



# Time Series Analysis and Forecasting

## Chapter 2: ARMA Models



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## Learning Objectives

By the end of this chapter, you will be able to:

1. **Define** and simulate  $AR(p)$ ,  $MA(q)$ , and  $ARMA(p, q)$  processes
2. **Verify** stationarity and invertibility conditions
3. **Identify** orders  $p$  and  $q$  through ACF/PACF analysis
4. **Estimate** parameters via Yule-Walker, MLE, and information criteria (AIC, BIC)
5. **Diagnose** the model through residual analysis and the Ljung-Box test
6. **Forecast** using ARMA models with confidence intervals
7. **Apply** the Box-Jenkins methodology to real data (sunspots)

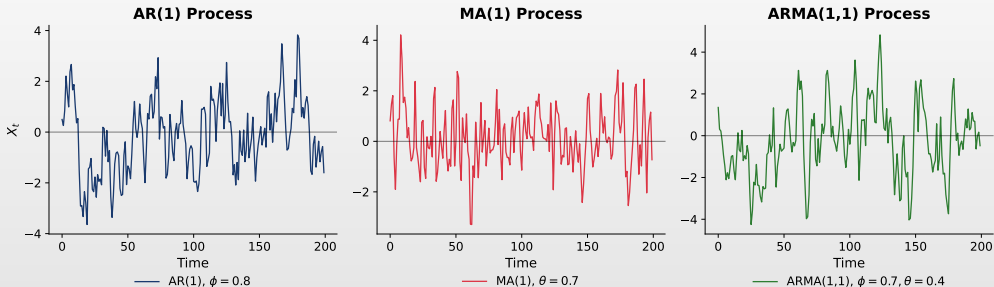
## Chapter Structure

- Motivation
- Introduction and the Lag Operator
- Autoregressive (AR) Models
- Moving Average (MA) Models
- ARMA Models
- Model Identification
- Parameter Estimation
- Model Diagnostics
- Forecasting with ARMA
- Practical Implementation
- Case Study: Real Data
- Summary
- Quiz



## Why ARMA Models?

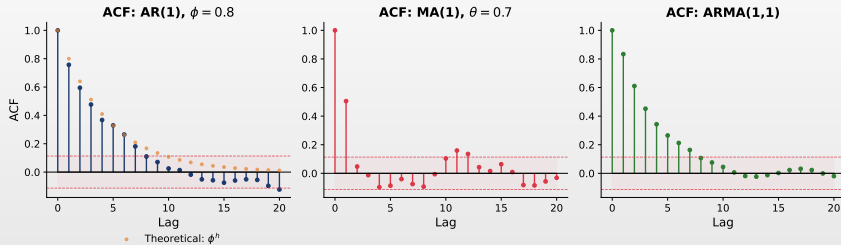
### Stationary processes: AR, MA and ARMA



- ▣ **AR processes:** Current value depends on past values  $\succ$  mean-reverting behavior
- ▣ **MA processes:** Current value depends on past shocks  $\succ$  short memory
- ▣ **ARMA:** Combines both mechanisms for flexible modeling

## Model Identification Through ACF Patterns

Distinct ACF patterns for different models



### ACF Reflects Model Structure

- ▣ **Distinct patterns:** AR: exponential decay; MA: sharp cutoff; ARMA: mixed decay
- ▣ **Identification:** Visual analysis of ACF/PACF guides the selection of orders  $p$  and  $q$

## Recap: Stationarity

### From Chapter 1

- A process  $\{X_t\}$  is **weakly stationary** if:
  1.  $\mathbb{E}[X_t] = \mu$  (constant mean)
  2.  $\text{Var}(X_t) = \sigma^2 < \infty$  (constant, finite variance)
  3.  $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$  (covariance depends only on lag  $h$ )

### Why Stationarity Matters for ARMA

- ARMA models assume stationarity
  - ▶ Parameters remain stable over time
  - ▶ Autocorrelation structure is maintained
- Non-stationary data  $\succ$  difference first (ARIMA, Ch. 3)

### Chapter Objective

- Parametric models for stationary series  $\succ$  combining dependence on past observations (AR) with the influence of random shocks (MA)

## The Lag Operator (Backshift Operator)

### Definition 1 (Lag Operator)

- The **lag operator**  $L$  (or backshift operator  $B$ ) shifts a time series back by one period:  $LX_t = X_{t-1}$

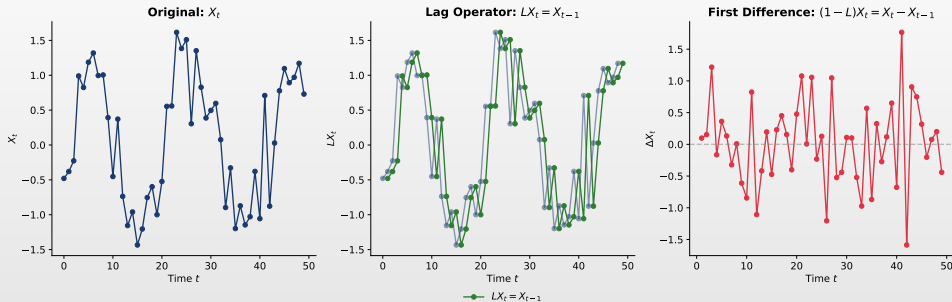
### Properties

- $L^k X_t = X_{t-k}$  (shift back by  $k$  periods)
- $L^0 X_t = X_t$  (identity)
- $(1 - L)X_t = X_t - X_{t-1} = \Delta X_t$  (first difference)
- $(1 - L)^d X_t = \Delta^d X_t$  (difference of order  $d$ )

### Lag Polynomials

- **AR polynomial:**  $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$
- **MA polynomial:**  $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$

## The Lag Operator: Visual Illustration



### Role of the Lag Operator

- ▣ **Notation foundation:** Enables compact writing of difference equations
- ▣ **Utility:** Facilitates algebraic manipulation of ARMA models



## The White Noise Process

### Definition 2 (White Noise)

- A process  $\{\varepsilon_t\}$  is **white noise**, denoted  $\varepsilon_t \sim WN(0, \sigma^2)$ , if:
  1.  $\mathbb{E}[\varepsilon_t] = 0$  for all  $t$
  2.  $\text{Var}(\varepsilon_t) = \sigma^2$  for all  $t$
  3.  $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$  for all  $t \neq s$

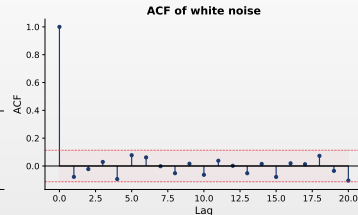
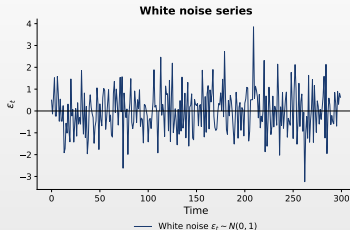
### Properties

- **Building block:** White noise underlies all ARMA models
- **ACF:**  $\rho(0) = 1$ ,  $\rho(h) = 0$  for  $h \neq 0$ ; PACF: same pattern
- **Gaussian white noise:**  $\varepsilon_t \sim N(0, \sigma^2)$  i.i.d.
- **Unpredictable:** White noise is *not* predictable  $\succ$  it is purely random

## White Noise: Visual Illustration

### Key Characteristics

- **Top:** Random fluctuations, no patterns, constant variance
- **Bottom:** ACF only a spike at lag 0; others within significance bounds  $\succ$  no linear dependence



 TSA\_ch2\_white\_noise

## The AR(1) Model: Definition

### Definition 3 (AR(1) Process)

- An **autoregressive process of order 1** is:  $X_t = c + \phi X_{t-1} + \varepsilon_t$
- $\varepsilon_t \sim WN(0, \sigma^2)$  and  $|\phi| < 1$  for stationarity

### Interpretation

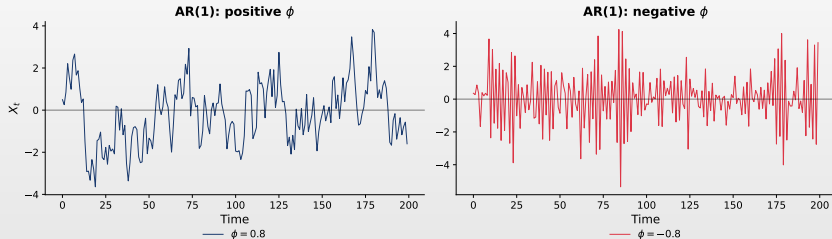
- $c$ : constant (intercept)
- $\phi$ : autoregressive coefficient
  - ▶ Measures the persistence of the series
- $\varepsilon_t$ : innovation (shock)

### Lag Operator Notation

- $(1 - \phi L)X_t = c + \varepsilon_t$
- $\phi(L)X_t = c + \varepsilon_t$
- $\phi(L) = 1 - \phi L$

## AR(1): Visual Illustration

AR(1): different behavior for positive vs negative  $\phi$



### Visual Interpretation

- ▣ **Positive  $\phi$ :** Persistent fluctuations, gradual mean reversion
- ▣ **Negative  $\phi$ :** Oscillating behavior, alternating around the mean
- ▣ **Larger  $|\phi|$   $\succ$  greater persistence, slower reversion**

## AR(1) Stationarity Condition

Necessary and Sufficient Condition:  $|\phi| < 1$

- The root of the characteristic equation must lie outside the unit circle

- Shocks diminish over time
  - ▶ Process reverts to the mean
  - ▶ Finite, stable variance

Non-stationary ( $|\phi| \geq 1$ )

- $|\phi| = 1$ : random walk
  - ▶ Unit root, variance  $\rightarrow \infty$
- $|\phi| > 1$ : explosive process

### Characteristic Equation

- $\phi(z) = 1 - \phi z = 0 \implies z = 1/\phi$
- Stationarity  $\Leftrightarrow$  root outside the unit circle ( $|z| > 1$ )

## AR(1) Properties

### Stationary AR(1) with $|\phi| < 1$

□ Moment properties:

**Mean:**  $\mu = \mathbb{E}[X_t] = \frac{c}{1-\phi}$

**Variance:**  $\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$

**Autocovariance:**  $\gamma(h) = \frac{\phi^h \sigma^2}{1-\phi^2}$

**Autocorrelation (ACF):**  $\rho(h) = \phi^h$

### Key Observation

□ **AR(1) signature:** ACF decays exponentially with factor  $\phi$

- ▶  $\phi > 0$ : monotone decay towards zero
- ▶  $\phi < 0$ : damped oscillations (alternating signs)

## Proof: AR(1) Mean

### Claim

- For AR(1):  $X_t = c + \phi X_{t-1} + \varepsilon_t$ , the mean is  $\mu = \frac{c}{1-\phi}$

### Proof

- Take expectations of both sides:  $\mathbb{E}[X_t] = c + \phi\mathbb{E}[X_{t-1}] + \mathbb{E}[\varepsilon_t]$
- By stationarity,  $\mathbb{E}[X_t] = \mathbb{E}[X_{t-1}] = \mu$ , and  $\mathbb{E}[\varepsilon_t] = 0$ :  $\mu = c + \phi\mu$
- Solving:  $\mu - \phi\mu = c \implies \mu(1 - \phi) = c \implies \boxed{\mu = \frac{c}{1 - \phi}}$

### Requirement

- **Necessary condition:**  $\phi \neq 1$  for the mean to be defined
  - ▶ If  $\phi = 1$  (unit root), the mean is undefined
  - ▶ The process becomes a random walk (non-stationarity)

## Proof: AR(1) Variance

### Claim

$$\square \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$$

### Proof

$\square$  Assume  $c = 0$ . Take the variance of  $X_t = \phi X_{t-1} + \varepsilon_t$ :

$$\square \text{Var}(X_t) = \phi^2 \text{Var}(X_{t-1}) + \text{Var}(\varepsilon_t) + 2\phi \underbrace{\text{Cov}(X_{t-1}, \varepsilon_t)}_{=0}$$

$\square$  By stationarity,  $\text{Var}(X_t) = \text{Var}(X_{t-1}) = \gamma(0)$ :

$$\square \gamma(0) = \phi^2 \gamma(0) + \sigma^2 \implies \gamma(0)(1 - \phi^2) = \sigma^2 \implies \boxed{\gamma(0) = \frac{\sigma^2}{1 - \phi^2}}$$

### Note

$\square$  Requires  $|\phi| < 1$  for positive variance. When  $|\phi| \rightarrow 1$ , variance  $\rightarrow \infty$



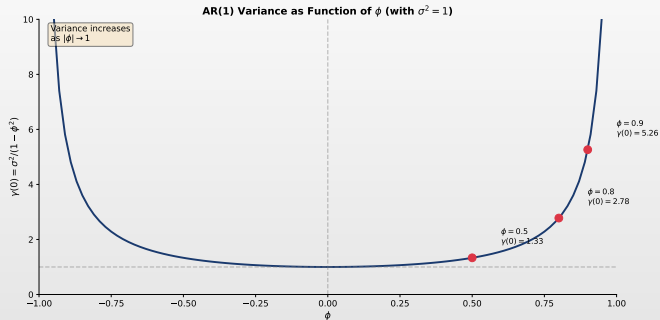
## Proof: AR(1) Autocorrelation Function

Claim:  $\rho(h) = \phi^h$  for  $h \geq 0$

- Find the autocovariance  $\gamma(h) = \text{Cov}(X_t, X_{t-h})$

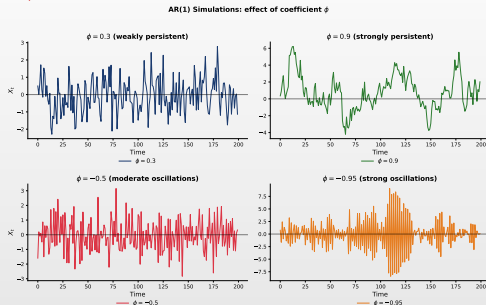
### Proof

- Multiply  $X_t = \phi X_{t-1} + \varepsilon_t$  by  $X_{t-h}$  and take expectations:
- $\mathbb{E}[X_t X_{t-h}] = \phi \mathbb{E}[X_{t-1} X_{t-h}] + \mathbb{E}[\varepsilon_t X_{t-h}]$
- For  $h \geq 1$ :  $\mathbb{E}[\varepsilon_t X_{t-h}] = 0 \succ \gamma(h) = \phi \gamma(h-1)$
- Recursive relation from  $\gamma(0)$ :  $\gamma(1) = \phi \gamma(0)$ ,  $\gamma(2) = \phi^2 \gamma(0)$ , ...  $\gamma(h) = \phi^h \gamma(0)$
- ACF:  $\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\phi^h \gamma(0)}{\gamma(0)} = \boxed{\phi^h}$

AR(1) Variance as a Function of  $\phi$ 

## Observations

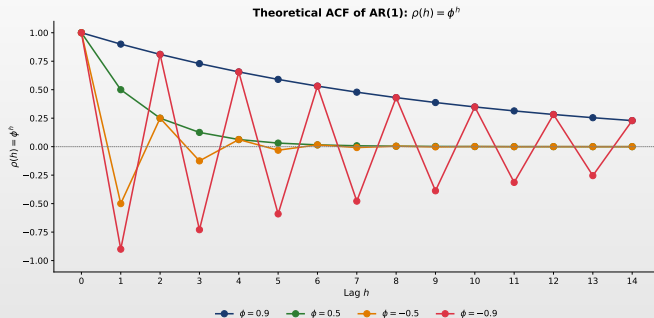
- As  $|\phi| \rightarrow 1$ , the variance explodes  $\succ$  non-stationarity
- For  $\phi = 0$ :  $\gamma(0) = \sigma^2$  (white noise); variance increases monotonically with  $|\phi|$

AR(1) Simulations: Effect of  $\phi$ 

## Interpretation

- Different values of  $\phi$  produce distinct behaviors: larger  $|\phi|$   $\succ$  more persistence
- Positive  $\phi$  creates smooth trajectories; negative  $\phi$  creates oscillations
- As  $|\phi| \rightarrow 1$ , the process approaches non-stationarity

## Theoretical AR(1) ACF



## ACF Pattern

- **Formula:**  $\rho(h) = \phi^h$   $\succ$  exponential decay
- $\phi > 0$ : monotone decay;  $\phi < 0$ : alternating signs

## Proof: AR(1) Stationarity Condition

### Claim

- AR(1) is stationary if and only if  $|\phi| < 1$

### Proof

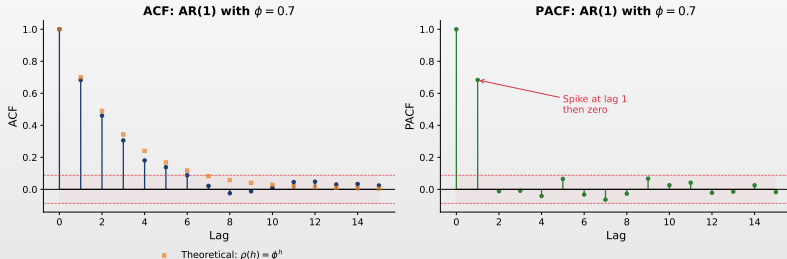
- Recursive substitution:  $X_t = \phi X_{t-1} + \varepsilon_t = \phi(\phi X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \dots$
- After  $n$  steps:  $X_t = \phi^n X_{t-n} + \sum_{j=0}^{n-1} \phi^j \varepsilon_{t-j}$
- If  $|\phi| < 1$ :  $\phi^n \rightarrow 0$  as  $n \rightarrow \infty$ , so  $X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$
- Finite variance:  $\text{Var}(X_t) = \sigma^2 \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma^2}{1-\phi^2} < \infty$  (geometric series)

### Conclusion

- Converges  $\iff |\phi| < 1$ . For  $|\phi| \geq 1$ , the term  $\phi^n X_{t-n}$  does not vanish  $\Rightarrow$  infinite variance

## AR(1) ACF and PACF: Theory vs Sample

ACF and PACF for AR(1): theory vs sample



### Interpretation

- **ACF:** Exponential decay with factor  $\phi$ ; formula:  $\rho(h) = \phi^h$
- **PACF:** A single spike at lag 1, then cuts off  $\succ$  identifies AR(1)
- Sample estimates fluctuate around theoretical values

## The Partial Autocorrelation Function (PACF)

### Definition 4 (PACF)

- The **partial autocorrelation** of order  $k$ , denoted  $\pi_k$ , measures the correlation between  $X_t$  and  $X_{t-k}$  **after removing** the linear effects of the intermediate variables  $X_{t-1}, \dots, X_{t-k+1}$

### Formal Definition

- $\pi_1 = \rho(1)$
- For  $k \geq 2$ :  $\pi_k$  is the last coefficient in:  
$$X_t = \alpha_1 X_{t-1} + \dots + \alpha_k X_{t-k} + e_t$$
- $\pi_k = \alpha_k$  (coefficient of  $X_{t-k}$ )

### Computation via Yule-Walker

- Solve the Yule-Walker equations of order  $k$
- $\pi_k$  = last element of the solution vector

### Utility

- **Identification:** PACF determines the order  $p$  of an AR model
  - PACF cuts off after lag  $p$

## AR(1) ACF and PACF Patterns

### ACF of AR(1)

- Decays exponentially:  $\rho(h) = \phi^h$ 
  - $\phi > 0$ : all positive
  - $\phi < 0$ : alternating signs

### PACF of AR(1)

- Cuts off after lag 1**
  - $\pi_1 = \phi$ ,  $\pi_k = 0$  for  $k > 1$

	ACF	PACF
AR(1)	Exponential decay	Cuts off at lag 1

### Key Pattern

- This is the key identification pattern for AR(1)!



## The AR(p) Model: General Form

### Definition 5 (AR(p) Process)

- An **autoregressive process of order p** is:  $X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + \varepsilon_t$
- **Lag operator**:  $\phi(L)X_t = c + \varepsilon_t$ , where  $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$

### Stationarity Condition

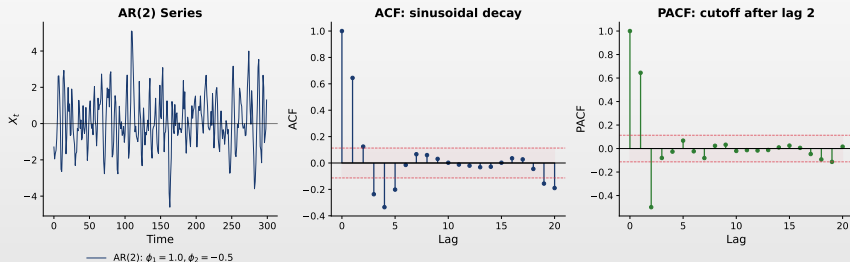
- All roots of  $\phi(z) = 0$  must lie **outside** the unit circle
- Equivalently: all roots have modulus  $> 1$

### PACF Pattern

- PACF cuts off after lag  $p$
- ACF decays (exponentially or with damped oscillations)

## AR(p): Visual Illustration

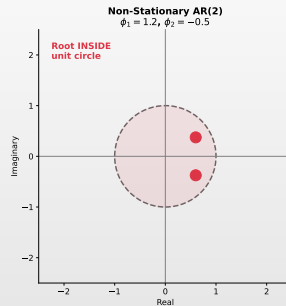
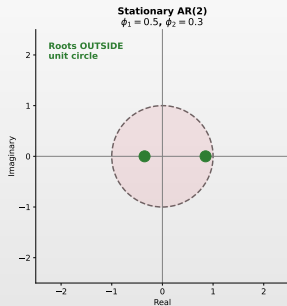
AR(2) Process: pseudo-cyclic behavior



### Observations

- AR(2) can exhibit pseudo-cyclic behavior (complex roots); damped sinusoidal ACF
- PACF cuts off after lag 2  $\succ$  key identification pattern

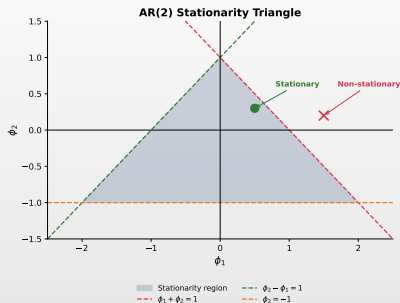
## AR(2) Stationarity: Unit Circle Visualization



### Characteristic Polynomial and Unit Circle Condition

- **Characteristic polynomial** of an  $AR(p)$  process:  $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$
- All roots of  $\phi(z) = 0$  must lie **outside** the unit circle ( $|z| > 1$ )
- Roots on the circle: non-stationary; roots inside: explosive process

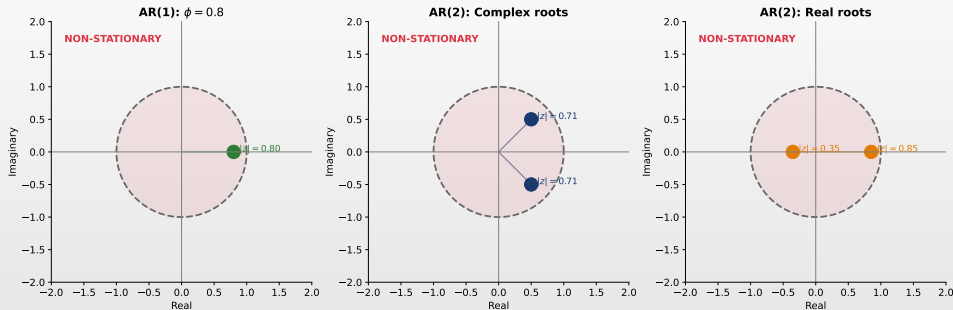
## The AR(2) Stationarity Triangle



### Stationarity Region

- The triangular region defines the stationary AR(2) parameter combinations
- **Boundaries:**  $\phi_1 + \phi_2 < 1$ ,  $\phi_2 - \phi_1 < 1$  and  $|\phi_2| < 1$
- Points outside the region  $\succ$  non-stationary or explosive processes

## Characteristic Polynomial Roots



### Types of Roots

- Real roots: exponential decay in ACF
- Complex roots: damped oscillations (pseudo-cycles)
- All roots must lie outside the unit circle



## The AR(2) Model

### Definition 6 (AR(2) Process)

$$\square X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

### Stationarity Conditions

$$\square \phi_1 + \phi_2 < 1; \quad \phi_2 - \phi_1 < 1; \quad |\phi_2| < 1$$

### ACF Behavior

- $\square$  **Real roots:** mixture of two exponential decays
- $\square$  **Complex roots:** damped sinusoidal pattern (pseudo-cycles)
- $\square$  **PACF:** Cuts off after lag 2 ( $\pi_k = 0$  for  $k > 2$ )

## The MA(1) Model: Definition

### Definition 7 (MA(1) Process)

- ▣ A **moving average process of order 1** is:  $X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$
- ▣  $\varepsilon_t \sim WN(0, \sigma^2)$

### Interpretation

- ▣  $\mu$ : process mean
- ▣  $\theta$ : MA coefficient
  - Measures the impact of the past shock
- ▣ Depends on  $\varepsilon_t$  and  $\varepsilon_{t-1}$

### Lag Operator Notation

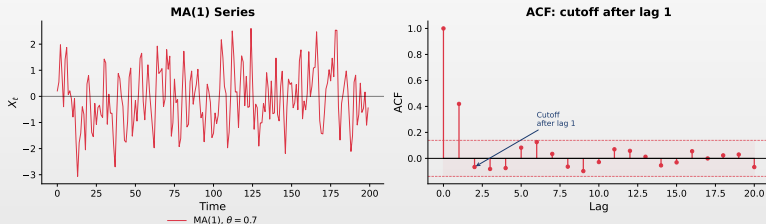
- ▣  $X_t = \mu + \theta(L)\varepsilon_t$
- ▣  $\theta(L) = 1 + \theta L$

### Key Property

- ▣ **Guaranteed stationarity:** MA processes are always stationary
  - Does not depend on the value of  $\theta$

## MA(1): Visual Illustration

MA(1): short memory series with ACF cutoff



### Visual Interpretation

- Left panel: MA(1) series  $\succ$  rapid mean reversion
- Right panel: ACF with **cutoff after lag 1**; PACF exponential decay



## MA(1) Properties

$$\text{MA}(1): X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$$

- **Mean:**  $\mathbb{E}[X_t] = \mu$ ;    **Variance:**  $\gamma(0) = \sigma^2(1 + \theta^2)$
- **Autocovariance:**  $\gamma(1) = \theta\sigma^2$ ,  $\gamma(h) = 0$  ( $h > 1$ )
- **ACF:**  $\rho(1) = \frac{\theta}{1+\theta^2}$ ,  $\rho(h) = 0$  ( $h > 1$ )

### Key Observation

- **MA(1) signature:** ACF cuts off after lag 1
  - ▶  $\rho(1) \neq 0$ , but  $\rho(h) = 0$  for  $h > 1$ ; opposite pattern to AR(1)

## Proof: MA(1) Variance and Autocovariance

Starting point:  $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$  (assuming  $\mu = 0$ )

▣ **Variance:**

$$\gamma(0) = \text{Var}(\varepsilon_t + \theta\varepsilon_{t-1}) = \sigma^2 + \theta^2\sigma^2 + 0 = \boxed{\sigma^2(1 + \theta^2)}$$

Autocovariance at lag 1

$$\square \gamma(1) = \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-1} + \theta\varepsilon_{t-2})$$

$$\square = \text{Cov}(\varepsilon_t, \varepsilon_{t-1}) + \theta\text{Cov}(\varepsilon_t, \varepsilon_{t-2}) + \theta\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-1}) + \theta^2\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-2})$$

$$\square = 0 + 0 + \theta\sigma^2 + 0 = \boxed{\theta\sigma^2}$$

Autocovariance at lag  $h \geq 2$

$$\square \text{ No common } \varepsilon \text{ terms } \succ \gamma(h) = 0$$

## Proof: Maximum ACF for MA(1)

Claim:  $|\rho(1)| \leq 0.5$  for any value of  $\theta$

- ACF at lag 1:  $\rho(1) = \frac{\theta}{1+\theta^2}$
- Differentiate:  $\frac{d\rho(1)}{d\theta} = \frac{1-\theta^2}{(1+\theta^2)^2} = 0 \succ \theta = \pm 1$
- At these values:  $\rho(1)|_{\theta=1} = \frac{1}{2}$ ,  $\rho(1)|_{\theta=-1} = -\frac{1}{2}$

### Implication

- **Practical test:** If  $|\hat{\rho}(1)| > 0.5$  from data, the process is **not** MA(1)
  - ▶ The maximum  $|\rho(1)| = 0.5$  is reached at  $\theta = \pm 1$
  - ▶ Consider AR or ARMA models as alternatives

## Proof: ACF for MA(1)

Claim:  $\rho(1) = \frac{\theta}{1+\theta^2}$  and  $\rho(h) = 0$  for  $h > 1$

- MA(1) has non-zero autocorrelation **only** at lag 1

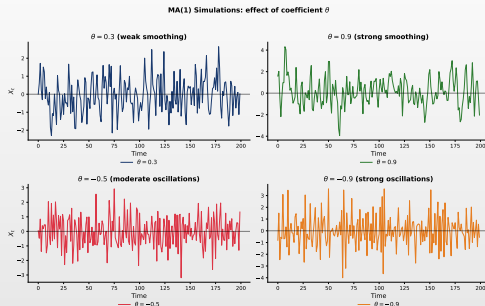
### Proof

- Let  $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ . Autocorrelation at lag 1:
- $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta\sigma^2}{\sigma^2(1+\theta^2)} = \frac{\theta}{1+\theta^2}$
- For  $h > 1$ :  $\gamma(h) = \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-h} + \theta\varepsilon_{t-h-1})$
- The terms  $\varepsilon_t, \varepsilon_{t-1}$  do not overlap with  $\varepsilon_{t-h}, \varepsilon_{t-h-1}$  when  $h > 1$ , so  $\gamma(h) = 0$

### Practical Consequence

- ACF cuts off sharply after lag 1  $\Rightarrow$  distinctive signature of MA(1) processes

## MA(1) Simulations: Effect of $\theta$



### Interpretation

- MA(1) is always stationary regardless of  $\theta$   $\succ$  finite memory of only one lag
- Positive  $\theta$  smooths the series; negative  $\theta$  creates faster fluctuations
- Unlike AR(1), MA(1) shocks affect the process for only one period

## MA(1) ACF and PACF Patterns

### ACF of MA(1)

- ▣ **Cuts off after lag 1**
  - ▶  $\rho(1) = \frac{\theta}{1+\theta^2}$
  - ▶  $\rho(h) = 0$  for  $h > 1$
  - ▶  $|\rho(1)| \leq 0.5$  always

### PACF of MA(1)

- ▣ **Decays exponentially**
  - ▶ Or with alternating signs
  - ▶ Does *not* cut off

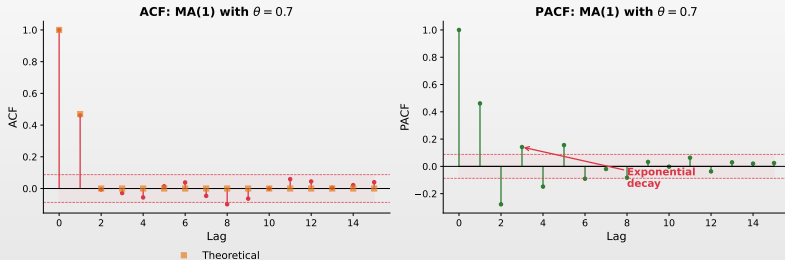
	ACF	PACF
MA(1)	Cuts off at lag 1	Exponential decay

### Observation

- ▣ **Opposite pattern to AR(1)!**

## MA(1) ACF and PACF: Visual Comparison

ACF and PACF for MA(1): opposite pattern to AR(1)



### Interpretation

- **ACF:** A single spike at lag 1, then cuts off  $\succ$  key MA(1) signature
- **PACF:** Exponential decay  $\succ$  opposite pattern to AR(1)
- This reversal differentiates MA processes from AR processes

## Invertibility of MA Models

### Definition 8 (Invertibility)

- ▣ An MA process is **invertible** if it can be written as an infinite AR process:
- ▣  $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$

### Invertibility Conditions

- ▣ **MA(1)**: Invertible if  $|\theta| < 1$
- ▣ **MA(q)**: Roots of  $\theta(z) = 0$  outside the unit circle

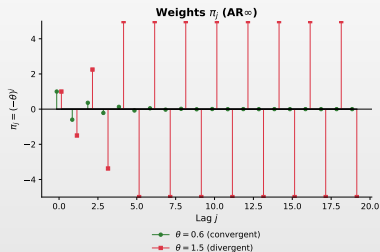
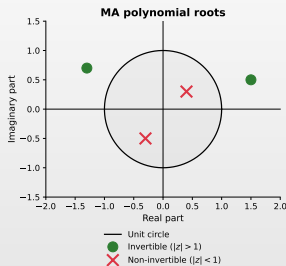
### Why Invertibility Matters

- ▣ Ensures unique representation (without invertibility, multiple MA models describe the same data)
- ▣ Necessary for forecasting and estimation
- ▣ **Stationarity**  $\succ$  AR; **Invertibility**  $\succ$  MA



## Invertibility: Visual Illustration

Invertibility of MA models



### Interpretation

- Left: invertibility requires roots outside the unit circle
- Right:  $AR(\infty)$  weights decay only when  $|\theta| < 1$

## Proof: MA(1) Invertibility

### Claim

- MA(1) is invertible if and only if  $|\theta| < 1$

### Proof

- From  $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ , isolate:  $\varepsilon_t = X_t - \theta\varepsilon_{t-1}$
- Recursive back-substitution:  $\varepsilon_t = X_t - \theta(X_{t-1} - \theta\varepsilon_{t-2}) = X_t - \theta X_{t-1} + \theta^2\varepsilon_{t-2}$
- Continuing:  $\varepsilon_t = \sum_{j=0}^n (-\theta)^j X_{t-j} + (-\theta)^{n+1} \varepsilon_{t-n-1}$
- If  $|\theta| < 1$ :  $(-\theta)^{n+1} \rightarrow 0$ , so

$$\varepsilon_t = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j}$$

### Conclusion

- Geometric series converges  $\iff |\theta| < 1 \Rightarrow$  MA(1) can be written as AR( $\infty$ )

## The MA(q) Model: General Form

### Definition 9 (MA(q) Process)

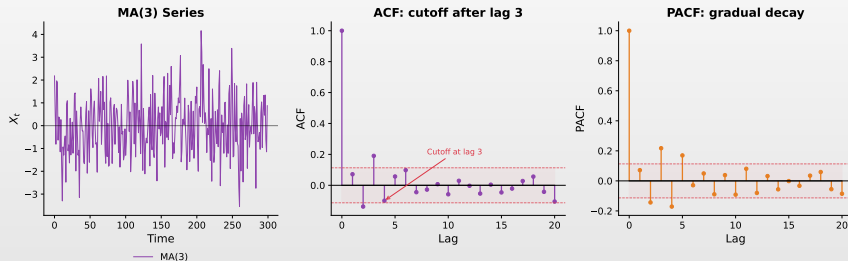
- ▣ A **moving average process of order q**:  $X_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q}$
- ▣ **Lag operator**:  $X_t = \mu + \theta(L)\varepsilon_t$ , where  $\theta(L) = 1 + \theta_1L + \cdots + \theta_qL^q$

### Properties

- ▣ Always stationary (finite variance)
- ▣ ACF cuts off after lag  $q$ :  $\rho(h) = 0$  for  $h > q$ ; PACF decays gradually
- ▣ Invertible if all roots of  $\theta(z) = 0$  lie outside the unit circle

## MA(q): Visual Illustration

MA(q) Process: ACF signature cuts off after lag  $q$



### Observation

- MA(3) process: key signature  $\succ$  ACF cuts off after lag  $q$  ( $\rho(h) = 0$  for  $h > 3$ )

## The ARMA(p,q) Model: Definition

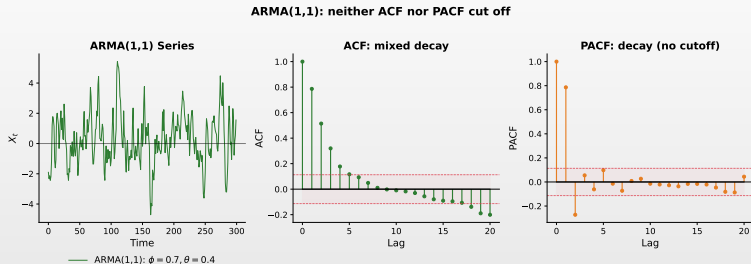
### Definition 10 (ARMA(p,q) Process)

- $X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$
- **Compact form:**  $\phi(L)X_t = c + \theta(L)\varepsilon_t$ , where  $\mu = \frac{c}{1 - \phi_1 - \cdots - \phi_p}$

### Key Idea

- **Flexibility:** Combines AR and MA components
  - ▶ AR captures persistence; MA captures shock response
- **Parsimony:** ARMA(1,1) can be better than AR(5) or MA(5)
  - ▶ Fewer parameters, less risk of overfitting

## ARMA: Visual Illustration



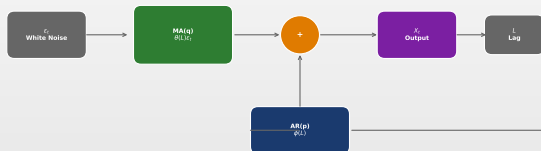
### ARMA(1,1) Interpretation

- ▣ **Combines** AR persistence with MA shock response
- ▣ **ACF pattern:** Decay after the first lag (lags decay geometrically)
- ▣ **PACF pattern:** Also decays (no sharp cutoff as in pure AR)
- ▣ Neither ACF nor PACF cuts off  $\succ$  key identifier for mixed models

## ARMA Model Structure

ARMA(p,q) Model Structure

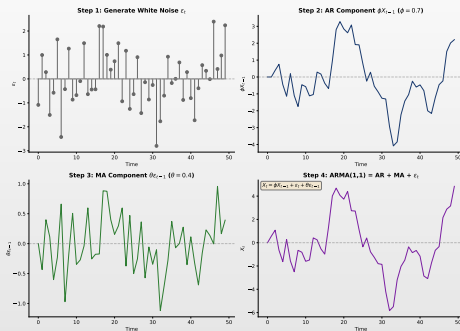
$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$



### Components

- ▣ **AR component:** influence of past values of the series
- ▣ **MA component:** impact of past random shocks

## How ARMA Simulation Works

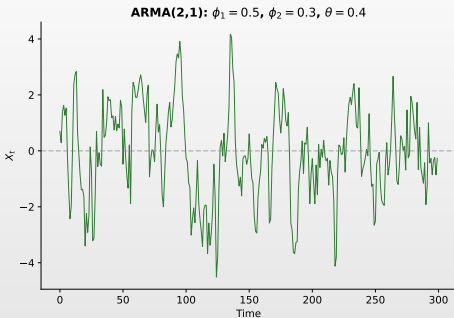
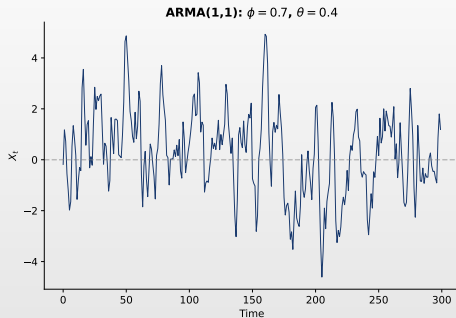


### Steps

- Generate white noise, apply the ARMA equation recursively, obtain simulated series



## ARMA Examples



### Observation

- Different combinations of orders  $(p, q)$  produce distinct behaviors

## The ARMA(1,1) Model

### Definition 11 (ARMA(1,1) Process)

$$\boxed{\cdot} \quad X_t = c + \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

### Properties (stationarity and invertibility)

$$\boxed{\cdot} \quad \text{Mean: } \mu = \frac{c}{1-\phi}; \quad \text{Variance: } \gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$$

### ACF

$$\boxed{\cdot} \quad \rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}; \quad \rho(h) = \phi \cdot \rho(h-1) \text{ for } h \geq 2$$

$\boxed{\cdot}$  ACF decays exponentially after lag 1 (starting point depends on  $\phi$  and  $\theta$ )

## Proof: ARMA(1,1) Variance

### Claim

$$\square \gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$$

### Proof

- Let  $Y_t = X_t - \mu$ :  $Y_t = \phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$
- Square:  $Y_t^2 = \phi^2 Y_{t-1}^2 + \varepsilon_t^2 + \theta^2 \varepsilon_{t-1}^2 + 2\phi Y_{t-1} \varepsilon_t + 2\phi\theta Y_{t-1} \varepsilon_{t-1} + 2\theta \varepsilon_t \varepsilon_{t-1}$
- Take expectations;  $\mathbb{E}[\varepsilon_t Y_{t-1}] = 0$ ,  $\mathbb{E}[\varepsilon_t \varepsilon_{t-1}] = 0$ :
- $\gamma(0) = \phi^2 \gamma(0) + \sigma^2 + \theta^2 \sigma^2 + 2\phi\theta \mathbb{E}[\varepsilon_{t-1} Y_{t-1}]$
- From  $Y_{t-1} = \phi Y_{t-2} + \varepsilon_{t-1} + \theta \varepsilon_{t-2}$ : only  $\varepsilon_{t-1}^2$  contributes  $\Rightarrow \mathbb{E}[\varepsilon_{t-1} Y_{t-1}] = \sigma^2$
- $\gamma(0)(1 - \phi^2) = (1 + 2\phi\theta + \theta^2)\sigma^2 \implies \boxed{\gamma(0) = \frac{(1 + 2\phi\theta + \theta^2)\sigma^2}{1 - \phi^2}}$

## Proof: ARMA(1,1) ACF at Lag 1

### Claim

$$\square \quad \rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}; \quad \rho(h) = \phi \rho(h-1) \text{ for } h \geq 2$$

### Proof

□ Multiply  $Y_t$  by  $Y_{t-1}$  and take expectations:

$$\square \quad \gamma(1) = \phi\gamma(0) + \underbrace{\mathbb{E}[\varepsilon_t Y_{t-1}]}_{=0} + \theta \underbrace{\mathbb{E}[\varepsilon_{t-1} Y_{t-1}]}_{=\sigma^2} = \phi\gamma(0) + \theta\sigma^2$$

□ Divide by  $\gamma(0)$ :  $\rho(1) = \phi + \frac{\theta\sigma^2}{\gamma(0)}$ . Substitute  $\gamma(0)$ :

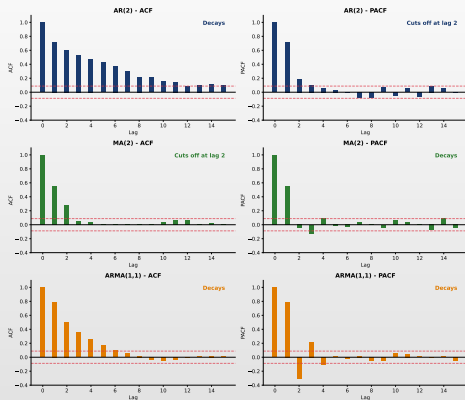
$$\square \quad \rho(1) = \phi + \frac{\theta(1-\phi^2)}{1+2\phi\theta+\theta^2} = \frac{\phi(1+2\phi\theta+\theta^2)+\theta(1-\phi^2)}{1+2\phi\theta+\theta^2}$$

$$\square \quad \text{Numerator: } \phi + \theta + \phi^2\theta + \phi\theta^2 = (\phi + \theta)(1 + \phi\theta), \text{ so } \boxed{\rho(1) = \frac{(1 + \phi\theta)(\phi + \theta)}{1 + 2\phi\theta + \theta^2}}$$

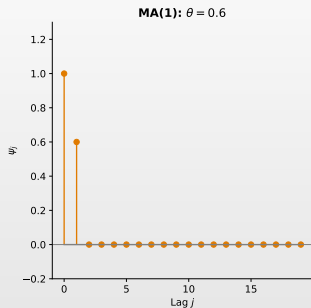
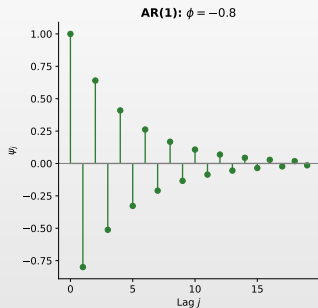
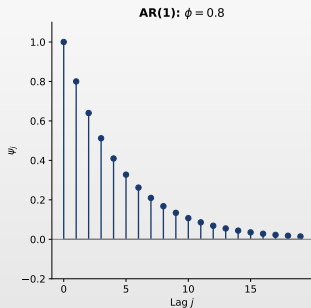
### Recursion

□ For  $h \geq 2$ :  $\gamma(h) = \phi\gamma(h-1)$ , so  $\rho(h) = \phi \rho(h-1) \Rightarrow$  exponential decay from lag 1

## ACF/PACF Patterns: AR vs MA vs ARMA



## Impulse Response Functions



### Shock Propagation

- ▣ Shows how a unit shock propagates through the system over time
- ▣ **AR**: exponential or oscillating decay; **MA**: effect limited to  $q$  periods

## Stationarity and Invertibility Summary

### Conditions for a Valid ARMA(p,q) Model

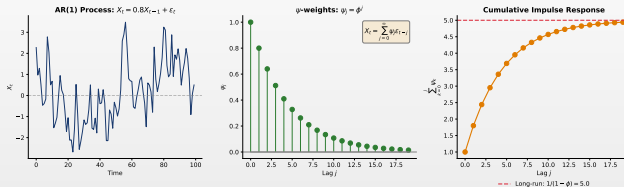
#### ▣ Requirements summary:

Condition	Requirement
Stationarity	Roots of $\phi(z) = 0$ outside the unit circle
Invertibility	Roots of $\theta(z) = 0$ outside the unit circle

### Implications

- ▣ **Stationarity:** Can be written as  $MA(\infty)$ :  $X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- ▣ **Invertibility:** Can be written as  $AR(\infty)$ :  $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$
- ▣ **Causal representation:**  $X_t$  depends only on *past* shocks  $\succ$  necessary for forecasting

## Wold's Decomposition Theorem

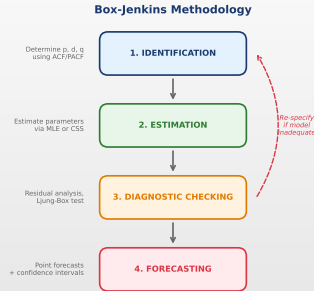


### Wold's Theorem

- Any purely non-deterministic stationary process can be written as MA( $\infty$ ):
- $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$  with  $\sum \psi_j^2 < \infty$
- Theoretical justification for ARMA modeling



# The Box-Jenkins Methodology



## Iterative Approach

□ Identification  $\succ$  estimation  $\succ$  validation; repeat until residuals are white noise

## Model Identification Summary Table

Model Identification: ACF/PACF Patterns

Model	ACF Pattern	PACF Pattern
AR(p)	Exponential decay or damped oscillation	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponential decay or damped oscillation
ARMA(p,q)	Exponential decay after lag q-p	Exponential decay after lag p-q

### Parsimony Principle

- Start simple (small  $p, q$ ), increase order if checks are not satisfied
- Simpler models are preferred



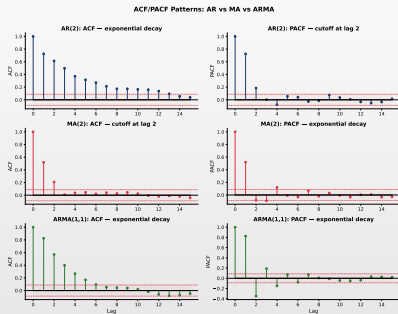
## ACF/PACF Identification Rules

### Theoretical Patterns for Stationary Processes

- The table summarizes ACF/PACF patterns for model identification:

Model	ACF Pattern	PACF Pattern
AR(1)	Exponential decay	Spike at lag 1, then 0
AR(2)	Exp./damped sinusoid	Spikes at lags 1-2, then 0
AR(p)	Gradual decay	Cuts off after lag $p$
MA(1)	Spike at lag 1, then 0	Exponential decay
MA(2)	Spikes at lags 1-2, then 0	Exp./damped sinusoid
MA(q)	Cuts off after lag $q$	Gradual decay
ARMA(p,q)	Decays	Decays

## ACF/PACF Patterns: Visual Guide



### Identification Rules

- **AR**: ACF decays, PACF cuts off  $\succ$  use PACF for order  $p$
- **MA**: ACF cuts off, PACF decays  $\succ$  use ACF for order  $q$
- **ARMA**: Both decay  $\succ$  requires information criteria

## Information Criteria

### AIC (Akaike)

- $AIC = -2 \ln(\hat{L}) + 2k$
- Moderate penalty
  - ▶ Tends to select larger models
  - ▶ Optimal for forecasting

### BIC (Bayesian)

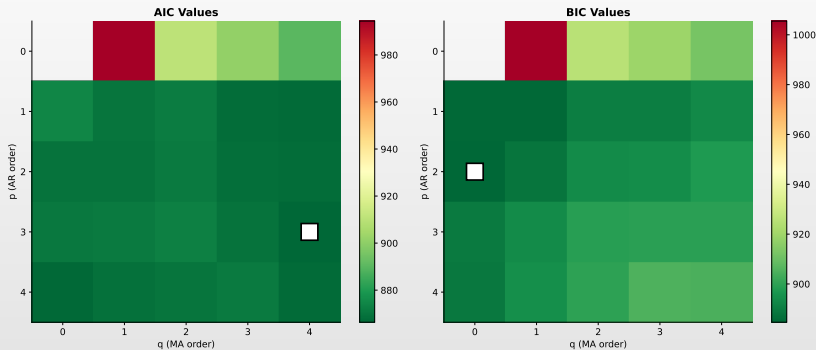
- $BIC = -2 \ln(\hat{L}) + k \ln(n)$
- Stronger penalty
  - ▶ Prefers parsimonious models
  - ▶ Consistent for identification

where:  $\hat{L}$  = maximum of the likelihood function,  $k$  = number of estimated parameters,  $n$  = sample size

### Rules

- Lower values = better model. Compare models on the *same data*

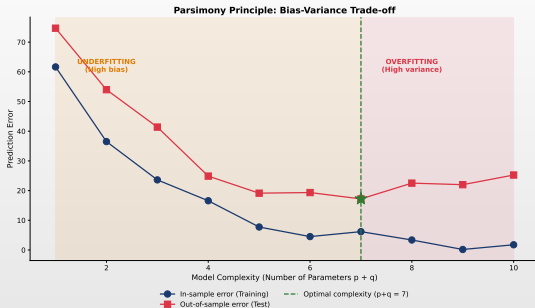
## AIC vs BIC: Model Selection



### Interpretation

- White square marks the best model; lower values (green) are better

## Parsimony Principle: Bias-Variance Trade-off



### Bias-Variance Trade-off

- Too simple model  $\succ$  high bias (underfitting)
- Too complex model  $\succ$  high variance (overfitting)
- The optimum lies at the intersection of the two curves

## Automatic Model Selection

### Grid Search Approach

- ▣ Estimate ARMA( $p, q$ ) for  $p = 0, \dots, p_{max}$  and  $q = 0, \dots, q_{max}$
- ▣ Select the model with the lowest AIC or BIC; verify with validation tests

### In Python

- ▣ `pm.auto_arima()` from the `pmdarima` package
- ▣ Automatically tests stationarity, iterates over orders  $(p, q)$ , returns the best model

### Caution

- ▣ Automatic selection is not the final answer  $\succ$  verify model validity
- ▣ Full Auto-ARIMA (including selection of  $d$ )  $\succ$  Chapter 3



## Estimation Methods Overview

### 1. Method of Moments / Yule-Walker (AR only)

- ▣ Equates sample autocorrelations with theoretical values
- ▣ Simple, closed-form for AR models; not efficient for MA

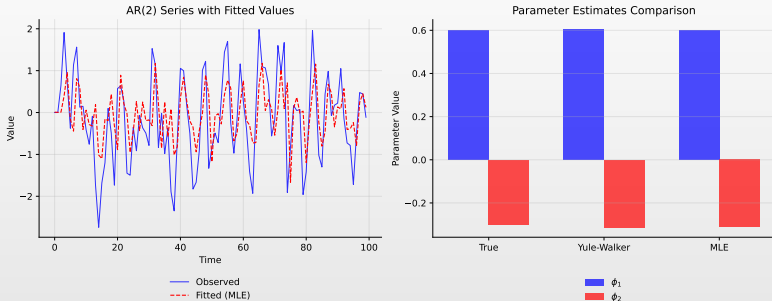
### 2. Maximum Likelihood Estimation (MLE)

- ▣ Most common approach; requires distributional assumption (Gaussian)
- ▣ Efficient and consistent

### 3. Conditional Least Squares

- ▣ Minimizes the sum of squared residuals
- ▣ Conditional on initial observations; algorithmically simpler than exact MLE

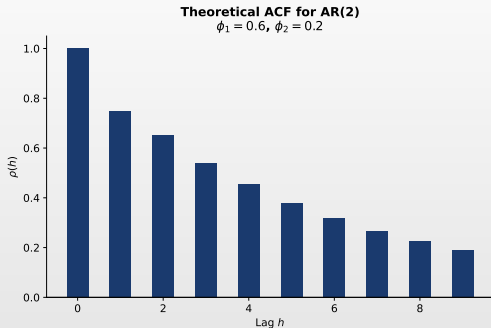
## Estimation Methods Comparison



### Comparison

- **MLE:** most efficient, but requires distributional assumption
- **Yule-Walker:** closed-form, only for AR models
- **CLS:** compromise between MLE and Yule-Walker

## The Yule-Walker Equations for AR(p)



### Yule-Walker Equations

$$\rho(1) = \phi_1 + \phi_2 \rho(1)$$

$$\rho(2) = \phi_1 \rho(1) + \phi_2$$

Matrix form:  $R \cdot \phi = \rho$

$R$  = autocorrelation matrix

$$\text{Solution: } \hat{\phi} = R^{-1} \rho$$

### Main Idea

- Linear relationship between autocorrelations and AR parameters
- Allows closed-form estimation (no numerical optimization)

## The Yule-Walker Equations: Matrix Form

### Yule-Walker Equations for AR(p)

$$\square \quad \rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p), \quad k = 1, 2, \dots, p$$

### Matrix Form

$$\square \quad \begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{pmatrix}$$

$\square$  **Estimation:** Replace  $\rho(k)$  with  $\hat{\rho}(k)$ ; the Toeplitz matrix is symmetric and positive definite

## Numerical Example: Yule-Walker for AR(2)

Sample Data ( $T = 100$ )

▣ **Estimated autocorrelations:**  $\hat{\rho}(1) = 0.75$ ,  $\hat{\rho}(2) = 0.65$

▶ Estimated variance:  $\hat{\gamma}(0) = 4.0$

### Step 1: Matrix System

▣ **Yule-Walker:**  $R\hat{\phi} = \rho$

▶ 
$$\begin{pmatrix} 1 & 0.75 \\ 0.75 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.65 \end{pmatrix}$$

### Step 2: Solution (Cramer's Rule)

▣  $\det(R) = 1 - 0.75^2 = 0.4375$

▣  $\hat{\phi}_1 = \frac{0.75 \times 1 - 0.75 \times 0.65}{0.4375} = \frac{0.2625}{0.4375} = \boxed{0.600}$        $\hat{\phi}_2 = \frac{0.65 \times 1 - 0.75 \times 0.75}{0.4375} = \frac{0.0875}{0.4375} = \boxed{0.200}$

### Step 3: Noise Variance

▣  $\hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\phi}_1\hat{\rho}(1) - \hat{\phi}_2\hat{\rho}(2)) = 4.0(1 - 0.45 - 0.13) = \boxed{1.68}$

**Stationarity check:**  $\hat{\phi}_1 + \hat{\phi}_2 = 0.8 < 1 \checkmark$      $|\hat{\phi}_2| = 0.2 < 1 \checkmark$      $\hat{\phi}_2 - \hat{\phi}_1 = -0.4 > -1 \checkmark$

## Proof: The Yule-Walker Equations

Goal: Derive  $\rho(k) = \phi_1\rho(k-1) + \cdots + \phi_p\rho(k-p)$

- Start from AR(p):  $X_t = \phi_1X_{t-1} + \cdots + \phi_pX_{t-p} + \varepsilon_t$
- Multiply by  $X_{t-k}$  and take expectations:
- $\mathbb{E}[X_tX_{t-k}] = \phi_1\mathbb{E}[X_{t-1}X_{t-k}] + \cdots + \phi_p\mathbb{E}[X_{t-p}X_{t-k}] + \mathbb{E}[\varepsilon_tX_{t-k}]$
- For  $k \geq 1$ :  $\mathbb{E}[\varepsilon_tX_{t-k}] = 0 \succ \gamma(k) = \phi_1\gamma(k-1) + \cdots + \phi_p\gamma(k-p)$
- Dividing by  $\gamma(0)$ :  $\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \cdots + \phi_p\rho(k-p)$

### Special Case AR(1)

- $\rho(k) = \phi_1\rho(k-1) = \phi_1^k$  (using  $\rho(0) = 1$ )

## Maximum Likelihood Estimation

ARMA(p,q) Log-Likelihood (Gaussian errors:  $\varepsilon_t \sim N(0, \sigma^2)$ )

- ▣  $\ell(\phi, \theta, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^n \varepsilon_t^2$
- ▣  $\varepsilon_t$  are the innovations computed recursively

### Estimation Procedure

- ▣ Initialization: use method of moments or OLS for starting values
- ▣ Optimization: numerical methods (BFGS, Newton-Raphson)
- ▣ Iterate until convergence

### In Practice

- ▣ `statsmodels.tsa.arima.model.ARIMA` implements exact MLE with automatic initialization

## Standard Errors and Inference

### Asymptotic Distribution of MLE

- ▣  $\hat{\theta} \xrightarrow{d} N(\theta_0, \frac{1}{n}I(\theta_0)^{-1})$ , where  $I(\theta)$  is the **Fisher information matrix**
- ▣  $I(\theta) = -E\left[\frac{\partial^2 \ln L(\theta)}{\partial \theta \partial \theta'}\right] \succ$  average curvature of the log-likelihood
- ▣ Estimated variance-covariance matrix:  $\hat{V} = \frac{1}{n}\hat{I}^{-1}$

### What is the Standard Error (SE)?

- ▣  $SE(\hat{\theta}_j) = \sqrt{\hat{V}_{jj}} = \sqrt{\text{diag}_j\left(\frac{1}{n}\hat{I}^{-1}\right)} \succ$  measures estimation uncertainty
- ▣ **Example AR(1):**  $SE(\hat{\phi}) \approx \sqrt{(1 - \hat{\phi}^2)/n}$ ; for  $\hat{\phi} = 0.8$ ,  $n = 100$ :  $SE \approx 0.06$
- ▣ **Interpretation:** small SE  $\Rightarrow$  parameter is estimated with high precision

### Testing Parameter Significance

- ▣  $H_0 : \theta_j = 0$     Statistic:  $z = \frac{\hat{\theta}_j}{SE(\hat{\theta}_j)} \sim N(0, 1)$  asymptotically
- ▣ Reject if  $|z| > 1.96$  at 5%     $\Rightarrow$  **CI:**  $\hat{\theta}_j \pm 1.96 \cdot SE(\hat{\theta}_j)$



## Residual Analysis

If the model is correctly specified, residuals must be white noise

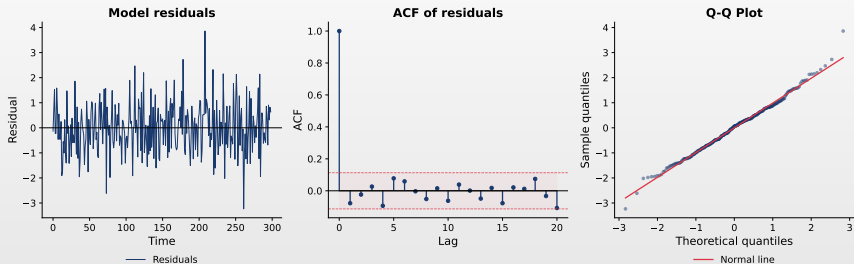
- ▣ **Residual time plot**
  - ▶ Fluctuates around zero, no obvious patterns; constant variance
- ▣ **Residual ACF**
  - ▶ All correlations within significance bounds; no significant spikes  $\succ$  white noise
- ▣ **Histogram / Q-Q plot**
  - ▶ Approximately normal distribution; heavy tails  $\succ$  non-normal errors

## Decision

- ▣ ✓ All checks OK  $\succ$  adequate model
- ▣ × Not satisfied  $\succ$  return to identification

## Residual Diagnostics: Example

AR(1) Model Diagnostics: white noise residuals



### Interpretation

- ▣ **Residual plot:** random fluctuations around zero, constant variance
- ▣ **Residual ACF:** no significant spikes  $\leadsto$  white noise
- ▣ **Q-Q plot:** points on the diagonal  $\leadsto$  normally distributed residuals

## The Ljung-Box Test

### Definition 12 (Ljung-Box Test)

- ▣ Tests whether residuals are independently distributed (no autocorrelation)
- ▣ **Statistic:**  $Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}$

### Hypotheses and Distribution

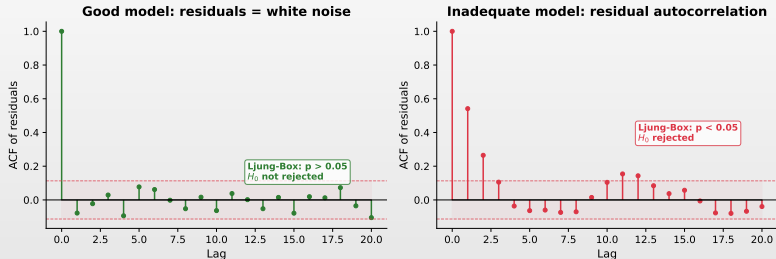
- ▣  $H_0$ : Residuals are white noise;  $H_1$ : Residuals are autocorrelated
- ▣ Under  $H_0$ ,  $Q(m) \sim \chi^2(m-p-q)$  approximately

### Decision

- ▣ **p-value**  $> 0.05$   $\succ$  do not reject  $H_0$   $\succ$  residuals are white noise
- ▣ **p-value**  $< 0.05$   $\succ$  residual autocorrelation  $\succ$  inadequate model

## The Ljung-Box Test: Visual Illustration

Ljung-Box Test: good model vs inadequate model



### Interpretation

- Left: good model  $\gamma$  white noise residuals
- Right: inadequate model  $\gamma$  residual autocorrelation  $\gamma$  re-specification needed

## Model Checklist

### A Good ARMA Model Should Satisfy

- ▣ **Stationarity:** AR roots outside the unit circle (arroots)
- ▣ **Invertibility:** MA roots outside the unit circle (maroots)
- ▣ **White noise residuals:** No significant ACF (Ljung-Box test)
- ▣ **Normal residuals:** Q-Q plot, Jarque-Bera test
- ▣ **No heteroscedasticity:** Constant variance (ARCH test)
- ▣ **Simple:** Lowest AIC/BIC among adequate models

### If Checks Are Not Satisfied

- ▣ Return to identification, try different orders

## Point Forecasts

Optimal Forecast:  $\hat{X}_{n+h|n} = \mathbb{E}[X_{n+h}|X_n, X_{n-1}, \dots]$

- The conditional expectation minimizes MSE

AR(1):  $X_t = c + \phi X_{t-1} + \varepsilon_t$

- $\hat{X}_{n+1|n} = c + \phi X_n$ ;  $\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu)$
- Forecasts converge to the mean  $\mu$  as  $h \rightarrow \infty$  (mean reversion)

MA(1):  $X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

- $\hat{X}_{n+1|n} = \mu + \theta \varepsilon_n$ ;  $\hat{X}_{n+h|n} = \mu$  for  $h > 1$

## Forecast Uncertainty

### Mean Square Forecast Error (MSFE)

- **Error:**  $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- **MSFE:**  $\text{MSFE}(h) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$ , where  $\psi_j$  are the  $\text{MA}(\infty)$  coefficients

For AR(1):  $\psi_j = \phi^j$

- $\text{MSFE}(h) = \sigma^2 \frac{1-\phi^{2h}}{1-\phi^2} \rightarrow \frac{\sigma^2}{1-\phi^2} = \text{Var}(X_t)$

### Key Observation

- Forecast uncertainty increases with the horizon
- Converges to the unconditional variance  $\text{Var}(X_t)$

## Proof: MSFE for AR(1)

### Claim

$$\square \text{ MSFE}(h) = \sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2} \quad \text{and} \quad \text{MSFE}(\infty) = \gamma(0)$$

### Proof

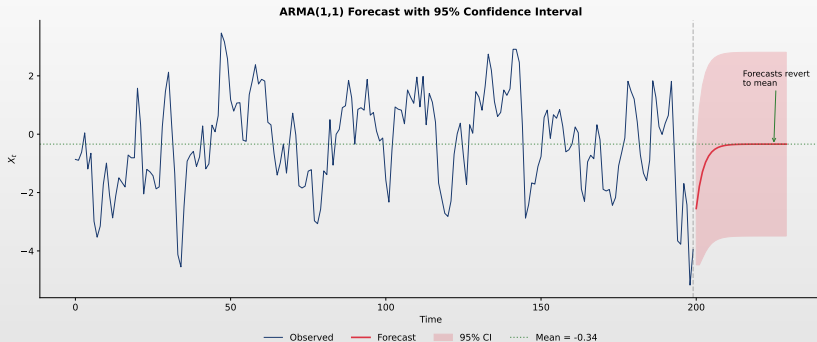
- Forecast error at horizon  $h$ :  $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- By recursive substitution:  $e_{n+h|n} = \sum_{j=0}^{h-1} \phi^j \varepsilon_{n+h-j}$
- $\text{MSFE}(h) = \mathbb{E}[e_{n+h|n}^2] = \sigma^2 \sum_{j=0}^{h-1} \phi^{2j} = \boxed{\sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2}}$
- Limit:  $\text{MSFE}(\infty) = \frac{\sigma^2}{1 - \phi^2} = \gamma(0) \Rightarrow$  forecast converges to unconditional mean

### Interpretation

- At long horizons, we do no better than the unconditional mean:  $\text{CI} \rightarrow 2 \times 1.96 \sqrt{\gamma(0)}$



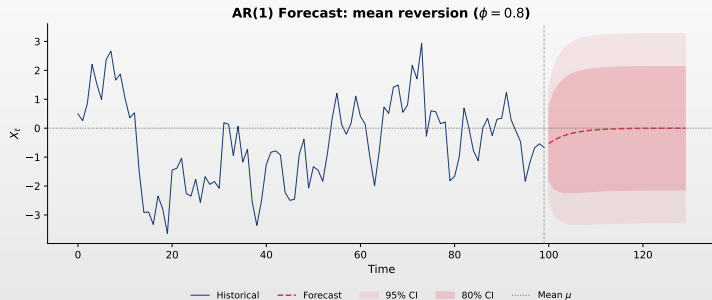
## ARMA Forecast with Confidence Intervals



### Observation

- The confidence band widens with the horizon  $\nearrow$  convergence to the unconditional interval

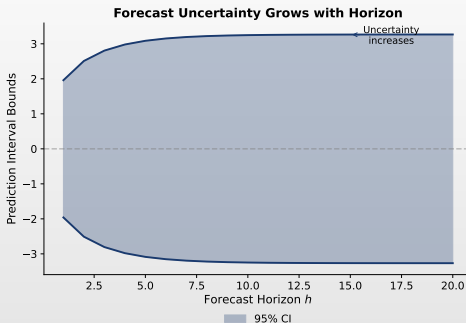
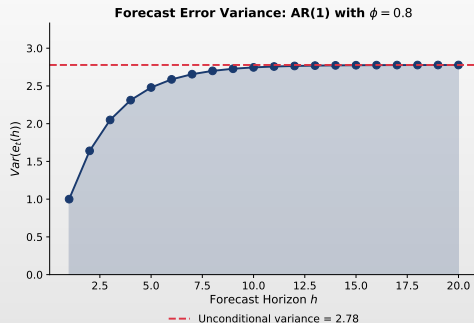
## AR(1) Forecast: Mean Reversion



### Properties

- ▣ Forecasts converge to the unconditional mean  $\mu$  as the horizon increases
- ▣ Larger  $|\phi|$   $\succ$  slower reversion; CIs widen with the horizon

## Forecast Error Variance by Horizon



### Observation

- MSFE increases monotonically with horizon  $h$   $\rightarrow$  convergence to  $\text{Var}(X_t)$  (predictability limit)

## Confidence Intervals for Forecasts

### Formulas

- ▣  $X_{n+h}|X_n, \dots \sim N(\hat{X}_{n+h|n}, \text{MSFE}(h))$
- ▣ **CI**  $(1 - \alpha)$ :  $\hat{X}_{n+h|n} \pm z_{\alpha/2} \cdot \sqrt{\text{MSFE}(h)}$ , where  $z_{\alpha/2} = 1.96$  for 95%

### Properties

- ▣ Intervals widen as the horizon increases
  - ▶ Converge to the unconditional interval:  $\mu \pm z_{\alpha/2} \sigma_X$
- ▣ Width depends on model parameters
  - ▶ Larger AR coefficients  $\succ$  wider intervals
- ▣ **Python**: `model.get_forecast(h).conf_int()`

## Forecast Evaluation

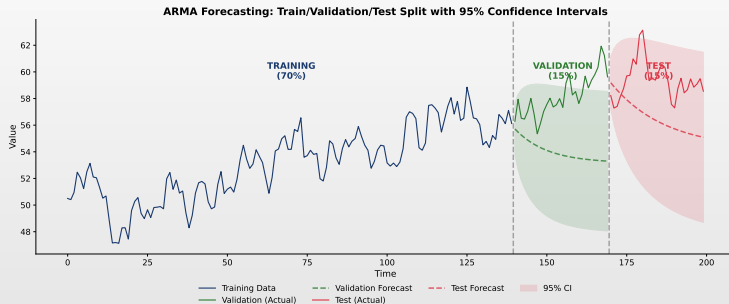
### Out-of-Sample Testing

- Split data: training + test
- Generate forecasts on test
- Compare with actual values
- **Rolling window**: re-estimate as new data arrives

### Error Metrics

- **MAE** =  $\frac{1}{n} \sum |e_t|$ 
  - ▶ Robust to outliers
- **RMSE** =  $\sqrt{\frac{1}{n} \sum e_t^2}$ 
  - ▶ Penalizes large errors
- **MAPE** =  $\frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$ 
  - ▶ Percentage-based, interpretable

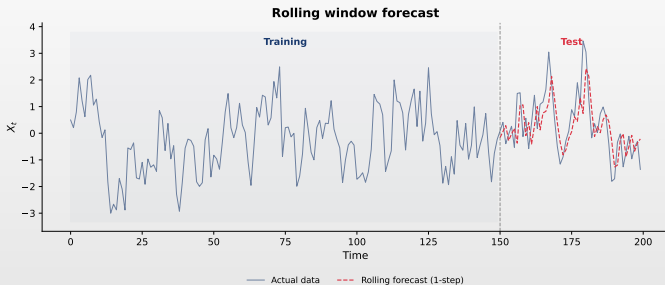
## Train/Validation/Test Forecast Example



### Best Practice

- Always evaluate forecasts on data not used for estimation (train/validation/test split)

## Rolling Window Forecast

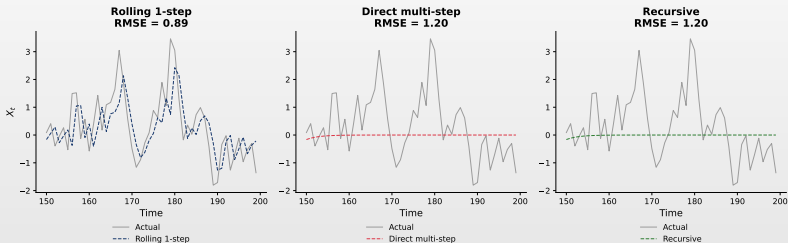


### Rolling Forecast Methodology

- Fixed window (last  $w$  obs.) vs expanding (all data); generate 1-step forecast, repeat

## Rolling vs Multi-Step Forecast

Comparison: Rolling vs Multi-step vs Recursive

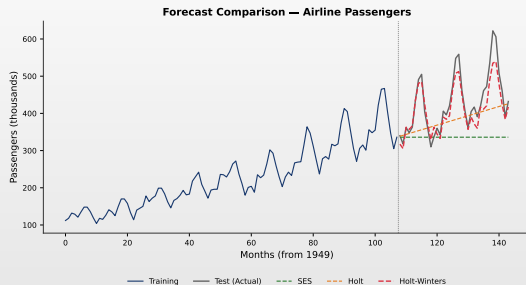


### Key Differences

- Rolling 1-step (accurate); Multi-step direct (separate model/horizon); Recursive (error accumulation)



## Real Data Application: Forecast Comparison



### Practical Considerations

- Real data: non-stationarity, structural breaks; compare models; use rolling window validation

## Python Implementation: Estimating ARMA Models

### Using statsmodels

```
from statsmodels.tsa.arima.model import ARIMA
model = ARIMA(data, order=(2, 0, 1)) # ARMA(2,1)
results = model.fit()
print(results.summary())
```

### Forecasting

```
forecast = results.get_forecast(steps=10)
print(forecast.predicted_mean)
print(forecast.conf_int())
```

### Note

ARIMA with  $d = 0$  is equivalent to ARMA

## Python: Model Selection with pmdarima

### Automatic ARIMA Selection

```
import pmdarima as pm
model = pm.auto_arima(data,
    start_p=0, max_p=5, start_q=0, max_q=5,
    d=0, seasonal=False,
    information_criterion='aic', trace=True)
print(model.summary())
```

### Result

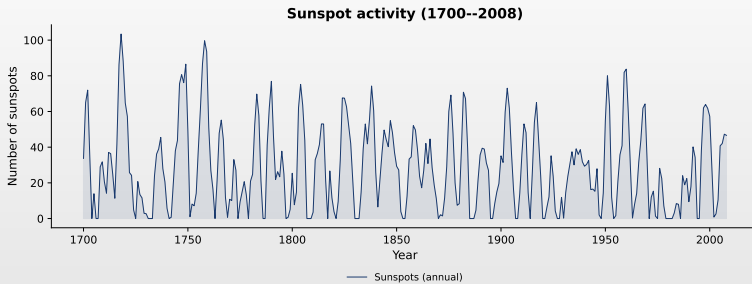
- Best model order and estimated parameters

## Workflow Summary

### Box-Jenkins Methodology Steps

- 1. **Data preparation:** Check for missing values, outliers; transform if necessary
- 2. **Stationarity check:** Visual inspection, formal tests (ADF, KPSS); difference if non-stationary
- 3. **Model identification:** ACF/PACF patterns; grid search with information criteria
- 4. **Estimation and validation:** Estimate model, check significance; residual analysis, Ljung-Box test
- 5. **Forecasting:** Point forecasts with confidence intervals; out-of-sample validation

## Case Study: Sunspots

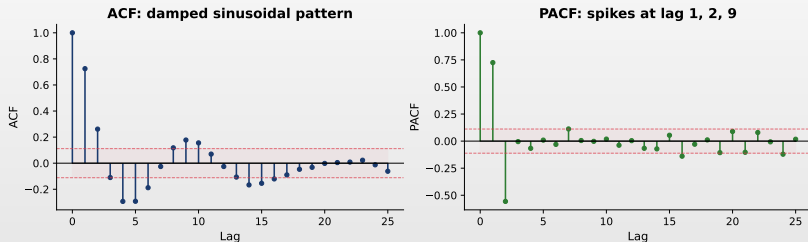


### Data Description

- Annual sunspots (1700–2008): stationary series with  $\sim 11$ -year cycles; Box-Jenkins methodology

## Step 1: ACF/PACF Analysis

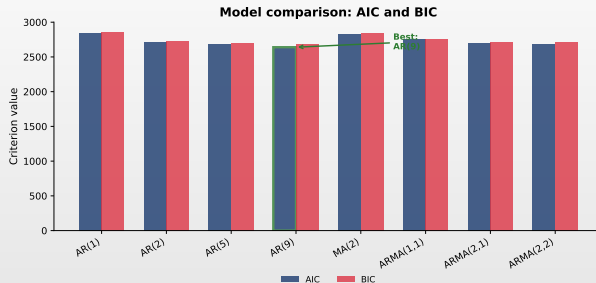
ACF/PACF analysis for sunspots



### Identification

- Sinusoidal ACF (AR); PACF with spikes at lags 1, 2, 9  $\succ$  AR(2) or AR(9); stationary series ( $d = 0$ )

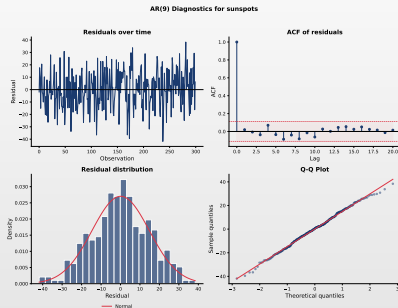
## Step 2: Model Comparison



### Model Selection

- Compare multiple candidate models using the AIC criterion
- The **AR(9)** model has the lowest AIC, capturing the 11-year solar cycle

## Step 3: Model Diagnostics

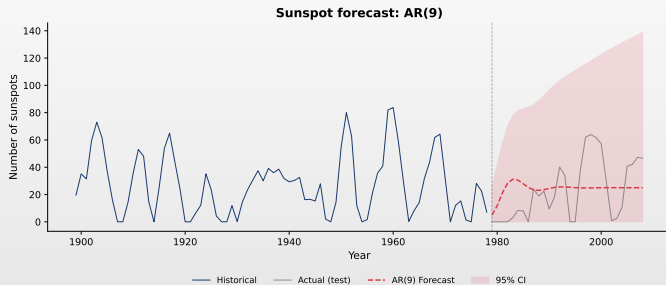


### AR(9) Diagnostics

- Residuals: white noise, zero mean, constant variance, ACF without structure,  $\approx$  normal



## Step 4: Forecasting



### Results

- AR(9) captures the cyclicity; 95% CI covers actual values; RMSE  $\approx 30$

## Key Takeaways

### Chapter Summary

- ▣ **AR( $p$ )**: Depends on  $p$  past values; stationarity: roots outside the unit circle; PACF cuts off at lag  $p$
- ▣ **MA( $q$ )**: Depends on  $q$  past shocks; always stationary; ACF cuts off at lag  $q$
- ▣ **ARMA( $p, q$ )**: Combines AR and MA; both ACF and PACF decay
- ▣ **Box-Jenkins**: Identification  $\succ$  Estimation  $\succ$  Validation  $\succ$  Forecasting
- ▣ **Validation**: Residuals must be white noise
- ▣ **Forecasts**: Converge to the mean; uncertainty increases with the horizon

## Next Chapter Preview

### Chapter 3: ARIMA Models for Non-Stationary Data

- ▣ Non-stationarity: types, unit root tests (ADF, PP, KPSS)
- ▣ Differencing and the difference operator
- ▣ ARIMA(p,d,q): integrated models for non-stationary data
- ▣ The Auto-ARIMA algorithm: automatic model selection
- ▣ Case study: US GDP Forecasting

### Reading

- ▣ Hyndman & Athanasopoulos, *Forecasting: Principles and Practice*, Ch. 9
- ▣ Box, Jenkins, Reinsel & Ljung, *Time Series Analysis*, Ch. 3-4

## Question 1

### Question

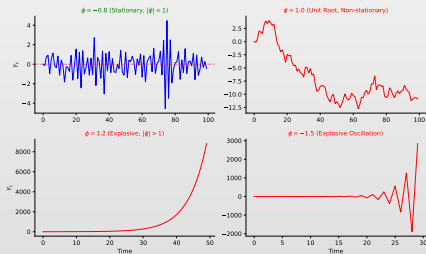
□ For which value of  $\phi$  is the AR(1) process  $X_t = c + \phi X_{t-1} + \varepsilon_t$  stationary?

- (A)  $\phi = 1.2$
- (B)  $\phi = 1.0$
- (C)  $\phi = -0.8$
- (D)  $\phi = -1.5$

## Question 1: Answer

Correct Answer: (C)  $\phi = -0.8$

- ☐ AR(1) is stationary if and only if  $|\phi| < 1$
- ☐ Only  $|-0.8| = 0.8 < 1$



## Question 2

### Question

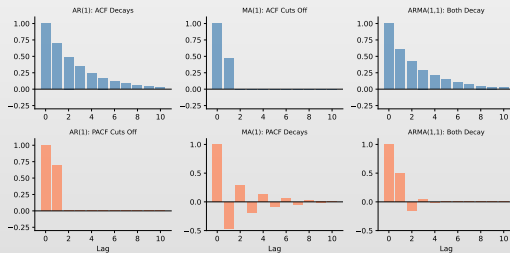
☐ You observe: ACF has a spike at lag 1, then cuts off. PACF decays gradually. What model?

- (A) AR(1)
- (B) MA(1)
- (C) ARMA(1,1)
- (D) White noise

## Question 2: Answer

Correct Answer: (B) MA(1)

- ACF cuts off  $\succ$  MA process
- PACF decays  $\succ$  confirms MA(1)



### Question 3

#### Question

□ Is the MA(1)  $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$  invertible?

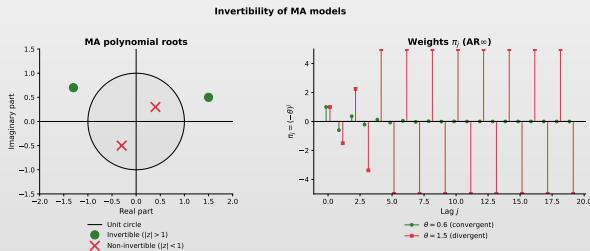
- (A) Yes, MA processes are always invertible
- (B) Yes, because  $1.5 > 0$
- (C) No, because  $|\theta| = 1.5 > 1$
- (D) No, MA processes are never invertible



## Question 3: Answer

Correct Answer: (C) No, because  $|\theta| = 1.5 > 1$

- Invertibility requires  $|\theta| < 1$
- Here  $|\theta| = 1.5 > 1$ , so it is not invertible



## Question 4

### Question

☐ The compact form  $\phi(L)X_t = \theta(L)\varepsilon_t$  represents which model?

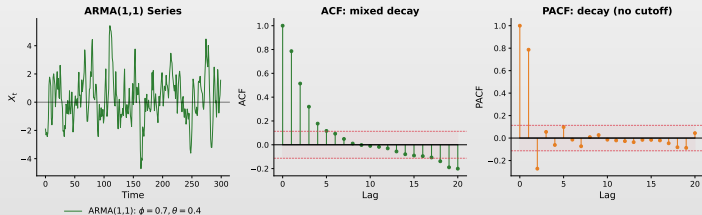
- (A) Pure AR model
- (B) Pure MA model
- (C) ARMA model
- (D) None of the above

## Question 4: Answer

Correct Answer: (C) ARMA model

□  $\phi(L)$  is the AR polynomial,  $\theta(L)$  is the MA polynomial  $\succ$  ARMA(p,q)

ARMA(1,1): neither ACF nor PACF cut off



## Question 5

## Question

What is  $(1 - L)^2 X_t$ ?

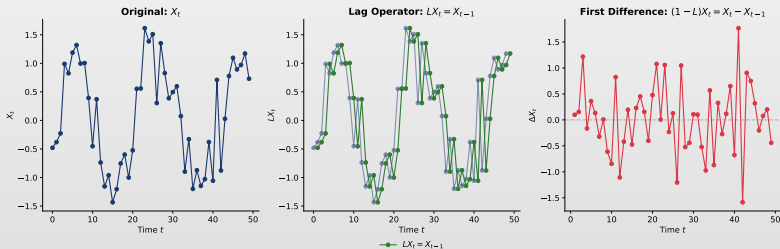
- (A)  $X_t - X_{t-1}$
- (B)  $X_t - 2X_{t-1} + X_{t-2}$
- (C)  $X_t + X_{t-1} + X_{t-2}$
- (D)  $X_t - X_{t-2}$

## Question 5: Answer

Correct Answer: (B)  $X_t - 2X_{t-1} + X_{t-2}$

☐  $(1 - L)^2 = 1 - 2L + L^2$

☐  $(1 - L)^2 X_t = X_t - 2X_{t-1} + X_{t-2}$



## Question 6

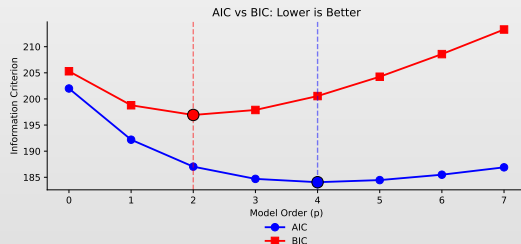
### Question

- ☐ Comparing ARMA(1,1) vs ARMA(2,1) using BIC, which is correct?
- (A) Lower BIC always means better forecasts
  - (B) BIC penalizes complexity less than AIC
  - (C) The model with lower BIC is preferred
  - (D) BIC can only compare models with the same number of parameters

## Question 6: Answer

Correct Answer: (C) The model with lower BIC is preferred

- ☐ Lower BIC indicates a better balance between estimation quality and complexity
- ☐ BIC penalizes complexity *more* than AIC



## Question 7

### Question

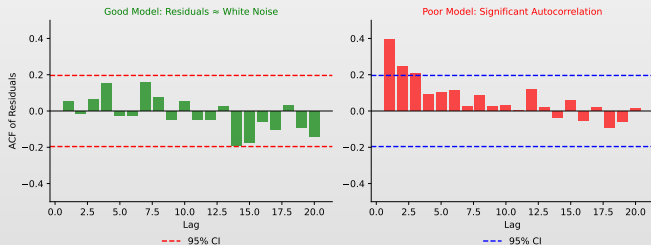
- ☐ After estimating an ARMA model, you run the Ljung-Box test on residuals and obtain  $p\text{-value} = 0.03$ . What does this mean?
- (A) The model is adequate, residuals are white noise
  - (B) The model is inadequate, residuals have autocorrelation
  - (C) You need to increase the sample size
  - (D) The test is inconclusive



## Question 7: Answer

Correct Answer: (B) The model is inadequate

- ☐ p-value  $< 0.05$  rejects  $H_0$  (white noise)
- ☐ Indicates remaining residual autocorrelation



## Question 8

### Question

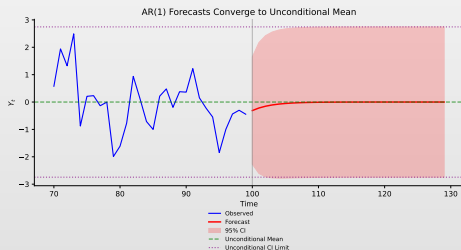
□ For a stationary AR(1) model, what happens to forecasts as the horizon  $h \rightarrow \infty$ ?

- (A) Forecasts increase without bound
- (B) Forecasts oscillate indefinitely
- (C) Forecasts converge to the unconditional mean  $\mu$
- (D) Forecasts become more precise

## Question 8: Answer

Correct Answer: (C) Forecasts converge to  $\mu$

$$\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu) \rightarrow \mu \text{ as } h \rightarrow \infty \text{ (since } |\phi| < 1)$$



## Question 9

## Question

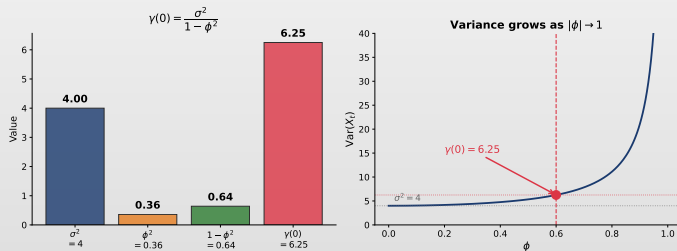
□ Consider an AR(1) process with  $\phi = 0.6$  and  $\sigma^2 = 4$ . What is  $\text{Var}(X_t)$ ?

- (A) 4.0
- (B) 5.56
- (C) 6.25
- (D) 10.0

## Question 9: Answer

Correct Answer: (C) 6.25

- ☐  $\text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2} = \frac{4}{1-0.36} = \frac{4}{0.64} = 6.25$
- ☐ The process variance exceeds  $\sigma^2$  due to persistence



## Question 10

## Question

□ Consider an MA(1) process with  $\theta = 0.5$ . What is  $\rho(1)$ ?

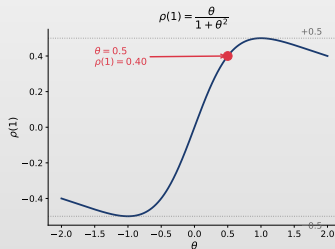
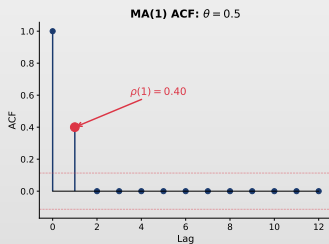
- (A) 0.50
- (B) 0.40
- (C) 0.25
- (D) 0.33

## Question 10: Answer

Correct Answer: (B) 0.40

□  $\rho(1) = \frac{\theta}{1+\theta^2} = \frac{0.5}{1+0.25} = \frac{0.5}{1.25} = 0.40$

□ Note that  $\rho(1) < \theta$  — the autocorrelation is **always** attenuated



## Question 11

## Question

□ Which statement about the ACF of an ARMA(1,1) process is **true**?

- (A) It cuts off after lag 1
- (B) Exponential decay starting from lag 1, with  $\rho(1) \neq \phi$
- (C) It is zero for all lags
- (D) It exactly follows the pattern  $\phi^h$  for all  $h \geq 0$

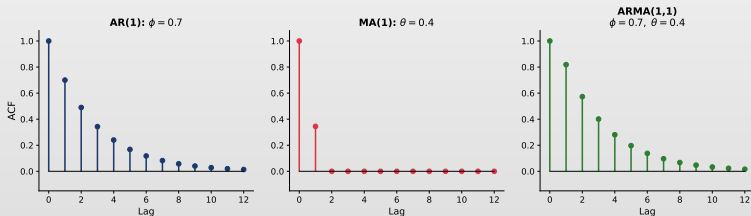


## Question 11: Answer

Correct Answer: (B) Exponential decay from lag 1, with  $\rho(1) \neq \phi$

- $\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2} \neq \phi$  (the MA component modifies lag 1)
- For  $h \geq 2$ :  $\rho(h) = \phi \rho(h-1)$  — exponential decay as in AR(1)

ACF Comparison: AR(1) vs MA(1) vs ARMA(1,1)



## Question 12

## Question

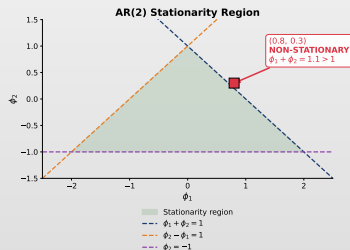
□ An AR(2) process has  $\phi_1 = 0.8$  and  $\phi_2 = 0.3$ . Is it stationary?

- (A) Yes, it is stationary
- (B) No, because  $\phi_1 + \phi_2 = 1.1 > 1$
- (C) Cannot be determined without data
- (D) Depends on the value of  $\sigma^2$

## Question 12: Answer

Correct Answer: (B) No, because  $\phi_1 + \phi_2 = 1.1 > 1$

- Necessary conditions for AR(2) stationarity:
- $\phi_1 + \phi_2 < 1$ ,  $\phi_2 - \phi_1 < 1$ ,  $|\phi_2| < 1$
- Here  $0.8 + 0.3 = 1.1 > 1 \Rightarrow$  the first condition is violated



## Question 13

### Question

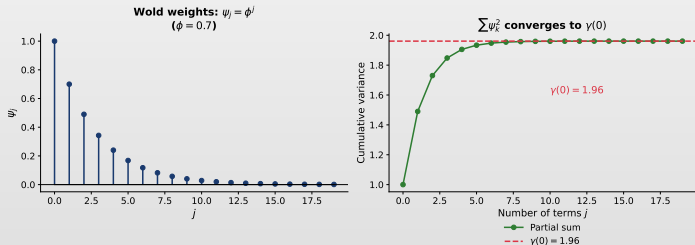
□ What does the Wold decomposition theorem guarantee?

- (A) Any time series is an AR process
- (B) Any stationary process can be written as  $MA(\infty)$ :  $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- (C) Any process has finite variance
- (D) ARMA models are always invertible

## Question 13: Answer

Correct Answer: (B) Any stationary process =  $MA(\infty)$

- Wold's theorem:  $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + D_t$ , where  $D_t$  is the deterministic component
- This justifies ARMA models: they are parsimonious approximations of  $MA(\infty)$



## Question 14

## Question

□ AR(1) with  $\phi = 0.9$ ,  $\sigma^2 = 1$ . What happens to the CI width as  $h \rightarrow \infty$ ?

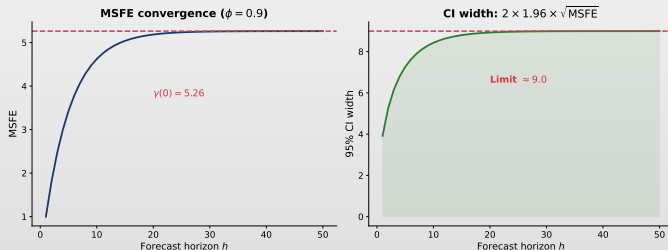
- (A) It remains constant
- (B) It decreases to zero
- (C) It grows toward  $2 \times 1.96 \times \sqrt{1/(1 - 0.81)} \approx 9.0$
- (D) It grows to infinity

## Question 14: Answer

Correct Answer: (C) Grows toward  $\approx 9.0$

□  $MSFE(\infty) = \frac{\sigma^2}{1-\phi^2} = \frac{1}{1-0.81} = \frac{1}{0.19} \approx 5.26$

□  $CI \text{ width} = 2 \times 1.96 \sqrt{5.26} \approx 2 \times 1.96 \times 2.29 \approx 9.0$



## Data Sources and Software

### Software Packages

- ▣ `statsmodels` > Statistical models for Python, including ARIMA
- ▣ `pmdarima` > Automatic ARIMA selection for Python
- ▣ `scipy` > Optimization and statistical functions
- ▣ `numpy`, `pandas` > Data manipulation
- ▣ `matplotlib` > Visualization

### Data and Examples

- ▣ Simulated time series for illustrations
- ▣ Examples based on Hyndman & Athanasopoulos (2021)



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- ▣ Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.
- ▣ Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

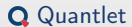
### Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> → Code repository for statistics
- ▣ **Quantinar**: <https://quantinar.com> → Learning platform for quantitative methods
- ▣ **GitHub TSA**: <https://github.com/QuantLet/TSA> → Python code for this course

# Thank You!

## Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar