Chapter 4: SARIMA Models

Seminar

Seminar Outline

1. Review Quiz
2. Practice Problems
3. Worked Examples
4. Real Data Analysis
5. Discussion Topics
6. Exercises for Self-Study

Quiz 1: Seasonal Differencing

Question

For monthly data with annual seasonality, what does the operator $(1 - L^{12})$ do?

- A) Takes 12 consecutive differences
- B) Computes $Y_t - Y_{t-12}$
- C) Averages over 12 months
- D) Removes the first 12 observations

Quiz 1: Seasonal Differencing

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- A) Takes 12 consecutive differences
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- C) Averages over 12 months
- D) Removes the first 12 observations

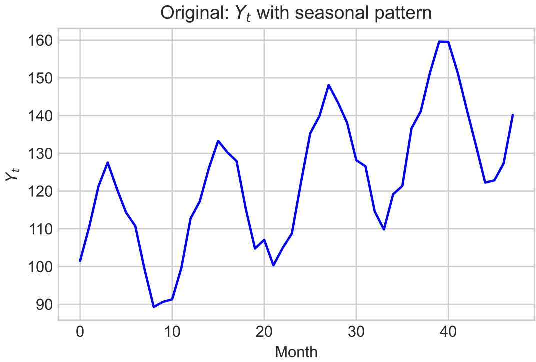
Answer: B – Computes $Y_t - Y_{t-12}$

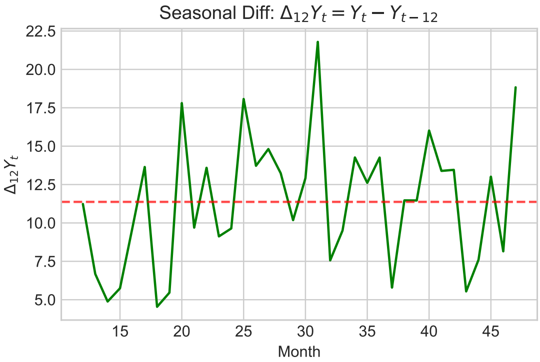
Seasonal difference operator:

$$(1 - L^{12})Y_t = Y_t - L^{12}Y_t = Y_t - Y_{t-12}$$

Example (January sales): $Y_{Jan2025} - Y_{Jan2024}$
Effect: Removes stable annual seasonal pattern
Note: $(1 - L^s)$ for any seasonal period s (quarterly: $s = 4$, weekly: $s = 52$)

Visual: Seasonal Difference





Seasonal differencing removes annual patterns by comparing same periods across years.

Quiz 2: SARIMA Notation

Question

What does $\text{SARIMA}(1, 1, 1) \times (1, 1, 1)_{12}$ represent?

- A) 12 different ARIMA models
- B) ARIMA with 12 AR and 12 MA terms
- C) ARIMA(1,1,1) with seasonal ARIMA(1,1,1) at period 12
- D) A model requiring 12 years of data

Quiz 2: Answer

Answer: C – ARIMA(1,1,1) with seasonal ARIMA(1,1,1) at period 12

SARIMA$(p, d, q) \times (P, D, Q)_s$ **Notation**

$$\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D Y_t = \theta(L)\Theta(L^s)\varepsilon_t$$

Regular (Non-Seasonal)

p	= AR order	(Number of AR lags)
d	= Differencing	(Regular differences)
q	= MA order	(Number of MA lags)

Seasonal

P	= Seasonal AR	(SAR lags at $s, 2s, \ldots$)
D	= Seasonal Diff	($(1-L^s)^D$)
Q	= Seasonal MA	(SMA lags at $s, 2s, \ldots$)
s	= Period	(Seasonal period)

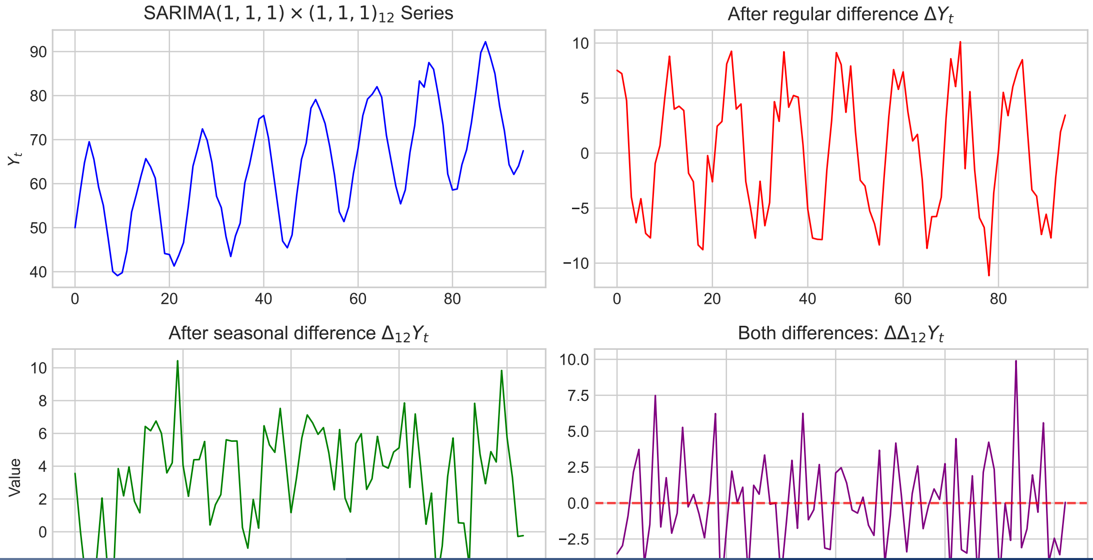
Example: SARIMA$(1, 1, 1) \times (0, 1, 1)_{12}$

Monthly data with: AR(1), MA(1), one regular diff,

one seasonal diff at lag 12, seasonal MA(1)

$$(1-\phi_1 L)(1-\Phi_1 L^{12})(1-L)(1-L^{12})Y_t = (1+\theta_1 L)(1+\Theta_1 L^{12})\varepsilon_t$$

Visual: SARIMA Model Structure



Quiz 3: The Airline Model

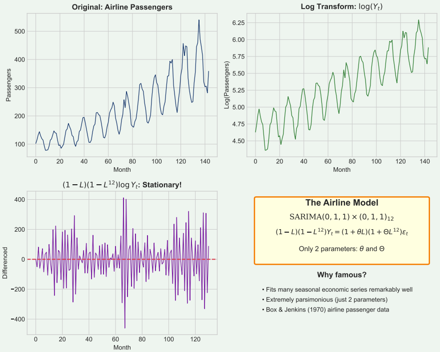
Question

The "airline model" refers to SARIMA$(0,1,1) \times (0,1,1)_{12}$. How many parameters does it have (excluding variance)?

A) 2 parameters
B) 4 parameters
C) 6 parameters
D) 12 parameters

Quiz 3: Answer

Answer: A – 2 parameters (θ_1 and Θ_1)



Airline model: $(1-L)(1-L^{12})Y_t = (1+\theta_1 L)(1+\Theta_1 L^{12})\varepsilon_t$
Remarkably fits many seasonal economic series (Box & Jenkins, 1970)

Quiz 4: ACF of Seasonal Data

Question

For monthly data with strong seasonality, where would you expect to see significant ACF spikes?

A) Only at lag 1
B) Only at lag 12
C) At lags 12, 24, 36, ...
D) Randomly distributed

Quiz 4: ACF of Seasonal Data

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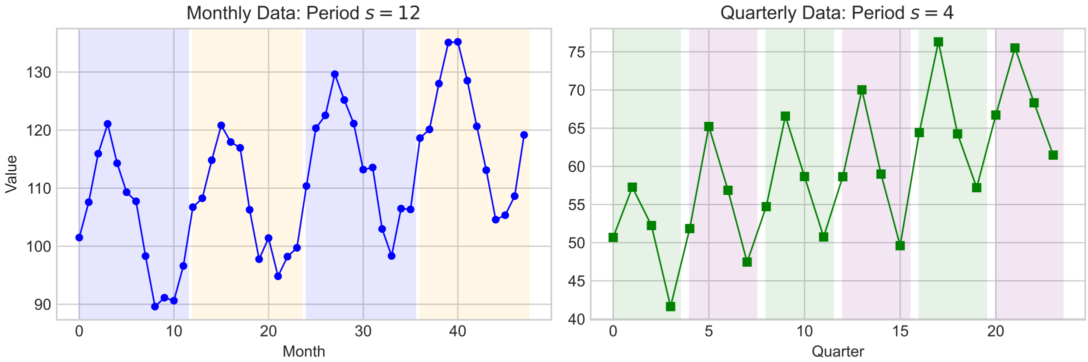
- C) At lags 12, 24, 36, ...
- D) Randomly distributed

Answer: C – At lags 12, 24, 36, ...

Intuition: January 2024 is similar to January 2023, 2022, etc.
ACF pattern: Spikes at lags $s, 2s, 3s, \ldots$ ($\rho_{12}, \rho_{24}, \rho_{36} \neq 0$)
Diagnostic: Slow decay at seasonal lags $\Rightarrow D = 1$; Cutoff after lag $s \Rightarrow Q = 1$

Visual: Seasonality Patterns



Seasonal patterns repeat at regular intervals (monthly, quarterly, etc.) and may be additive or multiplicative.

Quiz 5: Multiplicative Structure

Question

In SARIMA, what does “multiplicative structure” mean?

- A) The seasonal amplitude grows proportionally
- B) Regular and seasonal polynomials are multiplied
- C) We multiply the data by seasonal factors
- D) The model is estimated using multiplication

Quiz 5: Multiplicative Structure

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- B) Regular and seasonal polynomials are multiplied
- C) We multiply the data by seasonal factors
- D) The model is estimated using multiplication

Answer: B – Regular and seasonal polynomials are multiplied

Multiplicative SARIMA: $\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D Y_t = \theta(L)\Theta(L^s)\varepsilon_t$
Example: $(1-\phi_1 L)(1-\Phi_1 L^{12}) = 1 - \phi_1 L - \Phi_1 L^{12} + \phi_1\Phi_1 L^{13}$
Cross-term $\phi_1\Phi_1 L^{13}$: Captures interaction between short and long dynamics

Quiz 6: Seasonal vs Regular Differencing

Question

When would you apply both regular ($d = 1$) and seasonal ($D = 1$) differencing?

- A) When data has only a trend
- B) When data has only seasonality
- C) When data has both trend and seasonal non-stationarity
- D) Never – they cancel each other

Quiz 6: Seasonal vs Regular Differencing

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- C) When data has both trend and seasonal non-stationarity
- D) Never – they cancel each other

Answer: C – Both trend and seasonal non-stationarity

Combined: $W_t = (1 - L)(1 - L^{12})Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$
When needed: ACF slow decay at lags 1,2,3... $\Rightarrow d = 1$; at lags 12,24,36... $\Rightarrow D = 1$

Examples: Airline passengers, retail sales, energy demand

Quiz 7: Detecting Seasonality from ACF

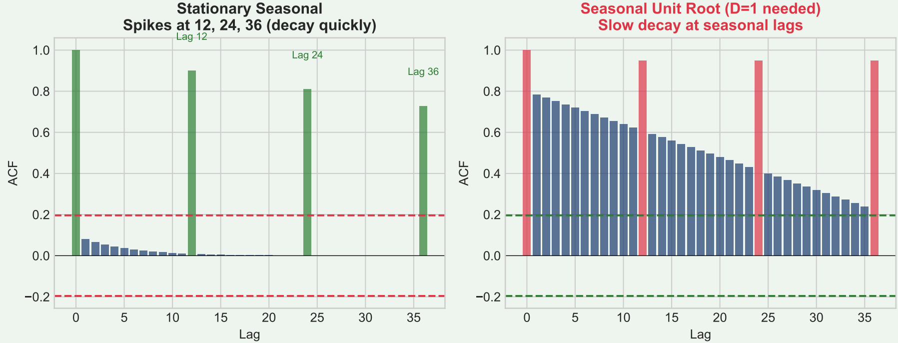
Question

The ACF of a monthly time series shows slow decay at lags 12, 24, and 36. What does this suggest?

- A) The series is stationary
- B) The series needs regular differencing only
- C) The series has a seasonal unit root requiring $D = 1$
- D) The series is white noise

Quiz 7: Answer

Answer: C – Seasonal unit root requiring $D = 1$



Left: Stationary seasonal (fast decay at seasonal lags)
Right: Seasonal unit root (slow decay $\Rightarrow$ need $D = 1$)

Quiz 8: Multiplicative vs Additive Seasonality

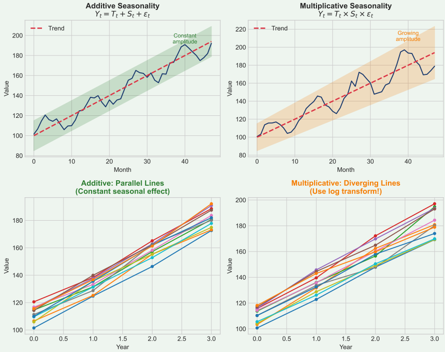
Question

If the seasonal amplitude of a time series grows proportionally with the level, this indicates:

- A) Additive seasonality – use $(1 - L^s)$
- B) Multiplicative seasonality – use log transformation
- C) No seasonality present
- D) Need for regular differencing only

Quiz 8: Answer

Answer: B – Multiplicative seasonality, use log transformation



Multiplicative: Seasonal amplitude grows with level (diverging lines)
Solution: Apply log transformation before fitting SARIMA

Quiz 9: Seasonal Subseries Plot

Question

In a seasonal subseries plot, what indicates multiplicative seasonality?

- A) Lines for each month are parallel
- B) Lines for each month diverge (spread increases over time)
- C) All months have the same mean
- D) Lines are horizontal

Quiz 9: Seasonal Subseries Plot

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In a seasonal subseries plot, what indicates multiplicative seasonality?

- A) Lines for each month are parallel
- B) Lines for each month diverge (spread increases over time)
- C) All months have the same mean
- D) Lines are horizontal

Answer: B – Lines diverge (spread increases over time)

Subseries plot: Groups data by month, plots each month's values across years
Parallel $\Rightarrow$ Additive; **Diverging** $\Rightarrow$ Multiplicative; **Horizontal** $\Rightarrow$ No trend
Action: If multiplicative, apply log before fitting SARIMA

Quiz 10: Invertibility in SARIMA

Question

For SARIMA$(0,1,1) \times (0,1,1)_{12}$ to be invertible, which condition must hold?

- A) $|\theta_1| < 1$ only
- B) $|\Theta_1| < 1$ only
- C) Both $|\theta_1| < 1$ and $|\Theta_1| < 1$
- D) No invertibility condition exists for MA models

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- C) Both $|\theta_1| < 1$ and $|\Theta_1| < 1$
- D) No invertibility condition exists for MA models

Answer: C – Both $|\theta_1| < 1$ and $|\Theta_1| < 1$

Invertibility: All MA roots outside unit circle
Multiplicative MA: $(1+\theta_1 L)(1+\Theta_1 L^{12})$
Roots: Regular $|z| = |-1/\theta_1| > 1 \Leftrightarrow |\theta_1| < 1$; Seasonal $|\Theta_1| < 1$

Both conditions required for overall invertibility!

Quiz 11: HEGY Test

Question

The HEGY test is used to:

- A) Estimate SARIMA parameters
- B) Test for unit roots at different frequencies (trend and seasonal)
- C) Check residual normality
- D) Compare SARIMA models using information criteria

Quiz 11: HEGY Test

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The HEGY test is used to:

- A) Estimate SARIMA parameters
- B) Test for unit roots at different frequencies (trend and seasonal)
- C) Check residual normality
- D) Compare SARIMA models using information criteria

Answer: B – Test for unit roots at different frequencies

HEGY test (Hylleberg-Engle-Granger-Yoo, 1990):
Tests at: Zero freq ($\omega = 0$) $\Rightarrow d = 1$; Nyquist ($\omega = \pi$); Seasonal $\Rightarrow D = 1$

Decision: Reject all $\Rightarrow$ seasonal dummies; Don't reject seasonal $\Rightarrow$ seasonal differencing

Quiz 12: Seasonal MA Identification

Question

After applying $(1 - L)(1 - L^{12})$, the ACF shows a single significant spike at lag 12 only (no spike at lag 1). The PACF decays at seasonal lags. This suggests:

- A) SARIMA$(0, 1, 0) \times (0, 1, 1)_{12}$
- B) SARIMA$(0, 1, 1) \times (0, 1, 0)_{12}$
- C) SARIMA$(1, 1, 0) \times (1, 1, 0)_{12}$
- D) SARIMA$(0, 1, 1) \times (0, 1, 1)_{12}$

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- B) SARIMA$(0,1,1) \times (0,1,0)_{12}$
- C) SARIMA$(1,1,0) \times (1,1,0)_{12}$
- D) SARIMA$(0,1,1) \times (0,1,1)_{12}$

Answer: A – SARIMA$(0,1,0) \times (0,1,1)_{12}$

Pattern: Regular lags – no spikes in ACF/PACF; Seasonal lags – ACF cuts off at s, PACF decays
Interpretation: No regular MA ($q=0$); Seasonal MA(1) indicated ($Q=1$)
Model: $(1-L)(1-L^{12})Y_t = (1+\Theta_1 L^{12})\varepsilon_t$

Quiz 13: Over-differencing

Question

After differencing, the ACF shows a large negative spike at lag 1 or lag s. This typically indicates:

- A) The model needs more AR terms
- B) The series has been over-differenced
- C) The series is perfectly stationary
- D) Heteroskedasticity is present

Quiz 13: Over-differencing

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- C) The series is perfectly stationary
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Answer: B – The series has been over-differenced

Signature: ACF at lag 1 $\approx -0.5 \Rightarrow$ over-diff at d; ACF at lag $s \approx -0.5 \Rightarrow$ over-diff at D
Why? $\Delta^2 Y_t = \varepsilon_t - \varepsilon_{t-1}$ is MA(1) with $\theta = -1$, giving $\rho_1 = -0.5$
Fix: Reduce d or D by one and re-examine ACF/PACF

Quiz 14: Forecasting Horizon

Question

For a SARIMA model with $D = 1$, what happens to forecast confidence intervals as the horizon $h \to \infty$?

- A) They converge to a fixed width
- B) They grow without bound
- C) They shrink to zero
- D) They oscillate seasonally

Quiz 14: Forecasting Horizon

Question

For a SARIMA model with $D = 1$, what happens to forecast confidence intervals as the horizon $h \to \infty$?

- A) They converge to a fixed width
- B) They grow without bound
- C) They shrink to zero
- D) They oscillate seasonally

Answer: B – They grow without bound

Unit root property: Any unit root causes unbounded forecast variance
For SARIMA with $D = 1$: $\text{Var}(\hat{Y}_{T+h} - Y_{T+h}) \to \infty$ as $h \to \infty$
Intuition: Seasonal shocks accumulate; long-range forecasts have wide CIs

Quiz 15: Seasonal Period Selection

Question

You have daily data showing clear weekly patterns. What seasonal period s should you use in a SARIMA model?

- A) $s = 12$ (monthly)
- B) $s = 7$ (weekly)
- C) $s = 365$ (yearly)
- D) $s = 24$ (hourly)

Quiz 15: Seasonal Period Selection

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- A) $s = 12$ (monthly)
- B) $s = 7$ (weekly)
- C) $s = 365$ (yearly)
- D) $s = 24$ (hourly)

Answer: B – $s = 7$ (weekly)

Data	**Pattern**	**Period s**
Daily	Weekly	7
Monthly	Annual	12
Quarterly	Annual	4

Rule: $s =$ observations per cycle of dominant pattern

Quiz 16: Seasonal AR Component

Question

In the seasonal component $\Phi(L^s) = 1 - \Phi_1 L^s$, what does the coefficient $\Phi_1 = 0.8$ tell us?

- A) 80% of this period's value comes from the previous period
- B) There is 80% correlation between consecutive observations
- C) 80% of this period's value is explained by the same period last year
- D) The seasonal pattern explains 80% of variance

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- B) There is 80% correlation between consecutive observations
- C) 80% of this period's value is explained by the same period last year
- D) The seasonal pattern explains 80% of variance

Answer: C – 80% explained by same period last year

SAR(1): $Y_t = \Phi_1 Y_{t-12} + \varepsilon_t$
With $\Phi_1 = 0.8$: $Y_{Jan2024} = 0.8 \cdot Y_{Jan2023} + \varepsilon_t$
Interpretation: Strong seasonal persistence – 80% explained by same month last year
Stationarity: Requires $|\Phi_1| < 1$ (satisfied here)

Quiz 17: Seasonal Stationarity

Question

A seasonal process with $\Phi_1 = 1$ in SARIMA$(0,0,0) \times (1,0,0)_{12}$ is:

- A) Stationary
- B) Has a seasonal unit root (seasonally integrated)
- C) Explosive
- D) Undefined

Quiz 17: Seasonal Stationarity

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A seasonal process with $\Phi_1 = 1$ in SARIMA$(0,0,0) \times (1,0,0)_{12}$ is:

- A) Stationary
- B) Has a seasonal unit root (seasonally integrated)
- C) Explosive
- D) Undefined

Answer: B – Has a seasonal unit root

Model: $Y_t = Y_{t-12} + \varepsilon_t$ (seasonal random walk)
Properties: Variance grows with time; each month follows its own RW; need $D = 1$
Analogy: Like regular random walk but at seasonal frequency

Quiz 18: Model Comparison

Question

Model A: $\text{SARIMA}(1,1,1) \times (1,1,1)_{12}$ has $\text{AIC} = 520$. Model B: $\text{SARIMA}(0,1,1) \times (0,1,1)_{12}$ has $\text{AIC} = 525$. Which statement is most accurate?

- A) Model A is always better since it has lower AIC
- B) Model B should be preferred due to parsimony despite higher AIC
- C) The AIC difference of 5 suggests Model A is substantially better
- D) We cannot compare models with different orders

Quiz 18: Model Comparison

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Model A: SARIMA$(1,1,1) \times (1,1,1)_{12}$ has AIC $= 520$. Model B: SARIMA$(0,1,1) \times (0,1,1)_{12}$ has AIC $= 525$. Which statement is most accurate?

- A) Model A is always better since it has lower AIC
- B) Model B should be preferred due to parsimony despite higher AIC
- C) The AIC difference of 5 suggests Model A is substantially better
- D) We cannot compare models with different orders

Answer: C – AIC difference of 5 suggests Model A is substantially better

Rule of thumb: ΔAIC < 2: equivalent; 2–10: some evidence; > 10: strong evidence
Here: ΔAIC $= 5$ suggests Model A meaningfully better

Always: Also check residual diagnostics and forecast performance!

Quiz 19: Seasonal Patterns in Residuals

Question

After fitting a SARIMA model, you notice significant ACF spikes at lags 12 and 24 in the residuals. What does this indicate?

A) The model is correctly specified
B) The seasonal component is inadequate
C) The data is not seasonal
D) Overfitting has occurred

Quiz 19: Seasonal Patterns in Residuals

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- A) The model is correctly specified
- B) The seasonal component is inadequate
- C) The data is not seasonal
- D) Overfitting has occurred

Answer: B – The seasonal component is inadequate

Diagnostics: Good residuals should be white noise (no significant ACF)
Seasonal ACF in residuals: Model hasn't captured seasonal structure; try increasing P or Q; verify D is correct
Action: Try SARIMA with higher seasonal order, check Ljung-Box at seasonal lags

Quiz 20: Practical Forecasting

Question

You're forecasting monthly retail sales with SARIMA$(0,1,1) \times (0,1,1)_{12}$. For the 13-month-ahead forecast, which historical observations are most influential?

- A) Only the most recent observation
- B) The observation from the same month last year
- C) All observations equally
- D) Only observations from the same month in all previous years

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- A) Only the most recent observation
- B) The observation from the same month last year
- C) All observations equally
- D) Only observations from the same month in all previous years

Answer: B – The observation from the same month last year

For 13-month ahead: Most influential is Y_{T-11} (same month last year), also Y_T and Y_{T-12}
Intuition: "Next January looks like last January, adjusted for recent trend"

True/False Questions (1-6)

Determine whether each statement is True or False:

1. The seasonal period s for quarterly data with annual patterns is $s = 4$.
2. SARIMA models can only handle one seasonal frequency.
3. If AIC selects SARIMA$(1,1,1) \times (1,1,1)_{12}$ and BIC selects the airline model, BIC is always wrong.
4. The Kruskal-Wallis test can detect seasonality without assuming normality.
5. After fitting a SARIMA model, residuals should show no significant ACF at seasonal lags.
6. Log transformation converts multiplicative seasonality to additive.

True/False Solutions (1-6)

1. **TRUE**: Quarterly data with annual cycle has $s = 4$ quarters per year.
2. **TRUE**: Standard SARIMA handles one s; multiple seasonalities need TBATS or Fourier terms.
3. **FALSE**: BIC penalizes complexity more; simpler model may be better for interpretation/forecasting.
4. **TRUE**: Kruskal-Wallis is nonparametric, comparing distributions across seasons.
5. **TRUE**: Residual ACF should be within confidence bands at ALL lags including seasonal.
6. **TRUE**: $\log(T \times S \times \varepsilon) = \log T + \log S + \log \varepsilon$ (additive form).

Problem 1: Expanding the Seasonal Difference

Exercise

Expand $(1 - L)(1 - L^{12})Y_t$ fully. What observations are involved?

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Exercise

Expand $(1-L)(1-L^{12})Y_t$ fully. What observations are involved?

Solution

$(1-L)(1-L^{12}) = 1 - L - L^{12} + L^{13}$

Therefore: $(1-L)(1-L^{12})Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$

Interpretation: This is the difference of differences:

- First seasonal difference: $Y_t - Y_{t-12}$ (this year vs last year)
- Then regular difference: $(Y_t - Y_{t-12}) - (Y_{t-1} - Y_{t-13})$

Problem 2: Airline Model Expansion

Exercise

Write out the full equation for the airline model SARIMA$(0, 1, 1) \times (0, 1, 1)_{12}$:
$(1 - L)(1 - L^{12})Y_t = (1 + \theta_1 L)(1 + \Theta_1 L^{12})\varepsilon_t$

Problem 2: Airline Model Expansion

Exercise

Write out the full equation for the airline model SARIMA$(0,1,1) \times (0,1,1)_{12}$:
$(1-L)(1-L^{12})Y_t = (1+\theta_1 L)(1+\Theta_1 L^{12})\varepsilon_t$

Solution

Expand the MA side: $(1+\theta_1 L)(1+\Theta_1 L^{12}) = 1 + \theta_1 L + \Theta_1 L^{12} + \theta_1\Theta_1 L^{13}$

Full model: $Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13} = \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-12} + \theta_1\Theta_1\varepsilon_{t-13}$

Note: The cross-term $\theta_1\Theta_1 L^{13}$ is the multiplicative interaction between regular and seasonal MA components.

Problem 3: Parameter Count

Exercise

How many parameters (excluding σ^2) are in SARIMA$(2, 1, 1) \times (1, 0, 1)_4$?

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Exercise

How many parameters (excluding σ^2) are in SARIMA$(2, 1, 1) \times (1, 0, 1)_4$?

Solution

- Regular AR($p = 2$): $\phi_1, \phi_2 \Rightarrow 2$ parameters
- Regular MA($q = 1$): $\theta_1 \Rightarrow 1$ parameter
- Seasonal AR($P = 1$): $\Phi_1 \Rightarrow 1$ parameter
- Seasonal MA($Q = 1$): $\Theta_1 \Rightarrow 1$ parameter

Total: 5 parameters

Note: The differencing orders ($d = 1$, $D = 0$) don't add parameters – they're transformations applied to the data.

Problem 4: SARIMA Forecasting

Exercise

Given the airline model with $\theta_1 = -0.4$ and $\Theta_1 = -0.6$, and:

- $Y_T = 500, \; Y_{T-1} = 495, \; Y_{T-11} = 480, \; Y_{T-12} = 470$
- $\varepsilon_T = 5$, $\varepsilon_{T-11} = -3$, $\varepsilon_{T-12} = 2$

Forecast Y_{T+1}.

Problem 4: SARIMA Forecasting

Exercise

Given the airline model with $\theta_1 = -0.4$ and $\Theta_1 = -0.6$, and:

- $Y_T = 500,\ Y_{T-1} = 495,\ Y_{T-11} = 480,\ Y_{T-12} = 470$
- $\varepsilon_T = 5$, $\varepsilon_{T-11} = -3$, $\varepsilon_{T-12} = 2$

Forecast Y_{T+1}.

Solution

From the model: $Y_{T+1} = Y_T + Y_{T-11} - Y_{T-12} + \varepsilon_{T+1} + \theta_1\varepsilon_T + \Theta_1\varepsilon_{T-11} + \theta_1\Theta_1\varepsilon_{T-12}$
Setting $\mathbb{E}[\varepsilon_{T+1}] = 0$:
$\hat{Y}_{T+1} = 500 + 480 - 470 + 0 + (-0.4)(5) + (-0.6)(-3) + (-0.4)(-0.6)(2)$
$= 510 - 2 + 1.8 + 0.48 = \mathbf{510.28}$

Problem 5: Identifying Seasonal Period

Exercise

Match each data type with its typical seasonal period s:

1. Quarterly GDP data
2. Monthly retail sales
3. Weekly restaurant reservations
4. Daily electricity demand

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Exercise

Match each data type with its typical seasonal period s:

1. Quarterly GDP data
2. Monthly retail sales
3. Weekly restaurant reservations
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Solution

1. Quarterly GDP: $s = 4$ (annual cycle over 4 quarters)
2. Monthly retail sales: $s = 12$ (annual cycle over 12 months)
3. Weekly restaurant reservations: $s = 7$ (weekly cycle) or $s = 52$ (annual)
4. Daily electricity demand: $s = 7$ (weekly pattern) or $s = 365$ (annual)

Note: Some series have multiple seasonal patterns (e.g., daily data may have weekly AND annual cycles).

Example: Monthly Retail Sales Analysis

Scenario

You have 5 years of monthly retail sales data showing clear December peaks and January troughs. Build an appropriate SARIMA model.

Step-by-step Approach

1. **Visual inspection**: Plot shows upward trend + strong December spikes
2. **Seasonal period**: Monthly data with annual pattern $\Rightarrow s = 12$
3. **Transformation**: Consider $\log(Y_t)$ if seasonal amplitude grows with level
4. **Differencing**: Try $(1 - L)(1 - L^{12})Y_t$ – check ACF/PACF
5. **Model selection**: Start with airline model, compare via AIC

Example: ACF/PACF Interpretation for Seasonal Data

Observed Patterns (after differencing)

- ACF: Significant at lags 1, 12; cuts off after lag 1 and lag 12
- PACF: Significant at lags 1, 12, 13; decays at multiples of 12

Interpretation

Regular component: ACF cuts off at 1 $\Rightarrow$ MA(1)
Seasonal component: ACF significant only at lag 12 $\Rightarrow$ seasonal MA(1)
Suggested model: SARIMA$(0, d, 1) \times (0, D, 1)_{12}$ – the airline model!

Alternative check: If PACF showed cutoff at seasonal lags instead of ACF, consider seasonal AR terms.

Example: Python Implementation

Fitting SARIMA in Python

```
from statsmodels.tsa.statespace.sarimax import SARIMAX
import pmdarima as pm

# Manual fit
model = SARIMAX(y, order=(0,1,1), seasonal_order=(0,1,1,12))
results = model.fit()
print(results.summary())

# Automatic selection
auto_model = pm.auto_arima(y, seasonal=True, m=12,
                           start_p=0, max_p=2,
                           start_q=0, max_q=2,
                           d=1, D=1,
                           trace=True)
```

Example: Interpreting SARIMA Output

Sample statsmodels Output

```
                          SARIMAX Results
==============================================================
Model:            SARIMAX(0,1,1)x(0,1,1,12)   AIC:     1348.52
                                               BIC:     1358.21
==============================================================
                 coef     std err        z      P>|z|
--------------------------------------------------------------
ma.L1         -0.4018       0.072    -5.58      0.000
ma.S.L12      -0.5521       0.081    -6.82      0.000
sigma2      1254.3201     142.856     8.78      0.000
```

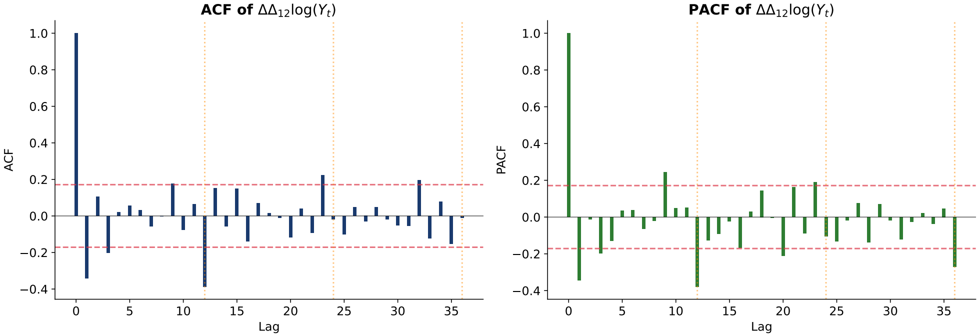
Interpretation

- $\hat{\theta}_1 = -0.40$: Negative MA means positive shocks reduce next period's value
- $\hat{\Theta}_1 = -0.55$: Same-season correlation is captured
- Both coefficients significant ($p < 0.001$); $|\theta|, |\Theta| < 1$ – invertible

Case Study: Airline Passengers (1949–1960)

- Classic Box-Jenkins dataset: 144 monthly observations
- Clear **upward trend** and **seasonal pattern** (summer peaks)
- Seasonal amplitude **grows with level** $\Rightarrow$ multiplicative seasonality
- Suggests: log transformation + SARIMA modeling

ACF/PACF Analysis After Differencing



- After $(1-L)(1-L^{12})\log(Y_t)$: series appears stationary
- Significant spike at lag 1 in ACF $\Rightarrow$ MA(1) component
- Significant spike at lag 12 in ACF $\Rightarrow$ Seasonal MA(1) component
- Pattern suggests: **SARIMA**$(0,1,1)(0,1,1)_{12}$ (airline model)

SARIMA Estimation Results: Airline Data

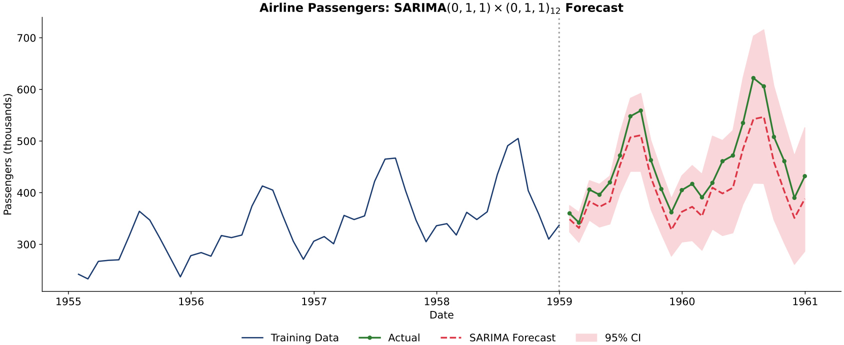
Model: SARIMA$(0,1,1)(0,1,1)_{12}$ on log(Passengers)

Parameter	Estimate	Std. Error	z-stat	p-value
θ_1 (MA.L1)	-0.4018	0.0896	-4.48	< 0.001
Θ_1 (MA.S.L12)	-0.5569	0.0731	-7.62	< 0.001
σ^2	0.00135	–	–	–

Model Fit Statistics

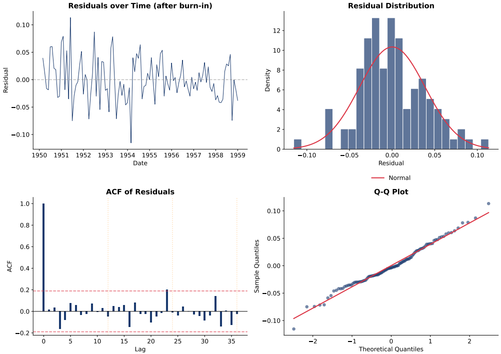
- Log-Likelihood: 244.70
- AIC: -483.40, BIC: -474.53
- Both MA coefficients significant and within invertibility bounds

Forecast: 24 Months Ahead



- Forecasts capture both trend and seasonal pattern
- 95% confidence intervals widen over forecast horizon
- Seasonal peaks (July-August) and troughs (February) clearly visible
- Model successfully extrapolates the multiplicative seasonal pattern

Model Diagnostics



- Residuals appear random with no systematic patterns
- Distribution approximately normal (Q-Q plot close to diagonal)
- ACF of residuals within confidence bounds – no significant autocorrelation
- Ljung-Box test: $p > 0.05$ at all tested lags $\Rightarrow$ adequate model

Discussion: Deterministic vs Stochastic Seasonality

Key Question

When should you use seasonal dummies vs SARIMA for seasonal data?

Considerations

Seasonal dummies (deterministic):

- Fixed, repeating pattern each year
- Same December effect every year
- Appropriate when seasonality is stable

SARIMA (stochastic):

- Evolving seasonal pattern
- This year's December depends on last year's December
- Better when seasonal amplitude varies

Discussion: Log Transformation

Key Question

When should you take logarithms before fitting SARIMA?

Guidelines

Use log transformation when:

- Seasonal fluctuations grow with the level (multiplicative seasonality)
- Variance increases over time
- Data is strictly positive (prices, sales, counts)

Avoid log when:

- Seasonal pattern is additive (constant amplitude)
- Data contains zeros or negatives
- Already on a rate/ratio scale

Tip: Compare AIC of models with and without log transformation.

Discussion: Multiple Seasonalities

Challenge

Daily sales data may have both weekly (7-day) and annual (365-day) seasonal patterns. How do you handle this?

Approaches

1. **Nested SARIMA**: Model at shorter frequency, include longer as exogenous
2. **TBATS/BATS models**: Explicitly handle multiple seasonalities
3. **Fourier terms**: Add sin/cos terms for each seasonal frequency
4. **Prophet/similar**: Modern tools designed for multiple seasonalities

Note: Standard SARIMA handles only one seasonal period. For complex seasonality, consider specialized methods.

Discussion: Forecasting Seasonal Data

Key Question

What are the unique challenges of forecasting seasonal time series?

Challenges and Solutions

- **Horizon matters**: 12-month forecast means predicting a full cycle
- **Uncertainty grows**: Seasonal forecasts compound regular uncertainty
- **Turning points**: Capturing when seasons peak/trough
- **Structural breaks**: COVID-19 disrupted many seasonal patterns

Best practices:

- Use rolling-origin cross-validation
- Compare against seasonal naive benchmark
- Report forecast intervals, especially at seasonal horizons

Take-Home Exercises

1. **Theoretical**: Show that $(1 - L)(1 - L^4)$ can be written as $(1 - L - L^4 + L^5)$ and explain what this transformation does to quarterly data with annual seasonality.

2. **Computation**: For SARIMA$(1,0,0) \times (1,0,0)_4$ with $\phi_1 = 0.5$ and $\Phi_1 = 0.8$, write out the full AR polynomial and identify all non-zero coefficients.

3. **Applied**: Download monthly airline passenger data and:
   - Plot the series and identify trend/seasonality
   - Apply appropriate transformations
   - Fit the airline model and interpret coefficients
   - Generate 24-month forecasts with confidence intervals

4. **Comparison**: Fit both SARIMA$(0,1,1) \times (0,1,1)_{12}$ and SARIMA$(1,1,0) \times (1,1,0)_{12}$ to the airline data. Compare using AIC, BIC, and residual diagnostics. Which is preferred?

Exercise Solutions Hints

Hints

1. Expand $(1 - L)(1 - L^4) = 1 \cdot 1 - 1 \cdot L^4 - L \cdot 1 + L \cdot L^4 = 1 - L - L^4 + L^5$
2. AR polynomial: $(1 - \phi_1 L)(1 - \Phi_1 L^4) = 1 - 0.5L - 0.8L^4 + 0.4L^5$
3. For airline data:
   - Use log transformation (multiplicative seasonality)
   - Both $d = 1$ and $D = 1$ needed
   - Typical estimates: $\theta_1 \approx -0.4$, $\Theta_1 \approx -0.6$
4. The MA-based airline model typically fits better than pure AR seasonal model for this data (lower AIC).

Key Takeaways from This Seminar

Main Points

1. Seasonal differencing $(1 - L^s)$ removes stochastic seasonality
2. SARIMA notation: $(p, d, q) \times (P, D, Q)_s$ separates regular and seasonal
3. The airline model is surprisingly effective for many datasets
4. Multiplicative structure creates interaction terms
5. ACF/PACF show patterns at both regular and seasonal lags
6. Log transformation often needed for multiplicative seasonality

Next Steps

Chapter 5 will cover multivariate time series: VAR models, Granger causality, and cointegration.