



Time Series Analysis and Forecasting

Chapter 7: Cointegration and VECM



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Learning Objectives

By the end of this chapter, you will be able to:

- Understand the problem of spurious regression with non-stationary data
- Test for cointegration using Engle-Granger and Johansen methods
- Estimate Vector Error Correction Models (VECM)
- Interpret error correction mechanisms and adjustment speeds



Outline

Motivation

Spurious Regression

Cointegration Concept

Engle-Granger Method

Johansen Method

VECM Estimation

Practical Considerations

Real-World Examples

Case Study: Interest Rates

Summary

Quiz



Why Cointegration Matters

The Challenge

- Many economic/financial time series are **non-stationary ($I(1)$)**
- GDP, stock prices, exchange rates, interest rates all have unit roots
- Standard regression with $I(1)$ variables \Rightarrow **spurious results**
- Differencing removes non-stationarity but loses **long-run information**

The Solution: Cointegration

Some non-stationary series share a **common stochastic trend**—they move together in the long run.

Nobel Prize 2003

Granger & Engle received the Nobel Prize for “methods for analyzing economic time series with common trends.”



Real-World Applications

Finance

- Pairs Trading:** Cointegrated stocks
- Term Structure:** Interest rates
- Spot-Futures:** Arbitrage

Policy Analysis

- Fiscal:** Spending & taxes
- Monetary:** Rate pass-through
- Labor:** Wages & productivity

Macroeconomics

- Consumption & Income**
- Money & Prices**
- PPP:** Exchange rates



The Spurious Regression Problem

Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk: $Y_t = \alpha + \beta X_t + u_t$ where Y_t and X_t are independent $I(1)$ processes.

Symptoms of Spurious Regression

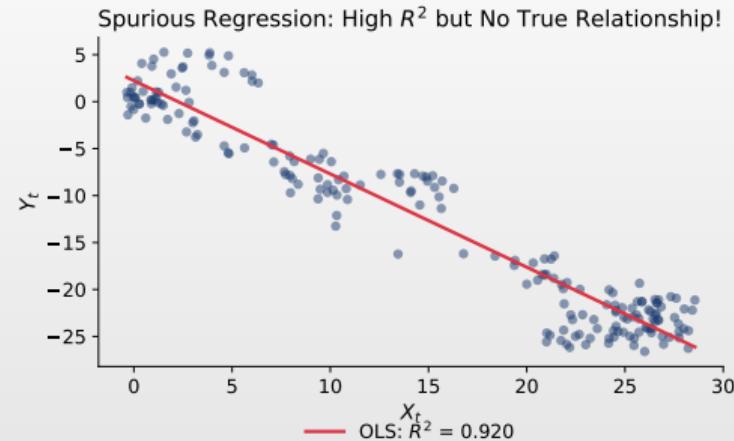
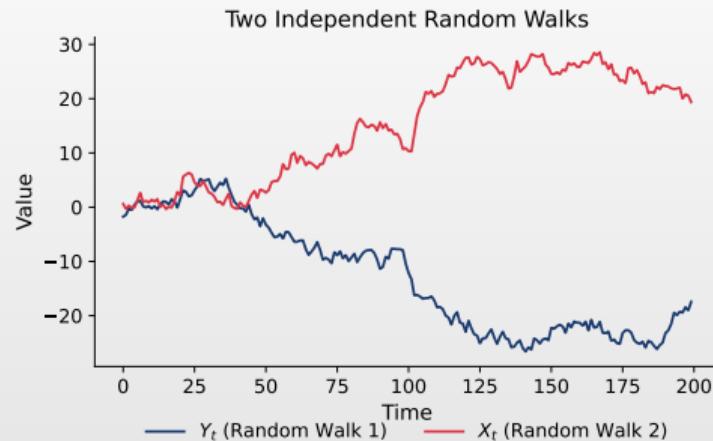
- High R^2 (often > 0.9) even though variables are **unrelated**
- Highly significant t -statistics (reject $H_0 : \beta = 0$)
- Very low Durbin-Watson statistic ($DW \approx 0$)
- Residuals are non-stationary (have unit root)

Rule of Thumb

If $R^2 > DW$, suspect spurious regression!



Spurious Regression: Visual Example



Warning: Two independent random walks show high correlation ($R^2 > 0.8$) by chance! This is why we need cointegration analysis.

Q TSA_ch7_spurious_regression



Spurious Correlations in the Real World

Data Mining Can Produce Meaningless Correlations

With enough variables and long time series, purely coincidental patterns emerge:

- Distance between Neptune and Uranus ↔ SAP SE stock price (2002–2023)
- GMO corn use in South Dakota ↔ Google searches for “i cant even” (2004–2023)
- Two and a Half Men* season ratings ↔ Jet fuel used in Serbia (2006–2015)
- “Its Wednesday my dudes” meme popularity ↔ Boeing stock price (2006–2023)

Lesson

High correlation \neq causation. Non-stationary series with common trends produce high R^2 by construction. Always test for stationarity and cointegration before interpreting regression results!

Explore more examples: tylervigen.com/spurious-correlations



Definition of Cointegration

Definition 1 (Cointegration (Engle & Granger, 1987))

Variables $Y_{1t}, Y_{2t}, \dots, Y_{kt}$ are **cointegrated of order (d, b)** , written $CI(d, b)$, if:

1. All variables are integrated of order d : $Y_{it} \sim I(d)$
2. There exists a linear combination $\beta' Y_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$ that is integrated of order $(d - b)$, where $b > 0$

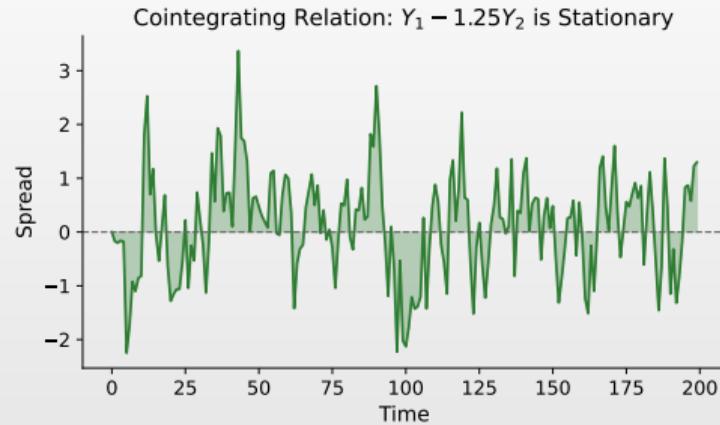
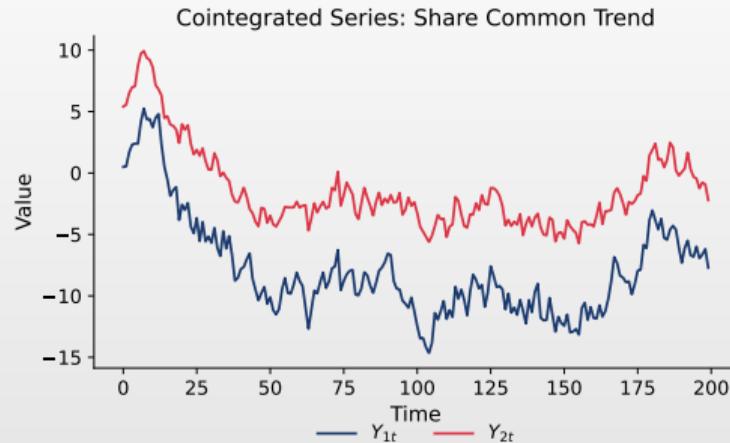
Most Common Case: $CI(1, 1)$

- Variables are $I(1)$ (have unit roots)
- Linear combination is $I(0)$ (stationary)
- Vector $\beta = (\beta_1, \dots, \beta_k)'$ is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized: $\beta_1 = 1$.



Cointegration: Visual Example



Key insight: Both series are $I(1)$ and trend together, but their linear combination (spread) is stationary—this is cointegration!

Q [TSA_ch7_cointegrated_series](#)

Intuition: Common Stochastic Trends

Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**: $Y_{1t} = \gamma_1 \tau_t + S_{1t}$, $Y_{2t} = \gamma_2 \tau_t + S_{2t}$ where τ_t is a common random walk and S_{it} are stationary.

Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

Economic Interpretation

- Cointegration = **long-run equilibrium relationship**
- Variables may deviate in the short run, but are “pulled back”
- The cointegrating vector defines the equilibrium



Cointegrating Rank

How Many Cointegrating Relationships?

For k variables that are $I(1)$:

- Maximum possible cointegrating relationships: $r = k - 1$
- If $r = 0$: No cointegration (variables drift apart)
- If $r = k$: All variables are $I(0)$ (contradiction)

Example: 3 Variables

- $r = 0$: No cointegration
- $r = 1$: One cointegrating relationship
- $r = 2$: Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends = $k - r$



Engle-Granger Two-Step Method

Step 1: Estimate Cointegrating Regression

Run OLS: $Y_t = \alpha + \beta X_t + e_t$. Save residuals: $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

Step 2: Test Residuals for Stationarity

Test if \hat{e}_t is $I(0)$ using ADF: $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$

- $H_0: \rho = 0$ (unit root \Rightarrow no cointegration)
- $H_1: \rho < 0$ (stationary \Rightarrow cointegration)

Important

Use **Engle-Granger critical values**, not standard ADF! (More negative because residuals are estimated)



Engle-Granger Critical Values

Critical Values for Cointegration Test

Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

MacKinnon (1991), $T = 100$

Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on choice of dependent variable
- Small sample bias; cannot test hypotheses on cointegrating vector



Johansen Cointegration Test

Advantages over Engle-Granger

- Tests for **multiple** cointegrating relationships
- Maximum likelihood estimation (more efficient)
- Can test restrictions on cointegrating vectors
- Does not require choosing a dependent variable

Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...



VECM Representation

Vector Error Correction Model

$$\Delta \mathbf{Y}_t = \mathbf{c} + \boldsymbol{\Pi} \mathbf{Y}_{t-1} + \sum_{j=1}^{p-1} \boldsymbol{\Gamma}_j \Delta \mathbf{Y}_{t-j} + \boldsymbol{\varepsilon}_t$$

- $\boldsymbol{\Pi} = \sum_i \mathbf{A}_i - \mathbf{I}$ (long-run impact); $\boldsymbol{\Gamma}_j$ (short-run dynamics)

Key Insight: Rank of $\boldsymbol{\Pi}$

The **rank of $\boldsymbol{\Pi}$** determines cointegration:

- $\text{rank}(\boldsymbol{\Pi}) = 0$: No cointegration (VAR in differences)
- $\text{rank}(\boldsymbol{\Pi}) = k$: All variables are $I(0)$ (VAR in levels)
- $0 < \text{rank}(\boldsymbol{\Pi}) = r < k$: r cointegrating vectors



Derivation: From VAR to VECM

Starting Point: VAR(p) in Levels

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Step 1: Subtract Y_{t-1} from Both Sides

$$Y_t - Y_{t-1} = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} - Y_{t-1} + \varepsilon_t$$

$$\Delta Y_t = (A_1 - I) Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Goal

Rewrite so that all terms are either in **levels** (Y_{t-1}) or **differences** (ΔY_{t-j}).



