



Time Series Analysis and Forecasting

Chapter 3: ARIMA Models



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Learning Objectives

By the end of this chapter, you will be able to:

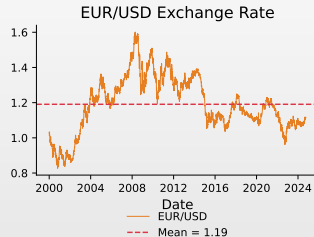
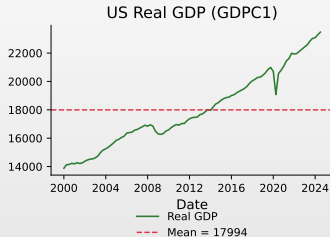
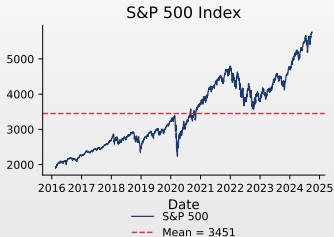
1. **Understand** the concept and implications of non-stationarity
2. **Apply** differencing to achieve stationarity in time series
3. **Use** the Augmented Dickey-Fuller (ADF) test for unit root detection
4. **Build**, estimate, and forecast with ARIMA models

Outline

- ▣ Motivation
- ▣ Non-Stationarity in Time Series
- ▣ Differencing and the Difference Operator
- ▣ ARIMA(p,d,q) Models
- ▣ Unit Root Tests
- ▣ ARIMA Model Identification
- ▣ ARIMA Estimation
- ▣ Diagnostic Checking
- ▣ Forecasting with ARIMA
- ▣ Case Study
- ▣ Summary
- ▣ AI Use Case
- ▣ Quiz

Motivating Example: Non-Stationary Data Is Everywhere

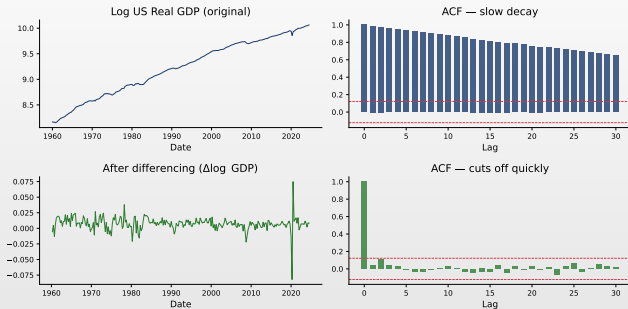
Non-stationary data: sample mean is meaningless



Key Observations

- Stock prices, GDP, exchange rates all exhibit **trends** or **wandering behavior**
- The sample mean (red line) is not a consistent estimator for a non-stationary process
- Standard ARMA models **cannot** handle these series directly

The Solution: Differencing



Key Insight

Differencing transforms a non-stationary series into a stationary one: $\Delta Y_t = Y_t - Y_{t-1}$. The ACF changes from slow decay to quick decay!

Why Non-Stationarity Matters

The Problem

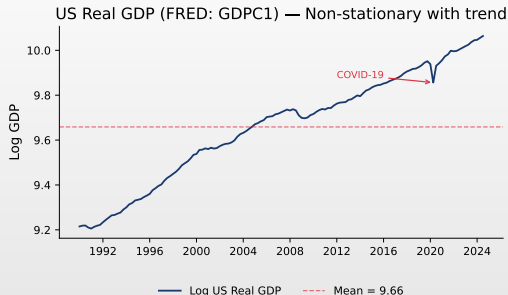
Many economic and financial time series are **non-stationary**:

- ▣ GDP, stock prices, exchange rates, inflation indices
- ▣ They exhibit trends, changing means, or growing variance

Consequences of Non-Stationarity

- ▣ Standard ARMA models assume stationarity
- ▣ OLS regression with non-stationary data leads to **spurious regression**
- ▣ Sample moments (mean, variance, ACF) are not consistent estimators
- ▣ Statistical inference becomes invalid

Example: US Real GDP



Key Observations

- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data

Types of Non-Stationarity

Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- ▣ Trend is a deterministic function of time
- ▣ Can be removed by **detrending**
- ▣ Shocks have temporary effects

Stochastic Trend (Unit Root)

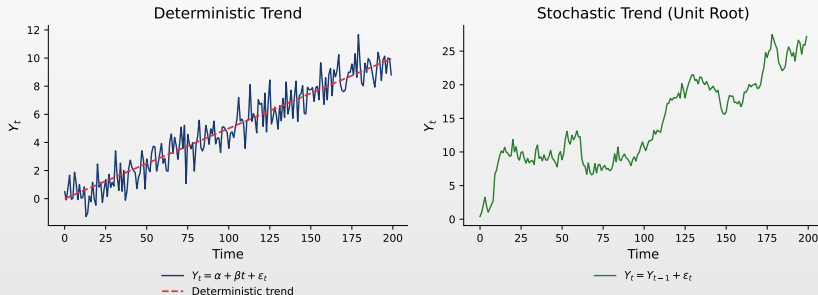
$$Y_t = Y_{t-1} + \varepsilon_t$$

- ▣ Random walk process
- ▣ Must be removed by **differencing**
- ▣ Shocks have permanent effects

Key Distinction

Correct identification is crucial: detrending a unit root \Rightarrow misspecification; differencing trend-stationary \Rightarrow misspecification.

Visualizing the Difference



Key Distinction

- **Left:** Deterministic trend – deviations from trend are temporary
- **Right:** Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!

The Random Walk Process

Definition 1 (Random Walk)

A **random walk** is defined as:

$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

With initial condition $Y_0 = 0$, we have: $Y_t = \sum_{i=1}^t \varepsilon_i$

Properties of Random Walk

- ▣ $\mathbb{E}[Y_t] = 0$ (constant mean)
- ▣ $\text{Var}(Y_t) = t\sigma^2$ (variance grows with time!)
- ▣ $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$ for $k \leq t$
- ▣ ACF: $\rho_k = \sqrt{\frac{t-k}{t}} \rightarrow 1$ as $t \rightarrow \infty$

Proof: Random Walk Variance

Claim: For $Y_t = Y_{t-1} + \varepsilon_t$ with $Y_0 = 0$: $\text{Var}(Y_t) = t\sigma^2$

Proof: By recursive substitution: $Y_t = \sum_{i=1}^t \varepsilon_i$

Taking variance:

$$\text{Var}(Y_t) = \text{Var}\left(\sum_{i=1}^t \varepsilon_i\right) = \sum_{i=1}^t \text{Var}(\varepsilon_i) + 2 \sum_{i < j} \text{Cov}(\varepsilon_i, \varepsilon_j)$$

