

Time Series Analysis and Forecasting

Chapter 0: Fundamentals



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Learning Objectives

By the end of this chapter, you will be able to:

1. **Define** time series and distinguish from cross-sectional and panel data
2. **Decompose** time series into trend-cycle, seasonal, and residual components
3. **Apply** exponential smoothing methods (SES, Holt, Holt-Winters, ETS)
4. **Evaluate** forecasts using MAE, RMSE, MAPE, sMAPE
5. **Implement** train/validation/test splits and cross-validation
6. **Model** seasonality using dummy variables or Fourier terms
7. **Remove** trend and seasonality through appropriate methods
8. **Distinguish** between deterministic and stochastic trends



Chapter Outline

- Motivation
- What is a Time Series?
- Time Series Decomposition
- Exponential Smoothing Methods
- Forecast Evaluation
- Modeling Seasonality
- Handling Trend and Seasonality
- Summary and Quiz



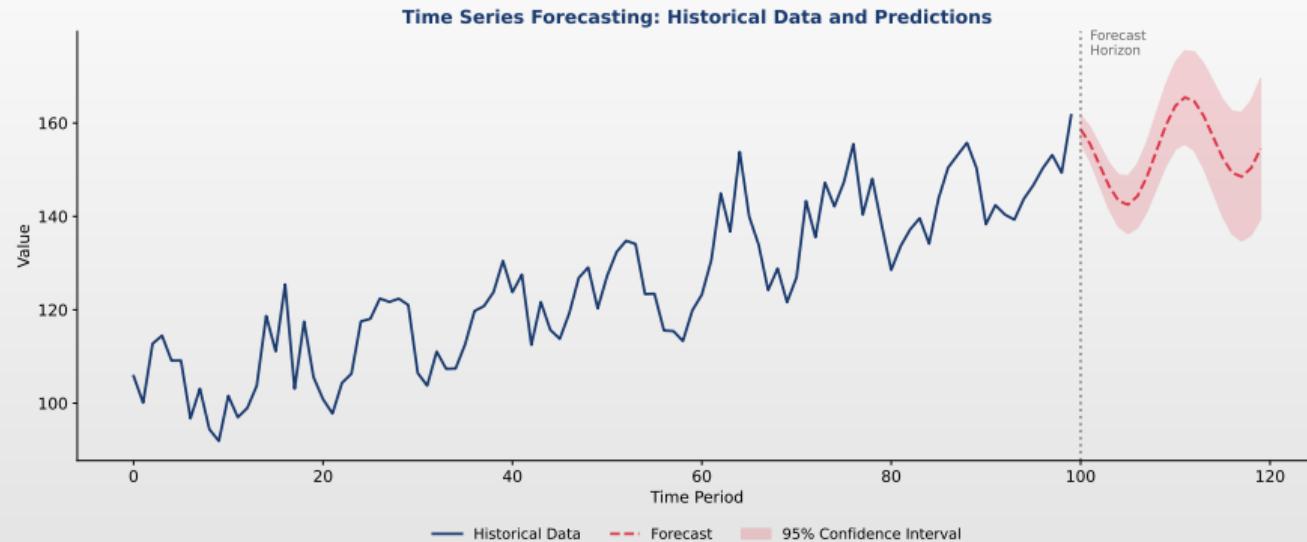
Time Series Are Everywhere



- Finance**: Stock prices, exchange rates, trading volumes
- Economics**: GDP, unemployment, inflation rates
- Business**: Sales, website traffic, customer demand
- Science**: Temperature, pollution levels, patient vitals



Why Study Time Series?

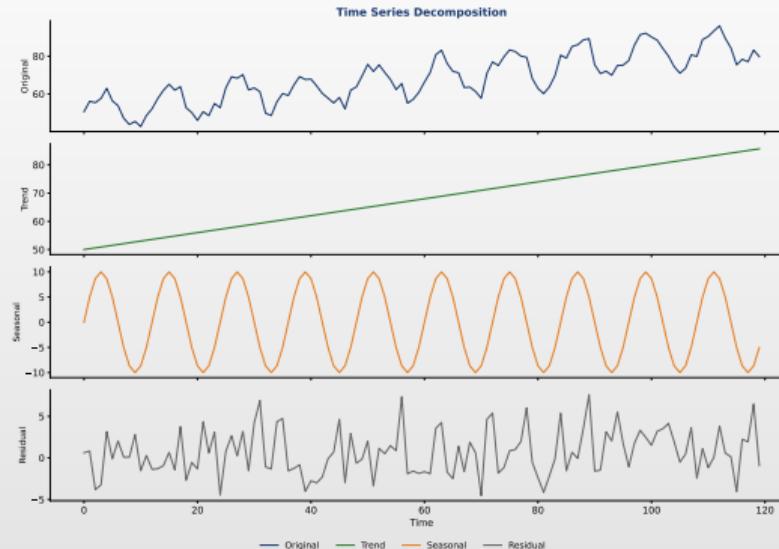


Key Goal: Forecasting

Use historical patterns to predict future values — critical for business planning, risk management, and policy decisions.



Understanding Time Series Structure



Decomposition

Every time series can be decomposed into interpretable components: trend-cycle, seasonality, and noise.



Definition of a Time Series

Definition 1 (Time Series)

A **time series** is a sequence of observations $\{X_t\}$ indexed by time:

$$\{X_t : t \in \mathcal{T}\}$$

where \mathcal{T} is an index set representing time points.

Key Characteristics

- Ordered:** Natural temporal ordering
- Dependent:** Consecutive observations correlated
- Discrete/Continuous:** $t = 1, 2, 3, \dots$

Notation

- X_t = observation at time t
- $\{X_t\}_{t=1}^T$ = series with T observations

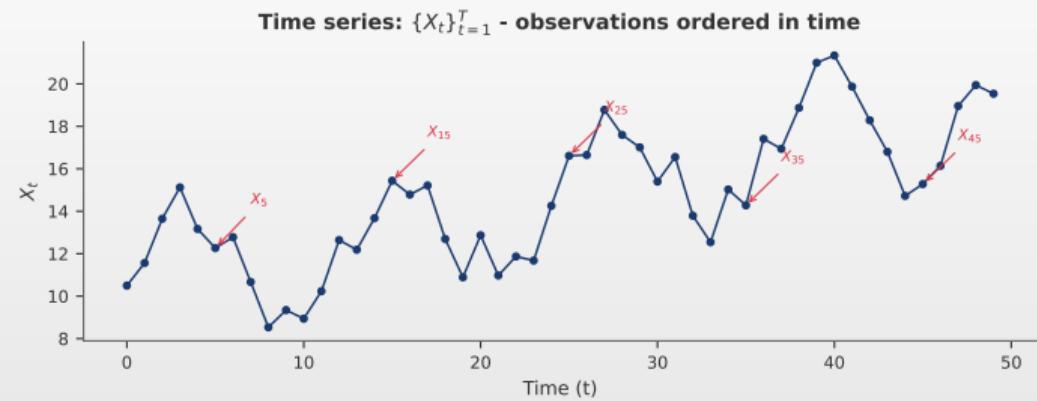


Time Series: Visual Illustration

Interpretation

Each point X_t represents an observation at time t . The sequence is ordered and consecutive observations are typically correlated.

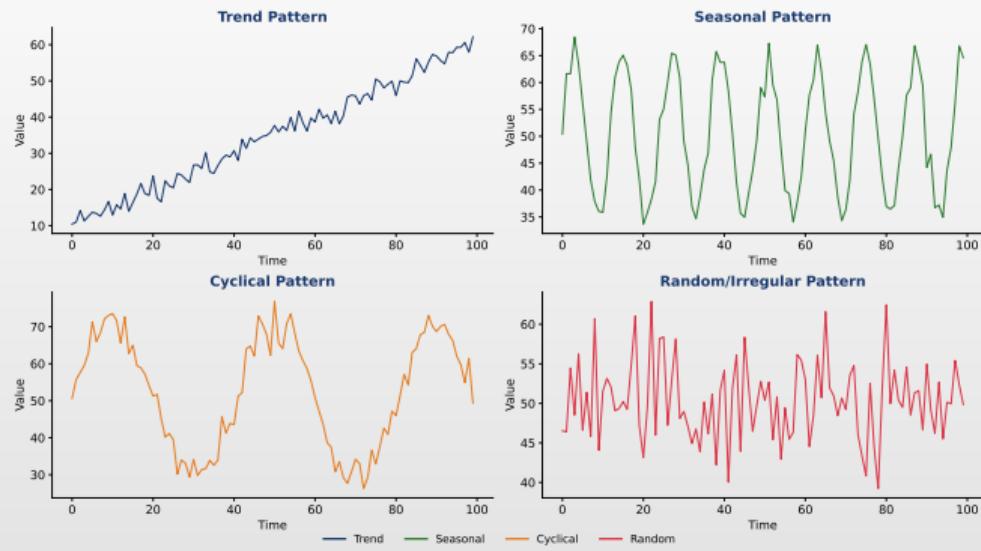
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Common Time Series Patterns

Pattern Types

- Trend:** Long-term increase or decrease
- Seasonal:** Regular periodic patterns
- Cyclical:** Medium-term fluctuations (2–10 years)
- Random:** Unpredictable fluctuations



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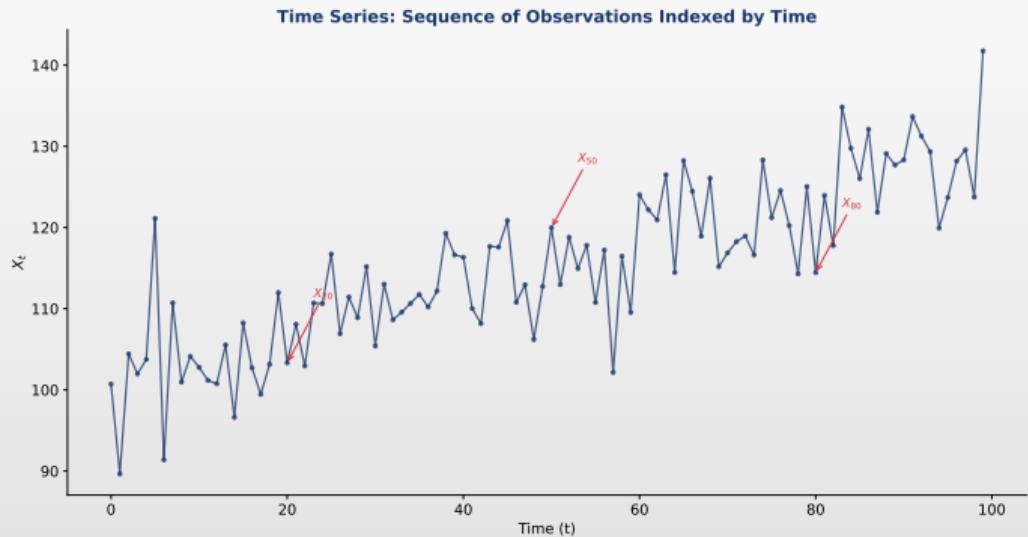


Time Series: Visual Definition

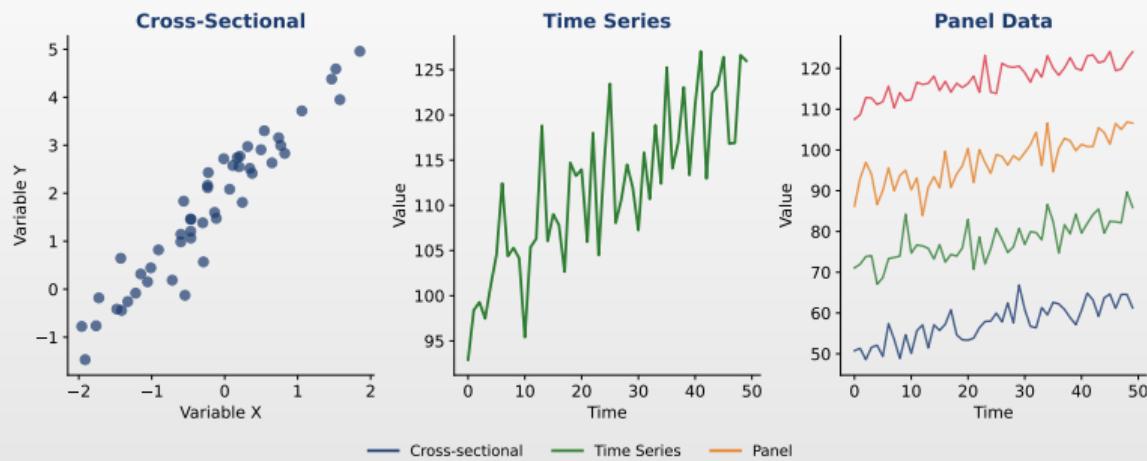
Interpretation

Each point X_t represents a measurement at discrete time t . The temporal ordering creates dependence between observations.
Data: S&P 500 (2024).

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Types of Data: Comparison



Data Type	Units (N)	Time (T)	Example
Cross-sectional	Many	1	Survey of 1000 households
Time series	1	Many	Daily S&P 500 prices
Panel	Many	Many	GDP of 50 countries, 20 years

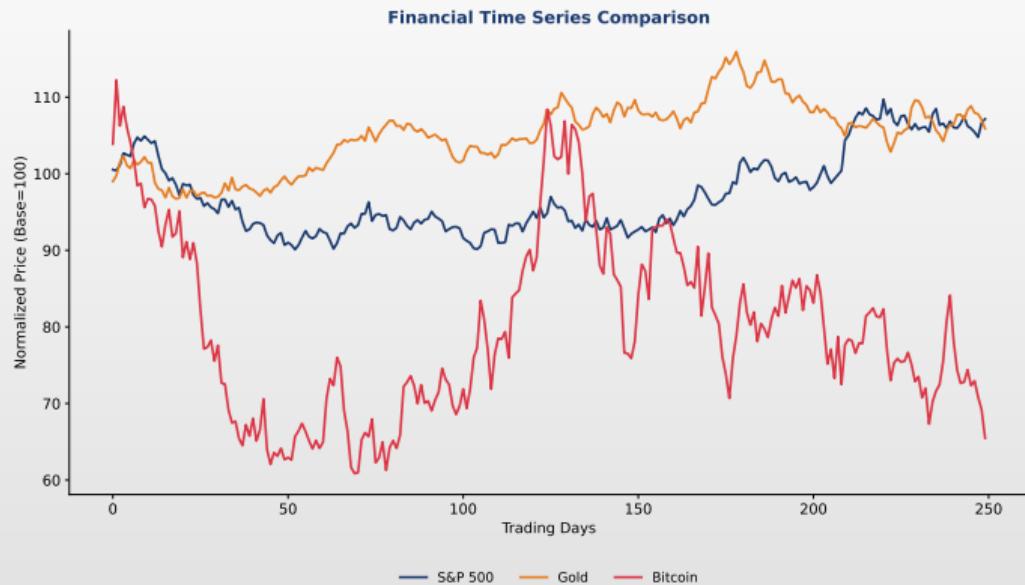


Examples of Time Series Data

Real Financial Data

Yahoo Finance (2019–2025),
normalized to base 100. Notice
different volatility patterns: Bitcoin
most volatile, Gold most stable.

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Why Decompose a Time Series?

Decomposition separates a time series into interpretable components:

Goals:

- Understand underlying patterns
- Remove seasonality for modeling
- Identify trend direction
- Isolate irregular fluctuations
- Improve forecasting accuracy

Components:

- T_t = **Trend-Cycle**: Long-term movement
- S_t = **Seasonal**: Regular periodic pattern
- ε_t = **Residual**: Random noise

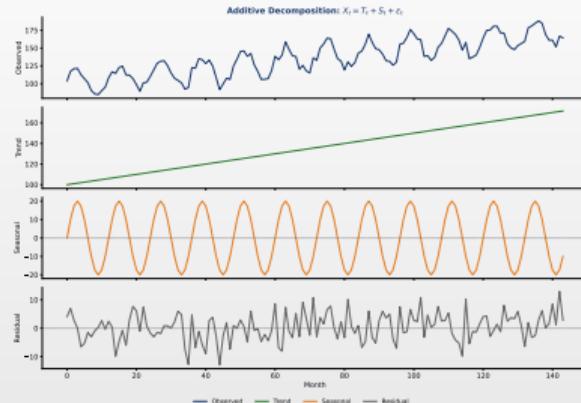
Note: Cyclical component is typically absorbed into T_t

Classical Decomposition Models

- Additive**: $X_t = T_t + S_t + \varepsilon_t$
- Multiplicative**: $X_t = T_t \times S_t \times \varepsilon_t$



Time Series Decomposition: Visual Example

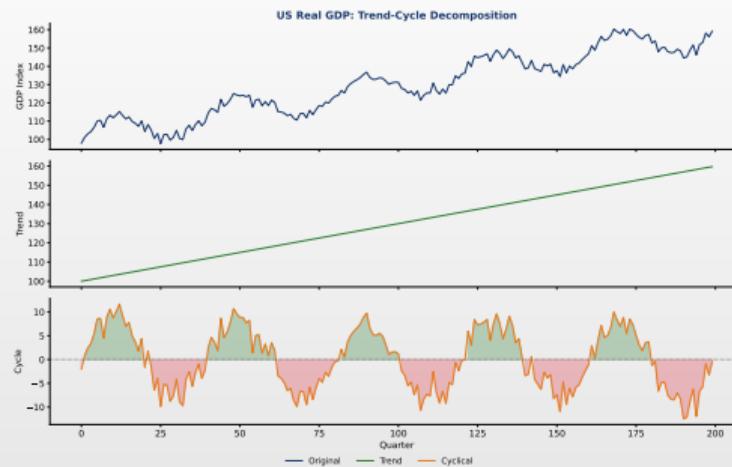


Components Explained

- Original:** observed series
- Trend-Cycle:** long-term movement
- Seasonal:** periodic pattern
- Residual:** random noise



The Cyclical Component



Characteristics

- Medium-term fluctuations (2–10 years)
- No fixed period (unlike seasonal)
- Reflects expansions/recessions

In Practice

- Cycle is often combined with trend
- Difficult to identify in short series
- Usually not modeled separately



Additive Decomposition Model

Model

$$X_t = T_t + S_t + \varepsilon_t \quad (1)$$

When to Use

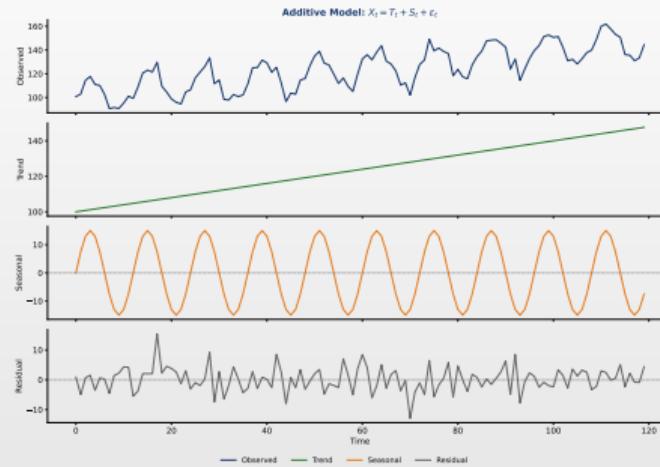
- Seasonal fluctuations are **constant** over time
- Variance of the series is **stable**

Properties

- $\mathbb{E}[\varepsilon_t] = 0$ (zero mean)
- $\sum_{j=1}^s S_j = 0$ (seasonal sums to zero)
- Units of S_t same as X_t



Additive Decomposition: US Retail Sales (FRED)



Interpretation

Original = Trend + Seasonal + Residual. Seasonal amplitude stays constant. Data: US Retail Sales (RSXFS) from FRED.



Multiplicative Decomposition Model

Model

$$X_t = T_t \times S_t \times \varepsilon_t \quad (2)$$

When to Use

- Seasonal fluctuations **grow** with series level
- Variance **increases** over time

Properties

- $\mathbb{E}[\varepsilon_t] = 1$ (centered at 1)
- $\frac{1}{s} \sum S_j = 1$ (averages to 1)
- S_t is dimensionless ratio

Tip

Log transform converts multiplicative to additive model: $\log X_t = \log T_t + \log S_t + \log \varepsilon_t$



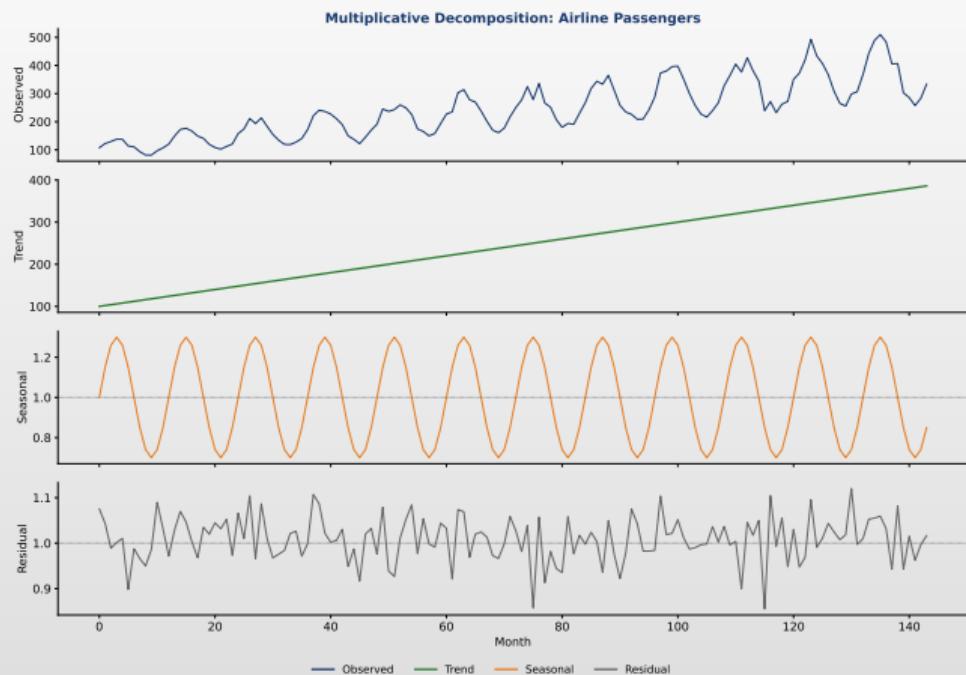
Multiplicative Decomposition: Real Data

Example

Classic Box-Jenkins airline passengers (1949–1960).

Seasonal amplitude grows with level.

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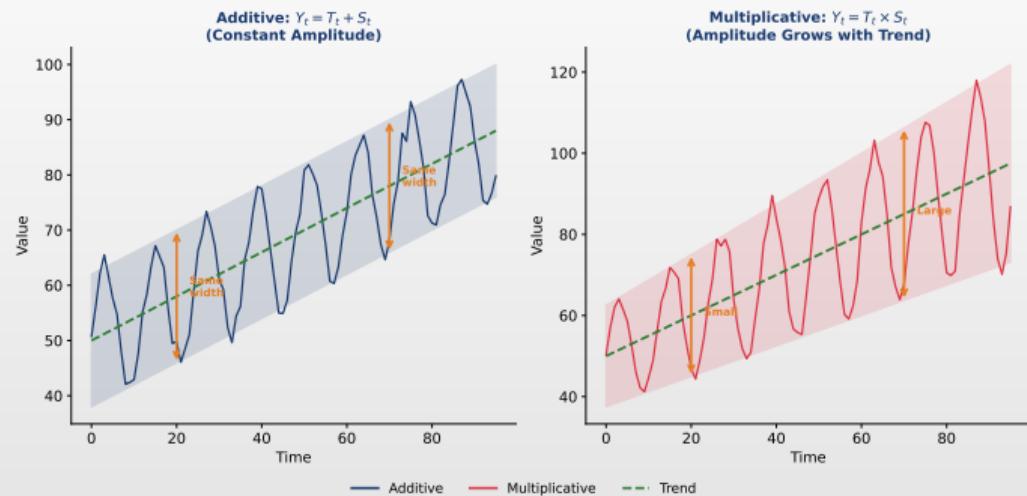


Additive vs Multiplicative: Comparison

Key Difference

- Multiplicative:** seasonal is a *ratio* (centered at 1)
- Additive:** seasonal in *absolute units* (centered at 0)

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Trend Estimation: Moving Average

Definition 2 (Centered Moving Average)

The **centered moving average** of order $2q + 1$ is:

$$\hat{T}_t = \frac{1}{2q+1} \sum_{j=-q}^q X_{t+j} \quad (3)$$

For Seasonal Data

- Period s **odd**: simple average
- Period s **even**: $2 \times s$ MA with half-weights

Properties

- Smooths seasonal & random
- Larger window \Rightarrow smoother
- Trade-off: lose endpoints

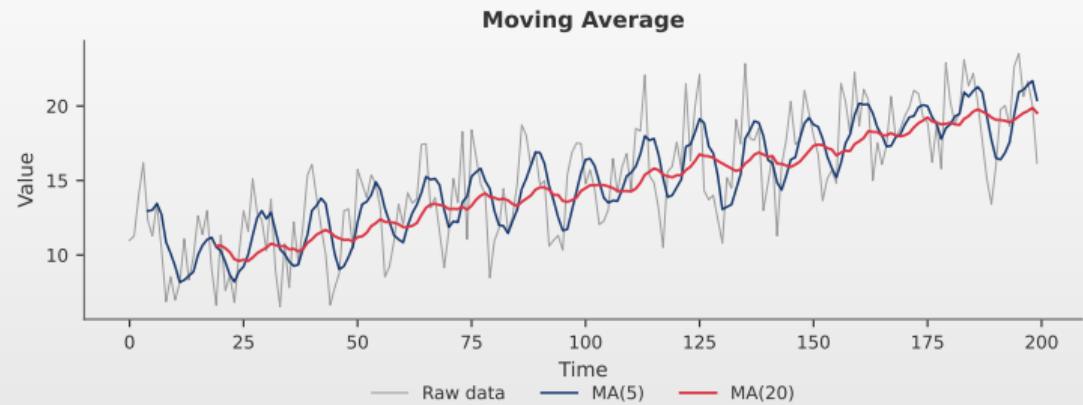


Centered Moving Average: Visual Illustration

Interpretation

The moving average smooths out short-term fluctuations, revealing the underlying trend.

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Classical Decomposition Algorithm

Steps for Multiplicative Decomposition

1. **Estimate Trend:** $\hat{T}_t = MA_s(X_t)$
2. **Detrend:** $D_t = X_t / \hat{T}_t$
3. **Estimate Seasonal:** $\hat{S}_j = \text{mean}(D_t \text{ for season } j)$
4. **Normalize:** Scale so $\frac{1}{s} \sum_{j=1}^s \hat{S}_j = 1$
5. **Compute Residuals:** $\hat{\varepsilon}_t = X_t / (\hat{T}_t \times \hat{S}_t)$

Note

For **additive** decomposition: replace division with subtraction and multiplication with addition.



Seasonal Indices: Interpretation

Interpretation

- $S_t > 1$: above-average activity
- $S_t < 1$: below-average activity
- Airline data shows peak travel in July–August

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STL Decomposition: A Modern Approach

Definition 3 (STL - Seasonal-Trend decomposition using LOESS)

STL uses locally weighted regression (LOESS): $X_t = T_t + S_t + R_t$

Advantages

- Any seasonal period
- Seasonal can change over time
- Robust to outliers
- Smooth trend estimates

Key Parameters

- `period`: Seasonal period
- `seasonal`: Smoothing window
- `robust`: Downweight outliers

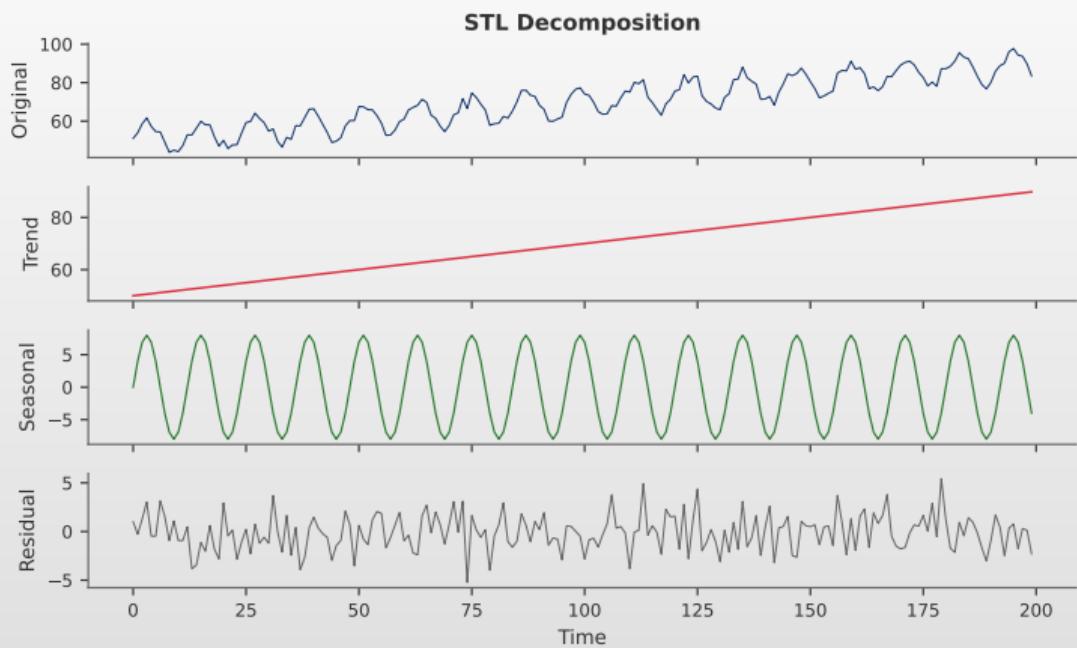


STL Decomposition: Visual Illustration

Key Insight

STL separates the series into trend, seasonal, and remainder using LOESS.

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Exponential Smoothing: Overview

Definition

Exponential smoothing produces forecasts based on weighted averages of past observations, with weights decaying exponentially.

Why Exponential Smoothing?

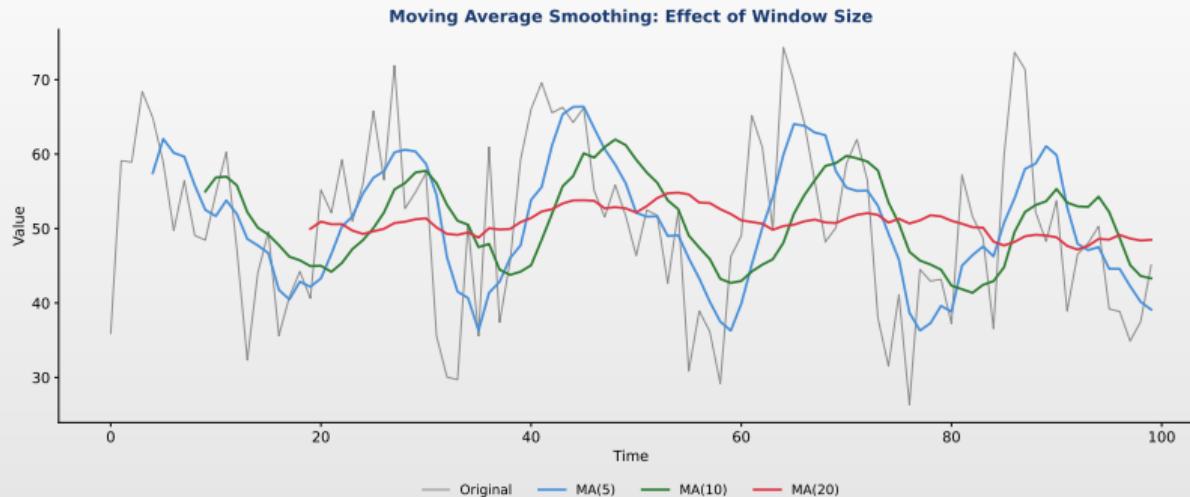
- Simple yet effective
- Recent obs. get higher weights
- Handles trend & seasonality
- Foundation for ETS models

Three Main Methods

1. **SES:** Level only
2. **Holt:** Level + Trend
3. **Holt-Winters:** + Seasonality



Moving Average Smoothing



Window Size Trade-off

- Small window:** Responsive but noisy
- Large window:** Smoother but slower to react



Simple Exponential Smoothing (SES)

Model

$$\hat{X}_{t+1|t} = \alpha X_t + (1 - \alpha) \hat{X}_{t|t-1} \quad (4)$$

where $\alpha \in (0, 1)$ is the **smoothing parameter**.

How It Works

- Weights decay exponentially
- Large α : responsive
- Small α : smoother

Level Form

$$\ell_t = \alpha X_t + (1 - \alpha) \ell_{t-1}$$

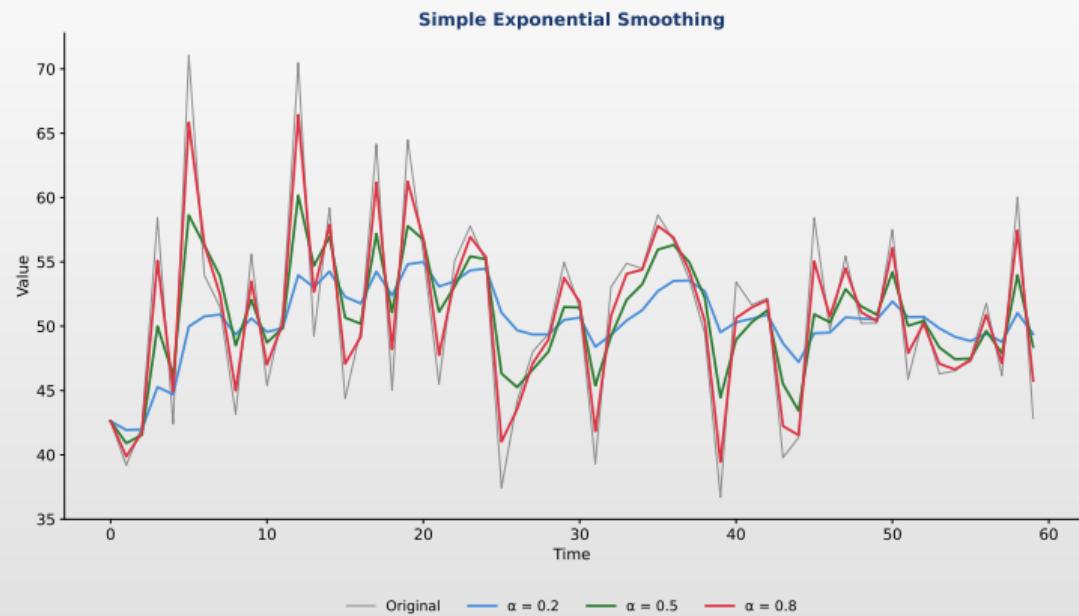


Simple Exponential Smoothing: Effect of α

Trade-off

Smaller α produces smoother forecasts; larger α follows data more closely.

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Holt's Linear Trend Method

Equations

- Level:** $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$
- Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$
- Forecast:** $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t$

Parameters

- α : Level smoothing
- β^* : Trend smoothing

Components

- ℓ_t : Estimated level
- b_t : Estimated trend (slope)

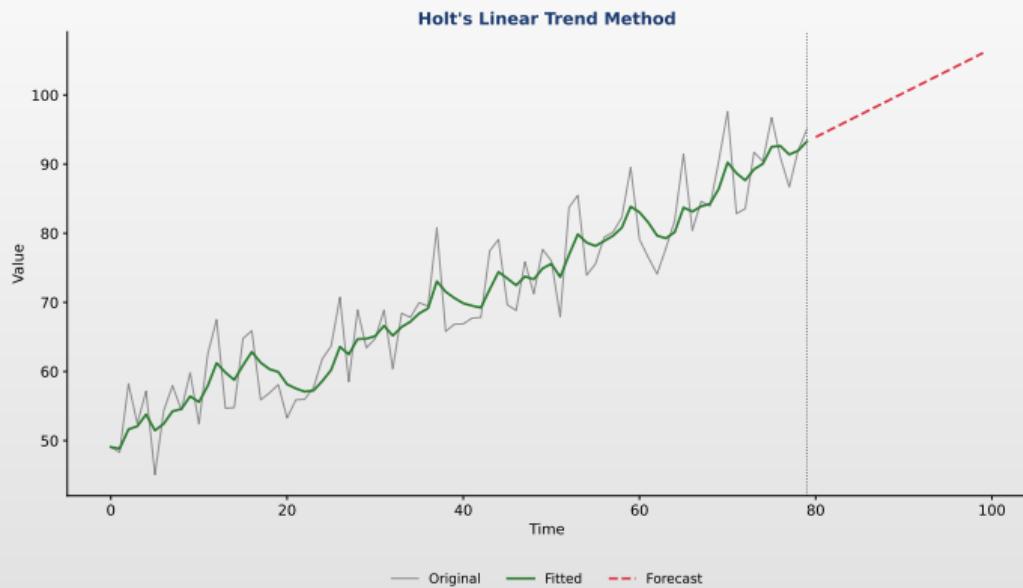


Holt's Method: Visualization

Interpretation

- Holt's method captures both level and trend
- Projects them into the forecast horizon
- α controls level changes
- β^* controls trend changes

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Holt-Winters Seasonal Method

Equations (Additive Seasonality)

- **Level:** $\ell_t = \alpha(X_t - S_{t-s}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})$
- **Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$
- **Seasonal:** $S_t = \gamma(X_t - \ell_t) + (1 - \gamma)S_{t-s}$
- **Forecast:** $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t + S_{t+h-s(k+1)}$

Parameters

- α : Level smoothing
- β^* : Trend smoothing
- γ : Seasonal smoothing
- s : Seasonal period

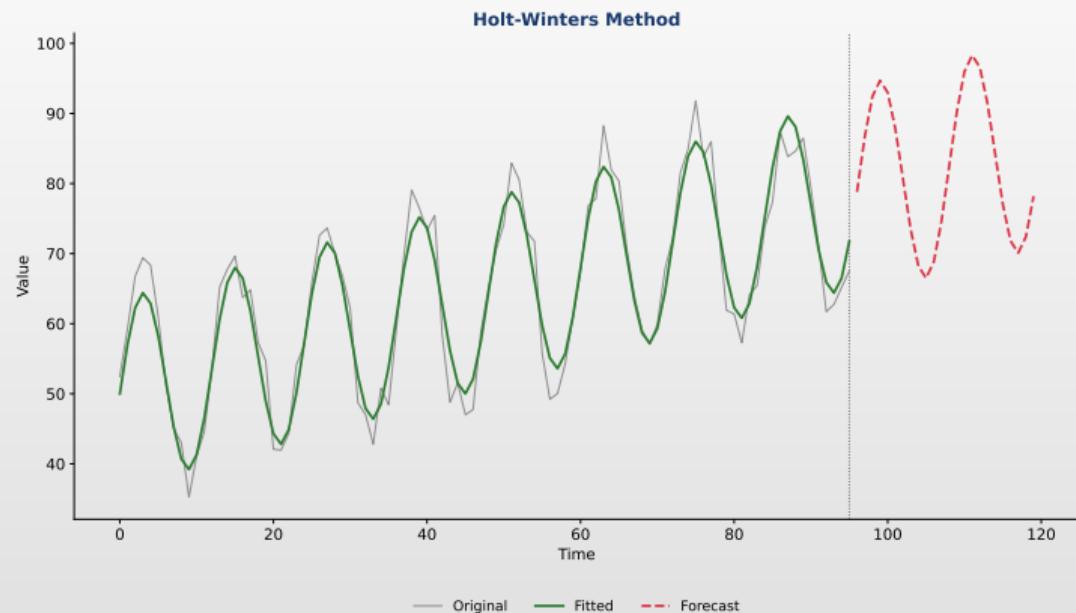


Holt-Winters: Capturing Seasonality

Key Feature

Holt-Winters decomposes the series and produces seasonal forecasts with trend.

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ETS Framework: Error-Trend-Seasonal

Definition 4 (ETS Models)

The **ETS framework** generalizes exponential smoothing: $\text{ETS}(E, T, S)$

Component	N	A	M
Error (E)	–	Additive	Multiplicative
Trend (T)	None	Additive	Multiplicative
Seasonal (S)	None	Additive	Multiplicative

Examples

- $\text{ETS}(A, N, N) = \text{Simple Exponential Smoothing}$
- $\text{ETS}(A, A, N) = \text{Holt's Linear Method}$
- $\text{ETS}(A, A, A) = \text{Holt-Winters Additive}$

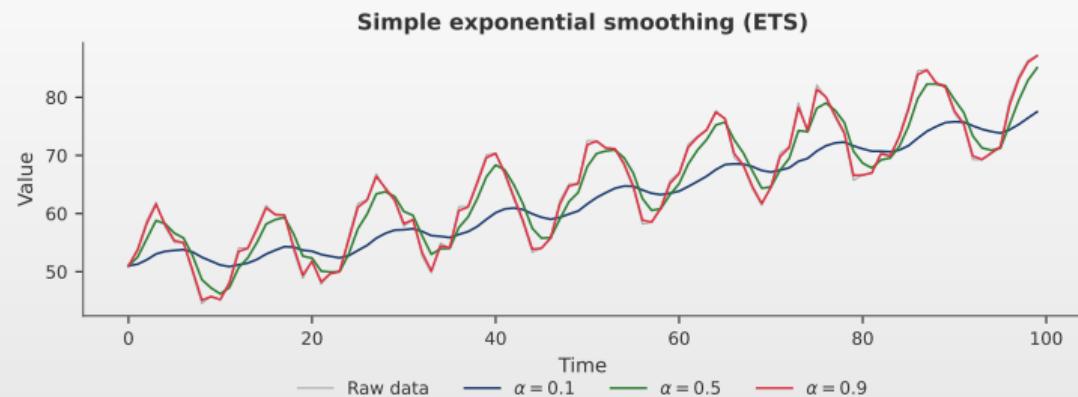


ETS: Exponential Smoothing Illustration

Interpretation

ETS models use exponentially weighted observations for forecasting. Weights decay as observations get older.

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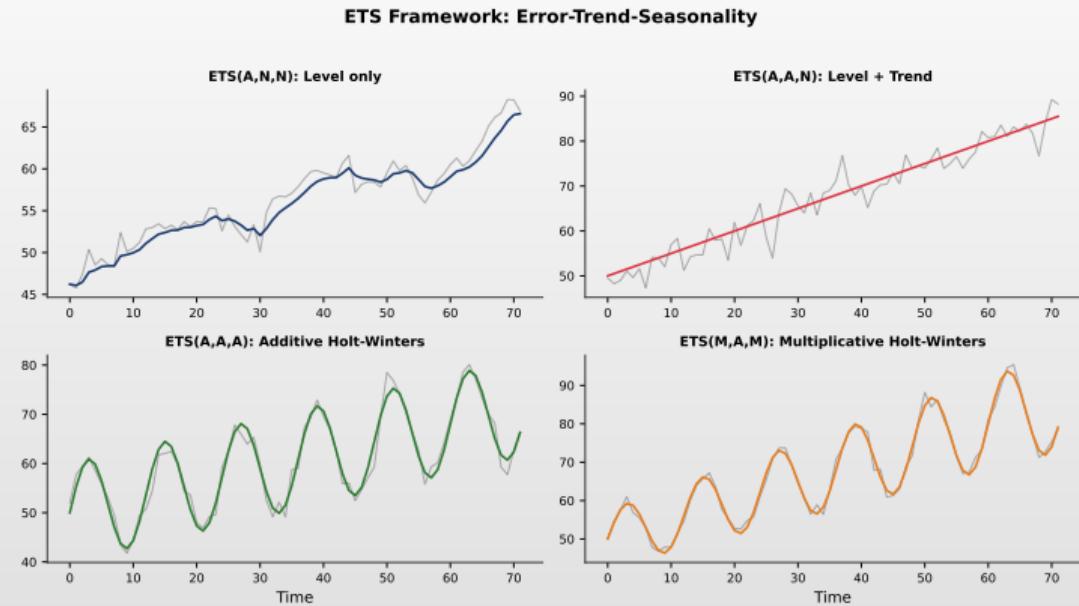


ETS Model Selection

Interpretation

The ETS framework provides a systematic way to choose the best model using AIC/BIC.

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Damped Trend Methods

Damping Parameter

Introduces $\phi \in (0, 1)$ to prevent over-projection

Equations

- **Level:** $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$
- **Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$
- **Forecast:** $\hat{X}_{t+h|t} = \ell_t + \phi^{\frac{1-\phi^h}{1-\phi}} b_t$

Key Insight

- As $h \rightarrow \infty$: forecast \rightarrow constant
- Prevents unrealistic long-term extrapolation
- Often best for longer horizons



Forecast Accuracy Metrics

Forecast Error

- $e_t = X_t - \hat{X}_t$ (actual minus predicted)

Scale-Dependent

- $MAE = \frac{1}{n} \sum |e_t|$
- $MSE = \frac{1}{n} \sum e_t^2$
- $RMSE = \sqrt{MSE}$

Scale-Independent

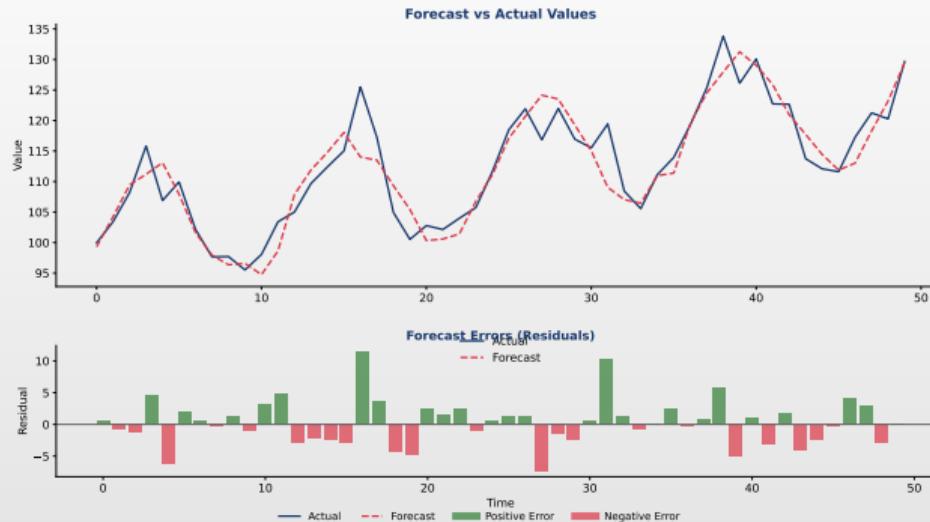
- $MAPE = \frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$
- $sMAPE = \frac{100}{n} \sum \frac{|e_t|}{(|X_t| + |\hat{X}_t|)/2}$

Which to use?

- Same series: RMSE, MAE
- Compare across series: MAPE, sMAPE



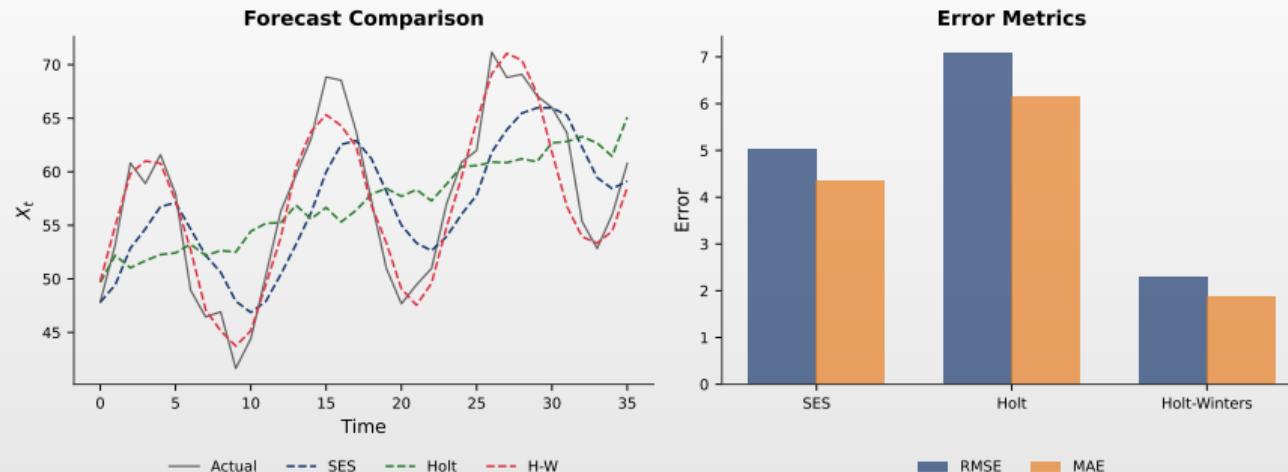
Forecast Evaluation: Visual Example



- ☐ **Top:** Actual values vs. forecasted values – visual assessment of fit
- ☐ **Bottom:** Residuals should be centered around zero with no pattern
- ☐ Good forecasts have small, random residuals with constant variance



Comparing Forecast Methods



Interpretation

- Left:** Comparing SES, Holt, and Holt-Winters forecasts
- Right:** Error metrics for each method



Residual Diagnostics

Residual Properties

Good forecasts should have residuals that are:

1. **Zero mean:** $\mathbb{E}[e_t] = 0$
2. **Uncorrelated:** $\text{Cov}(e_t, e_{t-k}) = 0$
3. **Constant variance:** $\text{Var}(e_t) = \sigma^2$
4. **Normally distributed**

Diagnostic Tests

Ljung-Box test (autocorrelation):

$$Q = T(T+2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{T-k} \sim \chi_h^2$$

Jarque-Bera test (normality):

$$JB = \frac{T}{6} \left(S^2 + \frac{(K-3)^2}{4} \right) \sim \chi_2^2$$

S = skewness, K = kurtosis

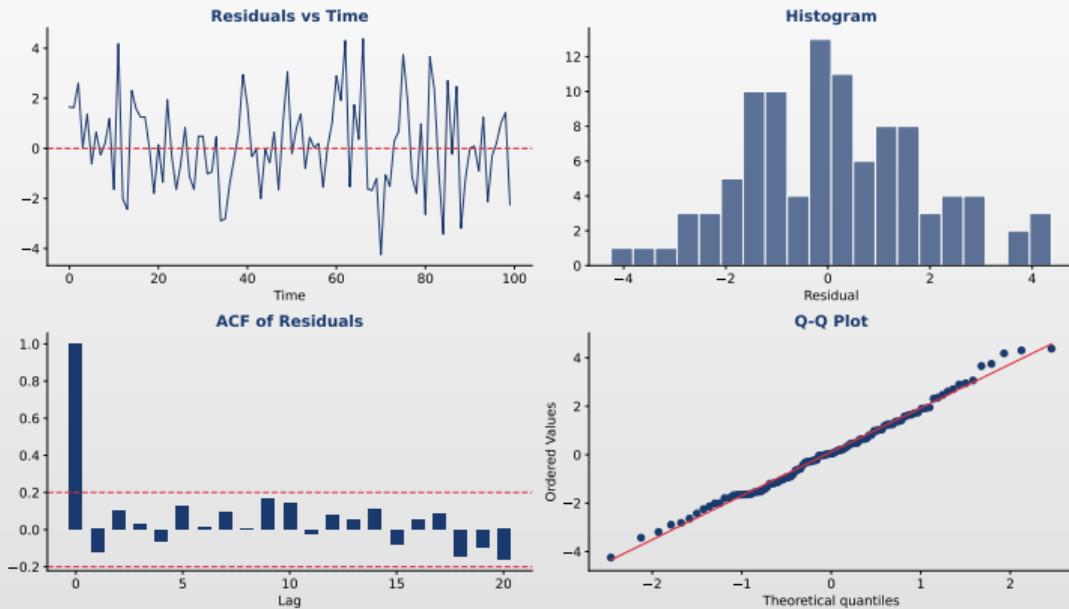


Residual Diagnostics: Visualization

What to Check

- Time plot (no patterns)
- Histogram (normality)
- ACF (no autocorrelation)
- Q-Q plot (normality)

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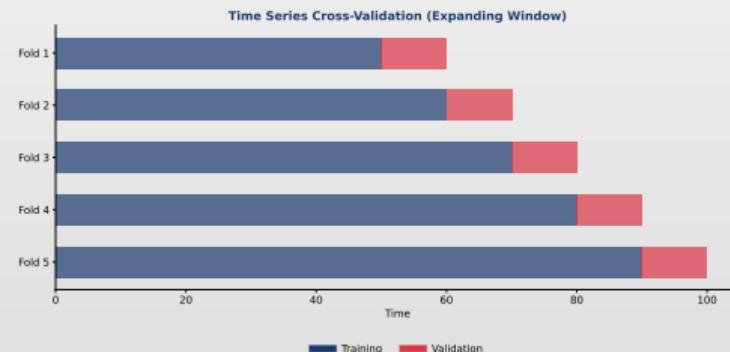
Time Series Cross-Validation

Why Not Standard CV?

- Time series have temporal dependence
- Future data cannot predict the past
- Standard k-fold causes data leakage

Rolling Origin CV

1. Train on $\{X_1, \dots, X_t\}$
2. Forecast \hat{X}_{t+h}
3. Increment t , repeat



Train / Validation / Test Split

Three-way split for model development:

Training Set

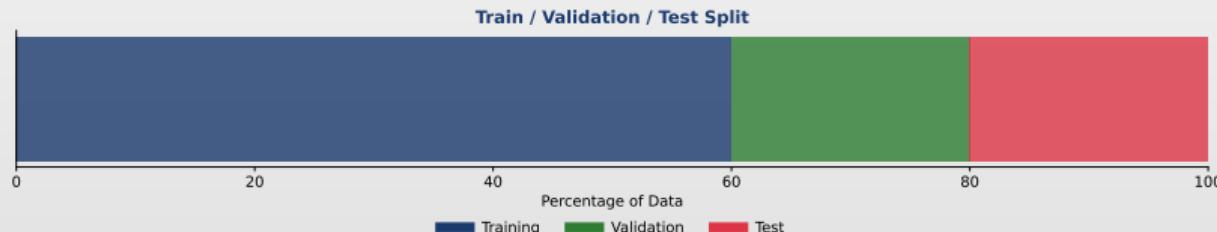
- Fit model parameters
- Largest portion (60–80%)
- Used for estimation

Validation Set

- Tune hyperparameters
- Compare models
- Select best approach

Test Set

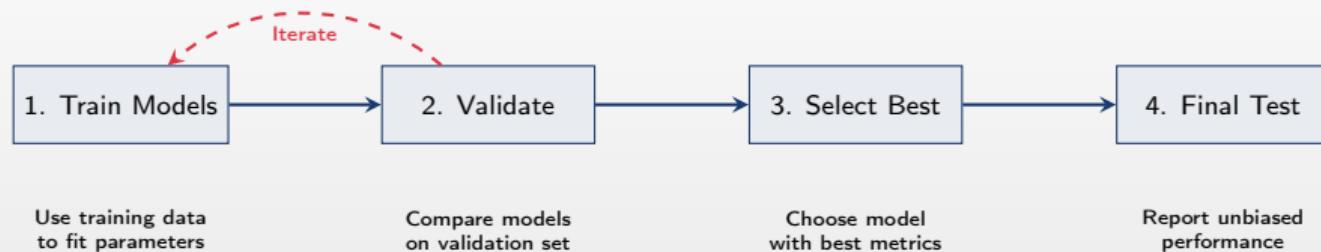
- Final evaluation only
- Never used for tuning
- Unbiased performance



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Model Development Workflow



Critical Rule

Never use test set for model selection! This causes *data leakage* and overly optimistic performance estimates.

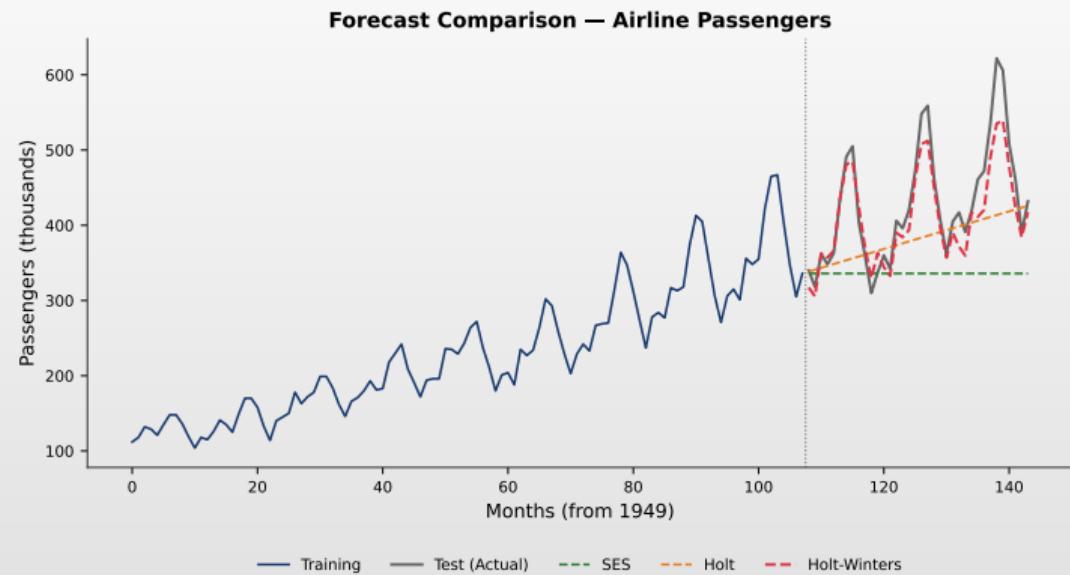


Real Data: Forecast Comparison

Interpretation

Airline passengers data:
Holt-Winters Multiplicative
performs best for seasonal data.

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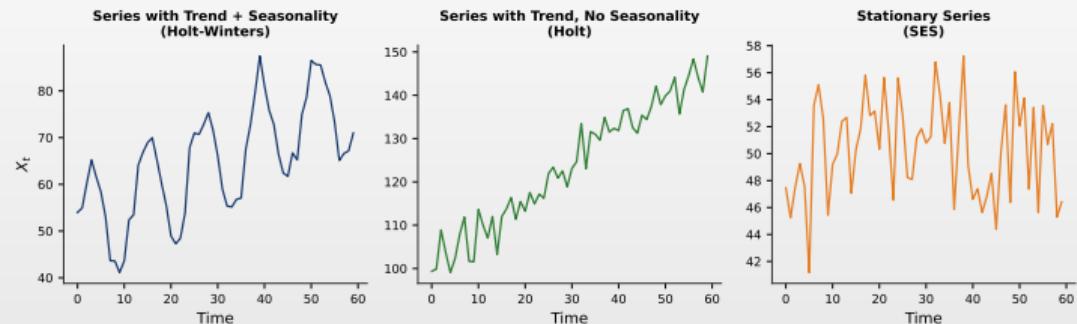
Forecast Performance Across Datasets

Interpretation

Different series require different models. Seasonal data needs seasonal methods.

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Different Series Require Different Models



Modeling Seasonality: Two Approaches

1. Dummy Variables:

$$X_t = \mu + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$$

- $D_{jt} = 1$ if t in season j
- $s - 1$ parameters
- Any seasonal pattern

2. Fourier Terms:

$$X_t = \mu + \sum_{k=1}^K [\alpha_k \sin(\cdot) + \beta_k \cos(\cdot)]$$

- Sinusoidal functions
- $2K$ parameters
- Smooth patterns

Trade-off

- Dummies:** any pattern, more parameters
- Fourier:** smooth, fewer parameters

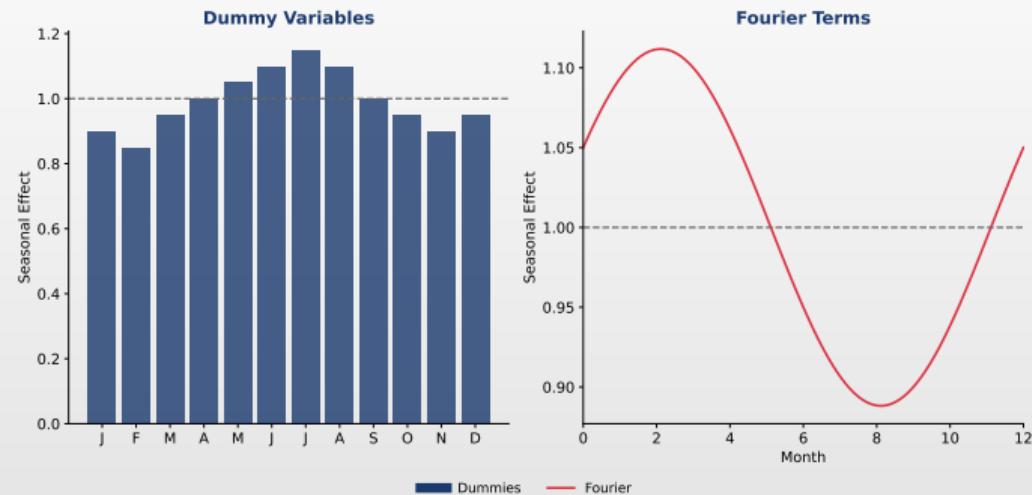


Dummy Variables vs Fourier Terms

Comparison

- **Dummies:** capture any shape but need $s - 1$ parameters
- **Fourier:** uses $2K$ parameters for smooth patterns

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Choosing Between Dummies and Fourier

Criterion	Dummies	Fourier
Parameters (monthly)	11	$2K$ (often 4–6)
Seasonal pattern	Any shape	Smooth/sinusoidal
Interpretation	Direct (month effects)	Frequency components
High-frequency seasons	Many parameters	Efficient
Multiple seasonality	Complex	Easy (add terms)

Guidelines

- Use **dummies**: irregular patterns, interpretable coefficients
- Use **Fourier**: smooth patterns, high-frequency seasonality, multiple periods
- Fourier terms** are used in TBATS and Facebook Prophet



Why Remove Trend and Seasonality?

Before modeling, we often need to make series stationary:

Reasons to detrend:

- Stationarity requirement
- Focus on fluctuations
- Avoid spurious regression
- Enable valid inference

Reasons to deseasonalize:

- Reveal underlying trend
- Compare across seasons
- Simplify modeling
- Focus on irregular component

Important

After modeling the detrended/deseasonalized series, we must **reverse the transformation** for forecasting.



Trend Removal Methods

Six Common Detrending Approaches

1. **Differencing:** $\Delta X_t = X_t - X_{t-1}$
2. **Linear regression:** $\hat{T}_t = \hat{\beta}_0 + \hat{\beta}_1 t$
3. **Polynomial:** Higher-order polynomial
4. **HP Filter:** Balance fit vs smoothness
5. **Moving average:** $\hat{T}_t = MA_q(X_t)$
6. **LOESS:** Local polynomial regression

Choice Depends On

- Nature of trend (deterministic vs stochastic)
- Purpose (forecasting vs analysis)

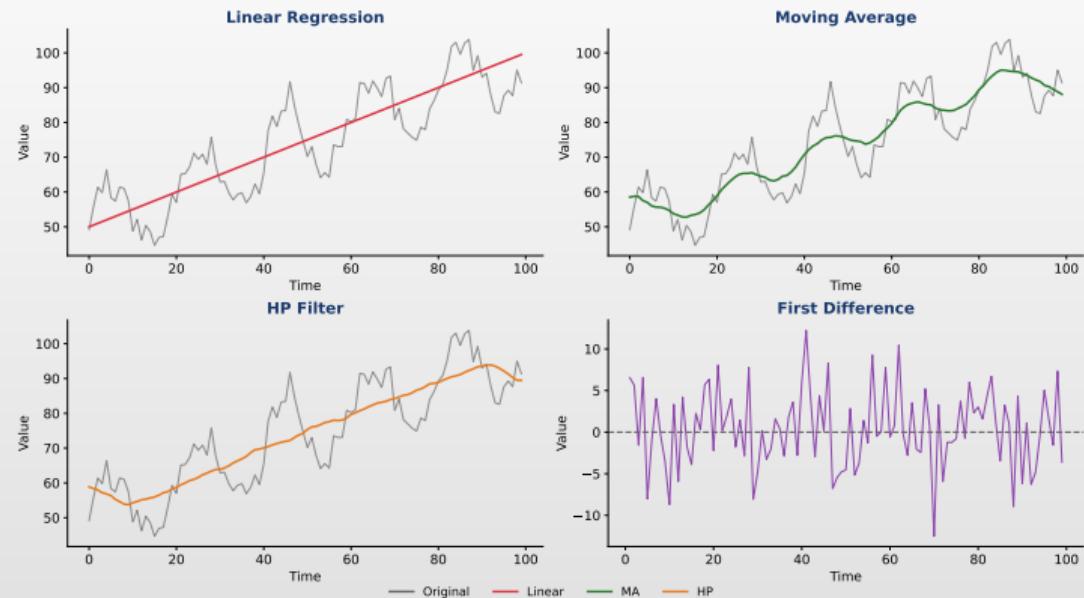


Detrending Methods: Comparison

Key Insight

Different methods produce different residuals. Choose based on trend type and analysis goals.

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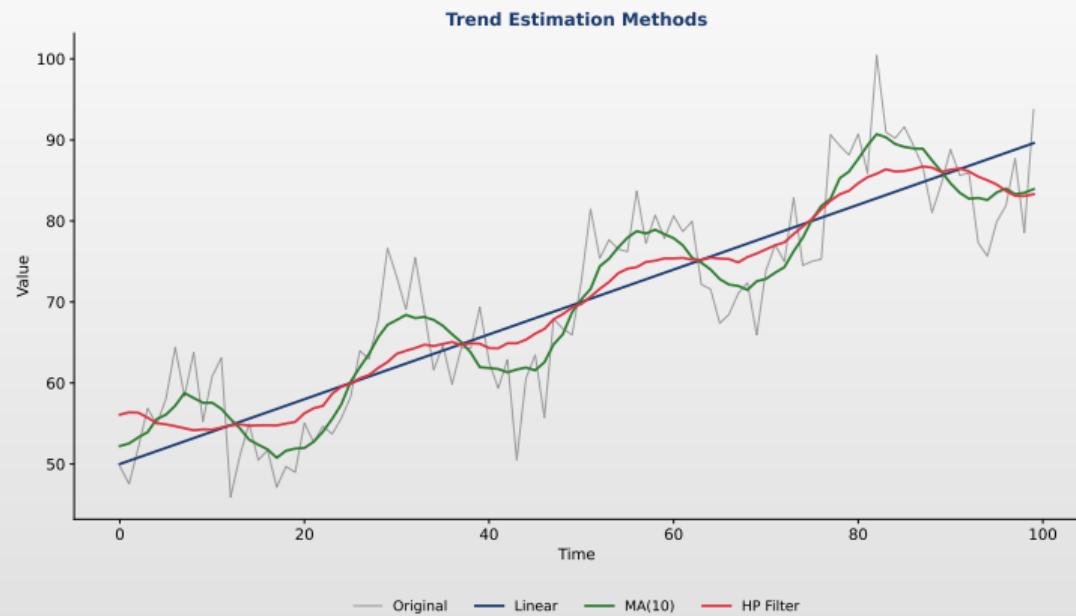


Trend Estimation: Multiple Approaches

Interpretation

Different methods capture trend at varying levels of smoothness.

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Hodrick-Prescott (HP) Filter

Definition 5 (HP Filter)

The **HP filter** decomposes X_t into trend τ_t and cycle c_t : $X_t = \tau_t + c_t$, by minimizing:

$$\min_{\{\tau_t\}} \left\{ \sum_{t=1}^T (\tau_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right\}$$

Interpretation

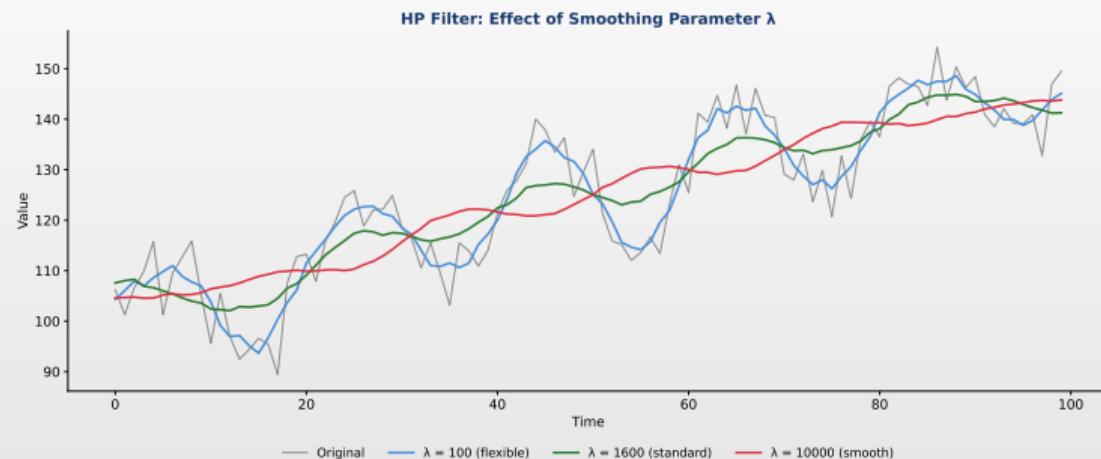
- First term: fit to data
- Second term: smoothness penalty
- λ : trade-off parameter

Standard λ Values

- Annual: $\lambda = 6.25$
- Quarterly: $\lambda = 1600$
- Monthly: $\lambda = 129600$



HP Filter: Effect of λ

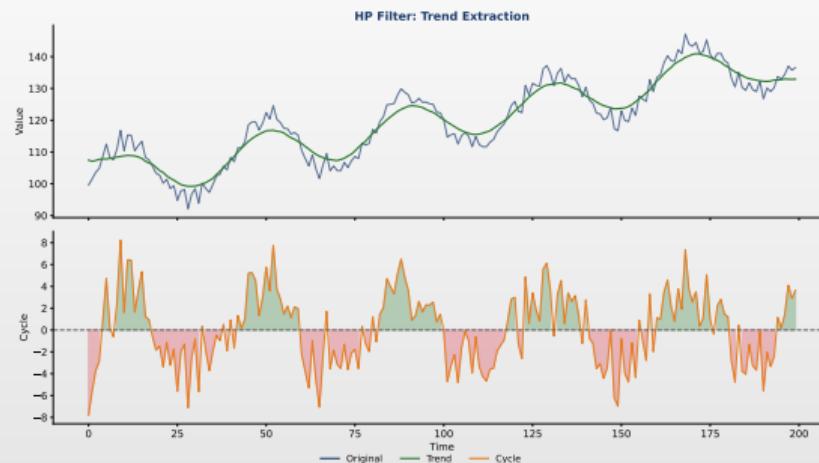


Trade-off

- Small λ :** Trend follows data closely (more flexible)
- Large λ :** Trend becomes smoother (approaches linear trend)



HP Filter: Business Cycle Extraction



Application

HP filter is widely used in macroeconomics to extract business cycles from GDP and other economic series.

Q TSA_ch0_hp_cycle



HP Filter: Limitations

Known Issues

- End-point problem:** Trend estimates unreliable at endpoints
- Spurious cycles:** Can create artificial dynamics
- λ **choice:** Results sensitive to parameter
- Non-stationary:** Assumes trend is smooth

Alternatives

- Band-pass filters:** Baxter-King, Christiano-Fitzgerald
- Hamilton filter:** Regression-based
- Unobserved components:** State-space models

Hamilton (2018) Critique

"Why You Should Never Use the Hodrick-Prescott Filter" — suggests using regression on lagged values instead.



Seasonality Removal Methods

Four Approaches to Remove Seasonality

1. **Seasonal differencing:** $\Delta_s X_t = X_t - X_{t-s}$
2. **Division** (multiplicative): $X_t^{adj} = X_t / \hat{S}_t$
3. **Subtraction** (additive): $X_t^{adj} = X_t - \hat{S}_t$
4. **X-13ARIMA-SEATS:** Government statistical method

Seasonal Period s

- Monthly $\Rightarrow s = 12$
- Quarterly $\Rightarrow s = 4$



Seasonal Adjustment: Visualization

Result

Seasonally adjusted series reveals underlying trend without periodic fluctuations.

Q TSA_ch0_seasonal_adj



Deterministic vs Stochastic Trend

Deterministic Trend:

$$X_t = \beta_0 + \beta_1 t + \varepsilon_t$$

Stochastic Trend:

$$X_t = X_{t-1} + \varepsilon_t$$

- Trend is a function of time
- Detrend by regression
- ε_t is stationary
- Random walk component
- Detrend by differencing
- ΔX_t is stationary

Wrong Method = Problems

- Differencing deterministic trend \Rightarrow over-differencing
- Regression on stochastic trend \Rightarrow spurious regression

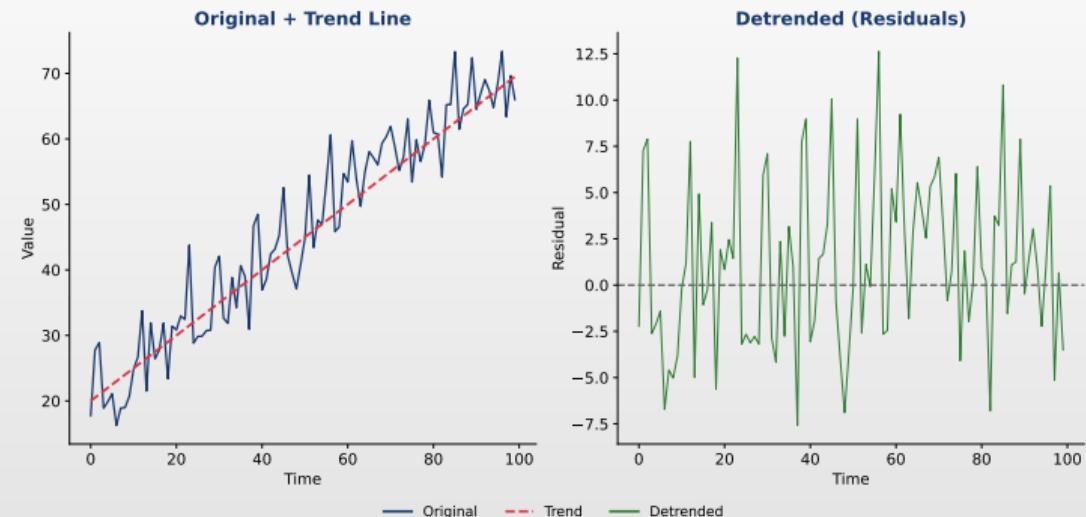


Example: Deterministic Trend

Key

Use **regression** to remove trend
→ residuals are stationary (ACF decays quickly).

Q TSA_ch0_det_trend

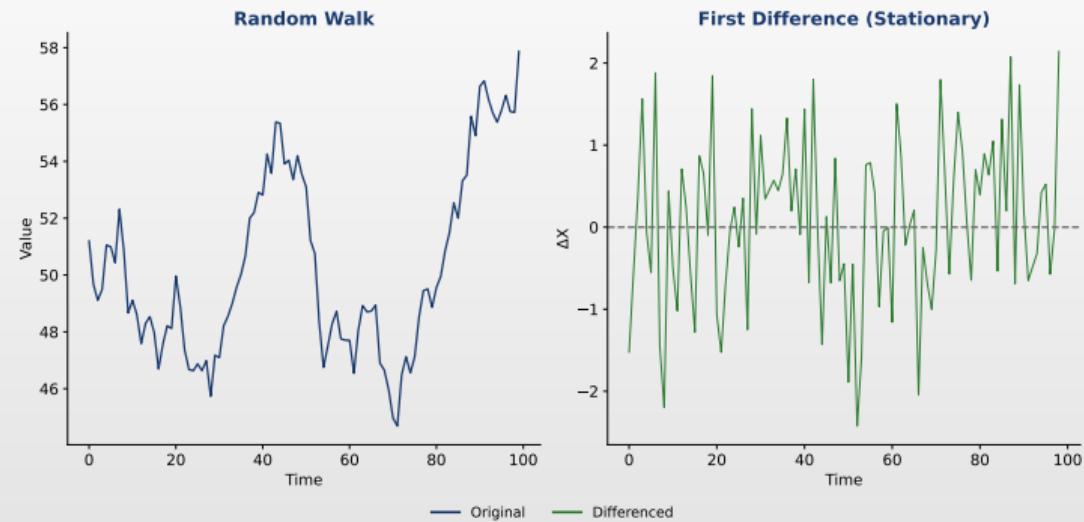


Example: Stochastic Trend (Random Walk)

Key

Use **differencing** to remove trend
→ differences are stationary
(white noise).

Q [TSA_ch0_stoch_trend](#)

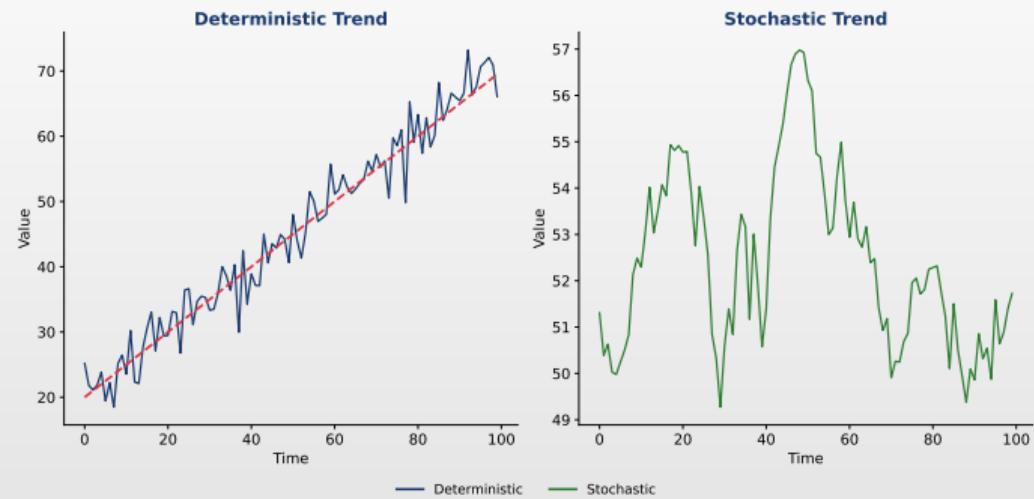


Side-by-Side Comparison

Remember

- Deterministic trend → regression
- Stochastic trend → differencing

Q TSA_ch0_trend_compare



Summary

What We Learned

- **Time Series Definition:** Sequence of observations indexed by time
- **Decomposition:** Trend-Cycle + Seasonal + Residual components
- **Exponential Smoothing:** SES, Holt, Holt-Winters, ETS framework
- **Forecast Evaluation:** MAE, RMSE, MAPE; train/validation/test splits

Key Takeaway

- **Understand Before Modeling:**
 - ▶ Always visualize and decompose your data first
 - ▶ Choose additive vs multiplicative based on variance behavior



Quick Quiz

1. What is the difference between additive and multiplicative decomposition?
2. When should you use Holt-Winters instead of simple exponential smoothing?
3. Why can't we use standard k-fold cross-validation for time series?
4. What does $\alpha = 0.9$ mean in exponential smoothing?
5. How do you distinguish between deterministic and stochastic trend?



Quiz Answers

1. **Additive vs Multiplicative:** Additive when seasonal amplitude is constant; multiplicative when it grows with the level.
2. **Holt-Winters:** When data has both trend AND seasonality. SES only handles level.
3. **Time Series CV:** Standard k-fold ignores temporal order — would use future data to predict the past (data leakage).
4. $\alpha = 0.9$: High weight on recent observations, forecast reacts quickly to changes but is more volatile.
5. **Trend type:** Deterministic — predictable function of time (use regression). Stochastic — random walk component (use differencing).



What Comes Next?

Chapter 1: Stochastic Processes and Stationarity

- Stochastic Processes:** Mathematical foundation for time series
 - ▶ Random variables indexed by time
 - ▶ Strict vs weak (covariance) stationarity
- Key Processes:** White noise and random walk
 - ▶ Building blocks for ARIMA models
 - ▶ Understanding mean reversion vs unit roots
- ACF and PACF:** Tools for model identification
 - ▶ Detecting autocorrelation structure
 - ▶ Choosing AR and MA orders

Questions?

