Time Series Analysis and Forecasting

Bucharest University of Economic Studies

Academic Year 2025-2026

Outline

Recap: Stationarity

From Chapter 1: A process $\{X_t\}$ is **weakly stationary** if:

1. $\mathbb{E}[X_t] = \mu$ (constant mean)
2. $\text{Var}(X_t) = \sigma^2 < \infty$ (constant, finite variance)
3. $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ (covariance depends only on lag h)

Why stationarity matters for ARMA:

- ARMA models assume the underlying process is stationary
- Non-stationary data must be differenced first (ARIMA)
- Stationarity ensures stable model parameters

Today: We build models for stationary time series using past values and past errors.

The Lag Operator (Backshift Operator)

Definition 1 (Lag Operator)

The **lag operator** L (or backshift operator B) shifts a time series back by one period:

$$LX_t = X_{t-1}$$

Properties:

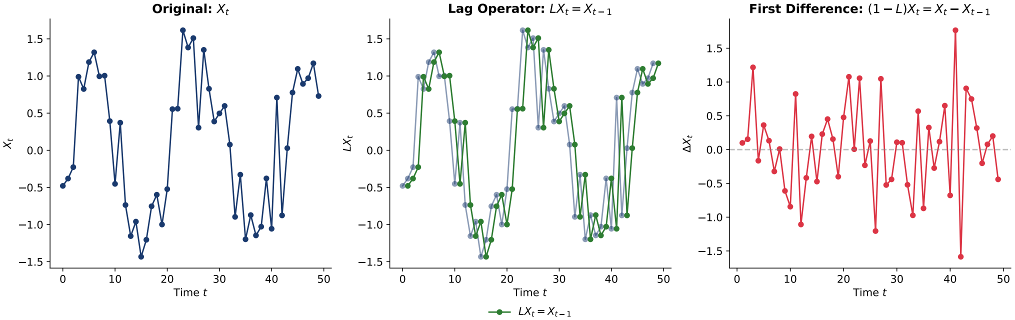
- $L^k X_t = X_{t-k}$ (shift back k periods)
- $L^0 X_t = X_t$ (identity)
- $(1 - L)X_t = X_t - X_{t-1} = \Delta X_t$ (first difference)
- $(1 - L)^d X_t = \Delta^d X_t$ (d-th difference)

Lag Polynomials:

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$$

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$

Lag Operator: Visual Illustration



Key insight: The lag operator is the foundation of ARMA model notation

White Noise Process

Definition 2 (White Noise)

A process $\{\varepsilon_t\}$ is **white noise**, denoted $\varepsilon_t \sim WN(0, \sigma^2)$, if:

1. $\mathbb{E}[\varepsilon_t] = 0$ for all t
2. $\text{Var}(\varepsilon_t) = \sigma^2$ for all t
3. $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ for all $t \neq s$

Properties:

- White noise is the "building block" of ARMA models
- ACF: $\rho(0) = 1$, $\rho(h) = 0$ for $h \neq 0$
- PACF: same pattern
- **Gaussian white noise:** additionally $\varepsilon_t \sim N(0, \sigma^2)$

Note: White noise is *not* predictable — it's pure randomness.

AR(1) Model: Definition

Definition 3 (AR(1) Process)

An **autoregressive process of order 1** is:

$$X_t = c + \phi X_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim WN(0, \sigma^2)$ and $|\phi| < 1$ for stationarity.

Interpretation:

- c: constant (intercept)
- ϕ: autoregressive coefficient — measures persistence
- ε_t: innovation (unpredictable shock)

Using lag operator:

$$(1 - \phi L)X_t = c + \varepsilon_t$$

$$\phi(L)X_t = c + \varepsilon_t \quad \text{where } \phi(L) = 1 - \phi L$$

AR(1) Stationarity Condition

For AR(1) to be stationary: $|\phi| < 1$

Intuition:

- If $|\phi| < 1$: shocks decay over time $\rightarrow$ stationary
- If $|\phi| = 1$: random walk $\rightarrow$ non-stationary (unit root)
- If $|\phi| > 1$: explosive process $\rightarrow$ non-stationary

Characteristic equation:

$$\phi(z) = 1 - \phi z = 0 \implies z = \frac{1}{\phi}$$

Stationarity requires the root $z = 1/\phi$ to lie **outside the unit circle**, i.e., $|z| > 1$, which means $|\phi| < 1$.

AR(1) Properties

For a stationary AR(1) with $|\phi| < 1$:

Mean:

$$\mu = \mathbb{E}[X_t] = \frac{c}{1-\phi}$$

Variance:

$$\gamma(0) = \mathrm{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$$

Autocovariance:

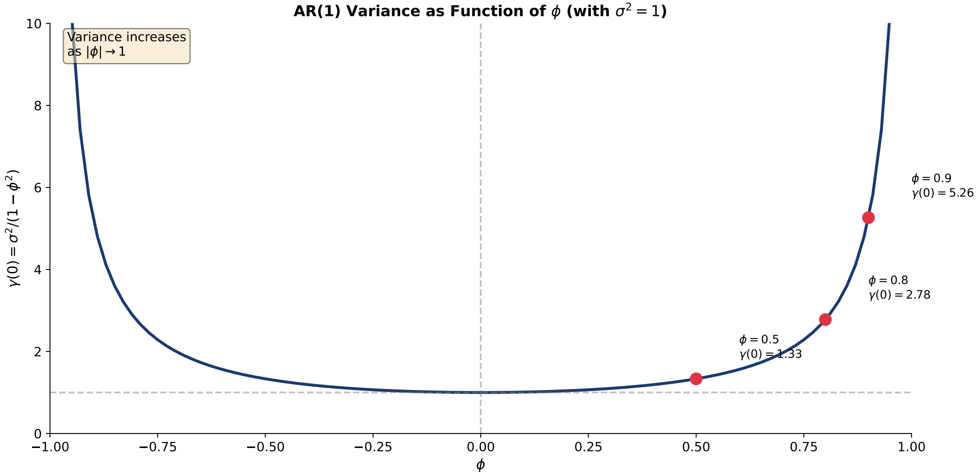
$$\gamma(h) = \phi^h \gamma(0) = \frac{\phi^h \sigma^2}{1-\phi^2}$$

Autocorrelation (ACF):

$$\rho(h) = \phi^h$$

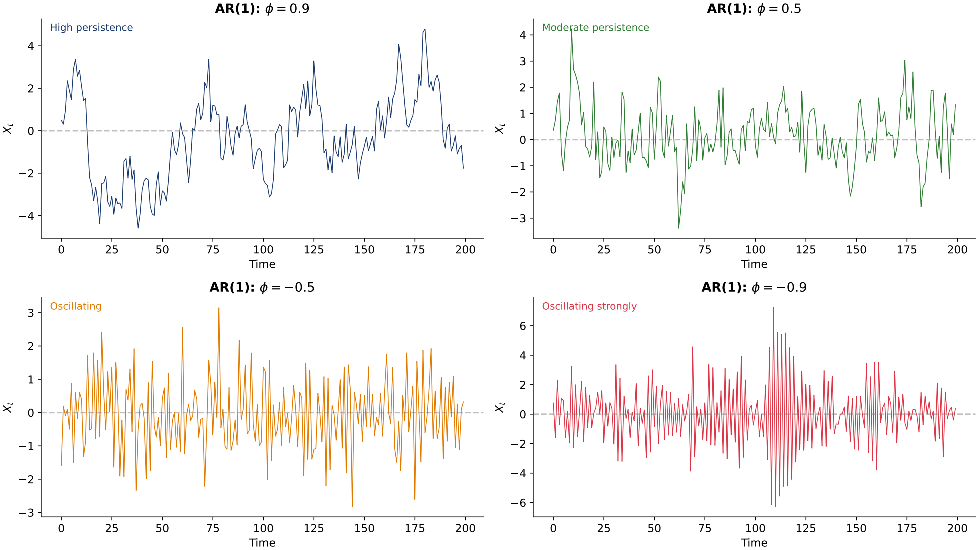
Key insight: ACF decays exponentially at rate ϕ

AR(1) Variance as Function of ϕ

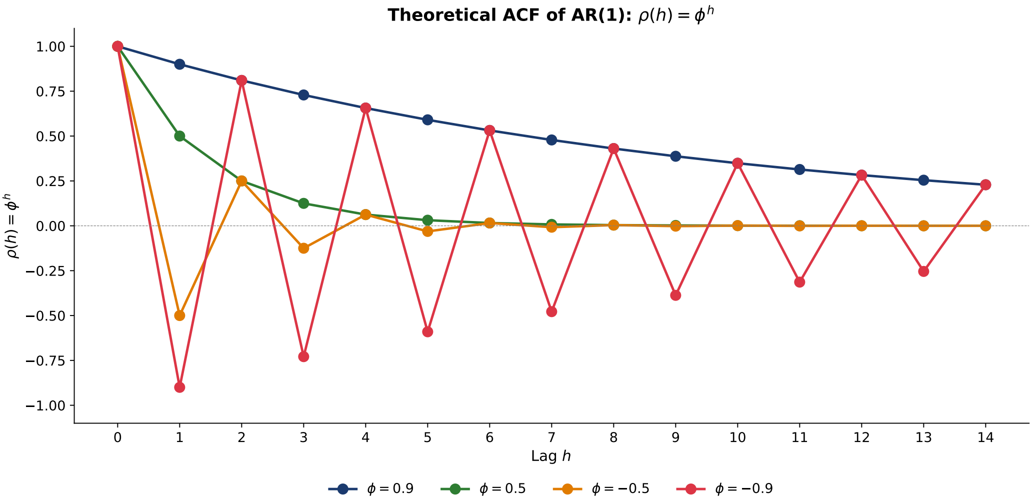


Key insight: As $|\phi| \to 1$, variance explodes $\to$ non-stationarity

AR(1) Simulations: Effect of ϕ



AR(1) Theoretical ACF



Pattern: $\rho(h) = \phi^h$ — exponential decay (or alternating for $\phi < 0$)

AR(1) ACF and PACF Patterns

ACF of AR(1):

- Decays exponentially: $\rho(h) = \phi^h$
- If $\phi > 0$: all positive, gradual decay
- If $\phi < 0$: alternating signs, decay in magnitude

PACF of AR(1):

- **Cuts off after lag 1**
- $\pi_1 = \phi$, $\pi_k = 0$ for $k > 1$

	ACF	**PACF**
AR(1)	Exponential decay	Cuts off at lag 1

This is the key identification pattern for AR(1)!

AR(p) Model: General Form

Definition 4 (AR(p) Process)

An **autoregressive process of order p** is:

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \varepsilon_t$$

Using lag operator:

$$\phi(L)X_t = c + \varepsilon_t$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$

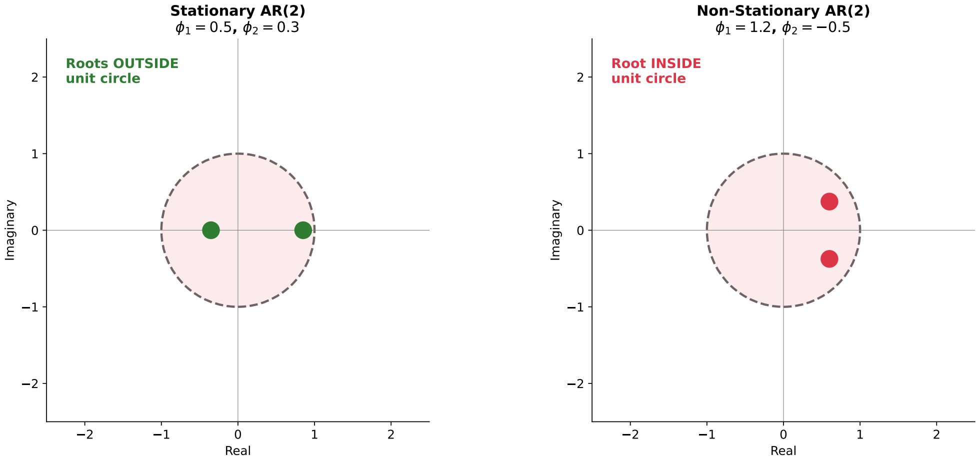
Stationarity condition:

- All roots of $\phi(z) = 0$ must lie **outside** the unit circle
- Equivalently: all roots have modulus > 1

PACF pattern:

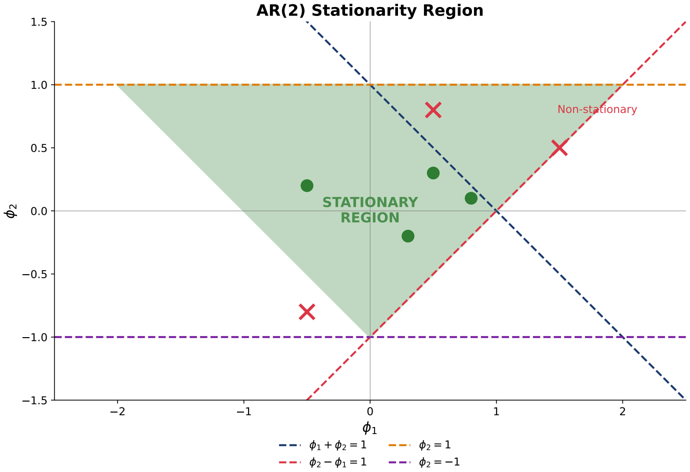
- PACF cuts off after lag p
- ACF decays (exponentially or with damped oscillations)

