



# Time Series Analysis and Forecasting

## Chapter 7: Cointegration and VECM



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## Learning Objectives

By the end of this chapter, you will be able to:

- Understand the problem of spurious regression with non-stationary data
- Test for cointegration using Engle-Granger and Johansen methods
- Estimate Vector Error Correction Models (VECM)
- Interpret error correction mechanisms and adjustment speeds

## Outline

Motivation

Spurious Regression

Cointegration Concept

Engle-Granger Method

Johansen Method

VECM Estimation

Practical Considerations

Real-World Examples

Case Study: Interest Rates

Summary

Quiz

## Why Cointegration Matters

### The Challenge

- Many economic/financial time series are **non-stationary** ( $I(1)$ )
- GDP, stock prices, exchange rates, interest rates all have unit roots
- Standard regression with  $I(1)$  variables  $\Rightarrow$  **spurious results**
- Differencing removes non-stationarity but loses **long-run information**

### The Solution: Cointegration

Some non-stationary series share a **common stochastic trend**—they move together in the long run.

### Nobel Prize 2003

Granger & Engle received the Nobel Prize for “methods for analyzing economic time series with common trends.”

## Real-World Applications

### Finance

- ▣ **Pairs Trading:** Cointegrated stocks
- ▣ **Term Structure:** Interest rates
- ▣ **Spot-Futures:** Arbitrage

### Macroeconomics

- ▣ **Consumption & Income**
- ▣ **Money & Prices**
- ▣ **PPP:** Exchange rates

### Policy Analysis

- ▣ **Fiscal:** Spending & taxes
- ▣ **Monetary:** Rate pass-through
- ▣ **Labor:** Wages & productivity

## The Spurious Regression Problem

### Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk:  $Y_t = \alpha + \beta X_t + u_t$  where  $Y_t$  and  $X_t$  are independent  $I(1)$  processes.

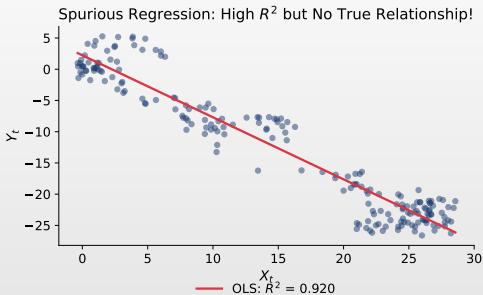
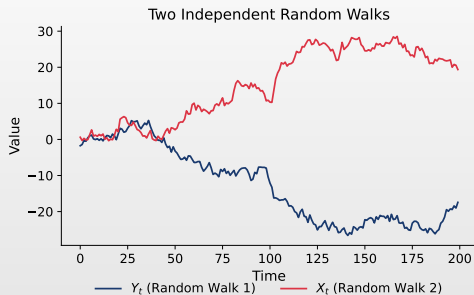
### Symptoms of Spurious Regression

- ▣ High  $R^2$  (often  $> 0.9$ ) even though variables are **unrelated**
- ▣ Highly significant  $t$ -statistics (reject  $H_0 : \beta = 0$ )
- ▣ Very low Durbin-Watson statistic ( $DW \approx 0$ )
- ▣ Residuals are non-stationary (have unit root)

### Rule of Thumb

If  $R^2 > DW$ , suspect spurious regression!

## Spurious Regression: Visual Example



**Warning:** Two independent random walks show high correlation ( $R^2 > 0.8$ ) by chance! This is why we need cointegration analysis.

 [TSA\\_ch7\\_spurious\\_regression](#)

## Spurious Correlations in the Real World

### Data Mining Can Produce Meaningless Correlations

With enough variables and long time series, purely coincidental patterns emerge:

- Distance between Neptune and Uranus  $\leftrightarrow$  SAP SE stock price (2002–2023)
- GMO corn use in South Dakota  $\leftrightarrow$  Google searches for “i cant even” (2004–2023)
- Two and a Half Men* season ratings  $\leftrightarrow$  Jet fuel used in Serbia (2006–2015)
- “Its Wednesday my dudes” meme popularity  $\leftrightarrow$  Boeing stock price (2006–2023)

### Lesson

High correlation  $\neq$  causation. Non-stationary series with common trends produce high  $R^2$  by construction. Always test for stationarity and cointegration before interpreting regression results!

🌐 Explore more examples: [tylervigen.com/spurious-correlations](https://tylervigen.com/spurious-correlations)



## Definition of Cointegration

### Definition 1 (Cointegration (Engle & Granger, 1987))

Variables  $Y_{1t}, Y_{2t}, \dots, Y_{kt}$  are **cointegrated of order  $(d, b)$** , written  $CI(d, b)$ , if:

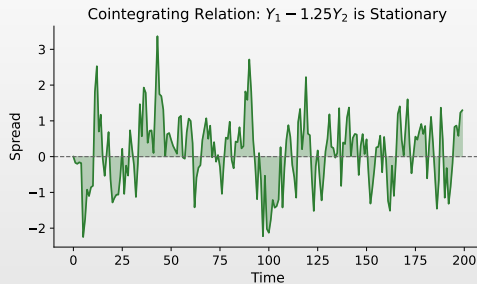
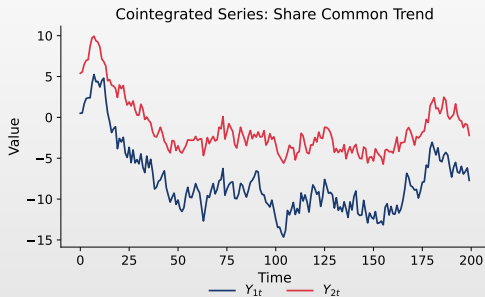
1. All variables are integrated of order  $d$ :  $Y_{it} \sim I(d)$
2. There exists a linear combination  $\beta'Y_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$  that is integrated of order  $(d - b)$ , where  $b > 0$

### Most Common Case: $CI(1, 1)$

- ▣ Variables are  $I(1)$  (have unit roots)
- ▣ Linear combination is  $I(0)$  (stationary)
- ▣ Vector  $\beta = (\beta_1, \dots, \beta_k)'$  is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized:  $\beta_1 = 1$ .

## Cointegration: Visual Example



**Key insight:** Both series are  $I(1)$  and trend together, but their linear combination (spread) is stationary—this is cointegration!

 [TSA\\_ch7\\_cointegrated\\_series](#)

## Intuition: Common Stochastic Trends

### Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**:  $Y_{1t} = \gamma_1 \tau_t + S_{1t}$ ,  $Y_{2t} = \gamma_2 \tau_t + S_{2t}$  where  $\tau_t$  is a common random walk and  $S_{it}$  are stationary.

### Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

### Economic Interpretation

- Cointegration = **long-run equilibrium relationship**
- Variables may deviate in the short run, but are “pulled back”
- The cointegrating vector defines the equilibrium

## Cointegrating Rank

### How Many Cointegrating Relationships?

For  $k$  variables that are  $I(1)$ :

- Maximum possible cointegrating relationships:  $r = k - 1$
- If  $r = 0$ : No cointegration (variables drift apart)
- If  $r = k$ : All variables are  $I(0)$  (contradiction)

### Example: 3 Variables

- $r = 0$ : No cointegration
- $r = 1$ : One cointegrating relationship
- $r = 2$ : Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends =  $k - r$

## Engle-Granger Two-Step Method

### Step 1: Estimate Cointegrating Regression

Run OLS:  $Y_t = \alpha + \beta X_t + e_t$ . Save residuals:  $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

### Step 2: Test Residuals for Stationarity

Test if  $\hat{e}_t$  is  $I(0)$  using ADF:  $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$

- ▣  $H_0: \rho = 0$  (unit root  $\Rightarrow$  no cointegration)
- ▣  $H_1: \rho < 0$  (stationary  $\Rightarrow$  cointegration)

### Important

Use **Engle-Granger critical values**, not standard ADF! (More negative because residuals are estimated)

## Engle-Granger Critical Values

### Critical Values for Cointegration Test

Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

MacKinnon (1991),  $T = 100$

### Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on choice of dependent variable
- Small sample bias; cannot test hypotheses on cointegrating vector

## Johansen Cointegration Test

### Advantages over Engle-Granger

- ▣ Tests for **multiple** cointegrating relationships
- ▣ Maximum likelihood estimation (more efficient)
- ▣ Can test restrictions on cointegrating vectors
- ▣ Does not require choosing a dependent variable

### Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...

## VECM Representation

### Vector Error Correction Model

$$\Delta Y_t = c + \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

- $\Pi = \sum_i A_i - I$  (long-run impact);  $\Gamma_j$  (short-run dynamics)

### Key Insight: Rank of $\Pi$

The **rank of  $\Pi$**  determines cointegration:

- $\text{rank}(\Pi) = 0$ : No cointegration (VAR in differences)
- $\text{rank}(\Pi) = k$ : All variables are  $I(0)$  (VAR in levels)
- $0 < \text{rank}(\Pi) = r < k$ :  $r$  cointegrating vectors



## Derivation: From VAR to VECM

Starting Point: VAR(p) in Levels

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Step 1: Subtract  $Y_{t-1}$  from Both Sides

$$Y_t - Y_{t-1} = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} - Y_{t-1} + \varepsilon_t$$

$$\Delta Y_t = (A_1 - I) Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Goal

Rewrite so that all terms are either in **levels** ( $Y_{t-1}$ ) or **differences** ( $\Delta Y_{t-j}$ ).

## Derivation: From VAR to VECM (cont.)

### Step 2: Add and Subtract Terms Strategically

Add  $A_2 Y_{t-1}$  and subtract  $A_2 Y_{t-1}$ :

$$\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} - A_2(Y_{t-1} - Y_{t-2}) + A_3 Y_{t-3} + \cdots + \varepsilon_t$$

Continue adding  $A_3 Y_{t-1}$ , etc., until all lagged **levels** are collected in one term.

### Step 3: General Pattern

After algebraic manipulation, we obtain:

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

## The Key Matrices

Time Series Analysis and Forecasting

$$\Pi = \sum_{i=1}^p A_i - I = -(I - A_1 - A_2 - \cdots - A_p)$$



## Derivation: Verifying the $\Gamma_j$ Formula

### Example: VAR(2)

Starting from:  $Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \epsilon_t$

Subtract  $Y_{t-1}$ :

$$\Delta Y_t = (A_1 - I)Y_{t-1} + A_2 Y_{t-2} + \epsilon_t$$

Add and subtract  $A_2 Y_{t-1}$ :

$$\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} + A_2(Y_{t-2} - Y_{t-1}) + \epsilon_t$$

$$\Delta Y_t = \underbrace{(A_1 + A_2 - I)}_{\Pi} Y_{t-1} \underbrace{- A_2}_{\Gamma_1} \Delta Y_{t-1} + \epsilon_t$$

### Verification

For VAR(2):  $\Pi = A_1 + A_2 - I$  and  $\Gamma_1 = -A_2$

Using our formula:  $\Gamma_1 = -\sum_{i=2}^2 A_i = -A_2 \quad \checkmark$

## Economic Interpretation of Error Correction

### The VECM with Cointegration

When  $\text{rank}(\Pi) = r$ , we write  $\Pi = \alpha\beta'$ :

$$\Delta Y_t = \alpha \underbrace{(\beta' Y_{t-1})}_{\text{equilibrium error}} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

### Economic Interpretation

- ▣  $\beta' Y_{t-1} = \text{equilibrium error}$ : deviation from long-run relationship
- ▣  $\alpha = \text{adjustment speeds}$ : how fast variables correct deviations
- ▣  $\Gamma_j = \text{short-run dynamics}$ : transitory effects

### Error Correction Mechanism

If  $\beta' Y_{t-1} > 0$  (above equilibrium) and  $\alpha_i < 0$ , then  $\Delta Y_{it}$  decreases.

**The system self-corrects toward equilibrium!**

## Decomposition of $\Pi$

When  $\text{rank}(\Pi) = r < k$

$\Pi = \alpha\beta'$  where  $\beta$  ( $k \times r$ ) = **cointegrating vectors**,  $\alpha$  ( $k \times r$ ) = **adjustment coefficients**

### Interpretation

- ▣  $\beta'Y_{t-1}$  = deviations from equilibrium (error correction terms)
- ▣  $\alpha$  = speed of adjustment; rows show each variable's response

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta'Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

## Johansen Test Statistics

### Two Test Statistics

Based on eigenvalues  $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$  of a certain matrix:

#### Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests  $H_0: \text{rank} \leq r$  vs  $H_1: \text{rank} > r$

#### Maximum Eigenvalue Test:

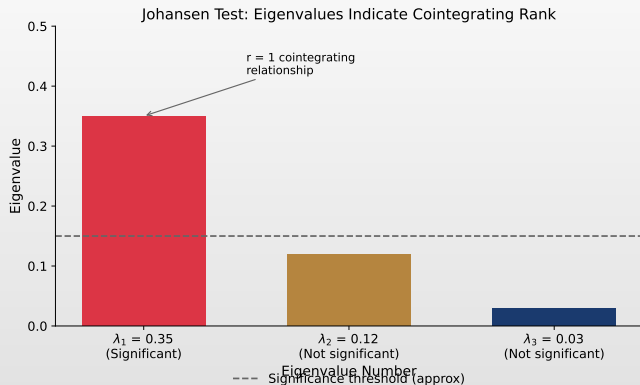
$$\lambda_{\text{max}}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

Tests  $H_0: \text{rank} = r$  vs  $H_1: \text{rank} = r+1$

Critical values from Johansen & Juselius (1990), depend on:

- ▣ Number of variables  $k$
- ▣ Deterministic components (constant, trend)

## Johansen Test: Visual Interpretation



Significant eigenvalues (above threshold) indicate cointegrating relationships. First eigenvalue significant  $\Rightarrow r = 1$ .

[TSA\\_ch7\\_johansen\\_eigenvalues](#)



## Testing Procedure

### Sequential Testing (Trace Test)

1. Test  $H_0: r = 0$ . If rejected  $\Rightarrow$  continue
2. Test  $H_0: r \leq 1$ . If not rejected  $\Rightarrow r = 1$
3. Continue until  $H_0$  is not rejected

### Deterministic Components

- ☐ No constant, no trend (rarely used)
- ☐ Constant in cointegrating relation only
- ☐ **Constant in both** (most common)
- ☐ Constant + trend in cointegrating relation



## VECM Structure

### Full VECM Specification

For  $k = 2$  variables with  $r = 1$  cointegrating relation:

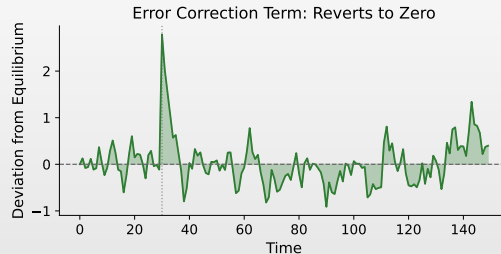
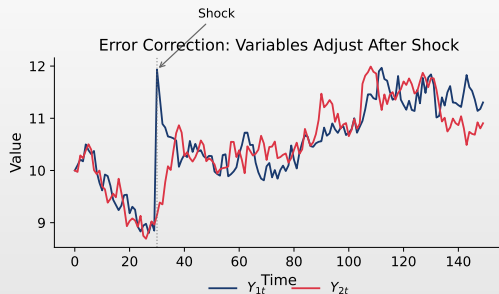
$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

### Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$  = error correction term (deviation from equilibrium)
- $\alpha_1, \alpha_2$  = adjustment speeds (should have opposite signs)
- $\gamma_{ij}$  = short-run dynamics
- $\varepsilon_{it}$  = innovations

## Error Correction Mechanism: Visual



**Error correction in action:** When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment. [TSA\\_ch7\\_error\\_correction](#)

## Interpreting Adjustment Coefficients

### The $\alpha$ Coefficients

If the cointegrating relation is  $Y_1 - \beta Y_2 = 0$  (equilibrium):

- ▣  $\alpha_1 < 0$ :  $Y_1$  adjusts downward when above equilibrium
- ▣  $\alpha_2 > 0$ :  $Y_2$  adjusts upward when  $Y_1$  is above equilibrium

### Weak Exogeneity

If  $\alpha_i = 0$ , variable  $Y_i$  does **not** respond to disequilibrium.

- ▣  $Y_i$  is **weakly exogenous** for the long-run parameters
- ▣ The other variable does all the adjusting
- ▣ Can simplify estimation (single-equation approach)

Test weak exogeneity:  $H_0 : \alpha_i = 0$  using likelihood ratio test.

## VECM vs VAR in Differences

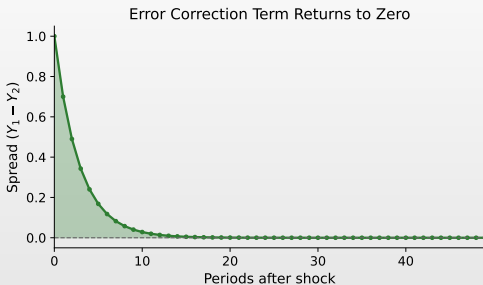
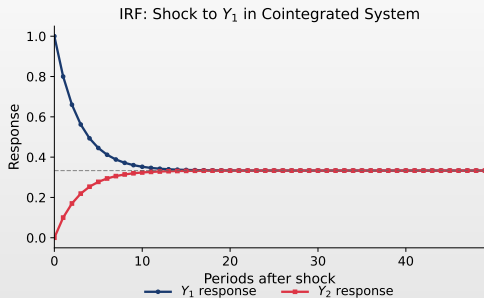
### When Variables are Cointegrated

	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

### Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

## VECM Impulse Response Functions



**IRF interpretation:** In a cointegrated system, shocks have **permanent effects** on levels but the system returns to equilibrium. Unlike stationary VAR, effects don't decay to zero—they converge to a new long-run value.

 [TSA\\_ch7\\_vecm\\_irf](#)

## Practical Workflow

### Step-by-Step Procedure

1. **Unit Root Tests:** Verify all variables are  $I(1)$ 
  - ▶ ADF, KPSS on levels and first differences
2. **Lag Length Selection:** Choose  $p$  for VAR in levels
  - ▶ Use AIC, BIC, or sequential LR tests
3. **Cointegration Test:** Johansen trace/max-eigenvalue tests
  - ▶ Determine cointegrating rank  $r$
4. **Estimate VECM:** If  $0 < r < k$ 
  - ▶ Estimate  $\alpha$ ,  $\beta$ ,  $\Gamma_j$
5. **Diagnostics:** Check residuals for autocorrelation, normality
6. **Analysis:** IRF, FEVD, hypothesis tests

## Common Pitfalls

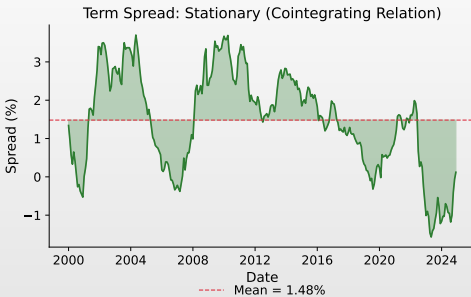
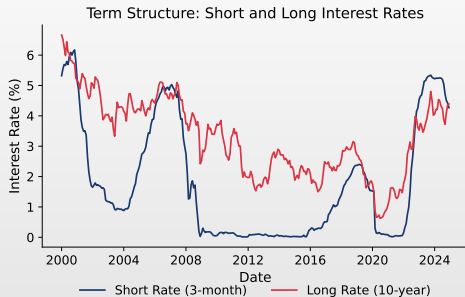
### Things to Watch Out For

- ▣ **Structural breaks:** Cause spurious unit roots or cointegration
- ▣ **Near-unit-root:** Tests have low power
- ▣ **Lag selection:** Too many/few lags bias results
- ▣ **Small samples:** Johansen test oversized

### Recommendation

Always check: residual diagnostics, stability of cointegrating relationship, sensitivity to specification

## Example 1: Term Structure of Interest Rates

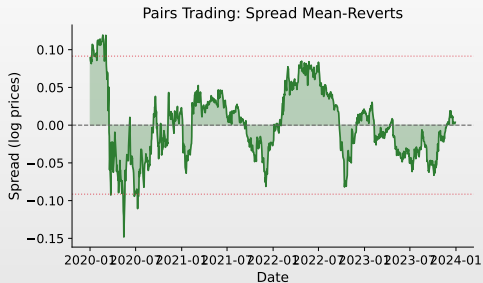
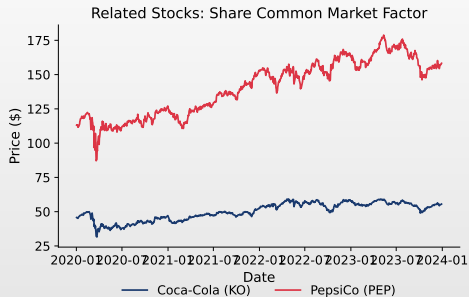


**Expectations Hypothesis:** Short and long rates share common trend. The spread (term premium) is stationary—evidence of cointegration!

 TSA\_ch7\_interest\_rates\_coint



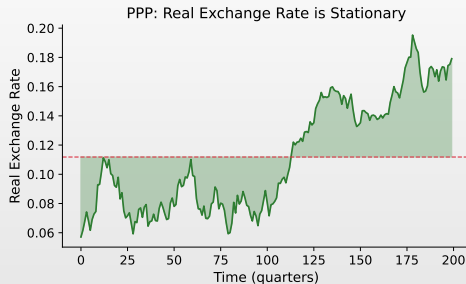
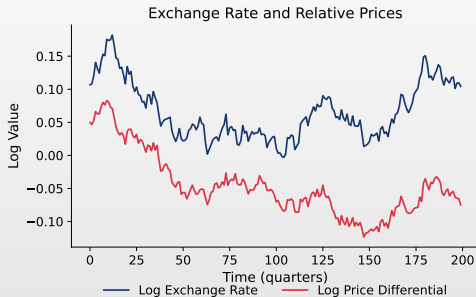
## Example 2: Pairs Trading in Finance



**Strategy:** Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When spread deviates from mean, trade expecting mean reversion. Sell spread when high, buy when low.

 TSA\_ch7\_pairs\_trading

### Example 3: Purchasing Power Parity (PPP)



**PPP Theory:**  $e_t = p_t - p_t^*$  (log exchange rate equals price differential). Real exchange rate should be stationary in the long run.

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## Case Study: Cointegration of Interest Rates

### Research Question

Are short-term and long-term interest rates cointegrated? Does the expectations hypothesis of the term structure hold?

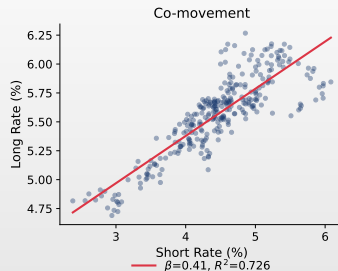
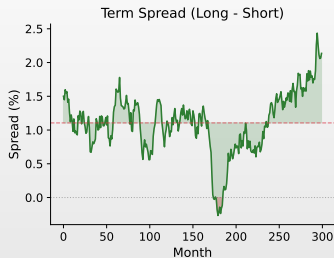
### Data

- ▣ US Monthly Data (1962-2023)
- ▣ 3-Month Treasury Bill Rate
- ▣ 10-Year Treasury Bond Yield
- ▣ Source: FRED Database

### Methodology

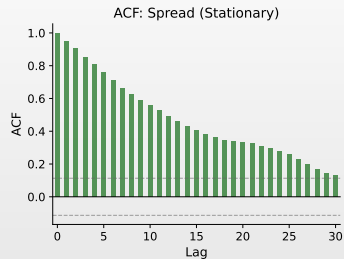
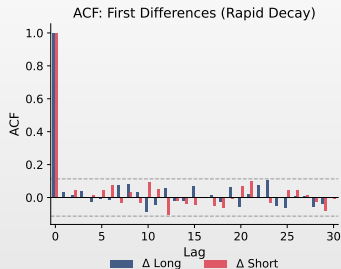
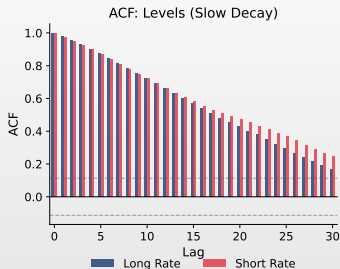
- ▣ Unit root tests (ADF, PP)
- ▣ Engle-Granger cointegration test
- ▣ Johansen procedure
- ▣ VECM estimation
- ▣ Impulse response analysis

## Step 1: Data Visualization



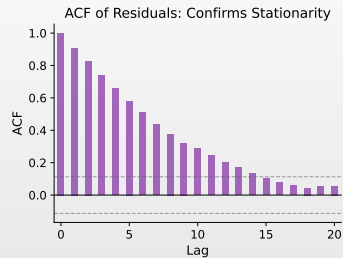
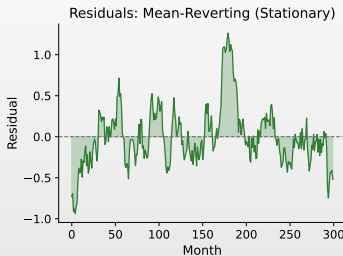
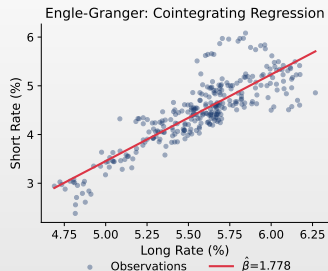
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## Step 2: Unit Root Tests



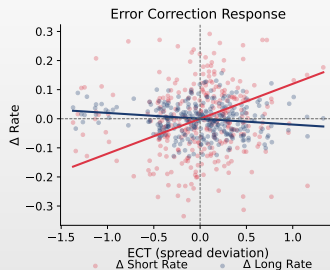
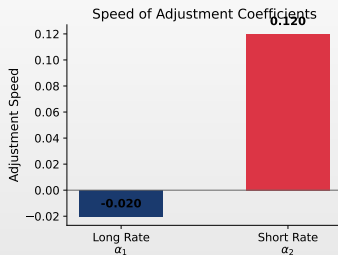
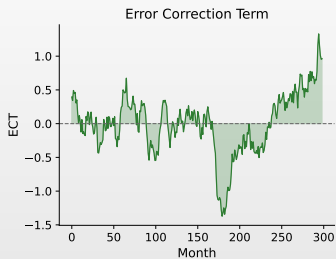
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## Step 3: Engle-Granger Cointegration Test



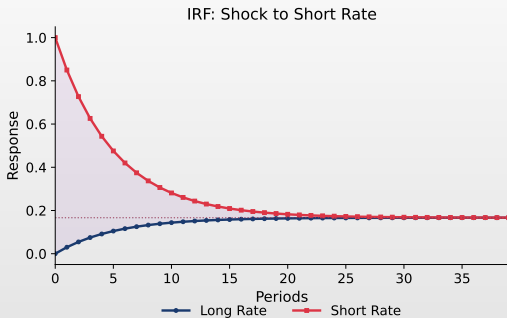
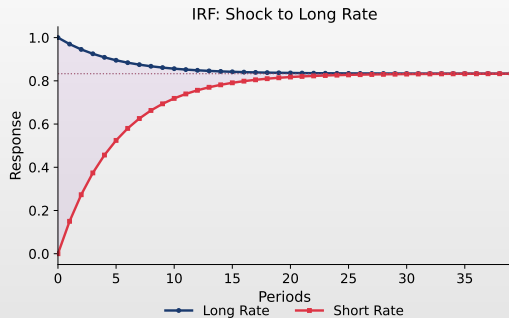
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## Step 4: VECM Estimation



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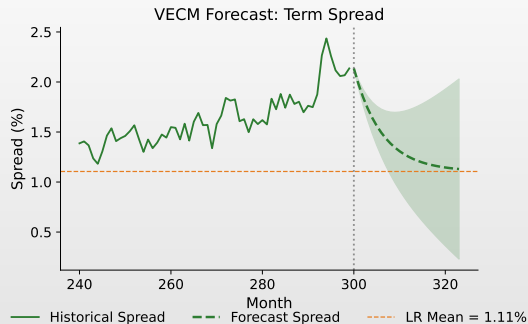
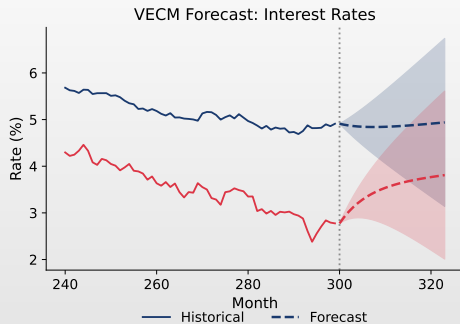
## Step 5: Impulse Response Functions



TSA\_ch7\_case\_irf



## Step 6: Forecasting



TSA\_ch7\_case\_forecast

## Key Takeaways

### Main Concepts

- ▣ **Cointegration:**  $I(1)$  variables with stationary linear combination
- ▣ **Spurious regression:** High  $R^2$  with unrelated  $I(1)$  variables
- ▣ **VECM:** VAR with error correction for cointegrated systems

### Testing Methods

- ▣ **Engle-Granger:** Simple, one vector only
- ▣ **Johansen:** Multiple vectors, MLE-based

### Remember

Tests have low power in small samples. Theory should guide specification.

## What's Next?

### Extensions and Related Topics

- ▣ **Structural VECM:** Identifying structural shocks
- ▣ **Threshold cointegration:** Nonlinear adjustment
- ▣ **Panel cointegration:** Multiple cross-sections
- ▣ **Fractional cointegration:** Long memory
- ▣ **Time-varying cointegration:** Regime changes

Questions?

## Quick Quiz

1. What does it mean for two  $I(1)$  variables to be cointegrated?
2. What is the “spurious regression” problem?
3. In a VECM, what do the  $\alpha$  coefficients represent?
4. What is the main advantage of Johansen over Engle-Granger?
5. If  $\alpha_i = 0$  for variable  $Y_i$ , what does this imply?

## Quiz Answers

1. **Cointegration:** A linear combination of the variables is  $I(0)$  (stationary). They share a common stochastic trend.
2. **Spurious regression:** Regressing one  $I(1)$  variable on another unrelated  $I(1)$  variable gives high  $R^2$  and significant coefficients even though there's no true relationship.
3.  **$\alpha$  coefficients:** Speed of adjustment—how quickly each variable responds to deviations from long-run equilibrium.
4. **Johansen advantage:** Can test for multiple cointegrating relationships, uses MLE (more efficient), doesn't require choosing dependent variable.
5.  **$\alpha_i = 0$ :** Variable  $Y_i$  is weakly exogenous—it doesn't respond to disequilibrium. Other variables do all the adjusting.

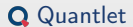
## Online Resources and Code

- ▣ **Quantlet:** <https://quantlet.com> → Code repository for statistics
- ▣ **Quantinar:** <https://quantinar.com> → Learning platform for quantitative methods
- ▣ **GitHub TSA\_ch7:** [https://github.com/QuantLet/TSA/tree/main/TSA\\_ch7](https://github.com/QuantLet/TSA/tree/main/TSA_ch7)

# Thank You!

## Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar