



# Time Series Analysis and Forecasting

## Chapter 10: Comprehensive Review



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## Learning Objectives

By the end of this chapter, you will be able to:

- Apply the complete forecasting workflow from data to evaluation
- Select appropriate models based on data characteristics
- Evaluate forecast accuracy using proper metrics and cross-validation
- Integrate knowledge from all previous chapters in practice



## Outline

Forecasting Methodology

Case Study 1: Bitcoin Volatility (GARCH)

Case Study 2: Sunspot Cycles (Fourier)

Case Study 3: Unemployment (Prophet)

Case Study 4: Multivariate Analysis (VAR)

Synthesis and Guidelines

AI Use Case

Quiz

Summary



## The Scientific Approach to Forecasting

### Research Question

How do we **rigorously evaluate** forecast performance while avoiding overfitting?

### The Fundamental Problem

- In-sample fit  $\neq$  Out-of-sample performance
- Models can “memorize” training data without learning patterns
- **Solution:**
  - ▶ Proper train/validation/test methodology

### Key Principle

“The test set must remain **untouched** until final evaluation.”

— Standard practice in machine learning and econometrics



## Train/Validation/Test Framework

### Time Series Train/Validation/Test Split



Q TSA\_ch10\_train\_val\_test\_split

## Evaluation Metrics

### Definition 1 (Forecast Error Metrics)

Let  $y_t$  be actual,  $\hat{y}_t$  forecast:

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_t (y_t - \hat{y}_t)^2}, \quad \text{MAE} = \frac{1}{n} \sum_t |y_t - \hat{y}_t|, \quad \text{MAPE} = \frac{100\%}{n} \sum_t \left| \frac{y_t - \hat{y}_t}{y_t} \right|$$

#### When to Use Each

- RMSE**: Penalizes large errors
- MAE**: Robust to outliers
- MAPE**: Scale-independent (%)

#### Caution

- MAPE undefined when  $y_t = 0$
- Compare on **same** test set
- Report **out-of-sample** metrics



## Forecast Evaluation Beyond RMSE

### Alternative metrics

- **MASE** (Mean Absolute Scaled Error):  $\frac{\text{MAE}_{\text{model}}}{\text{MAE}_{\text{naïve}}} ; < 1 \Rightarrow \text{beats naïve}$
- **DA** (Directional Accuracy):  $\frac{1}{h} \sum_{t=1}^h \mathbb{1}(\text{sgn } \Delta \hat{y}_t = \text{sgn } \Delta y_t)$
- **QL** (Quantile Loss): asymmetric penalty  $\alpha$  vs  $1-\alpha$

$$QL_\alpha = \begin{cases} \alpha(y_t - \hat{q}_t), & y_t > \hat{q}_t \\ (1 - \alpha)(\hat{q}_t - y_t), & y_t \leq \hat{q}_t \end{cases}$$

- **CRPS** (Continuous Ranked Probability Score):  $\int_{-\infty}^{\infty} (F(x) - \mathbb{1}_{x \geq y})^2 dx$

## Forecast Evaluation: Bitcoin Results

### Bitcoin Results (GARCH volatility)

Metric	Value
RMSE	2.21
MAE	1.89
MASE	0.98
Dir. Accuracy	28.7%

- MASE < 1: GARCH beats naïve
- DA 28.7%: volatility direction is hard

### Interpretation

- RMSE/MAE**: absolute volatility forecast error
- MASE < 1**: GARCH outperforms naïve benchmark
- DA 28.7%**: volatility direction is extremely hard to predict
- Evaluation must be done on the **test set**



## Formal Forecast Comparison: Diebold–Mariano

### Definition 2 (Diebold–Mariano Test)

Loss differential:  $d_t = L(e_{1t}) - L(e_{2t})$ , Statistic:  $DM = \frac{\bar{d}}{\sqrt{\widehat{\text{Var}}(\bar{d})}} \xrightarrow{d} N(0, 1)$

### Hypotheses

- $H_0$ : equal predictive accuracy
- $H_1$ : one model is significantly better
- Large  $|DM|$   $\Rightarrow$  reject  $H_0$

### Bitcoin Result (GARCH volatility)

- Normal vs Student-t:  $DM = -0.51$
- $p = 0.612$  — **do not reject  $H_0$**
- Similar accuracy, but Student-t preferred by AIC ( $\Delta = 509$ )

### Key message

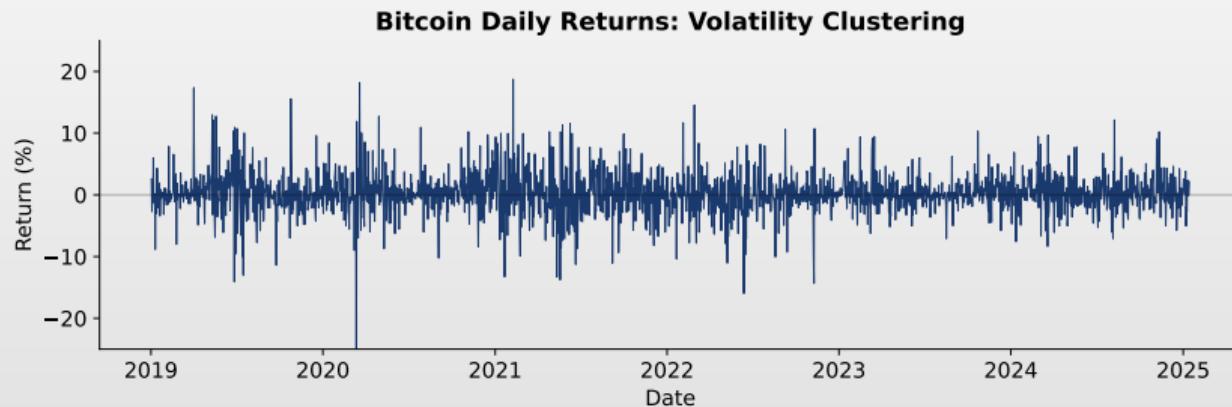
- Lower RMSE  $\neq$  significant difference — formal testing is **mandatory**



## Bitcoin: Volatility Clustering

### Observation

- Large returns follow large returns, small follow small—**volatility clustering**



Q TSA\_ch10\_btc\_returns



## Bitcoin: Problem Statement

### Research Question

Can we forecast Bitcoin's **volatility** using GARCH models?

### Data Characteristics

- Source: Yahoo Finance (BTC-USD)
- Period: Jan 2019 – Jan 2025
- Frequency: Daily
- Observations:  $\approx 2,200$  days

### Stylized Facts

- Returns: near-zero mean
- Fat tails ( $kurtosis > 3$ )
- Volatility clustering

### Key Insight

Financial returns are typically:

- **Unpredictable** in mean
- **Predictable** in variance

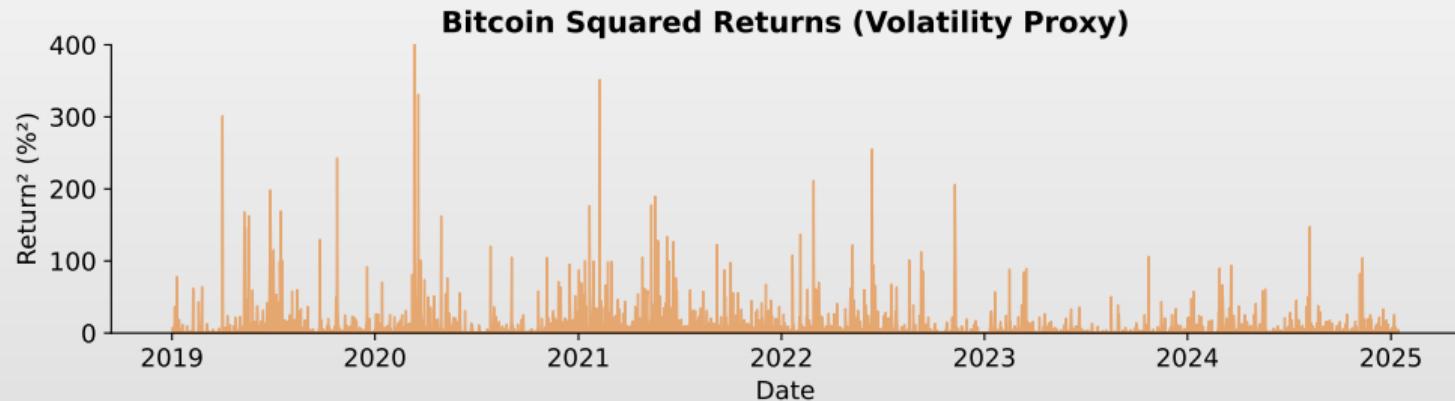
⇒ Focus on **volatility forecasting**



## Bitcoin: Evidence for GARCH

### Observation

- Squared returns  $r_t^2$  exhibit significant autocorrelation  $\succ$  GARCH effects
- Slow ACF decay  $\succ$  high volatility persistence



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## GARCH Model Specification

### Definition 3 (GARCH(p,q) Model)

Let  $r_t$  denote returns. The GARCH(p,q) model:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \stackrel{iid}{\sim} N(0, 1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^p \beta_j \sigma_{t-j}^2$$

where  $\omega > 0$ ,  $\alpha_i \geq 0$ ,  $\beta_j \geq 0$ , and  $\sum_{i=1}^q \alpha_i + \sum_{j=1}^p \beta_j < 1$ .

### Model Variants

- GARCH(1,1)**: Most common
- GJR-GARCH**: Leverage effect
- EGARCH**: Log-variance, asymmetric

### Interpretation

- $\alpha$ : Shock impact (ARCH effect)
- $\beta$ : Volatility persistence
- $\alpha + \beta \approx 1$ : High persistence



## GARCH: Stationarity and Unconditional Variance

### Theorem 1 (Covariance Stationarity of GARCH(1,1))

If  $\alpha_1 + \beta_1 < 1$ , then  $\{\varepsilon_t\}$  is covariance stationary with:

$$\bar{\sigma}^2 = \mathbb{E}[\sigma_t^2] = \frac{\omega}{1 - \alpha_1 - \beta_1}$$

### Derivation

Take expectations of both sides of the variance equation:

$$\begin{aligned}\mathbb{E}[\sigma_t^2] &= \omega + \alpha_1 \mathbb{E}[\varepsilon_{t-1}^2] + \beta_1 \mathbb{E}[\sigma_{t-1}^2] \\ \bar{\sigma}^2 &= \omega + (\alpha_1 + \beta_1) \bar{\sigma}^2 \quad (\text{stationarity}) \\ \bar{\sigma}^2 &= \frac{\omega}{1 - \alpha_1 - \beta_1}\end{aligned}$$

### Multi-Step Forecasts Converge to $\bar{\sigma}^2$

As  $h \rightarrow \infty$ :  $\mathbb{E}_t[\sigma_{t+h}^2] \rightarrow \bar{\sigma}^2$  at rate  $(\alpha_1 + \beta_1)^h$ .



## Bitcoin: Model Selection on Validation Set

### Methodology

Fit each model on training data, evaluate on validation set.

Model	AIC	BIC	Val MAE	Selection
GARCH(1,1)	6,994.8	7,020.6	2.638	Best
GARCH(2,1)	6,993.7	7,024.6	2.640	
GJR-GARCH(1,1)	6,983.7	7,014.6	2.669	
EGARCH(1,1)	—	—	—	Failed*

\*Analytic forecasts not available for  $h > 1$

### Result

GARCH(1,1) selected based on lowest validation MAE for volatility forecasts.