AR(2) Stationarity: Unit Circle Visualization

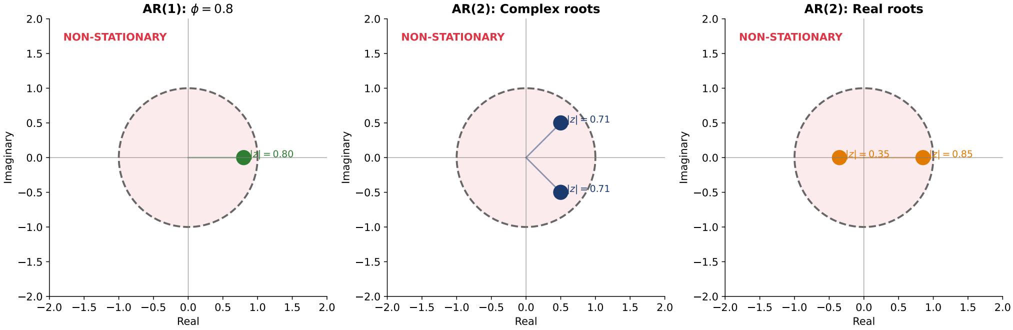


Rule: All roots of $\phi(z) = 0$ must lie **outside** the shaded unit circle

AR(2) Stationarity Triangle



Characteristic Polynomial Roots



AR(2) Model

Definition 5 (AR(2) Process)

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

Stationarity conditions for AR(2):

1. $\phi_1 + \phi_2 < 1$
2. $\phi_2 - \phi_1 < 1$
3. $|\phi_2| < 1$

ACF behavior depends on roots:

- **Real roots:** mixture of two exponential decays
- **Complex roots:** damped sinusoidal pattern (pseudo-cycles)

PACF: Cuts off after lag 2 ($\pi_k = 0$ for $k > 2$)

MA(1) Model: Definition

Definition 6 (MA(1) Process)

A **moving average process of order 1** is:

$$X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$$

where $\varepsilon_t \sim WN(0, \sigma^2)$.

Interpretation:

- μ: mean of the process
- θ: MA coefficient — measures impact of past shock
- Current value depends on current and one past shock

Using lag operator:

$$X_t = \mu + \theta(L)\varepsilon_t$$

where $\theta(L) = 1 + \theta L$

Key property: MA processes are **always stationary** for any finite θ

MA(1) Properties

For MA(1): $X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$

Mean:

$$\mathbb{E}[X_t] = \mu$$

Variance:

$$\gamma(0) = \text{Var}(X_t) = \sigma^2(1 + \theta^2)$$

Autocovariance:

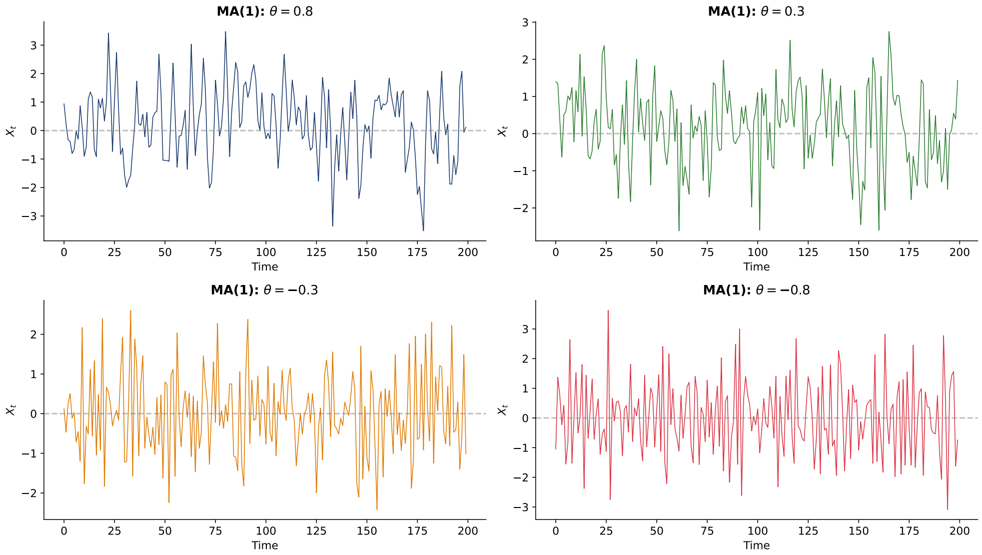
$$\gamma(1) = \theta\sigma^2, \quad \gamma(h) = 0 \text{ for } h > 1$$

Autocorrelation (ACF):

$$\rho(1) = \frac{\theta}{1 + \theta^2}, \quad \rho(h) = 0 \text{ for } h > 1$$

Key insight: ACF **cuts off** after lag 1

MA(1) Simulations: Effect of θ



MA(1) ACF and PACF Patterns

ACF of MA(1):

- **Cuts off after lag 1**
- $\rho(1) = \frac{\theta}{1+\theta^2}$, $\rho(h) = 0$ for $h > 1$
- Note: $|\rho(1)| \leq 0.5$ always (maximum at $\theta = \pm 1$)

PACF of MA(1):

- Decays exponentially (or with alternating signs)
- Does *not* cut off

	ACF	**PACF**
MA(1)	Cuts off at lag 1	Exponential decay

This is the opposite pattern from AR(1)!

Invertibility of MA Models

Definition 7 (Invertibility)

An MA process is **invertible** if it can be written as an infinite AR process:

$$X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$$

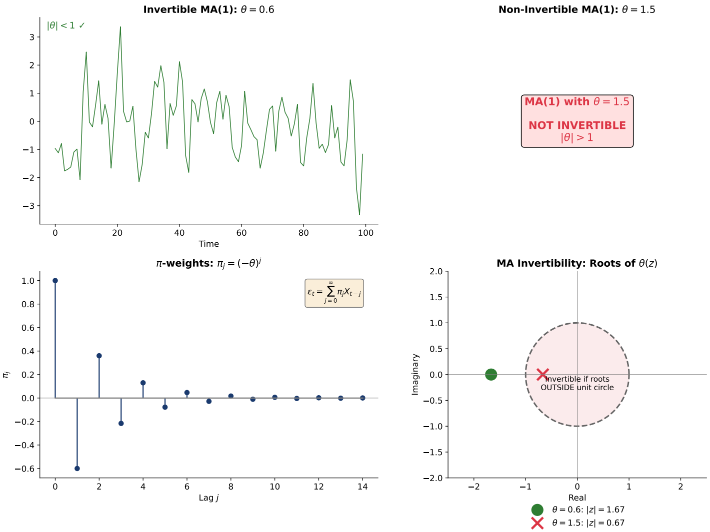
For MA(1): Invertible if $|\theta| < 1$
For MA(q): All roots of $\theta(z) = 0$ must lie outside the unit circle

Why invertibility matters:

- Ensures unique representation
- Required for forecasting and estimation
- Creates correspondence: AR(∞) $\leftrightarrow$ MA(q)

Note: Stationarity is for AR, Invertibility is for MA

Invertibility: Visualization



MA(q) Model: General Form

Definition 8 (MA(q) Process)

A **moving average process of order q** is:

$$X_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

Using lag operator:

$$X_t = \mu + \theta(L)\varepsilon_t$$

where $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$

Properties:

- Always stationary (finite variance)
- ACF cuts off after lag q: $\rho(h) = 0$ for $h > q$
- PACF decays gradually
- Invertible if all roots of $\theta(z) = 0$ lie outside unit circle

ARMA(p,q) Model: Definition

Definition 9 (ARMA(p,q) Process)

An **autoregressive moving average process** of order (p,q) is:

$$X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

Compact form using lag operators:

$$\phi(L)(X_t - \mu) = \theta(L)\varepsilon_t$$

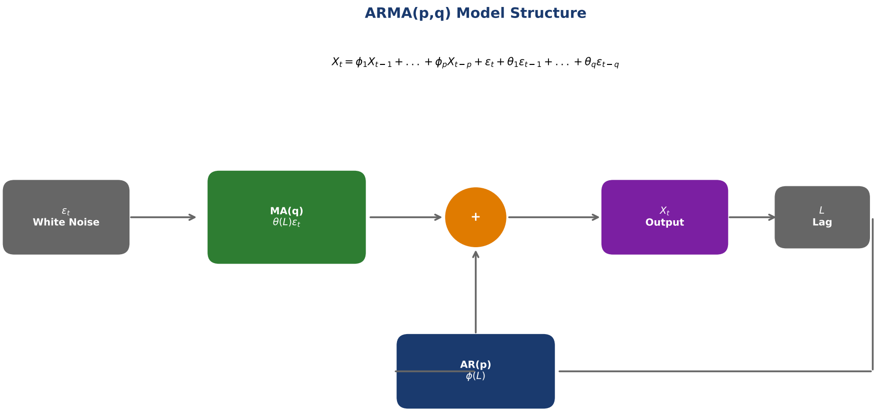
or equivalently:

$$\phi(L)X_t = c + \theta(L)\varepsilon_t$$

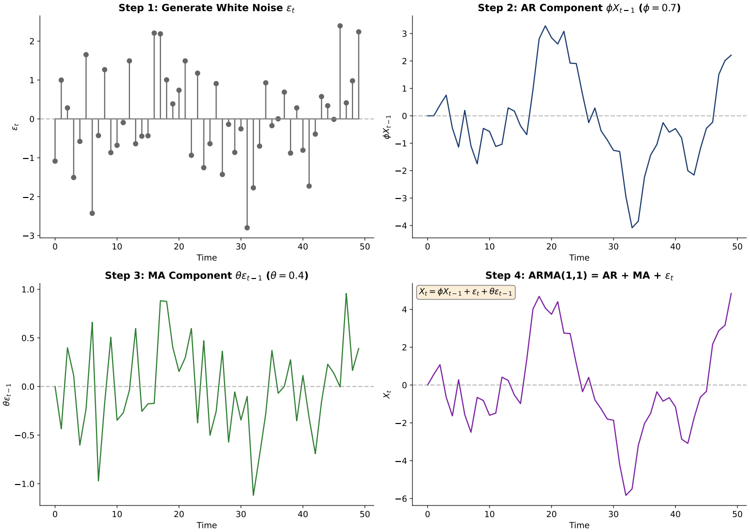
where $\mu = \frac{c}{1-\phi_1-\cdots-\phi_p}$

Key idea: Combines AR and MA components for more flexible modeling

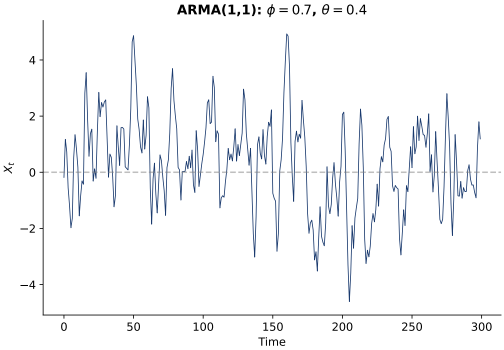
ARMA Model Structure

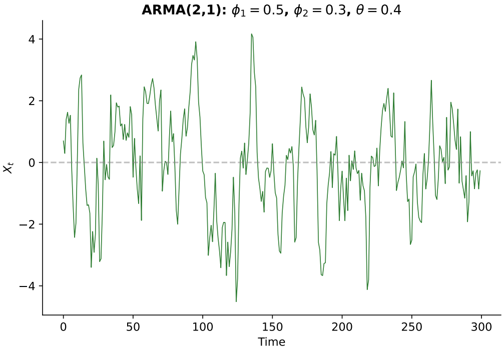


How ARMA Simulation Works



ARMA Examples





ARMA(1,1) Model

Definition 10 (ARMA(1,1) Process)

$$X_t = c + \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

Properties (assuming stationarity and invertibility):

- Mean: $\mu = \frac{c}{1-\phi}$
- Variance: $\gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$

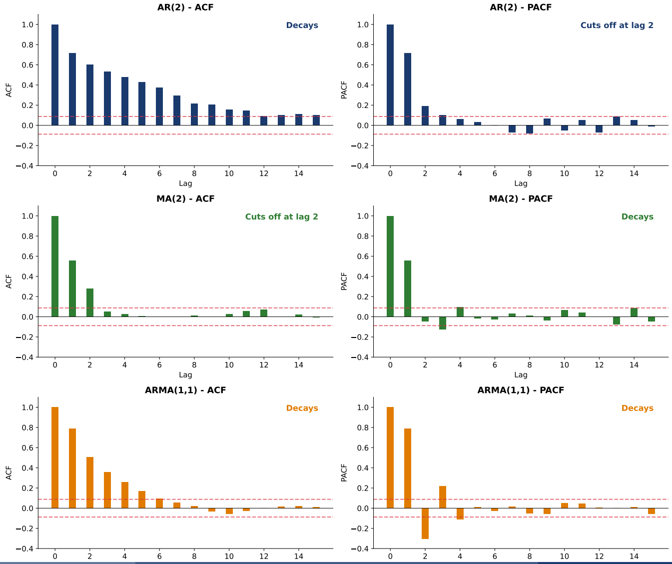
ACF:

$$\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}$$

$$\rho(h) = \phi \cdot \rho(h-1) \quad \text{for } h \geq 2$$

Pattern: ACF decays exponentially after lag 1 (like AR), but starting point depends on both ϕ and θ

ACF/PACF Patterns: AR vs MA vs ARMA



ARMA ACF and PACF Patterns

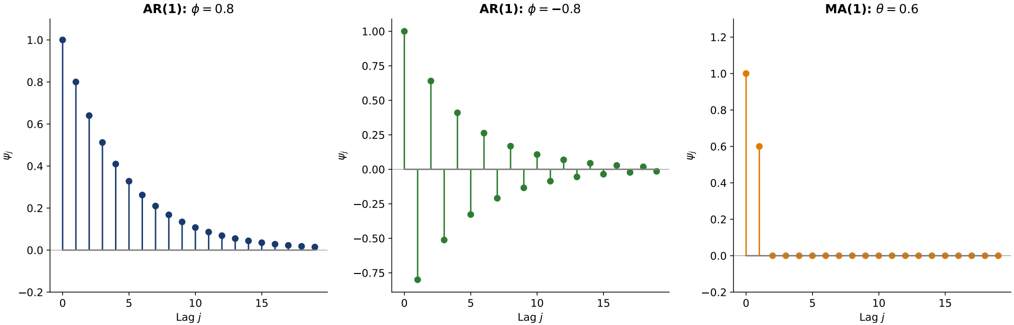
Model	**ACF**	**PACF**
AR(p)	Decays (exp./damped)	Cuts off at lag p
MA(q)	Cuts off at lag q	Decays (exp./damped)
ARMA(p,q)	Decays after lag $q-p$	Decays after lag $p-q$

Key identification rule:

- **PACF cuts off** $\rightarrow$ AR process (order = cutoff lag)
- **ACF cuts off** $\rightarrow$ MA process (order = cutoff lag)
- **Both decay** $\rightarrow$ ARMA process

Caution: In practice, sample ACF/PACF are noisy; use confidence bands

Impulse Response Functions



Interpretation: Shows how a unit shock propagates through the system over time

Stationarity and Invertibility Summary

For ARMA(p,q) to be well-behaved:

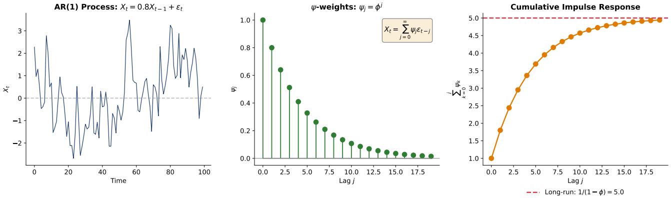
Condition	Requirement
Stationarity	Roots of $\phi(z) = 0$ outside unit circle
Invertibility	Roots of $\theta(z) = 0$ outside unit circle

Implications:

- **Stationarity:** Can write as MA(∞): $X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- **Invertibility:** Can write as AR(∞): $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$

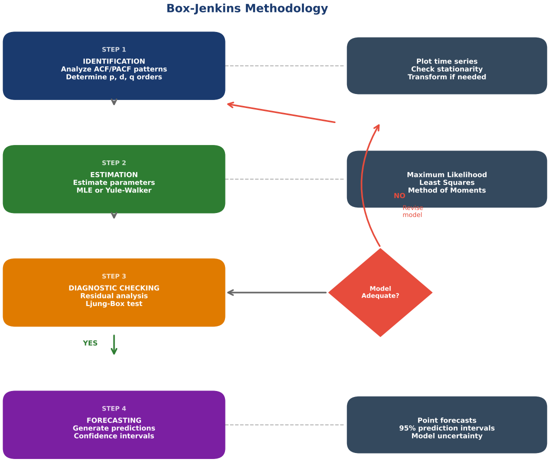
Causal representation: X_t depends only on *past* shocks (not future)

Wold's Decomposition Theorem



Any stationary process can be written as MA(∞): $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$

The Box-Jenkins Methodology



Model Identification Summary Table

Model Identification: ACF/PACF Patterns

Model	ACF Pattern	PACF Pattern
AR(p)	Exponential decay or damped oscillation	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponential decay or damped oscillation
ARMA(p,q)	Exponential decay after lag q-p	Exponential decay after lag p-q

Practical tip: Start simple (low p, q), increase if diagnostics fail

ACF/PACF Identification Rules

Theoretical patterns for stationary processes:

Model	ACF Pattern	PACF Pattern
AR(1)	Exponential decay	Spike at lag 1, then 0
AR(2)	Damped exponential/sine	Spikes at lags 1-2, then 0
AR(p)	Decays gradually	Cuts off after lag p
MA(1)	Spike at lag 1, then 0	Exponential decay
MA(2)	Spikes at lags 1-2, then 0	Damped exponential/sine
MA(q)	Cuts off after lag q	Decays gradually
ARMA(p,q)	Decays	Decays

Information Criteria

Purpose: Balance goodness-of-fit against model complexity

Akaike Information Criterion (AIC):

$$\text{AIC} = -2\ln(\hat{L}) + 2k$$

Bayesian Information Criterion (BIC/SBC):

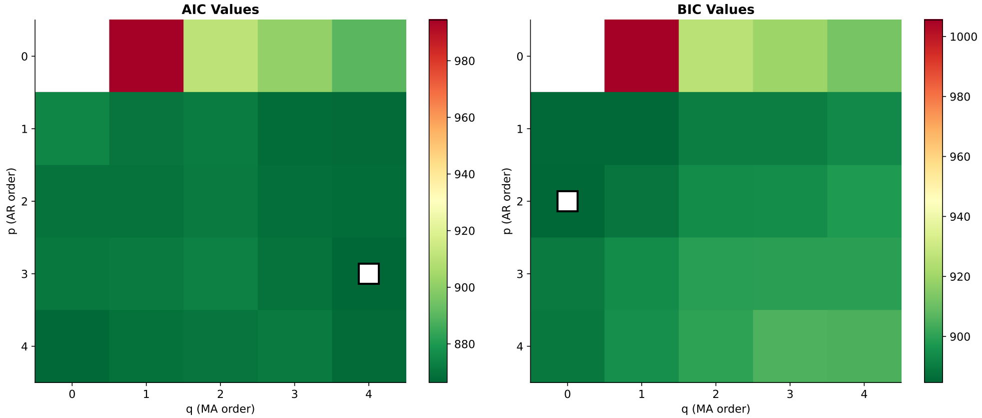
$$\text{BIC} = -2\ln(\hat{L}) + k\ln(n)$$

where $\hat{L}$ = maximized likelihood, k = number of parameters, n = sample size

Usage:

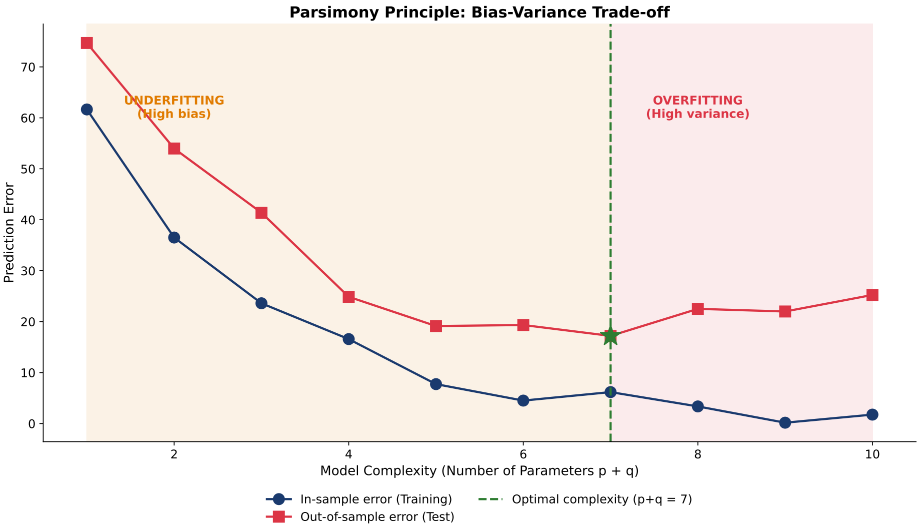
- Lower values are better
- BIC penalizes complexity more strongly than AIC
- AIC tends to choose larger models; BIC more parsimonious
- Compare models fit to the *same data*

AIC vs BIC: Model Selection



Note: White square marks the best model; lower values (green) are better

Parsimony Principle: Bias-Variance Trade-off



Automatic Model Selection

Grid search approach:

1. Fit ARMA(p,q) for $p = 0, 1, \ldots, p_{max}$ and $q = 0, 1, \ldots, q_{max}$
2. Select model with lowest AIC or BIC
3. Verify with diagnostic checks

In Python (statsmodels):

- `pm.auto_arima()` from `pmdarima` package
- Automatically tests stationarity, searches over orders
- Returns best model by AIC/BIC

Caution:

- Automatic selection is a starting point, not final answer
- Always check diagnostics
- Consider domain knowledge

Estimation Methods Overview

Three main approaches:

1. Method of Moments / Yule-Walker (AR only)

- Match sample autocorrelations to theoretical values
- Simple, closed-form for AR models
- Not efficient for MA components

2. Maximum Likelihood Estimation (MLE)

- Most common approach
- Requires distributional assumption (usually Gaussian)
- Efficient and consistent

3. Conditional Least Squares

- Minimize sum of squared residuals
- Conditioning on initial observations
- Computationally simpler than exact MLE

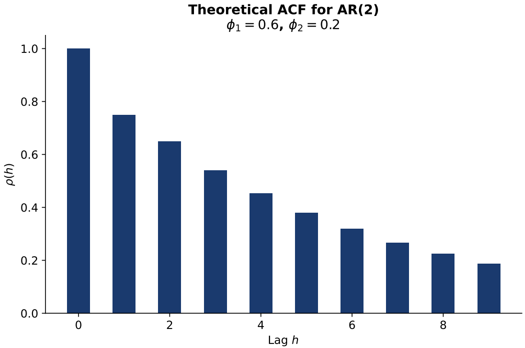
Estimation Methods Comparison

ARMA Parameter Estimation Methods

Yule-Walker	Maximum Likelihood	Conditional LS
Pros:	**Pros:**	**Pros:**
+ Simple computation	+ Most efficient	+ Simple to implement
+ Closed-form solution	+ Works for ARMA	+ Fast computation
Cons:	**Cons:**	**Cons:**
- AR only	- Iterative	- Biased for small n
- Less efficient	- Local optima risk	- Ignores initial values

**Recommendation: Use MLE for final estimation,
Yule-Walker for initial values**

Yule-Walker Equations for AR(p)



Yule-Walker Equations

$$\rho(1) = \phi_1 + \phi_2 \rho(1)$$

$$\rho(2) = \phi_1 \rho(1) + \phi_2$$

Matrix form: $R \cdot \phi = \rho$

R = autocorrelation matrix

Solution: $\hat{\phi} = R^{-1} \rho$

Yule-Walker Equations: Matrix Form

For AR(p): $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t$

Yule-Walker equations:

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p)$$

for $k = 1, 2, \ldots, p$

Matrix form:

$$\begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{pmatrix}$$

Estimation: Replace $\rho(k)$ with sample autocorrelations $\hat{\rho}(k)$

Maximum Likelihood Estimation

Assuming Gaussian errors: $\varepsilon_t \sim N(0, \sigma^2)$

Log-likelihood for ARMA(p,q):

$$\ell(\phi, \boldsymbol{\theta}, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{n} \varepsilon_t^2$$

where ε_t are the innovations computed recursively.

Estimation procedure:

1. Initialize: use method of moments or OLS for starting values
2. Optimize: numerical methods (e.g., BFGS, Newton-Raphson)
3. Iterate until convergence

In practice: Use `statsmodels.tsa.arima.model.ARIMA`

Standard Errors and Inference

Asymptotic distribution of MLE:

$$\hat{\boldsymbol{\theta}} \xrightarrow{d} N\left(\boldsymbol{\theta}_0, \frac{1}{n}\mathbf{I}(\boldsymbol{\theta}_0)^{-1}\right)$$

where $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher information matrix.

Standard errors: Square root of diagonal of $\frac{1}{n}\hat{\mathbf{I}}^{-1}$

Hypothesis testing:

- $H_0 : \phi_j = 0$ (or $\theta_j = 0$)
- Test statistic: $z = \frac{\hat{\phi}_j}{SE(\hat{\phi}_j)} \sim N(0,1)$ asymptotically
- Reject if $|z| > 1.96$ at 5% level

Confidence interval: $\hat{\phi}_j \pm 1.96 \cdot SE(\hat{\phi}_j)$

Residual Analysis

If model is correctly specified, residuals should be white noise:

1. Plot residuals over time

- Should fluctuate around zero
- No obvious patterns or trends
- Constant variance (no heteroskedasticity)

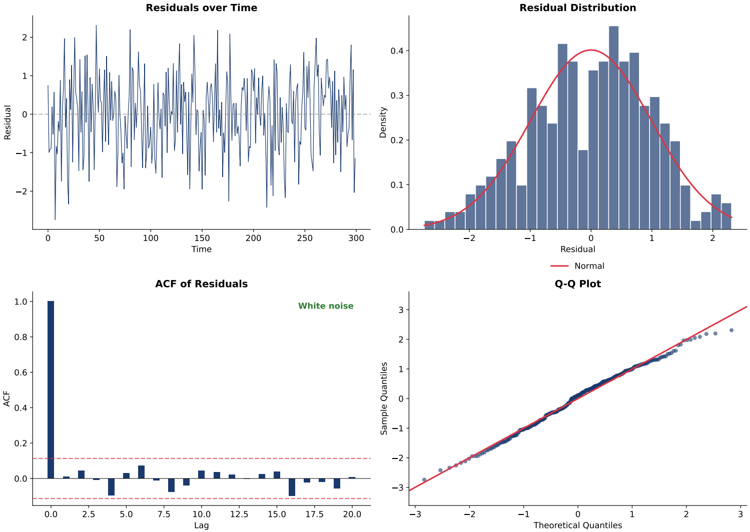
2. Check ACF of residuals

- All correlations should be within confidence bands
- No significant spikes $\rightarrow$ white noise

3. Check histogram / Q-Q plot

- Should be approximately normal (if assuming Gaussian)
- Heavy tails suggest non-normal errors

Residual Diagnostics: Example



Ljung-Box Test

Definition 11 (Ljung-Box Test)

Tests whether residuals are independently distributed (no autocorrelation).

Test statistic:

$$Q(m) = n(n+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{n-k}$$

Hypotheses:

- H_0: Residuals are white noise (no autocorrelation up to lag m)
- H_1: Residuals are autocorrelated

Distribution: Under H_0, $Q(m) \sim \chi^2(m-p-q)$ approximately

Decision:

- p-value $> 0.05 \rightarrow$ fail to reject $H_0 \rightarrow$ residuals look like white noise (good!)
- p-value $< 0.05 \rightarrow$ significant autocorrelation remains $\rightarrow$ model inadequate

Ljung-Box Test: Good vs Bad Model Fit

Diagnostic Checklist

A good ARMA model should satisfy:

1. **Stationarity:** AR roots outside unit circle
   ✓ Check with `arroots`
2. **Invertibility:** MA roots outside unit circle
   ✓ Check with `maroots`
3. **White noise residuals:** No significant ACF
   ✓ ACF plot, Ljung-Box test
4. **Normal residuals:** (if assumed)
   ✓ Q-Q plot, Jarque-Bera test
5. **No heteroskedasticity:** Constant variance
   ✓ Plot residuals, ARCH test
6. **Parsimonious:** Lowest AIC/BIC among adequate models

If diagnostics fail: Return to identification, try different orders

Point Forecasts

Optimal forecast: Conditional expectation minimizes MSE

$$\hat{X}_{n+h|n} = \mathbb{E}[X_{n+h}|X_n, X_{n-1}, \ldots]$$

For AR(1): $X_t = c + \phi X_{t-1} + \varepsilon_t$

$$\hat{X}_{n+1|n} = c + \phi X_n$$
$$\hat{X}_{n+2|n} = c + \phi \hat{X}_{n+1|n} = c(1 + \phi) + \phi^2 X_n$$
$$\hat{X}_{n+h|n} = \mu + \phi^h (X_n - \mu)$$

Key property: Forecasts converge to mean μ as $h \to \infty$

For MA(1): $X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

$$\hat{X}_{n+1|n} = \mu + \theta \varepsilon_n$$
$$\hat{X}_{n+h|n} = \mu \quad \text{for } h > 1$$

Forecast Uncertainty

Forecast error:

$$e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$$

Mean squared forecast error (MSFE):

$$\mathrm{MSFE}(h) = \mathbb{E}[e_{n+h|n}^2] = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$$

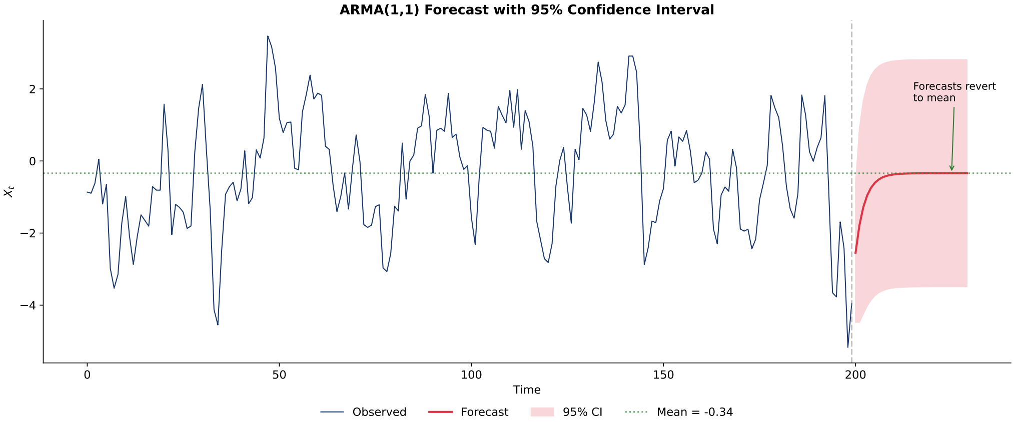
where ψ_j are the MA(∞) coefficients.

For AR(1): $\psi_j = \phi^j$

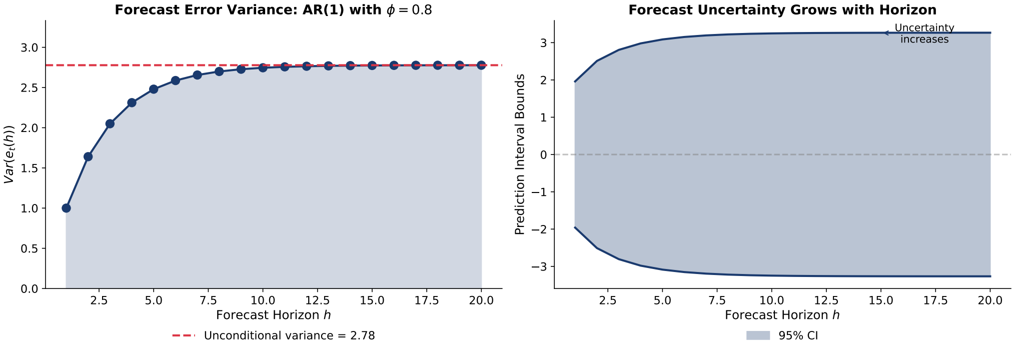
$$\mathrm{MSFE}(h) = \sigma^2 \frac{1-\phi^{2h}}{1-\phi^2} \to \frac{\sigma^2}{1-\phi^2} = \mathrm{Var}(X_t)$$

Key insight: Forecast uncertainty increases with horizon, eventually reaching unconditional variance

ARMA Forecasting with Confidence Intervals



Forecast Error Variance Over Horizon



Confidence Intervals for Forecasts

Assuming Gaussian errors:

$$X_{n+h}|X_n, \ldots \sim N\left(\hat{X}_{n+h|n}, \text{MSFE}(h)\right)$$

$(1-\alpha)$ **confidence interval:**

$$\hat{X}_{n+h|n} \pm z_{\alpha/2} \cdot \sqrt{\text{MSFE}(h)}$$

where $z_{\alpha/2} = 1.96$ for 95% CI.

Properties:

- Intervals widen as horizon increases
- Eventually converge to unconditional interval: $\mu \pm z_{\alpha/2}\sigma_X$
- Width depends on model parameters (AR coefficients, etc.)

In Python: `model.get_forecast(h).conf_int()`

Forecast Evaluation

Out-of-sample testing:

1. Split data: training set (fit model) and test set (evaluate)
2. Generate forecasts for test period
3. Compare forecasts to actual values

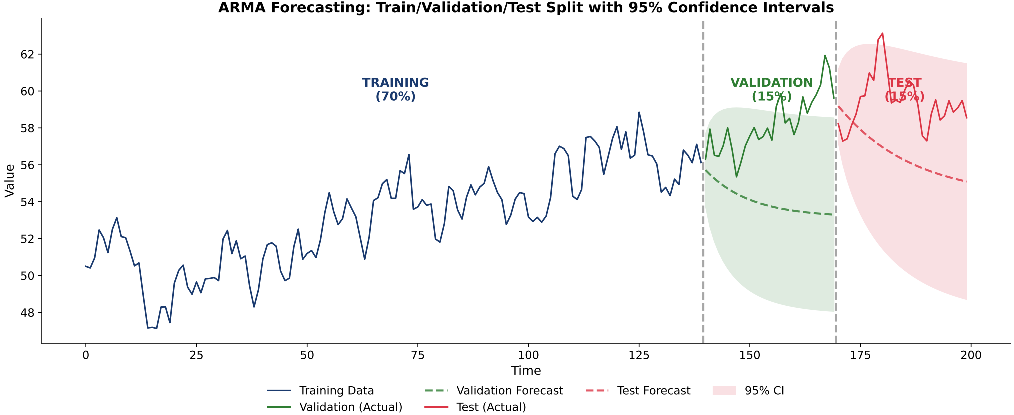
Metrics (from Chapter 1):

- $\text{MAE} = \frac{1}{n} \sum |e_t|$
- $\text{RMSE} = \sqrt{\frac{1}{n} \sum e_t^2}$
- $\text{MAPE} = \frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$

Rolling/expanding window:

- Re-estimate model as new data arrives
- More realistic assessment of forecast performance

Train/Validation/Test Forecasting Example



Python Implementation: Fitting ARMA

Using statsmodels:

```
from statsmodels.tsa.arima.model import ARIMA

# Fit ARMA(2,1) --- note:  ARIMA(p,d,q) with d=0
model = ARIMA(data, order=(2, 0, 1))
results = model.fit()

# Summary
print(results.summary())

# Forecasting
forecast = results.get_forecast(steps=10)
print(forecast.predicted_mean)
print(forecast.conf_int())
```

Note: ARIMA with $d = 0$ is equivalent to ARMA

Python: Model Selection with pmdarima

Automatic ARIMA selection:

```
import pmdarima as pm

# Auto ARIMA with AIC criterion
model = pm.auto_arima(data,
              start_p=0, max_p=5,
              start_q=0, max_q=5,
              d=0, # No differencing for stationary data
              seasonal=False,
              information_criterion='aic',
              trace=True)

print(model.summary())
```

Output: Best model order and fitted parameters

Workflow Summary

1. **Data preparation**
   - Check for missing values, outliers
   - Transform if necessary (log, differencing)
2. **Stationarity check**
   - Visual inspection: time plot, ACF
   - Formal tests: ADF, KPSS
   - Difference if non-stationary
3. **Model identification**
   - ACF/PACF patterns
   - Information criteria grid search
4. **Estimation and diagnostics**
   - Fit model, check significance
   - Residual analysis, Ljung-Box test
5. **Forecasting**
   - Point forecasts with confidence intervals
   - Out-of-sample validation

Key Takeaways

1. **AR(p) models:** Current value depends on p past values
   - Stationarity: roots of $\phi(z)$ outside unit circle
   - PACF cuts off at lag p
2. **MA(q) models:** Current value depends on q past shocks
   - Always stationary; invertibility: roots of $\theta(z)$ outside unit circle
   - ACF cuts off at lag q
3. **ARMA(p,q):** Combines AR and MA for flexible modeling
   - Both ACF and PACF decay
4. **Box-Jenkins:** Identify $\rightarrow$ Estimate $\rightarrow$ Diagnose $\rightarrow$ Forecast
5. **Diagnostics:** Residuals must be white noise
6. **Forecasts:** Converge to mean; uncertainty increases with horizon

Next Chapter Preview

Chapter 3: ARIMA and Seasonal Models

- ARIMA(p,d,q): Integrated models for non-stationary data
- Seasonal ARIMA: SARIMA(p,d,q)(P,D,Q)$_s$
- Seasonal differencing
- Real-world applications with seasonal patterns

Reading:

- Hyndman & Athanasopoulos, *Forecasting: Principles and Practice*, Ch. 9
- Box, Jenkins, Reinsel & Ljung, *Time Series Analysis*, Ch. 3-4