Time Series Analysis and Forecasting

Chapter 7: Cointegration and VECM

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Learning Objectives

By the end of this chapter, you will be able to:

- Understand the problem of spurious regression with non-stationary data
- Test for cointegration using Engle-Granger and Johansen methods
- Estimate Vector Error Correction Models (VECM)
- Interpret error correction mechanisms and adjustment speeds

Outline

Why Cointegration Matters

The Challenge

- Many economic/financial time series are **non-stationary** (I(1))
- GDP, stock prices, exchange rates, interest rates all have unit roots
- Standard regression with I(1) variables $\Rightarrow$ **spurious results**
- Differencing removes non-stationarity but loses **long-run information**

The Solution: Cointegration

Some non-stationary series share a **common stochastic trend**—they move together in the long run.

Nobel Prize 2003

Granger & Engle received the Nobel Prize for "methods for analyzing economic time series with common trends."

Real-World Applications

Finance

- **Pairs Trading**: Cointegrated stocks
- **Term Structure**: Interest rates
- **Spot-Futures**: Arbitrage

Policy Analysis

- **Fiscal**: Spending & taxes
- **Monetary**: Rate pass-through
- **Labor**: Wages & productivity

Macroeconomics

- **Consumption & Income**
- **Money & Prices**
- **PPP**: Exchange rates

The Spurious Regression Problem

Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk: $Y_t = \alpha + \beta X_t + u_t$ where Y_t and X_t are independent I(1) processes.

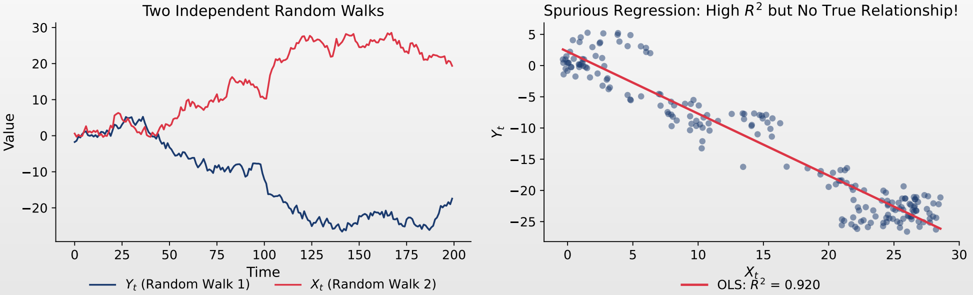
Symptoms of Spurious Regression

- High R^2 (often > 0.9) even though variables are **unrelated**
- Highly significant t-statistics (reject $H_0 : \beta = 0$)
- Very low Durbin-Watson statistic ($DW \approx 0$)
- Residuals are non-stationary (have unit root)

Rule of Thumb

If $R^2 > DW$, suspect spurious regression!

Spurious Regression: Visual Example



Warning: Two independent random walks show high correlation ($R^2 > 0.8$) by chance! This is why we need cointegration analysis.

TSA_ch7_spurious_regression

Spurious Correlations in the Real World

Data Mining Can Produce Meaningless Correlations

With enough variables and long time series, purely coincidental patterns emerge:

- Distance between Neptune and Uranus ↔ SAP SE stock price (2002–2023)
- GMO corn use in South Dakota ↔ Google searches for "i cant even" (2004–2023)
- *Two and a Half Men* season ratings ↔ Jet fuel used in Serbia (2006–2015)
- "Its Wednesday my dudes" meme popularity ↔ Boeing stock price (2006–2023)

Lesson

High correlation ≠ causation. Non-stationary series with common trends produce high R^2 by construction. Always test for stationarity and cointegration before interpreting regression results!

Explore more examples: `tylervigen.com/spurious-correlations`

Definition of Cointegration

Definition 1 (Cointegration (Engle & Granger, 1987))

Variables $Y_{1t}, Y_{2t}, \ldots, Y_{kt}$ are **cointegrated of order** (d, b), written $CI(d, b)$, if:

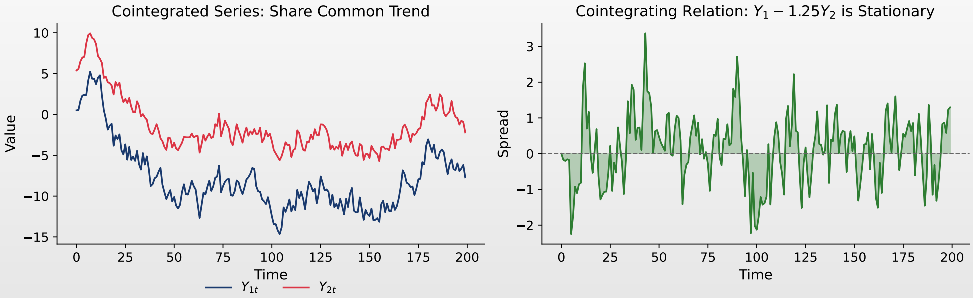
1. All variables are integrated of order d: $Y_{it} \sim I(d)$
2. There exists a linear combination $\boldsymbol{\beta}'\mathrm{Y}_t = \beta_1 Y_{1t} + \cdots + \beta_k Y_{kt}$ that is integrated of order $(d - b)$, where $b > 0$

Most Common Case: $CI(1, 1)$

- Variables are $I(1)$ (have unit roots)
- Linear combination is $I(0)$ (stationary)
- Vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)'$ is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized: $\beta_{\mathbf{1}} = 1$.

Cointegration: Visual Example



Key insight: Both series are I(1) and trend together, but their linear combination (spread) is stationary—this is cointegration!

TSA_ch7_cointegrated_series

Intuition: Common Stochastic Trends

Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**: $Y_{1t} = \gamma_1 \tau_t + S_{1t}$, $Y_{2t} = \gamma_2 \tau_t + S_{2t}$ where τ_t is a common random walk and S_{it} are stationary.

Linear Combination Eliminates the Trend

$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$

Economic Interpretation

- Cointegration = **long-run equilibrium relationship**
- Variables may deviate in the short run, but are "pulled back"
- The cointegrating vector defines the equilibrium

Cointegrating Rank

How Many Cointegrating Relationships?

For k variables that are $I(1)$:

- Maximum possible cointegrating relationships: $r = k - 1$
- If $r = 0$: No cointegration (variables drift apart)
- If $r = k$: All variables are $I(0)$ (contradiction)

Example: 3 Variables

- $r = 0$: No cointegration
- $r = 1$: One cointegrating relationship
- $r = 2$: Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends $= k - r$

Engle-Granger Two-Step Method

Step 1: Estimate Cointegrating Regression

Run OLS: $Y_t = \alpha + \beta X_t + e_t$. Save residuals: $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

Step 2: Test Residuals for Stationarity

Test if $\hat{e}_t$ is $I(0)$ using ADF: $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^{p} \gamma_j \Delta \hat{e}_{t-j} + v_t$

- H_0: $\rho = 0$ (unit root $\Rightarrow$ no cointegration)
- H_1: $\rho < 0$ (stationary $\Rightarrow$ cointegration)

Important

Use **Engle-Granger critical values**, not standard ADF! (More negative because residuals are estimated)

Engle-Granger Critical Values

Critical Values for Cointegration Test

Variables	**1%**	**5%**	**10%**
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

MacKinnon (1991), $T = 100$

Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on choice of dependent variable
- Small sample bias; cannot test hypotheses on cointegrating vector

Johansen Cointegration Test

Advantages over Engle-Granger

- Tests for **multiple** cointegrating relationships
- Maximum likelihood estimation (more efficient)
- Can test restrictions on cointegrating vectors
- Does not require choosing a dependent variable

Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...

