Time Series Analysis and Forecasting

Chapter 5: GARCH and Volatility

Daniel Traian PELE

Bucharest University of Economic Studies
IDA Institute Digital Assets
Blockchain Research Center
AI4EFin Artificial Intelligence for Energy Finance
Romanian Academy, Institute for Economic Forecasting
MSCA Digital Finance

Learning Objectives

By the end of this chapter, you will be able to:

- Understand volatility clustering and its importance in financial data
- Model conditional heteroskedasticity using ARCH and GARCH models
- Estimate GARCH models and interpret their parameters
- Forecast volatility and apply it to risk management

Outline

Why Model Volatility?

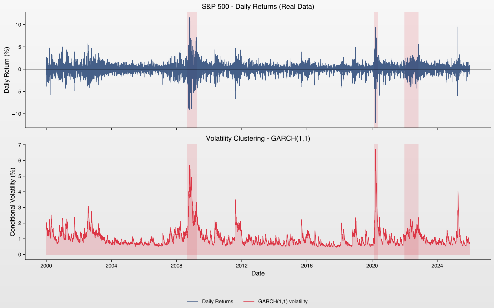
Empirical Observations in Financial Series

- Financial returns exhibit **volatility clustering** — periods of high volatility tend to be followed by periods of high volatility
- The distribution of returns has **fat tails** (leptokurtosis)
- Return correlation is nearly zero, but correlation of squares is significant
- Volatility responds **asymmetrically** to shocks (leverage effect)

Limitation of ARIMA Models

ARIMA models assume **constant variance** (homoskedasticity), which is not realistic for financial series!

Volatility Clustering



- High volatility periods are followed by high volatility; calm by calm
- This suggests that **conditional variance** is predictable

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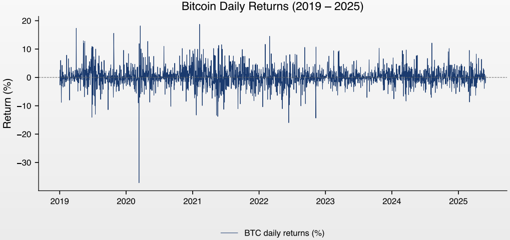
Example: Bitcoin ≻ Volatility Clustering



Observations

- Bitcoin daily returns (2019–2025): extremely pronounced volatility clustering
  - Returns of $\pm 20\%$ during crisis periods (COVID, Terra/Luna)
- Bitcoin volatility is significantly higher than traditional assets
  - Typical $\alpha \approx 0.10$–0.20 (fast reaction to news)

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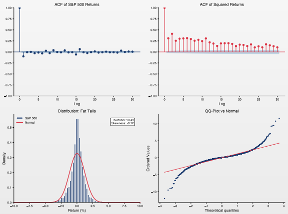
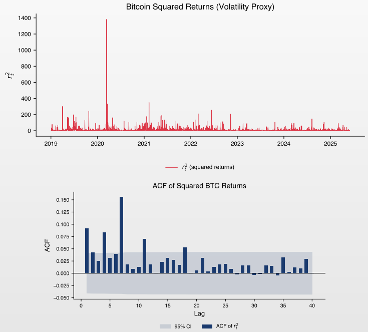
Stylized Facts of Financial Returns



Observed Properties

1. **No autocorrelation** in returns
2. **Autocorrelation** in r_t^2, $|r_t|$
3. **Fat tails** (kurtosis > 3)
4. **Leverage effect**
5. **Volatility clustering**

Example: Bitcoin ≻ Evidence for ARCH Effects



Interpretation

- **Top**: r_t^2 (volatility proxy) ≻ peaks coincide with market crises
- **Bottom**: ACF(r_t^2) significant ≻ ARCH effects present, variance is predictable

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Conditional Heteroskedasticity

Definition 1 (Conditional Variance)

For return series $\{r_t\}$, the **conditional variance** at time t is: $\sigma_t^2 = \text{Var}(r_t|\mathcal{F}_{t-1}) = \mathbb{E}[(r_t - \mu_t)^2|\mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the information available up to time $t-1$.

General Model

$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0, 1)$

- μ_t = conditional mean (ARMA); σ_t^2 = conditional variance (GARCH)
- z_t = standardized innovations (Normal, Student-t, GED)

The ARCH(q) Model — Engle (1982)

Definition 2 (ARCH(q))

The **Autoregressive Conditional Heteroskedasticity** model of order q:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1), \quad \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

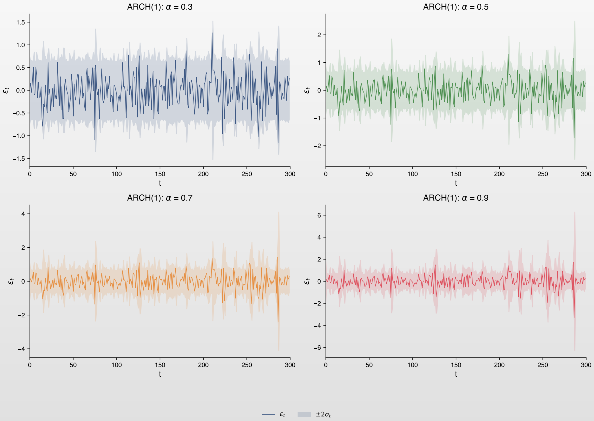
Stationarity Restrictions

- $\omega > 0$ (positive base variance), $\alpha_i \geq 0$ (non-negativity)
- $\sum_{i=1}^{q} \alpha_i < 1$ (stationarity)

Remark 1

Robert Engle received the **Nobel Prize in Economics** in 2003 for developing the ARCH model!

ARCH(1) Simulation: Effect of α Parameter



Higher α means volatility reacts more strongly to recent shocks.

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Properties of the ARCH(1) Model

ARCH(1): $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$

- **Unconditional variance**: $\mathbb{E}[\varepsilon_t^2] = \dfrac{\omega}{1 - \alpha_1}$ (if $\alpha_1 < 1$)
- **Kurtosis**: $\kappa = 3 \cdot \dfrac{1 - \alpha_1^2}{1 - 3\alpha_1^2}$ (if $\alpha_1^2 < 1/3$)
- Kurtosis > 3 for $\alpha_1 > 0 \Rightarrow$ **fat tails**!

Numerical Example

If $\omega = 0.0001$ and $\alpha_1 = 0.3$:

- Unconditional variance: $\sigma^2 = \frac{0.0001}{1-0.3} = 0.000143$
- Kurtosis: $\kappa = 3 \cdot \frac{1-0.09}{1-0.27} = 3.74 > 3$

Derivation: Unconditional Variance of ARCH(1)

Derivation.

Let $\varepsilon_t = \sigma_t z_t$ with $z_t \sim N(0,1)$ and $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$.

