Seminar: Volatility Models

ARCH, GARCH, EGARCH, TGARCH

Review Quiz and Practice Exercises

Seminar Outline

Question 1

What is "volatility clustering"?

- **A** Volatility is constant over time
- **B** Periods of high volatility tend to be followed by periods of high volatility
- **C** Returns are correlated over time
- **D** The return distribution is normal

Think about the behavior of financial markets during crisis periods...

Answer to Question 1

Correct Answer: (B)

Periods of high volatility tend to be followed by periods of high volatility

Explanation

- **Volatility clustering** is a stylized fact observed in financial time series
- "Turbulent" periods (with large movements) tend to persist
- "Calm" periods (with small movements) also tend to persist
- This implies that conditional variance σ_t^2 is **predictable**
- GARCH models capture exactly this phenomenon!

Question 2

In the GARCH(1,1) model: $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$

What does the parameter α represent?

- A. Volatility persistence
- B. Baseline volatility level
- C. Reaction to recent shocks (news coefficient)
- D. Unconditional variance

Answer to Question 2

Correct Answer: (C)

Reaction to recent shocks (news coefficient)

Interpretation of GARCH(1,1) Parameters

- $\omega =$ baseline (floor) volatility level
- $\alpha =$ **reaction** to squared innovations ("news")
- $\beta =$ volatility **persistence** (memory)
- $\alpha + \beta =$ total persistence

A large α means volatility reacts strongly to recent shocks.

Question 3

What is the stationarity condition for GARCH(1,1)?

- A. $\omega > 0$
- B. $\alpha + \beta = 1$
- C. $\alpha + \beta < 1$
- D. $\alpha > \beta$

Answer to Question 3

Correct Answer: (C)

$\alpha + \beta < 1$

Complete Conditions

For GARCH(1,1) stationarity:

- $\omega > 0$ (ensures positive variance)
- $\alpha \geq 0$, $\beta \geq 0$ (non-negativity)
- $\alpha + \beta < 1$ (**strict stationarity**)

If $\alpha + \beta = 1 \Rightarrow$ IGARCH (shocks have permanent effect)

Question 4

What is the formula for unconditional variance in GARCH(1,1)?

- A $\bar{\sigma}^2 = \omega$
- B $\bar{\sigma}^2 = \dfrac{\omega}{1-\alpha}$
- C $\bar{\sigma}^2 = \dfrac{\omega}{1-\alpha-\beta}$
- D $\bar{\sigma}^2 = \dfrac{\omega}{\alpha+\beta}$

Answer to Question 4

Correct Answer: (C)

$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$$

Derivation

Taking unconditional expectation of GARCH(1,1):

$$\begin{aligned}
\mathbb{E}[\sigma_t^2] &= \omega + \alpha\mathbb{E}[\varepsilon_{t-1}^2] + \beta\mathbb{E}[\sigma_{t-1}^2] \\
\bar{\sigma}^2 &= \omega + \alpha\bar{\sigma}^2 + \beta\bar{\sigma}^2 \\
\bar{\sigma}^2(1 - \alpha - \beta) &= \omega \\
\bar{\sigma}^2 &= \frac{\omega}{1 - \alpha - \beta}
\end{aligned}$$

Question 5

What is the “leverage effect”?

- A. Positive shocks increase volatility more than negative shocks
- B. Negative shocks increase volatility more than positive shocks
- C. Volatility is independent of shock sign
- D. Returns are asymmetric

Answer to Question 5

Correct Answer: (B)

Negative shocks increase volatility more than positive shocks

Explanation

- Empirically observed in stock markets
- When prices fall, firm leverage increases (debt becomes larger relative to equity)
- This makes the firm riskier $\Rightarrow$ higher volatility
- Standard GARCH **cannot** capture this effect (depends on ε^2)
- Solutions: **EGARCH**, **GJR-GARCH**, **TGARCH**

Question 6

In the EGARCH model, a negative γ parameter indicates:

- A. Absence of leverage effect
- B. Presence of leverage effect
- C. Constant volatility
- D. Non-stationary model

Answer to Question 6

Correct Answer: (B)

Presence of leverage effect

EGARCH(1,1)

$$\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

- $\gamma < 0$: negative shock ($z < 0$) $\Rightarrow$ increases $\ln(\sigma_t^2)$
- $\gamma > 0$: inverse effect (less common)
- $\gamma = 0$: symmetric effect (like GARCH)

Question 7

What is the main advantage of EGARCH over GARCH?

- A. Faster to estimate
- B. No non-negativity constraints needed
- C. Fewer parameters
- D. Easier to interpret

Answer to Question 7

Correct Answer: (B)

No non-negativity constraints needed

EGARCH Advantages

- Models $\ln(\sigma_t^2)$, not σ_t^2
- $\sigma_t^2 = e^{\ln(\sigma_t^2)} > 0$ **automatically**, regardless of parameter values
- GARCH requires $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$
- During estimation, these constraints can cause convergence problems

Question 8

Which test do we use to detect ARCH effects in residuals?

- A. Dickey-Fuller test
- B. Ljung-Box test on residuals
- C. Engle's ARCH-LM test
- D. Breusch-Pagan test

Answer to Question 8

Correct Answer: (C)

Engle's ARCH-LM test

ARCH-LM Test Procedure

1. Estimate mean model, obtain residuals $\hat{\varepsilon}_t$
2. Compute $\hat{\varepsilon}_t^2$
3. Regress: $\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \cdots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$
4. Test statistic: $LM = T \cdot R^2 \sim \chi^2(q)$ under H_0

H_0: No ARCH effects $\quad$ H_1: ARCH effects present

Question 9

For S&P 500, typical values of $\alpha + \beta$ in GARCH(1,1) are:

- A. 0.50 – 0.70
- B. 0.70 – 0.85
- C. 0.95 – 0.99
- D. Greater than 1

Answer to Question 9

Correct Answer: (C)

0.95 – 0.99

Highly Persistent Volatility

- Financial time series exhibit very persistent volatility
- $\alpha + \beta \approx 0.98$ for S&P 500
- Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)} \approx 35 - 60$ days
- This means a volatility shock dissipates over several months

Series	$\boldsymbol{\alpha + \beta}$
S&P 500	0.97–0.99
Bitcoin	0.90–0.98
EUR/USD	0.96–0.99

Question 10

Which distribution is most commonly used for GARCH innovations to capture fat tails?

- A. Normal
- B. Uniform
- C. Student-t
- D. Exponential

Answer to Question 10

Correct Answer: (C)

Student-t

Innovation Distributions

- **Normal**: standard, but underestimates extreme risk
- **Student-t**: fat tails, parameter ν (degrees of freedom)
- **GED**: Generalized Error Distribution, flexible
- **Skewed Student-t**: asymmetry + fat tails

For S&P 500: $\nu \approx 5 - 8$ (significantly fatter tails than normal)

True or False?

1. ARIMA models can capture volatility clustering.
2. In GARCH(1,1), if $\alpha + \beta = 1$, the model is called IGARCH.
3. GJR-GARCH uses an indicator variable for negative shocks.
4. GARCH volatility forecasts converge to zero in the long run.
5. EGARCH can have negative parameters without generating negative variance.
6. Value at Risk (VaR) can be calculated using GARCH volatility forecasts.

True/False Answers

1. **FALSE** — ARIMA assumes constant variance; GARCH models volatility.
2. **TRUE** — IGARCH = Integrated GARCH, volatility has unit root.
3. **TRUE** — $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$, else 0.
4. **FALSE** — Converges to unconditional variance $\bar{\sigma}^2$, not zero.
5. **TRUE** — Models $\ln(\sigma_t^2)$, so $\sigma_t^2 = e^{\ln(\sigma_t^2)} > 0$ always.
6. **TRUE** — $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{t+1}$ (for zero mean).

Problem 1: Calculating Unconditional Variance

Statement

A GARCH(1,1) model has estimated parameters:

- $\omega = 0.000002$
- $\alpha = 0.08$
- $\beta = 0.90$

Calculate:

a. Daily unconditional variance
b. Daily unconditional volatility (as percentage)
c. Annualized volatility (assuming 252 trading days)
d. Volatility half-life

Solution to Problem 1

Answers

a. $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta} = \frac{0.000002}{1-0.08-0.90} = \frac{0.000002}{0.02} = 0.0001$

b. $\bar{\sigma} = \sqrt{0.0001} = 0.01 = 1\%$ per day

c. $\sigma_{\text{annual}} = \bar{\sigma} \times \sqrt{252} = 0.01 \times 15.87 = 15.87\%$ per year

d. $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)} = \frac{\ln(0.5)}{\ln(0.98)} = \frac{-0.693}{-0.0202} \approx 34$ days

Interpretation

Volatility of 15.87% per year is typical for a stock index. Half-life of 34 days means a volatility shock is reduced by half after about 7 weeks.

Problem 2: Volatility Forecast

Statement

Using the GARCH(1,1) model from Problem 1:

- $\omega = 0.000002$, $\alpha = 0.08$, $\beta = 0.90$
- At time T: $\varepsilon_T = -0.03$ (3% drop), $\sigma_T^2 = 0.0004$

Calculate volatility forecasts for:

a. σ_{T+1}^2 (one step ahead)
b. σ_{T+5}^2 (five steps ahead)
c. σ_{T+100}^2 (one hundred steps ahead)

Solution to Problem 2

Answers

- **a** $\sigma^2_{T+1} = \omega + \alpha\varepsilon^2_T + \beta\sigma^2_T$

$$= 0.000002 + 0.08 \times (0.03)^2 + 0.90 \times 0.0004 = 0.000434$$

  Volatility: $\sqrt{0.000434} = 2.08\%$

- **b** $\mathbb{E}_T[\sigma^2_{T+5}] = \bar{\sigma}^2 + (0.98)^4(\sigma^2_{T+1} - \bar{\sigma}^2)$

$$= 0.0001 + 0.922 \times (0.000434 - 0.0001) = 0.000408$$

  Volatility: $\sqrt{0.000408} = 2.02\%$

- **c** $\mathbb{E}_T[\sigma^2_{T+100}] = 0.0001 + (0.98)^{99} \times 0.000334 \approx 0.000145$
  Volatility: $\sqrt{0.000145} = 1.20\%$ (close to $\bar{\sigma} = 1\%$)

Problem 3: Value at Risk

Statement

A portfolio of 1,000,000 EUR is invested in stocks with returns modeled by GARCH(1,1).
Tomorrow's volatility forecast: $\sigma_{T+1} = 2\%$ daily.
Assuming normally distributed returns with zero mean, calculate:

a. VaR at 95% (1 day)
b. VaR at 99% (1 day)
c. VaR at 99% (10 days), using "square root of time" rule

Quantiles: $z_{0.05} = 1.645$, $z_{0.01} = 2.326$

Solution to Problem 3

Answers

- (a) VaR 95% (1 day):

$$\text{VaR}_{95\%} = 1.645 \times 0.02 \times 1{,}000{,}000 = 32{,}900 \text{ EUR}$$

- (b) VaR 99% (1 day):

$$\text{VaR}_{99\%} = 2.326 \times 0.02 \times 1{,}000{,}000 = 46{,}520 \text{ EUR}$$

- (c) VaR 99% (10 days):

$$\text{VaR}_{99\%,10d} = \text{VaR}_{99\%,1d} \times \sqrt{10} = 46{,}520 \times 3.162 = 147{,}100 \text{ EUR}$$

Caution

In practice, for Student-t distribution, quantiles are larger (fatter tails)!

Problem 4: Model Identification

Statement

Analyze the following estimation results and identify the model:

Parameter	Estimate	Std. Error
ω	0.0000015	0.0000005
α	0.0550	0.0120
γ	0.0850	0.0180
β	0.9100	0.0150

a. What model is this?
b. Is leverage effect present?
c. What is the impact of negative vs positive shocks?
d. Is the model stationary?

Solution to Problem 4

Answers

a. **GJR-GARCH(1,1,1)** — presence of γ parameter (threshold/asymmetry)

b. **Yes, leverage effect present**: $\gamma = 0.085 > 0$ and significant

c. Impact:
   - Positive shock: impact $= \alpha = 0.055$
   - Negative shock: impact $= \alpha + \gamma = 0.055 + 0.085 = 0.140$
   - Negative shocks have **2.5x greater** impact!

d. Stationarity: $\alpha + \gamma/2 + \beta = 0.055 + 0.0425 + 0.91 = 1.0075$
   On the edge! Almost IGARCH. Very persistent model.

Step 1: Load and Prepare Data

```
import pandas as pd
import numpy as np
import yfinance as yf
from arch import arch_model
from arch.unitroot import ADF

# Download S&P 500 data
data = yf.download('^GSPC', start='2010-01-01', end='2024-01-01')
returns = 100 * data['Adj Close'].pct_change().dropna()

# Check stationarity
adf = ADF(returns)
print(f'ADF statistic: {adf.stat:.4f}')
print(f'p-value: {adf.pvalue:.4f}')
```

Step 2: Test for ARCH Effects

```
from statsmodels.stats.diagnostic import het_arch

# ARCH-LM test on residuals
residuals = returns - returns.mean()
lm_stat, lm_pvalue, f_stat, f_pvalue = het_arch(residuals, nlags=10)

print(f'ARCH-LM statistic: {lm_stat:.4f}')
print(f'p-value: {lm_pvalue:.4f}')

if lm_pvalue < 0.05:
    print('=> ARCH effects present! GARCH modeling justified.')
```

Step 3: Estimate Models

```
# GARCH(1,1) with Student-t distribution
model_garch = arch_model(returns, vol='Garch', p=1, q=1, dist='t')
res_garch = model_garch.fit(disp='off')
print(res_garch.summary())

# GJR-GARCH(1,1,1)
model_gjr = arch_model(returns, vol='Garch', p=1, o=1, q=1, dist='t')
res_gjr = model_gjr.fit(disp='off')

# EGARCH(1,1)
model_egarch = arch_model(returns, vol='EGARCH', p=1, q=1, dist='t')
res_egarch = model_egarch.fit(disp='off')

# Compare AIC
print(f'GARCH AIC: {res_garch.aic:.2f}')
print(f'GJR AIC: {res_gjr.aic:.2f}')
print(f'EGARCH AIC: {res_egarch.aic:.2f}')
```

Step 4: Diagnostics

```
# Standardized residuals
std_resid = res_gjr.std_resid

# Ljung-Box test on squared residuals
from statsmodels.stats.diagnostic import acorr_ljungbox
lb_test = acorr_ljungbox(std_resid**2, lags=10, return_df=True)
print(lb_test)

# Check for remaining ARCH effects
lm_stat2, lm_pval2, _, _ = het_arch(std_resid, nlags=5)
print(f'ARCH-LM residuals: stat={lm_stat2:.2f}, p={lm_pval2:.4f}')

if lm_pval2 > 0.05:
    print('=> No remaining ARCH effects. Model OK!')
```

Step 5: Forecast and VaR

```
# Forecast 10 days ahead
forecasts = res_gjr.forecast(horizon=10)
vol_forecast = np.sqrt(forecasts.variance.values[-1, :])

print('Volatility forecast (%):', vol_forecast)

# Value at Risk 99%
portfolio_value = 1_000_000
VaR_99 = 2.326 * vol_forecast[0] / 100 * portfolio_value
print(f'VaR 99% (1 day): {VaR_99:,.0f} EUR')

# 10-day VaR
VaR_99_10d = VaR_99 * np.sqrt(10)
print(f'VaR 99% (10 days): {VaR_99_10d:,.0f} EUR')
```

Seminar Summary

Key Concepts

- **ARCH**: conditional variance depends on past shocks
- **GARCH**: adds persistence through lagged variance
- **EGARCH/GJR**: capture leverage effect (asymmetry)
- **Stationarity**: $\alpha + \beta < 1$

Important Formulas

- Unconditional variance: $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$
- VaR: $\text{VaR}_\alpha = z_\alpha \cdot \sigma \cdot V$

Practical Tip

Use Student-t distribution to capture fat tails. Verify absence of ARCH effects in residuals!

Homework Exercises

Exercise 1

Download daily returns for BET (BVB index) and estimate a GARCH(1,1) model. Compare persistence ($\alpha + \beta$) with S&P 500.

Exercise 2

For Bitcoin, estimate GARCH, EGARCH, and GJR-GARCH. Is leverage effect present for cryptocurrencies?

Exercise 3

Calculate daily VaR for a portfolio of 100,000 EUR invested in EUR/USD, using GARCH-forecasted volatility.

Exercise 4

Compare GARCH(1,1) volatility forecast with realized volatility (sum of squared returns) for a 20-day period.