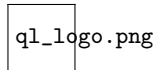
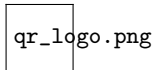
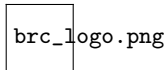
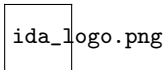


Seminar: Volatility Models

ARCH, GARCH, EGARCH, TGARCH

Review Quiz and Practice Exercises



Seminar Outline

Question 1

What is “volatility clustering”?

- ☐ A Volatility is constant over time
- ☐ B Periods of high volatility tend to be followed by periods of high volatility
- ☐ C Returns are correlated over time
- ☐ D The return distribution is normal

Think about the behavior of financial markets during crisis periods...

Answer to Question 1

Correct Answer: (B)

Periods of high volatility tend to be followed by periods of high volatility

Explanation

- **Volatility clustering** is a stylized fact observed in financial time series
- “Turbulent” periods (with large movements) tend to persist
- “Calm” periods (with small movements) also tend to persist
- This implies that conditional variance σ_t^2 is **predictable**
- GARCH models capture exactly this phenomenon!

Question 2

In the GARCH(1,1) model: $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$

What does the parameter α represent?

- ☐ A Volatility persistence
- ☐ B Baseline volatility level
- ☐ C Reaction to recent shocks (news coefficient)
- ☐ D Unconditional variance

Answer to Question 2

Correct Answer: (C)

Reaction to recent shocks (news coefficient)

Interpretation of GARCH(1,1) Parameters

- ω = baseline (floor) volatility level
- α = **reaction** to squared innovations (“news”)
- β = volatility **persistence** (memory)
- $\alpha + \beta$ = total persistence

A large α means volatility reacts strongly to recent shocks.

Question 3

What is the stationarity condition for GARCH(1,1)?

- ☐ A $\omega > 0$
- ☐ B $\alpha + \beta = 1$
- ☐ C $\alpha + \beta < 1$
- ☐ D $\alpha > \beta$

Answer to Question 3

Correct Answer: (C)

$$\alpha + \beta < 1$$

Complete Conditions

For GARCH(1,1) stationarity:

- $\omega > 0$ (ensures positive variance)
- $\alpha \geq 0, \beta \geq 0$ (non-negativity)
- $\alpha + \beta < 1$ (**strict stationarity**)

If $\alpha + \beta = 1 \Rightarrow$ IGARCH (shocks have permanent effect)

Question 4

What is the formula for unconditional variance in GARCH(1,1)?

- ☐ A $\bar{\sigma}^2 = \omega$
- ☐ B $\bar{\sigma}^2 = \frac{\omega}{1 - \alpha}$
- ☐ C $\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$
- ☐ D $\bar{\sigma}^2 = \frac{\omega}{\alpha + \beta}$

Answer to Question 4

Correct Answer: (C)

$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$$

Derivation

Taking unconditional expectation of GARCH(1,1):

$$\mathbb{E}[\sigma_t^2] = \omega + \alpha \mathbb{E}[\varepsilon_{t-1}^2] + \beta \mathbb{E}[\sigma_{t-1}^2]$$

$$\bar{\sigma}^2 = \omega + \alpha \bar{\sigma}^2 + \beta \bar{\sigma}^2$$

$$\bar{\sigma}^2(1 - \alpha - \beta) = \omega$$

$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$$

Question 5

What is the “leverage effect”?

- ☐ A Positive shocks increase volatility more than negative shocks
- ☐ B Negative shocks increase volatility more than positive shocks
- ☐ C Volatility is independent of shock sign
- ☐ D Returns are asymmetric

Answer to Question 5

Correct Answer: (B)

Negative shocks increase volatility more than positive shocks

Explanation

- Empirically observed in stock markets
- When prices fall, firm leverage increases (debt becomes larger relative to equity)
- This makes the firm riskier \Rightarrow higher volatility
- Standard GARCH **cannot** capture this effect (depends on ε^2)
- Solutions: **EGARCH, GJR-GARCH, TGARCH**

Question 6

In the EGARCH model, a negative γ parameter indicates:

- ☐ A Absence of leverage effect
- ☐ B Presence of leverage effect
- ☐ C Constant volatility
- ☐ D Non-stationary model

Answer to Question 6

Correct Answer: (B)

Presence of leverage effect

EGARCH(1,1)

$$\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

- $\gamma < 0$: negative shock ($z < 0$) \Rightarrow increases $\ln(\sigma_t^2)$
- $\gamma > 0$: inverse effect (less common)
- $\gamma = 0$: symmetric effect (like GARCH)

Question 7

What is the main advantage of EGARCH over GARCH?

- ☐ A Faster to estimate
- ☐ B No non-negativity constraints needed
- ☐ C Fewer parameters
- ☐ D Easier to interpret

Answer to Question 7

Correct Answer: (B)

No non-negativity constraints needed

EGARCH Advantages

- Models $\ln(\sigma_t^2)$, not σ_t^2
- $\sigma_t^2 = e^{\ln(\sigma_t^2)} > 0$ **automatically**, regardless of parameter values
- GARCH requires $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$
- During estimation, these constraints can cause convergence problems

Question 8

Which test do we use to detect ARCH effects in residuals?

- ☐ A Dickey-Fuller test
- ☐ B Ljung-Box test on residuals
- ☐ C Engle's ARCH-LM test
- ☐ D Breusch-Pagan test

Answer to Question 8

Correct Answer: (C)

Engle's ARCH-LM test

ARCH-LM Test Procedure

- 1 Estimate mean model, obtain residuals $\hat{\varepsilon}_t$
- 2 Compute $\hat{\varepsilon}_t^2$
- 3 Regress: $\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \dots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$
- 4 Test statistic: $LM = T \cdot R^2 \sim \chi^2(q)$ under H_0

H_0 : No ARCH effects H_1 : ARCH effects present

Question 9

For S&P 500, typical values of $\alpha + \beta$ in GARCH(1,1) are:

- ☐ A 0.50 – 0.70
- ☐ B 0.70 – 0.85
- ☐ C 0.95 – 0.99
- ☐ D Greater than 1

Answer to Question 9

Correct Answer: (C)

0.95 – 0.99

Highly Persistent Volatility

- Financial time series exhibit very persistent volatility
- $\alpha + \beta \approx 0.98$ for S&P 500
- Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha + \beta)} \approx 35 - 60$ days
- This means a volatility shock dissipates over several months

| Series | $\alpha + \beta$ |
|---------|------------------|
| S&P 500 | 0.97–0.99 |
| Bitcoin | 0.90–0.98 |
| EUR/USD | 0.96–0.99 |

Question 10

Which distribution is most commonly used for GARCH innovations to capture fat tails?

- ☐ A Normal
- ☐ B Uniform
- ☐ C Student-t
- ☐ D Exponential

Answer to Question 10

Correct Answer: (C)

Student-t

Innovation Distributions

- **Normal**: standard, but underestimates extreme risk
- **Student-t**: fat tails, parameter ν (degrees of freedom)
- **GED**: Generalized Error Distribution, flexible
- **Skewed Student-t**: asymmetry + fat tails

For S&P 500: $\nu \approx 5 - 8$ (significantly fatter tails than normal)

True or False?

- ❶ ARIMA models can capture volatility clustering.
- ❷ In GARCH(1,1), if $\alpha + \beta = 1$, the model is called IGARCH.
- ❸ GJR-GARCH uses an indicator variable for negative shocks.
- ❹ GARCH volatility forecasts converge to zero in the long run.
- ❺ EGARCH can have negative parameters without generating negative variance.
- ❻ Value at Risk (VaR) can be calculated using GARCH volatility forecasts.

True/False Answers

- ❶ **FALSE** — ARIMA assumes constant variance; GARCH models volatility.
- ❷ **TRUE** — IGARCH = Integrated GARCH, volatility has unit root.
- ❸ **TRUE** — $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$, else 0.
- ❹ **FALSE** — Converges to unconditional variance $\bar{\sigma}^2$, not zero.
- ❺ **TRUE** — Models $\ln(\sigma_t^2)$, so $\sigma_t^2 = e^{\ln(\sigma_t^2)} > 0$ always.
- ❻ **TRUE** — $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{t+1}$ (for zero mean).

Problem 1: Calculating Unconditional Variance

Statement

A GARCH(1,1) model has estimated parameters:

- $\omega = 0.000002$
- $\alpha = 0.08$
- $\beta = 0.90$

Calculate:

- a) Daily unconditional variance
- b) Daily unconditional volatility (as percentage)
- c) Annualized volatility (assuming 252 trading days)
- d) Volatility half-life

Solution to Problem 1

Answers

$$\text{a) } \bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta} = \frac{0.000002}{1 - 0.08 - 0.90} = \frac{0.000002}{0.02} = 0.0001$$

$$\text{b) } \bar{\sigma} = \sqrt{0.0001} = 0.01 = 1\% \text{ per day}$$

$$\text{c) } \sigma_{\text{annual}} = \bar{\sigma} \times \sqrt{252} = 0.01 \times 15.87 = 15.87\% \text{ per year}$$

$$\text{d) } HL = \frac{\ln(0.5)}{\ln(\alpha + \beta)} = \frac{\ln(0.5)}{\ln(0.98)} = \frac{-0.693}{-0.0202} \approx 34 \text{ days}$$

Interpretation

Volatility of 15.87% per year is typical for a stock index. Half-life of 34 days means a volatility shock is reduced by half after about 7 weeks.

Problem 2: Volatility Forecast

Statement

Using the GARCH(1,1) model from Problem 1:

- $\omega = 0.000002$, $\alpha = 0.08$, $\beta = 0.90$
- At time T : $\varepsilon_T = -0.03$ (3% drop), $\sigma_T^2 = 0.0004$

Calculate volatility forecasts for:

- a σ_{T+1}^2 (one step ahead)
- b σ_{T+5}^2 (five steps ahead)
- c σ_{T+100}^2 (one hundred steps ahead)

Solution to Problem 2

Answers

a $\sigma_{T+1}^2 = \omega + \alpha \varepsilon_T^2 + \beta \sigma_T^2$

$$= 0.000002 + 0.08 \times (0.03)^2 + 0.90 \times 0.0004 = 0.000434$$

Volatility: $\sqrt{0.000434} = 2.08\%$

b $\mathbb{E}_T[\sigma_{T+5}^2] = \bar{\sigma}^2 + (0.98)^4(\sigma_{T+1}^2 - \bar{\sigma}^2)$

$$= 0.0001 + 0.922 \times (0.000434 - 0.0001) = 0.000408$$

Volatility: $\sqrt{0.000408} = 2.02\%$

c $\mathbb{E}_T[\sigma_{T+100}^2] = 0.0001 + (0.98)^{99} \times 0.000334 \approx 0.000145$

Volatility: $\sqrt{0.000145} = 1.20\%$ (close to $\bar{\sigma} = 1\%$)

Problem 3: Value at Risk

Statement

A portfolio of 1,000,000 EUR is invested in stocks with returns modeled by GARCH(1,1).

Tomorrow's volatility forecast: $\sigma_{T+1} = 2\%$ daily.

Assuming normally distributed returns with zero mean, calculate:

- a VaR at 95% (1 day)
- b VaR at 99% (1 day)
- c VaR at 99% (10 days), using “square root of time” rule

Quantiles: $z_{0.05} = 1.645$, $z_{0.01} = 2.326$

Solution to Problem 3

Answers

a VaR 95% (1 day):

$$\text{VaR}_{95\%} = 1.645 \times 0.02 \times 1,000,000 = 32,900 \text{ EUR}$$

b VaR 99% (1 day):

$$\text{VaR}_{99\%} = 2.326 \times 0.02 \times 1,000,000 = 46,520 \text{ EUR}$$

c VaR 99% (10 days):

$$\text{VaR}_{99\%,10d} = \text{VaR}_{99\%,1d} \times \sqrt{10} = 46,520 \times 3.162 = 147,100 \text{ EUR}$$

Caution

In practice, for Student-t distribution, quantiles are larger (fatter tails)!

Problem 4: Model Identification

Statement

Analyze the following estimation results and identify the model:

| Parameter | Estimate | Std. Error |
|-----------|-----------|------------|
| ω | 0.0000015 | 0.0000005 |
| α | 0.0550 | 0.0120 |
| γ | 0.0850 | 0.0180 |
| β | 0.9100 | 0.0150 |

- a What model is this?
- b Is leverage effect present?
- c What is the impact of negative vs positive shocks?
- d Is the model stationary?

Answers

- a **GJR-GARCH(1,1,1)** — presence of γ parameter (threshold/asymmetry)
- b **Yes, leverage effect present:** $\gamma = 0.085 > 0$ and significant
- c **Impact:**
 - Positive shock: impact = $\alpha = 0.055$
 - Negative shock: impact = $\alpha + \gamma = 0.055 + 0.085 = 0.140$
 - Negative shocks have **2.5x greater** impact!
- d **Stationarity:** $\alpha + \gamma/2 + \beta = 0.055 + 0.0425 + 0.91 = 1.0075$
On the edge! Almost IGARCH. Very persistent model.

Step 1: Load and Prepare Data

```
import pandas as pd
import numpy as np
import yfinance as yf
from arch import arch_model
from arch.unitroot import ADF

# Download S&P 500 data
data = yf.download('^GSPC', start='2010-01-01', end='2024-01-01')
returns = 100 * data['Adj Close'].pct_change().dropna()

# Check stationarity
adf = ADF(returns)
print(f'ADF statistic: {adf.stat:.4f}')
print(f'p-value: {adf.pvalue:.4f}')
```

Step 2: Test for ARCH Effects

```
from statsmodels.stats.diagnostic import het_arch

# ARCH-LM test on residuals
residuals = returns - returns.mean()
lm_stat, lm_pvalue, f_stat, f_pvalue = het_arch(residuals, nlags=10)

print(f'ARCH-LM statistic: {lm_stat:.4f}')
print(f'p-value: {lm_pvalue:.4f}')

if lm_pvalue < 0.05:
    print('=> ARCH effects present! GARCH modeling justified.')
```

Step 3: Estimate Models

```
# GARCH(1,1) with Student-t distribution
model_garch = arch_model(returns, vol='Garch', p=1, q=1, dist='t')
res_garch = model_garch.fit(dispatch='off')
print(res_garch.summary())

# GJR-GARCH(1,1,1)
model_gjr = arch_model(returns, vol='Garch', p=1, o=1, q=1, dist='t')
res_gjr = model_gjr.fit(dispatch='off')

# EGARCH(1,1)
model_egarch = arch_model(returns, vol='EGARCH', p=1, q=1, dist='t')
res_egarch = model_egarch.fit(dispatch='off')

# Compare AIC
print(f'GARCH AIC: {res_garch.aic:.2f}')
print(f'GJR AIC: {res_gjr.aic:.2f}')
print(f'EGARCH AIC: {res_egarch.aic:.2f}')
```

Step 4: Diagnostics

```
# Standardized residuals
std_resid = res_gjr.std_resid

# Ljung-Box test on squared residuals
from statsmodels.stats.diagnostic import acorr_ljungbox
lb_test = acorr_ljungbox(std_resid**2, lags=10, return_df=True)
print(lb_test)

# Check for remaining ARCH effects
lm_stat2, lm_pval2, _, _ = het_arch(std_resid, nlags=5)
print(f'ARCH-LM residuals: stat={lm_stat2:.2f}, p={lm_pval2:.4f}')

if lm_pval2 > 0.05:
    print('=> No remaining ARCH effects. Model OK!')
```

Step 5: Forecast and VaR

```
# Forecast 10 days ahead
forecasts = res_gjr.forecast(horizon=10)
vol_forecast = np.sqrt(forecasts.variance.values[-1, :])

print('Volatility forecast (%):', vol_forecast)

# Value at Risk 99%
portfolio_value = 1_000_000
VaR_99 = 2.326 * vol_forecast[0] / 100 * portfolio_value
print(f'VaR 99% (1 day): {VaR_99:,.0f} EUR')

# 10-day VaR
VaR_99_10d = VaR_99 * np.sqrt(10)
print(f'VaR 99% (10 days): {VaR_99_10d:,.0f} EUR')
```

Key Concepts

- **ARCH**: conditional variance depends on past shocks
- **GARCH**: adds persistence through lagged variance
- **EGARCH/GJR**: capture leverage effect (asymmetry)
- **Stationarity**: $\alpha + \beta < 1$

Important Formulas

- Unconditional variance: $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$
- VaR: $VaR_{\alpha} = z_{\alpha} \cdot \sigma \cdot V$

Practical Tip

Use Student-t distribution to capture fat tails. Verify absence of ARCH effects in residuals!

Homework Exercises

Exercise 1

Download daily returns for BET (BVB index) and estimate a GARCH(1,1) model. Compare persistence ($\alpha + \beta$) with S&P 500.

Exercise 2

For Bitcoin, estimate GARCH, EGARCH, and GJR-GARCH. Is leverage effect present for cryptocurrencies?

Exercise 3

Calculate daily VaR for a portfolio of 100,000 EUR invested in EUR/USD, using GARCH-forecasted volatility.

Exercise 4

Compare GARCH(1,1) volatility forecast with realized volatility (sum of squared returns) for a 20-day period.