



Time Series Analysis and Forecasting

Chapter 4: SARIMA Models

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Outline

- Motivation
- Seasonality in Time Series
- Seasonal Differencing
- The SARIMA Model
- Seasonal ACF and PACF Patterns
- Estimation and Diagnostics
- Forecasting with SARIMA
- Case Study: Airline Passengers
- Summary
- Quiz



Motivating Example: Seasonality Is Everywhere

charts/ch4_motivation_seasonal.pdf

- Retail sales exhibit clear **annual patterns**: December peaks, January troughs
- Standard ARIMA models cannot capture these **repeating seasonal cycles**
- Ignoring seasonality leads to systematic forecast errors



Understanding Seasonal Components

charts/ch4_motivation_decomposition.pdf

- Seasonal time series = **Trend + Seasonal Pattern + Residuals**
- Decomposition helps visualize each component separately
- SARIMA models capture both trend dynamics and seasonal behavior



Real-World Application: Monthly Patterns

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- Energy demand shows strong **monthly seasonality**
 - ▶ Heating cycles in winter, cooling cycles in summer
- Pattern repeats predictably each year with slight weather variations
- Utility companies use SARIMA forecasts for capacity planning



Why Do We Need SARIMA?

charts/ch4_motivation_why_sarima.pdf

- Left:** Seasonal ACF patterns — spikes at lags 12, 24, 36 reveal annual cycle
- Right:** ARIMA residuals still show seasonal autocorrelation (incomplete model)
- SARIMA solution:** Adds seasonal AR/MA terms to capture periodic patterns

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What We'll Learn Today

Concepts

- Identifying seasonal patterns
- Seasonal differencing operator
- SARIMA(p, d, q)(P, D, Q) $_s$ notation
- The famous “Airline Model”
- Model selection for seasonal data

Skills

- Diagnose seasonality from ACF/PACF
- Determine seasonal period s
- Choose (P, D, Q) seasonal orders
- Implement SARIMA in Python/R
- Forecast seasonal time series

Key Insight

SARIMA = ARIMA applied at **two frequencies**: the regular (short-term) and seasonal (long-term) levels



What is Seasonality?

Definition 1 (Seasonality)

A time series exhibits **seasonality** when it shows regular, periodic fluctuations that repeat over a fixed period s (the seasonal period).

Common Seasonal Periods

- Monthly data: $s = 12$ (annual cycle)
- Quarterly data: $s = 4$ (annual cycle)
- Weekly data: $s = 52$ (annual) or $s = 7$ (weekly pattern)
- Daily data: $s = 7$ (weekly pattern)



Seasonality: Visual Illustration



charts/ch4_def_seasonality.pdf

Seasonal Periods

Left: Monthly data with $s = 12$ (annual cycle). Right: Quarterly data with $s = 4$. The pattern repeats every s periods — this regularity is exploited by SARIMA models.



Examples of Seasonal Data

Economic Series

- Retail sales (holiday peaks)
- Tourism (summer/winter)
- Agricultural production
- Energy consumption
- Employment (hiring cycles)

Other Domains

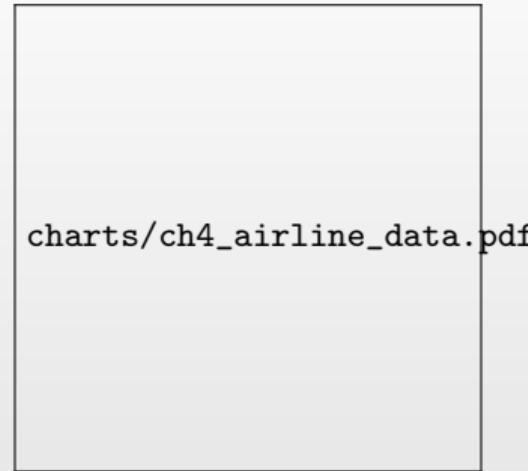
- Weather/temperature
- Website traffic
- Hospital admissions
- Transportation usage
- Electricity demand

Why It Matters

Ignoring seasonality leads to biased forecasts and invalid inference!



Example: Airline Passengers Data



- Monthly international airline passengers (1949–1960)
- Clear **upward trend** and **growing seasonal amplitude**
- Summer peaks reflect vacation travel patterns



Visualizing Seasonal Patterns



charts/ch4_seasonal_boxplot.pdf

- Box plot reveals consistent seasonal pattern across years
- July–August show highest passenger counts (summer travel)
- November–February show lowest counts (winter months)



Deterministic vs Stochastic Seasonality

Deterministic Seasonality

Fixed seasonal pattern: $Y_t = \sum_{j=1}^s \gamma_j D_{jt} + \varepsilon_t$
where D_{jt} are seasonal dummies.

Properties:

- Pattern constant over time
- Removed by regression

Stochastic Seasonality

Evolving seasonal pattern: $\Delta_s Y_t = Y_t - Y_{t-s}$
exhibits dependence structure.

Properties:

- Pattern evolves over time
- Requires seasonal differencing



Detecting Seasonality

Visual Methods (Primary Approach)

- Time series plot** – look for repeating patterns
- Seasonal boxplot** – compare distributions across seasons
- ACF plot** – spikes at seasonal lags ($s, 2s, 3s, \dots$)

ACF Signature of Seasonality

- Strong spikes at lags $s, 2s, 3s, \dots$ indicate seasonal pattern
- Slow decay at seasonal lags \Rightarrow stochastic seasonality (needs differencing)
- Quick cutoff at seasonal lags \Rightarrow deterministic seasonality (use dummies)



ACF Reveals Seasonal Structure



charts/ch4_acf_seasonality.pdf

- Slow decay at all lags indicates non-stationarity (trend)
- Spikes at lags 12, 24, 36 confirm seasonal pattern ($s = 12$)
- Slow decay at seasonal lags \Rightarrow needs seasonal differencing $(1 - L^{12})$



The Seasonal Difference Operator

Definition 2 (Seasonal Difference)

The **seasonal difference operator** Δ_s is defined as:

$$\Delta_s Y_t = (1 - L^s) Y_t = Y_t - Y_{t-s}$$

where $L^s Y_t = Y_{t-s}$ is the seasonal lag operator.

Examples

- Monthly data ($s = 12$): $\Delta_{12} Y_t = Y_t - Y_{t-12}$
Compares each month to the same month last year
- Quarterly data ($s = 4$): $\Delta_4 Y_t = Y_t - Y_{t-4}$
Compares each quarter to the same quarter last year



Seasonal Difference: Visual Illustration



charts/ch4_def_seasonal_diff.pdf

Effect of Seasonal Differencing

Left: Original series with clear seasonal pattern. Right: After $\Delta_{12} = (1 - L^{12})$, seasonal pattern is removed. Year-over-year comparison eliminates seasonal effects.



Proof: Seasonal Differencing Removes Deterministic Seasonality

Claim: If $Y_t = \mu_t + \varepsilon_t$ where $\mu_t = \mu_{t-s}$ (periodic mean), then $\Delta_s Y_t$ removes the seasonal mean.

Proof: Let $Y_t = \mu_t + \varepsilon_t$ where μ_t has period s . Apply seasonal difference:

$$\begin{aligned}\Delta_s Y_t &= Y_t - Y_{t-s} = (\mu_t + \varepsilon_t) - (\mu_{t-s} + \varepsilon_{t-s}) \\ &= \mu_t - \mu_{t-s} + \varepsilon_t - \varepsilon_{t-s} \\ &= 0 + \varepsilon_t - \varepsilon_{t-s} \quad (\text{since } \mu_t = \mu_{t-s})\end{aligned}$$

Properties of $\Delta_s Y_t = \varepsilon_t - \varepsilon_{t-s}$:

- $\mathbb{E}[\Delta_s Y_t] = 0$ (constant mean)
- $\text{Var}(\Delta_s Y_t) = 2\sigma^2$ (constant variance)
- Autocovariance: $\gamma(s) = -\sigma^2$, $\gamma(k) = 0$ for $k \neq 0, s$

Result

Seasonal differencing transforms periodic seasonal pattern into MA(1) at seasonal lag.



Combining Regular and Seasonal Differencing

Full Differencing

For series with both trend and seasonality:

$$\Delta\Delta_s Y_t = (1 - L)(1 - L^s)Y_t$$

Expansion

$$(1 - L)(1 - L^s)Y_t = Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}. \text{ For monthly: } \Delta\Delta_{12} Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$$

Order of Differencing

d : regular differences (trend removal); D : seasonal differences (seasonal trend removal)



Effect of Differencing Operations



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- Regular differencing removes trend but seasonal pattern remains
- Seasonal differencing removes seasonality but trend pattern remains
- Both differences** needed to achieve stationarity



ACF Before and After Differencing



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- Original ACF: slow decay indicates non-stationarity
- After Δ : seasonal spikes remain at lags 12, 24, 36
- After Δ_{12} : trend decay remains at early lags
- After $\Delta\Delta_{12}$: ACF cuts off \Rightarrow **stationary**



Seasonal Integration

Definition 3 (Seasonally Integrated Process)

A series Y_t is **seasonally integrated** of order $(d, D)_s$, written $Y_t \sim I(d, D)_s$, if:

$$(1 - L)^d (1 - L^s)^D Y_t$$

is stationary.

Common Cases

- $I(1, 0)_{12}$: Regular unit root only (monthly)
- $I(0, 1)_{12}$: Seasonal unit root only
- $I(1, 1)_{12}$:
 - ▶ Both regular and seasonal unit roots



SARIMA Model Definition

Definition 4 (SARIMA(p, d, q) \times (P, D, Q) $_s$)

The **Seasonal ARIMA** model is:

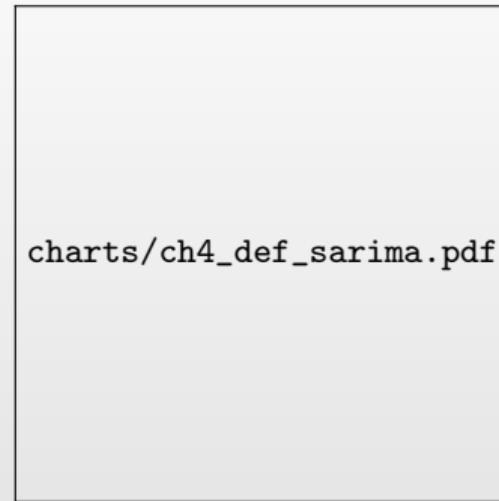
$$\phi(L)\Phi(L^s)(1 - L)^d(1 - L^s)^D Y_t = c + \theta(L)\Theta(L^s)\varepsilon_t$$

Components

- $\phi(L) = 1 - \phi_1L - \dots - \phi_pL^p$: Non-seasonal AR
- $\Phi(L^s) = 1 - \Phi_1L^s - \dots - \Phi_PL^{Ps}$: Seasonal AR
- $\theta(L) = 1 + \theta_1L + \dots + \theta_qL^q$: Non-seasonal MA
- $\Theta(L^s) = 1 + \Theta_1L^s + \dots + \Theta_QL^{Qs}$: Seasonal MA
- $(1 - L)^d$:
 - ▶ Regular differencing; $(1 - L^s)^D$: Seasonal differencing



SARIMA: Visual Illustration



charts/ch4_def_sarima.pdf

Differencing Strategy

Progressive transformation: Original → regular difference (removes trend) → seasonal difference (removes seasonality) → both. Apply minimum differencing needed to achieve stationarity.



Proof: Multiplicative Seasonal Structure

Why multiplicative? Consider SARIMA(1, 0, 0) \times (1, 0, 0)_s:

$$(1 - \phi L)(1 - \Phi L^s) Y_t = \varepsilon_t$$

Expand: $(1 - \phi L)(1 - \Phi L^s) Y_t = Y_t - \phi Y_{t-1} - \Phi Y_{t-s} + \phi \Phi Y_{t-s-1}$

Interpretation (Monthly, $s = 12$)

Y_t depends on: Y_{t-1} (last month), Y_{t-12} (same month last year), Y_{t-13} (interaction).

Parsimony: Multiplicative form uses 2 parameters (ϕ, Φ); additive would need 3+.



SARIMA Notation

Full Specification

SARIMA($p, d, q) \times (P, D, Q)_s$ has 7 parameters to specify:

Parameter	Meaning
p	Non-seasonal AR order
d	Non-seasonal differencing order
q	Non-seasonal MA order
P	Seasonal AR order
D	Seasonal differencing order
Q	Seasonal MA order
s	Seasonal period

Example

SARIMA(1, 1, 1) \times (1, 1, 1)₁₂: Monthly data with AR(1), MA(1), seasonal AR(1), seasonal MA(1), and both regular and seasonal differencing.



Common SARIMA Models

Airline Model: $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_s$

$(1 - L)(1 - L^s)Y_t = (1 + \theta L)(1 + \Theta L^s)\varepsilon_t$ – Classic model (Box & Jenkins, 1970)

$\text{SARIMA}(1, 0, 0) \times (1, 0, 0)_s$

$(1 - \phi L)(1 - \Phi L^s)Y_t = \varepsilon_t$ – Pure seasonal and non-seasonal AR

$\text{SARIMA}(0, 1, 1) \times (0, 1, 0)_s$

$(1 - L)(1 - L^s)Y_t = (1 + \theta L)\varepsilon_t$ – Random walk + seasonal diff + MA(1)

ACF/PACF for Seasonal Models

Key Insight

Seasonal models show patterns at both:

- Non-seasonal lags: $1, 2, 3, \dots$
- Seasonal lags: $s, 2s, 3s, \dots$

Model	ACF	PACF
SAR(P)	Decays at $s, 2s, \dots$	Cuts off after Ps
SMA(Q)	Cuts off after Qs	Decays at $s, 2s, \dots$
SARMA	Decays at seasonal lags	Decays at seasonal lags



Example: Airline Model ACF/PACF

SARIMA(0, 1, 1) \times (0, 1, 1)₁₂

After differencing $W_t = (1 - L)(1 - L^{12})Y_t$: $W_t = (1 + \theta L)(1 + \Theta L^{12})\varepsilon_t$

Expected ACF Pattern

Spikes at lag 1 (θ), lag 12 (Θ), lag 13 ($\theta \cdot \Theta$ interaction); all other lags near zero.

Expected PACF Pattern

Exponential decay at lags 1, 2, 3, ... and at lags 12, 24, 36, ...



Model Identification Guidelines

Step-by-Step Process

1. Examine ACF for slow decay at seasonal lags \Rightarrow seasonal differencing
2. After differencing, check ACF/PACF patterns
3. Non-seasonal behavior at lags $1, 2, \dots, s - 1$
4. Seasonal behavior at lags $s, 2s, 3s, \dots$

Practical Tips

- Start with $d \leq 1$ and $D \leq 1$
- Usually $P, Q \leq 2$ is sufficient
- Use information criteria (AIC, BIC) for final selection
- Auto-SARIMA algorithms can help



Estimation Methods

Maximum Likelihood Estimation

Standard approach for SARIMA:

- Conditional MLE (conditional on initial values)
- Exact MLE (via Kalman filter)

Computational Considerations

- More parameters than ARIMA \Rightarrow more data needed
- Seasonal parameters estimated from lags $s, 2s, \dots$
- Need sufficient seasonal cycles (at least 3-4 years of monthly data)



Stationarity and Invertibility

Stationarity Conditions

Both non-seasonal and seasonal AR polynomials must have roots outside the unit circle:

- $\phi(z) = 0 \Rightarrow |z| > 1$
- $\Phi(z^s) = 0 \Rightarrow |z| > 1$

Invertibility Conditions

Both non-seasonal and seasonal MA polynomials must have roots outside the unit circle:

- $\theta(z) = 0 \Rightarrow |z| > 1$
- $\Theta(z^s) = 0 \Rightarrow |z| > 1$



Diagnostic Checking

Residual Analysis

After fitting SARIMA, check that residuals are white noise:

1. Plot residuals over time (no patterns)
2. ACF of residuals (no significant spikes)
3. Ljung-Box test at multiple lags including seasonal
4. Normality tests (Q-Q plot, Jarque-Bera)

Important

Check ACF at **both** non-seasonal and seasonal lags!

Significant ACF at lag 12 suggests inadequate seasonal modeling.



Model Selection Criteria

Information Criteria

Compare competing SARIMA models using:

- $AIC = -2 \ln(L) + 2k$
- $BIC = -2 \ln(L) + k \ln(n)$
- $AICc = AIC + \frac{2k(k+1)}{n-k-1}$ (corrected for small samples)

where $k = p + q + P + Q + 1$ (plus 1 for variance).

Auto-SARIMA

Python's `pmdarima.auto_arima()` with `seasonal=True` automatically searches for optimal $(p, d, q) \times (P, D, Q)_s$.



Point Forecasts

Forecast Computation

SARIMA forecasts are computed recursively:

- Replace future ε_{T+h} with 0
- Replace future Y_{T+h} with forecasts $\hat{Y}_{T+h|T}$
- Use known past values Y_T, Y_{T-1}, \dots

Seasonal Pattern in Forecasts

SARIMA forecasts naturally capture seasonality:

- Short-term: influenced by recent values
- Long-term: revert to seasonal pattern



Forecast Intervals

Uncertainty Quantification

$(1 - \alpha)\%$ prediction interval:

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$$

Variance computed from $\text{MA}(\infty)$ representation.

Key Properties

- Intervals widen with forecast horizon
- For $I(1, 1)_s$ series: intervals grow without bound
- Seasonal pattern visible in point forecasts
- Uncertainty captures both trend and seasonal variation



Long-Horizon Forecasts

Behavior as $h \rightarrow \infty$

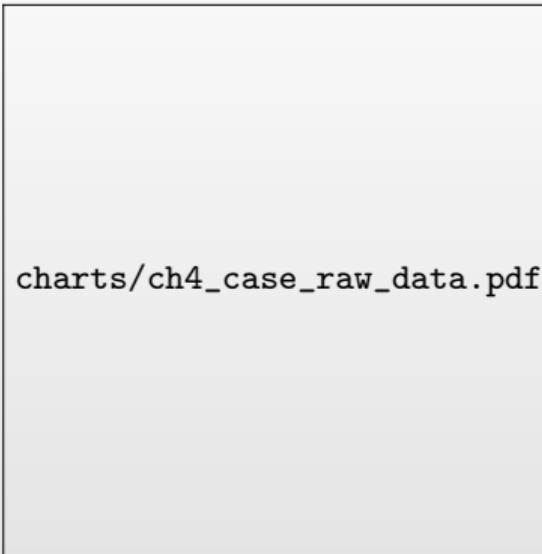
- Point forecasts converge to deterministic seasonal pattern
- If drift present: linear trend + seasonal pattern
- Forecast intervals continue to widen

Practical Implication

- Short-term: SARIMA captures both level and season
- Medium-term: Good seasonal forecasts, growing uncertainty
- Long-term:
 - ▶ Mainly reflects seasonal pattern, wide intervals



Case Study: Airline Passengers Data



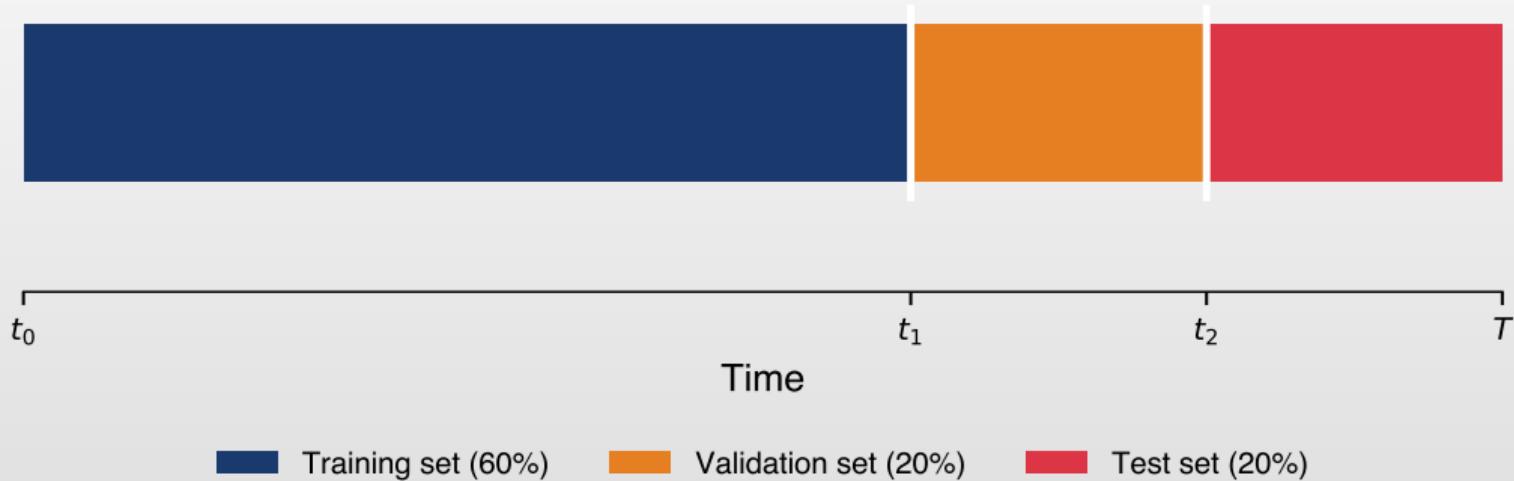
charts/ch4_case_raw_data.pdf

- Classic Box-Jenkins dataset: monthly airline passengers (1949-1960)
- Clear upward trend and increasing seasonal amplitude
- Multiplicative seasonality suggests log transformation



Data Splitting Strategy

Train / Validation / Test Split



- Training set (70%)** — Fit model parameters
 - ▶ Estimate SARIMA coefficients ($\phi, \theta, \Phi, \Theta$)
 - ▶ Largest portion ensures reliable parameter estimates
- Validation set (15%)** — Select best model



Step 1: Transformations

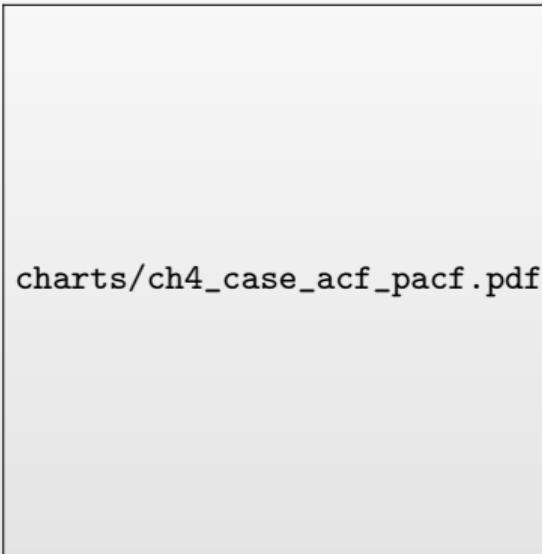


charts/ch4_case_transformations.pdf

- Log transform stabilizes variance (multiplicative → additive)
- First difference removes trend; seasonal difference removes seasonality
- Double-differenced series appears stationary



Step 2: ACF/PACF Analysis



charts/ch4_case_acf_pacf.pdf

- ACF: Significant spike at lag 1 and lag 12 \Rightarrow MA(1), SMA(1)
- PACF: Exponential decay pattern confirms MA structure
- Suggests SARIMA(0, 1, 1) \times (0, 1, 1)₁₂ (airline model)

Step 3: Model Comparison

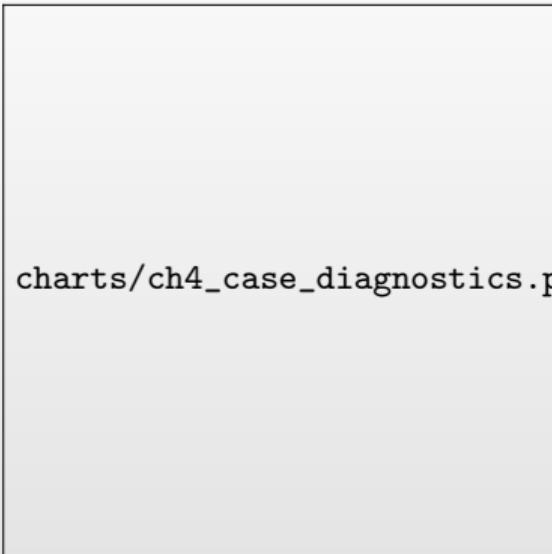


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- Compare candidate SARIMA models using AIC criterion
- SARIMA(0, 1, 1) \times (0, 1, 1)₁₂ provides best fit (lowest AIC)
- This is the famous “airline model” identified by Box & Jenkins



Step 4: Residual Diagnostics



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- Residuals appear random with no remaining autocorrelation
- Q-Q plot shows approximate normality
- Model adequately captures both trend and seasonal structure



Step 5: Forecasting



- 24-month forecast with 95% confidence interval
- Model captures seasonal pattern and upward trend
- Prediction intervals widen appropriately with forecast horizon

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Key Takeaways

Main Points

1. **Seasonality** is common in economic and business data
2. **Seasonal differencing** ($1 - L^s$) removes stochastic seasonality
3. **SARIMA**(p, d, q) \times (P, D, Q)_s extends ARIMA for seasonal data
4. **Multiplicative structure** captures seasonal-trend interactions
5. **ACF/PACF** show patterns at both regular and seasonal lags
6. **Model selection:** Use AIC/BIC or auto-SARIMA algorithms

Next Steps

Chapter 5 will cover multivariate time series: VAR models, Granger causality, and cointegration.



Quiz Question 1

Question

For monthly data with annual seasonality, what is the seasonal period s ?

- (A) $s = 4$
- (B) $s = 7$
- (C) $s = 12$
- (D) $s = 52$



Quiz Question 1: Answer

Correct Answer: (C) $s = 12$ (12 months per year)

Common periods: Quarterly=4, Monthly=12, Weekly=52, Daily=7, Hourly=24

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Quiz Question 2

Question

What does the seasonal difference operator $(1 - L^{12})$ do to a monthly series?

- (A) Computes $Y_t - Y_{t-1}$ (month-to-month change)
- (B) Computes $Y_t - Y_{t-12}$ (year-over-year change)
- (C) Computes the 12-month moving average
- (D) Removes the trend component only



Quiz Question 2: Answer

Correct Answer: (B) Year-over-year change

$(1 - L^{12})Y_t = Y_t - Y_{t-12}$ removes the seasonal pattern by comparing same months.

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Quiz Question 3

Question

In SARIMA(1, 1, 1) \times (1, 1, 1)₁₂ notation, what does the (1, 1, 1)₁₂ part represent?

- (A) AR(1), differencing once, MA(1) at the regular level
- (B) Seasonal AR(1), seasonal differencing once, seasonal MA(1)
- (C) 12 AR terms, 12 differences, 12 MA terms
- (D) The model has 12 parameters in total



Quiz Question 3: Answer

Correct Answer: (B)

Seasonal AR(1), seasonal differencing once, seasonal MA(1)

SARIMA Notation Breakdown

SARIMA(p, d, q) \times (P, D, Q)_s:

(p, d, q) Non-seasonal: AR(p), d differences, MA(q)
(P, D, Q)_s Seasonal: SAR(P), D seasonal diffs, SMA(Q)

For $(1, 1, 1) \times (1, 1, 1)_{12}$:

- Non-seasonal: AR(1), one regular difference, MA(1)
- Seasonal: SAR(1) at lag 12, one Δ_{12} , SMA(1) at lag 12



Quiz Question 4

Question

The “Airline Model” is SARIMA(0, 1, 1) \times (0, 1, 1)₁₂. How many parameters need to be estimated (excluding variance)?

- (A) 1
- (B) 2
- (C) 4
- (D) 12



Quiz Question 4: Answer

Correct Answer: (B) — 2 parameters

SARIMA(0, 1, 1) \times (0, 1, 1)₁₂: $(1 - L)(1 - L^{12})Y_t = (1 + \theta_1 L)(1 + \Theta_1 L^{12})\varepsilon_t$

Parameters: θ_1 (non-seasonal MA) and Θ_1 (seasonal MA), plus σ^2 .

Why “Airline Model”?

Box & Jenkins (1970) used this model to forecast international airline passengers. Remarkably effective for many seasonal economic series!



Quiz Question 5

Question

You observe significant ACF spikes at lags 12, 24, and 36 in a monthly series. What does this suggest?

- (A) The series has a unit root
- (B) The series has annual seasonality that needs seasonal differencing
- (C) The series follows an AR(36) process
- (D) The series is already stationary



Quiz Question 5: Answer

Correct Answer: (B) Needs seasonal differencing

ACF spikes at 12, 24, 36 = stochastic seasonality. Apply $(1 - L^{12})$ to remove it.

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Quiz Question 6

Question

After applying $(1 - L)(1 - L^{12})$ to a monthly series, the ACF shows a significant spike only at lag 1 and lag 12. What SARIMA model is suggested?

- (A) SARIMA(1, 1, 0) \times (1, 1, 0)₁₂
- (B) SARIMA(0, 1, 1) \times (0, 1, 1)₁₂
- (C) SARIMA(1, 1, 1) \times (1, 1, 1)₁₂
- (D) SARIMA(0, 1, 0) \times (0, 1, 0)₁₂



Quiz Question 6: Answer

Correct Answer: (B)

SARIMA(0, 1, 1) \times (0, 1, 1)₁₂ (The Airline Model)

ACF/PACF Identification Rules

For MA processes, ACF **cuts off** after lag q :

Pattern	Suggests
ACF spike at lag 1 only	MA(1) for non-seasonal part
ACF spike at lag 12 only	SMA(1) for seasonal part

Combined: MA(1) \times SMA(1) = (0, d , 1) \times (0, D , 1)₁₂

With $d = 1$ and $D = 1$ (already differenced): (0, 1, 1) \times (0, 1, 1)₁₂



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