

Chapter 2: ARMA Models

Seminar



Seminar Outline

Quiz 1: Lag Operator

Question: What is the result of applying $(1 - L)^2$ to X_t ?

- ☐ A. $X_t - X_{t-1}$
- ☐ B. $X_t - 2X_{t-1} + X_{t-2}$
- ☐ C. $X_t + X_{t-1} + X_{t-2}$
- ☐ D. $X_t - X_{t-2}$

Quiz 1: Solution

Answer: B

Explanation:

$$\begin{aligned}(1 - L)^2 X_t &= (1 - 2L + L^2) X_t \\ &= X_t - 2LX_t + L^2 X_t \\ &= X_t - 2X_{t-1} + X_{t-2}\end{aligned}$$

This is the **second difference** of X_t .

Note: $(1 - L)$ is the first difference operator, $(1 - L)^2$ is the second difference.

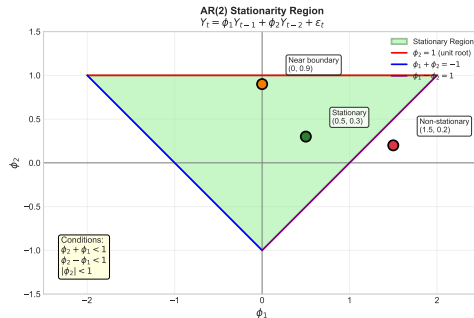
Quiz 2: AR(1) Stationarity

Question: For which value of ϕ is the AR(1) process $X_t = 0.5 + \phi X_{t-1} + \varepsilon_t$ stationary?

- ☐ A. $\phi = 1.2$
- ☐ B. $\phi = 1.0$
- ☐ C. $\phi = -0.8$
- ☐ D. $\phi = -1.5$

Quiz 2: Solution

Answer: $C - \phi = -0.8$ (Stationary)



AR(1) stationarity: $|\phi| < 1$ (root outside unit circle). Only C satisfies: $|-0.8| = 0.8 < 1$

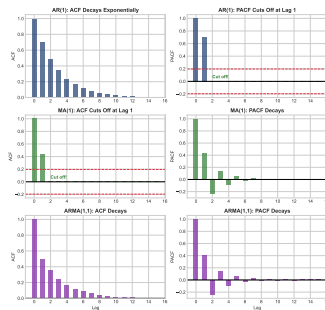
Quiz 3: ACF Pattern

Question: You observe the following ACF pattern: significant spike at lag 1, then all other lags are within confidence bands. The PACF shows gradual decay. What model is suggested?

- ☐ A. AR(1)
- ☐ B. MA(1)
- ☐ C. ARMA(1,1)
- ☐ D. White noise

Quiz 3: Solution

Answer: B – MA(1)



Identification Rule: ACF cuts off \Rightarrow MA(p); PACF cuts off \Rightarrow AR(p); Both decay \Rightarrow ARMA(p, q)

Pattern: ACF cuts off after lag 1 \Rightarrow MA(1); PACF decays \Rightarrow confirms MA structure (not AR)

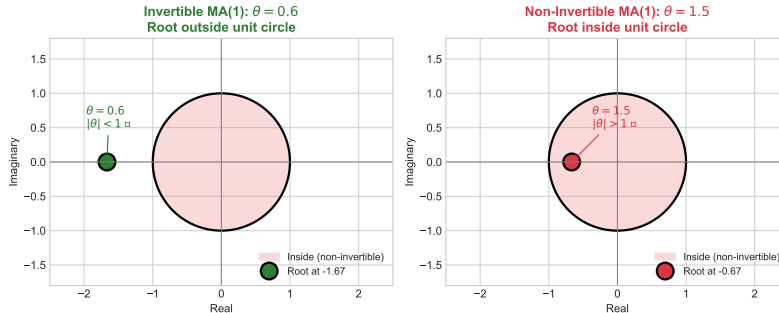
Quiz 4: MA Invertibility

Question: For the MA(1) process $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$, is the process invertible?

- ☐ A. Yes, because MA processes are always invertible
- ☐ B. Yes, because $1.5 > 0$
- ☐ C. No, because $|\theta| = 1.5 > 1$
- ☐ D. No, because MA processes are never invertible

Quiz 4: Solution

Answer: C – Not invertible ($|\theta| = 1.5 > 1$)



MA invertibility: Root $z = -1/\theta$ must be outside unit circle $\Leftrightarrow |\theta| < 1$. Here $z = -0.67$ is inside!

Quiz 5: ARMA Representation

Question: The compact form $\phi(L)X_t = \theta(L)\varepsilon_t$ represents which model?

- ☐ A. Pure AR model
- ☐ B. Pure MA model
- ☐ C. ARMA model
- ☐ D. None of the above

Quiz 5: Solution

Answer: C

Explanation:

- $\phi(L) = 1 - \phi_1 L - \dots - \phi_p L^p$ is the AR polynomial
- $\theta(L) = 1 + \theta_1 L + \dots + \theta_q L^q$ is the MA polynomial

The equation $\phi(L)X_t = \theta(L)\varepsilon_t$ expands to:

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

This is the general **ARMA(p,q)** model.

Special cases:

- $\theta(L) = 1$ (no MA): Pure AR
- $\phi(L) = 1$ (no AR): Pure MA

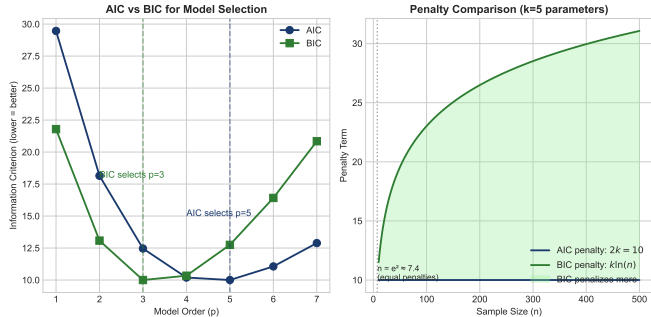
Quiz 6: Information Criteria

Question: When comparing ARMA(1,1) and ARMA(2,1) using BIC, which statement is correct?

- ☒ A. Lower BIC always means better forecasts
- ☐ B. BIC penalizes complexity less than AIC
- ☐ C. The model with lower BIC is preferred
- ☐ D. BIC can only compare models with same number of parameters

Quiz 6: Solution

Answer: C – Lower BIC is preferred



AIC: $-2 \ln(\hat{L}) + 2k$ **BIC:** $-2 \ln(\hat{L}) + k \ln(n)$ BIC penalizes complexity more \Rightarrow more parsimonious models

Quiz 7: Ljung-Box Test

Question: After fitting an ARMA(2,1) model, you run the Ljung-Box test on residuals and get $p\text{-value} = 0.02$. What do you conclude?

- ☐ A. The model is adequate
- ☐ B. Residuals are white noise
- ☐ C. There is significant autocorrelation in residuals
- ☐ D. The model has too many parameters

Quiz 7: Solution

Answer: C

Explanation: The Ljung-Box test has:

- H_0 : Residuals are white noise (no autocorrelation)
- H_1 : Residuals have significant autocorrelation

With $p\text{-value} = 0.02 < 0.05$:

- We **reject** H_0
- Conclusion: residuals are **not** white noise
- The model is **inadequate** — significant structure remains

Next step: Try a different model (e.g., increase p or q)

Quiz 8: Forecasting

Question: For an AR(1) model with $\phi = 0.6$ and mean $\mu = 10$, what happens to forecasts as horizon $h \rightarrow \infty$?

- ☐ A. Forecasts grow without bound
- ☐ B. Forecasts converge to 0
- ☐ C. Forecasts converge to $\mu = 10$
- ☐ D. Forecasts oscillate forever

Quiz 8: Solution

Answer: C

Explanation: For AR(1), the h -step ahead forecast is:

$$\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu)$$

Since $|\phi| = 0.6 < 1$:

$$\lim_{h \rightarrow \infty} \phi^h = 0$$

Therefore:

$$\lim_{h \rightarrow \infty} \hat{X}_{n+h|n} = \mu + 0 \cdot (X_n - \mu) = \mu = 10$$

Key insight: Long-run forecasts from stationary ARMA models always converge to the unconditional mean.

Quiz 9: AR(2) Roots

Question: An AR(2) process has characteristic roots $z_1 = 0.8$ and $z_2 = -0.5$. Is it stationary?

- ☒ A. Yes, because both roots are inside the unit circle
- ☐ B. No, because one root is negative
- ☐ C. No, because roots must be outside the unit circle
- ☐ D. Cannot determine without more information

Quiz 9: Solution

Answer: C

Explanation: For AR stationarity, roots of $\phi(z) = 0$ must lie **outside** the unit circle, i.e., $|z| > 1$.

Here: $|z_1| = 0.8 < 1$ and $|z_2| = 0.5 < 1$ – both **inside** unit circle.

→ **Non-stationary** (actually explosive)

Note: Equivalent condition: coefficients ϕ_1, ϕ_2 must satisfy stationarity triangle.

Quiz 10: MA(q) Properties

Question: For an MA(2) process, the ACF:

- ☐ A. Decays exponentially
- ☐ B. Cuts off after lag 2
- ☐ C. Cuts off after lag 1
- ☐ D. Never cuts off

Quiz 10: Solution

Answer: B

Explanation: For $MA(q)$, the ACF is exactly zero for lags $> q$.

- $MA(1)$: ACF cuts off after lag 1
- $MA(2)$: ACF cuts off after lag 2
- $MA(q)$: ACF cuts off after lag q

This is the key identification feature: ACF cutoff \Rightarrow MA order.
Meanwhile, PACF of MA processes decays (doesn't cut off).

Quiz 11: ARMA Parsimony

Question: Why might ARMA(1,1) be preferred over AR(5) even if both fit equally well?

- ☐ A. ARMA models are always better
- ☐ B. Fewer parameters reduce overfitting risk
- ☐ C. AR models cannot capture trends
- ☐ D. MA components are more stable

Quiz 11: Solution

Answer: B

Explanation: Parsimony principle: prefer simpler models.

- ARMA(1,1): 2 parameters (ϕ_1, θ_1)
- AR(5): 5 parameters (ϕ_1, \dots, ϕ_5)

Fewer parameters means:

- Less risk of overfitting
- Better out-of-sample forecasts
- More interpretable model

BIC penalizes complexity more than AIC, often selecting more parsimonious models.

Quiz 12: Residual Diagnostics

Question: After fitting an ARMA model, the residual ACF shows a significant spike at lag 5. This suggests:

- ☐ A. The model is adequate
- ☐ B. The model may need higher order terms
- ☐ C. Residuals are white noise
- ☐ D. The data is non-stationary

Quiz 12: Solution

Answer: B

Explanation: Good residuals should be white noise with no significant ACF.

A significant spike at lag 5 indicates remaining autocorrelation structure not captured by the model.

Actions:

- Consider adding AR or MA terms
- Check if AR(5) or MA(5) component helps
- Re-run Ljung-Box test after modification

Quiz 13: Wold Decomposition

Question: The Wold decomposition theorem states that any stationary process can be written as:

- ☐ A. A finite AR process
- ☐ B. A finite MA process
- ☐ C. An infinite MA process plus a deterministic component
- ☐ D. An ARIMA process

Quiz 13: Solution

Answer: C

Explanation: Wold's theorem: Any stationary process can be written as:

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \eta_t$$

where η_t is deterministic and $\sum \psi_j^2 < \infty$.

Implication: $MA(\infty)$ is the most general representation. ARMA models are parsimonious approximations to this infinite MA.

Quiz 14: Unit Root vs Trend Stationary

Question: How do you make a unit root process stationary?

- ☐ A. Subtract a linear trend
- ☐ B. Take first differences
- ☐ C. Apply moving average
- ☐ D. Use seasonal adjustment

Quiz 14: Solution

Answer: B

Explanation:

- **Unit root** (stochastic trend): Use **differencing**
- **Trend stationary** (deterministic trend): Use **detrending** (regression)

For random walk $X_t = X_{t-1} + \varepsilon_t$:

$$\Delta X_t = X_t - X_{t-1} = \varepsilon_t$$

which is stationary white noise.

True/False Questions

Determine if each statement is True or False:

- ① An AR(2) process can exhibit pseudo-cyclical behavior.
- ② MA processes require a stationarity condition.
- ③ The PACF of an AR(p) process cuts off after lag p .
- ④ If AIC selects ARMA(2,1) and BIC selects ARMA(1,1), they cannot both be correct.
- ⑤ Forecast confidence intervals narrow as the forecast horizon increases.
- ⑥ The Yule-Walker equations can be used to estimate MA parameters.

True/False: Solutions

- ① **TRUE**: AR(2) with complex roots shows damped oscillations
- ② **FALSE**: MA processes are always stationary; they need *invertibility* condition
- ③ **TRUE**: This is the key identification feature of AR(p)
- ④ **FALSE**: Both can be “correct” — they optimize different criteria (AIC favors fit, BIC favors parsimony)
- ⑤ **FALSE**: Confidence intervals *widen* as horizon increases (more uncertainty)
- ⑥ **FALSE**: Yule-Walker is for AR models only; MA uses MLE

Exercise 1: AR(1) Properties

Problem: Consider the AR(1) process:

$$X_t = 2 + 0.7X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, 9)$$

Calculate:

- ① The mean μ
- ② The variance $\gamma(0)$
- ③ The autocovariance $\gamma(1)$ and $\gamma(2)$
- ④ The autocorrelation $\rho(1)$ and $\rho(2)$

Exercise 1: Solution

Given: $c = 2$, $\phi = 0.7$, $\sigma^2 = 9$

1. Mean:

$$\mu = \frac{c}{1 - \phi} = \frac{2}{1 - 0.7} = \frac{2}{0.3} = 6.67$$

2. Variance:

$$\gamma(0) = \frac{\sigma^2}{1 - \phi^2} = \frac{9}{1 - 0.49} = \frac{9}{0.51} = 17.65$$

3. Autocovariance:

$$\gamma(1) = \phi \cdot \gamma(0) = 0.7 \times 17.65 = 12.35$$

$$\gamma(2) = \phi^2 \cdot \gamma(0) = 0.49 \times 17.65 = 8.65$$

4. Autocorrelation:

$$\rho(1) = \phi = 0.7, \quad \rho(2) = \phi^2 = 0.49$$

Exercise 2: MA(1) Properties

Problem: Consider the MA(1) process:

$$X_t = 5 + \varepsilon_t - 0.4\varepsilon_{t-1}, \quad \varepsilon_t \sim WN(0, 4)$$

Calculate:

- ① The mean μ
- ② The variance $\gamma(0)$
- ③ The autocovariance $\gamma(1)$
- ④ The autocorrelation $\rho(1)$
- ⑤ Is this process invertible?

Exercise 2: Solution

Given: $\mu = 5$, $\theta = -0.4$, $\sigma^2 = 4$

1. Mean:

$$\mathbb{E}[X_t] = \mu = 5$$

2. Variance:

$$\gamma(0) = \sigma^2(1 + \theta^2) = 4(1 + 0.16) = 4 \times 1.16 = 4.64$$

3. Autocovariance at lag 1:

$$\gamma(1) = \theta\sigma^2 = -0.4 \times 4 = -1.6$$

4. Autocorrelation:

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{-1.6}{4.64} = -0.345$$

5. Invertibility: $|\theta| = 0.4 < 1 \rightarrow$ **Yes, invertible**

Exercise 3: Characteristic Roots

Problem: Consider the AR(2) process:

$$X_t = 0.5X_{t-1} + 0.3X_{t-2} + \varepsilon_t$$

- ① Write the characteristic equation
- ② Find the characteristic roots
- ③ Is this process stationary?

Exercise 3: Solution

1. Characteristic equation:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 1 - 0.5z - 0.3z^2 = 0$$

Or: $0.3z^2 + 0.5z - 1 = 0$

2. Roots (using quadratic formula):

$$z = \frac{-0.5 \pm \sqrt{0.25 + 1.2}}{0.6} = \frac{-0.5 \pm 1.204}{0.6}$$

$$z_1 = \frac{0.704}{0.6} = 1.17, \quad z_2 = \frac{-1.704}{0.6} = -2.84$$

3. Stationarity check:

Both roots have $|z| > 1$: $|z_1| = 1.17 > 1$ and $|z_2| = 2.84 > 1$

→ **Stationary** (roots outside unit circle)

Exercise 4: Forecasting

Problem: You have fit an AR(1) model:

$$X_t = 3 + 0.8X_{t-1} + \varepsilon_t, \quad \sigma^2 = 4$$

Given $X_{100} = 20$, calculate:

- ① The 1-step ahead forecast $\hat{X}_{101|100}$
- ② The 2-step ahead forecast $\hat{X}_{102|100}$
- ③ The long-run forecast $\hat{X}_{100+h|100}$ as $h \rightarrow \infty$
- ④ The 95% confidence interval for $\hat{X}_{101|100}$

Exercise 4: Solution

Given: $c = 3$, $\phi = 0.8$, $\sigma^2 = 4$, $X_{100} = 20$

Mean: $\mu = \frac{3}{1-0.8} = 15$

1. One-step forecast:

$$\hat{X}_{101|100} = c + \phi X_{100} = 3 + 0.8 \times 20 = 19$$

2. Two-step forecast:

$$\hat{X}_{102|100} = c + \phi \hat{X}_{101|100} = 3 + 0.8 \times 19 = 18.2$$

3. Long-run forecast:

$$\lim_{h \rightarrow \infty} \hat{X}_{100+h|100} = \mu = 15$$

4. 95% CI for 1-step:

$$\text{MSFE}(1) = \sigma^2 = 4, \quad \sqrt{\text{MSFE}(1)} = 2$$

$$CI : 19 \pm 1.96 \times 2 = [15.08, 22.92]$$

Python Exercise 1: Simulate and Fit AR(1)

Task:

- 1 Simulate 500 observations from AR(1) with $\phi = 0.7$
- 2 Plot the series and ACF/PACF
- 3 Fit an AR(1) model and check if $\hat{\phi} \approx 0.7$
- 4 Examine residual diagnostics

Hint code:

```
np.random.seed(42)
n = 500
phi = 0.7
x = np.zeros(n)
for t in range(1, n):
    x[t] = phi * x[t-1] + np.random.randn()
```

Python Exercise 2: Model Selection

Task:

- 1 Load a real time series (e.g., stock returns)
- 2 Check stationarity using ADF test
- 3 Compare AIC/BIC for ARMA(1,0), ARMA(0,1), ARMA(1,1), ARMA(2,1)
- 4 Select the best model
- 5 Generate forecasts with confidence intervals

Key functions:

- `adfuller()` for stationarity test
- `ARIMA(data, order=(p,0,q)).fit()` for fitting
- `results.aic`, `results.bic` for criteria
- `results.get_forecast(h)` for predictions

Python Exercise 3: Diagnostic Checking

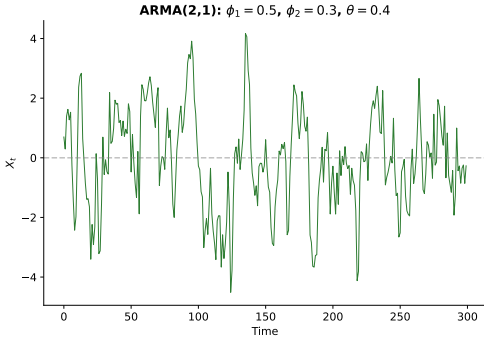
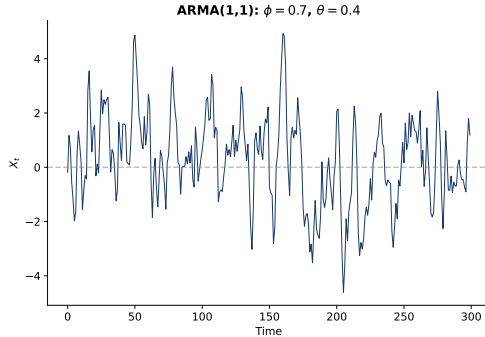
Task: After fitting a model, perform complete diagnostics:

- ➊ Plot residuals over time
- ➋ Plot ACF of residuals
- ➌ Create Q-Q plot
- ➍ Run Ljung-Box test
- ➎ Check if AR/MA roots are outside unit circle

Key functions:

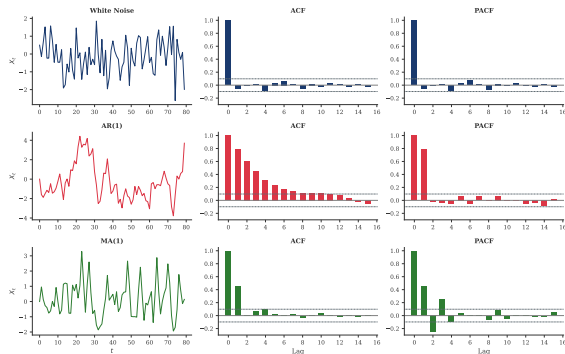
- `results.resid` for residuals
- `plot_acf(resid)` for ACF plot
- `stats.probplot(resid)` for Q-Q plot
- `acorr_ljungbox(resid)` for portmanteau test
- `results.arroots`, `results.marroots` for roots

Case Study: Industrial Production Index



- US Industrial Production: monthly data, already stationary (growth rates)
- Shows typical ARMA patterns: mean-reverting with short-term dependence
- Volatility clustering visible – ARMA captures the conditional mean
- Suitable for ARMA modeling without differencing

ACF/PACF Pattern Recognition



- ACF shows gradual decay – suggests AR component
- PACF cuts off after lag 2 – suggests AR(2) might be appropriate
- Some significant lags in ACF beyond lag 2 – MA terms may help
- Pattern consistent with ARMA(2,1) or similar low-order models

ARMA Estimation Results

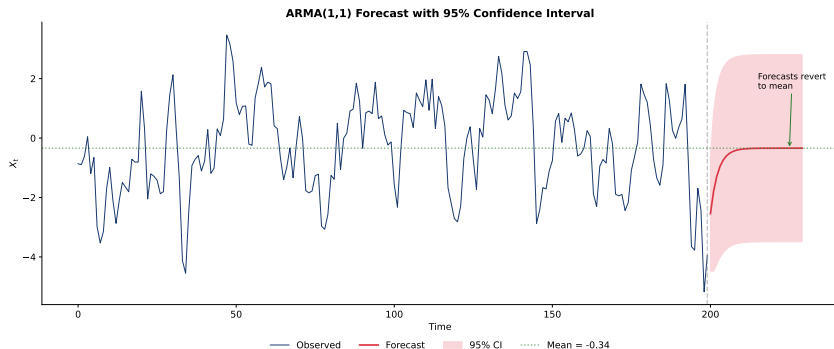
Model: ARMA(2,1) for Industrial Production Growth

| Parameter | Estimate | Std. Error | z-stat | p-value |
|--------------------|----------|------------|--------|---------|
| c (const) | 0.156 | 0.048 | 3.25 | 0.001 |
| ϕ_1 (AR.L1) | 0.423 | 0.089 | 4.75 | < 0.001 |
| ϕ_2 (AR.L2) | 0.187 | 0.072 | 2.60 | 0.009 |
| θ_1 (MA.L1) | -0.156 | 0.091 | -1.71 | 0.087 |

Model Selection

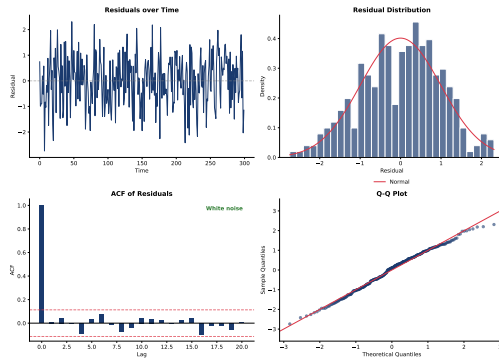
AIC: -412.5, BIC: -398.2. Model passes stationarity and invertibility checks.

Forecast Performance



- ARMA forecasts mean-revert toward unconditional mean
- Short-term forecasts capture recent dynamics
- Confidence intervals expand with forecast horizon
- Comparison with naive forecast shows ARMA improvement

Model Diagnostics



- Residuals appear random with no systematic patterns
- ACF of residuals within confidence bounds
- Q-Q plot shows approximate normality
- Ljung-Box test: $p > 0.05$ – no significant autocorrelation in residuals

Discussion 1: Model Selection

Scenario: You're modeling monthly inflation rates. After checking stationarity (passed), you find:

- ACF: significant at lags 1, 2, 3, then decays
- PACF: significant at lags 1, 2, then cuts off
- AIC selects ARMA(2,3)
- BIC selects ARMA(2,0) = AR(2)

Questions:

- 1 What does the ACF/PACF pattern suggest?
- 2 Why do AIC and BIC disagree?
- 3 Which model would you choose and why?
- 4 What additional checks would you perform?

Discussion 2: Forecast Evaluation

Scenario: You fit an ARMA(1,1) model to daily stock returns. The in-sample fit looks good (Ljung-Box p-value = 0.45), but out-of-sample RMSE is worse than a simple random walk forecast.

Questions:

- 1 Is this surprising? Why or why not?
- 2 What does this tell us about stock return predictability?
- 3 Should you conclude the ARMA model is useless?
- 4 What alternatives might you consider?

Hint: Think about the Efficient Market Hypothesis and what ARMA captures vs. what it doesn't (e.g., volatility clustering).

Discussion 3: Real-World Application

Scenario: A central bank economist asks you to forecast quarterly GDP growth for policy planning.

Questions:

- ➊ What preliminary analysis would you do before fitting ARMA?
- ➋ GDP is often non-stationary — how would you handle this?
- ➌ Would you use AIC or BIC for model selection? Why?
- ➍ How would you communicate forecast uncertainty to policymakers?
- ➎ What limitations of ARMA models should you mention?

Key Takeaways from Today's Seminar

- ① **AR models:** Current value depends on past values
 - Stationarity: $|\phi| < 1$ for AR(1)
 - PACF cuts off at lag p
- ② **MA models:** Current value depends on past shocks
 - Always stationary; invertibility: $|\theta| < 1$ for MA(1)
 - ACF cuts off at lag q
- ③ **Model selection:** Use ACF/PACF patterns + information criteria
- ④ **Diagnostics:** Residuals must be white noise (Ljung-Box test)
- ⑤ **Forecasting:** Point forecasts converge to mean; uncertainty grows

Next Seminar: ARIMA and Seasonal Models