



Time Series Analysis and Forecasting

Chapter 7: Cointegration and VECM



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Learning Objectives

By the end of this chapter, you will be able to:

- Understand the problem of spurious regression with non-stationary data
- Test for cointegration using Engle-Granger and Johansen methods
- Estimate Vector Error Correction Models (VECM)
- Interpret error correction mechanisms and adjustment speeds

Outline

- Motivation
- Spurious Regression
- Cointegration Concept
- Engle-Granger Method
- Johansen Method
- VECM Estimation
- Practical Considerations
- Real-World Examples
- Case Study: Interest Rates
- AI Use Case
- Summary
- Quiz

Why Cointegration Matters

The Challenge

- ▣ Many economic/financial time series are **non-stationary** ($I(1)$)
- ▣ GDP, stock prices, exchange rates, interest rates all have unit roots
- ▣ Standard regression with $I(1)$ variables \Rightarrow **spurious results**
- ▣ Differencing removes non-stationarity but loses **long-run information**

The Solution: Cointegration

Some non-stationary series share a **common stochastic trend**—they move together in the long run.

Nobel Prize 2003

Granger & Engle received the Nobel Prize for “methods for analyzing economic time series with common trends.”

Real-World Applications

Finance

- ▣ **Pairs Trading:** Cointegrated stocks
- ▣ **Term Structure:** Interest rates
- ▣ **Spot-Futures:** Arbitrage

Macroeconomics

- ▣ **Consumption & Income**
- ▣ **Money & Prices**
- ▣ **PPP:** Exchange rates

Policy Analysis

- ▣ **Fiscal:** Spending & taxes
- ▣ **Monetary:** Rate pass-through
- ▣ **Labor:** Wages & productivity

The Spurious Regression Problem

Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk: $Y_t = \alpha + \beta X_t + u_t$ where Y_t and X_t are independent $I(1)$ processes.

Symptoms of Spurious Regression

- ▣ High R^2 (often > 0.9) even though variables are **unrelated**
- ▣ Highly significant t -statistics (reject $H_0 : \beta = 0$)
- ▣ Very low Durbin-Watson statistic ($DW \approx 0$)
- ▣ Residuals are non-stationary (have unit root)

Rule of Thumb

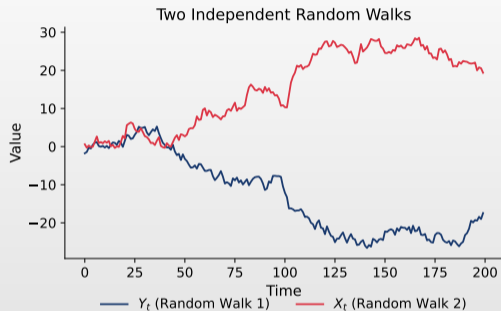
If $R^2 > DW$, suspect spurious regression!

Spurious Regression: Visual Example

Warning

- **Result:** Two completely independent random walks show high correlation ($R^2 > 0.8$) purely by chance!
This is why we need cointegration analysis

Spurious Regression: Visual Example



Spurious Correlations in the Real World

Data Mining Can Produce Meaningless Correlations

With enough variables and long time series, purely coincidental patterns emerge:

- Distance between Neptune and Uranus \leftrightarrow SAP SE stock price (2002–2023)
- GMO corn use in South Dakota \leftrightarrow Google searches for “i cant even” (2004–2023)
- Two and a Half Men* season ratings \leftrightarrow Jet fuel used in Serbia (2006–2015)
- “Its Wednesday my dudes” meme popularity \leftrightarrow Boeing stock price (2006–2023)

Lesson

High correlation \neq causation. Non-stationary series with common trends produce high R^2 by construction. Always test for stationarity and cointegration before interpreting regression results!

🌐 Explore more examples: tylervigen.com/spurious-correlations

Definition of Cointegration

Definition 1 (Cointegration (Engle & Granger, 1987))

Variables $Y_{1t}, Y_{2t}, \dots, Y_{kt}$ are **cointegrated of order** (d, b) , written $CI(d, b)$, if:

1. All variables are integrated of order d : $Y_{it} \sim I(d)$
2. There exists a linear combination $\beta'Y_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$ that is integrated of order $(d - b)$, where $b > 0$

Most Common Case: $CI(1, 1)$

- ▣ Variables are $I(1)$ (have unit roots)
- ▣ Linear combination is $I(0)$ (stationary)
- ▣ Vector $\beta = (\beta_1, \dots, \beta_k)'$ is the **cointegrating vector**

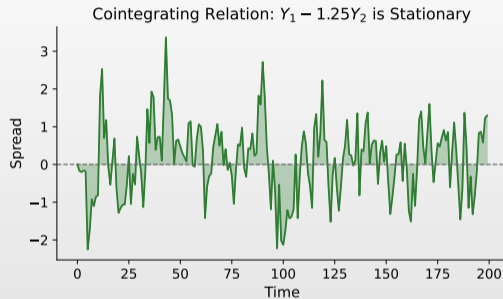
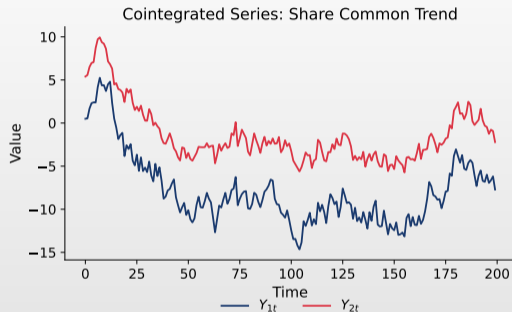
The cointegrating vector is unique only up to scalar multiplication. Usually normalized: $\beta_1 = 1$.

Cointegration: Visual Example

Key Insight

- **Cointegration:** Both series are $I(1)$ and trend together, but their linear combination (spread) is stationary — this is cointegration!

Cointegration: Visual Example



TSA_ch7_cointegrated_series

Intuition: Common Stochastic Trends

Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**: $Y_{1t} = \gamma_1 \tau_t + S_{1t}$, $Y_{2t} = \gamma_2 \tau_t + S_{2t}$ where τ_t is a common random walk and S_{it} are stationary.

Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

Economic Interpretation

- Cointegration = **long-run equilibrium relationship**
- Variables may deviate in the short run, but are “pulled back”
- The cointegrating vector defines the equilibrium

Cointegrating Rank

How Many Cointegrating Relationships?

For k variables that are $I(1)$:

- ▣ Maximum possible cointegrating relationships: $r = k - 1$
- ▣ If $r = 0$: No cointegration (variables drift apart)
- ▣ If $r = k$: All variables are $I(0)$ (contradiction)

Example: 3 Variables

- ▣ $r = 0$: No cointegration
- ▣ $r = 1$: One cointegrating relationship
- ▣ $r = 2$: Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends = $k - r$

Engle-Granger Two-Step Method

Step 1: Estimate Cointegrating Regression

Run OLS: $Y_t = \alpha + \beta X_t + e_t$. Save residuals: $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

Step 2: Test Residuals for Stationarity

Test if \hat{e}_t is $I(0)$ using ADF: $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$

- $H_0: \rho = 0$ (unit root \Rightarrow no cointegration)
- $H_1: \rho < 0$ (stationary \Rightarrow cointegration)

Important

Use Engle-Granger critical values, not standard ADF! (More negative because residuals are estimated)

Engle-Granger Critical Values

Critical Values for Cointegration Test

| Variables | 1% | 5% | 10% |
|-----------|-------|-------|-------|
| 2 | -3.90 | -3.34 | -3.04 |
| 3 | -4.29 | -3.74 | -3.45 |
| 4 | -4.64 | -4.10 | -3.81 |
| 5 | -4.96 | -4.42 | -4.13 |

MacKinnon (1991), $T = 100$

Limitations of Engle-Granger

- ▣ Only tests for **one** cointegrating relationship
- ▣ Results depend on choice of dependent variable
- ▣ Small sample bias; cannot test hypotheses on cointegrating vector

Researcher Spotlight: Søren Johansen



*1939

 Wikipedia

Biography

- Danish statistician and econometrician, Professor Emeritus at University of Copenhagen
- Known for his rigorous mathematical approach to econometrics
- Fellow of the Econometric Society; recipient of numerous honors in statistical science

Key Contributions

- **Johansen cointegration test** (1988, 1991) — maximum likelihood approach to testing for multiple cointegrating vectors
- **Trace and maximum eigenvalue** statistics for determining cointegration rank
- **VECM estimation** — linking cointegration with error correction models
- Standard framework for multivariate cointegration analysis in economics and finance

Johansen Cointegration Test

Advantages over Engle-Granger

- ▣ Tests for **multiple** cointegrating relationships
- ▣ Maximum likelihood estimation (more efficient)
- ▣ Can test restrictions on cointegrating vectors
- ▣ Does not require choosing a dependent variable

Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...

VECM Representation

Vector Error Correction Model

$$\Delta Y_t = c + \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

▣ $\Pi = \sum_i A_i - I$ (long-run impact); Γ_j (short-run dynamics)

Key Insight: Rank of Π

The **rank of Π** determines cointegration:

- ▣ $\text{rank}(\Pi) = 0$: No cointegration (VAR in differences)
- ▣ $\text{rank}(\Pi) = k$: All variables are $I(0)$ (VAR in levels)
- ▣ $0 < \text{rank}(\Pi) = r < k$: r cointegrating vectors

Derivation: From VAR to VECM

Starting Point: VAR(p) in Levels

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \epsilon_t$$

Step 1: Subtract Y_{t-1} from Both Sides

$$Y_t - Y_{t-1} = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} - Y_{t-1} + \epsilon_t$$

$$\Delta Y_t = (A_1 - I) Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \epsilon_t$$

Goal

Rewrite so that all terms are either in levels (Y_{t-1}) or differences (ΔY_{t-j}).

Derivation: From VAR to VECM (cont.)

Step 2: Add and Subtract Terms Strategically

Add $A_2 Y_{t-1}$ and subtract $A_2 Y_{t-1}$: $\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} - A_2(Y_{t-1} - Y_{t-2}) + A_3 Y_{t-3} + \dots + \epsilon_t$
Continue adding $A_3 Y_{t-1}$, etc., until all lagged **levels** are collected in one term.

Step 3: General Pattern

After algebraic manipulation: $\Delta Y_t = \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$

The Key Matrices

$$\Pi = \sum_{i=1}^p A_i - I = -(I - A_1 - A_2 - \dots - A_p)$$

$$\Gamma_j = - \sum_{i=j+1}^p A_i \text{ for } j = 1, \dots, p-1$$

Derivation: Verifying the Γ_j Formula

Example: VAR(2)

Starting from: $Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$

Subtract Y_{t-1} :

$$\Delta Y_t = (A_1 - I)Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$$

Add and subtract $A_2 Y_{t-1}$:

$$\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} + A_2(Y_{t-2} - Y_{t-1}) + \varepsilon_t$$

$$\Delta Y_t = \underbrace{(A_1 + A_2 - I)}_{\Pi} Y_{t-1} - \underbrace{A_2}_{\Gamma_1} \Delta Y_{t-1} + \varepsilon_t$$

Verification

For VAR(2): $\Pi = A_1 + A_2 - I$ and $\Gamma_1 = -A_2$

Using our formula: $\Gamma_1 = -\sum_{i=2}^2 A_i = -A_2 \quad \checkmark$

Economic Interpretation of Error Correction

The VECM with Cointegration

When $\text{rank}(\Pi) = r$, we write $\Pi = \alpha\beta'$: $\Delta Y_t = \alpha \underbrace{(\beta' Y_{t-1})}_{\text{equilibrium error}} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$

Economic Interpretation

- ▣ $\beta' Y_{t-1} = \text{equilibrium error}$: deviation from long-run relationship
- ▣ $\alpha = \text{adjustment speeds}$: how fast variables correct deviations
- ▣ $\Gamma_j = \text{short-run dynamics}$: transitory effects

Error Correction Mechanism

If $\beta' Y_{t-1} > 0$ (above equilibrium) and $\alpha_i < 0$, then ΔY_{it} decreases. **The system self-corrects toward equilibrium!**

Decomposition of Π

When $\text{rank}(\Pi) = r < k$

$\Pi = \alpha\beta'$ where β ($k \times r$) = **cointegrating vectors**, α ($k \times r$) = **adjustment coefficients**

Interpretation

- ▣ $\beta'Y_{t-1}$ = deviations from equilibrium (error correction terms)
- ▣ α = speed of adjustment; rows show each variable's response

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta'Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$$

Johansen Test Statistics

Two Test Statistics

Based on eigenvalues $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$ of a certain matrix:

Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests H_0 : rank $\leq r$ vs H_1 : rank $> r$

Maximum Eigenvalue Test:

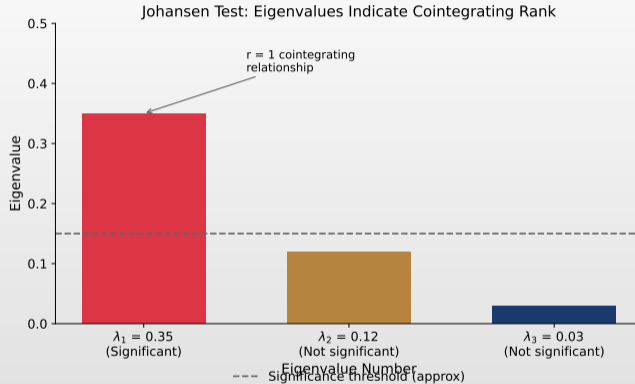
$$\lambda_{\max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

Tests H_0 : rank $= r$ vs H_1 : rank $= r+1$

Critical values from Johansen & Juselius (1990), depend on:

- ☐ Number of variables k
- ☐ Deterministic components (constant, trend)

Johansen Test: Visual Interpretation



Significant eigenvalues (above threshold) indicate cointegrating relationships. First eigenvalue significant $\Rightarrow r = 1$.

[TSA_ch7_johansen_eigenvalues](#)



Testing Procedure

Sequential Testing (Trace Test)

1. Test $H_0: r = 0$. If rejected \Rightarrow continue
2. Test $H_0: r \leq 1$. If not rejected $\Rightarrow r = 1$
3. Continue until H_0 is not rejected

Deterministic Components

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- **Constant in both** (most common)
- Constant + trend in cointegrating relation

VECM Structure

Full VECM Specification

For $k = 2$ variables with $r = 1$ cointegrating relation:

$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

Components

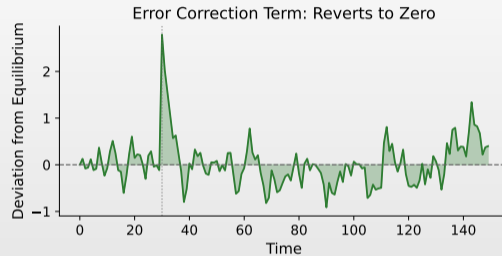
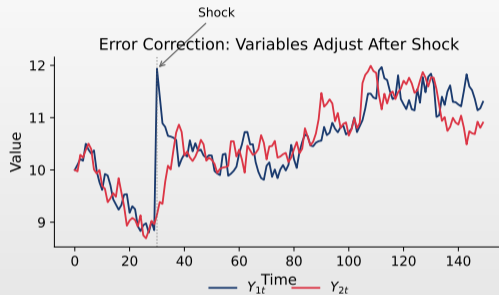
- ▣ $(Y_{1,t-1} - \beta Y_{2,t-1})$ = error correction term (deviation from equilibrium)
- ▣ α_1, α_2 = adjustment speeds (should have opposite signs)
- ▣ γ_{ij} = short-run dynamics
- ▣ ε_{it} = innovations

Error Correction Mechanism: Visual

Interpretation

- **Error correction:** When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment

Error Correction Mechanism: Visual



 TSA_ch7_error_correction

Interpreting Adjustment Coefficients

The α Coefficients

If the cointegrating relation is $Y_1 - \beta Y_2 = 0$ (equilibrium):

- $\alpha_1 < 0$: Y_1 adjusts downward when above equilibrium
- $\alpha_2 > 0$: Y_2 adjusts upward when Y_1 is above equilibrium

Weak Exogeneity

If $\alpha_i = 0$, variable Y_i does **not** respond to disequilibrium.

- Y_i is **weakly exogenous** for the long-run parameters
- The other variable does all the adjusting
- Can simplify estimation (single-equation approach)

Test weak exogeneity: $H_0 : \alpha_i = 0$ using likelihood ratio test.

VECM vs VAR in Differences

When Variables are Cointegrated

| | VAR in Differences | VECM |
|--------------------|--------------------|-----------|
| Long-run info | Lost | Preserved |
| Short-run dynamics | Yes | Yes |
| Error correction | No | Yes |
| Forecasting | Poor (long-run) | Better |
| IRF interpretation | Short-run only | Both |

Granger Representation Theorem

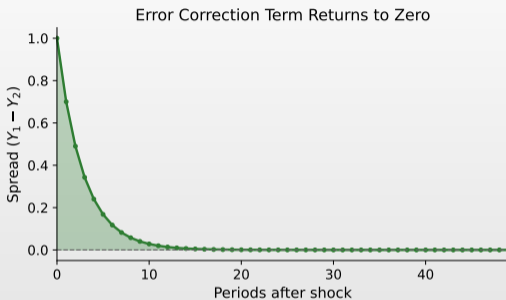
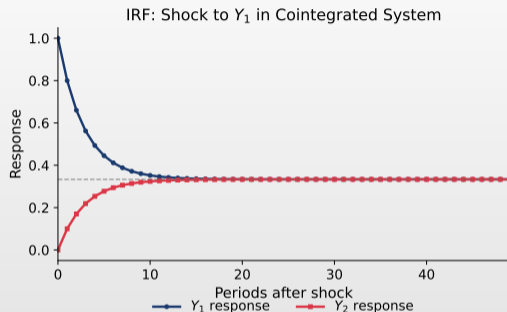
If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

VECM Impulse Response Functions

IRF Interpretation

- **Permanent effects:** In a cointegrated system, shocks have permanent effects on levels, but the system returns to equilibrium — they converge to a new long-run value

VECM Impulse Response Functions



Practical Workflow

Step-by-Step Procedure

1. **Unit Root Tests:** Verify all variables are $I(1)$
 - ▶ ADF, KPSS on levels and first differences
2. **Lag Length Selection:** Choose p for VAR in levels
 - ▶ Use AIC, BIC, or sequential LR tests
3. **Cointegration Test:** Johansen trace/max-eigenvalue tests
 - ▶ Determine cointegrating rank r
4. **Estimate VECM:** If $0 < r < k$
 - ▶ Estimate α , β , Γ_j
5. **Diagnostics:** Check residuals for autocorrelation, normality
6. **Analysis:** IRF, FEVD, hypothesis tests

Common Pitfalls

Things to Watch Out For

- ▣ **Structural breaks:** Cause spurious unit roots or cointegration
- ▣ **Near-unit-root:** Tests have low power
- ▣ **Lag selection:** Too many/few lags bias results
- ▣ **Small samples:** Johansen test oversized

Recommendation

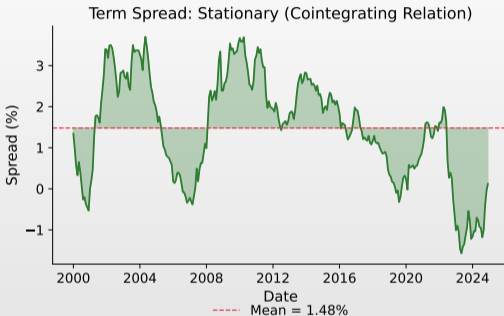
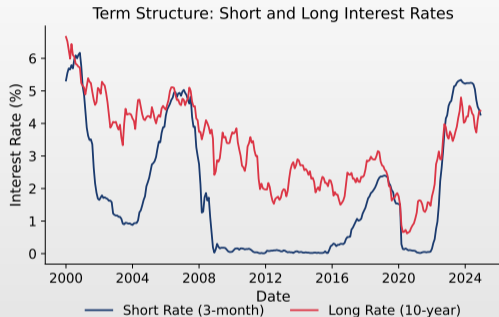
Always check: residual diagnostics, stability of cointegrating relationship, sensitivity to specification

Example 1: Term Structure of Interest Rates

Expectations Hypothesis

- **Conclusion:** Short and long rates share a common trend. The spread (term premium) is stationary — evidence of cointegration!

Example 1: Term Structure of Interest Rates



 TSA_ch7_interest_rates_coint

Interest Rates: Economic Theory

Expectations Hypothesis of the Term Structure

- ▣ **Formula:** Long-term rate as average of expected future rates
 - ▶ $R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{term premium}$
- ▣ **Implication:** If the term premium is constant, r_t and R_t are cointegrated
 - ▶ Cointegrating vector: $(1, -1)$

Empirical Results

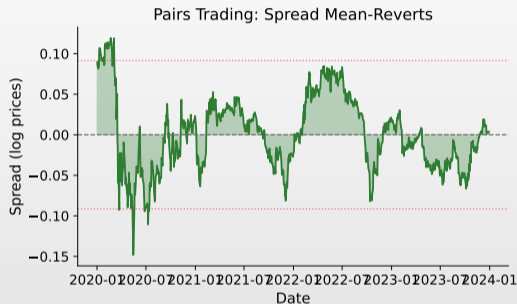
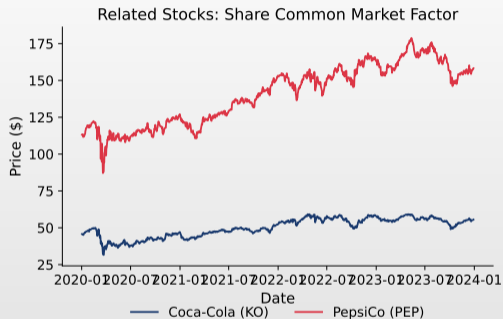
- ▣ **Unit root tests:** Both rates are $I(1)$
 - ▶ One cointegrating relationship (Johansen test)
- ▣ **Cointegrating vector:** $\approx (1, -1)$, the spread is stationary
 - ▶ The short rate adjusts to disequilibrium (the long rate is weakly exogenous)

Example 2: Pairs Trading in Finance

Strategy

- ▣ **Pairs trading:** Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When the spread deviates from the mean, trade expecting mean reversion

Example 2: Pairs Trading in Finance



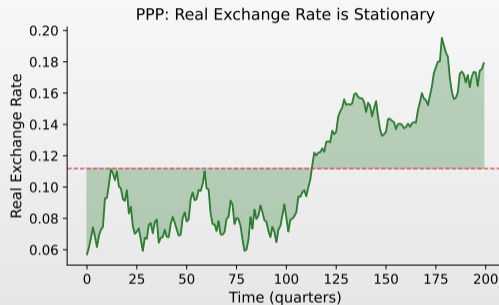
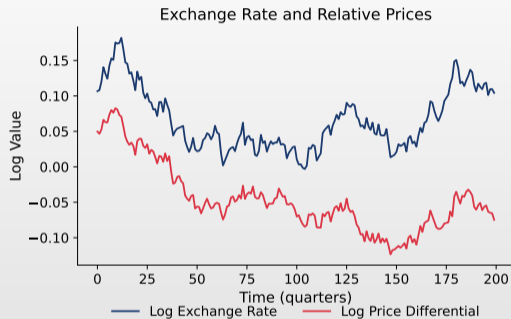
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Example 3: Purchasing Power Parity (PPP)

PPP Theory

- **Formula:** $e_t = p_t - p_t^*$ (log exchange rate equals price differential). The real exchange rate should be stationary in the long run

Example 3: Purchasing Power Parity (PPP)



VECM Results for Interest Rates

Typical Results

- ▣ **Integration:** Both rates are $I(1)$, one cointegrating relationship identified
 - ▶ Cointegrating vector close to $(1, -1)$: the spread is stationary
- ▣ **Adjustment:** The short rate adjusts to the long rate
 - ▶ The long rate does not adjust (weakly exogenous)

VECM Equations (Stylized)

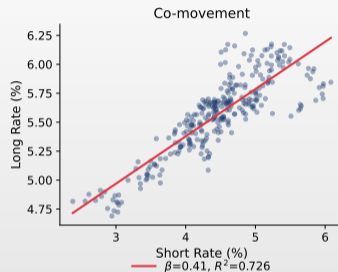
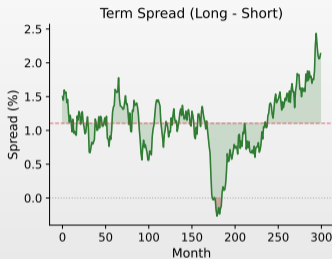
- ▣ **Estimated system:**
 - ▶ $\Delta r_t = 0.02 - 0.15(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{1t}$
 - ▶ $\Delta R_t = 0.01 - 0.02(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{2t}$
- ▣ **Interpretation:** The short rate adjusts faster ($\alpha_1 = -0.15$)
 - ▶ The long rate is nearly weakly exogenous ($\alpha_2 \approx 0$)

Case Study: Cointegration of Interest Rates

Data

- ▣ **US Interest Rates:** Long-term (10 years) and short-term (3 months)
- ▣ **Observation:** Both series are $I(1)$, but the spread appears stationary

Case Study: Cointegration of Interest Rates



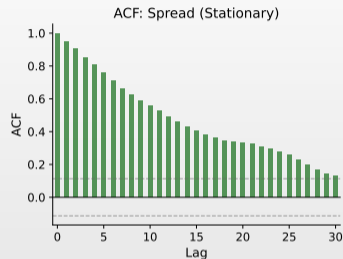
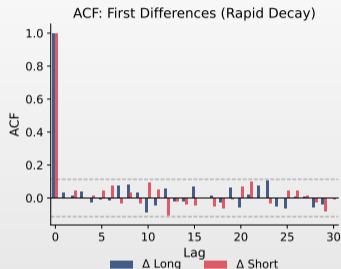
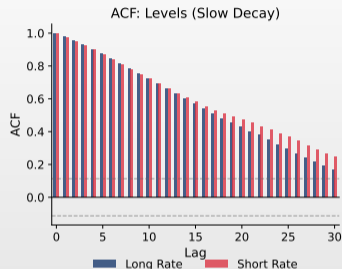
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Step 1: Unit Root Tests

Results

- **ACF levels:** Slow decay — non-stationarity; after differencing: rapid decay — $I(1)$
- **ACF spread:** Stationary — possible cointegration!

Step 1: Unit Root Tests



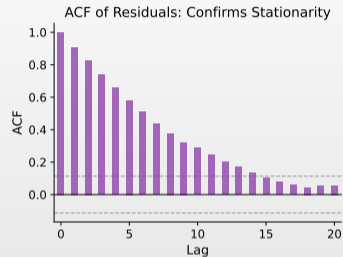
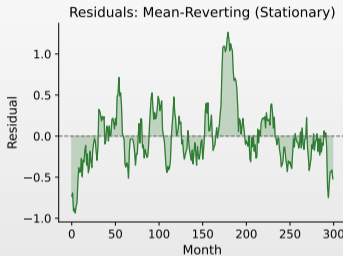
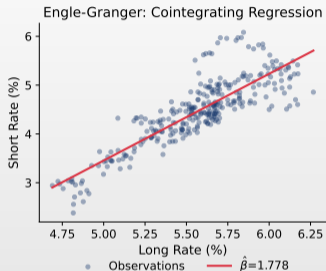
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Step 2: Engle-Granger Cointegration Test

Results

- **Engle-Granger regression:** Short rate = $\alpha + \beta \times \text{Long rate} + \varepsilon_t$
- **Conclusion:** The series are cointegrated — a long-run equilibrium relationship exists

Step 2: Engle-Granger Cointegration Test



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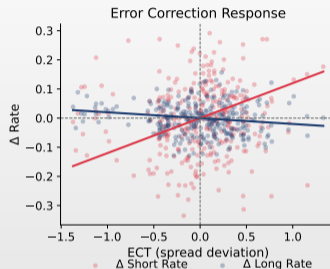
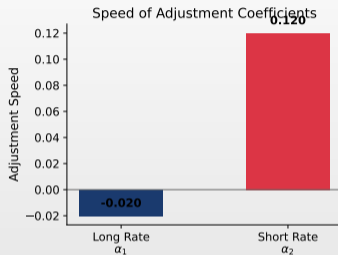
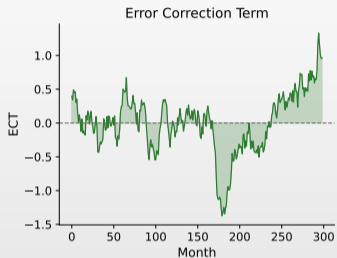


Step 3: VECM Estimation

Model

- ▣ **VECM(2)**: Cointegration rank = 1
- ▣ **Adjustment**: The α coefficients indicate the speed of return to equilibrium

Step 3: VECM Estimation



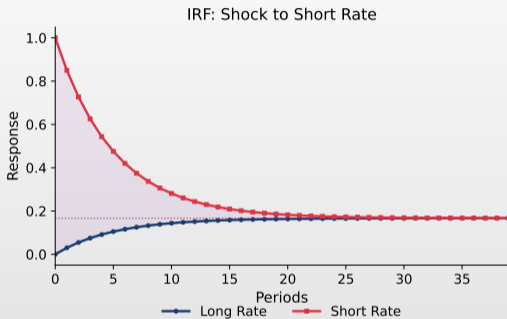
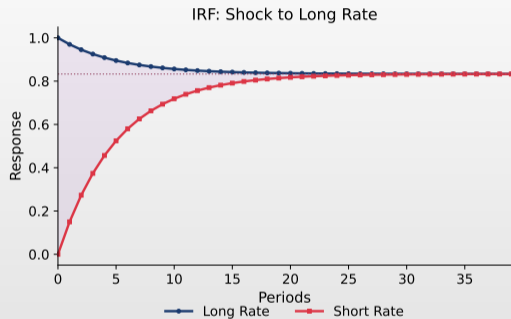
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Step 4: Impulse Response Functions

Interpretation

- ▣ **Permanent effects:** Shocks to the long rate persistently affect both rates
- ▣ **Cointegration:** Effects do not converge to zero — characteristic of cointegrated series

Step 4: Impulse Response Functions



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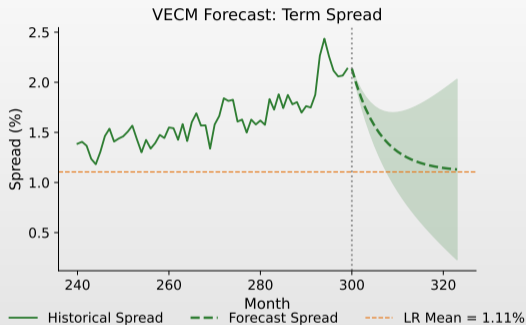
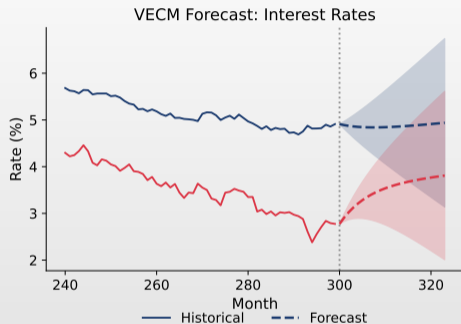


Step 5: VECM Forecast

Forecast

- **Horizon:** 24 months for both rates simultaneously
- **Advantage:** VECM maintains the cointegrating relationship in the forecast

Step 5: VECM Forecast



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AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download daily prices for gold and silver using yfinance. Test for cointegration and estimate a VECM model. Analyze the speed of adjustment parameters. Give me complete Python code."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it verify that each series is $I(1)$ before testing for cointegration?
3. Does it use both Engle-Granger and Johansen tests? What are the trade-offs?
4. How does it determine the cointegration rank? Trace vs max-eigenvalue statistics?
5. Does it correctly interpret the α (adjustment) coefficients?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Key Takeaways

Main Concepts

- ▣ **Cointegration:** $I(1)$ variables with stationary linear combination
- ▣ **Spurious regression:** High R^2 with unrelated $I(1)$ variables
- ▣ **VECM:** VAR with error correction for cointegrated systems

Testing Methods

- ▣ **Engle-Granger:** Simple, one vector only
- ▣ **Johansen:** Multiple vectors, MLE-based

Remember

Tests have low power in small samples. Theory should guide specification.

What's Next?

Extensions and Related Topics

- ▣ **Structural VECM:** Identifying structural shocks
- ▣ **Threshold cointegration:** Nonlinear adjustment
- ▣ **Panel cointegration:** Multiple cross-sections
- ▣ **Fractional cointegration:** Long memory
- ▣ **Time-varying cointegration:** Regime changes

- ▣ **Questions?**

Key Formulas – Summary

Cointegration

- ▣ **Definition:** $Y_t - \beta X_t = u_t \sim I(0)$
- ▣ **Interpretation:** Long-run equilibrium

Engle-Granger Test

- ▣ **Step 1:** $Y_t = \alpha + \beta X_t + u_t$
- ▣ **Step 2:** ADF test on \hat{u}_t
- ▣ **Note:** Special critical values

Cointegration Rank

- ▣ **Rank r :** $0 \leq r \leq K - 1$ relationships

VECM Model

- ▣ **Equation:** $\Delta Y_t = \Pi Y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta Y_{t-i} + \varepsilon_t$
- ▣ **Factorization:** $\Pi = \alpha \beta'$

Interpretation of α and β

- ▣ β : Cointegrating vectors
- ▣ α : Speed of adjustment

Johansen Test

- ▣ **Trace:** $\lambda_{trace} = -T \sum_{i=r+1}^K \ln(1 - \hat{\lambda}_i)$
- ▣ **Max-Eigen:** $\lambda_{max} = -T \ln(1 - \hat{\lambda}_{r+1})$

Question 1

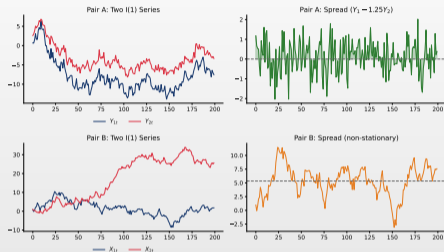
Question

□ Analyze the two pairs of $I(1)$ series below. Which pair is cointegrated?

Answer choices

- (A) Pair A, because the series have the same trend
- (B) Pair B, because the series are uncorrelated
- (C) Pair A, because their spread is stationary
- (D) Both pairs are cointegrated

Question 1: Answer



Correct answer: (C) Pair A – stationary spread

- Cointegration = stationary linear combination, not just correlation
- Pair B's spread is non-stationary → not cointegrated

Question 2

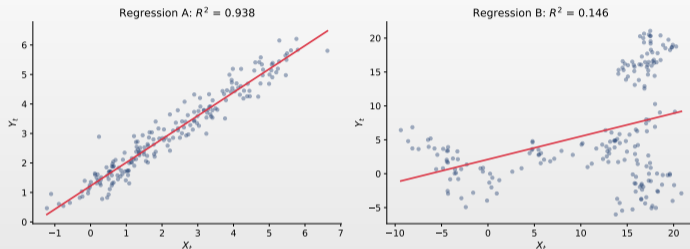
Question

- Both regressions below have high R^2 . How can you distinguish a spurious regression from a genuine one?

Answer choices

- (A) Cannot distinguish – both have high R^2
- (B) Test the residuals: stationary residuals = genuine cointegration
- (C) Check the significance of the β coefficient
- (D) Compare R^2 values: higher = more real relationship

Question 2: Answer



Correct answer: (B) Test the stationarity of residuals

- ▣ Engle-Granger test: if OLS residuals are stationary (ADF), the relationship is genuine
- ▣ High R^2 does NOT imply a real relationship between $I(1)$ variables!

Question 3

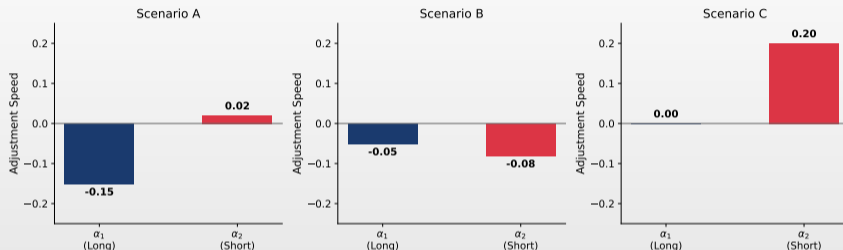
Question

- ☐ In which scenario is the long rate weakly exogenous (does not adjust to disequilibrium)?

Answer choices

- (A) Scenario A: $\alpha_1 = -0.15$, $\alpha_2 = 0.02$
- (B) Scenario B: $\alpha_1 = -0.05$, $\alpha_2 = -0.08$
- (C) Scenario C: $\alpha_1 = 0.00$, $\alpha_2 = 0.20$
- (D) No scenario – both variables must adjust

Question 3: Answer



Correct answer: (C) Scenario C – $\alpha_1 = 0$

- $\alpha_1 = 0$: the long rate does not respond to disequilibrium (weakly exogenous)
- All adjustment is done by the short rate ($\alpha_2 = 0.20$)

Question 4

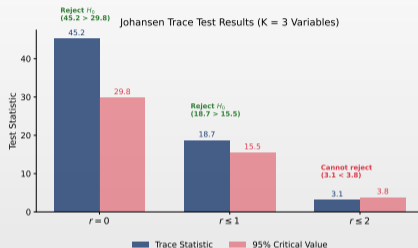
Question

□ Given the Johansen Trace test results for $K = 3$ variables, what is the cointegrating rank?

Answer choices

- (A) $r = 0$ (no cointegrating relationships)
- (B) $r = 1$ (one cointegrating relationship)
- (C) $r = 2$ (two cointegrating relationships)
- (D) $r = 3$ (fully stationary system)

Question 4: Answer



Correct answer: (C) $r = 2$ cointegrating relationships

- Reject $H_0 : r = 0$ (45.2 > 29.8) and $H_0 : r \leq 1$ (18.7 > 15.5)
- Cannot reject $H_0 : r \leq 2$ (3.1 < 3.8) \rightarrow rank is $r = 2$

Question 5

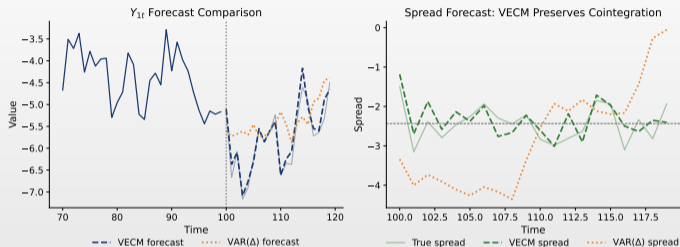
Question

- ☐ What is the main advantage of VECM over VAR in differences for forecasting?

Answer choices

- (A) VECM has fewer parameters to estimate
- (B) VECM preserves the cointegrating relationship in long-run forecasts
- (C) VAR in differences cannot produce forecasts
- (D) No advantage – both are equivalent

Question 5: Answer



Correct answer: (B) VECM preserves cointegration

- VAR(Δ) loses the level relationship \rightarrow spread diverges
- VECM incorporates long-run equilibrium \rightarrow forecast stays coherent

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- ▣ Johansen, S. (1991). Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models, *Econometrica*, 59(6), 1551–1580.

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- ▣ Juselius, K. (2006). *The Cointegrated VAR Model: Methodology and Applications*, Oxford University Press.
- ▣ Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*, Springer.

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- ▣ Hamilton, J.D. (1994). *Time Series Analysis*, Princeton University Press.
- ▣ Banerjee, A., Dolado, J.J., Galbraith, J.W., & Hendry, D.F. (1993). *Co-Integration, Error-Correction, and the Econometric Analysis of Non-Stationary Data*, Oxford University Press.

Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> → Code repository for statistics
- ▣ **Quantinar**: <https://quantinar.com> → Learning platform for quantitative methods
- ▣ **GitHub TSA**: <https://github.com/QuantLet/TSA> → Python code for this course

Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar