



Time Series Analysis and Forecasting

Chapter 10: Comprehensive Review



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Learning Objectives

By the end of this chapter, you will be able to:

- ▣ Apply the complete forecasting workflow from data to evaluation
- ▣ Select appropriate models based on data characteristics
- ▣ Evaluate forecast accuracy using proper metrics and cross-validation
- ▣ Integrate knowledge from all previous chapters in practice

Outline

Forecasting Methodology

Case Study 1: Bitcoin Volatility (GARCH)

Case Study 2: Sunspot Cycles (Fourier)

Case Study 3: Unemployment (Prophet)

Case Study 4: Multivariate Analysis (VAR)

Synthesis and Guidelines

AI Use Case

Quiz

Summary

The Scientific Approach to Forecasting

Research Question

How do we **rigorously evaluate** forecast performance while avoiding overfitting?

The Fundamental Problem

- In-sample fit \neq Out-of-sample performance
- Models can “memorize” training data without learning patterns
- **Solution:**
 - ▶ Proper train/validation/test methodology

Key Principle

“The test set must remain **untouched** until final evaluation.”
— Standard practice in machine learning and econometrics

Train/Validation/Test Framework

Time Series Train/Validation/Test Split



 TSA_ch10_train_val_test_split

Evaluation Metrics

Definition 1 (Forecast Error Metrics)

Let y_t be actual, \hat{y}_t forecast:

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_t (y_t - \hat{y}_t)^2}, \quad \text{MAE} = \frac{1}{n} \sum_t |y_t - \hat{y}_t|, \quad \text{MAPE} = \frac{100\%}{n} \sum_t \left| \frac{y_t - \hat{y}_t}{y_t} \right|$$

When to Use Each

- ▣ **RMSE**: Penalizes large errors
- ▣ **MAE**: Robust to outliers
- ▣ **MAPE**: Scale-independent (%)

Caution

- ▣ MAPE undefined when $y_t = 0$
- ▣ Compare on **same** test set
- ▣ Report **out-of-sample** metrics

Forecast Evaluation Beyond RMSE

Alternative metrics

- **MASE** (Mean Absolute Scaled Error): $\frac{\text{MAE}_{\text{model}}}{\text{MAE}_{\text{naïve}}}$; $< 1 \Rightarrow$ beats naïve
- **DA** (Directional Accuracy): $\frac{1}{h} \sum_{t=1}^h 1(\text{sgn } \Delta \hat{y}_t = \text{sgn } \Delta y_t)$
- **QL** (Quantile Loss): asymmetric penalty α vs $1-\alpha$

$$QL_{\alpha} = \begin{cases} \alpha(y_t - \hat{q}_t), & y_t > \hat{q}_t \\ (1 - \alpha)(\hat{q}_t - y_t), & y_t \leq \hat{q}_t \end{cases}$$

- **CRPS** (Continuous Ranked Probability Score): $\int_{-\infty}^{\infty} (F(x) - 1_{x \geq y})^2 dx$

Forecast Evaluation: Bitcoin Results

Bitcoin Results (GARCH volatility)

Metric	Value
--------	-------

RMSE	2.21
------	------

MAE	1.89
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MASE	0.98
------	------

Dir. Accuracy	28.7%
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- $MASE < 1$: GARCH beats naïve
- DA 28.7%: volatility direction is hard

Interpretation

- **RMSE/MAE**: absolute volatility forecast error
- **MASE** < 1 : GARCH outperforms naïve benchmark
- **DA 28.7%**: volatility direction is extremely hard to predict
- Evaluation must be done on the **test set**

Formal Forecast Comparison: Diebold–Mariano

Definition 2 (Diebold–Mariano Test)

Loss differential: $d_t = L(e_{1t}) - L(e_{2t})$, Statistic: $DM = \frac{\bar{d}}{\sqrt{\widehat{\text{Var}}(\bar{d})}} \xrightarrow{d} N(0, 1)$

Hypotheses

- ▣ H_0 : equal predictive accuracy
- ▣ H_1 : one model is significantly better
- ▣ Large $|DM| \Rightarrow$ reject H_0

Bitcoin Result (GARCH volatility)

- ▣ Normal vs Student-t: $DM = -0.51$
- ▣ $p = 0.612$ — **do not reject** H_0
- ▣ Similar accuracy, but Student-t preferred by AIC ($\Delta = 509$)

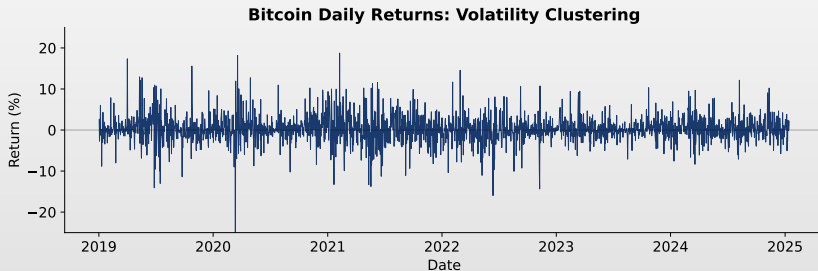
Key message

- ▣ Lower RMSE \neq significant difference — formal testing is **mandatory**

Bitcoin: Volatility Clustering

Observation

- Large returns follow large returns, small follow small—**volatility clustering**



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Bitcoin: Problem Statement

Research Question

Can we forecast Bitcoin's **volatility** using GARCH models?

Data Characteristics

- ▣ Source: Yahoo Finance (BTC-USD)
- ▣ Period: Jan 2019 – Jan 2025
- ▣ Frequency: Daily
- ▣ Observations: $\approx 2,200$ days

Stylized Facts

- ▣ Returns: near-zero mean
- ▣ Fat tails (kurtosis > 3)
- ▣ Volatility clustering

Key Insight

Financial returns are typically:

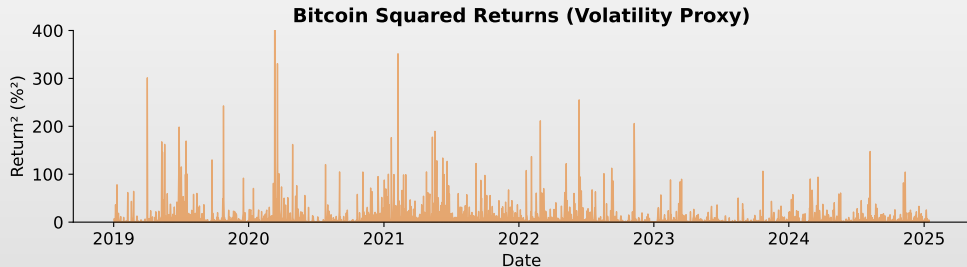
- ▣ **Unpredictable** in mean
- ▣ **Predictable** in variance

⇒ Focus on **volatility forecasting**

Bitcoin: Evidence for GARCH

Observation

- Squared returns r_t^2 exhibit significant autocorrelation \Rightarrow GARCH effects
- Slow ACF decay \Rightarrow high volatility persistence



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GARCH Model Specification

Definition 3 (GARCH(p,q) Model)

Let r_t denote returns. The GARCH(p,q) model:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \stackrel{iid}{\sim} N(0, 1)$$
$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^p \beta_j \sigma_{t-j}^2$$

where $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^q \alpha_i + \sum_{j=1}^p \beta_j < 1$.

Model Variants

- ▣ **GARCH(1,1)**: Most common
- ▣ **GJR-GARCH**: Leverage effect
- ▣ **EGARCH**: Log-variance, asymmetric

Interpretation

- ▣ α : Shock impact (ARCH effect)
- ▣ β : Volatility persistence
- ▣ $\alpha + \beta \approx 1$: High persistence

GARCH: Stationarity and Unconditional Variance

Theorem 1 (Covariance Stationarity of GARCH(1,1))

If $\alpha_1 + \beta_1 < 1$, then $\{\varepsilon_t\}$ is covariance stationary with:

$$\bar{\sigma}^2 = \mathbb{E}[\sigma_t^2] = \frac{\omega}{1 - \alpha_1 - \beta_1}$$

Derivation

Take expectations of both sides of the variance equation:

$$\begin{aligned}\mathbb{E}[\sigma_t^2] &= \omega + \alpha_1 \mathbb{E}[\varepsilon_{t-1}^2] + \beta_1 \mathbb{E}[\sigma_{t-1}^2] \\ \bar{\sigma}^2 &= \omega + (\alpha_1 + \beta_1) \bar{\sigma}^2 \quad (\text{stationarity}) \\ \bar{\sigma}^2 &= \frac{\omega}{1 - \alpha_1 - \beta_1}\end{aligned}$$

Multi-Step Forecasts Converge to $\bar{\sigma}^2$

As $h \rightarrow \infty$: $\mathbb{E}_t[\sigma_{t+h}^2] \rightarrow \bar{\sigma}^2$ at rate $(\alpha_1 + \beta_1)^h$.

Bitcoin: Model Selection on Validation Set

Methodology

Fit each model on training data, evaluate on validation set.

Model	AIC	BIC	Val MAE	Selection
GARCH(1,1)	6,994.8	7,020.6	2.638	Best
GARCH(2,1)	6,993.7	7,024.6	2.640	
GJR-GARCH(1,1)	6,983.7	7,014.6	2.669	
EGARCH(1,1)	—	—	—	Failed*

* Analytic forecasts not available for $h > 1$

Result

GARCH(1,1) selected based on lowest validation MAE for volatility forecasts.

Bitcoin: Data Split and Stationarity

Data Split

Set	Period	N
Training (70%)	2019-01 to 2023-03	1,543
Validation (15%)	2023-04 to 2024-02	333
Test (15%)	2024-03 to 2025-01	329
Total		2,205

Stationarity Tests

Series	ADF	Result
Prices	$p = 0.50$	Non-stationary
Returns	$p < 0.01$	Stationary

⇒ Model **returns**, not prices

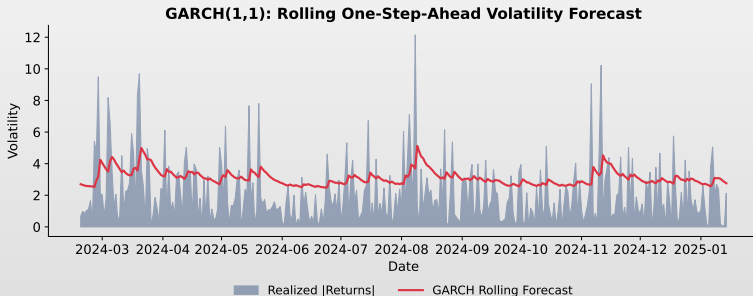
Why Stationarity Matters

- GARCH requires weakly stationary input
- Prices follow random walk; returns are stationary

Bitcoin: Volatility Forecast

Interpretation

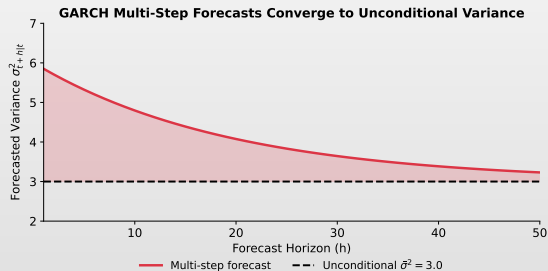
- Shaded area: 95% confidence interval of the volatility forecast
- GARCH(1,1) captures Bitcoin's volatility dynamics well



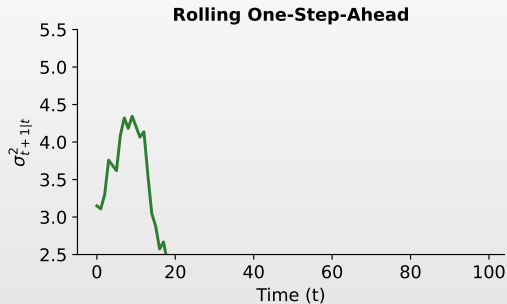
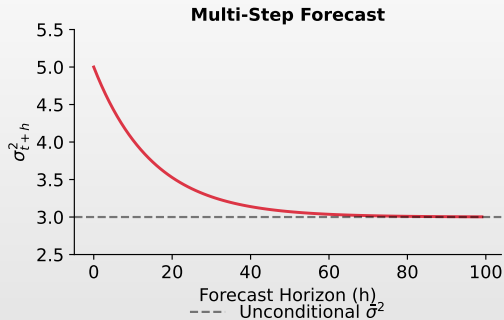
GARCH: Multi-Step Forecasts Converge

Key Insight

- Multi-step forecasts converge to $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- Use rolling forecasts



GARCH: Rolling One-Step-Ahead Solution



 TSA_ch10_rolling_vs_multistep

GARCH: Innovation Distributions

Model

$$r_t = \mu + \sigma_t Z_t$$

- Options for z_t : $\mathcal{N}(0, 1)$ (normal) or t_ν (fat tails)

Bitcoin: empirical evidence

- Residual kurtosis: **13.81** (Normal = 3)
- Skewness: -0.29
- Jarque-Bera: 9085, $p < 0.001$
- Normality **underestimates** tail risk

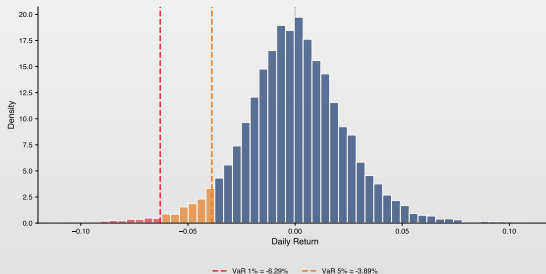
Student-t: the right choice

- $\hat{\nu} = 2.96$ degrees of freedom
- AIC Normal: 9769 vs Student-t: **9260**
- $\Delta\text{AIC} = 509$ — **overwhelming** evidence
- Fat tails = **more realistic** VaR estimates

VaR and ES: Graphical Illustration

Interpretation

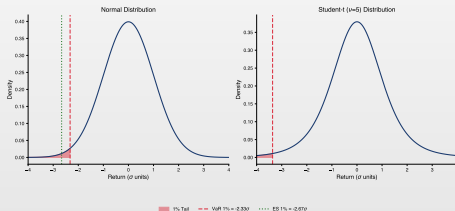
- VaR 1% = loss exceeded only in 1% of cases
- Red area = extreme losses (beyond VaR)



VaR vs Expected Shortfall: Normal vs Student-t

Interpretation

- ES measures average loss when VaR is exceeded; Student-t: VaR and ES are larger than under normal



TSA_ch5_var_es

Formal Definitions

- VaR**: $\text{VaR}_\alpha = \inf\{l : P(r_t \leq -l) \leq \alpha\}$
- ES** (preferred under Basel III/IV): $\text{ES}_\alpha = \mathbb{E}[-r_t \mid r_t \leq -\text{VaR}_\alpha] = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_u \, du$
- ES is **coherent** (subadditive); VaR is **not** subadditive

Value at Risk — Numerical Example

VaR Calculation

Portfolio: **1,000,000 EUR**, forecasted volatility $\hat{\sigma}_{T+1} = 1.5\%$

VaR with Normal Distribution

Level	z_{α}	VaR (%)	VaR (EUR)
5% (1 day)	1.645	2.47%	24,675
1% (1 day)	2.326	3.49%	34,890

Scaling for Longer Periods

$\text{VaR}_{h \text{ days}} = \text{VaR}_{1 \text{ day}} \cdot \sqrt{h}$ — assumes i.i.d. returns

Value at Risk — Student-t Distribution

Why Student-t?

Normal distribution **underestimates** tail risk. Student-t with ν degrees of freedom better captures fat tails (kurtosis > 3).

VaR 1% (1 day) Comparison: $\sigma = 1.5\%$, Portfolio = 1M EUR

Distribution	Quantile	VaR (EUR)
Normal	2.326	34,890
Student-t ($\nu = 6$)	3.143	47,145
Student-t ($\nu = 4$)	3.747	56,205

Observation

With $\nu = 6$ (typical for stocks), VaR is **35% higher** than normal!

VaR — Complete Example with GARCH

VaR Calculation Procedure

1. Estimate GARCH(1,1) model with Student-t distribution
2. Obtain volatility forecast: $\hat{\sigma}_{T+1}$
3. Calculate VaR: $\text{VaR}_\alpha = t_\alpha(\nu) \cdot \hat{\sigma}_{T+1}$

Example: S&P 500

- ▣ Estimated parameters: $\alpha = 0.088$, $\beta = 0.900$, $\nu = 6.4$
- ▣ Forecasted volatility: $\hat{\sigma}_{T+1} = 1.2\%$
- ▣ Portfolio: 10,000,000 EUR

VaR 1% (1 day): $\text{VaR} = 3.05 \times 0.012 \times 10,000,000 = \mathbf{366,000 \text{ EUR}}$

What is VaR Backtesting?

Definition

- **Backtesting** = ex-post verification of VaR model quality
- Compares realized losses with the forecasted VaR threshold
 - ▶ A **violation** occurs when $r_t < -\text{VaR}_t$

Backtesting Principle

- Violation indicator: $I_t = 1(r_t < -\text{VaR}_{\alpha,t})$
- For a correctly specified model at level α :
 - ▶ Frequency: $\hat{p} = \frac{1}{T} \sum I_t \approx \alpha$; violations **independent**
- VaR 1% over 250 days \Rightarrow expect ~ 2.5 violations/year

Importance

- Regulatory requirement under **Basel III/IV** for banks: backtesting is mandatory

Kupiec Test (1995) — Unconditional Coverage

Hypotheses

- ▣ H_0 : Violation rate equals the VaR level ($p = \alpha$)
- ▣ H_1 : Violation rate differs from the VaR level ($p \neq \alpha$)

Test Statistic (Likelihood Ratio)

- ▣ **Formula:** $LR_{uc} = -2 \ln \left[\frac{\alpha^x (1-\alpha)^{T-x}}{\hat{p}^x (1-\hat{p})^{T-x}} \right] \sim \chi^2(1)$
- ▣ **Notation:** x = no. violations, T = no. observations, $\hat{p} = x/T$

Example

- ▣ VaR 1%, $T = 250$ days, $x = 5$ violations: $\hat{p} = 2\%$
 - ▶ Too many violations \Rightarrow model **underestimates** risk
- ▣ VaR 1%, $T = 250$ days, $x = 1$ violation: $\hat{p} = 0.4\% \Rightarrow$ acceptable

Christoffersen Test (1998) — Conditional Coverage

Motivation

- Kupiec only tests the **frequency** of violations
- Does not detect **clustering** of violations (consecutive violations)
 - ▶ If violations cluster \Rightarrow model fails to capture volatility dynamics

Independence + Conditional Coverage Test

- **Formula:** $LR_{cc} = LR_{uc} + LR_{ind} \sim \chi^2(2)$
- LR_{ind} tests whether $P(I_t = 1 | I_{t-1} = 1) = P(I_t = 1 | I_{t-1} = 0)$
- A good model: violations are rare **and** uniformly distributed over time

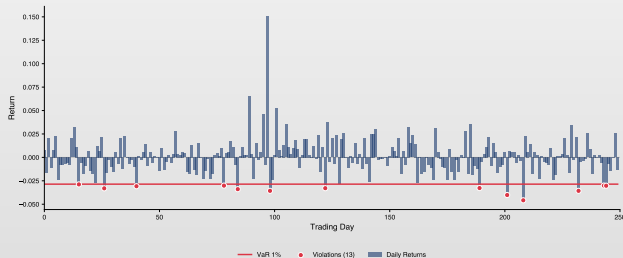
Recommendation

- Use **both** tests: Kupiec (frequency) + Christoffersen (independence)

VaR Backtesting: Visualization

Interpretation

- ▣ Red line: VaR 1% threshold estimated with GARCH(1,1)
- ▣ Red dots: 13 violations out of 250 days ($\hat{p} = 5.2\%$)
 - ▶ **Basel red zone** \Rightarrow model significantly underestimates risk
 - ▶ Solutions: Student-t distribution, EGARCH model, or more conservative VaR level



VaR Backtesting: Basel Traffic Light

Basel III/IV Traffic Light Zones

Zone	Violations/250 days	Interpretation	Penalty
Green	0–4	Model acceptable	No penalty
Yellow	5–9	Needs investigation	Factor k increases
Red	≥ 10	Model inadequate	Maximum penalty

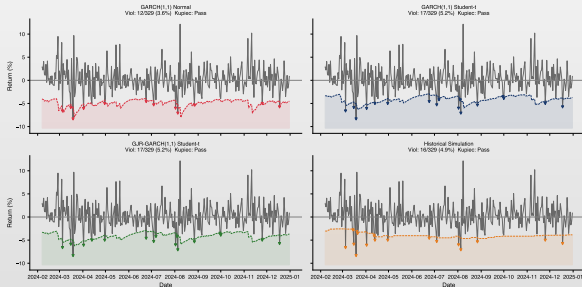
Practical Example

- Portfolio with VaR 1%: 250 days of backtesting
- 3 violations \Rightarrow **Green zone** \Rightarrow model acceptable
- 7 violations \Rightarrow **Yellow zone** \Rightarrow revision needed
- 13 violations \Rightarrow **Red zone** \Rightarrow model rejected

Application: Rolling VaR with Multiple Models

Methodology

- Rolling one-step-ahead VaR: $\text{VaR}_{t+1}^{\alpha} = \mu + \hat{\sigma}_{t+1} \cdot z_{\alpha}$
- 4 models compared on Bitcoin test set (329 days, 2024)



VaR Backtesting: Model Comparison

Bitcoin Results — VaR 5% (T = 329 days, expected: 16.4 violations)

Model	Violations	Rate	Kupiec p	Chr. p	Conclusion
GARCH(1,1)-N	12	3.6%	0.238	1.000	Too conservative
GARCH(1,1)-t	17	5.2%	0.890	0.272	\approx 5% target
GJR-GARCH(1,1)-t	17	5.2%	0.890	0.272	$\gamma \approx 0$ (symmetric)
Hist. Simulation	16	4.9%	0.909	0.216	\approx 5% target

Conclusions

- **Student-t**: perfect coverage (5.2% \approx 5%)
- Normal: too conservative (3.6% < 5%)
- GJR \approx GARCH: no leverage for Bitcoin
- All pass both Kupiec **and** Christoffersen

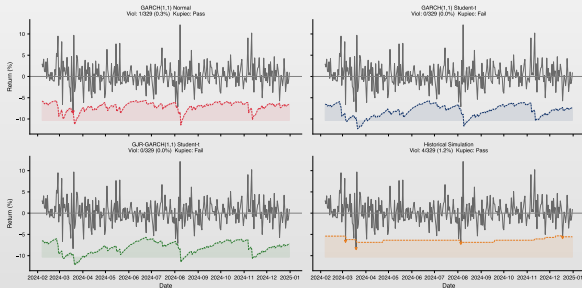
Practical Lessons

- GARCH-t and HistSim: both \approx 5% target
- Historical simulation: simple and effective alternative
- Formal statistical testing (Kupiec, Christoffersen) is **mandatory**

Application: Rolling VaR 1% on Multiple Models

Methodology

- Rolling one-step-ahead VaR at $\alpha = 1\%$ (extreme risk)
- Same 4 models; expected: $T \times 0.01 = 3.3$ violations



VaR 1% Backtesting: Model Comparison

Bitcoin Results — VaR 1% ($T = 329$ days, expected: 3.3 violations)

Model	Violations	Rate	Kupiec p	Chr. p	Conclusion
GARCH(1,1)-N	1	0.3%	0.137	1.000	Acceptable
GARCH(1,1)-t	0	0.0%	0.010	1.000	Rejected
GJR-GARCH(1,1)-t	0	0.0%	0.010	1.000	Rejected
Hist. Simulation	4	1.2%	0.704	1.000	\approx 1% target

VaR 1% Conclusions

- GARCH-t/GJR-t: 0 violations \Rightarrow Kupiec **rejects** ($p = 0.010$)
- Student-t tails too heavy ($\nu \approx 3$) \Rightarrow VaR 1% **too conservative**
- **Hist. Simulation**: 1.2% \approx 1% — only accurate model

Lesson: 5% vs 1%

- At 5%: GARCH-t and HistSim both excellent
- At 1%: GARCH-t **rejected** (too conservative)
- **Optimal model depends on the α level!**

GARCH Limitations and Modern Extensions

Limitations

- Does not capture **jumps**
- Constant parameters over time
- Sensitive to chosen distribution
- Does not model different **regimes**

Extensions

- **GJR-GARCH**: leverage effect
- **EGARCH**: asymmetric shocks
- **Markov-Switching GARCH**: regimes
- Realized volatility (HAR)
- Hybrid GARCH + ML

Key message

- GARCH is a **starting point**, not the end of risk modeling

Bitcoin: Key Findings

Summary

1. **Returns are stationary**; prices are not
2. **GARCH(1,1)** outperforms more complex variants
3. **High persistence** ($\alpha + \beta = 0.93$)
4. Volatility is **predictable** even when returns are not

Practical Implications

- ▣ Risk management: VaR, Expected Shortfall
- ▣ Option pricing requires volatility forecasts
- ▣ Portfolio optimization with time-varying risk

Limitations

- ▣ GARCH assumes **symmetric** shocks
- ▣ Does not capture **jumps**
- ▣ Normal distribution may be restrictive

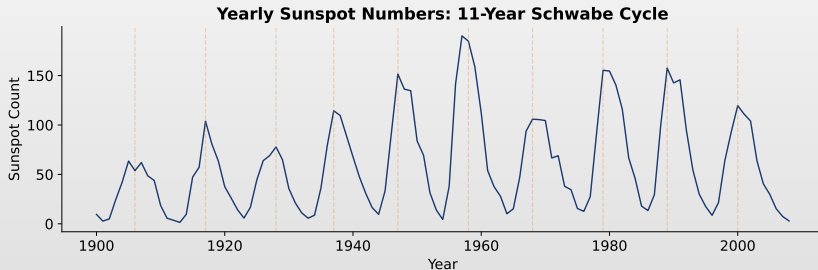
Extensions

- ▣ Student-t innovations
- ▣ Realized volatility
- ▣ HAR models

Sunspots: The 11-Year Solar Cycle

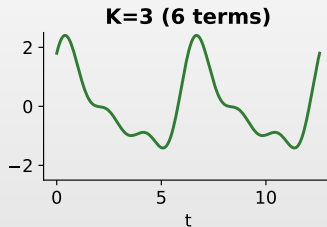
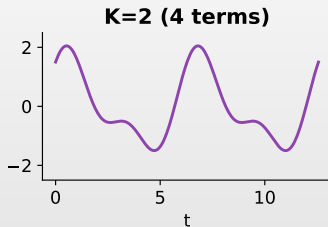
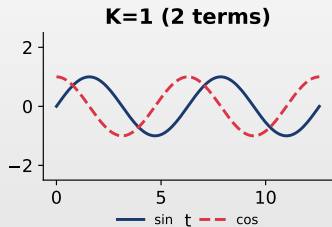
Observation

- Clear ≈ 11 -year solar cycle; variable amplitude across cycles
- Periodic ACF \Rightarrow long seasonality, ideal for Fourier terms



Fourier Terms for Seasonality

Fourier Terms: More K = More Flexibility



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Sunspots: Model Selection

Methodology

Compare $K = 1, 2, 3, 4$ Fourier harmonics on validation set.

Data Split	Set	Period	N
	Training (70%)	1900–1975	76
	Validation (20%)	1976–1997	22
	Test (10%)	1998–2008	11
	Total		109

Model Comparison			
K	AIC	Val RMSE	
1	665.9	87.15	
2	668.0	86.92	
3	671.8	86.81	Best
4	674.5	87.93	

Result

$K = 3$ Fourier harmonics selected (6 parameters for 11-year cycle).

Overfitting in Choosing K

Overfitting risk

- ▣ K too large = memorizing historical cycle
- ▣ Model fits noise, not signal
- ▣ Test performance **degrades**

Fourier \approx periodic regression

- ▣ Each harmonic adds 2 parameters (sin, cos)
- ▣ $K = 3$: 6 extra parameters
- ▣ $K = 6$: 12 parameters — overfitting risk

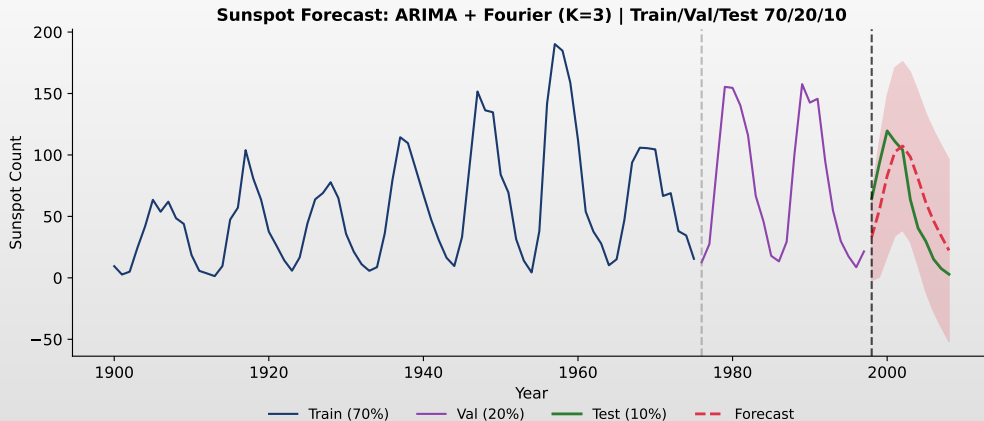
Solution: validation

- ▣ Select K on **validation** set
- ▣ Evaluate on **test** — untouched
- ▣ Trade-off: complexity vs generalization

Our results

- ▣ $K = 3$ minimizes Val RMSE
- ▣ $K = 4$ increases error \Rightarrow overfitting

Sunspots: Forecast Results



Sunspots: Key Takeaways

When to Use Fourier Terms

- Seasonal period s is **long** (e.g., 11 years, 52 weeks)
- SARIMA would require too many seasonal lags
- Pattern is **smooth and periodic**
- Multiple cycles need to be captured

Choosing K

- Start with $K = 1$, increase until validation error stops improving
- Too high K = overfitting

Fourier vs SARIMA

	Fourier	SARIMA
Long seasons	✓	×
Short seasons	OK	✓
Parameters	$2K$	Many
Flexibility	Fixed	Adaptive

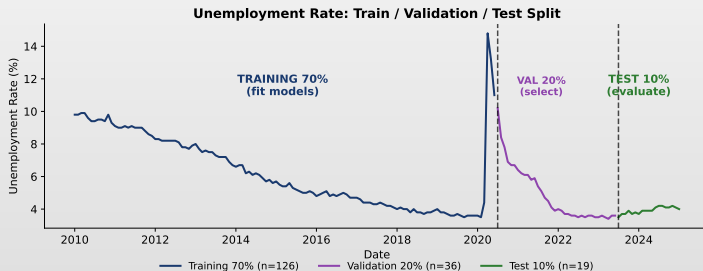
Applications

Climate cycles, business cycles, astronomical phenomena

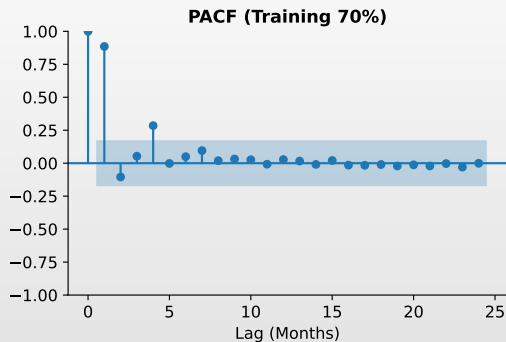
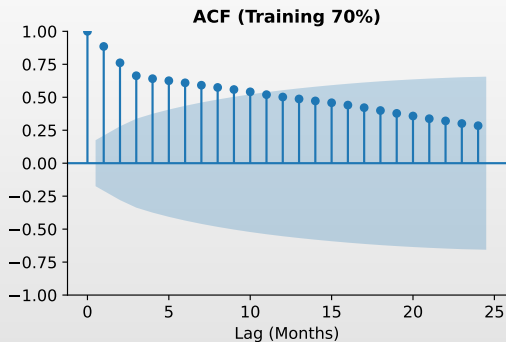
Unemployment: Train / Validation / Test Split

Methodology

- **Training:** Fit models
- **Validation:** Select best
- **Test:** Final evaluation

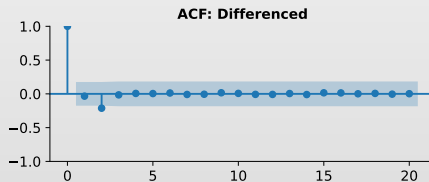
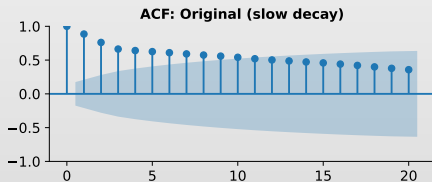
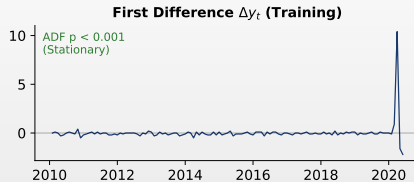
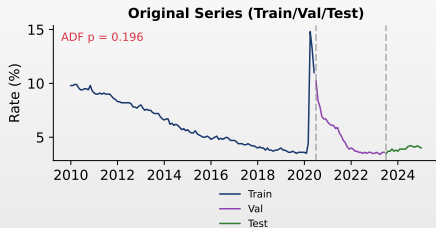


Unemployment: Preliminary Analysis



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Unemployment: Stationarity Tests



Structural Breaks: Formal Approach

Classical methods

- ▣ **Chow Test**: break at known point
- ▣ **Bai–Perron**: multiple unknown breaks
- ▣ **CUSUM**: sequential detection

Problem

- ▣ ADF can confuse **break** with **unit root**
- ▣ Zivot–Andrews test: ADF with endogenous break

Result: Unemployment at COVID (March 2020)

- ▣ Chow Test: $F = 21.73$, $p < 0.001$
- ▣ Structural break **confirmed**
- ▣ SARIMA: constant parameters — risk
- ▣ Prophet: detects changepoints automatically

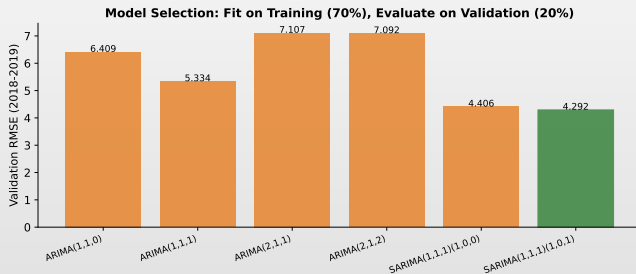
Key message

- ▣ Model must be adapted to **parameter stability**

Unemployment: Model Selection (Validation Set)

Best: SARIMA(1,1,1)(1,0,0)₁₂

Selected by lowest validation RMSE



Unemployment: SARIMA Parameters

SARIMA(1,1,1)(1,0,0)₁₂ fitted on Train+Val (2010–2023)

- AR(1): $\phi_1 = -0.86$
- MA(1): $\theta_1 = 0.78$
- SAR(12): $\Phi_1 = -0.08$ (n.s.)

SARIMA(1,1,1)(1,0,1) - Fitted on Train+Val (85%)

Parameter	Coef	Std Err	P-value	Sig
ar.L1	0.8423	0.2084	0.0001	***
ma.L1	-0.9540	0.1973	0.0000	***
ar.S.L12	0.0326	4.5951	0.9943	
ma.S.L12	-0.0113	4.6087	0.9980	
sigma2	0.8122	0.0608	0.0000	***

Ljung-Box Test for Residual Autocorrelation

Definition 4 (Ljung-Box Test)

For residuals $\hat{\varepsilon}_t$ with sample autocorrelations $\hat{\rho}_k$, the test statistic:

$$Q(h) = n(n+2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{n-k} \stackrel{H_0}{\sim} \chi^2(h-p-q)$$

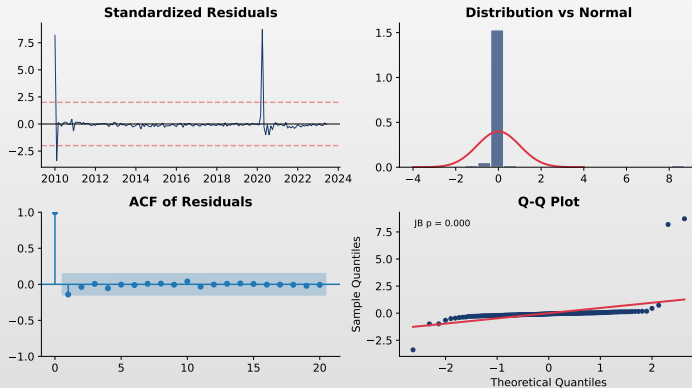
where p, q are ARMA orders. H_0 : Residuals are white noise.

Interpretation

- ▣ Large Q (small p-value): Reject H_0 , residuals have structure
- ▣ Small Q (large p-value): Fail to reject H_0 , model is adequate
- ▣ Rule of thumb: Use $h = \min(10, n/5)$ for lag order

Unemployment: SARIMA Diagnostics

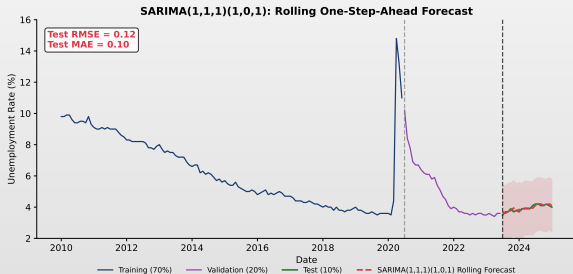
SARIMA(1,1,1)(1,0,1) Diagnostics on Train+Val (85%) | Ljung-Box $p = 1.00$



Unemployment: SARIMA Rolling Forecast

Problem: Structural Break

- Rolling one-step-ahead forecast (re-estimate at each t)
- Test RMSE = 0.12**



Prophet Model

Definition 5 (Prophet Decomposition)

$$y_t = g(t) + s(t) + h(t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$$

where $g(t)$ = trend, $s(t)$ = seasonality, $h(t)$ = holidays, σ^2 = noise variance (estimated).

Changepoint Detection

- Automatic location selection
- `changepoint_prior_scale` controls flexibility

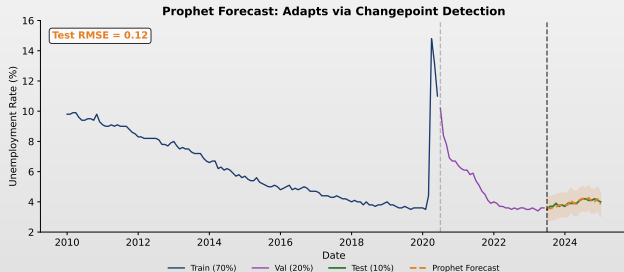
Advantages

- Handles missing data
- Interpretable components
- Robust to outliers

Unemployment: Prophet Forecast Results

Key Finding

- Prophet adapts via changepoint detection
- Test RMSE = 0.58**



 TSA_ch10_unemployment_forecast

Unemployment: Model Tuning

Hyperparameter Tuning

Tune `changepoint_prior_scale` on validation set.

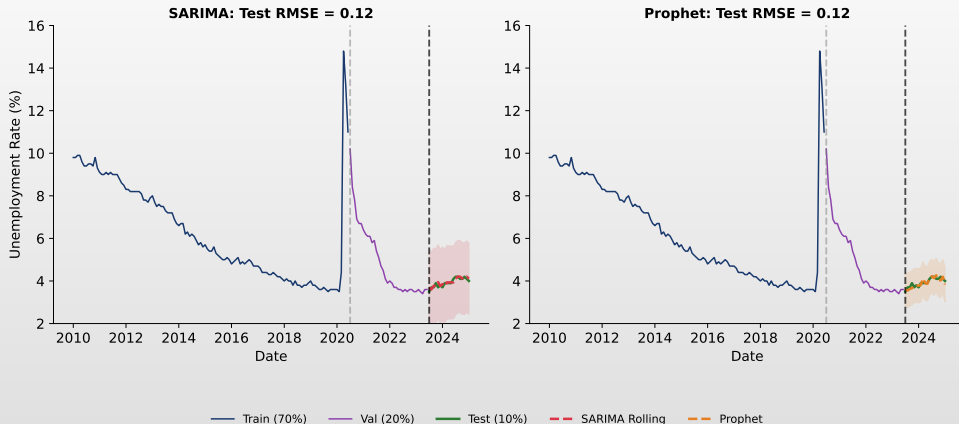
Data Split	Set	Period	N
	Training (70%)	2010-01 to 2020-06	126
	Validation (20%)	2020-07 to 2023-06	36
	Test (10%)	2023-07 to 2025-01	19
	Total		181

Scale Comparison	Scale	Val RMSE	
	0.01	4.21	
	0.05	3.89	
	0.10	3.52	Best
	0.30	3.67	
	0.50	3.81	

Interpretation

Scale = 0.10 balances flexibility (capturing COVID shock) with stability.

Unemployment: SARIMA vs Prophet Comparison



Prophet: When to Use It

Ideal Use Cases

- Business data with **holidays**
- **Missing values** present
- Need **interpretable** components
- Forecasts with **uncertainty bands**

Caveat: Structural Breaks

Prophet handles breaks via changepoints, but **SARIMA outperformed** it on unemployment (0.12 vs 0.58). Always validate!

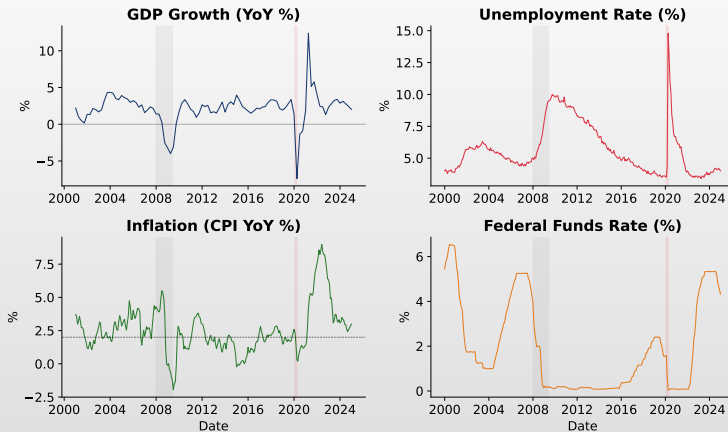
Prophet vs ARIMA

	Prophet	ARIMA
Changepoints	✓	×
Missing data	✓	×
Holidays	✓	×
Speed	Fast	Moderate
Interpretable	✓	×

Key Parameters

`changepoint_prior_scale`: flexibility
`seasonality_prior_scale`: smoothness

VAR: Multivariate Economic Data



VAR Model Specification

Definition 6 (Vector Autoregression VAR(p))

For K variables $y_t = (y_{1t}, \dots, y_{Kt})'$:

$$y_t = c + A_1 y_{t-1} + A_2 y_{t-2} + \dots + A_p y_{t-p} + u_t$$

where A_i are $K \times K$ coefficient matrices, $u_t \sim N(0, \Sigma)$, Σ = covariance matrix.

For Our 4-Variable System

VAR(2) has:

- ▣ 4 intercepts
- ▣ $2 \times 4 \times 4 = 32$ AR coefficients
- ▣ **36 parameters total**

Lag Selection

Use information criteria:

- ▣ AIC: Tends to overfit
- ▣ **BIC**: More parsimonious
- ▣ Cross-validation on held-out data

Information Criteria for Model Selection

Definition 7 (Akaike and Bayesian Information Criteria)

For a model with log-likelihood \mathcal{L} , k parameters, and n observations:

$$\text{AIC} = -2\mathcal{L} + 2k$$

$$\text{BIC} = -2\mathcal{L} + k \ln(n)$$

AIC

- Asymptotically efficient
- May overfit with small n
- Minimizes prediction error

BIC

- Consistent (finds true model)
- Heavier penalty: $\ln(n) > 2$ if $n \geq 8$
- More parsimonious

VAR: Lag Selection and Estimation

BIC by Lag Order

Lag	BIC
1	-4.810
2	-5.178 Best
3	-4.633
4	-4.614

Data Split

Set	Period	N
Training (70%)	2001-Q1 to 2017-Q4	67
Validation (20%)	2018-Q1 to 2022-Q4	20
Test (10%)	2023-Q1 to 2025-Q2	10
Total		97

Validation Check

VAR(2) also achieves lowest validation RMSE.

VAR Model Stability

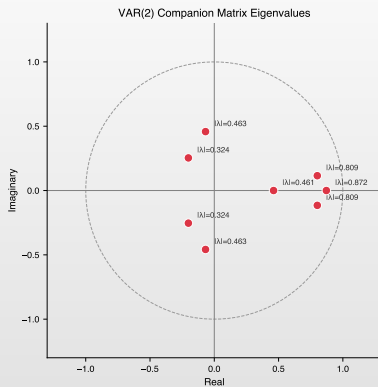
Stability condition

- All eigenvalues of the companion matrix:
 $|\lambda_i| < 1, \forall i$

VAR(2) Results — economic data

$ \lambda_1 , \lambda_2 $	0.324
$ \lambda_3 , \lambda_4 $	0.463
$ \lambda_5 $	0.461
$ \lambda_6 $	0.872
$ \lambda_7 , \lambda_8 $	0.810

- $\text{Max } |\lambda| = 0.872 < 1$ — **stable**



VAR vs VECM: Cointegration

Problem

- ▣ If variables are $I(1) \Rightarrow$ VAR on levels produces spurious regressions

Definition 8 (VECM)

$$\Delta y_t = \Pi y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + u_t, \quad \Pi = \alpha \beta'$$

Key message

- ▣ VAR on differences: loses long-run relationship; VECM: preserves it through $\Pi = \alpha \beta'$

Johansen Test — economic data

r	Trace	CV 5%	Reject?
0	64.09	47.85	Yes
1	24.03	29.80	No
2	11.89	15.49	No
3	1.28	3.84	No

- ▣ **1 cointegrating relation** found
- ▣ VECM more appropriate than VAR on levels

Granger Causality: Empirical Results

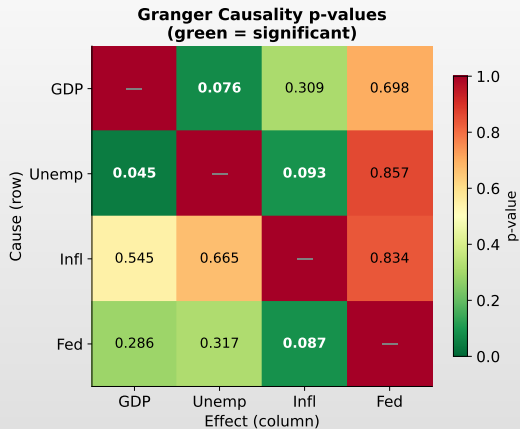
Interpretation

Each cell shows p-value for testing whether the row variable Granger-causes the column variable. Green: $p < 0.10$. Read: row causes column.

Economic Findings

- Unemp \rightarrow GDP ($p = 0.045$): Okun's Law
- Fed \rightarrow Inflation ($p = 0.087$): Monetary policy transmission
- GDP \rightarrow Unemp: Weak evidence

Granger Causality: p -value Heatmap



Granger Causality: Formal Definition

Definition 9 (Granger Causality)

X **Granger-causes** Y if, for some $h > 0$:

$$\mathbb{E} \left[(Y_{t+h} - \mathbb{E}[Y_{t+h} | \mathcal{F}_t^{X,Y}])^2 \right] < \mathbb{E} \left[(Y_{t+h} - \mathbb{E}[Y_{t+h} | \mathcal{F}_t^Y])^2 \right]$$

where $\mathcal{F}_t^{X,Y}$ includes past of both X and Y , while \mathcal{F}_t^Y includes only past Y .

Important Caveat

- Granger causality = **predictive causality**, not true causality
- “ X Granger-causes Y ” \Rightarrow X contains useful info for forecasting Y
- Does **not** imply X causes Y in a structural sense

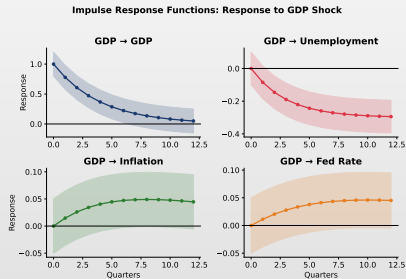
Test Procedure

- F-test (or Wald): H_0 : coefficients on lagged X are jointly zero in Y equation
- Reject $H_0 \Rightarrow X$ Granger-causes Y

Impulse Response Functions (IRF)

Effects

□ \uparrow GDP \Rightarrow \downarrow Unemployment (Okun), \uparrow Inflation (demand), Fed raises rate

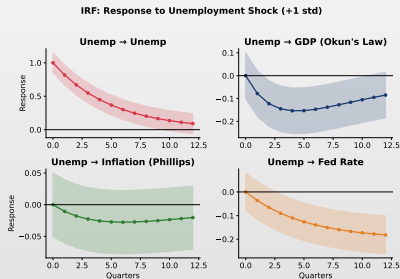


TSA_ch10_irf_gdp_shock

IRF: Unemployment Shock

Effects

□ \uparrow Unemp \Rightarrow \downarrow GDP, \downarrow Infl, Fed cuts rates

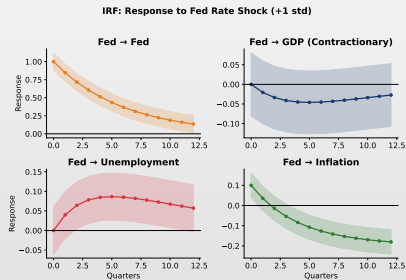


 TSA_ch10_irf_unemp_shock

IRF: Fed Rate Shock

Monetary Policy

☐ Rate hike \Rightarrow GDP \downarrow , Unemp \uparrow , Infl \downarrow

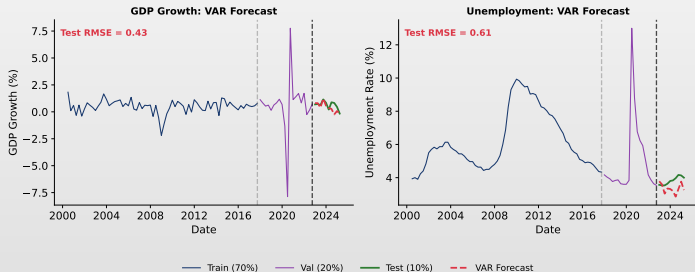


 TSA_ch10_irf_fed_shock

VAR: Forecast (Train/Val/Test)

Rolling One-Step-Ahead Forecast

- VAR captures GDP-Unemployment dynamics
- COVID shock visible in test period



VAR: Test Set Results

Test Set Performance by Variable

Variable	RMSE	MAE	Dir. Acc.
GDP Growth	1.33	0.99	50%
Unemployment	0.64	0.52	50%
Inflation	1.56	1.12	60%
Fed Rate	2.59	2.45	80%
Average	1.53	1.27	60%

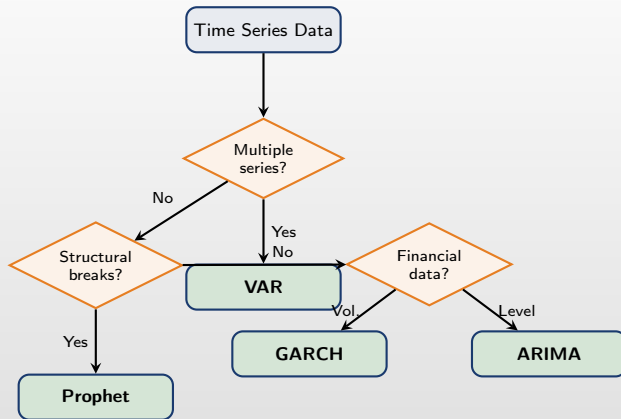
Strengths

- Cross-variable dynamics
- Good directional accuracy

Limitations

- Many parameters
- Sensitive to lag selection

Model Selection Framework



Comprehensive Model Comparison

Feature	GARCH	Fourier	Prophet	VAR
Target	Volatility	Level	Level	Multiple
Seasonality	No	Yes (long)	Yes (multi)	No
Structural breaks	No	No	Yes	No
Multiple series	No	No	No	Yes
Interpretable	Medium	High	High	High
Parameters	Few	2K	Auto	Many
Missing data	No	No	Yes	No
Best for	Finance	Cycles	Business	Macro

Empirical Conclusions

- ▣ **GARCH**: Student-t $>$ Normal ($\Delta\text{AIC} = 509$)
- ▣ **Fourier**: $K = 3$ harmonics, validated on validation set
- ▣ **Prophet**: adapts to breaks via changepoints
- ▣ **VAR**: significant macro interactions (Granger)

Key Insight

- ▣ RMSE cannot be compared across different datasets!
- ▣ Each model excels in its domain
- ▣ The art: matching model \leftrightarrow data

Best Practices for Applied Forecasting

Methodology

1. **Explore** data
2. **Test** stationarity
3. **Split** train/val/test
4. **Compare** on validation
5. **Report** test metrics

Common Mistakes

- Peeking at test data
- Over-fitting
- Ignoring assumptions

Practical Tips

- Start simple (naive)
- Add complexity if needed
- Check residuals
- Report CIs

Remember

"All models are wrong, but some are useful." — Box

Forecasting vs Causality vs Decision

Objective	Model	Focus
Pure prediction	ARIMA / ML	Out-of-sample accuracy
Financial risk	GARCH	Volatility, VaR
Macro dynamics	VAR	Multivariate interactions
Structural relations	SVAR / VECM	Causal identification
Regimes	Markov Switching	Regime changes

Key Message

- ▣ There is no universal model
- ▣ There is **fit between model and problem**

Key Takeaways

1. Rigorous Methodology

- ▶ Train/validation/test split prevents overfitting
- ▶ Test set must remain untouched until final evaluation

2. Match Model to Data

- ▶ Financial volatility \Rightarrow GARCH
- ▶ Long seasonality \Rightarrow Fourier terms
- ▶ Structural breaks \Rightarrow Prophet
- ▶ Multiple series \Rightarrow VAR

3. Interpret Results Carefully

- ▶ Granger causality \neq true causality
- ▶ Out-of-sample performance matters most
- ▶ Simpler models often work better

The Role of AI in Time Series Modeling

AI can

- ▣ Generate code for estimation and forecasting
- ▣ Select models (AutoML, grid search)
- ▣ Combine forecasts (ensemble)
- ▣ Detect anomalies and patterns

But cannot

- ▣ Replace statistical validation
- ▣ Automatically detect **data leakage**
- ▣ Guarantee correct economic interpretation
- ▣ Verify model assumptions

Principle

- ▣ AI is a **tool**, not an authority
- ▣ Statistical validation remains the researcher's responsibility

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download monthly US Retail Sales from FRED (series RSXFS) for 2010-01 to 2024-12 (180 observations). Perform a complete time series analysis: decomposition, stationarity tests, model selection (compare ETS, SARIMA, and Prophet), 12-month forecast, and evaluation using RMSE/MAE/MASE on a 70/15/15 temporal split. Give me publication-quality Python code."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it follow the correct workflow? (plot → decompose → test → model → diagnose → forecast)
3. Does it compare multiple models (ETS, ARIMA, SARIMA) with proper benchmarks?
4. Is the train/test split done properly? Is there any data leakage?
5. Does it discuss limitations and assumptions of the chosen model?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Synthesis: From Theory to Practice

Chapter	Concept	Formula	Application
Ch1	Stationarity	$\gamma(h) = \text{Cov}(X_t, X_{t+h})$	ADF/KPSS Test
Ch2	ARMA	$\phi(L)X_t = \theta(L)\varepsilon_t$	Box-Jenkins
Ch3	ARIMA	$(1 - L)^d X_t \sim \text{ARMA}$	Differencing
Ch4	SARIMA	$(1 - L^s)^D X_t$	Seasonality
Ch5	GARCH	$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$	VaR
Ch6	VAR	$\mathbf{Y}_t = \mathbf{A}_1 \mathbf{Y}_{t-1} + \varepsilon_t$	Granger
Ch7	VECM	$\Delta \mathbf{Y}_t = \Pi \mathbf{Y}_{t-1} + \dots$	Cointegration
Ch8	Extensions	ARFIMA, TAR, MSAR	Long memory
Ch9	Prophet	$y(t) = g(t) + s(t) + h(t) + \varepsilon_t$	Forecasting

Question 1

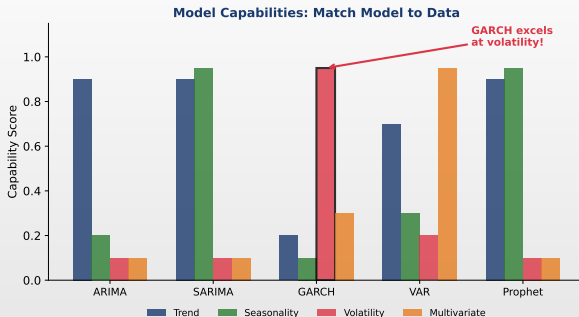
Question

☐ Which model would you choose to forecast the volatility of financial returns?

Answer Choices

- (A) ARIMA — captures trends and autocorrelations
- (B) GARCH — models conditional variance
- (C) Prophet — detects changepoints and seasonality
- (D) VAR — multivariate model for interdependencies

Question 1: Answer



Answer: (B)

- GARCH captures volatility clustering and time-varying risk. ARIMA models the level, Prophet handles seasonality, VAR captures cross-series dynamics — none model variance directly.

Question 2

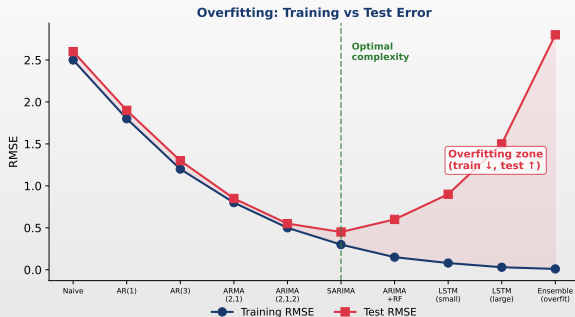
Question

- ☐ A SARIMA model achieves $\text{RMSE} = 0.05$ on training but $\text{RMSE} = 2.30$ on test. What does this indicate?

Answer Choices

- (A) The model is excellent — low training error confirms quality
- (B) The model suffers from overfitting — it memorizes noise
- (C) The test set is faulty and should be replaced
- (D) The difference is normal — all models have higher test error

Question 2: Answer



Answer: (B)

- A $46\times$ ratio between test and training RMSE signals severe overfitting. The model fits noise in the training data and fails to generalize. Solution: simpler model, proper validation.

Question 3

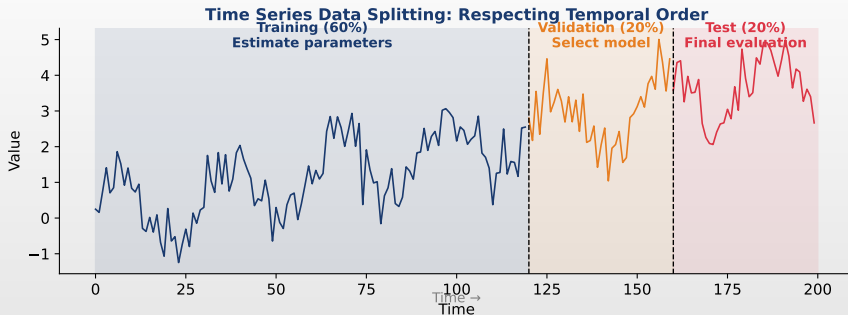
Question

☐ Why is it important to separate data into train/validation/test sets?

Answer Choices

- (A) To have more training data
- (B) To prevent overfitting and evaluate correctly
- (C) It is just a convention with no real importance
- (D) To reduce computation time

Question 3: Answer



Answer: (B)

- Train: estimate parameters. Validation: select model/hyperparameters. Test: final unbiased evaluation. Mixing these roles leads to optimistic performance estimates.

Question 4

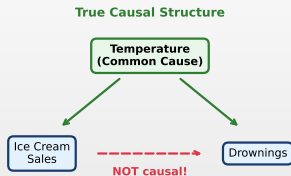
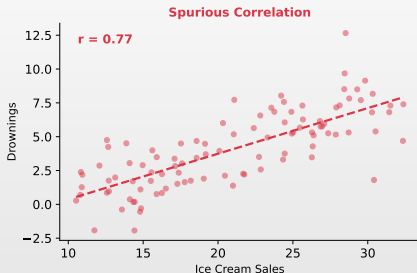
Question

☐ Is Granger causality equivalent to true (structural) causality?

Answer Choices

- (A) Yes — if X predicts Y , then X causes Y
- (B) No — it only tests predictive content, not causation
- (C) It depends on the number of lags selected
- (D) Yes, if the p-value is below 0.05

Question 4: Answer

Granger Causality \neq True Causality

Answer: (B)

- Granger causality tests whether past X improves forecasts of Y . Spurious correlations (e.g., ice cream sales and drownings) can pass the test due to common causes.

Question 5

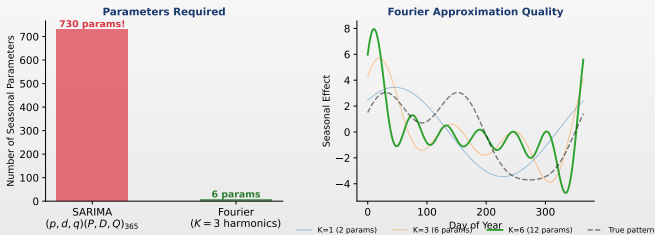
Question

□ What model do you use for a series with long seasonality (e.g., $s = 365$ days)?

Answer Choices

- (A) $\text{SARIMA}(p, d, q)(P, D, Q)_{365}$
- (B) GARCH — models variation
- (C) ARIMA + Fourier terms or Prophet/TBATS
- (D) VAR with 365 lags

Question 5: Answer

Long Seasonality ($s = 365$): Fourier Terms vs SARIMA

Answer: (C)

- SARIMA₃₆₅ requires lag polynomials of order 365 — infeasible. Fourier terms ($K=3$) use only 6 parameters. Prophet and TBATS handle multiple seasonalities automatically.

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- ▣ Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.

Domain-Specific References

- ▣ Tsay, R.S. (2010). *Analysis of Financial Time Series*, 3rd ed., Wiley. (GARCH, VAR)
- ▣ Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*, Springer. (VAR, VECM)
- ▣ Francq, C., & Zakoïan, J.-M. (2019). *GARCH Models*, 2nd ed., Wiley. (Volatility)

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- ▣ Makridakis, S., Spiliotis, E., & Assimakopoulos, V. (2020). The M4 Competition, *International Journal of Forecasting*, 36(1), 54–74.
- ▣ Taylor, S.J., & Letham, B. (2018). Forecasting at Scale, *The American Statistician*, 72(1), 37–45.

Online Resources and Code

- ▣ **Quantlet:** <https://quantlet.com> – Code platform for quantitative methods
- ▣ **Quantinar:** <https://quantinar.com> – Learning platform for quantitative methods
- ▣ **GitHub TSA:** https://github.com/QuantLet/TSA/tree/main/TSA_ch10 – Python code for this chapter

Course Summary

What We Learned

- Model selection depends on data characteristics: stationarity, seasonality, volatility
- The Box-Jenkins methodology provides a systematic framework for time series modeling
- Proper evaluation requires out-of-sample testing and time series cross-validation

Important

No single model wins everywhere. Match the model to the data: ARIMA for trends, SARIMA for seasonality, GARCH for volatility, VAR/VECM for multivariate dynamics, Prophet/TBATS for complex patterns. Always validate out-of-sample!

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Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31(3), 307-327.



Sims, C.A. (1980). Macroeconomics and Reality. *Econometrica*, 48(1), 1-48.

Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar