# Robustified Markowitz approach for diversified portfolios with crypto-assets

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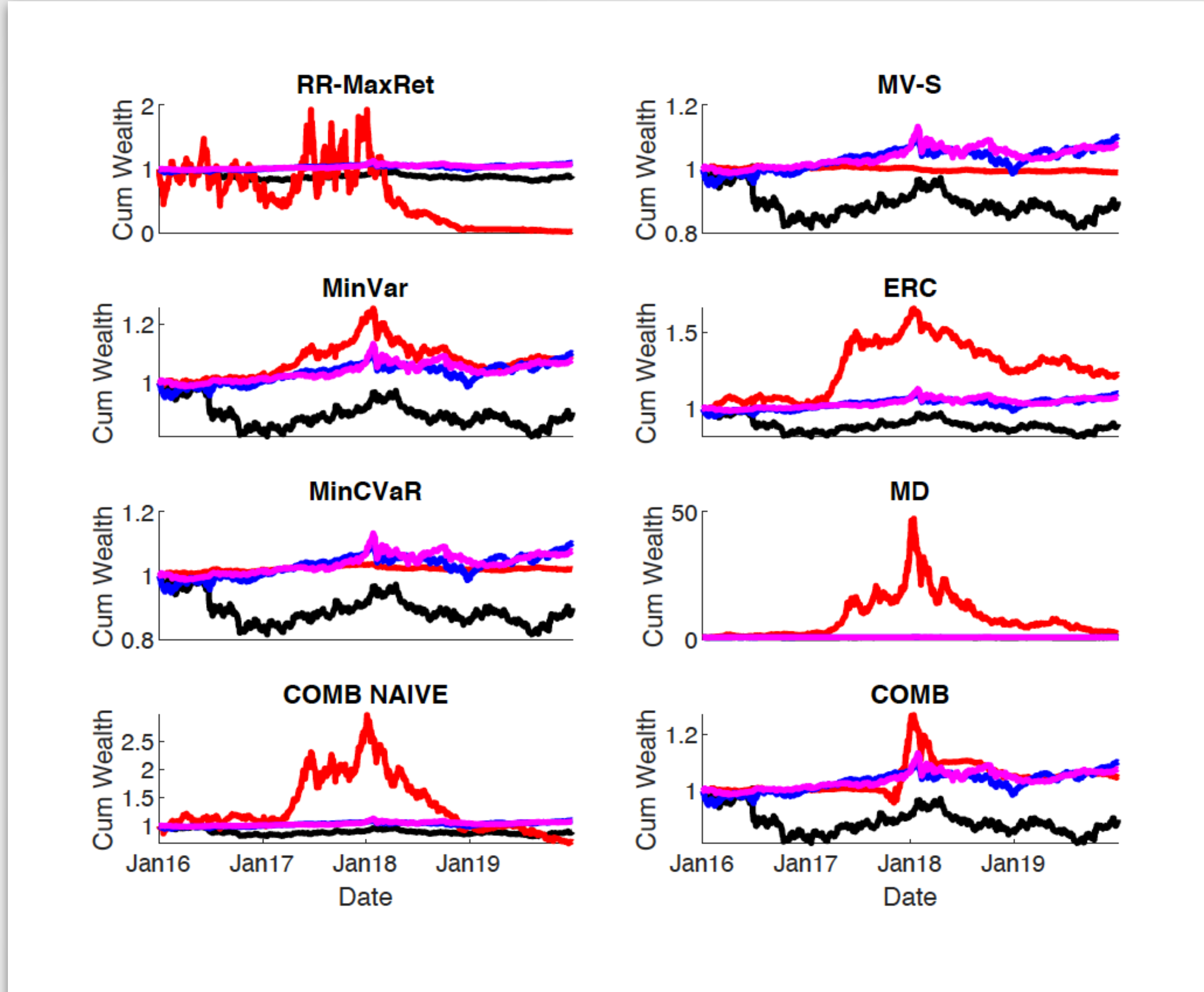
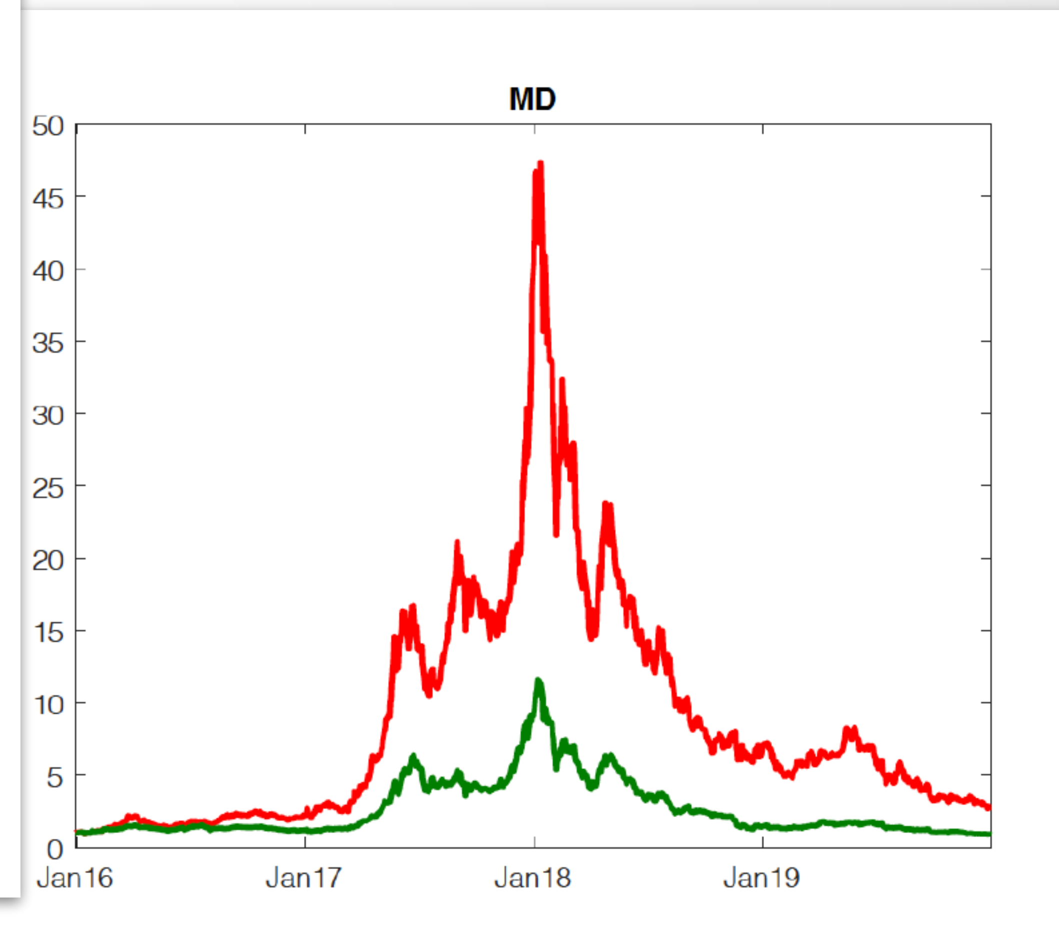
# Portfolio allocation with cryptos

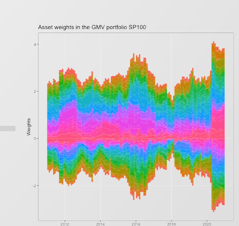


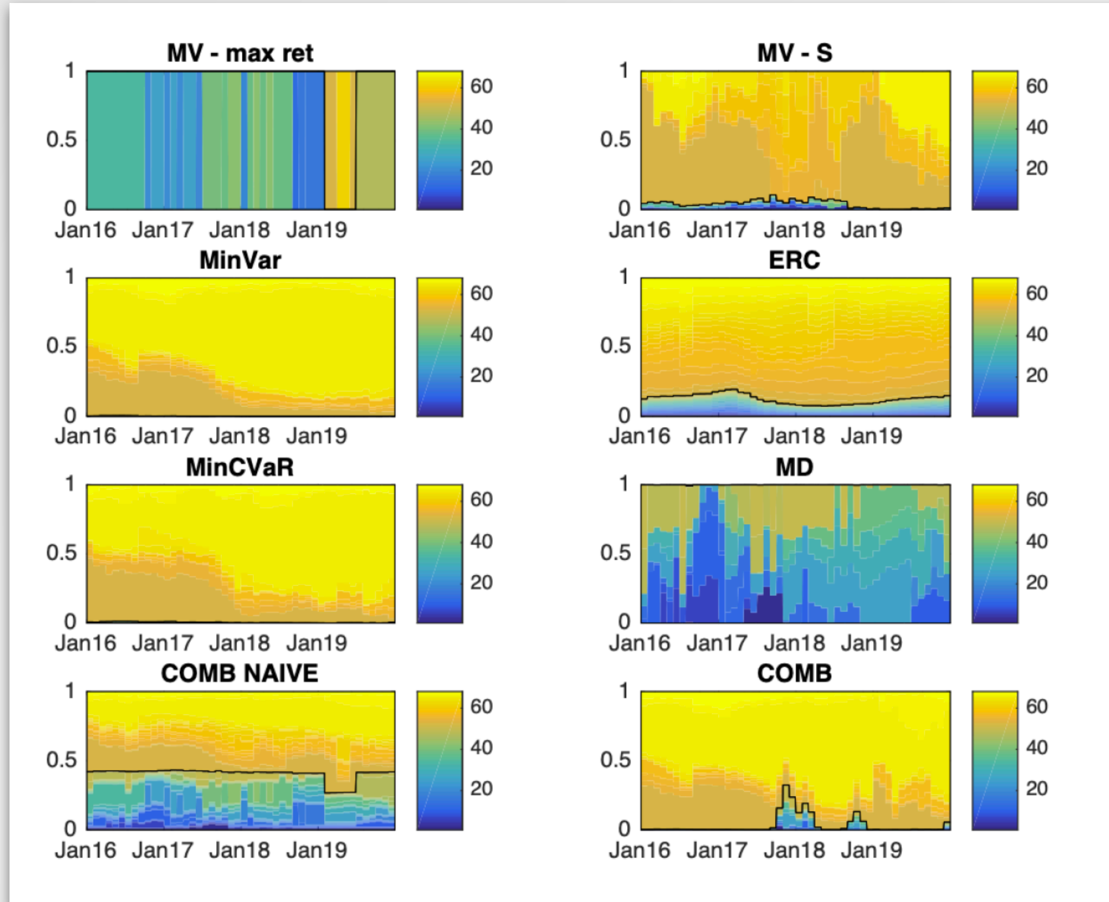

Performance of portfolios: S&P100, EW, EW–TrA, RR-Max ret–TrA and corresponding Allocation strategy Data range: 20160101 to 20191231

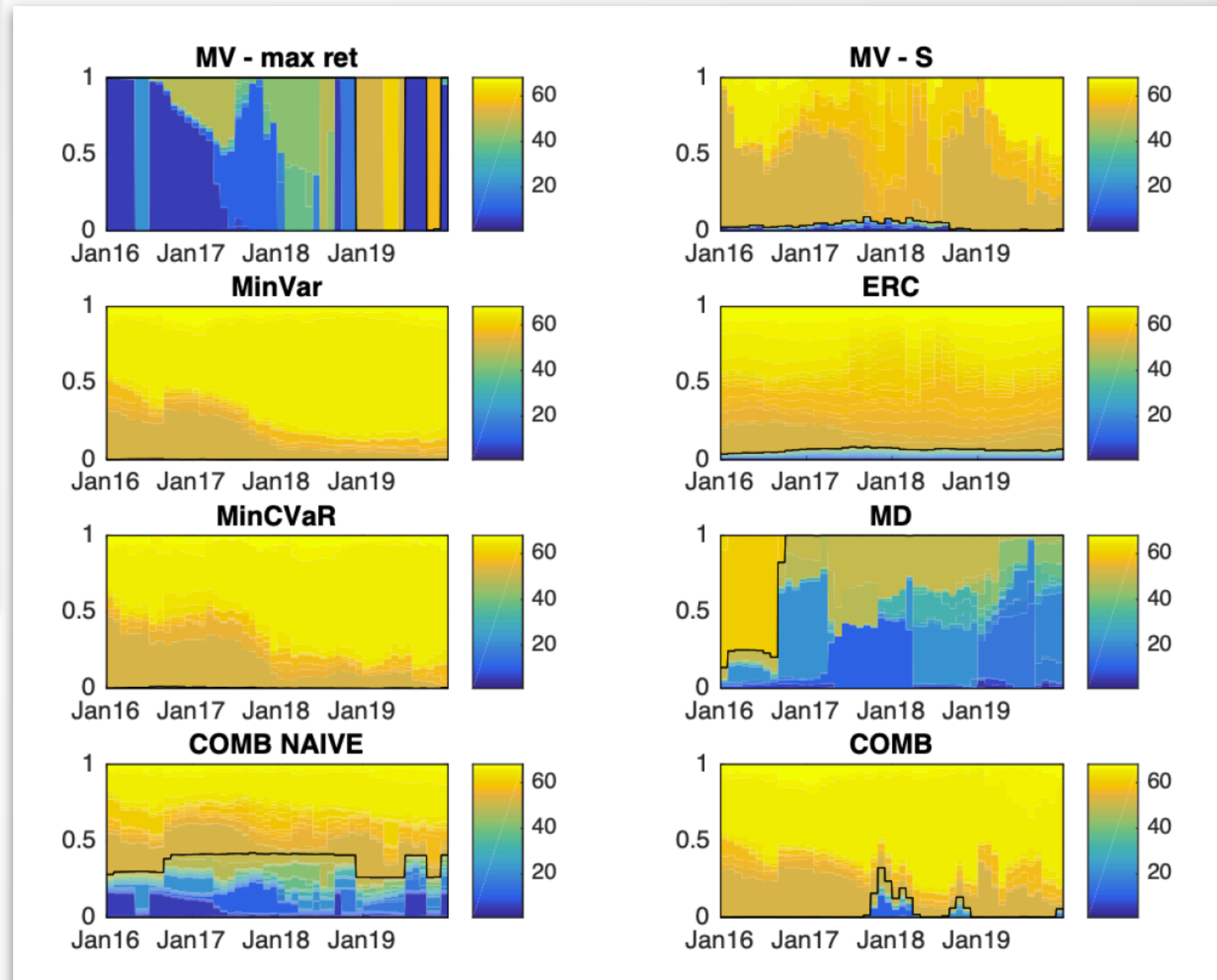
CCPPerformance

Petukhina et al. (2021)



# Capital allocation with CAs

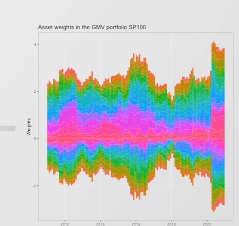


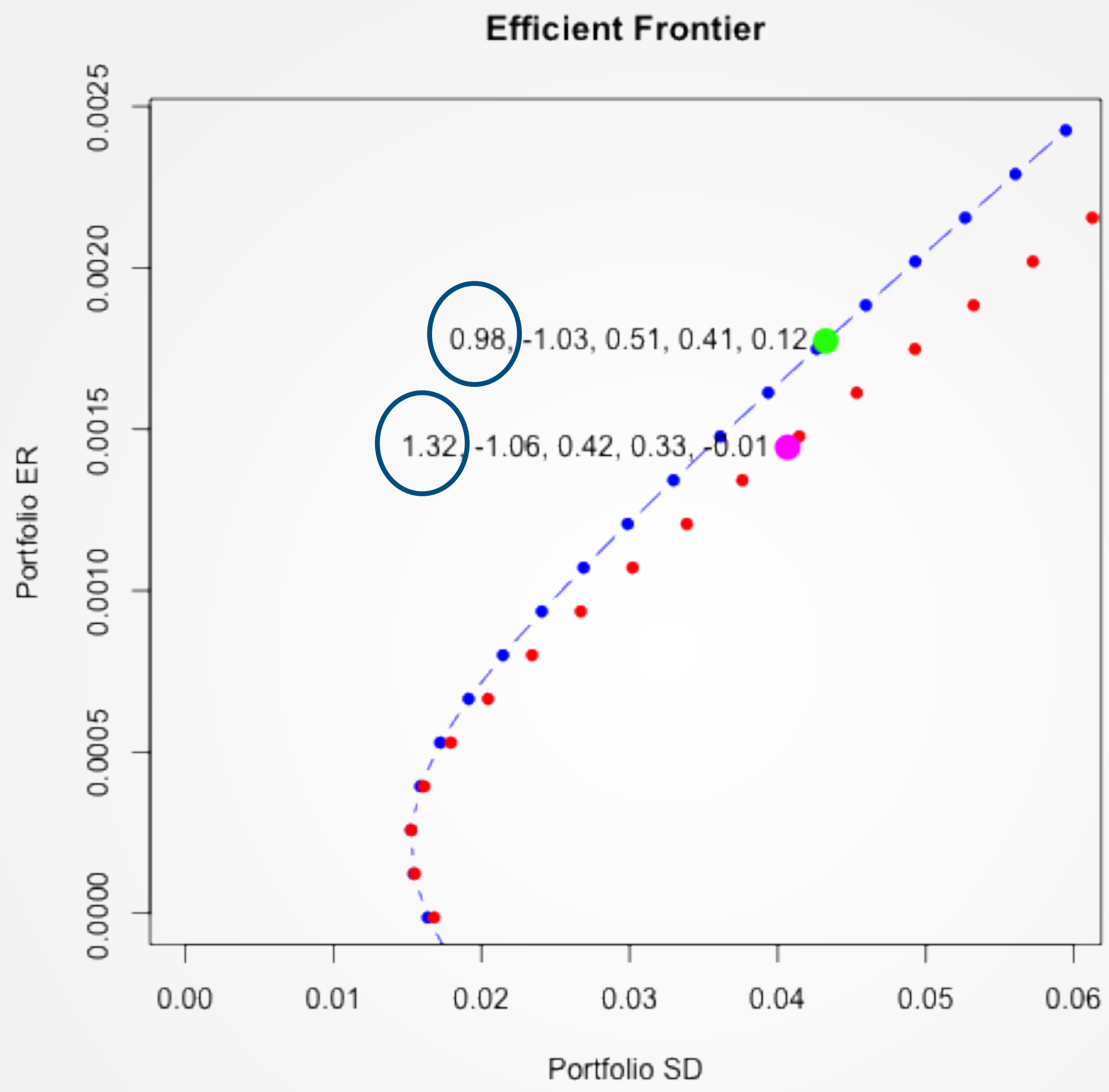

Dynamics in the capital composition Data range: 20160101 to 20191231

[CCPWeights]

Petukhina et al. (2021)



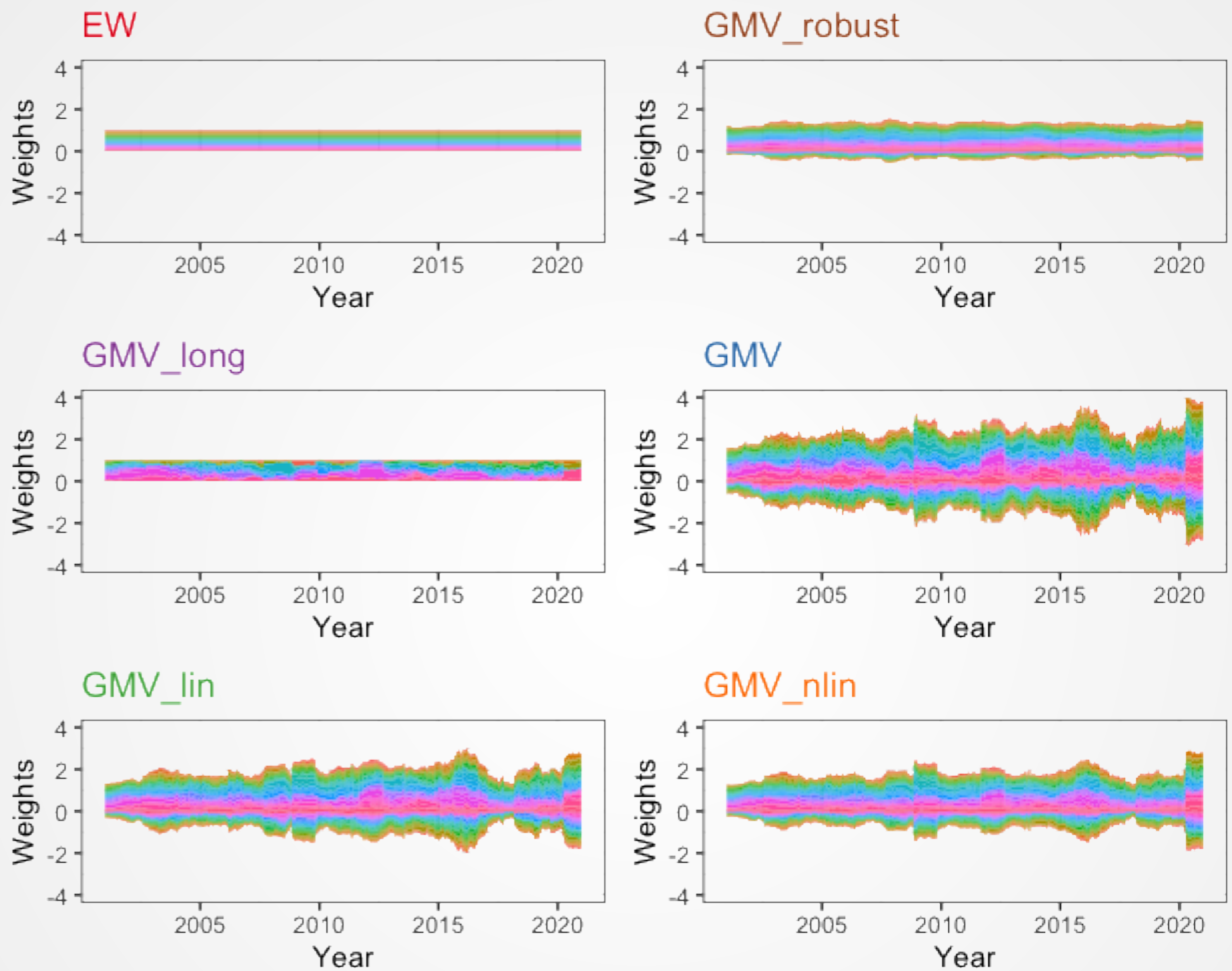
# "M" curse: sensitivity to input




Efficient frontiers for constituents: AAPL, IBM, NVDA, AMZN, MSFT (blue) and with change of mean of AAPL by -0.02% (2 BP)

RobustM_Example

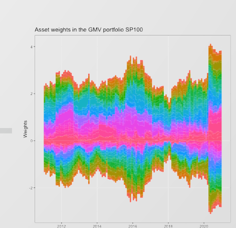
# Robustifying Markowitz




Dynamics of weights for MV portfolios for 81 stocks from S&P100, 20010101-20201231

RobustM_PerformanceSP100

Petukhina et al. (2022)

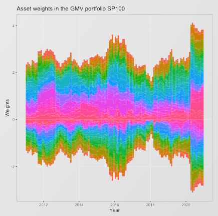


# Math to solve Markowitz enigma

- Improve estimators of the mean and covariance matrix
  - Linear shrinkage Ledoit & Wolf (2004)
  - Nonlinear shrinkage Ledoit & Wolf (2017)
  - Couillet & McKay (2014)
  - Petukhina et al. (2022)
- Constraints of portfolio weights
  - Short-sale constraints: Jagannathan & Ma (2003)
  - Gross-exposure constraint: Fan et al. (2012) extend the no-short-sale constraint to varying degrees of leverage constraints
  - Liquidity constraints: Trimborn et al. (2019)
- Other Risk-based portfolios
  - Equal Risk Contribution Roncalli et al. (2010)
  - Maximum Diversification Rudin and Morgan(2006)
  - Hierarchical Risk Parity Lopez de Prado(2016)

# GMV portfolios

- Traditional assets
- One input (Covariance M) only
- Better than mean variance algos
  - Haugen & Baker (1991)
  - Nielsen & Aylursubramanian (2008)
- Cryptoassets
  - Trimborn (2019 )&Petukhina et al. (2021): very low share of CCs in GMV and MinCvaR portfolios
- This research
  - **Map Robustified GMV with other risk-based portfolio rules for cryptos without constraints**

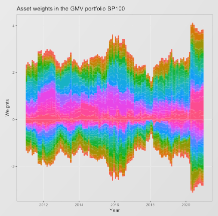


# Outline

1. Motivation ✓
2. Methodology Robust M
3. Data
4. Empirical results
5. Conclusion

# Objectives

- Portfolio investment strategies crypto-assets
- I Out-of-sample performance analysis – is there the best asset
- allocation model?
- I Does portfolio concentration depend on the optimization
- procedure used?
- I Effect of liquidity constraints for the performance
- I Diversification effects

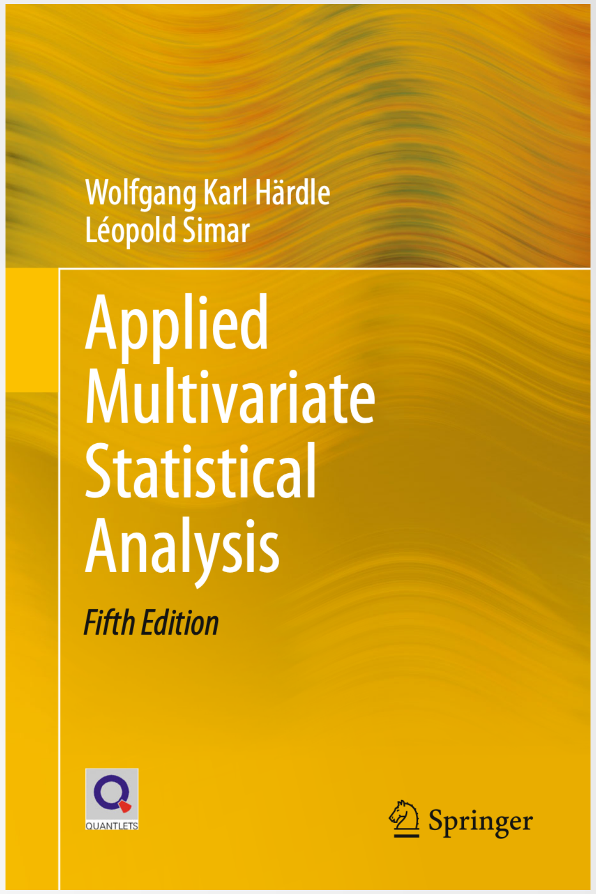


# "M" curse: extreme weight changes = high costs


Portfolio weights of monthly rebalanced GMV for 81 stocks

# Markowitz portfolio

- MV Mean-Variance portfolio: max Utility




$$M_\gamma(w;\mu,\Sigma) = \mu^\top w - \frac{\gamma}{2} w^\top \Sigma w$$

$$\widehat{w} = \arg \max_{w^\top 1 = 1} M_\gamma(w;\widehat{\mu},\widehat{\Sigma})$$

$$w^* = \arg \max_{w^\top 1 = 1} M_\gamma(w;\mu,\Sigma)$$

$\mu, \Sigma$ mean, cov of multivariate log-returns vector $X = (r_1 \ldots r_N)^\top$

$\widehat{\mu}, \widehat{\Sigma}$ mean, cov (generic) estimators

$w \in \mathbb{R}^N$ vector of weights, $N$ - # of assets, $T$-#of observations

# Markowitz portfolio

- GMV: min risk

$$R(w;\Sigma) = \frac{1}{2} w^\top \Sigma w$$

$$\widehat{w} = \arg \min_{w^\top 1 = 1} R(w;\widehat{\Sigma})$$

$$w^* = \arg \min_{w^\top 1 = 1} R(w;\Sigma)$$

- $\mu, \Sigma$ mean, cov of multivariate log-returns vector $X = (r_1 \ldots r_N)^\top$
- $\widehat{\mu}, \widehat{\Sigma}$ mean, cov (generic) estimators
  - $w \in \mathbb{R}^N$ vector of weights, $N$ - # of assets, $T$-#of observations

Gradient descent without constraint:

$$w_s = \left(w_{s-1} - \eta \Sigma w_{s-1}\right), s = 1,2, \ldots$$

*mem it*

# Robust estimation

- Ideal situation $X_i \sim \mathcal{N}(\mu, \Sigma)$ and $\hat{\mu} = T^{-1} \sum_{i=1}^{T} X_i$

$$\| \hat{\mu} - \mu \| \leq \sqrt{\frac{\mathrm{Tr}(\Sigma)}{T}} + \sqrt{\frac{2 \| \Sigma \| \log(1/\delta)}{T}}$$

*with Prob = 1-$\delta$*

- DeMiguel and Nogales (2009): "A robust estimator is one that gives meaningful information about asset returns even when the empirical (sample) distribution deviates from the assumed (normal) distribution …"
- Construct an estimator that works “normally” in non normal situations!

# Robust estimation: current state of the art

- Heavy tails: only limited moments exist
- Robust covariance estimation
  - Mendelson & Zhivotovskiy (2020): “Gaussian” performance
  - But NO real practical algorithm
- Robust mean estimation
  - Lugosi & Mendelson (2019) estimator with “Gaussian” performance
  - Hopkins et al. (2020) nearly linear algorithm, same performance

# Well conditioned case: how good can it be?

- Bounded kurtosis: $\mathsf{E}^{1/4}|u^\top(X-\mu)|^4 \le K\mathsf{E}^{1/2}|u^\top(X-\mu)|^2$

- Large enough sample, $\kappa = \lambda_{max}(\Sigma)/\lambda_{min}(\Sigma)$ (condition number)

$$\|\Sigma\|\sqrt{\frac{N\log N + \log(1/\delta)}{T}} \le c\lambda_{min}(\Sigma)$$

- Step size $\eta \le 1/\gamma\lambda_{max}$, # steps $S \sim \log(T)$

$$M_\gamma(w^*;\mu,\Sigma) - M_\gamma(\widehat{w}_\delta;\mu,\Sigma) \lesssim \frac{N\log N + \log(1/\delta)}{T}$$

# Not so nice *N/T* situation - ill-conditioned case

- GMV requires a few more steps!
- Bounded kurtosis: $\mathsf{E}^{1/4}|u^\top(X-\mu)|^4 \leq K\mathsf{E}^{1/2}|u^\top(X-\mu)|^2$
- Define effective rank: $r(\Sigma) = \mathrm{Tr}(\Sigma)/\parallel \Sigma \parallel$
- Step size $\eta \leq 1/\lambda_{max}$, # steps $S \sim T\{\mathbf{r}(\Sigma)\log\mathbf{r}(\Sigma) + \log(1/\delta)\}$
- Risk excess bound:

$$R(\widehat{w}_\delta; \Sigma) - R(w^*; \Sigma) \lesssim \sqrt{\frac{\mathbf{r}(\Sigma)\log\mathbf{r}(\Sigma) + \log(1/\delta)}{T}}$$

