



Time Series Analysis and Forecasting

Chapter 2: ARMA Models



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Learning Objectives

By the end of this chapter, you will be able to:

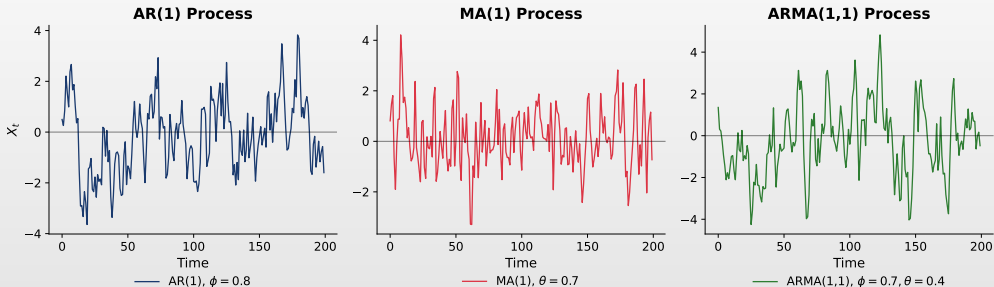
1. **Define** and simulate $AR(p)$, $MA(q)$, and $ARMA(p, q)$ processes
2. **Verify** stationarity and invertibility conditions
3. **Identify** orders p and q through ACF/PACF analysis
4. **Estimate** parameters via Yule-Walker, MLE, and information criteria (AIC, BIC)
5. **Diagnose** the model through residual analysis and the Ljung-Box test
6. **Forecast** using ARMA models with confidence intervals
7. **Apply** the Box-Jenkins methodology to real data (sunspots)

Chapter Structure

- Motivation
- Introduction and the Lag Operator
- Autoregressive (AR) Models
- Moving Average (MA) Models
- ARMA Models
- Model Identification
- Parameter Estimation
- Model Diagnostics
- Forecasting with ARMA
- Practical Implementation
- Case Study: Real Data
- Summary
- Quiz

Why ARMA Models?

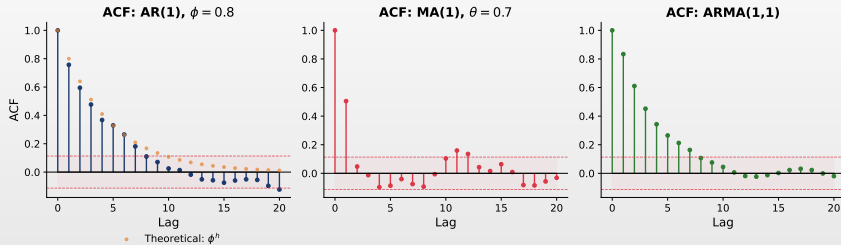
Stationary processes: AR, MA and ARMA



- ▣ **AR processes:** Current value depends on past values \succ mean-reverting behavior
- ▣ **MA processes:** Current value depends on past shocks \succ short memory
- ▣ **ARMA:** Combines both mechanisms for flexible modeling

Model Identification Through ACF Patterns

Distinct ACF patterns for different models



ACF Reflects Model Structure

- ▣ **Distinct patterns:** AR: exponential decay; MA: sharp cutoff; ARMA: mixed decay
- ▣ **Identification:** Visual analysis of ACF/PACF guides the selection of orders p and q

Recap: Stationarity

From Chapter 1

- A process $\{X_t\}$ is **weakly stationary** if:
 1. $\mathbb{E}[X_t] = \mu$ (constant mean)
 2. $\text{Var}(X_t) = \sigma^2 < \infty$ (constant, finite variance)
 3. $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ (covariance depends only on lag h)

Why Stationarity Matters for ARMA

- ARMA models assume stationarity
 - ▶ Parameters remain stable over time
 - ▶ Autocorrelation structure is maintained
- Non-stationary data \succ difference first (ARIMA, Ch. 3)

Chapter Objective

- Parametric models for stationary series \succ combining dependence on past observations (AR) with the influence of random shocks (MA)

The Lag Operator (Backshift Operator)

Definition 1 (Lag Operator)

- The **lag operator** L (or backshift operator B) shifts a time series back by one period: $LX_t = X_{t-1}$

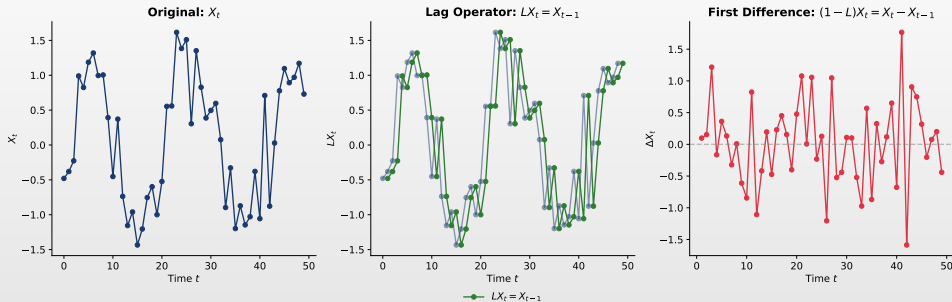
Properties

- $L^k X_t = X_{t-k}$ (shift back by k periods)
- $L^0 X_t = X_t$ (identity)
- $(1 - L)X_t = X_t - X_{t-1} = \Delta X_t$ (first difference)
- $(1 - L)^d X_t = \Delta^d X_t$ (difference of order d)

Lag Polynomials

- **AR polynomial:** $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$
- **MA polynomial:** $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$

The Lag Operator: Visual Illustration



Role of the Lag Operator

- ▣ **Notation foundation:** Enables compact writing of difference equations
- ▣ **Utility:** Facilitates algebraic manipulation of ARMA models

The White Noise Process

Definition 2 (White Noise)

- A process $\{\varepsilon_t\}$ is **white noise**, denoted $\varepsilon_t \sim WN(0, \sigma^2)$, if:
 1. $\mathbb{E}[\varepsilon_t] = 0$ for all t
 2. $\text{Var}(\varepsilon_t) = \sigma^2$ for all t
 3. $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ for all $t \neq s$

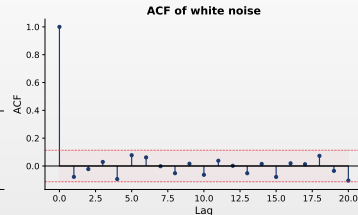
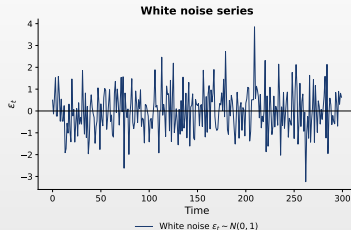
Properties

- **Building block:** White noise underlies all ARMA models
- **ACF:** $\rho(0) = 1$, $\rho(h) = 0$ for $h \neq 0$; PACF: same pattern
- **Gaussian white noise:** $\varepsilon_t \sim N(0, \sigma^2)$ i.i.d.
- **Unpredictable:** White noise is *not* predictable \succ it is purely random

White Noise: Visual Illustration

Key Characteristics

- **Top:** Random fluctuations, no patterns, constant variance
- **Bottom:** ACF only a spike at lag 0; others within significance bounds \succ no linear dependence



 TSA_ch2_white_noise

The AR(1) Model: Definition

Definition 3 (AR(1) Process)

- An **autoregressive process of order 1** is: $X_t = c + \phi X_{t-1} + \varepsilon_t$
- $\varepsilon_t \sim WN(0, \sigma^2)$ and $|\phi| < 1$ for stationarity

Interpretation

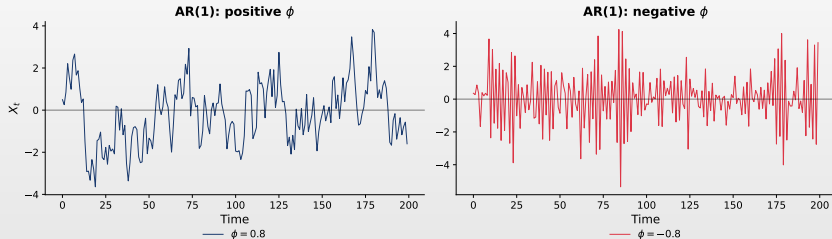
- c : constant (intercept)
- ϕ : autoregressive coefficient
 - ▶ Measures the persistence of the series
- ε_t : innovation (shock)

Lag Operator Notation

- $(1 - \phi L)X_t = c + \varepsilon_t$
- $\phi(L)X_t = c + \varepsilon_t$
- $\phi(L) = 1 - \phi L$

AR(1): Visual Illustration

AR(1): different behavior for positive vs negative ϕ



Visual Interpretation

- ▣ **Positive ϕ :** Persistent fluctuations, gradual mean reversion
- ▣ **Negative ϕ :** Oscillating behavior, alternating around the mean
- ▣ **Larger $|\phi|$ \succ greater persistence, slower reversion**

AR(1) Stationarity Condition

Necessary and Sufficient Condition: $|\phi| < 1$

- The root of the characteristic equation must lie outside the unit circle

- Shocks diminish over time
 - ▶ Process reverts to the mean
 - ▶ Finite, stable variance

Non-stationary ($|\phi| \geq 1$)

- $|\phi| = 1$: random walk
 - ▶ Unit root, variance $\rightarrow \infty$
- $|\phi| > 1$: explosive process

Characteristic Equation

- $\phi(z) = 1 - \phi z = 0 \implies z = 1/\phi$
- Stationarity \Leftrightarrow root outside the unit circle ($|z| > 1$)

AR(1) Properties

Stationary AR(1) with $|\phi| < 1$

□ Moment properties:

Mean: $\mu = \mathbb{E}[X_t] = \frac{c}{1-\phi}$

Variance: $\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$

Autocovariance: $\gamma(h) = \frac{\phi^h \sigma^2}{1-\phi^2}$

Autocorrelation (ACF): $\rho(h) = \phi^h$

Key Observation

□ **AR(1) signature:** ACF decays exponentially with factor ϕ

- ▶ $\phi > 0$: monotone decay towards zero
- ▶ $\phi < 0$: damped oscillations (alternating signs)

Proof: AR(1) Mean

Claim

- For AR(1): $X_t = c + \phi X_{t-1} + \varepsilon_t$, the mean is $\mu = \frac{c}{1-\phi}$

Proof

- Take expectations of both sides: $\mathbb{E}[X_t] = c + \phi \mathbb{E}[X_{t-1}] + \mathbb{E}[\varepsilon_t]$
- By stationarity, $\mathbb{E}[X_t] = \mathbb{E}[X_{t-1}] = \mu$, and $\mathbb{E}[\varepsilon_t] = 0$: $\mu = c + \phi \mu$
- Solving: $\mu - \phi \mu = c \implies \mu(1 - \phi) = c \implies \boxed{\mu = \frac{c}{1 - \phi}}$

Requirement

- **Necessary condition:** $\phi \neq 1$ for the mean to be defined
 - ▶ If $\phi = 1$ (unit root), the mean is undefined
 - ▶ The process becomes a random walk (non-stationarity)

Proof: AR(1) Variance

Claim

$$\square \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$$

Proof

□ Assume $c = 0$. Take the variance of $X_t = \phi X_{t-1} + \varepsilon_t$:

$$\square \text{Var}(X_t) = \phi^2 \text{Var}(X_{t-1}) + \text{Var}(\varepsilon_t) + 2\phi \underbrace{\text{Cov}(X_{t-1}, \varepsilon_t)}_{=0}$$

□ By stationarity, $\text{Var}(X_t) = \text{Var}(X_{t-1}) = \gamma(0)$:

$$\square \gamma(0) = \phi^2 \gamma(0) + \sigma^2 \implies \gamma(0)(1 - \phi^2) = \sigma^2 \implies \boxed{\gamma(0) = \frac{\sigma^2}{1 - \phi^2}}$$

Note

□ Requires $|\phi| < 1$ for positive variance. When $|\phi| \rightarrow 1$, variance $\rightarrow \infty$

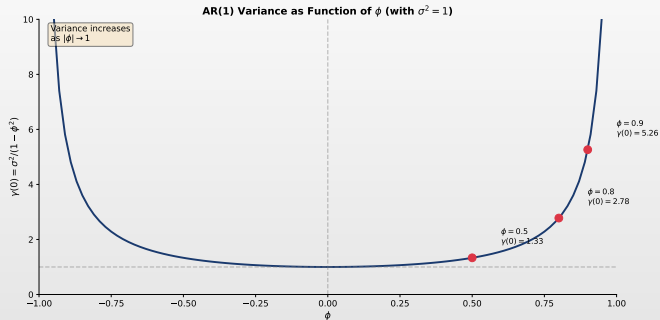
Proof: AR(1) Autocorrelation Function

Claim: $\rho(h) = \phi^h$ for $h \geq 0$

- Find the autocovariance $\gamma(h) = \text{Cov}(X_t, X_{t-h})$

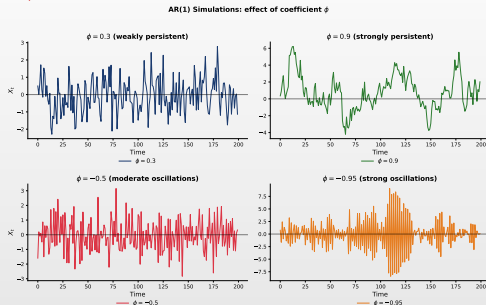
Proof

- Multiply $X_t = \phi X_{t-1} + \varepsilon_t$ by X_{t-h} and take expectations:
- $\mathbb{E}[X_t X_{t-h}] = \phi \mathbb{E}[X_{t-1} X_{t-h}] + \mathbb{E}[\varepsilon_t X_{t-h}]$
- For $h \geq 1$: $\mathbb{E}[\varepsilon_t X_{t-h}] = 0 \succ \gamma(h) = \phi \gamma(h-1)$
- Recursive relation from $\gamma(0)$: $\gamma(1) = \phi \gamma(0)$, $\gamma(2) = \phi^2 \gamma(0)$, ... $\gamma(h) = \phi^h \gamma(0)$
- ACF: $\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\phi^h \gamma(0)}{\gamma(0)} = \boxed{\phi^h}$

AR(1) Variance as a Function of ϕ 

Observations

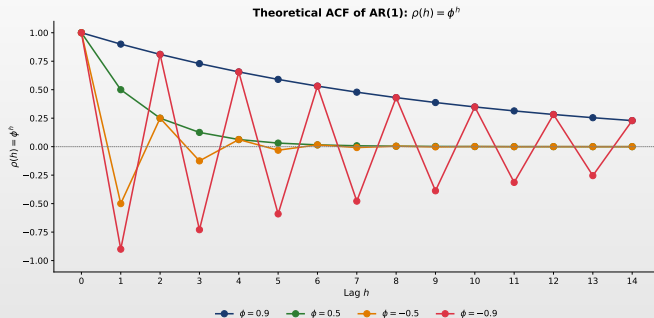
- As $|\phi| \rightarrow 1$, the variance explodes \succ non-stationarity
- For $\phi = 0$: $\gamma(0) = \sigma^2$ (white noise); variance increases monotonically with $|\phi|$

AR(1) Simulations: Effect of ϕ 

Interpretation

- Different values of ϕ produce distinct behaviors: larger $|\phi|$ \succ more persistence
- Positive ϕ creates smooth trajectories; negative ϕ creates oscillations
- As $|\phi| \rightarrow 1$, the process approaches non-stationarity

Theoretical AR(1) ACF



ACF Pattern

- Formula: $\rho(h) = \phi^h$ \succ exponential decay
- $\phi > 0$: monotone decay; $\phi < 0$: alternating signs

Proof: AR(1) Stationarity Condition

Claim

- AR(1) is stationary if and only if $|\phi| < 1$

Proof

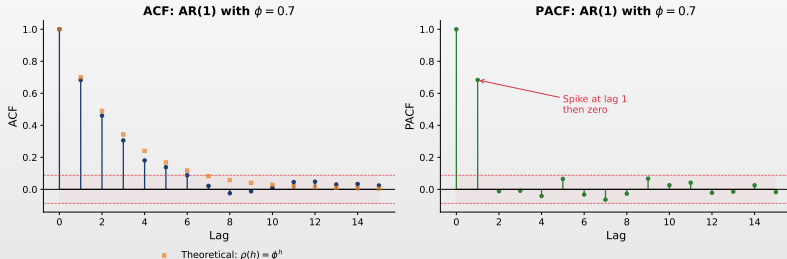
- Recursive substitution: $X_t = \phi X_{t-1} + \varepsilon_t = \phi(\phi X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \dots$
- After n steps: $X_t = \phi^n X_{t-n} + \sum_{j=0}^{n-1} \phi^j \varepsilon_{t-j}$
- If $|\phi| < 1$: $\phi^n \rightarrow 0$ as $n \rightarrow \infty$, so $X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$
- Finite variance: $\text{Var}(X_t) = \sigma^2 \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma^2}{1-\phi^2} < \infty$ (geometric series)

Conclusion

- Converges $\iff |\phi| < 1$. For $|\phi| \geq 1$, the term $\phi^n X_{t-n}$ does not vanish \Rightarrow infinite variance

AR(1) ACF and PACF: Theory vs Sample

ACF and PACF for AR(1): theory vs sample



Interpretation

- **ACF:** Exponential decay with factor ϕ ; formula: $\rho(h) = \phi^h$
- **PACF:** A single spike at lag 1, then cuts off \succ identifies AR(1)
- Sample estimates fluctuate around theoretical values

The Partial Autocorrelation Function (PACF)

Definition 4 (PACF)

- The **partial autocorrelation** of order k , denoted π_k , measures the correlation between X_t and X_{t-k} **after removing** the linear effects of the intermediate variables $X_{t-1}, \dots, X_{t-k+1}$

Formal Definition

- $\pi_1 = \rho(1)$
- For $k \geq 2$: π_k is the last coefficient in:
$$X_t = \alpha_1 X_{t-1} + \dots + \alpha_k X_{t-k} + e_t$$
- $\pi_k = \alpha_k$ (coefficient of X_{t-k})

Computation via Yule-Walker

- Solve the Yule-Walker equations of order k
- π_k = last element of the solution vector

Utility

- **Identification:** PACF determines the order p of an AR model
 - PACF cuts off after lag p

AR(1) ACF and PACF Patterns

ACF of AR(1)

- Decays exponentially: $\rho(h) = \phi^h$
 - $\phi > 0$: all positive
 - $\phi < 0$: alternating signs

PACF of AR(1)

- Cuts off after lag 1**
 - $\pi_1 = \phi$, $\pi_k = 0$ for $k > 1$

	ACF	PACF
AR(1)	Exponential decay	Cuts off at lag 1

Key Pattern

- This is the key identification pattern for AR(1)!

The AR(p) Model: General Form

Definition 5 (AR(p) Process)

- ▣ An **autoregressive process of order p** is: $X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \varepsilon_t$
- ▣ **Lag operator**: $\phi(L)X_t = c + \varepsilon_t$, where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$

Stationarity Condition

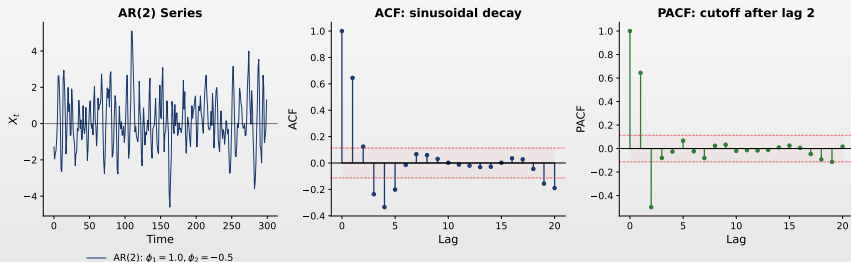
- ▣ All roots of $\phi(z) = 0$ must lie **outside** the unit circle
- ▣ Equivalently: all roots have modulus > 1

PACF Pattern

- ▣ PACF cuts off after lag p
- ▣ ACF decays (exponentially or with damped oscillations)

AR(p): Visual Illustration

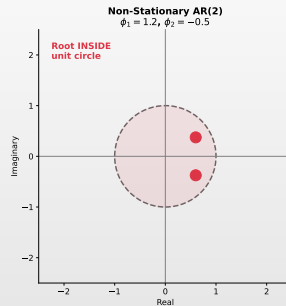
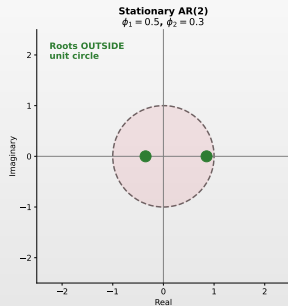
AR(2) Process: pseudo-cyclic behavior



Observations

- AR(2) can exhibit pseudo-cyclic behavior (complex roots); damped sinusoidal ACF
- PACF cuts off after lag 2 \succ key identification pattern

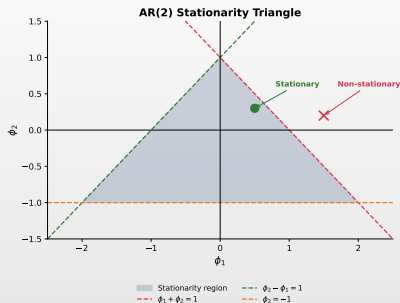
AR(2) Stationarity: Unit Circle Visualization



Characteristic Polynomial and Unit Circle Condition

- **Characteristic polynomial** of an $AR(p)$ process: $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$
- All roots of $\phi(z) = 0$ must lie **outside** the unit circle ($|z| > 1$)
- Roots on the circle: non-stationary; roots inside: explosive process

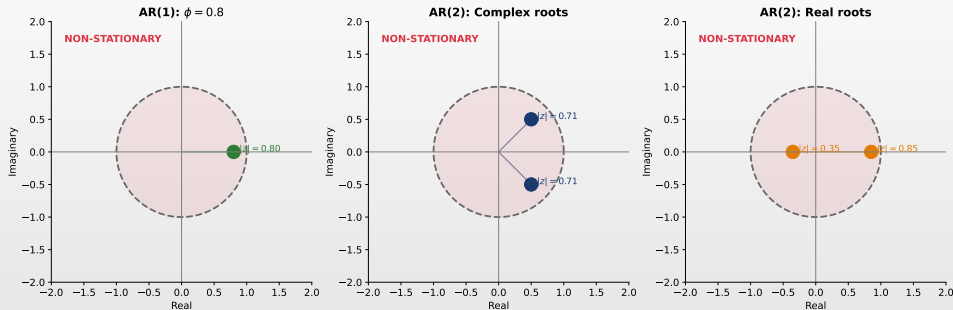
The AR(2) Stationarity Triangle



Stationarity Region

- The triangular region defines the stationary AR(2) parameter combinations
- **Boundaries:** $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$ and $|\phi_2| < 1$
- Points outside the region \succ non-stationary or explosive processes

Characteristic Polynomial Roots



Types of Roots

- Real roots: exponential decay in ACF
- Complex roots: damped oscillations (pseudo-cycles)
- All roots must lie outside the unit circle

