



Chapter 4: SARIMA Models

Seasonal Time Series



Outline

- 1 Seasonality in Time Series
- 2 Seasonal Differencing
- 3 The SARIMA Model
- 4 Seasonal ACF and PACF Patterns
- 5 Estimation and Diagnostics
- 6 Forecasting with SARIMA
- 7 Real Data Application: Airline Passengers
- 8 Summary

What is Seasonality?

Definition 1 (Seasonality)

A time series exhibits **seasonality** when it shows regular, periodic fluctuations that repeat over a fixed period s (the seasonal period).

Common Seasonal Periods

- Monthly data: $s = 12$ (annual cycle)
- Quarterly data: $s = 4$ (annual cycle)
- Weekly data: $s = 52$ (annual) or $s = 7$ (weekly pattern)
- Daily data: $s = 7$ (weekly pattern)

Examples of Seasonal Data

Economic Series

- Retail sales (holiday peaks)
- Tourism (summer/winter)
- Agricultural production
- Energy consumption
- Employment (hiring cycles)

Other Domains

- Weather/temperature
- Website traffic
- Hospital admissions
- Transportation usage
- Electricity demand

Why It Matters

Ignoring seasonality leads to biased forecasts and invalid inference!

Deterministic vs Stochastic Seasonality

Deterministic Seasonality

Fixed seasonal pattern:

$$Y_t = \sum_{j=1}^s \gamma_j D_{jt} + \varepsilon_t$$

where D_{jt} are seasonal dummies.

Properties:

- Pattern is constant over time
- Can be removed by regression

Stochastic Seasonality

Evolving seasonal pattern:

$$\Delta_s Y_t = Y_t - Y_{t-s}$$

exhibits dependence structure.

Properties:

- Pattern evolves over time
- Requires seasonal differencing

Detecting Seasonality

Visual Methods

- Time series plot – look for repeating patterns
- Seasonal subseries plot – compare same seasons across years
- ACF plot – spikes at seasonal lags ($s, 2s, 3s, \dots$)

Statistical Tests

- Seasonal unit root tests (HEGY, CH, OCSB)
- F-test for seasonal dummies
- Kruskal-Wallis test (nonparametric)

ACF Signature

Strong seasonality: ACF shows significant spikes at lags $s, 2s, 3s, \dots$

Definition 2 (Seasonal Difference)

The **seasonal difference operator** Δ_s is defined as:

$$\Delta_s Y_t = (1 - L^s) Y_t = Y_t - Y_{t-s}$$

where $L^s Y_t = Y_{t-s}$ is the seasonal lag operator.

Examples

- Monthly data ($s = 12$): $\Delta_{12} Y_t = Y_t - Y_{t-12}$
Compares each month to the same month last year
- Quarterly data ($s = 4$): $\Delta_4 Y_t = Y_t - Y_{t-4}$
Compares each quarter to the same quarter last year

Combining Regular and Seasonal Differencing

Full Differencing

For series with both trend and seasonality:

$$\Delta\Delta_s Y_t = (1 - L)(1 - L^s)Y_t$$

Expansion

$$(1 - L)(1 - L^s)Y_t = Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}$$

For monthly data ($s = 12$):

$$\Delta\Delta_{12} Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$$

Order of Differencing

- d : number of regular differences (trend removal)
- D : number of seasonal differences (seasonal trend removal)

Definition 3 (Seasonally Integrated Process)

A series Y_t is **seasonally integrated** of order $(d, D)_s$, written $Y_t \sim I(d, D)_s$, if:

$$(1 - L)^d (1 - L^s)^D Y_t$$

is stationary.

Common Cases

- $I(1, 0)_{12}$: Regular unit root only (monthly)
- $I(0, 1)_{12}$: Seasonal unit root only
- $I(1, 1)_{12}$: Both regular and seasonal unit roots

Definition 4 (SARIMA(p, d, q) \times (P, D, Q) $_s$)

The **Seasonal ARIMA** model is:

$$\phi(L)\Phi(L^s)(1 - L)^d(1 - L^s)^D Y_t = c + \theta(L)\Theta(L^s)\varepsilon_t$$

Components

- $\phi(L) = 1 - \phi_1L - \dots - \phi_pL^p$: Non-seasonal AR
- $\Phi(L^s) = 1 - \Phi_1L^s - \dots - \Phi_PL^{Ps}$: Seasonal AR
- $\theta(L) = 1 + \theta_1L + \dots + \theta_qL^q$: Non-seasonal MA
- $\Theta(L^s) = 1 + \Theta_1L^s + \dots + \Theta_QL^{Qs}$: Seasonal MA
- $(1 - L)^d$: Regular differencing
- $(1 - L^s)^D$: Seasonal differencing

SARIMA Notation

Full Specification

SARIMA($p, d, q) \times (P, D, Q)_s$ has 7 parameters to specify:

Parameter	Meaning
p	Non-seasonal AR order
d	Non-seasonal differencing order
q	Non-seasonal MA order
P	Seasonal AR order
D	Seasonal differencing order
Q	Seasonal MA order
s	Seasonal period

Example

SARIMA(1, 1, 1) \times (1, 1, 1)₁₂: Monthly data with AR(1), MA(1), seasonal AR(1), seasonal MA(1), regular and seasonal differencing.

Common SARIMA Models

Airline Model: $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_s$

$$(1 - L)(1 - L^s)Y_t = (1 + \theta L)(1 + \Theta L^s)\varepsilon_t$$

Classic model for many economic series (Box & Jenkins, 1970).

$\text{SARIMA}(1, 0, 0) \times (1, 0, 0)_s$

$$(1 - \phi L)(1 - \Phi L^s)Y_t = \varepsilon_t$$

Pure seasonal and non-seasonal autoregressive model.

$\text{SARIMA}(0, 1, 1) \times (0, 1, 0)_s$

$$(1 - L)(1 - L^s)Y_t = (1 + \theta L)\varepsilon_t$$

Random walk with seasonal differencing and $\text{MA}(1)$ errors.

The Multiplicative Structure

Why Multiplicative?

The seasonal and non-seasonal parts **multiply**:

$$\phi(L)\Phi(L^s) \quad \text{and} \quad \theta(L)\Theta(L^s)$$

Example: SARIMA(1, 0, 0) \times (1, 0, 0)₁₂

$$(1 - \phi L)(1 - \Phi L^{12}) Y_t = \varepsilon_t$$

$$\text{Expanding: } Y_t - \phi Y_{t-1} - \Phi Y_{t-12} + \phi\Phi Y_{t-13} = \varepsilon_t$$

The cross-term $\phi\Phi Y_{t-13}$ captures interaction!

Interpretation

Multiplicative structure allows parsimonious modeling of complex seasonal patterns with few parameters.

Key Insight

Seasonal models show patterns at both:

- Non-seasonal lags: $1, 2, 3, \dots$
- Seasonal lags: $s, 2s, 3s, \dots$

Model	ACF	PACF
SAR(P)	Decays at $s, 2s, \dots$	Cuts off after Ps
SMA(Q)	Cuts off after Qs	Decays at $s, 2s, \dots$
SARMA	Decays at seasonal lags	Decays at seasonal lags

Example: Airline Model ACF/PACF

SARIMA(0, 1, 1) \times (0, 1, 1)₁₂

After differencing $W_t = (1 - L)(1 - L^{12})Y_t$:

$$W_t = (1 + \theta L)(1 + \Theta L^{12})\varepsilon_t$$

Expected ACF Pattern

- Spike at lag 1 (from θ)
- Spike at lag 12 (from Θ)
- Spike at lag 13 (from $\theta \cdot \Theta$ interaction)
- All other lags near zero

Expected PACF Pattern

- Exponential decay at lags 1, 2, 3, ...
- Exponential decay at lags 12, 24, 36, ...

Step-by-Step Process

- ① Examine ACF for slow decay at seasonal lags \Rightarrow seasonal differencing
- ② After differencing, check ACF/PACF patterns
- ③ Non-seasonal behavior at lags $1, 2, \dots, s - 1$
- ④ Seasonal behavior at lags $s, 2s, 3s, \dots$

Practical Tips

- Start with $d \leq 1$ and $D \leq 1$
- Usually $P, Q \leq 2$ is sufficient
- Use information criteria (AIC, BIC) for final selection
- Auto-SARIMA algorithms can help

Maximum Likelihood Estimation

Standard approach for SARIMA:

- Conditional MLE (conditional on initial values)
- Exact MLE (via Kalman filter)

Computational Considerations

- More parameters than ARIMA \Rightarrow more data needed
- Seasonal parameters estimated from lags $s, 2s, \dots$
- Need sufficient seasonal cycles (at least 3-4 years of monthly data)

Stationarity and Invertibility

Stationarity Conditions

Both non-seasonal and seasonal AR polynomials must have roots outside the unit circle:

- $\phi(z) = 0 \Rightarrow |z| > 1$
- $\Phi(z^s) = 0 \Rightarrow |z| > 1$

Invertibility Conditions

Both non-seasonal and seasonal MA polynomials must have roots outside the unit circle:

- $\theta(z) = 0 \Rightarrow |z| > 1$
- $\Theta(z^s) = 0 \Rightarrow |z| > 1$

Residual Analysis

After fitting SARIMA, check that residuals are white noise:

- ① Plot residuals over time (no patterns)
- ② ACF of residuals (no significant spikes)
- ③ Ljung-Box test at multiple lags including seasonal
- ④ Normality tests (Q-Q plot, Jarque-Bera)

Important

Check ACF at **both** non-seasonal and seasonal lags!

Significant ACF at lag 12 suggests inadequate seasonal modeling.

Information Criteria

Compare competing SARIMA models using:

- $AIC = -2 \ln(L) + 2k$
- $BIC = -2 \ln(L) + k \ln(n)$
- $AICc = AIC + \frac{2k(k+1)}{n-k-1}$ (corrected for small samples)

where $k = p + q + P + Q + 1$ (plus 1 for variance).

Auto-SARIMA

Python's `pmdarima.auto_arima()` with `seasonal=True` automatically searches for optimal $(p, d, q) \times (P, D, Q)_s$.

Forecast Computation

SARIMA forecasts are computed recursively:

- Replace future ε_{T+h} with 0
- Replace future Y_{T+h} with forecasts $\hat{Y}_{T+h|T}$
- Use known past values Y_T, Y_{T-1}, \dots

Seasonal Pattern in Forecasts

SARIMA forecasts naturally capture seasonality:

- Short-term: influenced by recent values
- Long-term: revert to seasonal pattern

Uncertainty Quantification

$(1 - \alpha)\%$ prediction interval:

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$$

Variance computed from $\text{MA}(\infty)$ representation.

Key Properties

- Intervals widen with forecast horizon
- For $I(1, 1)_s$ series: intervals grow without bound
- Seasonal pattern visible in point forecasts
- Uncertainty captures both trend and seasonal variation

Behavior as $h \rightarrow \infty$

- Point forecasts converge to deterministic seasonal pattern
- If drift present: linear trend + seasonal pattern
- Forecast intervals continue to widen

Practical Implication

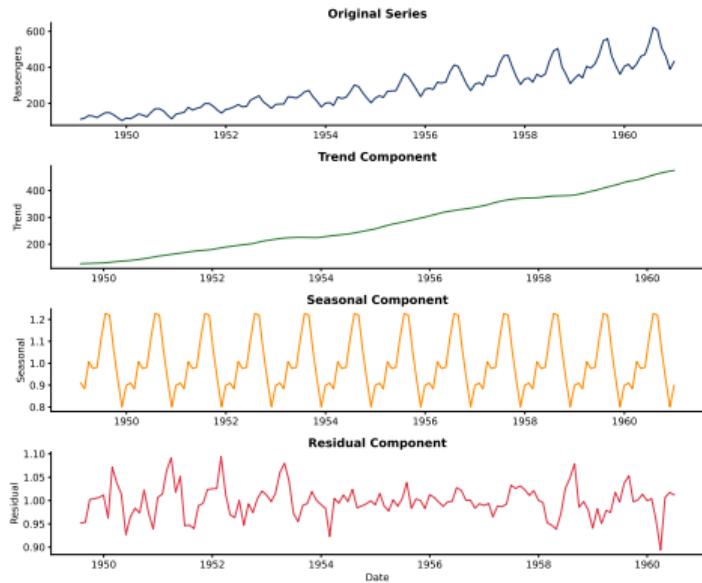
- Short-term: SARIMA captures both level and season
- Medium-term: Good seasonal forecasts, growing uncertainty
- Long-term: Mainly reflects seasonal pattern, wide intervals

Airline Passengers Data



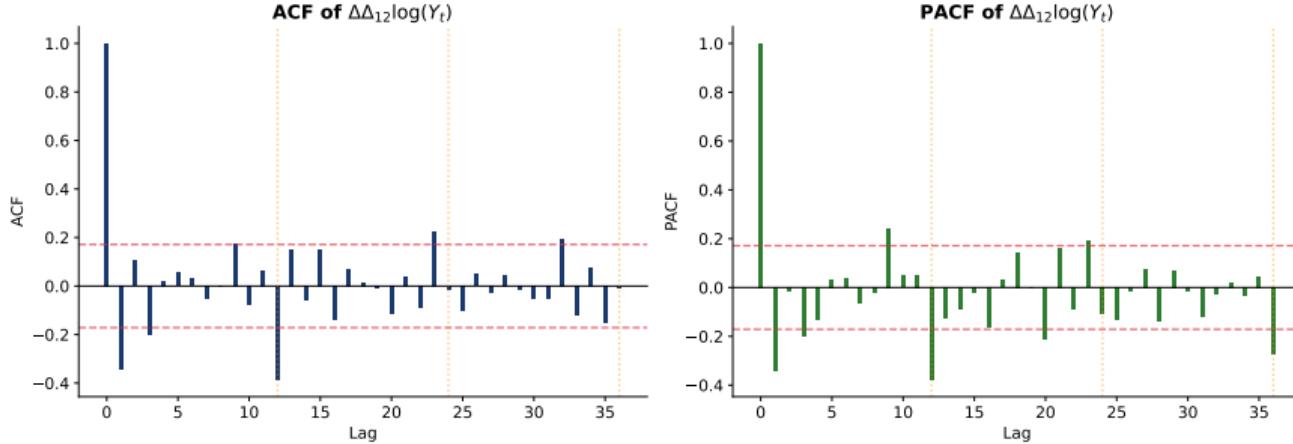
- Classic dataset: Monthly international airline passengers (1949-1960)
- Clear upward trend and growing seasonal amplitude

Seasonal Decomposition



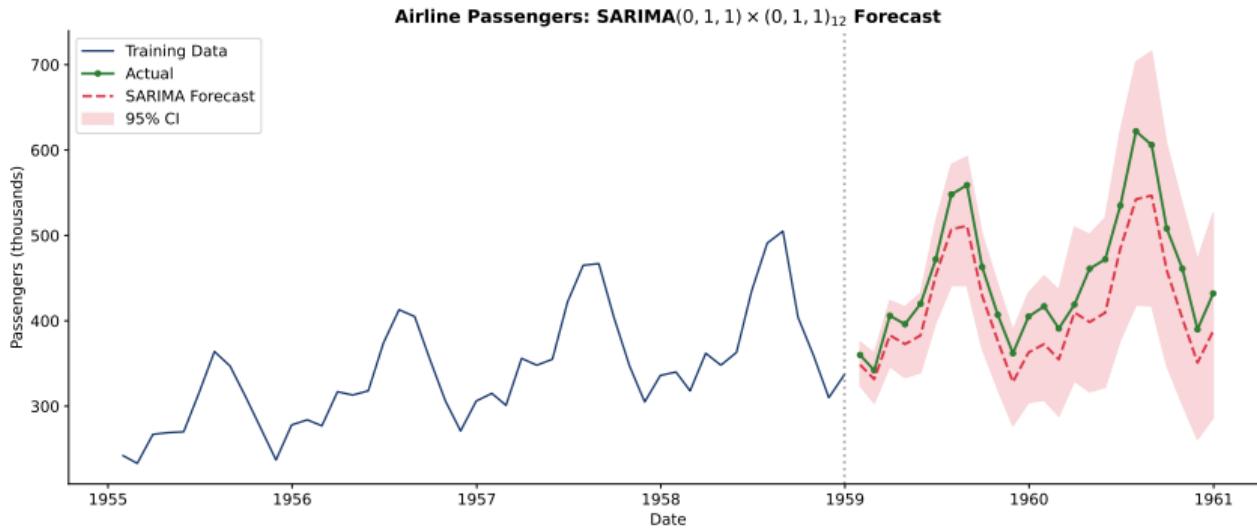
- Trend: Strong upward growth
- Seasonality: Summer peaks (vacation travel)
- Residual: Random variation after removing trend and season

ACF/PACF Analysis



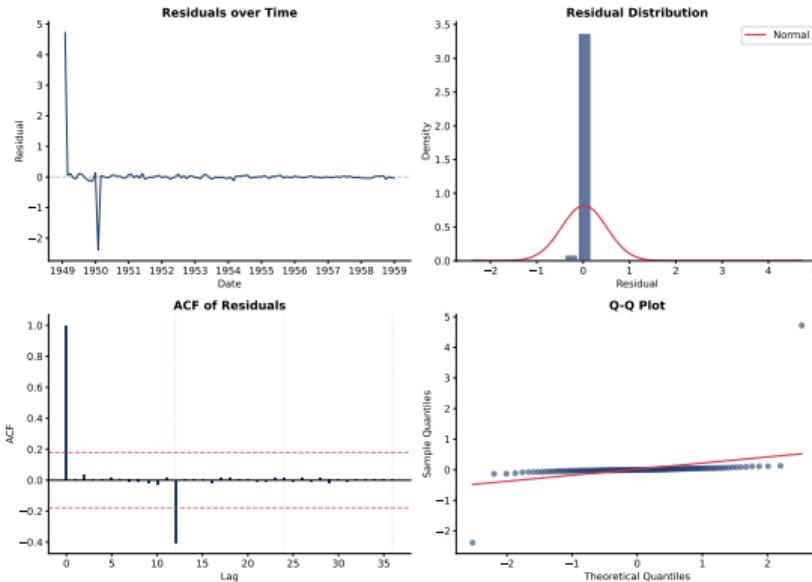
- After $\Delta\Delta_{12}$ differencing: spikes at lags 1 and 12
- Suggests SARIMA(0, 1, 1) \times (0, 1, 1)₁₂ (Airline model)

SARIMA Forecast Results



- SARIMA captures both trend and seasonal pattern
- Forecasts show appropriate seasonal peaks and troughs

Model Diagnostics



- Residuals appear random; ACF within bounds at all lags
- Model adequately captures seasonal structure

```
Fit SARIMA(0,1,1)(0,1,1)[12] model = SARIMAX(y, order=(0,1,1),  
seasonal_order = (0, 1, 1, 12))results = model.fit()print(results.summary())  
Forecast forecast = results.get_forecast(steps = 24)
```

Key Takeaways

Main Points

- ① Seasonality is common in economic and business data
- ② Seasonal differencing $(1 - L^s)$ removes stochastic seasonality
- ③ SARIMA(p, d, q) \times (P, D, Q)_s extends ARIMA for seasonal data
- ④ Multiplicative structure captures seasonal-trend interactions
- ⑤ ACF/PACF show patterns at both regular and seasonal lags
- ⑥ Model selection: Use AIC/BIC or auto-SARIMA algorithms

Next Steps

Chapter 5 will cover multivariate time series: VAR models, Granger causality, and cointegration.

References

-  Box, G.E.P., Jenkins, G.M., Reinsel, G.C., & Ljung, G.M. (2015). *Time Series Analysis: Forecasting and Control*. 5th ed. Wiley.
-  Hyndman, R.J. & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*. 3rd ed. OTexts.
-  Hamilton, J.D. (1994). *Time Series Analysis*. Princeton University Press.
-  Brockwell, P.J. & Davis, R.A. (2016). *Introduction to Time Series and Forecasting*. 3rd ed. Springer.