# Robustifying Markowitz via “managable” iteration

- Simple projected gradient descent (PGD) procedure:

$$w_s = \Pi_1\left[w_{s-1} - \eta \nabla_w R(w_{s-1}; \Sigma)\right], s = 1, \ldots, S$$

$\eta \leq 1/\lambda_{max}$ - step size; $\Pi_1 x = \left(I - N^{-1} 1 1^\top\right)x + N^{-1} 1$ orthogonal projector onto the restricted (convex) set $\left\{w^\top 1 = 1\right\}$

$$\nabla_w R(w; \Sigma) = \Sigma w$$

- For any $w$ estimate

$$\Sigma w = \mathsf{E} X X^\top w$$

- Apply robust mean algorithm to $X_i X_i^\top w$ and obtain a robust estimator $\hat{a}(w)$ of $\Sigma w$
- Plug into update steps

$$w_s = \Pi_1\left\{w_{s-1} - \eta \hat{a}(w_{s-1})\right\}, s = 1, \ldots, S$$

# Data

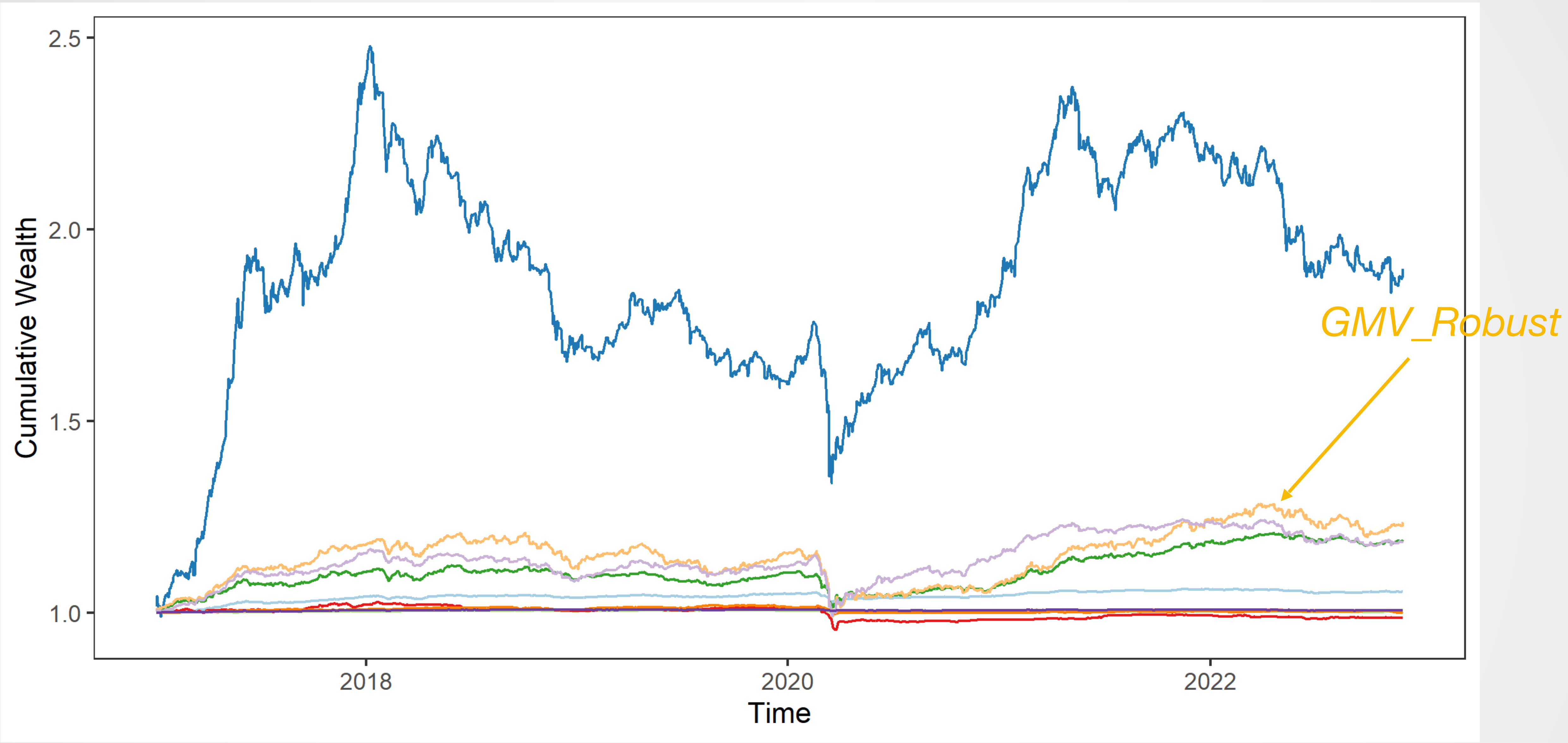
- Sample $N = 102$
  - 35 cryptos
  - 27 stocks (S&P 500)
  - 7 bonds
  - 2 real estates
  - 13 exchange rates
  - 18 commodities (Bloomberg Commodity Index)
  - Time span: 20150101- 20221130
    1. In-sample: 20150101- 20151231
    2. Out-of-sample: 20160101- 20221130
  - Monthly rebalancing
  - Rolling window size, months: $K = 24$

RobustM_Data

# The horse race

| Model | Reference | Abbreviation |
|---|---|---|
| Equally weighted | DeMiguel et al. (2009) | EW |
| Robust Global Minimum Variance | Petukhina et al. (2023) | GMV_robust |
| GMV with sample covariance | Merton (1980) | GMV |
| GMV with linear shrinkage cov estimator | Ledoit&Wolf (2004) | GMV_lin |
| GMV with non-linear shrinkage cov estimator | Ledoit&Wolf (2017) | GMV_nlin |
| GMV with short sale constraints | Jagannathan&Ma (2003) | GMV_long |
| Inverse volatility | Roncalli et al. (2010) | IV |
| Equally risk contribution | Roncalli et al. (2010) | ERC |
| Maximum diversification | Rudin and Morgan(2006) | MD |
| Hiracical Risk Parity | De Prado (2016) | HRP |

# Cumulative wealth




Cumulative wealth of benchmark portfolios for 102 assets, 20160101-20221130

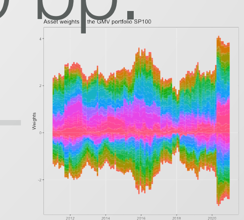
RobustM_PerformanceCC

# Out-of-sample performance with TC

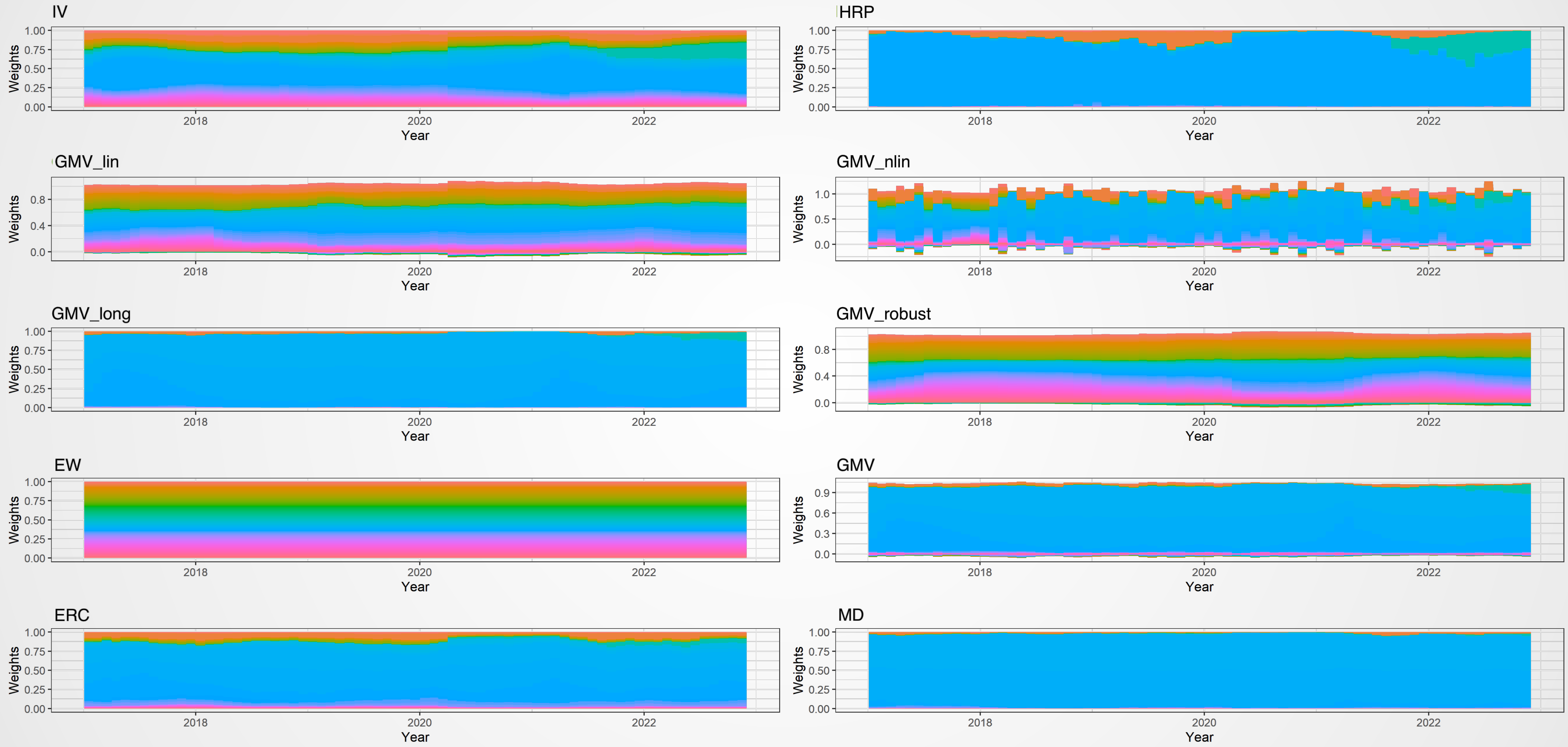
| | EW | GMV_long | GMV | GMV_lin | GMV_nlin | IV | ERC | MD | GMV_robust | HRP |
|---|---|---|---|---|---|---|---|---|---|---|
| CumWealth-TC | 1.8974 | 1.0057 | 1.0025 | 1.1916 | 0.9870 | 1.1884 | 1.0554 | 1.0065 | 1.2361 | 1.0004 |
| Sharpe.Ann-TC | 38.4813 | 38.7533 | 13.8729 | 81.8522 | -16.0864 | 55.0620 | 99.6344 | 41.9987 | 53.0337 | 1.0359 |
| TTO | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0001 |
| TO | 0.5663 | 0.0612 | 0.0795 | 0.2061 | 0.1125 | 0.1467 | 0.0807 | 0.0745 | 0.2794 | 0.0092 |
| AverageDrawdown | 0.0723 | 0.0004 | 0.0008 | 0.0069 | 0.0079 | 0.0089 | 0.0018 | 0.0004 | 0.0107 | 0.0010 |
| StdDev.annualized | 0.2970 | 0.0025 | 0.0030 | 0.0367 | 0.0138 | 0.0538 | 0.0092 | 0.0026 | 0.0688 | 0.0061 |
| CVaR | -0.0782 | -0.0004 | -0.0004 | -0.0055 | -0.0019 | -0.0110 | -0.0016 | -0.0004 | -0.0161 | -0.0021 |
| AverageLength | 66.7619 | 23.8983 | 49.3448 | 25.5000 | 102.9286 | 32.3810 | 23.0678 | 25.0351 | 24.0172 | 24.9643 |
| AverageRecovery | 37.9524 | 17.4068 | 35.4138 | 10.6667 | 59.8571 | 10.7857 | 13.2542 | 17.5439 | 11.5690 | 18.0536 |
| BernardoLedoitRatio | 1.1005 | 1.0690 | 1.0240 | 1.1726 | 0.9627 | 1.1253 | 1.2075 | 1.0743 | 1.1115 | 1.0022 |
| BurkeRatio | 0.1629 | 0.1571 | 0.0532 | 0.3193 | -0.0416 | 0.2281 | 0.3545 | 0.1913 | 0.2171 | 0.0022 |
| CDD | 0.1502 | 0.0007 | 0.0017 | 0.0119 | 0.0115 | 0.0108 | 0.0047 | 0.0008 | 0.0172 | 0.0014 |
| DownsideDeviation | 0.0140 | 0.0001 | 0.0001 | 0.0016 | 0.0007 | 0.0027 | 0.0004 | 0.0001 | 0.0032 | 0.0003 |
| DownsideFrequency | 0.4572 | 0.4757 | 0.4887 | 0.4463 | 0.5024 | 0.4298 | 0.4476 | 0.4846 | 0.4497 | 0.4394 |
| DownsidePotential | 0.0061 | 0.0001 | 0.0001 | 0.0008 | 0.0002 | 0.0010 | 0.0002 | 0.0001 | 0.0014 | 0.0001 |
| DRatio | 0.7654 | 0.8488 | 0.9335 | 0.6873 | 1.0488 | 0.6700 | 0.6711 | 0.8752 | 0.7352 | 0.7822 |
| HurstIndex | 0.3634 | 0.2983 | 0.3282 | 0.3877 | 0.4058 | 0.4185 | 0.4180 | 0.3173 | 0.3974 | 0.3645 |
| KellyRatio | 0.8772 | 79.8179 | 23.5345 | 12.2487 | -5.9259 | 5.6232 | 56.6486 | 82.0719 | 4.3067 | 0.8652 |
| MartinRatio | 0.2087 | 0.4241 | 0.1312 | 0.9953 | -0.0757 | 0.7015 | 1.4635 | 0.7217 | 0.5278 | 0.0037 |
| OmegaSharpeRatio | 0.1005 | 0.0690 | 0.0240 | 0.1726 | -0.0373 | 0.1253 | 0.2075 | 0.0743 | 0.1115 | 0.0022 |
| ProspectRatio | -0.5019 | -0.5976 | -0.6449 | -0.5021 | -0.4662 | -0.4356 | -0.4566 | -0.6130 | -0.5160 | -0.4756 |
| SemiDeviation | 0.0143 | 0.0001 | 0.0001 | 0.0017 | 0.0007 | 0.0027 | 0.0004 | 0.0001 | 0.0033 | 0.0003 |
| SkewnessKurtosisRatio | -0.0918 | -0.0521 | -0.0167 | -0.0145 | -0.0484 | -0.0731 | -0.0453 | -0.0112 | -0.0551 | -0.1122 |
| SortinoRatio | 0.0439 | 0.0349 | 0.0126 | 0.0805 | -0.0135 | 0.0485 | 0.0909 | 0.0387 | 0.0505 | 0.0008 |
| UpsidePotentialRatio | 0.5987 | 0.7115 | 0.7364 | 0.6593 | 0.4966 | 0.5011 | 0.6406 | 0.7566 | 0.6138 | 0.4517 |
| UpsideRisk | 0.0124 | 0.0001 | 0.0001 | 0.0016 | 0.0006 | 0.0021 | 0.0004 | 0.0001 | 0.0029 | 0.0002 |
| VolatilitySkewness | 0.7891 | 0.9531 | 0.9903 | 1.0200 | 0.7264 | 0.6241 | 0.9311 | 1.0467 | 0.8349 | 0.5365 |

**Performance metrics**

Out-of-sample performance, 102 assets, monthly rebalancing, TC = 50 bp.
Time period: 20160101- 20221130



# Dynamics of weights




Portfolio weights for 102 assets, 20160101- 20221130

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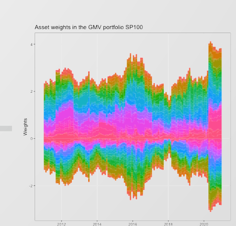
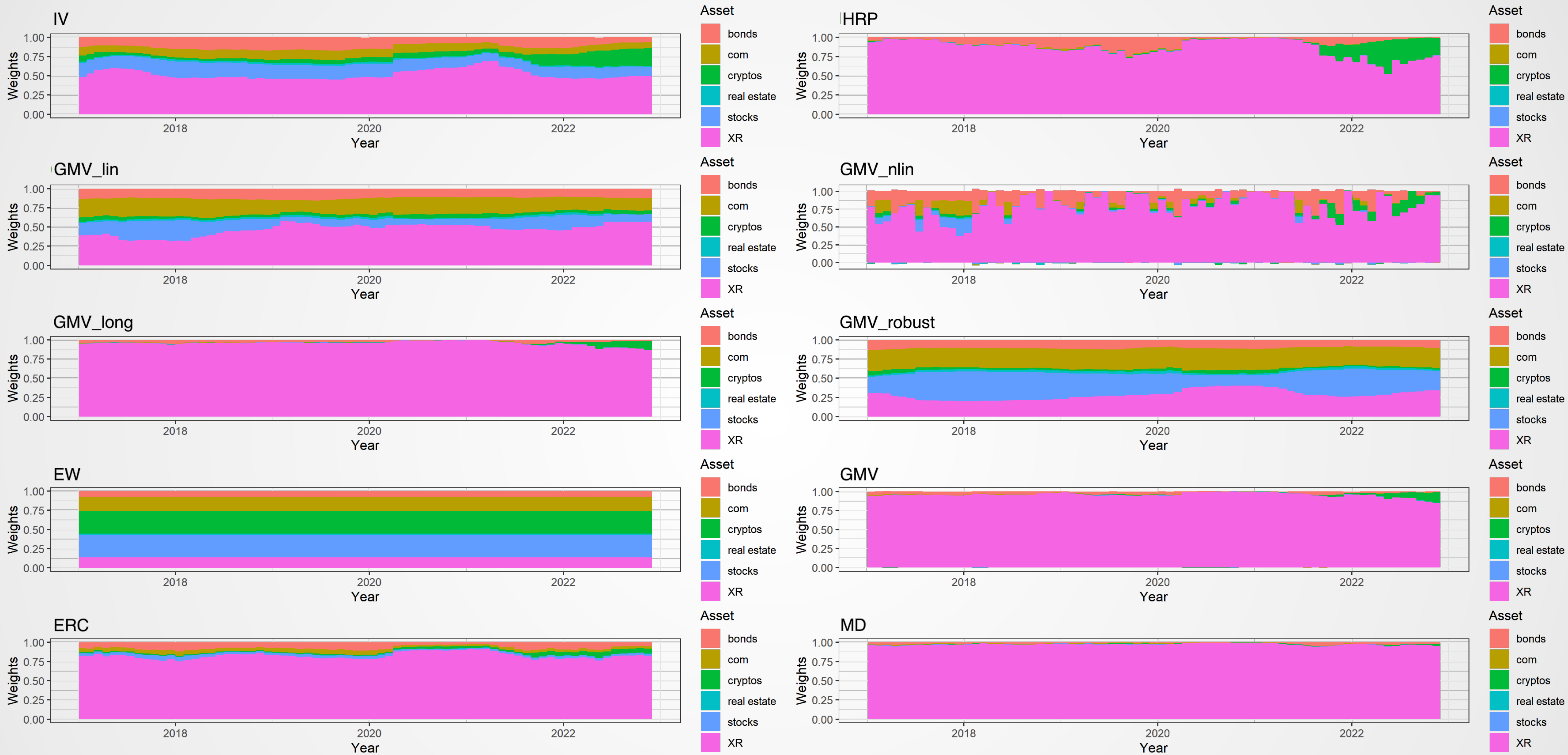


# Dynamics of weights: asset classes




Aggregated portfolio weights for 6 asset classes, 20160101- 20221130

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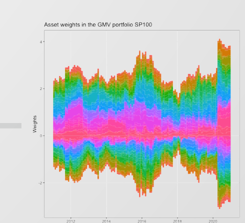
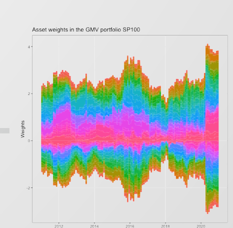


# Descriptive statistics for weights

| | min | max | sd | mad-ew | max-min |
|---|---|---|---|---|---|
| EW | 0.0106 | 0.0106 | 0.0000 | 0.0000 | 0.0000 |
| GMV_long | 0.0000 | 0.4035 | 0.0595 | 0.0197 | 0.4035 |
| GMV_lin | -0.0061 | 0.0448 | 0.0136 | 0.0109 | 0.0509 |
| GMV_nlin | -0.0198 | 0.2521 | 0.0435 | 0.0184 | 0.2719 |
| GMV | -0.0069 | 0.4069 | 0.0600 | 0.0198 | 0.4138 |
| ERC | 0.0001 | 0.2723 | 0.0395 | 0.0162 | 0.2722 |
| MD | 0.0000 | 0.4625 | 0.0665 | 0.0195 | 0.4625 |
| GMV_robust | -0.0062 | 0.0262 | $\boxed{0.0089}$ | $\boxed{0.0076}$ | 0.0324 |
| HRP | 0.0000 | 0.8181 | 0.0858 | 0.0196 | 0.8181 |
| invvol | 0.0005 | 0.2718 | 0.0300 | 0.0107 | 0.2712 |

RobustM_PerformanceCC

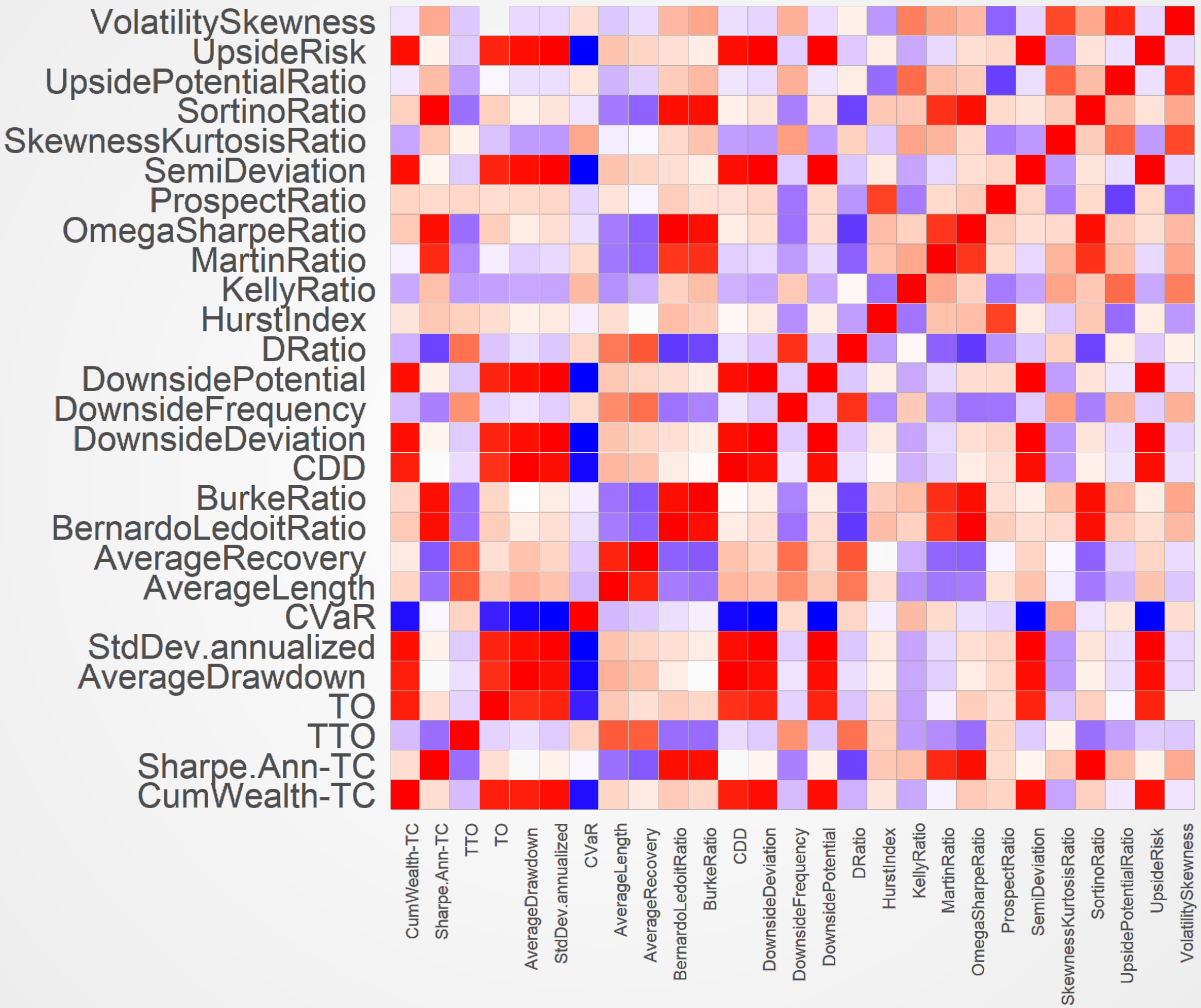


# Linear Factor model

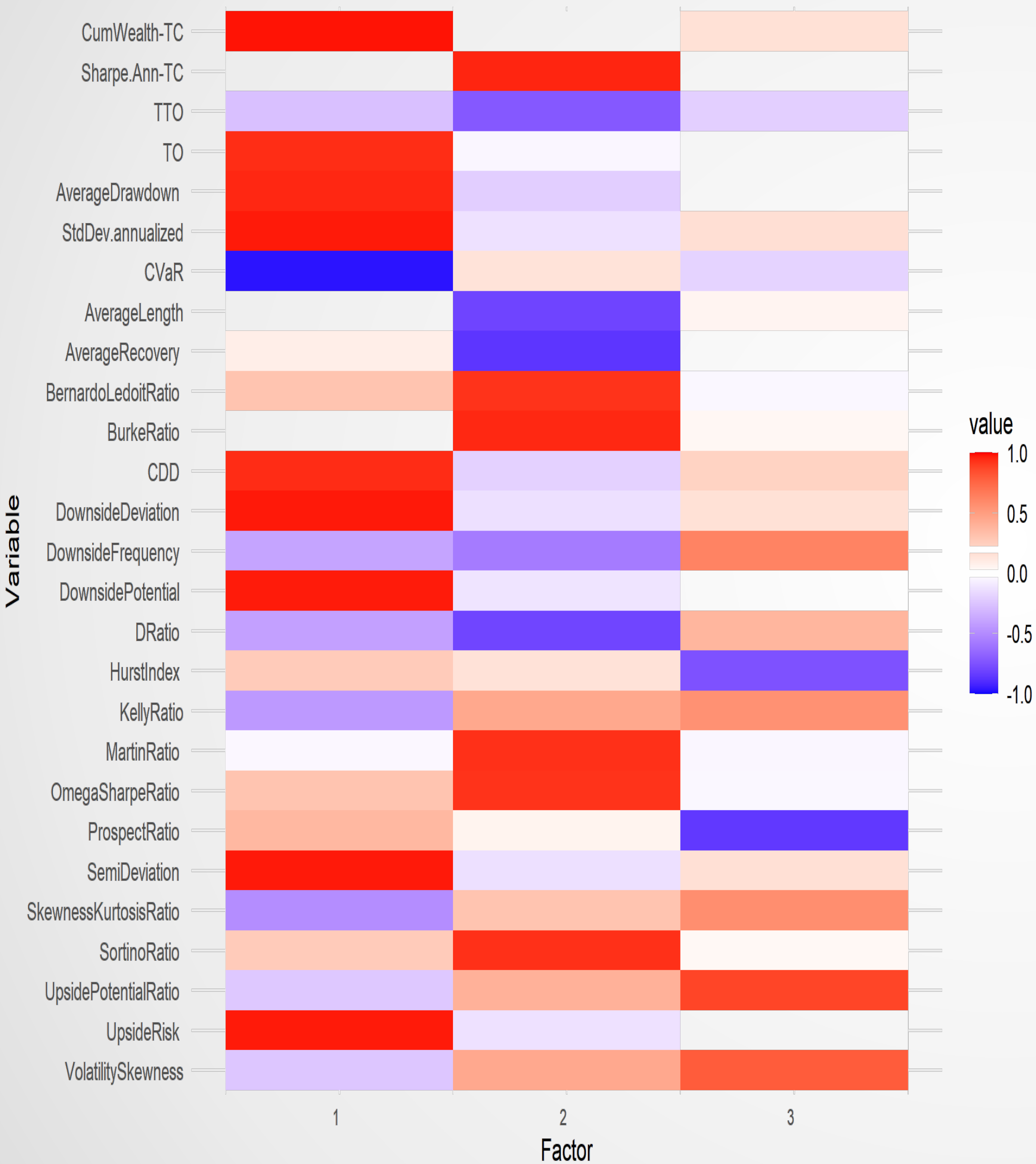
**Performance metrics**

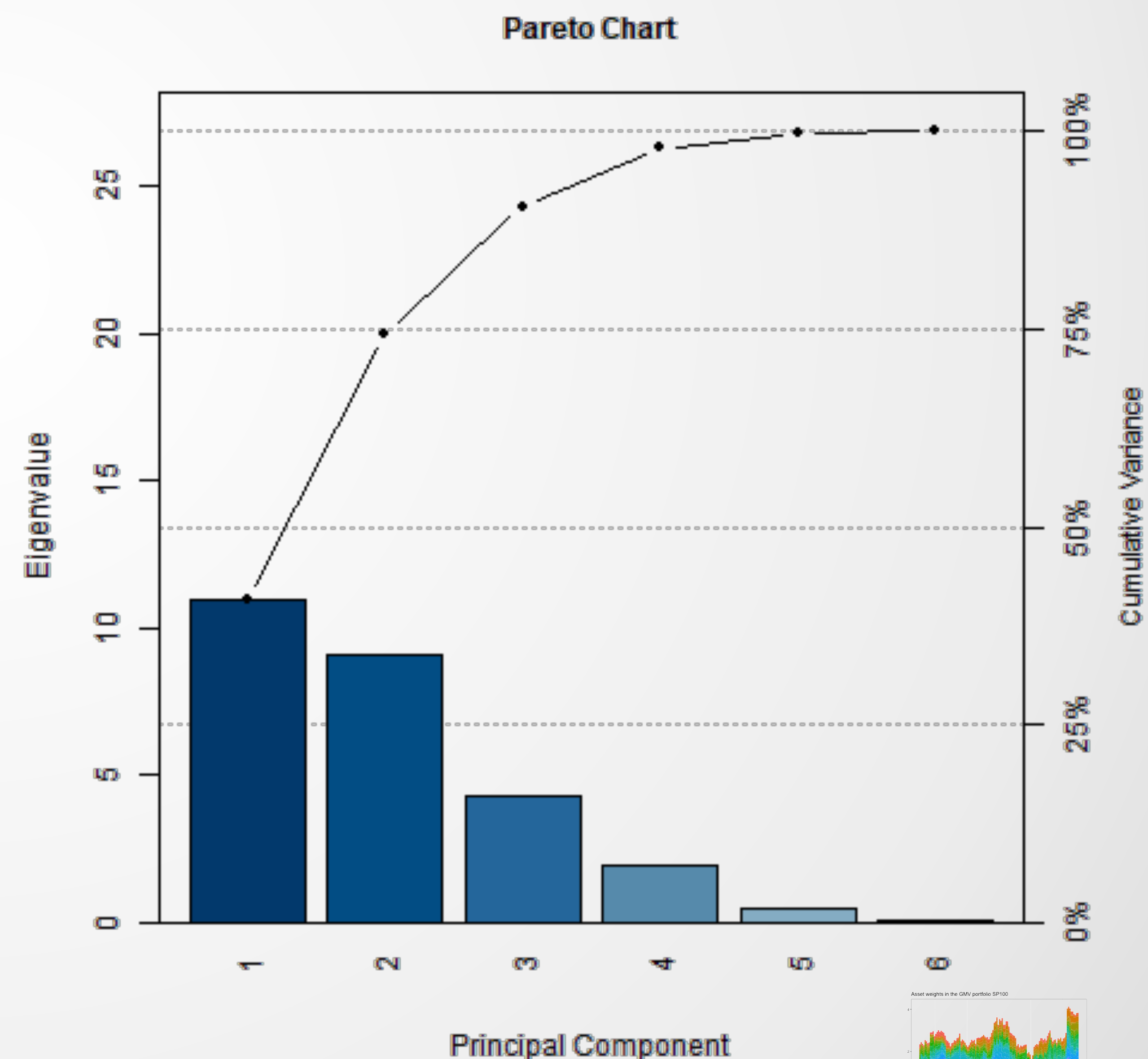
- $X = QF + \mu + \varepsilon, \varepsilon \sim G()$
  - $X$ is the initial matrix of $p$ variables - Portfolio performance metrics ( # 27)
  - $Q$ is a matrix of the non-random loadings
  - $F$ are the common $k$ factors ($k < p$)
  - $\mu$ is the vector of the means of initial $p$ variables
  - $\varepsilon$ is a matrix of the random specific factors
  - Random vectors $F$ and $\varepsilon$ are unobservable and uncorrelated

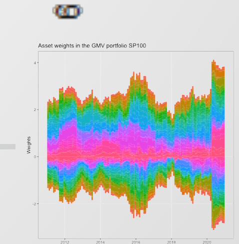
# Correlation matrix




# Factors loadings and scree plot



# Mappings of the factors

| | Correlation | p-value |
|---|---|---|
| CumWealth-TC | 0.99 | 0.00 |
| SemiDeviation | 0.98 | 0.00 |
| DownsideDeviation | 0.98 | 0.00 |
| StdDev.annualized | 0.98 | 0.00 |
| UpsideRisk | 0.98 | 0.00 |
| DownsidePotential | 0.97 | 0.00 |
| AverageDrawdown | 0.95 | 0.00 |
| CDD | 0.94 | 0.00 |
| TO | 0.94 | 0.00 |
| CVaR | -0.97 | 0.00 |

1. Wealth factor – 41% of the total variance

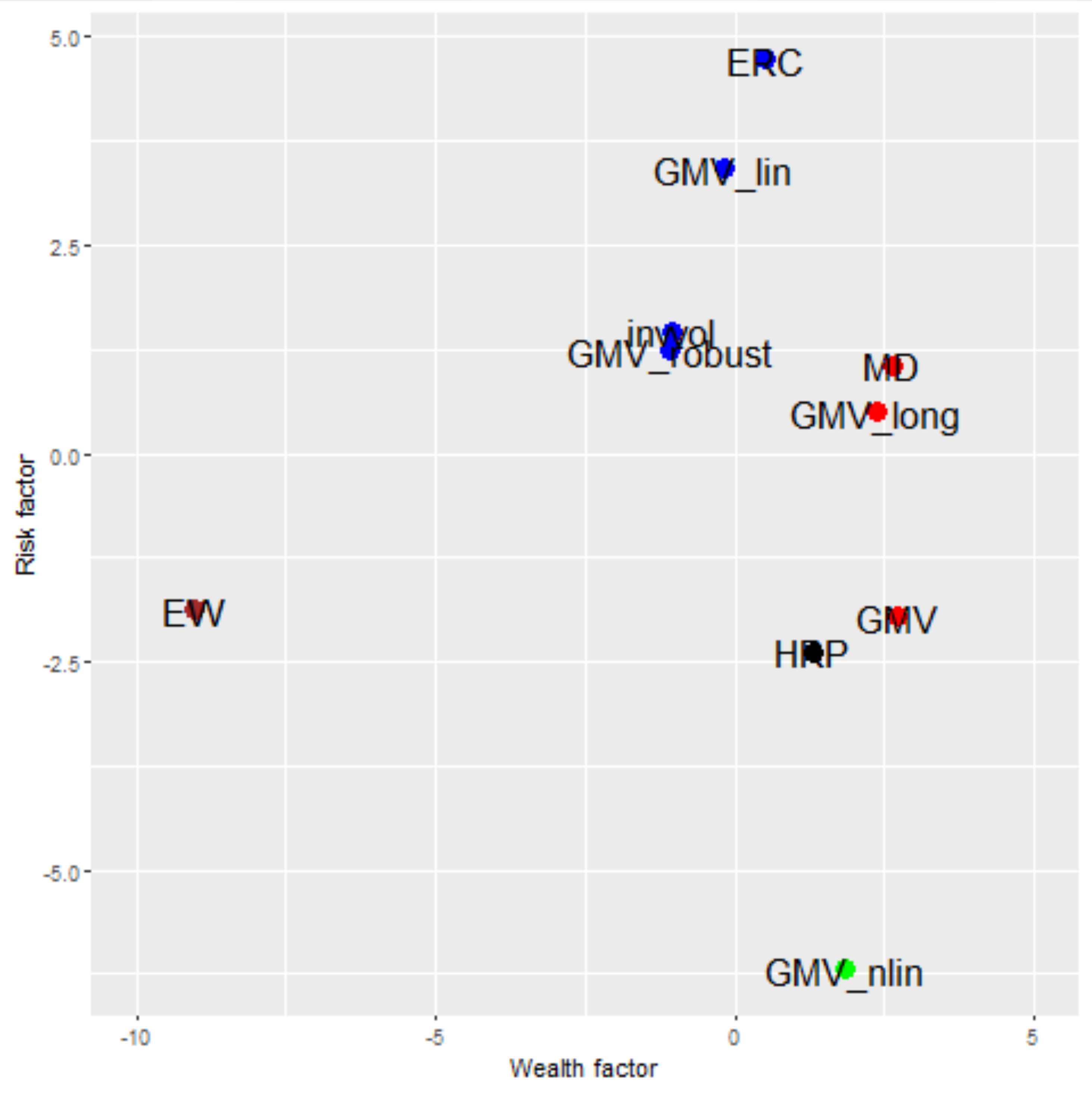
| | Correlation | p-value |
|---|---|---|
| Sharpe.Ann-TC | 0.96 | 0.00 |
| BurkeRatio | 0.95 | 0.00 |
| SortinoRatio | 0.94 | 0.00 |
| MartinRatio | 0.94 | 0.00 |
| BernardoLedoitRatio | 0.93 | 0.00 |
| OmegaSharpeRatio | 0.93 | 0.00 |
| TTO | -0.72 | 0.02 |
| DRatio | -0.80 | 0.01 |
| AverageLength | -0.80 | 0.00 |
| AverageRecovery | -0.86 | 0.00 |

2. Risk factor – 34% of the total variance

| | Correlation | p-value |
|---|---|---|
| UpsidePotentialRatio | 0.88 | 0.00 |
| VolatilitySkewness | 0.80 | 0.01 |
| HurstIndex | -0.75 | 0.01 |
| ProspectRatio | -0.85 | 0.00 |

3. Risk-adjusted return factor– 16% of the total variance

# Mappings of the factors

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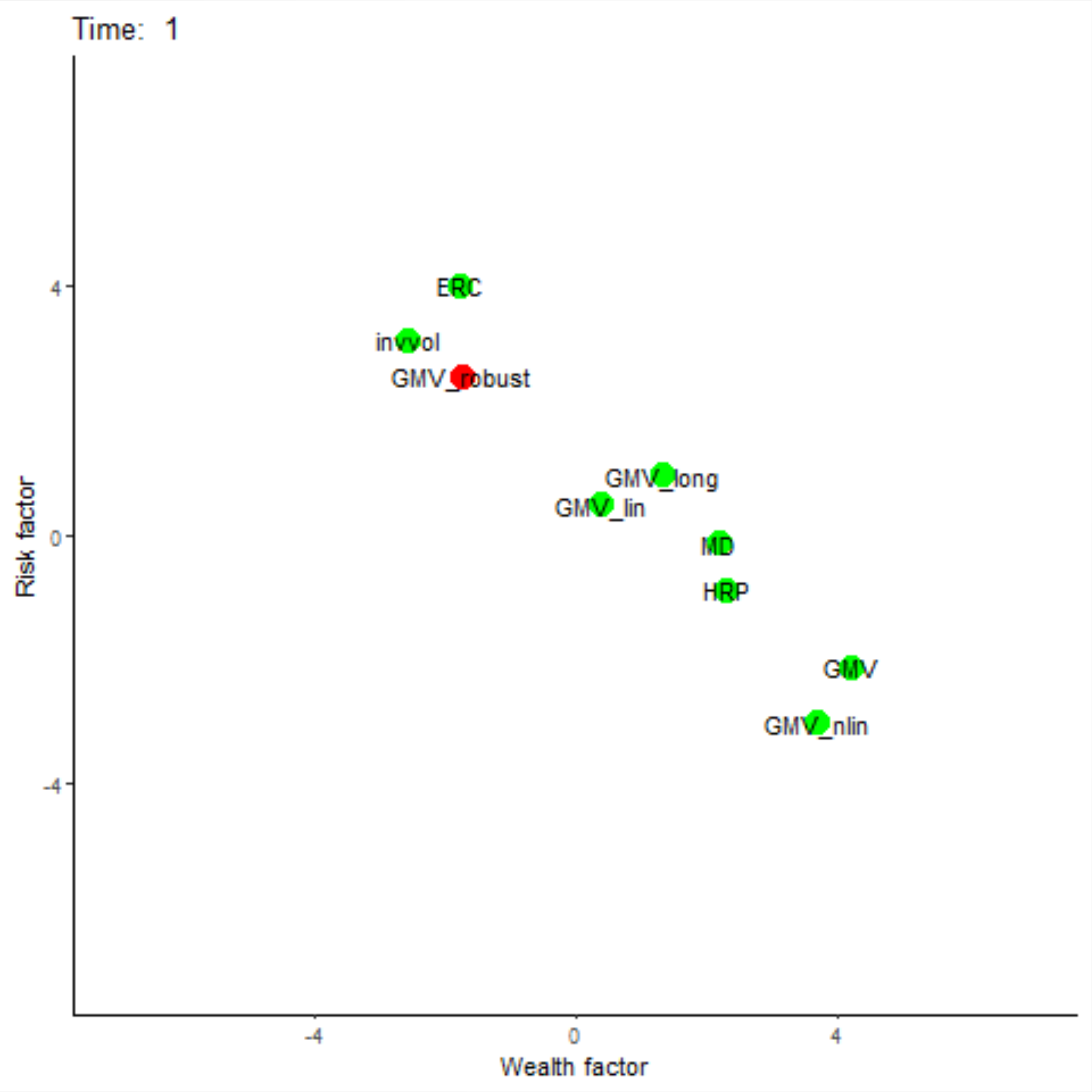
# Dynamic performance

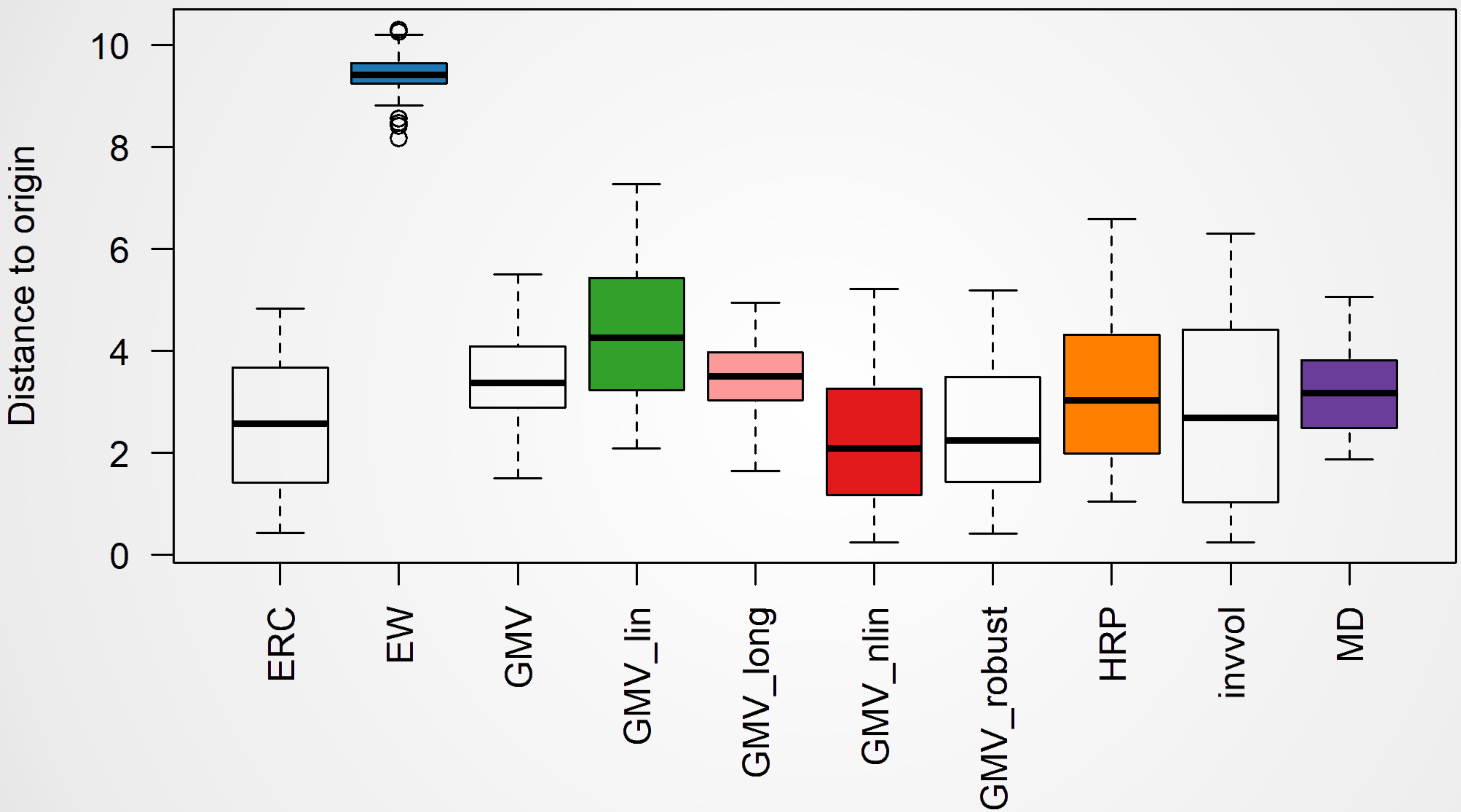
- Expanding window estimation
- Starting window 20160101- 20171231 (2 years of data)
- Increases monthly up to full window 20160101- 20221130 (60 windows)



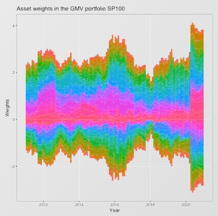

# Distances to the origin

| | Benchmark | Mean | Median | SD | Min | Max |
|---|---|---|---|---|---|---|
| 1 | ERC | 2.62 | 2.58 | 1.29 | 0.44 | 4.82 |
| 2 | EW | 9.45 | 9.43 | 0.47 | 8.18 | 10.31 |
| 3 | GMV | 3.48 | 3.37 | 0.93 | 1.50 | 5.50 |
| 4 | GMV_lin | 4.28 | 4.26 | 1.42 | 2.09 | 7.28 |
| 5 | GMV_long | 3.45 | 3.50 | 0.75 | 1.64 | 4.95 |
| 6 | GMV_nlin | 2.22 | 2.08 | 1.17 | 0.24 | 5.22 |
| 7 | GMV_robust | 2.41 | 2.25 | 1.29 | 0.41 | 5.18 |
| 8 | HRP | 3.31 | 3.02 | 1.58 | 1.04 | 6.58 |
| 9 | IV | 2.81 | 2.69 | 1.89 | 0.24 | 6.30 |
| 10 | MD | 3.24 | 3.18 | 0.83 | 1.88 | 5.06 |

# Distances to the origin



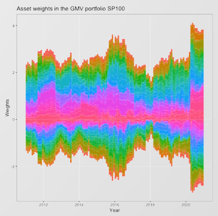

Conv_Portfolios



# Conclusion & Outlook

- Features of Robust Markowitz portfolio for a diversified investment universe
  - More stable weights
  - Stable risk-return profile (factor-model based mapping)
  - Better diversification potential
  - Out-of-sample performance and TO are in the same ballpark

- Outlook
  - Add NFTs to the investment universe
  - Check diversification benefits
  - ICA for building of the strategies' map



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# Robustified Markowitz approach diversified portfolios with crypto-assets

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# Performance measures

- TTO-Target turnover

$$TTO = L^{-1} \sum_{l=1}^{L} \sum_{j=1}^{N} \left| \widehat{w}_{j,l+1} - \widehat{w}_{j,l} \right|$$

- TO - Turnover

$$TO = L^{-1} \sum_{l=1}^{L} \sum_{j=1}^{N} \left| \widehat{w}_{j,l+1} - \widehat{w}_{j,l+} \right|$$

where $w_{i,j,l}$ and $w_{i,j,l+1}$ are the weights assigned to asset $j$ for periods $l$ and $l+1$,

$w_{i,j,l+}$ - weight just before rebalancing at $l+1$,

$L$ the number of rebalancing periods

$T$ - length of the estimation window

Back

# Performance measures

- Transactional Costs

$$TC_i = \sum_{t=1}^{L-T} \sum_{j=1}^{N} tc \left| \widehat{w}_{i,j,l+1} - \widehat{w}_{i,j,l} \right|$$

where $tc = 50bp$ is a proportional transactions cost per transaction

- Performance Analytics package for R

Back

## Performance measures

- Sharpe Ratio

$$SR = \frac{AV}{SD}$$

where $AV_i$ and $SD_i$ are annualised mean and standard deviation of out-of-sample returns of strategy $i$.

- CumWealth: Cumulative Wealth

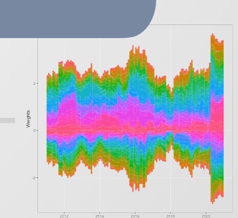
$$W_{l+1} = W_l + \widehat{w}_l^{\top} X_{l+1}$$

where $W_l$ wealth at the rebalancing period $l$ and initial portfolio $W_0 = 1\$$

- $mad - ew$: Mean absolute deviation from EW portfolio

$$mad - ew_l = N^{-1} \sum_{i=1}^{N} \left| \widehat{w}_{i,l} - N^{-1} \right|$$

Back



# M-estimator (DeMiguel & Nogalles, 2009)

$$m = \arg\min_{m} \frac{1}{T} \sum_{t=1}^{T} \rho(\mathrm{w}^{\top} r_t - m).$$

Particular cases of M-estimators are the sample mean and variance, which are obtained for $\rho(r) = 0.5r^2$, and the median and MAD, for $\rho(r) = |r|$. In our numerical experiments we focus on the M-estimators derived from Huber's loss function

$$\rho(r) = \begin{cases} r^2/2, & |r| \leqslant c, \\ c(|r| - c/2), & |r| > c, \end{cases} \tag{8}$$