



Time Series Analysis and Forecasting

# Chapter 6: VAR & Granger Causality

Seminar



# Seminar Outline

- 1 Review Quiz
- 2 True/False Questions
- 3 Practice Problems
- 4 Worked Examples
- 5 Real Data Analysis
- 6 Discussion Topics
- 7 Exercises for Self-Study

## Quiz 1: VAR Definition

### Question

In a VAR(2) model with 3 variables, how many coefficient matrices  $\mathbf{A}_i$  are there?

- A) 2
- B) 3
- C) 6
- D) 9

*Answer on next slide...*

## Quiz 1: Answer

Answer: A – 2 coefficient matrices

**VAR( $p$ ) model:**  $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$

**VAR(2) with  $K = 3$ :**

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \\ Y_{3t} \end{pmatrix} = \mathbf{c} + \underbrace{\mathbf{A}_1}_{3 \times 3} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{pmatrix} + \underbrace{\mathbf{A}_2}_{3 \times 3} \begin{pmatrix} Y_{1,t-2} \\ Y_{2,t-2} \\ Y_{3,t-2} \end{pmatrix} + \boldsymbol{\varepsilon}_t$$

**Key:**  $p = \text{number of lags} = \text{number of matrices}$

## Quiz 2: Number of Parameters

### Question

A VAR(2) with  $K = 3$  variables (including constants) has how many parameters to estimate per equation?

- A) 3
- B) 6
- C) 7
- D) 9

*Answer on next slide...*

## Quiz 2: Answer

Answer: C – 7 parameters per equation

### VAR(p) Parameter Count: The Curse of Dimensionality

Parameters per equation:  $1 + K \times p$

Total parameters:  $K(1 + Kp) + K(K + 1)/2$

(coefficients + covariance matrix)

Model	Coefficients	Total
K=2, p=1	$2(1+2 \times 1) = 6$	<b>+ 3 = 9</b>
K=3, p=2	$3(1+3 \times 2) = 21$	<b>+ 6 = 27</b>
K=5, p=4	$5(1+5 \times 4) = 105$	<b>+ 15 = 120</b>
K=10, p=4	$10(1+10 \times 4) = 410$	<b>+ 55 = 465</b>

Warning: Parameters grow as  $K^2 \times p$  — need lots of data!

**Formula:** Per equation =  $1 + K \times p = 1 + 3 \times 2 = 7$ . **Total:**  $K(1 + Kp) = 3(1 + 6) = 21$  parameters

## Quiz 3: Granger Causality

### Question

"X Granger-causes Y" means:

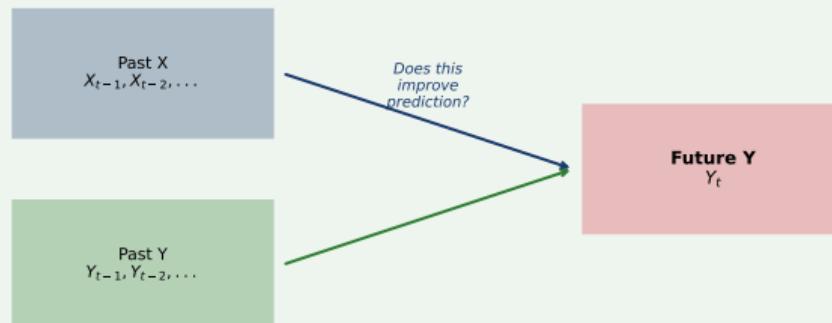
- (A) X is the economic cause of Y
- (B) Past X helps predict future Y
- (C) X and Y are contemporaneously correlated
- (D) X always increases when Y increases

*Answer on next slide...*

## Quiz 3: Answer

Answer: B – Past  $X$  helps predict future  $Y$

### Granger Causality: Predictive, Not Causal!



**Key:** Predictive relationship, NOT true causation!

## Quiz 4: Granger Causality Test

### Question

To test if  $Y_2$  Granger-causes  $Y_1$  in a VAR(p), we test:

- A) All coefficients in the  $Y_1$  equation equal zero
- B) Coefficients on lagged  $Y_2$  in the  $Y_1$  equation equal zero
- C) Coefficients on lagged  $Y_1$  in the  $Y_2$  equation equal zero
- D) The error covariance equals zero

*Answer on next slide...*

## Quiz 4: Answer

Answer: B – Coefficients on lagged  $Y_2$  in  $Y_1$  equation = 0

**Null hypothesis:**  $H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$

**Test statistic:** Wald or F-test with  $p$  restrictions

**Interpretation:**

- Reject  $H_0$ :  $Y_2$  Granger-causes  $Y_1$
- Don't reject: No evidence of predictive relationship

**Note:** Test  $Y_1 \rightarrow Y_2$  separately (different coefficients in  $Y_2$  equation)

## Quiz 5: VAR Stability

### Question

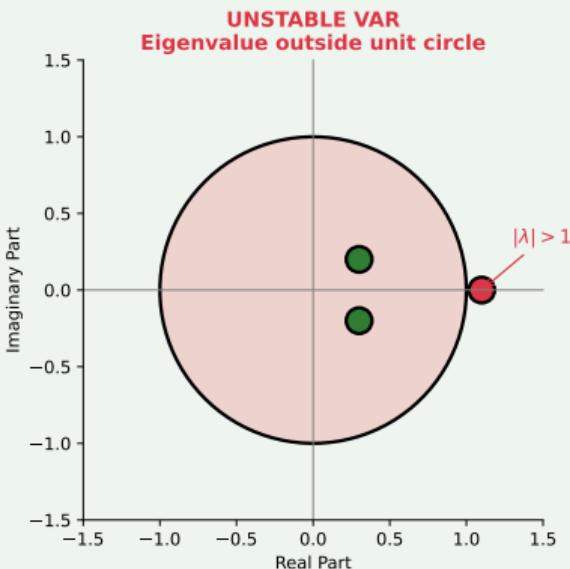
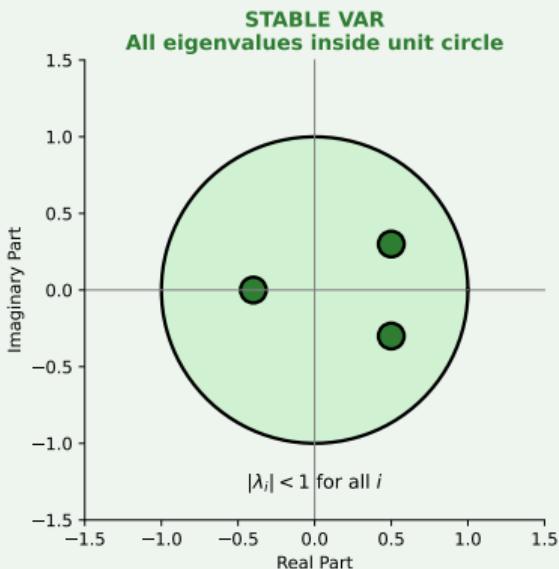
A VAR(1) model is stable (stationary) if:

- A All diagonal elements of  $\mathbf{A}_1$  are less than 1
- B The determinant of  $\mathbf{A}_1$  is less than 1
- C All eigenvalues of  $\mathbf{A}_1$  are less than 1 in absolute value
- D The trace of  $\mathbf{A}_1$  equals zero

*Answer on next slide...*

## Quiz 5: Answer

Answer: C – All eigenvalues of  $\mathbf{A}_1$  inside unit circle



**Stable:** All  $|\lambda_i| < 1$  (inside unit circle)  $\Rightarrow$  shocks die out over time

## Quiz 6: Impulse Response Functions

### Question

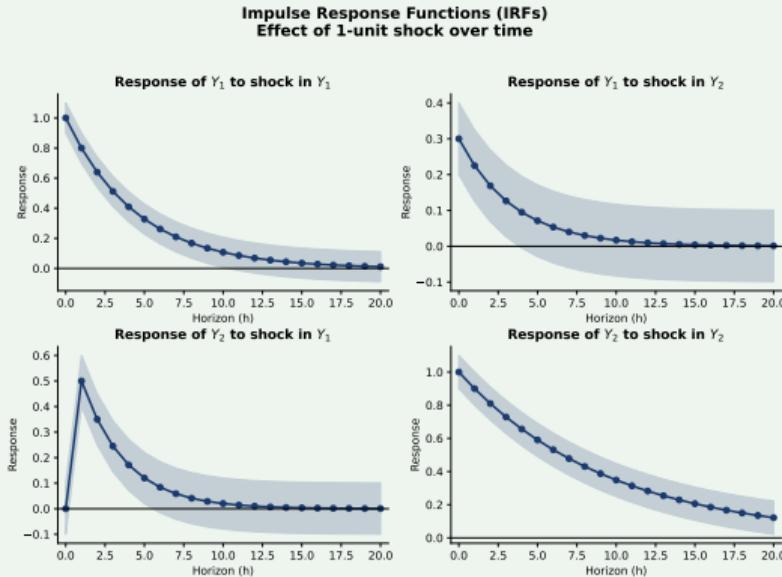
An impulse response function shows:

- (A) The correlation between two variables
- (B) The effect of a shock to one variable on all variables over time
- (C) The forecast accuracy of the model
- (D) The p-values of coefficient tests

*Answer on next slide...*

## Quiz 6: Answer

Answer: B – Effect of shock on all variables over time



$\text{IRF}_{ij}(h)$ : Response of variable  $i$  at horizon  $h$  to shock in variable  $j$

## Quiz 7: Lag Order Selection

### Question

Which criterion typically selects the most parsimonious VAR model?

- (A) AIC (Akaike Information Criterion)
- (B) BIC (Bayesian Information Criterion)
- (C) FPE (Final Prediction Error)
- (D) Adjusted  $R^2$

*Answer on next slide...*

## Quiz 7: Answer

Answer: B – BIC (Bayesian Information Criterion)

**Penalty comparison** (for  $k$  parameters,  $n$  observations):

- AIC:  $-2 \ln L + 2k$
- BIC:  $-2 \ln L + k \ln n$

Since  $\ln n > 2$  for  $n > 8$ , BIC penalizes complexity more heavily

**Practical guidance:**

- Forecasting: AIC may perform better
- Inference/parsimony: BIC preferred
- Large samples: BIC consistent, AIC tends to overfit

## Quiz 8: Granger Causality Interpretation

### Question

"X Granger-causes Y" means:

- (A)  $X$  is the true cause of  $Y$
- (B) Past values of  $X$  help predict  $Y$  beyond  $Y$ 's own past
- (C)  $X$  and  $Y$  are correlated
- (D)  $Y$  depends only on  $X$

*Answer on next slide...*

## Quiz 8: Answer

Answer: B

Granger causality is about **predictive** content, not true causation.  $X$  Granger-causes  $Y$  if lagged  $X$  terms are jointly significant in the equation for  $Y$ , after controlling for lagged  $Y$ .

## Quiz 9: Forecast Error Variance Decomposition

### Question

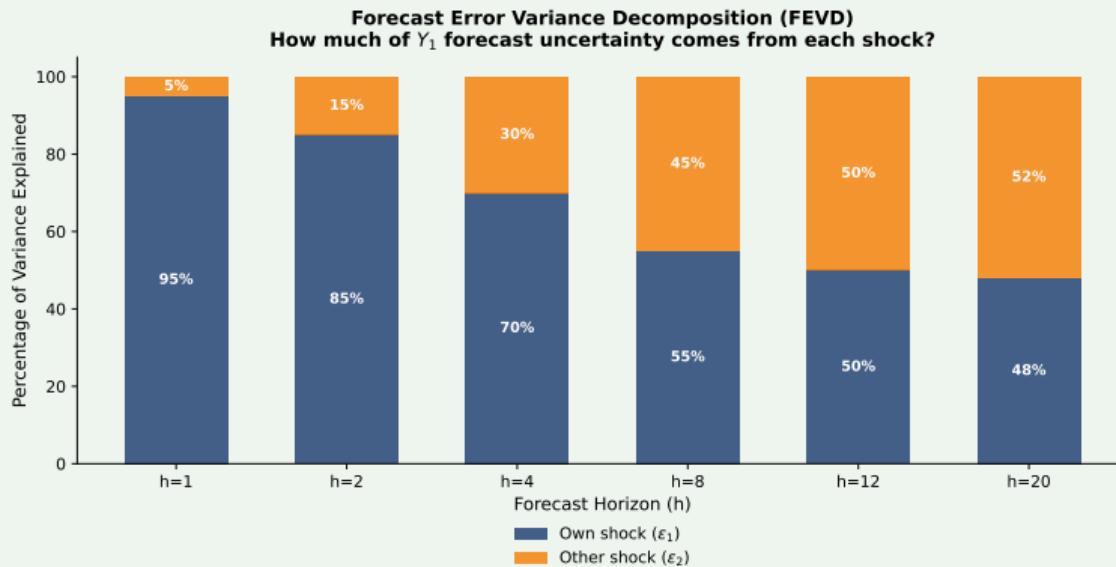
FEVD (Forecast Error Variance Decomposition) tells us:

- A) The correlation between variables
- B) What proportion of forecast error variance comes from each shock
- C) The optimal forecast horizon
- D) Which variables to include in the model

*Answer on next slide...*

## Quiz 9: Answer

Answer: B – Proportion of forecast error variance from each shock



FEVD: Shows how much forecast uncertainty comes from each shock at different horizons

## Quiz 10: Structural vs Reduced Form VAR

### Question

The difference between structural VAR (SVAR) and reduced-form VAR is:

- A) SVAR has more variables
- B) SVAR allows contemporaneous effects between variables
- C) SVAR uses different estimation methods
- D) There is no difference

*Answer on next slide...*

## Quiz 10: Answer

Answer: B

Reduced-form VAR: shocks are correlated, no contemporaneous effects in equations. SVAR: imposes identifying restrictions to recover structural shocks with economic interpretation (e.g., monetary policy shock).

## Quiz 11: Cholesky Decomposition

### Question

Cholesky ordering in IRF analysis assumes:

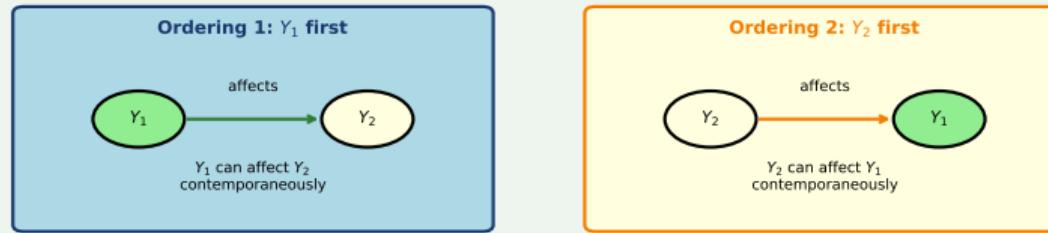
- (A) All variables are equally important
- (B) Variables ordered first affect later variables contemporaneously, not vice versa
- (C) Shocks are uncorrelated
- (D) No restrictions are needed

*Answer on next slide...*

## Quiz 11: Answer

Answer: B – Variables ordered first affect later ones contemporaneously

### Cholesky Ordering: Order Matters!



#### Key Points:

- Different orderings give DIFFERENT IRFs!
- Ordering should be based on economic theory
- "Fast-moving" variables should come first

**Cholesky:** Recursive structure. Ordering matters – justify by economic theory (most exogenous first)!

## Quiz 12: VAR Residual Diagnostics

### Question

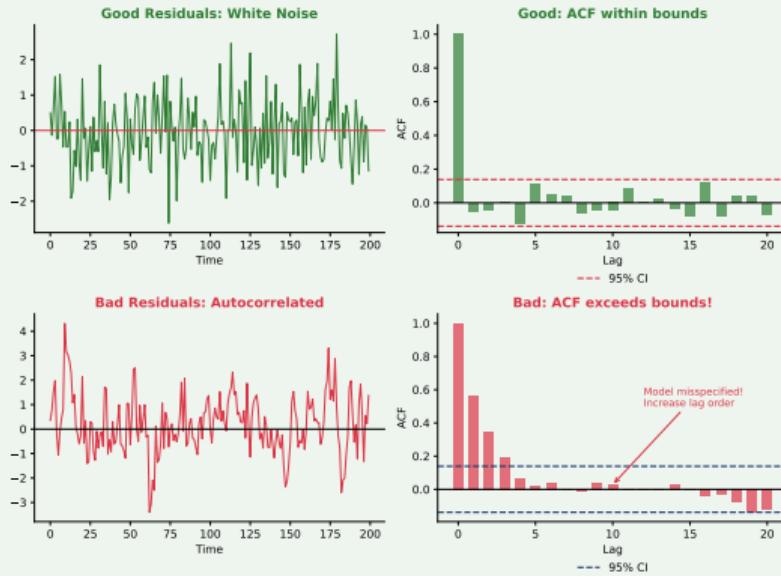
In a well-specified VAR, residuals should be:

- (A) Autocorrelated but homoskedastic
- (B) White noise (no autocorrelation)
- (C) Normally distributed only
- (D) Correlated across equations

*Answer on next slide...*

## Quiz 12: Answer

Answer: B – White noise (no autocorrelation)



**Diagnostics:** Residuals should be white noise. Use Portmanteau/LM test. Cross-equation correlation allowed ( $\Sigma_u$ ).

## Quiz 13: Cointegration and VAR

### Question

If variables are  $I(1)$  and cointegrated, you should use:

- A) VAR in levels
- B) VAR in first differences
- C) Vector Error Correction Model (VECM)
- D) Univariate ARIMA models

*Answer on next slide...*

## Quiz 13: Answer

Answer: C

With cointegration, VAR in differences loses long-run information, while VAR in levels may be inefficient. VECM incorporates both short-run dynamics and long-run equilibrium relationships through the error correction term.

## Quiz 14: Instantaneous Causality

### Question

Instantaneous causality differs from Granger causality because it tests:

- (A) Lagged relationships only
- (B) Contemporaneous correlation of residuals
- (C) Long-run relationships
- (D) Model stability

*Answer on next slide...*

## Quiz 14: Answer

Answer: B

Instantaneous causality tests whether shocks to  $X$  and  $Y$  are correlated within the same period (correlation of VAR residuals). Granger causality tests whether *lagged* values help predict.

## True/False Questions

Determine if each statement is True or False:

- ① VAR models treat all variables as endogenous.
- ② Granger causality proves true economic causation.
- ③ A stable VAR always has eigenvalues inside the unit circle.
- ④ FEVD results depend on the ordering of variables.
- ⑤ VAR can be estimated by OLS equation by equation.
- ⑥ Impulse responses eventually die out in a stable VAR.

*Answers on next slide...*

## True/False: Solutions

- ① VAR models treat all variables as endogenous.

Each variable is regressed on lags of all variables, including itself.

TRUE

- ② Granger causality proves true economic causation.

It only shows predictive content, not structural causation.

FALSE

- ③ A stable VAR always has eigenvalues inside the unit circle.

Stability condition: all eigenvalues of companion matrix satisfy  $|\lambda_i| < 1$ .

TRUE

- ④ FEVD results depend on the ordering of variables.

Under Cholesky identification, different orderings give different results.

TRUE

- ⑤ VAR can be estimated by OLS equation by equation.

With same regressors in each equation, OLS = GLS = ML (under normality).

TRUE

- ⑥ Impulse responses eventually die out in a stable VAR.

Stability ensures shocks have transitory effects; IRFs  $\rightarrow 0$  as  $h \rightarrow \infty$ .

TRUE

## Problem 1: Writing VAR Equations

### Exercise

Write out the two equations for a bivariate VAR(1) model with variables  $Y_t$  (GDP growth) and  $X_t$  (inflation).

*Answer on next slide...*

## Problem 1: Solution

### Solution

$$Y_t = c_1 + a_{11} Y_{t-1} + a_{12} X_{t-1} + \varepsilon_{1t}$$

$$X_t = c_2 + a_{21} Y_{t-1} + a_{22} X_{t-1} + \varepsilon_{2t}$$

#### Interpretation:

- $a_{12}$ : Effect of past inflation on current GDP growth
- $a_{21}$ : Effect of past GDP growth on current inflation

## Problem 2: Parameter Count

### Exercise

How many total parameters need to be estimated in a VAR(3) with  $K = 4$  variables (including constants)?

*Answer on next slide...*

## Problem 2: Solution

### Solution

Per equation:  $1 + K \times p = 1 + 4 \times 3 = 13$  parameters

Total for  $K = 4$  equations:  $4 \times 13 = 52$  parameters

Plus covariance matrix  $\Sigma$ :  $K(K + 1)/2 = 4 \times 5/2 = 10$  unique elements

**Grand total: 62 parameters**

*This is why VARs can be “over-parameterized” with limited data!*

## Problem 3: Granger Causality Interpretation

### Exercise

A Granger causality test yields:

- $H_0$ : Money does not Granger-cause GDP.  $p\text{-value} = 0.02$
- $H_0$ : GDP does not Granger-cause Money.  $p\text{-value} = 0.35$

Interpret these results.

*Answer on next slide...*

## Problem 3: Solution

### Solution

- Reject  $H_0$  at 5%: Money **Granger-causes** GDP
- Fail to reject  $H_0$ : GDP does **not** Granger-cause Money

**Conclusion:** Unidirectional causality: Money  $\rightarrow$  GDP

*Interpretation:* Past money supply helps predict GDP growth. This is consistent with monetarist views, but remember: Granger causality  $\neq$  structural causality!

## Problem 4: Stability Check

### Exercise

For VAR(1) with  $\mathbf{A}_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.1 & 0.5 \end{pmatrix}$ , check stability.

*Answer on next slide...*

## Problem 4: Solution

### Solution

Find eigenvalues:  $\det(\mathbf{A}_1 - \lambda\mathbf{I}) = 0$

$$(0.7 - \lambda)(0.5 - \lambda) - (0.2)(0.1) = 0$$

$$\lambda^2 - 1.2\lambda + 0.33 = 0$$

$$\lambda = \frac{1.2 \pm \sqrt{1.44 - 1.32}}{2} = \frac{1.2 \pm 0.346}{2}$$

$$\lambda_1 = 0.773, \quad \lambda_2 = 0.427$$

Both  $|\lambda_i| < 1 \Rightarrow \mathbf{Stable!}$

## Problem 5: IRF Computation

### Exercise

For VAR(1) with  $\mathbf{A} = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$ , compute  $\Phi_2$  (response at  $h = 2$ ).

*Answer on next slide...*

## Problem 5: Solution

### Solution

$$\begin{aligned}\Phi_2 &= \mathbf{A}^2 = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix} \\ &= \begin{pmatrix} 0.25 + 0 & 0.10 + 0.12 \\ 0 + 0 & 0 + 0.36 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.22 \\ 0 & 0.36 \end{pmatrix}\end{aligned}$$

**Interpretation:** A unit shock to  $Y_2$  at  $t$  increases  $Y_1$  by 0.22 at  $t + 2$ .

## Example: Stock Returns and Trading Volume

### Scenario

Daily data on stock returns ( $R_t$ ) and trading volume ( $V_t$ ). Test Granger causality both directions.

### Typical Findings in Finance Literature

- Returns often Granger-cause volume (price changes trigger trading)
- Volume sometimes Granger-causes returns (volume as leading indicator)
- Results: Often **bidirectional** causality  $R \leftrightarrow V$

### Practical Issue

Stock returns are typically stationary, but volume may need transformation (log or difference).

## Example: Interest Rates and Inflation

### Taylor Rule Context

Central banks set interest rates ( $i_t$ ) in response to inflation ( $\pi_t$ ):  $i_t = r^* + \pi^* + 1.5(\pi_t - \pi^*) + 0.5(y_t - y^*)$

### VAR Analysis Questions

- Does inflation Granger-cause interest rates? (Central bank reaction)
- Do interest rates Granger-cause inflation? (Monetary policy transmission)

### Expected Results

Bidirectional causality: Quick  $\pi \rightarrow i$  (policy reaction), Delayed  $i \rightarrow \pi$  (policy effect)

# Python VAR Analysis: Key Functions

## Essential Libraries

```
from statsmodels.tsa.api import VAR  
from statsmodels.tsa.stattools import grangercausalitytests
```

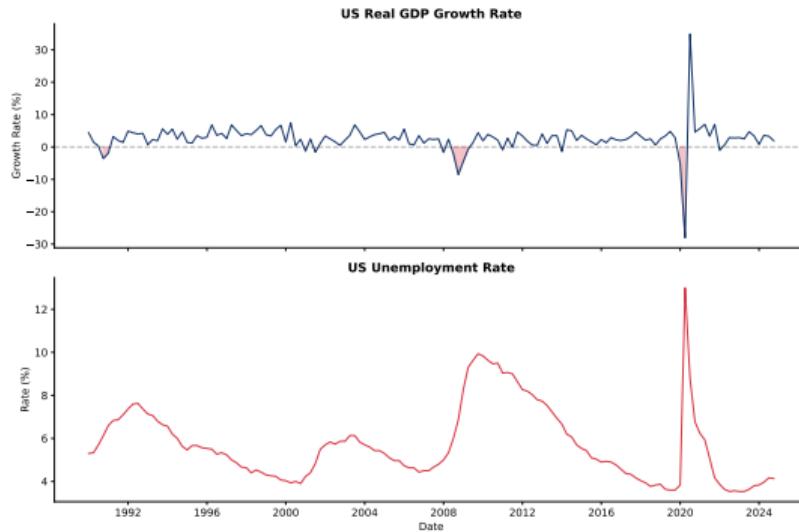
## Workflow

- ① Create DataFrame: `data = pd.DataFrame({'gdp': ... , 'unemp': ...})`
- ② Fit VAR: `model = VAR(data); results = model.fit(maxlags=8, ic='aic')`
- ③ Get IRF: `irf = results.irf(periods=20)`
- ④ Get FEVD: `fevd = results.fevd(periods=20)`
- ⑤ Granger tests: `grangercausalitytests(data[['y', 'x']], maxlag=4)`

## Note

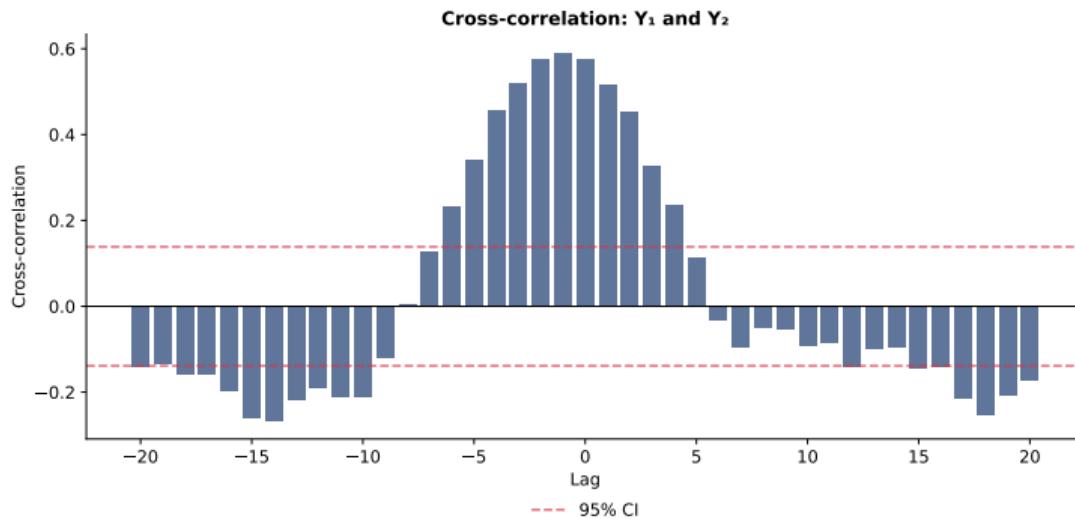
Complete working examples are provided in the Jupyter notebooks.

## Case Study: GDP and Unemployment



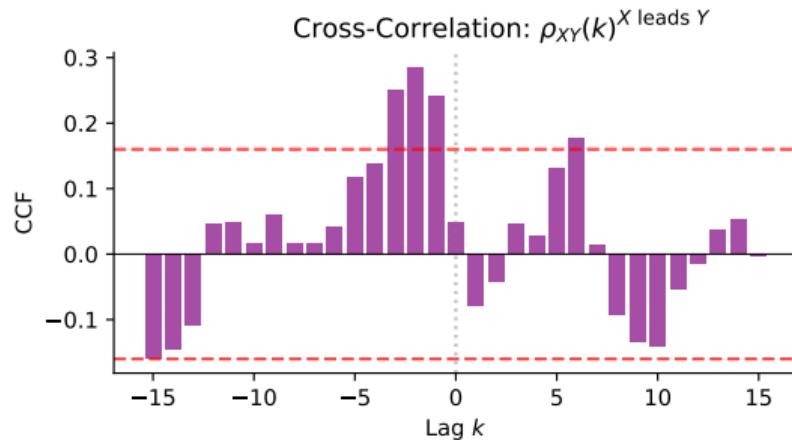
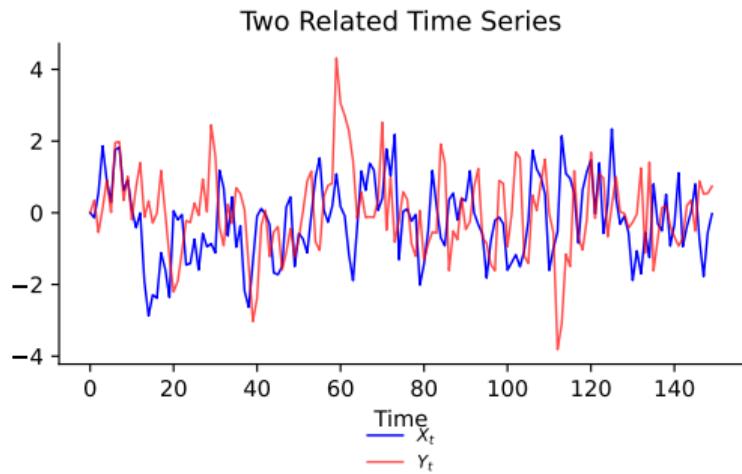
- **Top:** US Real GDP growth rate (quarterly)
- **Bottom:** US Unemployment rate
- Clear negative relationship (Okun's Law)
- VAR model can capture dynamic interactions between these variables

## Cross-Correlation Analysis



- Cross-correlation measures lead-lag relationships
- Negative correlation at lag 0: contemporaneous inverse relationship
- Asymmetric pattern suggests unemployment responds to GDP with lag
- Helps inform VAR lag order selection

## Visual: Cross-Correlation Function



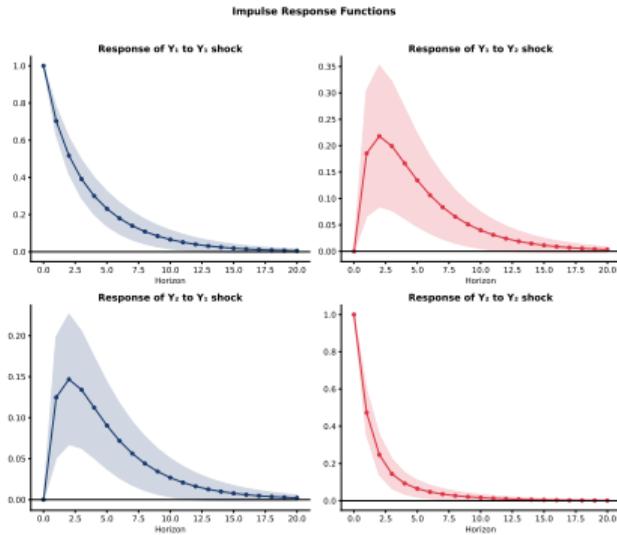
The CCF measures correlation between two series at different lags, revealing lead-lag relationships.

## VAR Estimation Results

Model: VAR(2) for GDP Growth and Unemployment

Equation	Variable	Coef.	Std. Error	t-stat
$\Delta GDP_t$	$\Delta GDP_{t-1}$	0.312	0.087	3.59
	$\Delta GDP_{t-2}$	0.145	0.082	1.77
	$U_{t-1}$	-0.421	0.156	-2.70
	$U_{t-2}$	0.198	0.148	1.34
$U_t$	$\Delta GDP_{t-1}$	-0.087	0.032	-2.72
	$\Delta GDP_{t-2}$	-0.045	0.030	-1.50
	$U_{t-1}$	1.456	0.058	25.1
	$U_{t-2}$	-0.521	0.055	-9.47

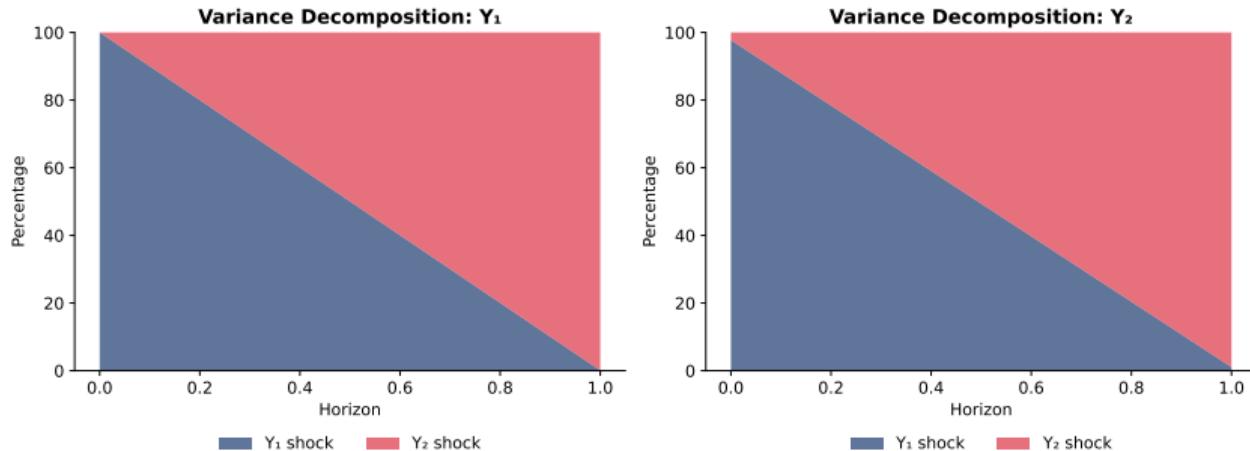
# Impulse Response Functions



- IRFs show dynamic response to one-unit shocks
- GDP shock: temporary positive effect on GDP, negative on unemployment
- Unemployment shock: negative effect on GDP, persistent on unemployment
- 95% confidence bands show uncertainty in responses

# Forecast Error Variance Decomposition

Forecast Error Variance Decomposition



- FEVD shows proportion of variance explained by each shock
- GDP variance: mostly explained by own shocks, some by unemployment
- Unemployment variance: highly persistent (own shocks dominant)
- Provides insight into relative importance of different shocks

# Discussion: Granger Causality vs True Causality

## Key Question

If  $X$  Granger-causes  $Y$ , does that mean  $X$  actually causes  $Y$ ? NO!

## Why Granger Causality Can Fail

- **Omitted variable bias:**  $Z$  might cause both  $X$  and  $Y$  (e.g., weather → ice cream & drownings)
- **Anticipation effects:** Markets anticipate future events (stock prices → earnings)
- **Aggregation issues:** Timing of data collection matters

## Conclusion

Granger causality is about **prediction**, not **mechanism**. For structural causality, need theory + identification strategy.

## Discussion: Variable Ordering in IRFs

### Key Question

Why does variable ordering matter for orthogonalized IRFs?

### Cholesky Decomposition Assumes

- First variable: Affects all others contemporaneously
- Second variable: Affected by first, affects remaining
- Last variable: Affected by all, affects none contemporaneously

### Example: Monetary Policy VAR Ordering

1. Oil prices (exogenous) → 2. GDP (slow) → 3. Inflation → 4. Interest rates (fast)

### Rule

Order from “most exogenous” to “most endogenous” — justify with economic theory!

## Take-Home Exercises

- ① **Theoretical:** Show that a VAR(1) can be written as MA( $\infty$ ):  $\mathbf{Y}_t = \sum_{i=0}^{\infty} \mathbf{A}^i \boldsymbol{\epsilon}_{t-i}$  when stable.
- ② **Computation:** For VAR(1) with  $\mathbf{A} = \begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$ :
  - Check stability; Compute IRFs for  $h = 0, 1, 2, 3$
  - Plot the response of  $Y_1$  to a shock in  $Y_2$
- ③ **Applied:** Download US GDP growth and unemployment data:
  - Test stationarity; Estimate VAR (select optimal lag)
  - Test Granger causality; Compute and interpret IRFs
- ④ **Critical Thinking:** Why might stock prices “Granger-cause” GDP even though GDP is determined by real factors?

## Hints

- ① Use recursive substitution:  $\mathbf{Y}_t = \mathbf{A}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t = \mathbf{A}(\mathbf{A}\mathbf{Y}_{t-2} + \boldsymbol{\varepsilon}_{t-1}) + \boldsymbol{\varepsilon}_t = \dots$
- ② Eigenvalues of  $\begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$ :
  - Characteristic equation:  $\lambda^2 - 1.2\lambda + 0.35 = 0$
  - $\lambda_1 \approx 0.85$ ,  $\lambda_2 \approx 0.41$  (both  $< 1$ , stable)
- ③ For GDP/Unemployment:
  - GDP growth is usually I(0), unemployment may be I(1)
  - Use unemployment rate changes if needed
  - Expect GDP growth  $\rightarrow$  unemployment (Okun's Law)
- ④ Stock prices anticipate future economic conditions—they reflect expectations about future GDP, so they “lead” GDP in the data even though causation runs the other way.

# Key Takeaways from This Seminar

## Main Points

- ① VAR models capture **interdependencies** between multiple time series
- ② Parameter count grows quickly:  $K^2 p + K$  per system
- ③ **Granger causality** tests predictive content, not true causation
- ④ IRFs show dynamic propagation of shocks; ordering matters

## Practical Points

- Always check stationarity before estimating VAR
- Use information criteria (AIC/BIC) for lag selection
- Report Granger tests in both directions
- Justify variable ordering with economic theory

## Remember

Granger causality is about **prediction**, not **mechanism**!