Time Series Analysis and Forecasting

Chapter 1: Stochastic Processes and Stationarity

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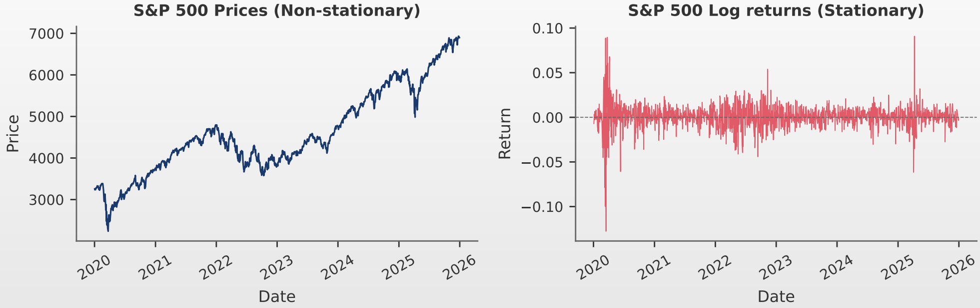
Learning Objectives

By the end of this chapter, you will be able to:

1. **Define** stochastic processes and understand their properties
2. **Distinguish** between strict and weak (covariance) stationarity
3. **Identify** white noise and random walk processes
4. **Compute** and interpret ACF and PACF
5. **Apply** the lag operator and differencing
6. **Conduct** stationarity tests (ADF, KPSS)
7. **Analyze** financial time series data
8. **Distinguish** between unit root and trend-stationary processes

Outline

Examples: stationary vs. non-stationary series



Observations

- **Prices** (left) are non-stationary: trend, the mean changes over time
- **Returns** (right) are stationary: mean ≈ 0, approximately constant variance
- Log returns: $r_t = \ln P_t - \ln P_{t-1} \Rightarrow$ non-stationary $\rightarrow$ stationary

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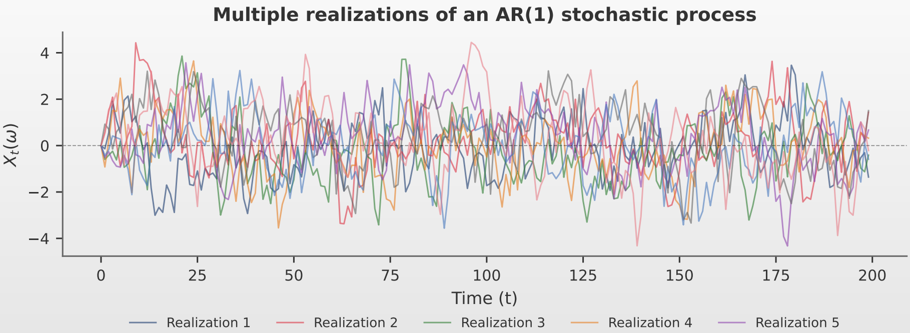
Stochastic process: visual illustration



Interpretation

- Each line is a **different realization** from the same underlying stochastic process
- We observe only **one realization**, yet aim to understand the properties of the process

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Stochastic process: definition

Definition 1 (Stochastic Process)

- A **stochastic process** is a collection of random variables indexed by time
  - $\{X_t(\omega) : t \in \mathcal{T}, \omega \in \Omega\}$
  - Ω is the sample space of possible outcomes

Two Perspectives

- **Fixed** ω: A *realization* $\{X_t(\omega)\}_{t\in\mathcal{T}}$
- **Fixed** t: A *random variable* X_t

Key Insight

- A time series we observe is **one realization** of the underlying stochastic process

Moments of a stochastic process

The First Two Moments Characterize the Process

- **Mean Function**: $\mu_t = \mathbb{E}[X_t]$
- **Autocovariance (ACVF)**: $\gamma(t, s) = \text{Cov}(X_t, X_s)$
  - $\gamma(t, s) = \mathbb{E}[(X_t - \mu_t)(X_s - \mu_s)]$
- **Autocorrelation (ACF)**:
  - $\rho(t, s) = \gamma(t, s)/\sqrt{\text{Var}(X_t) \cdot \text{Var}(X_s)}$

ACF Properties

- **Range**: $\rho(t, s) \in [-1, 1]$
- **Normalization**: $\rho(t, t) = 1$ (perfect correlation with itself)

Key Point

- **General**: μ_t and $\gamma(t, s)$ may depend on t
- **Stationary**: Removes this dependence

Why stationarity matters

Without Stationarity

- Mean, variance change over time
  - Estimates are inconsistent
- Past may not predict the future
- Standard methods fail
- Spurious correlations

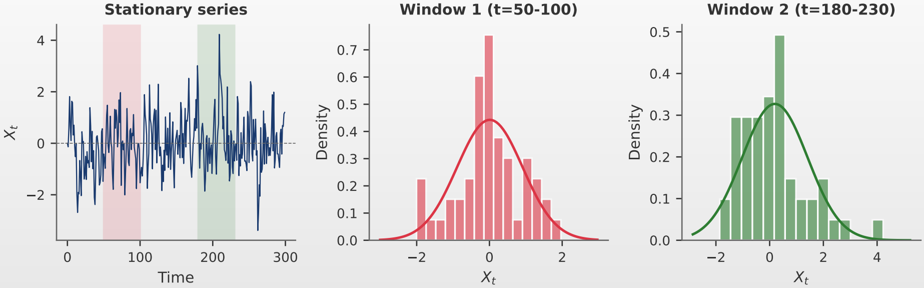
With Stationarity

- Statistical properties constant
  - Ergodicity justified
- Can estimate from a single realization
- Valid inference possible
- Models are meaningful

Key Principle

- Most time series models (ARMA, ARIMA, etc.) require stationarity
- Non-stationary series must be transformed (e.g., differencing) before modeling

Strict stationarity: visual illustration



Interpretation

- Time translation does not change the joint distribution of the variables
- Any two time windows have the same statistical properties
- In practice: we only check the first moments (weak stationarity)

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Strict stationarity

Definition 2 (Strict (Strong) Stationarity)

- A process $\{X_t\}$ is **strictly stationary** if for all k, all $t_1, \ldots, t_k$, and all h:
  - $(X_{t_1}, \ldots, X_{t_k}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_k+h})$
- **Notation**: $X \stackrel{d}{=} Y$ means *equality in distribution*
  - $P(X \leq x) = P(Y \leq x)$

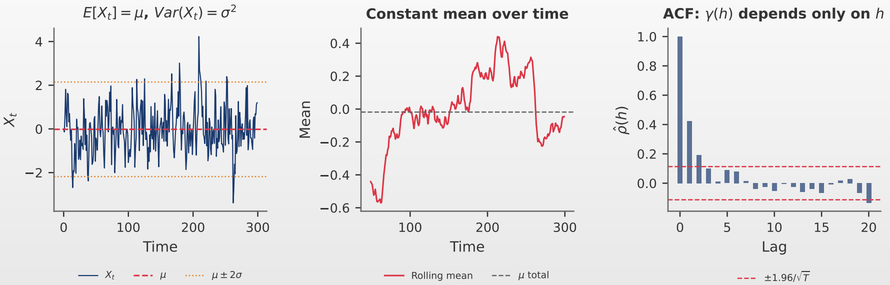
Implications

- **Identical distributions**: $F_{X_t}(x)$ does not depend on t
  - $\mathbb{E}[X_t] = \mu$ (constant mean, if it exists)
  - $\mathrm{Var}(X_t) = \sigma^2$ (constant variance, if it exists)
- **Lag dependence**: Joint distributions depend only on lag

Note

- Strict stationarity is a strong condition, often impossible to verify in practice

Weak stationarity: visual illustration



The Three Conditions

- $\mathbb{E}[X_t] = \mu$ constant $\Rightarrow$ mean does not depend on time
- $\mathrm{Var}(X_t) = \sigma^2$ constant $\Rightarrow$ variance does not depend on time
- $\mathrm{Cov}(X_t, X_{t+h}) = \gamma(h)$ $\Rightarrow$ autocovariance depends only on lag h

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Weak (covariance) stationarity

Definition 3 (Weak Stationarity)

- A process $\{X_t\}$ is **weakly stationary** (or covariance stationary) if:
  - $\mathbb{E}[X_t^2] < \infty$ for all t — finite second-order moments
  - $\mathbb{E}[X_t] = \mu$ for all t — constant mean
  - $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ — covariance depends only on lag h, not on t

Key Properties

- **Autocovariance**: $\gamma(h) = \text{Cov}(X_t, X_{t+h}) = \mathbb{E}[(X_t - \mu)(X_{t+h} - \mu)]$
- **Autocorrelation**: $\rho(h) = \gamma(h)/\gamma(0) = \text{Cov}(X_t, X_{t+h})/\text{Var}(X_t)$
- **Note**: $\rho(0) = 1$, $|\rho(h)| \leq 1$, $\rho(h) = \rho(-h)$ (symmetry)

Relationship between strict and weak stationarity

Theorem 1 (Fundamental Implication)

If $\{X_t\}$ is **strictly stationary** and $\mathbb{E}[X_t^2] < \infty$, then $\{X_t\}$ is also **weakly stationary**.

Proof.

- Let t_1, t_2 be arbitrary and h any time shift
- From joint distribution invariance: $(X_{t_1}, X_{t_2}) \stackrel{d}{=} (X_{t_1+h}, X_{t_2+h})$
- $\mathbb{E}[X_{t_1}] = \mathbb{E}[X_{t_1+h}] = \mu$ (constant mean)
- $\text{Cov}(X_{t_1}, X_{t_2}) = \text{Cov}(X_{t_1+h}, X_{t_2+h})$
- Thus autocovariance depends only on the difference $t_2 - t_1 = h$, not on t_1

$\square$

Warning: The Converse is NOT True!

- There exist weakly stationary processes that are **not** strictly stationary

Example: AR(1) is weakly stationary

Model

- $X_t = \phi X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2), \quad |\phi| < 1$

Verification of the three conditions

1. **Constant mean**: $\mathbb{E}[X_t] = \phi\,\mathbb{E}[X_{t-1}] + 0 = \phi\,\mathbb{E}[X_t] \quad \Rightarrow \quad \mathbb{E}[X_t] = 0$
2. **Constant variance**: $\mathrm{Var}(X_t) = \phi^2\,\mathrm{Var}(X_t) + \sigma^2 \quad \Rightarrow \quad \mathrm{Var}(X_t) = \dfrac{\sigma^2}{1-\phi^2}$
3. **Autocovariance depends only on lag**: $\gamma(h) = \phi^{|h|} \cdot \dfrac{\sigma^2}{1-\phi^2}, \quad \rho(h) = \phi^{|h|}$

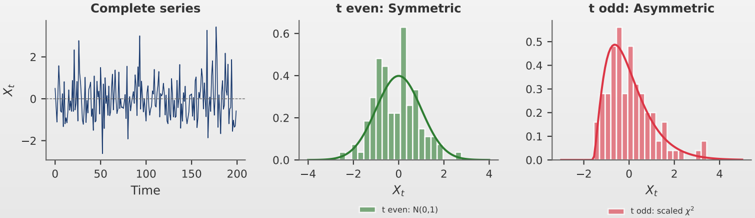
Numerical example: $\phi = 0.8$, $\sigma^2 = 1$

- $\mathbb{E}[X_t] = 0, \quad \mathrm{Var}(X_t) = \frac{1}{1-0.64} = 2.78, \quad \rho(1) = 0.8, \quad \rho(2) = 0.64, \quad \rho(5) = 0.33$

Counterexample: weakly stationary but NOT strictly stationary

Construction

- Let $\{X_t\}$ be **independent** random variables with: t even: $X_t \sim N(0,1)$; t odd: $X_t \sim \frac{\chi^2(5)-5}{\sqrt{10}}$



Weakly stationary ✓

- $\mathbb{E}[X_t] = 0,\ \mathrm{Var}(X_t) = 1,\ \mathrm{Cov}(X_t, X_{t+h}) = 0$

NOT strictly stationary ×

- Skewness differs (0 vs > 0) $\Rightarrow X_1 \overset{d}{\neq} X_2$

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Properties of the autocovariance function

Proposition 1

For a weakly stationary process, the ACVF $\gamma(h)$ satisfies:

- **Symmetry:** $\gamma(h) = \gamma(-h)$
- **Maximum at zero:** $|\gamma(h)| \leq \gamma(0) = \text{Var}(X_t)$
- **Non-negative definiteness:** $\sum_{i,j} a_i a_j \gamma(i-j) \geq 0$ for any $a_1, \ldots, a_n$

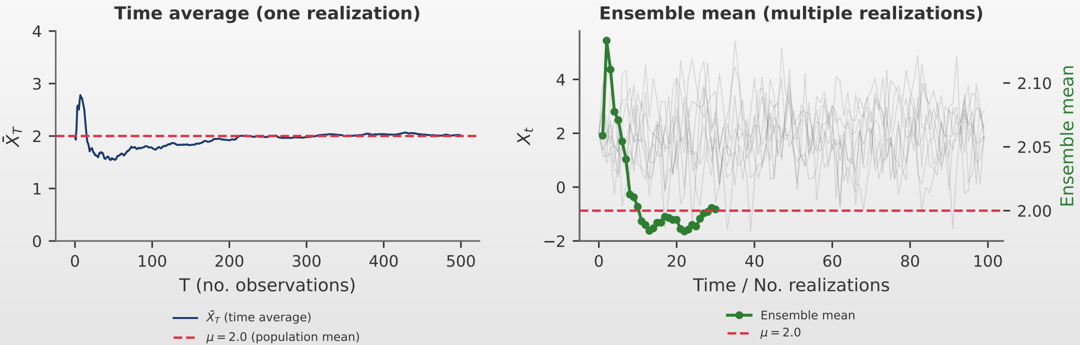
Proof (property 3)

- $\text{Var}\left(\sum_{i=1}^{n} a_i X_{t+i}\right) = \sum_{i,j} a_i a_j \gamma(i-j) \geq 0 \quad (\text{variance} \geq 0)$

Implication

- Not every function can be a valid autocovariance function

Ergodicity: visual illustration



- **Time average** (single realization) and **ensemble average** (multiple realizations) both converge to μ
- Ergodicity guarantees that we can estimate μ from a **single sufficiently long time series**

TSA_ch1_stationarity

Ergodicity: the foundation of inference from data

Definition 4 (Ergodicity for Mean)

- A stationary process $\{X_t\}$ is **ergodic for the mean** if $\bar{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow{P} \mathbb{E}[X_t] = \mu$ as $T \to \infty$

Why does ergodicity matter?

- **Problem**: We have only **one realization** of the stochastic process
- **Solution**: Ergodicity allows estimating μ from $\bar{X}_T$ — the time average converges to the population mean. Without ergodicity, inference is not possible!

Theorem 2 (Sufficient Condition)

If $\sum_{h=0}^{\infty} |\gamma(h)| < \infty$ (absolutely summable autocovariances), the process is ergodic.

Counterexample: stationary but non-ergodic

- Let $Z \sim N(0,1)$, define $X_t = Z \; \forall\, t$. Strictly stationary, but $\bar{X}_T = Z \; \forall\, T \Rightarrow$ time average does **not** converge to $\mu = 0$
- **Conclusion**: ergodicity is an **additional** assumption, stronger than stationarity

Spectral Density: The Frequency Domain

Definition 5 (Power Spectral Density)

- For a stationary process with $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, the **spectral density** is:

$$S(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h)\, e^{-i\omega h} = \frac{1}{2\pi} \left[\gamma(0) + 2 \sum_{h=1}^{\infty} \gamma(h) \cos(\omega h) \right], \quad \omega \in [-\pi, \pi]$$

- $S(\omega)$ decomposes the variance across frequencies: $\gamma(0) = \int_{-\pi}^{\pi} S(\omega)\, d\omega$

Interpretation

- Large $S(\omega)$ at low $\omega \Rightarrow$ dominant long cycle
- White noise: $S(\omega) = \frac{\sigma^2}{2\pi}$ (flat)
- AR(1) $\phi > 0$: power at low freq.
- MA(1) $\theta > 0$: power at high freq.

Connections

- **Fourier pair**: $S(\omega) \leftrightarrow \gamma(h)$ (equivalent)
- Time domain (ACF) $\equiv$ freq. domain (spectrum)
- **Periodogram**: empirical estimator of $S(\omega)$
- Useful for detecting hidden seasonality

The Wold decomposition theorem

Theorem 3 (Wold, 1938)

Any **covariance stationary** process $\{X_t\}$ can be written as: $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \eta_t$

- $\varepsilon_t \sim WN(0, \sigma^2) \Rightarrow$ white noise
  - $\psi_0 = 1, \sum \psi_j^2 < \infty$
- $\eta_t \Rightarrow$ deterministic component (perfectly predictable)

Significance of the Wold Theorem

- **Decomposition**: Any stationary process = **MA(∞)** + deterministic component
  - Theoretically justifies MA(q) and ARMA(p, q) models
  - Coefficients ψ_j measure the impact of past shocks

Proof of the Wold theorem (sketch)

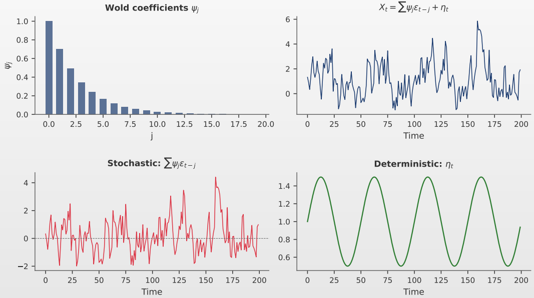
Proof sketch.

1. **Hilbert space of the past**: Define $\mathcal{H}_t = \overline{\mathrm{sp}}\{X_s : s \le t\}$ — the closed linear span of past and present values, with inner product $\langle X, Y\rangle = \mathrm{Cov}(X, Y)$.
2. **Innovation**: Define $\varepsilon_t = X_t - \hat{X}_t$, where $\hat{X}_t = \mathrm{Proj}_{\mathcal{H}_{t-1}}(X_t)$ is the orthogonal projection. By construction, $\varepsilon_t \perp \mathcal{H}_{t-1}$, so $\varepsilon_t \perp \varepsilon_s$ for $t \neq s \Rightarrow \{\varepsilon_t\}$ is white noise.
3. **Iterative representation**: Applying the projection recursively:

$$X_t = \varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \cdots + \eta_t = \sum_{j=0}^{\infty} \psi_j\varepsilon_{t-j} + \eta_t$$

   where ψ_j arise from successive projections, and $\eta_t \in \mathcal{H}_{-\infty} = \bigcap_t \mathcal{H}_t$.
4. **Convergence**: $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ because $\mathrm{Var}(X_t) < \infty$ (stationarity).
5. **Deterministic component**: $\eta_t \in \mathcal{H}_{-\infty} \Rightarrow \eta_t$ is *perfectly predictable* from the infinite past. $\square$

The Wold theorem: visual illustration

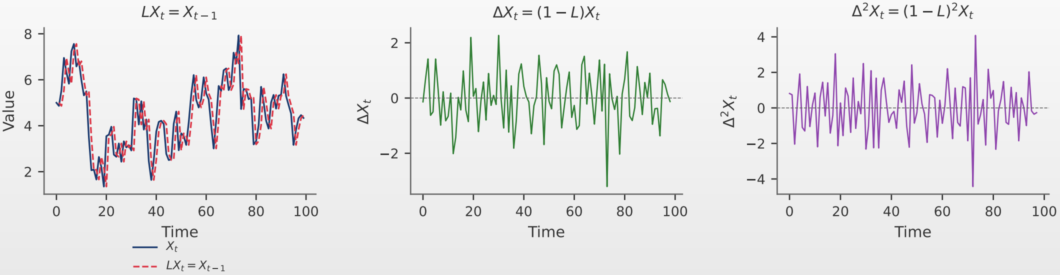


Interpretation

- X_t decomposes into a **stochastic** component (MA(∞)) and a **deterministic** component (η_t)
- Coefficients ψ_j decay $\Rightarrow$ recent shocks have greater impact than distant ones

TSA_ch1_operators

Lag operator: visual illustration



Properties

- $LX_t = X_{t-1} \Rightarrow$ the lag operator shifts the series back by one period
- $L^k X_t = X_{t-k} \Rightarrow$ shift by k periods; $L^0 = I$ (identity)
- **Difference operator**: $\Delta = (1 - L)$, so $\Delta X_t = X_t - X_{t-1}$

TSA_ch1_operators

The lag operator

Definition 6 (Lag Operator)

- The **lag operator** (or backshift operator) L is defined by: $LX_t = X_{t-1}$

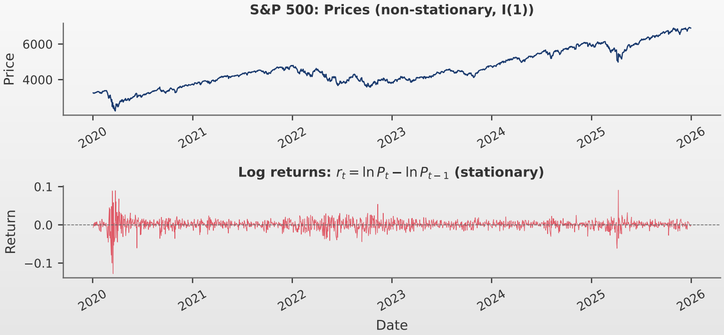
Properties

- **Powers**: $L^k X_t = X_{t-k}$ (lag by k periods)
  - Compact notation for models
- **Identity**: $L^0 = I$
- **Polynomial**: $(1 - \phi L)X_t = X_t - \phi X_{t-1}$

Examples

- **First difference**: $(1 - L)X_t = X_t - X_{t-1}$
- **Second difference**: $(1 - L)^2 X_t = \Delta^2 X_t$
- **Seasonal**: $(1 - L^{12})X_t$

Effect of differencing: S&P 500



Interpretation

- **Top**: S&P 500 prices $\Rightarrow$ clear trend, non-stationary ($I(1)$)
- **Bottom**: Log returns $r_t = \ln P_t - \ln P_{t-1}$ $\Rightarrow$ fluctuates around mean ≈ 0, stationary

TSA_ch1_operators

Differencing

Why Do We Difference?

- **First Difference**: $\Delta X_t = X_t - X_{t-1} = (1 - L)X_t$
  - Removes trend and unit root
  - Random walk: $\Delta X_t = \varepsilon_t$

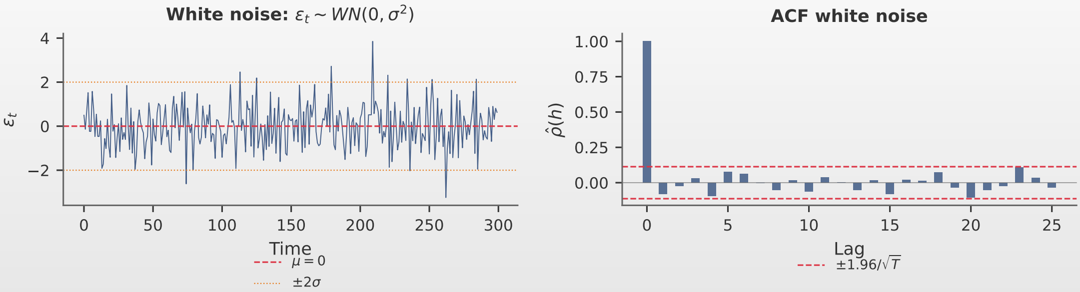
Definition 7 (Integrated Process of Order d)

- A process $\{X_t\}$ is **integrated of order** d, denoted $X_t \sim I(d)$, if:
  - $\Delta^d X_t = (1 - L)^d X_t$ is stationary ($I(0)$ process)
  - $\Delta^{d-1} X_t$ is **not** stationary

Examples

- $I(0)$: Stationary process (white noise, stationary AR)
- $I(1)$: Random walk $\Rightarrow \Delta X_t = \varepsilon_t$ is stationary
- $I(2)$: Requires two differences for stationarity

White noise: visual illustration



TSA_ch1_white_noise

White noise process

Definition 8 (White Noise)

- A process $\{\varepsilon_t\}$ is **white noise**, denoted $\varepsilon_t \sim WN(0, \sigma^2)$, if:
  - $\mathbb{E}[\varepsilon_t] = 0$ for all t (zero mean)
  - $\text{Var}(\varepsilon_t) = \sigma^2$ for all t (constant variance)
  - $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ for $t \neq s$ (uncorrelated)

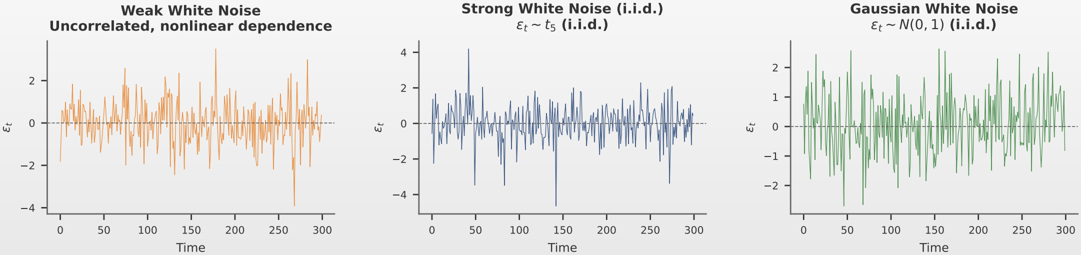
ACF of White Noise

- By definition: $\gamma(0) = \sigma^2$ and $\gamma(h) = 0$ for $h \neq 0$; $\rho(h) = \begin{cases} 1 & h = 0 \\ 0 & h \neq 0 \end{cases}$

Types of white noise (in order of increasing restrictions)

- **Weak**: uncorrelated, but nonlinear dependencies may exist
- **Strong**: ε_t are *independent* and identically distributed (i.i.d.)
- **Gaussian**: $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$
  - Uncorrelated $\Rightarrow$ independent

The three types of white noise

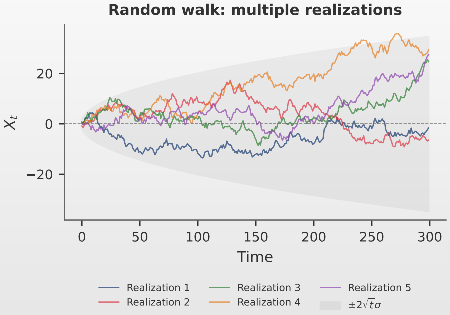


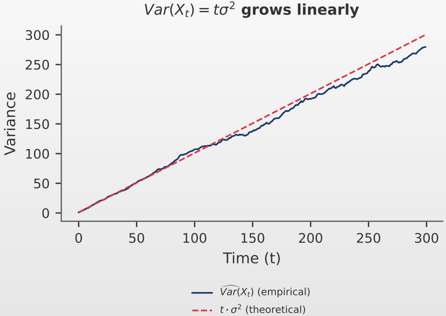
Inclusion relationship: Gaussian $\subset$ Strong (i.i.d.) $\subset$ Weak (uncorrelated)

- **Weak**: $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$, but nonlinear dependencies may exist (e.g. GARCH)
- **Strong**: ε_t are i.i.d. — any distribution (e.g. Student-t)
- **Gaussian**: $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$ — uncorrelated $\Leftrightarrow$ independent

TSA_ch1_white_noise

Random walk: visualization





Observations

- Each shock has a **permanent effect**; $\text{Var}(X_t) = t\sigma^2$ grows linearly with time
- **Solution** — differencing transforms into white noise, $\Delta X_t = \varepsilon_t$

TSA_ch1_random_walk

Random walk process

Definition 9 (Random Walk)

$X_t = X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2),\ X_0 = 0 \quad \Rightarrow$ **Explicit form**: $X_t = \sum_{i=1}^{t} \varepsilon_i$

Proposition 2 (Properties)

- $\mathbb{E}[X_t] = 0$
- $\text{Var}(X_t) = t\sigma^2$ (grows with time!)
- $\text{Cov}(X_t, X_s) = \min(t, s) \cdot \sigma^2$

Proofs.

- $\mathbb{E}[X_t] = \mathbb{E}\left[\sum_{i=1}^{t} \varepsilon_i\right] = 0$
- $\text{Var}(X_t) = \text{Var}\left(\sum_{i=1}^{t} \varepsilon_i\right) = \sum_{i=1}^{t} \text{Var}(\varepsilon_i) = t\sigma^2$ (independence)
- $\text{Cov}(X_t, X_s) = \min(t, s)\,\sigma^2$ (for $s \leq t$)

$\square$

Non-Stationary!

$\text{Var}(X_t) = t\sigma^2$ depends on $t \Rightarrow$ random walk is **not stationary**

Random walk with drift

Definition 10 (Random Walk with Drift)

$X_t = c + X_{t-1} + \varepsilon_t, \quad c \neq 0$ is the **drift** $\Rightarrow$ **Explicit form**: $X_t = ct + \sum_{i=1}^{t} \varepsilon_i$

Proposition 3 (Properties)

- $\mathbb{E}[X_t] = ct$ (linear trend)
- $\text{Var}(X_t) = t\sigma^2$ (grows with time)

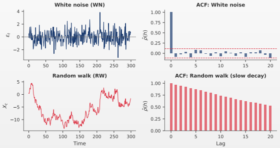
Differencing

$\Delta X_t = c + \varepsilon_t$ — constant plus white noise $\Rightarrow$ the differenced series is stationary

Practical Importance

- Nominal GDP, stock prices $\Rightarrow$ often modeled as RW with drift
- The ADF test includes variants: without constant, with constant, with constant and trend

White noise vs random walk: comparison



TSA_ch1_random_walk

White Noise

- Stationary; Var $= \sigma^2$ (const.); ACF $= 0$, $h \neq 0$; no memory

Random Walk

- Non-stationary; Var $= t\sigma^2$ (grows); ACF ≈ 1 (slow); permanent shocks

Link

- $\Delta X_t = \varepsilon_t$

Trend-stationary vs. difference-stationary

Trend-Stationary (TS)

- **Model**: $Y_t = \alpha + \beta t + \varepsilon_t$
  - **Deterministic** trend
  - Deviations from the trend are temporary
- **Solution**: regression on t, extract residuals
- **Effect**: Shocks do NOT have a permanent effect

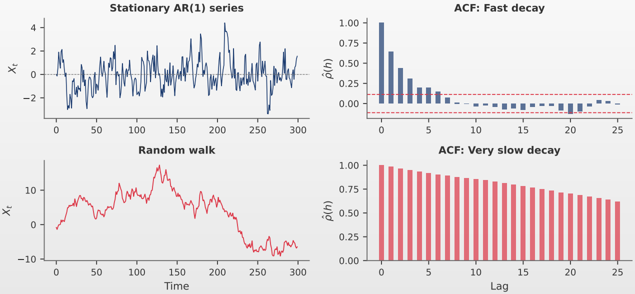
Difference-Stationary (DS)

- **Model**: $Y_t = c + Y_{t-1} + \varepsilon_t$
  - **Stochastic** trend
  - Deviations from the trend are permanent
- **Solution**: differencing (ΔY_t)
- **Effect**: Shocks HAVE a permanent effect

Why does the distinction matter?

- **Differencing a TS process**: introduces an artificial unit root in the MA part
- **Regression on a DS process**: produces residuals that are **still non-stationary**
- **Solution**: ADF and KPSS tests help distinguish between the two

ACF comparison: stationary vs random walk

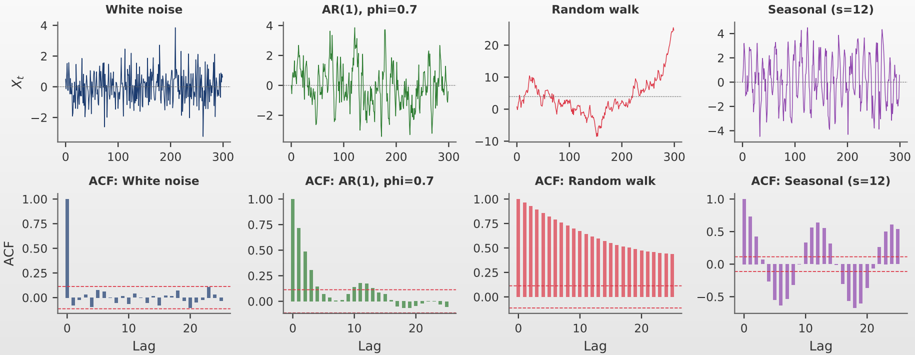


Interpretation

- **Stationary**: ACF decays rapidly (exponentially or oscillating) toward zero
- **Random walk**: ACF decays very slowly, stays close to 1
- **Rule of thumb**: Slow ACF decay ⇒ suspect unit root ⇒ ADF test

TSA_ch1_random_walk

ACF patterns for different processes



Interpretation

- **White noise**: ACF = 0; **Stationary**: decays fast; **Non-stationary**: decays slowly
- **Seasonal**: Spikes at seasonal lags (12, 24 for monthly data)

TSA_ch1_acf_patterns

Sample autocorrelation function

Sample ACF at Lag h

- $\hat{\rho}(h) = \frac{\sum_{t=1}^{T-h}(x_t - \bar{x})(x_{t+h} - \bar{x})}{\sum_{t=1}^{T}(x_t - \bar{x})^2}$
  - Properties: $\hat{\rho}(0) = 1$, $|\hat{\rho}(h)| \leq 1$

Theorem 4 (Bartlett, 1946)

Under H_0: white noise, for large T: $\hat{\rho}(h) \approx N(0, 1/T)$

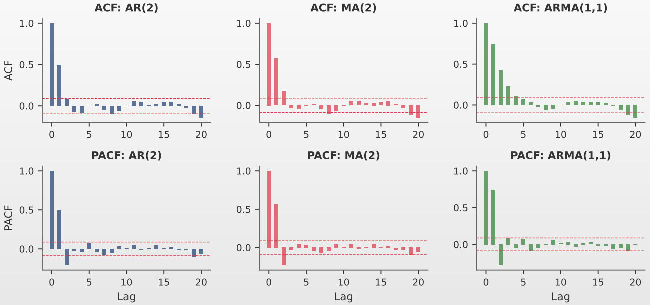
95% Confidence Interval

- $\pm 1.96/\sqrt{T}$ (the bands in ACF plots)

Caution

- Bartlett's formula is valid **only under H_0: white noise**
- For AR/MA, the asymptotic variance differs

ACF and PACF patterns



Identification Rules

- **AR(p)**: ACF decays exponentially, PACF cuts off after lag p
- **MA(q)**: ACF cuts off after lag q, PACF decays exponentially
- **ARMA(p, q)**: Both decay exponentially $\Rightarrow$ identification requires information criteria

TSA_ch1_acf_patterns

Partial autocorrelation function (PACF)

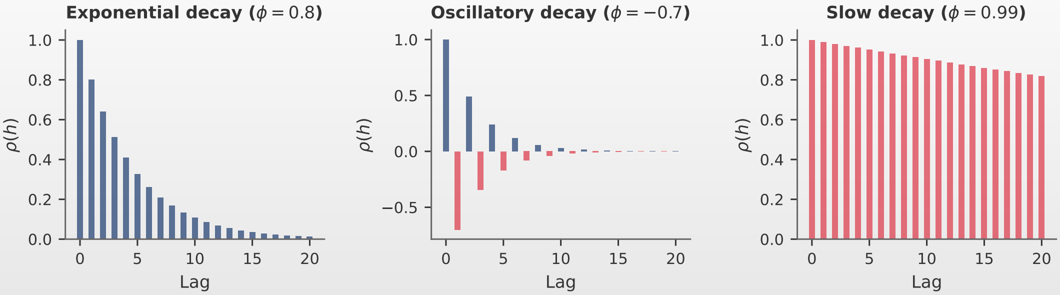
Definition 11 (Partial Autocorrelation)

- **PACF** at lag h, denoted ϕ_{hh}: the last coefficient in the regression:
  - $X_t = \phi_{h1}X_{t-1} + \phi_{h2}X_{t-2} + \cdots + \phi_{hh}X_{t-h} + e_t$
- **Alternatively**:
  - $\phi_{hh} = \text{Corr}(X_t - \hat{X}_t^{(h-1)}, X_{t-h} - \hat{X}_{t-h}^{(h-1)})$
- **Interpretation**: *Direct* dependence at lag h
  - Removes the effect of intermediate lags

Key Application: Model Order Identification

- **AR(p)**: PACF **cuts off** after lag p
  - ACF decays exponentially or oscillates
- **MA(q)**: ACF **cuts off** after lag q
  - PACF decays exponentially or oscillates

ACF decay patterns



Interpretation

- **Exponential decay**: Persistent positive dependence (AR with $\phi > 0$)
- **Oscillating decay**: Alternating dependence (AR with $\phi < 0$)
- The decay rate indicates the strength of the process memory

TSA_ch1_acf_patterns

Augmented Dickey-Fuller (ADF) test

ADF Model

$\Delta X_t = \alpha + \gamma X_{t-1} + \sum_{i=1}^{p} \delta_i \Delta X_{t-i} + \varepsilon_t, \quad \gamma = \rho - 1, \quad H_0 : \gamma = 0 \Leftrightarrow \rho = 1$

Hypotheses

- H_0: $\gamma = 0$ (unit root)
- H_1: $\gamma < 0$ (stationary)

Test Statistic

- $\tau_{ADF} = \hat{\gamma} \,/\, SE(\hat{\gamma})$
- $\hat{\gamma}$ = OLS coefficient of X_{t-1}
- $SE(\hat{\gamma})$ from the OLS regression

Decision Rule

- $\tau_{ADF} <$ critical value $\Rightarrow$ Reject H_0 $\Rightarrow$ Stationary
- $\tau_{ADF} \geq$ critical value $\Rightarrow$ Non-stationary (unit root)
- Critical values follow the Dickey-Fuller distribution (**not** t-Student!)

KPSS test

Model

- $X_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$

Hypotheses (opposite of ADF)

- H_0: $\sigma_u^2 = 0$ (stationary)
- H_1: $\sigma_u^2 > 0$ (unit root)

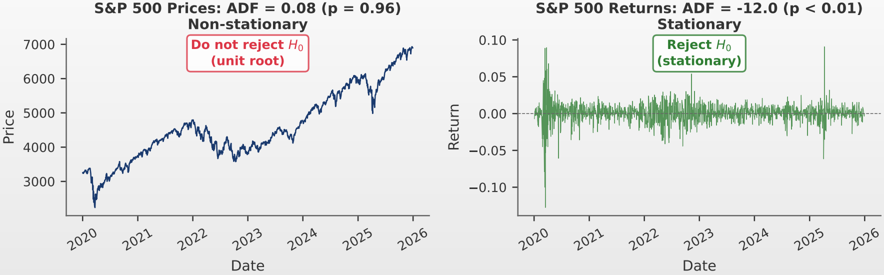
Test Statistic

- $LM = \frac{\sum_{t=1}^{T} S_t^2}{T^2 \hat{\sigma}_{LR}^2}$
- $S_t = \sum_{i=1}^{t} \hat{e}_i$, $\hat{\sigma}_{LR}^2 =$ long-run variance

Decision Rule

- $LM >$ critical value $\Rightarrow$ Reject H_0 $\Rightarrow$ Non-stationary
- $LM \leq$ critical value $\Rightarrow$ Stationary

ADF test: visualization with S&P 500



TSA_ch1_unit_root_tests

Interpreting the ADF Test

- **Hypothesis**: H_0: Unit root
  - Critical values: -3.43 (1%), -2.86 (5%), -2.57 (10%)
  - $\tau <$ critical value $\Rightarrow$ reject H_0 $\Rightarrow$ stationary series
- **S&P 500**: Prices non-stationary; Returns stationary

Using ADF and KPSS together

Confirmatory Testing

- **ADF rejects H_0 + KPSS fails to reject**: Stationary
- **ADF fails to reject + KPSS rejects H_0**: Unit Root
- **Both reject or both fail to reject**: Inconclusive
  - Additional tests required (PP, DF-GLS)

Workflow

- **Step 1**: ADF test (H_0: unit root)
- **Step 2**: KPSS test (H_0: stationary)
- **Step 3**: Concordant results $\Rightarrow$ OK
  - Otherwise: PP, DF-GLS tests

The Phillips-Perron (PP) Test

Definition 12 (Phillips-Perron, 1988)

- Tests the same hypothesis as ADF: H_0: unit root ($\gamma = 0$)
- Base model (no augmented lags): $\Delta y_t = \alpha + \gamma\, y_{t-1} + e_t$
- Corrects for autocorrelation and heteroskedasticity in e_t via **nonparametric correction** (Newey-West) of the t-statistic

PP Test Statistic

- $Z_t = t_\gamma \cdot \sqrt{\frac{s_e^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - s_e^2)}{2\hat{\lambda}^2 \cdot SE(\hat{\gamma})}$
- $\hat{\lambda}^2$: long-run variance (Newey-West kernel)
- s_e^2: OLS residual variance
- Critical values: same as ADF (Dickey-Fuller distribution)

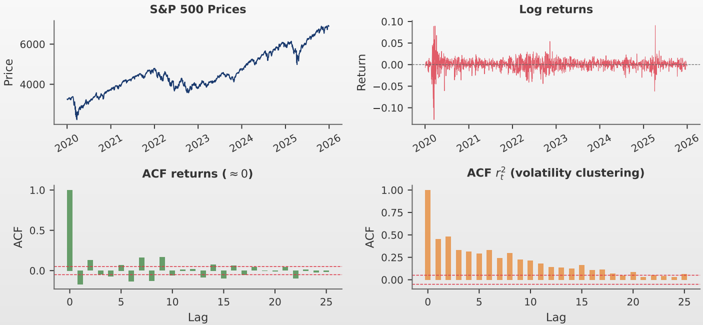
PP vs ADF

- **ADF**: adds lagged Δy_{t-j} $\Rightarrow$ parametric
- **PP**: corrects t-statistic $\Rightarrow$ nonparametric
- PP more robust to heteroskedasticity
- ADF more robust to MA roots near -1

Python

```
from statsmodels.tsa.stattools import PhillipsPerron
```

S&P 500 analysis: overview

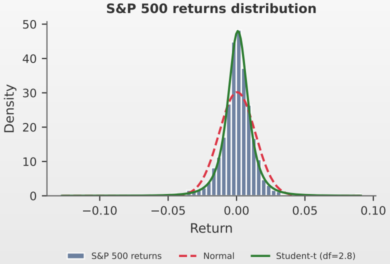


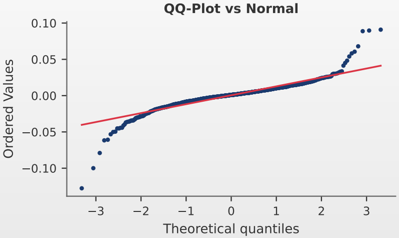
Observations

- **Prices**: Upward trend, non-stationary; **Returns**: Mean ≈ 0, stationary
- **ACF returns**: ≈ 0 (efficient); **ACF** r_t^2: Significant (volatility clustering)

TSA_ch1_ts_basics

Stylized facts of financial returns





Observed Properties

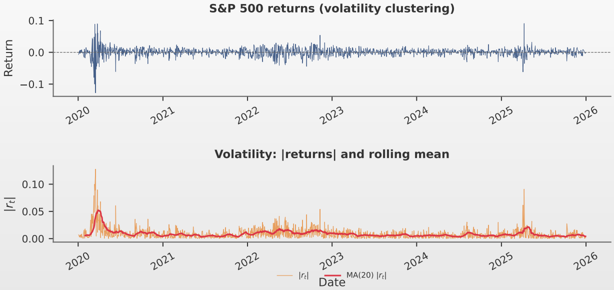
- Negative skewness (left tail)
- Excess kurtosis ($\gg 3$)
- Heavy tails (fat tails)

Implications

- Normal distribution inadequate
- Extreme events more likely
- Student-t or GED required

TSA_ch1_ts_basics

Volatility clustering

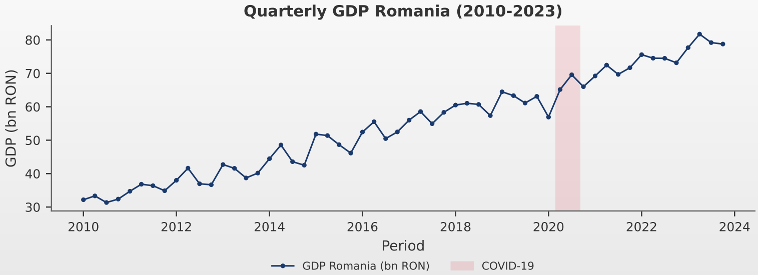


Observations

- Large returns (in absolute value) followed by large returns
- Calm periods followed by calm periods
- **Time-varying volatility** ⇒ ARCH/GARCH models (Ch. 5)

TSA_ch1_ts_basics

Case study: Romanian quarterly GDP



TSA_ch1_case_gdp

Initial Analysis

- **Data**: Romanian quarterly GDP 2010–2023 (56 obs., INS/Eurostat)
- **Observations**: Upward trend, possibly seasonal
  - COVID-19 structural shock visible
- **Hypothesis**: Non-stationary series $\Rightarrow$ test with ADF and KPSS

Stationarity testing: ADF and KPSS

ADF Test

- **Hypothesis**: H_0: Unit root
- **Result**: ADF stat.: -1.23
  - Critical value: -2.89
  - Fail to reject H_0

KPSS Test

- **Hypothesis**: H_0: Stationary
- **Result**: KPSS stat.: 0.89
  - Critical value: 0.46
  - Reject H_0

Conclusion: Both Tests Agree

- The GDP series is **non-stationary** $\Rightarrow$ requires differencing

Differencing: transformation to stationarity

After Differencing

- **Tests**: Both confirm stationarity
  - ADF: -4.56 ($p < 0.01$)
  - KPSS: 0.21 ($p > 0.10$)

Conclusion

- **GDP level**: non-stationary
- **ΔGDP**: stationary
  - Use ΔGDP_t for modeling

Final Result

- GDP requires one differencing to become stationary

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Using yfinance, download daily EUR/RON exchange rate data (EURRON=X) from 2020-01-01 to 2024-12-31 (approx. 1,250 observations). Test whether the series is stationary using ADF and KPSS tests. Fit an appropriate model and forecast the exchange rate for the next 5 trading days. Tell me if the forecast is reliable."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Download real EUR/RON data and reproduce the analysis. Do the results match?
3. Is the ADF test correctly specified (trend, lags)? What changes if you modify the options?
4. Compare the AI model's forecast against a naïve benchmark ($\hat{X}_{t+1} = X_t$).
5. If the series is a random walk, does fitting an ARMA model make sense?

Warning: Low RMSE and significant coefficients *do not guarantee* a useful forecast.

Key takeaways

Summary

- **Stochastic process**: collection of random variables indexed by time
- **Weak stationarity**: constant mean, variance, autocovariance
- **White noise**: $\varepsilon_t \sim WN(0, \sigma^2)$
  - Stationary, ACF $= 0$ for $h \neq 0$
- **Random walk**: $X_t = X_{t-1} + \varepsilon_t$
  - Non-stationary, $\text{Var}(X_t) = t\sigma^2$
- **ACF/PACF**: key tools for identifying structure
- **Differencing**: transforms non-stationary series into stationary ones
- **Unit root tests**:
  - ADF (H_0: unit root) vs KPSS (H_0: stationary)

Important formulas

Weak Stationarity

- **Constant moments**:
  - $\mathbb{E}[X_t] = \mu$ (constant mean)
  - $\mathrm{Var}(X_t) = \sigma^2$ (constant variance)
- **Autocovariance**: $\gamma(h) = \mathrm{Cov}(X_t, X_{t+h})$
- **Autocorrelation**: $\rho(h) = \gamma(h)/\gamma(0)$

Lag Operator

- **Lag**: $LX_t = X_{t-1}$
- **Difference**: $\Delta X_t = (1 - L)X_t$

White Noise (WN)

- **Model**: $\varepsilon_t \sim WN(0, \sigma^2)$
- **ACF**: $\rho(h) = 0$ for $h \neq 0$

Random Walk (RW)

- **Model**: $X_t = X_{t-1} + \varepsilon_t$
- **Variance**: $\mathrm{Var}(X_t) = t\sigma^2$ (grows!)

Next chapter preview

Chapter 2: ARMA Models

- **AR(p)**: Autoregressive Models
- **MA(q)**: Moving Average Models
- **ARMA(p, q)**: Combined Models
- **Identification**: Using ACF/PACF

What We Will Learn

- **Estimation**: Model parameters
- **Diagnostics**: Model validation
- **Forecasting**: Confidence intervals
- **Selection**: AIC, BIC

Question 1

Question

- What are the three conditions for weak (covariance) stationarity?

Answer Choices

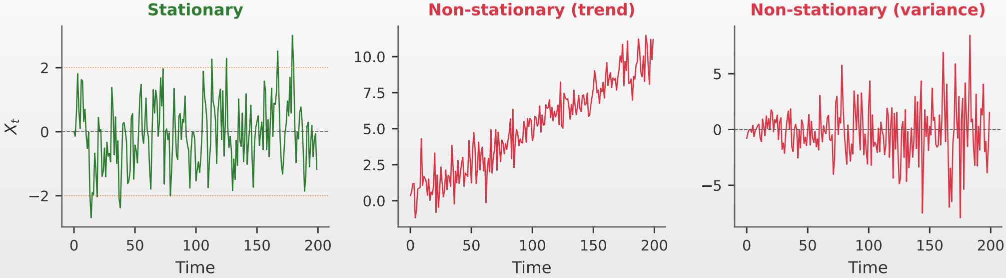
(A) Zero mean, infinite variance, time-dependent covariance

(B) Constant mean, constant variance, autocovariance depends only on lag

(C) Normal distribution, independence, unit variance

(D) Linear trend, constant seasonality, white residuals

Question 1: Answer



Answer: (B)

- $\mathbb{E}[X_t] = \mu$, $\text{Var}(X_t) = \sigma^2$, $\gamma(t, s) = \gamma(|t - s|)$

TSA_ch1_stationarity

Question 2

Question

- What is the null hypothesis (H_0) of the ADF (Augmented Dickey-Fuller) test?

Answer Choices

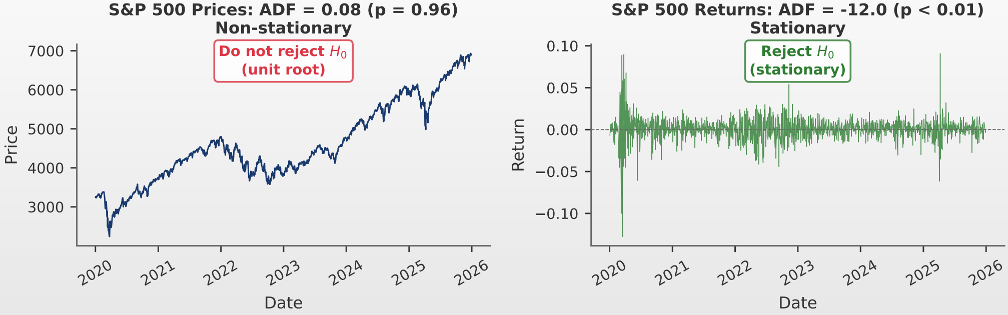
(A) The series is stationary

(B) The series has a unit root (is non-stationary)

(C) The series has no autocorrelation

(D) The series has a normal distribution

Question 2: Answer



Answer: (B)

- H_0: unit root; $\tau <$ critical value $\Rightarrow$ stationary

TSA_ch1_unit_root_tests

Question 3

Question

- What is the null hypothesis (H_0) of the KPSS test?

Answer Choices

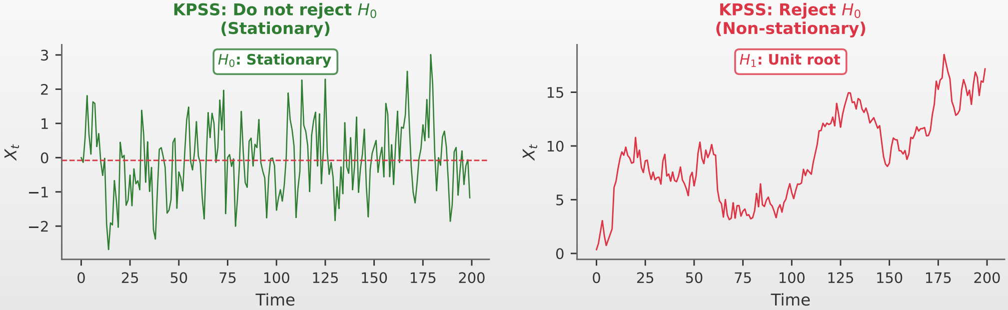
(A) The series has a unit root (non-stationary)

(B) The series is stationary

(C) The series is a random walk

(D) The series has a deterministic trend

Question 3: Answer



Answer: (B)

- KPSS: H_0 stationary (opposite of ADF). Use both tests!

TSA_ch1_unit_root_tests

Question 4

Question

What is the key property of the variance of a random walk $X_t = X_{t-1} + \varepsilon_t$?

Answer Choices

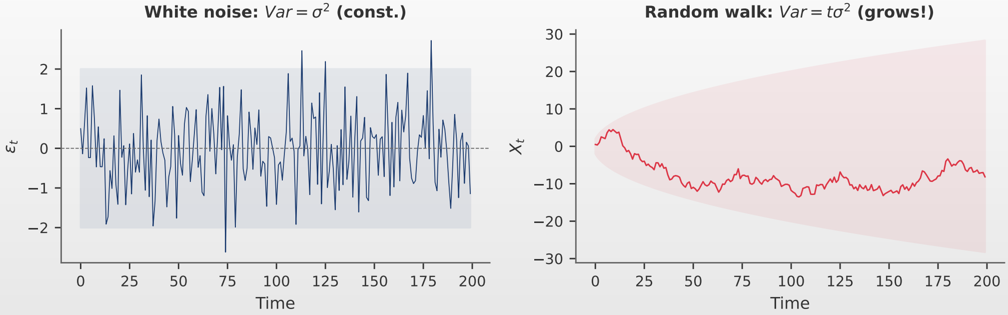
(A) Variance is constant: $\text{Var}(X_t) = \sigma^2$

(B) Variance grows linearly with time: $\text{Var}(X_t) = t\sigma^2$

(C) Variance decreases with time

(D) Variance is zero

Question 4: Answer



Answer: (B)

- $\text{Var}(X_t) = t\sigma^2$ grows linearly $\Rightarrow$ non-stationary

TSA_ch1_random_walk

Question 5

Question

☐ What does the ACF of a random walk (non-stationary series with unit root) look like?

Answer Choices

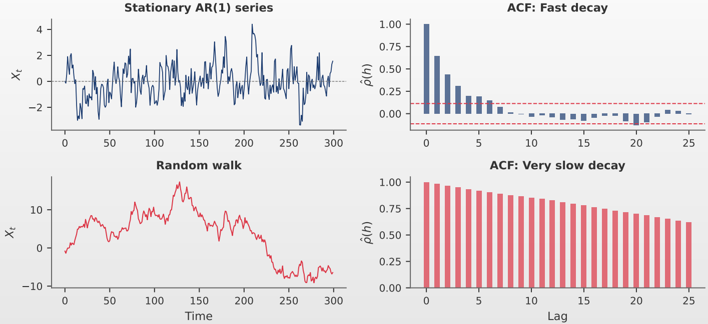
(A) All values are zero after lag 0

(B) Decays exponentially fast

(C) Decays very slowly (high persistence)

(D) Oscillates between positive and negative

Question 5: Answer



Answer: (C)

- ACF $\approx$ 1 for many lags, slow decay $\Rightarrow$ ADF test

TSA_ch1_random_walk

Question 6

Question

- How do we obtain stationary returns from a financial price series P_t?

Answer Choices

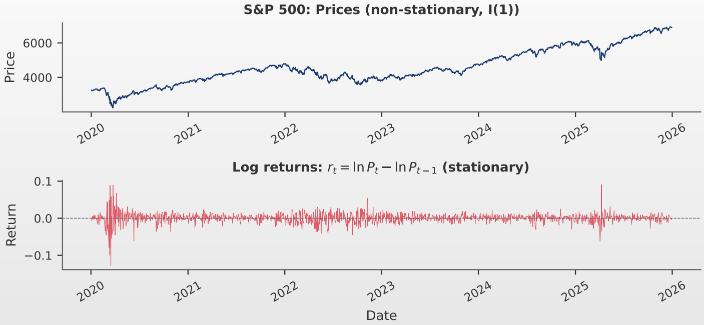
(A) Simple differencing: $\Delta P_t = P_t - P_{t-1}$

(B) Log then differencing: $r_t = \ln P_t - \ln P_{t-1}$

(C) Log only: $\ln P_t$

(D) Standardization: $(P_t - \bar{P})/s_P$

Question 6: Answer



Answer: (B)

- Log returns: $r_t = \ln P_t - \ln P_{t-1}$
- First ln (stabilizes variance), then Δ (removes trend) $\Rightarrow$ stationary series

TSA_ch1_operators

Bibliography I

Core Textbooks

- Hamilton, J.D. (1994). *Time Series Analysis*, Princeton University Press.
- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.
- Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.

Classic References

- Wold, H. (1938). *A Study in the Analysis of Stationary Time Series*, Almqvist & Wiksell.
- Bartlett, M.S. (1946). On the Theoretical Specification and Sampling Properties of Autocorrelated Time-Series, *JRSS Supplement*, 8(1), 27–41.
- Box, G.E.P., & Jenkins, G.M. (1970). *Time Series Analysis: Forecasting and Control*, Holden-Day.

Bibliography II

Stationarity Tests

- Dickey, D.A., & Fuller, W.A. (1979). Distribution of the Estimators for Autoregressive Time Series with a Unit Root, *JASA*, 74(366), 427–431.
- Kwiatkowski, D., et al. (1992). Testing the Null Hypothesis of Stationarity, *Journal of Econometrics*, 54(1–3), 159–178.

Online Resources and Code

- **Quantlet**: `https://quantlet.com` – Code platform for quantitative methods
- **Quantinar**: `https://quantinar.com` – Learning platform for quantitative methods
- **GitHub TSA**: `https://github.com/QuantLet/TSA/tree/main/TSA_ch1` – Python code for this chapter

Thank You!

Questions?

Course materials available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar