± 0.94
	3	94.81 ± 0.52	94.87 ± 0.53	89.60 ± 0.76	90.08 ± 0.60	65.2 ± 3.8	87.85 ± 0.53
	4	95.16 ± 0.39	95.32 ± 0.33	90.82 ± 0.62	92.07 ± 0.54	67.2 ± 3.4	86.91 ± 0.30
glass	2	53.6 ± 1.4	54.2 ± 1.5	55.8 ± 1.1	57.2 ± 1.7	58.5 ± 1.2	59.4 ± 1.5
	3	61.8 ± 1.1	60.9 ± 1.2	63.2 ± 1.4	56.3 ± 1.8	61.5 ± 1.2	63.5 ± 1.1
	4	62.7 ± 1.4	62.2 ± 1.5	62.0 ± 1.4	54.5 ± 1.6	65.2 ± 1.2	65.2 ± 1.0
	5	64.5 ± 1.2	63.46 ± 0.97	64.9 ± 1.1	57.1 ± 1.3	67.5 ± 1.2	67.2 ± 1.3
	9	66.5 ± 1.8	65.5 ± 1.8	68.9 ± 1.5	57.8 ± 1.5	68.2 ± 1.6	64.1 ± 1.3
ionosphere	2	86.23 ± 0.80	86.18 ± 0.75	82.83 ± 0.74	85.33 ± 0.93	85.90 ± 0.74	84.20 ± 0.74
	3	87.69 ± 0.62	87.64 ± 0.62	85.42 ± 0.62	83.44 ± 0.69	88.11 ± 0.74	84.91 ± 0.52
	4	87.64 ± 0.55	87.64 ± 0.57	86.08 ± 0.90	83.6 ± 1.1	88.73 ± 0.92	83.7 ± 1.1
	5	89.39 ± 0.70	89.39 ± 0.70	88.87 ± 0.61	81.46 ± 0.76	86.65 ± 0.86	82.08 ± 0.98
	33	87.97 ± 0.68	87.97 ± 0.68	86.56 ± 0.92	66.09 ± 0.95	87.26 ± 0.45	84.76 ± 0.83

Table B.1: luil

Data set	d	NCA		NCA CS		NCA CS BACK	
		1-NN	NCA	1-NN	NCA	1-NN	NCA
iris	2	94.44 ± 0.51	94.11 ± 0.48	95.22 ± 0.59	94.78 ± 0.72	95.22 ± 0.67	95.11 ± 0.60
	3	95.44 ± 0.73	95.00 ± 0.75	95.33 ± 0.93	95.0 ± 1.0	94.89 ± 0.80	94.2 ± 1.1
	4	96.56 ± 0.46	96.44 ± 0.46	95.56 ± 0.54	95.67 ± 0.53	95.2 ± 1.0	93.11 ± 0.95
wine	2	96.67 ± 0.38	96.57 ± 0.38	97.04 ± 0.51	96.94 ± 0.44	97.22 ± 0.48	97.59 ± 0.46
	3	96.39 ± 0.48	96.39 ± 0.48	96.48 ± 0.63	95.93 ± 0.65	97.22 ± 0.53	97.50 ± 0.40
	4	96.20 ± 0.40	96.48 ± 0.41	98.06 ± 0.36	98.06 ± 0.28	98.24 ± 0.33	98.89 ± 0.30
	5	96.57 ± 0.60	96.57 ± 0.60	96.20 ± 0.71	95.00 ± 0.81	97.78 ± 0.43	97.87 ± 0.40
	13	96.20 ± 0.48	96.20 ± 0.48	95.09 ± 0.70	92.59 ± 0.92	94.63 ± 0.81	98.43 ± 0.30
yeast	2	42.37 ± 0.81	43.20 ± 0.84	42.57 ± 0.63	48.31 ± 0.90	44.38 ± 0.66	48.24 ± 0.79
	3	48.06 ± 0.67	48.30 ± 0.65	46.63 ± 0.72	51.6 ± 1.0	47.33 ± 0.61	50.67 ± 0.55
	4	49.06 ± 0.47	49.14 ± 0.47	49.26 ± 0.51	53.06 ± 0.94	49.93 ± 0.36	53.08 ± 0.30
	5	50.09 ± 0.39	50.09 ± 0.38	50.16 ± 0.59	52.96 ± 0.93	49.79 ± 0.62	54.54 ± 0.56
	8	50.66 ± 0.52	50.70 ± 0.52	50.93 ± 0.44	54.89 ± 0.43	48.73 ± 0.65	56.93 ± 0.62

Table B.2: lals

Data set	d	Conjugate gradients		Bold driver	
		1-NN	NCA	1-NN	NCA
balance	2	92.74 ± 0.57	92.29 ± 0.56	92.05 ± 0.93	92.45 ± 0.81
	3	94.81 ± 0.52	94.87 ± 0.53	92.79 ± 0.95	92.87 ± 0.97
	4	95.16 ± 0.39	95.32 ± 0.33	94.18 ± 0.76	94.36 ± 0.69
glass	2	53.6 ± 1.4	54.2 ± 1.5	58.7 ± 1.5	62.5 ± 1.4
	3	61.8 ± 1.1	60.9 ± 1.2	66.3 ± 1.1	67.1 ± 1.1
	4	62.7 ± 1.4	62.2 ± 1.5	65.8 ± 1.0	66.6 ± 1.3
	5	64.5 ± 1.2	63.46 ± 0.97	66.69 ± 0.93	67.00 ± 0.87
	9	66.5 ± 1.8	65.5 ± 1.8	66.0 ± 1.7	65.6 ± 1.7
ionosphere	2	86.23 ± 0.80	86.18 ± 0.75	82.78 ± 0.98	82.78 ± 0.91
	3	87.69 ± 0.62	87.64 ± 0.62	86.04 ± 0.83	86.08 ± 0.85
	4	87.64 ± 0.55	87.64 ± 0.57	86.51 ± 0.73	86.51 ± 0.73
	5	89.39 ± 0.70	89.39 ± 0.70	87.88 ± 0.64	87.88 ± 0.64
	33	87.97 ± 0.68	87.97 ± 0.68	88.44 ± 0.59	88.44 ± 0.59

Table B.3: Comparison in terms of accuracy between two possible optimization methods for NCA: conjugate gradients and gradient ascent with the "bold driver" heuristic

| Data set | d | Conjugate gradient | | Bold-driver | |
		1-NN	NCA	1-NN	NCA
iris	2	94.44 ± 0.51	94.11 ± 0.48	93.67 ± 0.82	93.78 ± 0.84
	3	95.44 ± 0.73	95.00 ± 0.75	95.22 ± 0.72	95.33 ± 0.74
	4	96.56 ± 0.46	96.44 ± 0.46	95.44 ± 0.76	95.56 ± 0.77
wine	2	96.67 ± 0.38	96.57 ± 0.38	96.30 ± 0.54	96.30 ± 0.54
	3	96.39 ± 0.48	96.39 ± 0.48	96.57 ± 0.59	96.57 ± 0.59
	4	96.20 ± 0.40	96.48 ± 0.41	97.13 ± 0.52	97.13 ± 0.52
	5	96.57 ± 0.60	96.57 ± 0.60	96.76 ± 0.55	96.76 ± 0.55
	13	96.20 ± 0.48	96.20 ± 0.48	97.22 ± 0.48	97.22 ± 0.48
yeast	2	42.37 ± 0.81	43.20 ± 0.84	44.65 ± 0.56	46.68 ± 0.57
	3	48.06 ± 0.67	48.30 ± 0.65	47.06 ± 0.53	47.95 ± 0.54
	4	49.06 ± 0.47	49.14 ± 0.47	49.48 ± 0.42	49.83 ± 0.38
	5	50.09 ± 0.39	50.09 ± 0.38	50.16 ± 0.61	50.41 ± 0.62
	8	50.66 ± 0.52	50.70 ± 0.52	50.19 ± 0.58	50.28 ± 0.60

Table B.4: Comparison in terms of accuracy between two possible optimization methods for NCA: conjugate gradients and gradient ascent with the "bold driver" heuristic

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