



Time Series Analysis and Forecasting

## Chapter 3: ARIMA Models

Seminar



# Seminar Outline

- 1 Review Quiz
- 2 True/False Questions
- 3 Practice Problems
- 4 Worked Examples
- 5 Real Data Analysis
- 6 Discussion Topics
- 7 Summary

## Quiz 1: Integration Order

### Question

A time series  $Y_t$  requires two differences to become stationary. What is its order of integration?

- ☐ A)  $I(0)$
- ☐ B)  $I(1)$
- ☐ C)  $I(2)$
- ☐ D) Cannot be determined

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Answer: C –  $I(2)$

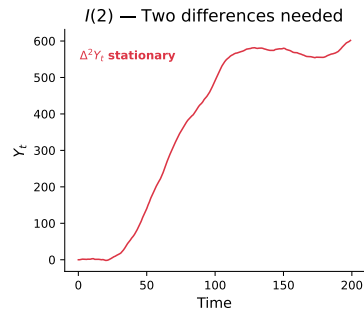
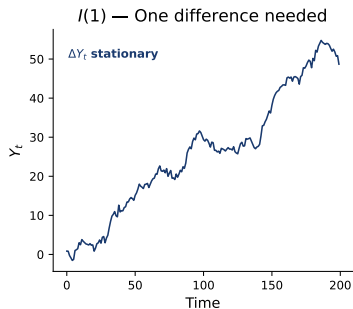
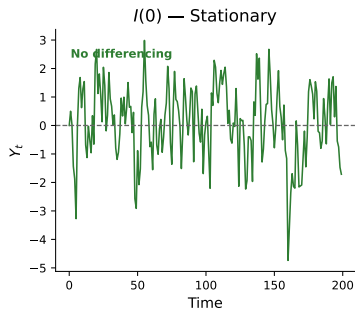
**Definition:**  $Y_t \sim I(d)$  if  $\Delta^d Y_t$  is stationary but  $\Delta^{d-1} Y_t$  is not.

**Example:** If  $Y_t$  follows  $\Delta^2 Y_t = \varepsilon_t$ , then:

- $\Delta Y_t = \Delta Y_{t-1} + \varepsilon_t$  (still has unit root)
- $\Delta^2 Y_t = \varepsilon_t$  (white noise, stationary)

**Real-world:** Price levels may be  $I(2)$  when inflation itself is non-stationary.

## Visual: Integrated Processes



$I(0)$ : stationary.  $I(1)$ : one difference needed.  $I(2)$ : two differences needed to become stationary.

## Quiz 2: Random Walk Properties

### Question

For a random walk  $Y_t = Y_{t-1} + \varepsilon_t$  with  $\text{Var}(\varepsilon_t) = \sigma^2$ , what is  $\text{Var}(Y_t)$ ?

- ☐ A)  $\sigma^2$
- ☐ B)  $t \cdot \sigma^2$
- ☐ C)  $\sigma^2/t$
- ☐ D)  $\sigma^2/(1 - \phi^2)$

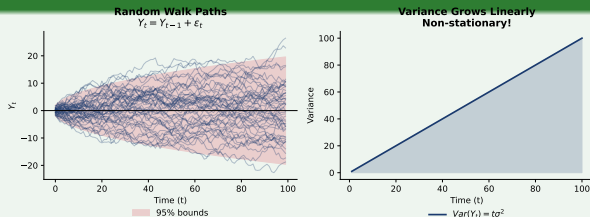
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- ☐ D)  $\sigma^2/(1 - \phi^2)$

Answer: B –  $t \cdot \sigma^2$



**Proof:**  $Y_t = \sum_{i=1}^t \varepsilon_i \Rightarrow \text{Var}(Y_t) = t\sigma^2$  (grows linearly  $\Rightarrow$  non-stationary!)

## Quiz 3: ADF Test Hypotheses

### Question

In the Augmented Dickey-Fuller test, what is the null hypothesis?

- ☐ A) The series is stationary
- ☐ B) The series has a unit root
- ☐ C) The series has no autocorrelation
- ☐ D) The series is normally distributed



## Quiz 3: ADF Test Hypotheses

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Answer: B – The series has a unit root

**ADF regression:**  $\Delta Y_t = \alpha + \gamma Y_{t-1} + \sum_{j=1}^p \delta_j \Delta Y_{t-j} + \varepsilon_t$

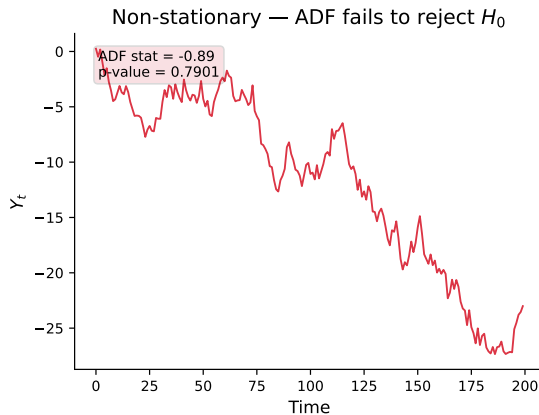
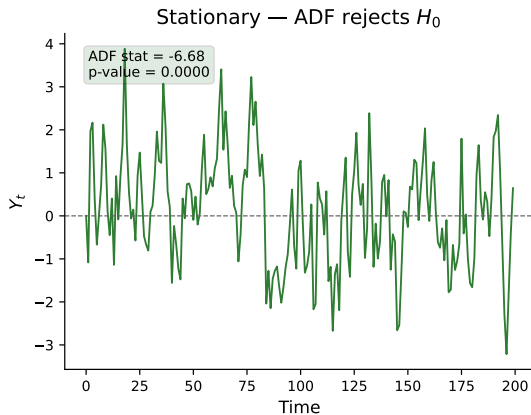
**Hypotheses:**

- $H_0 : \gamma = 0$  (unit root, non-stationary)
- $H_1 : \gamma < 0$  (stationary)

**Decision:** Reject  $H_0$  if  $t$ -statistic  $<$  critical value (e.g.,  $-2.86$  at 5%)

**Note:** Uses special Dickey-Fuller distribution, not standard  $t$ .

## Visual: ADF Test



Left: stationary – ADF rejects unit root. Right: non-stationary – ADF fails to reject.

## Quiz 4: ARIMA Notation

### Question

What does ARIMA(2,1,1) mean?

- ☐ A) AR(2) on differenced data with MA(1) errors
- ☐ B) AR(1) with 2 differences and MA(1)
- ☐ C) MA(2) with 1 difference and AR(1)
- ☐ D) 2 lags, 1 trend, 1 seasonal component

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- ☐ D) 2 lags, 1 trend, 1 seasonal component

Answer: A – AR(2) on differenced data with MA(1) errors

**ARIMA**( $p, d, q$ ):  $\phi(L)(1-L)^d Y_t = \theta(L)\varepsilon_t$

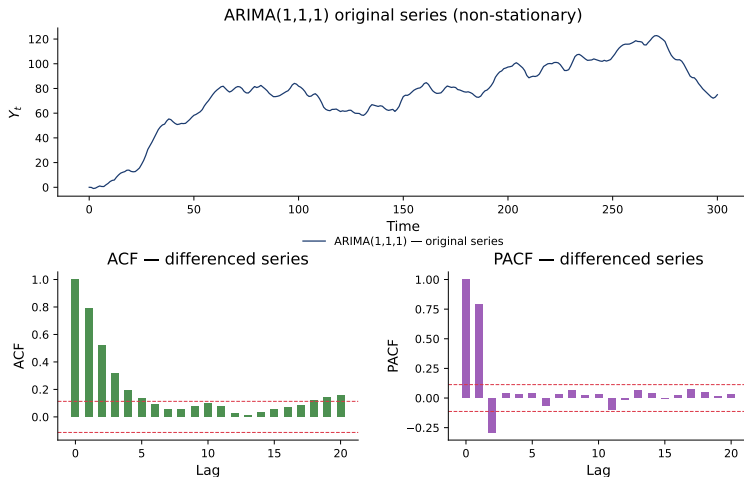
**ARIMA(2,1,1) expands to:**

$$(1 - \phi_1 L - \phi_2 L^2)(1 - L)Y_t = (1 + \theta_1 L)\varepsilon_t$$

Or equivalently:  $(1 - \phi_1 L - \phi_2 L^2)\Delta Y_t = (1 + \theta_1 L)\varepsilon_t$

**Interpretation:** First difference the series, then fit ARMA(2,1) to  $\Delta Y_t$ .

## Visual: ARIMA Process



Top: original ARIMA series. Bottom: after differencing, use ACF/PACF to identify AR and MA orders.