Derivation: From VAR to VECM (cont.)

Step 2: Add and Subtract Terms Strategically

Add $A_2 Y_{t-1}$ and subtract $A_2 Y_{t-1}$:

$$\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} - A_2(Y_{t-1} - Y_{t-2}) + A_3 Y_{t-3} + \dots + \epsilon_t$$

Continue adding $A_3 Y_{t-1}$, etc., until all lagged **levels** are collected in one term.

Step 3: General Pattern

After algebraic manipulation, we obtain:

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$$

The Key Matrices

Time Series Analysis and Forecasting

$$\Pi = \sum_{i=1}^p A_i - I = -(I - A_1 - A_2 - \dots - A_p)$$



Derivation: Verifying the Γ_j Formula

Example: VAR(2)

Starting from: $Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$

Subtract Y_{t-1} :

$$\Delta Y_t = (A_1 - I)Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$$

Add and subtract $A_2 Y_{t-1}$:

$$\begin{aligned}\Delta Y_t &= (A_1 + A_2 - I)Y_{t-1} + A_2(Y_{t-2} - Y_{t-1}) + \varepsilon_t \\ \Delta Y_t &= \underbrace{(A_1 + A_2 - I)}_{\Pi} Y_{t-1} - \underbrace{A_2}_{\Gamma_1} \Delta Y_{t-1} + \varepsilon_t\end{aligned}$$

Verification

For VAR(2): $\Pi = A_1 + A_2 - I$ and $\Gamma_1 = -A_2$

Using our formula: $\Gamma_1 = -\sum_{i=2}^2 A_i = -A_2 \quad \checkmark$



Economic Interpretation of Error Correction

The VECM with Cointegration

When $\text{rank}(\boldsymbol{\Pi}) = r$, we write $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$:

$$\Delta Y_t = \boldsymbol{\alpha} \underbrace{(\boldsymbol{\beta}' Y_{t-1})}_{\text{equilibrium error}} + \sum_{j=1}^{p-1} \boldsymbol{\Gamma}_j \Delta Y_{t-j} + \boldsymbol{\varepsilon}_t$$

Economic Interpretation

- $\boldsymbol{\beta}' Y_{t-1}$ = equilibrium error:** deviation from long-run relationship
- $\boldsymbol{\alpha}$ = adjustment speeds:** how fast variables correct deviations
- $\boldsymbol{\Gamma}_j$ = short-run dynamics:** transitory effects

Error Correction Mechanism

If $\boldsymbol{\beta}' Y_{t-1} > 0$ (above equilibrium) and $\alpha_i < 0$, then ΔY_{it} decreases.

The system self corrects toward equilibrium!



Decomposition of Π

When $\text{rank}(\Pi) = r < k$

$\Pi = \alpha\beta'$ where β ($k \times r$) = cointegrating vectors, α ($k \times r$) = adjustment coefficients

Interpretation

- $\beta'Y_{t-1}$ = deviations from equilibrium (error correction terms)
- α = speed of adjustment; rows show each variable's response

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta' Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$$



Johansen Test Statistics

Two Test Statistics

Based on eigenvalues $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$ of a certain matrix:

Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests $H_0: \text{rank} \leq r$ vs $H_1: \text{rank} > r$

Maximum Eigenvalue Test:

$$\lambda_{\max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

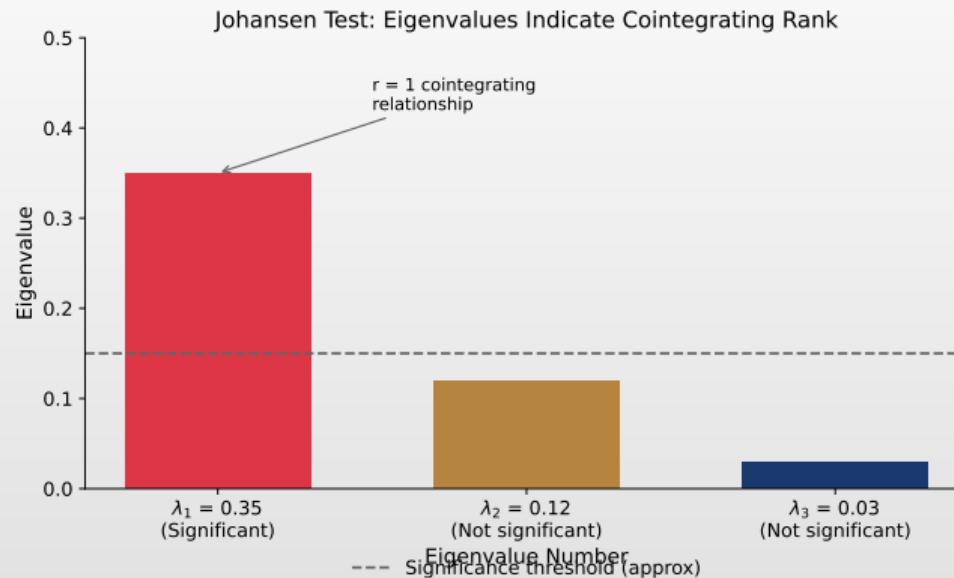
Tests $H_0: \text{rank} = r$ vs $H_1: \text{rank} = r + 1$

Critical values from Johansen & Juselius (1990), depend on:

- Number of variables k
- Deterministic components (constant, trend)



Johansen Test: Visual Interpretation



Significant eigenvalues (above threshold) indicate cointegrating relationships. First eigenvalue significant $\Rightarrow r = 1$.

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Testing Procedure

Sequential Testing (Trace Test)

1. Test $H_0: r = 0$. If rejected \Rightarrow continue
2. Test $H_0: r \leq 1$. If not rejected $\Rightarrow r = 1$
3. Continue until H_0 is not rejected

Deterministic Components

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- Constant in both** (most common)
- Constant + trend in cointegrating relation



VECM Structure

Full VECM Specification

For $k = 2$ variables with $r = 1$ cointegrating relation:

$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

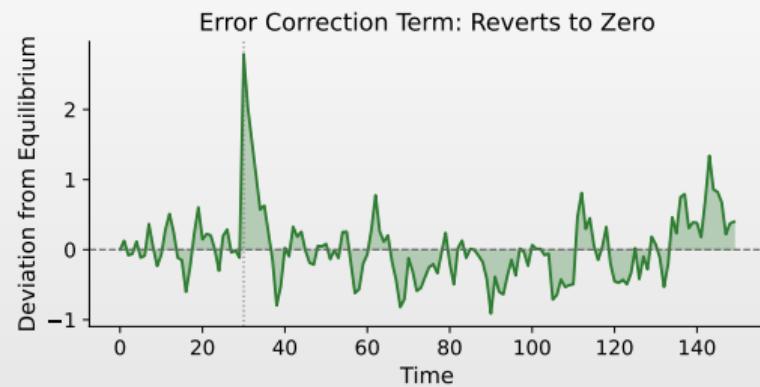
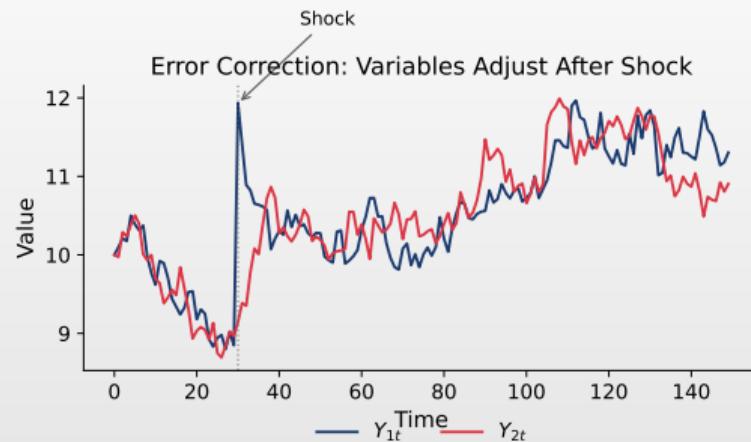
$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

Components

- ◻ $(Y_{1,t-1} - \beta Y_{2,t-1})$ = error correction term (deviation from equilibrium)
- ◻ α_1, α_2 = adjustment speeds (should have opposite signs)
- ◻ γ_{ij} = short-run dynamics
- ◻ ε_{it} = innovations



Error Correction Mechanism: Visual



Error correction in action: When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment.

[TSA_ch7_error_correction](#)

Interpreting Adjustment Coefficients

The α Coefficients

If the cointegrating relation is $Y_1 - \beta Y_2 = 0$ (equilibrium):

- $\alpha_1 < 0$: Y_1 adjusts downward when above equilibrium
- $\alpha_2 > 0$: Y_2 adjusts upward when Y_1 is above equilibrium

Weak Exogeneity

If $\alpha_i = 0$, variable Y_i does **not** respond to disequilibrium.

- Y_i is **weakly exogenous** for the long-run parameters
- The other variable does all the adjusting
- Can simplify estimation (single-equation approach)

Test weak exogeneity: $H_0 : \alpha_i = 0$ using likelihood ratio test.



VECM vs VAR in Differences

When Variables are Cointegrated

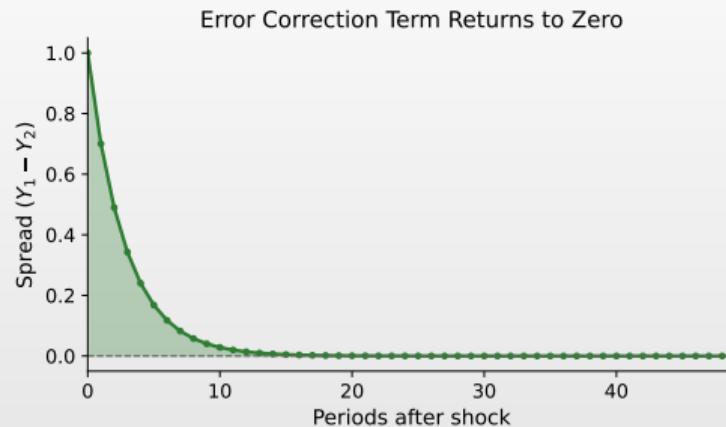
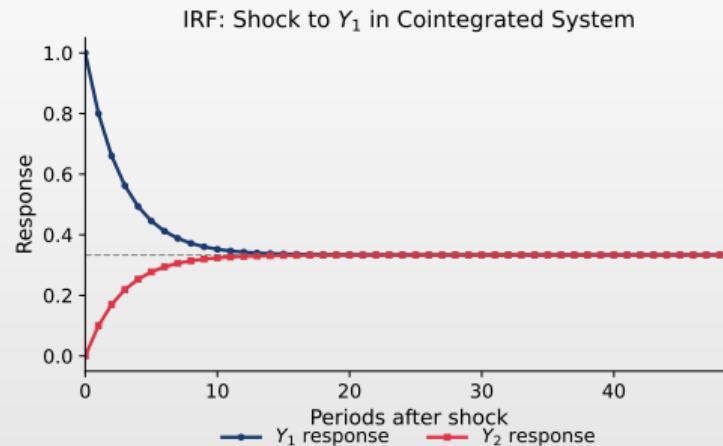
	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!



VECM Impulse Response Functions



IRF interpretation: In a cointegrated system, shocks have **permanent effects** on levels but the system returns to equilibrium. Unlike stationary VAR, effects don't decay to zero—they converge to a new long-run value.

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Practical Workflow

Step-by-Step Procedure

1. **Unit Root Tests:** Verify all variables are $I(1)$
 - ▶ ADF, KPSS on levels and first differences
2. **Lag Length Selection:** Choose p for VAR in levels
 - ▶ Use AIC, BIC, or sequential LR tests
3. **Cointegration Test:** Johansen trace/max-eigenvalue tests
 - ▶ Determine cointegrating rank r
4. **Estimate VECM:** If $0 < r < k$
 - ▶ Estimate α, β, Γ_j
5. **Diagnostics:** Check residuals for autocorrelation, normality
6. **Analysis:** IRF, FEVD, hypothesis tests



Common Pitfalls

Things to Watch Out For

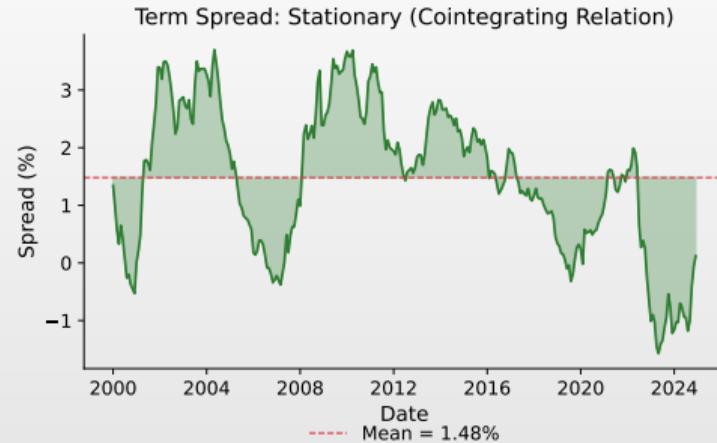
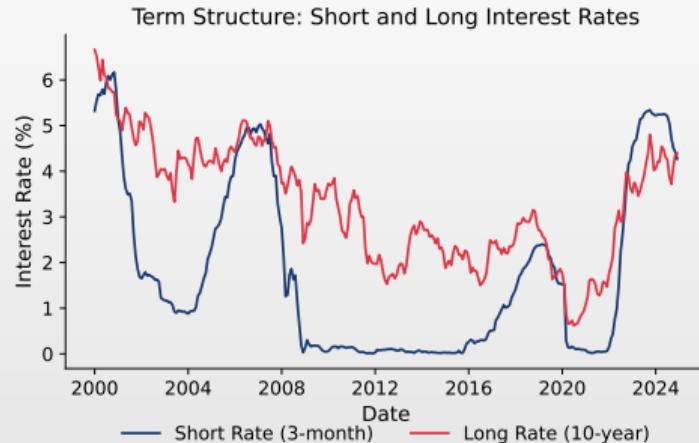
- Structural breaks:** Cause spurious unit roots or cointegration
- Near-unit-root:** Tests have low power
- Lag selection:** Too many/few lags bias results
- Small samples:** Johansen test oversized

Recommendation

Always check: residual diagnostics, stability of cointegrating relationship, sensitivity to specification



Example 1: Term Structure of Interest Rates

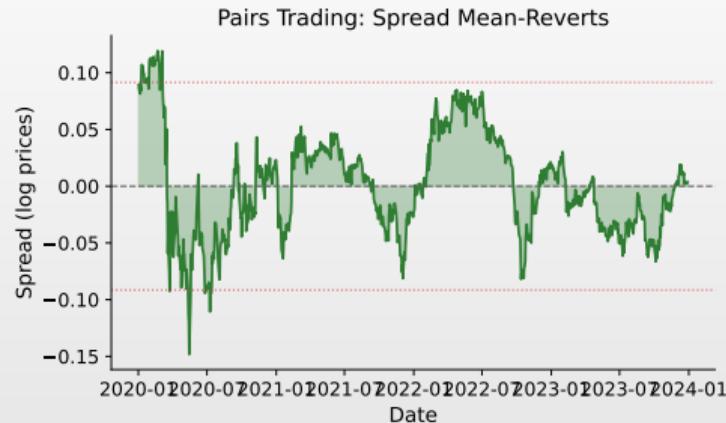


Expectations Hypothesis: Short and long rates share common trend. The spread (term premium) is stationary—evidence of cointegration!

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Example 2: Pairs Trading in Finance

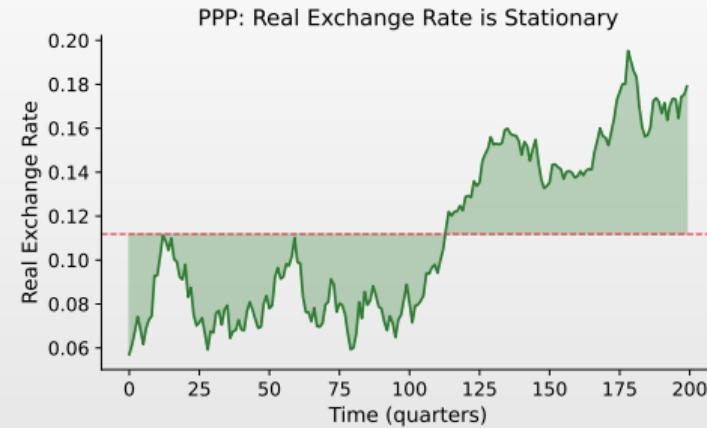


Strategy: Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When spread deviates from mean, trade expecting mean reversion. Sell spread when high, buy when low.

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Example 3: Purchasing Power Parity (PPP)



PPP Theory: $e_t = p_t - p_t^*$ (log exchange rate equals price differential). Real exchange rate should be stationary in the long run.

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Case Study: Cointegration of Interest Rates

Research Question

Are short-term and long-term interest rates cointegrated? Does the expectations hypothesis of the term structure hold?

Data

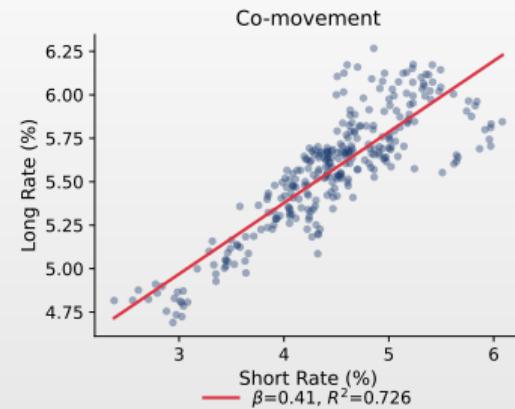
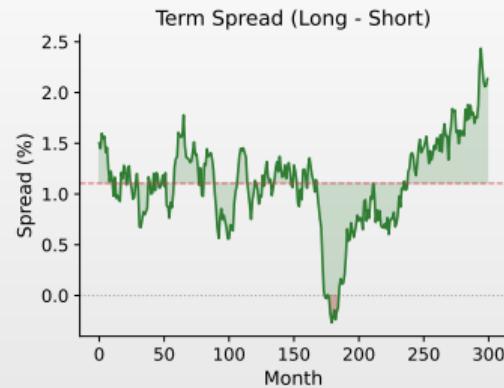
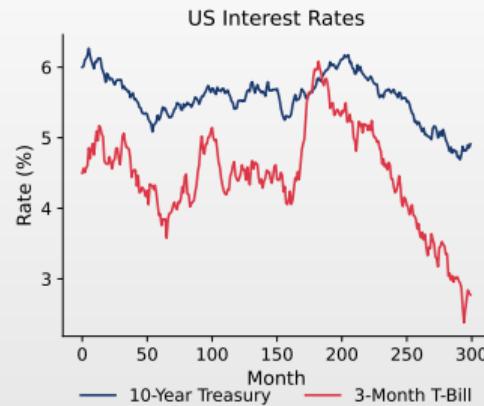
- US Monthly Data (1962-2023)
- 3-Month Treasury Bill Rate
- 10-Year Treasury Bond Yield
- Source: FRED Database

Methodology

- Unit root tests (ADF, PP)
- Engle-Granger cointegration test
- Johansen procedure
- VECM estimation
- Impulse response analysis



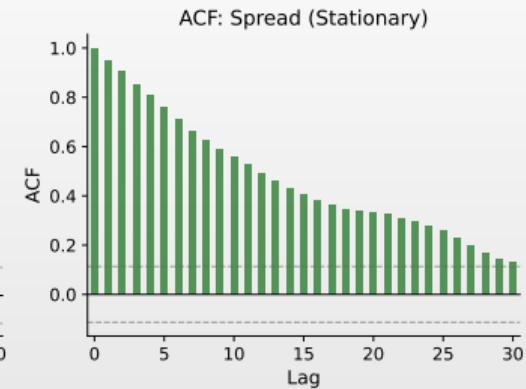
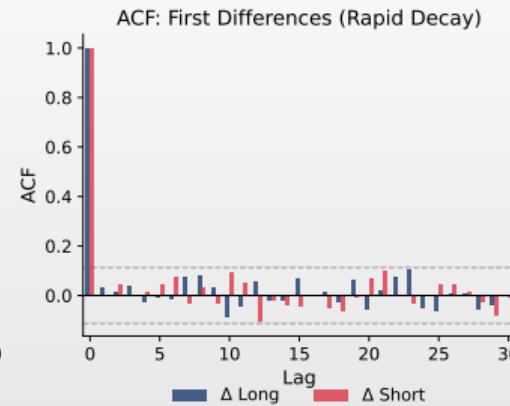
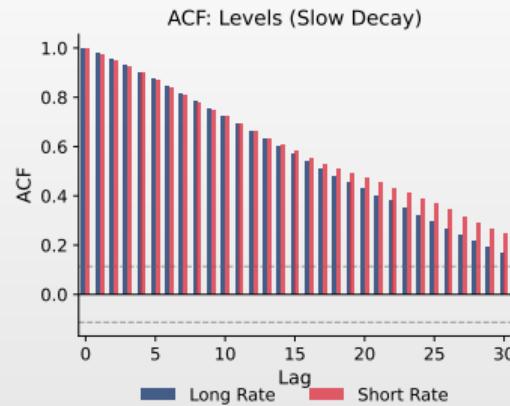
Step 1: Data Visualization



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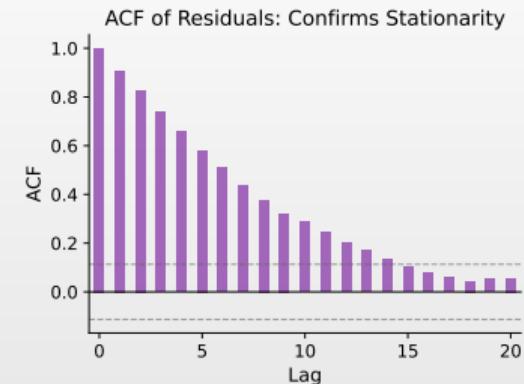
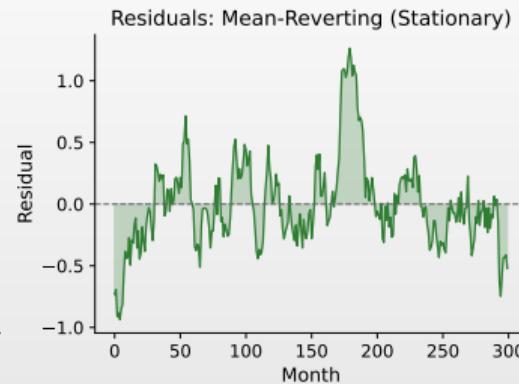
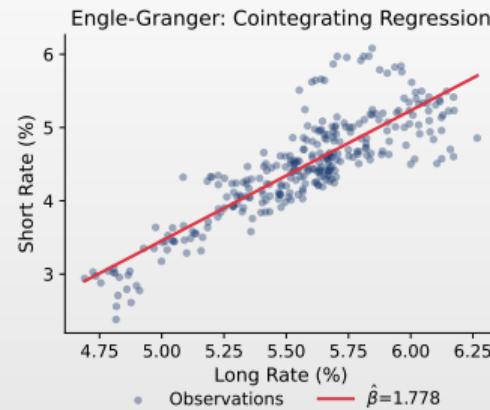


Step 2: Unit Root Tests



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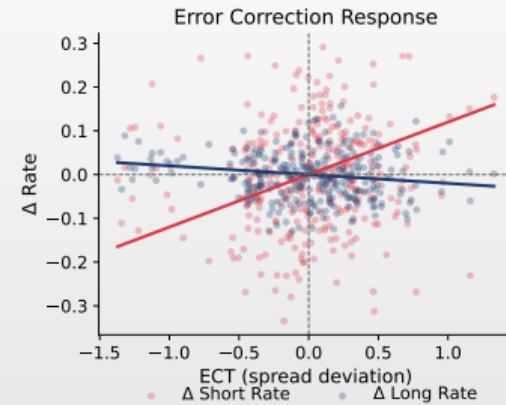
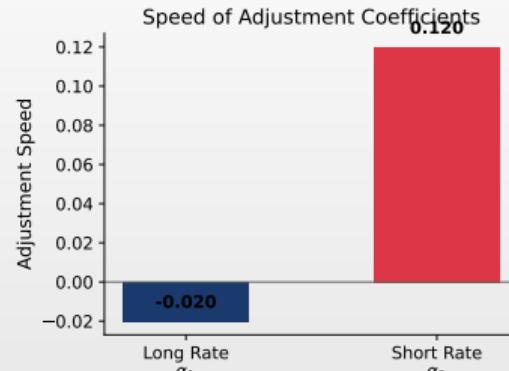
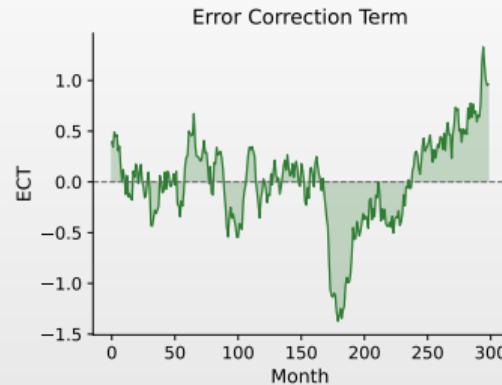
Step 3: Engle-Granger Cointegration Test



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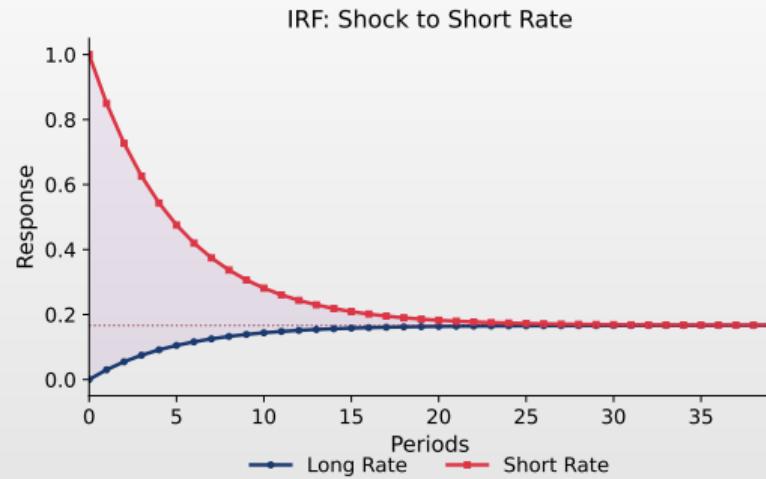
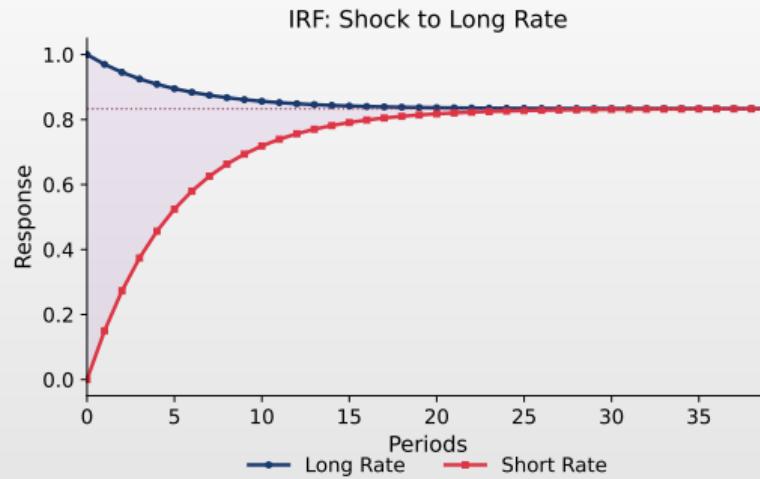
Step 4: VECM Estimation