## Bitcoin: Data Split and Stationarity

### Data Split

Set	Period	N
Training (70%)	2019-01 to 2023-03	1,543
Validation (20%)	2023-03 to 2024-06	441
Test (10%)	2024-06 to 2025-01	221
<b>Total</b>	<b>2,205</b>	

### Stationarity Tests

Series	ADF	Result
Prices	$p = 0.50$	Non-stationary
Returns	$p < 0.01$	Stationary

⇒ Model **returns**, not prices

### Why Stationarity Matters

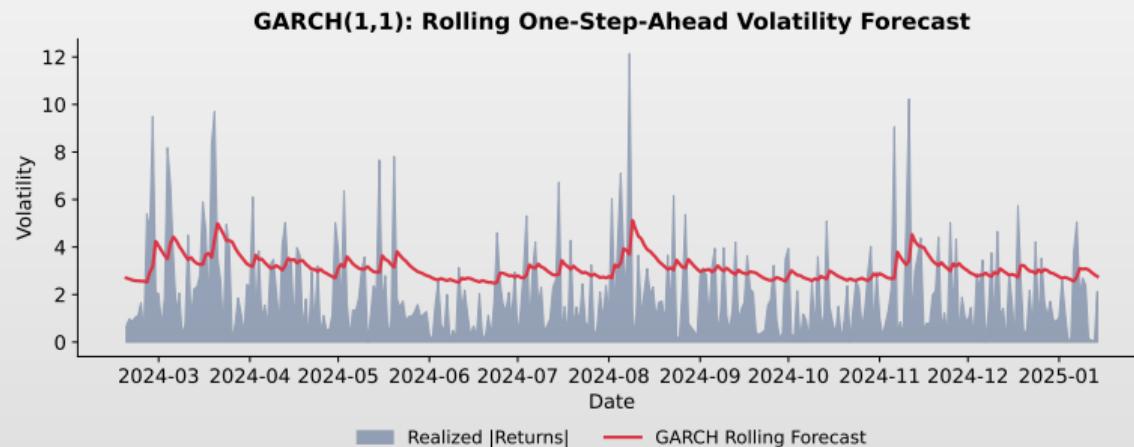
- ◻ GARCH requires weakly stationary input
- ◻ Prices follow random walk; returns are stationary



## Bitcoin: Volatility Forecast

### Interpretation

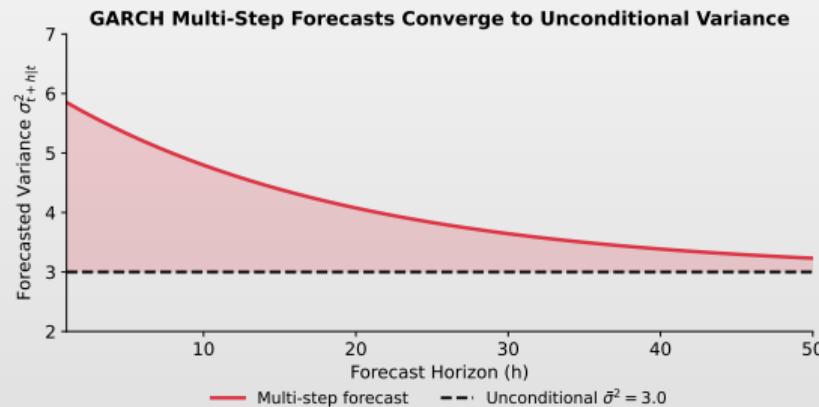
- Shaded area: 95% confidence interval of the volatility forecast
- GARCH(1,1) captures Bitcoin's volatility dynamics well



## GARCH: Multi-Step Forecasts Converge

### Key Insight

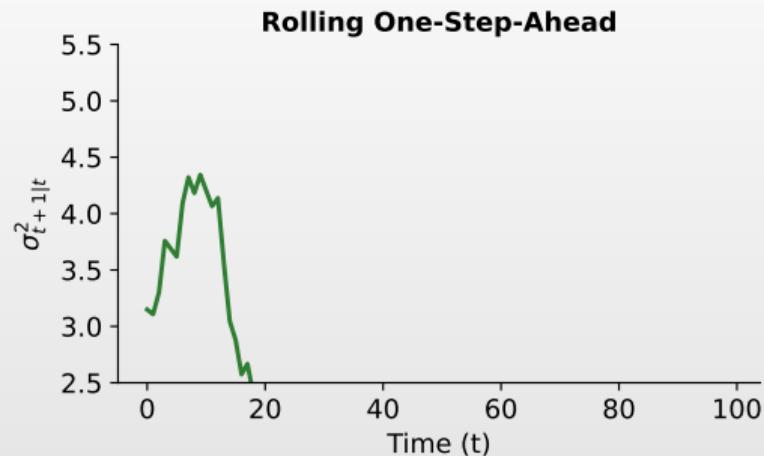
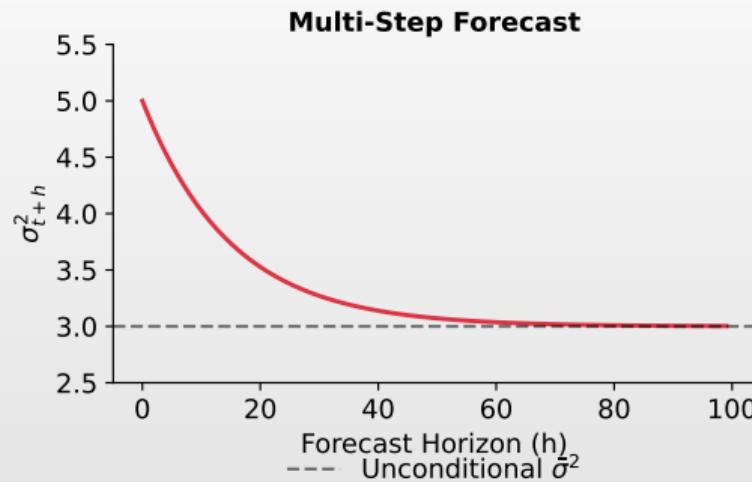
- Multi-step forecasts converge to  $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- Use rolling forecasts



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## GARCH: Rolling One-Step-Ahead Solution



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## GARCH: Innovation Distributions

### Model

$$r_t = \mu + \sigma_t z_t$$

- Options for  $z_t$ :  $\mathcal{N}(0, 1)$  (normal) or  $t_\nu$  (fat tails)

### Bitcoin: empirical evidence

- Residual kurtosis: **13.81** (Normal = 3)
- Skewness: -0.29
- Jarque-Bera: 9085,  $p < 0.001$
- Normality **underestimates** tail risk

### Student-t: the right choice

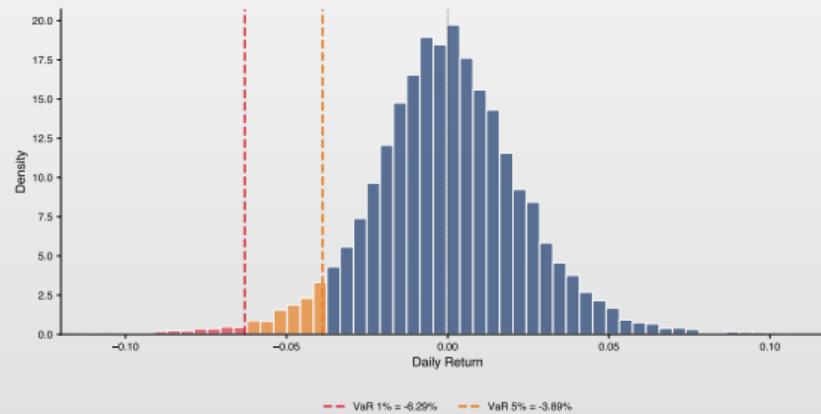
- $\hat{\nu} = 2.96$  degrees of freedom
- AIC Normal: 9769 vs Student-t: **9260**
- $\Delta\text{AIC} = 509$  — **overwhelming** evidence
- Fat tails = **more realistic** VaR estimates



## VaR and ES: Graphical Illustration

### Interpretation

- VaR 1% = loss exceeded only in 1% of cases
- Red area = extreme losses (beyond VaR)



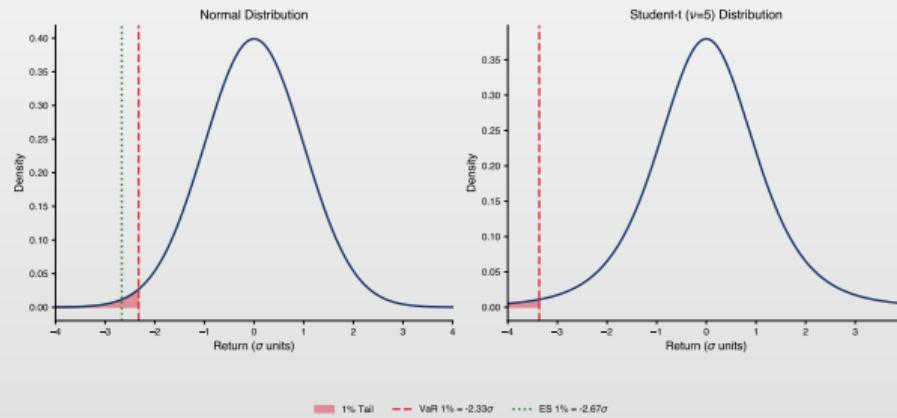
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## VaR vs Expected Shortfall: Normal vs Student-t

### Interpretation

- ES measures average loss when VaR is exceeded
- Student-t: VaR and ES are larger than under normal distribution



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## Value at Risk — Numerical Example

### VaR Calculation

Portfolio: **1,000,000 EUR**, forecasted volatility  $\hat{\sigma}_{T+1} = 1.5\%$

### VaR with Normal Distribution

Level	$z_\alpha$	VaR (%)	VaR (EUR)
5% (1 day)	1.645	2.47%	24,675
1% (1 day)	2.326	3.49%	34,890

### Scaling for Longer Periods

$$\text{VaR}_{h \text{ days}} = \text{VaR}_{1 \text{ day}} \cdot \sqrt{h} \quad \text{— assumes i.i.d. returns}$$



## Value at Risk — Student-t Distribution

### Why Student-t?

Normal distribution **underestimates** tail risk. Student-t with  $\nu$  degrees of freedom better captures fat tails ( $kurtosis > 3$ ).

VaR 1% (1 day) Comparison:  $\sigma = 1.5\%$ , Portfolio = 1M EUR

Distribution	Quantile	VaR (EUR)
Normal	2.326	34,890
Student-t ( $\nu = 6$ )	3.143	47,145
Student-t ( $\nu = 4$ )	3.747	56,205

### Observation

With  $\nu = 6$  (typical for stocks), VaR is **35% higher** than normal!



## VaR — Complete Example with GARCH

### VaR Calculation Procedure

1. Estimate GARCH(1,1) model with Student-t distribution
2. Obtain volatility forecast:  $\hat{\sigma}_{T+1}$
3. Calculate VaR:  $\text{VaR}_\alpha = t_\alpha(\nu) \cdot \hat{\sigma}_{T+1} \cdot \sqrt{\frac{\nu-2}{\nu}}$

### Example: S&P 500

- Estimated parameters:  $\alpha = 0.088$ ,  $\beta = 0.900$ ,  $\nu = 6.4$
- Forecasted volatility:  $\hat{\sigma}_{T+1} = 1.2\%$
- Portfolio: 10,000,000 EUR

**VaR 1% (1 day):**  $\text{VaR} = 3.05 \times 0.012 \times 10,000,000 = 366,000 \text{ EUR}$



## What is VaR Backtesting?

### Definition

- Backtesting** = ex-post verification of VaR model quality
- Compares realized losses with the forecasted VaR threshold
  - ▶ A **violation** occurs when  $r_t < -\text{VaR}_t$

### Backtesting Principle

- Violation indicator:  $I_t = 1(r_t < -\text{VaR}_{\alpha,t})$
- For a correctly specified model at level  $\alpha$ :
  - ▶ Frequency:  $\hat{\rho} = \frac{1}{T} \sum I_t \approx \alpha$ ; violations **independent**
- VaR 1% over 250 days  $\Rightarrow$  expect  $\sim 2.5$  violations/year

### Importance

- Regulatory requirement under **Basel III/IV** for banks: backtesting is mandatory



## Kupiec Test (1995) — Unconditional Coverage

### Hypotheses

- $H_0$ : Violation rate equals the VaR level ( $p = \alpha$ )
- $H_1$ : Violation rate differs from the VaR level ( $p \neq \alpha$ )

### Test Statistic (Likelihood Ratio)

- Formula:**  $LR_{uc} = -2 \ln \left[ \frac{\alpha^x (1-\alpha)^{T-x}}{\hat{p}^x (1-\hat{p})^{T-x}} \right] \sim \chi^2(1)$
- Notation:**  $x$  = no. violations,  $T$  = no. observations,  $\hat{p} = x/T$

### Example

- VaR 1%,  $T = 250$  days,  $x = 5$  violations:  $\hat{p} = 2\%$ 
  - ▶ Too many violations  $\Rightarrow$  model **underestimates** risk
- VaR 1%,  $T = 250$  days,  $x = 1$  violation:  $\hat{p} = 0.4\% \Rightarrow$  acceptable



## Christoffersen Test (1998) — Conditional Coverage

### Motivation

- Kupiec only tests the **frequency** of violations
- Does not detect **clustering** of violations (consecutive violations)
  - If violations cluster  $\Rightarrow$  model fails to capture volatility dynamics

