Time Series Analysis and Forecasting

Chapter 1: Introduction to Time Series

Fundamentals and Concepts

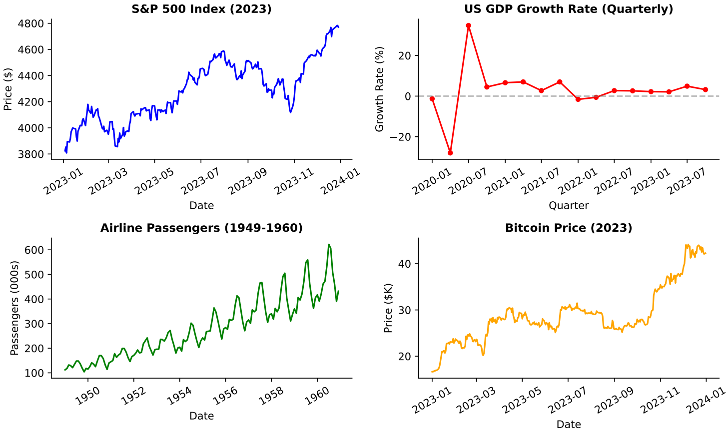
Learning Objectives

By the end of this chapter, you will be able to:

1. **Define** time series and distinguish from cross-sectional and panel data
2. **Decompose** time series into trend, seasonal, and residual components
3. **Apply** exponential smoothing (SES, Holt, Holt-Winters, ETS)
4. **Evaluate** forecasts using MAE, RMSE, MAPE; train/validation/test splits
5. **Model** seasonality using dummy variables or Fourier terms
6. **Handle** trend and seasonality through detrending and adjustment
7. **Understand** stochastic processes and stationarity
8. **Compute** ACF/PACF and conduct stationarity tests (ADF, KPSS)

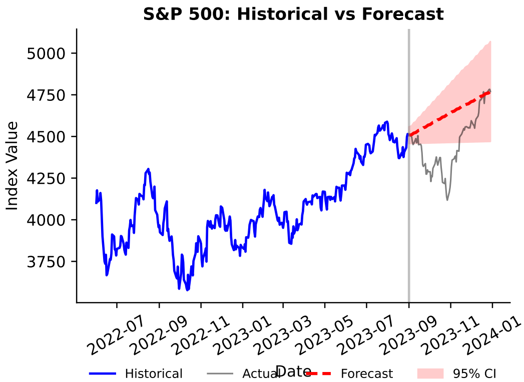
Chapter Outline

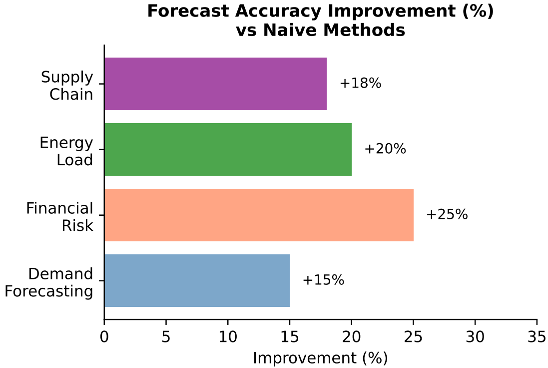
Time Series Are Everywhere



- **Finance**: Stock prices, exchange rates, trading volumes
- **Economics**: GDP, unemployment, inflation rates
- **Business**: Sales, website traffic, customer demand
- **Science**: Temperature, pollution levels, patient vitals

Why Study Time Series?



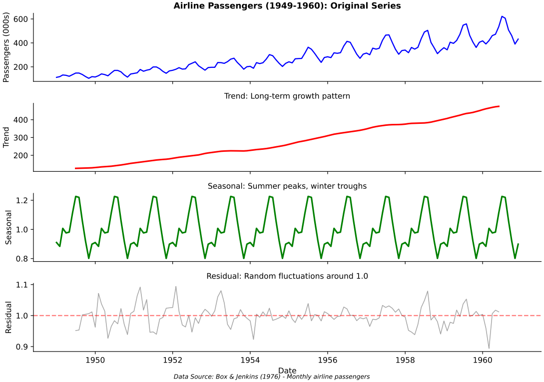


Source: M-Competition benchmarks (Makridakis et al.)

Key Goal: Forecasting

Use historical patterns to predict future values — critical for business planning, risk management, and policy decisions.

Understanding Time Series Structure



Decomposition

Every time series can be decomposed into interpretable components: trend, seasonality, and noise.

Definition of a Time Series

Definition 1 (Time Series)

A **time series** is a sequence of observations $\{X_t\}$ indexed by time:

$$\{X_t : t \in \mathcal{T}\}$$

where $\mathcal{T}$ is an index set representing time points.

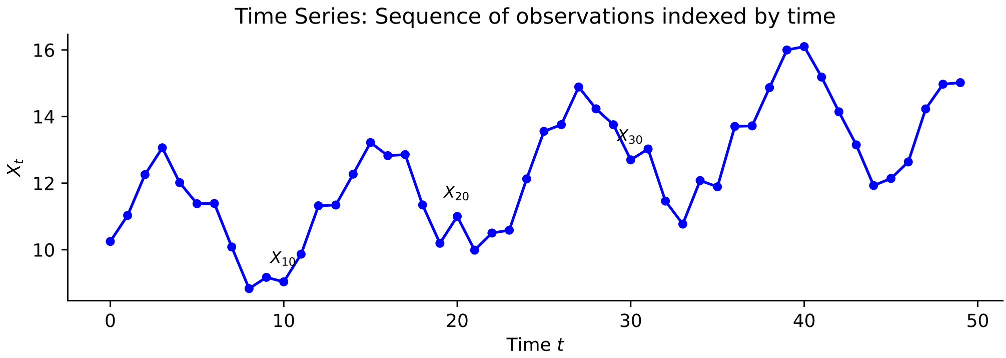
Key Characteristics

- **Ordered**: Natural temporal ordering
- **Dependent**: Consecutive observations correlated
- **Discrete/Continuous**: $t = 1, 2, 3, \ldots$

Notation

- X_t = observation at time t
- $\{X_t\}_{t=1}^{T}$ = series with T observations

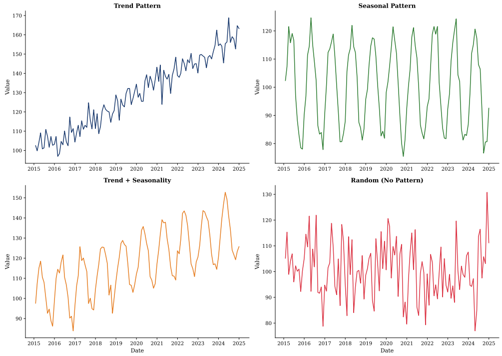
Time Series: Visual Illustration



Interpretation

Each point X_t represents an observation at time t. The sequence is ordered and consecutive observations are typically correlated.

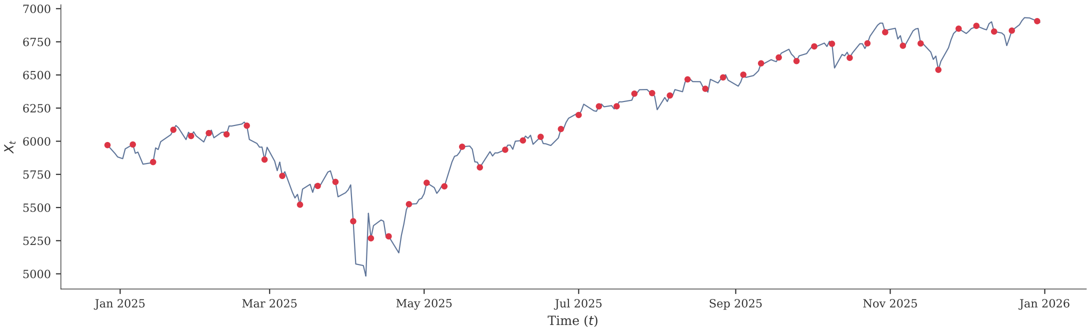
Common Time Series Patterns



Pattern Types

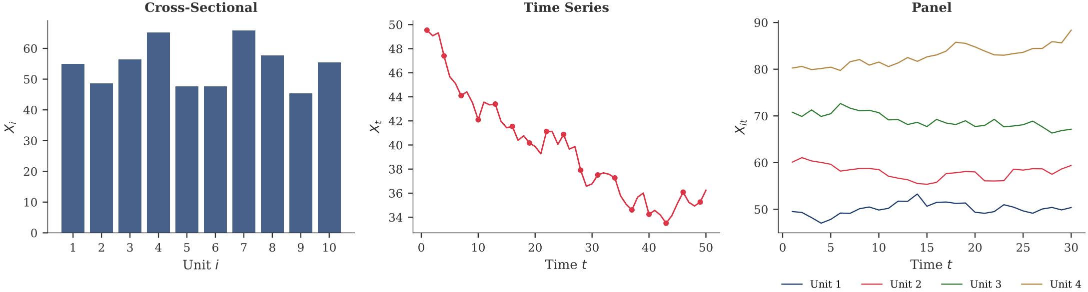
- **Trend**: Long-term increase or decrease in the data
- **Seasonal**: Regular periodic patterns (e.g., monthly, quarterly)
- **Random**: No systematic pattern – unpredictable fluctuations

Time Series: Visual Definition



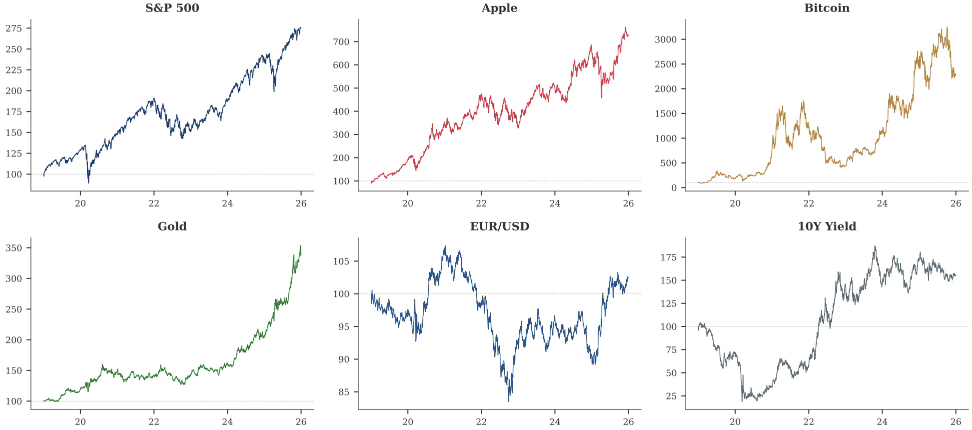
Each point X_t represents a measurement at discrete time t. Data: S&P 500 (2024).

Types of Data: Comparison



Data Type	Units (N)	Time (T)	Example
Cross-sectional	Many	1	Survey of 1000 households
Time series	1	Many	Daily S&P 500 prices
Panel	Many	Many	GDP of 50 countries, 20 years

Examples of Time Series Data



Real financial data from Yahoo Finance (2019–2025). Normalized to base 100.

Why Decompose a Time Series?

Decomposition separates a time series into interpretable components:

Goals:

- Understand underlying patterns
- Remove seasonality for modeling
- Identify trend direction
- Isolate irregular fluctuations
- Improve forecasting accuracy

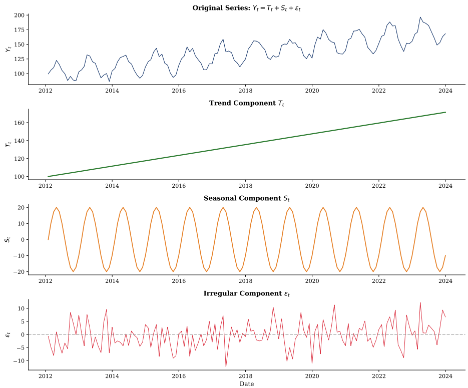
Components:

- T_t = **Trend**: Long-term movement
- S_t = **Seasonal**: Regular periodic pattern
- C_t = **Cyclical**: Business cycle fluctuations
- ε_t = **Residual**: Random noise

Classical Decomposition Models

- **Additive**: $X_t = T_t + S_t + \varepsilon_t$
- **Multiplicative**: $X_t = T_t \times S_t \times \varepsilon_t$

Time Series Decomposition: Visual Example



Components Explained

- **Original**: The observed time series with all components
- **Trend**: Underlying long-term movement extracted via smoothing
- **Seasonal**: Regular periodic pattern that repeats each cycle
- **Residual**: Random noise after removing trend and seasonality

The Cyclical Component

Definition

Cyclical component C_t: Medium-term fluctuations (2–10 years)

Characteristics

- Business cycle fluctuations
- No fixed period (unlike seasonal)
- Duration varies: 2–10 years
- Amplitude varies over time

Examples

- Economic expansions/recessions
- Credit cycles
- Real estate cycles
- Commodity price cycles

Practical Note

Often combined with trend as **trend-cycle** component because it's difficult to separate from trend with short data.

