



# Time Series Analysis and Forecasting

## Chapter 7: Cointegration and VECM



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## Outline

- Motivation
- Spurious Regression
- Cointegration Concept
- Engle-Granger Method
- Johansen Method
- VECM Estimation
- Practical Considerations
- Real-World Examples
- Case Study: Interest Rates
- Summary
- Quiz



## Why Cointegration Matters

### The Challenge

- Many economic/financial time series are **non-stationary ( $I(1)$ )**
- GDP, stock prices, exchange rates, interest rates all have unit roots
- Standard regression with  $I(1)$  variables ⇒ **spurious results**
- Differencing removes non-stationarity but loses **long-run information**

### The Solution: Cointegration

Some non-stationary series share a **common stochastic trend**—they move together in the long run.

### Nobel Prize 2003

Granger & Engle received the Nobel Prize for “methods for analyzing economic time series with common trends.”



## Real-World Applications

### Finance

- Pairs Trading:** Cointegrated stocks
- Term Structure:** Interest rates
- Spot-Futures:** Arbitrage

### Policy Analysis

- Fiscal:** Spending & taxes
- Monetary:** Rate pass-through
- Labor:** Wages & productivity

### Macroeconomics

- Consumption & Income**
- Money & Prices**
- PPP:** Exchange rates



## The Spurious Regression Problem

Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk:  $Y_t = \alpha + \beta X_t + u_t$  where  $Y_t$  and  $X_t$  are independent I(1) processes.

### Symptoms of Spurious Regression

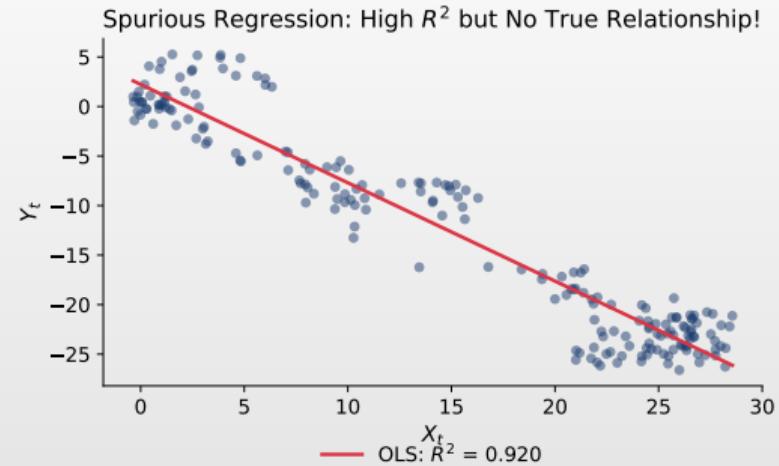
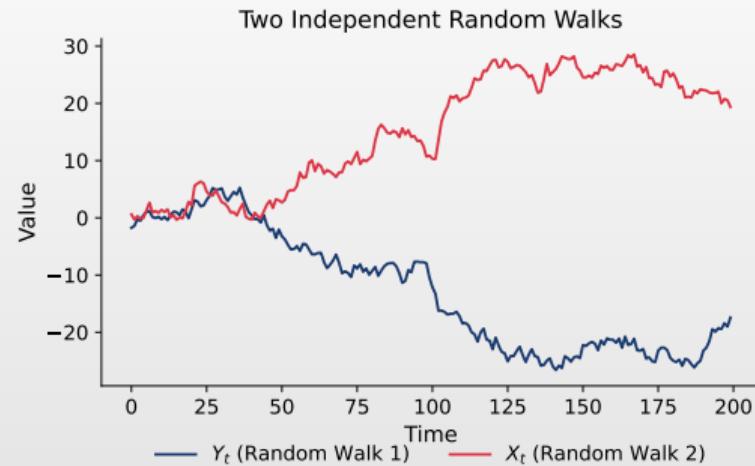
- High  $R^2$  (often  $> 0.9$ ) even though variables are **unrelated**
- Highly significant  $t$ -statistics (reject  $H_0 : \beta = 0$ )
- Very low Durbin-Watson statistic ( $DW \approx 0$ )
- Residuals are non-stationary (have unit root)

### Rule of Thumb

If  $R^2 > DW$ , suspect spurious regression!



## Spurious Regression: Visual Example



**Warning:** Two completely independent random walks show high correlation ( $R^2 > 0.8$ ) purely by chance! This is why we need cointegration analysis.



## Definition of Cointegration

### Definition 1 (Cointegration (Engle & Granger, 1987))

Variables  $Y_{1t}, Y_{2t}, \dots, Y_{kt}$  are **cointegrated of order  $(d, b)$** , written  $CI(d, b)$ , if:

1. All variables are integrated of order  $d$ :  $Y_{it} \sim I(d)$
2. There exists a linear combination  $\beta' Y_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$  that is integrated of order  $(d - b)$ , where  $b > 0$

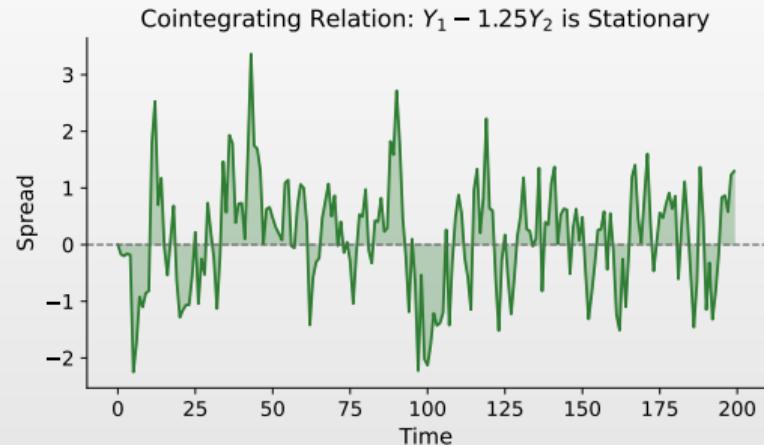
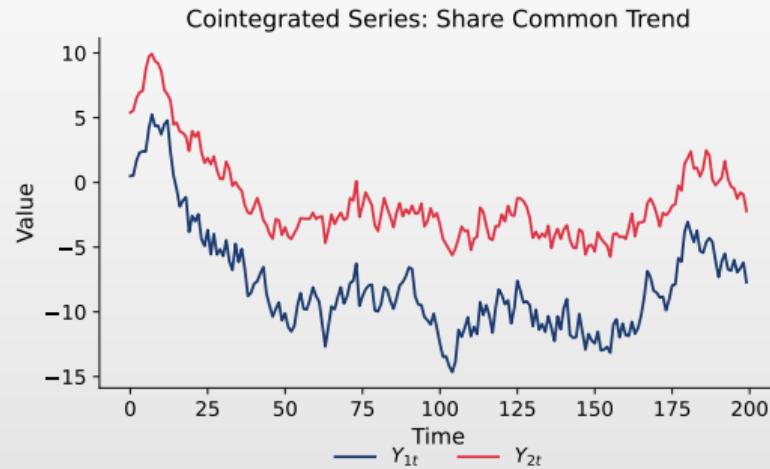
### Most Common Case: $CI(1, 1)$

- Variables are  $I(1)$  (have unit roots)
- Linear combination is  $I(0)$  (stationary)
- Vector  $\beta = (\beta_1, \dots, \beta_k)'$  is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized:  $\beta_1 = 1$ .



## Cointegration: Visual Example



**Key insight:** Both series are I(1) and trend together, but their linear combination (spread) is stationary—this is cointegration!



## Intuition: Common Stochastic Trends

### Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**:  $Y_{1t} = \gamma_1 \tau_t + S_{1t}$ ,  $Y_{2t} = \gamma_2 \tau_t + S_{2t}$  where  $\tau_t$  is a common random walk and  $S_{it}$  are stationary.

### Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

### Economic Interpretation

- Cointegration = **long-run equilibrium relationship**
- Variables may deviate in the short run, but are “pulled back”
- The cointegrating vector defines the equilibrium



## Cointegrating Rank

### How Many Cointegrating Relationships?

For  $k$  variables that are  $I(1)$ :

- Maximum possible cointegrating relationships:  $r = k - 1$
- If  $r = 0$ : No cointegration (variables drift apart)
- If  $r = k$ : All variables are  $I(0)$  (contradiction)

### Example: 3 Variables

- $r = 0$ : No cointegration
- $r = 1$ : One cointegrating relationship
- $r = 2$ : Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends =  $k - r$



## Engle-Granger Two-Step Method

### Step 1: Estimate Cointegrating Regression

Run OLS:  $Y_t = \alpha + \beta X_t + e_t$ . Save residuals:  $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

### Step 2: Test Residuals for Stationarity

Test if  $\hat{e}_t$  is  $I(0)$  using ADF:  $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$

- $H_0: \rho = 0$  (unit root  $\Rightarrow$  no cointegration)
- $H_1: \rho < 0$  (stationary  $\Rightarrow$  cointegration)

### Important

Use **Engle-Granger critical values**, not standard ADF! (More negative because residuals are estimated)



## Engle-Granger Critical Values

### Critical Values for Cointegration Test

Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

MacKinnon (1991),  $T = 100$

### Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on choice of dependent variable
- Small sample bias; cannot test hypotheses on cointegrating vector



## Johansen Cointegration Test

### Advantages over Engle-Granger

- Tests for **multiple** cointegrating relationships
- Maximum likelihood estimation (more efficient)
- Can test restrictions on cointegrating vectors
- Does not require choosing a dependent variable

### Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...



## VECM Representation

### Vector Error Correction Model

$$\Delta \mathbf{Y}_t = \mathbf{c} + \boldsymbol{\Pi} \mathbf{Y}_{t-1} + \sum_{j=1}^{p-1} \boldsymbol{\Gamma}_j \Delta \mathbf{Y}_{t-j} + \boldsymbol{\varepsilon}_t$$

- $\boldsymbol{\Pi} = \sum_i \mathbf{A}_i - \mathbf{I}$  (long-run impact);  $\boldsymbol{\Gamma}_j$  (short-run dynamics)

### Key Insight: Rank of $\boldsymbol{\Pi}$

The **rank of  $\boldsymbol{\Pi}$**  determines cointegration:

- $\text{rank}(\boldsymbol{\Pi}) = 0$ : No cointegration (VAR in differences)
- $\text{rank}(\boldsymbol{\Pi}) = k$ : All variables are  $I(0)$  (VAR in levels)
- $0 < \text{rank}(\boldsymbol{\Pi}) = r < k$ :  $r$  cointegrating vectors



## Decomposition of $\Pi$

When  $\text{rank}(\Pi) = r < k$

$\Pi = \alpha\beta'$  where  $\beta$  ( $k \times r$ ) = cointegrating vectors,  $\alpha$  ( $k \times r$ ) = adjustment coefficients

### Interpretation

- ◻  $\beta'Y_{t-1}$  = deviations from equilibrium (error correction terms)
- ◻  $\alpha$  = speed of adjustment; rows show each variable's response

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta' Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$$



## Johansen Test Statistics

### Two Test Statistics

Based on eigenvalues  $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$  of a certain matrix:

#### Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests  $H_0: \text{rank} \leq r$  vs  $H_1: \text{rank} > r$

#### Maximum Eigenvalue Test:

$$\lambda_{\max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

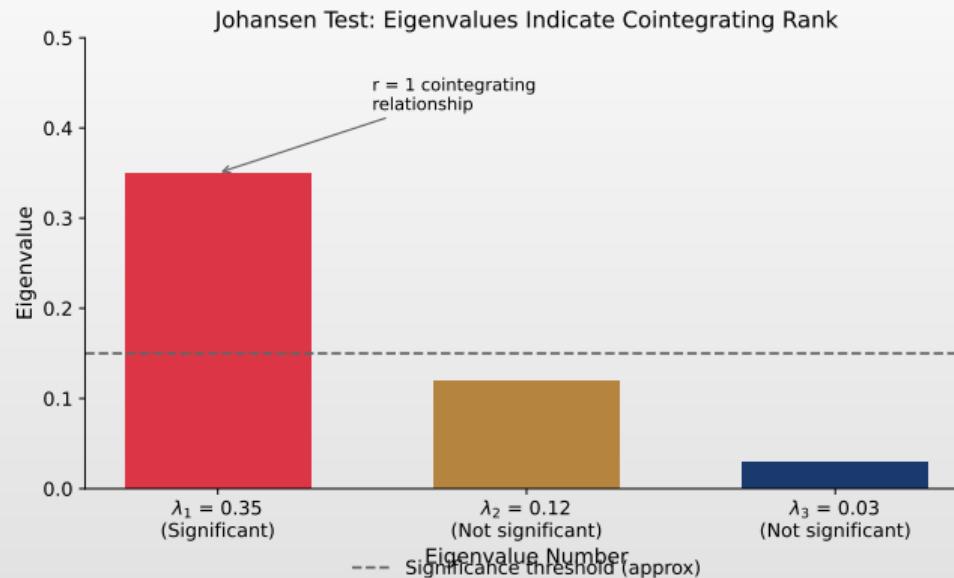
Tests  $H_0: \text{rank} = r$  vs  $H_1: \text{rank} = r + 1$

Critical values from Johansen & Juselius (1990), depend on:

- Number of variables  $k$
- Deterministic components (constant, trend)



## Johansen Test: Visual Interpretation



Significant eigenvalues (above threshold) indicate cointegrating relationships. First eigenvalue significant  $\Rightarrow r = 1$ .

Q TSA\_ch7\_johansen\_eigenvalues



## Testing Procedure

### Sequential Testing (Trace Test)

1. Test  $H_0: r = 0$ . If rejected  $\Rightarrow$  continue
2. Test  $H_0: r \leq 1$ . If not rejected  $\Rightarrow r = 1$
3. Continue until  $H_0$  is not rejected

### Deterministic Components

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- Constant in both** (most common)
- Constant + trend in cointegrating relation



## VECM Structure

### Full VECM Specification

For  $k = 2$  variables with  $r = 1$  cointegrating relation:

$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

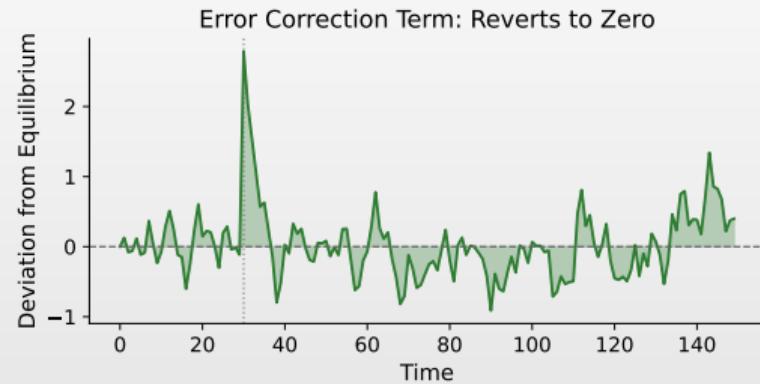
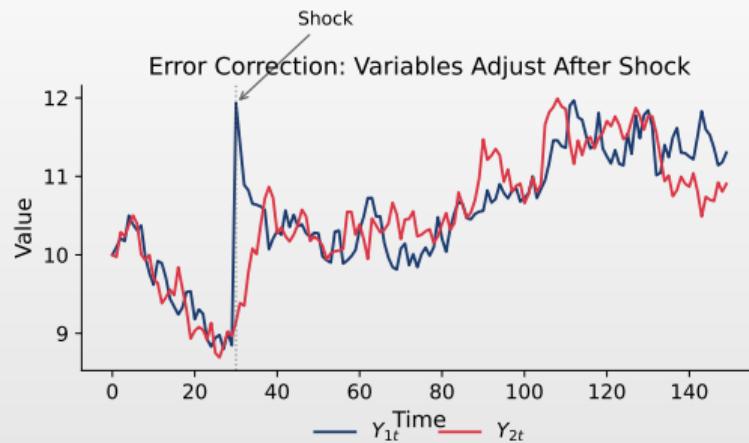
$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

### Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$  = error correction term (deviation from equilibrium)
- $\alpha_1, \alpha_2$  = adjustment speeds (should have opposite signs)
- $\gamma_{ij}$  = short-run dynamics
- $\varepsilon_{it}$  = innovations



## Error Correction Mechanism: Visual



**Error correction in action:** When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment.



## Interpreting Adjustment Coefficients

### The $\alpha$ Coefficients

If the cointegrating relation is  $Y_1 - \beta Y_2 = 0$  (equilibrium):

- $\alpha_1 < 0$ :  $Y_1$  adjusts downward when above equilibrium
- $\alpha_2 > 0$ :  $Y_2$  adjusts upward when  $Y_1$  is above equilibrium

### Weak Exogeneity

If  $\alpha_i = 0$ , variable  $Y_i$  does **not** respond to disequilibrium.

- $Y_i$  is **weakly exogenous** for the long-run parameters
- The other variable does all the adjusting
- Can simplify estimation (single-equation approach)

Test weak exogeneity:  $H_0 : \alpha_i = 0$  using likelihood ratio test.



## VECM vs VAR in Differences

### When Variables are Cointegrated

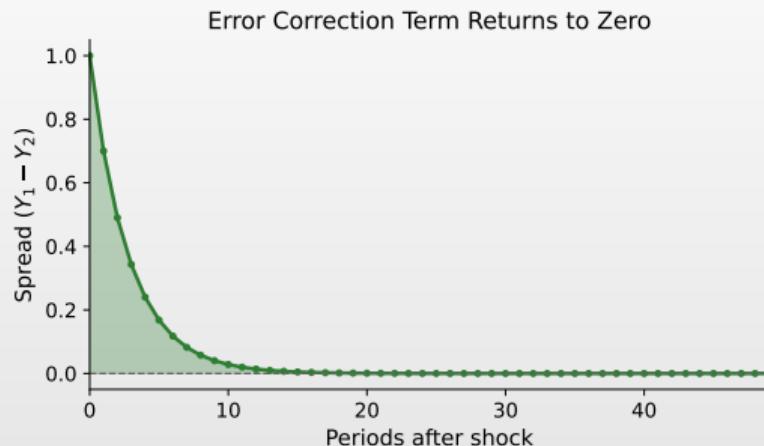
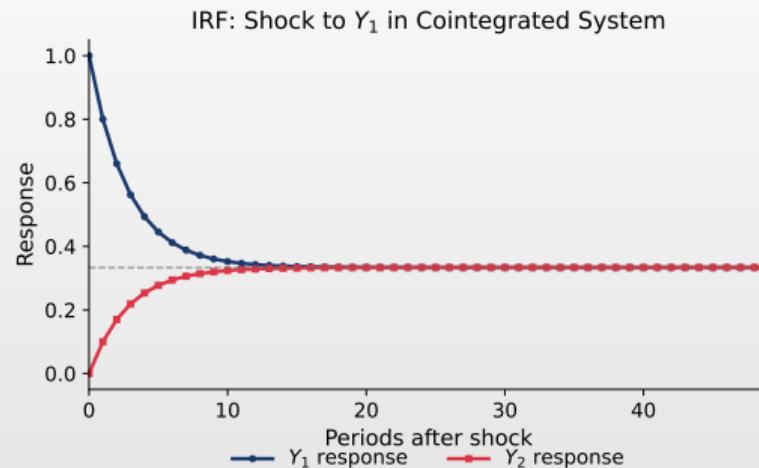
	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

### Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!



## VECM Impulse Response Functions



**IRF interpretation:** In a cointegrated system, shocks have **permanent effects** on levels but the system returns to equilibrium. Unlike stationary VAR, effects don't decay to zero—they converge to a new long-run value.



## Practical Workflow

### Step-by-Step Procedure

1. **Unit Root Tests:** Verify all variables are  $I(1)$ 
  - ▶ ADF, KPSS on levels and first differences
2. **Lag Length Selection:** Choose  $p$  for VAR in levels
  - ▶ Use AIC, BIC, or sequential LR tests
3. **Cointegration Test:** Johansen trace/max-eigenvalue tests
  - ▶ Determine cointegrating rank  $r$
4. **Estimate VECM:** If  $0 < r < k$ 
  - ▶ Estimate  $\alpha, \beta, \Gamma_j$
5. **Diagnostics:** Check residuals for autocorrelation, normality
6. **Analysis:** IRF, FEVD, hypothesis tests



## Common Pitfalls

### Things to Watch Out For

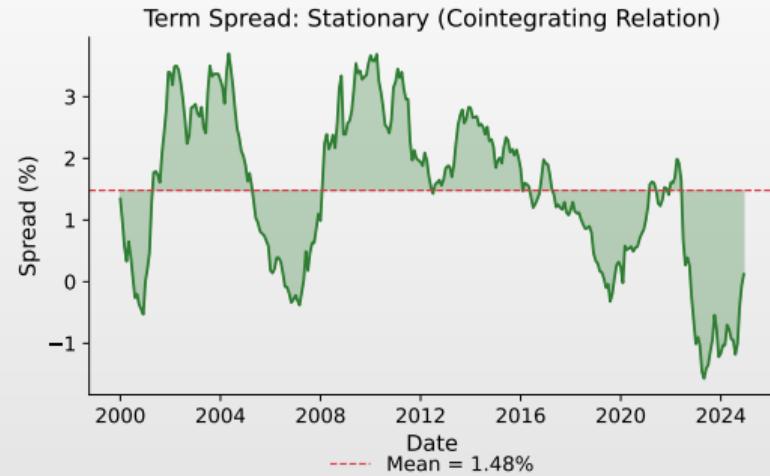
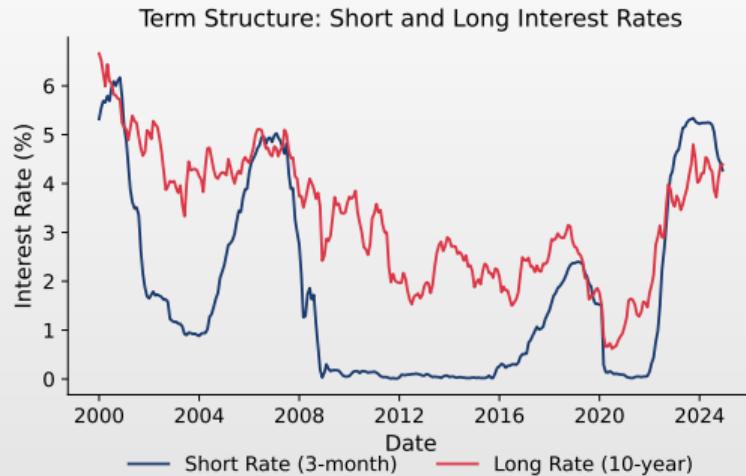
- Structural breaks:** Cause spurious unit roots or cointegration
- Near-unit-root:** Tests have low power
- Lag selection:** Too many/few lags bias results
- Small samples:** Johansen test oversized

### Recommendation

Always check: residual diagnostics, stability of cointegrating relationship, sensitivity to specification



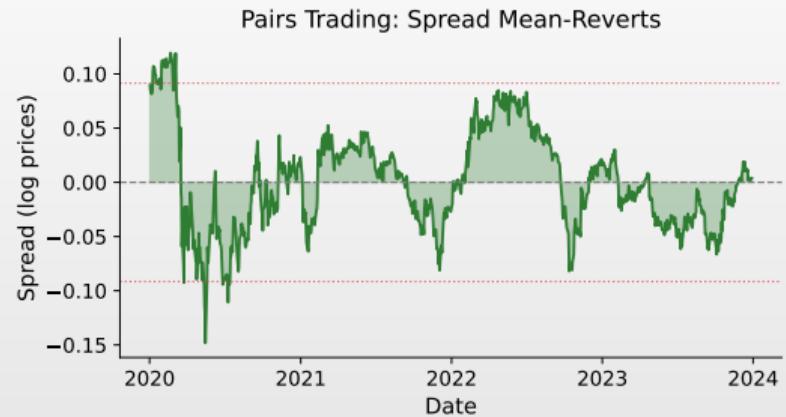
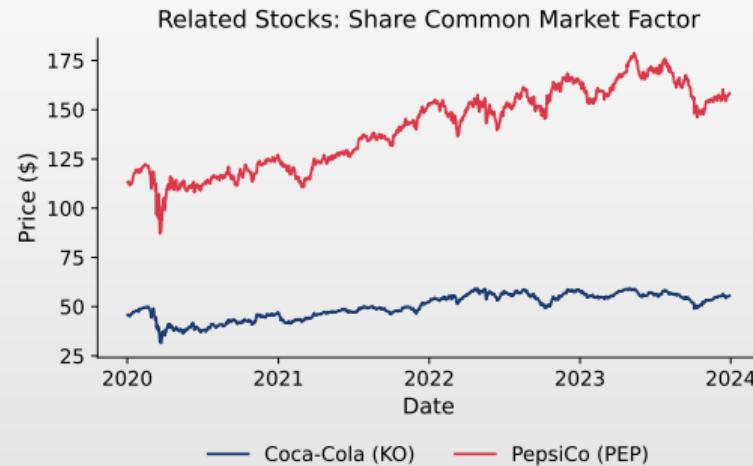
## Example 1: Term Structure of Interest Rates



**Expectations Hypothesis:** Short and long rates share common trend. The spread (term premium) is stationary—evidence of cointegration!



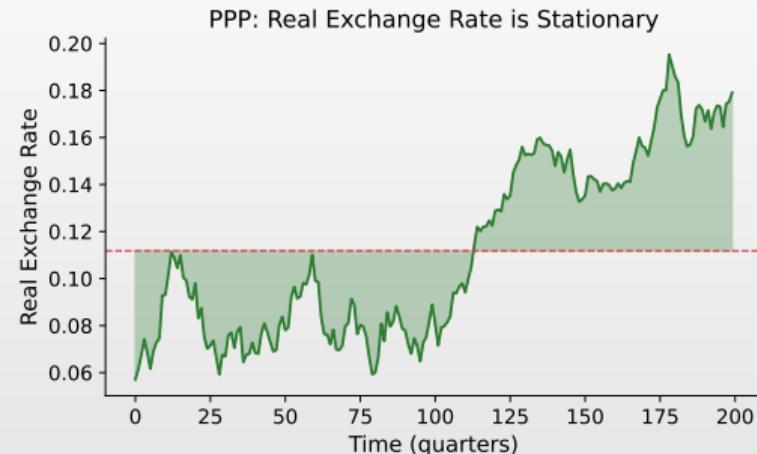
## Example 2: Pairs Trading in Finance



**Strategy:** Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When spread deviates from mean, trade expecting mean reversion. Sell spread when high, buy when low.



### Example 3: Purchasing Power Parity (PPP)



**PPP Theory:**  $e_t = p_t - p_t^*$  (log exchange rate equals price differential). Real exchange rate should be stationary in the long run.



## Case Study: Cointegration of Interest Rates

### Research Question

Are short-term and long-term interest rates cointegrated? Does the expectations hypothesis of the term structure hold?

### Data

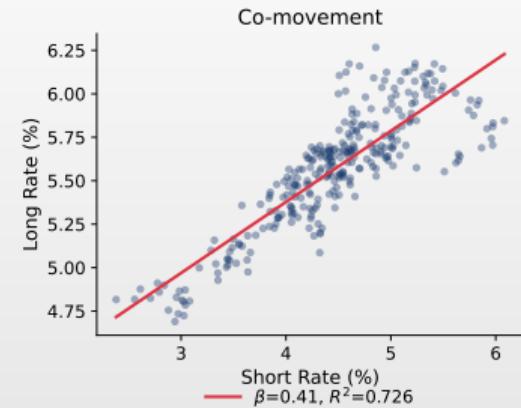
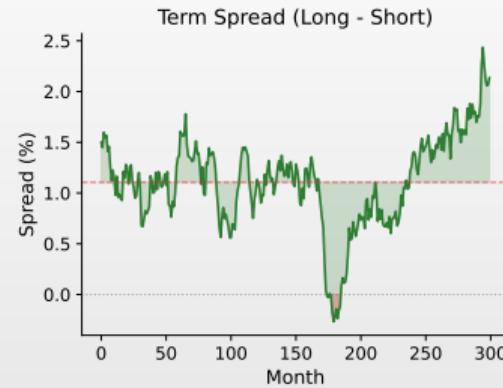
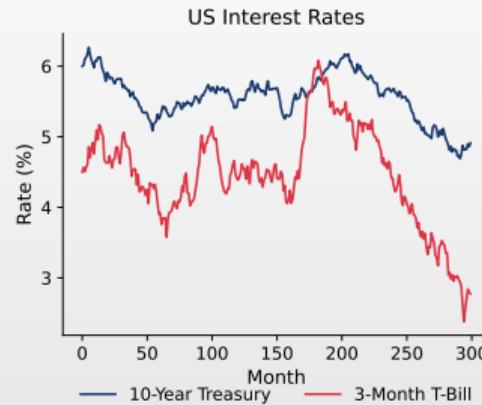
- US Monthly Data (1962-2023)
- 3-Month Treasury Bill Rate
- 10-Year Treasury Bond Yield
- Source: FRED Database

### Methodology

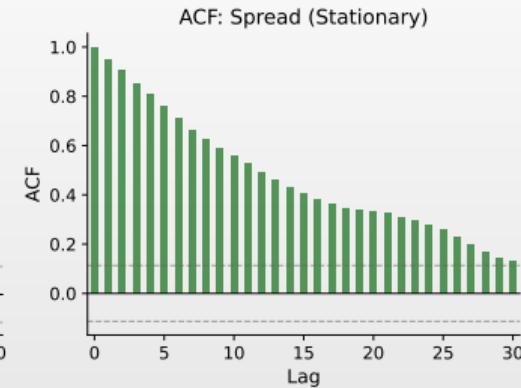
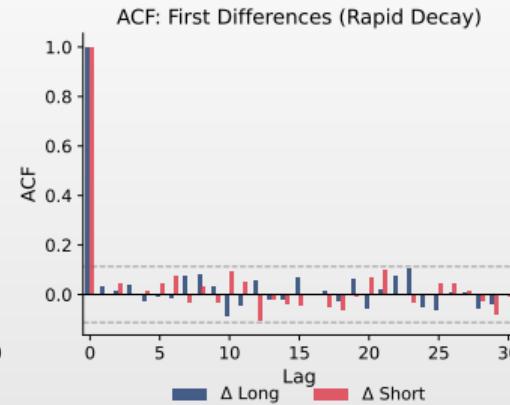
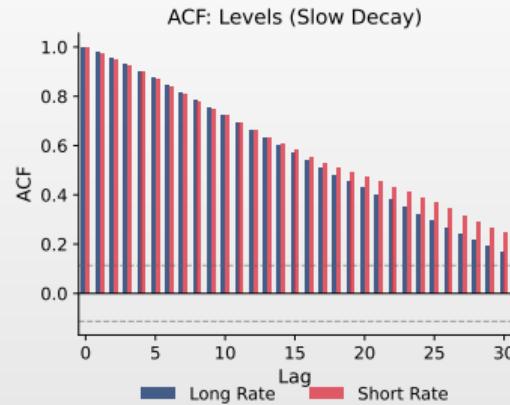
- Unit root tests (ADF, PP)
- Engle-Granger cointegration test
- Johansen procedure
- VECM estimation
- Impulse response analysis



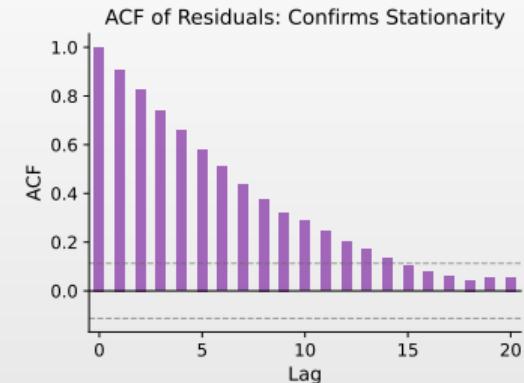
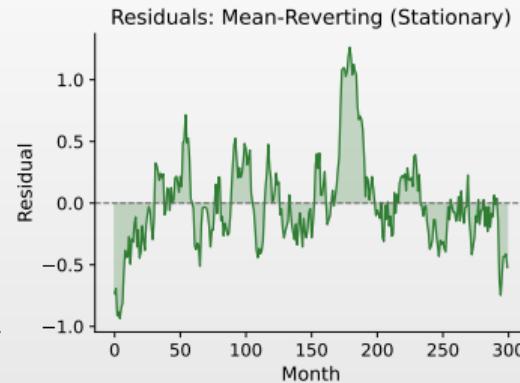
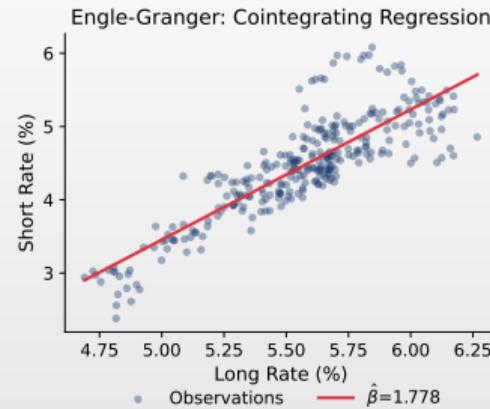
## Step 1: Data Visualization



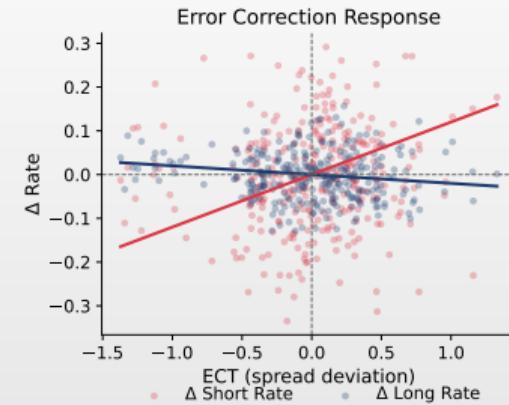
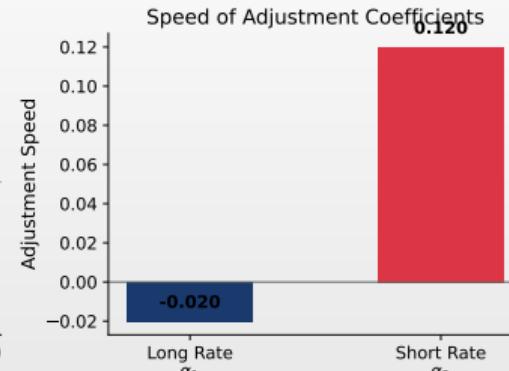
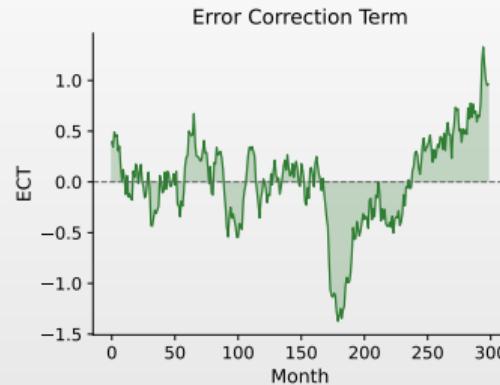
## Step 2: Unit Root Tests



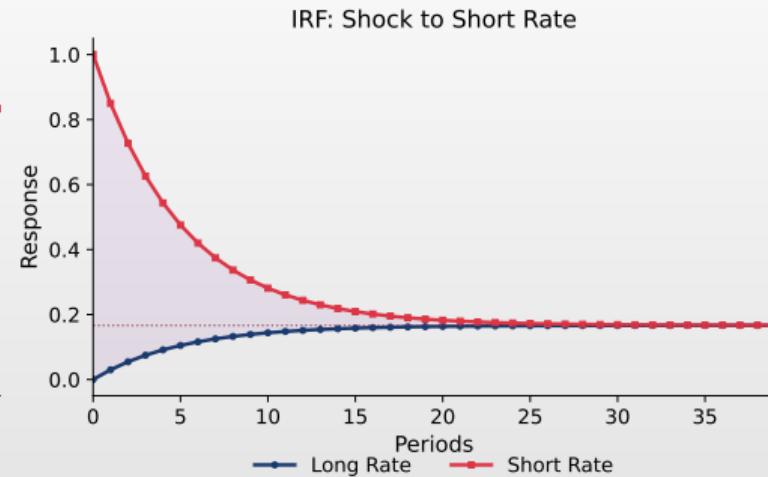
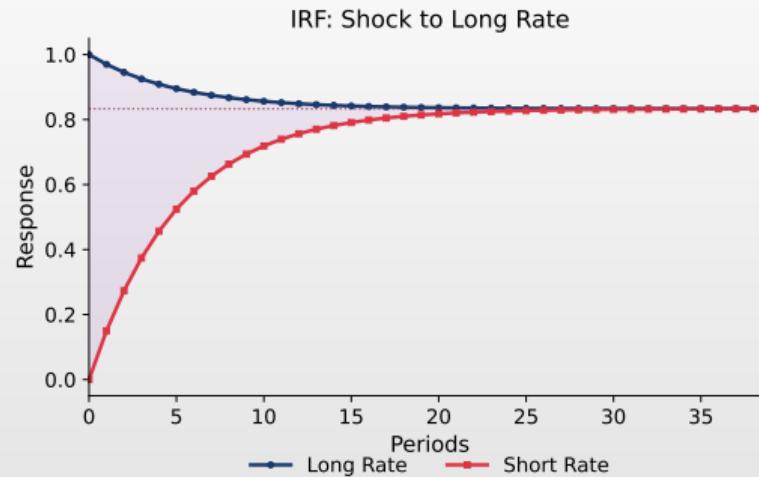
## Step 3: Engle-Granger Cointegration Test



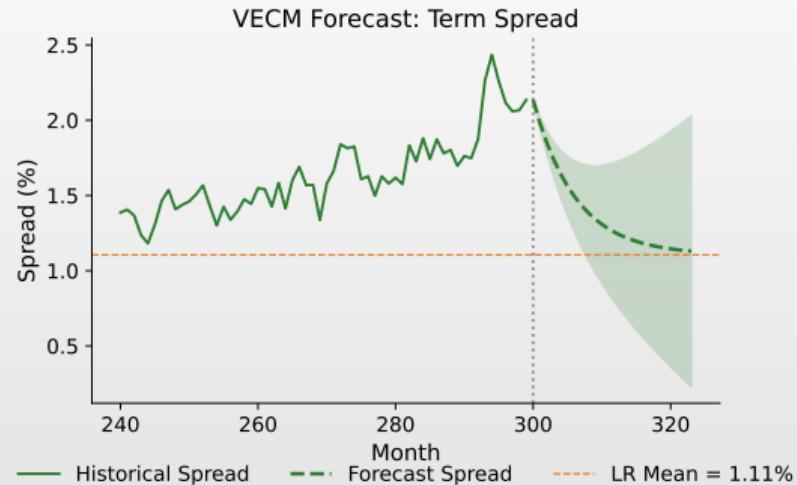
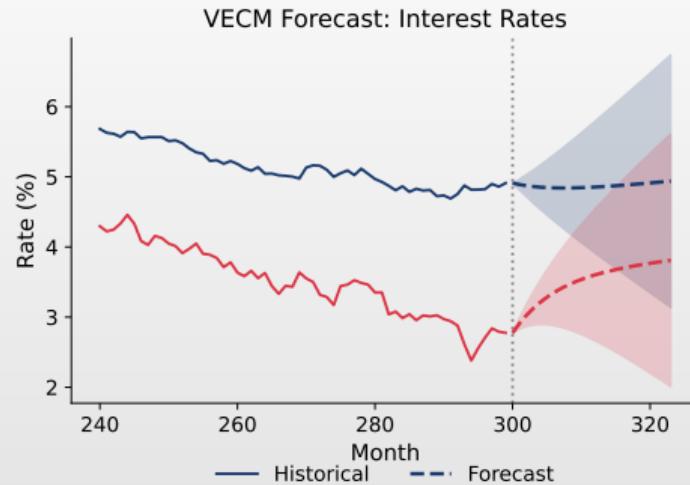
## Step 4: VECM Estimation



## Step 5: Impulse Response Functions



## Step 6: Forecasting



## Key Takeaways

### Main Concepts

- Cointegration:**  $I(1)$  variables with stationary linear combination
- Spurious regression:** High  $R^2$  with unrelated  $I(1)$  variables
- VECM:** VAR with error correction for cointegrated systems

### Testing Methods

- Engle-Granger:** Simple, one vector only
- Johansen:** Multiple vectors, MLE-based

### Remember

Tests have low power in small samples. Theory should guide specification.



## What's Next?

### Extensions and Related Topics

- **Structural VECM:** Identifying structural shocks
- **Threshold cointegration:** Nonlinear adjustment
- **Panel cointegration:** Multiple cross-sections
- **Fractional cointegration:** Long memory
- **Time-varying cointegration:** Regime changes

Questions?



## Quick Quiz

1. What does it mean for two  $I(1)$  variables to be cointegrated?
2. What is the “spurious regression” problem?
3. In a VECM, what do the  $\alpha$  coefficients represent?
4. What is the main advantage of Johansen over Engle-Granger?
5. If  $\alpha_i = 0$  for variable  $Y_i$ , what does this imply?



## Quiz Answers

1. **Cointegration:** A linear combination of the variables is  $I(0)$  (stationary). They share a common stochastic trend.
2. **Spurious regression:** Regressing one  $I(1)$  variable on another unrelated  $I(1)$  variable gives high  $R^2$  and significant coefficients even though there's no true relationship.
3.  $\alpha$  **coefficients:** Speed of adjustment—how quickly each variable responds to deviations from long-run equilibrium.
4. **Johansen advantage:** Can test for multiple cointegrating relationships, uses MLE (more efficient), doesn't require choosing dependent variable.
5.  $\alpha_i = 0$ : Variable  $Y_i$  is weakly exogenous—it doesn't respond to disequilibrium. Other variables do all the adjusting.

