



# Time Series Analysis and Forecasting

## Chapter 3: ARIMA Models



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## Learning Objectives

By the end of this chapter, you will be able to:

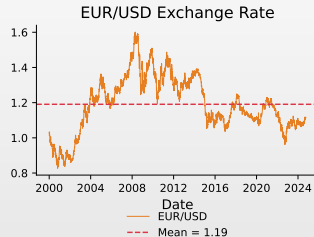
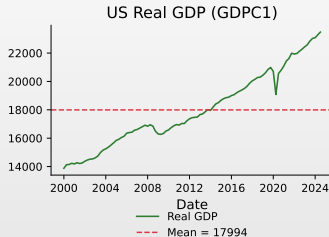
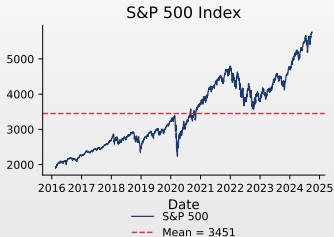
- ▣ Understand the concept and implications of non-stationarity
- ▣ Apply differencing to achieve stationarity in time series
- ▣ Use the Augmented Dickey-Fuller (ADF) test for unit root detection
- ▣ Build, estimate, and forecast with ARIMA models

## Outline

- ▣ Motivation
- ▣ Non-Stationarity in Time Series
- ▣ Differencing and the Difference Operator
- ▣ ARIMA(p,d,q) Models
- ▣ Unit Root Tests
- ▣ ARIMA Model Identification
- ▣ ARIMA Estimation
- ▣ Diagnostic Checking
- ▣ Forecasting with ARIMA
- ▣ Case Study
- ▣ Summary
- ▣ AI Use Case
- ▣ Quiz

## Motivating Example: Non-Stationary Data Is Everywhere

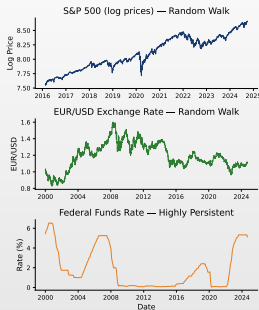
Non-stationary data: sample mean is meaningless



### Key Observations

- Stock prices, GDP, exchange rates all exhibit **trends** or **wandering behavior**
- The sample mean (red line) is meaningless for a random walk
- Standard ARMA models **cannot** handle these series directly

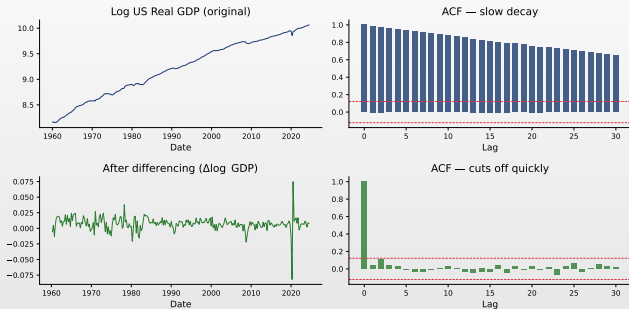
## Real-World Applications



### The Challenge

Financial/economic data are typically  $I(1)$ : stock prices, exchange rates, interest rates.

## The Solution: Differencing



### Key Insight

**Differencing** transforms a non-stationary series into a stationary one:  $\Delta Y_t = Y_t - Y_{t-1}$ . The ACF changes from slow decay to quick decay!

## What We'll Learn Today

### Core Concepts

1. **Non-Stationarity:** Why it matters and how to detect it
2. **Unit Root Tests:** ADF, PP, KPSS tests
3. **Differencing:** The key transformation
4. **ARIMA Models:** Combining differencing with ARMA
5. **Box-Jenkins Methodology:** Identify  $\rightarrow$  Estimate  $\rightarrow$  Diagnose

### By the End of This Lecture

You will be able to model and forecast non-stationary time series like stock prices, GDP, and exchange rates using ARIMA models.

## Why Non-Stationarity Matters

### The Problem

Many economic and financial time series are **non-stationary**:

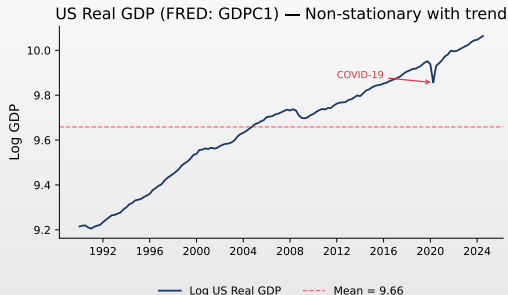
- ▣ GDP, stock prices, exchange rates, inflation indices
- ▣ They exhibit trends, changing means, or growing variance

### Consequences of Non-Stationarity

- ▣ Standard ARMA models assume stationarity
- ▣ OLS regression with non-stationary data leads to **spurious regression**
- ▣ Sample moments (mean, variance, ACF) are not consistent estimators
- ▣ Statistical inference becomes invalid



## Example: US Real GDP



### Key Observations

- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data

## Types of Non-Stationarity

### Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- ▣ Trend is a deterministic function of time
- ▣ Can be removed by **detrending**
- ▣ Shocks have temporary effects

### Stochastic Trend (Unit Root)

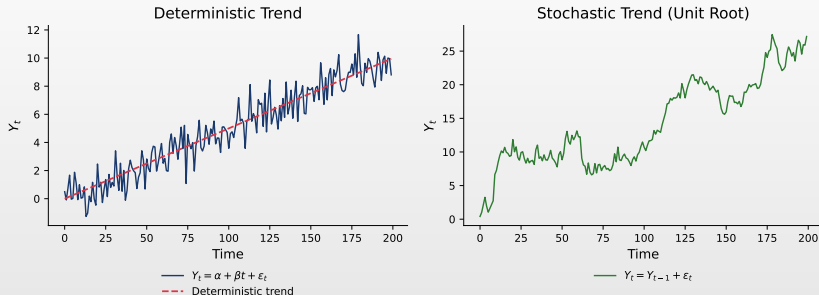
$$Y_t = Y_{t-1} + \varepsilon_t$$

- ▣ Random walk process
- ▣ Must be removed by **differencing**
- ▣ Shocks have permanent effects

### Key Distinction

Correct identification is crucial: detrending a unit root → misspecification; differencing trend-stationary → misspecification.

## Visualizing the Difference



### Key Distinction

- **Left:** Deterministic trend – deviations from trend are temporary
- **Right:** Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!

## The Random Walk Process

### Definition 1 (Random Walk)

A **random walk** is defined as:

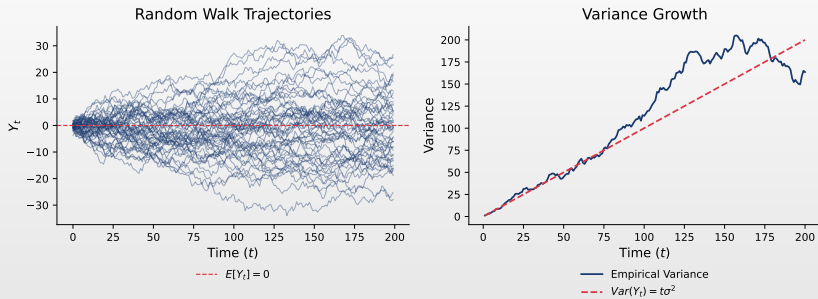
$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

With initial condition  $Y_0 = 0$ , we have:  $Y_t = \sum_{i=1}^t \varepsilon_i$

### Properties of Random Walk

- ▣  $\mathbb{E}[Y_t] = 0$  (constant mean)
- ▣  $\text{Var}(Y_t) = t\sigma^2$  (variance grows with time!)
- ▣  $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$  for  $k \leq t$
- ▣ ACF:  $\rho_k = \sqrt{\frac{t-k}{t}} \rightarrow 1$  as  $t \rightarrow \infty$

## Random Walk: Visual Illustration



### Key Properties

**Left:** Paths wander unpredictably, no mean reversion. **Right:**  $Var(Y_t) = t\sigma^2$  grows linearly  $\Rightarrow$  non-stationary.

## Proof: Random Walk Variance

**Claim:** For  $Y_t = Y_{t-1} + \varepsilon_t$  with  $Y_0 = 0$ :  $\text{Var}(Y_t) = t\sigma^2$

**Proof:** By recursive substitution:  $Y_t = \sum_{i=1}^t \varepsilon_i$

Taking variance:

$$\text{Var}(Y_t) = \text{Var}\left(\sum_{i=1}^t \varepsilon_i\right) = \sum_{i=1}^t \text{Var}(\varepsilon_i) + 2 \sum_{i < j} \text{Cov}(\varepsilon_i, \varepsilon_j)$$

Since  $\varepsilon_t$  independent (white noise):  $\text{Var}(Y_t) = \sum_{i=1}^t \sigma^2 = \boxed{t\sigma^2}$

Variance depends on  $t \Rightarrow$  non-stationary

## Proof: Random Walk Autocovariance

**Claim:**  $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$  for  $k \leq t$

**Proof:** Using  $Y_t = \sum_{i=1}^t \varepsilon_i$  and  $Y_{t-k} = \sum_{i=1}^{t-k} \varepsilon_i$ :

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}\left(\sum_{i=1}^t \varepsilon_i, \sum_{j=1}^{t-k} \varepsilon_j\right) \\ &= \sum_{i=1}^t \sum_{j=1}^{t-k} \text{Cov}(\varepsilon_i, \varepsilon_j) = \sum_{i=1}^{t-k} \text{Var}(\varepsilon_i) = \boxed{(t-k)\sigma^2}\end{aligned}$$

Only terms with  $i = j$  survive (when  $i \leq t-k$ ).

**ACF:**

$$\rho(k) = \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} = \frac{(t-k)\sigma^2}{\sqrt{t\sigma^2 \cdot (t-k)\sigma^2}} = \sqrt{\frac{t-k}{t}}$$

## Random Walk with Drift

### Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

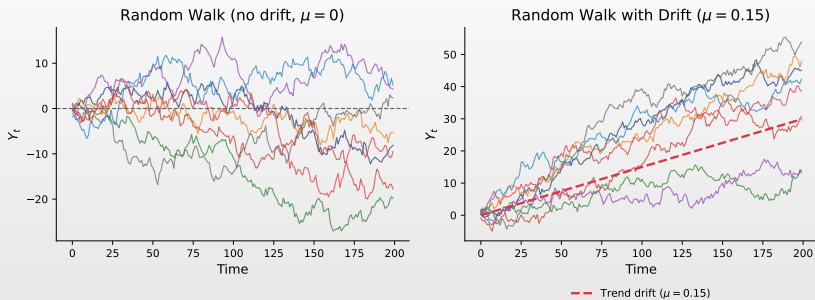
Equivalently:  $Y_t = Y_0 + \mu t + \sum_{i=1}^t \varepsilon_i$

### Properties

- ▣  $\mathbb{E}[Y_t] = Y_0 + \mu t$  (mean grows linearly)
- ▣  $\text{Var}(Y_t) = t\sigma^2$  (variance still grows)
- ▣ The drift  $\mu$  creates an upward or downward trend
- ▣ Still non-stationary despite having a “trend”



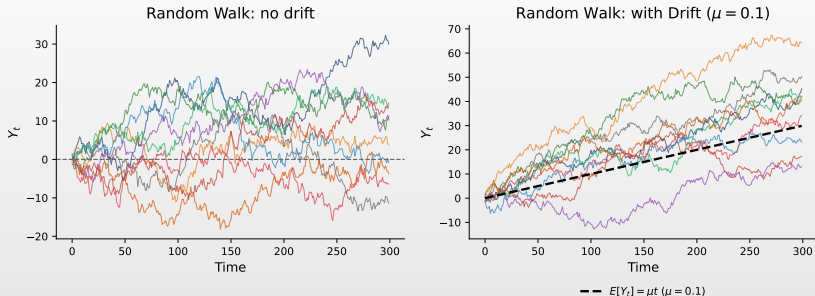
## Random Walk with Drift: Visual Illustration



### Comparison

Without drift (blue): wanders around zero with no direction. With drift  $\mu > 0$  (red): systematic upward trend. Both are non-stationary — drift adds deterministic trend to stochastic wandering.

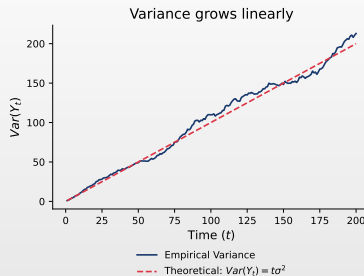
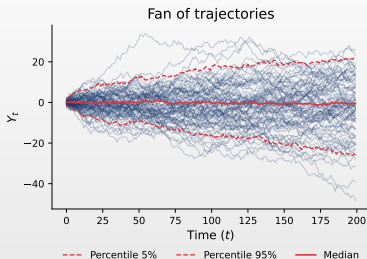
## Simulating Random Walks



### Random Walk Types

- Left: Pure random walks – no drift, wander unpredictably
- Right: Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time

## Variance Growth: Why Random Walks Are Non-Stationary



### Key Observations

- Left: Fan of paths shows uncertainty growing over time
- Right: Variance grows linearly:  $\text{Var}(Y_t) = t\sigma^2$
- This violates stationarity (variance should be constant)



## Integrated Processes

### Definition 3 (Integrated Process of Order $d$ )

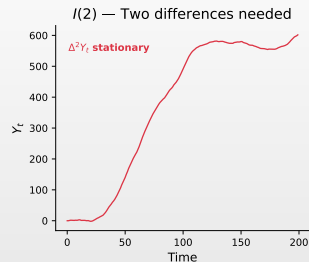
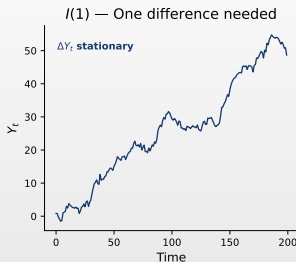
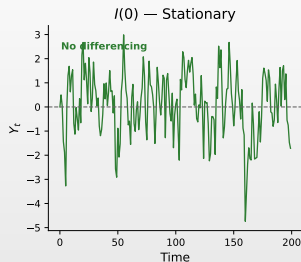
A time series  $\{Y_t\}$  is **integrated of order  $d$** , written  $Y_t \sim I(d)$ , if:

- ▣  $Y_t$  is non-stationary
- ▣  $(1 - L)^d Y_t = \Delta^d Y_t$  is stationary
- ▣  $(1 - L)^{d-1} Y_t$  is still non-stationary

### Common Cases

- ▣  $I(0)$ : Stationary process (e.g., ARMA)
- ▣  $I(1)$ : First difference is stationary (most common for economic data)
- ▣  $I(2)$ :
  - ▶ Second difference is stationary (less common)

## Integrated Process: Visual Illustration



### Order of Integration

- $I(0)$ : Stationary  $\Rightarrow$  no differencing needed
- $I(1)$ : One difference needed (random walk)
- $I(2)$ : Two differences needed
- Most economic series are  $I(0)$  or  $I(1)$

## The Difference Operator

### Definition 4 (First Difference)

The **first difference operator**  $\Delta$  is defined as:  $\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$ , where  $L$  is the lag operator ( $LY_t = Y_{t-1}$ ).

