Time Series Analysis and Forecasting

Chapter 5: GARCH and Volatility

Daniel Traian PELE

Bucharest University of Economic Studies
IDA Institute Digital Assets
Blockchain Research Center
AI4EFin Artificial Intelligence for Energy Finance
Romanian Academy, Institute for Economic Forecasting
MSCA Digital Finance

Learning Objectives

By the end of this chapter, you will be able to:

1. Understand **volatility clustering** and stylized facts of financial returns
2. Estimate and interpret **ARCH** and **GARCH** models
3. Apply asymmetric models (**EGARCH**, **GJR-GARCH**) for the leverage effect
4. Perform model validation and selection
5. Forecast volatility and calculate **Value at Risk (VaR)**

Practical Skills

- Python implementation with the `arch` package
  - Estimation, forecasting, and automatic diagnostics
- Interpreting parameters and volatility persistence
- VaR calculation for risk management
  - Backtesting and forecast validation

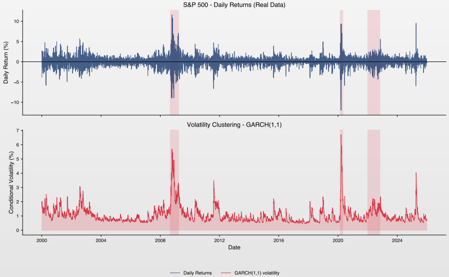
Outline

Foundations

Applications

Volatility Clustering

- High volatility periods are followed by high volatility; calm by calm
- This suggests that **conditional variance** is predictable



TSA_ch5_clustering

Why Model Volatility?

Empirical Observations in Financial Series

- Financial returns exhibit **volatility clustering** — periods of high volatility tend to be followed by periods of high volatility
- The distribution of returns has **fat tails** (leptokurtosis)
- Return correlation is nearly zero, but correlation of squares is significant
- Volatility responds **asymmetrically** to shocks (leverage effect)

Limitation of ARIMA Models

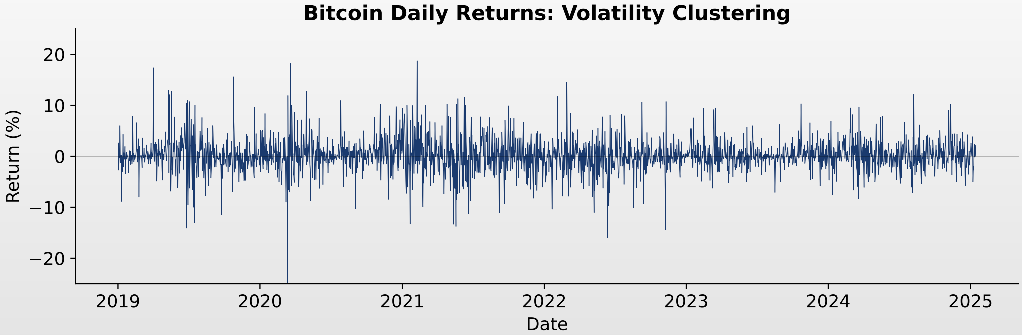
ARIMA models assume **constant variance** (homoskedasticity), which is not realistic for financial series!

Example: Bitcoin – Volatility Clustering

Observations

- Bitcoin daily returns (2019–2025): extremely pronounced volatility clustering
  - Returns of $\pm 20\%$ during crisis periods (COVID, Terra/Luna)
- Bitcoin volatility is significantly higher than traditional assets
  - Typical $\alpha \approx 0.10$–0.20 (fast reaction to news)

Example: Bitcoin – Volatility Clustering



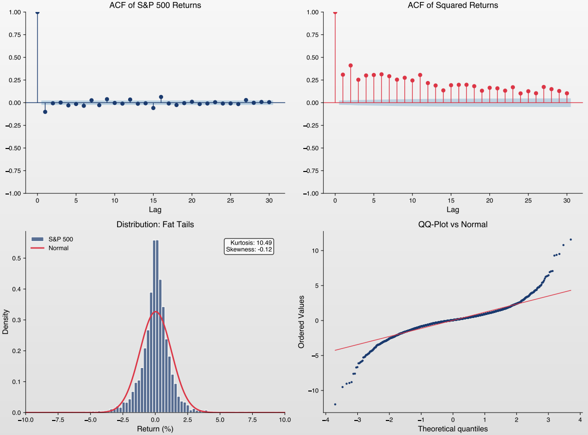
TSA_ch5_btc_returns

Stylized Facts of Financial Returns

Observed Properties

- **No autocorrelation** in returns
  - $\text{Corr}(r_t, r_{t-k}) \approx 0$ — returns are unpredictable
- **Autocorrelation** in r_t^2, $|r_t|$
  - $\text{Corr}(r_t^2, r_{t-k}^2) > 0$ — volatility is predictable
- **Fat tails** (kurtosis > 3)
  - More extreme values than the normal distribution predicts
- **Leverage effect**
  - Negative shocks increase volatility more than positive shocks
- **Volatility clustering**
  - High volatility periods are followed by high volatility

Stylized Facts of Financial Returns

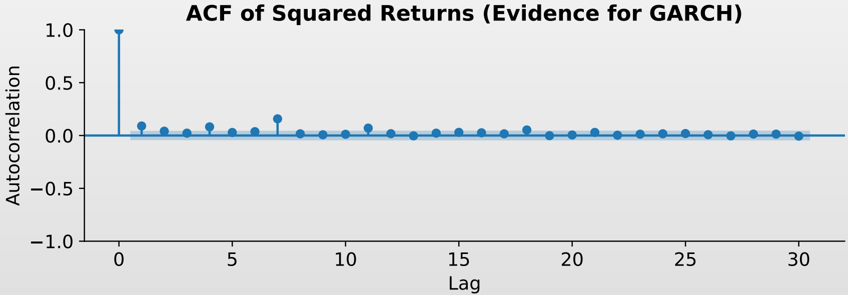


TSA_ch5_stylized

Example: Bitcoin – Evidence for ARCH Effects

Interpretation

- ACF(r_t^2) is significant at many lags $\Rightarrow$ **ARCH effects present**
- Conditional variance is predictable $\Rightarrow$ GARCH models are justified



TSA_ch5_btc_arch

Conditional Heteroskedasticity

Definition 1 (Conditional Variance)

For return series $\{r_t\}$, the **conditional variance** at time t is: $\sigma_t^2 = \text{Var}(r_t|\mathcal{F}_{t-1}) = \mathbb{E}[(r_t - \mu_t)^2|\mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the information available up to time $t-1$.

General Model

$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0, 1)$

- μ_t = conditional mean (ARMA); σ_t^2 = conditional variance (GARCH)
- z_t = standardized innovations (Normal, Student-t, GED)

Researcher Spotlight: Engle & Bollerslev

Robert Engle (*1942)

Nobel Prize 2003

W Wikipedia

Tim Bollerslev (*1958)

W Wikipedia

Biography

- **Robert Engle**: American economist at NYU Stern. Nobel Prize (2003) "for methods of analyzing economic time series with time-varying volatility (ARCH)"
- **Tim Bollerslev**: Danish-American economist at Duke University, PhD student of Engle

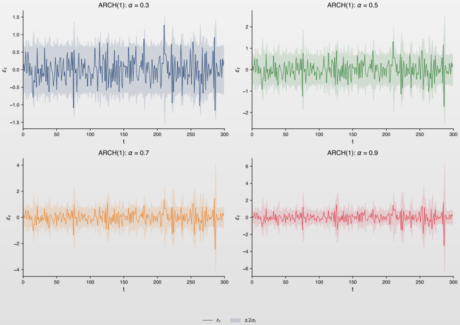
Key Contributions

- **ARCH model** (Engle, 1982) — autoregressive conditional heteroskedasticity
- **GARCH model** (Bollerslev, 1986) — generalized ARCH with persistent volatility
- **Realized volatility** measures and high-frequency econometrics
- Foundation for modern financial risk management (VaR, ES)

ARCH(1) Simulation: Effect of α Parameter

Interpretation

- Higher α means volatility reacts more strongly to recent shocks



TSA_ch5_arch_sim

The ARCH(q) Model — Engle (1982)

Definition 2 (ARCH(q))

The **Autoregressive Conditional Heteroskedasticity** model of order q:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1), \quad \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

Stationarity Restrictions

- $\omega > 0$ (positive base variance), $\alpha_i \geq 0$ (non-negativity)
- $\sum_{i=1}^{q} \alpha_i < 1$ (stationarity)

Properties of the ARCH(1) Model

ARCH(1): $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$

- **Unconditional variance**: $\mathbb{E}[\varepsilon_t^2] = \dfrac{\omega}{1-\alpha_1}$ (if $\alpha_1 < 1$)
- **Kurtosis**: $\kappa = 3 \cdot \dfrac{1-\alpha_1^2}{1-3\alpha_1^2}$ (if $\alpha_1^2 < 1/3$)
- Kurtosis > 3 for $\alpha_1 > 0 \Rightarrow$ **fat tails**!

Numerical Example

If $\omega = 0.0001$ and $\alpha_1 = 0.3$:

- Unconditional variance: $\sigma^2 = \frac{0.0001}{1-0.3} = 0.000143$
- Kurtosis: $\kappa = 3 \cdot \frac{1-0.09}{1-0.27} = 3.74 > 3$

Derivation: Unconditional Variance of ARCH(1)

Derivation.

Let $\varepsilon_t = \sigma_t z_t$ with $z_t \sim N(0,1)$ and $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$.

Step 1: Take unconditional expectation:

$$\mathbb{E}[\varepsilon_t^2] = \mathbb{E}[\sigma_t^2 z_t^2] = \mathbb{E}[\sigma_t^2] \cdot \mathbb{E}[z_t^2] = \mathbb{E}[\sigma_t^2]$$

Step 2: Apply expectation to variance equation:

$$\mathbb{E}[\sigma_t^2] = \mathbb{E}[\omega + \alpha_1 \varepsilon_{t-1}^2] = \omega + \alpha_1 \mathbb{E}[\varepsilon_{t-1}^2]$$

Step 3: By stationarity, $\mathbb{E}[\varepsilon_t^2] = \mathbb{E}[\varepsilon_{t-1}^2] = \sigma^2$:

$$\sigma^2 = \omega + \alpha_1 \sigma^2 \quad \Rightarrow \quad \sigma^2(1 - \alpha_1) = \omega$$

Result: $\boxed{\sigma^2 = \frac{\omega}{1 - \alpha_1}}$ (requires $\alpha_1 < 1$ for stationarity) $\square$

Derivation: Kurtosis of ARCH(1)

For $\varepsilon_t = \sigma_t z_t$ with $z_t \sim N(0,1)$:
Step 1: $\mathbb{E}[\varepsilon_t^4] = \mathbb{E}[\sigma_t^4] \cdot \mathbb{E}[z_t^4] = 3\mathbb{E}[\sigma_t^4]$ (since $\mathbb{E}[z^4] = 3$)
Step 2: Using $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$:

$$\mathbb{E}[\sigma_t^4] = \mathbb{E}[(\omega + \alpha_1 \varepsilon_{t-1}^2)^2] = \omega^2 + 2\omega\alpha_1\sigma^2 + \alpha_1^2 \mathbb{E}[\varepsilon_{t-1}^4]$$

Step 3: Solving the recursion yields:

$$\boxed{\kappa = \frac{\mathbb{E}[\varepsilon_t^4]}{(\mathbb{E}[\varepsilon_t^2])^2} = 3 \cdot \frac{1 - \alpha_1^2}{1 - 3\alpha_1^2}}$$

Interpretation

- $\kappa > 3$ for any $\alpha_1 > 0 \Rightarrow$ **fat tails** (leptokurtosis)
- Requires $\alpha_1 < 0.577$ for finite fourth moment
- ARCH naturally generates heavy-tailed distributions!

Theory vs Empirics: The Kurtosis Gap

ARCH(1) Kurtosis: Too Low!

- For typical stock returns: $\alpha_1 \approx 0.09$ (S&P 500)
- Theoretical kurtosis: $\kappa = 3 \cdot \frac{1-0.09^2}{1-3\times 0.09^2} \approx \mathbf{3.05}$
- Empirical kurtosis of S&P 500 daily returns: $\kappa \approx \mathbf{13.8}$

The Kurtosis Gap

$$\underbrace{3.05}_{\text{ARCH(1)-Normal}} \quad \ll \quad \underbrace{13.8}_{\text{Empirical}}$$

ARCH(1) with Gaussian innovations cannot match the extreme kurtosis observed in financial data.

Solutions

- **Student-t innovations**: $z_t \sim t_\nu$ adds kurtosis from the distribution itself
- **GARCH(1,1)**: persistence ($\beta > 0$) generates additional kurtosis
- Both combined: Student-t GARCH captures $\kappa \gg 3$ observed in practice

Testing for ARCH Effects

Engle's Test for ARCH Effects

Procedure:

1. Estimate the mean model and obtain residuals $\hat{\varepsilon}_t$
2. Calculate $\hat{\varepsilon}_t^2$
3. Regress $\hat{\varepsilon}_t^2$ on its lags:
$$\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \cdots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$$
4. Calculate the statistic $LM = T \cdot R^2 \sim \chi^2(q)$

Hypotheses

- H_0: No ARCH effects ($\alpha_1 = \cdots = \alpha_q = 0$)
- H_1: ARCH effects present (at least one $\alpha_i \neq 0$)

Limitations of the ARCH Model

Practical Problems

1. **High order** — many lags are usually needed (large q)
2. **Many parameters** — estimation difficulties
3. **Non-negativity constraints** — difficult to impose for large q
4. **Does not capture persistence** — observed volatility is very persistent

The Solution

The GARCH Model — introduces lags of conditional variance to capture persistence with fewer parameters!

The GARCH(1,1) Model

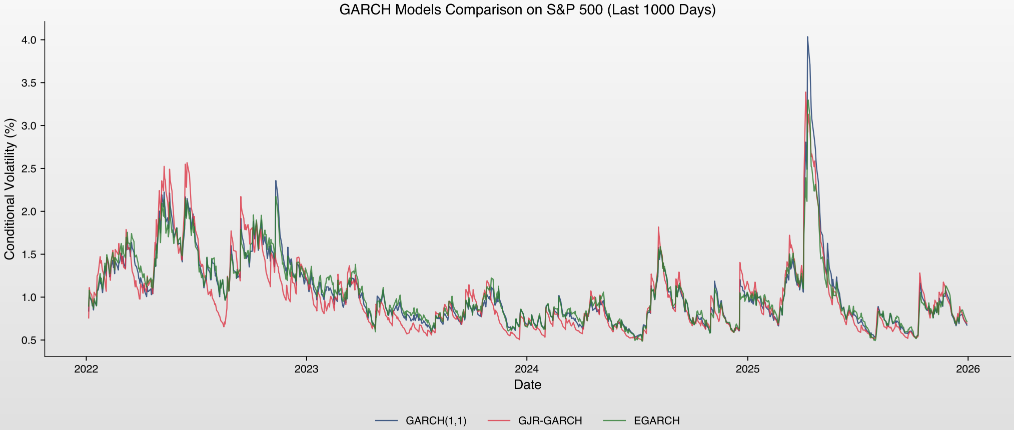
The Most Popular Volatility Model

$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$

Restrictions & Properties

- $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$; $\alpha + \beta < 1$ (stationarity)
- $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$; Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$

The GARCH(1,1) Model



The GARCH(p,q) Model — Bollerslev (1986)

Definition 3 (GARCH(p,q))

The **Generalized ARCH** model:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

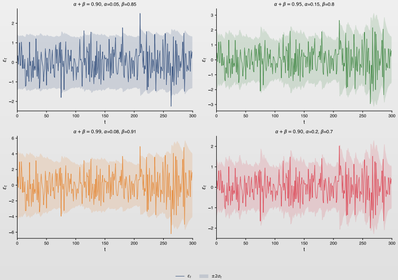
Interpretation

- ω = base level of volatility
- α_i = reaction to recent shocks (news coefficients)
- β_j = volatility persistence (memory)
- $\alpha + \beta$ = total persistence

GARCH(1,1) Simulation: Persistence Effect

Interpretation

- α controls reaction to shocks
- β controls persistence
- The sum $\alpha + \beta$ determines mean-reversion speed



TSA_ch5_garch_sim

Derivation: Unconditional Variance of GARCH(1,1)

Derivation.

For $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$:

Step 1: Take unconditional expectation: $\mathbb{E}[\sigma_t^2] = \omega + \alpha\mathbb{E}[\varepsilon_{t-1}^2] + \beta\mathbb{E}[\sigma_{t-1}^2]$

Step 2: By stationarity, $\mathbb{E}[\sigma_t^2] = \mathbb{E}[\sigma_{t-1}^2] = \bar{\sigma}^2$ and $\mathbb{E}[\varepsilon_t^2] = \bar{\sigma}^2$: $\bar{\sigma}^2 = \omega + (\alpha + \beta)\bar{\sigma}^2$

Step 3: Solve: $\bar{\sigma}^2(1 - \alpha - \beta) = \omega \Rightarrow \boxed{\bar{\sigma}^2 = \dfrac{\omega}{1 - \alpha - \beta}}$ □

Stationarity Condition

Requires $\alpha + \beta < 1$ for finite unconditional variance.

GARCH(1,1) as ARMA for ε_t^2

ARMA(1,1) Representation

Define $\nu_t = \varepsilon_t^2 - \sigma_t^2$ (variance shock). Then:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

This is an **ARMA(1,1)** for ε_t^2!

Implications

- ACF of ε_t^2 decays exponentially (like ARMA)
- Persistence is given by $\alpha + \beta$
- PACF can help identify the order

Derivation: ARMA Representation of GARCH(1,1)

Derivation.

Step 1: Define variance shock: $\nu_t = \varepsilon_t^2 - \sigma_t^2$

- $\mathbb{E}[\nu_t | \mathcal{F}_{t-1}] = \mathbb{E}[\varepsilon_t^2 | \mathcal{F}_{t-1}] - \sigma_t^2 = \sigma_t^2 - \sigma_t^2 = 0$
- ν_t is a martingale difference sequence

Step 2: Substitute $\sigma_t^2 = \varepsilon_t^2 - \nu_t$ into GARCH equation:

$$\varepsilon_t^2 - \nu_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta(\varepsilon_{t-1}^2 - \nu_{t-1})$$

Step 3: Rearrange:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

Result: ARMA(1,1) with AR coefficient $\phi = \alpha + \beta$ and MA coefficient $\theta = -\beta$. □

Derivation: Volatility Persistence and Half-Life

Multi-Step Forecast GARCH(1,1)

- $\mathbb{E}_t[\sigma^2_{t+h}] = \bar{\sigma}^2 + (\alpha + \beta)^{h-1}(\sigma^2_{t+1} - \bar{\sigma}^2)$

Derivation

- **Step 1**: Let $\phi = \alpha + \beta$ and $q_t = \sigma^2_t - \bar{\sigma}^2$ (deviation from mean)
- **Step 2**: From the GARCH equation: $\mathbb{E}_t[q_{t+1}] = \phi \cdot q_t$, so $\mathbb{E}_t[q_{t+h}] = \phi^h \cdot q_t$
- **Step 3**: Half-life = time until deviation halves: $\phi^{HL} = 0.5 \Rightarrow HL = \frac{\ln(0.5)}{\ln(\phi)} = \frac{-0.693}{\ln(\alpha+\beta)}$

Example: S&P 500

- With $\alpha + \beta = 0.988$: $HL = \frac{-0.693}{-0.012} \approx$ **58 days** (shocks persist $\sim$3 months!)

Estimation of GARCH Models

Maximum Likelihood Estimation (MLE)

Log-likelihood (normal): $\ell(\theta) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln(\sigma_t^2) + \frac{\varepsilon_t^2}{\sigma_t^2}\right]$

Alternative Distributions for z_t

- **Student-t**: captures fat tails — most common choice
- **GED**: flexibility for kurtosis
- **Skewed Student-t**: asymmetry and fat tails

Practical Note

Student-t distribution typically provides better fit for financial returns due to fat tails (kurtosis > 3).

Quasi-MLE and Robustness

What if the Distribution is Misspecified?

- Standard MLE assumes $z_t \sim N(0,1)$, but true distribution may differ
- **Quasi-MLE (QMLE)**: maximize the Gaussian log-likelihood even if z_t is non-normal
- Key insight: we only need the *first two conditional moments* to be correct

Definition 4 (QMLE Consistency — Bollerslev & Wooldridge (1992))

If the conditional mean and variance are correctly specified, QMLE is consistent for (ω, α, β) regardless of the true distribution of z_t.

Sandwich (Robust) Standard Errors

Standard errors must be corrected:

$$\hat{V}_{\textbf{robust}} = \hat{A}^{-1}\hat{B}\hat{A}^{-1} \quad \text{(White, 1982)}$$

where $\hat{A}$ = Hessian, $\hat{B}$ = outer product of scores. In Python: `result.summary()` reports robust SEs by default.

When to Use QMLE vs Full MLE

- **QMLE**: safe default — consistent even under misspecification
- **Full MLE** (Student-t): more *efficient* if the distribution is correct

Practical Estimation Challenges

Common Issues

- **Convergence failure**: Log-likelihood is non-convex; optimizer may get stuck
- **Boundary solutions**: $\hat{\alpha} \approx 0$ or $\hat{\alpha} + \hat{\beta} \approx 1$ (near-IGARCH)
- **Sensitivity to starting values**: Different initializations $\Rightarrow$ different estimates

Best Practices

1. Try multiple optimizers: **BFGS** (fast, gradient-based) vs **Nelder-Mead** (derivative-free)
2. Initialize σ_0^2 with sample variance (backcast)
3. Check $\alpha + \beta < 1$ post-estimation; if ≈ 1, consider IGARCH or regime-switching
4. Compare results across starting values; consistent estimates $\Rightarrow$ reliable model

Rule of Thumb

If BFGS and Nelder-Mead give *different* parameter estimates, the model may be misspecified or the data insufficient.

Typical Values for GARCH(1,1)

Series	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha + \beta}$
S&P 500 daily	0.05–0.10	0.85–0.95	0.95–0.99
EUR/USD daily	0.03–0.08	0.90–0.95	0.95–0.99
Bitcoin daily	0.10–0.20	0.75–0.85	0.90–0.98
Bonds	0.02–0.05	0.90–0.97	0.95–0.99

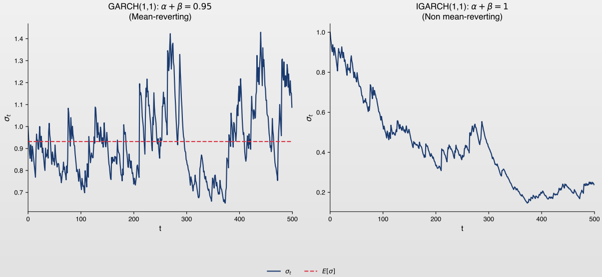
Observations

- $\alpha + \beta$ close to 1 $\Rightarrow$ **very persistent volatility**
- Small α, large β $\Rightarrow$ slow reaction to shocks, long memory
- Bitcoin: larger α $\Rightarrow$ faster reaction to news

GARCH vs IGARCH: Persistence Comparison

Interpretation

- Standard GARCH reverts to unconditional mean
- IGARCH has no finite mean $\Rightarrow$ shocks persist indefinitely



TSA_ch5_igarch

IGARCH — Integrated GARCH

Definition 5 (IGARCH(1,1))

When $\alpha + \beta = 1$:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2$$

Properties

- Unconditional variance does not exist (infinite)
- Shocks have **permanent** effect on volatility
- Used for series with extreme persistence
- Useful for **RiskMetrics** (J.P. Morgan): $\alpha = 0.06$, $\beta = 0.94$

Remark 1

IGARCH is analogous to a unit root in variance!

Volatility Regime Changes

Problem

GARCH assumes a **single regime** with constant parameters. But volatility often exhibits **structural breaks** (e.g., 2008 crisis, COVID-19).

Definition 6 (Markov-Switching GARCH)

Parameters (ω, α, β) switch between K regimes governed by a hidden Markov chain:

$$\sigma_t^2 = \omega_{s_t} + \alpha_{s_t}\varepsilon_{t-1}^2 + \beta_{s_t}\sigma_{t-1}^2, \quad s_t \in \{1, \ldots, K\}$$

Inclan-Tiao ICSS Test

- Detects **multiple change points** in unconditional variance
- Based on cumulative sum of squared observations (CUSUM-type)
- Typical findings: S&P 500 shows breaks at 2001, 2008, 2020

Practical Implications

- If structural breaks present, GARCH persistence $(\alpha + \beta)$ is **upward biased** $\Rightarrow$ spurious IGARCH
- Solution: estimate GARCH separately per regime, or use rolling windows
- Alternative: Markov-switching GARCH with $K = 2$ (low-vol / high-vol regimes)

Leverage Effect

Definition

Leverage effect: Negative shocks increase volatility **more** than positive shocks of the same magnitude.

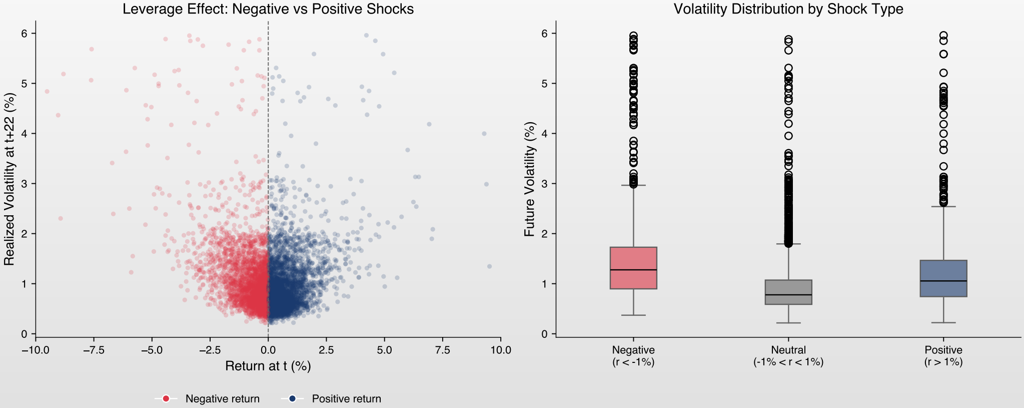
Problem with GARCH

- Standard GARCH: $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$ — only ε_{t-1}^2 matters, sign is lost!
- Economic intuition: Bad news $\Rightarrow$ stock price falls $\Rightarrow$ debt/equity ratio rises $\Rightarrow$ volatility increases

Empirical Evidence

- Black (1976): first documentation of the leverage effect in equity markets
- The effect is asymmetric: 5% drops increase volatility more than 5% rises
- Solutions: EGARCH, GJR-GARCH, TGARCH — models that distinguish the sign of shocks

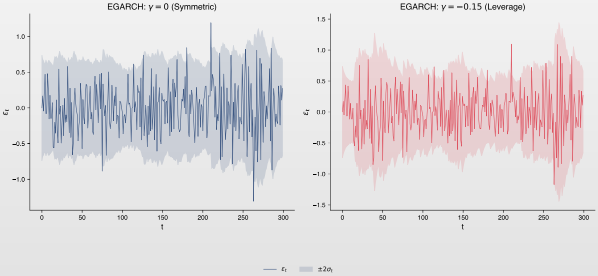
Leverage Effect



EGARCH Simulation: Symmetric vs Asymmetric

Interpretation

- When $\gamma < 0$, negative shocks increase volatility more than positive shocks



TSA_ch5_egarch_sim

The EGARCH Model — Nelson (1991)

Definition 7 (EGARCH(1,1))

Exponential GARCH:

$$\ln(\sigma_t^2) = \omega + \alpha\,(|z_{t-1}| - \mathbb{E}[|z_{t-1}|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

where $z_t = \varepsilon_t/\sigma_t$.

EGARCH Advantages

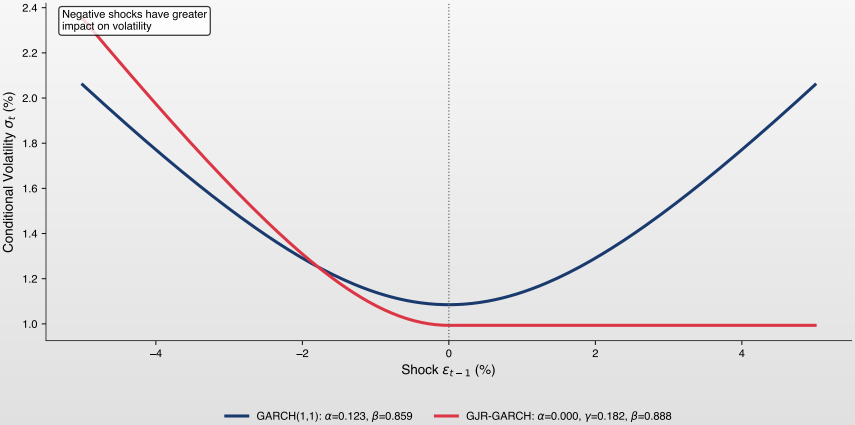
- **No non-negativity constraints required** — models $\ln(\sigma_t^2)$
- **Captures leverage effect** through parameter γ
  - $\gamma < 0$: negative shocks $\Rightarrow$ higher volatility
  - $\gamma = 0$: symmetric effect (like GARCH)
- Persistence is given by β

News Impact Curve — EGARCH

Interpretation

- **News Impact Curve**: relationship between ε_t and σ^2_{t+1}
- **GARCH**: symmetric curve (parabola)
  - Positive and negative shocks have the same impact
- **EGARCH**: asymmetric curve
  - Negative shocks have larger impact on volatility

News Impact Curve — EGARCH



TSA_ch5_nic

The GJR-GARCH Model

Definition 8 (GJR-GARCH(1,1))

Glosten, Jagannathan & Runkle (1993): $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 \cdot I_{t-1} + \beta\sigma_{t-1}^2$ where $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$, else 0.

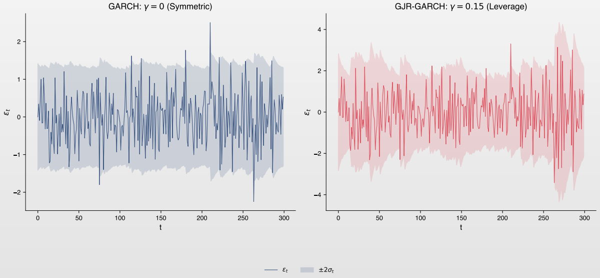
Interpretation

- Positive shocks: impact $= \alpha$;   Negative shocks: impact $= \alpha + \gamma$
- Leverage effect present if $\gamma > 0$
- Stationarity: $\alpha + \gamma/2 + \beta < 1$

GJR-GARCH/TGARCH Simulation

Interpretation

- GJR-GARCH adds an indicator term to capture asymmetric response to negative shocks



TSA_ch5_gjr_sim

TGARCH — Threshold GARCH

Definition 9 (TGARCH(1,1))

Zakoian (1994) models standard deviation: $\sigma_t = \omega + \alpha^+ \varepsilon_{t-1}^+ + \alpha^- \varepsilon_{t-1}^- + \beta \sigma_{t-1}$

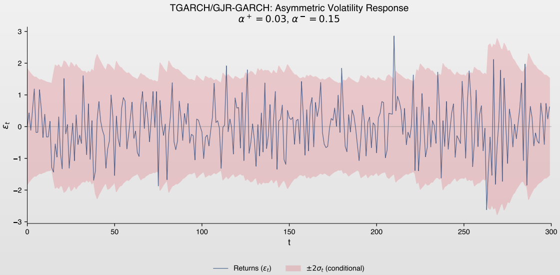
Comparison of Asymmetric Models

Model	**Specification**	**Leverage**
GARCH	σ_t^2	No
EGARCH	$\ln(\sigma_t^2)$	Yes ($\gamma < 0$)
GJR-GARCH	σ_t^2 with indicator	Yes ($\gamma > 0$)
TGARCH	σ_t	Yes ($\alpha^- > \alpha^+$)

TGARCH Simulation: Asymmetric Volatility Response

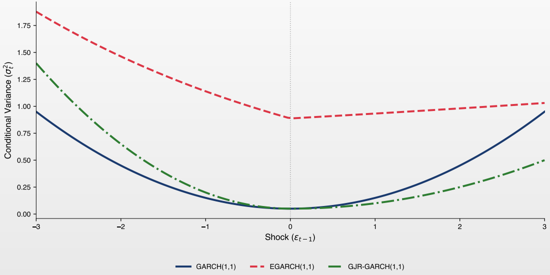
Interpretation

- TGARCH with $\alpha^{+} = 0.03$ and $\alpha^{-} = 0.15 \Rightarrow$ negative shocks amplify volatility by $5\times$
- Volatility bands $\pm 2\sigma$ widen asymmetrically during crisis periods



TSA_ch5_tgarch_sim

News Impact Curves Comparison



Interpretation

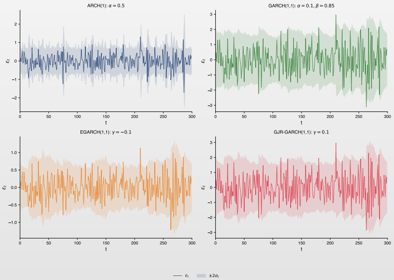
- **Standard GARCH**: symmetric
  - Treats positive and negative shocks identically
- **EGARCH** and **GJR-GARCH**: capture asymmetry
  - Leverage effect: negative shocks $\Rightarrow$ larger impact

TSA_ch5_nic_comp

GARCH Family Comparison

Interpretation

- All models capture volatility clustering, but differ in how they model asymmetry



TSA_ch5_family

GARCH-in-Mean (GARCH-M) — Engle, Lilien & Robins (1987)

Definition 10 (GARCH-M)

- **Model**: Volatility enters directly into the mean equation:

$$r_t = \mu + \delta \cdot g(\sigma_t^2) + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t$$
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

- **Function** g: can be σ_t^2, σ_t, or $\ln(\sigma_t^2)$

Economic Interpretation

- $\delta > 0$: **risk premium** $\Rightarrow$ higher returns when volatility is high
- Formalizes the risk-return relationship (CAPM, Merton ICAPM); test $H_0 : \delta = 0$

Typical Example: Equities

- $r_t = 0.02 + \underbrace{0.15}_{\delta} \cdot \sigma_t + \varepsilon_t \quad \Rightarrow$ At $\sigma_t = 2\%$: $\mathbb{E}[r_t] = 0.023$ (0.3% premium)

GARCH-M: Alternative Specifications

Common Specifications

Risk premium can enter in different forms: (1) $r_t = \mu + \lambda\sigma_t + \varepsilon_t$; (2) $r_t = \mu + \lambda\sigma_t^2 + \varepsilon_t$; (3) $r_t = \mu + \lambda \ln(\sigma_t^2) + \varepsilon_t$

Typical Results for Equity Markets

- Estimated λ often positive but small (0.01–0.10)
- Significance varies across markets and periods
- Variance specification yields larger λ estimates

Remark 2

GARCH-M is used in asset pricing, portfolio optimization, and CAPM testing.

Order Selection

Information Criteria

- **AIC** $= -2\ell + 2k$
- **BIC** $= -2\ell + k\ln(T)$
- **HQIC** $= -2\ell + 2k\ln(\ln(T))$

where $\ell =$ maximized log-likelihood, $k =$ number of parameters.

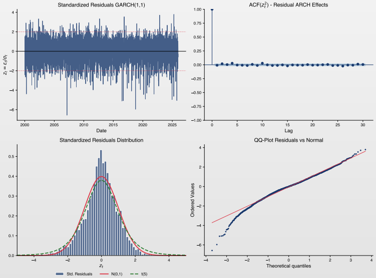
Practical Recommendations

- GARCH(1,1) is sufficient in **90% of cases**
- Check if asymmetric model significantly improves fit
- Choose innovation distribution that minimizes AIC/BIC

Diagnostic Example

Verification

- Standardized residuals should be i.i.d. with no residual ARCH effects



TSA_ch5_diagnostic

GARCH Model Diagnostics

Standardized Residuals

$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$$

If the model is correctly specified, $\hat{z}_t$ should be i.i.d.(0,1).

Diagnostic Checks

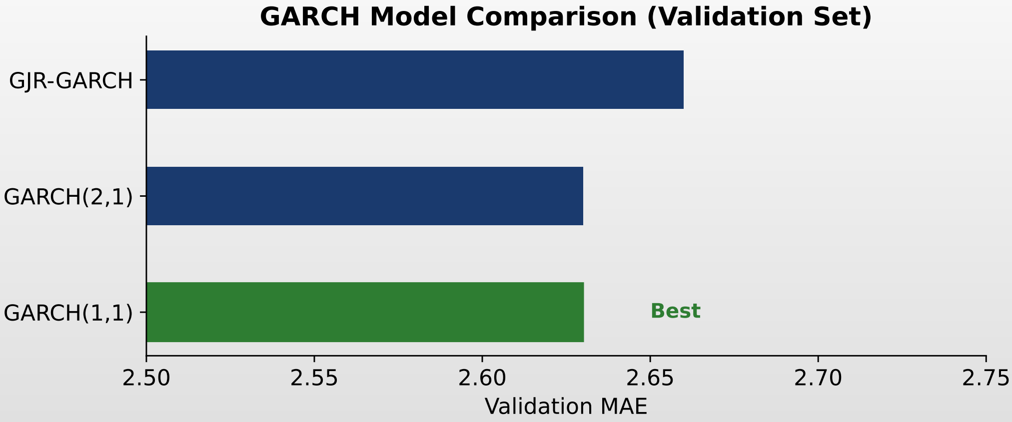
1. **Ljung-Box on $\hat{z}_t$**: check absence of autocorrelation in mean
2. **Ljung-Box on $\hat{z}_t^2$**: check absence of residual ARCH effects
3. **ARCH-LM test on $\hat{z}_t$**: confirm absence of heteroskedasticity
4. **Histogram + QQ-plot**: verify assumed distribution

GARCH Model Comparison: Validation

Interpretation

- GARCH(1,1) achieves the lowest MAE on the validation set
  - More parsimonious and stable than higher-order models
- GARCH(2,1) and GJR-GARCH: similar performance, but more parameters
- **Conclusion**: simplicity wins $\Rightarrow$ GARCH(1,1) is hard to beat

GARCH Model Comparison: Validation



TSA_ch5_comparison

Diebold-Mariano Test

Motivation

Given two competing volatility models, which forecasts better?

Definition 11 (Diebold-Mariano (1995))

Define the loss differential: $d_t = L(e_{1t}) - L(e_{2t})$ where $e_{it} = \sigma_t^2 - \hat{\sigma}_{it}^2$ and $L(\cdot)$ is a loss function (e.g., $L(e) = e^2$).

$$DM = \frac{\bar{d}}{\sqrt{\hat{V}(\bar{d})}} \xrightarrow{d} N(0,1) \quad \text{under } H_0 : \mathbb{E}[d_t] = 0$$

Practical Notes

- H_0: equal predictive accuracy; rejection $\Rightarrow$ one model is significantly better
- Common loss functions: MSE (e^2), MAE ($|e|$), QLIKE ($\log \hat{\sigma}^2 + \sigma^2/\hat{\sigma}^2$)
- QLIKE is robust to noisy volatility proxies (Patton, 2011)
- Use HAC standard errors (Newey-West) if d_t is autocorrelated

Example

EGARCH-t vs GARCH-N on S&P 500: DM test with QLIKE loss typically rejects H_0 in favor of EGARCH-t.

Mincer-Zarnowitz Regression

Forecast Efficiency Test

Regress realized volatility on forecasted volatility:

$$\sigma^2_{t,\textbf{realized}} = a + b\,\hat{\sigma}^2_t + u_t$$

Definition 12 (Hypotheses)

$H_0 : a = 0, \;\; b = 1 \quad$ (unbiased, efficient forecast)
Test via F-test or Wald test. Rejection $\Rightarrow$ forecast is biased or inefficient.

Interpretation

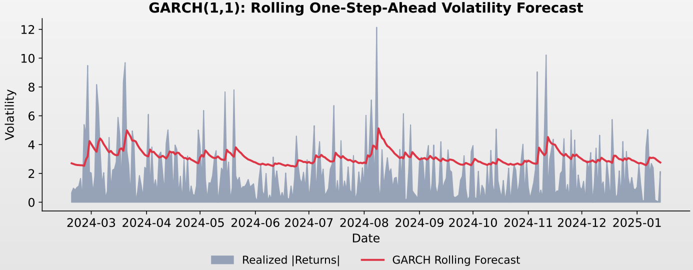
- R^2 measures fraction of variance explained by the forecast
- $b > 1$: forecasts underestimate volatility; $b < 1$: overestimate
- $a > 0$: systematic upward bias in realized volatility

Caveat

Realized variance (r_t^2) is a noisy proxy for $\sigma_t^2 \Rightarrow R^2$ typically low (< 0.10). Prefer RV_t when available.

Volatility Forecast — Visualization

- Forecast converges exponentially to $\bar{\sigma}^2$; speed depends on $\alpha + \beta$
- The closer $\alpha + \beta$ is to 1, the slower the convergence



TSA_ch5_vol_forecast

Forecasting with GARCH(1,1)

One-Step-Ahead Forecast

$\hat{\sigma}^2_{T+1} = \omega + \alpha\varepsilon^2_T + \beta\sigma^2_T$

Multi-Step Forecast

For $h > 1$: $\mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2 + (\alpha+\beta)^{h-1}(\sigma^2_{T+1} - \bar{\sigma}^2)$ where $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ = unconditional variance.

Convergence

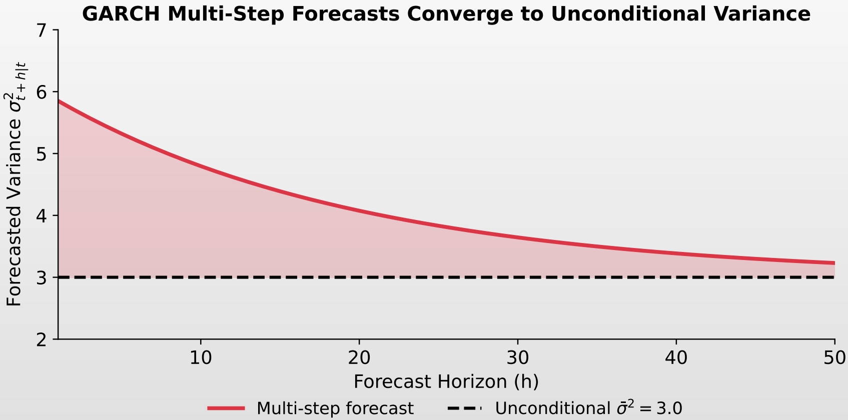
$\lim_{h\to\infty} \mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2$ — forecast converges to unconditional variance!

GARCH Forecast Convergence to Unconditional Variance

Interpretation

- Multi-step forecast converges exponentially to $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- The closer $\alpha + \beta$ is to 1, the slower the convergence
  - S&P 500: $\alpha + \beta \approx 0.99 \Rightarrow$ convergence in $\sim$50 days
  - Bitcoin: $\alpha + \beta \approx 0.95 \Rightarrow$ faster convergence

GARCH Forecast Convergence to Unconditional Variance

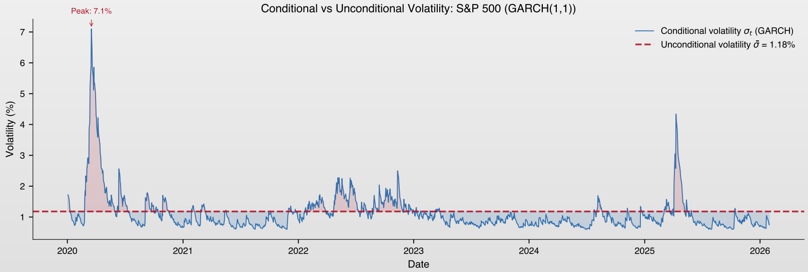


TSA_ch5_convergence

Conditional vs Unconditional Volatility

Interpretation

- **Conditional volatility** σ_t (GARCH): time-varying, reacts to news
- **Unconditional volatility** $\bar{\sigma} = \sqrt{\omega/(1-\alpha-\beta)}$: long-run constant level
- Red zones: $\sigma_t > \bar{\sigma}$ (high-stress periods); blue zones: $\sigma_t < \bar{\sigma}$ (calm periods)



TSA_ch5_cond_vs_uncond

Realized Volatility and HAR-RV

Realized Volatility (RV)

With intraday returns $r_{t,i}$ at frequency Δ (M observations per day):

$$RV_t = \sum_{i=1}^{M} r_{t,i}^2$$

RV_t is a **model-free**, **observable** measure of daily variance (vs. GARCH: latent).

Definition 13 (HAR-RV Model — Corsi (2009))

$$RV_t = \beta_0 + \beta_D\, RV_{t-1} + \beta_W\, RV_{t-1}^{(w)} + \beta_M\, RV_{t-1}^{(m)} + u_t$$

where $RV^{(w)} = \frac{1}{5}\sum_{j=1}^{5} RV_{t-j}$ (weekly) and $RV^{(m)} = \frac{1}{22}\sum_{j=1}^{22} RV_{t-j}$ (monthly).

Key Advantages

- Simple OLS regression that captures **long memory** in volatility
- Heterogeneous traders (daily/weekly/monthly) explain multi-scale dynamics
- Often outperforms GARCH when intraday data available (5-min returns, $M \approx 78$/day)
- Caveat: microstructure noise at very high frequency $\Rightarrow$ use kernel estimators

Realized GARCH

Motivation

Standard GARCH treats volatility as **latent**. With high-frequency data, we can **observe** it via RV_t.

Definition 14 (Realized GARCH — Hansen, Huang & Shek (2012))

Return equation: $r_t = \mu + \sigma_t z_t, \quad z_t \sim D(0,1)$

GARCH equation: $\log \sigma_t^2 = \omega + \beta \log \sigma_{t-1}^2 + \gamma \log RV_{t-1}$

Measurement equation: $\log RV_t = \xi + \varphi \log \sigma_t^2 + \tau(z_t) + u_t$

Benefits

- **Joint model**: links returns, latent volatility, and realized measures
- Leverage function $\tau(z_t)$ captures asymmetry directly
- Typically **more accurate** forecasts than standard GARCH (uses richer information set)

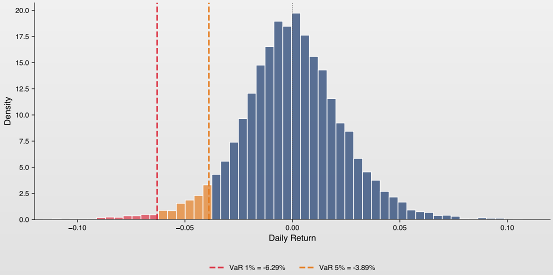
Comparison: GARCH vs HAR-RV vs Realized GARCH

- **GARCH**: no intraday data needed; most widely used
- **HAR-RV**: simple, model-free; requires 5-min data
- **Realized GARCH**: best of both worlds; requires 5-min data + parametric structure

VaR and ES: Graphical Illustration

Interpretation

- VaR 1% = loss exceeded only in 1% of cases
- Red area = extreme losses (beyond VaR)



TSA_ch5_var_plot

Applications of Volatility Forecasting

Value at Risk (VaR)

$\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$
Maximum loss with probability $1 - \alpha$.

Expected Shortfall (ES)

$\text{ES}_\alpha = \mathbb{E}[-r | r < -\text{VaR}_\alpha]$
Average loss when VaR is exceeded.

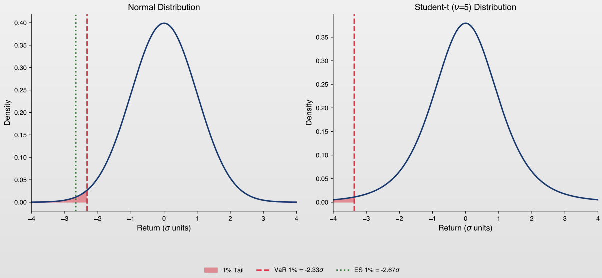
Other Applications

- Option pricing
- Dynamic hedging
- Portfolio allocation
- Stress testing

VaR vs Expected Shortfall: Normal vs Student-t

Interpretation

- ES measures average loss when VaR is exceeded
- Student-t: VaR and ES are larger than under normal distribution



TSA_ch5_var_es

Value at Risk — Numerical Example

VaR Calculation

Portfolio: **1,000,000 EUR**, forecasted volatility $\hat{\sigma}_{T+1} = 1.5\%$

VaR with Normal Distribution

Level	z_α	**VaR (%)**	**VaR (EUR)**
5% (1 day)	1.645	2.47%	24,675
1% (1 day)	2.326	3.49%	34,890

Scaling for Longer Periods

- $\text{VaR}_{h\text{ days}} = \text{VaR}_{1\text{ day}} \cdot \sqrt{h}$ — assumes i.i.d. returns
- **Warning**: Under GARCH models, returns are **not** i.i.d. (they are conditionally heteroskedastic) $\Rightarrow$ the $\sqrt{h}$ rule is only an **approximation**; for multi-period VaR, use iterated GARCH forecasts

Value at Risk — Student-t Distribution

Why Student-t?

Normal distribution **underestimates** tail risk. Student-t with ν degrees of freedom better captures fat tails (kurtosis > 3).

VaR 1% (1 day) Comparison: $\sigma = 1.5\%$, Portfolio $= 1\text{M EUR}$

Distribution	Quantile	VaR (EUR)
Normal	2.326	34,890
Student-t ($\nu = 6$)	3.143	47,145
Student-t ($\nu = 4$)	3.747	56,205

Observation

With $\nu = 6$ (typical for stocks), VaR is **35% higher** than normal!

VaR — Complete Example with GARCH

VaR Calculation Procedure

1. Estimate GARCH(1,1) model with Student-t distribution
2. Obtain volatility forecast: $\hat{\sigma}_{T+1}$
3. Calculate VaR: $\text{VaR}_\alpha = t_\alpha(\nu) \cdot \hat{\sigma}_{T+1} \cdot \sqrt{\frac{\nu-2}{\nu}}$

Example: S&P 500

- Estimated parameters: $\alpha = 0.088$, $\beta = 0.900$, $\nu = 6.4$
- Forecasted volatility: $\hat{\sigma}_{T+1} = 1.2\%$
- Portfolio: 10,000,000 EUR

VaR 1% (1 day): $\text{VaR} = 3.05 \times 0.012 \times 10{,}000{,}000 =$ **366,000 EUR**

What is VaR Backtesting?

Definition

- **Backtesting** = ex-post verification of VaR model quality
- Compares realized losses with the forecasted VaR threshold
  - A **violation** occurs when $r_t < -\text{VaR}_t$

Backtesting Principle

- Violation indicator: $I_t = 1(r_t < -\text{VaR}_{\alpha,t})$
- For a correctly specified model at level α:
  - Frequency: $\hat{p} = \frac{1}{T}\sum I_t \approx \alpha$; violations **independent**
- VaR 1% over 250 days $\Rightarrow$ expect $\sim$2.5 violations/year

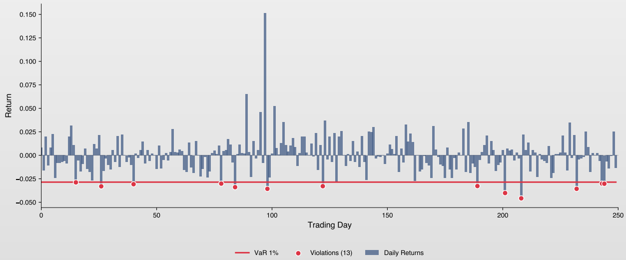
Importance

- Regulatory requirement under **Basel III/IV** for banks: backtesting is mandatory

VaR Backtesting: Visual Overview

Interpretation

- Red line: VaR 1% threshold estimated with GARCH(1,1)
- Red dots: 13 violations out of 250 days ($\hat{p} = 5.2\%$)
  - **Basel red zone** $\Rightarrow$ model significantly underestimates risk
  - Solutions: Student-t distribution, EGARCH model, or more conservative VaR level



TSA_ch5_backtest

VaR Backtesting: Basel Traffic Light

Basel III/IV Traffic Light Zones

Zone	Violations/250 days	Interpretation	Penalty
Green	0–4	Model acceptable	No penalty
Yellow	5–9	Needs investigation	Factor k increases
Red	≥ 10	Model inadequate	Maximum penalty

Practical Example

- Portfolio with VaR 1%: 250 days of backtesting
- 3 violations $\Rightarrow$ **Green zone** $\Rightarrow$ model acceptable
- 7 violations $\Rightarrow$ **Yellow zone** $\Rightarrow$ revision needed
- 13 violations $\Rightarrow$ **Red zone** $\Rightarrow$ model rejected

Rolling Window VaR Methodology

Rolling Window Concept

- A **rolling window** of fixed size W (e.g., 500 days) moves day by day
- At each step t: re-estimate GARCH on $[t-W,\ t-1]$, forecast $\hat{\sigma}_{t|t-1}$, compute VaR_t

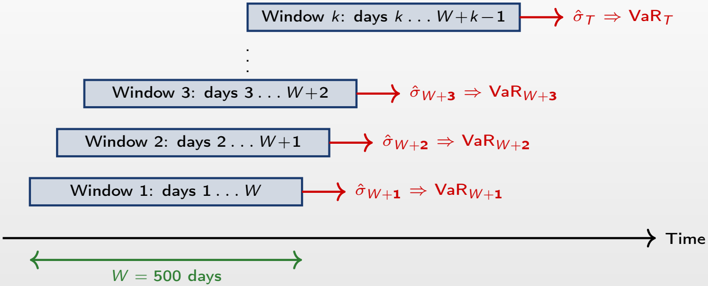
Step-by-Step Procedure (for each day $t = W+1, \ldots, T$)

1. Estimate GARCH on $\{r_{t-W}, \ldots, r_{t-1}\} \Rightarrow$ parameters $\hat{\omega}, \hat{\alpha}, \hat{\beta}, \hat{\nu}$
2. Forecast: $\hat{\sigma}^2_{t|t-1} = \hat{\omega} + \hat{\alpha} r^2_{t-1} + \hat{\beta}\hat{\sigma}^2_{t-1}$
3. Compute: $\text{VaR}_{\alpha,t} = -t_\alpha(\hat{\nu}) \cdot \sqrt{\frac{\hat{\nu}-2}{\hat{\nu}}} \cdot \hat{\sigma}_{t|t-1}$
4. Check violation: $I_t = 1(r_t < -\text{VaR}_{\alpha,t})$

Why Rolling and Not Expanding?

- Fixed window: parameters reflect the **current regime** of volatility
- Old data ($> W$ days) may be irrelevant (structural changes, crises)

Rolling Window VaR: Procedure Diagram



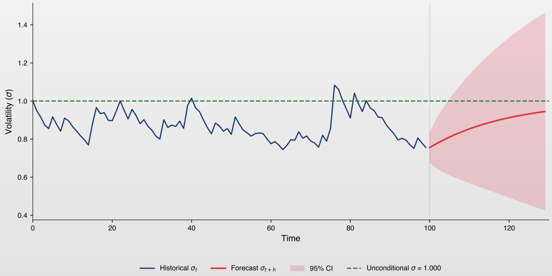
Result

- We obtain the series $\{\text{VaR}_{\alpha,t}\}_{t=W+1}^{T} \Rightarrow$ a **different** threshold every day
- VaR adapts to the current regime: increases during volatile periods, decreases during calm ones
- Compare r_t with $-\text{VaR}_{\alpha,t}$ to identify violations

Volatility Forecast with Confidence Intervals

Interpretation

- Forecast converges to $\bar{\sigma}$
- Uncertainty increases with forecast horizon



TSA_ch5_vol_ci

Rolling Forecast: Step-by-Step Prediction

Procedure

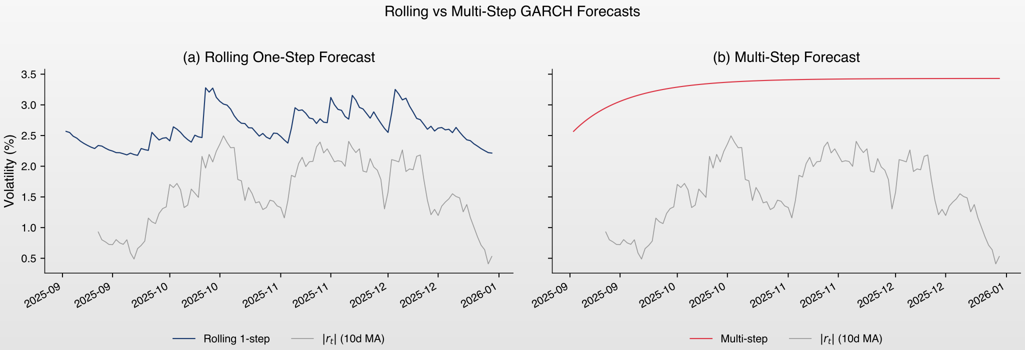
S&P 500, W=500, GARCH(1,1)-t

- Re-estimate GARCH on $[t-W,\ t-1]$; forecast $\hat{\sigma}_{t|t-1}$
- Compare with realized vol. (20-day rolling std.)

Results (2015 days OOS)

- $\rho = 0.938 \Rightarrow$ excellent tracking; MAE $= 0.15\%$, RMSE $= 0.24\%$
- COVID-19: temporary under-prediction, rapid adaptation

Rolling Forecast: Step-by-Step Prediction



TSA_ch5_rolling_forecast

VaR Backtesting: Kupiec Test

Unconditional Coverage Test

Tests whether the observed violation rate equals the expected rate p (e.g., 1% for VaR 1%).

Let N = number of VaR violations, T = total observations, $\hat{p} = N/T$.

Likelihood Ratio Statistic:

$$LR_{uc} = -2\ln\left[\frac{(1-p)^{T-N}p^N}{(1-\hat{p})^{T-N}\hat{p}^N}\right] \sim \chi^2(1)$$

Hypotheses

- H_0: $\hat{p} = p$ (VaR model is correctly calibrated)
- H_1: $\hat{p} \neq p$ (VaR model under- or over-estimates risk)

VaR Backtesting: Christoffersen Test

Conditional Coverage Test

Tests both **unconditional coverage** and **independence** of violations.
Violations should be independent — no clustering of exceptions!

Test Components

- **Independence test** (LR_{ind}): Tests if violations are serially independent
- **Conditional coverage**: $LR_{cc} = LR_{uc} + LR_{ind} \sim \chi^2(2)$

Interpretation

- Reject LR_{uc}: wrong frequency
- Reject LR_{ind}: clustered violations
- Reject LR_{cc}: model fails

Full Backtesting: Results and Decision

Application S&P 500 (T=500, VaR 1%)

- Rolling window: $W = 500$ days, EGARCH(1,1)-t model
- Test window: $T - W$ days out-of-sample
- Compare forecasted VaR with realized return each day

Typical Output

- Number of violations vs expected $\Rightarrow$ Basel zone (green/yellow/red)
- Kupiec (frequency): H_0: $\hat{p} = \alpha \Rightarrow$ p-value $> 0.05 \Rightarrow$ accepted
- Christoffersen (independence): H_0: violations i.i.d. $\Rightarrow$ p-value $> 0.05 \Rightarrow$ accepted
- Combined test: both conditions met $\Rightarrow$ **model validated**

TSA_ch5_backtest_full

Expected Shortfall Backtesting

Why Backtest ES?

- VaR only measures the *threshold* of extreme losses, not their *magnitude*
- ES = average loss *given* VaR is breached ⇒ more informative for tail risk

Definition 15 (Acerbi-Székely Z-test (2014))

$$Z_{\mathbf{ES}} = \frac{1}{n_\alpha} \sum_{t: r_t < -\mathbf{VaR}_t} \frac{r_t}{\mathbf{ES}_t} + 1$$

Under H_0 (correct ES model): $\mathbb{E}[Z_{\mathbf{ES}}] = 0$. Reject if $Z_{\mathbf{ES}} < z_{\alpha/2}$.

Key Insight

- Unlike VaR backtesting (binary hit/miss), ES backtesting uses the *magnitude* of losses
- More powerful but requires larger samples ($T > 1000$); alt.: McNeil & Frey (2000) bootstrap
- ES under Student-t: $\text{ES}_\alpha^{t_\nu} = -\mu + \sigma \cdot \frac{\nu + (t_{\alpha,\nu})^2}{\nu - 1} \cdot \frac{f_{t_\nu}(t_{\alpha,\nu})}{\alpha}$ — heavier tails ⇒ higher ES

Basel IV: From VaR to ES

Regulatory Shift

- **Basel II/III**: Capital based on VaR 99% (10-day)
- **Basel IV** (FRTB, 2023+): Capital based on **ES 97.5%** (10-day)
- Rationale: ES is **coherent** (subadditive), VaR is not

Equivalence under Normality

$$\text{ES}_{97.5\%}^{\text{Normal}} = \frac{\phi(z_{0.025})}{0.025} \cdot \sigma \approx 2.338\,\sigma \quad \approx \quad \text{VaR}_{99\%}^{\text{Normal}} = 2.326\,\sigma$$

$\Rightarrow$ Under normality, ES 97.5% $\approx$ VaR 99%. Under fat tails, ES $\gg$ VaR.

Practical Implication

- Student-*t* GARCH models become **essential** for accurate ES estimation
- Banks must now model the *entire tail*, not just a single quantile

ARMA-GARCH: Joint Mean and Variance Modeling

Why Joint Modeling?

Serial correlation $\Rightarrow$ ARMA for mean; **Volatility clustering** $\Rightarrow$ GARCH for variance.

Definition 16 (ARMA(p,q)-GARCH(r,s))

Mean equation: $r_t = \mu + \sum_{i=1}^{p} \phi_i (r_{t-i} - \mu) + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$

Variance equation: $\varepsilon_t = \sigma_t z_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$

ARMA-GARCH: Model Selection Strategy

Step-by-Step Approach

1. **Identify mean model**: Check ACF/PACF of returns for ARMA structure
2. **Test for ARCH effects**: Apply ARCH-LM test to residuals
3. **Specify variance model**: Usually GARCH(1,1) is sufficient
4. **Joint estimation**: Estimate both equations via MLE
5. **Diagnostic checking**: Standardized residuals should be i.i.d.

Common Specifications

- **Stock returns**: AR(1)-GARCH(1,1) or ARMA(1,1)-GARCH(1,1)
- **Exchange rates**: Often just GARCH(1,1) (no mean dynamics)
- **Interest rates**: AR(1)-EGARCH(1,1) for leverage effects

Python: GARCH Estimation with `arch`

Key Steps

```
from arch import arch_model
import numpy as np
from scipy.stats import norm

# Fit GARCH(1,1) with Student-t innovations
model = arch_model(returns, vol='Garch', p=1, q=1,
                   dist='StudentsT', mean='AR', lags=1)
result = model.fit(disp='off')
print(result.summary())

# Forecast 20-step ahead volatility
forecast = result.forecast(horizon=20)
sigma_forecast = np.sqrt(forecast.variance.iloc[-1])

# Compute 1-day VaR (1%) from conditional distribution
sigma_1 = sigma_forecast.iloc[0]
VaR_1pct = norm.ppf(0.01) * sigma_1  # Normal approx
```

Note: For Student-t VaR, replace `norm.ppf` with `scipy.stats.t.ppf(0.01, df=nu)` scaled by $\sqrt{(\nu-2)/\nu}$.

Step 1: Data — S&P 500 Daily Returns

Data Description

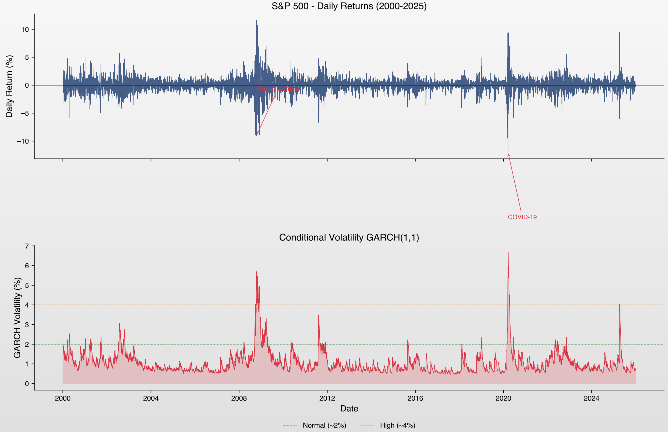
- **Source**: Yahoo Finance, S&P 500, daily data 2000–2024 ($T > 6000$)
- **Returns**: $r_t = \ln(P_t/P_{t-1}) \times 100$

Descriptive Statistics

Mean	Std. Dev.	Skewness	Kurtosis	Min	Max
0.034%	1.21%	−0.29	13.8	−12.8%	+11.0%

- Fat tails (kurtosis $\gg 3$) and negative skewness $\Rightarrow$ ARCH effects

Step 1: Data — S&P 500 Daily Returns



TSA_ch5_sp500

Step 2: Testing for ARCH Effects

Python Code — ARCH-LM and Ljung-Box on r_t^2

Results

Test	Statistic	p-value
ARCH-LM (10 lags)	892.4	< 0.0001
Ljung-Box r_t^2 (lag 20)	4217.6	< 0.0001

- **Conclusion**: Strong ARCH effects $\Rightarrow$ significant heteroskedasticity

Step 3: Estimated Parameters — Comparison

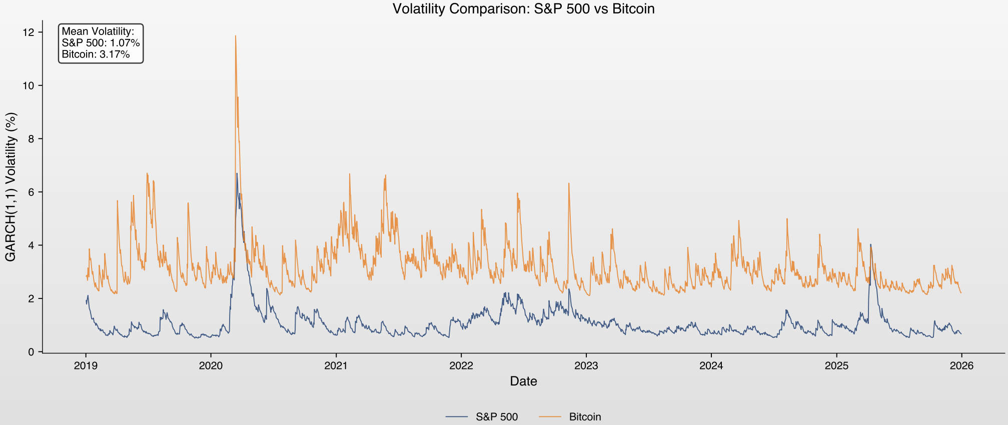
Estimated Parameters Table

Model	ω	α	β	γ	$\alpha+\beta$	ν	**HL**
GARCH-N	0.011	0.088	0.901	—	0.989	—	60 days
GARCH-t	0.011	0.088	0.900	—	0.989	6.42	60 days
EGARCH-t	0.003	0.103	0.987	-0.120	—	6.38	53 days
GJR-t	0.010	0.022	0.906	0.126	0.991	6.51	78 days

Interpretation

- EGARCH $\gamma = -0.12$ **significant** $\Rightarrow$ **leverage effect confirmed**
- GJR: $\alpha_{\text{neg}} = \alpha + \gamma = 0.148$ vs $\alpha_{\text{pos}} = 0.022 \Rightarrow$ strong asymmetry

Step 3: Estimated Conditional Volatility — S&P 500

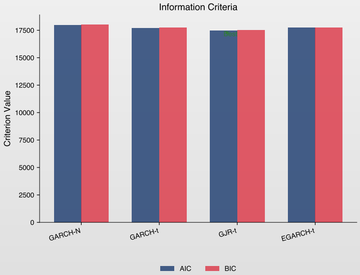


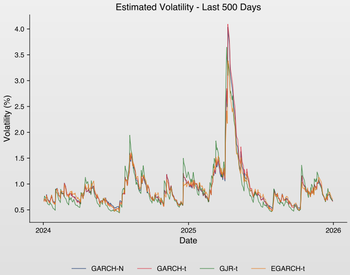
TSA_ch5_sp500_garch

Step 4: Leverage Effect — Visualization

GARCH vs EGARCH — Volatility Differences

- EGARCH produces **higher** volatility after negative shocks (2008, 2020)
- Symmetric GARCH **underestimates** risk during crisis periods
- Difference: up to 2–3 percentage points in daily volatility





TSA_ch5_sp500_comp

Step 5: Model Selection — AIC/BIC

Information Criteria

Model	Log-Lik	AIC	BIC	Rank
GARCH(1,1)-N	−8042.3	16090.6	16111.0	4
GARCH(1,1)-t	−7981.5	15971.0	15997.8	3
EGARCH(1,1)-t	−7964.2	**15938.4**	**15971.6**	**1**
GJR-GARCH(1,1)-t	−7968.1	15946.2	15979.4	2

Decision

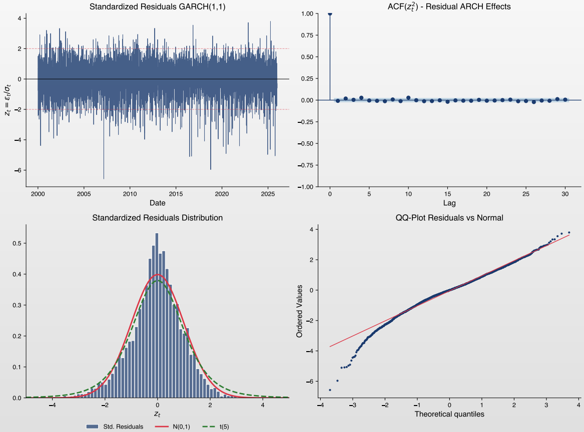
- **EGARCH(1,1)-t wins**: lowest AIC **and** BIC
- Student-t superior to Normal (ΔAIC $\approx$ 120) $\Rightarrow$ fat tails matter!
- Leverage effect justifies asymmetric models (ΔAIC $\approx$ 33 vs GARCH-t)

Step 6: Diagnostics — EGARCH(1,1)-t

Checks on Standardized Residuals $z_t = \varepsilon_t / \hat{\sigma}_t$

- **Ljung-Box** on z_t: p-value $= 0.38$ — no residual autocorrelation
- **Ljung-Box** on z_t^2: p-value $= 0.52$ — **ARCH effects eliminated**
- **Q-Q plot**: points follow the theoretical Student-t line
- **Conclusion**: EGARCH(1,1)-t adequately captures volatility dynamics

Step 6: Diagnostics — EGARCH(1,1)-t



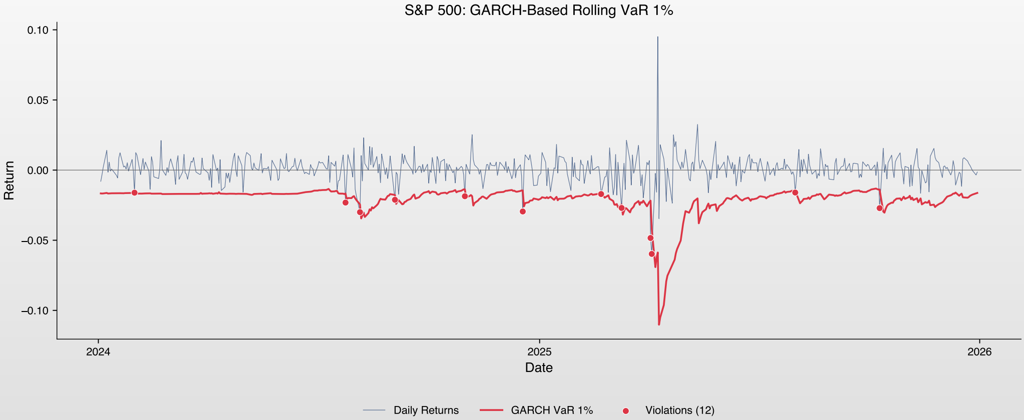
TSA_ch5_sp500_diag

Step 7: Backtesting Rolling VaR — S&P 500

Kupiec + Christoffersen Results (2015 days out-of-sample)

Test	**Statistic**	**p-value**	**Decision**
Violations	27/2015 ($\hat{p} = 1.34\%$)	—	**Green** zone
Kupiec (uc)	2.13	0.145	Accepted
Christoffersen (ind)	0.79	0.375	Accepted
Combined (cc)	2.91	0.233	Accepted

Step 7: Rolling VaR — S&P 500



TSA_ch5_sp500_var

Step 8: Conclusions — S&P 500 Case Study

Step-by-Step Methodology Summary

1. **Data**: log returns, descriptive statistics $\Rightarrow$ fat tails, skewness
2. **ARCH test**: ARCH-LM + Ljung-Box on r_t^2 $\Rightarrow$ significant ARCH effects
3. **Estimation**: 4 candidate models (symmetric/asymmetric $\times$ Normal/Student-t)
4. **Selection**: AIC/BIC $\Rightarrow$ **EGARCH(1,1)-t** winner
5. **Diagnostics**: standardized residuals $\Rightarrow$ model adequate
6. **VaR**: rolling window + Kupiec/Christoffersen backtesting $\Rightarrow$ model **validated**

Key Lessons

- Student-t distribution is **essential** for financial data
- Leverage effect: asymmetric models **mandatory** for equities
- Systematic backtesting: not just "looks good", but **statistically tested**

Step 1: Data — Bitcoin Daily Returns

Data Description

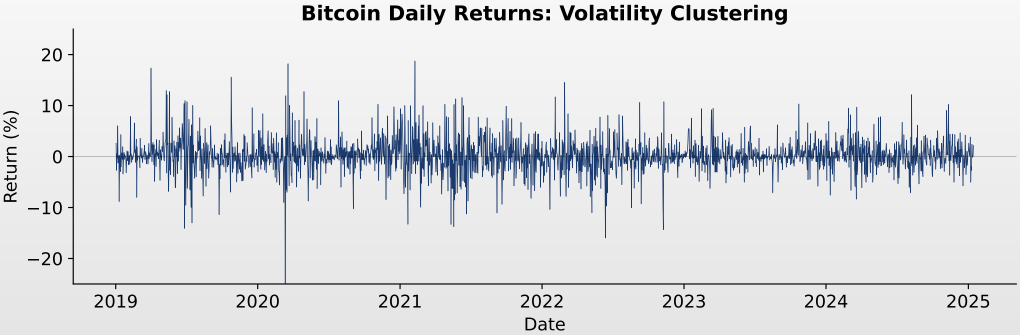
- **Source**: Yahoo Finance (BTC-USD), daily data 2018–2024
- Log returns: mean $\approx 0.05\%$, volatility $\approx 3.5\%$

Descriptive Statistics

Mean	Std. Dev.	Skewness	Kurtosis	Min	Max
0.05%	3.48%	-0.72	12.1	-46.5%	$+22.5\%$

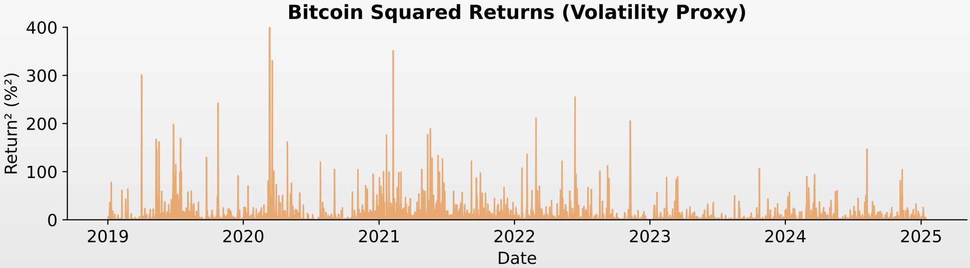
- Volatility $\sim 3\times$ higher than S&P 500
- Extreme kurtosis — high risk of large losses

Step 1: Data — Bitcoin Daily Returns



TSA_ch5_btc

Step 2: Testing for ARCH Effects — Bitcoin

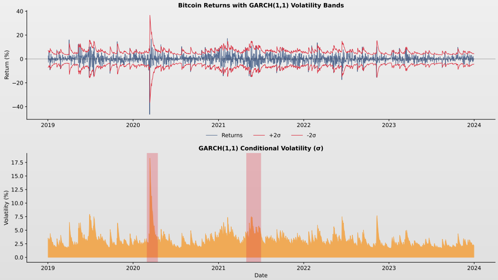


TSA_ch5_btc_arch

Step 3: Conditional Volatility — Bitcoin

GJR-GARCH(1,1)-t Diagnostics

- Ljung-Box on z_t^2: p-value $= 0.41$ — **ARCH effects eliminated**
- Volatility peaks: March 2020 (COVID), May 2022 (Terra/Luna)
- Daily volatility: from 1% (calm periods) to >15% (crises)



TSA_ch5_btc_garch

Step 4: Estimation and Model Selection — Bitcoin

Estimated Parameters

Model	ω	α	β	γ	$\alpha+\beta$	ν	**AIC**
GARCH-t	0.42	0.131	0.848	—	0.979	4.82	9284
EGARCH-t	0.08	0.184	0.976	-0.061	—	4.79	9276
GJR-t	0.40	0.088	0.854	0.078	0.976	4.85	**9271**

Interpretation

- **GJR-GARCH-t wins** (lowest AIC)
- $\nu \approx 4.8$: **much heavier tails** than S&P 500 ($\nu = 6.4$)
- $\alpha = 0.131$ (BTC) vs 0.088 (S&P) — Bitcoin reacts faster to news
- Leverage effect weaker than for stocks ($\gamma_{\text{BTC}} = 0.078$ vs 0.126)

Step 5: Backtesting Rolling VaR — Bitcoin

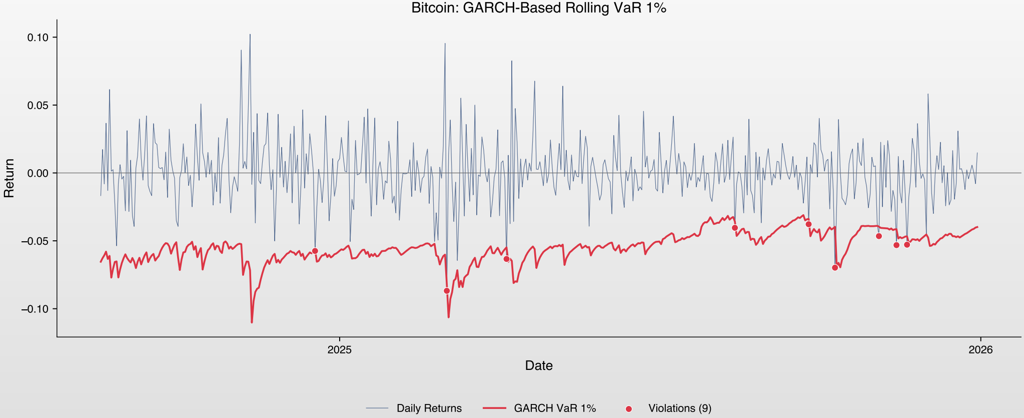
Statistical Tests (2421 days out-of-sample)

Test	Statistic	p-value	Decision
Violations	28/2421 ($\hat{p} = 1.16\%$)	—	Green zone
Kupiec (uc)	0.57	0.450	Accepted
Christoffersen (ind)	0.94	0.333	Accepted
Combined (cc)	1.51	0.471	Accepted

Interpretation

- Volatility ranges from 3% to 38% — rolling window is **essential**
- All tests **accepted**: model valid for risk management

Step 5: Rolling VaR — Bitcoin



TSA_ch5_btc_var

Step 5: VaR Rolling Window — Bitcoin

Rolling Window GJR-GARCH-t (W=500 days, VaR 1%)

Rolling VaR Characteristics Bitcoin (2018–2024)

- Mean VaR: 9.34% ($\approx$ EUR 93,400 / 1M EUR)
- Max VaR: 37.54% $\Rightarrow$ COVID crash March 2020
- Min VaR: 2.90% $\Rightarrow$ calm period
- Bitcoin: rolling VaR $\sim 4\times$ larger than S&P 500 at same exposure

Final Comparison: S&P 500 vs Bitcoin

Comparative Summary

	S&P 500	Bitcoin
Average volatility	1.2%	3.5%
Kurtosis	13.8	12.1
Student-t ν	6.42	4.82
Best model	EGARCH(1,1)-t	GJR-GARCH(1,1)-t
Leverage effect	Strong ($\gamma = -0.12$)	Moderate ($\gamma = 0.078$)
Half-life	$\sim$60 days	$\sim$42 days
Rolling VaR 1% mean	**2.53%**	**9.34%**
Rolling VaR 1% max	**22.02%** (COVID)	**37.54%** (COVID)
Rolling ES 1% mean	**3.42%**	**12.87%**
Kupiec	**Accepted** (p=0.145)	**Accepted** (p=0.450)
Christoffersen (ind)	**Accepted** (p=0.375)	**Accepted** (p=0.333)

General Conclusion

- Re-estimating GARCH at each step: Kupiec + Christoffersen **accepted**
- Rolling window VaR: **mandatory** — static VaR is completely inadequate
- Student-t + asymmetric model: **essential** for both markets

Key Formulas

Volatility Models

- **ARCH(q):** $\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$
- **GARCH(1,1):** $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$
- **EGARCH:** $\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$
- **GJR-GARCH:** $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \gamma \varepsilon_{t-1}^2 I_{t-1} + \beta \sigma_{t-1}^2$

Properties and Measures

- **Unconditional variance:** $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ **Half-life:** $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$
- **VaR:** $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$ **Stationarity:** $\alpha + \beta < 1$
- **ARCH-LM:** $LM = T \cdot R^2 \sim \chi^2(q)$

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Using yfinance, download daily closing prices for S&P 500 (^GSPC) from 2018-01-01 to 2024-12-31 (approx. 1,750 observations). Compute daily log returns. Test for ARCH effects, fit a GARCH(1,1) model, and forecast volatility for the next 20 trading days. Give me complete Python code."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it compute log returns correctly? Does it remove the mean before fitting GARCH?
3. How does it test for ARCH effects? Does it use Engle's LM test?
4. Does it separate the mean equation from the variance equation?
5. Does it discuss asymmetric effects (GJR-GARCH, EGARCH)? Are VaR estimates computed from the conditional distribution?
6. Identify at least **two factual errors** or questionable assumptions in the LLM output.
7. Does the code verify the stationarity constraint $\alpha + \beta < 1$? If not, add it.
8. Compare the LLM's VaR estimate with the results from this chapter. Are they consistent?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Summary – Chapter 5: Volatility Models

Key Concepts

- **ARCH(q)**: conditional variance depends on past squared errors (Nobel 2003)
- **GARCH(p,q)**: adds variance lags for persistence (GARCH(1,1) in 90% of cases)
- **EGARCH**: allows leverage effect, no positivity constraints
- **GJR-GARCH/TGARCH**: captures asymmetry with indicator variables

Applications

- Risk measurement and forecasting (VaR, ES)
- Derivative pricing, dynamic hedging, portfolio management

Practical Tip

- Start with GARCH(1,1), check for leverage, choose distribution minimizing AIC/BIC!
  - Student-t often superior to normal distribution

Question 1

Question

- What best describes the phenomenon of *volatility clustering* in financial series?

Answer Choices

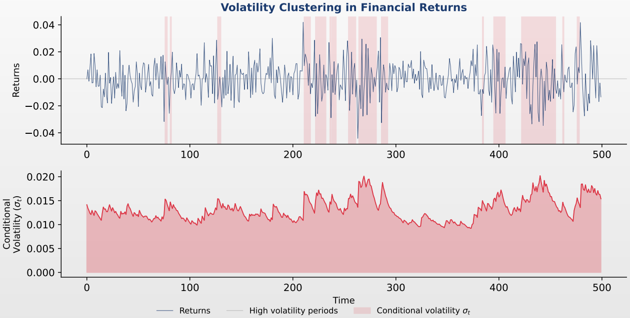
(A) Financial returns are normally distributed and independent

(B) Periods of high volatility are followed by high volatility, and vice versa

(C) Volatility is constant over time (homoscedasticity)

(D) Correlation between returns is always positive

Question 1: Answer



Answer: (B)

- Volatility clustering is a fundamental stylized fact of financial series. It implies that conditional variance is **predictable**, motivating ARCH/GARCH models.

TSA_ch5_quiz1_volatility_clustering

Question 2

Question

What is the main difference between an ARCH(q) and a GARCH(p,q) model?

Answer Choices

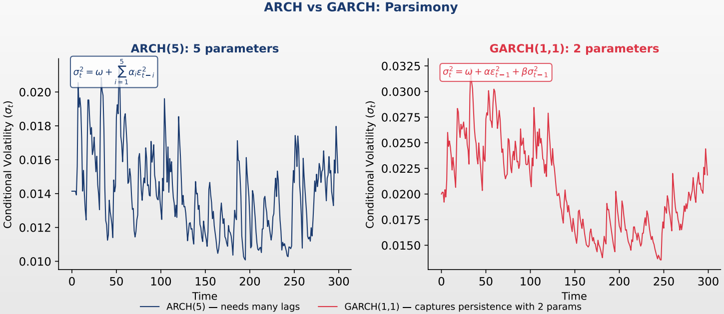
(A) GARCH models the conditional mean, ARCH models the variance

(B) ARCH includes lags of conditional variance, GARCH does not

(C) GARCH adds lags of conditional variance (σ^2_{t-j}) beyond squared errors

(D) ARCH is more parsimonious than GARCH

Question 2: Answer



Answer: (C)

- GARCH(1,1) captures the same persistence as ARCH(q) with only 2 parameters instead of q. In practice, GARCH(1,1) is sufficient in 90% of cases.

TSA_ch5_quiz2_arch_vs_garch

Question 3

Question

- What is the *leverage effect* and which GARCH models capture it?

Answer Choices

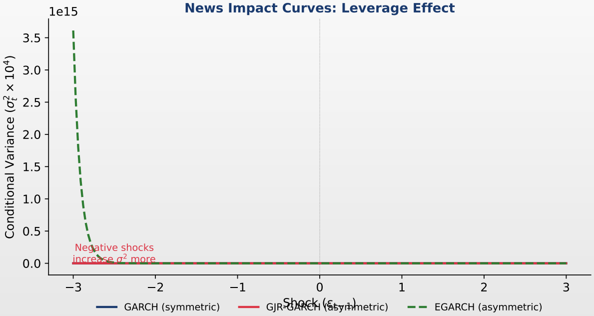
(A) Positive shocks increase volatility more; captured by standard GARCH

(B) Negative shocks increase volatility more; captured by EGARCH and GJR-GARCH

(C) Volatility is symmetric; captured by all GARCH models

(D) Financial leverage effect on stock prices; captured by IGARCH

Question 3: Answer



Answer: (B)

- Price drops increase volatility **more** than equivalent price increases. Standard GARCH uses ε_{t-1}^2, losing the sign information.

TSA_ch5_quiz3_leverage_effect

Question 4

Question

- What is the stationarity condition for a GARCH(1,1) model?

Answer Choices

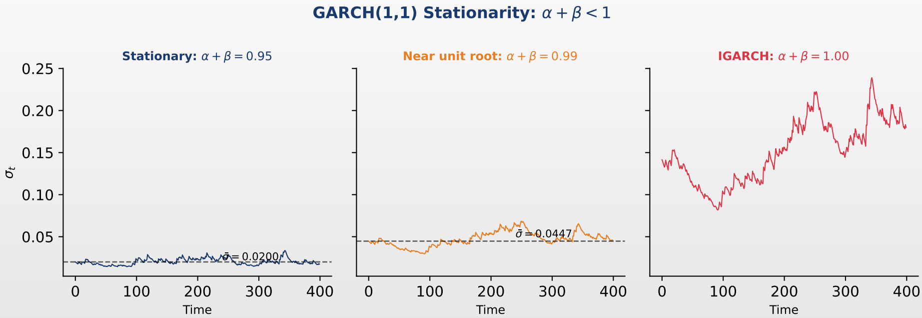
(A) $\alpha + \beta = 1$

(B) $\alpha > 0$ and $\beta > 0$

(C) $\alpha + \beta < 1$, with $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$

(D) $\alpha \cdot \beta < 1$

Question 4: Answer



Answer: (C)

- This ensures a finite unconditional variance $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$. When $\alpha + \beta = 1$ (IGARCH), variance is infinite.

TSA_ch5_quiz4_stationarity

Question 5

Applied Computational Question

A GARCH(1,1) model estimated on daily log returns (%) gives: $\omega = 0.05$, $\alpha = 0.15$, $\beta = 0.82$.

(a) Compute the unconditional variance $\bar{\sigma}^2$ and annualized volatility.

(b) Compute the half-life of a volatility shock.

(c) For a EUR 2 000 000 portfolio, compute the 1-day VaR at 1% under:
- Normal distribution
- Student-t with $\nu = 5$ degrees of freedom

(Use $\sigma_{T+1} = \bar{\sigma}$ for simplicity.)

Question 5: Answer

Solution

(a) $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta} = \frac{0.05}{1-0.97} = 1.667 \quad \Rightarrow \bar{\sigma} = 1.291\%$ daily $\quad \sigma_{\text{ann}} = 1.291\% \times \sqrt{252} \approx 20.5\%$

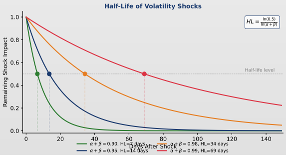
(b) $HL = \frac{\ln 0.5}{\ln(\alpha+\beta)} = \frac{-0.693}{\ln(0.97)} = \frac{-0.693}{-0.0305} \approx \mathbf{23\ days}$

(c) Portfolio value $V = 2{,}000{,}000$ EUR, $\sigma_{T+1} = 1.291\%$:

Normal: $\text{VaR}_{1\%} = 2.326 \times 0.01291 \times 2{,}000{,}000 = \mathbf{60{,}066}$ EUR

Student-t(5): $t_{0.01,5} = 3.365$ scaled by $\sqrt{(\nu-2)/\nu} = \sqrt{3/5}$:

$\text{VaR}_{1\%} = \frac{3.365}{\sqrt{5/3}} \times 0.01291 \times 2{,}000{,}000 = \mathbf{67{,}300}$ EUR $\quad$ (**+12%** vs Normal!)



TSA_ch5_quiz5_halflife

References I

Fundamental ARCH/GARCH Papers

- Engle, R.F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of UK Inflation, *Econometrica*, 50(4), 987–1007.
- Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics*, 31(3), 307–327.
- Nelson, D.B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica*, 59(2), 347–370.

Asymmetric Models and Extensions

- Glosten, L.R., Jagannathan, R., & Runkle, D.E. (1993). On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance*, 48(5), 1779–1801.
- Francq, C., & Zakoïan, J.-M. (2019). *GARCH Models: Structure, Statistical Inference and Financial Applications*, 2nd ed., Wiley.

References II

Textbooks and Financial Applications

- Tsay, R.S. (2010). *Analysis of Financial Time Series*, 3rd ed., Wiley.
- Franke, J., Härdle, W.K., & Hafner, C.M. (2019). *Statistics of Financial Markets*, 4th ed., Springer.
- McNeil, A.J., Frey, R., & Embrechts, P. (2015). *Quantitative Risk Management*, 2nd ed., Princeton University Press.

Online Resources and Code

- **Quantlet**: `https://quantlet.com` – Code platform for quantitative methods
- **Quantinar**: `https://quantinar.com` – Learning platform for quantitative methods
- **GitHub TSA**: `https://github.com/QuantLet/TSA/tree/main/TSA_ch5` – Python code for this chapter

Multivariate GARCH: Extensions for Portfolios

Why Multivariate GARCH?

- Portfolios require the conditional covariance matrix $\mathbf{H}_t = \mathrm{Cov}(\mathbf{r}_t \mid \mathcal{F}_{t-1})$
- K assets $\Rightarrow K(K+1)/2$ unique elements in $\mathbf{H}_t \Rightarrow$ dimensionality explosion

DCC (Engle, 2002)

- $\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$
- $\mathbf{D}_t = \mathrm{diag}(\sigma_{1,t}, \ldots, \sigma_{K,t})$ — univariate GARCH
- $\mathbf{R}_t$ — dynamic correlations (2 parameters)
- **Advantage**: 2-stage estimation, scalable
- **Limitation**: symmetric correlations

BEKK (Engle-Kroner, 1995)

- $\mathbf{H}_t = \mathbf{C}'\mathbf{C} + \mathbf{A}'\boldsymbol{\epsilon}_{t-1}\boldsymbol{\epsilon}_{t-1}'\mathbf{A} + \mathbf{B}'\mathbf{H}_{t-1}\mathbf{B}$
- Guarantees $\mathbf{H}_t \succ 0$ (positive-definite)
- **Advantage**: fully flexible model
- **Limitation**: K^2 parameters per matrix $\Rightarrow$ practical only for $K \leq 3$

Python Implementation

`from arch.univariate import ARCHModel` — DCC via `rmgarch` (R) or `mgarch` (Python/Stata)

References III

Estimation and Evaluation

- White, H. (1982). Maximum Likelihood Estimation of Misspecified Models, *Econometrica*, 50(1), 1–25.
- Hansen, P.R., & Lunde, A. (2005). A Forecast Comparison of Volatility Models: Does Anything Beat a GARCH(1,1)?, *Journal of Applied Econometrics*, 20(7), 873–889.
- Hansen, P.R., Huang, Z., & Shek, H.H. (2012). Realized GARCH: A Joint Model for Returns and Realized Measures of Volatility, *Journal of Applied Econometrics*, 27(6), 877–906.

High-Frequency and Risk Management

- Corsi, F. (2009). A Simple Approximate Long-Memory Model of Realized Volatility, *Journal of Financial Econometrics*, 7(2), 174–196.
- Patton, A.J. (2011). Volatility Forecast Comparison Using Imperfect Volatility Proxies, *Journal of Econometrics*, 160(1), 246–256.
- Acerbi, C., & Székely, B. (2014). Backtesting Expected Shortfall, *Risk Magazine*, December.

Thank You!

Questions?

Course materials available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar