



Time Series Analysis and Forecasting

Chapter 0: Fundamentals



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Learning Objectives

By the end of this chapter, you will be able to:

1. Define time series and distinguish them from cross-sectional and panel data
2. Decompose time series into trend-cycle, seasonality, and residual components
3. Apply exponential smoothing methods (SES, Holt, Holt-Winters, ETS)
4. Evaluate forecasts using MAE, RMSE, MAPE, sMAPE
5. Implement train/validation/test splitting and cross-validation
6. Model seasonality using dummy variables or Fourier terms
7. Remove trend and seasonality through appropriate methods
8. Distinguish between deterministic and stochastic trends



Data Sources and Software Tools

Data Sources

- **Yahoo Finance**
 - ▶ Stock prices, cryptocurrencies, exchange rates
- **FRED (Federal Reserve)**
 - ▶ GDP, unemployment, interest rates
- **Eurostat / INS / BNR**
 - ▶ European and Romanian economic data
- **Classic datasets**
 - ▶ AirPassengers, Sunspots, CO₂

Python

- `yfinance` — Yahoo Finance data
- `pandas_datareader` — FRED, Eurostat
- `statsmodels` — statistical models
- `pandas` — data manipulation
- `matplotlib` — visualization

R

- `quantmod` — financial data
- `tseries` — time series tests
- `forecast` — forecasting models
- `fredr` — FRED API access



Chapter Outline



Time Series Are Everywhere

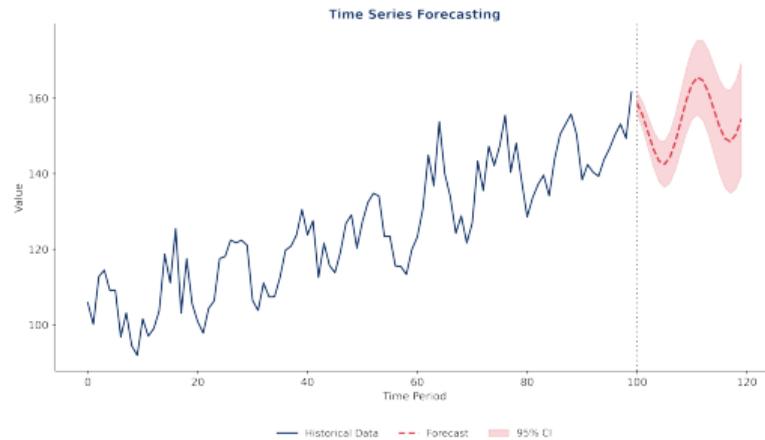


- **Finance**: Stock prices, exchange rates, volumes
- **Economics**: GDP, unemployment, inflation rates
- **Business**: Sales, website traffic, demand
- **Science**: Temperature, pollution, vital signs

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Why Do We Study Time Series?



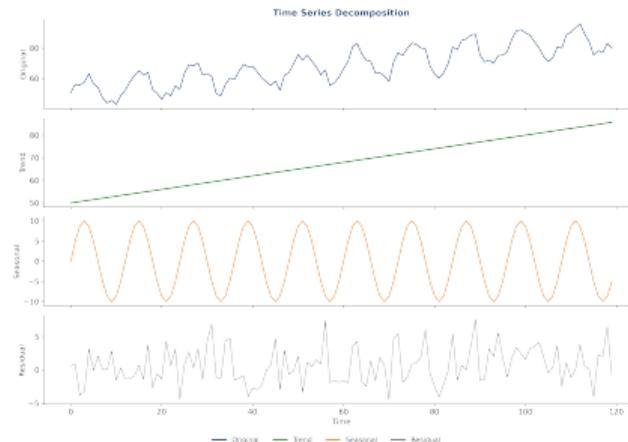
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Main objective: forecasting

- We use historical patterns to predict future values → essential for business planning, risk management, and policy decisions



Understanding Time Series Structure



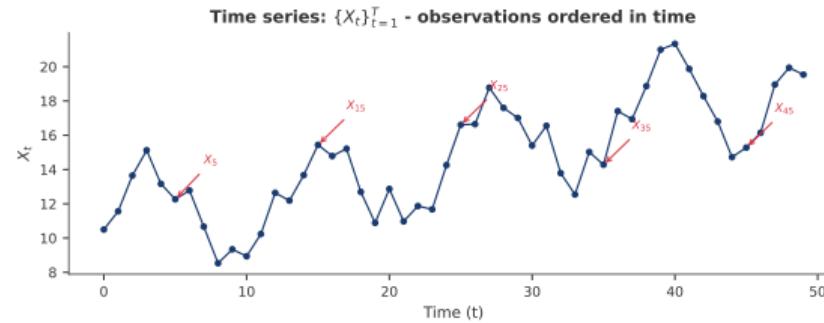
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Decomposition

- Any time series can be decomposed into: **trend-cycle + seasonality + noise**



Time Series: Conceptual Illustration



Q TSA_ch0_definition

Fundamental Elements

- Formal notation:** X_t = value at time t , $t \in \{1, 2, \dots, T\}$
- Autocorrelation:** $\rho_k = \text{Corr}(X_t, X_{t-k})$ — measures temporal dependence

Definition of a Time Series

Definition 1 (Time Series)

- **Time series:** a sequence of observations $\{X_t\}$ indexed by time: $\{X_t : t \in \mathcal{T}\}$ where \mathcal{T} is a set of indices representing time points

Key Characteristics

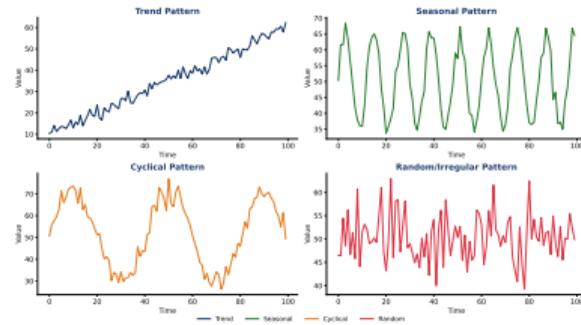
- **Ordered:** natural temporal order
- **Dependent:** consecutive observations are correlated
- **Discrete/Continuous:** $t = 1, 2, 3, \dots$

Notation

- X_t : observation at time t
- $\{X_t\}_{t=1}^T$: series with T observations



Common Patterns in Time Series



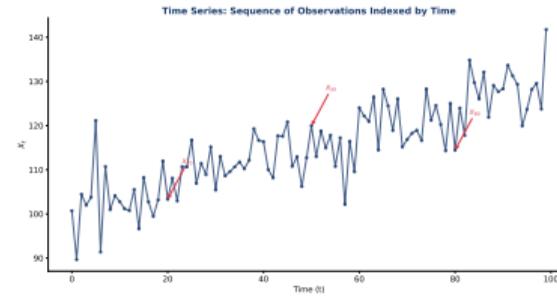
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Types of Patterns

- Trend:** long-term increase or decrease
- Seasonal:** regular periodic patterns
- Cyclical:** medium-term fluctuations (2–10 years)
- Random:** unpredictable fluctuations