### Independence + Conditional Coverage Test

- Formula:**  $LR_{cc} = LR_{uc} + LR_{ind} \sim \chi^2(2)$
- $LR_{ind}$  tests whether  $P(I_t = 1 | I_{t-1} = 1) = P(I_t = 1 | I_{t-1} = 0)$
- A good model: violations are rare **and** uniformly distributed over time

### Recommendation

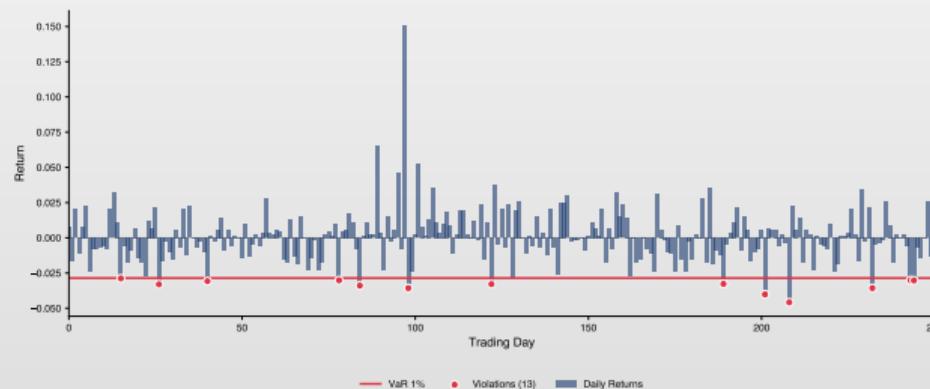
- Use **both** tests: Kupiec (frequency) + Christoffersen (independence)



## VaR Backtesting: Visualization

### Interpretation

- ◻ Red line: VaR 1% threshold estimated with GARCH(1,1)
- ◻ Red dots: 13 violations out of 250 days ( $\hat{p} = 5.2\%$ )
  - ▶ **Basel red zone** ⇒ model significantly underestimates risk
  - ▶ Solutions: Student-t distribution, EGARCH model, or more conservative VaR level



## VaR Backtesting: Basel Traffic Light

### Basel III/IV Traffic Light Zones

Zone	Violations/250 days	Interpretation	Penalty
Green	0–4	Model acceptable	No penalty
Yellow	5–9	Needs investigation	Factor $k$ increases
Red	$\geq 10$	Model inadequate	Maximum penalty

### Practical Example

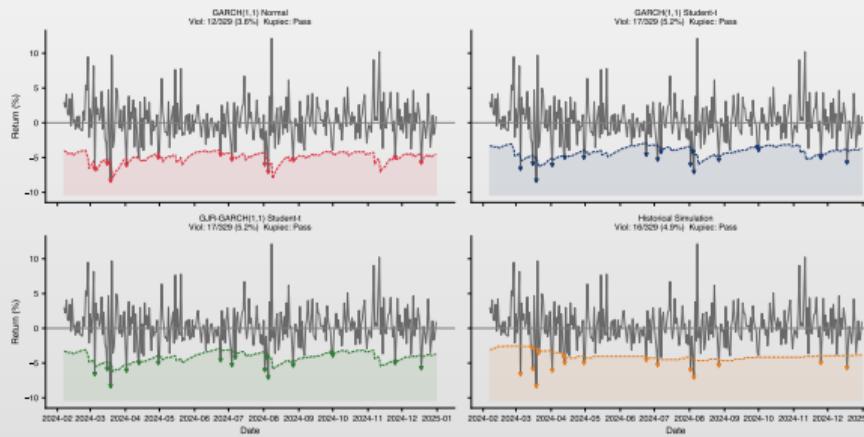
- Portfolio with VaR 1%: 250 days of backtesting
- 3 violations  $\Rightarrow$  **Green zone**  $\Rightarrow$  model acceptable
- 7 violations  $\Rightarrow$  **Yellow zone**  $\Rightarrow$  revision needed
- 13 violations  $\Rightarrow$  **Red zone**  $\Rightarrow$  model rejected



## Application: Rolling VaR with Multiple Models

### Methodology

- Rolling one-step-ahead VaR:  $\text{VaR}_{t+1}^{\alpha} = \mu + \hat{\sigma}_{t+1} \cdot z_{\alpha}$
- 4 models compared on Bitcoin test set (329 days, 2024)



## VaR Backtesting: Model Comparison

Bitcoin Results — VaR 5% ( $T = 329$  days, expected: 16.5 violations)

Model	Violations	Rate	Kupiec $p$	Chr. $p$	Conclusion
GARCH(1,1)-N	12	3.6%	0.238	1.000	Too conservative
GARCH(1,1)-t	17	5.2%	0.890	0.272	$\approx 5\%$ target
GJR-GARCH(1,1)-t	17	5.2%	0.890	0.272	$\gamma \approx 0$ (symmetric)
Hist. Simulation	16	4.9%	0.909	0.216	$\approx 5\%$ target

### Conclusions

- Student-t:** perfect coverage ( $5.2\% \approx 5\%$ )
- Normal: too conservative ( $3.6\% < 5\%$ )
- GJR  $\approx$  GARCH: no leverage for Bitcoin
- All pass both Kupiec **and** Christoffersen

### Practical Lessons

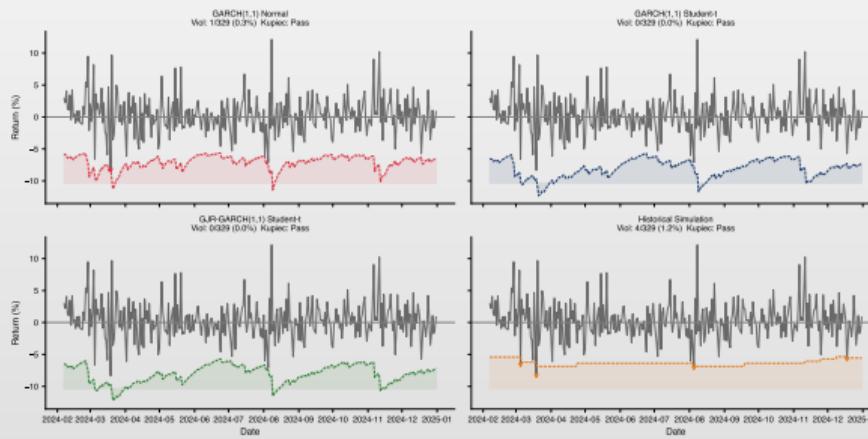
- GARCH-t and HistSim: both  $\approx 5\%$  target
- Historical simulation: simple and effective alternative
- Formal statistical testing (Kupiec, Christoffersen) is **mandatory**



## Application: Rolling VaR 1% on Multiple Models

### Methodology

- Rolling one-step-ahead VaR at  $\alpha = 1\%$  (extreme risk)
- Same 4 models; expected:  $T \times 0.01 = 3.3$  violations



## VaR 1% Backtesting: Model Comparison

Bitcoin Results — VaR 1% ( $T = 329$  days, expected: 3.3 violations)

Model	Violations	Rate	Kupiec $p$	Chr. $p$	Conclusion
GARCH(1,1)-N	1	0.3%	0.137	1.000	Acceptable
GARCH(1,1)-t	0	0.0%	1.000	1.000	Acceptable
GJR-GARCH(1,1)-t	0	0.0%	1.000	1.000	Acceptable
Hist. Simulation	4	1.2%	0.704	1.000	Acceptable

### VaR 1% Conclusions

- ☐ All models pass Kupiec ( $p > 0.05$ )
- ☐ With only 3.3 expected violations, test has **low statistical power**
- ☐ Cannot meaningfully distinguish between 0 and 4 violations

### Lesson: 5% vs 1%

- ☐ At 5% ( $\approx 16$  violations): powerful test, Student-t best
- ☐ At 1% ( $\approx 3$  violations): weak test, models **indistinguishable**
- ☐ **Conclusion:** test power increases with  $\alpha$  and  $T$



## GARCH Limitations and Modern Extensions

### Limitations

- Does not capture **jumps**
- Constant parameters over time
- Sensitive to chosen distribution
- Does not model different **regimes**

### Extensions

- GJR-GARCH:** leverage effect
- EGARCH:** asymmetric shocks
- Markov-Switching GARCH:** regimes
- Realized volatility (HAR)
- Hybrid GARCH + ML

### Key message

- GARCH is a **starting point**, not the end of risk modeling



## Bitcoin: Key Findings

### Summary

1. Returns are stationary; prices are not
2. GARCH(1,1) outperforms more complex variants
3. High persistence ( $\alpha + \beta = 0.93$ )
4. Volatility is predictable even when returns are not

### Limitations

- GARCH assumes symmetric shocks
- Does not capture jumps
- Normal distribution may be restrictive

### Practical Implications

- Risk management: VaR, Expected Shortfall
- Option pricing requires volatility forecasts
- Portfolio optimization with time-varying risk

### Extensions

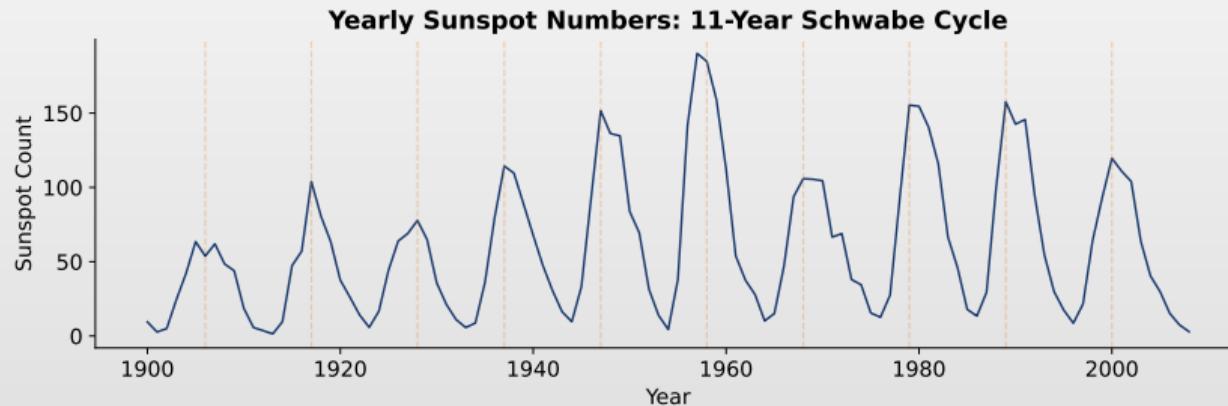
- Student-t innovations
- Realized volatility
- HAR models



## Sunspots: The 11-Year Solar Cycle

### Observation

- Clear  $\approx$  11-year solar cycle; variable amplitude across cycles
- Periodic ACF  $\succcurlyeq$  long seasonality, ideal for Fourier terms

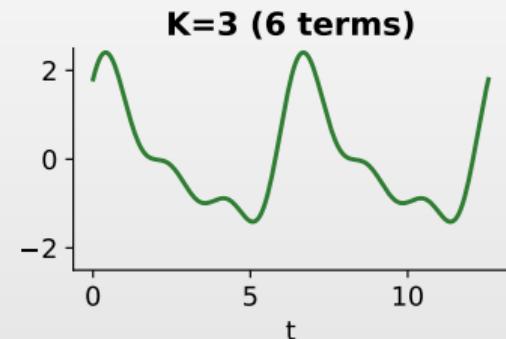
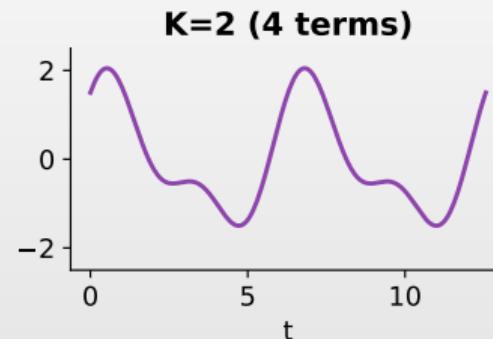
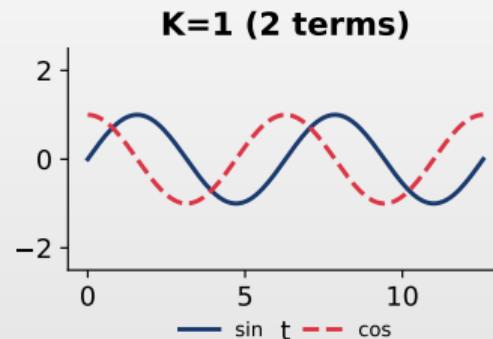


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## Fourier Terms for Seasonality

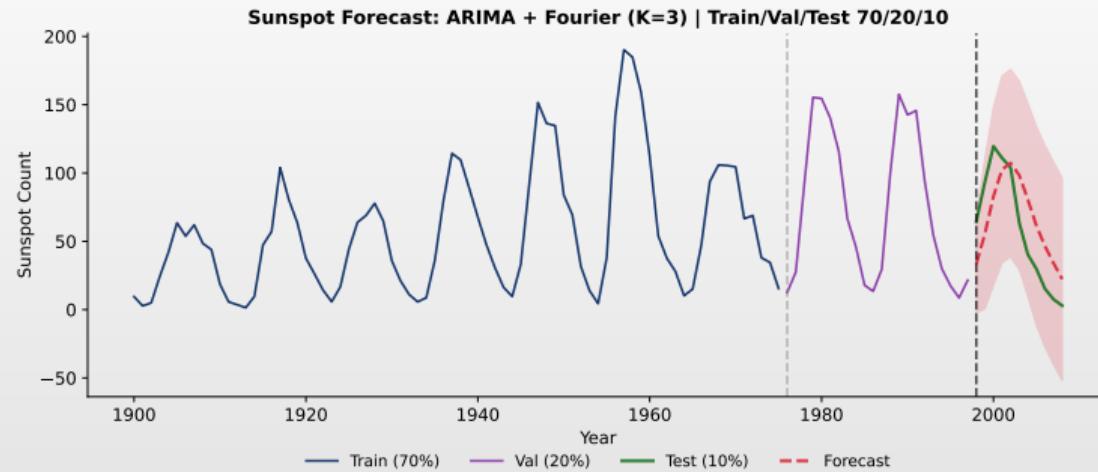
**Fourier Terms: More K = More Flexibility**



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## Sunspots: Forecast Results



## Sunspots: Model Selection

### Methodology

Compare  $K = 1, 2, 3, 4$  Fourier harmonics on validation set.

	Set	Period	N
Data Split	Training (70%)	1900–1975	76
	Validation (20%)	1976–1997	22
	Test (10%)	1998–2008	11
<b>Total</b>		<b>109</b>	

Model Comparison			
K	AIC	Val RMSE	
1	665.9	87.15	
2	668.0	86.92	
3	671.8	<b>86.81</b>	Best
4	674.5	87.93	

### Result

$K = 3$  Fourier harmonics selected (6 parameters for 11-year cycle).



## Overfitting in Choosing $K$

### Overfitting risk

- $K$  too large = memorizing historical cycle
- Model fits noise, not signal
- Test performance **degrades**

### Solution: validation

- Select  $K$  on **validation** set
- Evaluate on **test** — untouched
- Trade-off: complexity vs generalization

### Fourier $\approx$ periodic regression

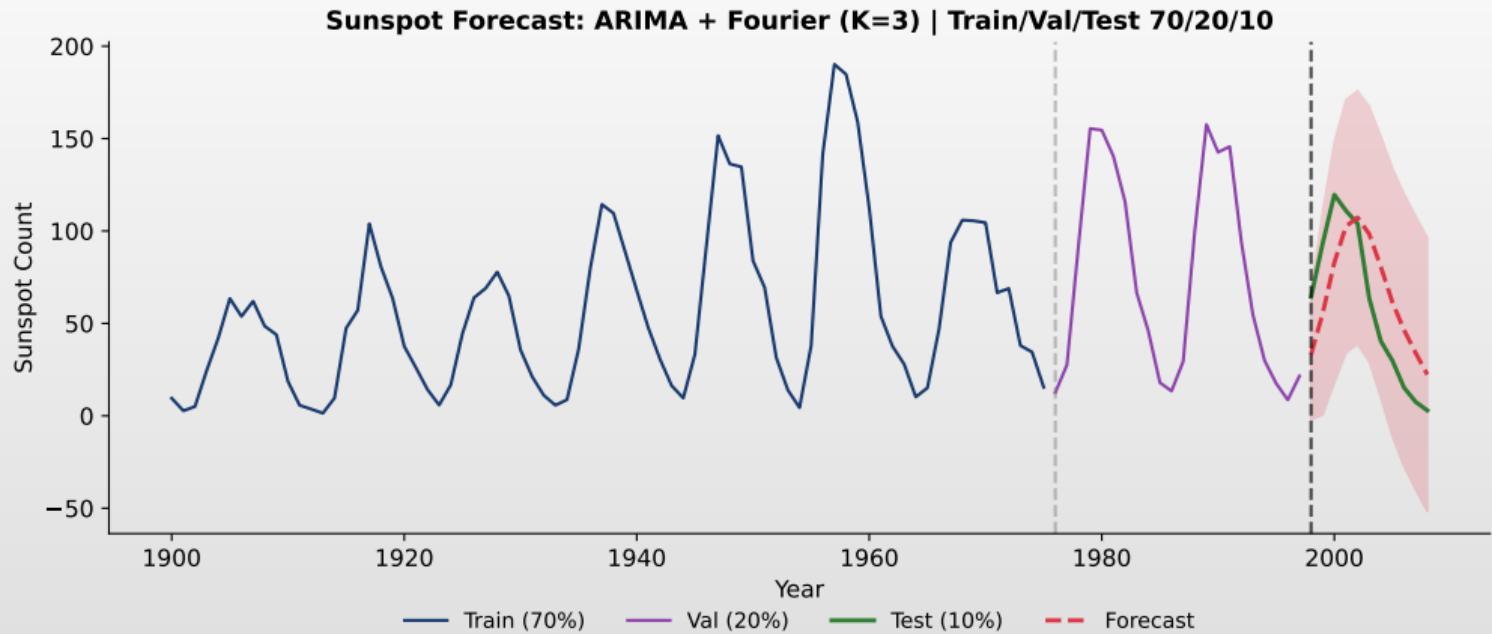
- Each harmonic adds 2 parameters (sin, cos)
- $K = 3$ : 6 extra parameters
- $K = 6$ : 12 parameters — overfitting risk

### Our results

- $K = 3$  minimizes Val RMSE
- $K = 4$  increases error  $\rightarrow$  overfitting



## Sunspots: Forecast Results



## Sunspots: Key Takeaways

### When to Use Fourier Terms

- Seasonal period  $s$  is **long** (e.g., 11 years, 52 weeks)
- SARIMA would require too many seasonal lags
- Pattern is **smooth and periodic**
- Multiple cycles need to be captured

### Choosing K

- Start with  $K = 1$ , increase until validation error stops improving
- Too high  $K$  = overfitting

### Fourier vs SARIMA

	Fourier	SARIMA
Long seasons	✓	✗
Short seasons	OK	✓
Parameters	$2K$	Many
Flexibility	Fixed	Adaptive

### Applications

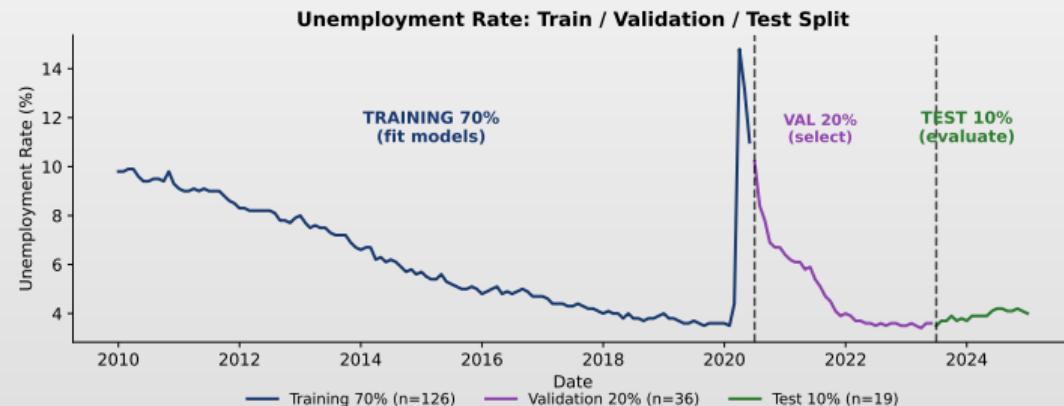
Climate cycles, business cycles, astronomical phenomena



## Unemployment: Train / Validation / Test Split

### Methodology

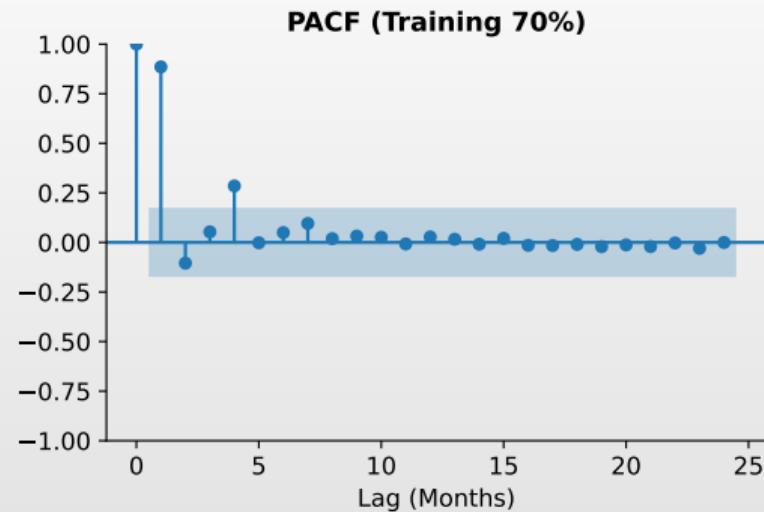
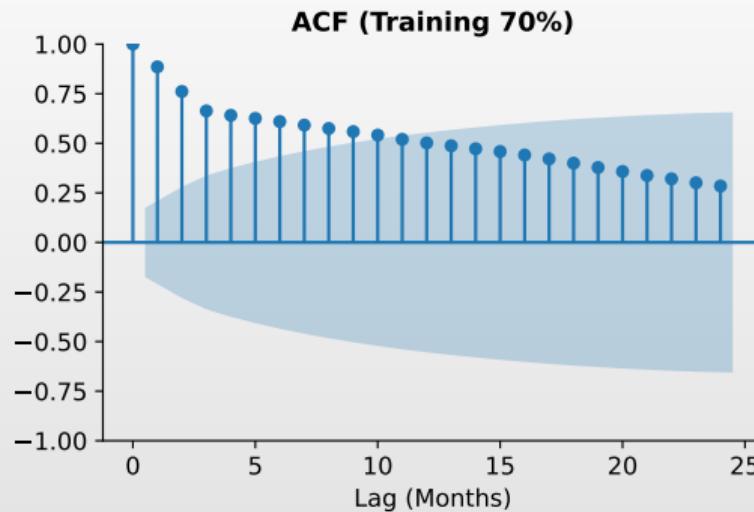
- Training:** Fit models
- Validation:** Select best
- Test:** Final evaluation



 TSA\_ch10\_unemployment\_train\_val\_test

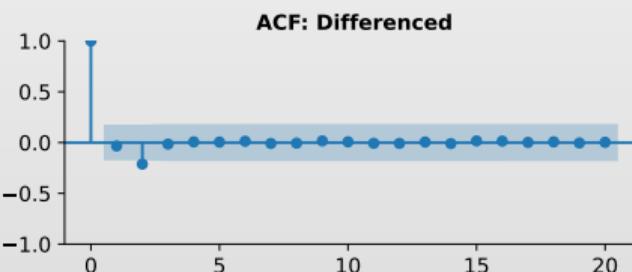
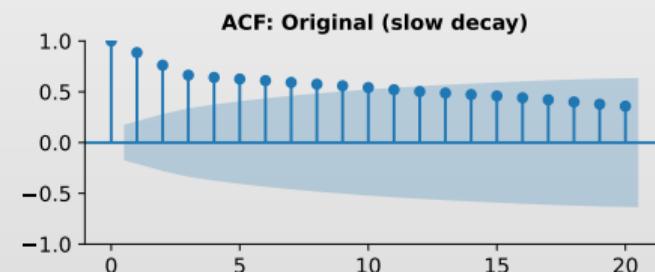
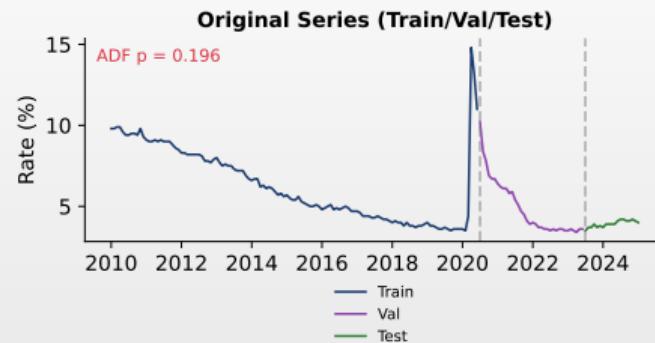


## Unemployment: Preliminary Analysis



Q TSA\_ch10\_unemployment\_acf\_pacf

## Unemployment: Stationarity Tests



## Structural Breaks: Formal Approach

### Classical methods

- Chow Test:** break at known point
- Bai–Perron:** multiple unknown breaks
- CUSUM:** sequential detection

### Problem

- ADF can confuse **break** with **unit root**
- Zivot–Andrews test: ADF with endogenous break

### Result: Unemployment at COVID (March 2020)

- Chow Test:  $F = 21.73, p < 0.001$
- Structural break **confirmed**
- SARIMA: constant parameters — risk
- Prophet: detects changepoints automatically

### Key message

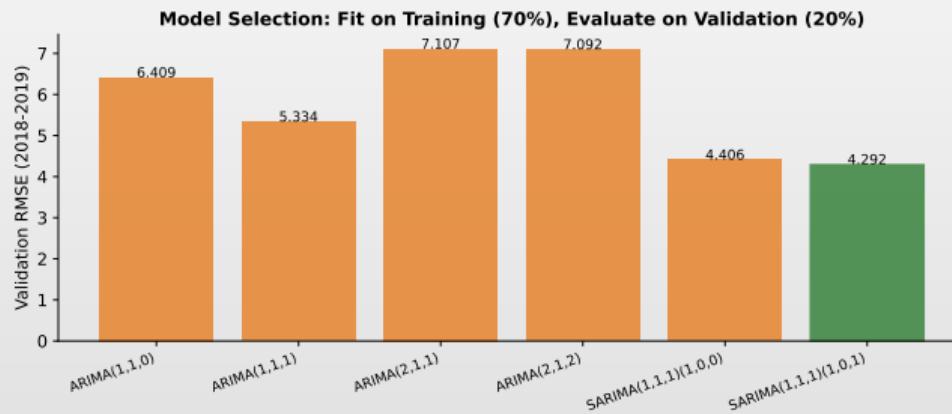
- Model must be adapted to **parameter stability**



## Unemployment: Model Selection (Validation Set)

Best: SARIMA(1,1,1)(1,0,0)<sub>12</sub>

- Selected by lowest validation RMSE



 TSA\_ch10\_sarima\_model\_selection



## Unemployment: SARIMA Parameters

SARIMA(1,1,1)(1,0,0)<sub>12</sub> fitted on Train+Val (2010-2019)

- AR(1):  $\phi_1 = -0.86$
- MA(1):  $\theta_1 = 0.78$
- SAR(12):  $\Phi_1 = -0.08$  (n.s.)

**SARIMA(1,1,1)(1,0,1) - Fitted on Train+Val (85%)**

Parameter	Coef	Std Err	P-value	Sig
ar.L1	0.8423	0.2084	0.0001	***
ma.L1	-0.9540	0.1973	0.0000	***
ar.S.L12	0.0326	4.5951	0.9943	
ma.S.L12	-0.0113	4.6087	0.9980	
sigma2	0.8122	0.0608	0.0000	***



## Ljung-Box Test for Residual Autocorrelation

### Definition 4 (Ljung-Box Test)

For residuals  $\hat{\varepsilon}_t$  with sample autocorrelations  $\hat{\rho}_k$ , the test statistic:

$$Q(h) = n(n + 2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{n - k} \stackrel{H_0}{\sim} \chi^2(h - p - q)$$

where  $p, q$  are ARMA orders.  $H_0$ : Residuals are white noise.

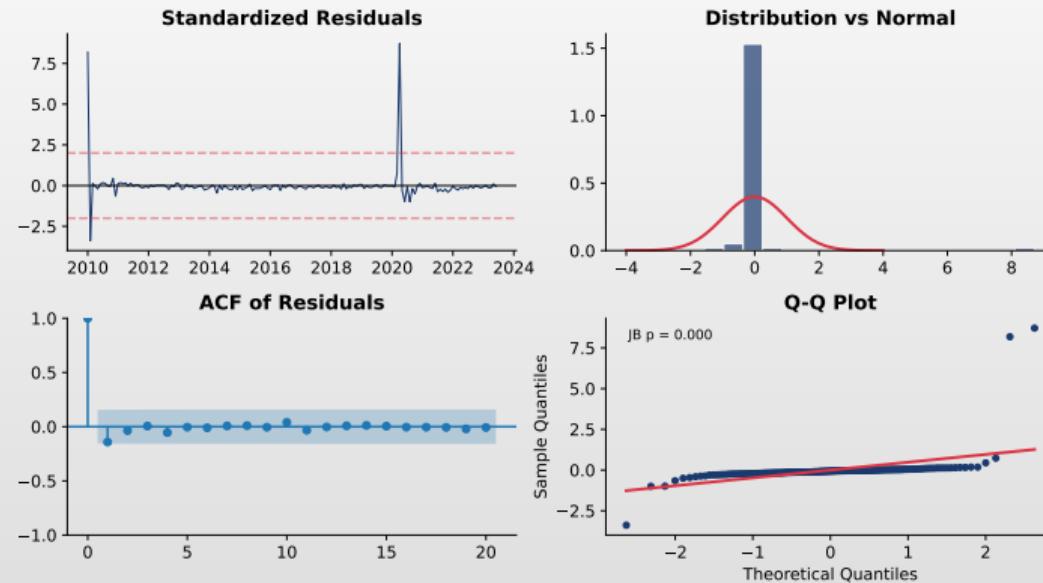
### Interpretation

- ◻ Large  $Q$  (small p-value): Reject  $H_0$ , residuals have structure
- ◻ Small  $Q$  (large p-value): Fail to reject  $H_0$ , model is adequate
- ◻ Rule of thumb: Use  $h = \min(10, n/5)$  for lag order



## Unemployment: SARIMA Diagnostics

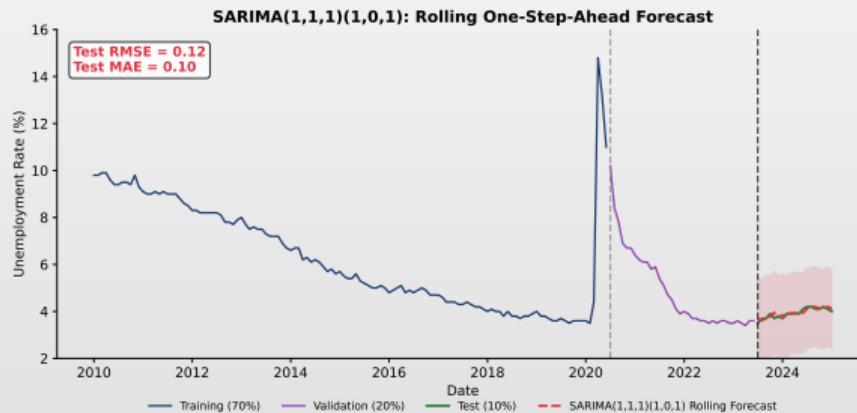
SARIMA(1,1,1)(1,0,1) Diagnostics on Train+Val (85%) | Ljung-Box p = 1.00



## Unemployment: SARIMA Rolling Forecast

### Problem: Structural Break

- Rolling one-step-ahead forecast (re-estimate at each  $t$ )
- Test RMSE = 0.12



Q TSA\_ch10\_sarima\_forecast



## Prophet Model

### Definition 5 (Prophet Decomposition)

$$y_t = g(t) + s(t) + h(t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$$

where  $g(t)$  = trend,  $s(t)$  = seasonality,  $h(t)$  = holidays,  $\sigma^2$  = noise variance (estimated).

### Changepoint Detection

- Automatic location selection
- `changepoint_prior_scale` controls flexibility

### Advantages

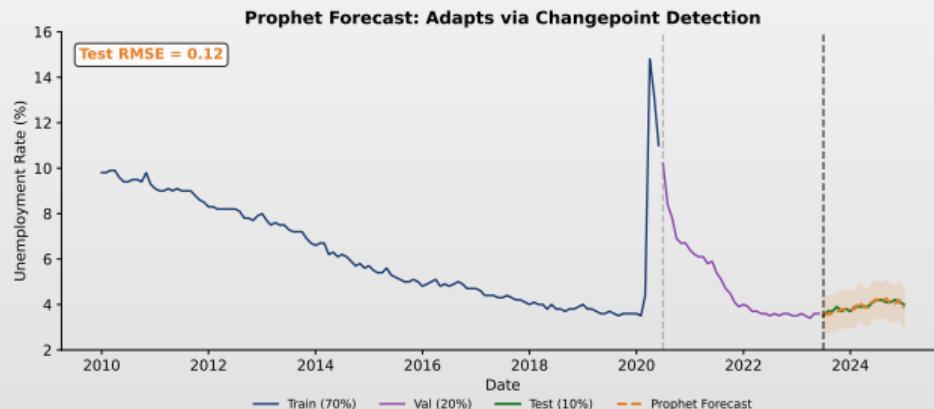
- Handles missing data
- Interpretable components
- Robust to outliers



## Unemployment: Prophet Forecast Results

### Key Finding

- Prophet adapts via changepoint detection
- Test RMSE = 0.58



 [TSA\\_ch10\\_unemployment\\_forecast](#)



## Unemployment: Model Tuning

### Hyperparameter Tuning

Tune `changepoint_prior_scale` on validation set.

Data Split	Set	Period	N
	Training (70%)	2010-01 to 2020-06	126
	Validation (20%)	2020-07 to 2023-06	36
	Test (10%)	2023-07 to 2025-01	19
	<b>Total</b>		<b>181</b>

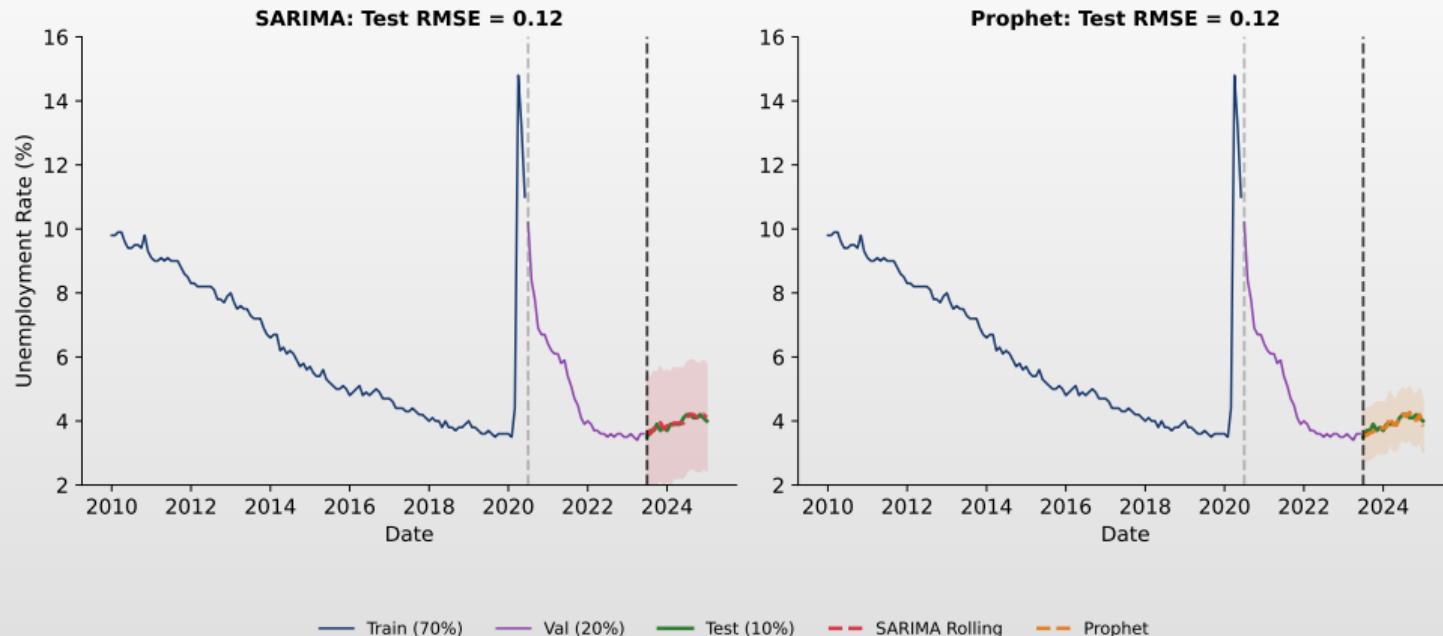
Scale Comparison	Scale	Val RMSE	Best
	0.01	4.21	
	0.05	3.89	
	0.10	<b>3.52</b>	
	0.30	3.67	
	0.50	3.81	

### Interpretation

$\text{Scale} = 0.10$  balances flexibility (capturing COVID shock) with stability.



## Unemployment: SARIMA vs Prophet Comparison



Q TSA\_ch10\_prophet\_vs\_sarima\_unemployment



## Prophet: When to Use It

### Ideal Use Cases

- Business data with **holidays**
- Missing values** present
- Need **interpretable** components
- Forecasts with **uncertainty bands**

### Caveat: Structural Breaks

Prophet handles breaks via changepoints, but **SARIMA outperformed** it on unemployment (0.12 vs 0.58). Always validate!

### Prophet vs ARIMA

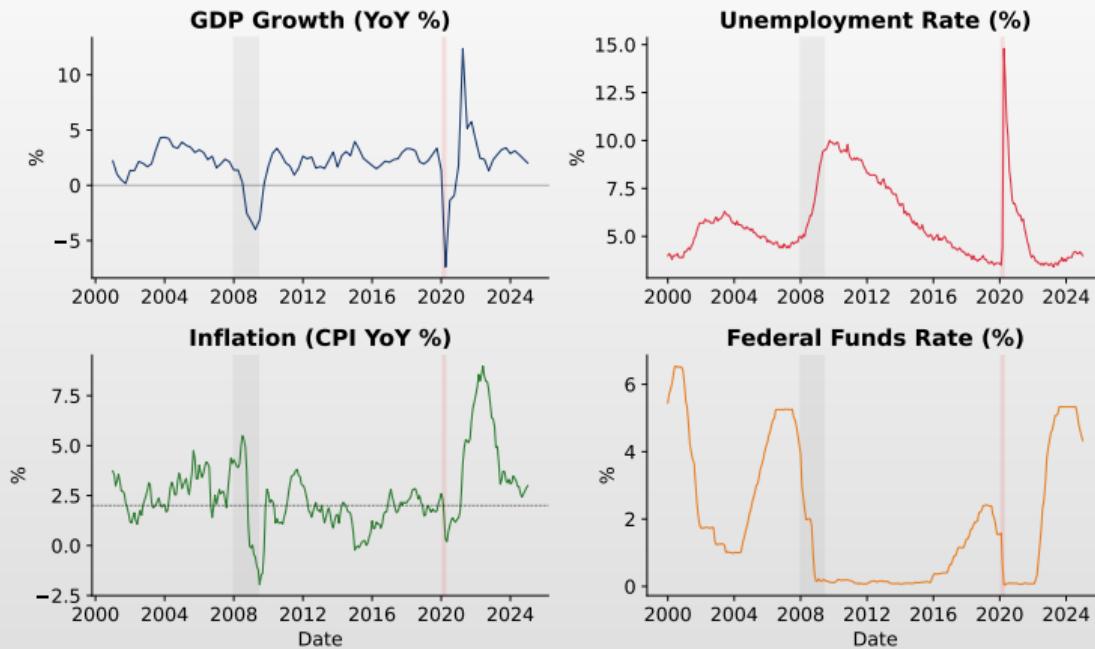
	Prophet	ARIMA
Changepoints	✓	✗
Missing data	✓	✗
Holidays	✓	✗
Speed	Fast	Moderate
Interpretable	✓	✗

### Key Parameters

`changepoint_prior_scale`: flexibility  
`seasonality_prior_scale`: smoothness



## VAR: Multivariate Economic Data



## VAR Model Specification

### Definition 6 (Vector Autoregression VAR(p))

For  $K$  variables  $y_t = (y_{1t}, \dots, y_{Kt})'$ :

$$y_t = c + A_1 y_{t-1} + A_2 y_{t-2} + \dots + A_p y_{t-p} + u_t$$

where  $A_i$  are  $K \times K$  coefficient matrices,  $u_t \sim N(0, \Sigma)$ ,  $\Sigma$  = covariance matrix.

### For Our 4-Variable System

VAR(2) has:

- 4 intercepts
- $2 \times 4 \times 4 = 32$  AR coefficients
- 36 parameters total**

### Lag Selection

Use information criteria:

- AIC**: Tends to overfit
- BIC**: More parsimonious
- Cross-validation on held-out data



## Information Criteria for Model Selection

### Definition 7 (Akaike and Bayesian Information Criteria)

For a model with log-likelihood  $\mathcal{L}$ ,  $k$  parameters, and  $n$  observations:

$$\text{AIC} = -2\mathcal{L} + 2k$$

$$\text{BIC} = -2\mathcal{L} + k \ln(n)$$

#### AIC

- Asymptotically efficient
- May overfit with small  $n$
- Minimizes prediction error

#### BIC

- Consistent (finds true model)
- Heavier penalty:  $\ln(n) > 2$  if  $n > 7$
- More parsimonious



## VAR: Lag Selection and Estimation

### BIC by Lag Order

Lag	BIC
1	-4.810
2	<b>-5.178</b>
3	-4.633
4	-4.614

### Data Split

Set	Period	N
Training (70%)	2001-Q1 to 2017-Q4	67
Validation (20%)	2018-Q1 to 2022-Q4	20
Test (10%)	2023-Q1 to 2025-Q1	10
<b>Total</b>		<b>97</b>

### Validation Check

VAR(2) also achieves lowest validation RMSE.



## VAR Model Stability

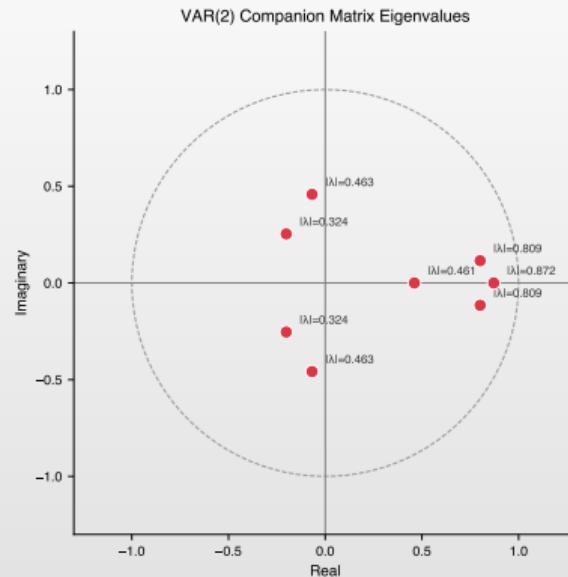
### Stability condition

- All eigenvalues of the companion matrix:  $|\lambda_i| < 1, \forall i$

### VAR(2) Results — economic data

$ \lambda_1 ,  \lambda_2 $	0.324
$ \lambda_3 ,  \lambda_4 $	0.463
$ \lambda_5 $	0.461
$ \lambda_6 $	<b>0.872</b>
$ \lambda_7 ,  \lambda_8 $	0.810

- Max  $|\lambda| = 0.872 < 1$  — **stable**



## VAR vs VECM: Cointegration

### Problem

- ◻ If variables are  $I(1)$  > VAR on levels produces spurious regressions

### Definition 8 (VECM)

$$\Delta y_t = \Pi y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + u_t, \quad \Pi = \alpha \beta'$$

### Johansen Test — economic data

r	Trace	CV 5%	Reject?
0	<b>64.09</b>	47.85	Yes
1	24.03	29.80	No
2	11.89	15.49	No
3	1.28	3.84	No

- ◻ 1 cointegrating relation found
- ◻ VECM more appropriate than VAR on levels

### Key message

- ◻ VAR on differences: loses long-run relationship; VECM: preserves it through  $\Pi = \alpha \beta'$



## Granger Causality: Empirical Results

### Interpretation

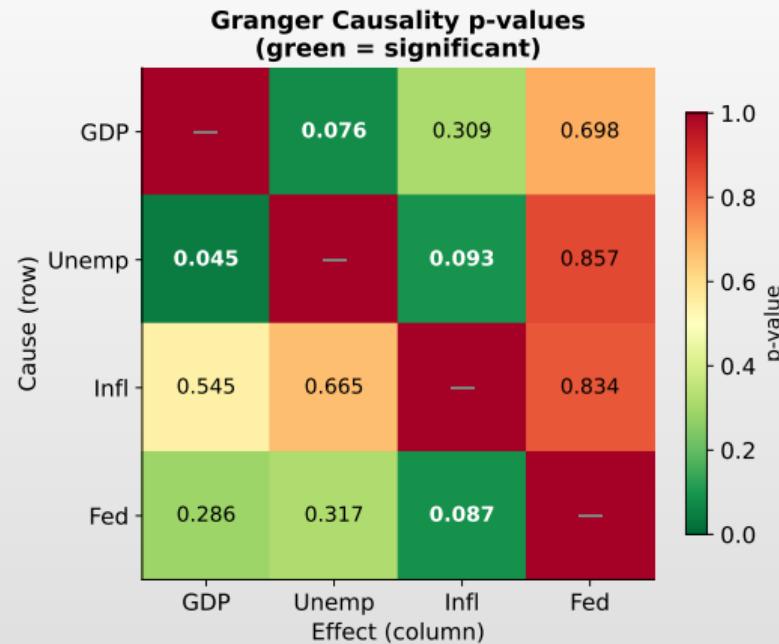
Each cell shows p-value for testing whether the row variable Granger-causes the column variable. Green:  $p < 0.10$ . Read: row causes column.

### Economic Findings

- Unemp → GDP ( $p = 0.045$ ): Okun's Law
- Fed → Inflation ( $p = 0.087$ ): Monetary policy transmission
- GDP → Unemp: Weak evidence



## Granger Causality: Empirical Results



## Granger Causality: Formal Definition

### Definition 9 (Granger Causality)

$X$  Granger-causes  $Y$  if, for some  $h > 0$ :

$$\text{MSE} \left[ \mathbb{E}[Y_{t+h} | \mathcal{F}_t^{X,Y}] \right] < \text{MSE} \left[ \mathbb{E}[Y_{t+h} | \mathcal{F}_t^Y] \right]$$

where  $\mathcal{F}_t^{X,Y}$  includes past values of both  $X$  and  $Y$ , while  $\mathcal{F}_t^Y$  includes only past  $Y$ .

### Important Caveat

- Granger causality = **predictive causality**, not true causality
- “ $X$  Granger-causes  $Y$ ”  $\Rightarrow X$  contains useful info for forecasting  $Y$
- Does **not** imply  $X$  causes  $Y$  in a structural sense

### Test Procedure

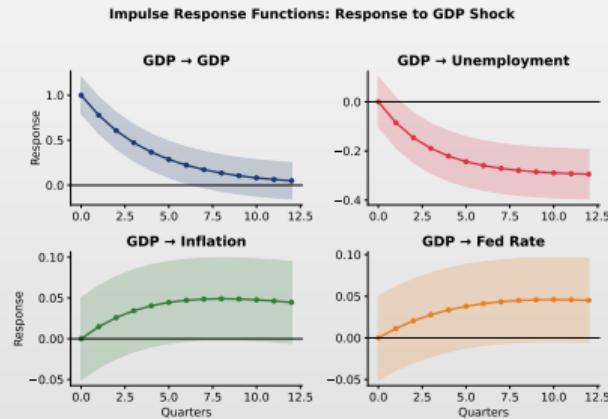
- F-test (or Wald):  $H_0$ : coefficients on lagged  $X$  are jointly zero in  $Y$  equation
- Reject  $H_0 \Rightarrow X$  Granger-causes  $Y$



## Impulse Response Functions (IRF)

### Effects

- $\uparrow$  GDP  $\succ \downarrow$  Unemployment (Okun),  $\uparrow$  Inflation (demand), Fed raises rate



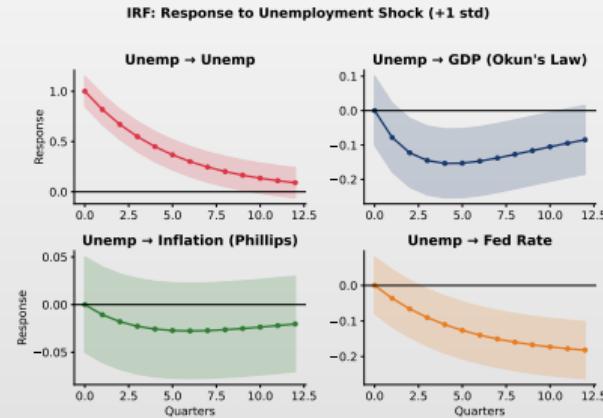
Q TSA\_ch10\_irf\_gdp\_shock



## IRF: Unemployment Shock

### Effects

- ↑ Unemp  $\Rightarrow$  ↓ GDP, ↓ Infl, Fed cuts rates



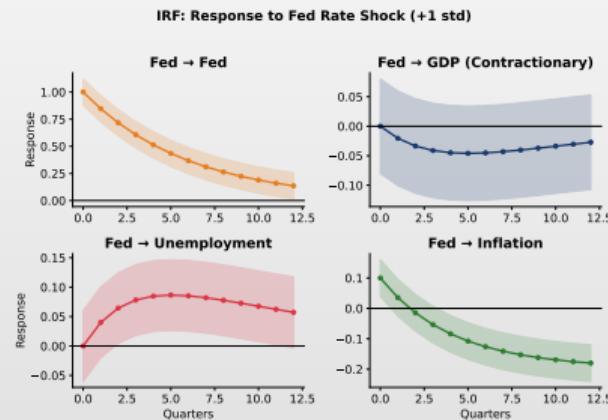
Q TSA\_ch10\_irf\_unemp\_shock



## IRF: Fed Rate Shock

### Monetary Policy

- Rate hike  $\Rightarrow$  GDP  $\downarrow$ , Unemp  $\uparrow$ , Infl  $\downarrow$



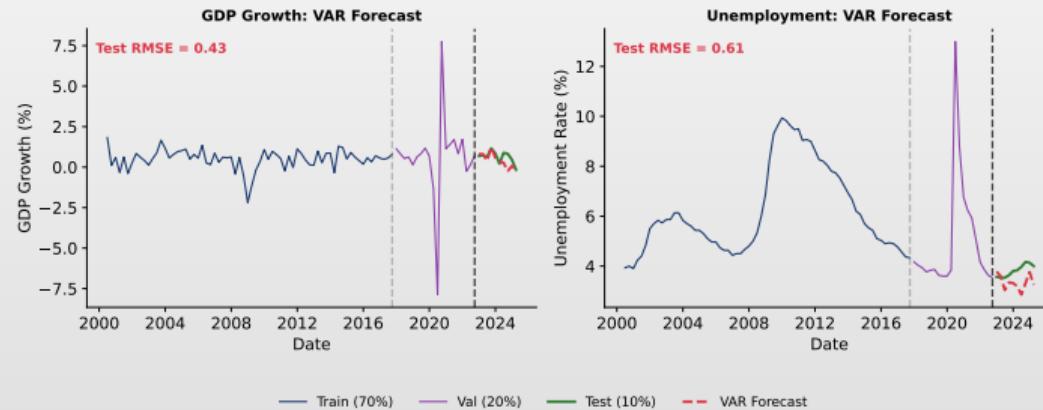
Q TSA\_ch10\_irf\_fed\_shock



## VAR: Forecast (Train/Val/Test)

### Rolling One-Step-Ahead Forecast

- VAR captures GDP-Unemployment dynamics
- COVID shock visible in test period



Q TSA\_ch10\_var\_forecast



## VAR: Test Set Results

### Test Set Performance by Variable

Variable	RMSE	MAE	Dir. Acc.
GDP Growth	1.33	0.99	50%
Unemployment	0.64	0.52	50%
Inflation	1.56	1.12	60%
Fed Rate	2.59	2.45	80%
Average	1.53	1.27	60%

### Strengths

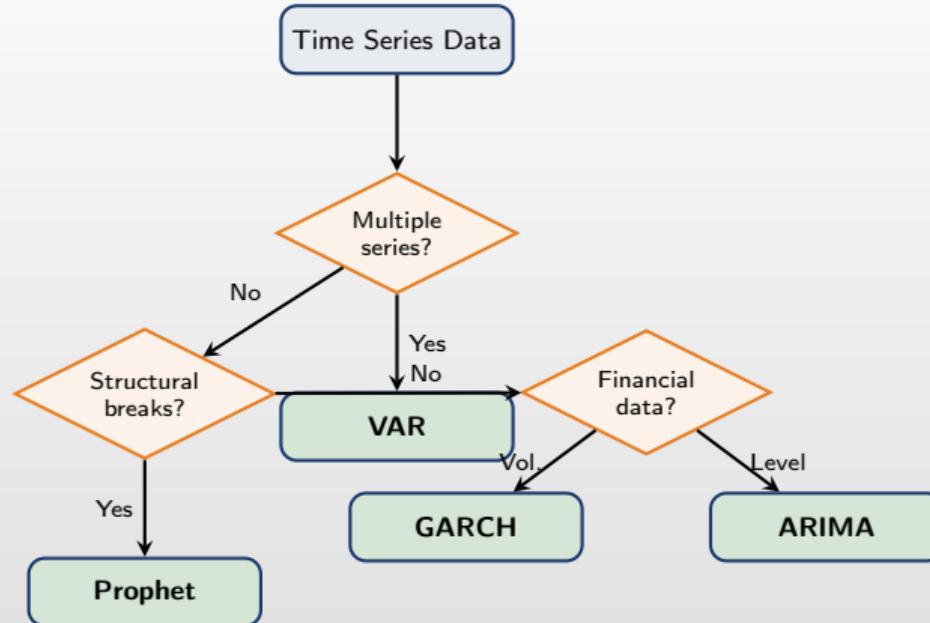
- Cross-variable dynamics
- Good directional accuracy

### Limitations

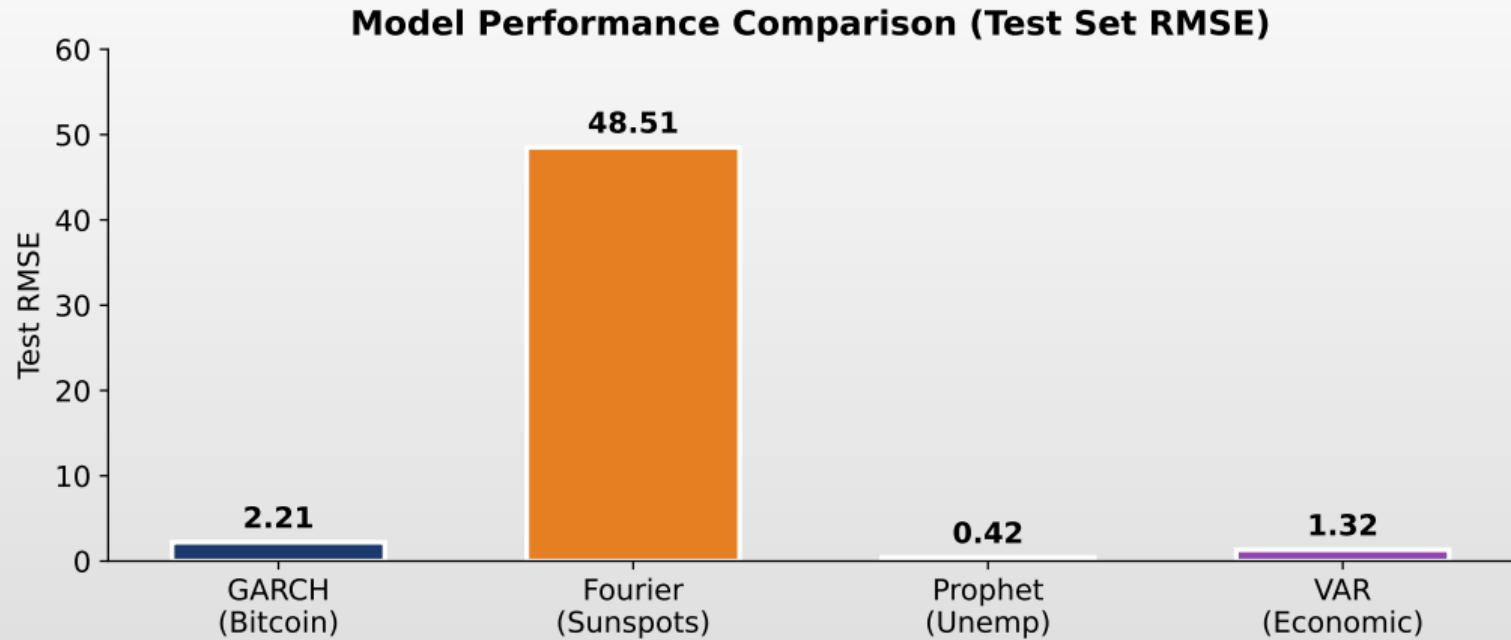
- Many parameters
- Sensitive to lag selection



## Model Selection Framework



## Summary: Model Comparison



Q TSA\_ch10\_model\_comparison



## Comprehensive Model Comparison

Feature	GARCH	Fourier	Prophet	VAR
<b>Target</b>	Volatility	Level	Level	Multiple
<b>Seasonality</b>	No	Yes (long)	Yes (multi)	No
<b>Structural breaks</b>	No	No	Yes	No
<b>Multiple series</b>	No	No	No	Yes
<b>Interpretable</b>	Medium	High	High	High
<b>Parameters</b>	Few	2K	Auto	Many
<b>Missing data</b>	No	No	Yes	No
<b>Best for</b>	Finance	Cycles	Business	Macro

### Empirical Conclusions

- **GARCH:** Student-t  $\succ$  Normal ( $\Delta AIC = 509$ )
- **Fourier:**  $K = 3$  harmonics, validated on test set
- **Prophet:** adapts to breaks via changepoints
- **VAR:** significant macro interactions (Granger)

### Key Insight

- RMSE cannot be compared across different datasets!
- Each model excels in its domain
- The art: matching model  $\leftrightarrow$  data



## Best Practices for Applied Forecasting

### Methodology

1. **Explore** data
2. **Test** stationarity
3. **Split** train/val/test
4. **Compare** on validation
5. **Report** test metrics

### Common Mistakes

- Peeking at test data
- Over-fitting
- Ignoring assumptions

### Practical Tips

- Start simple (naive)
- Add complexity if needed
- Check residuals
- Report CIs

### Remember

"All models are wrong, but some are useful." — Box



## Forecasting vs Causality vs Decision

Objective	Model	Focus
Pure prediction	ARIMA / ML	Out-of-sample accuracy
Financial risk	GARCH	Volatility, VaR
Macro dynamics	VAR	Multivariate interactions
Structural relations	SVAR / VECM	Causal identification
Regimes	Markov Switching	Regime changes

### Key Message

- There is no universal model
- There is **fit between model and problem**



## Key Takeaways

### 1. Rigorous Methodology

- ▶ Train/validation/test split prevents overfitting
- ▶ Test set must remain untouched until final evaluation

### 2. Match Model to Data

- ▶ Financial volatility → GARCH
- ▶ Long seasonality → Fourier terms
- ▶ Structural breaks → Prophet
- ▶ Multiple series → VAR

### 3. Interpret Results Carefully

- ▶ Granger causality  $\neq$  true causality
- ▶ Out-of-sample performance matters most
- ▶ Simpler models often work better



## The Role of AI in Time Series Modeling

### AI can

- Generate code for estimation and forecasting
- Select models (AutoML, grid search)
- Combine forecasts (ensemble)
- Detect anomalies and patterns

### But cannot

- Replace statistical validation
- Automatically detect **data leakage**
- Guarantee correct economic interpretation
- Verify model assumptions

### Principle

- AI is a **tool**, not an authority
- Statistical validation remains the researcher's responsibility



## AI Exercise: Critical Thinking

### Prompt to test in ChatGPT / Claude / Copilot

"Download monthly US Retail Sales from FRED (series RSXFS) for 2010-01 to 2024-12 (180 observations). Perform a complete time series analysis: decomposition, stationarity tests, model selection (compare ETS, SARIMA, and Prophet), 12-month forecast, and evaluation using RMSE/MAE/MASE on a 70/15/15 temporal split. Give me publication-quality Python code."

### Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it follow the correct workflow? (plot → decompose → test → model → diagnose → forecast)
3. Does it compare multiple models (ETS, ARIMA, SARIMA) with proper benchmarks?
4. Is the train/test split done properly? Is there any data leakage?
5. Does it discuss limitations and assumptions of the chosen model?

**Warning:** AI-generated code may run without errors and look professional. *That does not mean it is correct.*



## Question 1

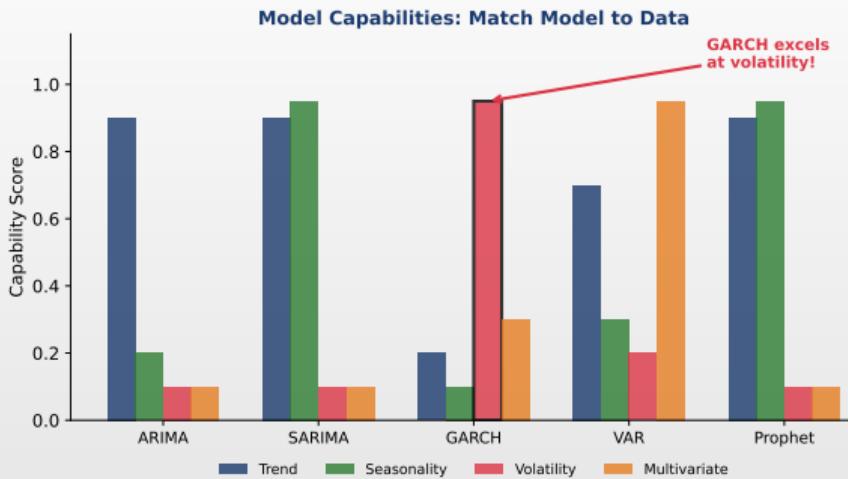
### Question

- Which model would you choose to forecast the volatility of financial returns?

### Answer Choices

- (A)** ARIMA — captures trends and autocorrelations
- (B)** GARCH — models conditional variance
- (C)** Prophet — detects changepoints and seasonality
- (D)** VAR — multivariate model for interdependencies

## Question 1: Answer



Answer: (B)

- GARCH captures volatility clustering and time-varying risk. ARIMA models the level, Prophet handles seasonality, VAR captures cross-series dynamics — none model variance directly.



## Question 2

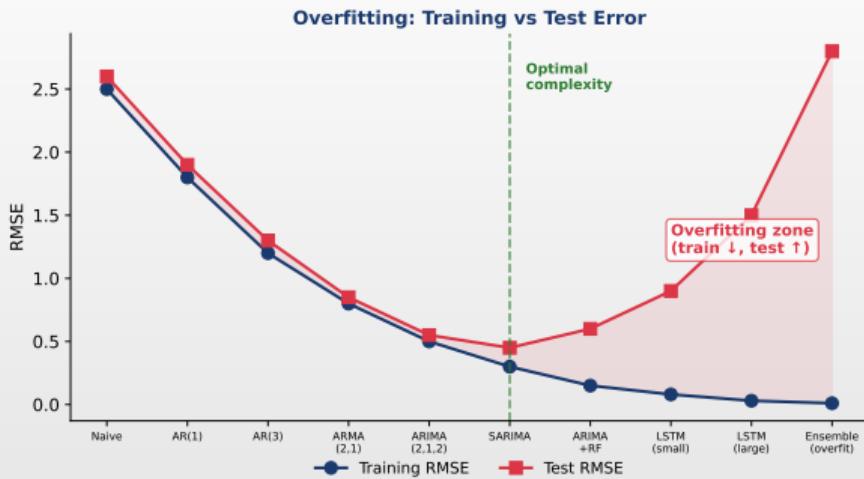
### Question

- A SARIMA model achieves RMSE = 0.05 on training but RMSE = 2.30 on test. What does this indicate?

### Answer Choices

- (A)** The model is excellent — low training error confirms quality
- (B)** The model suffers from overfitting — it memorizes noise
- (C)** The test set is faulty and should be replaced
- (D)** The difference is normal — all models have higher test error

## Question 2: Answer



Answer: (B)

- A  $46\times$  ratio between test and training RMSE signals severe overfitting. The model fits noise in the training data and fails to generalize. Solution: simpler model, proper validation.

Q TSA\_ch10\_quiz2\_overfitting



## Question 3

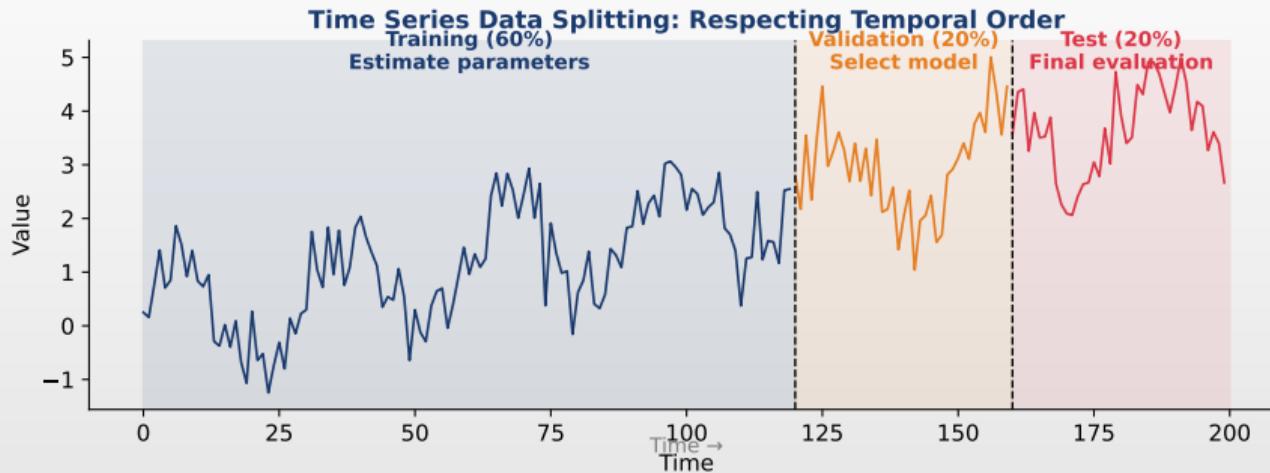
### Question

- Why is it important to separate data into train/validation/test sets?

### Answer Choices

- (A)** To have more training data
- (B)** To prevent overfitting and evaluate correctly
- (C)** It is just a convention with no real importance
- (D)** To reduce computation time

## Question 3: Answer



Answer: (B)

- Train: estimate parameters. Validation: select model/hyperparameters. Test: final unbiased evaluation. Mixing these roles leads to optimistic performance estimates.



## Question 4

### Question

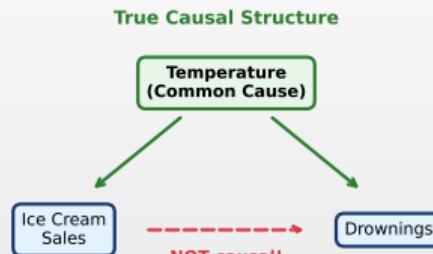
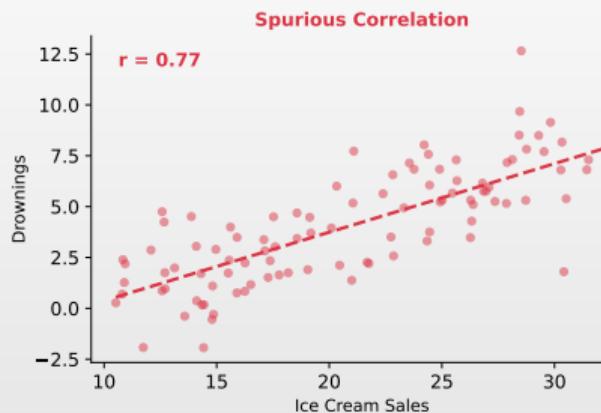
- Is Granger causality equivalent to true (structural) causality?

### Answer Choices

- (A)** Yes — if  $X$  predicts  $Y$ , then  $X$  causes  $Y$
- (B)** No — it only tests predictive content, not causation
- (C)** It depends on the number of lags selected
- (D)** Yes, if the p-value is below 0.05

## Question 4: Answer

Granger Causality ≠ True Causality



Answer: (B)

- Granger causality tests whether past  $X$  improves forecasts of  $Y$ . Spurious correlations (e.g., ice cream sales and drownings) can pass the test due to common causes.



## Question 5

### Question

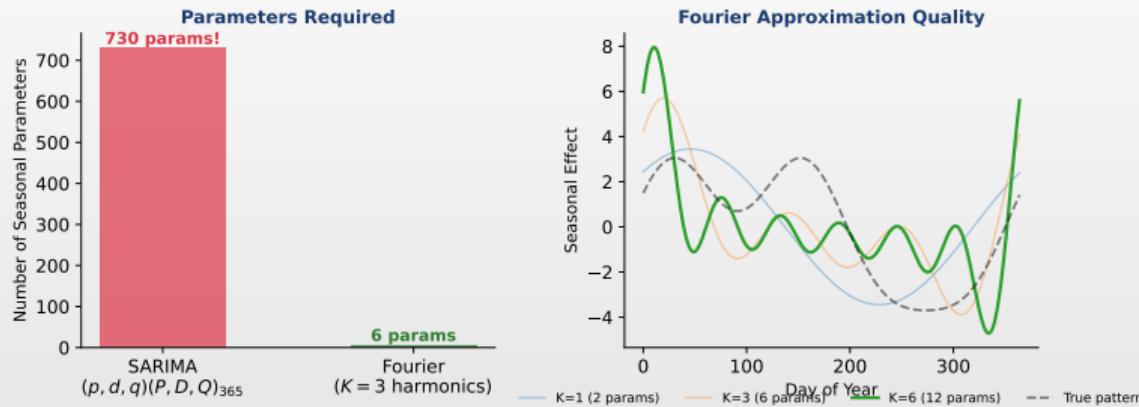
- What model do you use for a series with long seasonality (e.g.,  $s = 365$  days)?

### Answer Choices

- (A)** SARIMA( $p, d, q$ )( $P, D, Q$ )<sub>365</sub>
- (B)** GARCH — models variation
- (C)** ARIMA + Fourier terms or Prophet/TBATS
- (D)** VAR with 365 lags

## Question 5: Answer

Long Seasonality ( $s = 365$ ): Fourier Terms vs SARIMA



Answer: (C)

- SARIMA<sub>365</sub> needs  $\sim 730$  seasonal parameters — infeasible. Fourier terms with  $K = 3$  use only 6 parameters. Prophet and TBATS handle multiple seasonalities automatically.



## Bibliography I

### Fundamental Textbooks (common references across all chapters)

- Hamilton, J.D. (1994). *Time Series Analysis*, Princeton University Press.
- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.
- Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.

### Domain-Specific References

- Tsay, R.S. (2010). *Analysis of Financial Time Series*, 3rd ed., Wiley. (GARCH, VAR)
- Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*, Springer. (VAR, VECM)
- Francq, C., & Zakoïan, J.-M. (2019). *GARCH Models*, 2nd ed., Wiley. (Volatility)



## Bibliography II

### Modern Approaches and Forecasting Competitions

- Petropoulos, F., et al. (2022). Forecasting: Theory and Practice, *International Journal of Forecasting*, 38(3), 845–1054.
- Makridakis, S., Spiliotis, E., & Assimakopoulos, V. (2020). The M4 Competition, *International Journal of Forecasting*, 36(1), 54–74.
- Taylor, S.J., & Letham, B. (2018). Forecasting at Scale, *The American Statistician*, 72(1), 37–45.



## Key Takeaways

### What We Learned

- Model selection depends on data characteristics: stationarity, seasonality, volatility
- The Box-Jenkins methodology provides a systematic framework for time series modeling
- Proper evaluation requires out-of-sample testing and time series cross-validation

### Important

No single model wins everywhere. Match the model to the data: ARIMA for trends, SARIMA for seasonality, GARCH for volatility, VAR/VECM for multivariate dynamics, Prophet/TBATS for complex patterns. Always validate out-of-sample!



## References

-  Box, G.E.P., Jenkins, G.M., Reinsel, G.C., & Ljung, G.M. (2015). *Time Series Analysis: Forecasting and Control*. 5th ed., Wiley.
-  Hamilton, J.D. (1994). *Time Series Analysis*. Princeton University Press.
-  Tsay, R.S. (2010). *Analysis of Financial Time Series*. 3rd ed., Wiley.
-  Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*. 3rd ed., OTexts.
-  Taylor, S.J., & Letham, B. (2018). Forecasting at Scale. *The American Statistician*, 72(1), 37-45.
-  Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31(3), 307-327.
-  Sims, C.A. (1980). Macroeconomics and Reality. *Econometrica*, 48(1), 1-48.



# Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>