The AR(2) Model

Definition 6 (AR(2) Process)

$$\square X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

Stationarity Conditions

$$\square \phi_1 + \phi_2 < 1; \quad \phi_2 - \phi_1 < 1; \quad |\phi_2| < 1$$

ACF Behavior

- \square **Real roots:** mixture of two exponential decays
- \square **Complex roots:** damped sinusoidal pattern (pseudo-cycles)
- \square **PACF:** Cuts off after lag 2 ($\pi_k = 0$ for $k > 2$)

The MA(1) Model: Definition

Definition 7 (MA(1) Process)

- ▣ A **moving average process of order 1** is: $X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$
- ▣ $\varepsilon_t \sim WN(0, \sigma^2)$

Interpretation

- ▣ μ : process mean
- ▣ θ : MA coefficient
 - Measures the impact of the past shock
- ▣ Depends on ε_t and ε_{t-1}

Lag Operator Notation

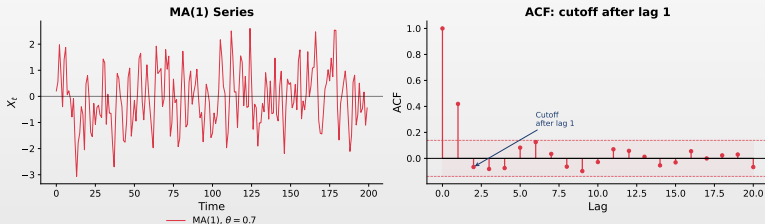
- ▣ $X_t = \mu + \theta(L)\varepsilon_t$
- ▣ $\theta(L) = 1 + \theta L$

Key Property

- ▣ **Guaranteed stationarity:** MA processes are always stationary
 - Does not depend on the value of θ

MA(1): Visual Illustration

MA(1): short memory series with ACF cutoff



Visual Interpretation

- Left panel: MA(1) series \succ rapid mean reversion
- Right panel: ACF with **cutoff after lag 1**; PACF exponential decay

MA(1) Properties

$$\text{MA}(1): X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$$

- **Mean:** $\mathbb{E}[X_t] = \mu$; **Variance:** $\gamma(0) = \sigma^2(1 + \theta^2)$
- **Autocovariance:** $\gamma(1) = \theta\sigma^2$, $\gamma(h) = 0$ ($h > 1$)
- **ACF:** $\rho(1) = \frac{\theta}{1+\theta^2}$, $\rho(h) = 0$ ($h > 1$)

Key Observation

- **MA(1) signature:** ACF cuts off after lag 1
 - ▶ $\rho(1) \neq 0$, but $\rho(h) = 0$ for $h > 1$; opposite pattern to AR(1)

Proof: MA(1) Variance and Autocovariance

Starting point: $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ (assuming $\mu = 0$)

▣ **Variance:**

$$\gamma(0) = \text{Var}(\varepsilon_t + \theta\varepsilon_{t-1}) = \sigma^2 + \theta^2\sigma^2 + 0 = \boxed{\sigma^2(1 + \theta^2)}$$

Autocovariance at lag 1

$$\square \gamma(1) = \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-1} + \theta\varepsilon_{t-2})$$

$$\square = \text{Cov}(\varepsilon_t, \varepsilon_{t-1}) + \theta\text{Cov}(\varepsilon_t, \varepsilon_{t-2}) + \theta\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-1}) + \theta^2\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-2})$$

$$\square = 0 + 0 + \theta\sigma^2 + 0 = \boxed{\theta\sigma^2}$$

Autocovariance at lag $h \geq 2$

$$\square \text{ No common } \varepsilon \text{ terms } \succ \gamma(h) = 0$$

Proof: Maximum ACF for MA(1)

Claim: $|\rho(1)| \leq 0.5$ for any value of θ

- ACF at lag 1: $\rho(1) = \frac{\theta}{1+\theta^2}$
- Differentiate: $\frac{d\rho(1)}{d\theta} = \frac{1-\theta^2}{(1+\theta^2)^2} = 0 \succ \theta = \pm 1$
- At these values: $\rho(1)|_{\theta=1} = \frac{1}{2}$, $\rho(1)|_{\theta=-1} = -\frac{1}{2}$

Implication

- **Practical test:** If $|\hat{\rho}(1)| > 0.5$ from data, the process is **not** MA(1)
 - ▶ The maximum $|\rho(1)| = 0.5$ is reached at $\theta = \pm 1$
 - ▶ Consider AR or ARMA models as alternatives

Proof: ACF for MA(1)

Claim: $\rho(1) = \frac{\theta}{1+\theta^2}$ and $\rho(h) = 0$ for $h > 1$

- MA(1) has non-zero autocorrelation **only** at lag 1

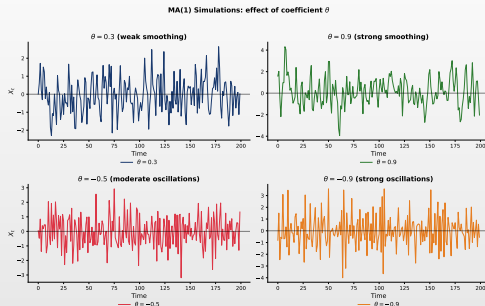
Proof

- Let $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$. Autocorrelation at lag 1:
- $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta\sigma^2}{\sigma^2(1+\theta^2)} = \frac{\theta}{1+\theta^2}$
- For $h > 1$: $\gamma(h) = \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-h} + \theta\varepsilon_{t-h-1})$
- The terms $\varepsilon_t, \varepsilon_{t-1}$ do not overlap with $\varepsilon_{t-h}, \varepsilon_{t-h-1}$ when $h > 1$, so $\gamma(h) = 0$

Practical Consequence

- ACF cuts off sharply after lag 1 \Rightarrow distinctive signature of MA(1) processes

MA(1) Simulations: Effect of θ



Interpretation

- MA(1) is always stationary regardless of θ \succ finite memory of only one lag
- Positive θ smooths the series; negative θ creates faster fluctuations
- Unlike AR(1), MA(1) shocks affect the process for only one period

MA(1) ACF and PACF Patterns

ACF of MA(1)

- ▣ **Cuts off after lag 1**
 - ▶ $\rho(1) = \frac{\theta}{1+\theta^2}$
 - ▶ $\rho(h) = 0$ for $h > 1$
 - ▶ $|\rho(1)| \leq 0.5$ always

PACF of MA(1)

- ▣ **Decays exponentially**
 - ▶ Or with alternating signs
 - ▶ Does *not* cut off

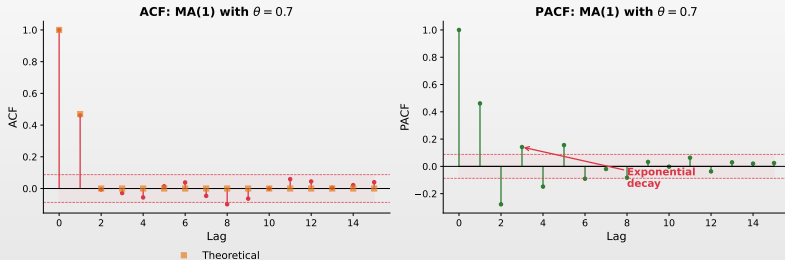
	ACF	PACF
MA(1)	Cuts off at lag 1	Exponential decay

Observation

- ▣ **Opposite pattern to AR(1)!**

MA(1) ACF and PACF: Visual Comparison

ACF and PACF for MA(1): opposite pattern to AR(1)



Interpretation

- ACF: A single spike at lag 1, then cuts off \succ key MA(1) signature
- PACF: Exponential decay \succ opposite pattern to AR(1)
- This reversal differentiates MA processes from AR processes

Invertibility of MA Models

Definition 8 (Invertibility)

- ▣ An MA process is **invertible** if it can be written as an infinite AR process:
- ▣ $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$

Invertibility Conditions

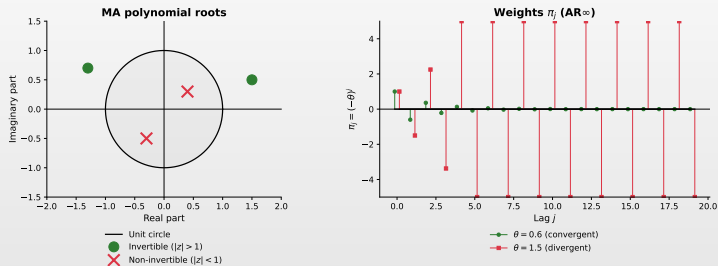
- ▣ **MA(1)**: Invertible if $|\theta| < 1$
- ▣ **MA(q)**: Roots of $\theta(z) = 0$ outside the unit circle

Why Invertibility Matters

- ▣ Ensures unique representation (without invertibility, multiple MA models describe the same data)
- ▣ Necessary for forecasting and estimation
- ▣ **Stationarity** \succ AR; **Invertibility** \succ MA

Invertibility: Visual Illustration

Invertibility of MA models



Interpretation

- Left: invertibility requires roots outside the unit circle
- Right: $AR(\infty)$ weights decay only when $|\theta| < 1$

Proof: MA(1) Invertibility

Claim

- MA(1) is invertible if and only if $|\theta| < 1$

Proof

- From $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$, isolate: $\varepsilon_t = X_t - \theta\varepsilon_{t-1}$
- Recursive back-substitution: $\varepsilon_t = X_t - \theta(X_{t-1} - \theta\varepsilon_{t-2}) = X_t - \theta X_{t-1} + \theta^2\varepsilon_{t-2}$
- Continuing: $\varepsilon_t = \sum_{j=0}^n (-\theta)^j X_{t-j} + (-\theta)^{n+1} \varepsilon_{t-n-1}$
- If $|\theta| < 1$: $(-\theta)^{n+1} \rightarrow 0$, so

$$\varepsilon_t = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j}$$

Conclusion

- Geometric series converges $\iff |\theta| < 1 \Rightarrow$ MA(1) can be written as AR(∞)

The MA(q) Model: General Form

Definition 9 (MA(q) Process)

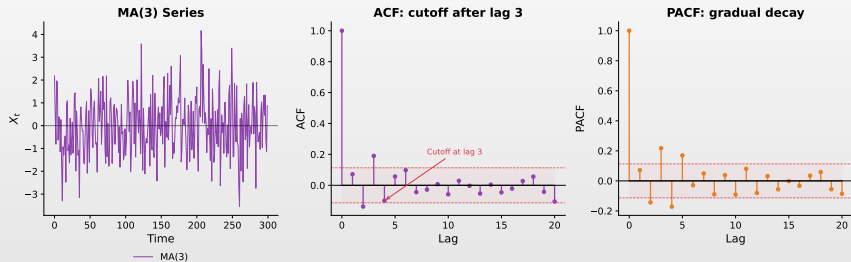
- ▣ A **moving average process of order q**: $X_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q}$
- ▣ **Lag operator**: $X_t = \mu + \theta(L)\varepsilon_t$, where $\theta(L) = 1 + \theta_1L + \cdots + \theta_qL^q$

Properties

- ▣ Always stationary (finite variance)
- ▣ ACF cuts off after lag q : $\rho(h) = 0$ for $h > q$; PACF decays gradually
- ▣ Invertible if all roots of $\theta(z) = 0$ lie outside the unit circle

MA(q): Visual Illustration

MA(q) Process: ACF signature cuts off after lag q



Observation

- MA(3) process: key signature \succ ACF cuts off after lag q ($\rho(h) = 0$ for $h > 3$)

The ARMA(p,q) Model: Definition

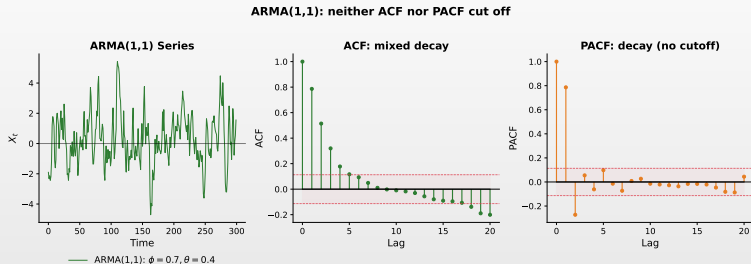
Definition 10 (ARMA(p,q) Process)

- $X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$
- **Compact form:** $\phi(L)X_t = c + \theta(L)\varepsilon_t$, where $\mu = \frac{c}{1 - \phi_1 - \cdots - \phi_p}$

Key Idea

- **Flexibility:** Combines AR and MA components
 - ▶ AR captures persistence; MA captures shock response
- **Parsimony:** ARMA(1,1) can be better than AR(5) or MA(5)
 - ▶ Fewer parameters, less risk of overfitting

ARMA: Visual Illustration



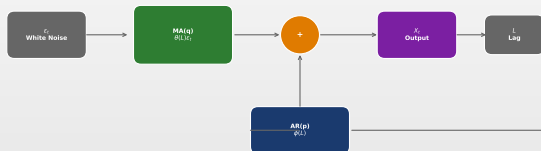
ARMA(1,1) Interpretation

- ▣ **Combines** AR persistence with MA shock response
- ▣ **ACF pattern:** Decay after the first lag (lags decay geometrically)
- ▣ **PACF pattern:** Also decays (no sharp cutoff as in pure AR)
- ▣ Neither ACF nor PACF cuts off \succ key identifier for mixed models

ARMA Model Structure

ARMA(p,q) Model Structure

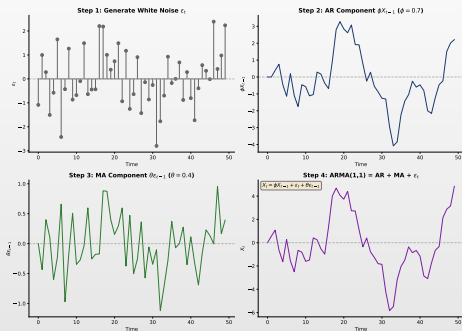
$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$



Components

- ▣ **AR component:** influence of past values of the series
- ▣ **MA component:** impact of past random shocks

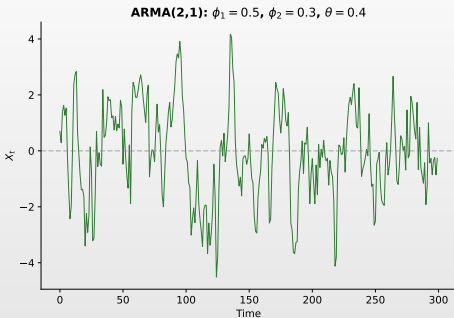
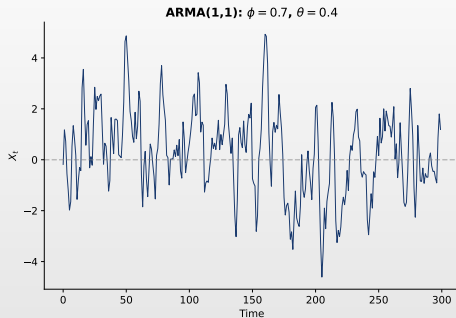
How ARMA Simulation Works



Steps

- Generate white noise, apply the ARMA equation recursively, obtain simulated series

ARMA Examples



Observation

- Different combinations of orders (p, q) produce distinct behaviors

The ARMA(1,1) Model

Definition 11 (ARMA(1,1) Process)

$$\square X_t = c + \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

Properties (stationarity and invertibility)

$$\square \text{Mean: } \mu = \frac{c}{1-\phi}; \text{ Variance: } \gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$$

ACF

$$\square \rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}; \quad \rho(h) = \phi \cdot \rho(h-1) \text{ for } h \geq 2$$

\square ACF decays exponentially after lag 1 (starting point depends on ϕ and θ)

Proof: ARMA(1,1) Variance

Claim

$$\square \gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$$

Proof

- Let $Y_t = X_t - \mu$: $Y_t = \phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$
- Square: $Y_t^2 = \phi^2 Y_{t-1}^2 + \varepsilon_t^2 + \theta^2 \varepsilon_{t-1}^2 + 2\phi Y_{t-1} \varepsilon_t + 2\phi\theta Y_{t-1} \varepsilon_{t-1} + 2\theta \varepsilon_t \varepsilon_{t-1}$
- Take expectations; $\mathbb{E}[\varepsilon_t Y_{t-1}] = 0$, $\mathbb{E}[\varepsilon_t \varepsilon_{t-1}] = 0$:
- $\gamma(0) = \phi^2 \gamma(0) + \sigma^2 + \theta^2 \sigma^2 + 2\phi\theta \mathbb{E}[\varepsilon_{t-1} Y_{t-1}]$
- From $Y_{t-1} = \phi Y_{t-2} + \varepsilon_{t-1} + \theta \varepsilon_{t-2}$: only ε_{t-1}^2 contributes $\Rightarrow \mathbb{E}[\varepsilon_{t-1} Y_{t-1}] = \sigma^2$
- $\gamma(0)(1 - \phi^2) = (1 + 2\phi\theta + \theta^2)\sigma^2 \implies \boxed{\gamma(0) = \frac{(1 + 2\phi\theta + \theta^2)\sigma^2}{1 - \phi^2}}$

Proof: ARMA(1,1) ACF at Lag 1

Claim

$$\square \quad \rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}; \quad \rho(h) = \phi \rho(h-1) \text{ for } h \geq 2$$

Proof

□ Multiply Y_t by Y_{t-1} and take expectations:

$$\square \quad \gamma(1) = \phi\gamma(0) + \underbrace{\mathbb{E}[\varepsilon_t Y_{t-1}]}_{=0} + \theta \underbrace{\mathbb{E}[\varepsilon_{t-1} Y_{t-1}]}_{=\sigma^2} = \phi\gamma(0) + \theta\sigma^2$$

□ Divide by $\gamma(0)$: $\rho(1) = \phi + \frac{\theta\sigma^2}{\gamma(0)}$. Substitute $\gamma(0)$:

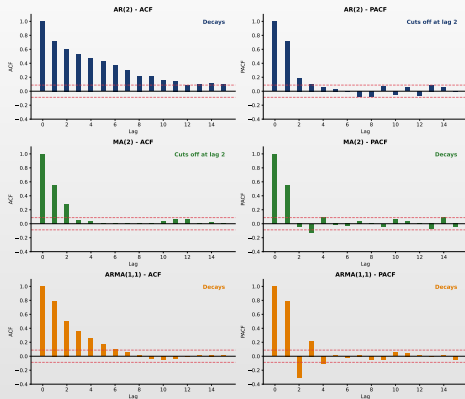
$$\square \quad \rho(1) = \phi + \frac{\theta(1-\phi^2)}{1+2\phi\theta+\theta^2} = \frac{\phi(1+2\phi\theta+\theta^2)+\theta(1-\phi^2)}{1+2\phi\theta+\theta^2}$$

$$\square \quad \text{Numerator: } \phi + \theta + \phi^2\theta + \phi\theta^2 = (\phi + \theta)(1 + \phi\theta), \text{ so } \boxed{\rho(1) = \frac{(1 + \phi\theta)(\phi + \theta)}{1 + 2\phi\theta + \theta^2}}$$

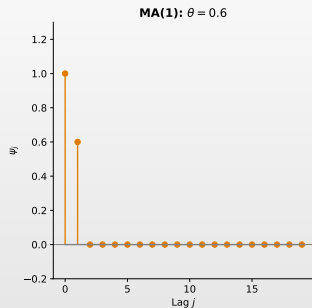
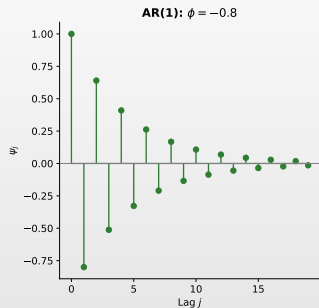
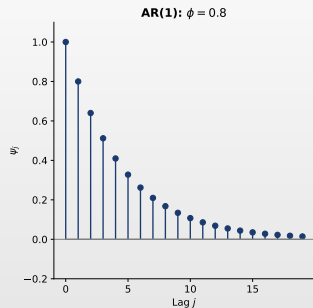
Recursion

□ For $h \geq 2$: $\gamma(h) = \phi\gamma(h-1)$, so $\rho(h) = \phi \rho(h-1) \Rightarrow$ exponential decay from lag 1

ACF/PACF Patterns: AR vs MA vs ARMA



Impulse Response Functions



Shock Propagation

- ▣ Shows how a unit shock propagates through the system over time
- ▣ **AR**: exponential or oscillating decay; **MA**: effect limited to q periods

Stationarity and Invertibility Summary

Conditions for a Valid ARMA(p,q) Model

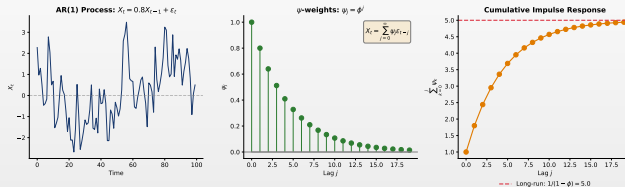
▣ Requirements summary:

Condition	Requirement
Stationarity	Roots of $\phi(z) = 0$ outside the unit circle
Invertibility	Roots of $\theta(z) = 0$ outside the unit circle

Implications

- ▣ **Stationarity:** Can be written as $MA(\infty)$: $X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- ▣ **Invertibility:** Can be written as $AR(\infty)$: $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$
- ▣ **Causal representation:** X_t depends only on *past* shocks \succ necessary for forecasting

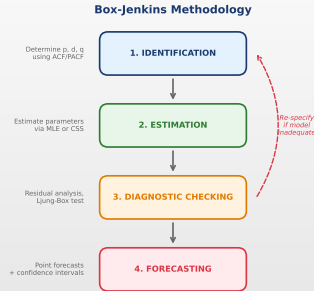
Wold's Decomposition Theorem



Wold's Theorem

- Any purely non-deterministic stationary process can be written as MA(∞):
- $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ with $\sum \psi_j^2 < \infty$
- Theoretical justification for ARMA modeling

The Box-Jenkins Methodology



Iterative Approach

□ Identification \succ estimation \succ validation; repeat until residuals are white noise

Model Identification Summary Table

Model Identification: ACF/PACF Patterns

Model	ACF Pattern	PACF Pattern
AR(p)	Exponential decay or damped oscillation	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponential decay or damped oscillation
ARMA(p,q)	Exponential decay after lag q-p	Exponential decay after lag p-q

Parsimony Principle

- Start simple (small p, q), increase order if checks are not satisfied
- Simpler models are preferred



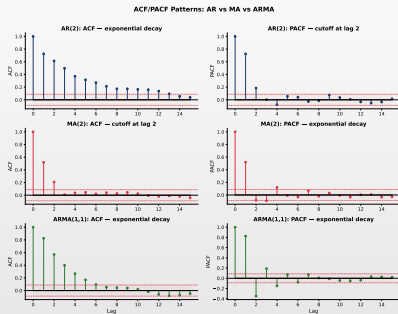
ACF/PACF Identification Rules

Theoretical Patterns for Stationary Processes

- The table summarizes ACF/PACF patterns for model identification:

Model	ACF Pattern	PACF Pattern
AR(1)	Exponential decay	Spike at lag 1, then 0
AR(2)	Exp./damped sinusoid	Spikes at lags 1-2, then 0
AR(p)	Gradual decay	Cuts off after lag p
MA(1)	Spike at lag 1, then 0	Exponential decay
MA(2)	Spikes at lags 1-2, then 0	Exp./damped sinusoid
MA(q)	Cuts off after lag q	Gradual decay
ARMA(p,q)	Decays	Decays

ACF/PACF Patterns: Visual Guide



Identification Rules

- **AR**: ACF decays, PACF cuts off \succ use PACF for order p
- **MA**: ACF cuts off, PACF decays \succ use ACF for order q
- **ARMA**: Both decay \succ requires information criteria

Information Criteria

AIC (Akaike)

- $AIC = -2 \ln(\hat{L}) + 2k$
- Moderate penalty
 - ▶ Tends to select larger models
 - ▶ Optimal for forecasting

BIC (Bayesian)

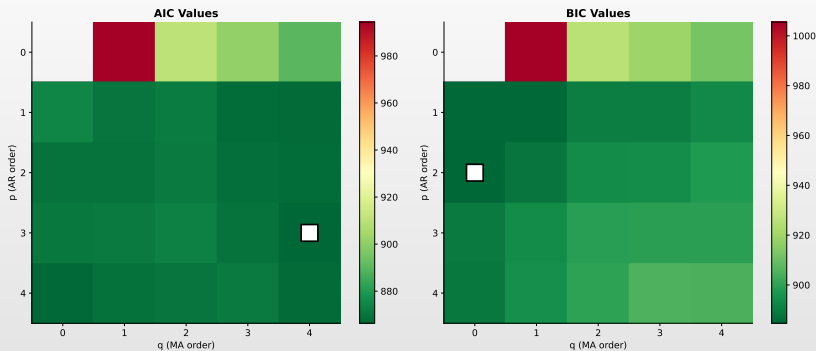
- $BIC = -2 \ln(\hat{L}) + k \ln(n)$
- Stronger penalty
 - ▶ Prefers parsimonious models
 - ▶ Consistent for identification

where: \hat{L} = maximum of the likelihood function, k = number of estimated parameters, n = sample size

Rules

- Lower values = better model. Compare models on the *same data*

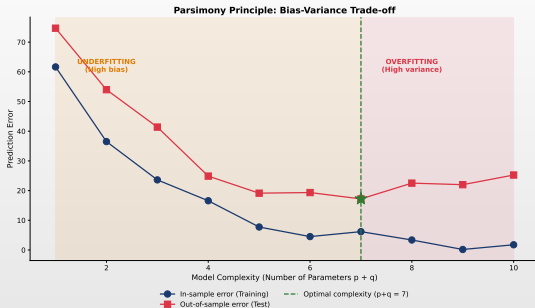
AIC vs BIC: Model Selection



Interpretation

- White square marks the best model; lower values (green) are better

Parsimony Principle: Bias-Variance Trade-off



Bias-Variance Trade-off

- Too simple model \succ high bias (underfitting)
- Too complex model \succ high variance (overfitting)
- The optimum lies at the intersection of the two curves

Automatic Model Selection

Grid Search Approach

- ▣ Estimate ARMA(p, q) for $p = 0, \dots, p_{max}$ and $q = 0, \dots, q_{max}$
- ▣ Select the model with the lowest AIC or BIC; verify with validation tests

In Python

- ▣ `pm.auto_arima()` from the `pmdarima` package
- ▣ Automatically tests stationarity, iterates over orders (p, q) , returns the best model

Caution

- ▣ Automatic selection is not the final answer \succ verify model validity
- ▣ Full Auto-ARIMA (including selection of d) \succ Chapter 3

Estimation Methods Overview

1. Method of Moments / Yule-Walker (AR only)

- ▣ Equates sample autocorrelations with theoretical values
- ▣ Simple, closed-form for AR models; not efficient for MA

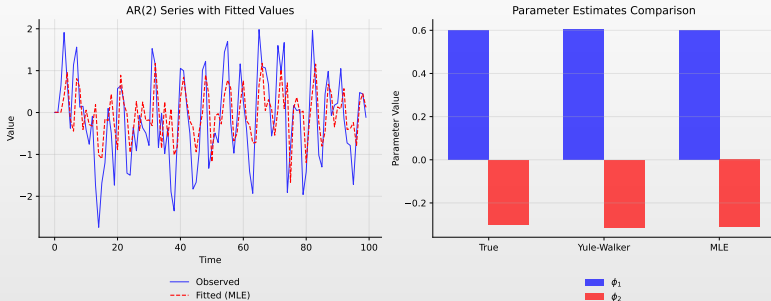
2. Maximum Likelihood Estimation (MLE)

- ▣ Most common approach; requires distributional assumption (Gaussian)
- ▣ Efficient and consistent

3. Conditional Least Squares

- ▣ Minimizes the sum of squared residuals
- ▣ Conditional on initial observations; algorithmically simpler than exact MLE

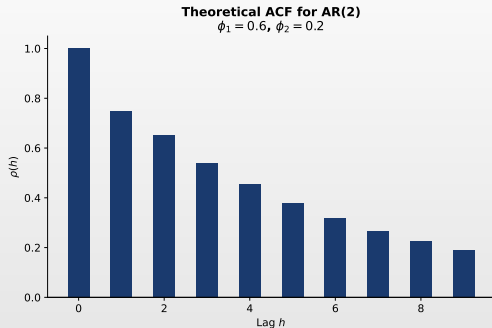
Estimation Methods Comparison



Comparison

- **MLE:** most efficient, but requires distributional assumption
- **Yule-Walker:** closed-form, only for AR models
- **CLS:** compromise between MLE and Yule-Walker

The Yule-Walker Equations for AR(p)



Yule-Walker Equations

$$\rho(1) = \phi_1 + \phi_2 \rho(1)$$

$$\rho(2) = \phi_1 \rho(1) + \phi_2$$

Matrix form: $R \cdot \phi = \rho$

R = autocorrelation matrix

$$\text{Solution: } \hat{\phi} = R^{-1} \rho$$

Main Idea

- Linear relationship between autocorrelations and AR parameters
- Allows closed-form estimation (no numerical optimization)

The Yule-Walker Equations: Matrix Form

Yule-Walker Equations for AR(p)

$$\square \quad \rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p), \quad k = 1, 2, \dots, p$$

Matrix Form

$$\square \quad \begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{pmatrix}$$

\square **Estimation:** Replace $\rho(k)$ with $\hat{\rho}(k)$; the Toeplitz matrix is symmetric and positive definite

Numerical Example: Yule-Walker for AR(2)

Sample Data ($T = 100$)

▣ **Estimated autocorrelations:** $\hat{\rho}(1) = 0.75$, $\hat{\rho}(2) = 0.65$

▶ Estimated variance: $\hat{\gamma}(0) = 4.0$

Step 1: Matrix System

▣ **Yule-Walker:** $R\hat{\phi} = \rho$

▶
$$\begin{pmatrix} 1 & 0.75 \\ 0.75 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.65 \end{pmatrix}$$

Step 2: Solution (Cramer's Rule)

▣ $\det(R) = 1 - 0.75^2 = 0.4375$

▣ $\hat{\phi}_1 = \frac{0.75 \times 1 - 0.75 \times 0.65}{0.4375} = \frac{0.2625}{0.4375} = \boxed{0.600}$ $\hat{\phi}_2 = \frac{0.65 \times 1 - 0.75 \times 0.75}{0.4375} = \frac{0.0875}{0.4375} = \boxed{0.200}$

Step 3: Noise Variance

▣ $\hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\phi}_1\hat{\rho}(1) - \hat{\phi}_2\hat{\rho}(2)) = 4.0(1 - 0.45 - 0.13) = \boxed{1.68}$

Stationarity check: $\hat{\phi}_1 + \hat{\phi}_2 = 0.8 < 1 \checkmark$ $|\hat{\phi}_2| = 0.2 < 1 \checkmark$ $\hat{\phi}_2 - \hat{\phi}_1 = -0.4 > -1 \checkmark$

Proof: The Yule-Walker Equations

Goal: Derive $\rho(k) = \phi_1\rho(k-1) + \dots + \phi_p\rho(k-p)$

- Start from AR(p): $X_t = \phi_1X_{t-1} + \dots + \phi_pX_{t-p} + \varepsilon_t$
- Multiply by X_{t-k} and take expectations:
- $\mathbb{E}[X_tX_{t-k}] = \phi_1\mathbb{E}[X_{t-1}X_{t-k}] + \dots + \phi_p\mathbb{E}[X_{t-p}X_{t-k}] + \mathbb{E}[\varepsilon_tX_{t-k}]$
- For $k \geq 1$: $\mathbb{E}[\varepsilon_tX_{t-k}] = 0 \succ \gamma(k) = \phi_1\gamma(k-1) + \dots + \phi_p\gamma(k-p)$
- Dividing by $\gamma(0)$: $\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \dots + \phi_p\rho(k-p)$

Special Case AR(1)

- $\rho(k) = \phi_1\rho(k-1) = \phi_1^k$ (using $\rho(0) = 1$)

Maximum Likelihood Estimation

ARMA(p,q) Log-Likelihood (Gaussian errors: $\varepsilon_t \sim N(0, \sigma^2)$)

- ▣ $\ell(\phi, \theta, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^n \varepsilon_t^2$
- ▣ ε_t are the innovations computed recursively

Estimation Procedure

- ▣ Initialization: use method of moments or OLS for starting values
- ▣ Optimization: numerical methods (BFGS, Newton-Raphson)
- ▣ Iterate until convergence

In Practice

- ▣ `statsmodels.tsa.arima.model.ARIMA` implements exact MLE with automatic initialization

Standard Errors and Inference

Asymptotic Distribution of MLE

- ▣ $\hat{\theta} \xrightarrow{d} N(\theta_0, \frac{1}{n}I(\theta_0)^{-1})$, where $I(\theta)$ is the **Fisher information matrix**
- ▣ $I(\theta) = -E\left[\frac{\partial^2 \ln L(\theta)}{\partial \theta \partial \theta'}\right] \succ$ average curvature of the log-likelihood
- ▣ Estimated variance-covariance matrix: $\hat{V} = \frac{1}{n}\hat{I}^{-1}$

What is the Standard Error (SE)?

- ▣ $SE(\hat{\theta}_j) = \sqrt{\hat{V}_{jj}} = \sqrt{\text{diag}_j\left(\frac{1}{n}\hat{I}^{-1}\right)} \succ$ measures estimation uncertainty
- ▣ **Example AR(1):** $SE(\hat{\phi}) \approx \sqrt{(1 - \hat{\phi}^2)/n}$; for $\hat{\phi} = 0.8$, $n = 100$: $SE \approx 0.06$
- ▣ **Interpretation:** small SE \Rightarrow parameter is estimated with high precision

Testing Parameter Significance

- ▣ $H_0 : \theta_j = 0$ Statistic: $z = \frac{\hat{\theta}_j}{SE(\hat{\theta}_j)} \sim N(0, 1)$ asymptotically
- ▣ Reject if $|z| > 1.96$ at 5% \Rightarrow **CI:** $\hat{\theta}_j \pm 1.96 \cdot SE(\hat{\theta}_j)$

Residual Analysis

If the model is correctly specified, residuals must be white noise

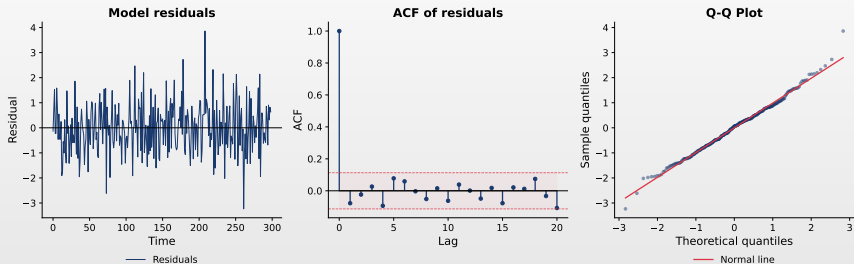
- **Residual time plot**
 - ▶ Fluctuates around zero, no obvious patterns; constant variance
- **Residual ACF**
 - ▶ All correlations within significance bounds; no significant spikes \succ white noise
- **Histogram / Q-Q plot**
 - ▶ Approximately normal distribution; heavy tails \succ non-normal errors

Decision

- ✓ **All checks OK** \succ adequate model
- ✗ **Not satisfied** \succ return to identification

Residual Diagnostics: Example

AR(1) Model Diagnostics: white noise residuals



Interpretation

- **Residual plot:** random fluctuations around zero, constant variance
- **Residual ACF:** no significant spikes \leadsto white noise
- **Q-Q plot:** points on the diagonal \leadsto normally distributed residuals

The Ljung-Box Test

Definition 12 (Ljung-Box Test)

- ▣ Tests whether residuals are independently distributed (no autocorrelation)
- ▣ **Statistic:** $Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}$

Hypotheses and Distribution

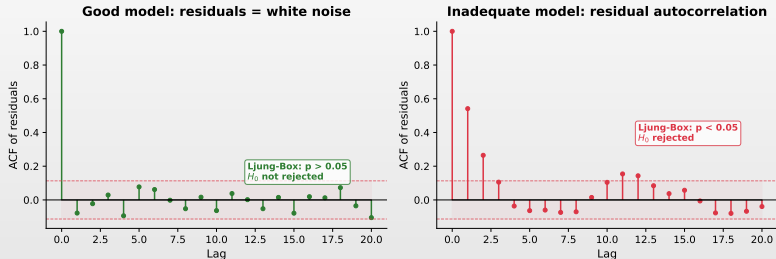
- ▣ H_0 : Residuals are white noise; H_1 : Residuals are autocorrelated
- ▣ Under H_0 , $Q(m) \sim \chi^2(m-p-q)$ approximately

Decision

- ▣ **p-value** > 0.05 \succ do not reject H_0 \succ residuals are white noise
- ▣ **p-value** < 0.05 \succ residual autocorrelation \succ inadequate model

The Ljung-Box Test: Visual Illustration

Ljung-Box Test: good model vs inadequate model



Interpretation

- Left: good model γ white noise residuals
- Right: inadequate model γ residual autocorrelation γ re-specification needed

Model Checklist

A Good ARMA Model Should Satisfy

- ▣ **Stationarity:** AR roots outside the unit circle (arroots)
- ▣ **Invertibility:** MA roots outside the unit circle (maroots)
- ▣ **White noise residuals:** No significant ACF (Ljung-Box test)
- ▣ **Normal residuals:** Q-Q plot, Jarque-Bera test
- ▣ **No heteroscedasticity:** Constant variance (ARCH test)
- ▣ **Simple:** Lowest AIC/BIC among adequate models

If Checks Are Not Satisfied

- ▣ Return to identification, try different orders

Point Forecasts

Optimal Forecast: $\hat{X}_{n+h|n} = \mathbb{E}[X_{n+h}|X_n, X_{n-1}, \dots]$

- The conditional expectation minimizes MSE

AR(1): $X_t = c + \phi X_{t-1} + \varepsilon_t$

- $\hat{X}_{n+1|n} = c + \phi X_n$; $\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu)$
- Forecasts converge to the mean μ as $h \rightarrow \infty$ (mean reversion)

MA(1): $X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

- $\hat{X}_{n+1|n} = \mu + \theta \varepsilon_n$; $\hat{X}_{n+h|n} = \mu$ for $h > 1$

Forecast Uncertainty

Mean Square Forecast Error (MSFE)

- **Error:** $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- **MSFE:** $\text{MSFE}(h) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$, where ψ_j are the $\text{MA}(\infty)$ coefficients

For AR(1): $\psi_j = \phi^j$

- $\text{MSFE}(h) = \sigma^2 \frac{1-\phi^{2h}}{1-\phi^2} \rightarrow \frac{\sigma^2}{1-\phi^2} = \text{Var}(X_t)$

Key Observation

- Forecast uncertainty increases with the horizon
- Converges to the unconditional variance $\text{Var}(X_t)$

Proof: MSFE for AR(1)

Claim

$$\square \text{ MSFE}(h) = \sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2} \quad \text{and} \quad \text{MSFE}(\infty) = \gamma(0)$$

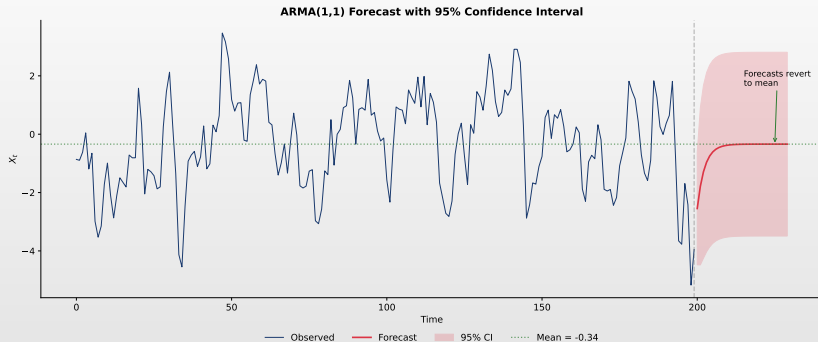
Proof

- \square Forecast error at horizon h : $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- \square By recursive substitution: $e_{n+h|n} = \sum_{j=0}^{h-1} \phi^j \varepsilon_{n+h-j}$
- \square $\text{MSFE}(h) = \mathbb{E}[e_{n+h|n}^2] = \sigma^2 \sum_{j=0}^{h-1} \phi^{2j} = \boxed{\sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2}}$
- \square Limit: $\text{MSFE}(\infty) = \frac{\sigma^2}{1 - \phi^2} = \gamma(0) \Rightarrow$ forecast converges to unconditional mean

Interpretation

- \square At long horizons, we do no better than the unconditional mean: $\text{CI} \rightarrow 2 \times 1.96 \sqrt{\gamma(0)}$

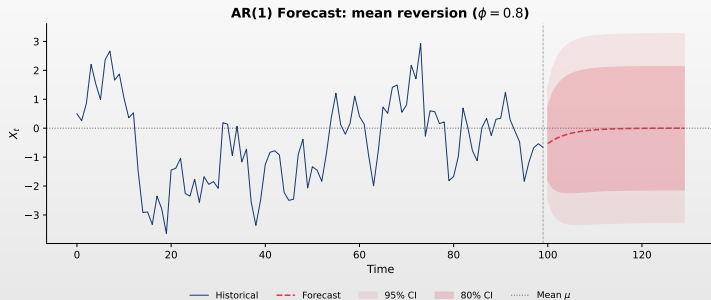
ARMA Forecast with Confidence Intervals



Observation

- The confidence band widens with the horizon \nearrow convergence to the unconditional interval

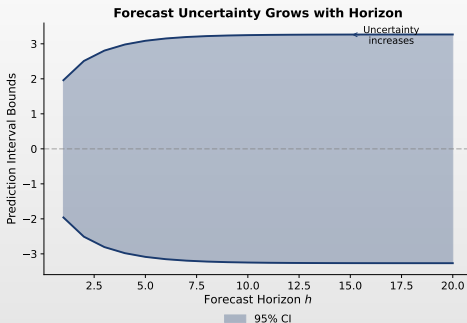
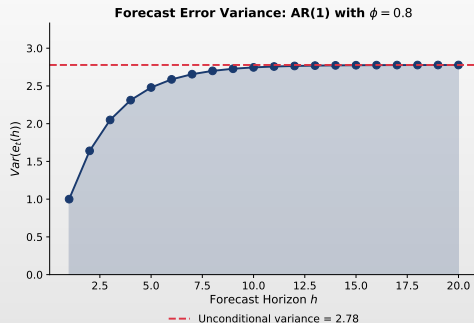
AR(1) Forecast: Mean Reversion



Properties

- ▣ Forecasts converge to the unconditional mean μ as the horizon increases
- ▣ Larger $|\phi|$ \succ slower reversion; CIs widen with the horizon

Forecast Error Variance by Horizon



Observation

- MSFE increases monotonically with horizon h \rightarrow convergence to $\text{Var}(X_t)$ (predictability limit)

Confidence Intervals for Forecasts

Formulas

- ▣ $X_{n+h}|X_n, \dots \sim N(\hat{X}_{n+h|n}, \text{MSFE}(h))$
- ▣ **CI** $(1 - \alpha)$: $\hat{X}_{n+h|n} \pm z_{\alpha/2} \cdot \sqrt{\text{MSFE}(h)}$, where $z_{\alpha/2} = 1.96$ for 95%

Properties

- ▣ Intervals widen as the horizon increases
 - ▶ Converge to the unconditional interval: $\mu \pm z_{\alpha/2} \sigma_X$
- ▣ Width depends on model parameters
 - ▶ Larger AR coefficients \succ wider intervals
- ▣ **Python**: `model.get_forecast(h).conf_int()`

Forecast Evaluation

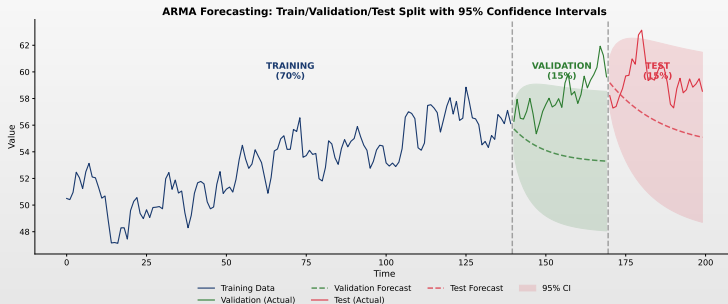
Out-of-Sample Testing

- Split data: training + test
- Generate forecasts on test
- Compare with actual values
- **Rolling window**: re-estimate as new data arrives

Error Metrics

- **MAE** = $\frac{1}{n} \sum |e_t|$
 - ▶ Robust to outliers
- **RMSE** = $\sqrt{\frac{1}{n} \sum e_t^2}$
 - ▶ Penalizes large errors
- **MAPE** = $\frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$
 - ▶ Percentage-based, interpretable

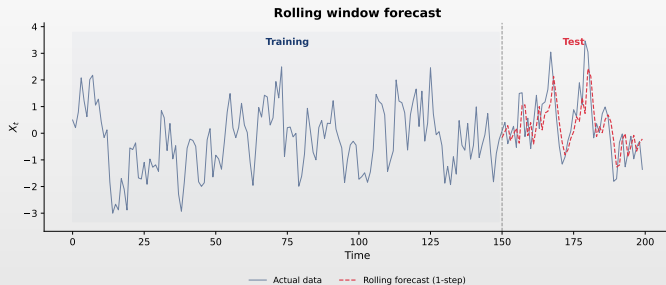
Train/Validation/Test Forecast Example



Best Practice

- Always evaluate forecasts on data not used for estimation (train/validation/test split)

Rolling Window Forecast

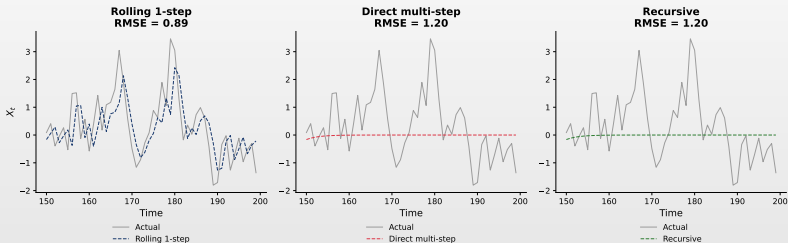


Rolling Forecast Methodology

- Fixed window (last w obs.) vs expanding (all data); generate 1-step forecast, repeat

Rolling vs Multi-Step Forecast

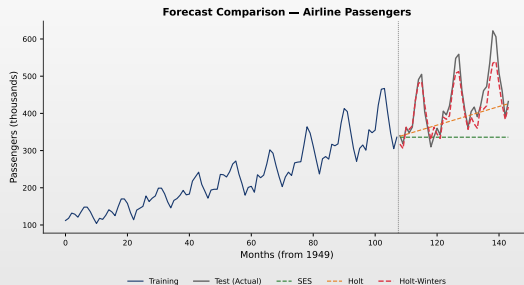
Comparison: Rolling vs Multi-step vs Recursive



Key Differences

- Rolling 1-step (accurate); Multi-step direct (separate model/horizon); Recursive (error accumulation)

Real Data Application: Forecast Comparison



Practical Considerations

- Real data: non-stationarity, structural breaks; compare models; use rolling window validation

Python Implementation: Estimating ARMA Models

Using statsmodels

```
from statsmodels.tsa.arima.model import ARIMA
model = ARIMA(data, order=(2, 0, 1)) # ARMA(2,1)
results = model.fit()
print(results.summary())
```

Forecasting

```
forecast = results.get_forecast(steps=10)
print(forecast.predicted_mean)
print(forecast.conf_int())
```

Note

ARIMA with $d = 0$ is equivalent to ARMA

Python: Model Selection with pmdarima

Automatic ARIMA Selection

```
□ import pmdarima as pm
□ model = pm.auto_arima(data,
□     start_p=0, max_p=5, start_q=0, max_q=5,
□     d=0, seasonal=False,
□     information_criterion='aic', trace=True)
□ print(model.summary())
```

Result

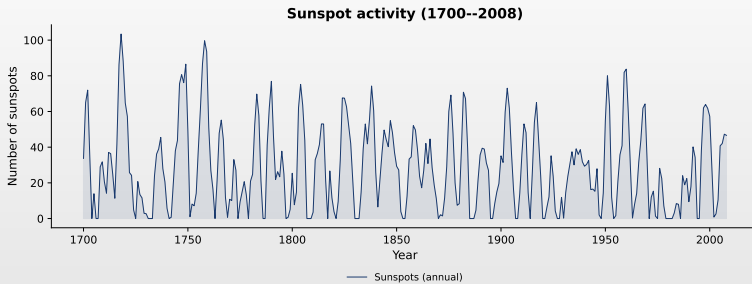
```
□ Best model order and estimated parameters
```

Workflow Summary

Box-Jenkins Methodology Steps

- 1. **Data preparation:** Check for missing values, outliers; transform if necessary
- 2. **Stationarity check:** Visual inspection, formal tests (ADF, KPSS); difference if non-stationary
- 3. **Model identification:** ACF/PACF patterns; grid search with information criteria
- 4. **Estimation and validation:** Estimate model, check significance; residual analysis, Ljung-Box test
- 5. **Forecasting:** Point forecasts with confidence intervals; out-of-sample validation

Case Study: Sunspots

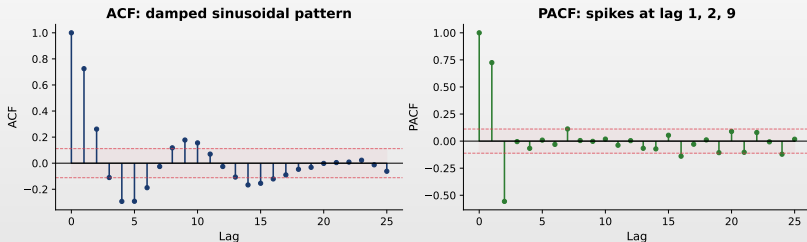


Data Description

- Annual sunspots (1700–2008): stationary series with ~ 11 -year cycles; Box-Jenkins methodology

Step 1: ACF/PACF Analysis

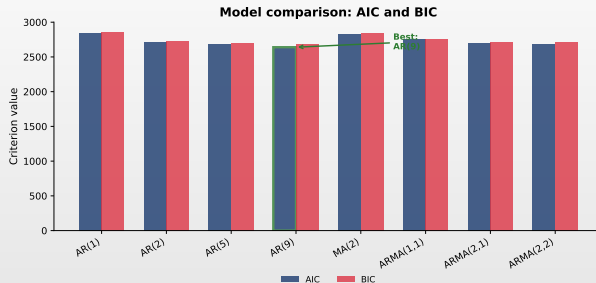
ACF/PACF analysis for sunspots



Identification

- Sinusoidal ACF (AR); PACF with spikes at lags 1, 2, 9 \succ AR(2) or AR(9); stationary series ($d = 0$)

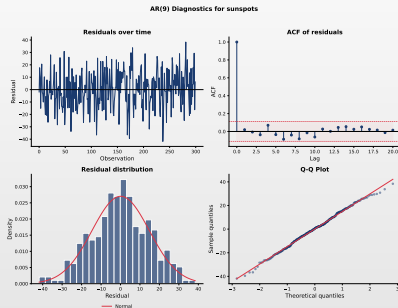
Step 2: Model Comparison



Model Selection

- Compare multiple candidate models using the AIC criterion
- The **AR(9)** model has the lowest AIC, capturing the 11-year solar cycle

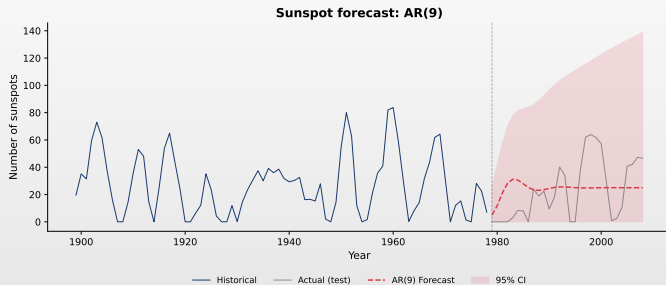
Step 3: Model Diagnostics



AR(9) Diagnostics

- Residuals: white noise, zero mean, constant variance, ACF without structure, \approx normal

Step 4: Forecasting



Results

- AR(9) captures the cyclical; 95% CI covers actual values; RMSE ≈ 30

Key Takeaways

Chapter Summary

- ▣ **AR(p)**: Depends on p past values; stationarity: roots outside the unit circle; PACF cuts off at lag p
- ▣ **MA(q)**: Depends on q past shocks; always stationary; ACF cuts off at lag q
- ▣ **ARMA(p,q)**: Combines AR and MA; both ACF and PACF decay
- ▣ **Box-Jenkins**: Identification \succ Estimation \succ Validation \succ Forecasting
- ▣ **Validation**: Residuals must be white noise
- ▣ **Forecasts**: Converge to the mean; uncertainty increases with the horizon

Next Chapter Preview

Chapter 3: ARIMA Models for Non-Stationary Data

- ▣ Non-stationarity: types, unit root tests (ADF, PP, KPSS)
- ▣ Differencing and the difference operator
- ▣ ARIMA(p,d,q): integrated models for non-stationary data
- ▣ The Auto-ARIMA algorithm: automatic model selection
- ▣ Case study: US GDP Forecasting

Reading

- ▣ Hyndman & Athanasopoulos, *Forecasting: Principles and Practice*, Ch. 9
- ▣ Box, Jenkins, Reinsel & Ljung, *Time Series Analysis*, Ch. 3-4

Question 1

Question

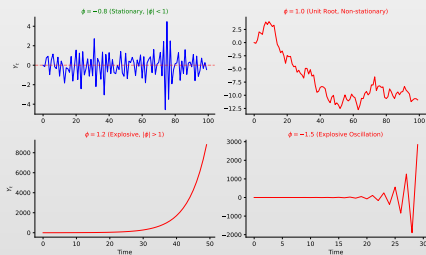
□ For which value of ϕ is the AR(1) process $X_t = c + \phi X_{t-1} + \varepsilon_t$ stationary?

- (A) $\phi = 1.2$
- (B) $\phi = 1.0$
- (C) $\phi = -0.8$
- (D) $\phi = -1.5$

Question 1: Answer

Correct Answer: (C) $\phi = -0.8$

- ☐ AR(1) is stationary if and only if $|\phi| < 1$
- ☐ Only $|-0.8| = 0.8 < 1$



Question 2

Question

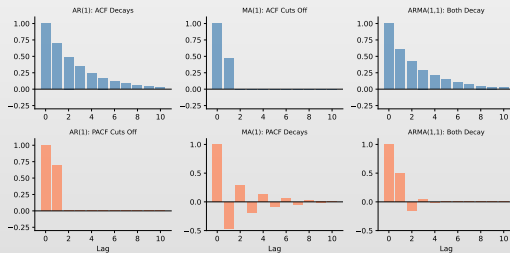
☐ You observe: ACF has a spike at lag 1, then cuts off. PACF decays gradually. What model?

- (A) AR(1)
- (B) MA(1)
- (C) ARMA(1,1)
- (D) White noise

Question 2: Answer

Correct Answer: (B) MA(1)

- ☐ ACF cuts off \succ MA process
- ☐ PACF decays \succ confirms MA(1)



Question 3

Question

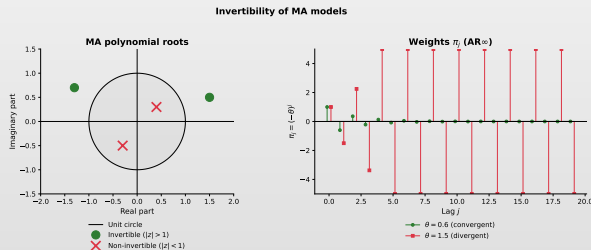
□ Is the MA(1) $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$ invertible?

- (A) Yes, MA processes are always invertible
- (B) Yes, because $1.5 > 0$
- (C) No, because $|\theta| = 1.5 > 1$
- (D) No, MA processes are never invertible

Question 3: Answer

Correct Answer: (C) No, because $|\theta| = 1.5 > 1$

- Invertibility requires $|\theta| < 1$
- Here $|\theta| = 1.5 > 1$, so it is not invertible



Question 4

Question

☐ The compact form $\phi(L)X_t = \theta(L)\varepsilon_t$ represents which model?

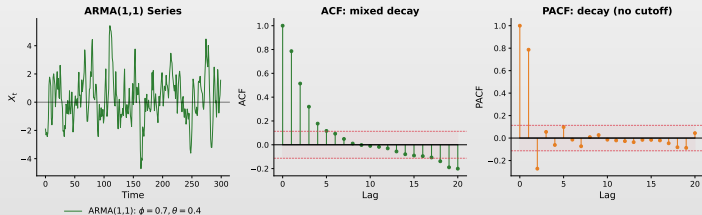
- (A) Pure AR model
- (B) Pure MA model
- (C) ARMA model
- (D) None of the above

Question 4: Answer

Correct Answer: (C) ARMA model

□ $\phi(L)$ is the AR polynomial, $\theta(L)$ is the MA polynomial \succ ARMA(p,q)

ARMA(1,1): neither ACF nor PACF cut off



Question 5

Question

What is $(1 - L)^2 X_t$?

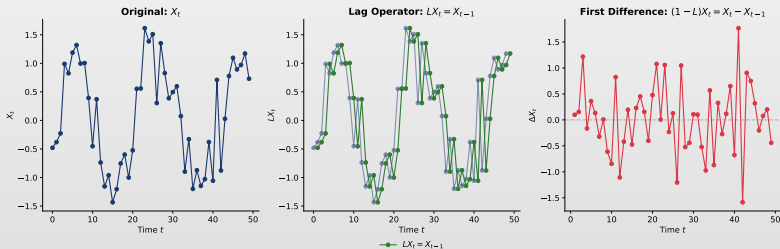
- (A) $X_t - X_{t-1}$
- (B) $X_t - 2X_{t-1} + X_{t-2}$
- (C) $X_t + X_{t-1} + X_{t-2}$
- (D) $X_t - X_{t-2}$

Question 5: Answer

Correct Answer: (B) $X_t - 2X_{t-1} + X_{t-2}$

☐ $(1 - L)^2 = 1 - 2L + L^2$

☐ $(1 - L)^2 X_t = X_t - 2X_{t-1} + X_{t-2}$



Question 6

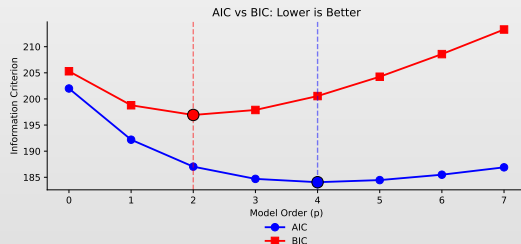
Question

- ☐ Comparing ARMA(1,1) vs ARMA(2,1) using BIC, which is correct?
- (A) Lower BIC always means better forecasts
 - (B) BIC penalizes complexity less than AIC
 - (C) The model with lower BIC is preferred
 - (D) BIC can only compare models with the same number of parameters

Question 6: Answer

Correct Answer: (C) The model with lower BIC is preferred

- ☐ Lower BIC indicates a better balance between estimation quality and complexity
- ☐ BIC penalizes complexity *more* than AIC



Question 7

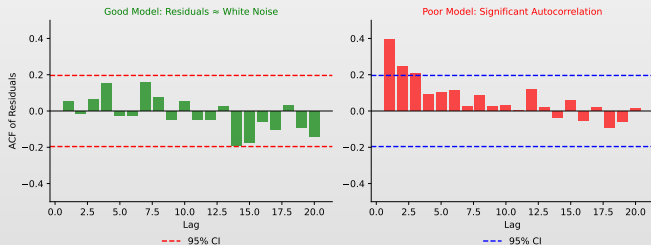
Question

- ☐ After estimating an ARMA model, you run the Ljung-Box test on residuals and obtain $p\text{-value} = 0.03$. What does this mean?
- (A) The model is adequate, residuals are white noise
 - (B) The model is inadequate, residuals have autocorrelation
 - (C) You need to increase the sample size
 - (D) The test is inconclusive

Question 7: Answer

Correct Answer: (B) The model is inadequate

- ☐ p-value < 0.05 rejects H_0 (white noise)
- ☐ Indicates remaining residual autocorrelation



Question 8

Question

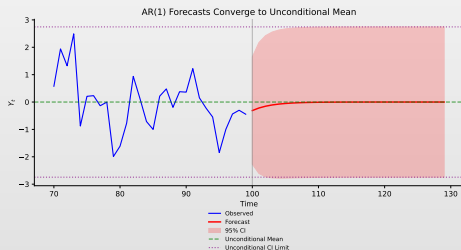
□ For a stationary AR(1) model, what happens to forecasts as the horizon $h \rightarrow \infty$?

- (A) Forecasts increase without bound
- (B) Forecasts oscillate indefinitely
- (C) Forecasts converge to the unconditional mean μ
- (D) Forecasts become more precise

Question 8: Answer

Correct Answer: (C) Forecasts converge to μ

$$\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu) \rightarrow \mu \text{ as } h \rightarrow \infty \text{ (since } |\phi| < 1)$$



Question 9

Question

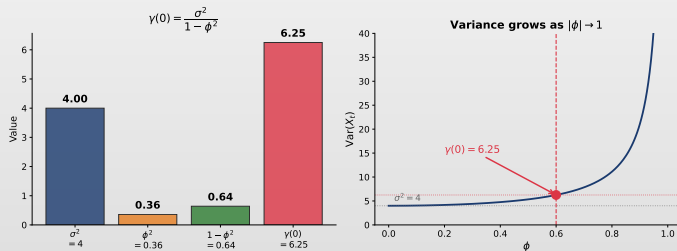
□ Consider an AR(1) process with $\phi = 0.6$ and $\sigma^2 = 4$. What is $\text{Var}(X_t)$?

- (A) 4.0
- (B) 5.56
- (C) 6.25
- (D) 10.0

Question 9: Answer

Correct Answer: (C) 6.25

- ☐ $\text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2} = \frac{4}{1-0.36} = \frac{4}{0.64} = 6.25$
- ☐ The process variance exceeds σ^2 due to persistence



Question 10

Question

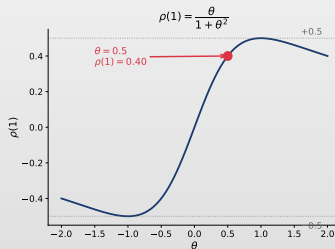
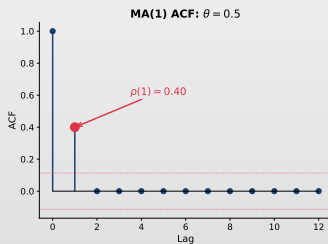
□ Consider an MA(1) process with $\theta = 0.5$. What is $\rho(1)$?

- (A) 0.50
- (B) 0.40
- (C) 0.25
- (D) 0.33

Question 10: Answer

Correct Answer: (B) 0.40

- $\rho(1) = \frac{\theta}{1+\theta^2} = \frac{0.5}{1+0.25} = \frac{0.5}{1.25} = 0.40$
- Note that $\rho(1) < \theta$ — the autocorrelation is **always** attenuated



Question 11

Question

☐ Which statement about the ACF of an ARMA(1,1) process is **true**?

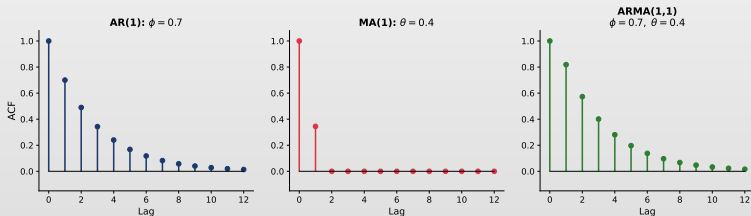
- (A) It cuts off after lag 1
- (B) Exponential decay starting from lag 1, with $\rho(1) \neq \phi$
- (C) It is zero for all lags
- (D) It exactly follows the pattern ϕ^h for all $h \geq 0$

Question 11: Answer

Correct Answer: (B) Exponential decay from lag 1, with $\rho(1) \neq \phi$

- $\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2} \neq \phi$ (the MA component modifies lag 1)
- For $h \geq 2$: $\rho(h) = \phi \rho(h-1)$ — exponential decay as in AR(1)

ACF Comparison: AR(1) vs MA(1) vs ARMA(1,1)



Question 12

Question

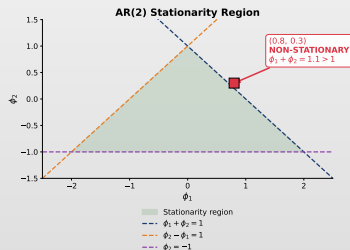
□ An AR(2) process has $\phi_1 = 0.8$ and $\phi_2 = 0.3$. Is it stationary?

- (A) Yes, it is stationary
- (B) No, because $\phi_1 + \phi_2 = 1.1 > 1$
- (C) Cannot be determined without data
- (D) Depends on the value of σ^2

Question 12: Answer

Correct Answer: (B) No, because $\phi_1 + \phi_2 = 1.1 > 1$

- Necessary conditions for AR(2) stationarity:
- $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, $|\phi_2| < 1$
- Here $0.8 + 0.3 = 1.1 > 1 \Rightarrow$ the first condition is violated



Question 13

Question

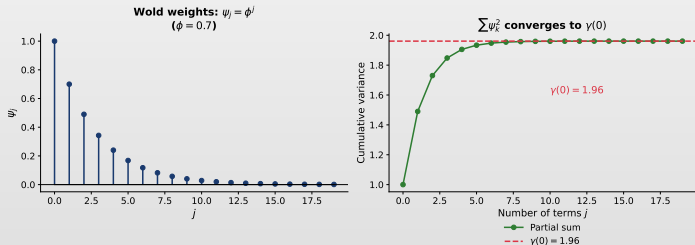
☐ What does the Wold decomposition theorem guarantee?

- (A) Any time series is an AR process
- (B) Any stationary process can be written as $MA(\infty)$: $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- (C) Any process has finite variance
- (D) ARMA models are always invertible

Question 13: Answer

Correct Answer: (B) Any stationary process = $MA(\infty)$

- Wold's theorem: $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + D_t$, where D_t is the deterministic component
- This justifies ARMA models: they are parsimonious approximations of $MA(\infty)$



Question 14

Question

▣ AR(1) with $\phi = 0.9$, $\sigma^2 = 1$. What happens to the CI width as $h \rightarrow \infty$?

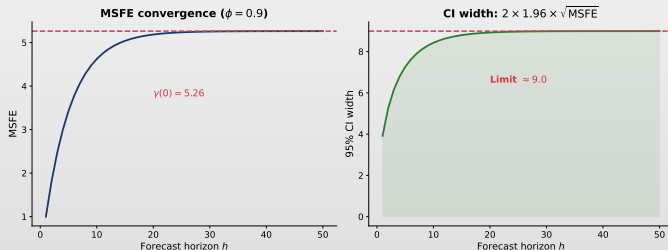
- (A) It remains constant
- (B) It decreases to zero
- (C) It grows toward $2 \times 1.96 \times \sqrt{1/(1 - 0.81)} \approx 9.0$
- (D) It grows to infinity

Question 14: Answer

Correct Answer: (C) Grows toward ≈ 9.0

□ $MSFE(\infty) = \frac{\sigma^2}{1-\phi^2} = \frac{1}{1-0.81} = \frac{1}{0.19} \approx 5.26$

□ $CI \text{ width} = 2 \times 1.96 \sqrt{5.26} \approx 2 \times 1.96 \times 2.29 \approx 9.0$



Data Sources and Software

Software Packages

- ▣ statsmodels \succ Statistical models for Python, including ARIMA
- ▣ pmdarima \succ Automatic ARIMA selection for Python
- ▣ scipy \succ Optimization and statistical functions
- ▣ numpy, pandas \succ Data manipulation
- ▣ matplotlib \succ Visualization

Data and Examples

- ▣ Simulated time series for illustrations
- ▣ Examples based on Hyndman & Athanasopoulos (2021)

Bibliography I

Fundamental ARMA Works

- Box, G.E.P., & Jenkins, G.M. (1970). *Time Series Analysis: Forecasting and Control*, Holden-Day.
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- Ljung, G.M., & Box, G.E.P. (1978). On a Measure of Lack of Fit in Time Series Models, *Biometrika*, 65(2), 297–303.
- Brockwell, P.J., & Davis, R.A. (2016). *Introduction to Time Series and Forecasting*, 3rd ed., Springer.

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- ▣ Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.
- ▣ Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

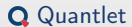
Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> → Code repository for statistics
- ▣ **Quantinar**: <https://quantinar.com> → Learning platform for quantitative methods
- ▣ **GitHub TSA**: <https://github.com/QuantLet/TSA> → Python code for this course

Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



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