Practical Example: Real Financial Data



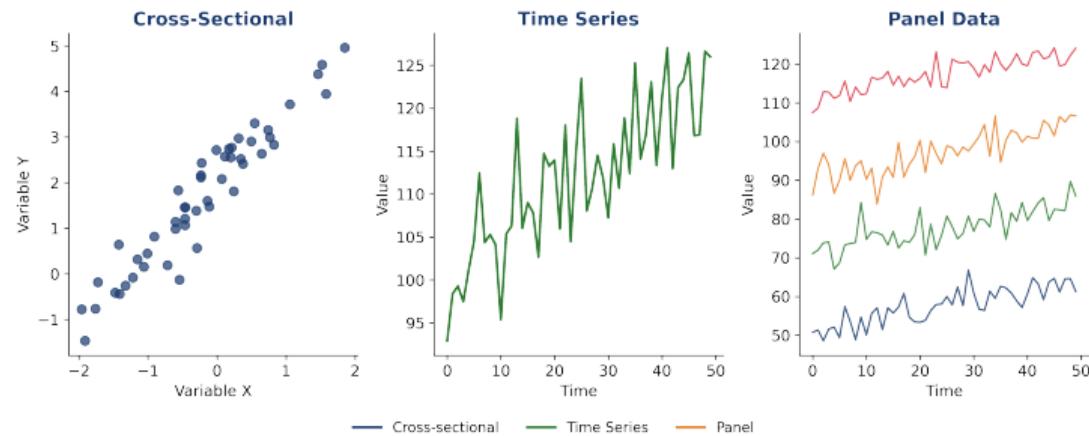
Q TSA_ch0_definition

S&P 500 (2024)

- Daily frequency:** ≈ 252 trading days/year
- Observed characteristics:** upward trend, volatility clustering, persistence (momentum)



Data Types: Comparison



| Data Type | Units (N) | Time (T) | Example |
|-----------------|---------------|--------------|-------------------------------|
| Cross-sectional | Many | 1 | Survey of 1000 households |
| Time series | 1 | Many | Daily S&P 500 prices |
| Panel | Many | Many | GDP of 50 countries, 20 years |

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Examples of Time Series Data



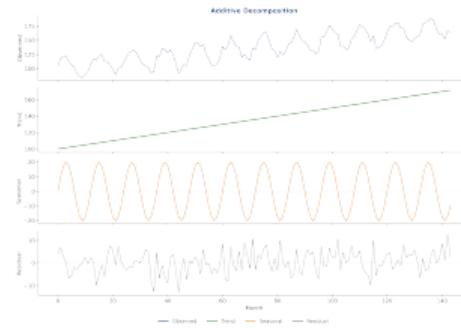
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Real financial data

- Source:** Yahoo Finance (2019–2025), normalized to base 100
- Bitcoin:** most volatile; **Gold:** most stable



Time Series Decomposition: Visual Example



TSA_ch0_decomposition

Components Explained

- Original:** observed series
- Trend-Cycle:** long-term movement
- Seasonal:** periodic pattern
- Residual:** random noise



Why Do We Decompose a Time Series?

Objectives

- ◻ Understanding underlying patterns
- ◻ Removing seasonality for modeling
- ◻ Identifying trend direction
- ◻ Isolating irregular fluctuations
- ◻ Improving forecast accuracy

Components

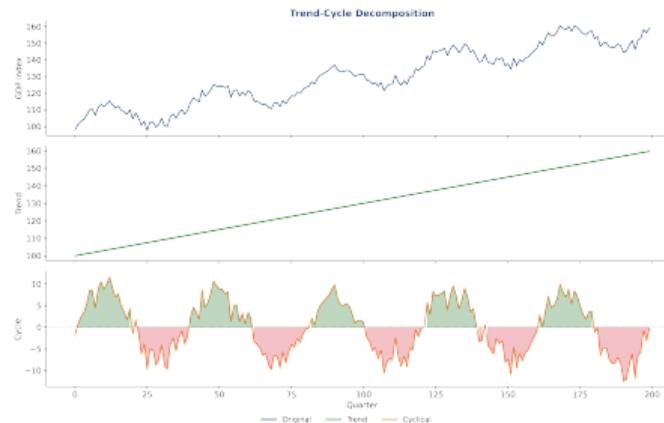
- ◻ T_t : Trend-Cycle
 - ▶ Long-term movement
- ◻ S_t : Seasonal
 - ▶ Regular periodic pattern
- ◻ ε_t : Residual
 - ▶ Random noise

Classical decomposition models

- ◻ **Additive:** $X_t = T_t + S_t + \varepsilon_t$
 - ▶ Constant seasonal amplitude
- ◻ **Multiplicative:** $X_t = T_t \times S_t \times \varepsilon_t$
 - ▶ Seasonal amplitude grows with the level



The Cyclical Component



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Characteristics

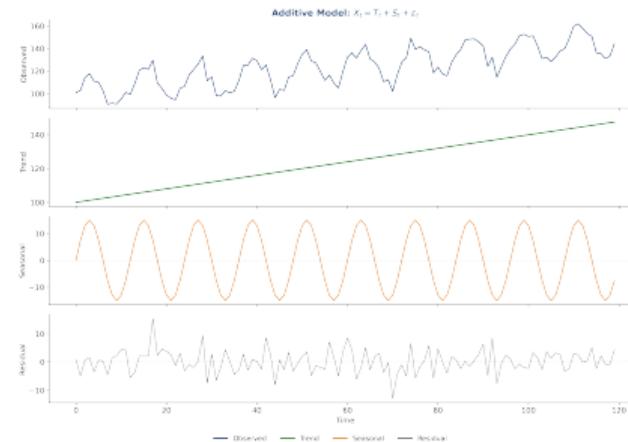
- **Duration:** medium-term fluctuations (2–10 years)
- **Aperiodic:** no fixed period (vs seasonality)
- **Origin:** reflects business cycles

In Practice

- **Combination:** cycle combined with trend
- **Difficulty:** hard to identify in short series
- **Solution:** usually absorbed into trend-cycle



Additive Decomposition: Visualization



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Interpretation

- **Decomposition:** Original = Trend + Seasonal + Residual
- **Property:** constant seasonal amplitude, does not depend on the level



The Additive Decomposition Model

Model

- **Equation:** $X_t = T_t + S_t + \varepsilon_t$
 - ▶ Components are added together to form the observed series

When to Use

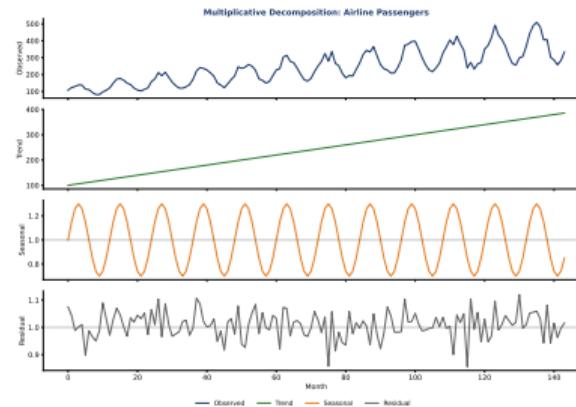
- **Constant seasonal fluctuations**
 - ▶ Amplitude does not depend on the level
- **Stable series variance**
 - ▶ Measures dispersion around the mean
 - ▶ Estimator: $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$

Properties

- **Error:** $\mathbb{E}[\varepsilon_t] = 0$ (zero mean)
- **Seasonal:** $\sum_{j=1}^s S_j = 0$ (seasonal sum is zero)
- **Units:** S_t are the same as X_t



