



# Chapter 3: ARIMA Models

Seminar



# Seminar Outline

1 Review Quiz

2 Practice Problems

3 Worked Examples

4 Discussion Topics

5 Summary

## Quiz 1: Integration Order

### Question

A time series  $Y_t$  requires two differences to become stationary. What is its order of integration?

- (A)  $I(0)$
- (B)  $I(1)$
- (C)  $I(2)$
- (D) Cannot be determined

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### Answer: C

By definition,  $Y_t \sim I(d)$  means  $d$  differences are needed to achieve stationarity. Since two differences are required,  $Y_t \sim I(2)$ .

## Quiz 2: Random Walk Properties

### Question

For a random walk  $Y_t = Y_{t-1} + \varepsilon_t$  with  $\text{Var}(\varepsilon_t) = \sigma^2$ , what is  $\text{Var}(Y_t)$ ?

- A)  $\sigma^2$
- B)  $t \cdot \sigma^2$
- C)  $\sigma^2/t$
- D)  $\sigma^2/(1 - \phi^2)$

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### Answer: B

Since  $Y_t = \sum_{i=1}^t \varepsilon_i$  (assuming  $Y_0 = 0$ ), we have  $\text{Var}(Y_t) = t \cdot \sigma^2$ . The variance grows linearly with time – a key feature of non-stationarity.

## Quiz 3: ADF Test Hypotheses

### Question

In the Augmented Dickey-Fuller test, what is the null hypothesis?

- (A) The series is stationary
- (B) The series has a unit root
- (C) The series has no autocorrelation
- (D) The series is normally distributed

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### Answer: B

The ADF test has  $H_0$ : unit root (non-stationary) vs  $H_1$ : stationary. We reject  $H_0$  if the test statistic is sufficiently negative (below critical value).

## Quiz 4: ARIMA Notation

### Question

What does ARIMA(2,1,1) mean?

- A) AR(2) on differenced data with MA(1) errors
- B) AR(1) with 2 differences and MA(1)
- C) MA(2) with 1 difference and AR(1)
- D) 2 lags, 1 trend, 1 seasonal component

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### Answer: A

ARIMA( $p,d,q$ ) means:  $p$ =AR order,  $d$ =differencing order,  $q$ =MA order. So ARIMA(2,1,1) has AR(2) and MA(1) components applied to the first-differenced series.

## Quiz 5: Difference Operator

### Question

What is  $(1 - L)^2 Y_t$  expanded?

- A)  $Y_t - Y_{t-1}$
- B)  $Y_t - 2Y_{t-1} + Y_{t-2}$
- C)  $Y_t + 2Y_{t-1} + Y_{t-2}$
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### Answer: B

$(1 - L)^2 = 1 - 2L + L^2$ , so  $(1 - L)^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$ . This is the second difference of  $Y_t$ .

## Quiz 6: KPSS vs ADF

### Question

How does the KPSS test differ from the ADF test?

- A) KPSS tests for seasonality, ADF tests for trends
- B) KPSS has stationarity as null, ADF has unit root as null
- C) KPSS is more powerful than ADF
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### Answer: B

The key difference is reversed hypotheses. KPSS:  $H_0$  = stationary. ADF:  $H_0$  = unit root. Using both tests together provides stronger evidence.

## Quiz 7: Overdifferencing

### Question

If  $Y_t \sim I(1)$  and we compute  $\Delta^2 Y_t$ , what happens?

- A) We get a better stationary series
- B) We introduce artificial negative autocorrelation
- C) The variance decreases
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### Answer: B

Overdifferencing creates an MA component at the invertibility boundary. If  $\Delta Y_t = \varepsilon_t$ , then  $\Delta^2 Y_t = \varepsilon_t - \varepsilon_{t-1}$ , which is MA(1) with  $\theta = -1$  (non-invertible).

## Quiz 8: Forecast Variance

### Question

For an ARIMA(0,1,0) model (random walk), how does forecast variance behave as horizon  $h$  increases?

- A) Stays constant
- B) Decreases to zero
- C) Grows linearly with  $h$
- D) Converges to a finite limit

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### Answer: C

For a random walk,  $\text{Var}(Y_{T+h} - \hat{Y}_{T+h|T}) = h\sigma^2$ . The forecast uncertainty grows without bound – a key characteristic of I(1) processes.

## Problem 1: Unit Root Testing

### Exercise

You have quarterly GDP data for 80 quarters. The ADF test (with constant and trend) gives a test statistic of  $-2.85$ . The 5% critical value is  $-3.41$ .

- ① What is your conclusion about stationarity?
- ② What would you do next?

## Problem 1: Unit Root Testing

### Exercise

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### Solution

- ① Since  $-2.85 > -3.41$ , we **fail to reject  $H_0$** . The data appears to have a unit root (non-stationary).
- ② Take the first difference  $\Delta Y_t$  and repeat the ADF test on the differenced series to confirm it is now stationary.

## Problem 2: Model Identification

### Exercise

After differencing a time series once, the ACF shows:

- Significant spike at lag 1 ( $\rho_1 = 0.4$ )
- All other lags insignificant

The PACF shows gradual decay.

What ARIMA model is suggested?

## Problem 2: Model Identification

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What ARIMA model is suggested?

### Solution

- ACF cuts off after lag 1  $\Rightarrow$  MA(1) component
- PACF decays  $\Rightarrow$  Confirms MA structure
- Since we differenced once:  $d = 1$

**Suggested model: ARIMA(0,1,1) or IMA(1,1)**

## Problem 3: ARIMA Equation

### Exercise

Write out the full equation for ARIMA(1,1,1):

$$(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$$

Expand this completely in terms of  $Y_t$ ,  $Y_{t-1}$ ,  $Y_{t-2}$ , etc.

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Expand this completely in terms of  $Y_t$ ,  $Y_{t-1}$ ,  $Y_{t-2}$ , etc.

### Solution

Expanding  $(1 - \phi_1 L)(1 - L) = 1 - L - \phi_1 L + \phi_1 L^2 = 1 - (1 + \phi_1)L + \phi_1 L^2$ :

$$Y_t - (1 + \phi_1)Y_{t-1} + \phi_1 Y_{t-2} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Or equivalently:

$$Y_t = c + (1 + \phi_1)Y_{t-1} - \phi_1 Y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

## Problem 4: Forecast Calculation

### Exercise

Given ARIMA(0,1,1):  $\Delta Y_t = \varepsilon_t + 0.3\varepsilon_{t-1}$

At time  $T$ :  $Y_T = 100$ ,  $\hat{\varepsilon}_T = 2$ ,  $\sigma^2 = 4$

Calculate:

- ①  $\hat{Y}_{T+1|T}$  (one-step forecast)
- ②  $\hat{Y}_{T+2|T}$  (two-step forecast)

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### Solution

- ①  $\hat{Y}_{T+1|T} = Y_T + 0.3\hat{\varepsilon}_T = 100 + 0.3(2) = \mathbf{100.6}$
- ②  $\hat{Y}_{T+2|T} = \hat{Y}_{T+1|T} + 0.3 \cdot 0 = 100.6 + 0 = \mathbf{100.6}$   
(Future shocks  $\varepsilon_{T+1}, \varepsilon_{T+2}$  are forecast as 0)

## Problem 5: Confidence Intervals

### Exercise

Continuing from Problem 4, calculate the 95% forecast intervals for  $\hat{Y}_{T+1|T}$  and  $\hat{Y}_{T+2|T}$ .

Recall:  $\sigma^2 = 4$ ,  $\theta_1 = 0.3$

## Problem 5: Confidence Intervals

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Recall:  $\sigma^2 = 4$ ,  $\theta_1 = 0.3$

### Solution

For IMA(1,1), the MA( $\infty$ ) weights are  $\psi_0 = 1$ ,  $\psi_j = 1 + \theta_1$  for  $j \geq 1$ .

**1-step:**  $\text{Var}(e_{T+1}) = \sigma^2 \psi_0^2 = 4$ , so  $SE = 2$

$$100.6 \pm 1.96(2) = [96.68, 104.52]$$

**2-step:**  $\text{Var}(e_{T+2}) = \sigma^2(\psi_0^2 + \psi_1^2) = 4(1 + 1.3^2) = 10.76$ ,  $SE = 3.28$

$$100.6 \pm 1.96(3.28) = [94.17, 107.03]$$

## Example: Testing for Unit Root in Stock Prices

### Scenario

You have daily closing prices for a stock over 500 days. You want to determine if prices follow a random walk.

### Step-by-step Approach

- ① **Visual inspection:** Plot prices – likely shows trend
- ② **ADF test on prices:** Expect to fail to reject  $H_0$  (unit root)
- ③ **Take log returns:**  $r_t = \ln(P_t/P_{t-1}) = \Delta \ln(P_t)$
- ④ **ADF test on returns:** Should reject  $H_0$  (stationary)
- ⑤ **Conclusion:** Log prices are  $I(1)$ , returns are  $I(0)$

## Example: Box-Jenkins for Inflation Data

### Scenario

Monthly inflation rates for 10 years. Build an ARIMA model.

### Workflow

- ① **Plot & test:** ADF suggests borderline – try both  $d = 0$  and  $d = 1$
- ② **If  $d = 0$ :** Fit ARMA models, compare AIC
- ③ **If  $d = 1$ :** Examine ACF/PACF of  $\Delta Y_t$ 
  - ACF: spike at lag 1, then cuts off
  - PACF: decays
  - $\Rightarrow$  Try ARIMA(0,1,1)
- ④ **Estimate:** Fit ARIMA(0,1,1), check coefficients
- ⑤ **Diagnose:** Ljung-Box on residuals (want  $p > 0.05$ )
- ⑥ **Compare:** AIC of ARIMA(0,1,1) vs ARMA(1,1) on levels

## Example: Interpreting Python Output

### statsmodels ARIMA Output

```
ARIMA Model Results
=====
Dep. Variable:      D.y    No. Observations:     99
Model:             ARIMA(1,1,1)    AIC:            285.32
                  BIC:            295.63
=====
                           coef    std err        z     P>|z|
-----
const            0.0521    0.048     1.085    0.278
ar.L1            0.4532    0.102     4.443    0.000
ma.L1           -0.2891    0.118    -2.450    0.014
sigma2          1.2340    0.176     7.011    0.000
```

### Interpretation

- AR coefficient (0.45) is significant, MA coefficient (-0.29) is significant
- Constant (0.052) not significant – could set  $c = 0$
- Check:  $|\phi_1| < 1$  (stationary),  $|\theta_1| < 1$  (invertible) – OK!

# Discussion: Deterministic vs Stochastic Trends

## Key Question

Why is it important to distinguish between deterministic and stochastic trends?

## Discussion Points

- **Wrong treatment consequences:**
  - Detrending a unit root  $\Rightarrow$  spurious stationarity
  - Differencing a trend-stationary  $\Rightarrow$  overdifferencing
- **Economic interpretation:**
  - Deterministic trend: shocks are temporary
  - Stochastic trend: shocks have permanent effects
- **Policy implications:**
  - Does a recession permanently lower GDP, or does the economy return to trend?

## Key Question

When should you use AIC vs BIC for ARIMA model selection?

## Considerations

- **AIC:** Minimizes prediction error, may overfit
  - Better for forecasting
  - Tends to select larger models
- **BIC:** Consistent model selection, more parsimonious
  - Better for identifying “true” model
  - Penalizes complexity more heavily
- **Practical advice:** Report both, prefer BIC if they disagree substantially

# Discussion: Limitations of ARIMA

## Key Question

What are the main limitations of ARIMA models?

## Discussion Points

- **Linearity:** Cannot capture nonlinear dynamics
- **Constant variance:** Assumes homoskedasticity (no GARCH effects)
- **No structural breaks:** Parameters assumed constant
- **Univariate:** Ignores relationships with other variables
- **Symmetric:** Treats positive and negative shocks equally
- **Long-horizon forecasts:** Uncertainty grows rapidly

## Extensions

These limitations motivate GARCH (volatility), VAR (multivariate), regime-switching models, etc.

## What We Covered

- ① **Integration and differencing:**  $I(d)$  processes require  $d$  differences
- ② **Unit root testing:** ADF tests  $H_0$ : unit root; KPSS tests  $H_0$ : stationary
- ③ **ARIMA(p,d,q):** Combines ARMA with differencing
- ④ **Model identification:** Use ACF/PACF patterns and information criteria
- ⑤ **Forecasting:** Point forecasts and growing confidence intervals

## Next Seminar

Hands-on Python exercises with real economic data:

- Unit root testing with `statsmodels`
- Auto-ARIMA with `pmdarima`
- Forecasting and model diagnostics