Step 1: Take unconditional expectation:

$$\mathbb{E}[\varepsilon_t^2] = \mathbb{E}[\sigma_t^2 z_t^2] = \mathbb{E}[\sigma_t^2] \cdot \mathbb{E}[z_t^2] = \mathbb{E}[\sigma_t^2]$$

Step 2: Apply expectation to variance equation:

$$\mathbb{E}[\sigma_t^2] = \mathbb{E}[\omega + \alpha_1 \varepsilon_{t-1}^2] = \omega + \alpha_1 \mathbb{E}[\varepsilon_{t-1}^2]$$

Step 3: By stationarity, $\mathbb{E}[\varepsilon_t^2] = \mathbb{E}[\varepsilon_{t-1}^2] = \sigma^2$:

$$\sigma^2 = \omega + \alpha_1 \sigma^2 \quad \Rightarrow \quad \sigma^2(1 - \alpha_1) = \omega$$

Result: $\boxed{\sigma^2 = \frac{\omega}{1 - \alpha_1}}$ (requires $\alpha_1 < 1$ for stationarity) $\square$

Derivation: Kurtosis of ARCH(1)

For $\varepsilon_t = \sigma_t z_t$ with $z_t \sim N(0,1)$:

Step 1: $\mathbb{E}[\varepsilon_t^4] = \mathbb{E}[\sigma_t^4] \cdot \mathbb{E}[z_t^4] = 3\mathbb{E}[\sigma_t^4]$ (since $\mathbb{E}[z^4] = 3$)

Step 2: Using $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$:

$$\mathbb{E}[\sigma_t^4] = \mathbb{E}[(\omega + \alpha_1 \varepsilon_{t-1}^2)^2] = \omega^2 + 2\omega\alpha_1\sigma^2 + \alpha_1^2 \mathbb{E}[\varepsilon_{t-1}^4]$$

Step 3: Solving the recursion yields:

$$\boxed{\kappa = \frac{\mathbb{E}[\varepsilon_t^4]}{(\mathbb{E}[\varepsilon_t^2])^2} = 3 \cdot \frac{1 - \alpha_1^2}{1 - 3\alpha_1^2}}$$

Interpretation

- $\kappa > 3$ for any $\alpha_1 > 0 \Rightarrow$ **fat tails** (leptokurtosis)
- Requires $\alpha_1 < 0.577$ for finite fourth moment
- ARCH naturally generates heavy-tailed distributions!

Testing for ARCH Effects

Engle's Test for ARCH Effects

Procedure:

1. Estimate the mean model and obtain residuals $\hat{\varepsilon}_t$
2. Calculate $\hat{\varepsilon}_t^2$
3. Regress $\hat{\varepsilon}_t^2$ on its lags:
$$\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \cdots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$$
4. Calculate the statistic $LM = T \cdot R^2 \sim \chi^2(q)$

Hypotheses

- H_0: No ARCH effects ($\alpha_1 = \cdots = \alpha_q = 0$)
- H_1: ARCH effects present (at least one $\alpha_i \neq 0$)

Limitations of the ARCH Model

Practical Problems

1. **High order** — many lags are usually needed (large q)
2. **Many parameters** — estimation difficulties
3. **Non-negativity constraints** — difficult to impose for large q
4. **Does not capture persistence** — observed volatility is very persistent

The Solution

The GARCH Model — introduces lags of conditional variance to capture persistence with fewer parameters!

The GARCH(p,q) Model — Bollerslev (1986)

Definition 3 (GARCH(p,q))

The **Generalized ARCH** model:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

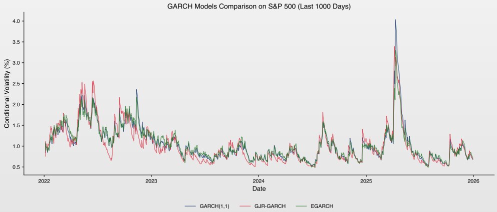
Interpretation

- ω = base level of volatility
- α_i = reaction to recent shocks (news coefficients)
- β_j = volatility persistence (memory)
- $\alpha + \beta$ = total persistence

The GARCH(1,1) Model

The Most Popular Volatility Model

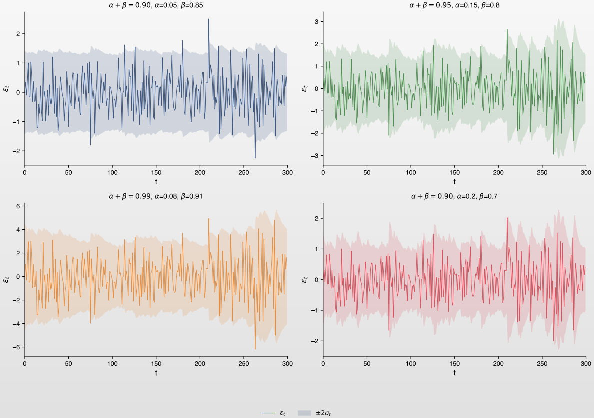
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$



Restrictions & Properties

- $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$; $\alpha + \beta < 1$ (stationarity)
- $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$; Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$

GARCH(1,1) Simulation: Persistence Effect



Parameter α controls reaction to shocks, β controls persistence. The sum $\alpha + \beta$ determines mean-reversion speed.

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Derivation: Unconditional Variance of GARCH(1,1)

Derivation.

For $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$:

Step 1: Take unconditional expectation: $\mathbb{E}[\sigma_t^2] = \omega + \alpha\mathbb{E}[\varepsilon_{t-1}^2] + \beta\mathbb{E}[\sigma_{t-1}^2]$

Step 2: By stationarity, $\mathbb{E}[\sigma_t^2] = \mathbb{E}[\sigma_{t-1}^2] = \bar{\sigma}^2$ and $\mathbb{E}[\varepsilon_t^2] = \bar{\sigma}^2$: $\bar{\sigma}^2 = \omega + (\alpha + \beta)\bar{\sigma}^2$

Step 3: Solve: $\bar{\sigma}^2(1 - \alpha - \beta) = \omega \Rightarrow \boxed{\bar{\sigma}^2 = \dfrac{\omega}{1 - \alpha - \beta}}$ $\square$

Stationarity Condition

Requires $\alpha + \beta < 1$ for finite unconditional variance.

GARCH(1,1) as ARMA for ε_t^2

ARMA(1,1) Representation

Define $\nu_t = \varepsilon_t^2 - \sigma_t^2$ (variance shock). Then:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

This is an **ARMA(1,1)** for ε_t^2!

Implications

- ACF of ε_t^2 decays exponentially (like ARMA)
- Persistence is given by $\alpha + \beta$
- PACF can help identify the order

Derivation: ARMA Representation of GARCH(1,1)

Derivation.

Step 1: Define variance shock: $\nu_t = \varepsilon_t^2 - \sigma_t^2$

- $\mathbb{E}[\nu_t|\mathcal{F}_{t-1}] = \mathbb{E}[\varepsilon_t^2|\mathcal{F}_{t-1}] - \sigma_t^2 = \sigma_t^2 - \sigma_t^2 = 0$
- ν_t is a martingale difference sequence

Step 2: Substitute $\sigma_t^2 = \varepsilon_t^2 - \nu_t$ into GARCH equation:

$$\varepsilon_t^2 - \nu_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta(\varepsilon_{t-1}^2 - \nu_{t-1})$$

Step 3: Rearrange:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

Result: ARMA(1,1) with AR coefficient $\phi = \alpha + \beta$ and MA coefficient $\theta = -\beta$. $\square$

Volatility Persistence and Half-Life

Persistence

$\alpha + \beta$ measures mean reversion speed:

- ≈ 1: very persistent
- $\ll 1$: quick reversion

Half-Life Formula

$$HL = \frac{-0.693}{\ln(\alpha + \beta)}$$

Example: S&P 500

$\alpha = 0.09$, $\beta = 0.90$
$\alpha + \beta = 0.99$

$$HL = \frac{-0.693}{\ln(0.99)} \approx 69$$

Shock halves in $\sim$69 trading days!

Estimation of GARCH Models

Maximum Likelihood Estimation (MLE)

Log-likelihood (normal): $\ell(\theta) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln(\sigma_t^2) + \frac{\varepsilon_t^2}{\sigma_t^2}\right]$

Alternative Distributions for z_t

- **Student-t**: captures fat tails — most common choice
- **GED**: flexibility for kurtosis
- **Skewed Student-t**: asymmetry and fat tails

Practical Note

Student-t distribution typically provides better fit for financial returns due to fat tails (kurtosis > 3).

Typical Values for GARCH(1,1)

Series	α	β	$\alpha + \beta$
S&P 500 daily	0.05–0.10	0.85–0.95	0.95–0.99
EUR/USD daily	0.03–0.08	0.90–0.95	0.95–0.99
Bitcoin daily	0.10–0.20	0.75–0.85	0.90–0.98
Bonds	0.02–0.05	0.90–0.97	0.95–0.99

Observations

- $\alpha + \beta$ close to 1 $\Rightarrow$ **very persistent volatility**
- Small α, large β $\Rightarrow$ slow reaction to shocks, long memory
- Bitcoin: larger α $\Rightarrow$ faster reaction to news

IGARCH — Integrated GARCH

Definition 4 (IGARCH(1,1))

When $\alpha + \beta = 1$:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2$$

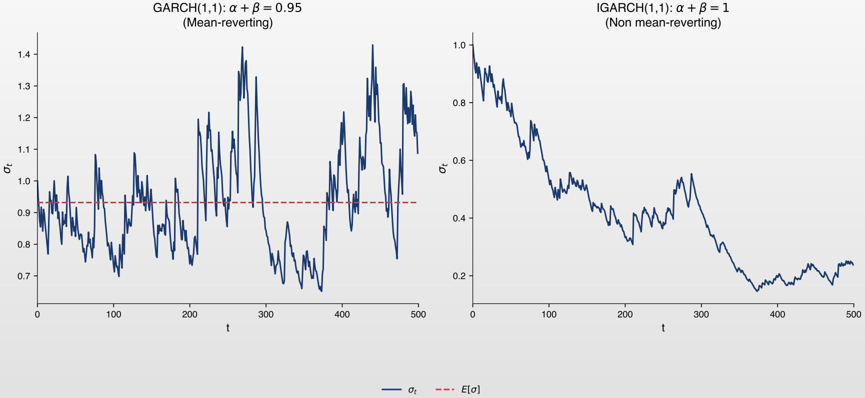
Properties

- Unconditional variance does not exist (infinite)
- Shocks have **permanent** effect on volatility
- Used for series with extreme persistence
- Useful for **RiskMetrics** (J.P. Morgan): $\alpha = 0.06$, $\beta = 0.94$

Remark 2

IGARCH is analogous to a unit root in variance!

GARCH vs IGARCH: Persistence Comparison



Standard GARCH reverts to unconditional mean, while IGARCH has no finite mean and shocks persist indefinitely.

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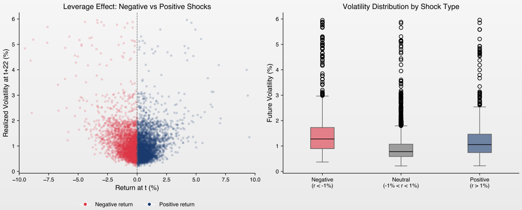
Leverage Effect



Definition

Leverage effect: Negative shocks increase volatility **more** than positive shocks of the same magnitude.

Problem with GARCH

Standard GARCH: $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$ — only ε_{t-1}^2 matters, sign is lost! Economic intuition: Bad news $\Rightarrow$ stock price falls $\Rightarrow$ debt/equity ratio rises $\Rightarrow$ volatility increases.

The EGARCH Model — Nelson (1991)

Definition 5 (EGARCH(1,1))

Exponential GARCH:

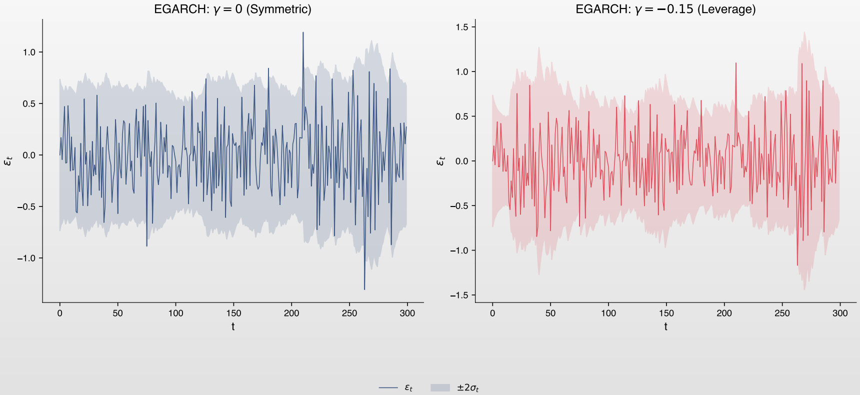
$$\ln(\sigma_t^2) = \omega + \alpha\left(|z_{t-1}| - \mathbb{E}[|z_{t-1}|]\right) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

where $z_t = \varepsilon_t/\sigma_t$.

EGARCH Advantages

- **No non-negativity constraints required** — models $\ln(\sigma_t^2)$
- **Captures leverage effect** through parameter γ
  - $\gamma < 0$: negative shocks $\Rightarrow$ higher volatility
  - $\gamma = 0$: symmetric effect (like GARCH)
- Persistence is given by β

EGARCH Simulation: Symmetric vs Asymmetric



When $\gamma < 0$, negative shocks (bad news) increase volatility more than positive shocks of the same magnitude.

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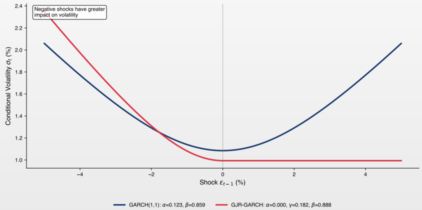
News Impact Curve — EGARCH



Definition

News Impact Curve: σ^2_{t+1} as function of ε_t, holding σ^2_t constant.

- **GARCH**: Symmetric V-shape (parabola); **EGARCH**: Asymmetric — steeper for negative shocks; **GJR**: Piecewise linear with kink at zero
- The asymmetry captures the leverage effect: bad news has larger impact on volatility than good news.

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The GJR-GARCH Model

Definition 6 (GJR-GARCH(1,1))

Glosten, Jagannathan & Runkle (1993): $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 \cdot I_{t-1} + \beta\sigma_{t-1}^2$ where $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$, else 0.

Interpretation

- Positive shocks: impact $= \alpha$; Negative shocks: impact $= \alpha + \gamma$
- Leverage effect present if $\gamma > 0$
- Stationarity: $\alpha + \gamma/2 + \beta < 1$

TGARCH — Threshold GARCH

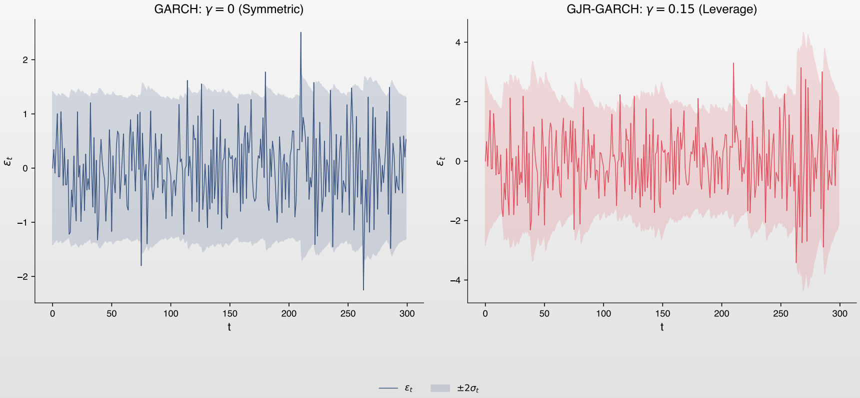
Definition 7 (TGARCH(1,1))

Zakoian (1994) models standard deviation: $\sigma_t = \omega + \alpha^+ \varepsilon_{t-1}^+ + \alpha^- \varepsilon_{t-1}^- + \beta \sigma_{t-1}$

Comparison of Asymmetric Models

Model	**Specification**	**Leverage**
GARCH	σ_t^2	No
EGARCH	$\ln(\sigma_t^2)$	Yes ($\gamma < 0$)
GJR-GARCH	σ_t^2 with indicator	Yes ($\gamma > 0$)
TGARCH	σ_t	Yes ($\alpha^- > \alpha^+$)

GJR-GARCH/TGARCH Simulation



GJR-GARCH adds an indicator term to capture asymmetric response to negative shocks.

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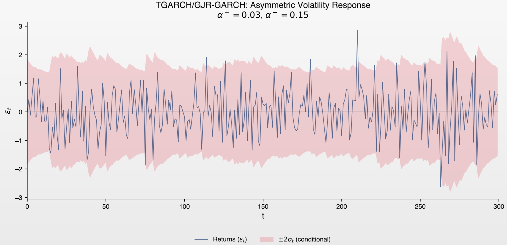
TGARCH Simulation: Asymmetric Volatility Response



Interpretation

- TGARCH with $\alpha^+ = 0.03$ and $\alpha^- = 0.15 \succ$ negative shocks amplify volatility by 5$\times$
- Volatility bands $\pm 2\sigma$ widen asymmetrically during crisis periods

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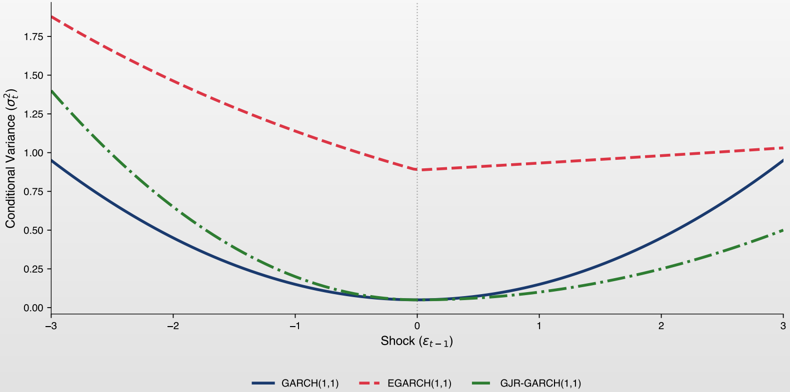
News Impact Curves Comparison



Standard GARCH is symmetric, while EGARCH and GJR-GARCH capture asymmetry (leverage effect).

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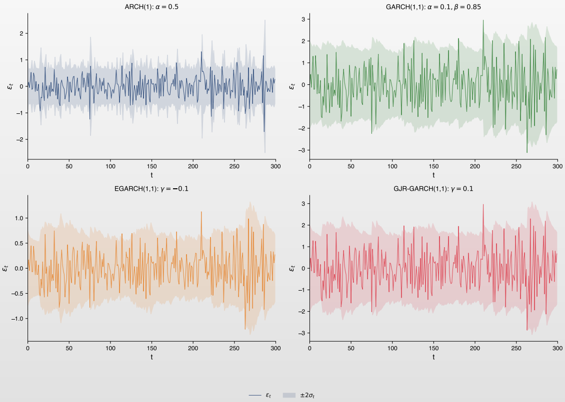
GARCH Family Comparison



All models capture volatility clustering, but differ in how they model asymmetry.

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GARCH-M: GARCH-in-Mean Model

Definition 8 (GARCH-M)

The **GARCH-in-Mean** model: $r_t = \mu + \lambda\sigma_t + \varepsilon_t$, where λ is the **risk premium**.

Interpretation

- $\lambda > 0$: Higher risk $\Rightarrow$ higher expected return
- $\lambda = 0$: Reduces to standard GARCH
- $\lambda < 0$: Higher risk $\Rightarrow$ lower return (rare)

Financial Intuition

Investors demand compensation for bearing risk — GARCH-M captures this **risk-return tradeoff**.

GARCH-M: Alternative Specifications

Common Specifications

Risk premium can enter in different forms: (1) $r_t = \mu + \lambda\sigma_t + \varepsilon_t$; (2) $r_t = \mu + \lambda\sigma_t^2 + \varepsilon_t$; (3) $r_t = \mu + \lambda \ln(\sigma_t^2) + \varepsilon_t$

Typical Results for Equity Markets

- Estimated λ often positive but small (0.01–0.10)
- Significance varies across markets and periods
- Variance specification yields larger λ estimates

Remark 3

GARCH-M is used in asset pricing, portfolio optimization, and CAPM testing.

Order Selection

Information Criteria

- **AIC** $= -2\ell + 2k$
- **BIC** $= -2\ell + k\ln(T)$
- **HQIC** $= -2\ell + 2k\ln(\ln(T))$

where ℓ = maximized log-likelihood, k = number of parameters.

Practical Recommendations

- GARCH(1,1) is sufficient in **90% of cases**
- Check if asymmetric model significantly improves fit
- Choose innovation distribution that minimizes AIC/BIC

GARCH Model Diagnostics

Standardized Residuals

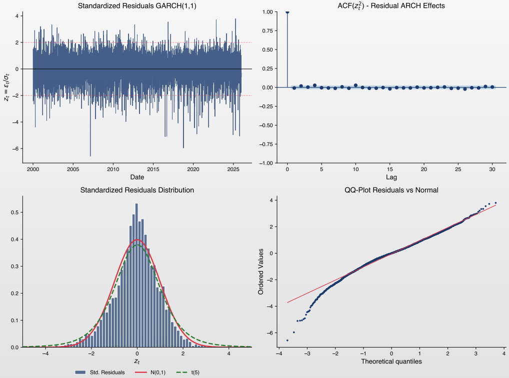
$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$$

If the model is correctly specified, $\hat{z}_t$ should be i.i.d.(0,1).

Diagnostic Checks

1. **Ljung-Box on** $\hat{z}_t$: check absence of autocorrelation in mean
2. **Ljung-Box on** $\hat{z}_t^2$: check absence of residual ARCH effects
3. **ARCH-LM test on** $\hat{z}_t$: confirm absence of heteroskedasticity
4. **Histogram + QQ-plot**: verify assumed distribution

Diagnostic Example



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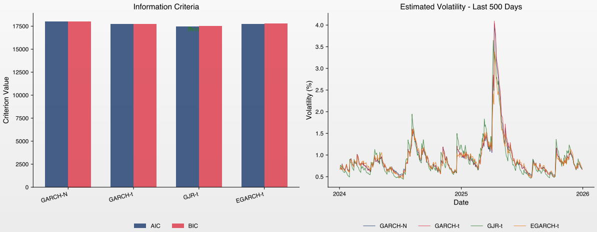
GARCH Model Comparison: Validation



Interpretation

- GARCH(1,1) achieves the lowest MAE on the validation set
  - More parsimonious and stable than higher-order models
- GARCH(2,1) and GJR-GARCH: similar performance, but more parameters
- **Conclusion**: simplicity wins $\succ$ GARCH(1,1) is hard to beat

TSA_ch5_comparison

Forecasting with GARCH(1,1)

One-Step-Ahead Forecast

$\hat{\sigma}^2_{T+1} = \omega + \alpha\varepsilon^2_T + \beta\sigma^2_T$

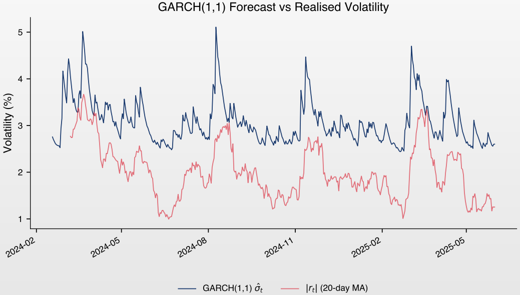
Multi-Step Forecast

For $h > 1$: $\mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2 + (\alpha+\beta)^{h-1}(\sigma^2_{T+1} - \bar{\sigma}^2)$ where $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ = unconditional variance.

Convergence

$\lim_{h\to\infty} \mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2$ — forecast converges to unconditional variance!

Volatility Forecast — Visualization



- Forecast converges exponentially to $\bar{\sigma}^2$; speed depends on $\alpha + \beta$
- The closer $\alpha + \beta$ is to 1, the slower the convergence

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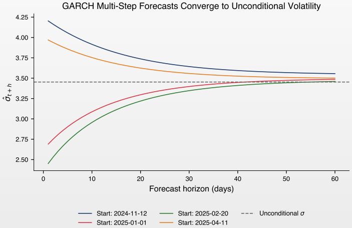
GARCH Forecast Convergence to Unconditional Variance



Interpretation

- Multi-step forecast converges exponentially to $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- The closer $\alpha + \beta$ is to 1, the slower the convergence
  - S&P 500: $\alpha + \beta \approx 0.99 \succ$ convergence in $\sim$50 days
  - Bitcoin: $\alpha + \beta \approx 0.95 \succ$ faster convergence

TSA_ch5_convergence

Applications of Volatility Forecasting

Value at Risk (VaR)

$\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$
Maximum loss with probability $1 - \alpha$.

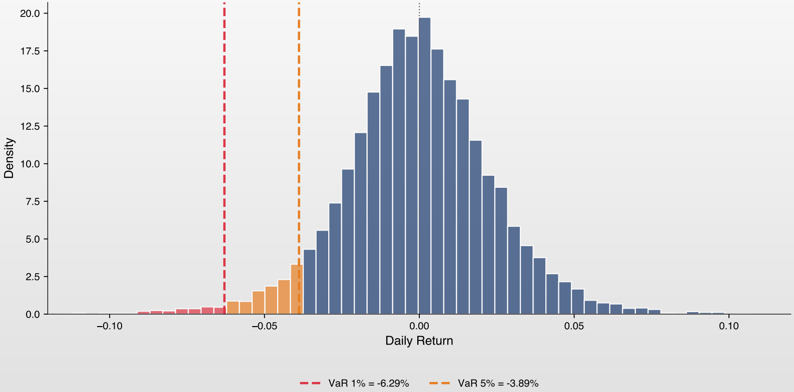
Expected Shortfall (ES)

$\text{ES}_\alpha = \mathbb{E}[-r | r < -\text{VaR}_\alpha]$
Average loss when VaR is exceeded.

Other Applications

- Option pricing
- Dynamic hedging
- Portfolio allocation
- Stress testing

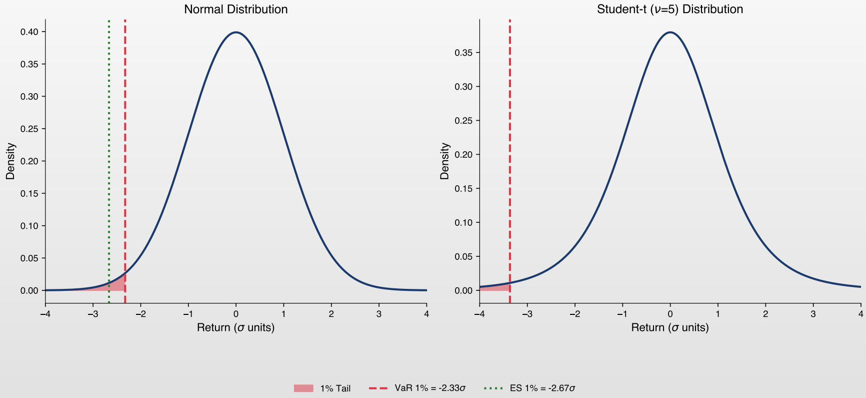
VaR and ES: Graphical Illustration



VaR 1% = loss exceeded only in 1% of cases. Red area = left tail (extreme losses).

TSA_ch5_var_plot

VaR vs Expected Shortfall: Normal vs Student-t



ES (green line) measures average loss when VaR is exceeded. Student-t has heavier tails ⇒ larger VaR and ES.

TSA_ch5_var_es

Value at Risk — Numerical Example

VaR Calculation

Portfolio: **1,000,000 EUR**, forecasted volatility $\hat{\sigma}_{T+1} = 1.5\%$

VaR with Normal Distribution

Level	z_α	**VaR (%)**	**VaR (EUR)**
5% (1 day)	1.645	2.47%	24,675
1% (1 day)	2.326	3.49%	34,890

Scaling for Longer Periods

$\text{VaR}_{h\ \text{days}} = \text{VaR}_{1\ \text{day}} \cdot \sqrt{h}$ — assumes i.i.d. returns

Value at Risk — Student-t Distribution

Why Student-t?

Normal distribution **underestimates** tail risk. Student-t with ν degrees of freedom better captures fat tails (kurtosis > 3).

VaR 1% (1 day) Comparison: $\sigma = 1.5\%$, Portfolio $=$ 1M EUR

Distribution	Quantile	VaR (EUR)
Normal	2.326	34,890
Student-t ($\nu = 6$)	3.143	47,145
Student-t ($\nu = 4$)	3.747	56,205

Observation

With $\nu = 6$ (typical for stocks), VaR is **35% higher** than normal!

VaR — Complete Example with GARCH

VaR Calculation Procedure

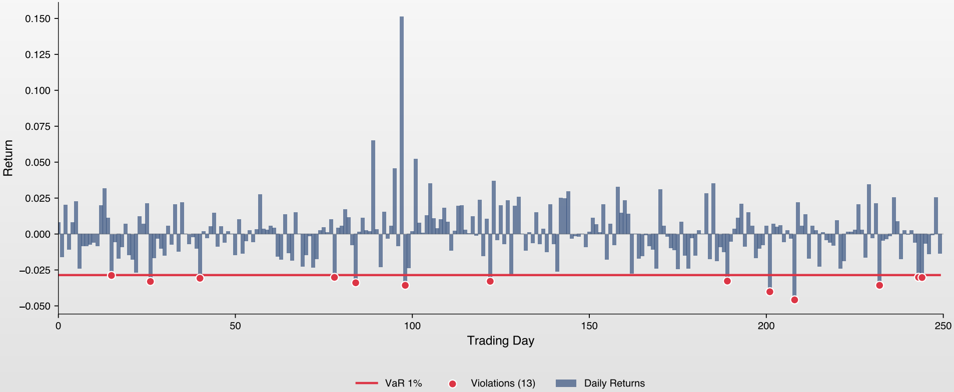
1. Estimate GARCH(1,1) model with Student-t distribution
2. Obtain volatility forecast: $\hat{\sigma}_{T+1}$
3. Calculate VaR: $\text{VaR}_\alpha = t_\alpha(\nu) \cdot \hat{\sigma}_{T+1} \cdot \sqrt{\frac{\nu-2}{\nu}}$

Example: S&P 500

- Estimated parameters: $\alpha = 0.088$, $\beta = 0.900$, $\nu = 6.4$
- Forecasted volatility: $\hat{\sigma}_{T+1} = 1.2\%$
- Portfolio: 10,000,000 EUR

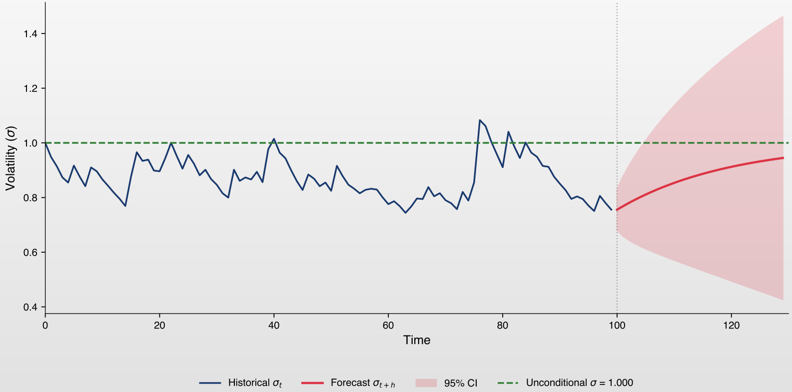
VaR 1% (1 day): VaR $= 3.05 \times 0.012 \times 10{,}000{,}000 =$ **366,000 EUR**

VaR Backtesting: Visual Overview



Backtesting checks if VaR violations match expected rate (e.g., 2.5 violations/year for VaR 1%). TSA_ch5_backtest

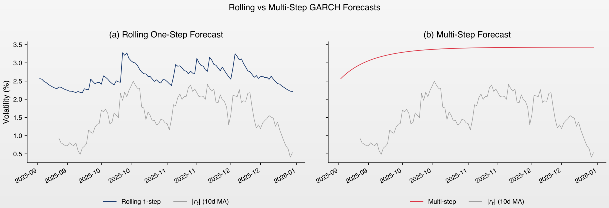
Volatility Forecast with Confidence Intervals



Forecast converges to unconditional volatility $\bar{\sigma}$. Uncertainty increases with forecast horizon.

TSA_ch5_vol_ci

Rolling Forecast: Step-by-Step Prediction



Procedure

S&P 500, W=500, GARCH(1,1)-t

- Re-estimate GARCH on $[t-W,\ t-1]$; forecast $\hat{\sigma}_{t|t-1}$
- Compare with realized vol. (20-day rolling std.)

Results (2015 days OOS)

- $\rho = 0.938 \succ$ excellent tracking; MAE $= 0.15\%$, RMSE $= 0.24\%$
- COVID-19: temporary under-prediction, rapid adaptation

GARCH Estimation in Python: arch Package

Python Code

```
pip install arch
from arch import arch_model

model = arch_model(returns,
    vol='Garch', p=1, q=1,
    dist='normal')
results = model.fit(disp='off')
print(results.summary())
```

Key Parameters

- **vol**: model type
  - 'Garch', 'EGARCH'
- **p, q**: GARCH order
  - p=1, q=1 standard
- **dist**: distribution
  - 'normal', 't'

TSA_ch5_garch

Asymmetric Models in Python

EGARCH and GJR-GARCH

```
# EGARCH
model_egarch = arch_model(returns, vol='EGARCH', p=1, q=1)
# GJR-GARCH (o=1 adds the asymmetric term)
model_gjr = arch_model(returns, vol='Garch', p=1, o=1, q=1)
```

Alternative Distributions

```
# Student-t for fat tails
model_t = arch_model(returns, vol='Garch', p=1, q=1, dist='t')
# Skewed Student-t for asymmetry and fat tails
model_skewt = arch_model(returns, vol='Garch', p=1, q=1,
                         dist='skewt')
```

TSA_ch5_egarch

Forecasting and Diagnostics

Volatility Forecast

```
forecasts = results.forecast(horizon=10)
vol_forecast = np.sqrt(forecasts.variance.values[-1,:])
```

Diagnostics and VaR

```
std_resid = results.std_resid
lb_test = acorr_ljungbox(std_resid**2, lags=10)
sigma = np.sqrt(forecasts.variance.values[-1, 0])
VaR_5pct = 1.645 * sigma
```

TSA_ch5_forecast

VaR Backtesting: Kupiec Test

Unconditional Coverage Test

Tests whether the observed violation rate equals the expected rate p (e.g., 1% for VaR 1%).

Let N = number of VaR violations, T = total observations, $\hat{p} = N/T$.

Likelihood Ratio Statistic:

$$LR_{uc} = -2\ln\left[\frac{(1-p)^{T-N}p^N}{(1-\hat{p})^{T-N}\hat{p}^N}\right] \sim \chi^2(1)$$

Hypotheses

- H_0: $\hat{p} = p$ (VaR model is correctly calibrated)
- H_1: $\hat{p} \neq p$ (VaR model under- or over-estimates risk)

VaR Backtesting: Christoffersen Test

Conditional Coverage Test

Tests both **unconditional coverage** and **independence** of violations.
Violations should be independent — no clustering of exceptions!

Test Components

- **Independence test** (LR_{ind}): Tests if violations are serially independent
- **Conditional coverage**: $LR_{cc} = LR_{uc} + LR_{ind} \sim \chi^2(2)$

Interpretation

Reject LR_{uc}: wrong frequency; Reject LR_{ind}: clustered violations; Reject LR_{cc}: model fails

VaR Backtesting: Python Implementation

Kupiec Test Implementation

```
import numpy as np
from scipy import stats
def kupiec_test(violations, T, p=0.01):
    N = np.sum(violations)
    p_hat = N / T
    if N == 0 or N == T:
        return np.nan, np.nan
    LR = -2 * (np.log((1-p)**(T-N) * p**N) -
               np.log((1-p_hat)**(T-N) * p_hat**N))
    return LR, 1 - stats.chi2.cdf(LR, df=1)
```

Usage

```
LR, pval = kupiec_test(violations, T=250, p=0.01)
```

Full Backtesting: Results and Decision

Application S&P 500 (T=500, VaR 1%)

```
violations = returns[window:] < -VaR_series
n_viol = violations.sum()
T = len(violations)
rate = n_viol / T
print(f"Violations: {n_viol}/{T} (rate = {rate:.2%})")
LR_uc, p_uc = kupiec_test(violations, T, alpha=0.01)
LR_ind, p_ind = christoffersen_test(violations)
LR_cc = LR_uc + LR_ind   # combined test ~ chi2(2)
p_cc = 1 - stats.chi2.cdf(LR_cc, df=2)
```

Typical Output

```
Violations: 13/500 (rate = 2.60%)
Kupiec    LR = 5.83,  p-value = 0.0157  => Rejected (p<0.05)
Independ. LR = 0.42,  p-value = 0.5171  => Accepted
Combined  LR = 6.25,  p-value = 0.0439  => Rejected
Basel Zone: RED (>=10 violations) => Inadequate model
```

TSA_ch5_backtest_full

ARMA-GARCH: Joint Mean and Variance Modeling

Why Joint Modeling?

Serial correlation $\Rightarrow$ ARMA for mean; **Volatility clustering** $\Rightarrow$ GARCH for variance.

Definition 9 (ARMA(p,q)-GARCH(r,s))

Mean equation: $r_t = \mu + \sum_{i=1}^{p} \phi_i (r_{t-i} - \mu) + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$

Variance equation: $\varepsilon_t = \sigma_t z_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$

ARMA-GARCH: Model Selection Strategy

Step-by-Step Approach

1. **Identify mean model**: Check ACF/PACF of returns for ARMA structure
2. **Test for ARCH effects**: Apply ARCH-LM test to residuals
3. **Specify variance model**: Usually GARCH(1,1) is sufficient
4. **Joint estimation**: Estimate both equations via MLE
5. **Diagnostic checking**: Standardized residuals should be i.i.d.

Common Specifications

- **Stock returns**: AR(1)-GARCH(1,1) or ARMA(1,1)-GARCH(1,1)
- **Exchange rates**: Often just GARCH(1,1) (no mean dynamics)
- **Interest rates**: AR(1)-EGARCH(1,1) for leverage effects

ARMA-GARCH: Python Implementation

Using the arch Package

```
from arch import arch_model
model = arch_model(returns,
                   mean='ARX',
                   lags=1,
                   vol='Garch',
                   p=1, q=1,
                   dist='t')
result = model.fit(disp='off')
print(result.summary())
```

Parameters

`mean='ARX'`: ARMA mean; `lags=1`: AR(1); `dist='t'`: Student-t

TSA_ch5_armagarch

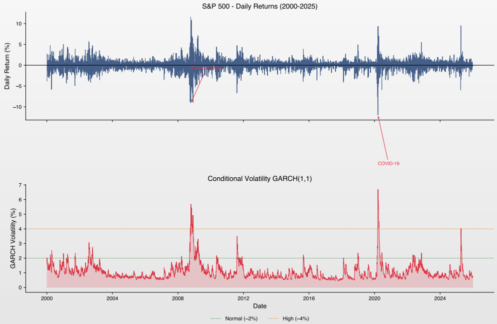
ARMA-GARCH: Complete Example

```
from arch import arch_model
model = arch_model(returns,
                   mean='ARX', lags=[1],
                   vol='EGARCH', p=1, q=1,
                   dist='skewt')
result = model.fit(update_freq=0)
cond_mean = result.conditional_mean
cond_vol = result.conditional_volatility
forecasts = result.forecast(horizon=5)
```

Note

For MA terms, use `mean='HARX'` or pre-filter with statsmodels ARIMA.

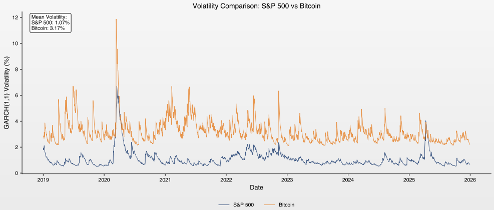
S&P 500 Volatility Analysis



- S&P 500 daily returns (2000–2024) — volatility clustering visible
- Crisis periods: 2008 (financial), 2020 (COVID-19), 2022 (inflation)

TSA_ch5_sp500

GARCH(1,1) Estimation — S&P 500

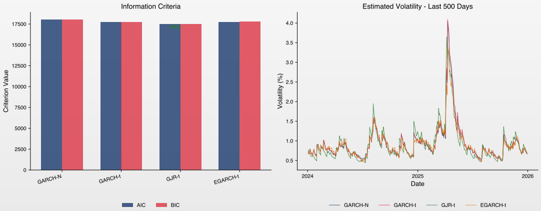


Estimation Results

Parameter	Value
ω	0.0108
α	0.0883
β	0.9002
$\alpha + \beta$	0.9885
ν (df)	6.42

Very persistent; Half-life $\approx$ 60 days

GARCH vs EGARCH Comparison — S&P 500

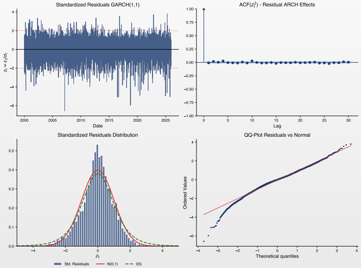


Leverage Effect Confirmed

EGARCH: $\gamma = -0.12$ (significantly negative) — negative shocks amplify volatility more than positive shocks. Both models capture volatility clustering, but EGARCH better fits crisis periods (2008, 2020).

TSA_ch5_sp500_comp

Step 5: Diagnostics — EGARCH(1,1)-t

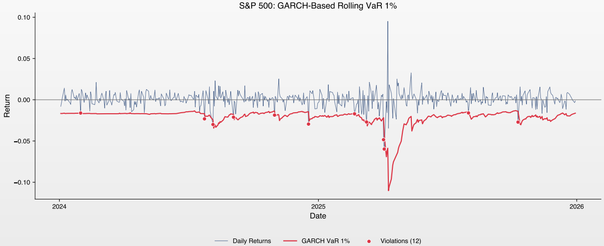


Checks on Standardized Residuals $z_t = \varepsilon_t / \hat{\sigma}_t$

- **Ljung-Box** on z_t: p-value $= 0.38$ — no residual autocorrelation
- **Ljung-Box** on z_t^2: p-value $= 0.52$ — **ARCH effects eliminated**
- **Q-Q plot**: points follow the theoretical Student-t line
- **Conclusion**: EGARCH(1,1)-t adequately captures volatility dynamics

TSA_ch5_sp500_diag

Step 6: Backtesting Rolling VaR — S&P 500

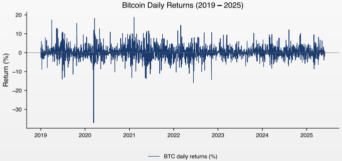


Kupiec + Christoffersen Results (2015 days out-of-sample)

Test	Statistic	p-value	Decision
Violations	27/2015 ($\hat{p} = 1.34\%$)	—	**Green** zone
Kupiec (uc)	2.13	0.145	Accepted
Christoffersen (ind)	0.79	0.375	Accepted
Combined (cc)	2.91	0.233	Accepted

TSA_ch5_sp500_var

Step 1: Data — Bitcoin Daily Returns



Data Description

- **Source**: Yahoo Finance (BTC-USD), daily data 2018–2024
- Log returns: mean ≈ 0.05%, volatility ≈ 3.5%

Descriptive Statistics

Mean	Std. Dev.	Skewness	Kurtosis	Min	Max
0.05%	3.48%	−0.72	12.1	−46.5%	+22.5%

- Volatility ∼3× higher than S&P 500
- Extreme kurtosis — high risk of large losses

Steps 3–4: Estimation and Model Selection — Bitcoin

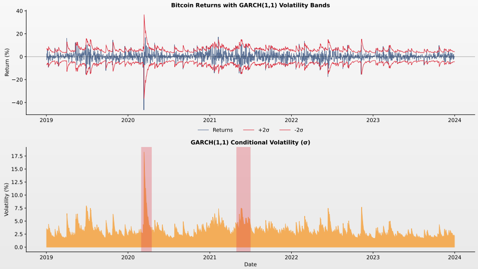
Estimated Parameters

Model	ω	α	β	γ	$\alpha+\beta$	ν	AIC
GARCH-t	0.42	0.131	0.848	—	0.979	4.82	9284
EGARCH-t	0.08	0.184	0.976	−0.061	—	4.79	9276
GJR-t	0.40	0.088	0.854	0.078	0.976	4.85	**9271**

Interpretation

- **GJR-GARCH-t wins** (lowest AIC)
- $\nu \approx 4.8$: **much heavier tails** than S&P 500 ($\nu = 6.4$)
- $\alpha = 0.131$ (BTC) vs 0.088 (S&P) — Bitcoin reacts faster to news
- Leverage effect weaker than for stocks ($\gamma_{\text{BTC}} = 0.078$ vs 0.126)

Step 5: Conditional Volatility — Bitcoin

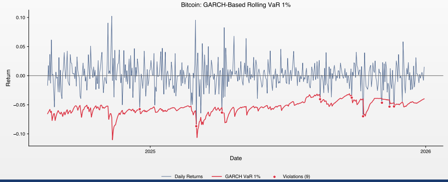


GJR-GARCH(1,1)-t Diagnostics

- Ljung-Box on z_t^2: p-value $= 0.41$ — **ARCH effects eliminated**
- Volatility peaks: March 2020 (COVID), May 2022 (Terra/Luna)
- Daily volatility: from 1% (calm periods) to >15% (crises)

TSA_ch5_btc_garch

Step 6: Backtesting Rolling VaR — Bitcoin



Statistical Tests (2421 days out-of-sample)

Test	Statistic	p-value	Decision
Violations	28/2421 ($\hat{p} = 1.16\%$)	—	Green zone
Kupiec (uc)	0.57	0.450	Accepted
Christoffersen (ind)	0.94	0.333	Accepted
Combined (cc)	1.51	0.471	Accepted

Interpretation

- Volatility ranges from 3% to 38% — rolling window is **essential**
- All tests **accepted**: model valid for risk management

TSA_ch5_btc_var

Final Comparison: S&P 500 vs Bitcoin

Comparative Summary

	S&P 500	**Bitcoin**
Average volatility	1.2%	3.5%
Kurtosis	13.8	12.1
Student-t ν	6.42	4.82
Best model	EGARCH(1,1)-t	GJR-GARCH(1,1)-t
Leverage effect	Strong ($\gamma = -0.12$)	Moderate ($\gamma = 0.078$)
Half-life	$\sim$60 days	$\sim$42 days
Rolling VaR 1% mean	2.53%	9.34%
Rolling VaR 1% max	22.02% (COVID)	37.54% (COVID)
Kupiec	Accepted (p=0.145)	Accepted (p=0.450)
Christoffersen (ind)	Accepted (p=0.375)	Accepted (p=0.333)

General Conclusion

- Re-estimating GARCH at each step: Kupiec + Christoffersen **accepted**
- Rolling window VaR: **mandatory** — static VaR is completely inadequate
- Student-t + asymmetric model: **essential** for both markets

Key Formulas

Volatility Models

- **ARCH(q):** $\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$
- **GARCH(1,1):** $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$
- **EGARCH:** $\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$
- **GJR-GARCH:** $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 I_{t-1} + \beta\sigma_{t-1}^2$

Properties and Measures

- **Unconditional variance:** $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ **Half-life:** $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$
- **VaR:** $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$ **Stationarity:** $\alpha + \beta < 1$
- **ARCH-LM:** $LM = T \cdot R^2 \sim \chi^2(q)$

Summary — Chapter 5: Volatility Models

Key Concepts

- **ARCH(q)**: conditional variance depends on past squared errors
- **GARCH(p,q)**: adds variance lags for persistence
- **EGARCH/GJR-GARCH**: capture leverage effect (asymmetric response)

Applications

Risk measurement (VaR, ES), derivative pricing, portfolio management

Practical Tip

Start with GARCH(1,1), check for leverage, choose distribution minimizing AIC/BIC!

References

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Online Resources and Code

- **Quantlet**: https://quantlet.com → Code repository for statistics
- **Quantinar**: https://quantinar.com → Learning platform for quantitative methods
- **GitHub TSA_ch5**: https://github.com/QuantLet/TSA/tree/main/TSA_ch5

Thank You!

Questions?

Course materials available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar