



# Chapter 5: VAR Models & Granger Causality

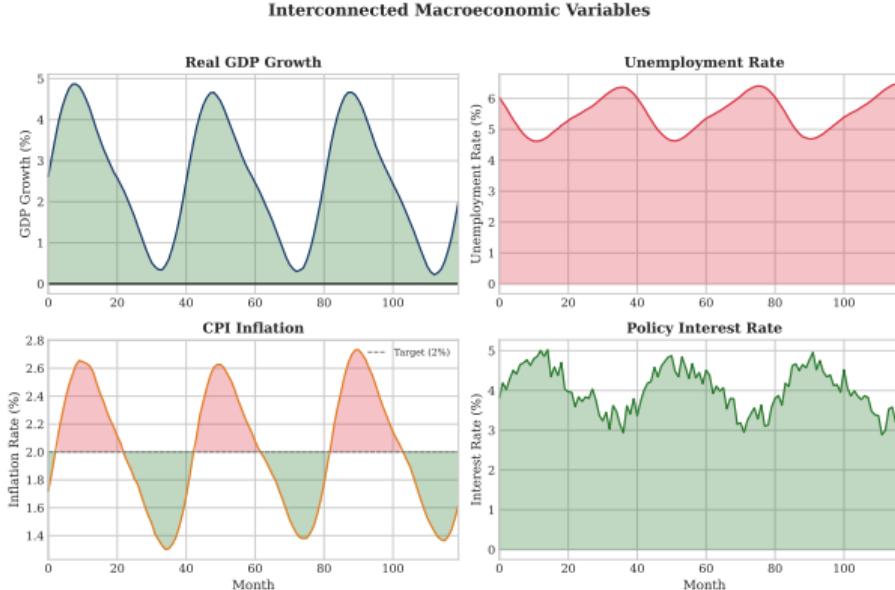
Multivariate Time Series



# Lecture Outline

- 1 Introduction to Multivariate Time Series
- 2 Vector Autoregression (VAR)
- 3 Granger Causality
- 4 Impulse Response Functions
- 5 Forecast Error Variance Decomposition
- 6 VAR Diagnostics
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- 8 Practical Example
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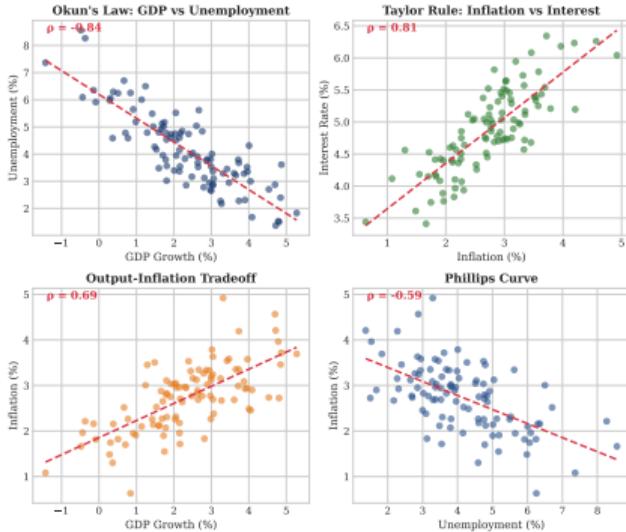
# Motivating Example: Macroeconomic Dynamics



- Economic variables are **interconnected**: GDP affects unemployment, inflation affects interest rates
- Changes in one variable **propagate** through the system
- Understanding these dynamics requires **multivariate** analysis

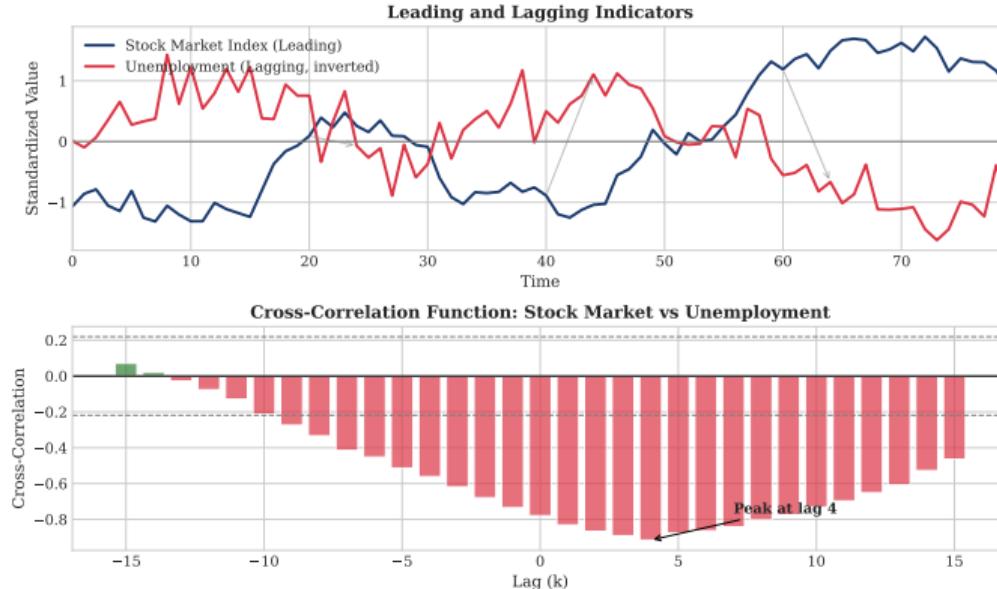
# The Key Insight: Variables Interact

Relationships Between Economic Variables



- **Okun's Law:** Higher GDP growth  $\Rightarrow$  lower unemployment
- **Taylor Rule:** Higher inflation  $\Rightarrow$  higher interest rates
- **Phillips Curve:** Unemployment-inflation tradeoff

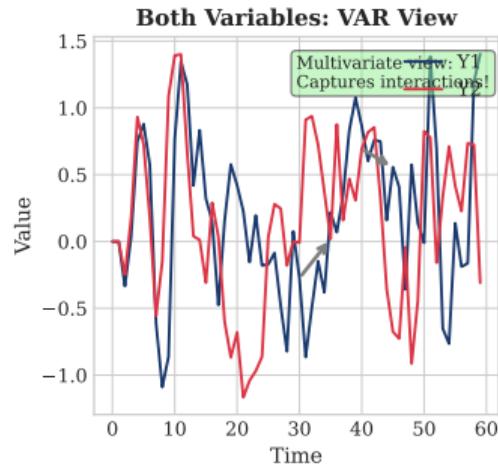
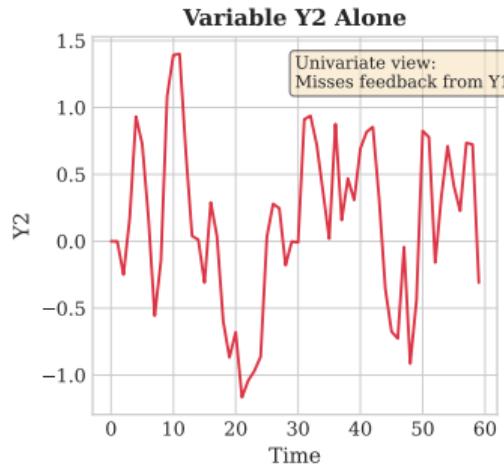
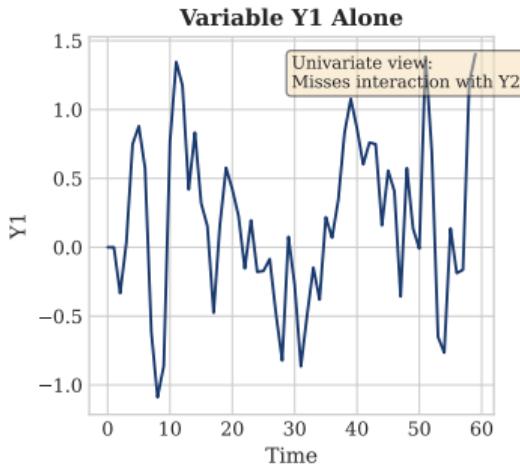
# Lead-Lag Relationships



- Some variables **lead** others: stock market predicts economic activity
- Cross-correlation reveals the **timing** of relationships
- Peak correlation at lag 4: stock market leads unemployment by  $\sim 4$  months

# Why Univariate Models Are Not Enough

## Why Multivariate Analysis? Capturing Interdependence



## The Problem

ARIMA models each variable **in isolation**—ignoring valuable information from other variables!

## The Solution

VAR models capture the joint dynamics and feedback effects between multiple time series.

# What We'll Learn Today

## Core Concepts

- ① **VAR Models:** How to model multiple time series jointly
- ② **Granger Causality:** Does  $X$  help predict  $Y$ ?
- ③ **Impulse Response Functions:** How do shocks propagate?
- ④ **Variance Decomposition:** What drives each variable?

## Applications

- Macroeconomic policy analysis (monetary policy effects)
- Financial market dynamics (stock-bond relationships)
- Business cycle analysis (leading indicators)
- Risk management (volatility transmission)

# Why Multivariate Analysis?

## Limitations of Univariate Models

- ARIMA models each variable **in isolation**
- Ignores potential **interactions** between variables
- Cannot capture **feedback effects**

## Economic Examples of Interdependence

- GDP and unemployment (Okun's law)
- Interest rates and inflation (Taylor rule)
- Stock prices and trading volume
- Exchange rates and trade balance

## Vector of Variables

Let  $\mathbf{Y}_t = (Y_{1t}, Y_{2t}, \dots, Y_{Kt})'$  be a  $K \times 1$  vector of time series.

Example with  $K = 2$ :

$$\mathbf{Y}_t = \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \text{GDP growth}_t \\ \text{Inflation}_t \end{pmatrix}$$

## Key Questions

- ① Does  $Y_1$  help predict  $Y_2$ ? (Granger causality)
- ② How do shocks to  $Y_1$  affect  $Y_2$ ? (Impulse responses)
- ③ What proportion of  $Y_2$ 's variance is due to  $Y_1$ ? (Variance decomposition)

## Definition: Weak Stationarity

A  $K$ -dimensional time series  $\mathbf{Y}_t$  is **weakly stationary** if:

- ①  $\mathbb{E}[\mathbf{Y}_t] = \boldsymbol{\mu}$  (constant mean vector)
- ②  $\text{Cov}(\mathbf{Y}_t, \mathbf{Y}_{t-h}) = \boldsymbol{\Gamma}(h)$  depends only on  $h$ , not  $t$

## Autocovariance Matrix

$$\boldsymbol{\Gamma}(h) = \mathbb{E}[(\mathbf{Y}_t - \boldsymbol{\mu})(\mathbf{Y}_{t-h} - \boldsymbol{\mu})'] = \begin{pmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{pmatrix}$$

Note:  $\boldsymbol{\Gamma}(-h) = \boldsymbol{\Gamma}(h)'$  (transpose, not equal!)

## Cross-Covariance Function

For variables  $Y_{it}$  and  $Y_{jt}$ :

$$\gamma_{ij}(h) = \text{Cov}(Y_{it}, Y_{j,t-h}) = \mathbb{E}[(Y_{it} - \mu_i)(Y_{j,t-h} - \mu_j)]$$

## Key Difference from Univariate Case

- In general:  $\gamma_{ij}(h) \neq \gamma_{ij}(-h)$
- But:  $\gamma_{ij}(h) = \gamma_{ji}(-h)$
- The cross-covariance matrix is **not symmetric** for  $h \neq 0$

## Example

If  $Y_1$  leads  $Y_2$ :  $\gamma_{12}(h) > 0$  for  $h > 0$  but  $\gamma_{12}(h) \approx 0$  for  $h < 0$

## Correlation Matrix Function

### Definition

The **autocorrelation matrix** at lag  $h$ :

$$\mathbf{R}(h) = \mathbf{D}^{-1} \boldsymbol{\Gamma}(h) \mathbf{D}^{-1}$$

where  $\mathbf{D} = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_K)$  and  $\sigma_i = \sqrt{\gamma_{ii}(0)}$

### For Bivariate Case

$$\mathbf{R}(h) = \begin{pmatrix} \rho_{11}(h) & \rho_{12}(h) \\ \rho_{21}(h) & \rho_{22}(h) \end{pmatrix}$$

where  $\rho_{ij}(h) = \frac{\gamma_{ij}(h)}{\sigma_i \sigma_j}$

Diagonal elements: usual ACFs; Off-diagonal: cross-correlations

# The VAR(p) Model

## Definition

A **VAR(p)** model for  $K$  variables:

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$$

where:

- $\mathbf{Y}_t$ :  $K \times 1$  vector of endogenous variables
- $\mathbf{c}$ :  $K \times 1$  vector of constants
- $\mathbf{A}_i$ :  $K \times K$  coefficient matrices
- $\boldsymbol{\varepsilon}_t$ :  $K \times 1$  vector of error terms with  $\mathbb{E}[\boldsymbol{\varepsilon}_t] = \mathbf{0}$ ,  $\mathbb{E}[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}'_t] = \Sigma$

# VAR(1) with Two Variables

## Bivariate VAR(1)

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

## Equation by Equation

$$Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$$

$$Y_{2t} = c_2 + a_{21} Y_{1,t-1} + a_{22} Y_{2,t-1} + \varepsilon_{2t}$$

**Key insight:** Each equation includes lags of **all** variables!

## Numerical Example: VAR(1)

### Specific VAR(1) Model

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} + \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

### Interpretation of Coefficients

- $a_{11} = 0.7$ : A 1-unit increase in  $Y_1$  at  $t - 1$  increases  $Y_1$  at  $t$  by 0.7
- $a_{12} = 0.2$ : A 1-unit increase in  $Y_2$  at  $t - 1$  increases  $Y_1$  at  $t$  by 0.2
- $a_{21} = -0.1$ : A 1-unit increase in  $Y_1$  at  $t - 1$  **decreases**  $Y_2$  at  $t$  by 0.1
- $a_{22} = 0.6$ : A 1-unit increase in  $Y_2$  at  $t - 1$  increases  $Y_2$  at  $t$  by 0.6

## VAR(2): Higher Order Dynamics

### VAR(2) Specification

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \boldsymbol{\varepsilon}_t$$

For  $K = 2$ , the full model has  $2 + 2 \times 4 + 2 \times 4 = 18$  parameters!

### Written Out

$$\begin{aligned} Y_{1t} &= c_1 + a_{11}^{(1)} Y_{1,t-1} + a_{12}^{(1)} Y_{2,t-1} + a_{11}^{(2)} Y_{1,t-2} + a_{12}^{(2)} Y_{2,t-2} + \varepsilon_{1t} \\ Y_{2t} &= c_2 + a_{21}^{(1)} Y_{1,t-1} + a_{22}^{(1)} Y_{2,t-1} + a_{21}^{(2)} Y_{1,t-2} + a_{22}^{(2)} Y_{2,t-2} + \varepsilon_{2t} \end{aligned}$$

### Curse of Dimensionality

VAR( $p$ ) with  $K$  variables has  $K + pK^2$  parameters. With  $K = 5$ ,  $p = 4$ :  $5 + 4 \times 25 = 105$  parameters!

# The Companion Form

## Converting VAR( $p$ ) to VAR(1)

Any VAR( $p$ ) can be written as a VAR(1) in **companion form**:

$$\xi_t = \mathbf{A}\xi_{t-1} + \mathbf{v}_t$$

## For VAR(2)

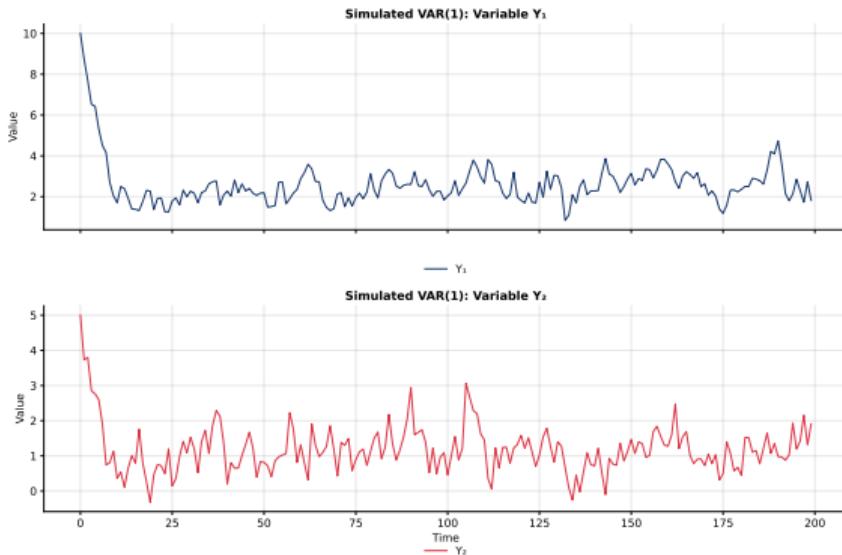
$$\underbrace{\begin{pmatrix} \mathbf{Y}_t \\ \mathbf{Y}_{t-1} \end{pmatrix}}_{\xi_t} = \underbrace{\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I}_K & \mathbf{0} \end{pmatrix}}_A \underbrace{\begin{pmatrix} \mathbf{Y}_{t-1} \\ \mathbf{Y}_{t-2} \end{pmatrix}}_{\xi_{t-1}} + \underbrace{\begin{pmatrix} \mathbf{e}_t \\ \mathbf{0} \end{pmatrix}}_{\mathbf{v}_t}$$

The companion matrix  $\mathbf{A}$  is  $Kp \times Kp$ .

## Why Useful?

Stationarity, forecasting, and IRFs are easier to analyze in companion form.

## Simulated VAR Process



- Two simulated series from a bivariate VAR(1) process showing interdependence
- Each variable responds to both its own past and the other variable's past
- Notice how the series co-move due to cross-equation dynamics

## Stationarity of VAR

### Stability Condition

VAR(p) is **stable** (stationary) if all roots of:

$$\det(\mathbf{I}_K - \mathbf{A}_1 z - \mathbf{A}_2 z^2 - \cdots - \mathbf{A}_p z^p) = 0$$

lie **outside** the unit circle (i.e.,  $|z| > 1$ ).

### For VAR(1)

The model is stable if all **eigenvalues** of  $\mathbf{A}_1$  are less than 1 in absolute value.

Example: For  $\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$ , eigenvalues are  $\lambda_1 = 0.6$  and  $\lambda_2 = 0.2$ .

Both  $< 1 \Rightarrow$  stable!

## Computing Eigenvalues: Example

For  $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$

Characteristic polynomial:  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det \begin{pmatrix} 0.7 - \lambda & 0.2 \\ -0.1 & 0.6 - \lambda \end{pmatrix} = (0.7 - \lambda)(0.6 - \lambda) + 0.02 = 0$$
$$\lambda^2 - 1.3\lambda + 0.44 = 0$$

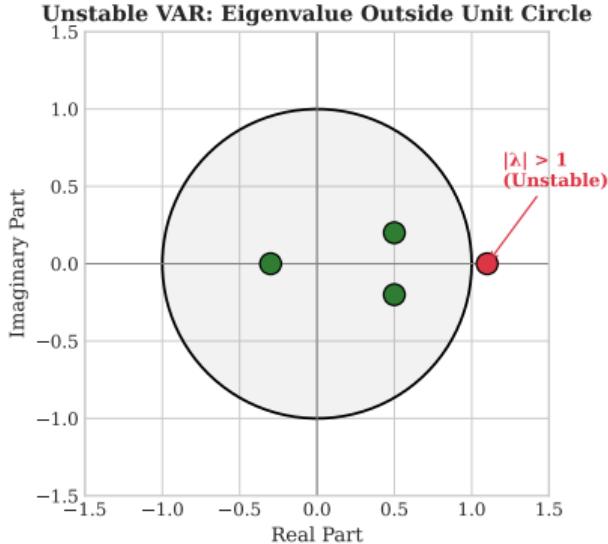
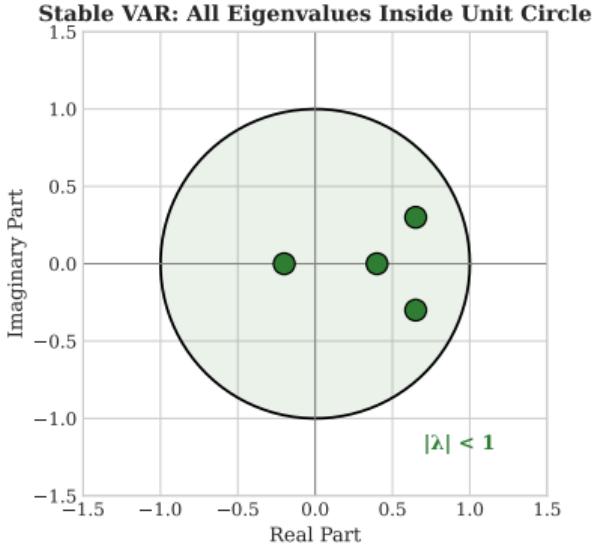
### Solution

Using quadratic formula:

$$\lambda = \frac{1.3 \pm \sqrt{1.69 - 1.76}}{2} = \frac{1.3 \pm \sqrt{-0.07}}{2} = 0.65 \pm 0.132i$$

$$|\lambda| = \sqrt{0.65^2 + 0.132^2} = \sqrt{0.44} = 0.663 < 1 \quad \checkmark \text{ Stable!}$$

## Stability Condition: Visual Interpretation



- Eigenvalues of the companion matrix must lie inside the unit circle
- Complex eigenvalues come in conjugate pairs
- If any eigenvalue is outside the circle, the VAR is explosive (non-stationary)

# Mean of a Stationary VAR

## Unconditional Mean

For a stationary VAR(1):  $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t$

Taking expectations:

$$\mathbb{E}[\mathbf{Y}_t] = \mathbf{c} + \mathbf{A}\mathbb{E}[\mathbf{Y}_{t-1}]$$

Since  $\mathbb{E}[\mathbf{Y}_t] = \mathbb{E}[\mathbf{Y}_{t-1}] = \boldsymbol{\mu}$  (stationarity):

$$\boldsymbol{\mu} = \mathbf{c} + \mathbf{A}\boldsymbol{\mu} \quad \Rightarrow \quad \boldsymbol{\mu} = (\mathbf{I}_K - \mathbf{A})^{-1}\mathbf{c}$$

## Example

If  $\mathbf{c} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}$  and  $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$ :

$$\boldsymbol{\mu} = \begin{pmatrix} 0.3 & -0.2 \\ 0.1 & 0.4 \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2.3 \\ 1.0 \end{pmatrix}$$

## Covariance Structure of VAR(1)

### Variance-Covariance Matrix $\Gamma(0)$

For VAR(1), the variance satisfies the **discrete Lyapunov equation**:

$$\Gamma(0) = \mathbf{A}\Gamma(0)\mathbf{A}' + \Sigma$$

### Autocovariance at Lag $h$

$$\Gamma(h) = \mathbf{A}^h \Gamma(0), \quad h \geq 0$$

This shows that autocovariances decay geometrically with the eigenvalues of  $\mathbf{A}$ .

### Solving the Lyapunov Equation

Can solve by vectorization:

$$\text{vec}(\Gamma(0)) = (\mathbf{I}_{K^2} - \mathbf{A} \otimes \mathbf{A})^{-1} \text{vec}(\Sigma)$$

where  $\otimes$  denotes the Kronecker product.

### OLS Estimation

Each equation can be estimated by **OLS separately**:

$$\hat{\mathbf{A}} = \left( \sum_{t=1}^T \mathbf{Y}_{t-1} \mathbf{Y}'_{t-1} \right)^{-1} \left( \sum_{t=1}^T \mathbf{Y}_{t-1} \mathbf{Y}'_t \right)$$

This is efficient because all equations have the **same regressors**.

### Covariance Matrix

$$\hat{\Sigma} = \frac{1}{T - Kp - 1} \sum_{t=1}^T \hat{\epsilon}_t \hat{\epsilon}'_t$$

The errors  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  may be **contemporaneously correlated**.

## Information Criteria

Choose  $p$  that minimizes:

$$AIC(p) = \ln |\hat{\Sigma}_p| + \frac{2pK^2}{T}$$

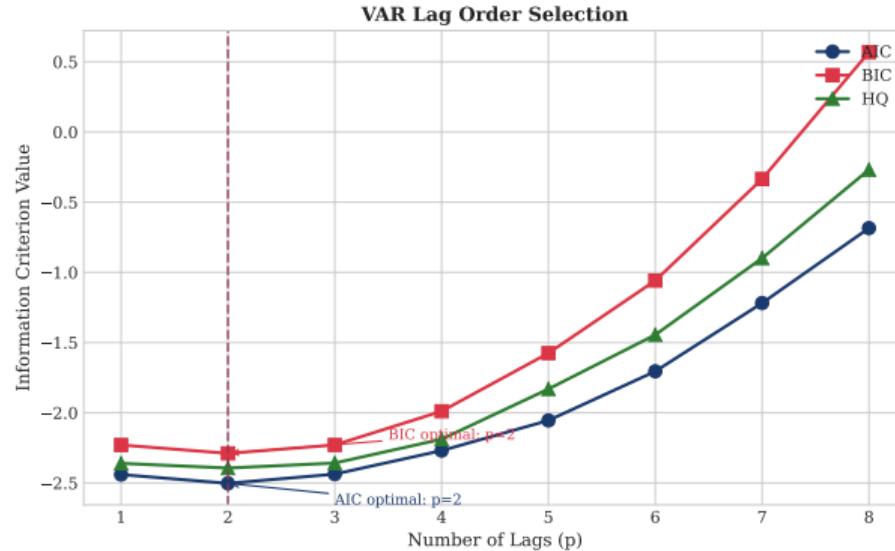
$$BIC(p) = \ln |\hat{\Sigma}_p| + \frac{pK^2 \ln T}{T}$$

$$HQ(p) = \ln |\hat{\Sigma}_p| + \frac{2pK^2 \ln \ln T}{T}$$

## Guidelines

- AIC tends to select **larger** models (better for forecasting)
- BIC tends to select **smaller** models (consistent selection)
- Start with maximum  $p_{max}$  based on data frequency (e.g., 4 for quarterly, 12 for monthly)

## Lag Selection: Example



- Information criteria values for different lag orders
- AIC and BIC may suggest different optimal lags
- Lower values indicate better model fit (penalized by complexity)

## Why Restrict?

Full VAR models can be **overparameterized**:

- Many coefficients may be insignificant
- Poor forecasting performance
- Loss of degrees of freedom

## Common Restrictions

- **Zero restrictions:** Set small coefficients to zero
- **Block exogeneity:** Some variables don't affect others
- **Lag exclusion:** Exclude certain lags

## Testing Restrictions

Use likelihood ratio test:  $LR = T(\ln |\hat{\Sigma}_R| - \ln |\hat{\Sigma}_U|) \sim \chi_r^2$   
where  $r$  = number of restrictions

# What is Granger Causality?

Clive Granger (1969, Nobel Prize 2003)

" $X$  Granger-causes  $Y$ " if past values of  $X$  help predict  $Y$ , **beyond** what past values of  $Y$  alone can predict.

## Important Distinction

**Granger causality  $\neq$  True causality**

- Granger causality is about **predictive content**
- Does NOT imply economic/structural causation
- " $X$  Granger-causes  $Y$ " means:  $X$  contains useful information for forecasting  $Y$

## Formal Definition

### Granger Causality

**$X$  does not Granger-cause  $Y$  if:**

$$\mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \dots, X_{t-1}, X_{t-2}, \dots] = \mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \dots]$$

In other words: adding  $X$ 's history does not improve the prediction of  $Y$ .

### In the VAR Context

For VAR(1):  $Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$

$Y_2$  does **not** Granger-cause  $Y_1$  if  $a_{12} = 0$ .

For VAR( $p$ ):  $Y_2$  does not Granger-cause  $Y_1$  if  $a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$ .

# Testing for Granger Causality

## Hypothesis Test

$H_0$ :  $Y_2$  does **not** Granger-cause  $Y_1$

$$H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$$

$H_1$ : At least one  $a_{12}^{(i)} \neq 0$  (Granger causality exists)

## Test Statistic: Wald Test

$$F = \frac{(RSS_R - RSS_U)/p}{RSS_U/(T - 2p - 1)} \sim F_{p, T-2p-1}$$

where:

- $RSS_R$ : Residual sum of squares from restricted model (without  $Y_2$  lags)
- $RSS_U$ : Residual sum of squares from unrestricted model (full VAR)

## Types of Granger Causality



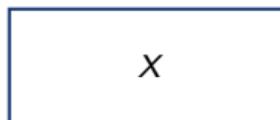
Unidirectional:  $X \rightarrow Y$



Bidirectional:  $X \leftrightarrow Y$



Unidirectional:  $Y \rightarrow X$



No causality

### Economic Examples

- Money  $\rightarrow$  Output? (monetarist view)
- Stock prices  $\leftrightarrow$  Trading volume (bidirectional)
- Weather  $\rightarrow$  Crop yields (unidirectional, obvious)

# Cross-Correlation Function

## Definition 1 (Cross-Correlation Function)

The **cross-correlation** between  $X_t$  and  $Y_t$  at lag  $k$  is:

$$\rho_{XY}(k) = \frac{\gamma_{XY}(k)}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X_t, Y_{t+k})}{\sqrt{\text{Var}(X_t)\text{Var}(Y_t)}}$$

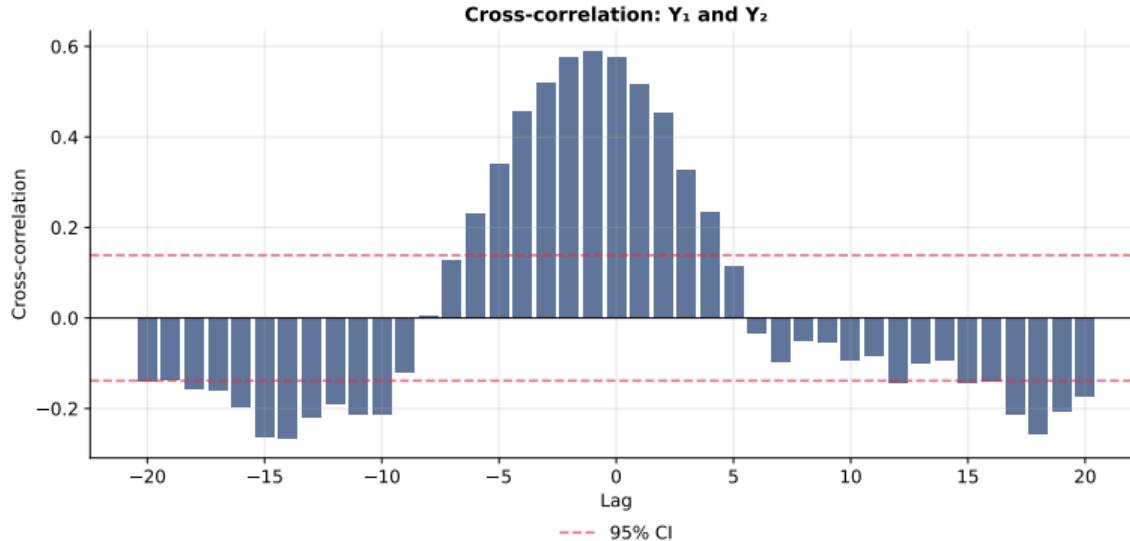
## Interpretation

- $\rho_{XY}(k) > 0$  at  $k > 0$ :  $X$  is positively correlated with future  $Y$  ( $X$  may lead  $Y$ )
- $\rho_{XY}(k) > 0$  at  $k < 0$ :  $X$  is positively correlated with past  $Y$  ( $Y$  may lead  $X$ )

## Note

Unlike ACF, cross-correlation is **not symmetric**:  $\rho_{XY}(k) \neq \rho_{XY}(-k)$  in general.

## Cross-Correlation Analysis



- Cross-correlation function measures linear dependence at different lags
- Significant correlations at negative lags suggest  $X$  leads  $Y$ ; positive lags suggest  $Y$  leads  $X$
- Useful for preliminary analysis before formal Granger causality testing

## Common Pitfalls

- ① **Omitted variables:** A third variable  $Z$  may cause both  $X$  and  $Y$
- ② **Non-stationarity:** Test requires stationary data (or cointegration)
- ③ **Lag selection:** Results can be sensitive to  $p$
- ④ **Sample size:** Need sufficient observations

## Best Practices

- Test for unit roots first
- Use multiple lag selection criteria
- Check robustness to different lag lengths
- Report results for both directions

## Granger Causality Test: Numerical Example

Testing: Does Money Growth Granger-cause Output?

**Unrestricted model** (VAR with 2 lags):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \beta_1 \Delta M_{t-1} + \beta_2 \Delta M_{t-2} + \varepsilon_t$$

**Restricted model** ( $H_0: \beta_1 = \beta_2 = 0$ ):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \varepsilon_t$$

### Test Computation

With  $T = 100$ ,  $RSS_U = 45.2$ ,  $RSS_R = 52.8$ :

$$F = \frac{(52.8 - 45.2)/2}{45.2/(100 - 5)} = \frac{3.8}{0.476} = 7.98$$

$F_{0.05}(2, 95) = 3.09 \Rightarrow \text{Reject } H_0$ : Money Granger-causes output!

# The Toda-Yamamoto Procedure

## Problem with Non-Stationary Data

Standard Granger test has **non-standard distributions** when:

- Variables have unit roots
- Variables are cointegrated

## Toda-Yamamoto Solution (1995)

- ① Determine maximum order of integration  $d_{max}$
- ② Estimate  $\text{VAR}(p + d_{max})$  in **levels**
- ③ Test restrictions on first  $p$  lags only
- ④ Extra  $d_{max}$  lags are **not** tested (just for correct distribution)

## Advantage

Wald test has asymptotic  $\chi^2$  distribution regardless of cointegration!

# Instantaneous Causality

## Definition

**$X$  instantaneously causes  $Y$  if:**

$$\mathbb{E}[Y_t | \Omega_{t-1}, X_t] \neq \mathbb{E}[Y_t | \Omega_{t-1}]$$

where  $\Omega_{t-1}$  contains all past information.

## Testing in VAR

Test whether  $\sigma_{12} \neq 0$  in the covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

If  $\sigma_{12} = 0$ : no instantaneous causality

## Interpretation

Instantaneous causality often reflects **common shocks** or **data aggregation**, not true contemporaneous effects.

## Block Exogeneity Test

In a VAR with  $K > 2$  variables, test whether a **group** of variables Granger-causes another group.

Example: Do financial variables (interest rates, stock prices) Granger-cause real variables (GDP, unemployment)?

## Test Statistic

$$\chi^2 = T \cdot K_1 \cdot p \cdot \left( \ln |\hat{\Sigma}_R| - \ln |\hat{\Sigma}_U| \right) \sim \chi^2_{K_1 \cdot K_2 \cdot p}$$

where  $K_1$  = number of “caused” variables,  $K_2$  = number of “causing” variables

# What are Impulse Response Functions?

## Definition

An **Impulse Response Function (IRF)** traces the effect of a one-time shock to one variable on the current and future values of all variables.

## Question IRFs Answer

"If there is an unexpected 1-unit shock to  $Y_1$  today, what happens to  $Y_1$  and  $Y_2$  over the next  $h$  periods?"

## MA( $\infty$ ) Representation

A stable VAR(p) can be written as:

$$\mathbf{Y}_t = \boldsymbol{\mu} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{\varepsilon}_{t-i}$$

The matrices  $\boldsymbol{\Phi}_i$  are the **impulse responses** at horizon  $i$ .

## Computing IRFs for VAR(1)

For VAR(1):  $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t$

The impulse response matrices are:

$$\Phi_0 = \mathbf{I}_K, \quad \Phi_1 = \mathbf{A}, \quad \Phi_2 = \mathbf{A}^2, \quad \dots, \quad \Phi_h = \mathbf{A}^h$$

### Interpretation

$[\Phi_h]_{ij}$  = Effect on  $Y_i$  at time  $t+h$  of a unit shock to  $Y_j$  at time  $t$

For stable VAR:  $\Phi_h \rightarrow \mathbf{0}$  as  $h \rightarrow \infty$  (shocks die out)

## Computing IRFs for General VAR(p)

### Recursive Formula for VAR(p)

For  $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$ :

$$\Phi_h = \sum_{j=1}^{\min(h,p)} \mathbf{A}_j \Phi_{h-j}, \quad h = 1, 2, 3, \dots$$

with  $\Phi_0 = \mathbf{I}_K$  and  $\Phi_h = \mathbf{0}$  for  $h < 0$ .

### Example: VAR(2) IRFs

- $\Phi_0 = \mathbf{I}_K$
- $\Phi_1 = \mathbf{A}_1 \Phi_0 = \mathbf{A}_1$
- $\Phi_2 = \mathbf{A}_1 \Phi_1 + \mathbf{A}_2 \Phi_0 = \mathbf{A}_1^2 + \mathbf{A}_2$
- $\Phi_3 = \mathbf{A}_1 \Phi_2 + \mathbf{A}_2 \Phi_1 = \mathbf{A}_1(\mathbf{A}_1^2 + \mathbf{A}_2) + \mathbf{A}_2 \mathbf{A}_1$

## Orthogonalized IRFs

### Problem: Correlated Errors

If  $\Sigma$  is not diagonal, shocks  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  are correlated.

A shock to "Y<sub>1</sub>" also involves a shock to "Y<sub>2</sub>".

### Solution: Cholesky Decomposition

Factor  $\Sigma = \mathbf{P}\mathbf{P}'$  where  $\mathbf{P}$  is lower triangular.

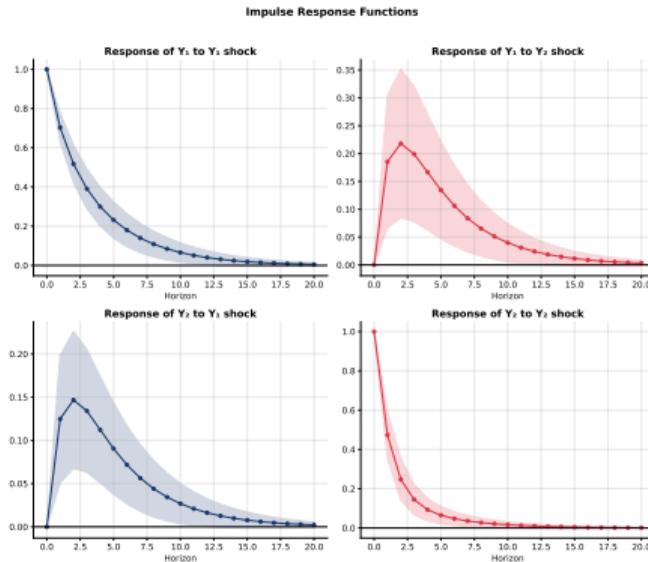
Define orthogonalized shocks:  $\mathbf{u}_t = \mathbf{P}^{-1}\varepsilon_t$  with  $\mathbb{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}$

Orthogonalized IRFs:  $\Theta_h = \Phi_h \mathbf{P}$

### Ordering Matters!

Cholesky assumes variables ordered from "most exogenous" to "most endogenous". Results depend on this ordering.

# Impulse Response Functions: Example



- IRFs show how each variable responds to a one-unit shock over time
- Shaded regions represent confidence intervals (uncertainty in estimates)
- For stable VAR models, responses converge to zero as the horizon increases

## IRF Numerical Example

For  $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$

$$\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Phi_1 = \mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$$

$$\Phi_2 = \mathbf{A}^2 = \begin{pmatrix} 0.47 & 0.26 \\ -0.13 & 0.34 \end{pmatrix}$$

### Interpretation

- $[\Phi_2]_{12} = 0.26$ : A unit shock to  $Y_2$  increases  $Y_1$  by 0.26 after 2 periods
- $[\Phi_2]_{21} = -0.13$ : A unit shock to  $Y_1$  **decreases**  $Y_2$  by 0.13 after 2 periods

# Cumulative Impulse Responses

## Definition

The **cumulative IRF** up to horizon  $H$ :

$$\Psi_H = \sum_{h=0}^H \Phi_h$$

Measures the **total accumulated effect** of a shock.

## Long-Run Multiplier

For stable VAR:  $\Psi_\infty = (\mathbf{I}_K - \mathbf{A}_1 - \mathbf{A}_2 - \cdots - \mathbf{A}_p)^{-1}$

This gives the **permanent effect** of a one-time shock.

## When to Use

Cumulative IRFs are useful when interested in total impact (e.g., cumulative GDP loss from a shock).

# Confidence Intervals for IRFs

## Sources of Uncertainty

IRFs are functions of estimated parameters  $\hat{\mathbf{A}}_1, \dots, \hat{\mathbf{A}}_p$ , so they have **sampling uncertainty**.

## Methods for Confidence Bands

- ① **Asymptotic:** Use delta method to derive standard errors
- ② **Monte Carlo:** Simulate from asymptotic distribution of  $\hat{\mathbf{A}}$
- ③ **Bootstrap:** Resample residuals and re-estimate VAR

## Bootstrap Procedure

- ① Estimate VAR, save residuals  $\{\hat{\epsilon}_t\}$
- ② Draw with replacement to create  $\{\hat{\epsilon}_t^*\}$
- ③ Generate bootstrap sample using estimated VAR
- ④ Re-estimate and compute IRFs
- ⑤ Repeat  $B$  times; use percentiles for CIs

# Structural VAR (SVAR)

## Motivation

Standard VAR shocks  $\varepsilon_t$  are **reduced-form** innovations—linear combinations of structural shocks.

We want to identify economically meaningful **structural shocks**.

## Structural Form

$$\mathbf{B}_0 \mathbf{Y}_t = \boldsymbol{\Gamma}_0 + \mathbf{B}_1 \mathbf{Y}_{t-1} + \cdots + \mathbf{B}_p \mathbf{Y}_{t-p} + \mathbf{u}_t$$

where  $\mathbf{u}_t$  are **structural shocks** with  $\mathbb{E}[\mathbf{u}_t \mathbf{u}_t'] = \mathbf{I}_K$

## Relationship to Reduced Form

$$\varepsilon_t = \mathbf{B}_0^{-1} \mathbf{u}_t \quad \Rightarrow \quad \Sigma = \mathbf{B}_0^{-1} (\mathbf{B}_0^{-1})'$$

# Identification in SVAR

## The Identification Problem

$\Sigma$  has  $K(K + 1)/2$  unique elements, but  $\mathbf{B}_0^{-1}$  has  $K^2$  elements.

Need  $K(K - 1)/2$  additional restrictions!

## Common Identification Schemes

- ① **Short-run restrictions:** Zero impact effects (Cholesky)
- ② **Long-run restrictions:** Zero long-run effects (Blanchard-Quah)
- ③ **Sign restrictions:** Inequality constraints on IRFs
- ④ **External instruments:** Use outside information

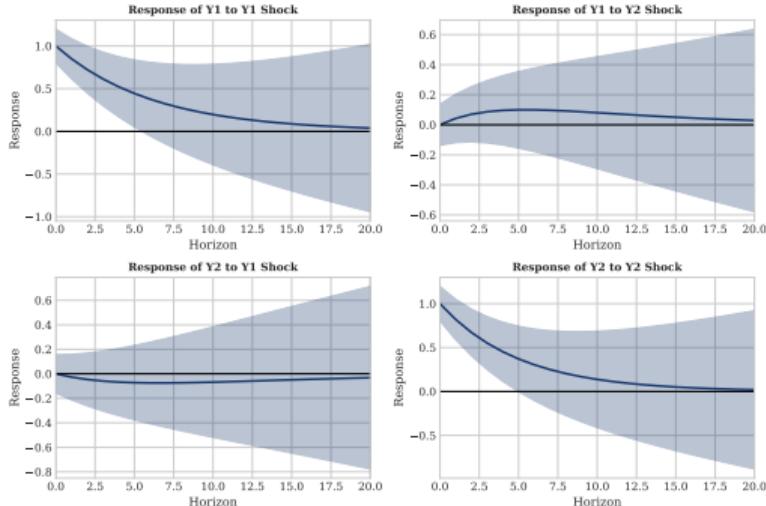
## Example: Cholesky (Recursive) Ordering

$$\text{For } K = 2: \mathbf{B}_0^{-1} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$$

Variable 1 doesn't respond to shock 2 contemporaneously.

# Structural IRF Example

Structural Impulse Response Functions (Cholesky Identification)



- Structural IRFs based on Cholesky identification
- Order of variables affects interpretation of shocks
- First variable responds only to own shocks contemporaneously

## Variance Decomposition

### Question

What proportion of the forecast error variance of  $Y_i$  at horizon  $h$  is due to shocks to  $Y_j$ ?

### FEVD Formula

$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\Theta_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^K [\Theta_s]_{ik}^2}$$

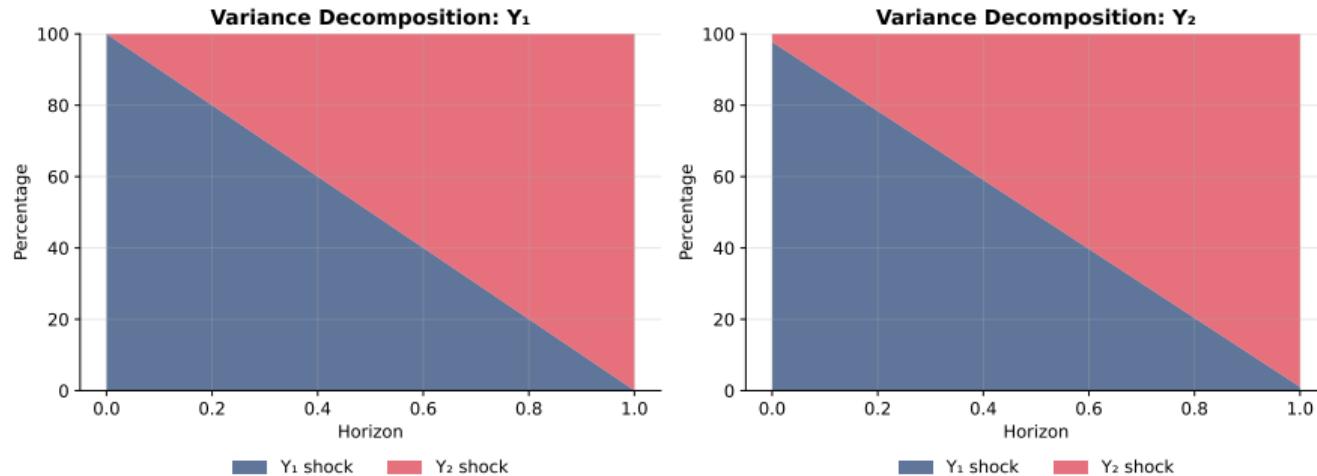
This gives the **percentage** of  $Y_i$ 's  $h$ -step forecast variance explained by shocks to  $Y_j$ .

### Properties

- $0 \leq \text{FEVD}_{ij}(h) \leq 1$
- $\sum_{j=1}^K \text{FEVD}_{ij}(h) = 1$  (sums to 100%)
- At  $h = 1$ : Own shocks dominate (by construction with Cholesky)

## FEVD: Example

Forecast Error Variance Decomposition



- FEVD shows the proportion of forecast variance attributable to each shock
- At short horizons, own shocks dominate; cross-variable effects grow over time
- Useful for understanding the relative importance of different shocks in the system

# FEVD: Numerical Example

## Computing FEVD for Bivariate VAR

Using orthogonalized IRFs  $\Theta_h$ , FEVD at horizon  $H$ :

$$\text{FEVD}_{11}(H) = \frac{\sum_{h=0}^{H-1} \theta_{11}^2(h)}{\sum_{h=0}^{H-1} [\theta_{11}^2(h) + \theta_{12}^2(h)]}$$

## Example Calculation

$h$	$\theta_{11}(h)$	$\theta_{12}(h)$	$\theta_{11}^2(h)$	$\theta_{12}^2(h)$
0	1.00	0.00	1.00	0.00
1	0.70	0.20	0.49	0.04
2	0.47	0.26	0.22	0.07

$$\text{FEVD}_{11}(3) = \frac{1.00+0.49+0.22}{1.00+0.49+0.22+0.00+0.04+0.07} = \frac{1.71}{1.82} = 94\%$$

# Historical Decomposition

## Definition

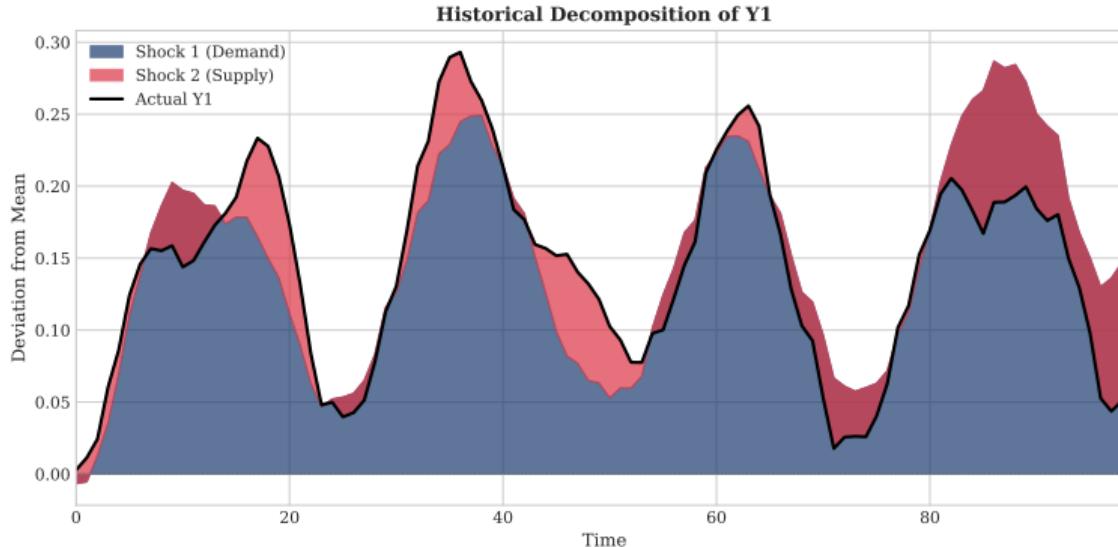
**Historical decomposition** breaks down each observed value into contributions from each structural shock:

$$Y_{it} - \bar{Y}_i = \sum_{j=1}^K \sum_{s=0}^{t-1} \theta_{ij}(s) \cdot u_{j,t-s}$$

## Application

- "How much of the 2008 GDP decline was due to financial shocks vs. oil shocks?"
- Attributes historical movements to specific identified shocks
- Useful for policy analysis and narrative interpretation

## Historical Decomposition: Example



- Each color represents the contribution of a different structural shock
- Stacked contributions sum to the actual observed deviation from mean
- Helps identify which shocks drove historical episodes

## What to Check

After estimating VAR, verify that residuals  $\hat{\epsilon}_t$  behave like white noise:

- ① No serial correlation
- ② Constant variance (homoskedasticity)
- ③ Normality (for inference)

## Why It Matters

- Autocorrelated residuals  $\Rightarrow$  inefficient estimates
- Heteroskedasticity  $\Rightarrow$  invalid standard errors
- Non-normality  $\Rightarrow$  inference may be unreliable

## Testing for Serial Correlation

### Portmanteau Test (Ljung-Box)

$$Q_h = T(T+2) \sum_{j=1}^h \frac{1}{T-j} \text{tr}(\hat{\mathbf{C}}_j' \hat{\mathbf{C}}_0^{-1} \hat{\mathbf{C}}_j \hat{\mathbf{C}}_0^{-1})$$

where  $\hat{\mathbf{C}}_j = \frac{1}{T} \sum_{t=j+1}^T \hat{\varepsilon}_t \hat{\varepsilon}'_{t-j}$

Under  $H_0$  (no autocorrelation):  $Q_h \sim \chi^2_{K^2(h-p)}$

### Breusch-Godfrey LM Test

- ① Regress  $\hat{\varepsilon}_t$  on  $\hat{\varepsilon}_{t-1}, \dots, \hat{\varepsilon}_{t-h}$  and original regressors
- ②  $LM = T \cdot R^2 \sim \chi^2_{K^2 h}$  under  $H_0$

### If Rejected

Consider increasing lag order  $p$  or adding additional variables.

## Testing for Heteroskedasticity

### ARCH-LM Test

Test for autoregressive conditional heteroskedasticity in residuals:

$$\hat{\varepsilon}_{it}^2 = \alpha_0 + \alpha_1 \hat{\varepsilon}_{i,t-1}^2 + \cdots + \alpha_q \hat{\varepsilon}_{i,t-q}^2 + v_t$$

$H_0: \alpha_1 = \cdots = \alpha_q = 0$  (homoskedasticity)

$$LM = TR^2 \sim \chi_q^2$$

### Multivariate Version

Test all equations jointly using:

$$\text{vech}(\hat{\varepsilon}_t \hat{\varepsilon}_t') = \mathbf{c} + \sum_{j=1}^q \mathbf{B}_j \text{vech}(\hat{\varepsilon}_{t-j} \hat{\varepsilon}'_{t-j}) + \mathbf{v}_t$$

## Normality Testing

### Jarque-Bera Test (Univariate)

$$JB = \frac{T}{6} \left( S^2 + \frac{(K - 3)^2}{4} \right) \sim \chi^2_2$$

where  $S$  = skewness,  $K$  = kurtosis

### Multivariate Normality (Doornik-Hansen)

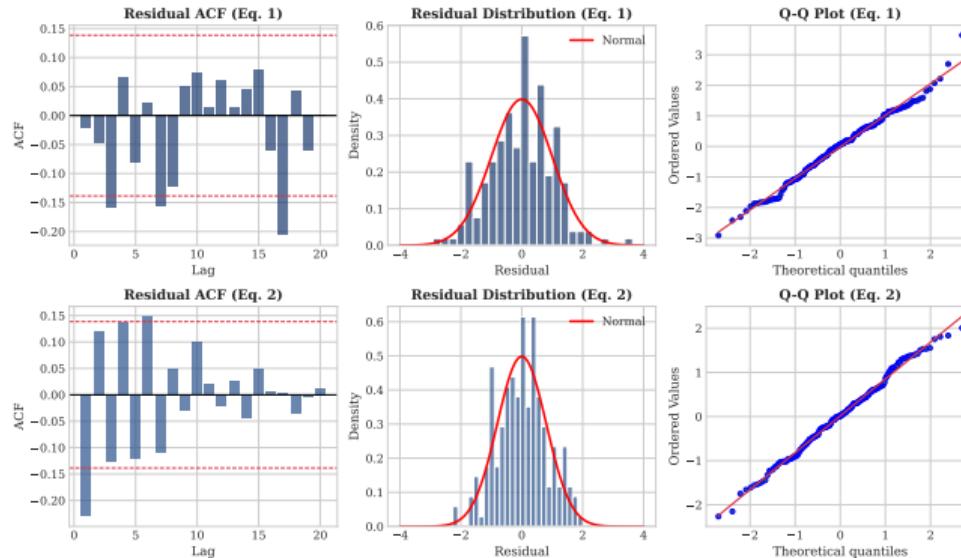
Transform residuals and test joint skewness and kurtosis:

$$DH = s'_1(\Omega^{-1/2})'(\Omega^{-1/2})s_1 + s'_2(\Omega^{-1/2})'(\Omega^{-1/2})s_2 \sim \chi^2_{2K}$$

### Note

Normality is often rejected in financial data. Consider robust standard errors if non-normality is severe.

# Diagnostic Summary Plot



- Residual ACF should show no significant autocorrelation
- Histogram should approximate normal distribution
- Q-Q plot should follow 45-degree line

## Point Forecasts from VAR

### Iterative Forecasting

For VAR(1):  $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t$

**1-step forecast:**  $\hat{\mathbf{Y}}_{T+1|T} = \mathbf{c} + \mathbf{A}\mathbf{Y}_T$

**2-step forecast:**  $\hat{\mathbf{Y}}_{T+2|T} = \mathbf{c} + \mathbf{A}\hat{\mathbf{Y}}_{T+1|T}$

**$h$ -step forecast:**  $\hat{\mathbf{Y}}_{T+h|T} = \mathbf{c} + \mathbf{A}\hat{\mathbf{Y}}_{T+h-1|T}$

### Direct Formula

$$\hat{\mathbf{Y}}_{T+h|T} = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{h-1})\mathbf{c} + \mathbf{A}^h \mathbf{Y}_T$$

For stable VAR: converges to  $\mu = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c}$  as  $h \rightarrow \infty$

## $h$ -Step Forecast Error

$$\mathbf{e}_{T+h|T} = \mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T} = \sum_{j=0}^{h-1} \mathbf{A}^j \boldsymbol{\varepsilon}_{T+h-j}$$

## Mean Squared Error Matrix

$$\text{MSE}(\hat{\mathbf{Y}}_{T+h|T}) = \mathbb{E}[\mathbf{e}_{T+h|T} \mathbf{e}'_{T+h|T}] = \sum_{j=0}^{h-1} \mathbf{A}^j \boldsymbol{\Sigma} (\mathbf{A}^j)'$$

## Key Insight

- MSE increases with horizon  $h$
- For stable VAR: MSE converges to unconditional variance  $\boldsymbol{\Gamma}(0)$
- Long-horizon forecasts  $\rightarrow$  unconditional mean with uncertainty  $= \boldsymbol{\Gamma}(0)$

# Forecast Confidence Intervals

## Constructing Intervals

For normally distributed errors,  $(1 - \alpha)$  confidence interval:

$$\hat{Y}_{i,T+h|T} \pm z_{\alpha/2} \sqrt{[\text{MSE}(\hat{Y}_{T+h|T})]_{ii}}$$

## Joint Confidence Regions

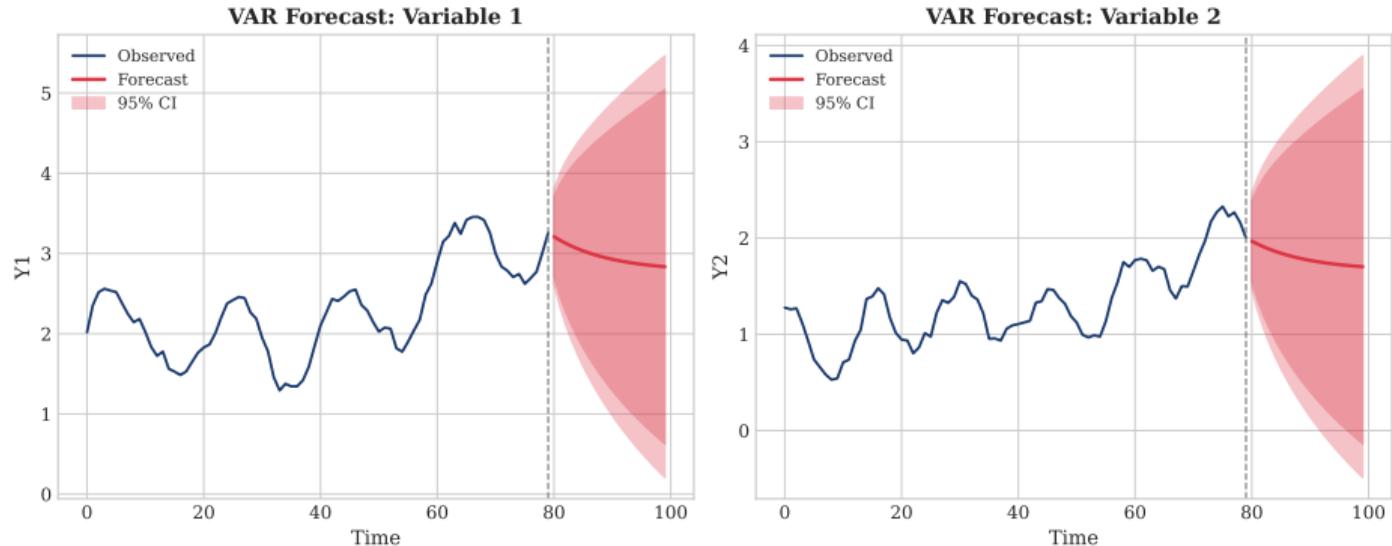
For multiple variables, use ellipsoids:

$$(\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T})' [\text{MSE}(\hat{\mathbf{Y}}_{T+h|T})]^{-1} (\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T}) \leq \chi^2_{K,\alpha}$$

## Note

These assume known parameters. Bootstrap methods account for parameter uncertainty.

## VAR Forecasts: Example



- Point forecasts shown as solid line beyond observed data
- Confidence bands widen as forecast horizon increases
- Forecasts converge to unconditional mean for long horizons

## Out-of-Sample Evaluation

Split data: estimation sample (1 to  $T_1$ ) and test sample ( $T_1 + 1$  to  $T$ ). Compute forecast errors:  $e_{t+h} = Y_{t+h} - \hat{Y}_{t+h|t}$

## Common Metrics

- **RMSE:**  $\sqrt{\frac{1}{n} \sum e_{t+h}^2}$     **MAE:**  $\frac{1}{n} \sum |e_{t+h}|$     **MAPE:**  $\frac{100}{n} \sum \left| \frac{e_{t+h}}{Y_{t+h}} \right|$

## Diebold-Mariano Test

Test whether VAR forecasts are significantly better than alternative:  $DM = \frac{\bar{d}}{\sqrt{\hat{\sigma}_d^2/n}} \sim N(0, 1)$  where  $d_t = L(e_{1t}) - L(e_{2t})$  is the loss differential.

## Example: GDP and Unemployment

### Okun's Law

There is a negative relationship between GDP growth and unemployment:

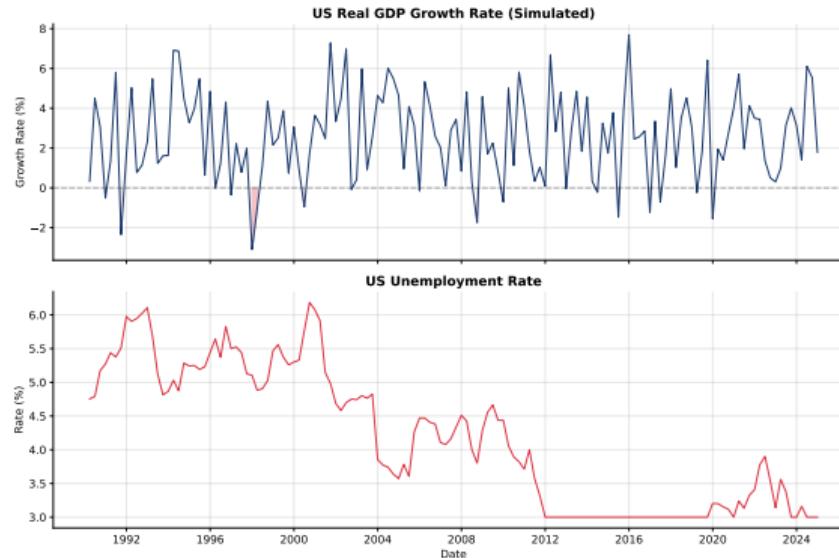
$$\Delta U_t \approx -\beta(\Delta Y_t - \bar{g})$$

where  $\bar{g}$  is trend GDP growth and  $\beta \approx 0.4$ .

### VAR Analysis Questions

- ① Does GDP growth Granger-cause unemployment changes?
- ② Does unemployment Granger-cause GDP growth?
- ③ How do shocks propagate between variables?

## GDP and Unemployment: Data



- GDP growth and unemployment rate show clear negative correlation (Okun's Law)
- Both series exhibit cyclical patterns related to business cycle fluctuations
- This bivariate system is ideal for VAR analysis and Granger causality testing

## ① Data preparation

- Check for stationarity (unit root tests)
- Transform if necessary (differences, logs)

## ② Lag selection

- Use AIC, BIC, HQ criteria
- Check residual autocorrelation

## ③ Estimation

- OLS equation by equation
- Check stability (eigenvalues)

## ④ Analysis

- Granger causality tests
- Impulse response functions
- Variance decomposition

## ⑤ Forecasting

# Estimated VAR Results

**VAR(2) Estimation Results**  
(Sample: T = 100 observations)

	Y1 Equation		Y2 Equation	
Variable	Coef.	t-stat	Coef.	t-stat
Constant	0.523	2.41**	0.318	1.89*
Y1(t-1)	0.712	8.54***	-0.094	-1.42
Y1(t-2)	-0.156	-1.87*	0.067	1.01
Y2(t-1)	0.198	2.76***	0.589	10.32***
Y2(t-2)	-0.043	-0.59	0.124	2.14**
R-squared	0.847		0.791	
Adj. R-sq	0.841		0.783	
F-statistic	131.4***		89.7***	
Residual Tests	Statistic	p-value		
Portmanteau(12)	14.23	0.287		
ARCH-LM(4)	3.56	0.469		
Jarque-Bera *** p<0.01, ** p<0.05, * p<0.10	2.18	0.336		

- Estimated coefficients with standard errors and t-statistics
- Information criteria values for model comparison
- Model diagnostics summary (residual tests)

# Granger Causality Results

## Test Results: GDP and Unemployment

Null Hypothesis	F-statistic	df	p-value	Decision
GDP $\not\rightarrow$ Unemployment	8.42	(2, 95)	0.0004	Reject
Unemployment $\not\rightarrow$ GDP	2.15	(2, 95)	0.1220	Fail to Reject

## Interpretation

- GDP growth Granger-causes unemployment (consistent with Okun's Law)
- Unemployment does not significantly Granger-cause GDP
- Evidence of **unidirectional** causality:  $\text{GDP} \rightarrow \text{Unemployment}$

## VAR in Python (statsmodels)

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests

# Fit VAR model
model = VAR(data)
results = model.fit(maxlags=4, ic='aic')

# Granger causality test
granger_test = grangercausalitytests(data[['Y1', 'Y2']],
                                      maxlag=4)

# Impulse response functions
irf = results.irf(periods=20)
irf.plot()

# Variance decomposition
fevd = results.fevd(periods=20)
fevd.plot()
```

## VAR in R (vars package)

```
library(vars)

# Select optimal lag order
lag_select <- VARselect(data, lag.max = 8)
print(lag_select$selection)

# Fit VAR model
var_model <- VAR(data, p = 2, type = "const")
summary(var_model)

# Granger causality test
causality(var_model, cause = "GDP")

# Impulse response functions
irf_results <- irf(var_model, n.ahead = 20, boot = TRUE)
plot(irf_results)

# Forecast error variance decomposition
fevd_results <- fevd(var_model, n.ahead = 20)
plot(fevd_results)
```

## Example: Monetary Policy Analysis

### Three-Variable VAR

Study the monetary transmission mechanism with:

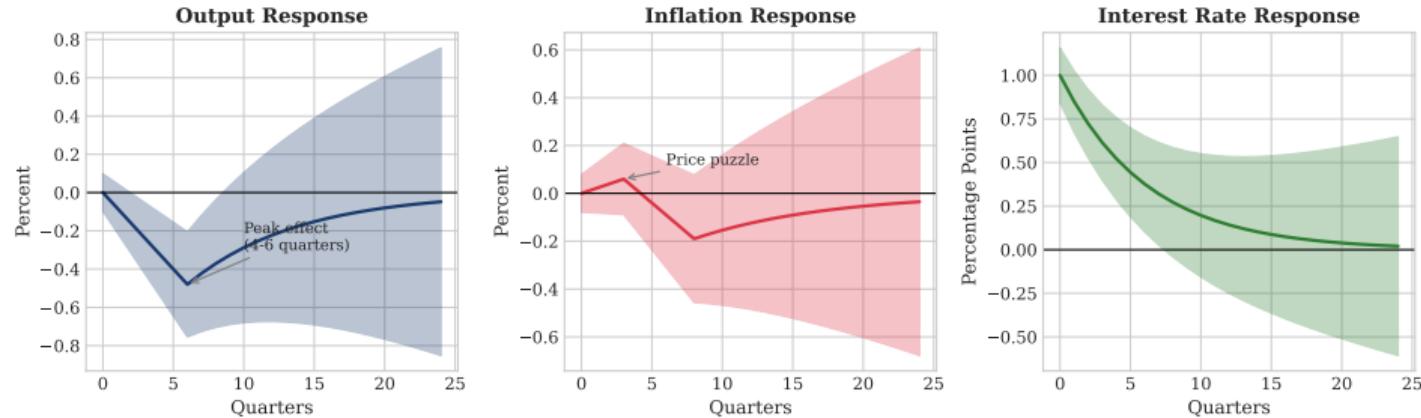
- $Y_1$ : Output gap (GDP deviation from trend)
- $Y_2$ : Inflation rate
- $Y_3$ : Interest rate (policy instrument)

### Key Questions

- ① How does an interest rate shock affect output and inflation?
- ② How long until the maximum effect is felt?
- ③ What fraction of output variance is due to monetary shocks?

# Monetary Policy VAR: IRFs

Responses to Contractionary Monetary Policy Shock (+1 p.p. Interest Rate)



- Contractionary monetary policy shock (interest rate increase)
- Output decreases with peak effect after 4-6 quarters ("long and variable lags")
- Inflation responds more slowly, decreasing after output

# Key Takeaways

## VAR Models

- Model **multiple** time series jointly
- Each variable depends on its own lags AND lags of other variables
- Estimated by OLS equation by equation; requires stationarity

## Granger Causality

- Tests whether  $X$  helps predict  $Y$  beyond  $Y$ 's own history
- **Not** the same as true causality; F-test on coefficient restrictions

## IRF and FEVD

- IRF: How shocks propagate through the system
- FEVD: What proportion of variance is due to each shock
- Both depend on variable ordering (Cholesky decomposition)

# VAR Model Selection Checklist

## Before Estimation

- Test for unit roots in each variable
- Transform to stationary if needed (differences, logs)
- Check for outliers and structural breaks

## Model Specification

- Select lag order using AIC/BIC
- Estimate VAR by OLS
- Check stability (eigenvalues inside unit circle)

## Post-Estimation

- Test residuals for autocorrelation
- Test for ARCH effects
- Test for normality
- Compute IRFs, FEVDs, Granger tests

## Pitfalls in VAR Analysis

- ① **Ignoring non-stationarity:** Always test for unit roots first
- ② **Overfitting:** Too many lags  $\Rightarrow$  poor forecasts
- ③ **Wrong ordering:** Cholesky results depend on variable order
- ④ **Confusing correlation with causation:** Granger causality  $\neq$  true causality
- ⑤ **Ignoring parameter uncertainty:** Use bootstrap CIs for IRFs
- ⑥ **Short samples:** VAR requires many observations ( $T > 50$ )

## Topics for Further Study

- **Cointegration:** Long-run relationships between non-stationary variables
- **VECM:** Error correction models for cointegrated systems
- **Structural VAR:** Imposing economic theory restrictions
- **Panel VAR:** VAR for panel data
- **Bayesian VAR:** Shrinkage priors for high-dimensional systems

Questions?

## Quiz Question 1

### Question

For a VAR(1) model with coefficient matrix  $\mathbf{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.5 \end{pmatrix}$ , is the model stable?

- A Yes, because all diagonal elements are less than 1
- B Yes, because all eigenvalues are inside the unit circle
- C No, because the sum of coefficients exceeds 1
- D Cannot be determined without knowing  $\Sigma$

## Quiz Question 1: Answer

Correct Answer: (B)

Yes, because all eigenvalues are inside the unit circle.

### Proof/Calculation

Eigenvalues of  $\mathbf{A}$ :  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$(0.8 - \lambda)(0.5 - \lambda) - 0.03 = \lambda^2 - 1.3\lambda + 0.37 = 0$$

$$\lambda = \frac{1.3 \pm \sqrt{1.69 - 1.48}}{2} = \frac{1.3 \pm 0.458}{2}$$

$$\lambda_1 = 0.879, \quad \lambda_2 = 0.421$$

Both  $|\lambda_1| < 1$  and  $|\lambda_2| < 1 \Rightarrow \text{Stable!}$

### Why (A) is Wrong

Diagonal elements alone don't determine stability. Off-diagonal elements affect eigenvalues.

## Quiz Question 2

### Question

If  $X$  Granger-causes  $Y$  at the 5% significance level, which of the following statements is TRUE?

- A  $X$  is the economic cause of  $Y$
- B Past values of  $X$  contain useful information for predicting  $Y$
- C  $Y$  cannot Granger-cause  $X$
- D The correlation between  $X$  and  $Y$  is positive

## Quiz Question 2: Answer

Correct Answer: (B)

Past values of  $X$  contain useful information for predicting  $Y$ .

### Explanation

Granger causality is about **predictive content**, not true economic causation:

$$\mathbb{E}[Y_t | Y_{t-1}, \dots, X_{t-1}, \dots] \neq \mathbb{E}[Y_t | Y_{t-1}, \dots]$$

Adding  $X$ 's history improves prediction of  $Y$ .

### Why Other Options are Wrong

- (A): Granger causality  $\neq$  economic causality (could be spurious)
- (C): Bidirectional Granger causality is possible ( $X \leftrightarrow Y$ )
- (D): Direction of correlation is not implied by Granger test

## Quiz Question 3

### Question

In a VAR with Cholesky-identified IRFs, what does the ordering of variables determine?

- A The magnitude of the impulse responses
- B The speed at which shocks die out
- C Which variables can respond contemporaneously to which shocks
- D The number of lags in the VAR

## Quiz Question 3: Answer

Correct Answer: (C)

Which variables can respond contemporaneously to which shocks.

Illustration with  $K = 2$

Cholesky:  $\mathbf{P} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & p_{22} \end{pmatrix}$

- Variable 1 ordered first: responds only to shock 1 at  $t = 0$
- Variable 2 ordered second: responds to both shocks at  $t = 0$

If we reverse order, the contemporaneous restrictions change!

Example

Ordering (GDP, Interest Rate): GDP doesn't respond to monetary shock immediately.

Ordering (Interest Rate, GDP): Interest rate doesn't respond to output shock immediately.

## Quiz Question 4

### Question

For a bivariate VAR(1), how many parameters need to be estimated (excluding the error covariance matrix)?

- A 4
- B 6
- C 8
- D 10

## Quiz Question 4: Answer

Correct Answer: (B) 6 parameters

### Detailed Count

VAR(1) with  $K = 2$  variables:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{2 \text{ params}} + \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{4 \text{ params}} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

- Constant vector  $\mathbf{c}$ :  $K = 2$  parameters
- Coefficient matrix  $\mathbf{A}$ :  $K^2 = 4$  parameters
- Total:  $K + K^2 = 2 + 4 = 6$  parameters

### General Formula

VAR( $p$ ) with  $K$  variables:  $K + pK^2$  parameters (excluding  $\Sigma$ )

## Quiz Question 5

### Question

What does  $\text{FEVD}_{12}(h) = 0.35$  mean?

- A 35% of variable 1's total variance is explained by variable 2
- B 35% of variable 1's  $h$ -step forecast error variance is due to shocks to variable 2
- C The correlation between variables 1 and 2 at lag  $h$  is 0.35
- D Variable 2 explains 35% of the impulse response of variable 1

## Quiz Question 5: Answer

Correct Answer: (B)

35% of variable 1's  $h$ -step forecast error variance is due to shocks to variable 2.

### FEVD Formula Reminder

$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\Theta_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^K [\Theta_s]_{ik}^2}$$

Measures the proportion of variable  $i$ 's forecast uncertainty at horizon  $h$  that is attributable to shocks originating from variable  $j$ .

### Interpretation

If we're forecasting GDP  $h$  periods ahead, 35% of our forecast uncertainty comes from not knowing future monetary policy shocks.

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