## Quiz 5: Difference Operator

### Question

What is  $(1 - L)^2 Y_t$  expanded?

- ☐ A)  $Y_t - Y_{t-1}$
- ☐ B)  $Y_t - 2Y_{t-1} + Y_{t-2}$
- ☐ C)  $Y_t + 2Y_{t-1} + Y_{t-2}$
- ☐ D)  $Y_t - Y_{t-2}$

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- ☐ D)  $Y_t - Y_{t-2}$

Answer: B -  $Y_t - 2Y_{t-1} + Y_{t-2}$

**Binomial expansion:**  $(1 - L)^2 = 1 - 2L + L^2$

**Applied:**  $(1 - L)^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$  (the “change in changes”)

## Quiz 6: KPSS vs ADF

### Question

How does the KPSS test differ from the ADF test?

- ☐ A) KPSS tests for seasonality, ADF tests for trends
- ☐ B) KPSS has stationarity as null, ADF has unit root as null
- ☐ C) KPSS is more powerful than ADF
- ☐ D) There is no difference



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- ☐ C) KPSS is more powerful than ADF
- ☐ D) There is no difference

Answer: B – Reversed null hypotheses

ADF Test	KPSS Test
$H_0$ : Unit Root	$H_0$ : Stationary
$H_1$ : Stationary	$H_1$ : Unit Root
Reject if $t\text{-stat} < \text{critical}$	Reject if $LM > \text{critical}$

Decision Matrix		
ADF rejects	KPSS fails to reject	→ Stationary
ADF fails to reject	KPSS rejects	→ Unit Root
Both reject	or both fail	→ Inconclusive

**Strategy:** Use both tests together for robust inference!

## Quiz 7: Overdifferencing

### Question

If  $Y_t \sim I(1)$  and we compute  $\Delta^2 Y_t$ , what happens?

- ☐ A) We get a better stationary series
- ☐ B) We introduce artificial negative autocorrelation
- ☐ C) The variance decreases
- ☐ D) Nothing changes

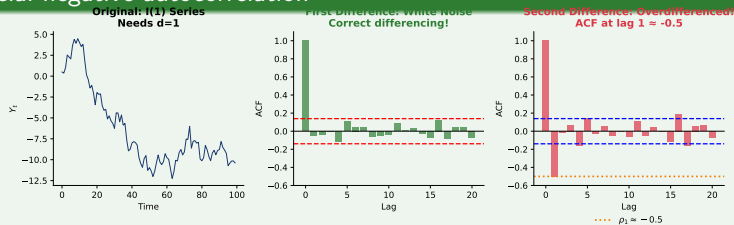
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Answer: B – Artificial negative autocorrelation



Diagnostic: ACF at lag 1  $\approx -0.5$  signals overdifferencing. Reduce  $d$  by 1!

## Quiz 8: Forecast Variance

### Question

For an ARIMA(0,1,0) model (random walk), how does forecast variance behave as horizon  $h$  increases?

- ☐ A) Stays constant
- ☐ B) Decreases to zero
- ☐ C) Grows linearly with  $h$
- ☐ D) Converges to a finite limit

## Quiz 8: Forecast Variance

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- ☐ D) Converges to a finite limit

Answer: C – Grows linearly with  $h$

**Random walk forecast:**  $\hat{Y}_{T+h|T} = Y_T$  (best forecast is current value)

**Forecast error:**  $Y_{T+h} - \hat{Y}_{T+h|T} = \sum_{i=1}^h \varepsilon_{T+i}$

**Variance:**

$$\text{Var}(Y_{T+h} - \hat{Y}_{T+h|T}) = h\sigma^2$$

**95% CI:**  $Y_T \pm 1.96\sqrt{h}\sigma$  (widens with  $\sqrt{h}$ )

## Quiz 9: Unit Root Test Power

### Question

The ADF test has low power when:

- ☐ A) Sample size is very large
- ☐ B) The true root is close to but not equal to 1
- ☐ C) The series has no trend
- ☐ D) The series is clearly stationary

## Quiz 9: Unit Root Test Power

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- ☐ A) Sample size is very large
- ☒ B) The true root is close to but not equal to 1
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- ☐ D) The series is clearly stationary

Answer: B – Root close to but not equal to 1

**Example:** AR(1) with  $\phi = 0.95$  vs random walk ( $\phi = 1$ ) – similar ACF patterns!

**Low power:** High probability of Type II error (failing to reject false  $H_0$ )

**Solutions:** Larger samples, Phillips-Perron test, panel unit root tests

## Quiz 10: ARIMA Model Selection

### Question

After differencing once, the ACF shows a spike at lag 1 only, and PACF decays. The appropriate model is:

- ☐ A) ARIMA(1,1,0)
- ☐ B) ARIMA(0,1,1)
- ☐ C) ARIMA(1,1,1)
- ☐ D) ARIMA(0,2,1)



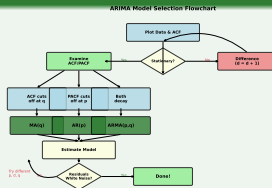
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- B) ARIMA(0,1,1)
- C) ARIMA(1,1,1)
- D) ARIMA(0,2,1)

Answer: B – ARIMA(0,1,1)



Pattern: ACF cuts off at lag 1, PACF decays  $\Rightarrow$  MA(1). Full model: ARIMA(0,1,1) = IMA(1,1)

## Quiz 11: Trend Stationarity vs Difference Stationarity

### Question

A trend-stationary process is made stationary by:

- ☐ A) Taking first differences
- ☐ B) Removing the deterministic trend via regression
- ☐ C) Taking second differences
- ☐ D) Applying seasonal adjustment

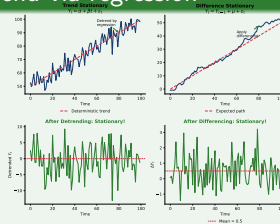
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Answer: B – Removing deterministic trend via regression



**Trend-stationary:** Detrend (shocks temporary). **Difference-stationary:** Difference (shocks permanent).

## Quiz 12: ARIMA Invertibility

### Question

ARIMA(0,1,1) with  $\theta_1 = 1.2$  is:

- ☐ A) Stationary and invertible
- ☐ B) Non-stationary but invertible
- ☐ C) Non-stationary and non-invertible
- ☐ D) Stationary but non-invertible

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Answer: C – Non-stationary and non-invertible

**Stationarity:**  $d = 1$  (unit root)  $\Rightarrow$  **Non-stationary**

**Invertibility:** MA root  $z = -1/1.2 = -0.833$  is inside unit circle;  $|\theta_1| > 1 \Rightarrow$  **Non-invertible**

**Fix:** Rewrite with  $\theta^* = 1/1.2 = 0.833$  and adjust variance.

## Quiz 13: Spurious Regression

### Question

Regressing one random walk on another independent random walk typically shows:

- ☐ A) No significant relationship
- ☒ B) High  $R^2$  and significant t-statistics (spuriously)
- ☐ C) Negative correlation
- ☐ D) Perfect multicollinearity

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- ☐ C) Negative correlation
- ☐ D) Perfect multicollinearity

Answer: B – High  $R^2$  and significant  $t$ -statistics (spuriously)

**Granger & Newbold (1974):** Spurious regression phenomenon

**Symptoms:** High  $R^2$  ( $> 0.9$ ), significant  $t$ -stats, low Durbin-Watson ( $\ll 2$ ), non-stationary residuals

**Solutions:** Difference both series, or test for cointegration

## Quiz 14: Long-Run Forecast

### Question

The long-run forecast from ARIMA(1,1,0) with  $\phi_1 = 0.7$  converges to:

- ☐ A) Zero
- ☐ B) The unconditional mean
- ☐ C) A linear trend extrapolation
- ☐ D) The last observed value



## Quiz 14: Long-Run Forecast

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- ☒ C) A linear trend extrapolation
- ☐ D) The last observed value

Answer: C – A linear trend extrapolation

**Model:**  $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$ ; **Long-run:**  $\hat{Y}_{T+h} \approx Y_T + h \cdot \frac{c}{1-\phi_1}$

**Key differences:** Stationary ARMA  $\rightarrow$  mean; I(1) no drift  $\rightarrow$  last value; I(1) with drift  $\rightarrow$  linear

### Question

Determine if each statement is True or False:

- ❶ An  $I(2)$  process requires two differences to become stationary.
- ❷ The ADF test always includes a constant term.
- ❸  $ARIMA(0,1,0)$  is another name for a random walk.
- ❹ Differencing a stationary series makes it “more stationary.”
- ❺ The KPSS test has stationarity as the null hypothesis.
- ❻  $ARIMA$  models can only capture linear patterns.

*Answer on next slide...*

### Answers

- ① An  $I(2)$  process requires two differences to become stationary.  
 $I(d)$  means  $d$  differences needed.  $I(2)$  = two unit roots. TRUE
- ② The ADF test always includes a constant term.  
You choose: no constant, constant only, or constant + trend. FALSE
- ③  $ARIMA(0,1,0)$  is another name for a random walk.  
 $(1 - L)Y_t = \varepsilon_t \Rightarrow Y_t = Y_{t-1} + \varepsilon_t$ . TRUE
- ④ Differencing a stationary series makes it “more stationary.”  
Over-differencing creates non-invertible MA; hurts model performance. FALSE
- ⑤ The KPSS test has stationarity as the null hypothesis.  
KPSS:  $H_0$  = stationary. Opposite of ADF. TRUE
- ⑥  $ARIMA$  models can only capture linear patterns.  
 $ARIMA$  is linear in parameters. Nonlinear patterns need GARCH, neural nets, etc. TRUE

## Problem 1: Unit Root Testing

### Exercise

You have quarterly GDP data for 80 quarters. The ADF test (with constant and trend) gives a test statistic of  $-2.85$ . The 5% critical value is  $-3.41$ .

- 1 What is your conclusion about stationarity?
- 2 What would you do next?

## Problem 1: Unit Root Testing

### Exercise

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- 1 What is your conclusion about stationarity?
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### Solution

- 1 Since  $-2.85 > -3.41$ , we **fail to reject**  $H_0$ . The data appears to have a unit root (non-stationary).
- 2 Take the first difference  $\Delta Y_t$  and repeat the ADF test on the differenced series to confirm it is now stationary.

## Problem 2: Model Identification

### Exercise

After differencing a time series once, the ACF shows:

- Significant spike at lag 1 ( $\rho_1 = 0.4$ )
- All other lags insignificant

The PACF shows gradual decay.

What ARIMA model is suggested?

## Problem 2: Model Identification

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The PACF shows gradual decay.

What ARIMA model is suggested?

### Solution

- ACF cuts off after lag 1  $\Rightarrow$  MA(1) component
- PACF decays  $\Rightarrow$  Confirms MA structure
- Since we differenced once:  $d = 1$

**Suggested model: ARIMA(0,1,1) or IMA(1,1)**

## Problem 3: ARIMA Equation

### Exercise

Write out the full equation for ARIMA(1,1,1):

$$(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$$

Expand this completely in terms of  $Y_t$ ,  $Y_{t-1}$ ,  $Y_{t-2}$ , etc.



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Expand this completely in terms of  $Y_t$ ,  $Y_{t-1}$ ,  $Y_{t-2}$ , etc.

### Solution

Expanding  $(1 - \phi_1 L)(1 - L) = 1 - L - \phi_1 L + \phi_1 L^2 = 1 - (1 + \phi_1)L + \phi_1 L^2$ :

$$Y_t - (1 + \phi_1)Y_{t-1} + \phi_1 Y_{t-2} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Or equivalently:

$$Y_t = c + (1 + \phi_1)Y_{t-1} - \phi_1 Y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

## Problem 4: Forecast Calculation

### Exercise

Given ARIMA(0,1,1):  $\Delta Y_t = \varepsilon_t + 0.3\varepsilon_{t-1}$

At time  $T$ :  $Y_T = 100$ ,  $\hat{\varepsilon}_T = 2$ ,  $\sigma^2 = 4$

Calculate:

- ①  $\hat{Y}_{T+1|T}$  (one-step forecast)
- ②  $\hat{Y}_{T+2|T}$  (two-step forecast)

## Problem 4: Forecast Calculation

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Calculate:

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### Solution

- ①  $\hat{Y}_{T+1|T} = Y_T + 0.3\hat{\varepsilon}_T = 100 + 0.3(2) = \mathbf{100.6}$
- ②  $\hat{Y}_{T+2|T} = \hat{Y}_{T+1|T} + 0.3 \cdot 0 = 100.6 + 0 = \mathbf{100.6}$   
(Future shocks  $\varepsilon_{T+1}, \varepsilon_{T+2}$  are forecast as 0)

## Problem 5: Confidence Intervals

### Exercise

Continuing from Problem 4, calculate the 95% forecast intervals for  $\hat{Y}_{T+1|T}$  and  $\hat{Y}_{T+2|T}$ .

Recall:  $\sigma^2 = 4$ ,  $\theta_1 = 0.3$

## Problem 5: Confidence Intervals

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Recall:  $\sigma^2 = 4$ ,  $\theta_1 = 0.3$

### Solution

For IMA(1,1), the MA( $\infty$ ) weights are  $\psi_0 = 1$ ,  $\psi_j = 1 + \theta_1$  for  $j \geq 1$ .

**1-step:**  $\text{Var}(e_{T+1}) = \sigma^2 \psi_0^2 = 4$ , so  $SE = 2$

$$100.6 \pm 1.96(2) = [96.68, 104.52]$$

**2-step:**  $\text{Var}(e_{T+2}) = \sigma^2(\psi_0^2 + \psi_1^2) = 4(1 + 1.3^2) = 10.76$ ,  $SE = 3.28$

$$100.6 \pm 1.96(3.28) = [94.17, 107.03]$$

## Example: Testing for Unit Root in Stock Prices

### Scenario

You have daily closing prices for a stock over 500 days. You want to determine if prices follow a random walk.

### Step-by-step Approach

- ➊ **Visual inspection:** Plot prices – likely shows trend
- ➋ **ADF test on prices:** Expect to fail to reject  $H_0$  (unit root)
- ➌ **Take log returns:**  $r_t = \ln(P_t/P_{t-1}) = \Delta \ln(P_t)$
- ➍ **ADF test on returns:** Should reject  $H_0$  (stationary)
- ➎ **Conclusion:** Log prices are  $I(1)$ , returns are  $I(0)$

## Example: Box-Jenkins for Inflation Data

### Scenario

Monthly inflation rates for 10 years. Build an ARIMA model.

### Workflow

- ❶ **Plot & test:** ADF suggests borderline – try both  $d = 0$  and  $d = 1$
- ❷ **If  $d = 0$ :** Fit ARMA models, compare AIC
- ❸ **If  $d = 1$ :** Examine ACF/PACF of  $\Delta Y_t$ 
  - ACF: spike at lag 1, then cuts off
  - PACF: decays
  - $\Rightarrow$  Try ARIMA(0,1,1)
- ❹ **Estimate:** Fit ARIMA(0,1,1), check coefficients
- ❺ **Diagnose:** Ljung-Box on residuals (want  $p > 0.05$ )
- ❻ **Compare:** AIC of ARIMA(0,1,1) vs ARMA(1,1) on levels

## Example: Interpreting Python Output

### statsmodels ARIMA Output

```

                        ARIMA Model Results
=====
Dep. Variable:          D.y    No. Observations:   99
Model:                 ARIMA(1,1,1)    AIC          285.32
                                   BIC          295.63
=====

```

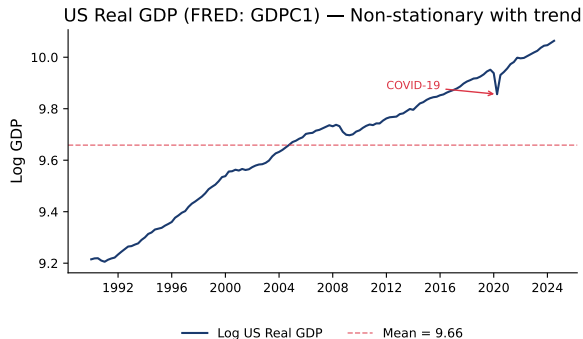
	coef	std err	z	P> z
const	0.0521	0.048	1.085	0.278
ar.L1	0.4532	0.102	4.443	0.000
ma.L1	-0.2891	0.118	-2.450	0.014
sigma2	1.2340	0.176	7.011	0.000

### Interpretation

- AR coefficient (0.45) is significant, MA coefficient (-0.29) is significant
- Constant (0.052) not significant – could set  $c = 0$
- Check:  $|\phi_1| < 1$  (stationary),  $|\theta_1| < 1$  (invertible) – OK!

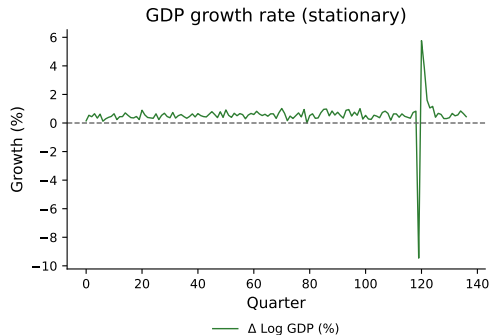
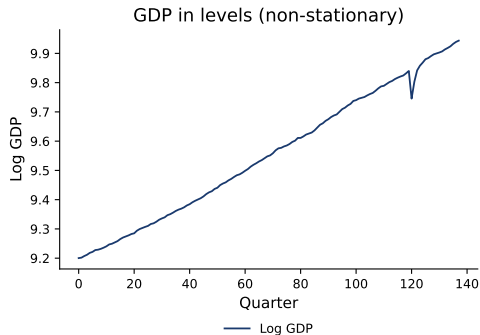


## Case Study: US Real GDP (1990–2024)



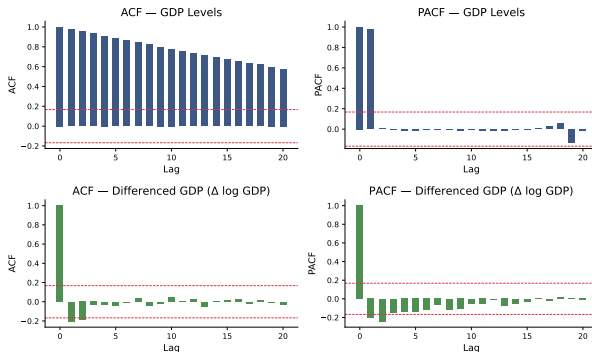
- US Real GDP in billions of 2017 dollars (quarterly data)
- Clear **upward trend** – typical of macroeconomic series
- Notable drops during recessions (2008–2009, 2020)
- Non-stationary: needs differencing before ARIMA modeling

# Stationarity Through Differencing



- **Left:** GDP in levels – clear upward trend (non-stationary)
- **Right:** GDP growth rate =  $\Delta \log(Y_t) \times 100$  – stationary
- First differencing of log GDP removes the stochastic trend
- Growth rate fluctuates around a constant mean ( $\approx 0.6\%$  quarterly)

# ACF/PACF: Levels vs Differenced



- **Top row:** ACF/PACF of GDP levels – slow decay indicates non-stationarity
- **Bottom row:** ACF/PACF of GDP growth – mostly within confidence bands
- Pattern suggests low-order ARIMA model is appropriate

## ARIMA Estimation Results: US GDP Growth

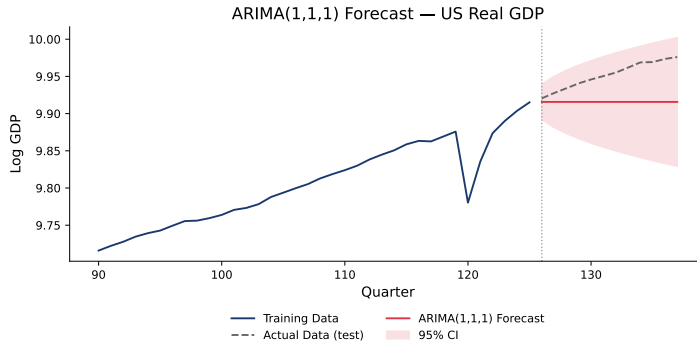
Model: ARIMA(1,1,1) on log(GDP)

Parameter	Estimate	Std. Error	z-stat	p-value
$\phi_1$ (AR.L1)	0.312	0.185	1.69	0.091
$\theta_1$ (MA.L1)	-0.087	0.203	-0.43	0.668
$\sigma^2$	0.00012	—	—	—

### Interpretation

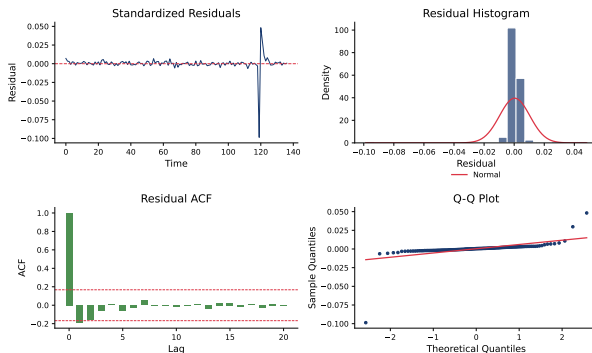
- Low-order ARIMA captures GDP dynamics reasonably well
- AR(1) coefficient positive – GDP growth shows persistence
- Alternative: simpler random walk (ARIMA(0,1,0)) often competitive

# Forecast: ARIMA vs Actual



- Blue: historical training data; Green: actual test data
- Red dashed: ARIMA forecasts with 95% confidence interval
- Forecasts capture the general trend direction
- Confidence intervals widen as forecast horizon increases

# Model Diagnostics: Residual Analysis



- Residuals show no systematic patterns over time
- Distribution approximately normal (histogram and Q-Q plot)
- ACF of residuals within bounds – no significant autocorrelation remaining
- Model adequately captures the data generating process

# Discussion: Deterministic vs Stochastic Trends

## Key Question

Why is it important to distinguish between deterministic and stochastic trends?

## Discussion Points

- **Wrong treatment consequences:**
  - Detrending a unit root  $\Rightarrow$  spurious stationarity
  - Differencing a trend-stationary  $\Rightarrow$  over differencing
- **Economic interpretation:**
  - Deterministic trend: shocks are temporary
  - Stochastic trend: shocks have permanent effects
- **Policy implications:**
  - Does a recession permanently lower GDP, or does the economy return to trend?

### Key Question

When should you use AIC vs BIC for ARIMA model selection?

### Considerations

- **AIC:** Minimizes prediction error, may overfit
  - Better for forecasting
  - Tends to select larger models
- **BIC:** Consistent model selection, more parsimonious
  - Better for identifying “true” model
  - Penalizes complexity more heavily
- **Practical advice:** Report both, prefer BIC if they disagree substantially



# Discussion: Limitations of ARIMA

## Key Question

What are the main limitations of ARIMA models?

## Discussion Points

- **Linearity:** Cannot capture nonlinear dynamics
- **Constant variance:** Assumes homoskedasticity (no GARCH effects)
- **No structural breaks:** Parameters assumed constant
- **Univariate:** Ignores relationships with other variables
- **Symmetric:** Treats positive and negative shocks equally
- **Long-horizon forecasts:** Uncertainty grows rapidly

## Extensions

These limitations motivate GARCH (volatility), VAR (multivariate), regime-switching models, etc.

# Key Points from Today's Seminar

## What We Covered

- 1 **Integration and differencing:**  $I(d)$  processes require  $d$  differences
- 2 **Unit root testing:** ADF tests  $H_0$ : unit root; KPSS tests  $H_0$ : stationary
- 3 **ARIMA(p,d,q):** Combines ARMA with differencing
- 4 **Model identification:** Use ACF/PACF patterns and information criteria
- 5 **Forecasting:** Point forecasts and growing confidence intervals

## Next Seminar

Hands-on Python exercises with real economic data:

- Unit root testing with `statsmodels`
- Auto-ARIMA with `pmdarima`
- Forecasting and model diagnostics