



Analiza și Prognoză Seriilor de Timp

# Capitolul 7: Cointegrare & VECM

Relații de Echilibru pe Termen Lung



# Cuprins

- 1 Motivație
- 2 Regresia Falsă
- 3 Conceptul de Cointegrare
- 4 Metoda Engle-Granger
- 5 Metoda Johansen
- 6 Estimarea VECM
- 7 Considerații Practice
- 8 Exemple Practice

La finalul acestui capitol, veți fi capabili să:

1. Înțelegeți conceptul de **cointegrare** și relații de echilibru pe termen lung
2. Recunoașteți și evitați problemă **regresiei false**
3. Aplicați metoda **Engle-Granger** în doi pași
4. Efectuați testul **Johansen** pentru cointegrare multiplă
5. Eștimați și interpretați modele **VECM**
6. Analizați viteza de ajustare și vectori de cointegrare
7. Implementați analiza de cointegrare în **Python**

# De ce contează cointegrarea?

## Provocarea

- Multe serii de timp economice/financiare sunt **nestaționare** ( $I(1)$ )
- PIB, prețuri acțiuni, cursuri valutare, rate ale dobânzii au rădăcini unitare
- Regresia standard cu variabile  $I(1) \Rightarrow$  **rezultate false**
- Diferențierea elimină nestaționaritatea dar pierde **informația pe termen lung**

## Soluția: Cointegrarea

Unele serii nestaționare au un **trend stochastic comun**—se mișcă împreună pe termen lung. Această relație pe termen lung poate fi modelată!

## Premiul Nobel 2003

Clive Granger a primit Premiul Nobel în Economie (împreună cu Robert Engle) pentru dezvoltarea analizei de cointegrare—“metode pentru analiza seriilor de timp economice cu tendințe comune.”

## Finanțe

- **Pairs Trading:** Tranzacționarea spread-ului între acțiuni cointegrate
- **Structura pe Termene:** Rate dobânzi pe termen scurt și lung
- **Spot-Futures:** Relații de arbitraj

## Macroeconomie

- **Consum și Venit:** Ipoteza venitului permanent
- **Bani și Prețuri:** Teoria cantitativă a banilor
- **PPP:** Cursuri valutare și niveluri de prețuri

## Analiza Politicilor

- **Politica Fiscală:** Cheltuieli guvernamentale și venituri fiscale
- **Politica Monetară:** Transmiterea ratelor dobânzii
- **Piața Muncii:** Salarii și productivitate

# The Spurious Regression Problem

## Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk:

$$Y_t = \alpha + \beta X_t + u_t$$

where  $Y_t$  and  $X_t$  are independent  $I(1)$  processes.

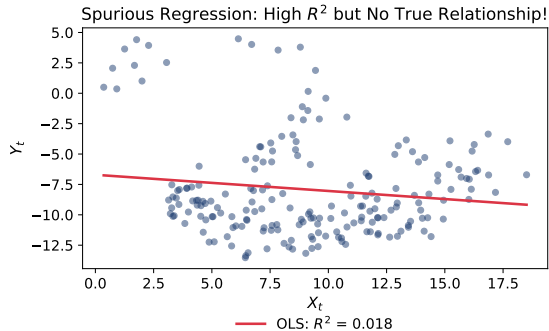
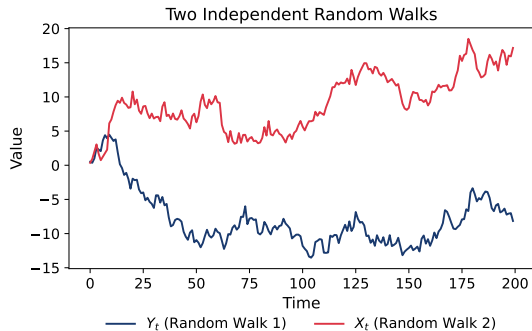
## Symptoms of Spurious Regression

- High  $R^2$  (often  $> 0.9$ ) even though variables are **unrelated**!
- Highly significant  $t$ -statistics (reject  $H_0 : \beta = 0$ )
- Very low Durbin-Watson statistic ( $DW \approx 0$ )
- Residuals are non-stationary (have unit root)

## Rule of Thumb (Granger)

If  $R^2 > DW$ , suspect spurious regression!

# Spurious Regression: Visual Example



**Warning:** Two completely independent random walks show high correlation ( $R^2 > 0.8$ ) purely by chance! This is why we need cointegration analysis.

# Definition of Cointegration

## Definiție 1 (Cointegration (Engle & Granger, 1987))

Variables  $Y_{1t}, Y_{2t}, \dots, Y_{kt}$  are **cointegrated of order  $(d, b)$** , written  $CI(d, b)$ , if:

- ❶ All variables are integrated of order  $d$ :  $Y_{it} \sim I(d)$
- ❷ There exists a linear combination  $'_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$  that is integrated of order  $(d - b)$ , where  $b > 0$

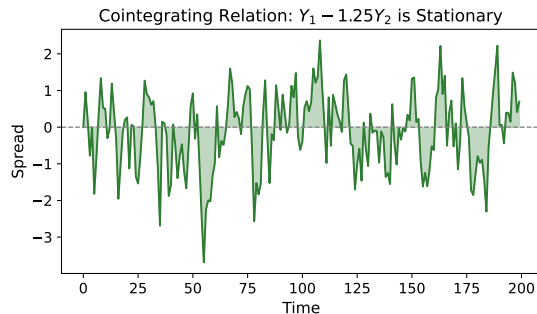
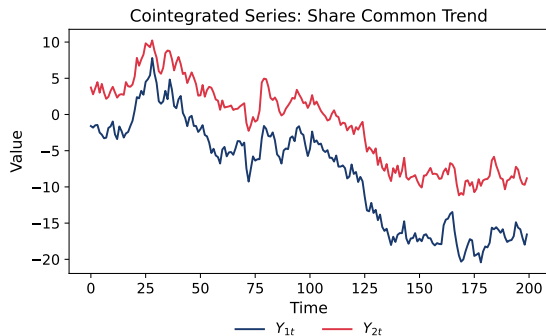
## Most Common Case: $CI(1, 1)$

- Variables are  $I(1)$  (have unit roots)
- Linear combination is  $I(0)$  (stationary)
- Vector  $= (\beta_1, \dots, \beta_k)'$  is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized:  $\beta_1 = 1$ .



## Cointegration: Visual Example



**Key insight:** Both series are  $I(1)$  and trend together, but their linear combination (spread) is stationary—this is cointegration!

# Intuition: Common Stochastic Trends

## Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**:

$$Y_{1t} = \gamma_1 \tau_t + S_{1t}, \quad Y_{2t} = \gamma_2 \tau_t + S_{2t}$$

where  $\tau_t$  is a common random walk and  $S_{it}$  are stationary components.

## Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

## Economic Interpretation

- Cointegration represents a **long-run equilibrium relationship**
- Variables may deviate in the short run
- But they are “pulled back” to equilibrium over time
- The cointegrating vector defines the equilibrium

# Cointegrating Rank

## How Many Cointegrating Relationships?

For  $k$  variables that are  $I(1)$ :

- Maximum possible cointegrating relationships:  $r = k - 1$
- If  $r = 0$ : No cointegration (variables drift apart)
- If  $r = k$ : All variables are  $I(0)$  (contradiction)

## Example: 3 Variables

- $r = 0$ : No cointegration
- $r = 1$ : One cointegrating relationship
- $r = 2$ : Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends =  $k - r$

# Engle-Granger Two-Step Method

## Step 1: Estimate Cointegrating Regression

Run OLS regression (assuming  $Y_t$  is the dependent variable):

$$Y_t = \alpha + \beta X_t + e_t$$

Save the residuals:  $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta}X_t$

## Step 2: Test Residuals for Stationarity

Test if  $\hat{e}_t$  is  $I(0)$  using ADF test:

$$\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$$

- $H_0: \rho = 0$  (residuals have unit root  $\Rightarrow$  no cointegration)
- $H_1: \rho < 0$  (residuals are stationary  $\Rightarrow$  cointegration)

Important

# Engle-Granger Critical Values

## Critical Values for Cointegration Test

Number of Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

Based on MacKinnon (1991) response surface estimates,  $T = 100$

## Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on which variable is chosen as dependent
- Small sample bias in estimated cointegrating vector
- Cannot test hypotheses on the cointegrating vector

# Johansen Cointegration Test

## Advantages over Engle-Granger

- Tests for **multiple** cointegrating relationships
- Maximum likelihood estimation (more efficient)
- Can test restrictions on cointegrating vectors
- Does not require choosing a dependent variable

## Starting Point: VAR in Levels

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t$$

Rewrite in **Vector Error Correction** form...

## Vector Error Correction Model

$$\Delta_t = \alpha + \beta_{t-1} + \sum_{j=1}^{p-1} \gamma_j \Delta_{t-j} + \epsilon_t$$

where:

- $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_p - I$  (long-run impact matrix)
- $\gamma_j = -(\beta_{j+1} + \dots + \beta_p)$  (short-run dynamics)

## Key Insight: Rank of

The **rank of** determines cointegration:

- $\text{rank}() = 0$ : No cointegration (VAR in differences)
- $\text{rank}() = k$ : All variables are  $I(0)$  (VAR in levels)
- $0 < \text{rank}() = r < k$ : Cointegration with  $r$  cointegrating vectors

## Decomposition of

When  $\text{rank}() = r < k$

The matrix can be decomposed as:

$$= \alpha \beta'$$

where:

- is  $k \times r$  matrix of **cointegrating vectors**
- is  $k \times r$  matrix of **adjustment coefficients**

## Interpretation

- $\epsilon'_{t-1}$  = deviations from long-run equilibrium (error correction terms)
- $\alpha$  = speed of adjustment to equilibrium
- Each row of  $\alpha$  shows how each variable responds to disequilibrium

$$\text{VECM: } \Delta_t = c + (\alpha'_{t-1}) + \sum_{j=1}^{p-1} \beta_j \Delta_{t-j} + \epsilon_t$$



## Two Test Statistics

Based on eigenvalues  $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$  of a certain matrix:

**Trace Test:**

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests  $H_0: \text{rank} \leq r$  vs  $H_1: \text{rank} > r$

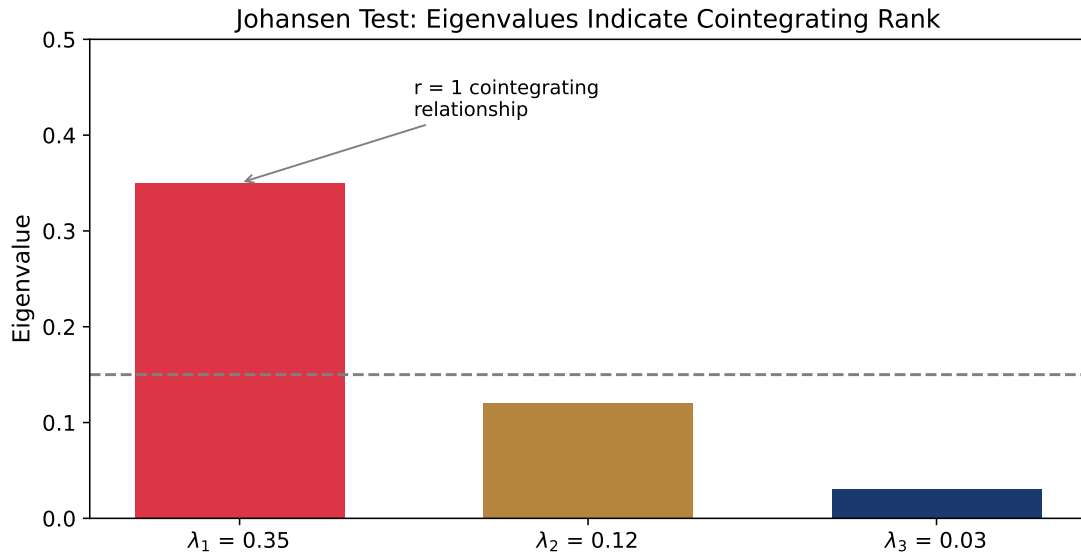
**Maximum Eigenvalue Test:**

$$\lambda_{\text{max}}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

Tests  $H_0: \text{rank} = r$  vs  $H_1: \text{rank} = r+1$

Critical values from Johansen & Juselius (1990), depend on:

- Number of variables  $k$
- Deterministic components (constant, trend)



## Sequential Testing (Trace Test)

- ❶ Test  $H_0: r = 0$  vs  $H_1: r > 0$ 
  - If not rejected: No cointegration. Stop.
  - If rejected: At least one cointegrating vector. Continue.
- ❷ Test  $H_0: r \leq 1$  vs  $H_1: r > 1$ 
  - If not rejected:  $r = 1$ . Stop.
  - If rejected: At least two cointegrating vectors. Continue.
- ❸ Continue until  $H_0$  is not rejected...

