Chapter 4: SARIMA Models

Seasonal Time Series

Outline

What is Seasonality?

Definition 1 (Seasonality)

A time series exhibits **seasonality** when it shows regular, periodic fluctuations that repeat over a fixed period s (the seasonal period).

Common Seasonal Periods

- Monthly data: $s = 12$ (annual cycle)
- Quarterly data: $s = 4$ (annual cycle)
- Weekly data: $s = 52$ (annual) or $s = 7$ (weekly pattern)
- Daily data: $s = 7$ (weekly pattern)

Examples of Seasonal Data

Economic Series

- Retail sales (holiday peaks)
- Tourism (summer/winter)
- Agricultural production
- Energy consumption
- Employment (hiring cycles)

Other Domains

- Weather/temperature
- Website traffic
- Hospital admissions
- Transportation usage
- Electricity demand

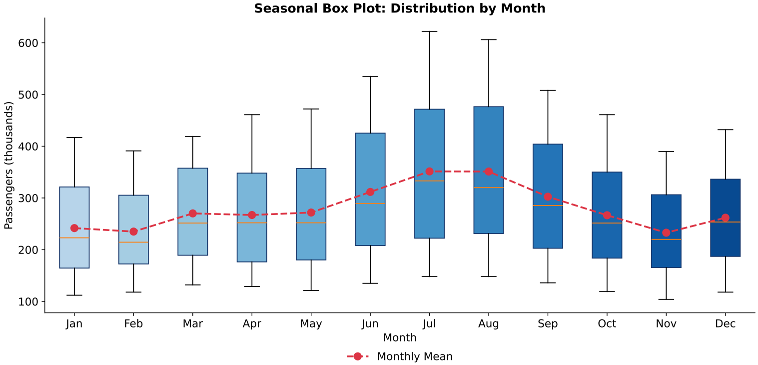
Why It Matters

Ignoring seasonality leads to biased forecasts and invalid inference!

Example: Airline Passengers Data

- Monthly international airline passengers (1949–1960)
- Clear **upward trend** and **growing seasonal amplitude**
- Summer peaks reflect vacation travel patterns

Visualizing Seasonal Patterns



- Box plot reveals consistent seasonal pattern across years
- July–August show highest passenger counts (summer travel)
- November–February show lowest counts (winter months)

Deterministic vs Stochastic Seasonality

Deterministic Seasonality

Fixed seasonal pattern: $Y_t = \sum_{j=1}^{s} \gamma_j D_{jt} + \varepsilon_t$ where D_{jt} are seasonal dummies.

Properties:

- Pattern constant over time
- Removed by regression

Stochastic Seasonality

Evolving seasonal pattern: $\Delta_s Y_t = Y_t - Y_{t-s}$ exhibits dependence structure.

Properties:

- Pattern evolves over time
- Requires seasonal differencing

Detecting Seasonality

Visual Methods

- Time series plot – look for repeating patterns
- Seasonal subseries plot – compare same seasons across years
- ACF plot – spikes at seasonal lags $(s, 2s, 3s, \ldots)$

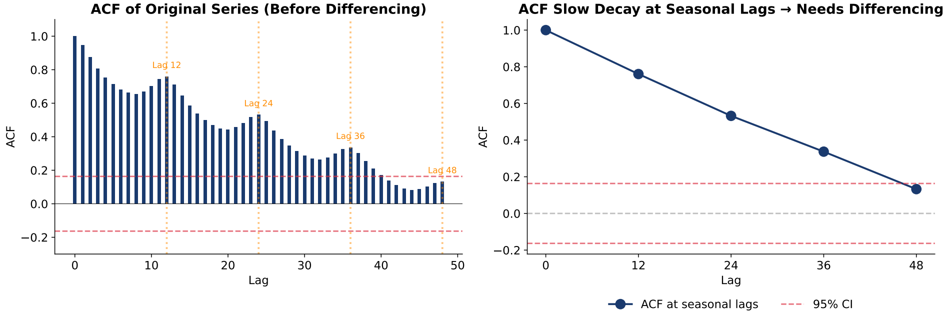
Statistical Tests

- Seasonal unit root tests (HEGY, CH, OCSB)
- F-test for seasonal dummies
- Kruskal-Wallis test (nonparametric)

ACF Signature

Strong seasonality: ACF shows significant spikes at lags $s, 2s, 3s, \ldots$

ACF Reveals Seasonal Structure



- **Slow decay** at all lags indicates non-stationarity (trend)
- **Spikes at lags 12, 24, 36** confirm seasonal pattern ($s = 12$)
- ACF at seasonal lags shows slow decay $\Rightarrow$ needs seasonal differencing

F-test for Seasonal Dummies: Intuition

What Does This Test Do?

Tests whether the **mean values differ significantly across seasons**.

- If January mean $\neq$ February mean $\neq$ ... $\neq$ December mean $\Rightarrow$ seasonality
- Compares a model WITH seasonal dummies vs. a model WITHOUT

The Models Being Compared

Restricted: $Y_t = \alpha + \varepsilon_t$ **Unrestricted**: $Y_t = \alpha + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$
where $D_{jt} = 1$ if observation t is in season j, 0 otherwise.

Key Idea

If adding seasonal dummies **significantly reduces** prediction errors, then seasonality is present.

F-test for Seasonal Dummies: Formula and Example

F-statistic Formula

$$F = \frac{(SSR_R - SSR_U)/(s-1)}{SSR_U/(n-s)} \sim F_{s-1,n-s}$$

- SSR_R = Sum of Squared Residuals from restricted model (no dummies)
- SSR_U = Sum of Squared Residuals from unrestricted model (with dummies)
- $s - 1$ = number of restrictions (monthly: 11, quarterly: 3)

Numerical Example (Monthly Data, n=120)

$SSR_R = 15000$, $SSR_U = 8500$, $s = 12$

$$F = \frac{(15000 - 8500)/11}{8500/108} = \frac{590.9}{78.7} = 7.51$$

Critical value $F_{0.05,11,108} \approx 1.87$. Since $7.51 > 1.87$: **Reject** $H_0 \Rightarrow$ Seasonality present!

Kruskal-Wallis Test: Intuition

What Does This Test Do?

A **nonparametric** test that checks if observations from different seasons come from the same distribution.

- Works by **ranking** all observations from smallest to largest
- Checks if ranks are evenly distributed across seasons
- If one season consistently has higher/lower ranks $\Rightarrow$ seasonality

Why Use It Instead of F-test?

- **No normality assumption** – works with any distribution
- **Robust to outliers** – extreme values don't distort results

Limitation

Less powerful than F-test when data IS normally distributed.

Kruskal-Wallis Test: Formula and Example

Test Statistic

$H = \frac{12}{N(N+1)} \sum_{j=1}^{s} \frac{R_j^2}{n_j} - 3(N+1)$ where N = total obs., n_j = obs. in season j, R_j = sum of ranks.

Example: Quarterly Sales (n=20, s=4)

Data ranked 1-20. Rank sums: Q1: $R_1 = 15$, Q2: $R_2 = 35$, Q3: $R_3 = 70$, Q4: $R_4 = 90$

$$H = \frac{12}{20 \times 21} \left(\frac{15^2}{5} + \frac{35^2}{5} + \frac{70^2}{5} + \frac{90^2}{5} \right) - 3(21) = 12.6$$

Critical value $\chi^2_{0.05,3} = 7.81$. Since $12.6 > 7.81$: **Reject** $H_0 \Rightarrow$ Seasonality!

In Python

```
scipy.stats.kruskal(q1, q2, q3, q4)
```

HEGY Test: What Problem Does It Solve?

The Key Question

Given a seasonal time series, we need to know:

1. Does it need **regular differencing** $(1 - L)$? $\Rightarrow$ set $d = 1$
2. Does it need **seasonal differencing** $(1 - L^s)$? $\Rightarrow$ set $D = 1$

HEGY tests for **both** types of unit roots simultaneously!

Why Not Just Use ADF?

ADF only tests for a **regular** unit root at frequency zero. Seasonal data can have unit roots at **seasonal frequencies** that ADF misses!

HEGY Tests Multiple Frequencies

Quarterly: tests at 0, π, $\pm\pi/2$. Monthly: tests at 0, π, $\pm\pi/6$, $\pm\pi/3$, $\pm\pi/2$, $\pm2\pi/3$, $\pm5\pi/6$.

HEGY Test: The Regression Formula (Quarterly)

HEGY Auxiliary Regression

For quarterly data ($s = 4$), estimate:

$$\Delta_4 y_t = \pi_1 z_{1,t-1} + \pi_2 z_{2,t-1} + \pi_3 z_{3,t-2} + \pi_4 z_{4,t-2} + \sum_{j=1}^{k} \phi_j \Delta_4 y_{t-j} + \varepsilon_t$$

Transformed Variables

$$z_{1t} = (1 + L + L^2 + L^3) y_t = y_t + y_{t-1} + y_{t-2} + y_{t-3}$$
$$z_{2t} = -(1 - L + L^2 - L^3) y_t = -y_t + y_{t-1} - y_{t-2} + y_{t-3}$$
$$z_{3t} = -(1 - L^2) y_t = -y_t + y_{t-2} \quad ; \quad z_{4t} = -(L - L^3) y_t = -y_{t-1} + y_{t-3}$$

Hypotheses

$H_0 : \pi_1 = 0$ (freq. 0), $H_0 : \pi_2 = 0$ (freq. π), $H_0 : \pi_3 = \pi_4 = 0$ (freq. $\pm\pi/2$)

HEGY Test: Decision Rules with Examples

HEGY Critical Values (5%, n=100, with constant)

Test	Statistic	Critical Value	If NOT rejected...
t_1 ($\pi_1 = 0$)	t-stat	-2.88	Need $d = 1$
t_2 ($\pi_2 = 0$)	t-stat	-2.88	Need $D = 1$
F_{34} ($\pi_3 = \pi_4 = 0$)	F-stat	6.57	Need $D = 1$

Example: Quarterly GDP

Suppose HEGY gives: $t_1 = -1.52$, $t_2 = -4.21$, $F_{34} = 2.15$

- $t_1 = -1.52 > -2.88$: Cannot reject $\Rightarrow$ **need** $d = 1$
- $t_2 = -4.21 < -2.88$: Reject $\Rightarrow$ no unit root at π
- $F_{34} = 2.15 < 6.57$: Cannot reject $\Rightarrow$ **need** $D = 1$

Conclusion: Use SARIMA with $d = 1, D = 1$

Canova-Hansen Test: The Opposite of HEGY

HEGY vs Canova-Hansen: Different Null Hypotheses!

	HEGY	**Canova-Hansen**
H_0	Seasonal unit root	**No** seasonal unit root
H_1	No seasonal unit root	Seasonal unit root
Reject H_0	Use seasonal dummies	Use $(1 - L^s)$ differencing
Don't reject	Use $(1 - L^s)$ differencing	Use seasonal dummies

Why Does This Matter?

- HEGY: "Prove there's NO unit root" (conservative toward differencing)
- CH: "Prove there IS a unit root" (conservative toward dummies)
- Use **both** tests for robust conclusions!

Canova-Hansen Test: Formula

Test Procedure

1. Regress y_t on seasonal dummies: $y_t = \sum_{j=1}^{s} \gamma_j D_{jt} + u_t$
2. Compute partial sums at seasonal frequency λ_i: $S_{it}^{(c)} = \sum_{j=1}^{t} \hat{u}_j \cos(\lambda_i j)$, $S_{it}^{(s)} = \sum_{j=1}^{t} \hat{u}_j \sin(\lambda_i j)$

LM Test Statistic

$$LM_i = \frac{1}{T^2 \hat{\omega}_i} \left[\sum_{t=1}^{T} (S_{it}^{(c)})^2 + \sum_{t=1}^{T} (S_{it}^{(s)})^2 \right]$$

where $\hat{\omega}_i$ = consistent estimate of spectral density at frequency λ_i.

Decision

Reject H_0 (stationarity) if $LM >$ critical value $\Rightarrow$ seasonal differencing needed.

Summary: Choosing the Right Seasonality Test

Test	H_0	**If Reject**	**Best For**
F-test	No seasonality	Seasonality exists	Normal data
Kruskal-Wallis	No seasonal diff.	Seasonality exists	Non-normal, outliers
HEGY	Unit root exists	Use dummies	Determining d, D
Canova-Hansen	No unit root	Use $(1 - L^s)$	Confirming stability

Key Insight

F-test/Kruskal-Wallis: "*Is there seasonality?*"
HEGY/Canova-Hansen: "*What type?*" (deterministic vs stochastic)

The Seasonal Difference Operator

Definition 2 (Seasonal Difference)

The **seasonal difference operator** Δ_s is defined as:

$$\Delta_s Y_t = (1 - L^s) Y_t = Y_t - Y_{t-s}$$

where $L^s Y_t = Y_{t-s}$ is the seasonal lag operator.

Examples

- Monthly data ($s = 12$): $\Delta_{12} Y_t = Y_t - Y_{t-12}$
  Compares each month to the same month last year
- Quarterly data ($s = 4$): $\Delta_4 Y_t = Y_t - Y_{t-4}$
  Compares each quarter to the same quarter last year

Combining Regular and Seasonal Differencing

Full Differencing

For series with both trend and seasonality:

$$\Delta\Delta_s Y_t = (1 - L)(1 - L^s)Y_t$$

Expansion

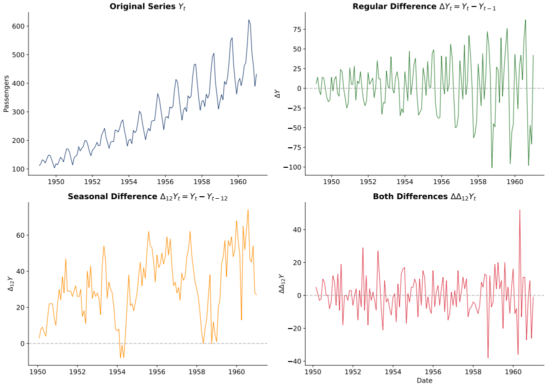
$(1 - L)(1 - L^s)Y_t = Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}$
For monthly data ($s = 12$):

$$\Delta\Delta_{12} Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$$

Order of Differencing

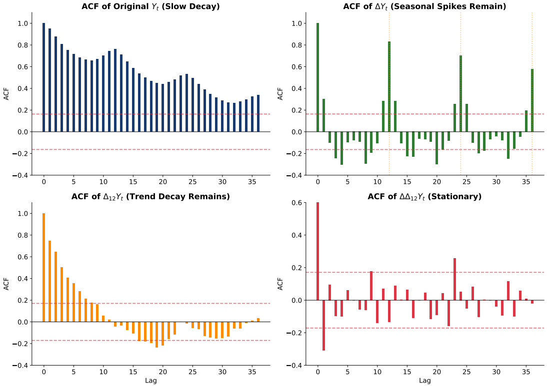
- d: number of regular differences (trend removal)
- D: number of seasonal differences (seasonal trend removal)

Effect of Differencing Operations



- Regular differencing removes trend but seasonal pattern remains
- Seasonal differencing removes seasonality but trend pattern remains
- **Both differences** needed to achieve stationarity

ACF Before and After Differencing



- Original ACF: slow decay indicates non-stationarity
- After Δ: seasonal spikes remain at lags 12, 24, 36
- After Δ_{12}: trend decay remains at early lags
- After $\Delta\Delta_{12}$: ACF cuts off $\Rightarrow$ **stationary**

Seasonal Integration

Definition 3 (Seasonally Integrated Process)

A series Y_t is **seasonally integrated** of order $(d, D)_s$, written $Y_t \sim I(d, D)_s$, if:

$$(1 - L)^d (1 - L^s)^D Y_t$$

is stationary.

Common Cases

- $I(1, 0)_{12}$: Regular unit root only (monthly)
- $I(0, 1)_{12}$: Seasonal unit root only
- $I(1, 1)_{12}$: Both regular and seasonal unit roots

SARIMA Model Definition

Definition 4 (SARIMA$(p, d, q) \times (P, D, Q)_s$)

The **Seasonal ARIMA** model is:

$$\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D Y_t = c + \theta(L)\Theta(L^s)\varepsilon_t$$

Components

- $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$: Non-seasonal AR
- $\Phi(L^s) = 1 - \Phi_1 L^s - \cdots - \Phi_P L^{Ps}$: Seasonal AR
- $\theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$: Non-seasonal MA
- $\Theta(L^s) = 1 + \Theta_1 L^s + \cdots + \Theta_Q L^{Qs}$: Seasonal MA
- $(1-L)^d$: Regular differencing; $(1-L^s)^D$: Seasonal differencing

SARIMA Notation

Full Specification

SARIMA$(p, d, q) \times (P, D, Q)_s$ has 7 parameters to specify:

Parameter	**Meaning**
p	Non-seasonal AR order
d	Non-seasonal differencing order
q	Non-seasonal MA order
P	Seasonal AR order
D	Seasonal differencing order
Q	Seasonal MA order
s	Seasonal period

Example

SARIMA$(1, 1, 1) \times (1, 1, 1)_{12}$: Monthly data with AR(1), MA(1), seasonal AR(1), seasonal MA(1), and both regular and seasonal differencing.

Common SARIMA Models

Airline Model: SARIMA$(0,1,1) \times (0,1,1)_s$

$(1-L)(1-L^s)Y_t = (1+\theta L)(1+\Theta L^s)\varepsilon_t$ – Classic model (Box & Jenkins, 1970)

SARIMA$(1,0,0) \times (1,0,0)_s$

$(1-\phi L)(1-\Phi L^s)Y_t = \varepsilon_t$ – Pure seasonal and non-seasonal AR

SARIMA$(0,1,1) \times (0,1,0)_s$

$(1-L)(1-L^s)Y_t = (1+\theta L)\varepsilon_t$ – Random walk + seasonal diff + MA(1)

The Multiplicative Structure

Why Multiplicative?

The seasonal and non-seasonal parts **multiply**:

$$\phi(L)\Phi(L^s) \quad \text{and} \quad \theta(L)\Theta(L^s)$$

Example: SARIMA$(1,0,0) \times (1,0,0)_{12}$

$(1-\phi L)(1-\Phi L^{12})Y_t = \varepsilon_t$
Expanding: $Y_t - \phi Y_{t-1} - \Phi Y_{t-12} + \phi\Phi Y_{t-13} = \varepsilon_t$
The cross-term $\phi\Phi Y_{t-13}$ captures interaction!

Interpretation

Multiplicative structure allows parsimonious modeling of complex seasonal patterns with few parameters.

ACF/PACF for Seasonal Models

Key Insight

Seasonal models show patterns at both:

- Non-seasonal lags: $1, 2, 3, \ldots$
- Seasonal lags: $s, 2s, 3s, \ldots$

Model	**ACF**	**PACF**
SAR(P)	Decays at $s, 2s, \ldots$	Cuts off after Ps
SMA(Q)	Cuts off after Qs	Decays at $s, 2s, \ldots$
SARMA	Decays at seasonal lags	Decays at seasonal lags

Example: Airline Model ACF/PACF

SARIMA$(0,1,1) \times (0,1,1)_{12}$

After differencing $W_t = (1-L)(1-L^{12})Y_t$:

$$W_t = (1+\theta L)(1+\Theta L^{12})\varepsilon_t$$

Expected ACF Pattern

- Spike at lag 1 (from θ)
- Spike at lag 12 (from Θ)
- Spike at lag 13 (from $\theta \cdot \Theta$ interaction)
- All other lags near zero

Expected PACF Pattern

- Exponential decay at lags $1, 2, 3, \ldots$
- Exponential decay at lags $12, 24, 36, \ldots$

Model Identification Guidelines

Step-by-Step Process

1. Examine ACF for slow decay at seasonal lags $\Rightarrow$ seasonal differencing
2. After differencing, check ACF/PACF patterns
3. Non-seasonal behavior at lags $1, 2, \ldots, s-1$
4. Seasonal behavior at lags $s, 2s, 3s, \ldots$

Practical Tips

- Start with $d \leq 1$ and $D \leq 1$
- Usually $P, Q \leq 2$ is sufficient
- Use information criteria (AIC, BIC) for final selection
- Auto-SARIMA algorithms can help

Estimation Methods

Maximum Likelihood Estimation

Standard approach for SARIMA:

- Conditional MLE (conditional on initial values)
- Exact MLE (via Kalman filter)

Computational Considerations

- More parameters than ARIMA $\Rightarrow$ more data needed
- Seasonal parameters estimated from lags $s, 2s, \ldots$
- Need sufficient seasonal cycles (at least 3-4 years of monthly data)

Stationarity and Invertibility

Stationarity Conditions

Both non-seasonal and seasonal AR polynomials must have roots outside the unit circle:

- $\phi(z) = 0 \Rightarrow |z| > 1$
- $\Phi(z^s) = 0 \Rightarrow |z| > 1$

Invertibility Conditions

Both non-seasonal and seasonal MA polynomials must have roots outside the unit circle:

- $\theta(z) = 0 \Rightarrow |z| > 1$
- $\Theta(z^s) = 0 \Rightarrow |z| > 1$

Diagnostic Checking

Residual Analysis

After fitting SARIMA, check that residuals are white noise:

1. Plot residuals over time (no patterns)
2. ACF of residuals (no significant spikes)
3. Ljung-Box test at multiple lags including seasonal
4. Normality tests (Q-Q plot, Jarque-Bera)

Important

Check ACF at **both** non-seasonal and seasonal lags!
Significant ACF at lag 12 suggests inadequate seasonal modeling.

Model Selection Criteria

Information Criteria

Compare competing SARIMA models using:

- $\text{AIC} = -2\ln(L) + 2k$
- $\text{BIC} = -2\ln(L) + k\ln(n)$
- $\text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1}$ (corrected for small samples)

where $k = p + q + P + Q + 1$ (plus 1 for variance).

Auto-SARIMA

Python's `pmdarima.auto_arima()` with `seasonal=True` automatically searches for optimal $(p, d, q) \times (P, D, Q)_s$.

Point Forecasts

Forecast Computation

SARIMA forecasts are computed recursively:

- Replace future ε_{T+h} with 0
- Replace future Y_{T+h} with forecasts $\hat{Y}_{T+h|T}$
- Use known past values $Y_T, Y_{T-1}, \ldots$

Seasonal Pattern in Forecasts

SARIMA forecasts naturally capture seasonality:

- Short-term: influenced by recent values
- Long-term: revert to seasonal pattern

Forecast Intervals

Uncertainty Quantification

$(1-\alpha)\%$ prediction interval:

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2}\sqrt{\mathrm{Var}(e_{T+h})}$$

Variance computed from MA(∞) representation.

Key Properties

- Intervals widen with forecast horizon
- For $I(1,1)_s$ series: intervals grow without bound
- Seasonal pattern visible in point forecasts
- Uncertainty captures both trend and seasonal variation

Long-Horizon Forecasts

Behavior as $h \to \infty$

- Point forecasts converge to deterministic seasonal pattern
- If drift present: linear trend $+$ seasonal pattern
- Forecast intervals continue to widen

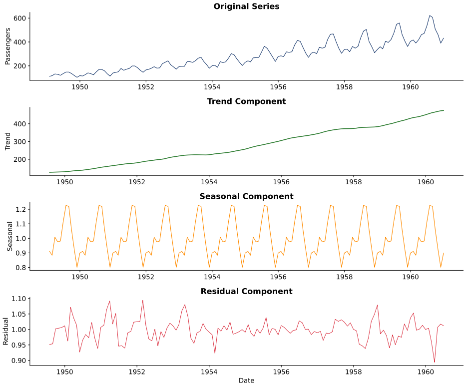
Practical Implication

- Short-term: SARIMA captures both level and season
- Medium-term: Good seasonal forecasts, growing uncertainty
- Long-term: Mainly reflects seasonal pattern, wide intervals

Airline Passengers Data

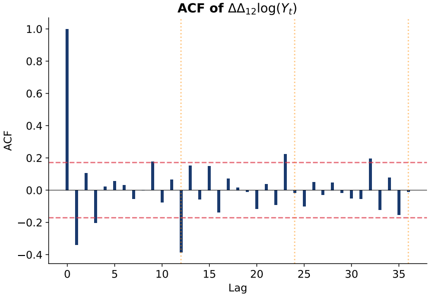
- Classic dataset: Monthly international airline passengers (1949-1960)
- Clear upward trend and growing seasonal amplitude

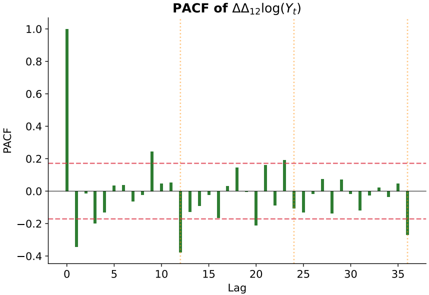
Seasonal Decomposition



- Trend: Strong upward growth
- Seasonality: Summer peaks (vacation travel)
- Residual: Random variation after removing trend and season

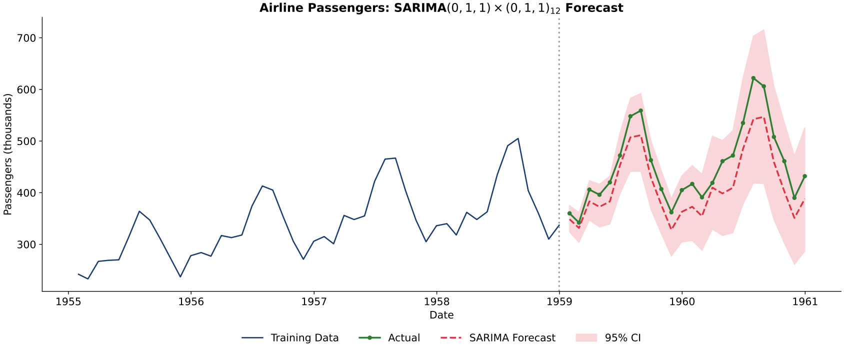
ACF/PACF Analysis





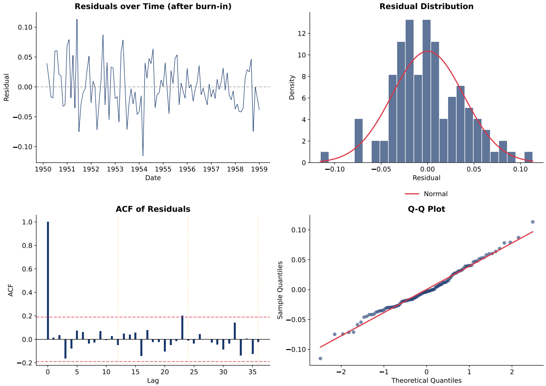
- After $\Delta\Delta_{12}$ differencing: spikes at lags 1 and 12
- Suggests SARIMA$(0,1,1) \times (0,1,1)_{12}$ (Airline model)

SARIMA Forecast Results



- SARIMA captures both trend and seasonal pattern
- Forecasts show appropriate seasonal peaks and troughs

Model Diagnostics



- Residuals appear random; ACF within bounds at all lags
- Model adequately captures seasonal structure

Python Implementation

Fitting SARIMA in Python

```
from statsmodels.tsa.statespace.sarimax import SARIMAX
model = SARIMAX(y, order=(0,1,1), seasonal_order=(0,1,1,12))
results = model.fit()
forecast = results.get_forecast(steps=24)
```

Note

Complete Python examples with comments are provided in the Jupyter notebooks.

Key Takeaways

Main Points

1. **Seasonality** is common in economic and business data
2. **Seasonal differencing** $(1 - L^s)$ removes stochastic seasonality
3. **SARIMA**$(p, d, q) \times (P, D, Q)_s$ extends ARIMA for seasonal data
4. **Multiplicative structure** captures seasonal-trend interactions
5. **ACF/PACF** show patterns at both regular and seasonal lags
6. **Model selection**: Use AIC/BIC or auto-SARIMA algorithms

Next Steps

Chapter 5 will cover multivariate time series: VAR models, Granger causality, and cointegration.

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