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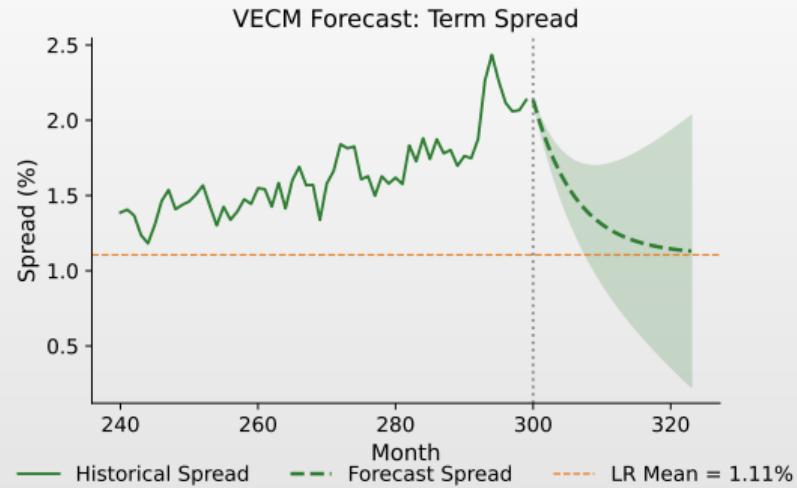
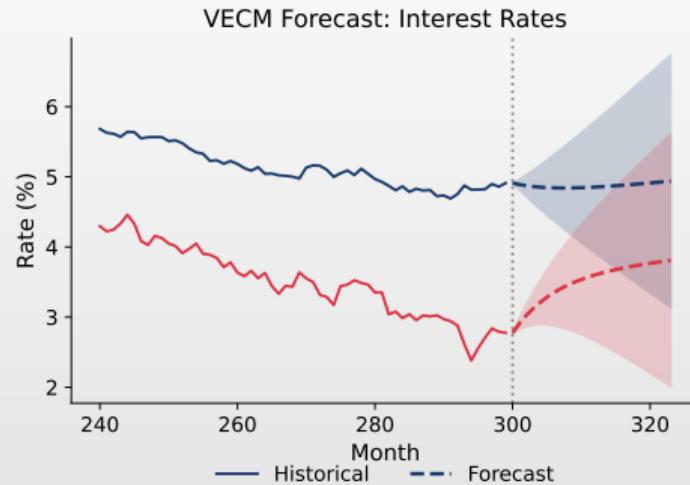
Step 5: Impulse Response Functions



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Step 6: Forecasting



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Key Takeaways

Main Concepts

- Cointegration:** $I(1)$ variables with stationary linear combination
- Spurious regression:** High R^2 with unrelated $I(1)$ variables
- VECM:** VAR with error correction for cointegrated systems

Testing Methods

- Engle-Granger:** Simple, one vector only
- Johansen:** Multiple vectors, MLE-based

Remember

Tests have low power in small samples. Theory should guide specification.



What's Next?

Extensions and Related Topics

- Structural VECM:** Identifying structural shocks
- Threshold cointegration:** Nonlinear adjustment
- Panel cointegration:** Multiple cross-sections
- Fractional cointegration:** Long memory
- Time-varying cointegration:** Regime changes

Questions?



Quick Quiz

1. What does it mean for two $I(1)$ variables to be cointegrated?
2. What is the “spurious regression” problem?
3. In a VECM, what do the α coefficients represent?
4. What is the main advantage of Johansen over Engle-Granger?
5. If $\alpha_i = 0$ for variable Y_i , what does this imply?



Quiz Answers

1. **Cointegration:** A linear combination of the variables is $I(0)$ (stationary). They share a common stochastic trend.
2. **Spurious regression:** Regressing one $I(1)$ variable on another unrelated $I(1)$ variable gives high R^2 and significant coefficients even though there's no true relationship.
3. α **coefficients:** Speed of adjustment—how quickly each variable responds to deviations from long-run equilibrium.
4. **Johansen advantage:** Can test for multiple cointegrating relationships, uses MLE (more efficient), doesn't require choosing dependent variable.
5. $\alpha_i = 0$: Variable Y_i is weakly exogenous—it doesn't respond to disequilibrium. Other variables do all the adjusting.



Online Resources and Code

- Quantlet:** <https://quantlet.com> → Code repository for statistics
- Quantinar:** <https://quantinar.com> → Learning platform for quantitative methods
- GitHub TSA_ch7:** https://github.com/QuantLet/TSA/tree/main/TSA_ch7



Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>