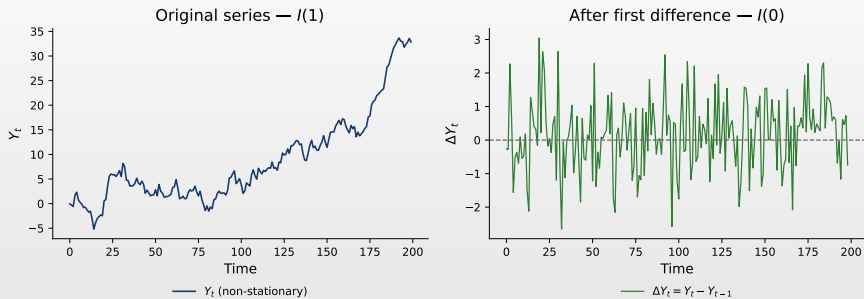
### Higher-Order Differences

- ▣ Second difference:  $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- ▣  $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- ▣  $d$ -th difference:  $\Delta^d Y_t = (1 - L)^d Y_t$

### Key Result

If  $Y_t \sim I(d)$ , then  $\Delta^d Y_t \sim I(0)$  (stationary).

## First Difference: Visual Illustration



### Observation

- Left: non-stationary series
- Right: after first difference, the series becomes stationary

## Example: Differencing a Random Walk

### Random Walk to White Noise

Let  $Y_t = Y_{t-1} + \varepsilon_t$  (random walk). Taking the first difference:

$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

### Interpretation

- ▣ A random walk is  $I(1)$
- ▣ One difference transforms it to  $I(0)$
- ▣ The “changes” in a random walk are stationary



## Proof: Differencing Induces Stationarity

**Claim:** If  $Y_t \sim I(1)$ , then  $\Delta Y_t = Y_t - Y_{t-1}$  is stationary.

**Proof for Random Walk with Drift:**  $Y_t = \mu + Y_{t-1} + \varepsilon_t$

The first difference is:

$$\Delta Y_t = Y_t - Y_{t-1} = \mu + \varepsilon_t$$

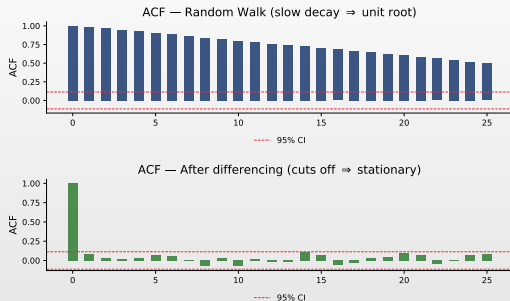
Check stationarity conditions:

1. **Mean:**  $\mathbb{E}[\Delta Y_t] = \mu$  (constant, does not depend on  $t$ ) ✓
2. **Variance:**  $\text{Var}(\Delta Y_t) = \text{Var}(\varepsilon_t) = \sigma^2$  (constant) ✓
3. **Autocovariance:**  $\text{Cov}(\Delta Y_t, \Delta Y_{t-k}) = \text{Cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$  for  $k \neq 0$  ✓

### General Principle

Differencing removes the “memory” that causes variance to accumulate. For  $I(d)$  processes,  $d$  differences are needed.

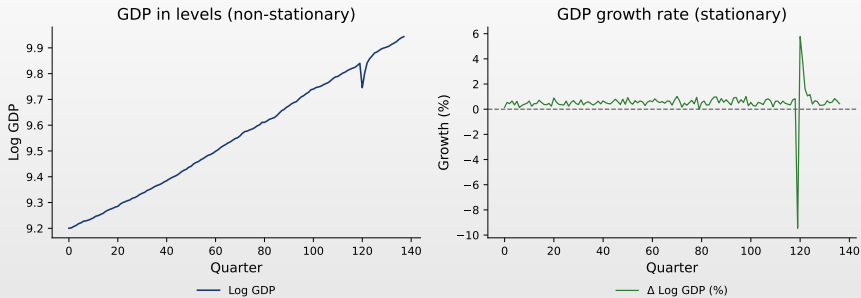
## ACF Diagnostic: Detecting Non-Stationarity



### ACF Patterns

- Top: Random walk ACF decays very slowly  $\Rightarrow$  unit root
- Bottom: After differencing, ACF cuts off  $\Rightarrow$  stationary

## Differencing in Practice: GDP Example



### Transformation

**Left:** GDP in levels with clear upward trend (non-stationary). **Right:** GDP growth rate  $\Delta \log(GDP_t)$  fluctuates around constant mean (stationary). One difference removes the stochastic trend.

## Overdifferencing

### Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
  - ▶ ACF shows spurious patterns
- Inflates variance
  - ▶ Reduces forecast accuracy
- Loses information
  - ▶ Cannot recover original level

### Example

If  $Y_t \sim I(1)$ , then  $\Delta Y_t \sim I(0)$ . But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with  $\theta = 1$  (non-invertible boundary)!

## Definition of ARIMA

### Definition 5 (ARIMA(p,d,q))

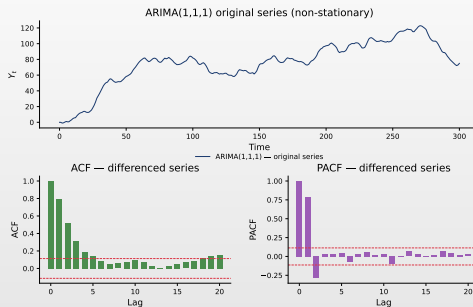
A time series  $\{Y_t\}$  follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$$

where:

- $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$  (AR polynomial)
- $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$  (MA polynomial)
- $d$  is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

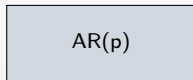
## ARIMA: Visual Illustration



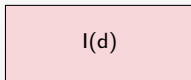
### Interpretation

Top: original ARIMA series (non-stationary). Bottom: after differencing  $d$  times, ACF/PACF reveal the AR and MA orders for the stationary component.

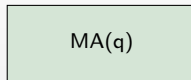
## ARIMA Components



Autoregressive  
Memory



Integration  
Differencing



Moving Average  
Shocks

### Special Cases

- ▣  $ARIMA(p,0,q) = ARMA(p,q)$  – stationary
- ▣  $ARIMA(0,1,0) =$  Random walk
- ▣  $ARIMA(0,1,1) = IMA(1,1)$  – exponential smoothing
- ▣  $ARIMA(1,1,0) = ARI(1,1)$  – differenced  $AR(1)$

## ARIMA(1,1,0) Example

### AR(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently:  $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$

### Interpretation

- ▣ The **changes** in  $Y_t$  follow an AR(1) process
- ▣ If  $|\phi_1| < 1$ , the changes are stationary
- ▣  $Y_t$  itself has a stochastic trend
- ▣ Common model for many economic time series



## ARIMA(0,1,1) Example

### IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently:  $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

### Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha) \hat{Y}_t$$

where  $\alpha = 1 + \theta_1$  (for  $-1 < \theta_1 < 0$ ).

## The Role of the Constant in ARIMA

### Constant Term in ARIMA(p,d,q)

When  $d > 0$ , the constant  $c$  has a different interpretation:  $\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$

### Important Implications

- For  $d = 1$ :  $c$  represents the **drift**
  - ▶ Average change:  $\mathbb{E}[\Delta Y_t] = \frac{c}{1-\phi_1-\dots-\phi_p}$
  - ▶ Linear trend in levels
- For  $d = 2$ :  $c$  affects the **curvature**
  - ▶ Quadratic trend in levels
- Often  $c = 0$  is assumed when  $d \geq 1$ 
  - ▶ No deterministic trend component

## Testing for Unit Roots

### Why Test?

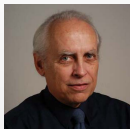
Before fitting an ARIMA model, we need to determine:

1. Is the series stationary? (Is  $d = 0$ ?)
2. If not, how many differences are needed? (What is  $d$ ?)

### Common Unit Root Tests

- **Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- **Phillips-Perron (PP)**
- **KPSS** (stationarity test – reversed null hypothesis)

## Researcher Spotlight: Dickey & Fuller



David Dickey (\*1945)



Wayne Fuller (1931–2022)



### Biography

- **David Dickey:** American statistician at NC State University. PhD student of Wayne Fuller at Iowa State
- **Wayne Fuller:** American statistician, professor at Iowa State University
- Together they developed the foundational test for unit roots in time series

### Key Contributions

- **Dickey-Fuller test** (1979) — the fundamental unit root test
- **Augmented Dickey-Fuller (ADF)** — extension with lagged differences
- **Critical value tables** — non-standard distributions under the null
- Enabled rigorous testing of integration order for ARIMA modeling

## The Dickey-Fuller Test

### Setup

Consider the AR(1) model:  $Y_t = \phi Y_{t-1} + \varepsilon_t$ . Subtract  $Y_{t-1}$ :  $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$ , where  $\gamma = \phi - 1$ .

### Hypotheses

- $H_0$ :  $\gamma = 0$  (unit root,  $\phi = 1$ , non-stationary)
- $H_1$ :  $\gamma < 0$  (stationary,  $|\phi| < 1$ )

### Key Issue

Under  $H_0$ , the  $t$ -statistic does **not** follow a standard  $t$ -distribution! Must use Dickey-Fuller critical values.

## Dickey-Fuller Test Variants

### Three Specifications

1. **No constant, no trend:**  $\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$
2. **With constant (drift):**  $\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$
3. **With constant and trend:**  $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$

### Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference

## Augmented Dickey-Fuller (ADF) Test

### The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

### Definition 6 (ADF Test)

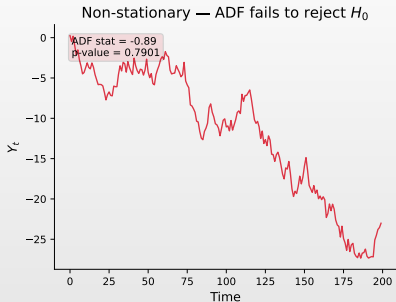
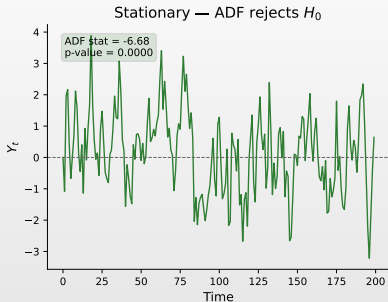
Add lagged differences:  $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^k \delta_j \Delta Y_{t-j} + \varepsilon_t$

Test  $H_0 : \gamma = 0$  using ADF critical values.

### Choosing Lag Length $k$

- Use information criteria (AIC, BIC)
- Start with  $k_{max}$ , reduce until last lag significant

## ADF Test: Visual Illustration



### Observation

- Left: stationary series  $\Rightarrow$  ADF rejects unit root
- Right: non-stationary  $\Rightarrow$  ADF fails to reject



## ADF Test Critical Values

Model	1%	5%	10%
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

### Decision Rule

- ▣ Test statistic  $<$  critical value  $\Rightarrow$  Reject  $H_0$  (stationary)
- ▣ Test statistic  $\geq$  critical value  $\Rightarrow$  Fail to reject (unit root)

## The Phillips-Perron (PP) Test

### Motivation

Like ADF, tests  $H_0$ : Unit root vs  $H_1$ : Stationary, but uses a **non-parametric correction** for serial correlation instead of adding lagged differences.

### Test Statistic

The PP test modifies the DF  $t$ -statistic:

$$Z_t = t_{\hat{\gamma}} \cdot \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - \hat{\sigma}^2)(se(\hat{\gamma}))}{2\hat{\lambda}^2 \cdot s}$$

where  $\hat{\lambda}^2$  is a consistent estimate of the long-run variance using Newey-West.

### Advantages over ADF

- ▣ Robust to heteroskedasticity and serial correlation
- ▣ No need to select lag length (uses bandwidth instead)

## The KPSS Test

### Reversed Hypotheses

Unlike ADF:  $H_0$ : Stationary vs  $H_1$ : Unit root

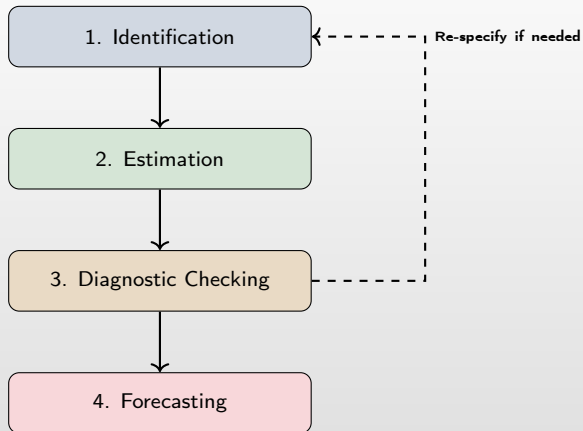
### KPSS Procedure

Decompose:  $Y_t = \xi t + r_t + \varepsilon_t$  where  $r_t = r_{t-1} + u_t$ . Test whether  $\text{Var}(u_t) = 0$ .

### Complementary Use with ADF

- ▣ ADF rejects, KPSS doesn't  $\Rightarrow$  Stationary
- ▣ ADF doesn't reject, KPSS rejects  $\Rightarrow$  Unit root
- ▣ Both reject or neither  $\Rightarrow$  Inconclusive

## The Box-Jenkins Methodology



## Step 1: Determining $d$

### Procedure

1. Plot the time series – look for trends, changing variance
2. Examine ACF – slow decay suggests non-stationarity
3. Apply unit root tests (ADF, KPSS)
4. If non-stationary, difference and repeat

### Practical Guidelines

- ▣ Most economic series:  $d = 1$  is sufficient
- ▣ Rarely need  $d > 2$
- ▣ If ACF of  $\Delta Y_t$  still decays slowly, try  $d = 2$
- ▣ Watch for overdifferencing (ACF with  $\rho_1 \approx -0.5$ )

## Step 2: Determining $p$ and $q$

### After Differencing

Once  $W_t = \Delta^d Y_t$  is stationary, use ACF/PACF to identify ARMA( $p, q$ ):

Model	ACF	PACF
AR( $p$ )	Decays exponentially	Cuts off after lag $p$
MA( $q$ )	Cuts off after lag $q$	Decays exponentially
ARMA( $p, q$ )	Decays	Decays

### Information Criteria

When patterns are unclear, compare models using:

□  $AIC = -2 \ln(L) + 2k$ ;     $BIC = -2 \ln(L) + k \ln(n)$

Lower is better. BIC penalizes complexity more.

## Auto-ARIMA Algorithms

### Automated Model Selection

Modern software can automatically select  $(p, d, q)$ :

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

### How Auto-ARIMA Works

1. Use unit root tests to determine  $d$
2. Fit models for various  $(p, q)$  combinations
3. Select model with lowest AIC/BIC
4. Optionally use stepwise search for efficiency

### Caution

Automated selection is helpful but not infallible. Always check diagnostics!

## Estimation Methods

### Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- ▣ Assumes  $\varepsilon_t \sim N(0, \sigma^2)$
- ▣ Maximizes the likelihood function
- ▣ Provides consistent, efficient estimators
- ▣ Yields standard errors for inference

### Conditional vs Exact MLE

- ▣ **Conditional MLE:** Conditions on initial values
- ▣ **Exact MLE:** Treats initial values as unknown
- ▣ Difference diminishes as sample size grows



## Conditional Log-Likelihood

### Gaussian Log-Likelihood Function

- ▣  $\ell(\boldsymbol{\theta}, \sigma^2) = -\frac{T}{2} \ln(2\pi) - \frac{T}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^T e_t^2(\boldsymbol{\theta})$
- ▣  $e_t(\boldsymbol{\theta}) = X_t - \hat{X}_{t|t-1}$  are the **one-step prediction errors**
- ▣  $\boldsymbol{\theta} = (\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q, c)$

### Example: ARIMA(1,1,1)

- ▣ Prediction errors:  $e_t = \Delta X_t - \phi_1 \Delta X_{t-1} - \theta_1 e_{t-1} - c$
- ▣ Conditional MLE: set  $e_0 = 0$ , compute  $e_1, \dots, e_T$ , maximize  $\ell$

### Estimating $\sigma^2$

- ▣ At optimal parameters  $\hat{\boldsymbol{\theta}}$ :  $\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^T e_t^2(\hat{\boldsymbol{\theta}})$

## Parameter Constraints

### Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- ▣ **AR stationarity:** Roots of  $\phi(z) = 0$  outside unit circle
- ▣ **MA invertibility:** Roots of  $\theta(z) = 0$  outside unit circle

### Checking in Practice

Most software reports:

- ▣ Estimated coefficients with standard errors
- ▣ Roots of AR and MA polynomials
- ▣ Warning if near-unit-root detected

## Residual Analysis

### What to Check

If the model is correct, residuals  $\hat{\varepsilon}_t$  should be white noise:

1. Zero mean
2. Constant variance
3. No autocorrelation
4. (Optional) Normality

### Diagnostic Tools

- ▣ **Residual ACF/PACF:** Should show no significant spikes
- ▣ **Ljung-Box test:** Tests for autocorrelation at multiple lags
- ▣ **Q-Q plot:** Checks normality assumption
- ▣ **Residual vs fitted:**
  - ▶ Checks for heteroskedasticity

## The Ljung-Box Test

### Definition 7 (Ljung-Box Q Statistic)

$Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}$ . Under  $H_0$  (no autocorrelation):  $Q(m) \sim \chi^2(m - p - q)$

### Usage

- Choose  $m \approx \ln(n)$  or  $m = 10$  for quarterly,  $m = 20$  for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if  $Q(m)$  exceeds critical value

### If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

## Point Forecasts

### Minimum MSE Forecast

The optimal  $h$ -step ahead forecast is the conditional expectation:  $\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \dots]$

### ARIMA(1,1,1) Forecasting

Model:  $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

One-step forecast:  $\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1 \hat{\varepsilon}_T$

For  $h > 1$ : replace unknown  $\varepsilon_{T+j}$  with 0, unknown  $Y_{T+j}$  with  $\hat{Y}_{T+j|T}$

## Forecast Intervals

### Forecast Uncertainty

The  $h$ -step forecast error variance:  $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$ , where  $\psi_j$  are  $\text{MA}(\infty)$  coefficients.

### Confidence Intervals

Under normality,  $(1 - \alpha)\%$  interval:  $\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$

### Key Property for $I(1)$ Series

For integrated processes, forecast variance grows without bound as  $h \rightarrow \infty$ . Intervals widen over time!

## Long-Run Forecasts for ARIMA

### Behavior as $h \rightarrow \infty$

For ARIMA(p,1,q) with drift  $c$ :

- ▣ Point forecasts: Linear trend with slope = drift
- ▣ Forecast intervals: Width grows with  $\sqrt{h}$

For ARIMA(p,1,q) without drift:

- ▣ Point forecasts: Converge to last level
- ▣ Forecast intervals: Still grow unboundedly

### Practical Implication

ARIMA forecasts are most reliable for short horizons. Long-term forecasts have very wide uncertainty bands.

## Rolling Forecasting: Concept

### What is Rolling Forecasting?

A technique to evaluate forecast accuracy out-of-sample:

1. Fix a **training window** of size  $w$
2. Estimate model on observations  $t = 1, \dots, w$
3. Forecast  $h$  steps ahead:  $\hat{Y}_{w+h|w}$
4. **Roll** the window forward by one period
5. Repeat until end of sample

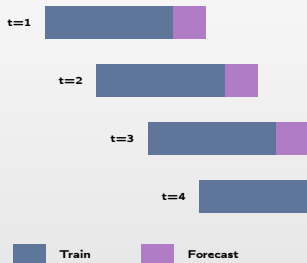
### Why Rolling Forecasts?

- Mimics real-time forecasting scenario
- Provides multiple forecast errors for evaluation
- Avoids overfitting to full sample

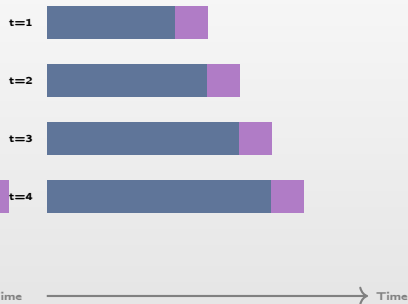


## Fixed vs Expanding Window

### Fixed Window (Rolling)



### Expanding Window



### Comparison

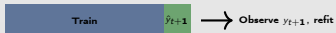
- ▣ **Fixed:** Window slides forward, constant size — adapts to regime changes
- ▣ **Expanding:** Window grows over time — uses all historical data

## 1-Step vs Multi-Step Forecasting

### 1-Step Ahead (Recursive)

- Forecast only next period
  - Refit model after each step
  - Use actual value once revealed
- Most accurate for short horizons

#### 1-Step Ahead



### Multi-Step (Direct)

- Forecast multiple periods ahead
  - No refit between steps
  - Uses forecasted values as inputs
- Uncertainty compounds over horizon

#### Multi-Step (h=3)



## Rolling Forecast: Step-by-Step Example

Setup: ARIMA(1,1,0) with  $\phi_1 = 0.6$

Model:  $\Delta Y_t = \phi_1 \Delta Y_{t-1} + \varepsilon_t$  where  $\Delta Y_t = Y_t - Y_{t-1}$

Given Data at Time  $T$

$Y_{T-2} = 100, \quad Y_{T-1} = 103, \quad Y_T = 108 \quad \Rightarrow \quad \Delta Y_{T-1} = 3, \quad \Delta Y_T = 5$

1-Step Ahead Point Forecast

$$\begin{aligned}\Delta \hat{Y}_{T+1|T} &= \phi_1 \cdot \Delta Y_T = 0.6 \times 5 = 3 \\ \hat{Y}_{T+1|T} &= Y_T + \Delta \hat{Y}_{T+1|T} = 108 + 3 = \boxed{111}\end{aligned}$$

## Multi-Step Point Forecasts

### 2-Step Ahead Forecast

$$\begin{aligned}\Delta \hat{Y}_{T+2|T} &= \phi_1 \cdot \Delta \hat{Y}_{T+1|T} = 0.6 \times 3 = 1.8 \\ \hat{Y}_{T+2|T} &= \hat{Y}_{T+1|T} + \Delta \hat{Y}_{T+2|T} = 111 + 1.8 = \boxed{112.8}\end{aligned}$$

### General Formula for $h$ -Step Forecast (ARIMA(1,1,0))

$$\begin{aligned}\Delta \hat{Y}_{T+h|T} &= \phi_1^h \cdot \Delta Y_T \\ \hat{Y}_{T+h|T} &= Y_T + \Delta Y_T \cdot \frac{\phi_1(1 - \phi_1^h)}{1 - \phi_1}\end{aligned}$$

### Numerical: 3-Step Forecast

$$\hat{Y}_{T+3|T} = 108 + 5 \times \frac{0.6(1-0.6^3)}{1-0.6} = 108 + 5 \times 1.092 = \boxed{113.46}$$

## Confidence Intervals: Formulas

### Forecast Error Variance

For ARIMA(1,1,0), the  $h$ -step forecast error variance:

$$\text{Var}(e_{T+h|T}) = \sigma^2 \left( 1 + \sum_{j=1}^{h-1} \psi_j^2 \right)$$

where  $\psi_j = \phi_1^{j-1}(1 + \phi_1 + \dots + \phi_1^{j-1}) = \phi_1^{j-1} \cdot \frac{1 - \phi_1^j}{1 - \phi_1}$

### $(1 - \alpha)\%$ Confidence Interval

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(e_{T+h|T})}$$

For 95% CI:  $z_{0.025} = 1.96$

## Confidence Interval: Numerical Example

Given:  $\sigma^2 = 4$ ,  $\phi_1 = 0.6$ ,  $\hat{Y}_{T+1|T} = 111$

### 1-Step Ahead CI

$$\text{Var}(e_{T+1|T}) = \sigma^2 = 4$$

$$95\% \text{ CI} = 111 \pm 1.96 \times \sqrt{4} = 111 \pm 3.92 = [107.08, 114.92]$$

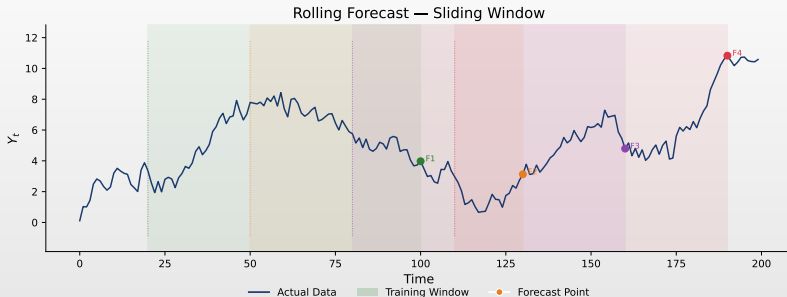
### 2-Step Ahead CI (for $\hat{Y}_{T+2|T} = 112.8$ )

$$\psi_1 = 1 + \phi_1 = 1.6, \quad \text{Var}(e_{T+2|T}) = 4(1 + 1.6^2) = 14.24$$

$$95\% \text{ CI} = 112.8 \pm 1.96 \times \sqrt{14.24} = 112.8 \pm 7.40 = [105.40, 120.20]$$

**Note:** CI widens as horizon increases!

## Rolling Window Illustration



### Rolling Procedure

- Each window produces a 1-step ahead forecast
- Compare forecasts to actuals to compute RMSE, MAE
- Rolling window keeps model estimation up-to-date

## Case Study: Complete ARIMA Analysis

### Objective

- Forecast US Real GDP using the Box-Jenkins methodology

### Steps

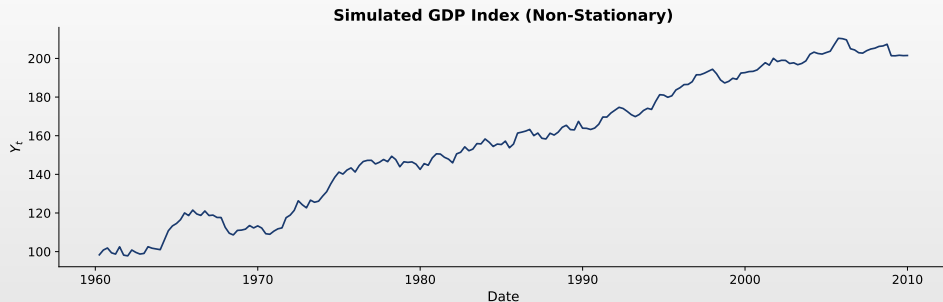
1. **Step 1:** Visualize data and check stationarity
2. **Step 2:** Apply unit root tests (ADF, KPSS)
3. **Step 3:** Difference if needed, identify  $p$  and  $q$
4. **Step 4:** Estimate the ARIMA model
5. **Step 5:** Model diagnostics
6. **Step 6:** Generate forecasts with confidence intervals
7. **Step 7:** Evaluate forecast accuracy

### Data

- US Real GDP (FRED: GDPC1), Quarterly, 1990Q1–2024Q2,  $n = 138$



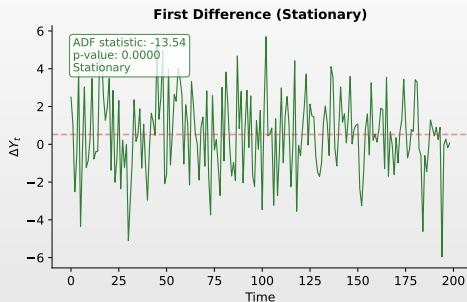
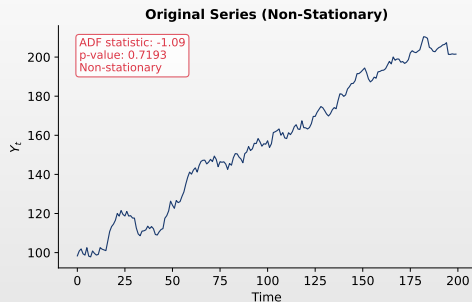
## Case Study: US Real GDP (FRED)



**Data: FRED GDPC1 (1960Q1–2024Q3)**

Quarterly Real GDP, seasonally adjusted, billions of chained 2017 dollars. Non-stationary series with upward trend  $\Rightarrow$  requires differencing.

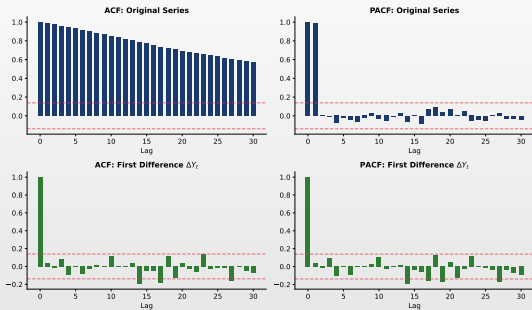
## Step 1: ADF Test for Stationarity



### ADF Test Results

**Original series:** Large p-value  $\Rightarrow$  fail to reject  $H_0$  (unit root present). **First difference:** p-value  $< 0.01 \Rightarrow$  reject  $H_0 \Rightarrow d = 1$  is sufficient.

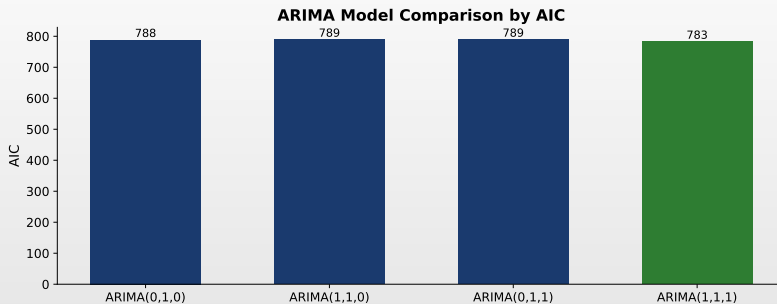
## Step 2: ACF/PACF Before and After Differencing



### ACF/PACF Analysis

Top: Slow ACF decay (non-stationary) | Bottom: After differencing, low-order ARMA

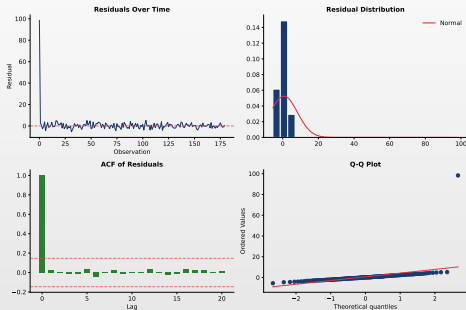
## Step 3: ARIMA Model Comparison



### Model Selection

Compare ARIMA(0,1,0), ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1). The model with lowest AIC is selected.

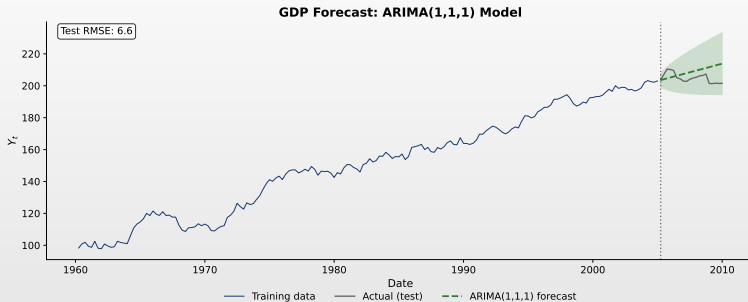
## Step 4: Diagnostic Checking



### ARIMA(1,1,1) Diagnostics

ACF: no autocorrelation ✓    Q-Q: **non-normal** (COVID-19 outlier)    JB test:  $p < 0.001$

## Step 5: Out-of-Sample Forecasting



**Train/Val/Test Split (70%/15%/15%)**

**Train 70% (blue):** Estimation | **Val 15% (green):** Tuning | **Test 15% (purple):** Evaluation with 95% CI

## Step 6: Rolling Forecast with Train/Val/Test



### Rolling 1-Step Ahead Forecast (Expanding Window, 95% CI)

Train 70% → Val 15% → Test 15% | Expanding window refits model at each step

## Summary

### What we learned in this chapter

- Non-stationarity in time series
  - ▶ Deterministic vs stochastic trend; consequences for statistical inference
- Differencing and integrated processes
  - ▶  $\Delta Y_t = Y_t - Y_{t-1}$ ; if  $Y_t \sim I(d)$ , then  $\Delta^d Y_t \sim I(0)$
- ARIMA( $p, d, q$ ) models and unit root tests
  - ▶ ADF, PP, KPSS; Box-Jenkins: identify  $\rightarrow$  estimate  $\rightarrow$  validate
- Forecasts with confidence intervals
  - ▶ For  $I(1)$ : CIs widen without bound ( $\propto \sqrt{h}$ )

### Key Insight

- **Difference carefully:** One difference is usually sufficient ( $d = 1$ ). Over-differencing creates artificial autocorrelation.



## AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"I have a quarterly GDP time series (80 observations). Test stationarity, difference if needed, estimate an ARIMA model, and forecast 8 quarters ahead. Give me complete Python code with plots."

### Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it test stationarity with ADF *before* estimating ARIMA? Does it also use KPSS?
3. How does it determine the differencing order  $d$ ? Does it check for over-differencing?
4. How does it choose  $p$  and  $q$ ? ACF/PACF or just `auto_arima`?
5. Do forecast confidence intervals widen with horizon? (key  $I(1)$  property)

**Warning:** AI-generated code may run without errors and look professional. *That does not mean it is correct.*

## What's Next?

### Chapter 4: SARIMA Models for Seasonal Data

- ▣ **Seasonality:** repetitive patterns at regular intervals
- ▣ **Seasonal differencing:** the  $(1 - L^s)$  operator
- ▣ **SARIMA( $p, d, q$ )( $P, D, Q$ )<sub>s</sub>:** seasonal extension of ARIMA
- ▣ **Model identification:** seasonal ACF/PACF
- ▣ **Case study:** Airline passengers forecast

Questions?

## Question 1

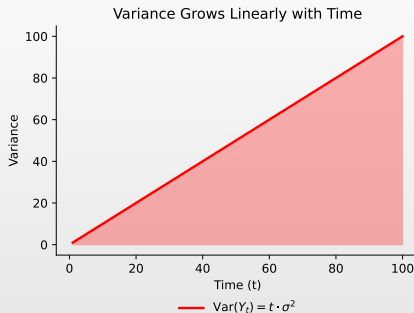
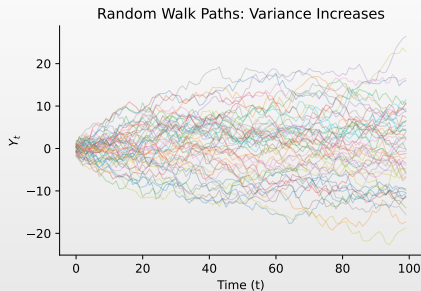
### Question

▣ A time series  $Y_t$  follows a random walk:  $Y_t = Y_{t-1} + \varepsilon_t$ . What is  $\text{Var}(Y_t)$ ?

### Answer Choices

- (A)  $\sigma^2$  (constant)
- (B)  $t \cdot \sigma^2$  (grows linearly with time)
- (C)  $\sigma^2/t$  (decreases with time)
- (D)  $\sigma^{2t}$  (grows exponentially)

## Question 1: Answer



Answer: (B)

- ☐ Random walk variance grows linearly with time — this is why random walks are non-stationary.

## Question 2

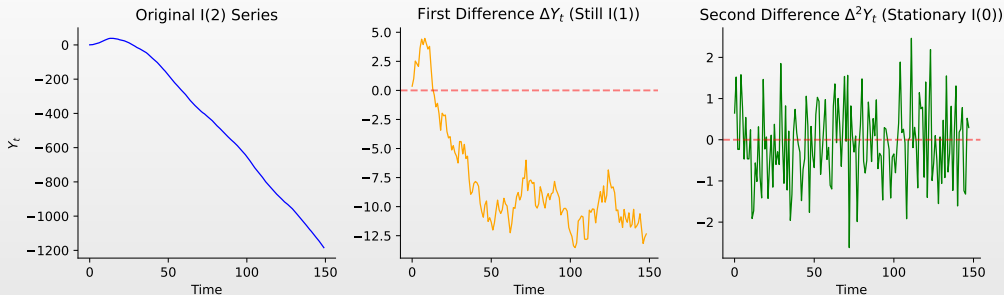
### Question

□ If a series  $Y_t$  is  $I(2)$ , how many times must you difference it to achieve stationarity?

### Answer Choices

- (A) 0 times (already stationary)
- (B) 1 time
- (C) 2 times
- (D) Cannot be made stationary by differencing

## Question 2: Answer



Answer: (C)

- $I(d)$  means “integrated of order  $d$ ” — requires  $d$  differences for stationarity.

### Question 3

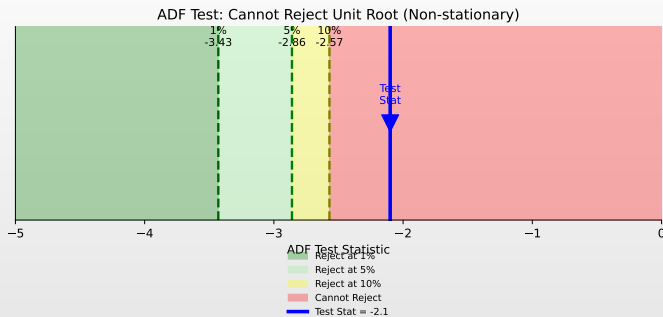
#### Question

- ☐ You run an ADF test and get a test statistic of  $-2.1$  with critical values:  $-3.43$  (1%),  $-2.86$  (5%),  $-2.57$  (10%). What do you conclude?

#### Answer Choices

- (A) Reject  $H_0$ : series is stationary at all levels
- (B) Reject  $H_0$ : series is stationary at 10% level only
- (C) Fail to reject  $H_0$ : series likely has a unit root
- (D) The test is inconclusive

## Question 3: Answer



Answer: (C)

□ Test stat  $-2.1 > -2.57$  (10% CV)  $\Rightarrow$  Cannot reject at any level. Consider differencing.



## Question 4

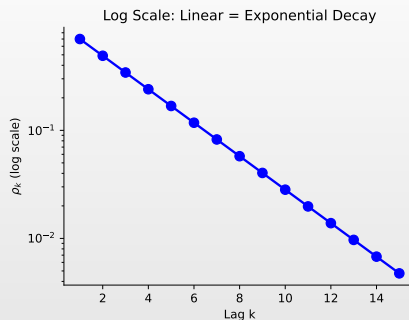
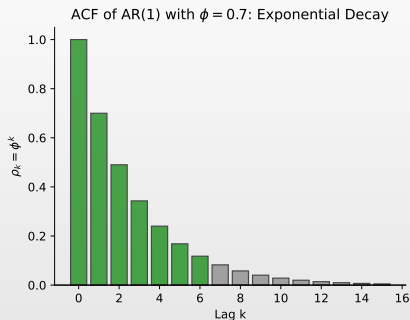
### Question

□ For an ARIMA(1,1,0) model, what is the ACF pattern of the **differenced** series  $\Delta Y_t$ ?

### Answer Choices

- (A) Cuts off after lag 1
- (B) Decays exponentially
- (C) Alternates in sign
- (D) Is zero at all lags

## Question 4: Answer



Answer: (B)

□ ARIMA(1,1,0)  $\Rightarrow \Delta Y_t$  follows AR(1) with ACF  $\rho_k = \phi_1^k$  (geometric decay).

## Question 5

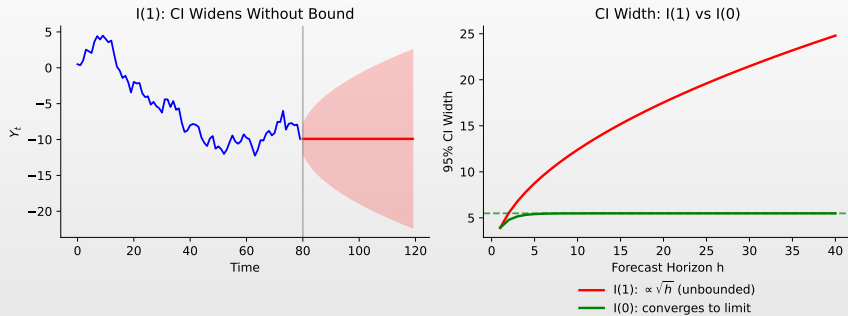
### Question

□ What happens to ARIMA forecast confidence intervals as the horizon  $h$  increases for an  $I(1)$  series?

### Answer Choices

- (A) They stay constant
- (B) They narrow (more precision)
- (C) They widen without bound
- (D) They widen but converge to a limit

## Question 5: Answer



Answer: (C)

□ For I(1): CI width  $\propto \sqrt{h}$  (unbounded). For I(0): CIs converge to a limit.

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### ARIMA Models and Automatic Selection

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- ▣ Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.
- ▣ Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.



### Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> > Code repository for statistics
- ▣ **Quantinar**: <https://quantinar.com> > Learning platform for quantitative methods
- ▣ **GitHub TSA**: [https://github.com/QuantLet/TSA/tree/main/TSA\\_ch3](https://github.com/QuantLet/TSA/tree/main/TSA_ch3) > Python code for this chapter

# Thank You!

## Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>

 Quantlet     Quantinar