Chapter 3: ARIMA Models

Seminar

Seminar Outline

Quiz 1: Integration Order

Question

A time series Y_t requires two differences to become stationary. What is its order of integration?

A) $I(0)$
B) $I(1)$
C) $I(2)$
D) Cannot be determined

Quiz 1: Integration Order

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- B) $I(1)$
- C) $I(2)$
- D) Cannot be determined

Answer: C – $I(2)$

Definition: $Y_t \sim I(d)$ if $\Delta^d Y_t$ is stationary but $\Delta^{d-1} Y_t$ is not.
Example: If Y_t follows $\Delta^2 Y_t = \varepsilon_t$, then:

- $\Delta Y_t = \Delta Y_{t-1} + \varepsilon_t$ (still has unit root)
- $\Delta^2 Y_t = \varepsilon_t$ (white noise, stationary)

Real-world: Price levels may be $I(2)$ when inflation itself is non-stationary.

Quiz 2: Random Walk Properties

Question

For a random walk $Y_t = Y_{t-1} + \varepsilon_t$ with $\mathrm{Var}(\varepsilon_t) = \sigma^2$, what is $\mathrm{Var}(Y_t)$?

- A) σ^2
- B) $t \cdot \sigma^2$
- C) σ^2 / t
- D) $\sigma^2 / (1 - \phi^2)$

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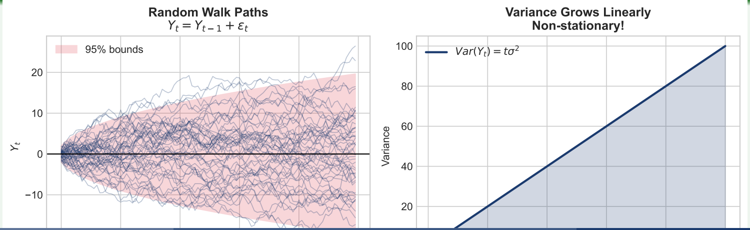
- C) σ^2/t
- D) $\sigma^2/(1-\phi^2)$

Answer: B – $t \cdot \sigma^2$



Quiz 3: ADF Test Hypotheses

Question

In the Augmented Dickey-Fuller test, what is the null hypothesis?

- A) The series is stationary
- B) The series has a unit root
- C) The series has no autocorrelation
- D) The series is normally distributed

Quiz 3: ADF Test Hypotheses

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Answer: B – The series has a unit root

ADF regression: $\Delta Y_t = \alpha + \gamma Y_{t-1} + \sum_{j=1}^{p} \delta_j \Delta Y_{t-j} + \varepsilon_t$
Hypotheses:

- $H_0 : \gamma = 0$ (unit root, non-stationary)
- $H_1 : \gamma < 0$ (stationary)

Decision: Reject H_0 if t-statistic $<$ critical value (e.g., -2.86 at 5%)
Note: Uses special Dickey-Fuller distribution, not standard t.

Quiz 4: ARIMA Notation

Question

What does ARIMA(2,1,1) mean?

- A) AR(2) on differenced data with MA(1) errors
- B) AR(1) with 2 differences and MA(1)
- C) MA(2) with 1 difference and AR(1)
- D) 2 lags, 1 trend, 1 seasonal component

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- B) AR(1) with 2 differences and MA(1)
- C) MA(2) with 1 difference and AR(1)
- D) 2 lags, 1 trend, 1 seasonal component

Answer: A – AR(2) on differenced data with MA(1) errors

ARIMA(p, d, q): $\phi(L)(1-L)^d Y_t = \theta(L)\varepsilon_t$
ARIMA(2,1,1) expands to:

$$(1 - \phi_1 L - \phi_2 L^2)(1 - L)Y_t = (1 + \theta_1 L)\varepsilon_t$$

Or equivalently: $(1 - \phi_1 L - \phi_2 L^2)\Delta Y_t = (1 + \theta_1 L)\varepsilon_t$
Interpretation: First difference the series, then fit ARMA(2,1) to ΔY_t.

Quiz 5: Difference Operator

Question

What is $(1 - L)^2 Y_t$ expanded?

- A) $Y_t - Y_{t-1}$
- B) $Y_t - 2Y_{t-1} + Y_{t-2}$
- C) $Y_t + 2Y_{t-1} + Y_{t-2}$
- D) $Y_t - Y_{t-2}$

Quiz 5: Difference Operator

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- C) $Y_t + 2Y_{t-1} + Y_{t-2}$
- D) $Y_t - Y_{t-2}$

Answer: B – $Y_t - 2Y_{t-1} + Y_{t-2}$

Expansion using binomial theorem:

$$(1 - L)^2 = 1 - 2L + L^2$$

Apply to Y_t:

$$(1 - L)^2 Y_t = Y_t - 2L \cdot Y_t + L^2 \cdot Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$$

Note: This equals $\Delta(\Delta Y_t) = \Delta Y_t - \Delta Y_{t-1}$, the "change in changes".

Quiz 6: KPSS vs ADF

Question

How does the KPSS test differ from the ADF test?

- A) KPSS tests for seasonality, ADF tests for trends
- B) KPSS has stationarity as null, ADF has unit root as null
- C) KPSS is more powerful than ADF
- D) There is no difference

Quiz 6: KPSS vs ADF

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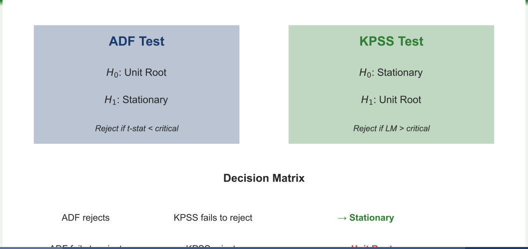
- B) KPSS has stationarity as null, ADF has unit root as null
- C) KPSS is more powerful than ADF
- D) There is no difference

Answer: B – Reversed null hypotheses



Quiz 7: Overdifferencing

Question

If $Y_t \sim I(1)$ and we compute $\Delta^2 Y_t$, what happens?

A) We get a better stationary series
B) We introduce artificial negative autocorrelation
C) The variance decreases
D) Nothing changes

Quiz 7: Overdifferencing

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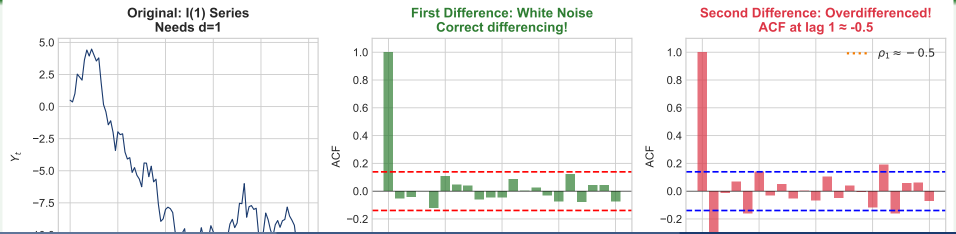
- B) We introduce artificial negative autocorrelation
- C) The variance decreases
- D) Nothing changes

Answer: B – Artificial negative autocorrelation



Quiz 8: Forecast Variance

Question

For an ARIMA(0,1,0) model (random walk), how does forecast variance behave as horizon h increases?

A) Stays constant
B) Decreases to zero
C) Grows linearly with h
D) Converges to a finite limit

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- (B) Decreases to zero
- (C) Grows linearly with h
- (D) Converges to a finite limit

Answer: C – Grows linearly with h

Random walk forecast: $\hat{Y}_{T+h|T} = Y_T$ (best forecast is current value)
Forecast error: $Y_{T+h} - \hat{Y}_{T+h|T} = \sum_{i=1}^{h} \varepsilon_{T+i}$
Variance:

$$\text{Var}(Y_{T+h} - \hat{Y}_{T+h|T}) = h\sigma^2$$

95% CI: $Y_T \pm 1.96\sqrt{h}\sigma$ (widens with $\sqrt{h}$)

Quiz 9: Unit Root Test Power

Question

The ADF test has low power when:

- A) Sample size is very large
- B) The true root is close to but not equal to 1
- C) The series has no trend
- D) The series is clearly stationary

Quiz 9: Unit Root Test Power

Question

The ADF test has low power when:

- A) Sample size is very large
- B) The true root is close to but not equal to 1
- C) The series has no trend
- D) The series is clearly stationary

Answer: B – Root close to but not equal to 1

Example: AR(1) with $\phi = 0.95$ vs random walk ($\phi = 1$)
Problem: Both have similar ACF patterns (slow decay), but one is stationary!
Low power means: High probability of Type II error (failing to reject false H_0)
Solutions:

- Larger sample sizes
- Phillips-Perron test (robust to heteroskedasticity)
- Panel unit root tests (multiple series)

Quiz 10: ARIMA Model Selection

Question

After differencing once, the ACF shows a spike at lag 1 only, and PACF decays. The appropriate model is:

A) ARIMA(1,1,0)
B) ARIMA(0,1,1)
C) ARIMA(1,1,1)
D) ARIMA(0,2,1)

Quiz 10: ARIMA Model Selection

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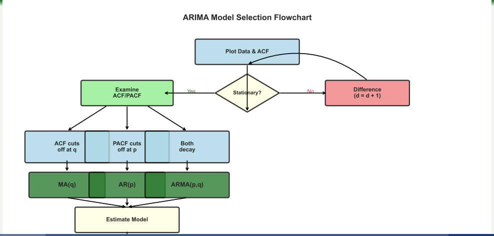
- B) ARIMA(0,1,1)
- C) ARIMA(1,1,1)
- D) ARIMA(0,2,1)

Answer: B – ARIMA(0,1,1)



Quiz 11: Trend Stationarity vs Difference Stationarity

Question

A trend-stationary process is made stationary by:

- A) Taking first differences
- B) Removing the deterministic trend via regression
- C) Taking second differences
- D) Applying seasonal adjustment

Quiz 11: Trend Stationarity vs Difference Stationarity

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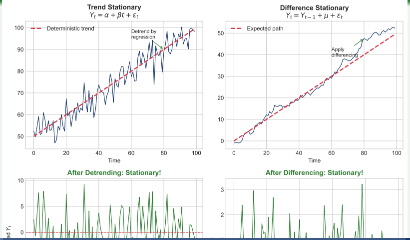
- A) Taking first differences
- B) Removing the deterministic trend via regression
- C) Taking second differences
- D) Applying seasonal adjustment

Answer: B – Removing deterministic trend via regression



Quiz 12: ARIMA Invertibility

Question

ARIMA(0,1,1) with $\theta_1 = 1.2$ is:

A) Stationary and invertible
B) Non-stationary but invertible
C) Non-stationary and non-invertible
D) Stationary but non-invertible

Quiz 12: ARIMA Invertibility

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- B) Non-stationary but invertible
- C) Non-stationary and non-invertible
- D) Stationary but non-invertible

Answer: C – Non-stationary and non-invertible

Check stationarity: $d = 1$ means one unit root $\Rightarrow$ Non-stationary
Check invertibility: MA polynomial is $\theta(z) = 1 + 1.2z$

- Root: $z = -1/1.2 = -0.833$ (inside unit circle)
- Invertibility requires root outside unit circle
- $|\theta_1| = 1.2 > 1 \Rightarrow$ Non-invertible

Fix: Rewrite with $\theta^* = 1/1.2 = 0.833$ and adjust variance.

Quiz 13: Spurious Regression

Question

Regressing one random walk on another independent random walk typically shows:

A) No significant relationship
B) High R^2 and significant t-statistics (spuriously)
C) Negative correlation
D) Perfect multicollinearity

Quiz 13: Spurious Regression

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Regressing one random walk on another independent random walk typically shows:

- A) No significant relationship
- B) High R^2 and significant t-statistics (spuriously)
- C) Negative correlation
- D) Perfect multicollinearity

Answer: B – High R^2 and significant t-statistics (spuriously)

Granger & Newbold (1974): Spurious regression phenomenon

Symptoms:

- High R^2 (often > 0.9) between unrelated series
- Significant t-statistics
- Very low Durbin-Watson statistic ($\ll 2$)
- Non-stationary residuals

Quiz 14: Long-Run Forecast

Question

The long-run forecast from ARIMA(1,1,0) with $\phi_1 = 0.7$ converges to:

- A) Zero
- B) The unconditional mean
- C) A linear trend extrapolation
- D) The last observed value

Quiz 14: Long-Run Forecast

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- A) Zero
- B) The unconditional mean
- C) A linear trend extrapolation
- D) The last observed value

Answer: C – A linear trend extrapolation

Model: $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$
Long-run forecast: For I(1) models with drift c:

$$\hat{Y}_{T+h} \approx Y_T + h \cdot \frac{c}{1 - \phi_1}$$

Key differences:

- Stationary ARMA: Forecasts $\rightarrow$ unconditional mean

True/False Questions

Determine if each statement is True or False:

1. An I(2) process requires two differences to become stationary.
2. The ADF test always includes a constant term.
3. ARIMA(0,1,0) is another name for a random walk.
4. Differencing a stationary series makes it “more stationary.”
5. The KPSS test has stationarity as the null hypothesis.
6. ARIMA models can only capture linear patterns.

Answers on next slide...

True/False: Solutions

1. An I(2) process requires two differences to become stationary. **TRUE**
   I(d) means d differences needed. I(2) = two unit roots.
2. The ADF test always includes a constant term. **FALSE**
   You choose: no constant, constant only, or constant + trend.
3. ARIMA(0,1,0) is another name for a random walk. **TRUE**
   $(1 - L)Y_t = \varepsilon_t \Rightarrow Y_t = Y_{t-1} + \varepsilon_t$.
4. Differencing a stationary series makes it "more stationary." **FALSE**
   Over-differencing creates non-invertible MA; hurts model performance.
5. The KPSS test has stationarity as the null hypothesis. **TRUE**
   KPSS: H_0 = stationary. Opposite of ADF.
6. ARIMA models can only capture linear patterns. **TRUE**
   ARIMA is linear in parameters. Nonlinear patterns need GARCH, neural nets, etc.

Problem 1: Unit Root Testing

Exercise

You have quarterly GDP data for 80 quarters. The ADF test (with constant and trend) gives a test statistic of -2.85. The 5% critical value is -3.41.

1. What is your conclusion about stationarity?
2. What would you do next?

Problem 1: Unit Root Testing

Exercise

You have quarterly GDP data for 80 quarters. The ADF test (with constant and trend) gives a test statistic of -2.85. The 5% critical value is -3.41.

1. What is your conclusion about stationarity?
2. What would you do next?

Solution

1. Since $-2.85 > -3.41$, we **fail to reject** H_0. The data appears to have a unit root (non-stationary).
2. Take the first difference ΔY_t and repeat the ADF test on the differenced series to confirm it is now stationary.

Problem 2: Model Identification

Exercise

After differencing a time series once, the ACF shows:

- Significant spike at lag 1 ($\rho_1 = 0.4$)
- All other lags insignificant

The PACF shows gradual decay.
What ARIMA model is suggested?

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- Significant spike at lag 1 ($\rho_1 = 0.4$)
- All other lags insignificant

The PACF shows gradual decay.
What ARIMA model is suggested?

Solution

- ACF cuts off after lag 1 $\Rightarrow$ MA(1) component
- PACF decays $\Rightarrow$ Confirms MA structure
- Since we differenced once: $d = 1$

Suggested model: ARIMA(0,1,1) or IMA(1,1)

Problem 3: ARIMA Equation

Exercise

Write out the full equation for ARIMA(1,1,1):

$$(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$$

Expand this completely in terms of Y_t, Y_{t-1}, Y_{t-2}, etc.

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Expand this completely in terms of Y_t, Y_{t-1}, Y_{t-2}, etc.

Solution

Expanding $(1 - \phi_1 L)(1 - L) = 1 - L - \phi_1 L + \phi_1 L^2 = 1 - (1 + \phi_1)L + \phi_1 L^2$:

$$Y_t - (1 + \phi_1)Y_{t-1} + \phi_1 Y_{t-2} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Or equivalently:

$$Y_t = c + (1 + \phi_1)Y_{t-1} - \phi_1 Y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Problem 4: Forecast Calculation

Exercise

Given ARIMA(0,1,1): $\Delta Y_t = \varepsilon_t + 0.3\varepsilon_{t-1}$
At time T: $Y_T = 100$, $\hat{\varepsilon}_T = 2$, $\sigma^2 = 4$
Calculate:

1. $\hat{Y}_{T+1|T}$ (one-step forecast)
2. $\hat{Y}_{T+2|T}$ (two-step forecast)

Problem 4: Forecast Calculation

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Calculate:

1. $\hat{Y}_{T+1|T}$ (one-step forecast)
2. $\hat{Y}_{T+2|T}$ (two-step forecast)

Solution

1. $\hat{Y}_{T+1|T} = Y_T + 0.3\hat{\varepsilon}_T = 100 + 0.3(2) = \mathbf{100.6}$
2. $\hat{Y}_{T+2|T} = \hat{Y}_{T+1|T} + 0.3 \cdot 0 = 100.6 + 0 = \mathbf{100.6}$
   (Future shocks $\varepsilon_{T+1}, \varepsilon_{T+2}$ are forecast as 0)

Problem 5: Confidence Intervals

Exercise

Continuing from Problem 4, calculate the 95% forecast intervals for $\hat{Y}_{T+1|T}$ and $\hat{Y}_{T+2|T}$.
Recall: $\sigma^2 = 4$, $\theta_1 = 0.3$

Problem 5: Confidence Intervals

Exercise

Continuing from Problem 4, calculate the 95% forecast intervals for $\hat{Y}_{T+1|T}$ and $\hat{Y}_{T+2|T}$.
Recall: $\sigma^2 = 4$, $\theta_1 = 0.3$

Solution

For IMA(1,1), the MA(∞) weights are $\psi_0 = 1$, $\psi_j = 1 + \theta_1$ for $j \geq 1$.
1-step: $\text{Var}(e_{T+1}) = \sigma^2 \psi_0^2 = 4$, so $SE = 2$
$100.6 \pm 1.96(2) = [\mathbf{96.68}, \mathbf{104.52}]$
2-step: $\text{Var}(e_{T+2}) = \sigma^2(\psi_0^2 + \psi_1^2) = 4(1 + 1.3^2) = 10.76$, $SE = 3.28$
$100.6 \pm 1.96(3.28) = [\mathbf{94.17}, \mathbf{107.03}]$

Example: Testing for Unit Root in Stock Prices

Scenario

You have daily closing prices for a stock over 500 days. You want to determine if prices follow a random walk.

Step-by-step Approach

1. **Visual inspection**: Plot prices – likely shows trend
2. **ADF test on prices**: Expect to fail to reject H_0 (unit root)
3. **Take log returns**: $r_t = \ln(P_t/P_{t-1}) = \Delta \ln(P_t)$
4. **ADF test on returns**: Should reject H_0 (stationary)
5. **Conclusion**: Log prices are $I(1)$, returns are $I(0)$

Example: Box-Jenkins for Inflation Data

Scenario

Monthly inflation rates for 10 years. Build an ARIMA model.

Workflow

1. **Plot & test**: ADF suggests borderline – try both $d = 0$ and $d = 1$
2. **If** $d = 0$: Fit ARMA models, compare AIC
3. **If** $d = 1$: Examine ACF/PACF of ΔY_t
   - ACF: spike at lag 1, then cuts off
   - PACF: decays
   - $\Rightarrow$ Try ARIMA(0,1,1)
4. **Estimate**: Fit ARIMA(0,1,1), check coefficients
5. **Diagnose**: Ljung-Box on residuals (want $p > 0.05$)
6. **Compare**: AIC of ARIMA(0,1,1) vs ARMA(1,1) on levels

Example: Interpreting Python Output

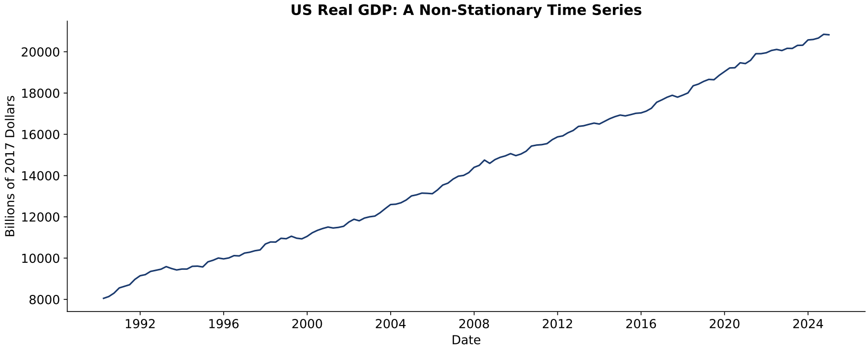
statsmodels ARIMA Output

```
                         ARIMA Model Results
==============================================================
Dep. Variable:          D.y   No. Observations:    99
Model:             ARIMA(1,1,1)   AIC                  285.32
                                  BIC                  295.63
==============================================================
                 coef    std err      z     P>|z|
--------------------------------------------------------------
const          0.0521    0.048    1.085   0.278
ar.L1          0.4532    0.102    4.443   0.000
ma.L1         -0.2891    0.118   -2.450   0.014
sigma2         1.2340    0.176    7.011   0.000
```

Interpretation

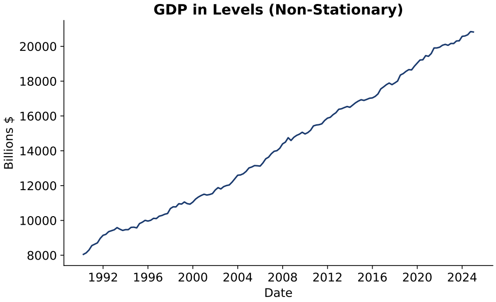
- AR coefficient (0.45) is significant, MA coefficient (-0.29) is significant
- Constant (0.052) not significant – could set $c = 0$
- Check: $|\phi_1| < 1$ (stationary), $|\theta_1| < 1$ (invertible) – OK!

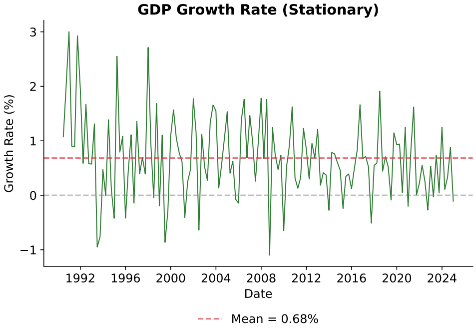
Case Study: US Real GDP (1990–2024)



- US Real GDP in billions of 2017 dollars (quarterly data)
- Clear **upward trend** – typical of macroeconomic series
- Notable drops during recessions (2008-2009, 2020)
- Non-stationary: needs differencing before ARIMA modeling

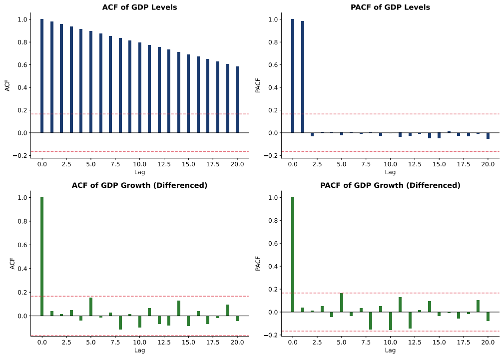
Stationarity Through Differencing





- **Left**: GDP in levels – clear upward trend (non-stationary)
- **Right**: GDP growth rate $= \Delta \log(Y_t) \times 100$ – stationary
- First differencing of log GDP removes the stochastic trend
- Growth rate fluctuates around a constant mean ($\approx 0.6\%$ quarterly)

ACF/PACF: Levels vs Differenced



- **Top row**: ACF/PACF of GDP levels – slow decay indicates non-stationarity
- **Bottom row**: ACF/PACF of GDP growth – mostly within confidence bands
- Pattern suggests low-order ARIMA model is appropriate

ARIMA Estimation Results: US GDP Growth

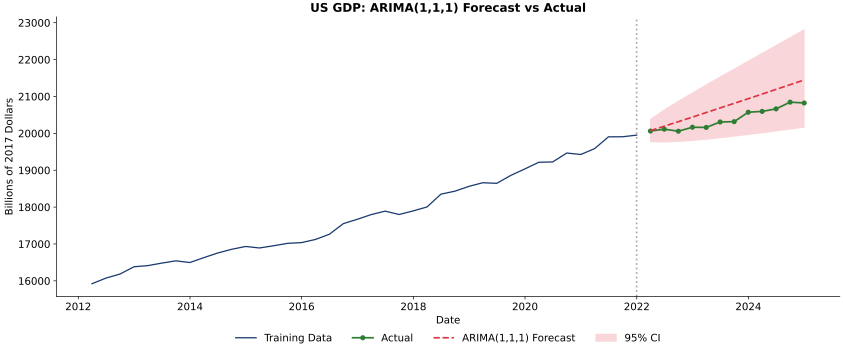
Model: ARIMA$(1, 1, 1)$ on log(GDP)

Parameter	Estimate	Std. Error	z-stat	p-value
ϕ_1 (AR.L1)	0.312	0.185	1.69	0.091
θ_1 (MA.L1)	-0.087	0.203	-0.43	0.668
σ^2	0.00012	–	–	–

Interpretation

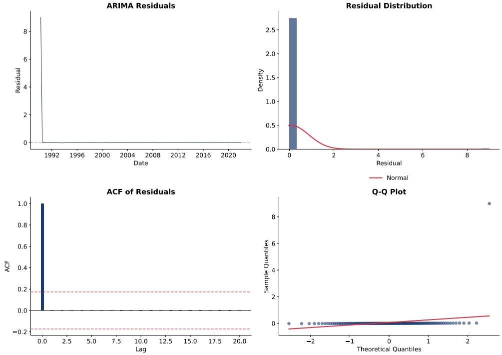
- Low-order ARIMA captures GDP dynamics reasonably well
- AR(1) coefficient positive – GDP growth shows persistence
- Alternative: simpler random walk (ARIMA(0,1,0)) often competitive

Forecast: ARIMA vs Actual



- Blue: historical training data; Green: actual test data
- Red dashed: ARIMA forecasts with 95% confidence interval
- Forecasts capture the general trend direction
- Confidence intervals widen as forecast horizon increases

Model Diagnostics: Residual Analysis



- Residuals show no systematic patterns over time
- Distribution approximately normal (histogram and Q-Q plot)
- ACF of residuals within bounds – no significant autocorrelation remaining
- Model adequately captures the data generating process

Discussion: Deterministic vs Stochastic Trends

Key Question

Why is it important to distinguish between deterministic and stochastic trends?

Discussion Points

- **Wrong treatment consequences**:
  - Detrending a unit root $\Rightarrow$ spurious stationarity
  - Differencing a trend-stationary $\Rightarrow$ overdifferencing
- **Economic interpretation**:
  - Deterministic trend: shocks are temporary
  - Stochastic trend: shocks have permanent effects
- **Policy implications**:
  - Does a recession permanently lower GDP, or does the economy return to trend?

Discussion: Model Selection Criteria

Key Question

When should you use AIC vs BIC for ARIMA model selection?

Considerations

- **AIC**: Minimizes prediction error, may overfit
  - Better for forecasting
  - Tends to select larger models
- **BIC**: Consistent model selection, more parsimonious
  - Better for identifying "true" model
  - Penalizes complexity more heavily
- **Practical advice**: Report both, prefer BIC if they disagree substantially

Discussion: Limitations of ARIMA

Key Question

What are the main limitations of ARIMA models?

Discussion Points

- **Linearity**: Cannot capture nonlinear dynamics
- **Constant variance**: Assumes homoskedasticity (no GARCH effects)
- **No structural breaks**: Parameters assumed constant
- **Univariate**: Ignores relationships with other variables
- **Symmetric**: Treats positive and negative shocks equally
- **Long-horizon forecasts**: Uncertainty grows rapidly

Extensions

These limitations motivate GARCH (volatility), VAR (multivariate), regime-switching models, etc.

Key Points from Today's Seminar

What We Covered

1. **Integration and differencing**: $I(d)$ processes require d differences
2. **Unit root testing**: ADF tests H_0: unit root; KPSS tests H_0: stationary
3. **ARIMA(p,d,q)**: Combines ARMA with differencing
4. **Model identification**: Use ACF/PACF patterns and information criteria
5. **Forecasting**: Point forecasts and growing confidence intervals

Next Seminar

Hands-on Python exercises with real economic data:

- Unit root testing with `statsmodels`
- Auto-ARIMA with `pmdarima`
- Forecasting and model diagnostics