Multiplicative Decomposition: Real Data



Example

- Box-Jenkins data: monthly passengers (1949–1960). Seasonal amplitude increases with the level

TSA_ch0_decomposition



The Multiplicative Decomposition Model

Model

- Equation: $X_t = T_t \times S_t \times \varepsilon_t \rightarrow$ components are multiplied

When to Use

- Growing fluctuations: seasonality increases with the level
- Heteroscedasticity: variance increases over time
- Examples: economic/financial data

Properties

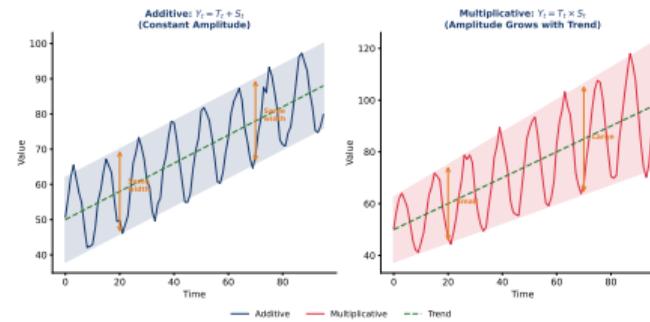
- Error: $\mathbb{E}[\varepsilon_t] = 1$ (centered at 1)
- Seasonal: $\frac{1}{s} \sum_{j=1}^s S_j = 1$ (mean is 1)
- Units: S_t is a dimensionless ratio

Tip

- Log transformation: multiplicative \rightarrow additive: $\log X_t = \log T_t + \log S_t + \log \varepsilon_t$



Additive vs Multiplicative: Comparison



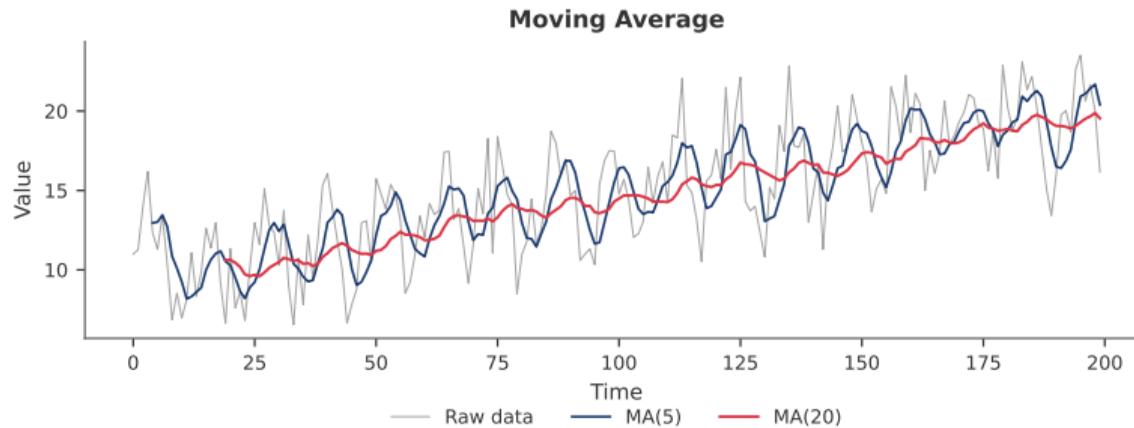
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Key Difference

- Multiplicative:** seasonal component is a *ratio*, centered at value 1
- Additive:** seasonal component in *absolute units*, centered at value 0



Centered Moving Average: Visual Illustration



Interpretation

- **Smoothing:** removes short-term fluctuations
- **Result:** reveals the underlying trend



Trend Estimation: Moving Average

Definition 2 (Centered Moving Average)

- **Centered moving average** of order $2q + 1$:

$$\hat{T}_t = \frac{1}{2q+1} \sum_{j=-q}^q X_{t+j} \quad (1)$$

For Seasonal Data

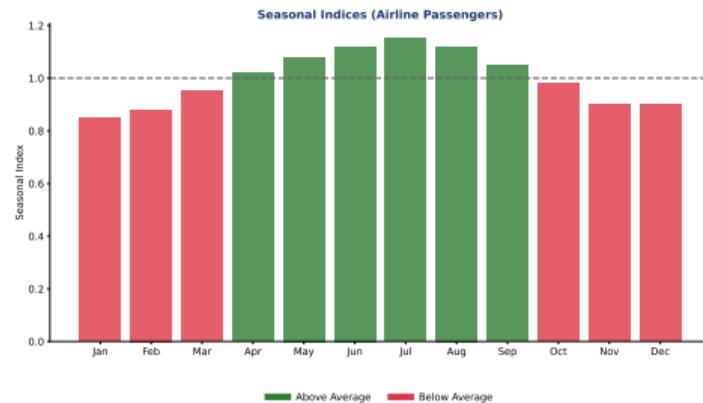
- **Odd period s**
 - ▶ Use simple average
- **Even period s**
 - ▶ $2 \times s$ MA with half weights

Properties

- **Smoothing**: removes seasonal & random components
- **Large window** → smoother estimate
- **Disadvantage**: data loss at endpoints



Seasonal Indices: Interpretation



Interpretation

- $S_t > 1$: above-average activity; $S_t < 1$: below average. Travel peak in July–August

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Classical Decomposition Algorithm

Steps for Multiplicative Decomposition

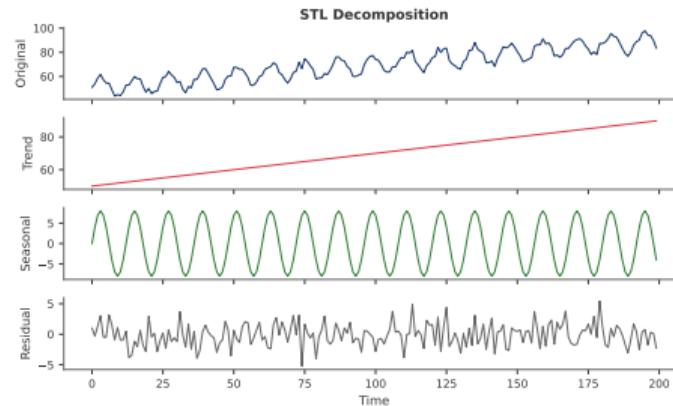
- **Step 1 → Estimate Trend:** $\hat{T}_t = MA_s(X_t)$
 - ▶ Centered moving average of order equal to the seasonal period
- **Step 2 → Detrend:** $D_t = X_t / \hat{T}_t$
- **Step 3 → Estimate Seasonal:** $\hat{S}_j = \text{mean}(D_t \text{ for season } j)$
- **Step 4 → Normalize:** scale so that $\frac{1}{s} \sum_{j=1}^s \hat{S}_j = 1$
- **Step 5 → Compute Residuals:** $\hat{\varepsilon}_t = X_t / (\hat{T}_t \times \hat{S}_t)$

Note

- **For additive decomposition:** operations change
 - ▶ Division → subtraction
 - ▶ Multiplication → addition



STL Decomposition: Visual Illustration



Key Idea

- ☐ STL (Seasonal-Trend-Loess): separates trend + seasonal + remainder using LOESS regression

 [TSA_ch0_decomposition](#)



STL Decomposition: A Modern Approach

Definition 3 (STL - Seasonal-Trend Decomposition using LOESS)

- **STL:** uses locally weighted regression (LOESS): $X_t = T_t + S_t + R_t$

Advantages

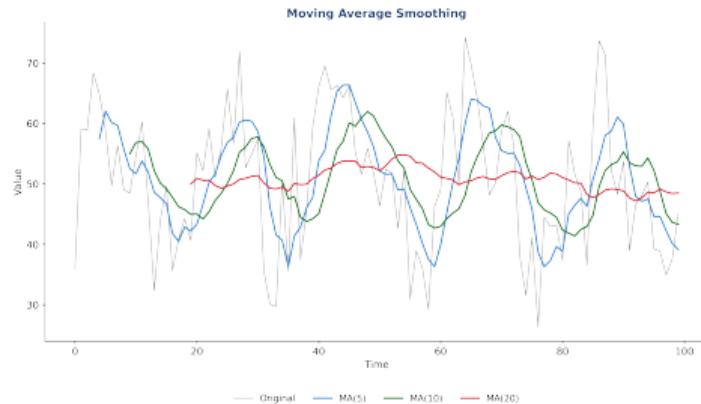
- **Flexibility:** any seasonal period
- **Variability:** seasonality can evolve over time
- **Robustness:** resistant to outliers
- **Smoothing:** smooth trend estimates

Key Parameters

- **period:** seasonal period
 - ▶ E.g.: 12 for monthly data, 4 for quarterly
- **seasonal:** smoothing window
- **robust:** reduced weight for outliers



Moving Average Smoothing



Window Size Trade-off

- Small window:** reactive but noisy
 - ▶ Captures rapid changes, but amplifies noise
- Large window:** smooth but lagging
 - ▶ Removes noise, but reacts slowly



Exponential Smoothing: Overview

Definition

- **Exponential smoothing:** weighted averages of past observations
 - ▶ Weights decrease exponentially over time
 - ▶ Recent observations receive higher weights

Why Exponential Smoothing?

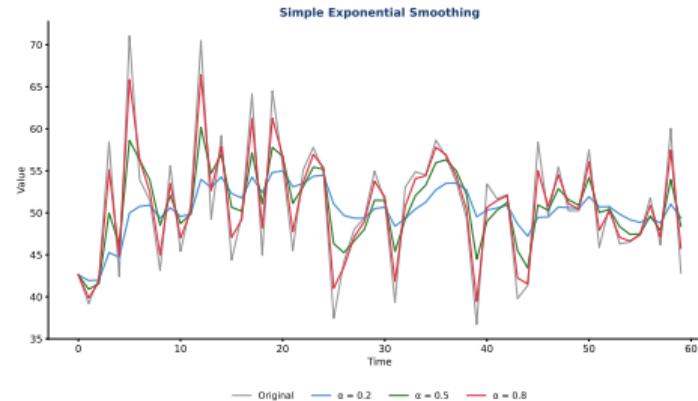
- **Simple:** easy to implement and understand
 - ▶ A single smoothing parameter
- **Adaptive:** higher weights for recent data
- **Versatile:** handles trend and seasonality

Three Main Methods

- **SES** (Simple Exponential Smoothing): level only
 - ▶ The simplest exponential method
- **Holt**: level + trend
 - ▶ Captures the direction of evolution
- **Holt-Winters**: + seasonality
 - ▶ Complete model with all components



Simple Exponential Smoothing: Effect of α



Trade-off

- Small $\alpha \rightarrow$ smooth forecasts
 - ▶ More weight on distant history
- Large $\alpha \rightarrow$ tracks the data
 - ▶ Fast reaction to recent changes



Simple Exponential Smoothing (SES)

Model

- **Equation:** $\hat{X}_{t+1|t} = \alpha X_t + (1 - \alpha)\hat{X}_{t|t-1}$
 - ▶ $\alpha \in (0, 1)$ is the smoothing parameter

How It Works

- **Principle:** weights decrease exponentially
- **Large α**
 - ▶ Forecast reactive to changes
- **Small α**
 - ▶ Smoother, more stable forecast

Level Form

- **Equation:** $\ell_t = \alpha X_t + (1 - \alpha)\ell_{t-1}$
 - ▶ ℓ_t = estimated level at time t
 - ▶ Forecast: $\hat{X}_{t+h|t} = \ell_t$ (constant)



SES: Step-by-Step Numerical Example

Data: Monthly Sales (thousands EUR)

□ Data: $X_1 = 100, X_2 = 110, X_3 = 105, X_4 = 115, X_5 = 120$ ($\alpha = 0.3, \hat{X}_{1|0} = 100$)

Iterative computation: $\hat{X}_{t+1|t} = \alpha X_t + (1 - \alpha) \hat{X}_{t|t-1}$

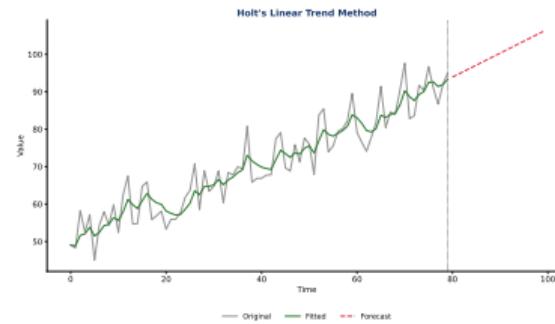
| t | X_t | $\hat{X}_{t t-1}$ | e_t | Computation $\hat{X}_{t+1 t}$ |
|-----|-------|-------------------|-------|---|
| 1 | 100 | 100.00 | 0.00 | $0.3 \times 100 + 0.7 \times 100 = 100.00$ |
| 2 | 110 | 100.00 | 10.00 | $0.3 \times 110 + 0.7 \times 100 = 103.00$ |
| 3 | 105 | 103.00 | 2.00 | $0.3 \times 105 + 0.7 \times 103 = 103.60$ |
| 4 | 115 | 103.60 | 11.40 | $0.3 \times 115 + 0.7 \times 103.6 = 107.02$ |
| 5 | 120 | 107.02 | 12.98 | $0.3 \times 120 + 0.7 \times 107.02 = 110.91$ |

Forecast and Evaluation

$\hat{X}_{6|5} = 110.91$ MAE = 7.28 RMSE = 8.97



Holt's Method: Visualization



 TSA_ch0_smoothing

Interpretation

- **Holt's method:** captures level and trend, projects them into the forecast horizon
- α : controls level changes; β^* : controls trend changes



Holt's Linear Trend Method

Equations

- Level:** $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$
- Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$
- Forecast:** $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t$
 - ▶ Extrapolates the linear trend over h steps

Parameters

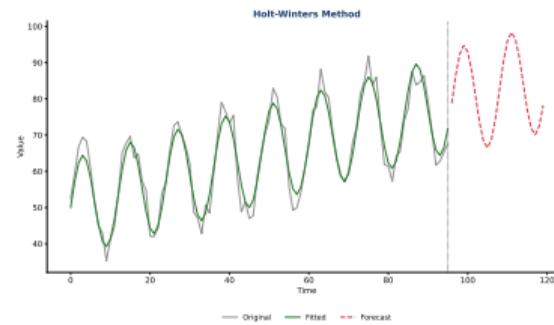
- α : level smoothing
 - ▶ Controls reactivity to level changes
- β^* : trend smoothing
 - ▶ Controls reactivity to slope changes

Components

- ℓ_t : estimated level
 - ▶ Local mean of the series
- b_t : estimated trend (slope)
 - ▶ Rate of increase/decrease



Holt-Winters: Capturing Seasonality



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Key Feature

- Complete decomposition:** separates level, trend, and seasonal
- Seasonal forecasts:** includes both trend and periodic pattern



Holt-Winters Seasonal Method

Equations (Additive Seasonality)

- **Level:** $\ell_t = \alpha(X_t - S_{t-s}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})$
- **Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$
- **Seasonal:** $S_t = \gamma(X_t - \ell_t) + (1 - \gamma)S_{t-s}$
- **Forecast:** $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t + S_{t+h-s(k+1)}$
 - ▶ Where $k = \lfloor (h-1)/s \rfloor$

Parameters

- α — level
- β^* — trend
- γ — seasonal
- s — seasonal period
 - ▶ All in $(0, 1)$; estimated by minimizing error



The ETS Framework: Error-Trend-Seasonality

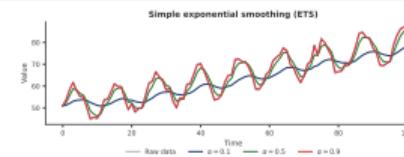
Definition 4 (ETS Models)

- **ETS framework:** generalizes exponential smoothing: $\text{ETS}(E, T, S)$

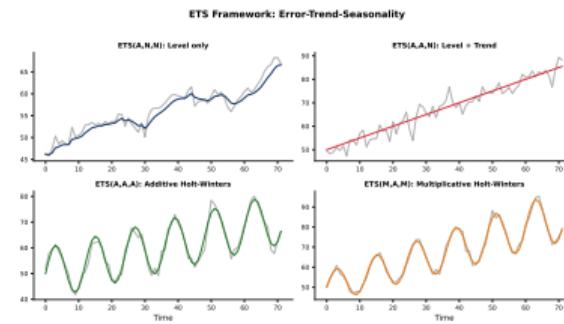
| Component | N | A | M |
|--------------|------|----------|----------------|
| Error (E) | — | Additive | Multiplicative |
| Trend (T) | None | Additive | Multiplicative |
| Seasonal (S) | None | Additive | Multiplicative |

Examples

- **ETS(A,N,N):** Simple Exponential Smoothing → level only, no trend or seasonality
- **ETS(A,A,N):** Holt's Linear Method → level + additive trend
- **ETS(A,A,A):** Additive Holt-Winters → level + trend + additive seasonality



ETS Model Selection



TSA_ch0_smoothing

Automatic Selection

- Information criteria:** AIC (Akaike) and BIC (Bayesian)
- Optimal selection:** balance between fit and complexity



Damped Trend Methods

Damping Parameter

- **Parameter:** $\phi \in (0, 1)$
 - ▶ Prevents over-projection of the trend
 - ▶ Trend converges to a constant

Equations

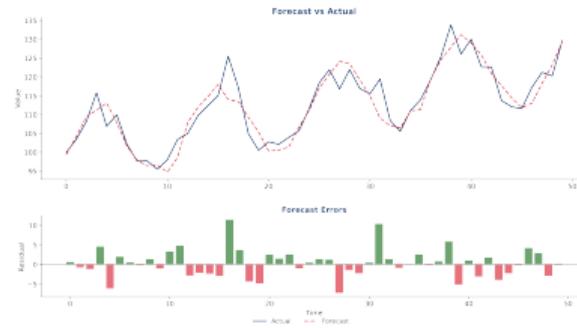
- **Level:** $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$
- **Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$
- **Forecast:** $\hat{X}_{t+h|t} = \ell_t + \phi^{\frac{1-\phi^h}{1-\phi}} b_t$

Key Idea

- **Asymptotic:** as $h \rightarrow \infty$, forecast \rightarrow constant
 - ▶ Prevents unrealistic long-term extrapolation
- **Advantage:** often better for long horizons



Forecast Evaluation: Visual Example



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Observations

- Top:** actual vs. forecast — visual assessment of forecast quality
- Bottom:** residuals — zero mean, constant variance, no pattern



Forecast Accuracy Metrics

Forecast Error

- **Definition:** $e_t = X_t - \hat{X}_t$ (actual minus predicted)
 - ▶ Positive \Rightarrow underestimates; Negative \Rightarrow overestimates

Scale-dependent

- **MAE:** $\frac{1}{n} \sum |e_t|$
- **MSE:** $\frac{1}{n} \sum e_t^2$
- **RMSE:** $\sqrt{\text{MSE}}$

Scale-independent

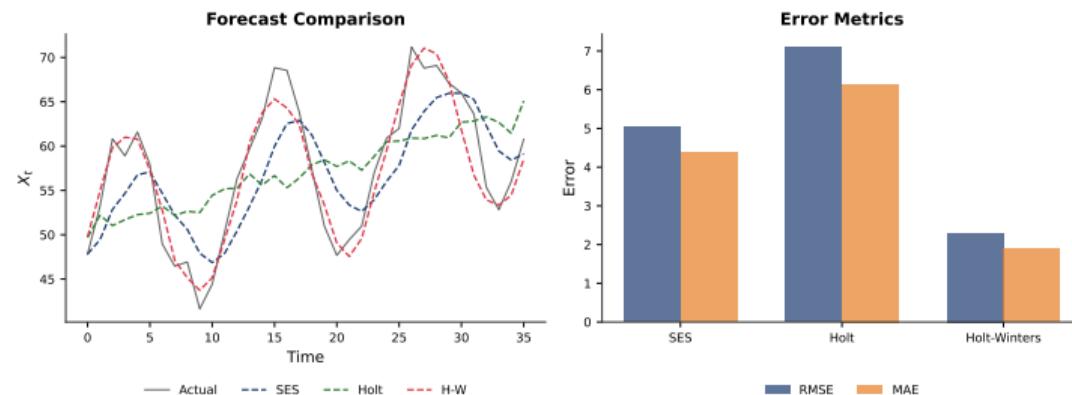
- **MAPE:** $\frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$
- **sMAPE:** $\frac{100}{n} \sum \frac{|e_t|}{(|X_t| + |\hat{X}_t|)/2}$

What to Use?

- **Same series:** RMSE, MAE \rightarrow compare models on the same data
- **Across different series:** MAPE, sMAPE \rightarrow percentage metrics, scale-independent



Comparing Forecast Methods



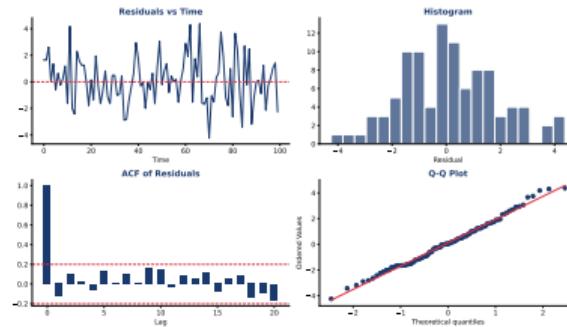
Interpretation

- Left: SES, Holt, Holt-Winters forecasts. Right: Error metrics. Visual and quantitative comparison

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Residual Diagnostics: Visualization



TSA_ch0_forecast_eval

What to Check

- Time plot:** no systematic patterns
- Histogram:** normality check
- ACF:** no significant autocorrelation
- Q-Q plot:** normality confirmation



Residual Diagnostics

Residual Properties

- Zero mean:** $\mathbb{E}[e_t] = 0$
 - ▶ Forecast has no systematic bias
- Uncorrelated:** $\text{Cov}(e_t, e_{t-k}) = 0$
 - ▶ No unexploited information remains
- Constant variance:** $\text{Var}(e_t) = \sigma^2$
- Normally distributed:** for confidence intervals

Diagnostic Tests

- Ljung-Box test** (autocorrelation):
 - ▶ $Q = T(T+2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{T-k} \sim \chi_h^2$
- Jarque-Bera test** (normality):
 - ▶ $JB = \frac{T}{6} \left(S^2 + \frac{(K-3)^2}{4} \right) \sim \chi_2^2$
 - ▶ $S = \text{skewness}$, $K = \text{kurtosis}$



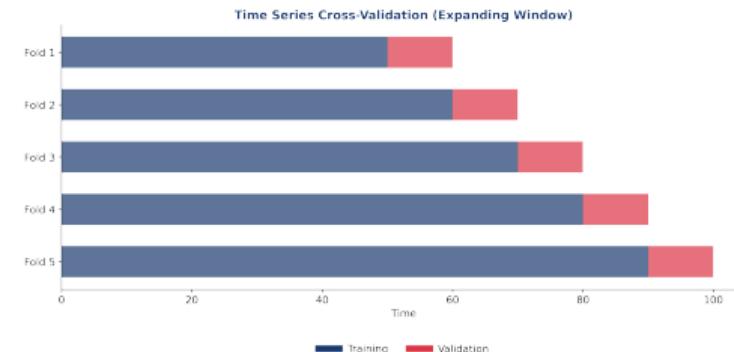
Cross-Validation for Time Series

Why Not Standard CV?

- Temporal dependence:** observations are correlated
- Order matters:** chronology must be respected
- Standard k-fold** → data leakage

CV with Rolling Origin

- Step 1:** train on $\{X_1, \dots, X_t\}$
- Step 2:** forecast \hat{X}_{t+h}
- Step 3:** increment t , repeat



Train / Validation / Test Split

Training Set

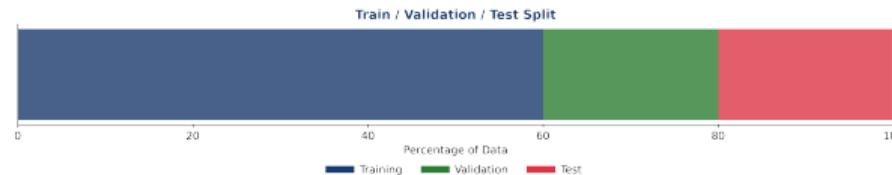
- Fitting model parameters
- Largest portion (60–80%)
- Used for estimation

Validation Set

- Hyperparameter tuning
- Comparing models
- Selecting the best approach

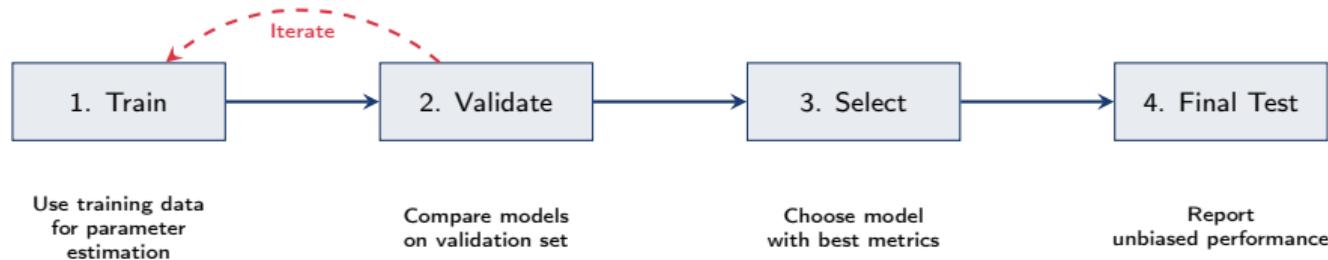
Test Set

- Final evaluation only
- Never used for tuning
- Unbiased performance



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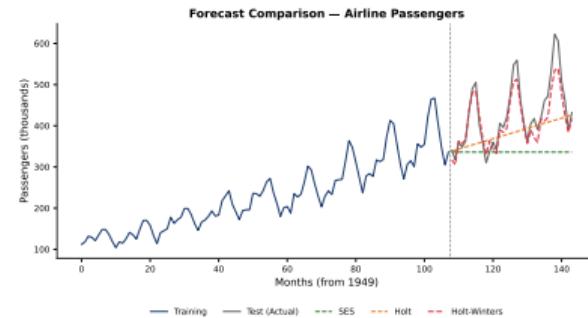
Model Development Workflow



Critical Rule

- Never use the test set for selection!**
 - ▶ Use it only for final evaluation
- Avoid data leakage**
 - ▶ Overly optimistic performance estimates

Real Data: Comparing Forecasts



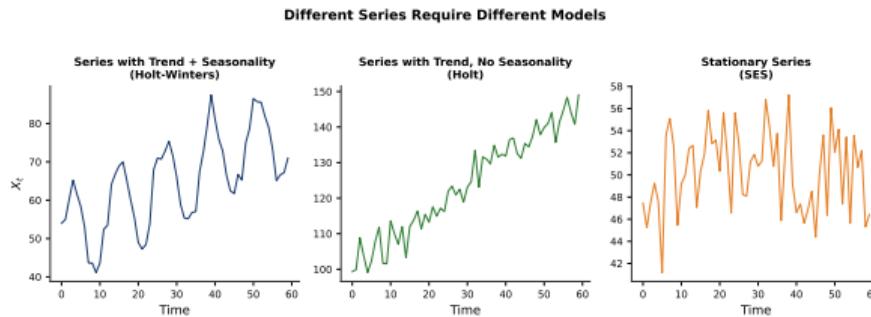
Q TSA_ch0_forecast_eval

Interpretation

- Data:** airline passengers
- Best:** multiplicative Holt-Winters — ideal for data with growing seasonality



Forecast Performance Across Different Datasets



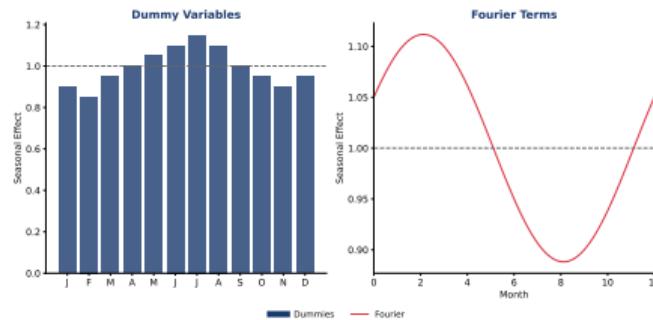
TSA_ch0_forecast_eval

Interpretation

- **Different series:** require different models
- **Seasonal data:** prefer seasonal methods
- **No universal model:** test multiple approaches



Dummy Variables vs Fourier Terms



Q TSA_ch0_seasonal

Comparison

- **Dummy variables:** capture any shape, require $s - 1$ parameters
- **Fourier terms:** only $2K$ parameters, smooth sinusoidal patterns

Modeling Seasonality: Two Approaches

1. Dummy Variables

- ◻ **Model:** $X_t = \mu + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$
- ◻ $D_{jt} = 1$ if t in season j
- ◻ $s - 1$ parameters
- ◻ Any seasonal pattern

2. Fourier Terms

- ◻ **Model:**
$$X_t = \mu + \sum_{k=1}^K \left[\alpha_k \sin\left(\frac{2\pi kt}{s}\right) + \beta_k \cos\left(\frac{2\pi kt}{s}\right) \right]$$
- ◻ Sinusoidal functions
- ◻ $2K$ parameters
- ◻ Smooth patterns

Trade-off

- ◻ **Dummy variables**
 - ▶ Any seasonal pattern, but more parameters
- ◻ **Fourier terms**
 - ▶ Smooth patterns, fewer parameters



Choosing Between Dummy and Fourier

| Criterion | Dummy | Fourier |
|------------------------|--------------------------|----------------------|
| Parameters (monthly) | 11 | $2K$ (often 4–6) |
| Seasonal pattern | Any shape | Smooth/sinusoidal |
| Interpretation | Direct (monthly effects) | Frequency components |
| High-frequency seasons | Many parameters | Efficient |
| Multiple seasonality | Complex | Easy (add terms) |

Recommendations

- Use Dummy**
 - ▶ Irregular patterns, interpretable coefficients
- Use Fourier**
 - ▶ Smooth patterns, high-frequency seasonality
 - ▶ Used in TBATS and Prophet



Why Do We Remove Trend and Seasonality?

Reasons for Detrending

- Stationarity requirement
- Focus on fluctuations
- Avoiding spurious regression
- Enabling valid inference

Reasons for Deseasonalizing

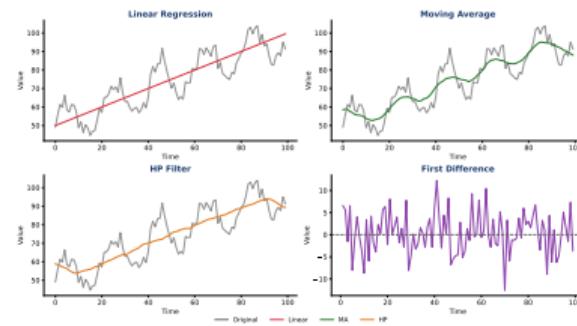
- Revealing the underlying trend
- Cross-season comparisons
- Simplifying modeling
- Focus on the irregular component

Important

- **We model the transformed series**
 - ▶ With trend and seasonality removed
- **We reverse the transformation**
 - ▶ Bring the forecast back to the original scale



Detrending Methods: Comparison



 [TSA_ch0_detrending](#)

Key Idea

- ☐ **Different methods:** produce different residuals
- ☐ **Choose by trend type:** consider the analysis objectives



Detrending Methods

Six Common Detrending Approaches

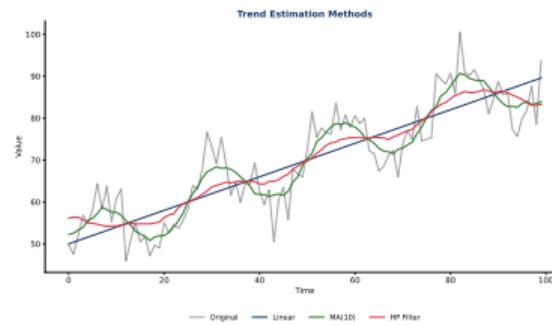
- Differencing:** $\Delta X_t = X_t - X_{t-1}$
 - ▶ Most commonly used, removes stochastic trend
- Linear regression:** $\hat{T}_t = \hat{\beta}_0 + \hat{\beta}_1 t$
- Polynomial:** higher-order polynomial
- HP filter:** balance between fit and smoothness
- Moving average:** $\hat{T}_t = MA_q(X_t)$
- LOESS:** local polynomial regression

The Choice Depends on

- Nature of the trend**
 - ▶ Deterministic vs stochastic
- Purpose of the analysis**
 - ▶ Forecasting vs descriptive analysis



Trend Estimation: Multiple Approaches



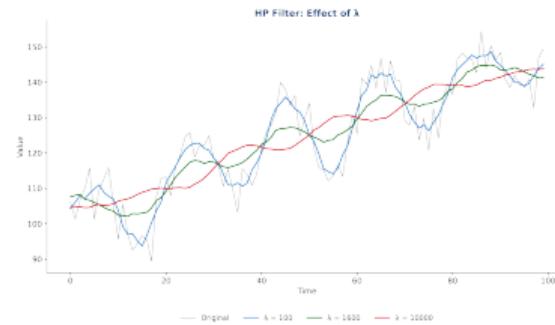
 [TSA_ch0_detrending](#)

Method Comparison

- Moving average:** simple but with lag
- Polynomial regression:** flexible, parametric
- HP filter:** macroeconomic standard



HP Filter: Effect of λ



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Trade-off

- Small λ : flexible trend, follows the data closely
- Large λ : smooth trend, approaches a linear trend



The Hodrick-Prescott (HP) Filter

Definition 5 (HP Filter)

- **HP filter:** decomposes X_t into trend τ_t and cycle c_t : $X_t = \tau_t + c_t$

$$\min_{\{\tau_t\}} \left\{ \sum_{t=1}^T (X_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right\}$$

Interpretation

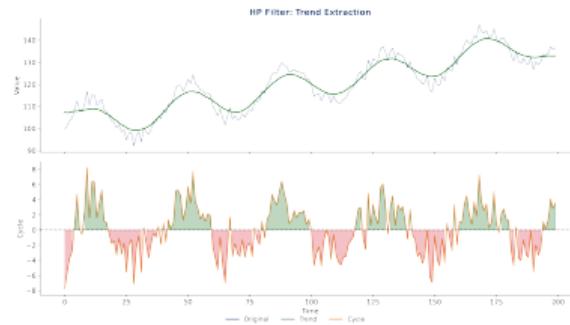
- **First term**
 - ▶ Goodness of fit
- **Second term**
 - ▶ Smoothness penalty
- λ
 - ▶ Controls the balance between fidelity and smoothness

Standard λ Values (Ravn-Uhlig)

- **Annual**
 - ▶ $\lambda = 6.25$
- **Quarterly**
 - ▶ $\lambda = 1600$ (macroeconomic standard)
- **Monthly**
 - ▶ $\lambda = 129600$



HP Filter: Business Cycle Extraction



 **TSA_ch0_detrending**

Application

- Macroeconomics:** business cycle extraction
- Common series:** GDP, unemployment, inflation



HP Filter: Limitations

Known Issues

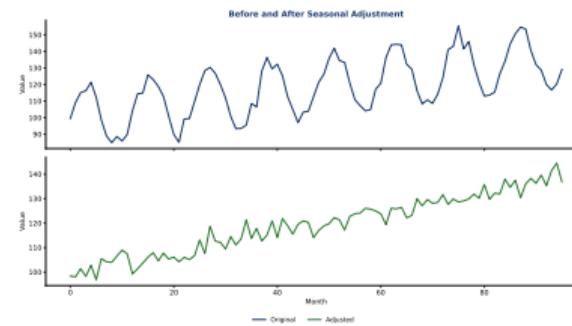
- Endpoint instability**
 - ▶ Trend estimates unreliable at the beginning and end
- Spurious cycles**
 - ▶ Can create artificial dynamics
- Choice of λ**
 - ▶ Results sensitive to the parameter

Alternatives

- Band-pass filters:** Baxter-King, Christiano-Fitzgerald
 - ▶ Isolate specific frequencies
- Hamilton filter:** regression-based
- Unobserved components:** state-space models



Seasonal Adjustment: Visualization



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Result

- Seasonally adjusted series:** reveals the underlying trend, removes periodic fluctuations



Seasonal Adjustment Methods

Four Approaches for Seasonal Adjustment

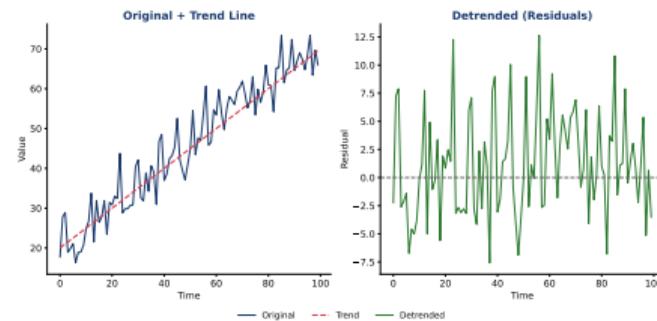
- Seasonal differencing:** $\Delta_s X_t = X_t - X_{t-s}$
 - ▶ Removes periodic pattern, simple to apply
- Division** (multiplicative): $X_t^{adj} = X_t / \hat{S}_t$
- Subtraction** (additive): $X_t^{adj} = X_t - \hat{S}_t$
- X-13ARIMA-SEATS**: official US Census Bureau standard
 - ▶ Sophisticated method, used by statistical institutes

Seasonal Period s

- Monthly: $s = 12$ | Quarterly: $s = 4$



Example: Deterministic Trend



 [TSA_ch0_detrending](#)

Key

- Method: regression
- Result: stationary residuals, ACF decays rapidly



Deterministic vs Stochastic Trend

Deterministic Trend

- Model:** $X_t = \beta_0 + \beta_1 t + \varepsilon_t$
- Characteristics:**
 - ▶ Trend is a function of time
 - ▶ ε_t is stationary
- Method:** detrend by regression

Stochastic Trend

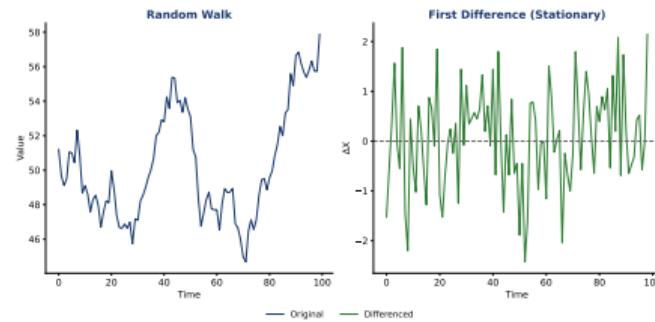
- Model:** $X_t = X_{t-1} + \varepsilon_t$
- Characteristics:**
 - ▶ Random walk component
 - ▶ ΔX_t is stationary
- Method:** detrend by differencing

Wrong Method = Problems

- Differencing a deterministic trend** → over-differencing
 - ▶ Introduces artificial dependence in the series
- Regression on a stochastic trend** → spurious regression
 - ▶ Invalid statistical results



Example: Stochastic Trend (Random Walk)



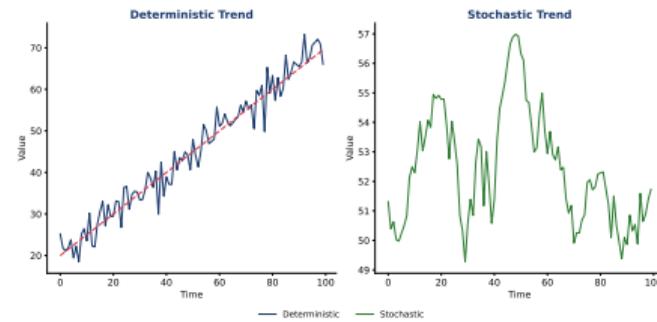
 [TSA_ch0_detrending](#)

Key

- Method: differencing
- Result: differences are stationary (white noise)



Side-by-Side Comparison



[TSA_ch0_detrending](#)

Remember

- Deterministic trend:** use regression — trend is a predictable function of time
- Stochastic trend:** use differencing — trend contains a random component



AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Using yfinance, download AAPL stock data. I want to see what patterns exist and predict where the price will go next year. Make it look professional with charts."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. What type of decomposition does the model choose? Is it correct? Justify.
3. How does it evaluate forecast quality? Is the metric computed correctly?
4. Check the residuals — do they show unexplained structure?
5. Rewrite the analysis correctly and compare with a seasonal naïve benchmark.

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*



Summary

What We Learned in This Chapter

- Time Series Definition and Characteristics
 - ▶ Sequence of temporally ordered observations with dependence
- Decomposition (Additive vs Multiplicative)
 - ▶ Components: Trend-Cycle + Seasonal + Residual
- Exponential Smoothing Methods
 - ▶ SES (level), Holt (+ trend), Holt-Winters (+ seasonality), ETS
- Forecast Evaluation and Validation
 - ▶ Metrics: MAE, RMSE, MAPE; Cross-Validation with rolling origin

Key Idea

- **Understand Before Modeling:**
 - ▶ Visualize and decompose the data first
 - ▶ Choose additive vs multiplicative based on variance behavior



Quick Quiz

Test Your Knowledge

1. What is the difference between additive and multiplicative decomposition?
2. When should you use Holt-Winters instead of SES?
3. Why can't we use standard k-fold CV for time series?
4. What does $\alpha = 0.9$ mean in exponential smoothing?
5. How do you distinguish between a deterministic and stochastic trend?



Quiz Answers

Answers

1. **Additive vs multiplicative:** additive when seasonal amplitude is constant; multiplicative when it grows with the level
2. **Holt-Winters:** when data have trend AND seasonality; SES handles only the level
3. **CV:** standard k-fold ignores temporal order → data leakage
4. $\alpha = 0.9$: high weight on recent observations, reacts quickly but more volatile
5. **Trend:** deterministic → function of time (regression); stochastic → random walk (differencing)



What's Next?

Chapter 1: Stochastic Processes and Stationarity

- Stochastic Processes:** mathematical foundation, random variables indexed by time
- Stationarity:** strict (invariant distribution) vs weak (invariant moments)
- Fundamental Processes:** white noise and random walk → building blocks for ARIMA
- ACF and PACF:** tools for model identification

Questions?



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Online Resources and Code

- **Quantlet:** <https://quantlet.com> → Code repository for statistics
- **Quantinar:** <https://quantinar.com> → Learning platform for quantitative methods
- **GitHub TSA:** https://github.com/QuantLet/TSA/tree/main/TSA_ch0 → Python code for this chapter



Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>