Additive Decomposition Model

Model

$$X_t = T_t + S_t + \varepsilon_t \tag{1}$$

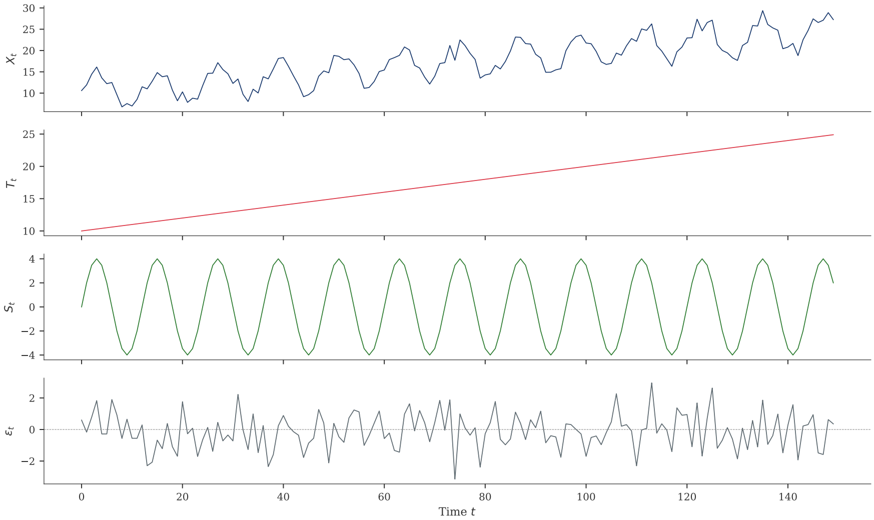
When to Use

- Seasonal fluctuations are **constant** over time
- Variance of the series is **stable**

Properties

- $\mathbb{E}[\varepsilon_t] = 0$ (zero mean)
- $\sum_{j=1}^{s} S_j = 0$ (seasonal sums to zero)
- Units of S_t same as X_t

Additive Decomposition: Visualization



Multiplicative Decomposition Model

Model

$$X_t = T_t \times S_t \times \varepsilon_t \tag{2}$$

When to Use

- Seasonal fluctuations **grow** with series level
- Variance **increases** over time

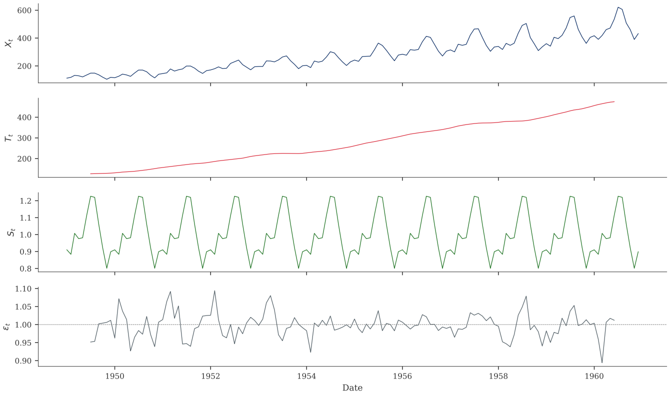
Properties

- $\mathbb{E}[\varepsilon_t] = 1$ (centered at 1)
- $\frac{1}{s} \sum S_j = 1$ (averages to 1)
- S_t is dimensionless ratio

Tip

Log transform converts multiplicative to additive model: $\log X_t = \log T_t + \log S_t + \log \varepsilon_t$

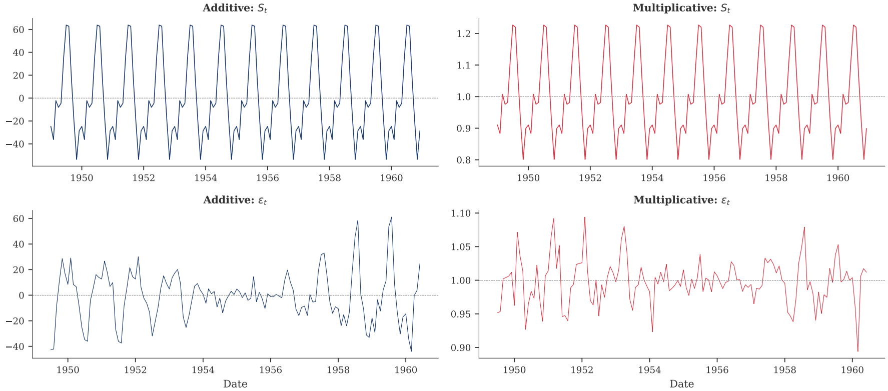
Multiplicative Decomposition: Real Data



Example

Classic Box-Jenkins airline passengers (1949–1960). Seasonal amplitude grows with level.

Additive vs Multiplicative: Comparison



Key Difference

Multiplicative: seasonal is a *ratio* (centered at 1). Additive: seasonal in *absolute units* (centered at 0).

Trend Estimation: Moving Average

Definition 2 (Centered Moving Average)

The **centered moving average** of order $2q+1$ is:

$$\hat{T}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t+j} \tag{3}$$

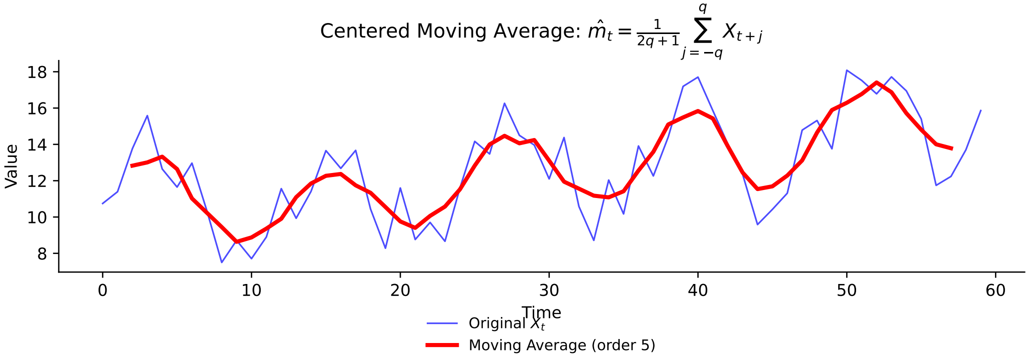
For Seasonal Data

- Period s **odd**: simple average
- Period s **even**: $2 \times s$ MA with half-weights

Properties

- Smooths seasonal & random
- Larger window $\Rightarrow$ smoother
- Trade-off: lose endpoints

Centered Moving Average: Visual Illustration



Interpretation

The moving average smooths out short-term fluctuations, revealing the underlying trend.

Classical Decomposition Algorithm

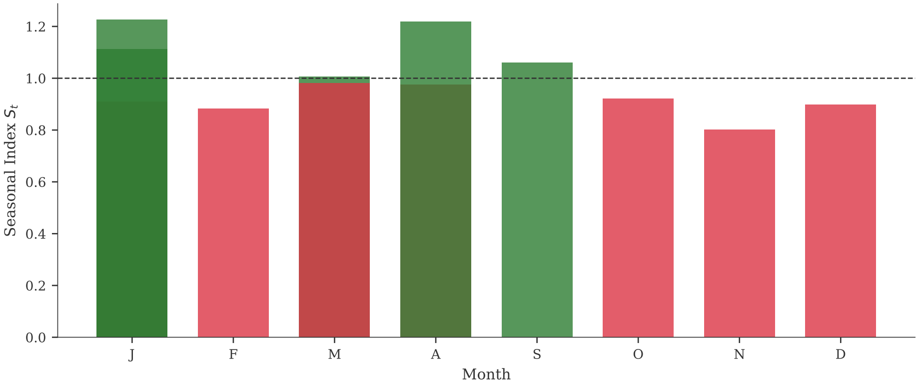
Steps for Multiplicative Decomposition

1. **Estimate Trend**: $\hat{T}_t = MA_s(X_t)$
2. **Detrend**: $D_t = X_t / \hat{T}_t$
3. **Estimate Seasonal**: $\hat{S}_j = \text{mean}(D_t \text{ for season } j)$
4. **Normalize**: Scale so $\frac{1}{s}\sum_{j=1}^{s} \hat{S}_j = 1$
5. **Compute Residuals**: $\hat{\varepsilon}_t = X_t / (\hat{T}_t \times \hat{S}_t)$

Note

For **additive** decomposition: replace division with subtraction and multiplication with addition.

Seasonal Indices: Interpretation



Interpretation

$S_t > 1$ means above-average activity; $S_t < 1$ means below-average. Airline data shows peak travel in July–August.

STL Decomposition: A Modern Approach

Definition 3 (STL - Seasonal-Trend decomposition using LOESS)

STL uses locally weighted regression (LOESS): $X_t = T_t + S_t + R_t$

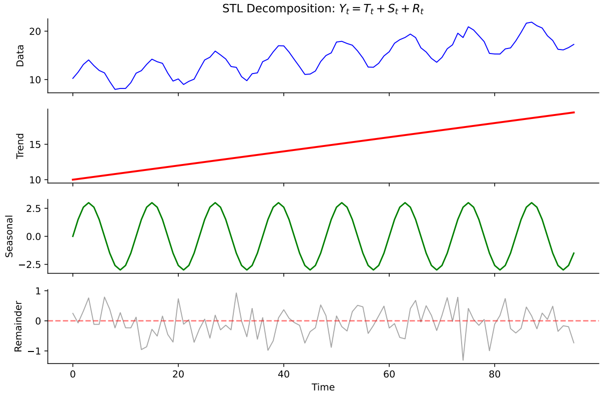
Advantages

- Any seasonal period
- Seasonal can change over time
- Robust to outliers
- Smooth trend estimates

Key Parameters

- `period`: Seasonal period
- `seasonal`: Smoothing window
- `robust`: Downweight outliers

STL Decomposition: Visual Illustration



Key Insight

STL separates the series into trend, seasonal, and remainder using LOESS.

Exponential Smoothing: Overview

Definition

Exponential smoothing produces forecasts based on weighted averages of past observations, with weights decaying exponentially.

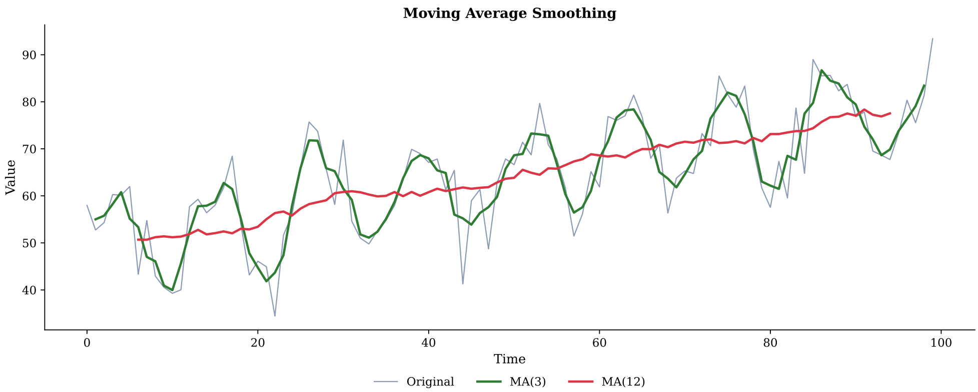
Why Exponential Smoothing?

- Simple yet effective
- Recent obs. get higher weights
- Handles trend & seasonality
- Foundation for ETS models

Three Main Methods

1. **SES**: Level only
2. **Holt**: Level + Trend
3. **Holt-Winters**: + Seasonality

Moving Average Smoothing



Window Size Trade-off

Small window: Responsive but noisy. **Large window**: Smoother but slower to react.

Simple Exponential Smoothing (SES)

Model

$$\hat{X}_{t+1|t} = \alpha X_t + (1-\alpha)\hat{X}_{t|t-1} \tag{4}$$

where $\alpha \in (0, 1)$ is the **smoothing parameter**.

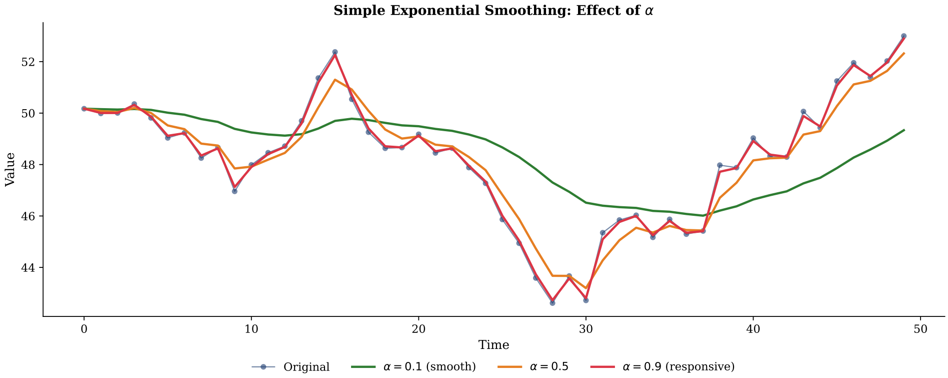
How It Works

- Weights decay exponentially
- Large α: responsive
- Small α: smoother

Level Form

$$\ell_t = \alpha X_t + (1-\alpha)\ell_{t-1}$$

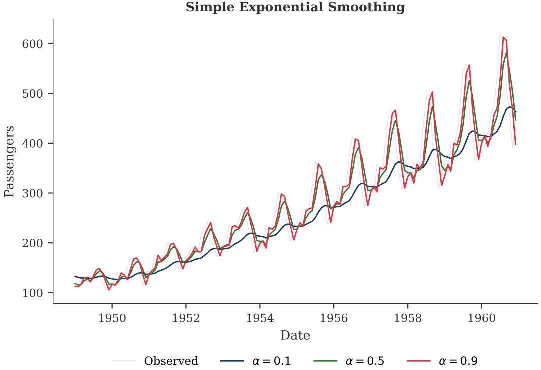
Exponential Smoothing: Effect of Alpha

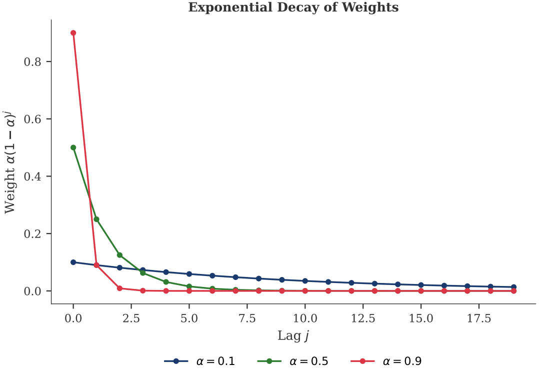


Choosing α

- **Low** α (0.1): More weight on past – smoother, slower adaptation
- **High** α (0.9): More weight on recent – responsive, volatile

Simple Exponential Smoothing: Effect of α





Trade-off

Smaller α produces smoother forecasts; larger α follows data more closely.

Holt's Linear Trend Method

Equations

Level: $\ell_t = \alpha X_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$

Trend: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$

Forecast: $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t$

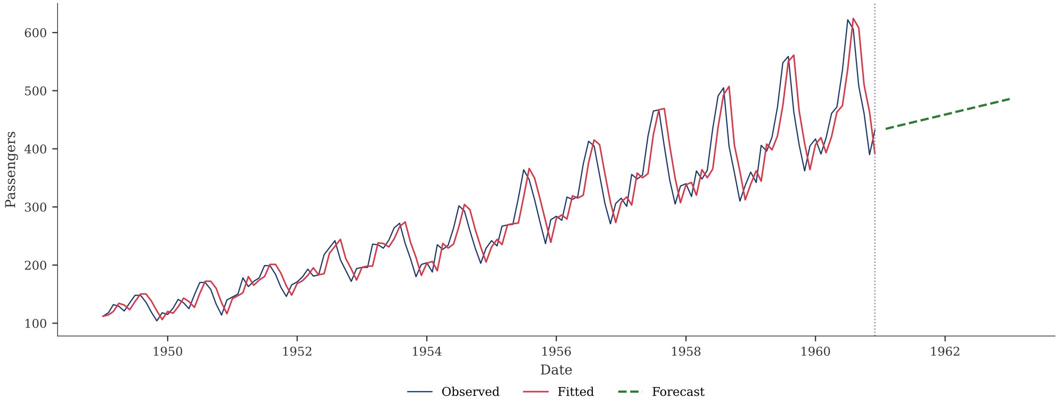
Parameters

- α: Level smoothing
- β^*: Trend smoothing

Components

- ℓ_t: Estimated level
- b_t: Estimated trend (slope)

Holt's Method: Visualization



Interpretation

Holt's method captures both level and trend, projecting them into the forecast horizon.

Holt-Winters Seasonal Method

Equations (Additive Seasonality)

Level: $\ell_t = \alpha(X_t - S_{t-s}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$

Trend: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$

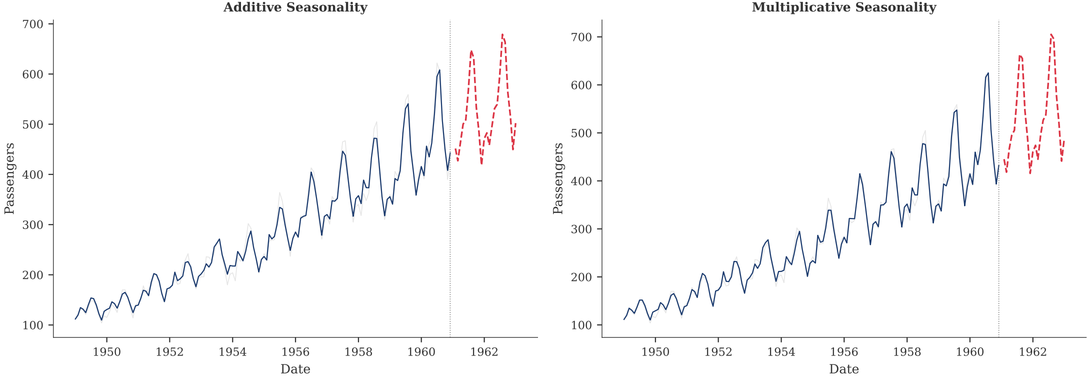
Seasonal: $S_t = \gamma(X_t - \ell_t) + (1-\gamma)S_{t-s}$

Forecast: $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t + S_{t+h-s(k+1)}$

Parameters

α: Level smoothing β^*: Trend smoothing γ: Seasonal smoothing s: Period

Holt-Winters: Capturing Seasonality



Key Feature

Holt-Winters decomposes the series and produces seasonal forecasts with trend.

ETS Framework: Error-Trend-Seasonal

Definition 4 (ETS Models)

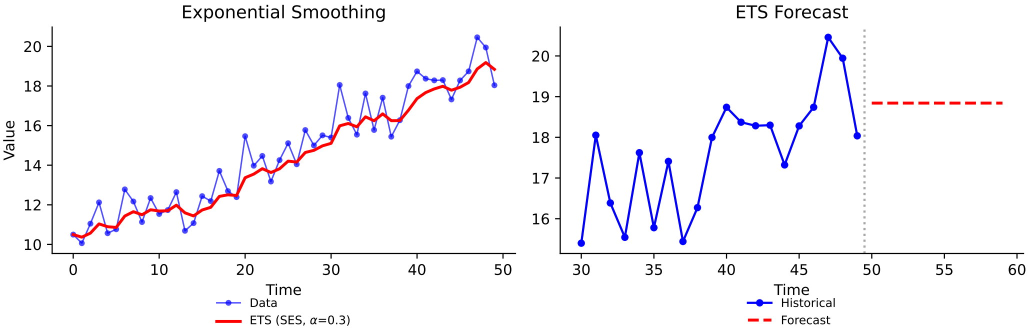
The **ETS framework** generalizes exponential smoothing: ETS(E, T, S)

Component	**N**	**A**	**M**
Error (E)	–	Additive	Multiplicative
Trend (T)	None	Additive	Multiplicative
Seasonal (S)	None	Additive	Multiplicative

Examples

- ETS(A,N,N) = Simple Exponential Smoothing
- ETS(A,A,N) = Holt's Linear Method
- ETS(A,A,A) = Holt-Winters Additive

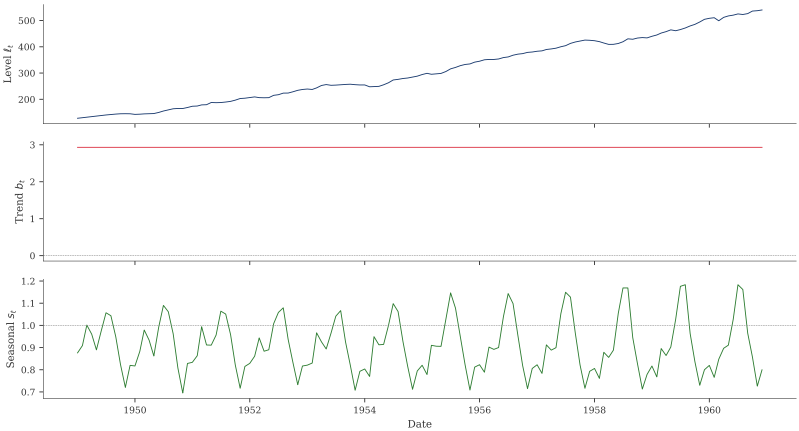
ETS: Exponential Smoothing Illustration



Interpretation

ETS models use exponentially weighted observations for forecasting. Weights decay as observations get older.

ETS Model Selection



Interpretation

The ETS framework provides a systematic way to choose the best model using AIC/BIC.

Damped Trend Methods

Damping Parameter

Introduces $\phi \in (0, 1)$ to prevent over-projection

Equations

Level: $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$
Trend: $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$
Forecast: $\hat{X}_{t+h|t} = \ell_t + \phi \frac{1-\phi^h}{1-\phi} b_t$

Key Insight

- As $h \to \infty$: forecast $\to$ constant
- Prevents unrealistic long-term extrapolation
- Often best for longer horizons

Forecast Accuracy Metrics

Forecast Error: $e_t = X_t - \hat{X}_t$ (actual minus predicted)

Scale-Dependent:

- $\text{MAE} = \frac{1}{n} \sum |e_t|$
- $\text{MSE} = \frac{1}{n} \sum e_t^2$
- $\text{RMSE} = \sqrt{\text{MSE}}$

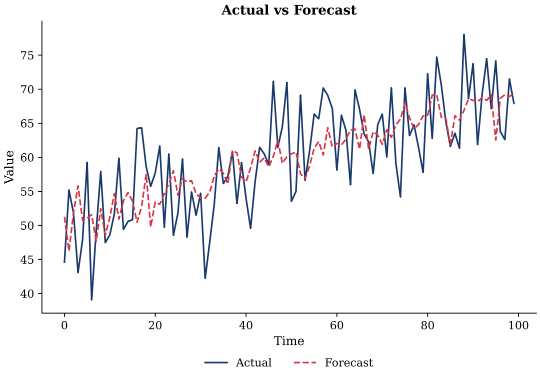
Scale-Independent:

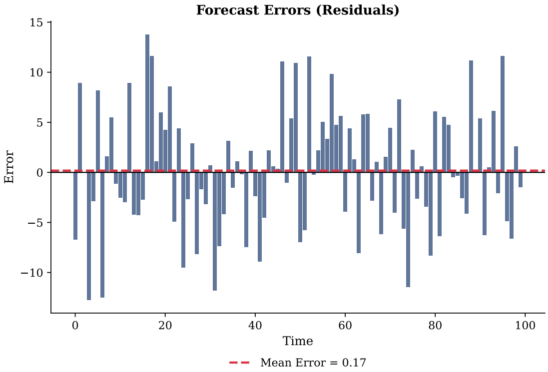
- $\text{MAPE} = \frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$
- sMAPE (symmetric)

Which to use?

- Same series: RMSE, MAE
- Compare across series: MAPE, sMAPE

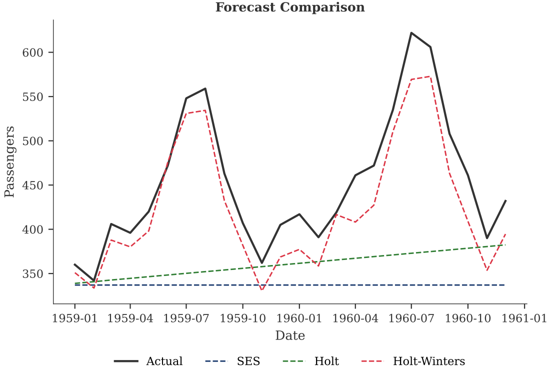
Forecast Evaluation: Visual Example

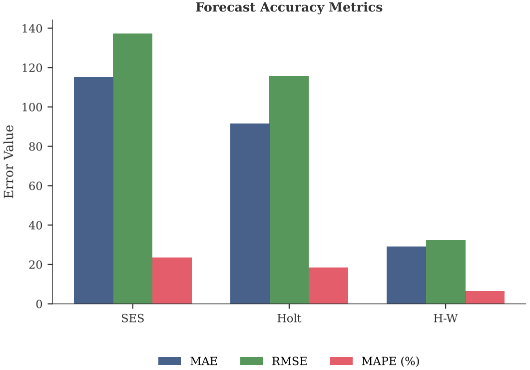




- **Top**: Actual values vs. forecasted values – visual assessment of fit
- **Bottom**: Residuals should be centered around zero with no pattern
- Good forecasts have small, random residuals with constant variance

Comparing Forecast Methods





Interpretation

Left: Comparing SES, Holt, and Holt-Winters forecasts. **Right:** Error metrics for each method.

Residual Diagnostics

Residual Properties

Good forecasts should have residuals that are:

1. **Zero mean**: $\mathbb{E}[e_t] = 0$
2. **Uncorrelated**: $\text{Cov}(e_t, e_{t-k}) = 0$
3. **Constant variance**: $\text{Var}(e_t) = \sigma^2$
4. **Normally distributed**

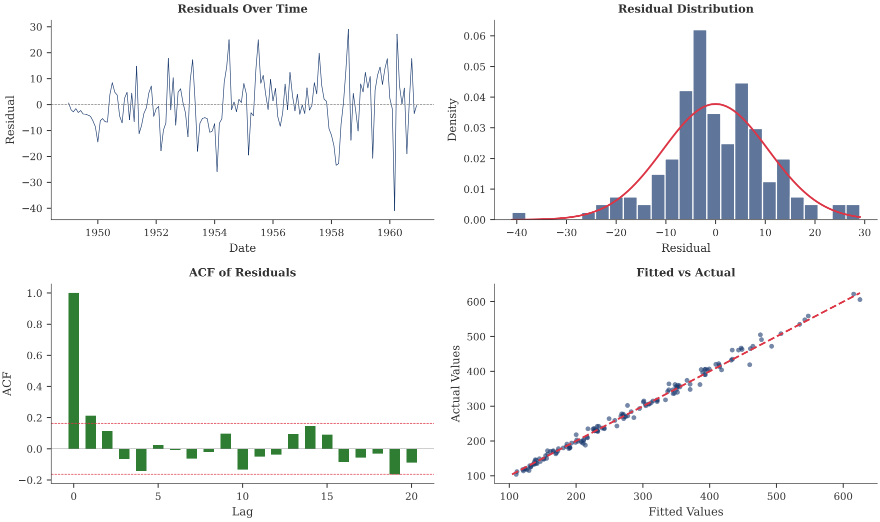
Diagnostic Tests

Ljung-Box test (autocorrelation):

$$Q = T(T+2)\sum_{k=1}^{h}\frac{\hat{\rho}_k^2}{T-k} \sim \chi_h^2$$

Jarque-Bera test (normality)

Residual Diagnostics: Visualization



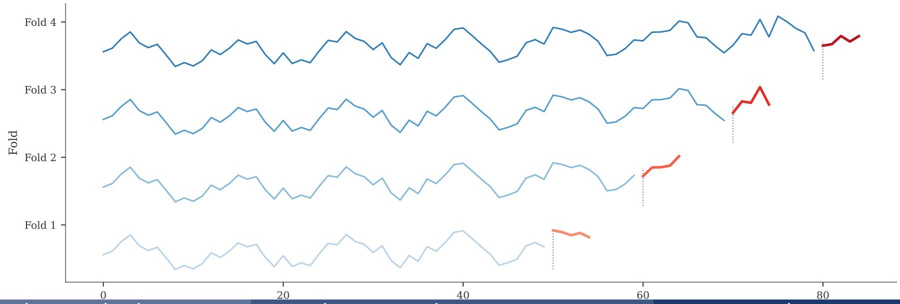
Time Series Cross-Validation

Important

Standard CV doesn't work for time series (temporal dependence).

Rolling Origin CV (Expanding Windows)

1. Train on $\{X_1, \ldots, X_t\}$, forecast $\hat{X}_{t+h}$
2. Increment t, repeat



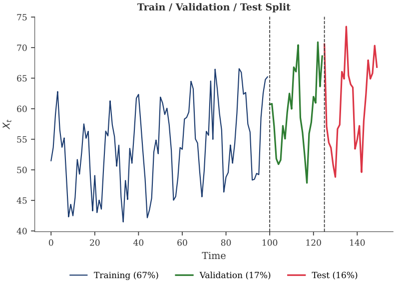
Train / Validation / Test Split

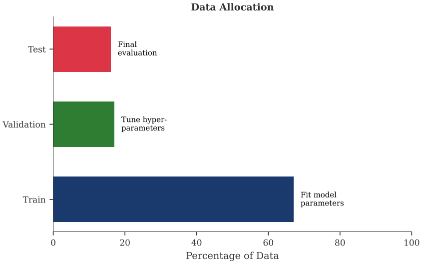
Three-way split for model development:

Training Set
- Fit model parameters
- Largest portion (60–80%)
- Used for estimation

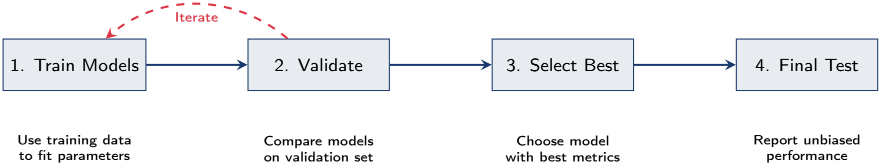
Validation Set
- Tune hyperparameters
- Compare models
- Select best approach

Test Set
- Final evaluation only
- Never used for tuning
- Unbiased performance





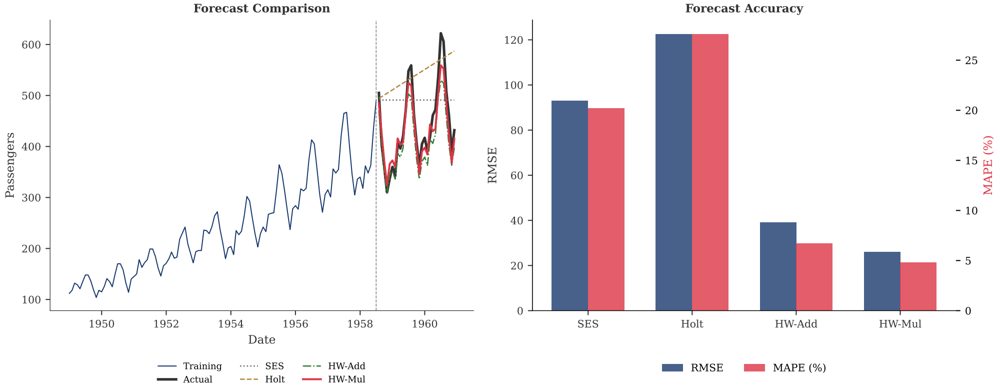
Model Development Workflow



Critical Rule

Never use test set for model selection! This causes *data leakage* and overly optimistic performance estimates.

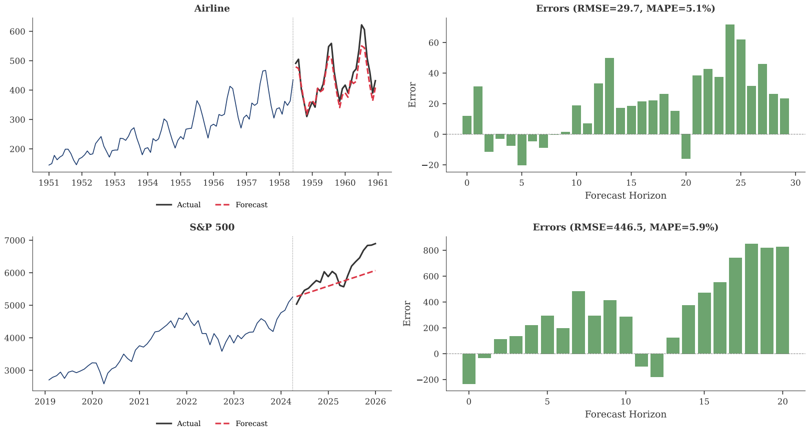
Real Data: Forecast Comparison



Interpretation

Airline passengers data: Holt-Winters Multiplicative performs best for seasonal data.

Forecast Performance Across Datasets



Interpretation

Different series require different models. Seasonal data needs seasonal methods.

Modeling Seasonality: Two Approaches

1. Dummy Variables:

$X_t = \mu + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$

- $D_{jt} = 1$ if t in season j
- $s - 1$ parameters
- Any seasonal pattern

2. Fourier Terms:

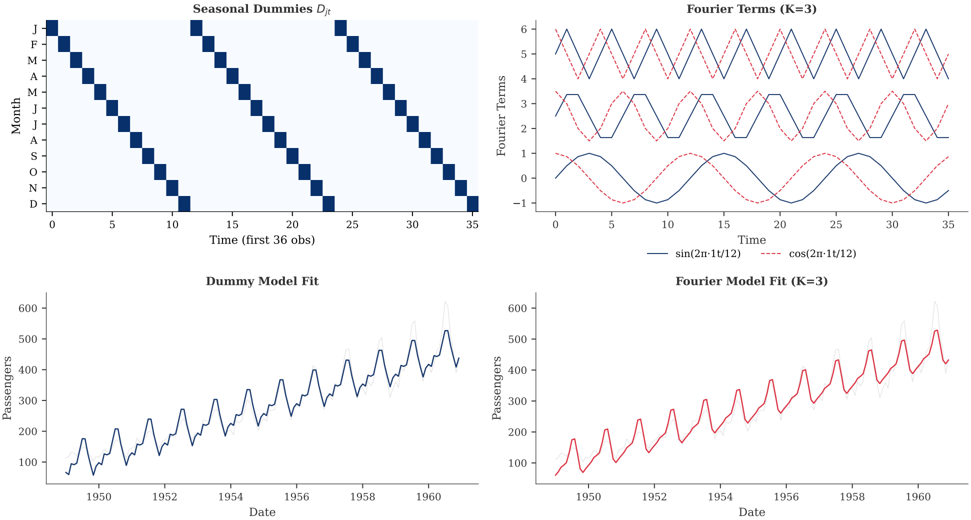
$X_t = \mu + \sum_{k=1}^{K} [\alpha_k \sin(\cdot) + \beta_k \cos(\cdot)]$

- Sinusoidal functions
- $2K$ parameters
- Smooth patterns

Trade-off

Dummies: any pattern, more parameters. Fourier: smooth, fewer parameters.

Dummy Variables vs Fourier Terms



Choosing Between Dummies and Fourier

Criterion	Dummies	Fourier
Parameters (monthly)	11	$2K$ (often 4–6)
Seasonal pattern	Any shape	Smooth/sinusoidal
Interpretation	Direct (month effects)	Frequency components
High-frequency seasons	Many parameters	Efficient
Multiple seasonality	Complex	Easy (add terms)

Guidelines

- Use **dummies**: irregular patterns, interpretable coefficients
- Use **Fourier**: smooth patterns, high-frequency seasonality, multiple periods
- **Fourier terms** are used in TBATS and Facebook Prophet

Why Remove Trend and Seasonality?

Before modeling, we often need to make series stationary:

Reasons to detrend:

- Stationarity requirement
- Focus on fluctuations
- Avoid spurious regression
- Enable valid inference

Reasons to deseasonalize:

- Reveal underlying trend
- Compare across seasons
- Simplify modeling
- Focus on irregular component

Important

After modeling the detrended/deseasonalized series, we must **reverse the transformation** for forecasting.

Trend Removal Methods

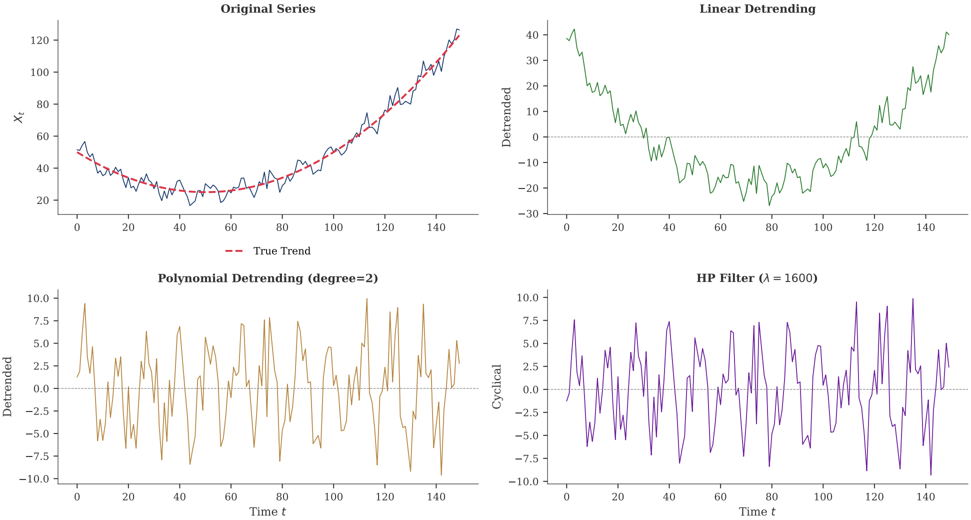
Six Common Detrending Approaches

1. **Differencing**: $\Delta X_t = X_t - X_{t-1}$
2. **Linear regression**: $\hat{T}_t = \hat{\beta}_0 + \hat{\beta}_1 t$
3. **Polynomial**: Higher-order polynomial
4. **HP Filter**: Balance fit vs smoothness
5. **Moving average**: $\hat{T}_t = MA_q(X_t)$
6. **LOESS**: Local polynomial regression

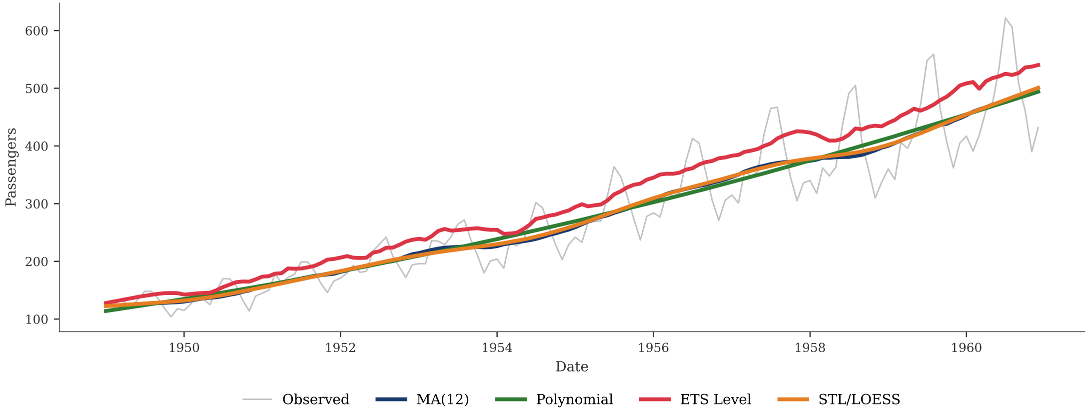
Choice Depends On

- Nature of trend (deterministic vs stochastic)
- Purpose (forecasting vs analysis)

Detrending Methods: Comparison



Trend Estimation: Multiple Approaches



Interpretation

Different methods capture trend at varying levels of smoothness.

Seasonality Removal Methods

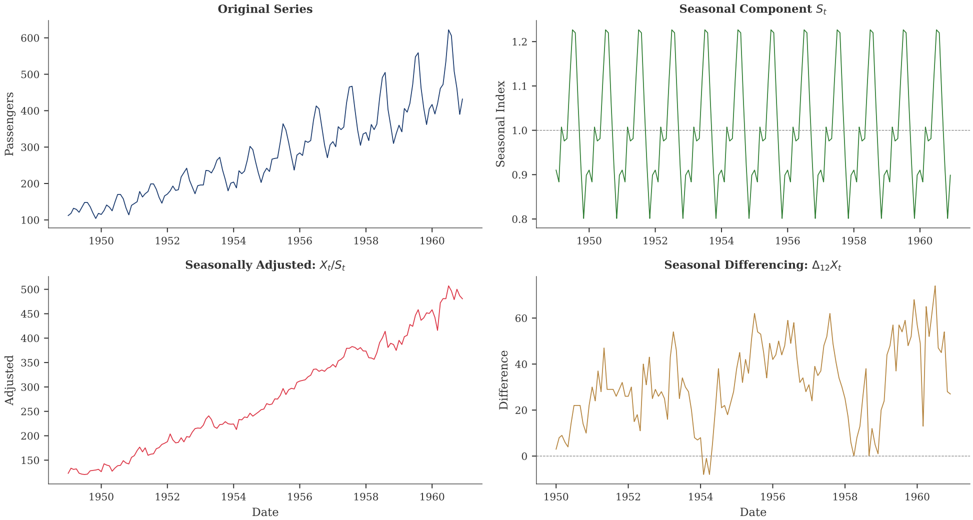
Four Approaches to Remove Seasonality

1. **Seasonal differencing**: $\Delta_s X_t = X_t - X_{t-s}$
2. **Division** (multiplicative): $X_t^{adj} = X_t / \hat{S}_t$
3. **Subtraction** (additive): $X_t^{adj} = X_t - \hat{S}_t$
4. **X-13ARIMA-SEATS**: Government statistical method

Seasonal Period s

Monthly $\Rightarrow s = 12$; Quarterly $\Rightarrow s = 4$

Seasonal Adjustment: Visualization



Deterministic vs Stochastic Trend

Deterministic Trend:

$$X_t = \beta_0 + \beta_1 t + \varepsilon_t$$

- Trend is a function of time
- Detrend by regression
- ε_t is stationary

Stochastic Trend:

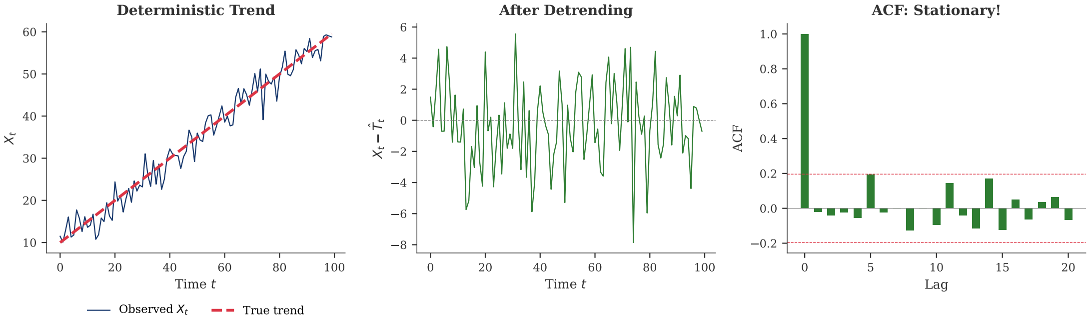
$$X_t = X_{t-1} + \varepsilon_t$$

- Random walk component
- Detrend by differencing
- ΔX_t is stationary

Wrong Method = Problems

- Differencing deterministic trend $\Rightarrow$ over-differencing
- Regression on stochastic trend $\Rightarrow$ spurious regression

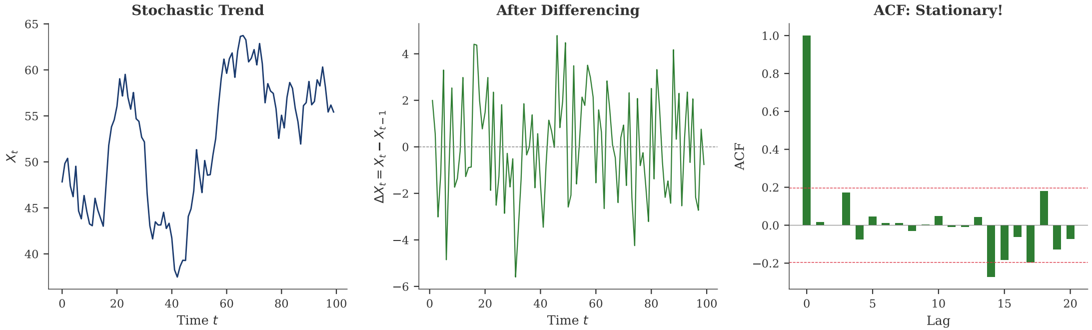
Example: Deterministic Trend



Key

Use regression to remove trend $\rightarrow$ residuals are stationary (ACF decays quickly).

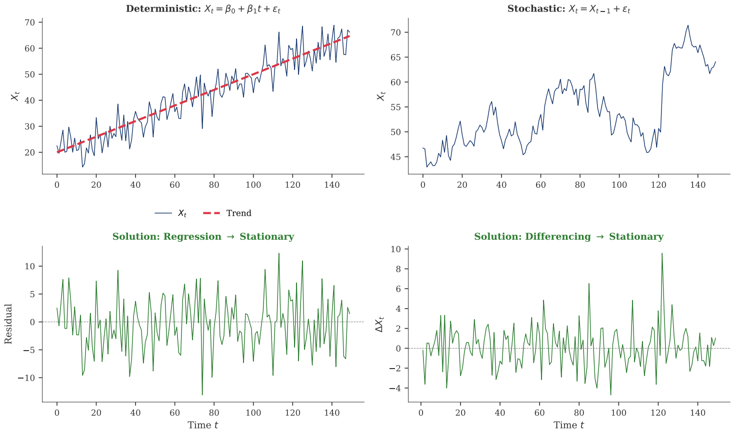
Example: Stochastic Trend (Random Walk)



Key

Use differencing to remove trend $\rightarrow$ differences are stationary (white noise).

Side-by-Side Comparison



Remember

Deterministic → regression. Stochastic → differencing.

Stochastic Process: Definition

Definition 5 (Stochastic Process)

A **stochastic process** is a collection of random variables indexed by time:

$$\{X_t(\omega) : t \in \mathcal{T}, \omega \in \Omega\}$$

where Ω is the sample space of possible outcomes.

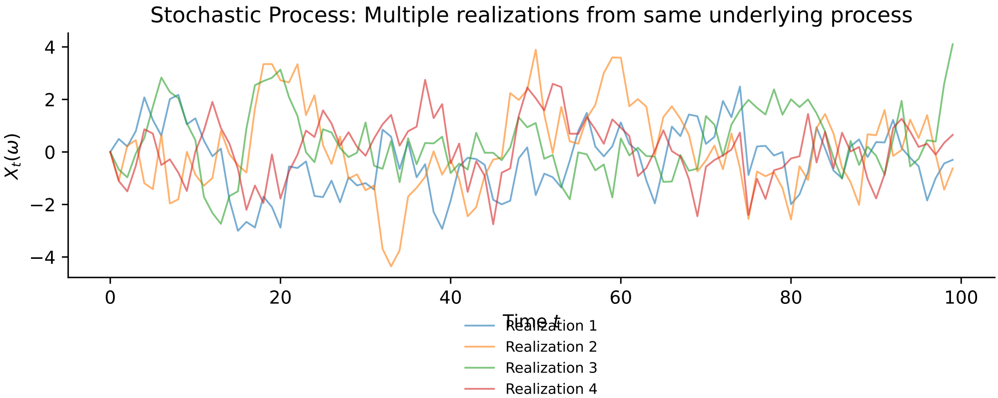
Two Perspectives

- **Fixed** ω: A *realization* $\{X_t(\omega)\}_{t \in \mathcal{T}}$
- **Fixed** t: A *random variable* X_t

Key Insight

A time series we observe is **one realization** of the underlying stochastic process.

Stochastic Process: Visual Illustration



Interpretation

Each line is a different realization from the same stochastic process. We observe only one realization but want to understand the process.

Moments of a Stochastic Process

First Two Moments Characterize Weak Properties

Mean Function: $\mu_t = \mathbb{E}[X_t]$

Autocovariance Function (ACVF):

$$\gamma(t,s) = \text{Cov}(X_t, X_s) = \mathbb{E}[(X_t - \mu_t)(X_s - \mu_s)]$$

Autocorrelation Function (ACF):

$$\rho(t,s) = \frac{\gamma(t,s)}{\sqrt{\text{Var}(X_t) \cdot \text{Var}(X_s)}}$$

Properties

$\rho(t,s) \in [-1,1]$ and $\rho(t,t) = 1$

Why Stationarity Matters

Stationarity is a fundamental assumption for time series analysis:

Without Stationarity:

- Mean, variance change over time
- Past may not predict future
- Standard methods fail
- Spurious correlations

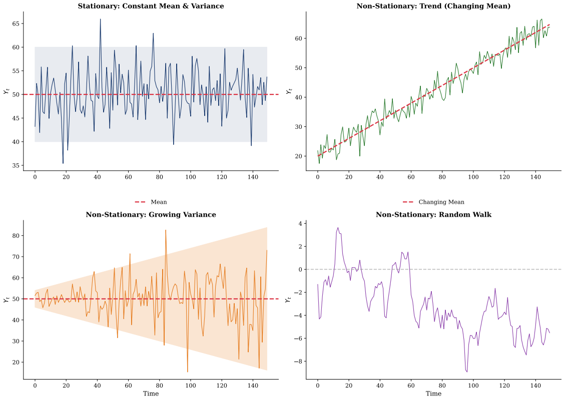
With Stationarity:

- Statistical properties constant
- Can estimate from one realization
- Valid inference possible
- Models are meaningful

Key Principle

Most time series models (ARMA, ARIMA, etc.) require stationarity. Non-stationary series must be transformed (e.g., differencing) before modeling.

Stationary vs Non-Stationary: Visual Comparison



- **Stationary**: Constant mean and variance – fluctuates around a fixed level
- **Non-stationary**: Mean and/or variance change over time
- Visual inspection is the first step; formal tests (ADF, KPSS) confirm

Strict Stationarity

Definition 6 (Strict (Strong) Stationarity)

A process $\{X_t\}$ is **strictly stationary** if for all k, all $t_1, \ldots, t_k$, and all h:

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \stackrel{d}{=} (X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h})$$

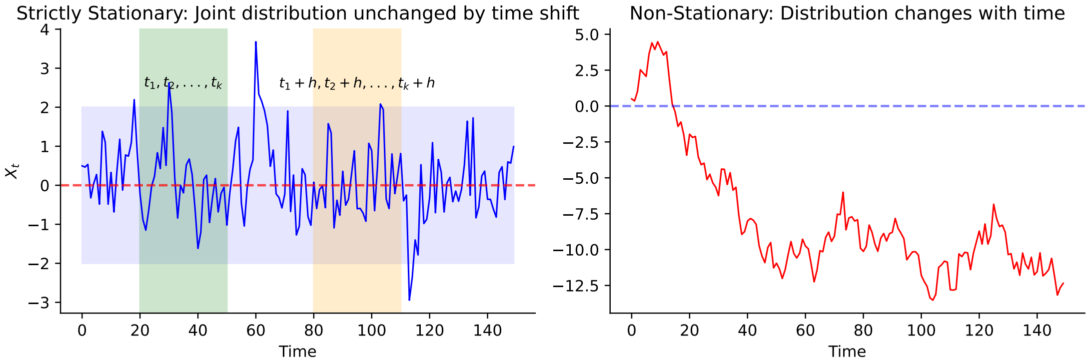
Implications

- All marginal distributions $F_{X_t}(x)$ identical
- $\mathbb{E}[X_t] = \mu$ (constant mean)
- $\text{Var}(X_t) = \sigma^2$ (constant variance)
- Joint distributions depend only on lag

Note

Strict stationarity is a strong condition, often impractical to verify.

Strict Stationarity: Visual Illustration



Interpretation

Stationary: any two windows have the same joint distribution. Non-stationary: distribution changes over time.

Weak (Covariance) Stationarity

Definition 7 (Weak Stationarity)

A process $\{X_t\}$ is **weakly stationary** (or covariance stationary) if:

1. $\mathbb{E}[X_t] = \mu$ (constant mean)
2. $\text{Var}(X_t) = \sigma^2 < \infty$ (constant, finite variance)
3. $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ (covariance depends only on lag h)

Key property: Autocovariance is a function of lag only:

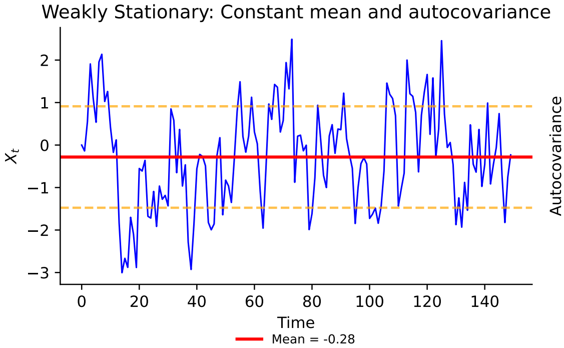
$$\gamma(h) = \text{Cov}(X_t, X_{t+h}) = \mathbb{E}[(X_t - \mu)(X_{t+h} - \mu)] \tag{5}$$

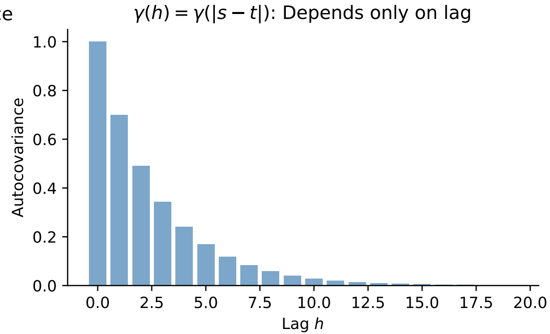
Autocorrelation function:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\text{Cov}(X_t, X_{t+h})}{\text{Var}(X_t)} \tag{6}$$

Note: $\rho(0) = 1$ and $\rho(h) = \rho(-h)$ (symmetry)

Weak Stationarity: Visual Illustration





Interpretation

Left: constant mean and variance. Right: autocovariance depends only on lag h, not time t.

Properties of the Autocovariance Function

ACVF Properties for Weakly Stationary Process

The ACVF $\gamma(h)$ satisfies:

1. **Symmetry:** $\gamma(h) = \gamma(-h)$
2. **Maximum at zero:** $|\gamma(h)| \leq \gamma(0)$
3. **Non-negative definiteness**

Implication

Not every function can be an autocovariance function.

White Noise Process

Definition 8 (White Noise)

A process $\{\varepsilon_t\}$ is **white noise**, denoted $\varepsilon_t \sim WN(0, \sigma^2)$, if:

1. $\mathbb{E}[\varepsilon_t] = 0$ for all t
2. $\text{Var}(\varepsilon_t) = \sigma^2$ for all t
3. $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ for $t \neq s$

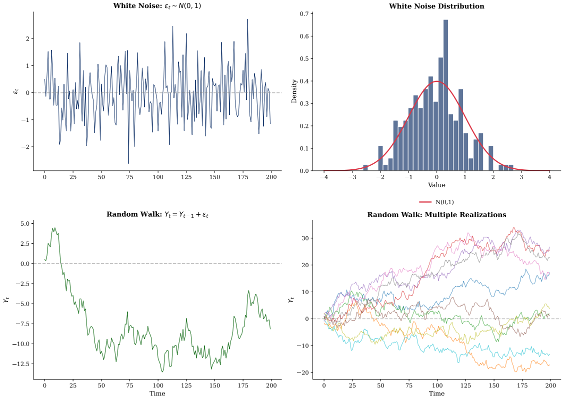
ACF of White Noise

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

Types

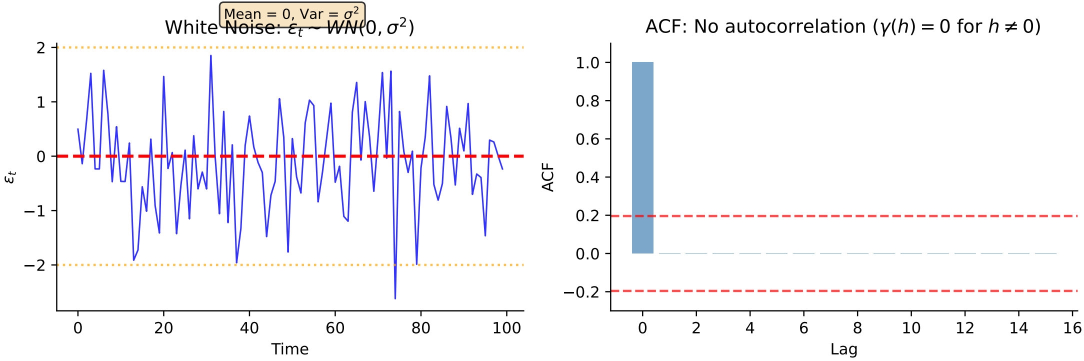
- **Weak**: Uncorrelated
- **Strong**: i.i.d.
- **Gaussian**: $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

White Noise vs Random Walk: Comparison



- **White noise**: Fluctuates around zero – stationary, constant variance
- **Random walk**: Cumulative sum of white noise – wanders away, non-stationary
- Random walk is the simplest non-stationary process (unit root)

White Noise: Visual Illustration



Interpretation

Left: white noise fluctuates around zero with constant variance. Right: ACF shows no autocorrelation (all zero after lag 0).

Random Walk Process

Definition

$X_t = X_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim WN(0, \sigma^2)$, $X_0 = 0$
Explicit form: $X_t = \sum_{i=1}^{t} \varepsilon_i$

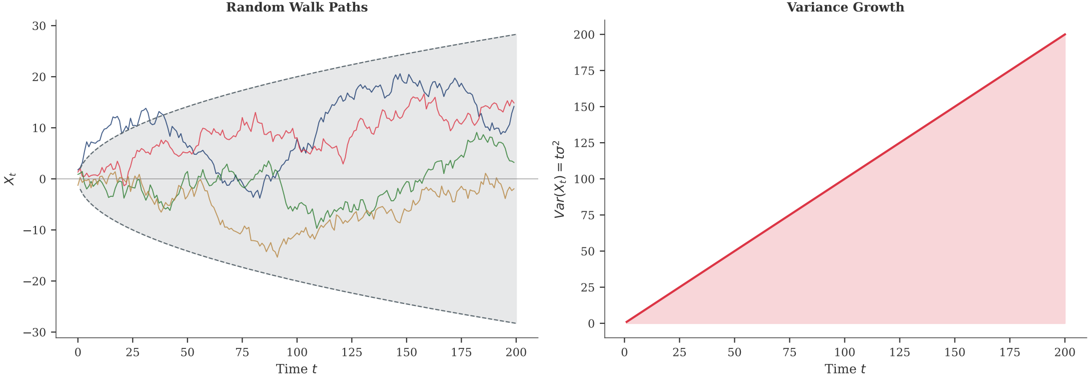
Properties

- $\mathbb{E}[X_t] = 0$ (constant mean)
- $\text{Var}(X_t) = t\sigma^2$ (grows with time!)
- $\text{Cov}(X_t, X_s) = \min(t, s) \cdot \sigma^2$

Non-Stationary!

Random walk is **not stationary** because variance depends on t.

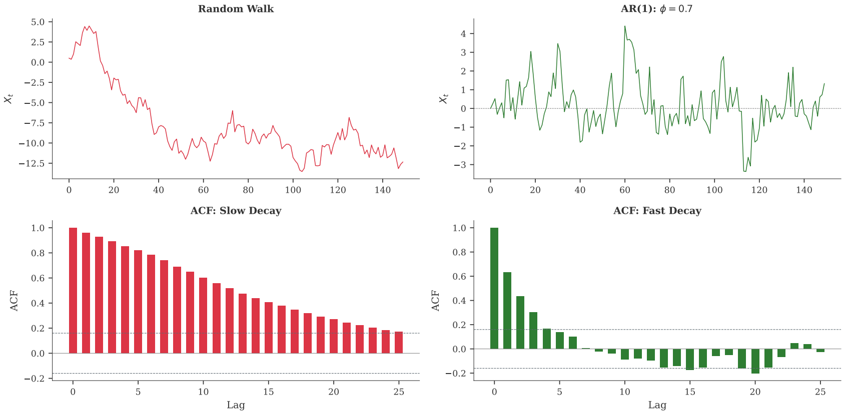
Random Walk: Visualization



Interpretation

Left: Multiple paths diverge over time. **Right:** Variance grows linearly: $\text{Var}(X_t) = t\sigma^2$.

Stationary vs Non-Stationary: Comparison



Key Diagnostic

ACF of stationary process decays quickly; ACF of random walk decays very slowly.

Sample Autocorrelation Function

Sample ACF at Lag h

$$\hat{\rho}(h) = \frac{\sum_{t=1}^{T-h}(x_t - \bar{x})(x_{t+h} - \bar{x})}{\sum_{t=1}^{T}(x_t - \bar{x})^2} \tag{7}$$

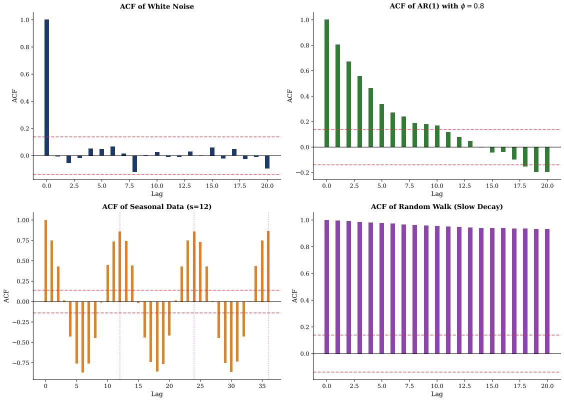
Properties

- $\hat{\rho}(0) = 1$ always
- $|\hat{\rho}(h)| \leq 1$

Significance Test

Under white noise: $\hat{\rho}(h) \approx N(0, 1/T)$
95% bounds: $\pm 1.96/\sqrt{T}$

ACF Patterns for Different Processes



- **White noise**: ACF drops to zero immediately (no dependence)
- **AR(1)**: ACF decays exponentially – indicates autoregressive structure
- **Seasonal**: ACF shows spikes at seasonal lags (e.g., 12, 24 for monthly)
- **Random walk**: ACF decays very slowly – sign of non-stationarity

Partial Autocorrelation Function (PACF)

Definition

PACF ϕ_{hh}: Correlation between X_t and X_{t+h} after removing the linear effect of $X_{t+1}, \ldots, X_{t+h-1}$.

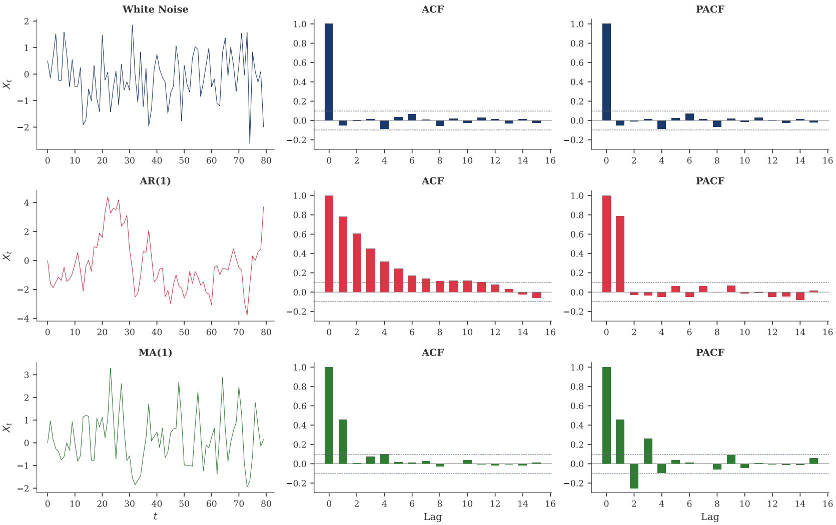
Interpretation

- $\phi_{11} = \rho(1)$ (same as ACF at lag 1)
- ϕ_{22}: correlation controlling for X_{t+1}
- Measures *direct* dependence

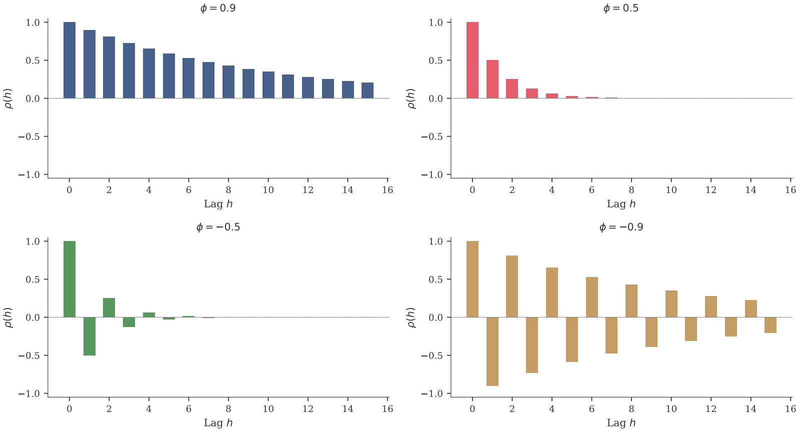
Key Application

- AR(p): PACF **cuts off** after lag p
- MA(q): ACF **cuts off** after lag q

ACF and PACF Patterns



Theoretical ACF for AR(1)



Interpretation

For AR(1): $X_t = \phi X_{t-1} + \varepsilon_t$, the theoretical ACF is $\rho(h) = \phi^h$.

The Lag Operator

Definition 9 (Lag Operator)

The **lag operator** (or backshift operator) L is defined by:

$$LX_t = X_{t-1}$$

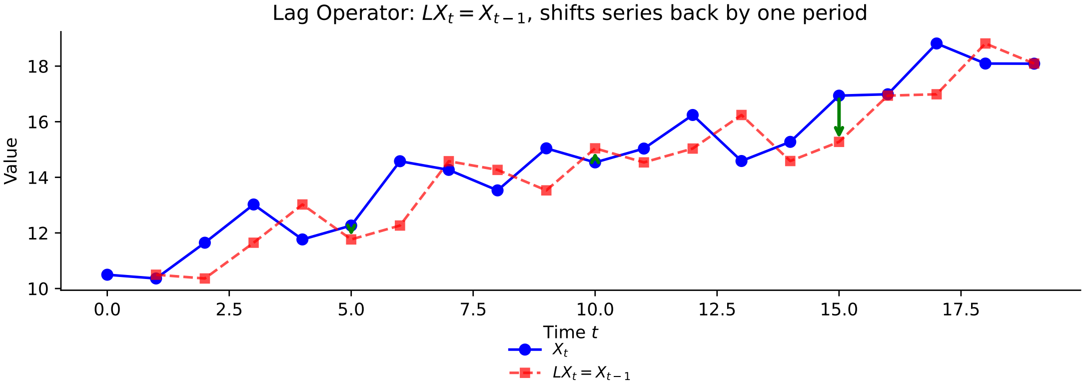
Properties

- $L^k X_t = X_{t-k}$ (lag by k periods)
- $L^0 = I$ (identity)
- $(1 - \phi L)X_t = X_t - \phi X_{t-1}$

Examples

- AR(1): $(1 - \phi L)X_t = \varepsilon_t$
- MA(1): $X_t = (1 + \theta L)\varepsilon_t$
- AR(p): $\phi(L)X_t = \varepsilon_t$

Lag Operator: Visual Illustration



Interpretation

The lag operator L shifts every observation back by one time period: $LX_t = X_{t-1}$.

Differencing

First Difference

$\Delta X_t = X_t - X_{t-1} = (1 - L)X_t$

Why Difference?

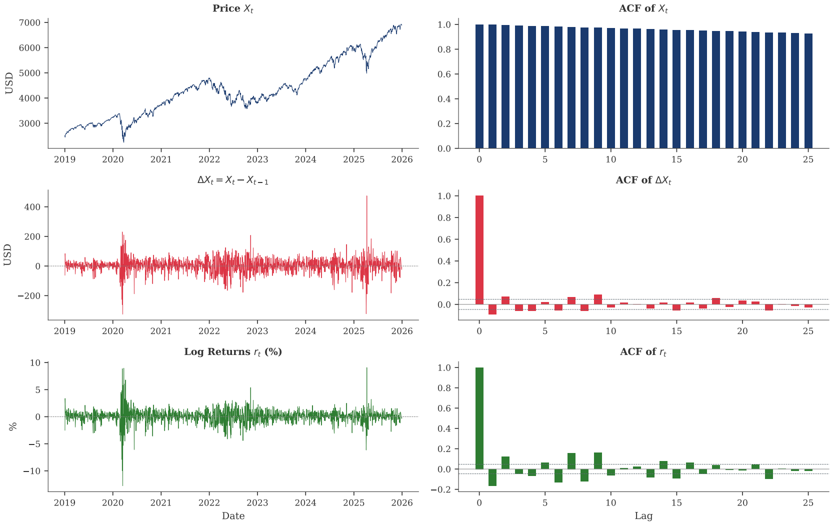
- Removes trend and unit root
- Random walk: $\Delta X_t = \varepsilon_t$ (white noise)

Integrated Process

$X_t \sim I(d)$ if $\Delta^d X_t$ is stationary

- $I(0)$: Stationary (no differencing)
- $I(1)$: One difference needed
- $I(2)$: Two differences needed

Effect of Differencing: S&P 500



Augmented Dickey-Fuller (ADF) Test

Model: $\Delta X_t = \alpha + \gamma X_{t-1} + \sum_{i=1}^{p} \delta_i \Delta X_{t-i} + \varepsilon_t$

Hypotheses:

- H_0: $\gamma = 0$ (unit root)
- H_1: $\gamma < 0$ (stationary)

Decision:

- $\tau <$ critical value $\Rightarrow$ Reject H_0 $\Rightarrow$ Stationary
- $\tau \geq$ critical value $\Rightarrow$ Non-stationary

Critical values: Dickey-Fuller distribution (not normal)

Test statistic:

$$\tau = \frac{\hat{\gamma}}{SE(\hat{\gamma})}$$

KPSS Test

Model: $X_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$

Hypotheses (opposite of ADF):

- H_0: $\sigma_u^2 = 0$ (stationary)
- H_1: $\sigma_u^2 > 0$ (unit root)

Test statistic:

$$LM = \frac{\sum_{t=1}^{T} S_t^2}{T^2 \hat{\sigma}^2}$$

where $S_t = \sum_{i=1}^{t} \hat{e}_i$

Decision:

- $LM >$ critical value $\Rightarrow$ Reject H_0 $\Rightarrow$ Non-stationary
- $LM \leq$ critical value $\Rightarrow$ Stationary

Note: KPSS complements ADF—use both for robust conclusions.

Using ADF and KPSS Together

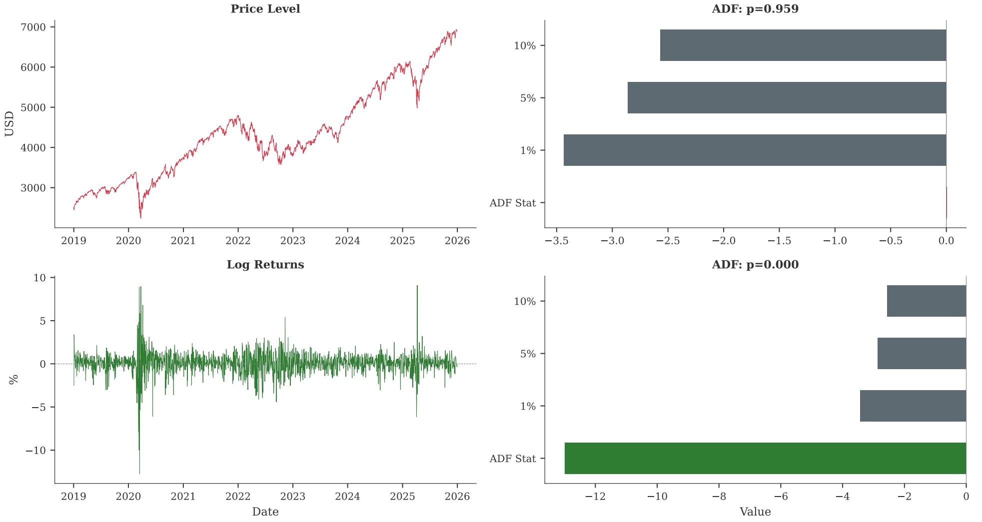
Confirmatory Testing for Robust Conclusions

ADF	KPSS	Conclusion
Reject H_0	Fail to reject H_0	Stationary
Fail to reject H_0	Reject H_0	Unit Root
Reject H_0	Reject H_0	Inconclusive
Fail to reject H_0	Fail to reject H_0	Inconclusive

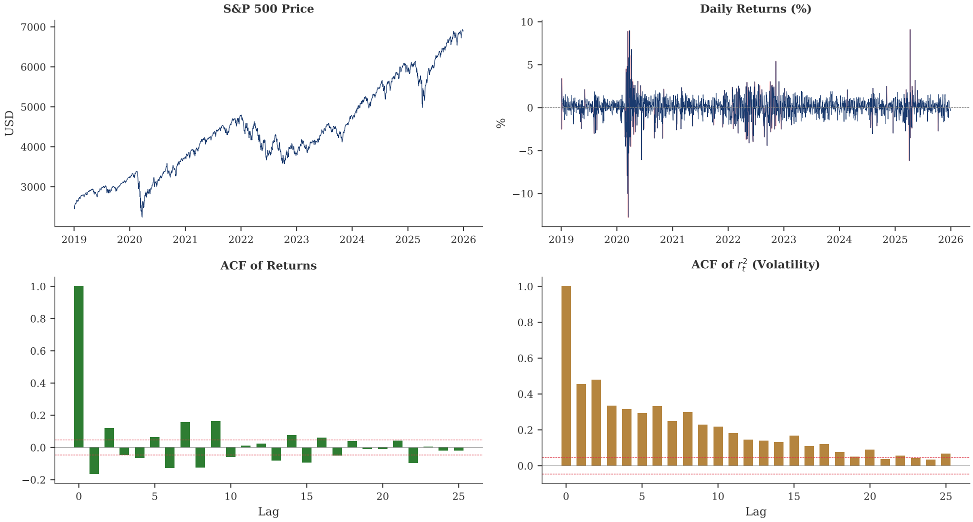
Recommended Workflow

1. Run ADF test (null = unit root)
2. Run KPSS test (null = stationary)
3. If results agree, proceed with confidence
4. If inconclusive, consider alternative tests (PP, DF-GLS)

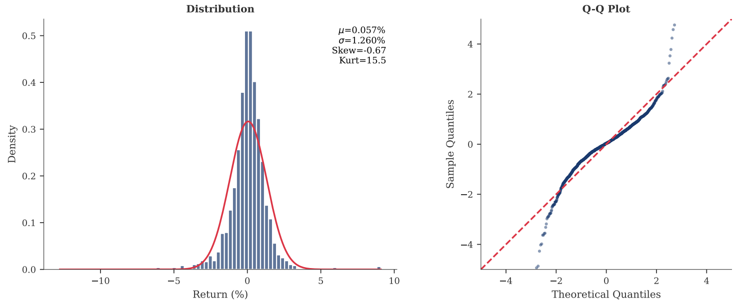
ADF Test: Visualization with S&P 500



S&P 500 Analysis: Overview



Stylized Facts of Financial Returns



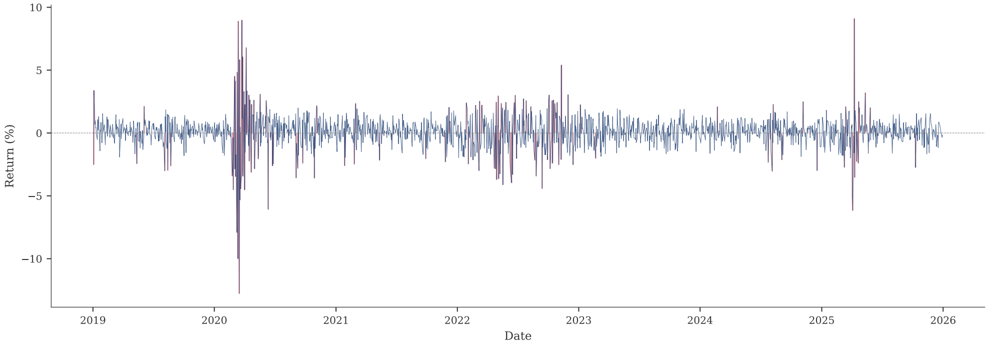
Observed properties:

- Negative skewness (left tail)
- Excess kurtosis ($\gg 3$)
- Heavy tails (fat tails)

Implications:

- Normal distribution inadequate
- Extreme events more likely
- Need Student-t or similar

Volatility Clustering



Stylized Fact

Large returns (positive or negative) tend to be followed by large returns. This **volatility clustering** motivates ARCH/GARCH models (future chapters).

Key Takeaways

Summary

1. **Time series** = observations indexed by time with temporal dependence
2. **Decomposition**: Additive $X_t = T_t + S_t + \varepsilon_t$ or Multiplicative
3. **Exponential Smoothing**: SES (level), Holt (trend), Holt-Winters (seasonal)
4. **Forecast Evaluation**: MAE, RMSE, MAPE; use train/validation/test splits
5. **Seasonality Modeling**: Dummy variables (any pattern) or Fourier terms (smooth)
6. **Trend Handling**: Differencing (stochastic) or regression (deterministic)
7. **Stationarity**: Mean, variance, autocovariance constant over time
8. **ACF/PACF**: Essential for identifying dependence structure
9. **Unit root tests**: ADF (H_0: unit root) vs KPSS (H_0: stationary)

Important Formulas I

Decomposition

Additive: $X_t = T_t + S_t + \varepsilon_t$ Multiplicative: $X_t = T_t \times S_t \times \varepsilon_t$

Simple Exponential Smoothing (SES)

$\hat{X}_{t+1|t} = \alpha X_t + (1-\alpha)\hat{X}_{t|t-1}$ where $\alpha \in (0,1)$

Holt's Linear Trend

$\ell_t = \alpha X_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$ $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$

Holt-Winters Additive

$\ell_t = \alpha(X_t - S_{t-s}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$ $S_t = \gamma(X_t - \ell_t) + (1-\gamma)S_{t-s}$

Important Formulas II

Moving Average (Trend Estimation)

$\hat{T}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t+j}$

Autocovariance and Autocorrelation

$\gamma(h) = \text{Cov}(X_t, X_{t+h}) \qquad \rho(h) = \frac{\gamma(h)}{\gamma(0)}$

Random Walk

$X_t = X_{t-1} + \varepsilon_t \quad \Rightarrow \quad \text{Var}(X_t) = t\sigma^2$ (non-stationary)

Differencing

$\Delta X_t = (1 - L)X_t = X_t - X_{t-1}$

Next Chapter Preview

Chapter 2: ARMA Models

- Autoregressive (AR) models
- Moving Average (MA) models
- Combined ARMA models
- Model identification using ACF/PACF
- Parameter estimation
- Model diagnostics
- Forecasting

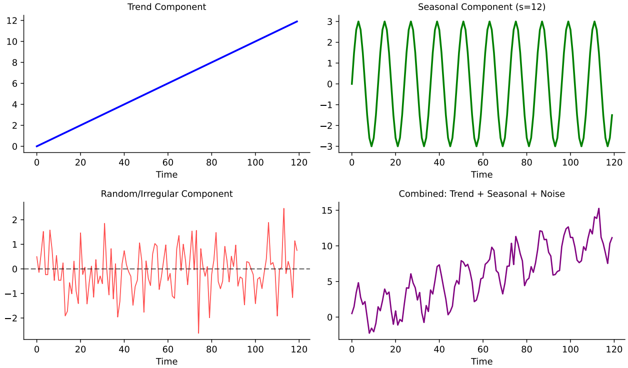
Quiz Question 1

Question

A time series Y_t shows upward movement over years plus repeating patterns each quarter. Which components are present?

- A. Trend only
- B. Seasonality only
- C. Trend and Seasonality
- D. Random noise only

Quiz Question 1: Answer



Correct Answer: (C) Trend and Seasonality

Upward movement = Trend; Quarterly patterns = Seasonality (s=4)

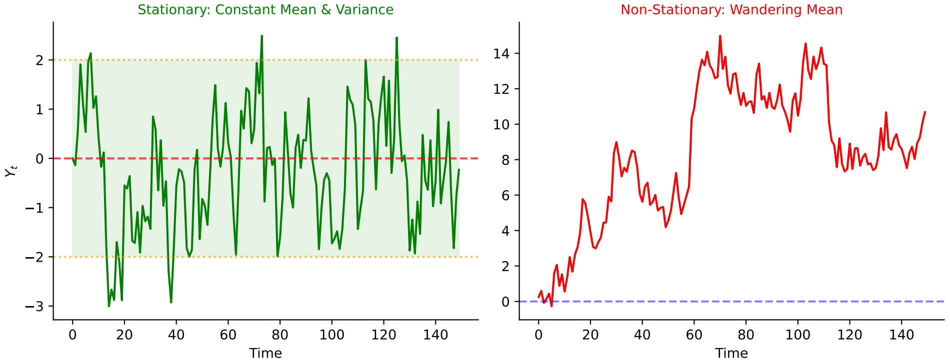
Quiz Question 2

Question

Which of the following is a characteristic of a stationary time series?

- A. Mean changes over time
- B. Variance increases with time
- C. Constant mean and variance over time
- D. Contains a trend component

Quiz Question 2: Answer



Correct Answer: (C) Constant mean and variance over time

Stationarity requires: constant mean, constant variance, and autocovariance depends only on lag.

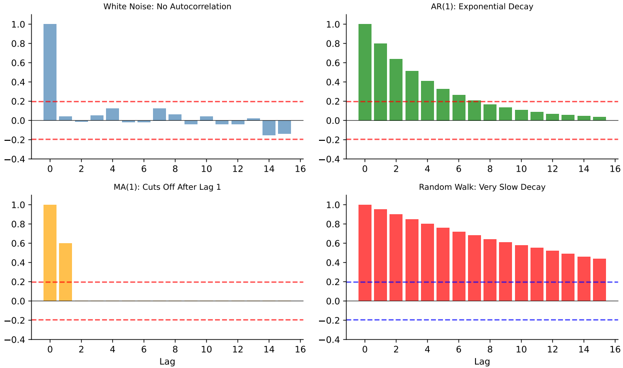
Quiz Question 3

Question

For a white noise process, what does the ACF look like at lags $k > 0$?

- A. Exponential decay
- B. All values significant and positive
- C. All values approximately zero (within confidence bands)
- D. Alternating positive and negative

Quiz Question 3: Answer



Correct Answer: (C) Approximately zero within confidence bands

White noise has no autocorrelation: $\rho_k = 0$ for all $k \neq 0$.

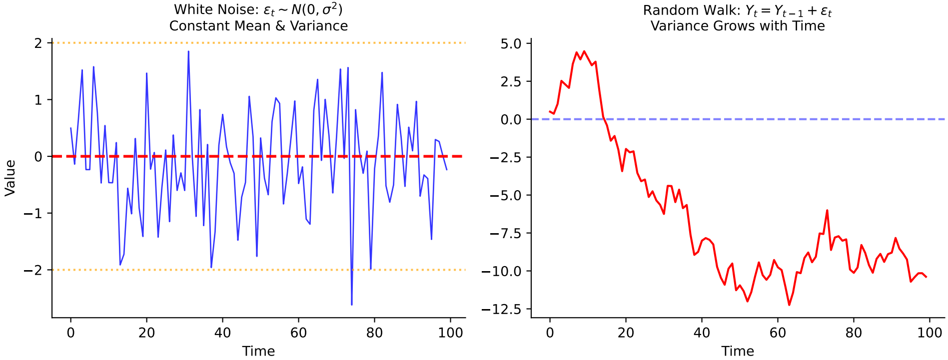
Quiz Question 4

Question

What is the key difference between white noise and a random walk?

- A. White noise has a trend, random walk doesn't
- B. Random walk is the cumulative sum of white noise
- C. Both are stationary processes
- D. White noise has higher variance

Quiz Question 4: Answer



Correct Answer: (B) Random walk = cumulative sum of white noise

$Y_t = Y_{t-1} + \varepsilon_t = \sum_{i=1}^{t} \varepsilon_i$ where ε_t is white noise.

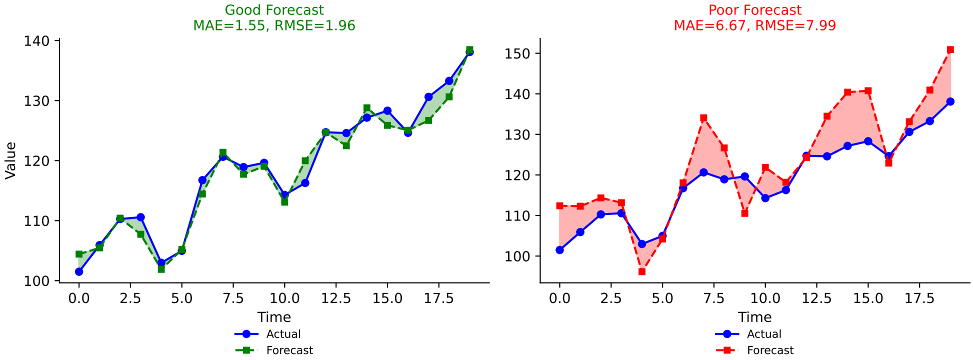
Quiz Question 5

Question

Which forecast error metric is most sensitive to large errors (outliers)?

- A. MAE (Mean Absolute Error)
- B. RMSE (Root Mean Squared Error)
- C. MAPE (Mean Absolute Percentage Error)
- D. All are equally sensitive

Quiz Question 5: Answer



Correct Answer: (B) RMSE

RMSE squares errors, so large errors have disproportionate impact: $\sqrt{\frac{1}{n}\sum e_t^2}$

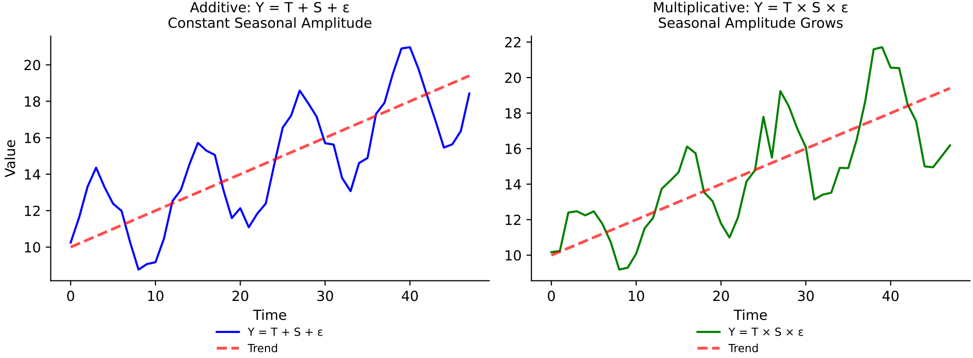
Quiz Question 6

Question

When should you use multiplicative decomposition instead of additive?

- A. When the series has no trend
- B. When seasonal amplitude is constant
- C. When seasonal amplitude grows with the level of the series
- D. When the series is stationary

Quiz Question 6: Answer



Correct Answer: (C) Seasonal amplitude grows with level

Multiplicative: $Y_t = T_t \times S_t \times \varepsilon_t$ — seasonal swings proportional to trend.

References

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- Box, G.E.P., Jenkins, G.M., Reinsel, G.C., & Ljung, G.M. (2015). *Time Series Analysis: Forecasting and Control*. 5th ed., Wiley.
- Tsay, R.S. (2010). *Analysis of Financial Time Series*. 3rd ed., Wiley.
- Cleveland, R.B., Cleveland, W.S., McRae, J.E., & Terpenning, I. (1990). STL: A Seasonal-Trend Decomposition. *Journal of Official Statistics*, 6(1), 3-73.

Data Sources

Real Data Used in This Chapter

- **Airline Passengers**: Box-Jenkins classic dataset, 1949–1960
- **S&P 500**: Yahoo Finance (SPY), historical data
- **Sunspots**: Statsmodels dataset, monthly observations

Software & Tools

- **Python**: statsmodels, pandas, matplotlib, yfinance
- **R**: forecast, tseries packages
- **Data Sources**: Yahoo Finance, FRED Economic Data

Thank You!

Questions?

Charts generated using Python (statsmodels, matplotlib)

Course materials available at: `https://github.com/danpele/Time-Series-Analysis`