## Deterministic Components

Choose specification carefully:

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- Constant in both (most common)
- Constant + trend in cointegrating relation
- Constant + trend in both

## Full VECM Specification

For  $k = 2$  variables with  $r = 1$  cointegrating relation:

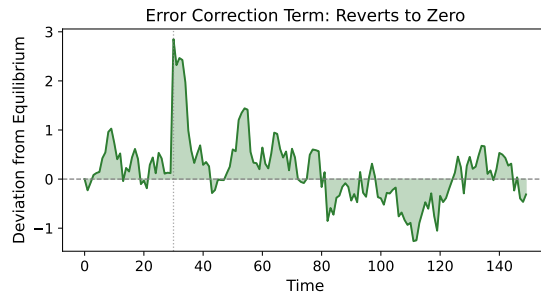
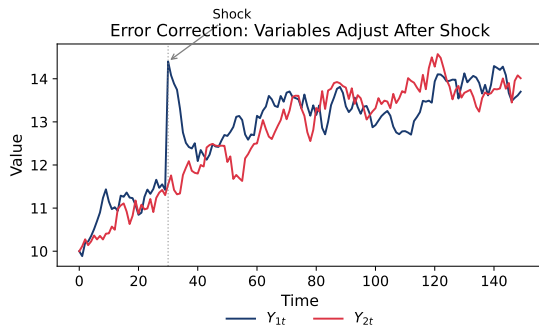
$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

## Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$  = error correction term (deviation from equilibrium)
- $\alpha_1, \alpha_2$  = adjustment speeds (should have opposite signs)
- $\gamma_{ij}$  = short-run dynamics
- $\varepsilon_{it}$  = innovations

# Error Correction Mechanism: Visual



**Error correction in action:** When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment.

# Interpreting Adjustment Coefficients

## The $\alpha$ Coefficients

If the cointegrating relation is  $Y_1 - \beta Y_2 = 0$  (equilibrium):

- $\alpha_1 < 0$ :  $Y_1$  adjusts downward when above equilibrium
- $\alpha_2 > 0$ :  $Y_2$  adjusts upward when  $Y_1$  is above equilibrium

## Weak Exogeneity

If  $\alpha_i = 0$ , variable  $Y_i$  does **not** respond to disequilibrium.

- $Y_i$  is **weakly exogenous** for the long-run parameters
- The other variable does all the adjusting
- Can simplify estimation (single-equation approach)

Test weak exogeneity:  $H_0 : \alpha_i = 0$  using likelihood ratio test.

## VECM vs VAR in Differences

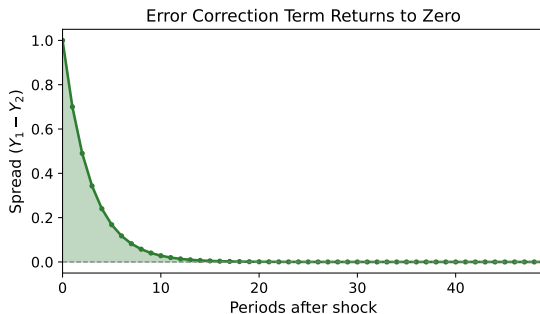
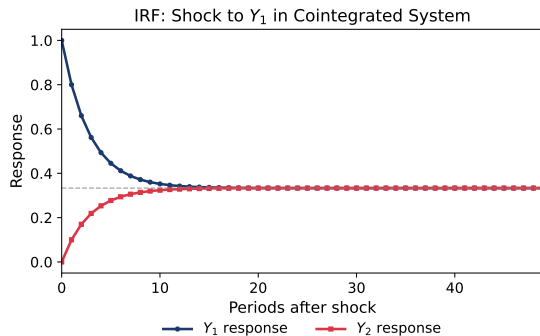
### When Variables are Cointegrated

	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

### Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

# VECM Impulse Response Functions



**IRF interpretation:** In a cointegrated system, shocks have **permanent effects** on levels but the system returns to equilibrium. Unlike stationary VAR, effects don't decay to zero—they converge to a new long-run value.



## Step-by-Step Procedure

- ➊ **Unit Root Tests:** Verify all variables are  $I(1)$ 
  - ADF, KPSS on levels and first differences
- ➋ **Lag Length Selection:** Choose  $p$  for VAR in levels
  - Use AIC, BIC, or sequential LR tests
- ➌ **Cointegration Test:** Johansen trace/max-eigenvalue tests
  - Determine cointegrating rank  $r$
- ➍ **Estimate VECM:** If  $0 < r < k$ 
  - Estimate  $\alpha, \beta, \gamma$
- ➎ **Diagnostics:** Check residuals for autocorrelation, normality
- ➏ **Analysis:** IRF, FEVD, hypothesis tests

## Things to Watch Out For

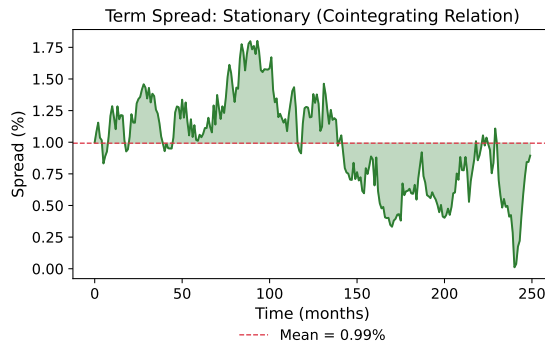
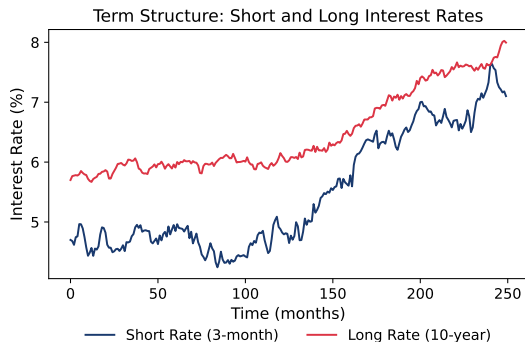
- **Structural breaks:** Can cause spurious unit roots or cointegration
- **Near-unit-root processes:** Tests have low power
- **Too many lags:** Over-parameterization, loss of efficiency
- **Too few lags:** Residual autocorrelation, biased estimates
- **Wrong deterministic specification:** Affects critical values
- **Small samples:** Johansen test oversized in small samples

## Recommendation

Always check:

- Residual diagnostics (Portmanteau test, normality)
- Stability of estimated cointegrating relationship over time
- Sensitivity to lag length and deterministic specification

## Example 1: Term Structure of Interest Rates



**Expectations Hypothesis:** Short and long rates share common trend. The spread (term premium) is stationary—evidence of cointegration!

## Expectations Hypothesis of Term Structure

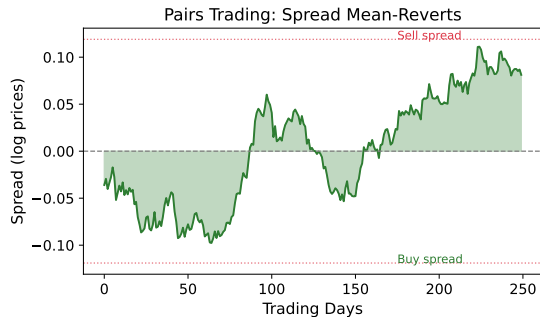
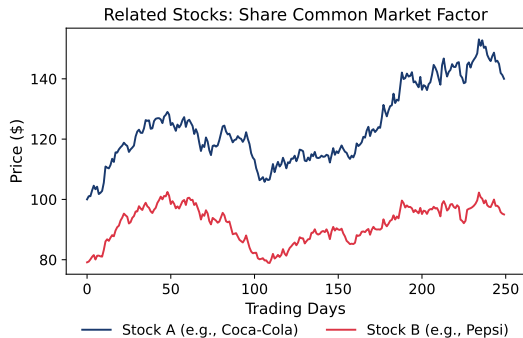
$$R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{term premium}$$

If term premium is constant, short rate  $r_t$  and long rate  $R_t$  should be cointegrated with vector  $(1, -1)$ .

## Empirical Findings

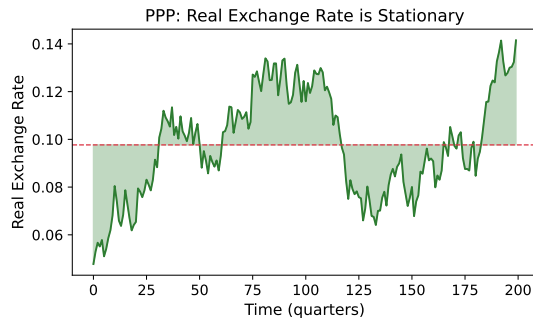
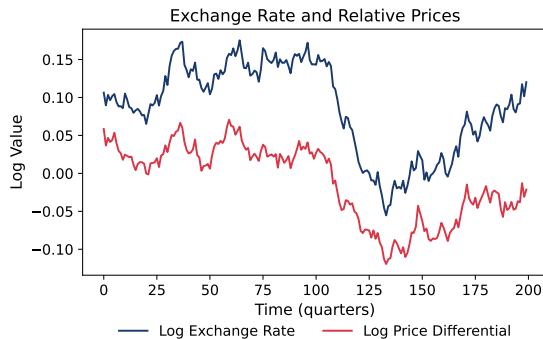
- 1 Both rates are  $I(1)$  (unit root tests)
- 2 One cointegrating relationship (Johansen test)
- 3 Cointegrating vector  $\approx (1, -1)$ : spread is stationary
- 4 Short rate adjusts to disequilibrium (long rate is weakly exogenous)

## Example 2: Pairs Trading in Finance



**Strategy:** Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When spread deviates from mean, trade expecting mean reversion. Sell spread when high, buy when low.

### Example 3: Purchasing Power Parity (PPP)



**PPP Theory:**  $e_t = p_t - p_t^*$  (log exchange rate equals price differential). Real exchange rate should be stationary in the long run.

## Typical Findings

- Both rates are  $I(1)$
- One cointegrating relationship found
- Cointegrating vector close to  $(1, -1)$ : spread is stationary
- Short rate adjusts to long rate (not vice versa)

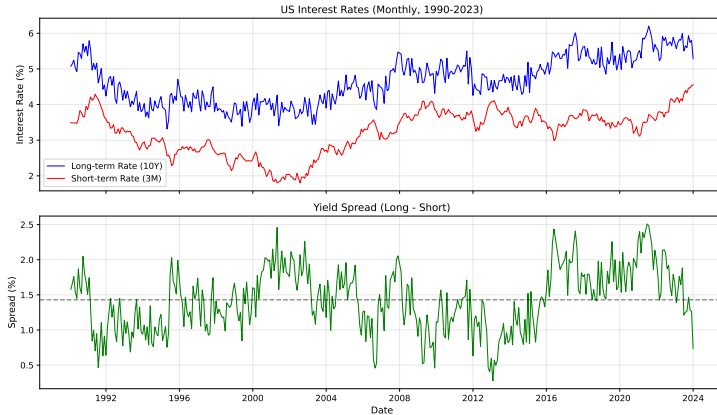
## VECM Equations (stylized)

$$\Delta r_t = 0.02 - 0.15(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{1t}$$

$$\Delta R_t = 0.01 - 0.02(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{2t}$$

- Short rate adjusts faster ( $\alpha_1 = -0.15$ )
- Long rate nearly weakly exogenous ( $\alpha_2 \approx 0$ )

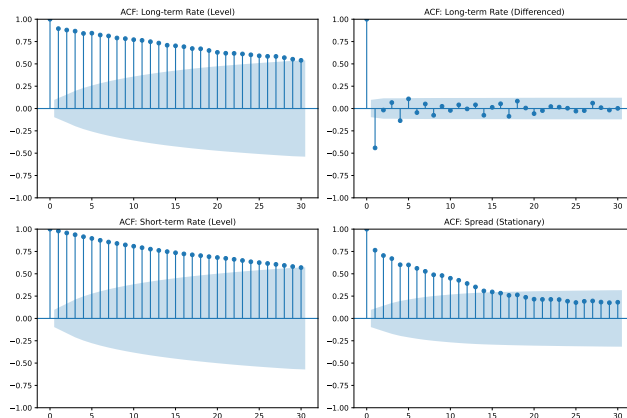
# Studiu de caz: Cointegrarea ratelor dobânzii



- Rate dobândă SUA: pe termen lung (10 ani) și scurt (3 luni)
- Ambele serii sunt nestaționare ( $I(1)$ ), dar spread-ul pare staționar
- Curba randamentelor inversată semnalează recesiuni

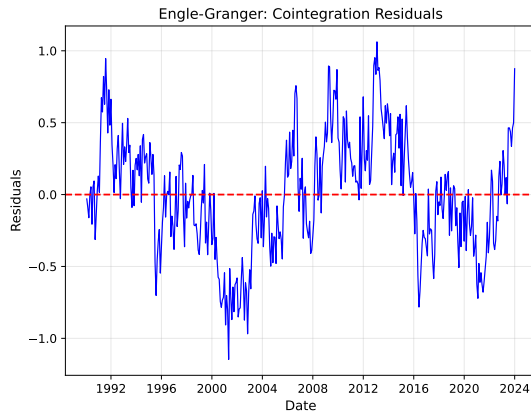


## Pasul 1: Teste de rădăcină unitară



- ACF ratelor în nivel: descreștere lentă  $\Rightarrow$  nestaționaritate
- ACF după diferențiere: scădere rapidă  $\Rightarrow$   $I(1)$
- ACF spread: staționar — posibilă cointegrare!

## Pasul 2: Testul Engle-Granger de cointegrare



### Engle-Granger Test Results

#### ADF Test on Residuals:

Test Statistic: -2.882  
p-value: 0.0475

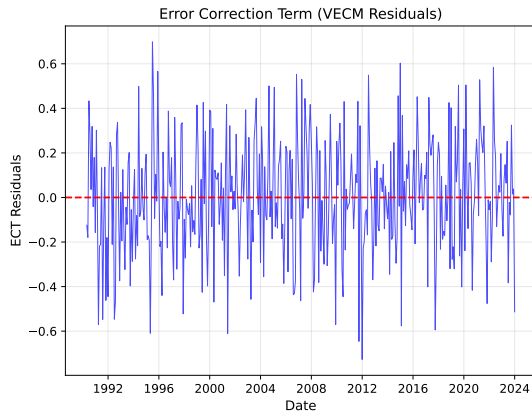
#### Critical Values:

1%: -3.447  
5%: -2.869  
10%: -2.571

Conclusion:  
Cointegrated

- Regresie:  $\text{Rata scurtă} = \alpha + \beta \times \text{Rata lungă} + \varepsilon_t$
- Test ADF pe reziduuri: respingem ipoteza de rădăcină unitară
- Concluzie: Seriile sunt cointegrate — există relație de echilibru pe termen lung

## Pasul 3: Estimare VECM



### VECM Parameters

#### VECM(2) Estimation Results:

Cointegrating Vector:  
 $\beta = [1.000, -0.877]'$

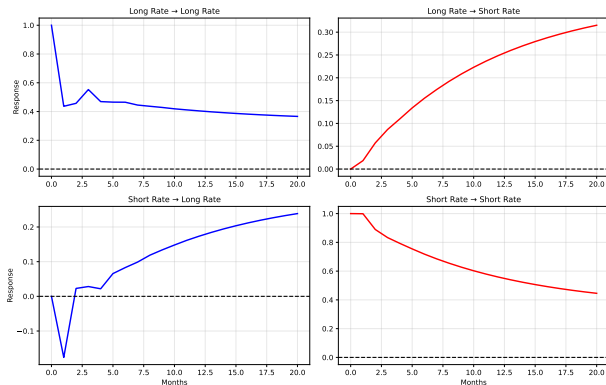
Adjustment Coefficients:  
 $\alpha_1 = -0.0532$   
 $\alpha_2 = 0.0588$

Interpretation:  
Both rates adjust to  
restore equilibrium

- VECM(2) cu rang de cointegrare = 1
- Coeficienții de ajustare  $\alpha$  indică viteza de revenire la echilibru
- Ambele rate se ajustează când spread-ul deviază de la medie

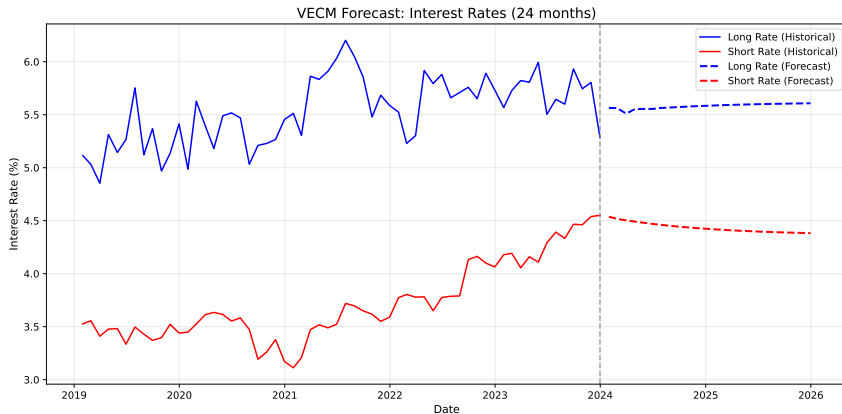
## Pasul 4: Funcții de răspuns la impuls

VECM Impulse Response Functions



- Șocuri în rata lungă afectează persistent ambele rate
- Efectele nu converg la zero — caracteristică a seriilor cointegrate
- VECM captează atât dinamica pe termen scurt cât și echilibrul pe termen lung

## Pasul 5: Prognoză VECM



- Prognoză pe 24 de luni pentru ambele rate simultan
- VECM menține relația de cointegrare în prognoză
- Spread-ul prognozat rămâne în jurul mediei istorice

# Key Takeaways

## Main Concepts

- **Cointegration:**  $I(1)$  variables with stationary linear combination
- **Spurious regression:** High  $R^2$  with unrelated  $I(1)$  variables
- **Error correction:** Adjustment toward long-run equilibrium
- **VECM:** VAR with error correction terms for cointegrated systems

## Testing Methods

- **Engle-Granger:** Simple, but only one cointegrating vector
- **Johansen:** Multiple vectors, more powerful, MLE-based

## Remember

Cointegration tests have low power in small samples. Economic theory should guide specification. Always check diagnostics!

# Ce urmează?

## Extensii și Subiecte Conexе

- **VECM Structural:** Identificarea șocurilor structurale
- **Cointegrare cu prag:** Ajustare neliniară
- **Cointegrare de panel:** Secțiuni transversale multiple
- **Cointegrare fracționară:** Memorie lungă
- **Cointegrare variabilă în timp:** Schimbări de regim

## Întrebări?

## Cointegrare

$$Y_t \sim I(1), X_t \sim I(1)$$

$$Y_t - \beta X_t = u_t \sim I(0)$$

Relație de echilibru pe termen lung

## Test Engle-Granger

Pas 1:  $Y_t = \alpha + \beta X_t + u_t$

Pas 2: Test ADF pe  $\hat{u}_t$

Valori critice speciale (nu standard ADF)

## Rang de Cointegrare

$r$  = numărul de relații de cointegrare

$$0 \leq r \leq K - 1 \text{ pentru } K \text{ variabile } I(1)$$

## Model VECM

$$\Delta_t = \Delta_{t-1} + \sum_{i=1}^{p-1} \alpha_i \Delta_{t-i} + \epsilon_t$$

= ' (factorizare)

## Interpretare și

: vectori de cointegrare (echilibru)

: viteza de ajustare

Corecția erorii:  $(\epsilon_{t-1})$

## Test Johansen

$$\text{Trace: } \lambda_{\text{trace}} = -T \sum_{i=r+1}^K \ln(1 - \hat{\lambda}_i)$$

$$\text{Max-Eigen: } \lambda_{\text{max}} = -T \ln(1 - \hat{\lambda}_{r+1})$$



## Quiz Rapid

- 1 Ce înseamnă că două variabile  $I(1)$  sunt cointegrate?
- 2 Care este problemă “regresiei false”?
- 3 În VECM, ce reprezintă coeficienții  $\alpha$ ?
- 4 Care este avantajul principal al metodei Johansen față de Engle-Granger?
- 5 Dacă  $\alpha_i = 0$  pentru variabila  $Y_i$ , ce implică aceasta?

## Răspunsuri Quiz

- ❶ **Cointegrare:** O combinație liniară a variabilelor este  $I(0)$  (staționară). Ele au un trend stochastic comun.
- ❷ **Regresie falsă:** Regresarea unei variabile  $I(1)$  pe alta  $I(1)$  necorelată dă  $R^2$  mare și coeficienți semnificativi deși nu există relație reală.
- ❸ **Coeficienții  $\alpha$ :** Viteza de ajustare—cât de repede răspunde fiecare variabilă la deviații de la echilibrul pe termen lung.
- ❹ **Avantajul Johansen:** Poate testa relații multiple de cointegrare, folosește MLE (mai eficient), nu necesită alegerea variabilei dependente.
- ❺  $\alpha_j = 0$ : Variabila  $Y_j$  este slab exogenă—nu răspunde la dezechilibru. Alte variabile fac toată ajustarea.