Time Series Analysis and Forecasting

Chapter 0: Fundamentals

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Learning Objectives

By the end of this chapter, you will be able to:

1. **Define** time series and distinguish them from cross-sectional and panel data
2. **Decompose** time series into trend-cycle, seasonality, and residual components
3. **Apply** exponential smoothing methods (SES, Holt, Holt-Winters, ETS)
4. **Evaluate** forecasts using MAE, RMSE, MAPE, sMAPE
5. **Implement** train/validation/test splitting and cross-validation
6. **Model** seasonality using dummy variables or Fourier terms
7. **Remove** trend and seasonality through appropriate methods
8. **Distinguish** between deterministic and stochastic trends

Data Sources and Software Tools

Data Sources

- **Yahoo Finance**
  - Stock prices, cryptocurrencies, exchange rates
- **FRED** (Federal Reserve)
  - GDP, unemployment, interest rates
- **Eurostat / INS / BNR**
  - European and Romanian economic data
- **Classic datasets**
  - AirPassengers, Sunspots, CO_2

Python

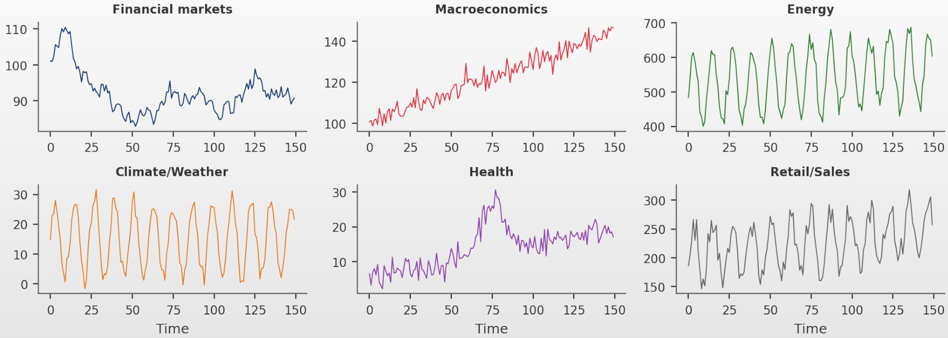
- `yfinance` — Yahoo Finance data
- `pandas_datareader` — FRED, Eurostat
- `statsmodels` — statistical models
- `pandas` — data manipulation
- `matplotlib` — visualization

R

- `quantmod` — financial data
- `tseries` — time series tests
- `forecast` — forecasting models
- `fredr` — FRED API access

Chapter Outline

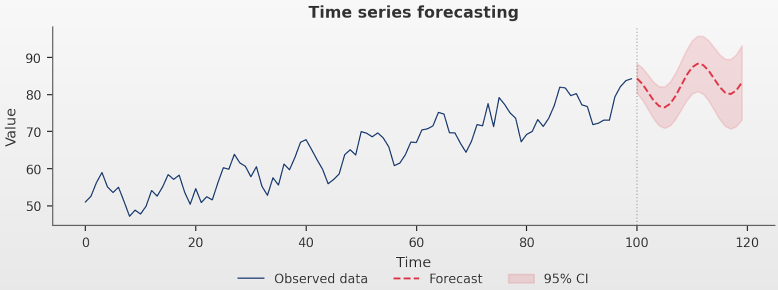
Time Series Are Everywhere



TSA_ch0_real_data

- **Finance**: Stock prices, exchange rates, volumes
- **Economics**: GDP, unemployment, inflation rates
- **Business**: Sales, website traffic, demand
- **Science**: Temperature, pollution, vital signs

Why Do We Study Time Series?



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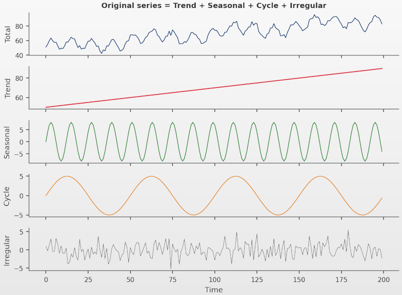
Main objective: forecasting

- We use historical patterns to predict future values $\succ$ essential for business planning, risk management, and policy decisions

Understanding Time Series Structure



TSA ch0 real data

Decomposition

- Any time series can be decomposed into: **trend-cycle** + **seasonality** + **noise**

Definition of a Time Series

Definition 1 (Time Series)

- **Time series**: a sequence of observations $\{X_t\}$ indexed by time: $\{X_t : t \in \mathcal{T}\}$ where $\mathcal{T}$ is a set of indices representing time points

Key Characteristics

- **Ordered**: natural temporal order
- **Dependent**: consecutive observations are correlated
- **Discrete/Continuous**: $t = 1, 2, 3, \ldots$

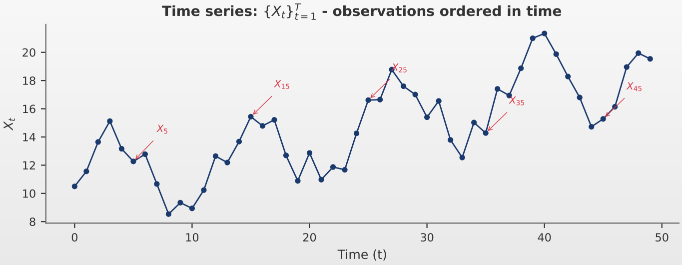
Notation

- X_t: observation at time t
- $\{X_t\}_{t=1}^{T}$: series with T observations

Time Series: Conceptual Illustration

Fundamental Elements

- **Formal notation**
  - X_t = value at time t
  - $t \in \{1, 2, \ldots, T\}$
- **Autocorrelation**
  - $\rho_k = \text{Corr}(X_t, X_{t-k})$
  - Measures temporal dependence



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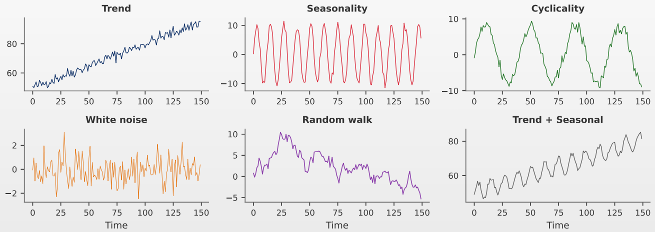
Common Patterns in Time Series

Types of Patterns

- **Trend**
  - Long-term increase or decrease
- **Seasonal**
  - Regular periodic patterns
- **Cyclical**
  - Medium-term fluctuations (2–10 years)
- **Random**
  - Unpredictable fluctuations



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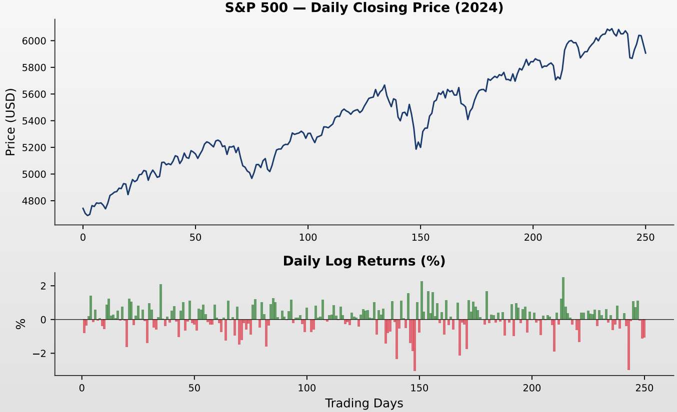
Practical Example: Real Financial Data

S&P 500 (2024)

- **Daily frequency**
  - $\approx$ 252 trading days/year
- **Observed characteristics**
  - Upward trend
  - Volatility clustering
  - Persistence (momentum)



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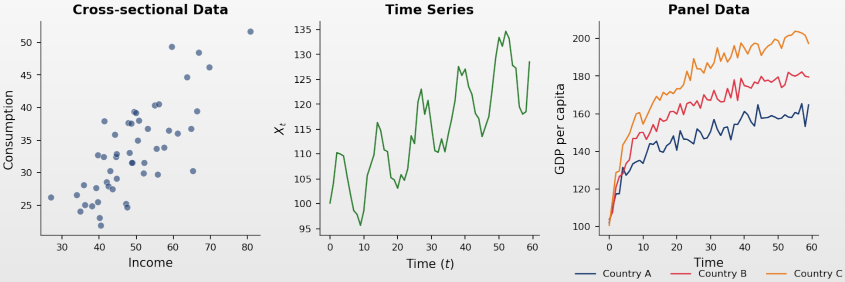
Data Types: Comparison



Data Type	Units (N)	Time (T)	Example
Cross-sectional	Many	1	Survey of 1000 households
Time series	1	Many	Daily S&P 500 prices
Panel	Many	Many	GDP of 50 countries, 20 years

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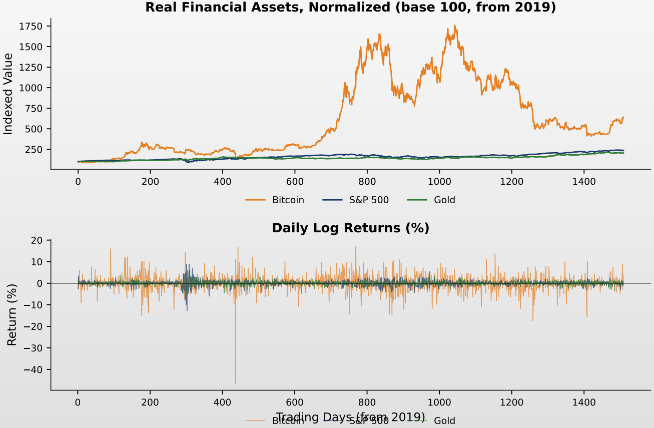
Examples of Time Series Data

Real financial data

- **Source**: Yahoo Finance (2019–2025)
  - Normalized to base 100
- **Bitcoin**: most volatile
- **Gold**: most stable



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Why Do We Decompose a Time Series?

Objectives

- Understanding underlying patterns
- Removing seasonality for modeling
- Identifying trend direction
- Isolating irregular fluctuations
- Improving forecast accuracy

Components

- T_t: Trend-Cycle
  - Long-term movement
- S_t: Seasonal
  - Regular periodic pattern
- ε_t: Residual
  - Random noise

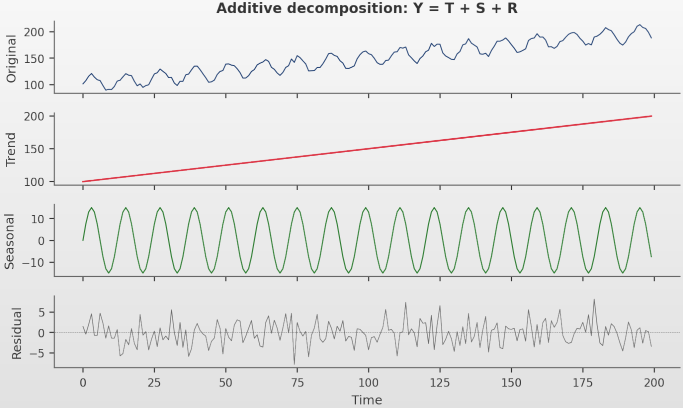
Classical decomposition models

- **Additive**: $X_t = T_t + S_t + \varepsilon_t$
  - Constant seasonal amplitude
- **Multiplicative**: $X_t = T_t \times S_t \times \varepsilon_t$
  - Seasonal amplitude grows with the level

Time Series Decomposition: Visual Example

Components Explained

- **Original**
  - Observed series
- **Trend-Cycle**
  - Long-term movement
- **Seasonal**
  - Periodic pattern
- **Residual**
  - Random noise



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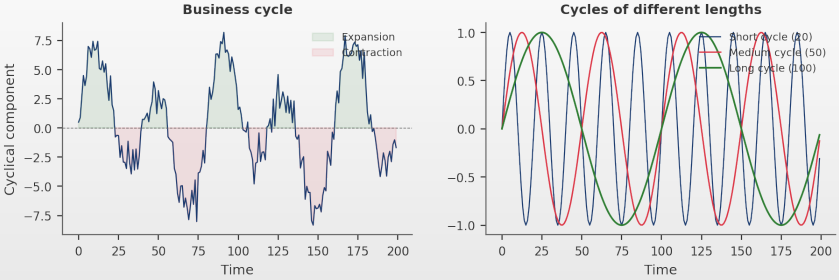
The Cyclical Component



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Characteristics

- **Duration**: medium-term fluctuations (2–10 years)
- **Aperiodic**: no fixed period (vs seasonality)
- **Origin**: reflects business cycles

In Practice

- **Combination**: cycle combined with trend
- **Difficulty**: hard to identify in short series
- **Solution**: usually absorbed into trend-cycle

The Additive Decomposition Model

Model

- **Equation**: $X_t = T_t + S_t + \varepsilon_t$
  - Components are added together to form the observed series

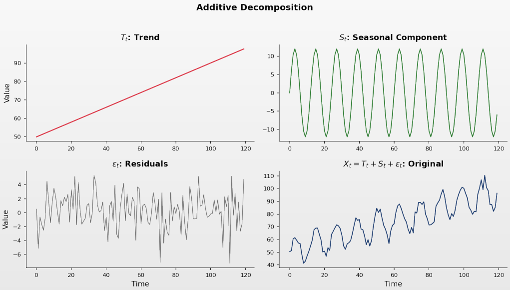
When to Use

- **Constant seasonal fluctuations**
  - Amplitude does not depend on the level
- **Stable series variance**
  - Measures dispersion around the mean
  - Estimator: $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Properties

- **Error**: $\mathbb{E}[\varepsilon_t] = 0$ (zero mean)
- **Seasonal**: $\sum_{j=1}^{s} S_j = 0$ (seasonal sum is zero)
- **Units**: S_t are the same as X_t

Additive Decomposition: Visualization



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Interpretation

- **Decomposition**: Original = Trend + Seasonal + Residual
- **Property**: constant seasonal amplitude, does not depend on the level

The Multiplicative Decomposition Model

Model

- **Equation**: $X_t = T_t \times S_t \times \varepsilon_t \succ$ components are multiplied

When to Use

- **Growing fluctuations**: seasonality increases with the level
- **Heteroscedasticity**: variance increases over time
- **Examples**: economic/financial data

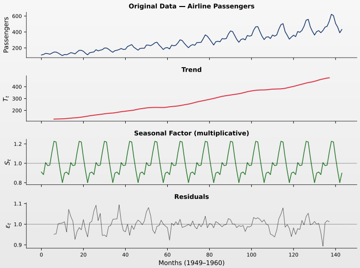
Properties

- **Error**: $\mathbb{E}[\varepsilon_t] = 1$ (centered at 1)
- **Seasonal**: $\frac{1}{s}\sum_{j=1}^{s} S_j = 1$ (mean is 1)
- **Units**: S_t is a dimensionless ratio

Tip

- **Log transformation**: multiplicative $\succ$ additive: $\log X_t = \log T_t + \log S_t + \log \varepsilon_t$

Multiplicative Decomposition: Real Data



Example

- Box-Jenkins data: monthly passengers (1949–1960). Seasonal amplitude increases with the level

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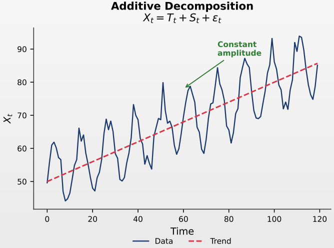
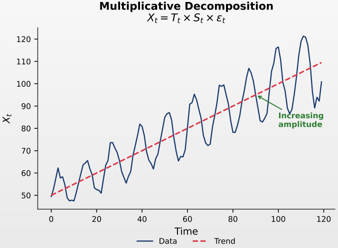
Additive vs Multiplicative: Comparison

Key Difference

- **Multiplicative**
  - Seasonal component is a *ratio*
  - Centered at value 1
- **Additive**
  - Seasonal component in *absolute units*
  - Centered at value 0





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Trend Estimation: Moving Average

Definition 2 (Centered Moving Average)

- **Centered moving average** of order $2q+1$:

$$\hat{T}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t+j} \tag{1}$$

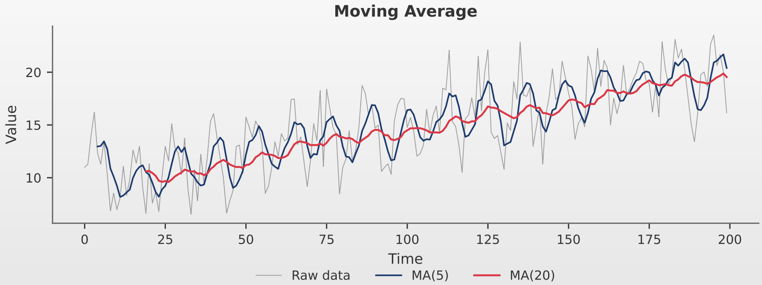
For Seasonal Data

- **Odd period** s
  - Use simple average
- **Even period** s
  - $2 \times s$ MA with half weights

Properties

- **Smoothing**: removes seasonal & random components
- **Large window** $\succ$ smoother estimate
- **Disadvantage**: data loss at endpoints

Centered Moving Average: Visual Illustration



Interpretation

- **Smoothing**: removes short-term fluctuations
- **Result**: reveals the underlying trend

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Classical Decomposition Algorithm

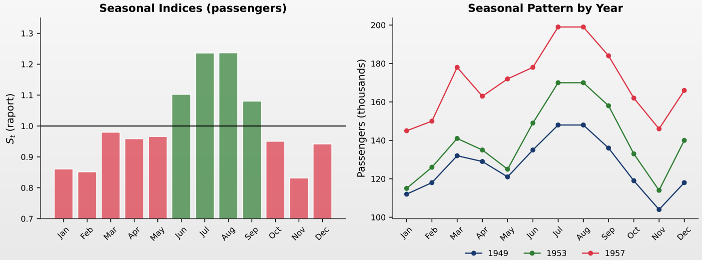
Steps for Multiplicative Decomposition

- **Step 1** $\succ$ **Estimate Trend**: $\hat{T}_t = MA_s(X_t)$
  - Centered moving average of order equal to the seasonal period
- **Step 2** $\succ$ **Detrend**: $D_t = X_t / \hat{T}_t$
- **Step 3** $\succ$ **Estimate Seasonal**: $\hat{S}_j = \text{mean}(D_t \text{ for season } j)$
- **Step 4** $\succ$ **Normalize**: scale so that $\frac{1}{s}\sum_{j=1}^{s} \hat{S}_j = 1$
- **Step 5** $\succ$ **Compute Residuals**: $\hat{\varepsilon}_t = X_t / (\hat{T}_t \times \hat{S}_t)$

Note

- **For additive decomposition**: operations change
  - Division $\succ$ subtraction
  - Multiplication $\succ$ addition

Seasonal Indices: Interpretation



Interpretation

- $S_t > 1$: above-average activity; $S_t < 1$: below average. Travel peak in July–August

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STL Decomposition: A Modern Approach

Definition 3 (STL - Seasonal-Trend Decomposition using LOESS)

- **STL**: uses locally weighted regression (LOESS): $X_t = T_t + S_t + R_t$

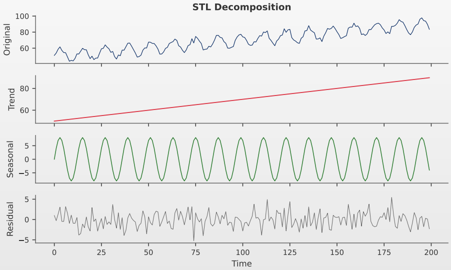
Advantages

- **Flexibility**: any seasonal period
- **Variability**: seasonality can evolve over time
- **Robustness**: resistant to outliers
- **Smoothing**: smooth trend estimates

Key Parameters

- `period`: seasonal period
  - E.g.: 12 for monthly data, 4 for quarterly
- `seasonal`: smoothing window
- `robust`: reduced weight for outliers

STL Decomposition: Visual Illustration



Key Idea

- STL (Seasonal-Trend-Loess): separates trend + seasonal + remainder using LOESS regression

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Exponential Smoothing: Overview

Definition

- **Exponential smoothing**: weighted averages of past observations
  - Weights decrease exponentially over time
  - Recent observations receive higher weights

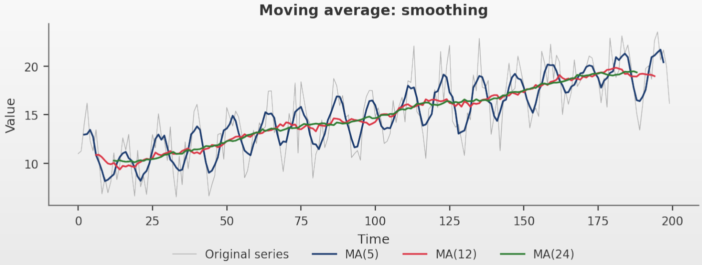
Why Exponential Smoothing?

- **Simple**: easy to implement and understand
  - A single smoothing parameter
- **Adaptive**: higher weights for recent data
- **Versatile**: handles trend and seasonality

Three Main Methods

- **SES** (Simple Exponential Smoothing): level only
  - The simplest exponential method
- **Holt**: level + trend
  - Captures the direction of evolution
- **Holt-Winters**: + seasonality
  - Complete model with all components

Moving Average Smoothing



Window Size Trade-off

- **Small window**: reactive but noisy
  - Captures rapid changes, but amplifies noise
- **Large window**: smooth but lagging
  - Removes noise, but reacts slowly

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Simple Exponential Smoothing (SES)

Model

- **Equation**: $\hat{X}_{t+1|t} = \alpha X_t + (1-\alpha)\hat{X}_{t|t-1}$
  - $\alpha \in (0,1)$ is the smoothing parameter

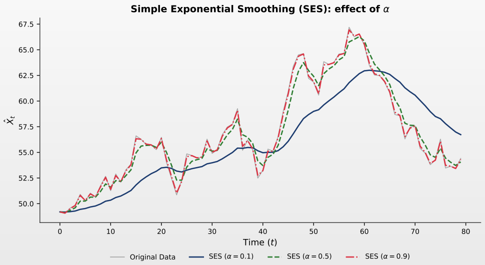
How It Works

- **Principle**: weights decrease exponentially
- **Large** α
  - Forecast reactive to changes
- **Small** α
  - Smoother, more stable forecast

Level Form

- **Equation**: $\ell_t = \alpha X_t + (1-\alpha)\ell_{t-1}$
  - ℓ_t = estimated level at time t
  - Forecast: $\hat{X}_{t+h|t} = \ell_t$ (constant)

Simple Exponential Smoothing: Effect of α



Trade-off

- **Small** $\alpha \succ$ smooth forecasts
  - More weight on distant history
- **Large** $\alpha \succ$ tracks the data
  - Fast reaction to recent changes

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SES: Step-by-Step Numerical Example

Data: Monthly Sales (thousands EUR)

- **Data**: $X_1 = 100,\ X_2 = 110,\ X_3 = 105,\ X_4 = 115,\ X_5 = 120 \quad (\alpha = 0.3,\ \hat{X}_{1|0} = 100)$

Iterative computation: $\hat{X}_{t+1|t} = \alpha X_t + (1-\alpha)\hat{X}_{t|t-1}$

t	X_t	$\hat{X}_{t\|t-1}$	e_t	Computation $\hat{X}_{t+1\|t}$
1	100	100.00	0.00	$0.3 \times 100 + 0.7 \times 100 = 100.00$
2	110	100.00	10.00	$0.3 \times 110 + 0.7 \times 100 = 103.00$
3	105	103.00	2.00	$0.3 \times 105 + 0.7 \times 103 = 103.60$
4	115	103.60	11.40	$0.3 \times 115 + 0.7 \times 103.6 = 107.02$
5	120	107.02	12.98	$0.3 \times 120 + 0.7 \times 107.02 = 110.91$

Forecast and Evaluation

$\hat{X}_{6|5} = 110.91$ MAE $= 7.28$ RMSE $= 8.97$

Holt's Linear Trend Method

Equations

- **Level**: $\ell_t = \alpha X_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
- **Trend**: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
- **Forecast**: $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t$
  - Extrapolates the linear trend over h steps

Parameters

- α: level smoothing
  - Controls reactivity to level changes
- β^*: trend smoothing
  - Controls reactivity to slope changes

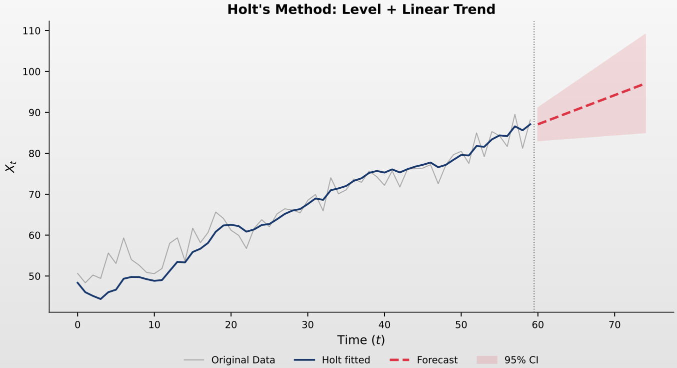
Components

- ℓ_t: estimated level
  - Local mean of the series
- b_t: estimated trend (slope)
  - Rate of increase/decrease

Holt's Method: Visualization

Interpretation

- **Holt's method**: captures level and trend
  - Projects them into the forecast horizon
- α: controls level changes
- β^*: controls trend changes



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Holt-Winters Seasonal Method

Equations (Additive Seasonality)

- **Level**: $\ell_t = \alpha(X_t - S_{t-s}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$
- **Trend**: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
- **Seasonal**: $S_t = \gamma(X_t - \ell_t) + (1-\gamma)S_{t-s}$
- **Forecast**: $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t + S_{t+h-s(k+1)}$
  - Where $k = \lfloor (h-1)/s \rfloor$

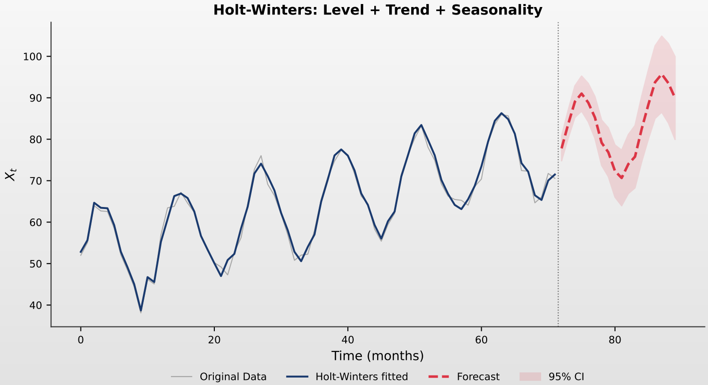
Parameters

- α — level
- β^* — trend
- γ — seasonal
- s — seasonal period
  - All in $(0,1)$; estimated by minimizing error

Holt-Winters: Capturing Seasonality

Key Feature

- **Complete decomposition**
  - Separates level, trend, and seasonal
- **Seasonal forecasts**
  - Includes both trend and periodic pattern



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The ETS Framework: Error-Trend-Seasonality

Definition 4 (ETS Models)

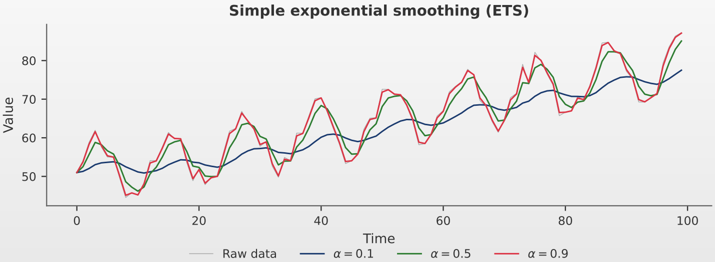
- **ETS framework**: generalizes exponential smoothing: ETS(E, T, S)

Component	**N**	**A**	**M**
Error (E)	–	Additive	Multiplicative
Trend (T)	None	Additive	Multiplicative
Seasonal (S)	None	Additive	Multiplicative

Examples

- **ETS(A,N,N)**: Simple Exponential Smoothing $\succ$ level only, no trend or seasonality
- **ETS(A,A,N)**: Holt's Linear Method $\succ$ level + additive trend
- **ETS(A,A,A)**: Additive Holt-Winters $\succ$ level + trend + additive seasonality

ETS: Exponential Smoothing Illustration



Interpretation

- Exponentially weighted observations: weights decrease with age; recent observations have greater importance

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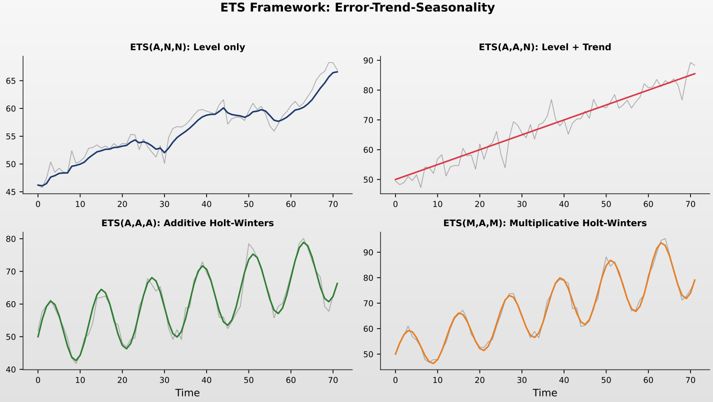
ETS Model Selection

Automatic Selection

- **Information criteria**
  - AIC (Akaike Information Criterion)
  - BIC (Bayesian Information Criterion)
- **Optimal selection**
  - Balance between fit and complexity



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Damped Trend Methods

Damping Parameter

- **Parameter**: $\phi \in (0, 1)$
  - Prevents over-projection of the trend
  - Trend converges to a constant

Equations

- **Level**: $\ell_t = \alpha X_t + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$
- **Trend**: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}$
- **Forecast**: $\hat{X}_{t+h|t} = \ell_t + \phi \frac{1-\phi^h}{1-\phi} b_t$

Key Idea

- **Asymptotic**: as $h \to \infty$, forecast $\to$ constant
  - Prevents unrealistic long-term extrapolation
- **Advantage**: often better for long horizons

Forecast Accuracy Metrics

Forecast Error

- **Definition**: $e_t = X_t - \hat{X}_t$ (actual minus predicted)
  - Positive $\Rightarrow$ underestimates; Negative $\Rightarrow$ overestimates

Scale-dependent

- **MAE**: $\frac{1}{n}\sum |e_t|$
- **MSE**: $\frac{1}{n}\sum e_t^2$
- **RMSE**: $\sqrt{\text{MSE}}$

Scale-independent

- **MAPE**: $\frac{100}{n}\sum \left|\frac{e_t}{X_t}\right|$
- **sMAPE**: $\frac{100}{n}\sum \frac{|e_t|}{(|X_t|+|\hat{X}_t|)/2}$

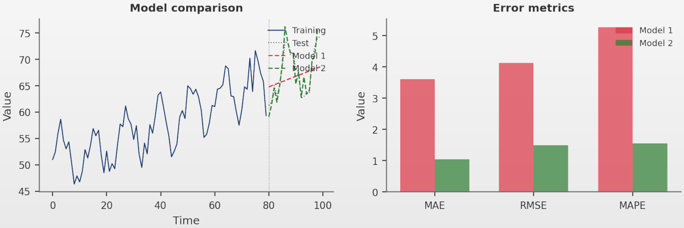
What to Use?

- **Same series**: RMSE, MAE $\succ$ compare models on the same data
- **Across different series**: MAPE, sMAPE $\succ$ percentage metrics, scale-independent

Forecast Evaluation: Visual Example

Observations

- **Top**: actual vs. forecast
  - Visual assessment of forecast quality
- **Bottom**: residuals
  - Zero mean
  - Constant variance
  - No pattern



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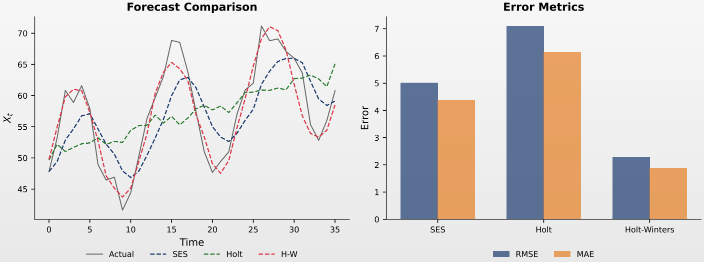
Comparing Forecast Methods



Interpretation

- **Left**: SES, Holt, Holt-Winters forecasts. **Right**: Error metrics. Visual and quantitative comparison

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Residual Diagnostics

Residual Properties

- **Zero mean**: $\mathbb{E}[e_t] = 0$
  - Forecast has no systematic bias
- **Uncorrelated**: $\text{Cov}(e_t, e_{t-k}) = 0$
  - No unexploited information remains
- **Constant variance**: $\text{Var}(e_t) = \sigma^2$
- **Normally distributed**: for confidence intervals

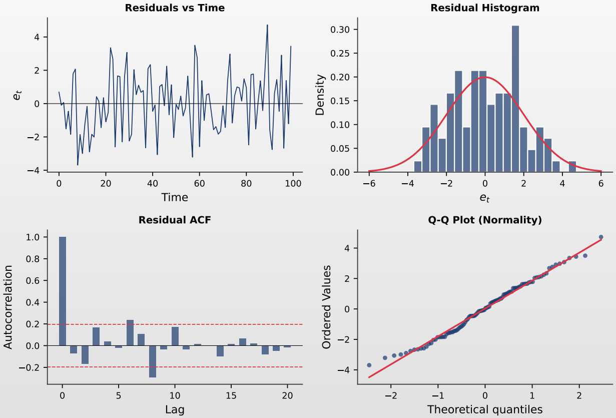
Diagnostic Tests

- **Ljung-Box test** (autocorrelation):
  - $Q = T(T+2)\sum_{k=1}^{h} \frac{\hat{\rho}_k^2}{T-k} \sim \chi_h^2$
- **Jarque-Bera test** (normality):
  - $JB = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right) \sim \chi_2^2$
  - S = skewness, K = kurtosis

Residual Diagnostics: Visualization

What to Check

- **Time plot**: no systematic patterns
- **Histogram**: normality check
- **ACF**: no significant autocorrelation
- **Q-Q plot**: normality confirmation



TSA_ch0_forecast_eval

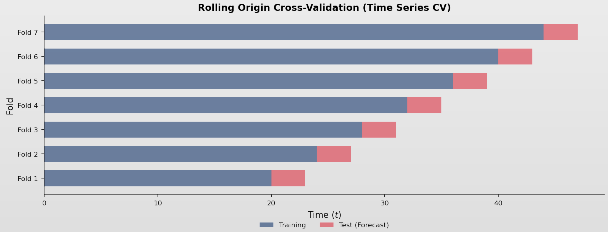
Cross-Validation for Time Series

Why Not Standard CV?

- **Temporal dependence**: observations are correlated
- **Order matters**: chronology must be respected
- **Standard k-fold** $\succ$ data leakage

CV with Rolling Origin

- **Step 1**: train on $\{X_1, \ldots, X_t\}$
- **Step 2**: forecast $\hat{X}_{t+h}$
- **Step 3**: increment t, repeat



TSA_ch0_forecast_eval

Train / Validation / Test Split

Training Set

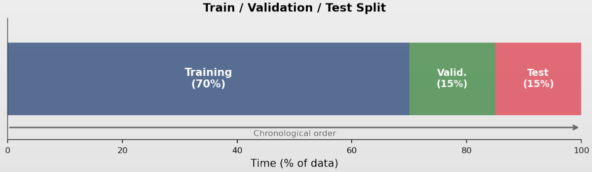
- Fitting model parameters
- Largest portion (60–80%)
- Used for estimation

Validation Set

- Hyperparameter tuning
- Comparing models
- Selecting the best approach

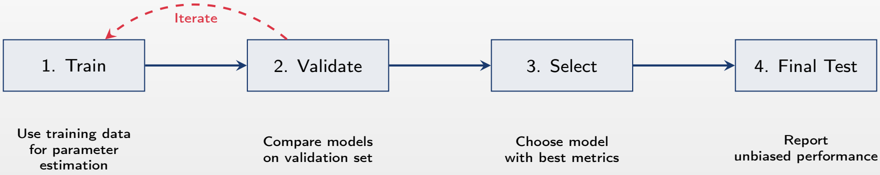
Test Set

- Final evaluation only
- Never used for tuning
- Unbiased performance



TSA_ch0_forecast_eval

Model Development Workflow



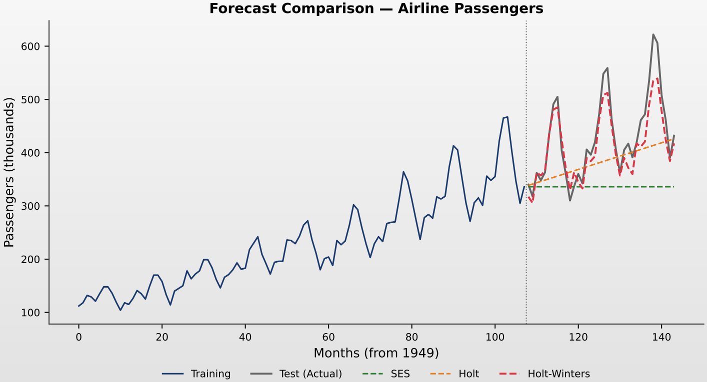
Critical Rule

- **Never use the test set for selection!**
  - Use it only for final evaluation
- **Avoid data leakage**
  - Overly optimistic performance estimates

Real Data: Comparing Forecasts

Interpretation

- **Data**: airline passengers
- **Best**: multiplicative Holt-Winters
  - Ideal for data with growing seasonality

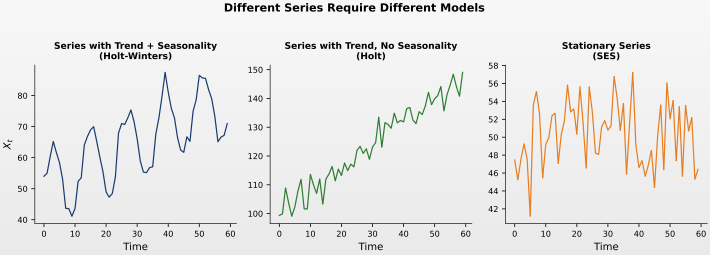


TSA_ch0_forecast_eval

Forecast Performance Across Different Datasets

Interpretation

- **Different series**
  - Require different models
- **Seasonal data**
  - Prefer seasonal methods
- **No universal model**
  - Test multiple approaches



TSA_ch0_forecast_eval

Modeling Seasonality: Two Approaches

1. Dummy Variables

- **Model**: $X_t = \mu + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$
- $D_{jt} = 1$ if t in season j
- $s - 1$ parameters
- Any seasonal pattern

2. Fourier Terms

- **Model**:
  $X_t = \mu + \sum_{k=1}^{K} \left[\alpha_k \sin\left(\frac{2\pi kt}{s}\right) + \beta_k \cos\left(\frac{2\pi kt}{s}\right)\right]$
- Sinusoidal functions
- $2K$ parameters
- Smooth patterns

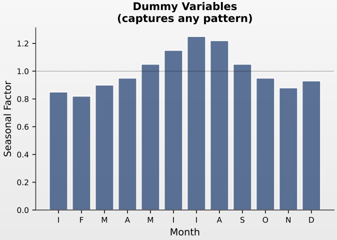
Trade-off

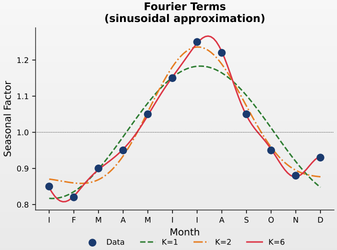
- **Dummy variables**
  - Any seasonal pattern, but more parameters
- **Fourier terms**
  - Smooth patterns, fewer parameters

Dummy Variables vs Fourier Terms

Comparison

- **Dummy variables**
  - Capture any shape
  - Require $s-1$ parameters
- **Fourier terms**
  - Only $2K$ parameters
  - Smooth, sinusoidal patterns





TSA_ch0_seasonal

Choosing Between Dummy and Fourier

Criterion	Dummy	Fourier
Parameters (monthly)	11	$2K$ (often 4–6)
Seasonal pattern	Any shape	Smooth/sinusoidal
Interpretation	Direct (monthly effects)	Frequency components
High-frequency seasons	Many parameters	Efficient
Multiple seasonality	Complex	Easy (add terms)

Recommendations

- **Use Dummy**
  - Irregular patterns, interpretable coefficients
- **Use Fourier**
  - Smooth patterns, high-frequency seasonality
  - Used in TBATS and Prophet

Why Do We Remove Trend and Seasonality?

Reasons for Detrending

- Stationarity requirement
- Focus on fluctuations
- Avoiding spurious regression
- Enabling valid inference

Reasons for Deseasonalizing

- Revealing the underlying trend
- Cross-season comparisons
- Simplifying modeling
- Focus on the irregular component

Important

- **We model the transformed series**
  - With trend and seasonality removed
- **We reverse the transformation**
  - Bring the forecast back to the original scale

Detrending Methods

Six Common Detrending Approaches

- **Differencing**: $\Delta X_t = X_t - X_{t-1}$
  - Most commonly used, removes stochastic trend
- **Linear regression**: $\hat{T}_t = \hat{\beta}_0 + \hat{\beta}_1 t$
- **Polynomial**: higher-order polynomial
- **HP filter**: balance between fit and smoothness
- **Moving average**: $\hat{T}_t = MA_q(X_t)$
- **LOESS**: local polynomial regression

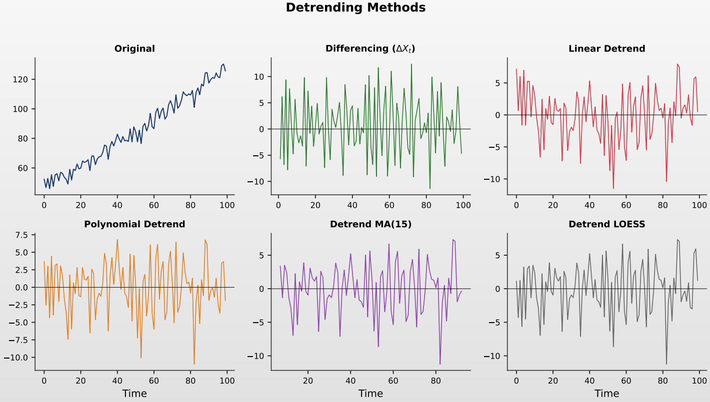
The Choice Depends on

- **Nature of the trend**
  - Deterministic vs stochastic
- **Purpose of the analysis**
  - Forecasting vs descriptive analysis

Detrending Methods: Comparison

Key Idea

- **Different methods**
  - Produce different residuals
- **Choose by trend type**
  - Consider the analysis objectives

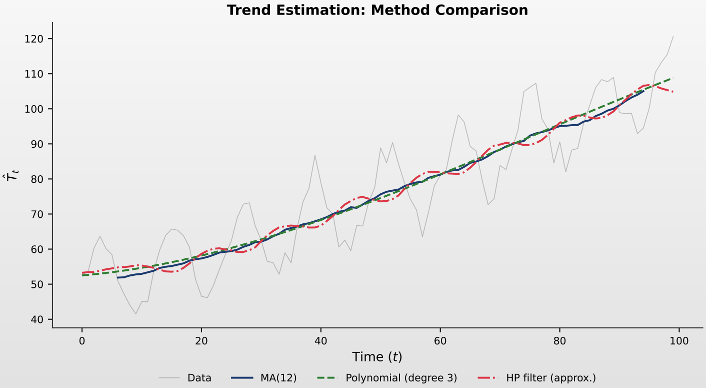


TSA_ch0_detrending

Trend Estimation: Multiple Approaches

Method Comparison

- **Moving average**
  - Simple but with lag
- **Polynomial regression**
  - Flexible, parametric
- **HP filter**
  - Macroeconomic standard



TSA_ch0_detrending

The Hodrick-Prescott (HP) Filter

Definition 5 (HP Filter)

- **HP filter**: decomposes X_t into trend τ_t and cycle c_t: $X_t = \tau_t + c_t$

$$\min_{\{\tau_t\}} \left\{ \sum_{t=1}^{T} (X_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right\}$$

Interpretation

- **First term**
  - Goodness of fit
- **Second term**
  - Smoothness penalty
- λ
  - Controls the balance between fidelity and smoothness

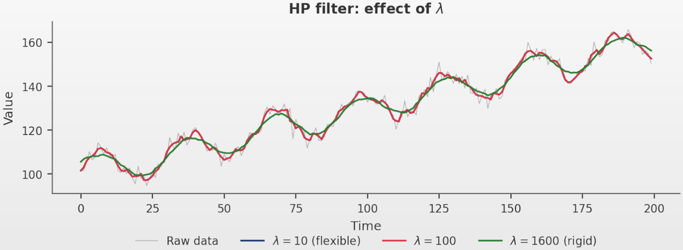
Standard λ Values (Ravn-Uhlig)

- **Annual**
  - $\lambda = 6.25$
- **Quarterly**
  - $\lambda = 1600$ (macroeconomic standard)
- **Monthly**
  - $\lambda = 129600$

HP Filter: Effect of λ

Trade-off

- **Small** λ: flexible trend
  - Follows the data closely
- **Large** λ: smooth trend
  - Approaches a linear trend

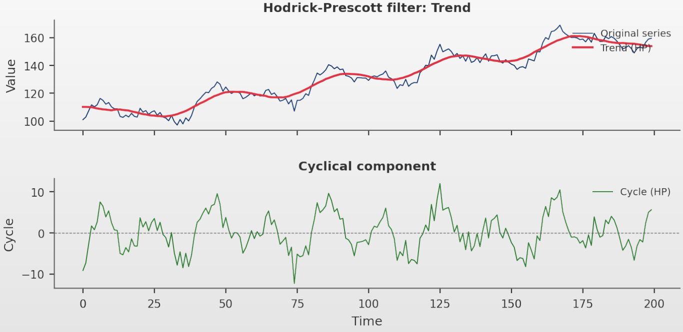


TSA_ch0_detrending

HP Filter: Business Cycle Extraction

Application

- **Macroeconomics**
  - Business cycle extraction
- **Common series**
  - GDP, unemployment, inflation



TSA_ch0_detrending

HP Filter: Limitations

Known Issues

- **Endpoint instability**
  - Trend estimates unreliable at the beginning and end
- **Spurious cycles**
  - Can create artificial dynamics
- **Choice of λ**
  - Results sensitive to the parameter

Alternatives

- **Band-pass filters**: Baxter-King, Christiano-Fitzgerald
  - Isolate specific frequencies
- **Hamilton filter**: regression-based
- **Unobserved components**: state-space models

Hamilton's Critique (2018)

- "Why You Should Never Use the Hodrick-Prescott Filter"
  - Suggests using regression on lagged values

Seasonal Adjustment Methods

Four Approaches for Seasonal Adjustment

- **Seasonal differencing**: $\Delta_s X_t = X_t - X_{t-s}$
  - Removes periodic pattern, simple to apply
- **Division** (multiplicative): $X_t^{adj} = X_t / \hat{S}_t$
- **Subtraction** (additive): $X_t^{adj} = X_t - \hat{S}_t$
- **X-13ARIMA-SEATS**: official US Census Bureau standard
  - Sophisticated method, used by statistical institutes

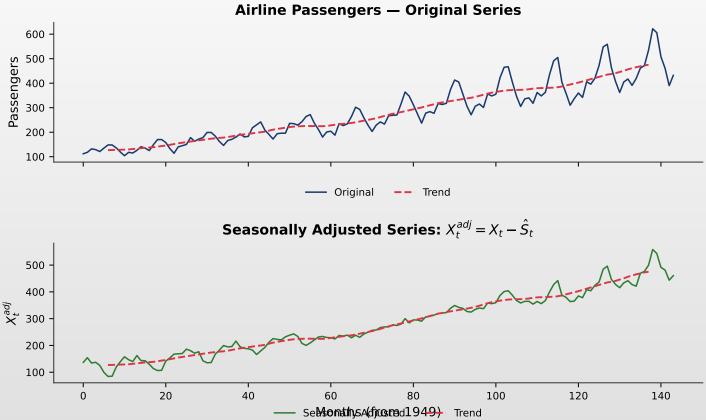
Seasonal Period s

- Monthly: $s = 12$ | Quarterly: $s = 4$

Seasonal Adjustment: Visualization

Result

- **Seasonally adjusted series**
  - Reveals the underlying trend
  - Removes periodic fluctuations



TSA_ch0_seasonal

Deterministic vs Stochastic Trend

Deterministic Trend

- **Model**: $X_t = \beta_0 + \beta_1 t + \varepsilon_t$
- **Characteristics**:
  - Trend is a function of time
  - ε_t is stationary
- **Method**: detrend by regression

Stochastic Trend

- **Model**: $X_t = X_{t-1} + \varepsilon_t$
- **Characteristics**:
  - Random walk component
  - ΔX_t is stationary
- **Method**: detrend by differencing

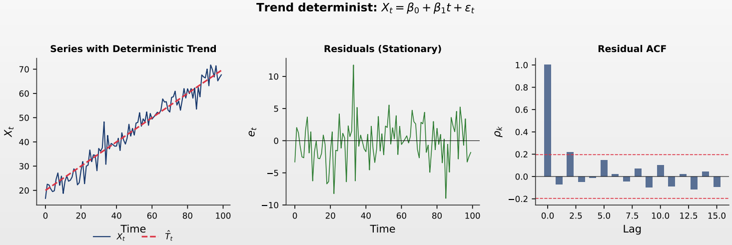
Wrong Method = Problems

- **Differencing a deterministic trend** $\succ$ over-differencing
  - Introduces artificial dependence in the series
- **Regression on a stochastic trend** $\succ$ spurious regression
  - Invalid statistical results

Example: Deterministic Trend

Key

- **Method**: regression
- **Result**: stationary residuals, ACF decays rapidly

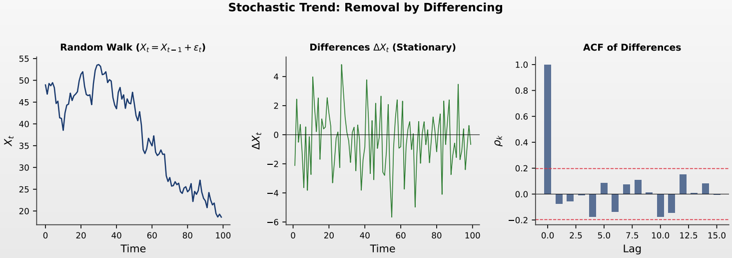


TSA_ch0_detrending

Example: Stochastic Trend (Random Walk)

Key

- **Method**: differencing
- **Result**: differences are stationary (white noise)

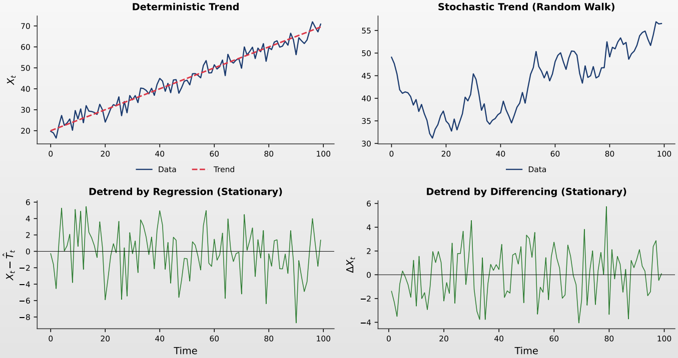


TSA_ch0_detrending

Side-by-Side Comparison

Remember

- **Deterministic trend**: use regression
  - Trend is a predictable function of time
- **Stochastic trend**: use differencing
  - Trend contains a random component



TSA_ch0_detrending

Experiment: ChatGPT vs Fundamentals

Prompt → Response

You: "I have monthly airline passenger data from 1949 to 1960. Analyze it and make a forecast."

ChatGPT: `seasonal_decompose(data, model='additive')`

"Trend extracted. Seasonal pattern identified. MAPE = 4.2%. *The model fits well.*"

Three errors a trained analyst catches immediately:

1. **Wrong decomposition type**: seasonal amplitude $\times 3$ from 1949 to 1960
   $\mathrm{Var}(S_t) \neq$ const $\succ$ additive violated $\succ$ use **multiplicative** or $\ln X_t$ first
2. **In-sample metric is meaningless**: MAPE $= 4.2\%$ is computed on **training data**
   Rolling-origin CV reveals true MAPE $= 8.7\%$ — the model is **twice as bad** as reported
3. **Residuals not checked**: ACF of residuals $\neq 0$ at lags 1–3
   Unexplained systematic pattern remains $\succ$ the decomposition is **misspecified**

Discussion: The code runs without errors. The output looks professional. *How do you know it's wrong?*

Summary

What We Learned in This Chapter

- Time Series Definition and Characteristics
  - Sequence of temporally ordered observations with dependence
- Decomposition (Additive vs Multiplicative)
  - Components: Trend-Cycle + Seasonal + Residual
- Exponential Smoothing Methods
  - SES (level), Holt (+ trend), Holt-Winters (+ seasonality), ETS
- Forecast Evaluation and Validation
  - Metrics: MAE, RMSE, MAPE; Cross-Validation with rolling origin

Key Idea

- **Understand Before Modeling**:
  - Visualize and decompose the data first
  - Choose additive vs multiplicative based on variance behavior

Quick Quiz

Test Your Knowledge

1. What is the difference between additive and multiplicative decomposition?
2. When should you use Holt-Winters instead of SES?
3. Why can't we use standard k-fold CV for time series?
4. What does $\alpha = 0.9$ mean in exponential smoothing?
5. How do you distinguish between a deterministic and stochastic trend?

Quiz Answers

Answers

1. **Additive vs multiplicative**: additive when seasonal amplitude is constant; multiplicative when it grows with the level
2. **Holt-Winters**: when data have trend AND seasonality; SES handles only the level
3. **CV**: standard k-fold ignores temporal order $\succ$ data leakage
4. $\alpha = 0.9$: high weight on recent observations, reacts quickly but more volatile
5. **Trend**: deterministic $\succ$ function of time (regression); stochastic $\succ$ random walk (differencing)

What's Next?

Chapter 1: Stochastic Processes and Stationarity

- **Stochastic Processes**: mathematical foundation, random variables indexed by time
- **Stationarity**: strict (invariant distribution) vs weak (invariant moments)
- **Fundamental Processes**: white noise and random walk $\succ$ building blocks for ARIMA
- **ACF and PACF**: tools for model identification

Questions?

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Online Resources and Code

- **Quantlet**: https://quantlet.com ≻ Code repository for statistics
- **Quantinar**: https://quantinar.com ≻ Learning platform for quantitative methods
- **GitHub TSA_ch0**: https://github.com/QuantLet/TSA/tree/main/TSA_ch0