VECM Representation

Vector Error Correction Model

$\Delta Y_t = c + \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$

- $\Pi = \sum_i A_i - I$ (long-run impact); Γ_j (short-run dynamics)

Key Insight: Rank of Π

The **rank of Π** determines cointegration:

- $\text{rank}(\Pi) = 0$: No cointegration (VAR in differences)
- $\text{rank}(\Pi) = k$: All variables are $I(0)$ (VAR in levels)
- $0 < \text{rank}(\Pi) = r < k$: r cointegrating vectors

Derivation: From VAR to VECM

Starting Point: VAR(p) in Levels

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Step 1: Subtract Y_{t-1} from Both Sides

$$Y_t - Y_{t-1} = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} - Y_{t-1} + \varepsilon_t$$

$$\Delta Y_t = (A_1 - I) Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Goal

Rewrite so that all terms are either in **levels** (Y_{t-1}) or **differences** (ΔY_{t-j}).

Derivation: From VAR to VECM (cont.)

Step 2: Add and Subtract Terms Strategically

Add $A_2 Y_{t-1}$ and subtract $A_2 Y_{t-1}$:

$$\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} - A_2(Y_{t-1} - Y_{t-2}) + A_3 Y_{t-3} + \cdots + \varepsilon_t$$

Continue adding $A_3 Y_{t-1}$, etc., until all lagged **levels** are collected in one term.

Step 3: General Pattern

After algebraic manipulation, we obtain:

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

The Key Matrices

$$\boxed{\Pi = \sum_{i=1}^{p} A_i - I = -(I - A_1 - A_2 - \cdots - A_p)}$$

Derivation: Verifying the $\boldsymbol{\Gamma}_j$ Formula

Example: VAR(2)

Starting from: $Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$

Subtract Y_{t-1}:

$$\Delta Y_t = (A_1 - I)Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$$

Add and subtract $A_2 Y_{t-1}$:

$$\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} + A_2(Y_{t-2} - Y_{t-1}) + \varepsilon_t$$

$$\Delta Y_t = \underbrace{(A_1 + A_2 - I)}_{\boldsymbol{\Pi}} Y_{t-1} \underbrace{-A_2}_{\boldsymbol{\Gamma}_1} \Delta Y_{t-1} + \varepsilon_t$$

Verification

For VAR(2): $\boldsymbol{\Pi} = A_1 + A_2 - I$ and $\boldsymbol{\Gamma}_1 = -A_2$

Using our formula: $\boldsymbol{\Gamma}_1 = -\sum_{i=2}^{2} A_i = -A_2 \quad \checkmark$

Economic Interpretation of Error Correction

The VECM with Cointegration

When $\text{rank}(\boldsymbol{\Pi}) = r$, we write $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$:

$$\Delta \mathbf{Y}_t = \boldsymbol{\alpha} \underbrace{(\boldsymbol{\beta}'\mathbf{Y}_{t-1})}_{\text{equilibrium error}} + \sum_{j=1}^{p-1} \boldsymbol{\Gamma}_j \Delta \mathbf{Y}_{t-j} + \boldsymbol{\varepsilon}_t$$

Economic Interpretation

- $\boldsymbol{\beta}'\mathbf{Y}_{t-1}$ = **equilibrium error**: deviation from long-run relationship
- $\boldsymbol{\alpha}$ = **adjustment speeds**: how fast variables correct deviations
- $\boldsymbol{\Gamma}_j$ = **short-run dynamics**: transitory effects

Error Correction Mechanism

If $\boldsymbol{\beta}'\mathbf{Y}_{t-1} > 0$ (above equilibrium) and $\alpha_i < 0$, then ΔY_{it} decreases.

The system self-corrects toward equilibrium!

Decomposition of $\mathbf{\Pi}$

When rank($\mathbf{\Pi}$) $= r < k$

$\mathbf{\Pi} = \alpha\beta'$ where β $(k \times r)$ = **cointegrating vectors**, α $(k \times r)$ = **adjustment coefficients**

Interpretation

- $\beta' \mathbf{Y}_{t-1}$ = deviations from equilibrium (error correction terms)
- $\boldsymbol{\alpha}$ = speed of adjustment; rows show each variable's response

VECM: $\Delta \mathbf{Y}_t = \mathbf{c} + \boldsymbol{\alpha}(\boldsymbol{\beta}' \mathbf{Y}_{t-1}) + \sum_{j=1}^{p-1} \boldsymbol{\Gamma}_j \Delta \mathbf{Y}_{t-j} + \boldsymbol{\varepsilon}_t$

Johansen Test Statistics

Two Test Statistics

Based on eigenvalues $\hat{\lambda}_1 > \hat{\lambda}_2 > \cdots > \hat{\lambda}_k$ of a certain matrix:

Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^{k} \ln(1 - \hat{\lambda}_i)$$

Tests H_0: rank $\leq r$ vs H_1: rank $> r$

Maximum Eigenvalue Test:

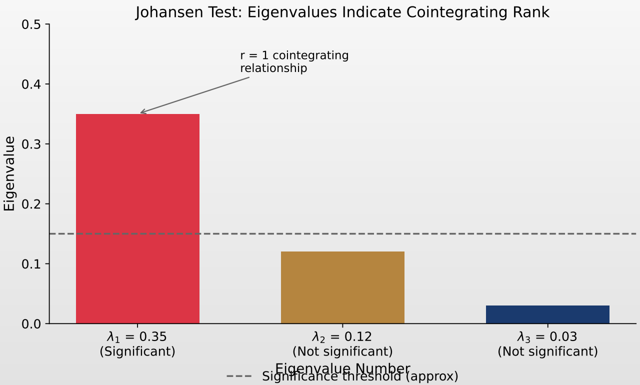
$$\lambda_{\max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

Tests H_0: rank $= r$ vs H_1: rank $= r + 1$

Critical values from Johansen & Juselius (1990), depend on:

- Number of variables k
- Deterministic components (constant, trend)

Johansen Test: Visual Interpretation



Significant eigenvalues (above threshold) indicate cointegrating relationships. First eigenvalue significant $\Rightarrow r = 1$.

TSA_ch7_johansen_eigenvalues

Testing Procedure

Sequential Testing (Trace Test)

1. Test H_0: $r = 0$. If rejected $\Rightarrow$ continue
2. Test H_0: $r \leq 1$. If not rejected $\Rightarrow$ $r = 1$
3. Continue until H_0 is not rejected

Deterministic Components

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- **Constant in both** (most common)
- Constant + trend in cointegrating relation

VECM Structure

Full VECM Specification

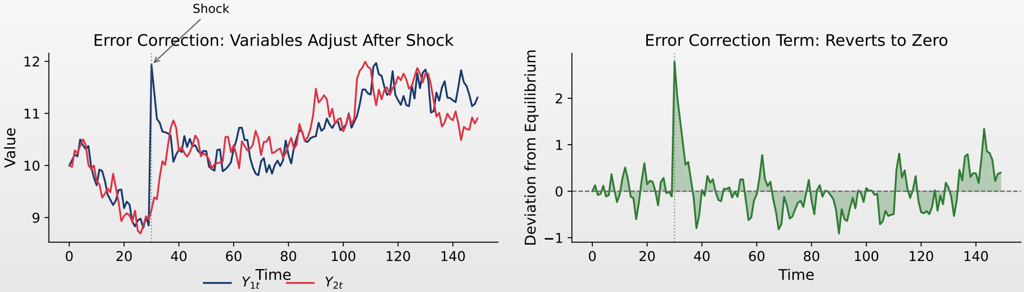
For $k = 2$ variables with $r = 1$ cointegrating relation:

$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$
$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$ = error correction term (deviation from equilibrium)
- α_1, α_2 = adjustment speeds (should have opposite signs)
- γ_{ij} = short-run dynamics
- ε_{it} = innovations

Error Correction Mechanism: Visual



Error correction in action: When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment. TSA_ch7_error_correction

Interpreting Adjustment Coefficients

The α Coefficients

If the cointegrating relation is $Y_1 - \beta Y_2 = 0$ (equilibrium):

- $\alpha_1 < 0$: Y_1 adjusts downward when above equilibrium
- $\alpha_2 > 0$: Y_2 adjusts upward when Y_1 is above equilibrium

Weak Exogeneity

If $\alpha_i = 0$, variable Y_i does **not** respond to disequilibrium.

- Y_i is **weakly exogenous** for the long-run parameters
- The other variable does all the adjusting
- Can simplify estimation (single-equation approach)

Test weak exogeneity: $H_0 : \alpha_i = 0$ using likelihood ratio test.

VECM vs VAR in Differences

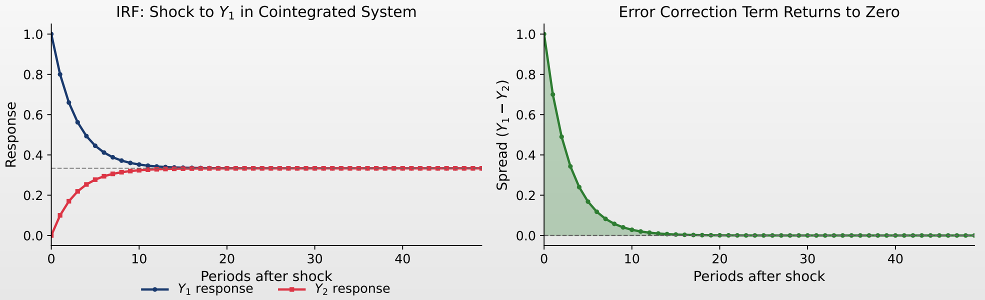
When Variables are Cointegrated

	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

VECM Impulse Response Functions



IRF interpretation: In a cointegrated system, shocks have **permanent effects** on levels but the system returns to equilibrium. Unlike stationary VAR, effects don't decay to zero—they converge to a new long-run value.

TSA_ch7_vecm_irf

Practical Workflow

Step-by-Step Procedure

1. **Unit Root Tests**: Verify all variables are $I(1)$
   - ADF, KPSS on levels and first differences
2. **Lag Length Selection**: Choose p for VAR in levels
   - Use AIC, BIC, or sequential LR tests
3. **Cointegration Test**: Johansen trace/max-eigenvalue tests
   - Determine cointegrating rank r
4. **Estimate VECM**: If $0 < r < k$
   - Estimate $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}_j$
5. **Diagnostics**: Check residuals for autocorrelation, normality
6. **Analysis**: IRF, FEVD, hypothesis tests

Common Pitfalls

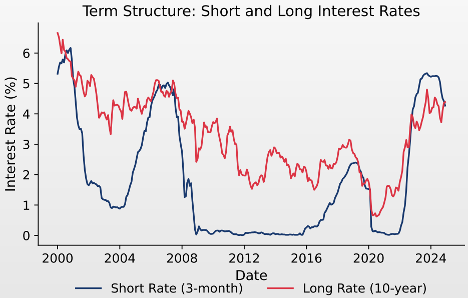
Things to Watch Out For

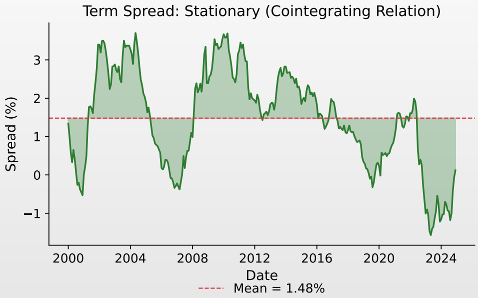
- **Structural breaks**: Cause spurious unit roots or cointegration
- **Near-unit-root**: Tests have low power
- **Lag selection**: Too many/few lags bias results
- **Small samples**: Johansen test oversized

Recommendation

Always check: residual diagnostics, stability of cointegrating relationship, sensitivity to specification

Example 1: Term Structure of Interest Rates

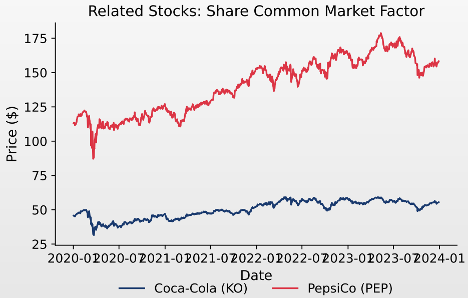


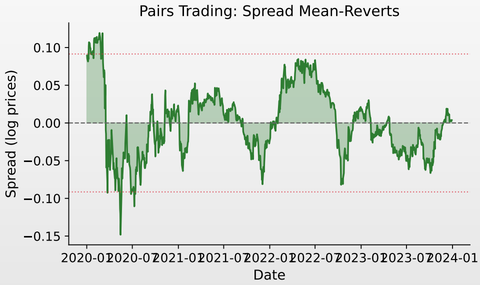


Expectations Hypothesis: Short and long rates share common trend. The spread (term premium) is stationary—evidence of cointegration!

TSA_ch7_interest_rates_coint

Example 2: Pairs Trading in Finance

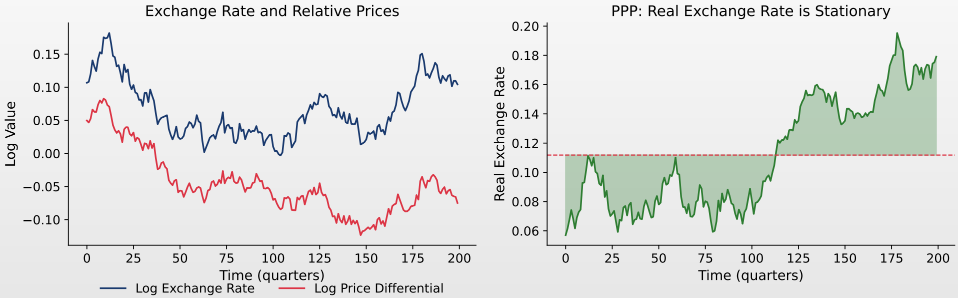




Strategy: Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When spread deviates from mean, trade expecting mean reversion. Sell spread when high, buy when low.

TSA_ch7_pairs_trading

Example 3: Purchasing Power Parity (PPP)



PPP Theory: $e_t = p_t - p_t^*$ (log exchange rate equals price differential). Real exchange rate should be stationary in the long run.

TSA_ch7_ppp_cointegration

Case Study: Cointegration of Interest Rates

Research Question

Are short-term and long-term interest rates cointegrated? Does the expectations hypothesis of the term structure hold?

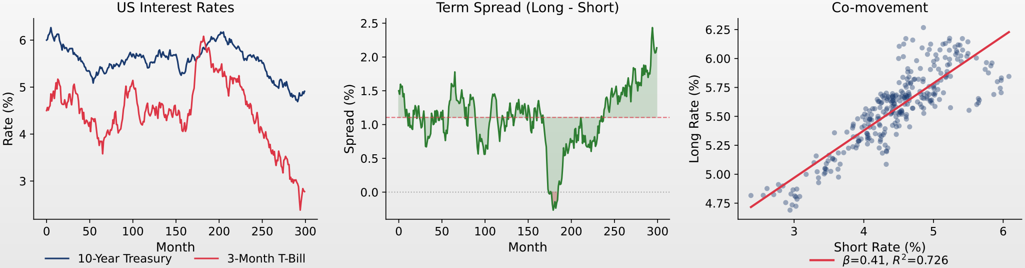
Data

- US Monthly Data (1962-2023)
- 3-Month Treasury Bill Rate
- 10-Year Treasury Bond Yield
- Source: FRED Database

Methodology

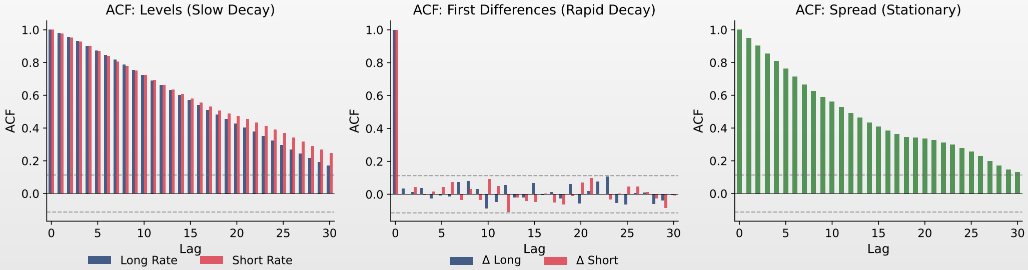
- Unit root tests (ADF, PP)
- Engle-Granger cointegration test
- Johansen procedure
- VECM estimation
- Impulse response analysis

Step 1: Data Visualization



TSA_ch7_case_raw_data

Step 2: Unit Root Tests



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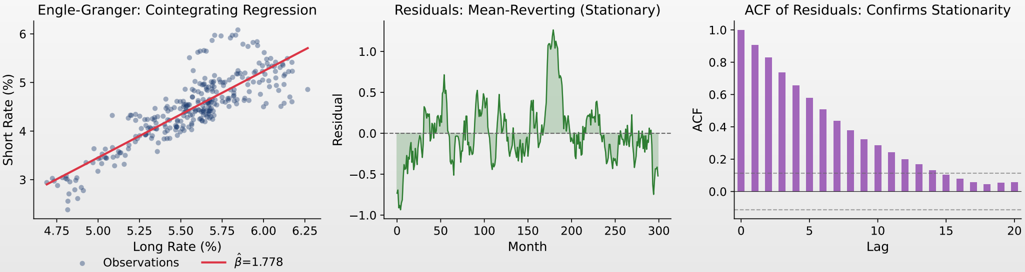
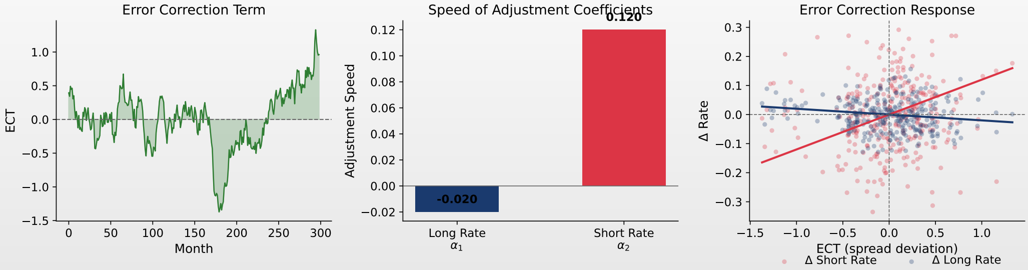
Step 3: Engle-Granger Cointegration Test



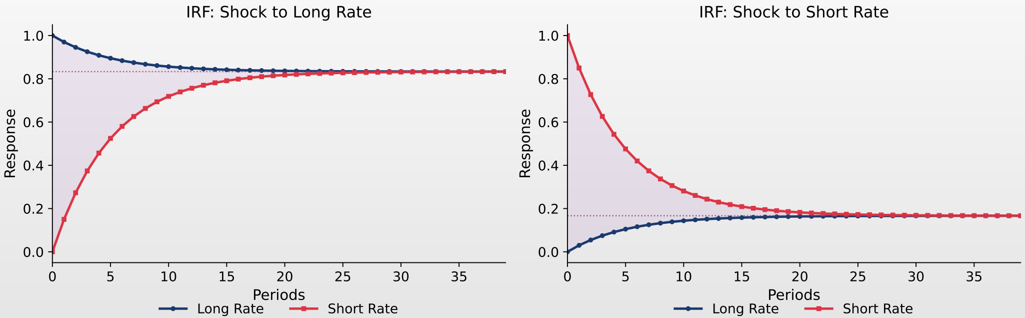
TSA_ch7_case_cointegration

Step 4: VECM Estimation



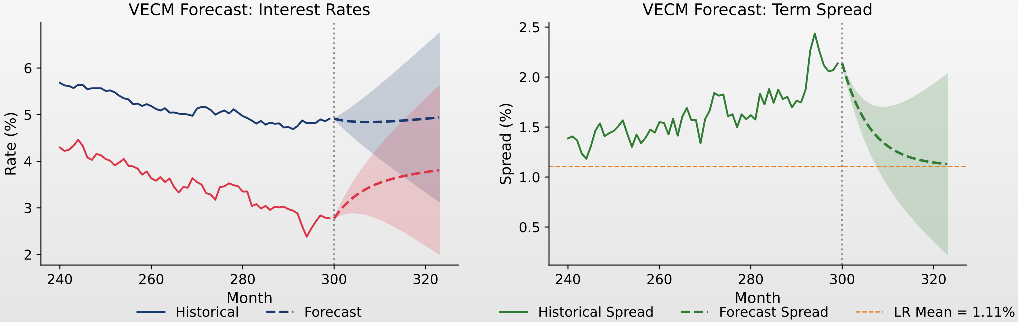
TSA_ch7_case_vecm

Step 5: Impulse Response Functions



TSA_ch7_case_irf

Step 6: Forecasting



TSA_ch7_case_forecast

Key Takeaways

Main Concepts

- **Cointegration**: $I(1)$ variables with stationary linear combination
- **Spurious regression**: High R^2 with unrelated $I(1)$ variables
- **VECM**: VAR with error correction for cointegrated systems

Testing Methods

- **Engle-Granger**: Simple, one vector only
- **Johansen**: Multiple vectors, MLE-based

Remember

Tests have low power in small samples. Theory should guide specification.

What's Next?

Extensions and Related Topics

- **Structural VECM**: Identifying structural shocks
- **Threshold cointegration**: Nonlinear adjustment
- **Panel cointegration**: Multiple cross-sections
- **Fractional cointegration**: Long memory
- **Time-varying cointegration**: Regime changes

Questions?

Quick Quiz

1. What does it mean for two $I(1)$ variables to be cointegrated?
2. What is the "spurious regression" problem?
3. In a VECM, what do the α coefficients represent?
4. What is the main advantage of Johansen over Engle-Granger?
5. If $\alpha_i = 0$ for variable Y_i, what does this imply?

Quiz Answers

1. **Cointegration**: A linear combination of the variables is $I(0)$ (stationary). They share a common stochastic trend.
2. **Spurious regression**: Regressing one $I(1)$ variable on another unrelated $I(1)$ variable gives high R^2 and significant coefficients even though there's no true relationship.
3. α **coefficients**: Speed of adjustment—how quickly each variable responds to deviations from long-run equilibrium.
4. **Johansen advantage**: Can test for multiple cointegrating relationships, uses MLE (more efficient), doesn't require choosing dependent variable.
5. $\alpha_i = 0$: Variable Y_i is weakly exogenous—it doesn't respond to disequilibrium. Other variables do all the adjusting.

Online Resources and Code

- **Quantlet**: https://quantlet.com → Code repository for statistics
- **Quantinar**: https://quantinar.com → Learning platform for quantitative methods
- **GitHub TSA_ch7**: https://github.com/QuantLet/TSA/tree/main/TSA_ch7

Thank You!

Questions?

Course materials available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar