Time Series Analysis and Forecasting

Seminar 4: SARIMA Models

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Seminar Outline

- **Multiple Choice Quiz** – Knowledge check
- **True/False** – Conceptual checks
- **Calculation Exercises** – Applied practice
- **Worked Examples** – Real-world applications
- **Real Data Analysis** – Case study with airline data
- **AI-Assisted Exercise** – Critical thinking
- **Summary** – Key takeaways

Quiz 1: Seasonal Differencing

Question

For monthly data with annual seasonality, what does the operator $(1 - L^{12})$ do?

Answer choices

(A) Takes 12 consecutive differences

(B) Computes $Y_t - Y_{t-12}$

(C) Averages over 12 months

(D) Removes the first 12 observations

Answer on next slide...

Quiz 1: Answer

Answer: B – Computes $Y_t - Y_{t-12}$

Question: What does the operator $(1 - L^{12})$ do?

Answer choices

(A) Takes 12 consecutive differences ×

(B) **Computes $Y_t - Y_{t-12}$** ✓

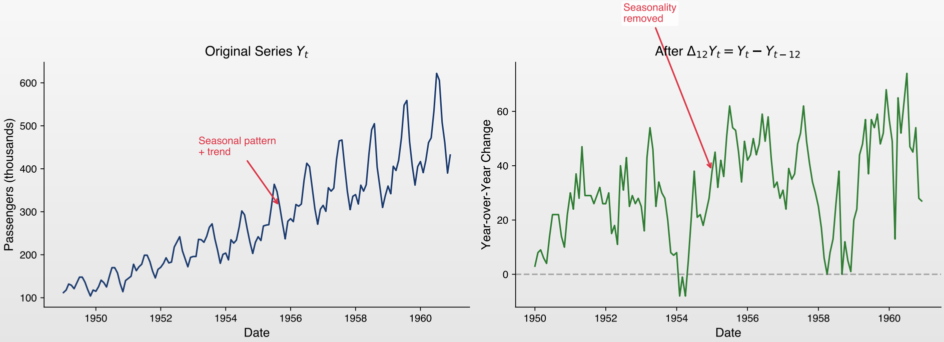
(C) Averages over 12 months ×

(D) Removes the first 12 observations ×

- **Seasonal difference operator**: $(1 - L^{12})Y_t = Y_t - L^{12}Y_t = Y_t - Y_{t-12}$
- **Example** (January sales): $Y_{Jan2025} - Y_{Jan2024}$
- **Effect**: Removes stable annual seasonal pattern

TSA_ch4_def_seasonal_diff

Visual: Seasonal Difference



Seasonal differencing removes annual patterns by comparing same periods across years.

TSA_ch4_def_seasonal_diff

Quiz 2: SARIMA Notation

Question

What does $\text{SARIMA}(1,1,1) \times (1,1,1)_{12}$ represent?

Answer choices

(A) 12 different ARIMA models

(B) ARIMA with 12 AR and 12 MA terms

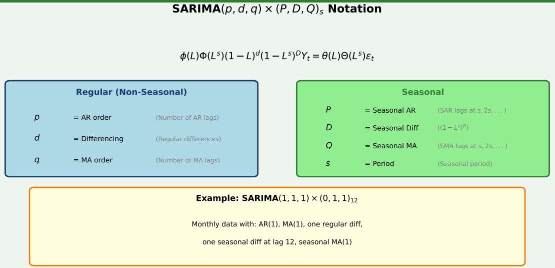
(C) ARIMA(1,1,1) with seasonal ARIMA(1,1,1) at period 12

(D) A model requiring 12 years of data

Answer on next slide...

Quiz 2: Answer

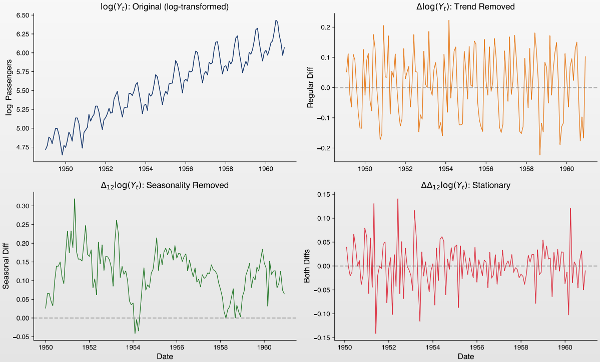
Answer: C – ARIMA(1,1,1) with seasonal ARIMA(1,1,1) at period 12



- $(1-\phi_1 L)(1-\Phi_1 L^{12})(1-L)(1-L^{12})Y_t = (1+\theta_1 L)(1+\Theta_1 L^{12})\varepsilon_t$

TSA_ch4_def_sarima

Visual: SARIMA Model Structure



SARIMA combines regular ARIMA components with seasonal components at lag s.

TSA_ch4_def_sarima

Quiz 3: The Airline Model

Question

The "airline model" refers to SARIMA$(0, 1, 1) \times (0, 1, 1)_{12}$. How many parameters does it have (excluding variance)?

Answer choices

(A) 2 parameters

(B) 4 parameters

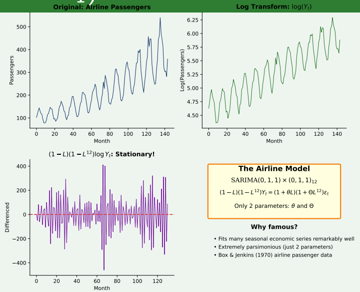
(C) 6 parameters

(D) 12 parameters

Answer on next slide...

Quiz 3: Answer

Answer: A – 2 parameters (θ_1 and Θ_1)



- **Airline model**: $(1-L)(1-L^{12})Y_t = (1+\theta_1 L)(1+\Theta_1 L^{12})\varepsilon_t$
- Remarkably fits many seasonal economic series (Box & Jenkins, 1970)

TSA_ch4_def_sarima

Quiz 4: ACF of Seasonal Data

Question

For monthly data with strong seasonality, where would you expect to see significant ACF spikes?

Answer choices

(A) Only at lag 1

(B) Only at lag 12

(C) At lags 12, 24, 36, ...

(D) Randomly distributed

Answer on next slide...

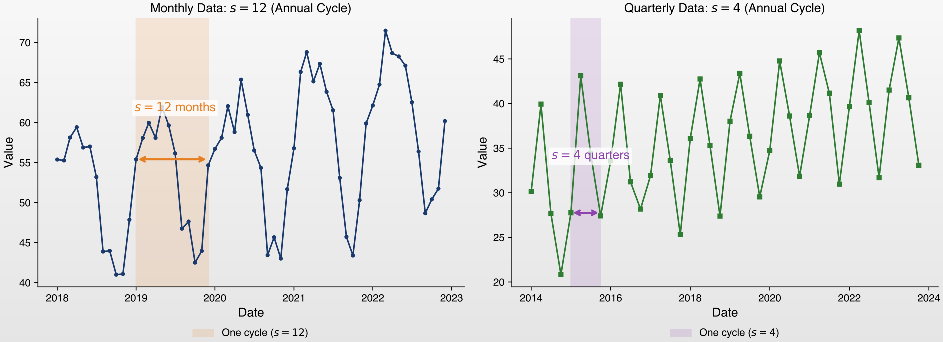
Quiz 4: Answer

Answer: C – At lags 12, 24, 36, ...

- **Intuition**: January 2024 is similar to January 2023, 2022, etc.
- **ACF pattern**: Spikes at lags $s, 2s, 3s, \ldots$ ($\rho_{12}, \rho_{24}, \rho_{36} \neq 0$)
- **Diagnostic**: Slow decay at seasonal lags $\Rightarrow$ $D = 1$; Cutoff after lag s $\Rightarrow$ $Q = 1$

TSA_ch4_def_seasonality

Visual: Seasonality Patterns



Seasonal patterns repeat at regular intervals (monthly, quarterly, etc.) and may be additive or multiplicative.

TSA_ch4_def_seasonality

Quiz 5: Multiplicative Structure

Question

In SARIMA, what does “multiplicative structure” mean?

Answer choices

(A) The seasonal amplitude grows proportionally

(B) Regular and seasonal polynomials are multiplied

(C) We multiply the data by seasonal factors

(D) The model is estimated using multiplication

Answer on next slide...

Quiz 5: Answer

Answer: B – Regular and seasonal polynomials are multiplied

- **Multiplicative SARIMA**: $\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D Y_t = \theta(L)\Theta(L^s)\varepsilon_t$
- **Example**: $(1-\phi_1 L)(1-\Phi_1 L^{12}) = 1 - \phi_1 L - \Phi_1 L^{12} + \phi_1 \Phi_1 L^{13}$
- **Cross-term** $\phi_1 \Phi_1 L^{13}$: Captures interaction between short and long dynamics

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Quiz 6: Seasonal vs Regular Differencing

Question

When would you apply both regular ($d = 1$) and seasonal ($D = 1$) differencing?

Answer choices

(A) When data has only a trend

(B) When data has only seasonality

(C) When data has both trend and seasonal non-stationarity

(D) Never – they cancel each other

Answer on next slide...

Quiz 6: Answer

Answer: C – Both trend and seasonal non-stationarity

- **Combined**: $W_t = (1 - L)(1 - L^{12})Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$
- **When needed**: ACF slow decay at lags 1,2,3... $\Rightarrow d = 1$; at lags 12,24,36... $\Rightarrow D = 1$
- **Examples**: Airline passengers, retail sales, energy demand

TSA_ch4_def_seasonal_diff

Quiz 7: Detecting Seasonality from ACF

Question

The ACF of a monthly time series shows slow decay at lags 12, 24, and 36. What does this suggest?

Answer choices

(A) The series is stationary

(B) The series needs regular differencing only

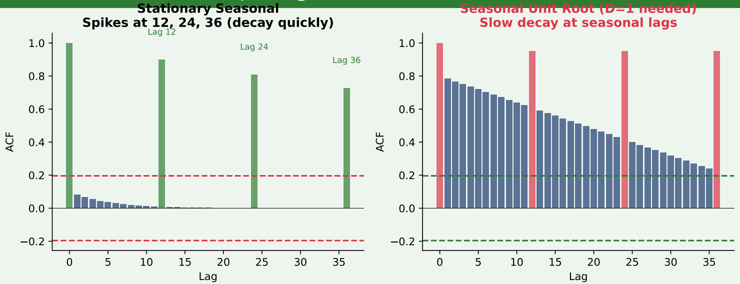
(C) The series has a seasonal unit root requiring $D = 1$

(D) The series is white noise

Answer on next slide...

Quiz 7: Answer

Answer: C – Seasonal unit root requiring $D = 1$



- **Left**: Stationary seasonal (fast decay at seasonal lags)
- **Right**: Seasonal unit root (slow decay $\Rightarrow$ need $D = 1$)

TSA_ch4_def_seasonality

Quiz 8: Multiplicative vs Additive Seasonality

Question

If the seasonal amplitude of a time series grows proportionally with the level, this indicates:

Answer choices

(A) Additive seasonality – use $(1 - L^s)$

(B) Multiplicative seasonality – use log transformation

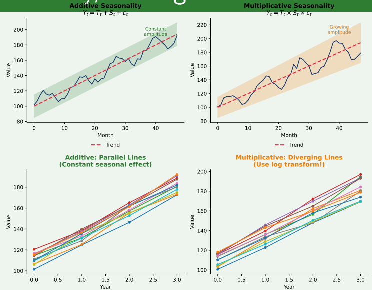
(C) No seasonality present

(D) Need for regular differencing only

Answer on next slide...

Quiz 8: Answer

Answer: B – Multiplicative seasonality, use log transformation



- **Multiplicative**: Seasonal amplitude grows with level (diverging lines)
- **Solution**: Apply log transformation before fitting SARIMA

TSA_ch4_def_seasonality

Quiz 9: Seasonal Subseries Plot

Question

In a seasonal subseries plot, what indicates multiplicative seasonality?

Answer choices

(A) Lines for each month are parallel

(B) Lines for each month diverge (spread increases over time)

(C) All months have the same mean

(D) Lines are horizontal

Answer on next slide...

Quiz 9: Answer

Answer: B – Lines diverge (spread increases over time)

- **Subseries plot**: Groups data by month, plots each month's values across years
- **Parallel** $\Rightarrow$ Additive; **Diverging** $\Rightarrow$ Multiplicative; **Horizontal** $\Rightarrow$ No trend
- **Action**: If multiplicative, apply log before fitting SARIMA

TSA_ch4_def_seasonality

Quiz 10: Invertibility in SARIMA

Question

For SARIMA$(0,1,1) \times (0,1,1)_{12}$ to be invertible, which condition must hold?

Answer choices

(A) $|\theta_1| < 1$ only

(B) $|\Theta_1| < 1$ only

(C) Both $|\theta_1| < 1$ and $|\Theta_1| < 1$

(D) No invertibility condition exists for MA models

Answer on next slide...

Quiz 10: Answer

Answer: C – Both $|\theta_1| < 1$ and $|\Theta_1| < 1$

- **Invertibility**: All MA roots outside unit circle
- **Multiplicative MA**: $(1 + \theta_1 L)(1 + \Theta_1 L^{12})$
- **Roots**: Regular $|z| = |-1/\theta_1| > 1 \Leftrightarrow |\theta_1| < 1$; Seasonal $|\Theta_1| < 1$
- **Both** conditions required for overall invertibility!

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Quiz 11: HEGY Test

Question

The HEGY test is used to:

Answer choices

(A) Estimate SARIMA parameters

(B) Test for unit roots at different frequencies (trend and seasonal)

(C) Check residual normality

(D) Compare SARIMA models using information criteria

Answer on next slide...

Quiz 11: Answer

Answer: B – Test for unit roots at different frequencies

- **HEGY test** (Hylleberg-Engle-Granger-Yoo, 1990)
- Tests at: Zero freq ($\omega = 0$) $\Rightarrow$ $d = 1$; Nyquist ($\omega = \pi$); Seasonal $\Rightarrow$ $D = 1$
- **Decision**: Reject all $\Rightarrow$ seasonal dummies; Don't reject seasonal $\Rightarrow$ seasonal differencing

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Quiz 12: Seasonal MA Identification

Question

After applying $(1-L)(1-L^{12})$, the ACF shows a single significant spike at lag 12 only (no spike at lag 1). The PACF decays at seasonal lags. This suggests:

Answer choices

(A) SARIMA$(0,1,0)\times(0,1,1)_{12}$

(B) SARIMA$(0,1,1)\times(0,1,0)_{12}$

(C) SARIMA$(1,1,0)\times(1,1,0)_{12}$

(D) SARIMA$(0,1,1)\times(0,1,1)_{12}$

Answer on next slide...

Quiz 12: Answer

Answer: A – SARIMA$(0, 1, 0) \times (0, 1, 1)_{12}$

- **Pattern**: Regular lags – no spikes in ACF/PACF; Seasonal lags – ACF cuts off at s, PACF decays
- **Interpretation**: No regular MA ($q = 0$); Seasonal MA(1) indicated ($Q = 1$)
- **Model**: $(1 - L)(1 - L^{12})Y_t = (1 + \Theta_1 L^{12})\varepsilon_t$

TSA_ch4_def_sarima

Quiz 13: Over-differencing

Question

After differencing, the ACF shows a large negative spike at lag 1 or lag s. This typically indicates:

Answer choices

(A) The model needs more AR terms

(B) The series has been over-differenced

(C) The series is perfectly stationary

(D) Heteroskedasticity is present

Answer on next slide...

Quiz 13: Answer

Answer: B – The series has been over-differenced

- **Signature**: ACF at lag 1 $\approx -0.5 \Rightarrow$ over-diff at d; ACF at lag $s \approx -0.5 \Rightarrow$ over-diff at D
- **Why?** $\Delta^2 Y_t = \varepsilon_t - \varepsilon_{t-1}$ is MA(1) with $\theta = -1$, giving $\rho_1 = -0.5$
- **Fix**: Reduce d or D by one and re-examine ACF/PACF

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Quiz 14: Forecasting Horizon

Question

For a SARIMA model with $D = 1$, what happens to forecast confidence intervals as the horizon $h \to \infty$?

Answer choices

(A) They converge to a fixed width

(B) They grow without bound

(C) They shrink to zero

(D) They oscillate seasonally

Answer on next slide...

Quiz 14: Answer

Answer: B – They grow without bound

- **Unit root property**: Any unit root causes unbounded forecast variance
- **For SARIMA with $D = 1$**: $\text{Var}(\hat{Y}_{T+h} - Y_{T+h}) \to \infty$ as $h \to \infty$
- **Intuition**: Seasonal shocks accumulate; long-range forecasts have wide CIs

TSA_ch4_sarima_forecast

Quiz 15: Seasonal Period Selection

Question

You have daily data showing clear weekly patterns. What seasonal period s should you use in a SARIMA model?

Answer choices

(A) $s = 12$ (monthly)

(B) $s = 7$ (weekly)

(C) $s = 365$ (yearly)

(D) $s = 24$ (hourly)

Answer on next slide...

Quiz 15: Answer

Answer: B – $s = 7$ (weekly)

Data	Pattern	Period s
Daily	Weekly	7
Monthly	Annual	12
Quarterly	Annual	4

- **Rule**: $s =$ observations per cycle of dominant pattern

TSA_ch4_def_seasonality

Quiz 16: Seasonal AR Component

Question

In the seasonal component $\Phi(L^s) = 1 - \Phi_1 L^s$, what does the coefficient $\Phi_1 = 0.8$ tell us?

Answer choices

(A) 80% of this period's value comes from the previous period

(B) There is 80% correlation between consecutive observations

(C) Strong seasonal persistence: the conditional expectation depends on 80% of last year's same-period value

(D) The seasonal pattern explains 80% of variance

Answer on next slide...

Quiz 16: Answer

Answer: C – Strong seasonal persistence ($\Phi_1 = 0.8$)

- **SAR(1)**: $Y_t = \Phi_1 Y_{t-12} + \varepsilon_t$
- **With** $\Phi_1 = 0.8$: $Y_{Jan2024} = 0.8 \cdot Y_{Jan2023} + \varepsilon_t$
- **Interpretation**: There is strong linear dependence ($\Phi_1 = 0.8$) on the same period last year. Note: Φ_1 is a regression coefficient, not R^2; the explained variance would be $\Phi_1^2 = 0.64$
- **Stationarity**: Requires $|\Phi_1| < 1$ (satisfied here)

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Quiz 17: Seasonal Stationarity

Question

A seasonal process with $\Phi_1 = 1$ in SARIMA$(0,0,0) \times (1,0,0)_{12}$ is:

Answer choices

(A) Stationary

(B) Has a seasonal unit root (seasonally integrated)

(C) Explosive

(D) Undefined

Answer on next slide...

Quiz 17: Answer

Answer: B – Has a seasonal unit root

- **Model**: $Y_t = Y_{t-12} + \varepsilon_t$ (seasonal random walk)
- **Properties**: Variance grows with time; each month follows its own RW; need $D = 1$
- **Analogy**: Like regular random walk but at seasonal frequency

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Quiz 18: Model Comparison

Question

Model A: SARIMA$(1,1,1)\times(1,1,1)_{12}$ has AIC $=520$. Model B: SARIMA$(0,1,1)\times(0,1,1)_{12}$ has AIC $=525$. Which statement is most accurate?

Answer choices

(A) Model A is always better since it has lower AIC

(B) Model B should be preferred due to parsimony despite higher AIC

(C) The AIC difference of 5 suggests Model A is substantially better

(D) We cannot compare models with different orders

Answer on next slide...

Quiz 18: Answer

Answer: C – AIC difference of 5 suggests Model A is substantially better

- **Rule of thumb**: ΔAIC < 2: equivalent; 2–10: some evidence; > 10: strong evidence
- **Here**: ΔAIC $= 5$ suggests Model A meaningfully better
- **Always**: Also check residual diagnostics and forecast performance!

TSA_ch4_def_sarima

Quiz 19: Seasonal Patterns in Residuals

Question

After fitting a SARIMA model, you notice significant ACF spikes at lags 12 and 24 in the residuals. What does this indicate?

Answer choices

(A) The model is correctly specified

(B) The seasonal component is inadequate

(C) The data is not seasonal

(D) Overfitting has occurred

Answer on next slide...

Quiz 19: Answer

Answer: B – The seasonal component is inadequate

- **Diagnostics**: Good residuals should be white noise (no significant ACF)
- **Seasonal ACF in residuals**: Model hasn't captured seasonal structure; try increasing P or Q; verify D is correct
- **Action**: Try SARIMA with higher seasonal order, check Ljung-Box at seasonal lags

TSA_ch4_diagnostics

Quiz 20: Practical Forecasting

Question

You're forecasting monthly retail sales with SARIMA$(0,1,1) \times (0,1,1)_{12}$. For the 13-month-ahead forecast, which historical observations are most influential?

Answer choices

(A) Only the most recent observation

(B) The observation from the same month last year

(C) All observations equally

(D) Only observations from the same month in all previous years

Answer on next slide...

Quiz 20: Answer

Answer: B – The observation from the same month last year

- **For 13-month ahead**: Most influential is Y_{T-11} (same month last year), also Y_T and Y_{T-12}
- **Intuition**: "Next January looks like last January, adjusted for recent trend"

TSA_ch4_sarima_forecast

True or False? — Questions

Statement	**T/F?**
1. The seasonal period s for quarterly data with annual patterns is $s = 4$.	?
2. SARIMA models can only handle one seasonal frequency.	?
3. If AIC selects SARIMA$(1,1,1) \times (1,1,1)_{12}$ and BIC selects the airline model, BIC is always wrong.	?
4. The Kruskal-Wallis test can detect seasonality without assuming normality.	?
5. After fitting a SARIMA model, residuals should show no significant ACF at seasonal lags.	?
6. Log transformation converts multiplicative seasonality to additive.	?

True or False? — Answers

Statement	T/F	Explanation
1. The seasonal period s for quarterly data with annual patterns is $s = 4$.	T	**4 quarters/year**
2. SARIMA models can only handle one seasonal frequency.	T	**Need TBATS for multiple**
3. If AIC selects SARIMA$(1,1,1) \times (1,1,1)_{12}$ and BIC selects the airline model, BIC is always wrong.	F	**BIC penalizes more**
4. The Kruskal-Wallis test can detect seasonality without assuming normality.	T	**Nonparametric test**
5. After fitting a SARIMA model, residuals should show no significant ACF at seasonal lags.	T	**White noise residuals**
6. Log transformation converts multiplicative seasonality to additive.	T	$\log(T \times S) = \log T + \log S$

Exercise 1: Expanding the Seasonal Difference

Problem

- **Task**: Expand $(1-L)(1-L^{12})Y_t$ fully. What observations are involved?

Solution

$(1-L)(1-L^{12}) = 1 - L - L^{12} + L^{13}$

Therefore: $(1-L)(1-L^{12})Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$

- First seasonal difference: $Y_t - Y_{t-12}$ (this year vs last year)
- Then regular difference: $(Y_t - Y_{t-12}) - (Y_{t-1} - Y_{t-13})$

Exercise 2: Airline Model Expansion

Problem

- **Task**: Write out the full equation for the airline model SARIMA$(0,1,1) \times (0,1,1)_{12}$:
- **Formula**: $(1-L)(1-L^{12})Y_t = (1+\theta_1 L)(1+\Theta_1 L^{12})\varepsilon_t$

Solution

Expand the MA side: $(1+\theta_1 L)(1+\Theta_1 L^{12}) = 1 + \theta_1 L + \Theta_1 L^{12} + \theta_1\Theta_1 L^{13}$

Full model: $Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13} = \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-12} + \theta_1\Theta_1\varepsilon_{t-13}$

- **Note**: The cross-term $\theta_1\Theta_1 L^{13}$ is the multiplicative interaction between regular and seasonal MA components.

Exercise 3: Parameter Count

Problem

- **Task**: How many parameters (excluding σ^2) are in SARIMA$(2, 1, 1) \times (1, 0, 1)_4$?

Solution

- Regular AR($p = 2$): $\phi_1, \phi_2 \Rightarrow 2$ parameters
- Regular MA($q = 1$): $\theta_1 \Rightarrow 1$ parameter
- Seasonal AR($P = 1$): $\Phi_1 \Rightarrow 1$ parameter
- Seasonal MA($Q = 1$): $\Theta_1 \Rightarrow 1$ parameter

Total: 5 parameters

Note: The differencing orders ($d = 1$, $D = 0$) don't add parameters – they're transformations applied to the data.

Exercise 4: SARIMA Forecasting

Problem

- **Data**: Airline model with $\theta_1 = -0.4$ and $\Theta_1 = -0.6$
- $Y_T = 500,\ Y_{T-1} = 495,\ Y_{T-11} = 480,\ Y_{T-12} = 470$
- $\varepsilon_T = 5$, $\varepsilon_{T-11} = -3$, $\varepsilon_{T-12} = 2$
- **Calculate**: Forecast Y_{T+1}

Solution

From the model: $Y_{T+1} = Y_T + Y_{T-11} - Y_{T-12} + \varepsilon_{T+1} + \theta_1\varepsilon_T + \Theta_1\varepsilon_{T-11} + \theta_1\Theta_1\varepsilon_{T-12}$
Setting $\mathbb{E}[\varepsilon_{T+1}] = 0$:
$\hat{Y}_{T+1} = 500 + 480 - 470 + 0 + (-0.4)(5) + (-0.6)(-3) + (-0.4)(-0.6)(2)$
$= 510 - 2 + 1.8 + 0.48 = 510.28$

Exercise 5: Identifying Seasonal Period

Problem

- **Task**: Match each data type with its typical seasonal period s:
- Quarterly GDP, Monthly retail sales, Weekly restaurant reservations, Daily electricity demand

Solution

- Quarterly GDP: $s = 4$ (annual cycle over 4 quarters)
- Monthly retail sales: $s = 12$ (annual cycle over 12 months)
- Weekly restaurant reservations: $s = 7$ (weekly cycle) or $s = 52$ (annual)
- Daily electricity demand: $s = 7$ (weekly pattern) or $s = 365$ (annual)

Note: Some series have multiple seasonal patterns (e.g., daily data may have weekly AND annual cycles).

Example: Monthly Retail Sales Analysis

Scenario

You have 5 years of monthly retail sales data showing clear December peaks and January troughs. Build an appropriate SARIMA model.

Step-by-step Approach

1. **Visual inspection**: Plot shows upward trend + strong December spikes
2. **Seasonal period**: Monthly data with annual pattern $\Rightarrow s = 12$
3. **Transformation**: Consider $\log(Y_t)$ if seasonal amplitude grows with level
4. **Differencing**: Try $(1 - L)(1 - L^{12})Y_t$ – check ACF/PACF
5. **Model selection**: Start with airline model, compare via AIC

Example: ACF/PACF Interpretation for Seasonal Data

Observed Patterns (after differencing)

- ACF: Significant at lags 1, 12; cuts off after lag 1 and lag 12
- PACF: Significant at lags 1, 12, 13; decays at multiples of 12

Interpretation

Regular component: ACF cuts off at 1 $\Rightarrow$ MA(1)
Seasonal component: ACF significant only at lag 12 $\Rightarrow$ seasonal MA(1)
Suggested model: SARIMA$(0, d, 1) \times (0, D, 1)_{12}$ – the airline model!

Alternative check: If PACF showed cutoff at seasonal lags instead of ACF, consider seasonal AR terms.

Example: Python Implementation

Fitting SARIMA in Python

```
from statsmodels.tsa.statespace.sarimax import SARIMAX
import pmdarima as pm

# Manual fit
model = SARIMAX(y, order=(0,1,1), seasonal_order=(0,1,1,12))
results = model.fit()
print(results.summary())

# Automatic selection
auto_model = pm.auto_arima(y, seasonal=True, m=12,
                           start_p=0, max_p=2,
                           start_q=0, max_q=2,
                           d=1, D=1,
                           trace=True)
```

Example: Interpreting SARIMA Output

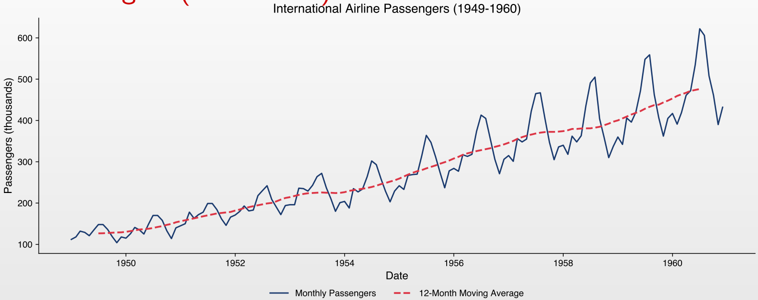
Sample statsmodels Output

```
                        SARIMAX Results
==============================================================
Model:              SARIMAX(0,1,1)x(0,1,1,12)    AIC:     1348.52
                                                  BIC:     1358.21
==============================================================
                 coef     std err        z      P>|z|
--------------------------------------------------------------
ma.L1          -0.4018      0.072    -5.58     0.000
ma.S.L12       -0.5521      0.081    -6.82     0.000
sigma2       1254.3201    142.856     8.78     0.000
```

Interpretation

- $\hat{\theta}_1 = -0.40$: Negative MA means positive shocks reduce next period's value
- $\hat{\Theta}_1 = -0.55$: Same-season correlation is captured
- Both coefficients significant ($p < 0.001$); $|\theta|, |\Theta| < 1$ – invertible

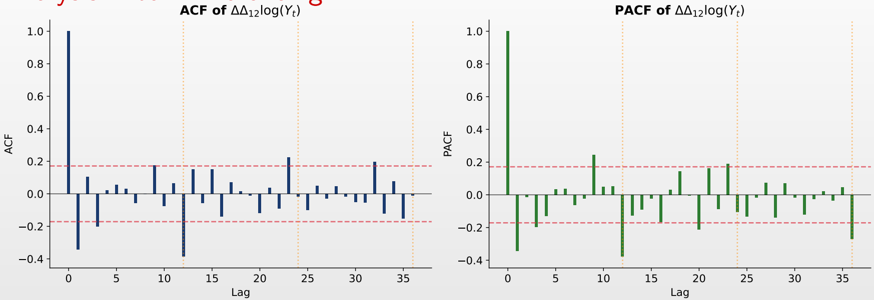
Case Study: Airline Passengers (1949–1960)



- Classic Box-Jenkins dataset: 144 monthly observations
- Clear **upward trend** and **seasonal pattern** (summer peaks)
- Seasonal amplitude **grows with level** $\Rightarrow$ multiplicative seasonality
- Suggests: log transformation + SARIMA modeling

TSA_ch4_airline_data

ACF/PACF Analysis After Differencing



- After $(1 - L)(1 - L^{12})\log(Y_t)$: series appears stationary
- Significant spike at lag 1 in ACF $\Rightarrow$ MA(1) component
- Significant spike at lag 12 in ACF $\Rightarrow$ Seasonal MA(1) component
- Pattern suggests: **SARIMA**$(0, 1, 1)(0, 1, 1)_{12}$ (airline model)

TSA_ch4_acf_pacf

SARIMA Estimation Results: Airline Data

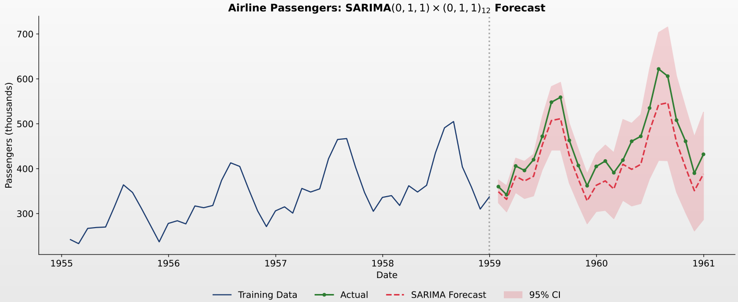
Model: SARIMA$(0,1,1)(0,1,1)_{12}$ on log(Passengers)

Parameter	Estimate	Std. Error	z-stat	p-value
θ_1 (MA.L1)	-0.4018	0.0896	-4.48	< 0.001
Θ_1 (MA.S.L12)	-0.5569	0.0731	-7.62	< 0.001
σ^2	0.00135	–	–	–

Model Fit Statistics

- Log-Likelihood: 244.70
- AIC: -483.40, BIC: -474.53
- Both MA coefficients significant and within invertibility bounds

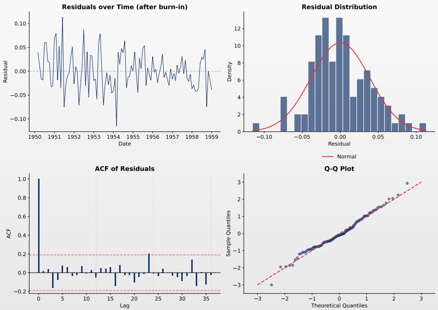
Forecast: 24 Months Ahead



- Forecasts capture both trend and seasonal pattern
- 95% confidence intervals widen over forecast horizon
- Seasonal peaks (July-August) and troughs (February) clearly visible
- Model successfully extrapolates the multiplicative seasonal pattern

TSA_ch4_sarima_forecast

Model Diagnostics



- Residuals appear random with no systematic patterns
- Distribution approximately normal (Q-Q plot close to diagonal)
- ACF of residuals within confidence bounds – no significant autocorrelation
- Ljung-Box test: $p > 0.05$ at all tested lags $\Rightarrow$ adequate model

TSA_ch4_diagnostics

Discussion: Deterministic vs Stochastic Seasonality

Key Question

When should you use seasonal dummies vs SARIMA for seasonal data?

Considerations

Seasonal dummies (deterministic):

- Fixed, repeating pattern each year
- Same December effect every year
- Appropriate when seasonality is stable

SARIMA (stochastic):

- Evolving seasonal pattern
- This year's December depends on last year's December
- Better when seasonal amplitude varies

Discussion: Log Transformation

Key Question

When should you take logarithms before fitting SARIMA?

Guidelines

Use log transformation when:

- Seasonal fluctuations grow with the level (multiplicative seasonality)
- Variance increases over time
- Data is strictly positive (prices, sales, counts)

Avoid log when:

- Seasonal pattern is additive (constant amplitude)
- Data contains zeros or negatives
- Already on a rate/ratio scale

Tip: Compare AIC of models with and without log transformation.

Discussion: Multiple Seasonalities

Challenge

Daily sales data may have both weekly (7-day) and annual (365-day) seasonal patterns. How do you handle this?

Approaches

1. **Nested SARIMA**: Model at shorter frequency, include longer as exogenous
2. **TBATS/BATS models**: Explicitly handle multiple seasonalities
3. **Fourier terms**: Add sin/cos terms for each seasonal frequency
4. **Prophet/similar**: Modern tools designed for multiple seasonalities

Note: Standard SARIMA handles only one seasonal period. For complex seasonality, consider specialized methods.

Discussion: Forecasting Seasonal Data

Key Question

What are the unique challenges of forecasting seasonal time series?

Challenges and Solutions

- **Horizon matters**: 12-month forecast means predicting a full cycle
- **Uncertainty grows**: Seasonal forecasts compound regular uncertainty
- **Turning points**: Capturing when seasons peak/trough
- **Structural breaks**: COVID-19 disrupted many seasonal patterns

Best practices:

- Use rolling-origin cross-validation
- Compare against seasonal naive benchmark
- Report forecast intervals, especially at seasonal horizons

Take-Home Exercises

1. **Theoretical**: Show that $(1-L)(1-L^4)$ can be written as $(1-L-L^4+L^5)$ and explain what this transformation does to quarterly data with annual seasonality.

2. **Computation**: For SARIMA$(1,0,0)\times(1,0,0)_4$ with $\phi_1=0.5$ and $\Phi_1=0.8$, write out the full AR polynomial and identify all non-zero coefficients.

3. **Applied**: Download monthly airline passenger data and:
   - Plot the series and identify trend/seasonality
   - Apply appropriate transformations
   - Fit the airline model and interpret coefficients
   - Generate 24-month forecasts with confidence intervals

4. **Comparison**: Fit both SARIMA$(0,1,1)\times(0,1,1)_{12}$ and SARIMA$(1,1,0)\times(1,1,0)_{12}$ to the airline data. Compare using AIC, BIC, and residual diagnostics. Which is preferred?

Exercise Solutions Hints

Hints

1. Expand $(1-L)(1-L^4) = 1 \cdot 1 - 1 \cdot L^4 - L \cdot 1 + L \cdot L^4 = 1 - L - L^4 + L^5$
2. AR polynomial: $(1-\phi_1 L)(1-\Phi_1 L^4) = 1 - 0.5L - 0.8L^4 + 0.4L^5$
3. For airline data:
   - Use log transformation (multiplicative seasonality)
   - Both $d = 1$ and $D = 1$ needed
   - Typical estimates: $\theta_1 \approx -0.4$, $\Theta_1 \approx -0.6$
4. The MA-based airline model typically fits better than pure AR seasonal model for this data (lower AIC).

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download the classic airline passenger dataset (Box-Jenkins, 1970). Check if seasonality is multiplicative (log transform if so), apply seasonal differencing, fit SARIMA using AIC, check residual ACF at seasonal lags. Compare against a seasonal naive benchmark using rolling 1-step forecasts on the last 24 observations. Show plots and a comparison table with RMSE and MAPE."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Did the AI check for multiplicative seasonality and apply log if needed?
3. Is the seasonal period s correctly identified from the data frequency?
4. Did it test for both regular and seasonal unit roots before choosing d and D?
5. Are residuals free of seasonal patterns (check ACF at lags $s, 2s, 3s$)?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

AI Exercise: Critique an AI SARIMA Analysis

Scenario

You asked an AI: "Fit the best SARIMA model to this monthly retail sales data." It returned:

- Fitted SARIMA$(2,1,2)(2,1,2)_{12}$ with AIC $= 1520$
- No check for multiplicative vs additive seasonality
- Applied $D = 1$ "because the data is monthly"
- Ljung-Box p-value $= 0.03$ on residuals
- 24-month forecast with constant-width confidence intervals

Your critique:

1. Why is the model likely over-parameterized? How many parameters does it have?
2. Why should multiplicative seasonality be checked before fitting?
3. Why is Ljung-Box p $= 0.03$ **not** acceptable at 5% level?
4. Why must SARIMA confidence intervals widen (not stay constant)?

AI Exercise: Prompt Refinement for SARIMA

Task

Iteratively improve prompts for fitting a SARIMA model to monthly airline data.

Round 1 (vague): *"Fit a seasonal model to monthly airline passengers"*

- What did the AI produce? Did it check for multiplicative seasonality?

Round 2 (better): *"Check if seasonality is multiplicative (log transform if so), apply seasonal differencing, fit SARIMA using AIC, check residual ACF at seasonal lags"*

- Did the AI follow the seasonal Box-Jenkins methodology?

Round 3 (expert): *"(1) Plot series, check if amplitude grows with level, (2) if multiplicative, apply log, (3) test ADF on levels & seasonal differences, (4) fit airline model + 2 alternatives, (5) compare AIC/BIC, (6) Ljung-Box on residuals, (7) 24-month rolling forecast with RMSE"*

- Compare results across all three rounds

AI Exercise: Model Selection Competition

Task

Download the classic airline passenger dataset (Box-Jenkins, 1970).

Your approach (manual):

- Check multiplicative vs additive seasonality $\rightarrow$ log if needed
- Determine d and D from ADF/KPSS and seasonal ACF
- ACF/PACF of transformed series $\rightarrow$ candidate models
- Compare airline model vs SARIMA$(1,1,0)(1,1,0)_{12}$ via AIC/BIC
- Rolling 1-step forecast on last 24 observations

AI approach:

- Ask AI to "find the best SARIMA model and forecast airline passengers"

Compare:

- Did the AI apply log transformation? Which model did each select?
- Compare out-of-sample RMSE; did the AI check seasonal residual patterns?
- **Submit:** 1-page reflection on AI strengths and weaknesses

Summary: Chapter 4

Key Concepts

1. **Seasonal differencing**: $(1 - L^s)$ removes stochastic seasonality
2. **SARIMA notation**: $(p, d, q) \times (P, D, Q)_s$ separates regular and seasonal
3. **The airline model**: $\text{SARIMA}(0, 1, 1)(0, 1, 1)_{12}$ is surprisingly effective
4. **Multiplicative structure**: Creates interaction terms between components
5. **ACF/PACF diagnostics**: Patterns at both regular and seasonal lags
6. **Log transformation**: Often needed for multiplicative seasonality

Questions?

Key Formulas Summary

Concept	Formula
Seasonal difference	$(1 - L^s)Y_t = Y_t - Y_{t-s}$
Combined differencing	$(1 - L)(1 - L^s)Y_t = Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}$
SARIMA$(p, d, q)(P, D, Q)_s$	$\phi(L)\Phi(L^s)(1 - L)^d(1 - L^s)^D Y_t = \theta(L)\Theta(L^s)\varepsilon_t$
Airline model	$(1 - L)(1 - L^{12})Y_t = (1 + \theta_1 L)(1 + \Theta_1 L^{12})\varepsilon_t$
Multiplicative AR	$(1 - \phi_1 L)(1 - \Phi_1 L^s) = 1 - \phi_1 L - \Phi_1 L^s + \phi_1 \Phi_1 L^{s+1}$
Multiplicative MA	$(1 + \theta_1 L)(1 + \Theta_1 L^s) = 1 + \theta_1 L + \Theta_1 L^s + \theta_1 \Theta_1 L^{s+1}$
Invertibility	$\lvert\theta_i\rvert < 1$ and $\lvert\Theta_j\rvert < 1$ for all i, j
Stationarity	$\lvert\Phi_j\rvert < 1$ and regular AR roots outside unit circle
Log transform	$\log(T \times S \times \varepsilon) = \log T + \log S + \log \varepsilon$

Notation: s = seasonal period, ϕ/Φ = regular/seasonal AR, θ/Θ = regular/seasonal MA, d/D = regular/seasonal differencing order

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Online Resources and Code

- **Quantlet**: `https://quantlet.com` — Code repository for statistics
- **Quantinar**: `https://quantinar.com` — Quantitative methods learning platform
- **GitHub TSA**: `https://github.com/QuantLet/TSA/tree/main/TSA_ch4` — Python code for this seminar

Thank You!

Questions?

Seminar materials are available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar