Time Series Analysis and Forecasting

Chapter 7: Cointegration & VECM

Seminar

Seminar Outline

Quiz 1: Cointegration Definition

Question

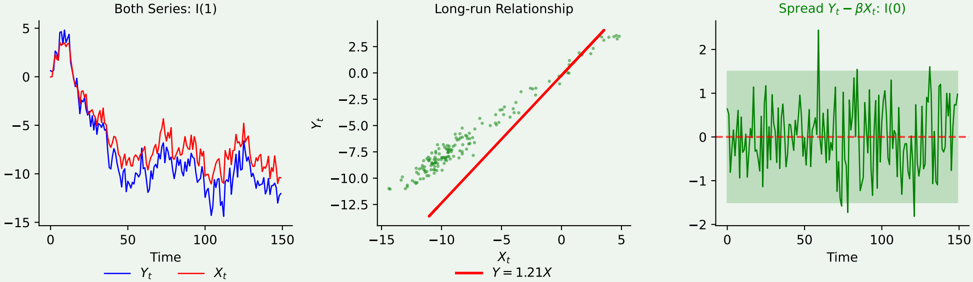
Two I(1) variables X_t and Y_t are cointegrated if:

- A) They are both stationary
- B) Their sum is I(2)
- C) A linear combination of them is I(0)
- D) They have the same mean

Answer on next slide...

Quiz 1: Answer

Answer: C – A linear combination is I(0)



Key: $Y_t - \beta X_t \sim I(0)$ means they share a common stochastic trend. The linear combination (spread) is stationary even though both series are non-stationary.

Quiz 2: Spurious Regression

Question

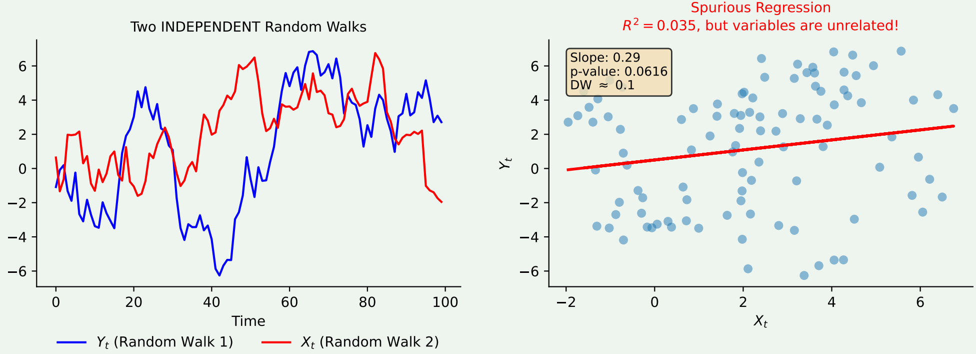
When regressing one independent random walk on another, you typically get:

A) Low R^2 and insignificant coefficients
B) High R^2 and significant coefficients (spurious!)
C) Zero coefficients
D) Undefined results

Answer on next slide...

Quiz 2: Answer

Answer: B – High R^2 and significant coefficients (spurious!)



Granger-Newbold (1974): Regressing unrelated I(1) series gives misleading results. Rule of thumb: If $R^2 > DW$, suspect spurious regression!

Quiz 3: Engle-Granger Test

Question

In the Engle-Granger two-step method, what do you test in step 2?

- A) Whether the original variables are stationary
- B) Whether the regression residuals have a unit root
- C) Whether the coefficients are significant
- D) Whether the R^2 is high enough

Answer on next slide...

Quiz 3: Answer

Answer: B – Whether residuals have unit root

Step 1: Run OLS: $Y_t = \alpha + \beta X_t + e_t$, save residuals $\hat{e}_t$

Step 2: ADF test on residuals: $\Delta\hat{e}_t = \rho\hat{e}_{t-1} + \dots$

- H_0: $\rho = 0$ (unit root $\Rightarrow$ no cointegration)
- H_1: $\rho < 0$ (stationary $\Rightarrow$ cointegration!)

Important: Use Engle-Granger critical values, not standard ADF!

Quiz 4: Johansen Test Advantage

Question

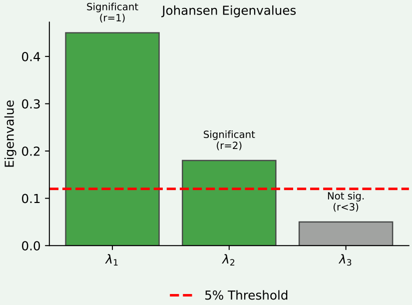
The main advantage of Johansen over Engle-Granger is:

- A) It's simpler to compute
- B) It can detect multiple cointegrating relationships
- C) It doesn't require data
- D) It always finds cointegration

Answer on next slide...

Quiz 4: Answer

Answer: B – Can detect multiple cointegrating relationships



Johansen vs Engle-Granger

Method	Engle-Granger	Johansen
# of CI vectors	1 only	Multiple
Dep. variable	Required	Not needed
Estimation	Two-step	MLE
Efficiency	Lower	Higher

Johansen advantages: Tests for $r = 0, 1, \ldots, k-1$ cointegrating vectors; MLE (more efficient); no need to choose dependent variable.

Quiz 5: Rank of $\boldsymbol{\Pi}$

Question

In a VECM with $k = 3$ variables, if $\text{rank}(\boldsymbol{\Pi}) = 2$, this means:

A) No cointegration
B) One cointegrating relationship
C) Two cointegrating relationships
D) All variables are stationary

Answer on next slide...

Quiz 5: Answer

Answer: C – Two cointegrating relationships

Rank interpretation for k variables:

- $\text{rank}(\mathbf{\Pi}) = 0$: No cointegration (use VAR in differences)
- $0 < \text{rank}(\mathbf{\Pi}) = r < k$: r cointegrating vectors (use VECM)
- $\text{rank}(\mathbf{\Pi}) = k$: All variables are I(0) (use VAR in levels)

With $k = 3$ **and** $r = 2$:

- Two equilibrium relationships
- Only $k - r = 1$ common stochastic trend

Quiz 6: VECM Structure

Question

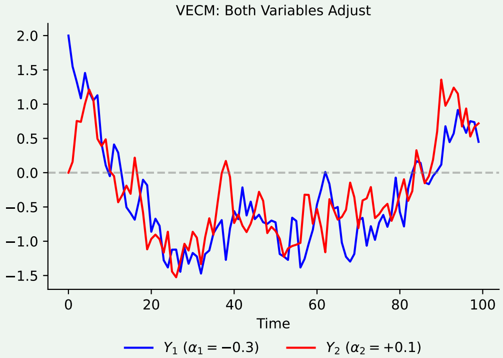
In the VECM equation $\Delta \mathbf{Y}_t = \mathbf{c} + \alpha\beta'\mathbf{Y}_{t-1} + \ldots$, what does α represent?

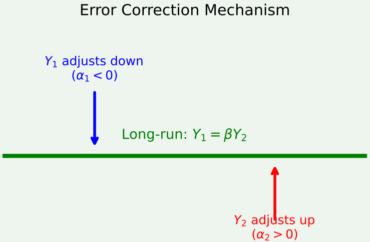
- A) The cointegrating vectors
- B) The adjustment (loading) coefficients
- C) The short-run dynamics
- D) The error variance

Answer on next slide...

Quiz 6: Answer

Answer: B – The adjustment (loading) coefficients





$\mathbf{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$:

- $\boldsymbol{\beta}$ = cointegrating vectors (define equilibrium)
- $\boldsymbol{\alpha}$ = adjustment speeds (how fast each variable corrects)

Quiz 7: Error Correction Term

Question

If $Y_t - \beta X_t$ is the cointegrating relation and this term is positive, what happens?

- A) Y is above equilibrium; Y should decrease (if $\alpha < 0$)
- B) Y is below equilibrium; Y should increase
- C) Nothing, error correction doesn't affect levels
- D) Both variables increase

Answer on next slide...

Quiz 7: Answer

Answer: A – Y above equilibrium; decreases if $\alpha < 0$

Error correction mechanism:

$$\Delta Y_t = \alpha(Y_{t-1} - \beta X_{t-1}) + \dots$$

- If $Y_{t-1} - \beta X_{t-1} > 0$: Y is "too high"
- With $\alpha < 0$: $\Delta Y_t < 0$ (Y decreases toward equilibrium)
- This is the "error correction" pulling Y back

Sign convention: α should be negative for the dependent variable to move back toward equilibrium.

Quiz 8: Weak Exogeneity

Question

If $\alpha_2 = 0$ in a bivariate VECM, this means:

- A) There is no cointegration
- B) Variable 2 does not adjust to disequilibrium (weakly exogenous)
- C) Variable 1 does not adjust
- D) Both variables are stationary

Answer on next slide...

Quiz 8: Answer

Answer: B – Variable 2 is weakly exogenous

Weak exogeneity: Variable doesn't respond to disequilibrium.

Example: Interest rates

- Long rate (R_t) often weakly exogenous ($\alpha_R \approx 0$)
- Short rate (r_t) adjusts to spread ($\alpha_r < 0$)
- Interpretation: Central bank adjusts short rate to maintain term structure

Implication: Can estimate single equation for the adjusting variable.

Quiz 9: Trace Test

Question

The Johansen trace test with $H_0 : r \leq 1$ vs $H_1 : r > 1$ tests whether:

A) There is exactly one cointegrating vector

B) There are at most one cointegrating vectors

C) There are more than one cointegrating vectors

D) All eigenvalues are zero

Answer on next slide...

Quiz 9: Answer

Answer: B/C – H_0: at most 1; H_1: more than 1

Sequential testing procedure:

1. Test $H_0 : r = 0$ vs $H_1 : r > 0$
2. If rejected, test $H_0 : r \leq 1$ vs $H_1 : r > 1$
3. Continue until fail to reject...

Trace statistic:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^{k} \ln(1 - \hat{\lambda}_i)$$

Reject H_0 if trace statistic $>$ critical value.

Quiz 10: VECM vs VAR in Differences

Question

If variables are cointegrated, using VAR in first differences instead of VECM:

- A) Gives identical results
- B) Is more efficient
- C) Loses long-run information (misspecified)
- D) Is the preferred approach

Answer on next slide...

Quiz 10: Answer

Answer: C – Loses long-run information

Granger Representation Theorem: If cointegrated, VECM representation exists and should be used.

	VAR(Δ)	**VECM**
Long-run equilibrium	Lost	Preserved
Error correction	No	Yes
Forecasts (long-run)	Poor	Better

Bottom line: Differencing removes the long-run relationship that cointegration represents!

True/False Questions

Determine if each statement is True or False:

1. Cointegration requires all variables to be I(1).
2. The cointegrating vector is unique.
3. Spurious regression has low Durbin-Watson statistic.
4. In VECM, both α coefficients must be non-zero.
5. Johansen test requires choosing a dependent variable.
6. The number of common trends $= k - r$.

Answers on next slide...

True/False: Solutions

1. Cointegration requires all variables to be I(1). **TRUE**
   Standard CI(1,1) case: all variables I(1), linear combination I(0).
2. The cointegrating vector is unique. **FALSE**
   Unique only up to scalar multiplication. Usually normalized ($\beta_1 = 1$).
3. Spurious regression has low Durbin-Watson statistic. **TRUE**
   $DW \approx 0$ indicates highly autocorrelated residuals (non-stationary).
4. In VECM, both α coefficients must be non-zero. **FALSE**
   One can be zero (weak exogeneity). At least one must be non-zero.
5. Johansen test requires choosing a dependent variable. **FALSE**
   That's Engle-Granger. Johansen treats all variables symmetrically.
6. The number of common trends $= k - r$. **TRUE**
   k variables, r cointegrating relations $\Rightarrow k - r$ common stochastic trends.

Problem 1: Cointegration Identification

Exercise

You have quarterly data on consumption (C_t) and income (Y_t). ADF tests show both are I(1). The regression $C_t = 0.85 Y_t + e_t$ gives residuals with ADF statistic $= -3.92$. The 5% Engle-Granger critical value for 2 variables is -3.34.

Are C_t and Y_t cointegrated?

Answer on next slide...

Problem 1: Solution

Solution: Yes, they are cointegrated

Test: H_0: No cointegration (residuals have unit root)

ADF statistic: -3.92
Critical value (5%): -3.34

Since $-3.92 < -3.34$, we **reject** H_0 at 5% level.

Conclusion: Residuals are stationary $\Rightarrow$ Cointegration exists!

Interpretation: Consumption and income share a common trend. The cointegrating vector is approximately $(1, -0.85)$, consistent with permanent income hypothesis.

Problem 2: VECM Interpretation

Exercise

A VECM for short rate (r_t) and long rate (R_t) gives:

$$\Delta r_t = 0.01 - 0.25(r_{t-1} - R_{t-1}) + \dots$$
$$\Delta R_t = 0.005 - 0.02(r_{t-1} - R_{t-1}) + \dots$$

Interpret the adjustment coefficients.

Answer on next slide...

Problem 2: Solution

Solution

Error correction term: $(r_{t-1} - R_{t-1}) =$ spread

Short rate ($\alpha_r = -0.25$):

- When spread is positive (short $>$ long), short rate decreases
- 25% of disequilibrium corrected per period
- Short rate actively adjusts

Long rate ($\alpha_R = -0.02$):

- Very small adjustment coefficient
- Long rate is nearly weakly exogenous
- Mostly driven by expectations, not error correction

Economic interpretation: Central bank (short rate) adjusts to maintain yield curve.

Problem 3: Johansen Test Results

Exercise

Johansen trace test for 3 variables gives:

H_0	Trace Stat	5% CV
$r = 0$	45.2	29.8
$r \leq 1$	18.1	15.5
$r \leq 2$	3.2	3.8

What is the cointegrating rank?

Answer on next slide...

Problem 3: Solution

Solution: Rank = 2

Sequential testing:

1. $H_0 : r = 0$: $45.2 > 29.8 \Rightarrow$ **Reject** (at least 1)
2. $H_0 : r \leq 1$: $18.1 > 15.5 \Rightarrow$ **Reject** (at least 2)
3. $H_0 : r \leq 2$: $3.2 < 3.8 \Rightarrow$ **Fail to reject**

Conclusion: $r = 2$ cointegrating relationships

Implications:

- Two equilibrium relationships among 3 variables
- Only $3 - 2 = 1$ common stochastic trend
- Use VECM with 2 error correction terms

Problem 4: Testing Weak Exogeneity

Exercise

In a VECM for prices (P) and exchange rate (E), you estimate $\alpha_P = -0.15$ (s.e. $= 0.04$) and $\alpha_E = 0.02$ (s.e. $= 0.03$).
Test whether the exchange rate is weakly exogenous at 5%.

Answer on next slide...

Problem 4: Solution

Solution: Exchange rate is weakly exogenous

Test: $H_0 : \alpha_E = 0$ (weak exogeneity)

t-statistic: $t = \frac{0.02}{0.03} = 0.67$
Critical value (5%, two-tailed): ± 1.96

Since $|0.67| < 1.96$, **fail to reject** H_0.

Conclusion: Exchange rate does not respond to PPP disequilibrium.

Implication: Prices do all the adjusting to restore PPP equilibrium. Can estimate single-equation model for prices.

Example: Term Structure of Interest Rates

Economic Theory

Expectations hypothesis: $R_t^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{premium}$
If premium is constant $\Rightarrow$ spread $(R_t - r_t)$ should be stationary.

Typical Findings

- Both rates are I(1) (confirmed by ADF)
- Johansen test: $r = 1$ cointegrating vector
- Cointegrating vector $\approx (1, -1)$: spread is stationary
- Short rate adjusts ($\alpha_r < 0$), long rate weakly exogenous

Policy Implication

Central bank controls short rate; long rate driven by expectations.

Example: Purchasing Power Parity

PPP Theory

$e_t = p_t - p_t^*$ (log exchange rate = price differential)
Real exchange rate: $q_t = e_t - p_t + p_t^*$ should be stationary (long-run PPP)

Empirical Challenges

- Unit root tests: e_t, p_t, p_t^* all I(1)
- Cointegration tests: Mixed results depending on sample
- Half-life of PPP deviations: 3-5 years (slow adjustment)
- Weak exogeneity: Exchange rate often doesn't adjust

PPP Puzzle

Real exchange rate is highly persistent—slow mean reversion is hard to explain with standard models.

Example: Pairs Trading Strategy

Idea

Find cointegrated stocks $\Rightarrow$ trade the stationary spread

Implementation Steps

1. **Identify pairs**: Test cointegration (e.g., Coca-Cola & Pepsi)
2. **Estimate spread**: $z_t = P_A - \beta P_B$
3. **Trading rules**:
   - $z_t > \mu + 2\sigma$: Sell A, Buy B (spread too wide)
   - $z_t < \mu - 2\sigma$: Buy A, Sell B (spread too narrow)
   - Exit when $z_t \approx \mu$

Risks

Cointegration can break down; spread may not revert; transaction costs.

Python Cointegration Analysis: Key Functions

Essential Libraries

```
from statsmodels.tsa.stattools import coint, adfuller
from statsmodels.tsa.vector_ar.vecm import coint_johansen, VECM
```

Workflow

1. Unit root tests: `adfuller(series)`
2. Engle-Granger: `coint(y, x)` returns test stat & p-value
3. Johansen: `coint_johansen(data, det_order, k_ar_diff)`
4. Fit VECM: `model = VECM(data, k_ar_diff=2, coint_rank=1)`
5. Results: `results = model.fit()`

Note

Complete working examples are provided in the Jupyter notebooks.

Discussion: Cointegration vs Correlation

Key Question

Two series are highly correlated. Are they cointegrated?

Answer: Not necessarily!

- **Correlation**: Measures co-movement (can be spurious for I(1))
- **Cointegration**: Requires stationary linear combination

Example

Two independent random walks can have correlation > 0.9 purely by chance (spurious correlation). But they're NOT cointegrated—their spread is also I(1).

Cointegration implies a meaningful long-run equilibrium relationship.

Discussion: Choosing Deterministic Components

Key Question

Johansen test has 5 cases for deterministics. Which to choose?

Guidelines

1. **No constant, no trend**: Rarely used (requires mean-zero data)
2. **Constant in CE only**: Level series, no drift
3. **Constant unrestricted**: Most common for economic data
4. **Trend in CE**: Series have deterministic trends
5. **Trend unrestricted**: Trending differences (uncommon)

Practical Advice

Start with Case 3 (constant unrestricted). Check sensitivity to specification. Use economic reasoning: do levels have trends?

Take-Home Exercises

1. **Theoretical**: Show that if Y_t and X_t are both random walks with the same innovation, they are cointegrated.
2. **Computation**: Given VECM estimates:

$$\Delta Y_t = 0.5 - 0.3(Y_{t-1} - 2X_{t-1}) + 0.2\Delta Y_{t-1}$$
$$\Delta X_t = 0.1 + 0.1(Y_{t-1} - 2X_{t-1}) + 0.4\Delta X_{t-1}$$

   - What is the cointegrating vector?
   - Which variable adjusts more quickly?
   - What is the long-run equilibrium relationship?
3. **Applied**: Download 10-year and 3-month Treasury rates:
   - Test for unit roots; Test for cointegration (Engle-Granger and Johansen)
   - Estimate VECM; Interpret adjustment coefficients
4. **Critical Thinking**: Why might PPP hold in the long run but not short run?

Exercise Solutions Hints

Hints

1. If $Y_t = Y_{t-1} + \varepsilon_t$ and $X_t = X_{t-1} + \varepsilon_t$ (same shock), then $Y_t - X_t = Y_0 - X_0$ is constant (stationary).
2. From the VECM:
   - Cointegrating vector: $(1, -2)$ (normalized on Y)
   - Y adjusts faster: $|\alpha_Y| = 0.3 > |\alpha_X| = 0.1$
   - Long-run: $Y = 2X$ (when EC term $= 0$)
3. For interest rates:
   - Both typically I(1); spread usually stationary
   - Expect one cointegrating vector with $(1, -1)$
   - Short rate typically adjusts; long rate often weakly exogenous
4. PPP deviations: Transportation costs, non-traded goods, sticky prices, tariffs, market segmentation all slow adjustment but don't prevent long-run convergence.

Key Takeaways from This Seminar

Main Points

1. **Cointegration**: I(1) variables with stationary linear combination
2. **Spurious regression**: High R^2 without cointegration is meaningless
3. **Engle-Granger**: Simple, but only one cointegrating vector
4. **Johansen**: Multiple vectors, MLE, more powerful

VECM Insights

- $\boldsymbol{\beta}$ defines equilibrium; $\boldsymbol{\alpha}$ determines adjustment speed
- Weak exogeneity ($\alpha = 0$): Variable doesn't respond to disequilibrium
- Always use VECM (not VAR in differences) when cointegrated

Remember

Cointegration is about **long-run equilibrium**, not just correlation!