Since ε_t independent (white noise): $\text{Var}(Y_t) = \sum_{i=1}^t \sigma^2 = \boxed{t\sigma^2}$

Variance depends on $t \Rightarrow$ non-stationary

Proof: Random Walk Autocovariance

Claim: $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$ for $k \leq t$

Proof: Using $Y_t = \sum_{i=1}^t \varepsilon_i$ and $Y_{t-k} = \sum_{i=1}^{t-k} \varepsilon_i$:

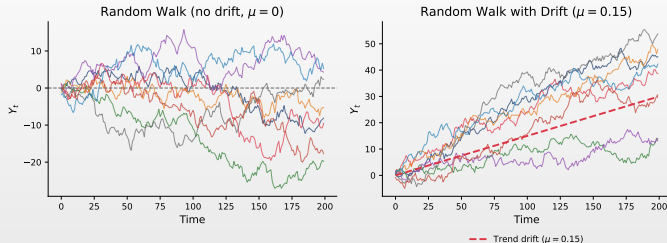
$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}\left(\sum_{i=1}^t \varepsilon_i, \sum_{j=1}^{t-k} \varepsilon_j\right) \\ &= \sum_{i=1}^t \sum_{j=1}^{t-k} \text{Cov}(\varepsilon_i, \varepsilon_j) = \sum_{i=1}^{t-k} \text{Var}(\varepsilon_i) = \boxed{(t-k)\sigma^2}\end{aligned}$$

Only terms with $i = j$ are nonzero (when $i \leq t-k$).

ACF:

$$\rho(k) = \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} = \frac{(t-k)\sigma^2}{\sqrt{t\sigma^2 \cdot (t-k)\sigma^2}} = \sqrt{\frac{t-k}{t}}$$

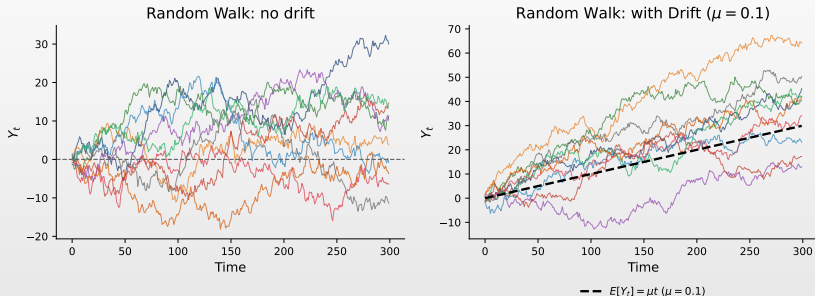
Random Walk with Drift



$$Y_t = \mu + Y_{t-1} + \varepsilon_t \quad \Leftrightarrow \quad Y_t = Y_0 + \mu t + \sum_{i=1}^t \varepsilon_i$$

- $\mathbb{E}[Y_t] = Y_0 + \mu t$ (mean grows linearly); $\text{Var}(Y_t) = t\sigma^2$ (variance still grows)
- **Without drift** (blue): oscillates randomly around zero; **With drift** $\mu > 0$ (red): systematic upward trend
- Both are non-stationary — drift adds deterministic trend to stochastic wandering

Simulating Random Walks



Random Walk Types

- Left: Pure random walks – no drift, wander unpredictably
- Right: Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time

Integrated Processes

Definition 2 (Integrated Process of Order d)

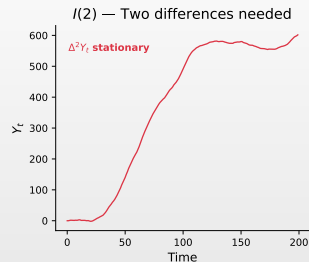
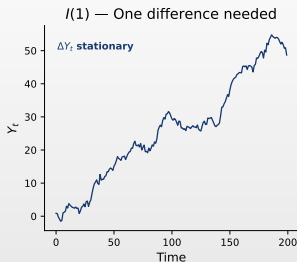
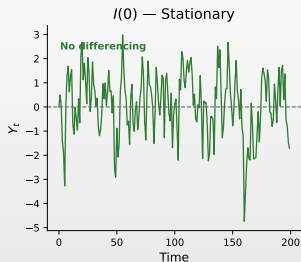
A time series $\{Y_t\}$ is **integrated of order d** , written $Y_t \sim I(d)$, if:

- ▣ Y_t is non-stationary
- ▣ $(1 - L)^d Y_t = \Delta^d Y_t$ is stationary
- ▣ $(1 - L)^{d-1} Y_t$ is still non-stationary

Common Cases

- ▣ $I(0)$: Stationary process (e.g., ARMA)
- ▣ $I(1)$: First difference is stationary (most common for economic data)
- ▣ $I(2)$:
 - ▶ Second difference is stationary (less common)

Integrated Process: Visual Illustration



Order of Integration

- $I(0)$: Stationary \Rightarrow no differencing needed
- $I(1)$: One difference needed (random walk)
- $I(2)$: Two differences needed
- Most economic series are $I(0)$ or $I(1)$

The Difference Operator

Definition 3 (First Difference)

The **first difference operator** Δ is defined as: $\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$, where L is the lag operator ($LY_t = Y_{t-1}$).

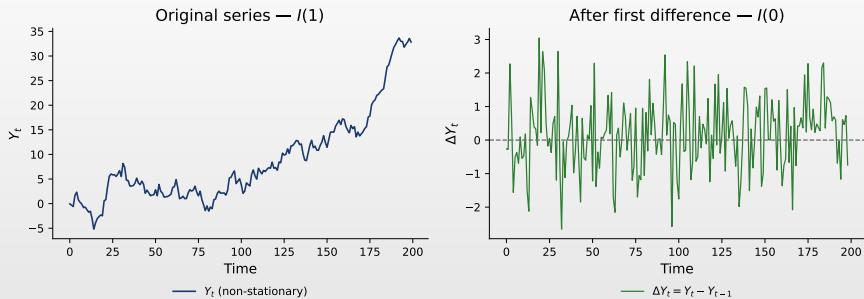
Higher-Order Differences

- ▣ Second difference: $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- ▣ $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- ▣ d -th difference: $\Delta^d Y_t = (1 - L)^d Y_t$

Key Result

If $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$ (stationary).

First Difference: Visual Illustration



Observation

- Left: non-stationary series
- Right: after first difference, the series becomes stationary

Example: Differencing a Random Walk

Random Walk to White Noise

Let $Y_t = Y_{t-1} + \varepsilon_t$ (random walk). Taking the first difference:

$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

Interpretation

- ▣ A random walk is $I(1)$
- ▣ One difference transforms it to $I(0)$
- ▣ The “changes” in a random walk are stationary

Proof: Differencing Induces Stationarity

Claim: If $Y_t \sim I(1)$, then $\Delta Y_t = Y_t - Y_{t-1}$ is stationary.

Proof for Random Walk with Drift: $Y_t = \mu + Y_{t-1} + \varepsilon_t$

The first difference is:

$$\Delta Y_t = Y_t - Y_{t-1} = \mu + \varepsilon_t$$

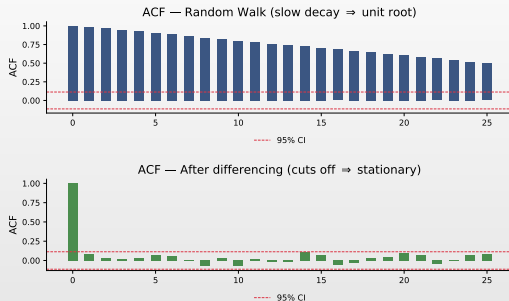
Check stationarity conditions:

1. **Mean:** $\mathbb{E}[\Delta Y_t] = \mu$ (constant, does not depend on t) ✓
2. **Variance:** $\text{Var}(\Delta Y_t) = \text{Var}(\varepsilon_t) = \sigma^2$ (constant) ✓
3. **Autocovariance:** $\text{Cov}(\Delta Y_t, \Delta Y_{t-k}) = \text{Cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$ for $k \neq 0$ ✓

General Principle

- Differencing removes the “memory” that causes variance to accumulate
- For $I(d)$ processes, d differences are needed

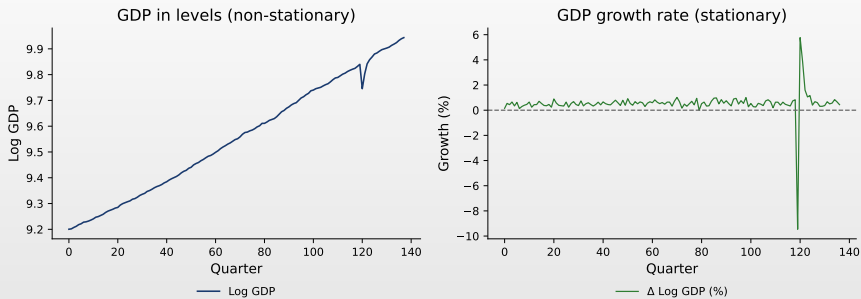
ACF Diagnostic: Detecting Non-Stationarity



ACF Patterns

- Top: Random walk ACF decays very slowly \Rightarrow unit root
- Bottom: After differencing, ACF cuts off \Rightarrow stationary

Differencing in Practice: GDP Example



Transformation

Left: GDP in levels with clear upward trend (non-stationary). **Right:** GDP growth rate $\Delta \log(GDP_t)$ fluctuates around constant mean (stationary). One difference removes the stochastic trend.

Overdifferencing

Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
 - ▶ ACF shows spurious patterns
- Inflates variance
 - ▶ Reduces forecast accuracy
- Loses information
 - ▶ Cannot recover original level

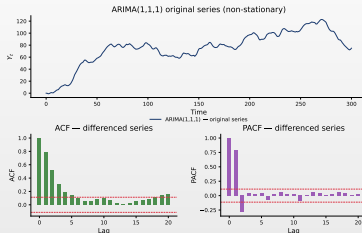
Example

If $Y_t \sim I(1)$, then $\Delta Y_t \sim I(0)$. But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with $\theta = 1$ (non-invertible boundary)!

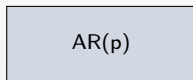
Definition of ARIMA



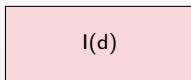
Definition 4 (ARIMA(p,d,q): $\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$)

- **AR:** $\phi(L) = 1 - \phi_1 L - \dots - \phi_p L^p$; **MA:** $\theta(L) = 1 + \theta_1 L + \dots + \theta_q L^q$; **I:** d differences
- **Top:** original ARIMA series (non-stationary); **Bottom:** after differencing — ACF/PACF reveal p, q

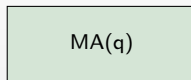
ARIMA Components



Autoregressive
Memory



Integration
Differencing



Moving Average
Shocks

Special Cases

- ▣ $ARIMA(p,0,q) = ARMA(p,q)$ – stationary
- ▣ $ARIMA(0,1,0) =$ Random walk
- ▣ $ARIMA(0,1,1) = IMA(1,1)$ – exponential smoothing
- ▣ $ARIMA(1,1,0) = ARI(1,1)$ – differenced $AR(1)$

ARIMA(1,1,0) Example

ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently: $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$

Interpretation

- ▣ The **changes** in Y_t follow an AR(1) process
- ▣ If $|\phi_1| < 1$, the changes are stationary
- ▣ Y_t itself has a stochastic trend
- ▣ Common model for many economic time series

ARIMA(0,1,1) Example

IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently: $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha) \hat{Y}_t$$

where $\alpha = 1 + \theta_1$ (for $-1 < \theta_1 < 0$).

The Role of the Constant in ARIMA

Constant Term in ARIMA(p,d,q)

When $d > 0$, the constant c has a different interpretation: $\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$

Important Implications

- For $d = 1$: c represents the **drift**
 - ▶ Average change: $\mathbb{E}[\Delta Y_t] = \frac{c}{1 - \phi_1 - \dots - \phi_p}$
 - ▶ Linear trend in levels
- For $d = 2$: c affects the **curvature**
 - ▶ Quadratic trend in levels
- Often $c = 0$ is assumed when $d \geq 1$
 - ▶ No deterministic trend component

Testing for Unit Roots

Why Test?

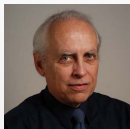
Before fitting an ARIMA model, we need to determine:

1. Is the series stationary? (Is $d = 0$?)
2. If not, how many differences are needed? (What is d ?)

Common Unit Root Tests

- **Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- **Phillips-Perron (PP)**
- **KPSS** (stationarity test – reversed null hypothesis)

Researcher Spotlight: Dickey & Fuller



David Dickey (*1945)



Wayne Fuller (1931–2022)



Biography

- **David Dickey:** American statistician at NC State University. PhD student of Wayne Fuller at Iowa State
- **Wayne Fuller:** American statistician, professor at Iowa State University
- Together they developed the foundational test for unit roots in time series

Key Contributions

- **Dickey-Fuller test** (1979) — the fundamental unit root test
- **Augmented Dickey-Fuller (ADF)** — extension with lagged differences
- **Critical value tables** — non-standard distributions under the null
- Enabled rigorous testing of integration order for ARIMA modeling

The Dickey-Fuller Test

Setup

Consider the AR(1) model: $Y_t = \phi Y_{t-1} + \varepsilon_t$. Subtract Y_{t-1} : $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$, where $\gamma = \phi - 1$.

Hypotheses

- H_0 : $\gamma = 0$ (unit root, $\phi = 1$, non-stationary)
- H_1 : $\gamma < 0$ (stationary, $|\phi| < 1$)

Key Issue

Under H_0 , the t -statistic does **not** follow a standard t -distribution! Must use Dickey-Fuller critical values.

Augmented Dickey-Fuller (ADF) Test

The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

Definition 5 (ADF Test)

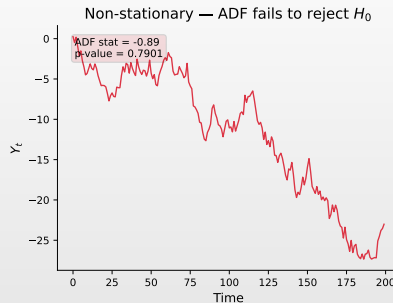
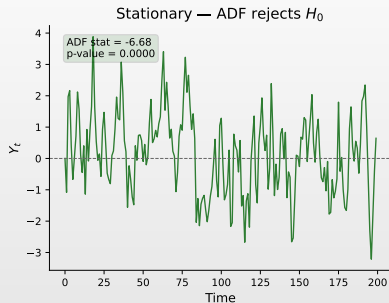
Add lagged differences: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^k \delta_j \Delta Y_{t-j} + \varepsilon_t$

Test $H_0 : \gamma = 0$ using ADF critical values.

Choosing Lag Length k

- Use information criteria (AIC, BIC)
- Start with k_{max} , reduce until last lag significant

ADF Test: Visual Illustration



Observation

- Left: stationary series \Rightarrow ADF rejects unit root
- Right: non-stationary \Rightarrow ADF fails to reject

ADF Test Critical Values

Model	1%	5%	10%
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

Decision Rule

- ▣ Test statistic $<$ critical value \Rightarrow Reject H_0 (stationary)
- ▣ Test statistic \geq critical value \Rightarrow Fail to reject (unit root)

The Phillips-Perron (PP) Test

Motivation

Like ADF, tests H_0 : Unit root vs H_1 : Stationary, but uses a **non-parametric correction** for serial correlation instead of adding lagged differences.

Test Statistic

The PP test modifies the DF t -statistic:

$$Z_t = t_{\hat{\gamma}} \cdot \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - \hat{\sigma}^2)(se(\hat{\gamma}))}{2\hat{\lambda}^2 \cdot s}$$

where $\hat{\lambda}^2$ is a consistent estimate of the long-run variance using Newey-West.

Advantages over ADF

- ▣ Robust to heteroskedasticity and serial correlation
- ▣ No need to select lag length (uses bandwidth instead)

The KPSS Test

Reversed Hypotheses

Unlike ADF: H_0 : Stationary vs H_1 : Unit root

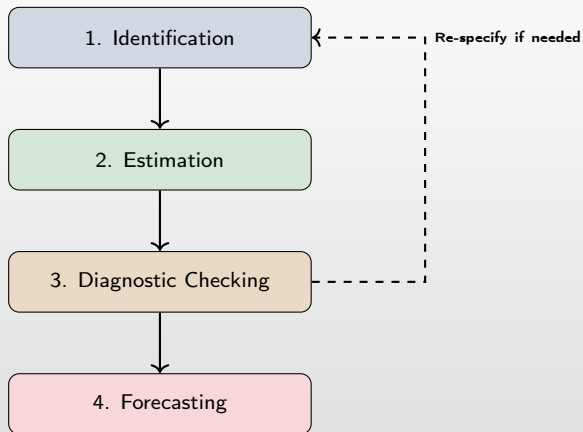
KPSS Procedure

Decompose: $Y_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$. Test whether $\text{Var}(u_t) = 0$.

Complementary Use with ADF

- ▣ ADF rejects, KPSS doesn't \Rightarrow Stationary
- ▣ ADF doesn't reject, KPSS rejects \Rightarrow Unit root
- ▣ Both reject or neither \Rightarrow Inconclusive

The Box-Jenkins Methodology



Step 1: Determining d

Procedure

1. Plot the time series – look for trends, changing variance
2. Examine ACF – slow decay suggests non-stationarity
3. Apply unit root tests (ADF, KPSS)
4. If non-stationary, difference and repeat

Practical Guidelines

- ▣ Most economic series: $d = 1$ is sufficient
- ▣ Rarely need $d > 2$
- ▣ If ACF of ΔY_t still decays slowly, try $d = 2$
- ▣ Watch for overdifferencing (ACF with $\rho_1 \approx -0.5$)

Step 2: Determining p and q

After Differencing

Once $W_t = \Delta^d Y_t$ is stationary, use ACF/PACF to identify ARMA(p, q):

Model	ACF	PACF
AR(p)	Decays exponentially	Cuts off after lag p
MA(q)	Cuts off after lag q	Decays exponentially
ARMA(p, q)	Decays	Decays

Information Criteria

When patterns are unclear, compare models using:

□ $AIC = -2 \ln(L) + 2k$; $BIC = -2 \ln(L) + k \ln(n)$

Lower is better. BIC penalizes complexity more.

Auto-ARIMA Algorithms

Automated Model Selection

Modern software can automatically select (p, d, q) :

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

How Auto-ARIMA Works

1. Use unit root tests to determine d
2. Fit models for various (p, q) combinations
3. Select model with lowest AIC/BIC
4. Optionally use stepwise search for efficiency

Caution

Automated selection is helpful but not infallible. Always check diagnostics!

Estimation Methods

Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- ▣ Assumes $\varepsilon_t \sim N(0, \sigma^2)$
- ▣ Maximizes the likelihood function
- ▣ Provides consistent, efficient estimators
- ▣ Yields standard errors for inference

Conditional vs Exact MLE

- ▣ **Conditional MLE:** Conditions on initial values
- ▣ **Exact MLE:** Treats initial values as unknown
- ▣ Difference diminishes as sample size grows

Conditional Log-Likelihood

Gaussian Log-Likelihood Function

- ▣ $\ell(\boldsymbol{\theta}, \sigma^2) = -\frac{T}{2} \ln(2\pi) - \frac{T}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^T e_t^2(\boldsymbol{\theta})$
- ▣ $e_t(\boldsymbol{\theta}) = X_t - \hat{X}_{t|t-1}$ are the **one-step prediction errors**
- ▣ $\boldsymbol{\theta} = (\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q, c)$

Example: ARIMA(1,1,1)

- ▣ Prediction errors: $e_t = \Delta X_t - \phi_1 \Delta X_{t-1} - \theta_1 e_{t-1} - c$
- ▣ Conditional MLE: set $e_0 = 0$, compute e_1, \dots, e_T , maximize ℓ

Estimating σ^2

- ▣ At optimal parameters $\hat{\boldsymbol{\theta}}$: $\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^T e_t^2(\hat{\boldsymbol{\theta}})$

Parameter Constraints

Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- ▣ **AR stationarity:** Roots of $\phi(z) = 0$ outside unit circle
- ▣ **MA invertibility:** Roots of $\theta(z) = 0$ outside unit circle

Checking in Practice

Most software reports:

- ▣ Estimated coefficients with standard errors
- ▣ Roots of AR and MA polynomials
- ▣ Warning if near-unit-root detected

Residual Analysis

What to Check

If the model is correct, residuals $\hat{\varepsilon}_t$ should be white noise:

1. Zero mean
2. Constant variance
3. No autocorrelation
4. (Optional) Normality

Diagnostic Tools

- ▣ **Residual ACF/PACF:** Should show no significant spikes
- ▣ **Ljung-Box test:** Tests for autocorrelation at multiple lags
- ▣ **Q-Q plot:** Checks normality assumption
- ▣ **Residual vs fitted:**
 - ▶ Checks for heteroskedasticity

The Ljung-Box Test

Definition 6 (Ljung-Box Q Statistic)

$Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}$. Under H_0 (no autocorrelation): $Q(m) \sim \chi^2(m - p - q)$

Usage

- Choose $m \approx \ln(n)$ or $m = 10$ for quarterly, $m = 20$ for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if $Q(m)$ exceeds critical value

If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

Point Forecasts

Minimum MSE Forecast

The optimal h -step ahead forecast is the conditional expectation: $\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \dots]$

ARIMA(1,1,1) Forecasting

Model: $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

One-step forecast: $\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1 \hat{\varepsilon}_T$

For $h > 1$: replace unknown ε_{T+j} with 0, unknown Y_{T+j} with $\hat{Y}_{T+j|T}$

Forecast Intervals

Forecast Uncertainty

The h -step forecast error variance: $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$, where ψ_j are $\text{MA}(\infty)$ coefficients.

Confidence Intervals

Under normality, $(1 - \alpha)\%$ interval: $\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$

Key Property for $I(1)$ Series

For integrated processes, forecast variance grows without bound as $h \rightarrow \infty$. Intervals widen over time!

Long-Run Forecasts for ARIMA

Behavior as $h \rightarrow \infty$

For ARIMA(p,1,q) with drift c :

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with \sqrt{h}

For ARIMA(p,1,q) without drift:

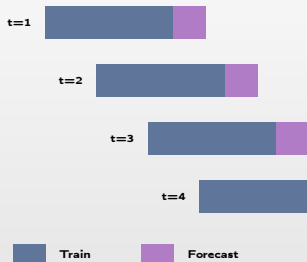
- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

Practical Implication

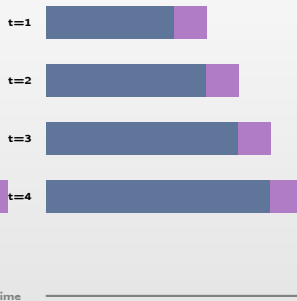
- ARIMA forecasts are most reliable for short horizons
- Long-term forecasts have very wide uncertainty bands

Rolling Forecasting: Fixed vs Expanding Window

Fixed Window (Rolling)



Expanding Window



Out-of-sample forecast accuracy evaluation

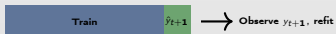
- ▣ **Fixed:** window slides forward, constant size w — adapts to regime changes
- ▣ **Expanding:** window grows over time — uses all historical data
- ▣ Mimics real-time forecasting scenario; provides multiple forecast errors for evaluation

1-Step vs Multi-Step Forecasting

1-Step Ahead (Recursive)

- Forecast only next period
 - Refit model after each step
 - Use actual value once revealed
- Most accurate for short horizons

1-Step Ahead



Multi-Step (Direct)

- Forecast multiple periods ahead
 - No refit between steps
 - Uses forecasted values as inputs
- Uncertainty compounds over horizon

Multi-Step (h=3)



Rolling Forecast: Step-by-Step Example

Setup: ARIMA(1,1,0) with $\phi_1 = 0.6$

Model: $\Delta Y_t = \phi_1 \Delta Y_{t-1} + \varepsilon_t$ where $\Delta Y_t = Y_t - Y_{t-1}$

Given Data at Time T

$$Y_{T-2} = 100, \quad Y_{T-1} = 103, \quad Y_T = 108 \quad \Rightarrow \quad \Delta Y_{T-1} = 3, \quad \Delta Y_T = 5$$

1-Step Ahead Point Forecast

$$\begin{aligned}\Delta \hat{Y}_{T+1|T} &= \phi_1 \cdot \Delta Y_T = 0.6 \times 5 = 3 \\ \hat{Y}_{T+1|T} &= Y_T + \Delta \hat{Y}_{T+1|T} = 108 + 3 = \boxed{111}\end{aligned}$$

Multi-Step Point Forecasts

2-Step Ahead Forecast

$$\begin{aligned}\Delta \hat{Y}_{T+2|T} &= \phi_1 \cdot \Delta \hat{Y}_{T+1|T} = 0.6 \times 3 = 1.8 \\ \hat{Y}_{T+2|T} &= \hat{Y}_{T+1|T} + \Delta \hat{Y}_{T+2|T} = 111 + 1.8 = \boxed{112.8}\end{aligned}$$

General Formula for h -Step Forecast (ARIMA(1,1,0))

$$\begin{aligned}\Delta \hat{Y}_{T+h|T} &= \phi_1^h \cdot \Delta Y_T \\ \hat{Y}_{T+h|T} &= Y_T + \Delta Y_T \cdot \frac{\phi_1(1 - \phi_1^h)}{1 - \phi_1}\end{aligned}$$

Numerical: 3-Step Forecast

$$\hat{Y}_{T+3|T} = 108 + 5 \times \frac{0.6(1-0.6^3)}{1-0.6} = 108 + 5 \times 1.092 = \boxed{113.46}$$

Confidence Intervals: Formulas

Forecast Error Variance

For ARIMA(1,1,0), the h -step forecast error variance:

$$\text{Var}(e_{T+h|T}) = \sigma^2 \left(1 + \sum_{j=1}^{h-1} \psi_j^2 \right)$$

where $\psi_j = \phi_1^{j-1}(1 + \phi_1 + \dots + \phi_1^{j-1}) = \phi_1^{j-1} \cdot \frac{1 - \phi_1^j}{1 - \phi_1}$

$(1 - \alpha)\%$ Confidence Interval

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(e_{T+h|T})}$$

For 95% CI: $z_{0.025} = 1.96$

Confidence Interval: Numerical Example

Given: $\sigma^2 = 4$, $\phi_1 = 0.6$, $\hat{Y}_{T+1|T} = 111$

1-Step Ahead CI

$$\text{Var}(e_{T+1|T}) = \sigma^2 = 4$$

$$95\% \text{ CI} = 111 \pm 1.96 \times \sqrt{4} = 111 \pm 3.92 = [107.08, 114.92]$$

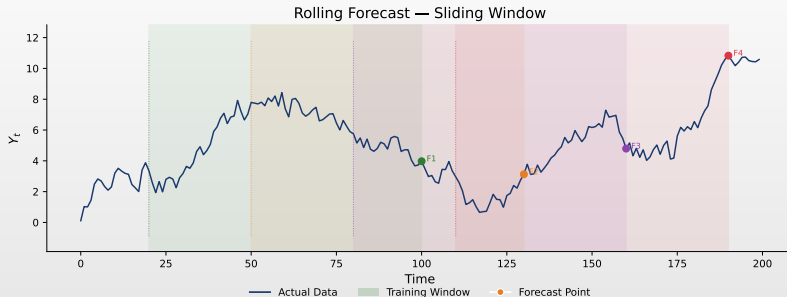
2-Step Ahead CI (for $\hat{Y}_{T+2|T} = 112.8$)

$$\psi_1 = 1 + \phi_1 = 1.6, \quad \text{Var}(e_{T+2|T}) = 4(1 + 1.6^2) = 14.24$$

$$95\% \text{ CI} = 112.8 \pm 1.96 \times \sqrt{14.24} = 112.8 \pm 7.40 = [105.40, 120.20]$$

Note: CI widens as horizon increases!

Rolling Window Illustration



Rolling Procedure

- Each window produces a 1-step ahead forecast
- Compare forecasts to actuals to compute RMSE, MAE
- Rolling window keeps model estimation up-to-date

Case Study: Complete ARIMA Analysis

Objective

- Forecast US Real GDP using the Box-Jenkins methodology

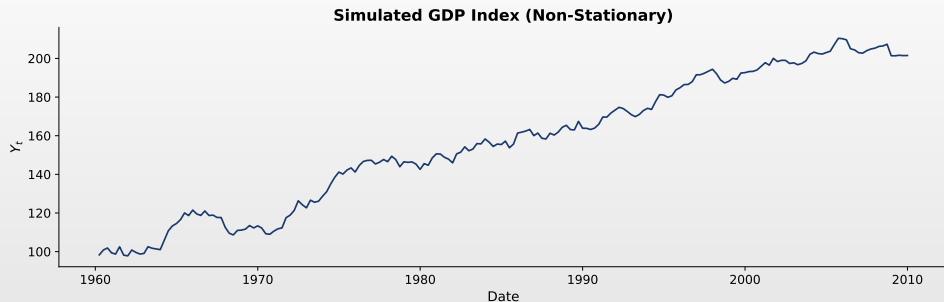
Steps

1. **Step 1:** Visualize data and check stationarity
2. **Step 2:** Apply unit root tests (ADF, KPSS)
3. **Step 3:** Difference if needed, identify p and q
4. **Step 4:** Estimate the ARIMA model
5. **Step 5:** Model diagnostics
6. **Step 6:** Generate forecasts with confidence intervals
7. **Step 7:** Evaluate forecast accuracy

Data

- US Real GDP (FRED: GDPC1), Quarterly, 1990Q1–2024Q2, $n = 138$

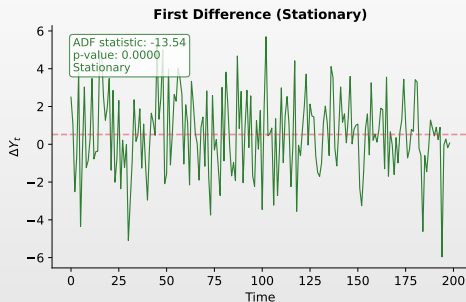
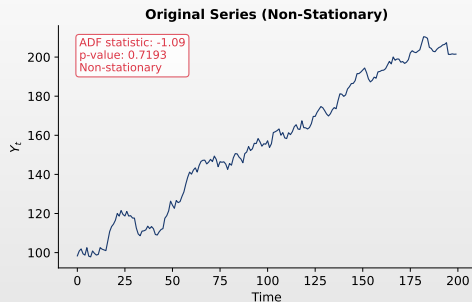
Case Study: US Real GDP (FRED)



Data: FRED GDPC1 (1960Q1–2024Q3)

Quarterly Real GDP, seasonally adjusted, billions of chained 2017 dollars. Non-stationary series with upward trend \Rightarrow requires differencing.

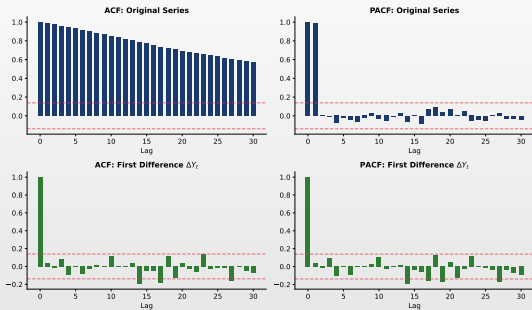
Step 1: ADF Test for Stationarity



ADF Test Results

Original series: Large p-value \Rightarrow fail to reject H_0 (unit root present). **First difference:** p-value $< 0.01 \Rightarrow$ reject $H_0 \Rightarrow d = 1$ is sufficient.

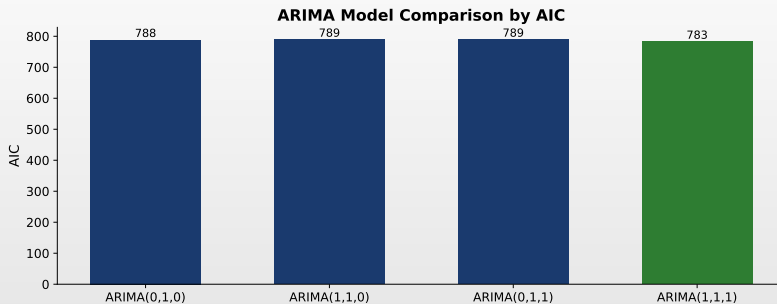
Step 2: ACF/PACF Before and After Differencing



ACF/PACF Analysis

Top: Slow ACF decay (non-stationary) | Bottom: After differencing, low-order ARMA

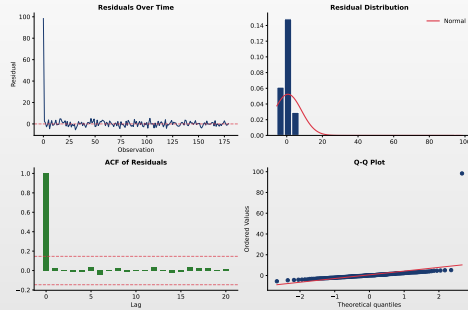
Step 3: ARIMA Model Comparison



Model Selection

Compare ARIMA(0,1,0), ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1). The model with lowest AIC is selected.

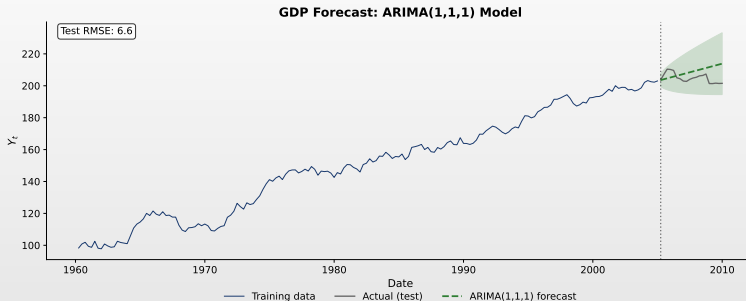
Step 4: Diagnostic Checking



ARIMA(1,1,1) Diagnostics

ACF: no autocorrelation ✓ Q-Q: **non-normal** (COVID-19 outlier) JB test: $p < 0.001$

Step 5: Out-of-Sample Forecasting



Train/Val/Test Split (70%/15%/15%)

Train 70% (blue): Estimation | **Val 15% (green):** Tuning | **Test 15% (purple):** Evaluation with 95% CI

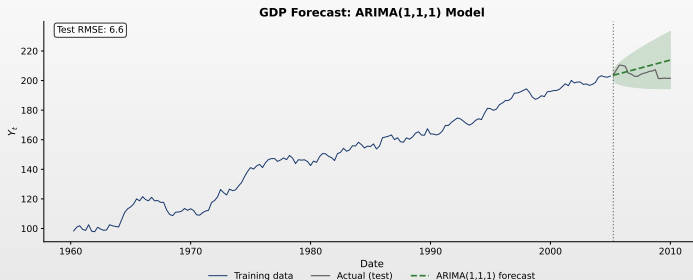
Step 6: Rolling Forecast with Train/Val/Test



Rolling 1-Step Ahead Forecast (Expanding Window, 95% CI)

Train 70% → Val 15% → Test 15% | Expanding window refits model at each step

Step 7: Forecast Evaluation



Out-of-sample performance (last 12 quarters)

- RMSE = 0.0486 \approx 4.86% error
- MAE = 0.0430 \approx 4.30% error
- Direction accuracy = 91% — correctly predicted growth/decline

Summary

What we learned in this chapter

- Non-stationarity in time series
 - ▶ Deterministic vs stochastic trend; consequences for statistical inference
- Differencing and integrated processes
 - ▶ $\Delta Y_t = Y_t - Y_{t-1}$; if $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$
- ARIMA(p, d, q) models and unit root tests
 - ▶ ADF, PP, KPSS; Box-Jenkins: identify \Rightarrow estimate \Rightarrow validate
- Forecasts with confidence intervals
 - ▶ For $I(1)$: CIs widen without bound ($\propto \sqrt{h}$)

Key Insight

- **Difference carefully:** One difference is usually sufficient ($d = 1$). Over-differencing creates artificial autocorrelation.

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

“Download quarterly US Real GDP from FRED (series GDPC1) for 2000-Q1 to 2024-Q4 (100 observations). Test stationarity, difference if needed, estimate an ARIMA model, and forecast 8 quarters ahead. Give me complete Python code with plots.”

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it test stationarity with ADF *before* estimating ARIMA? Does it also use KPSS?
3. How does it determine the differencing order d ? Does it check for over-differencing?
4. How does it choose p and q ? ACF/PACF or just `auto_arima`?
5. Do forecast confidence intervals widen with horizon? (key $I(1)$ property)

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

What's Next?

Chapter 4: SARIMA Models for Seasonal Data

- ▣ **Seasonality:** repetitive patterns at regular intervals
- ▣ **Seasonal differencing:** the $(1 - L^s)$ operator
- ▣ **SARIMA(p, d, q)(P, D, Q)_s:** seasonal extension of ARIMA
- ▣ **Model identification:** seasonal ACF/PACF
- ▣ **Case study:** Airline passengers forecast

Questions?

Question 1

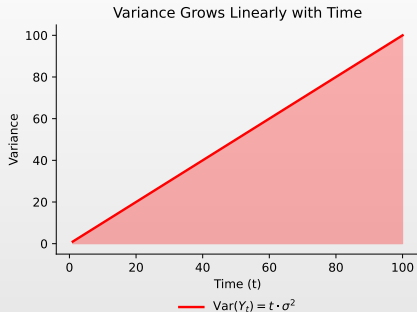
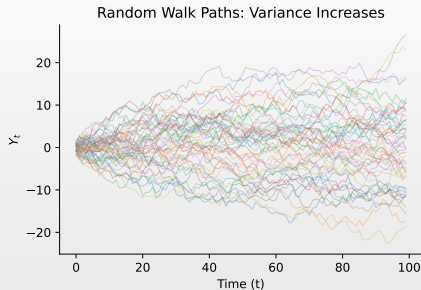
Question

□ A time series Y_t follows a random walk: $Y_t = Y_{t-1} + \varepsilon_t$. What is $\text{Var}(Y_t)$?

Answer Choices

- (A) σ^2 (constant)
- (B) $t \cdot \sigma^2$ (grows linearly with time)
- (C) σ^2/t (decreases with time)
- (D) σ^{2t} (grows exponentially)

Question 1: Answer



Answer: (B)

- ☐ Random walk variance grows linearly with time — this is why random walks are non-stationary.

Question 2

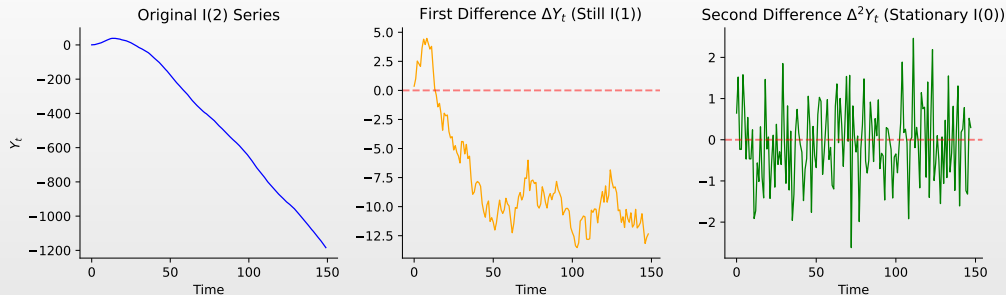
Question

□ If a series Y_t is $I(2)$, how many times must you difference it to achieve stationarity?

Answer Choices

- (A) 0 times (already stationary)
- (B) 1 time
- (C) 2 times
- (D) Cannot be made stationary by differencing

Question 2: Answer



Answer: (C)

- $I(d)$ means “integrated of order d ” — requires d differences for stationarity.

Question 3

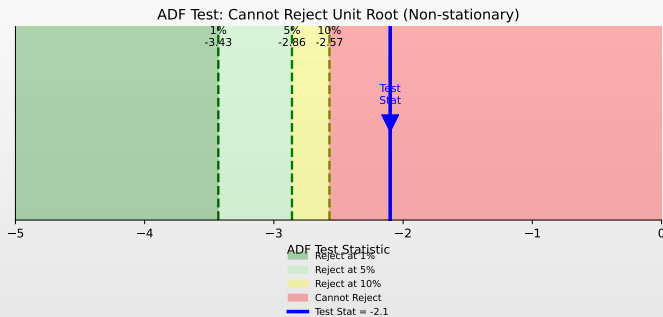
Question

- ☐ You run an ADF test and get a test statistic of -2.1 with critical values: -3.43 (1%), -2.86 (5%), -2.57 (10%). What do you conclude?

Answer Choices

- (A) Reject H_0 : series is stationary at all levels
- (B) Reject H_0 : series is stationary at 10% level only
- (C) Fail to reject H_0 : series likely has a unit root
- (D) The test is inconclusive

Question 3: Answer



Answer: (C)

□ Test stat $-2.1 > -2.57$ (10% CV) \Rightarrow Cannot reject at any level. Consider differencing.

Question 4

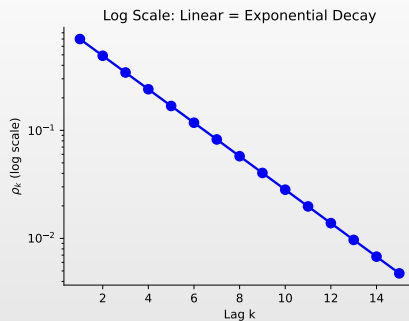
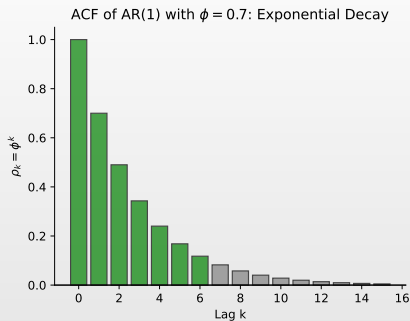
Question

□ For an ARIMA(1,1,0) model, what is the ACF pattern of the **differenced** series ΔY_t ?

Answer Choices

- (A) Cuts off after lag 1
- (B) Decays exponentially
- (C) Alternates in sign
- (D) Is zero at all lags

Question 4: Answer



Answer: (B)

□ ARIMA(1,1,0) $\Rightarrow \Delta Y_t$ follows AR(1) with ACF $\rho_k = \phi_1^k$ (geometric decay).

Question 5

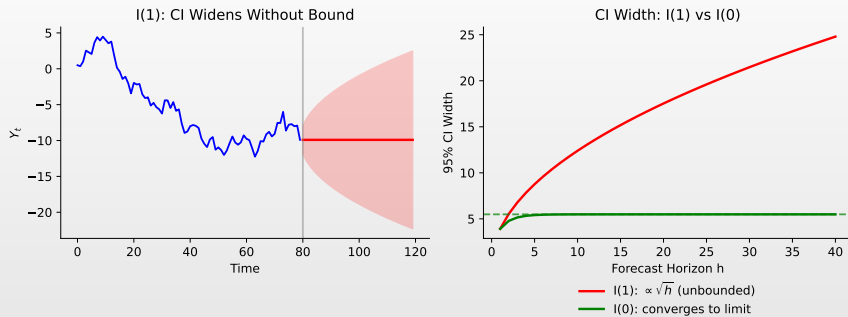
Question

□ What happens to ARIMA forecast confidence intervals as the horizon h increases for an $I(1)$ series?

Answer Choices

- (A) They stay constant
- (B) They narrow (more precision)
- (C) They widen without bound
- (D) They widen but converge to a limit

Question 5: Answer



Answer: (C)

□ For $I(1)$: CI width $\propto \sqrt{h}$ (unbounded). For $I(0)$: CIs converge to a limit.

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Unit Root Tests

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- Kwiatkowski, D., Phillips, P.C.B., Schmidt, P., & Shin, Y. (1992). Testing the Null Hypothesis of Stationarity, *Journal of Econometrics*, 54(1-3), 159–178.

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- Hyndman, R.J., & Khandakar, Y. (2008). Automatic Time Series Forecasting: The forecast Package for R, *Journal of Statistical Software*, 27(3), 1–22.

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- ▣ Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.
- ▣ Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> – Code platform for quantitative methods
- ▣ **Quantinar**: <https://quantinar.com> – Learning platform for quantitative methods
- ▣ **GitHub TSA**: https://github.com/QuantLet/TSA/tree/main/TSA_ch3 – Python code for this chapter

Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar