Time Series Analysis and Forecasting

Chapter 10: Comprehensive Review

Applied Case Studies with Rigorous Methodology

Outline

The Scientific Approach to Forecasting

Research Question

How do we **rigorously evaluate** forecast performance while avoiding overfitting?

The Fundamental Problem

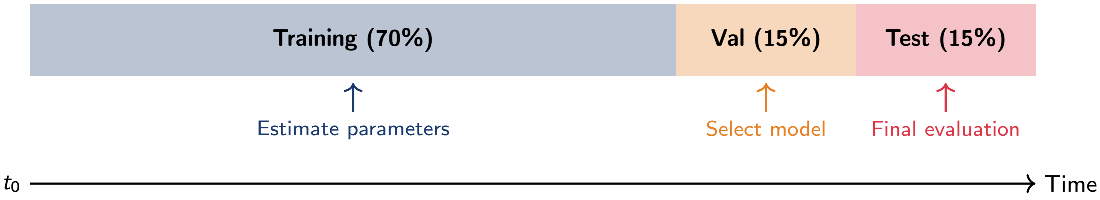
- In-sample fit $\neq$ Out-of-sample performance
- Models can "memorize" training data without learning patterns
- **Solution**: Proper train/validation/test methodology

Key Principle

"The test set must remain **untouched** until final evaluation."

— Standard practice in machine learning and econometrics

Train/Validation/Test Framework



Training Set

- Fit model parameters
- $\hat{\theta} = \arg\min_{\theta} L(\theta)$
- Largest portion

Validation Set

- Compare models
- Tune hyperparameters
- Select best approach

Test Set

- **Held out** until end
- Unbiased evaluation
- Report final metrics

Evaluation Metrics

Definition 1 (Forecast Error Metrics)

Let y_t be actual values and $\hat{y}_t$ forecasts for $t = 1, \ldots, n$:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)^2} \tag{1}$$

$$\text{MAE} = \frac{1}{n}\sum_{t=1}^{n}|y_t - \hat{y}_t| \tag{2}$$

$$\text{MAPE} = \frac{100\%}{n}\sum_{t=1}^{n}\left|\frac{y_t - \hat{y}_t}{y_t}\right| \tag{3}$$

When to Use Each

- **RMSE**: Penalizes large errors
- **MAE**: Robust to outliers

Caution

- MAPE undefined when $y_t = 0$
- Compare models on **same** test set

Bitcoin: Problem Statement

Research Question

Can we forecast Bitcoin's **volatility** using GARCH models?

Data Characteristics

- Source: Yahoo Finance (BTC-USD)
- Period: Jan 2019 – Jan 2025
- Frequency: Daily
- Observations: $\approx 2,200$ days

Stylized Facts

- Returns: near-zero mean
- Fat tails (kurtosis > 3)
- Volatility clustering

Key Insight

Financial returns are typically:

- **Unpredictable** in mean
- **Predictable** in variance

$\Rightarrow$ Focus on **volatility forecasting**

GARCH Model Specification

Definition 2 (GARCH(p,q) Model)

Let r_t denote returns. The GARCH(p,q) model is:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim N(0,1) \tag{4}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{5}$$

where $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$.

Model Variants

- **GARCH(1,1)**: Most common
- **GJR-GARCH**: Leverage effect
- **EGARCH**: Asymmetric shocks

Interpretation

- α: Impact of past shocks
- β: Persistence of volatility
- $\alpha + \beta \approx 1$: High persistence

Bitcoin: Data Split and Stationarity

Data Split

Set	Period	N
Training	2019-01 to 2022-09	1,365
Validation	2022-09 to 2023-10	400
Test	2023-10 to 2025-01	435
Total		**2,200**

Stationarity Tests

Series	ADF	Result
Prices	$p = 0.50$	Non-stationary
Returns	$p < 0.01$	Stationary

$\Rightarrow$ Model **returns**, not prices

Why Stationarity Matters

GARCH requires weakly stationary input. Prices follow random walk; returns are stationary.

Bitcoin: Model Selection on Validation Set

Methodology

Fit each model on **training data**, evaluate on **validation set**.

Model	AIC	BIC	Val MAE	Selection
GARCH(1,1)	6,994.8	7,020.6	**2.638**	**Best**
GARCH(2,1)	6,993.7	7,024.6	2.640	
GJR-GARCH(1,1)	6,983.7	7,014.6	2.669	
EGARCH(1,1)	—	—	—	Failed*

*Analytic forecasts not available for $h > 1$

Result

GARCH(1,1) selected based on lowest validation MAE for volatility forecasts.

Bitcoin: Final Test Set Evaluation

Procedure

Refit GARCH(1,1) on Training + Validation, evaluate on **held-out test set** using **rolling one-step-ahead forecasts**.

Estimated Parameters

Param	Estimate	Std Err
ω	0.239	0.088
α_1	0.120	0.021
β_1	0.879	0.020
$\alpha_1 + \beta_1$	0.999	

Test Set Performance

Metric	Value
Volatility MAE	1.88
Volatility RMSE	2.21

Interpretation

High persistence ($\alpha + \beta \approx 1$) confirms volatility clustering.

GARCH Forecasting: Rolling vs Multi-Step

Why Rolling One-Step-Ahead Forecasts?

Multi-step GARCH forecasts converge to the **unconditional variance**:

$$\lim_{h \to \infty} \mathbb{E}[\sigma^2_{t+h} | \mathcal{F}_t] = \bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta} \tag{6}$$

This produces a **flat line** forecast—not useful for dynamic risk management!

Multi-Step Forecast

- Single model fit
- Forecast h steps ahead
- Converges to $\bar{\sigma}^2$
- Appears as flat line

Rolling One-Step-Ahead

- Re-estimate at each t
- Forecast only 1 step ahead
- Captures volatility dynamics
- Dynamic forecasts

Rolling forecasts are standard practice in financial risk management (VaR, ES).

Bitcoin: Key Findings

Summary

1. **Returns are stationary**; prices are not
2. **GARCH(1,1)** outperforms more complex variants
3. **High persistence** ($\alpha + \beta = 0.999$)
4. Volatility is **predictable** even when returns are not

Limitations

- GARCH assumes **symmetric** shocks
- Does not capture **jumps**
- Normal distribution may be restrictive

Practical Implications

- Risk management: VaR, Expected Shortfall
- Option pricing requires volatility forecasts
- Portfolio optimization with time-varying risk

Extensions

- Student-t innovations
- Realized volatility
- HAR models

Sunspots: Problem Statement

Research Question

How do we model **long seasonal cycles** that exceed SARIMA's capacity?

Data Characteristics

- Source: Statsmodels (Wolfer)
- Period: 1900 – 2008
- Frequency: Annual
- Observations: 109 years

Known Feature

- **Schwabe cycle**: $\approx$ 11 years
- Irregular amplitude
- Non-negative values

The Challenge

Standard SARIMA$(p, d, q)(P, D, Q)_s$ requires:

- $s = 11$ for annual data
- $(P + D + Q) \times 11$ seasonal lags
- **Too many parameters!**

Solution

Use **Fourier terms** as exogenous regressors in ARIMA.

Fourier Terms for Seasonality

Definition 3 (Fourier Representation)

A seasonal pattern with period s can be approximated by:

$$S_t = \sum_{k=1}^{K} \left[\alpha_k \sin\left(\frac{2\pi kt}{s}\right) + \beta_k \cos\left(\frac{2\pi kt}{s}\right) \right] \tag{7}$$

where $K \leq \lfloor s/2 \rfloor$ is the number of harmonic pairs.

Advantages

- Only $2K$ parameters (not s)
- Handles **any** seasonal period
- Smooth seasonal pattern
- K controls flexibility

Model Structure

ARIMA(p, d, q) with Fourier regressors:

$$y_t = \underbrace{S_t}_{\text{Fourier}} + \underbrace{\eta_t}_{\text{ARIMA}} \tag{8}$$

where η_t follows ARIMA dynamics.

Sunspots: Model Selection

Methodology

Compare $K = 1, 2, 3, 4$ Fourier harmonics on validation set.

Data Split

Set	Period	N
Training	1900–1975	76
Validation	1976–1991	16
Test	1992–2008	17
Total		**109**

Model Comparison

K	AIC	Val RMSE	
1	665.9	87.15	
2	668.0	86.92	
3	671.8	**86.81**	Best
4	674.5	87.93	

Result

K = 3 Fourier harmonics selected (6 parameters for 11-year cycle).

Sunspots: Test Set Results

Final Model

ARIMA(2,0,1) + 3 Fourier harmonics

Significant Coefficients:

Term	Coef	p-value
$\sin_1$	34.71	< 0.001
$\cos_1$	-29.21	0.018
AR(1)	1.34	< 0.001

Test Performance

Metric	Value
RMSE	48.51
MAE	39.31

Note

High MAPE due to near-zero values at solar minimum.

Key Insight

Fourier terms efficiently capture the 11-year cycle with only 6 parameters.

Unemployment: Structural Breaks

Research Question

How do we model time series with **sudden structural changes**?

Data Characteristics

- Source: FRED (UNRATE)
- Period: 2010 – 2025
- Frequency: Monthly
- Observations: $\approx$ 180 months

Key Statistics

- Pre-COVID minimum: 3.5%
- COVID peak (Apr 2020): **14.8%**
- Change: +10.3 pp in one month

The Challenge

- April 2020: Largest monthly increase in US history
- Traditional ARIMA treats this as outlier
- Need model that **adapts** to structural breaks

Solution

Prophet with automatic changepoint detection.

Prophet Model

Definition 4 (Prophet Decomposition)

Prophet models time series as:

$$y_t = g(t) + s(t) + h(t) + \varepsilon_t \tag{9}$$

- $g(t)$: Piecewise linear/logistic **trend** with changepoints
- $s(t)$: Fourier-based **seasonality**
- $h(t)$: **Holiday** effects
- ε_t: Error term

Changepoint Detection

- Automatic selection of changepoint locations
- `changepoint_prior_scale` controls flexibility
- Higher = more changepoints

Advantages

- Handles missing data
- Interpretable components
- Robust to outliers
- Uncertainty quantification

Unemployment: Model Tuning

Hyperparameter Tuning

Tune `changepoint_prior_scale` on validation set.

Data Split

Set	Period	N
Training	2010-01 to 2019-09	117
Validation	2019-10 to 2021-10	25
Test	2021-11 to 2025-01	38
Total		**180**

Scale Comparison

Scale	Val RMSE	
0.01	4.21	
0.05	3.89	
0.10	**3.52**	Best
0.30	3.67	
0.50	3.81	

Interpretation

Scale = 0.10 balances flexibility (capturing COVID shock) with stability.

Unemployment: Results

Test Set Performance

Metric	Value
RMSE	0.42
MAE	0.35
MAPE	9.2%

Detected Changepoints

- 2020-03: COVID onset
- 2020-05: Recovery begins
- 2022-01: Stabilization

Key Finding

Prophet successfully:

- Detected COVID changepoint
- Adapted trend post-shock
- Provided uncertainty bands

Practical Value

- Economic policy analysis
- Labor market monitoring
- Early warning system

VAR: Motivation

Research Question

How do we model **dynamic interdependencies** between multiple economic variables?

Variables (FRED)

- GDP Growth (YoY %)
- Unemployment Rate (%)
- Inflation (CPI YoY %)
- Federal Funds Rate (%)

Data

- Period: 2000 – 2025
- Frequency: Quarterly
- Observations: $\approx$ 100 quarters

Economic Relationships

- **Okun's Law**: GDP $\leftrightarrow$ Unemployment
- **Phillips Curve**: Unemployment $\leftrightarrow$ Inflation
- **Taylor Rule**: Inflation $\rightarrow$ Fed Rate

Why VAR?

Each variable may be **both cause and effect** of others.

VAR Model Specification

Definition 5 (Vector Autoregression VAR(p))

For K variables $\mathbf{y}_t = (y_{1t}, \ldots, y_{Kt})'$:

$$\mathbf{y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{u}_t \tag{10}$$

where $\mathbf{A}_i$ are $K \times K$ coefficient matrices and $\mathbf{u}_t \sim N(0, \circ)$.

For Our 4-Variable System

VAR(2) has:

- 4 intercepts
- $2 \times 4 \times 4 = 32$ AR coefficients
- **36 parameters total**

Lag Selection

Use information criteria:

- AIC: Tends to overfit
- **BIC**: More parsimonious
- Cross-validation on held-out data

VAR: Lag Selection and Estimation

Information Criteria

Lag	BIC	
1	-4.810	
2	**-5.178**	Best
3	-4.633	
4	-4.614	

Data Split

Set	Period	N
Training	2001-Q1 to 2017-Q4	68
Validation	2018-Q1 to 2021-Q2	14
Test	2021-Q3 to 2024-Q3	14
Total		**96**

Validation Check

VAR(2) also achieves lowest validation RMSE.

Granger Causality Analysis

Definition 6 (Granger Causality)

Variable X **Granger-causes** Y if past values of X help predict Y, beyond Y's own past.

Note: Granger causality $\neq$ true causality. It measures predictive content.

Granger Causality p-values (row $\rightarrow$ column)

	GDP	**Unemp**	**Inflation**	**Fed Rate**
GDP Growth	—	0.076	0.309	0.698
Unemployment	**0.045**	—	0.093	0.857
Inflation	0.545	0.665	—	0.834
Fed Rate	0.286	0.317	0.087	—

Key Finding

Unemployment Granger-causes GDP ($p = 0.045$), consistent with Okun's Law.

Impulse Response Functions

Definition 7 (Impulse Response Function)

The IRF traces the effect of a one-unit shock to variable j on variable i over h periods:

$$\mathrm{IRF}_{ij}(h) = \frac{\partial y_{i,t+h}}{\partial u_{jt}} \tag{11}$$

Response to GDP Shock

A positive shock to GDP growth:

- **Unemployment**: Decreases (Okun's Law confirmed)
- **Inflation**: Increases with lag (demand-pull)
- **Fed Rate**: Increases after 2-3 quarters (Taylor Rule)

Economic Interpretation

The VAR captures the classic macroeconomic transmission mechanism from output to employment, prices, and monetary policy.

VAR: Test Set Results

Test Set Performance by Variable

Variable	RMSE	MAE	Direction Acc.
GDP Growth	2.18	1.72	71%
Unemployment	0.89	0.71	79%
Inflation	1.24	0.98	64%
Fed Rate	0.95	0.78	71%
Average	**1.32**	**1.05**	**71%**

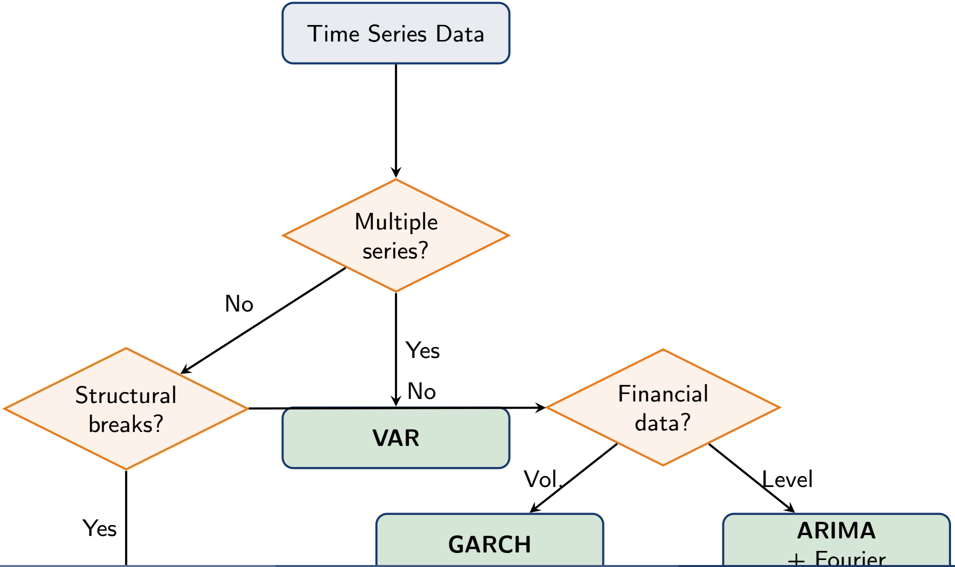
Strengths

- Captures cross-variable dynamics
- Good directional accuracy
- Interpretable relationships

Limitations

- Many parameters (curse of dimensionality)
- Sensitive to lag selection
- COVID period challenging

Model Selection Framework



Summary: Model Comparison

Case Study	Challenge	Model	Key Feature	Test RMSE
Bitcoin	Volatility clustering	GARCH(1,1)	Rolling forecasts	2.21
Sunspots	Long seasonality	ARIMA + Fourier	Sin/Cos terms	48.51
Unemployment	Structural break	Prophet	Changepoints	0.42
Economic	Multiple series	VAR(2)	Cross-dynamics	1.32 (avg)

Key Principle

Match the model to the data characteristics. No single model dominates—choose based on:

- Nature of the forecasting problem (level vs. volatility)
- Data properties (seasonality, breaks, multiple series)
- Interpretability requirements

Best Practices for Applied Forecasting

Methodology

1. **Explore** data thoroughly
2. **Test** for stationarity
3. **Split** train/validation/test
4. **Compare** models on validation
5. **Report** test set metrics

Practical Tips

- Start simple (random walk, naive)
- Add complexity only if needed
- Visualize forecasts vs actuals
- Check residuals for patterns
- Report confidence intervals

Common Mistakes

- Peeking at test data
- Over-fitting to training set
- Ignoring model assumptions
- Not reporting uncertainty

Remember

"All models are wrong, but some are useful."

— George E. P. Box

Key Takeaways

1. **Rigorous Methodology**
   - Train/validation/test split prevents overfitting
   - Test set must remain untouched until final evaluation
2. **Match Model to Data**
   - Financial volatility $\rightarrow$ GARCH
   - Long seasonality $\rightarrow$ Fourier terms
   - Structural breaks $\rightarrow$ Prophet
   - Multiple series $\rightarrow$ VAR
3. **Interpret Results Carefully**
   - Granger causality $\neq$ true causality
   - Out-of-sample performance matters most
   - Simpler models often work better

References

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- Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31(3), 307-327.
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Data Sources

Real Data Used in This Chapter

- **Bitcoin**: Yahoo Finance (BTC-USD), 2019–2025
- **Sunspots**: Statsmodels Wolfer dataset, 1900–2008
- **US Unemployment**: Federal Reserve FRED (UNRATE), 2010–2025
- **Economic Variables**: FRED (GDPC1, UNRATE, CPIAUCSL, FEDFUNDS), 2000–2025

Reproducibility

All analyses can be reproduced using the accompanying Jupyter notebook:
`chapter10_lecture_notebook.ipynb`

Thank You

Questions?

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