



Time Series Analysis and Forecasting

Chapter 7: Cointegration & VECM

Seminar



Seminar Outline

- 1 Review Quiz
- 2 True/False Questions
- 3 Practice Problems
- 4 Worked Examples
- 5 Discussion Topics
- 6 Exercises for Self-Study

Quiz 1: Cointegration Definition

Question

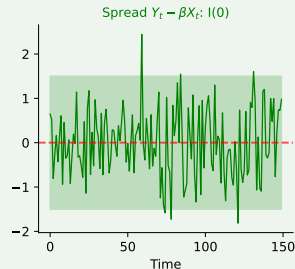
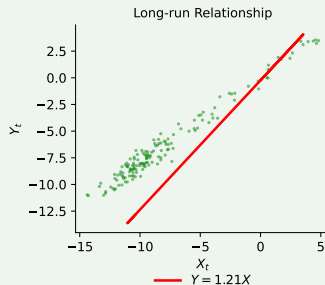
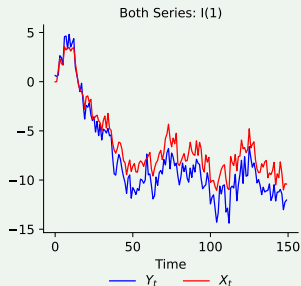
Two $I(1)$ variables X_t and Y_t are cointegrated if:

- ☐ A) They are both stationary
- ☐ B) Their sum is $I(2)$
- ☐ C) A linear combination of them is $I(0)$
- ☐ D) They have the same mean

Answer on next slide...

Quiz 1: Answer

Answer: C – A linear combination is $I(0)$



Key: $Y_t - \beta X_t \sim I(0)$ means they share a common stochastic trend. The linear combination (spread) is stationary even though both series are non-stationary.

Quiz 2: Spurious Regression

Question

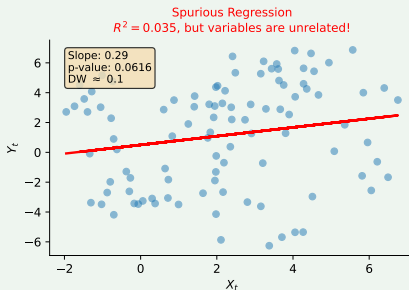
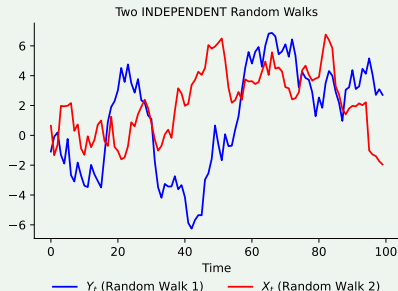
When regressing one independent random walk on another, you typically get:

- ☐ A) Low R^2 and insignificant coefficients
- ☐ B) High R^2 and significant coefficients (spurious!)
- ☐ C) Zero coefficients
- ☐ D) Undefined results

Answer on next slide...

Quiz 2: Answer

Answer: B – High R^2 and significant coefficients (spurious!)



Granger-Newbold (1974): Regressing unrelated $I(1)$ series gives misleading results. Rule of thumb: If $R^2 > DW$, suspect spurious regression!

Quiz 3: Engle-Granger Test

Question

In the Engle-Granger two-step method, what do you test in step 2?

- ☐ A) Whether the original variables are stationary
- ☐ B) Whether the regression residuals have a unit root
- ☐ C) Whether the coefficients are significant
- ☐ D) Whether the R^2 is high enough

Answer on next slide...

Quiz 3: Answer

Answer: B – Whether residuals have unit root

Step 1: Run OLS: $Y_t = \alpha + \beta X_t + e_t$, save residuals \hat{e}_t

Step 2: ADF test on residuals: $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \dots$

- $H_0: \rho = 0$ (unit root \Rightarrow no cointegration)
- $H_1: \rho < 0$ (stationary \Rightarrow cointegration!)

Important: Use Engle-Granger critical values, not standard ADF!

Quiz 4: Johansen Test Advantage

Question

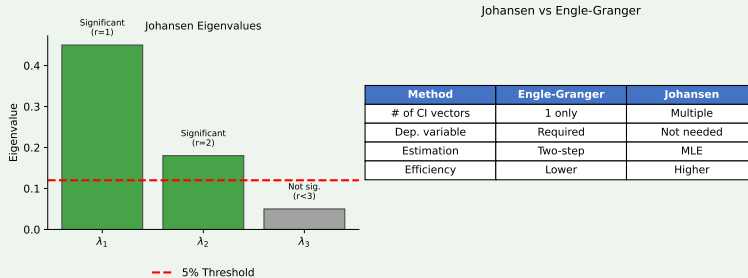
The main advantage of Johansen over Engle-Granger is:

- ☐ A) It's simpler to compute
- ☐ B) It can detect multiple cointegrating relationships
- ☐ C) It doesn't require data
- ☐ D) It always finds cointegration

Answer on next slide...

Quiz 4: Answer

Answer: B – Can detect multiple cointegrating relationships



Johansen advantages:

- Tests for $r = 0, 1, 2, \dots, k - 1$ cointegrating vectors
- Maximum likelihood (more efficient)
- No need to choose dependent variable

Quiz 5: Rank of Π

Question

In a VECM with $k = 3$ variables, if $\text{rank}(\Pi) = 2$, this means:

- ☐ A) No cointegration
- ☐ B) One cointegrating relationship
- ☐ C) Two cointegrating relationships
- ☐ D) All variables are stationary

Answer on next slide...

Quiz 5: Answer

Answer: C – Two cointegrating relationships

Rank interpretation for k variables:

- $\text{rank}(\Pi) = 0$: No cointegration (use VAR in differences)
- $0 < \text{rank}(\Pi) = r < k$: r cointegrating vectors (use VECM)
- $\text{rank}(\Pi) = k$: All variables are $I(0)$ (use VAR in levels)

With $k = 3$ and $r = 2$:

- Two equilibrium relationships
- Only $k - r = 1$ common stochastic trend

Quiz 6: VECM Structure

Question

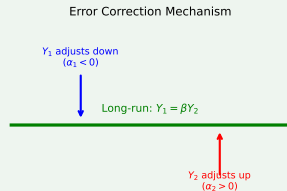
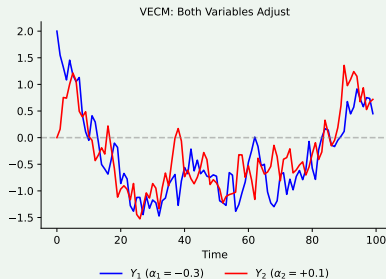
In the VECM equation $\Delta \mathbf{Y}_t = \mathbf{c} + \alpha \beta' \mathbf{Y}_{t-1} + \dots$, what does α represent?

- ☐ A) The cointegrating vectors
- ☐ B) The adjustment (loading) coefficients
- ☐ C) The short-run dynamics
- ☐ D) The error variance

Answer on next slide...

Quiz 6: Answer

Answer: B – The adjustment (loading) coefficients



$$\Pi = \alpha\beta'$$

- β = cointegrating vectors (define equilibrium)
- α = adjustment speeds (how fast each variable corrects)

Quiz 7: Error Correction Term

Question

If $Y_t - \beta X_t$ is the cointegrating relation and this term is positive, what happens?

- ☐ A) Y is above equilibrium; Y should decrease (if $\alpha < 0$)
- ☐ B) Y is below equilibrium; Y should increase
- ☐ C) Nothing, error correction doesn't affect levels
- ☐ D) Both variables increase

Answer on next slide...

Quiz 7: Answer

Answer: $A - Y$ above equilibrium; decreases if $\alpha < 0$

Error correction mechanism:

$$\Delta Y_t = \alpha(Y_{t-1} - \beta X_{t-1}) + \dots$$

- If $Y_{t-1} - \beta X_{t-1} > 0$: Y is “too high”
- With $\alpha < 0$: $\Delta Y_t < 0$ (Y decreases toward equilibrium)
- This is the “error correction” pulling Y back

Sign convention: α should be negative for the dependent variable to move back toward equilibrium.

Quiz 8: Weak Exogeneity

Question

If $\alpha_2 = 0$ in a bivariate VECM, this means:

- ☐ A) There is no cointegration
- ☐ B) Variable 2 does not adjust to disequilibrium (weakly exogenous)
- ☐ C) Variable 1 does not adjust
- ☐ D) Both variables are stationary

Answer on next slide...

Quiz 8: Answer

Answer: B – Variable 2 is weakly exogenous

Weak exogeneity: Variable doesn't respond to disequilibrium.

Example: Interest rates

- Long rate (R_t) often weakly exogenous ($\alpha_R \approx 0$)
- Short rate (r_t) adjusts to spread ($\alpha_r < 0$)
- Interpretation: Central bank adjusts short rate to maintain term structure

Implication: Can estimate single equation for the adjusting variable.

Quiz 9: Trace Test

Question

The Johansen trace test with $H_0 : r \leq 1$ vs $H_1 : r > 1$ tests whether:

- ☐ A) There is exactly one cointegrating vector
- ☐ B) There are at most one cointegrating vectors
- ☐ C) There are more than one cointegrating vectors
- ☐ D) All eigenvalues are zero

Answer on next slide...

Quiz 9: Answer

Answer: B/C – H_0 : at most 1; H_1 : more than 1

Sequential testing procedure:

- 1 Test $H_0 : r = 0$ vs $H_1 : r > 0$
- 2 If rejected, test $H_0 : r \leq 1$ vs $H_1 : r > 1$
- 3 Continue until fail to reject...

Trace statistic:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Reject H_0 if trace statistic > critical value.

Quiz 10: VECM vs VAR in Differences

Question

If variables are cointegrated, using VAR in first differences instead of VECM:

- ☐ A) Gives identical results
- ☐ B) Is more efficient
- ☐ C) Loses long-run information (misspecified)
- ☐ D) Is the preferred approach

Answer on next slide...

Quiz 10: Answer

Answer: C – Loses long-run information

Granger Representation Theorem: If cointegrated, VECM representation exists and should be used.

	VAR(Δ)	VECM
Long-run equilibrium	Lost	Preserved
Error correction	No	Yes
Forecasts (long-run)	Poor	Better

Bottom line: Differencing removes the long-run relationship that cointegration represents!

True/False Questions

Determine if each statement is True or False:

- ❶ Cointegration requires all variables to be $I(1)$.
- ❷ The cointegrating vector is unique.
- ❸ Spurious regression has low Durbin-Watson statistic.
- ❹ In VECM, both α coefficients must be non-zero.
- ❺ Johansen test requires choosing a dependent variable.
- ❻ The number of common trends $= k - r$.

Answers on next slide...

True/False: Solutions

- ① Cointegration requires all variables to be $I(1)$.

Standard $CI(1,1)$ case: all variables $I(1)$, linear combination $I(0)$.

TRUE

- ② The cointegrating vector is unique.

Unique only up to scalar multiplication. Usually normalized ($\beta_1 = 1$).

FALSE

- ③ Spurious regression has low Durbin-Watson statistic.

$DW \approx 0$ indicates highly autocorrelated residuals (non-stationary).

TRUE

- ④ In VECM, both α coefficients must be non-zero.

One can be zero (weak exogeneity). At least one must be non-zero.

FALSE

- ⑤ Johansen test requires choosing a dependent variable.

That's Engle-Granger. Johansen treats all variables symmetrically.

FALSE

- ⑥ The number of common trends = $k - r$.

k variables, r cointegrating relations $\Rightarrow k - r$ common stochastic trends.

TRUE

Problem 1: Cointegration Identification

Exercise

You have quarterly data on consumption (C_t) and income (Y_t). ADF tests show both are $I(1)$. The regression $C_t = 0.85Y_t + e_t$ gives residuals with ADF statistic $= -3.92$. The 5% Engle-Granger critical value for 2 variables is -3.34 .

Are C_t and Y_t cointegrated?

Answer on next slide...

Problem 1: Solution

Solution: Yes, they are cointegrated

Test: H_0 : No cointegration (residuals have unit root)

ADF statistic: -3.92

Critical value (5%): -3.34

Since $-3.92 < -3.34$, we **reject** H_0 at 5% level.

Conclusion: Residuals are stationary \Rightarrow Cointegration exists!

Interpretation: Consumption and income share a common trend. The cointegrating vector is approximately $(1, -0.85)$, consistent with permanent income hypothesis.

Problem 2: VECM Interpretation

Exercise

A VECM for short rate (r_t) and long rate (R_t) gives:

$$\Delta r_t = 0.01 - 0.25(r_{t-1} - R_{t-1}) + \dots$$

$$\Delta R_t = 0.005 - 0.02(r_{t-1} - R_{t-1}) + \dots$$

Interpret the adjustment coefficients.

Answer on next slide...

Problem 2: Solution

Solution

Error correction term: $(r_{t-1} - R_{t-1}) = \text{spread}$

Short rate ($\alpha_r = -0.25$):

- When spread is positive (short > long), short rate decreases
- 25% of disequilibrium corrected per period
- Short rate actively adjusts

Long rate ($\alpha_R = -0.02$):

- Very small adjustment coefficient
- Long rate is nearly weakly exogenous
- Mostly driven by expectations, not error correction

Economic interpretation: Central bank (short rate) adjusts to maintain yield curve.

Problem 3: Johansen Test Results

Exercise

Johansen trace test for 3 variables gives:

H_0	Trace Stat	5% CV
$r = 0$	45.2	29.8
$r \leq 1$	18.1	15.5
$r \leq 2$	3.2	3.8

What is the cointegrating rank?

Answer on next slide...

Problem 3: Solution

Solution: Rank = 2

Sequential testing:

- ① $H_0 : r = 0: 45.2 > 29.8 \Rightarrow$ **Reject** (at least 1)
- ② $H_0 : r \leq 1: 18.1 > 15.5 \Rightarrow$ **Reject** (at least 2)
- ③ $H_0 : r \leq 2: 3.2 < 3.8 \Rightarrow$ **Fail to reject**

Conclusion: $r = 2$ cointegrating relationships

Implications:

- Two equilibrium relationships among 3 variables
- Only $3 - 2 = 1$ common stochastic trend
- Use VECM with 2 error correction terms

Problem 4: Testing Weak Exogeneity

Exercise

In a VECM for prices (P) and exchange rate (E), you estimate $\alpha_P = -0.15$ (s.e. = 0.04) and $\alpha_E = 0.02$ (s.e. = 0.03).

Test whether the exchange rate is weakly exogenous at 5%.

Answer on next slide...

Problem 4: Solution

Solution: Exchange rate is weakly exogenous

Test: $H_0 : \alpha_E = 0$ (weak exogeneity)

t-statistic: $t = \frac{0.02}{0.03} = 0.67$

Critical value (5%, two-tailed): ± 1.96

Since $|0.67| < 1.96$, **fail to reject** H_0 .

Conclusion: Exchange rate does not respond to PPP disequilibrium.

Implication: Prices do all the adjusting to restore PPP equilibrium. Can estimate single-equation model for prices.

Example: Term Structure of Interest Rates

Economic Theory

Expectations hypothesis: $R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{premium}$
If premium is constant \Rightarrow spread $(R_t - r_t)$ should be stationary.

Typical Findings

- Both rates are $I(1)$ (confirmed by ADF)
- Johansen test: $r = 1$ cointegrating vector
- Cointegrating vector $\approx (1, -1)$: spread is stationary
- Short rate adjusts ($\alpha_r < 0$), long rate weakly exogenous

Policy Implication

Central bank controls short rate; long rate driven by expectations.

Example: Purchasing Power Parity

PPP Theory

$e_t = p_t - p_t^*$ (log exchange rate = price differential)

Real exchange rate: $q_t = e_t - p_t + p_t^*$ should be stationary (long-run PPP)

Empirical Challenges

- Unit root tests: e_t, p_t, p_t^* all $I(1)$
- Cointegration tests: Mixed results depending on sample
- Half-life of PPP deviations: 3-5 years (slow adjustment)
- Weak exogeneity: Exchange rate often doesn't adjust

PPP Puzzle

Real exchange rate is highly persistent—slow mean reversion is hard to explain with standard models.

Example: Pairs Trading Strategy

Idea

Find cointegrated stocks \Rightarrow trade the stationary spread

Implementation Steps

- ① **Identify pairs:** Test cointegration (e.g., Coca-Cola & Pepsi)
- ② **Estimate spread:** $z_t = P_A - \beta P_B$
- ③ **Trading rules:**
 - $z_t > \mu + 2\sigma$: Sell A, Buy B (spread too wide)
 - $z_t < \mu - 2\sigma$: Buy A, Sell B (spread too narrow)
 - Exit when $z_t \approx \mu$

Risks

Cointegration can break down; spread may not revert; transaction costs.

Python Cointegration Analysis: Key Functions

Essential Libraries

```
from statsmodels.tsa.stattools import coint, adfuller  
from statsmodels.tsa.vector_ar.vecm import coint_johansen, VECM
```

Workflow

- 1 Unit root tests: `adfuller(series)`
- 2 Engle-Granger: `coint(y, x)` returns test stat & p-value
- 3 Johansen: `coint_johansen(data, det_order, k_ar_diff)`
- 4 Fit VECM: `model = VECM(data, k_ar_diff=2, coint_rank=1)`
- 5 Results: `results = model.fit()`

Note

Complete working examples are provided in the Jupyter notebooks.

Discussion: Cointegration vs Correlation

Key Question

Two series are highly correlated. Are they cointegrated?

Answer: Not necessarily!

- **Correlation:** Measures co-movement (can be spurious for $I(1)$)
- **Cointegration:** Requires stationary linear combination

Example

Two independent random walks can have correlation > 0.9 purely by chance (spurious correlation). But they're NOT cointegrated—their spread is also $I(1)$.

Cointegration implies a meaningful long-run equilibrium relationship.

Discussion: Choosing Deterministic Components

Key Question

Johansen test has 5 cases for deterministics. Which to choose?

Guidelines

- ❶ **No constant, no trend:** Rarely used (requires mean-zero data)
- ❷ **Constant in CE only:** Level series, no drift
- ❸ **Constant unrestricted:** Most common for economic data
- ❹ **Trend in CE:** Series have deterministic trends
- ❺ **Trend unrestricted:** Trending differences (uncommon)

Practical Advice

Start with Case 3 (constant unrestricted). Check sensitivity to specification. Use economic reasoning: do levels have trends?

Take-Home Exercises

❶ **Theoretical:** Show that if Y_t and X_t are both random walks with the same innovation, they are cointegrated.

❷ **Computation:** Given VECM estimates:

$$\Delta Y_t = 0.5 - 0.3(Y_{t-1} - 2X_{t-1}) + 0.2\Delta Y_{t-1}$$

$$\Delta X_t = 0.1 + 0.1(Y_{t-1} - 2X_{t-1}) + 0.4\Delta X_{t-1}$$

- What is the cointegrating vector?
- Which variable adjusts more quickly?
- What is the long-run equilibrium relationship?

❸ **Applied:** Download 10-year and 3-month Treasury rates:

- Test for unit roots; Test for cointegration (Engle-Granger and Johansen)
- Estimate VECM; Interpret adjustment coefficients

❹ **Critical Thinking:** Why might PPP hold in the long run but not short run?

Hints

- ❶ If $Y_t = Y_{t-1} + \varepsilon_t$ and $X_t = X_{t-1} + \varepsilon_t$ (same shock), then $Y_t - X_t = Y_0 - X_0$ is constant (stationary).
- ❷ From the VECM:
 - Cointegrating vector: $(1, -2)$ (normalized on Y)
 - Y adjusts faster: $|\alpha_Y| = 0.3 > |\alpha_X| = 0.1$
 - Long-run: $Y = 2X$ (when EC term = 0)
- ❸ For interest rates:
 - Both typically $I(1)$; spread usually stationary
 - Expect one cointegrating vector with $(1, -1)$
 - Short rate typically adjusts; long rate often weakly exogenous
- ❹ PPP deviations: Transportation costs, non-traded goods, sticky prices, tariffs, market segmentation all slow adjustment but don't prevent long-run convergence.

Key Takeaways from This Seminar

Main Points

- 1 **Cointegration:** $I(1)$ variables with stationary linear combination
- 2 **Spurious regression:** High R^2 without cointegration is meaningless
- 3 **Engle-Granger:** Simple, but only one cointegrating vector
- 4 **Johansen:** Multiple vectors, MLE, more powerful

VECM Insights

- β defines equilibrium; α determines adjustment speed
- Weak exogeneity ($\alpha = 0$): Variable doesn't respond to disequilibrium
- Always use VECM (not VAR in differences) when cointegrated

Remember

Cointegration is about **long-run equilibrium**, not just correlation!