



Chapter 5: VAR & Granger Causality

Seminar



Seminar Outline

- 1 Review Quiz
- 2 True/False Questions
- 3 Practice Problems
- 4 Worked Examples
- 5 Real Data Analysis
- 6 Discussion Topics
- 7 Exercises for Self-Study

Quiz 1: VAR Definition

Question

In a VAR(2) model with 3 variables, how many coefficient matrices \mathbf{A}_i are there?

- A) 2
- B) 3
- C) 6
- D) 9

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- (B) 3
- (C) 6
- (D) 9

Answer: A – 2 coefficient matrices

VAR(p) model: $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$

VAR(2) with $K = 3$:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \\ Y_{3t} \end{pmatrix} = \mathbf{c} + \underbrace{\mathbf{A}_1}_{3 \times 3} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{pmatrix} + \underbrace{\mathbf{A}_2}_{3 \times 3} \begin{pmatrix} Y_{1,t-2} \\ Y_{2,t-2} \\ Y_{3,t-2} \end{pmatrix} + \boldsymbol{\varepsilon}_t$$

Key: $p = \text{number of lags} = \text{number of matrices}$

Quiz 2: Number of Parameters

Question

A VAR(2) with $K = 3$ variables (including constants) has how many parameters to estimate per equation?

- A) 3
- B) 6
- C) 7
- D) 9

Quiz 2: Number of Parameters

Question

A VAR(2) with $K = 3$ variables (including constants) has how many parameters to estimate per equation?

- (A) 3
- (B) 6
- (C) 7
- (D) 9

Answer: C – 7 parameters per equation

VAR(p) Parameter Count: The Curse of Dimensionality

Parameters per equation: $1 + K \times p$

Total parameters: $K(1 + Kp) + K(K + 1)/2$

(coefficients + covariance matrix)

Model	Coefficients	Total
$K=2, p=1$	$2(1+2\times 1) = 6$	$+ 3 = 9$
$K=3, p=2$	$3(1+3\times 2) = 21$	$+ 6 = 27$
$K=5, p=4$	$5(1+5\times 4) = 105$	$+ 15 = 120$
$K=10, p=4$	$10(1+10\times 4) = 410$	$+ 65 = 485$

Quiz 3: Granger Causality

Question

"X Granger-causes Y" means:

- A) X is the economic cause of Y
- B) Past X helps predict future Y
- C) X and Y are contemporaneously correlated
- D) X always increases when Y increases

Quiz 3: Granger Causality

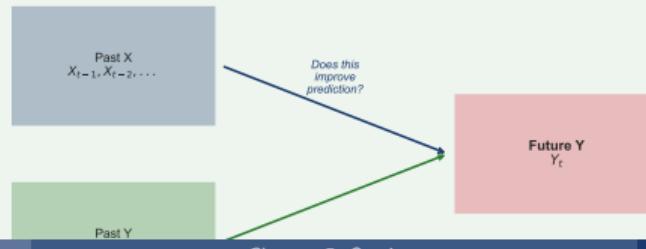
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Answer: B – Past X helps predict future Y

Granger Causality: Predictive, Not Causal!



Quiz 4: Granger Causality Test

Question

To test if Y_2 Granger-causes Y_1 in a VAR(p), we test:

- A) All coefficients in the Y_1 equation equal zero
- B) Coefficients on lagged Y_2 in the Y_1 equation equal zero
- C) Coefficients on lagged Y_1 in the Y_2 equation equal zero
- D) The error covariance equals zero

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- C) Coefficients on lagged Y_1 in the Y_2 equation equal zero
- D) The error covariance equals zero

Answer: B – Coefficients on lagged Y_2 in Y_1 equation = 0

Null hypothesis: $H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$

Test statistic: Wald or F-test with p restrictions

Interpretation:

- Reject H_0 : Y_2 Granger-causes Y_1
- Don't reject: No evidence of predictive relationship

Note: Test $Y_1 \rightarrow Y_2$ separately (different coefficients in Y_2 equation)

Quiz 5: VAR Stability

Question

A VAR(1) model is stable (stationary) if:

- (A) All diagonal elements of \mathbf{A}_1 are less than 1
- (B) The determinant of \mathbf{A}_1 is less than 1
- (C) All eigenvalues of \mathbf{A}_1 are less than 1 in absolute value
- (D) The trace of \mathbf{A}_1 equals zero

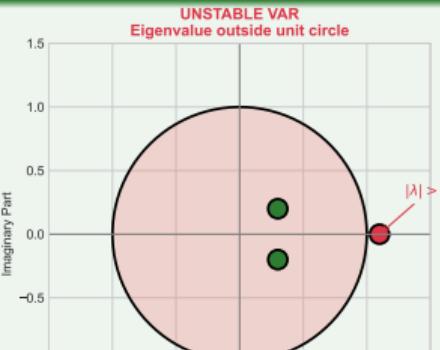
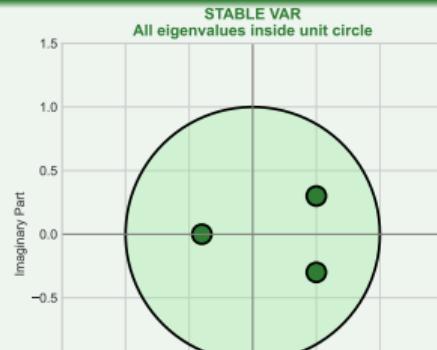
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Answer: C – All eigenvalues of \mathbf{A}_1 inside unit circle



Quiz 6: Impulse Response Functions

Question

An impulse response function shows:

- A) The correlation between two variables
- B) The effect of a shock to one variable on all variables over time
- C) The forecast accuracy of the model
- D) The p-values of coefficient tests

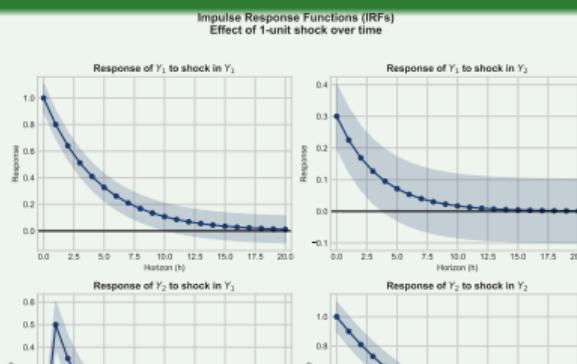
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- B) The effect of a shock to one variable on all variables over time
- C) The forecast accuracy of the model
- D) The p-values of coefficient tests

Answer: B – Effect of shock on all variables over time



Quiz 7: Lag Order Selection

Question

Which criterion typically selects the most parsimonious VAR model?

- A) AIC (Akaike Information Criterion)
- B) BIC (Bayesian Information Criterion)
- C) FPE (Final Prediction Error)
- D) Adjusted R^2

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- (D) Adjusted R^2

Answer: B – BIC (Bayesian Information Criterion)

Penalty comparison (for k parameters, n observations):

- AIC: $-2 \ln L + 2k$
- BIC: $-2 \ln L + k \ln n$

Since $\ln n > 2$ for $n > 8$, BIC penalizes complexity more heavily

Practical guidance:

- Forecasting: AIC may perform better

Quiz 8: Granger Causality Interpretation

Question

"X Granger-causes Y" means:

- (A) X is the true cause of Y
- (B) Past values of X help predict Y beyond Y 's own past
- (C) X and Y are correlated
- (D) Y depends only on X

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Answer: B

Granger causality is about **predictive** content, not true causation. X Granger-causes Y if lagged X terms are jointly significant in the equation for Y , after controlling for lagged Y .

Quiz 9: Forecast Error Variance Decomposition

Question

FEVD (Forecast Error Variance Decomposition) tells us:

- A) The correlation between variables
- B) What proportion of forecast error variance comes from each shock
- C) The optimal forecast horizon
- D) Which variables to include in the model

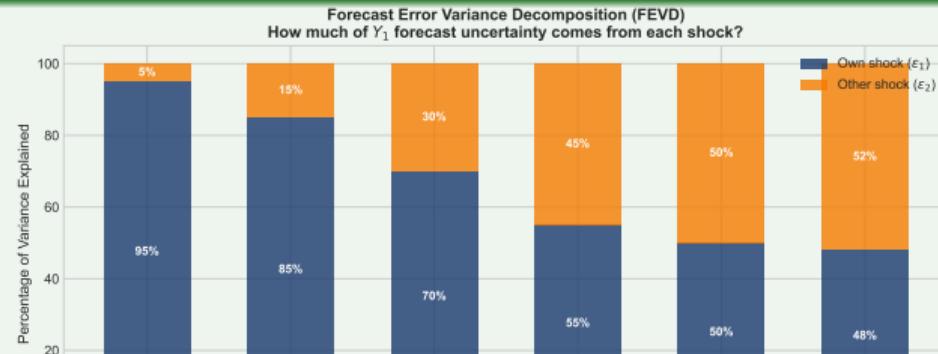
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Answer: B – Proportion of forecast error variance from each shock



Quiz 10: Structural vs Reduced Form VAR

Question

The difference between structural VAR (SVAR) and reduced-form VAR is:

- A) SVAR has more variables
- B) SVAR allows contemporaneous effects between variables
- C) SVAR uses different estimation methods
- D) There is no difference

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- B) SVAR allows contemporaneous effects between variables
- C) SVAR uses different estimation methods
- D) There is no difference

Answer: B

Reduced-form VAR: shocks are correlated, no contemporaneous effects in equations. SVAR: imposes identifying restrictions to recover structural shocks with economic interpretation (e.g., monetary policy shock).

Quiz 11: Cholesky Decomposition

Question

Cholesky ordering in IRF analysis assumes:

- (A) All variables are equally important
- (B) Variables ordered first affect later variables contemporaneously, not vice versa
- (C) Shocks are uncorrelated
- (D) No restrictions are needed

Quiz 11: Cholesky Decomposition

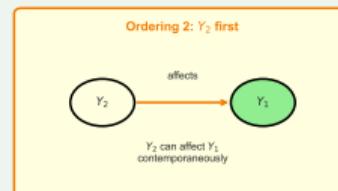
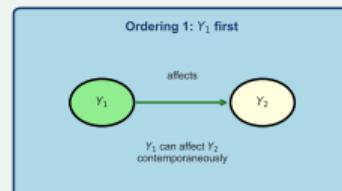
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- C) Shocks are uncorrelated
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Answer: B – Variables ordered first affect later ones contemporaneously

Cholesky Ordering: Order Matters!



Key Points:

- Different orderings give DIFFERENT IRFs!

Quiz 12: VAR Residual Diagnostics

Question

In a well-specified VAR, residuals should be:

- A) Autocorrelated but homoskedastic
- B) White noise (no autocorrelation)
- C) Normally distributed only
- D) Correlated across equations

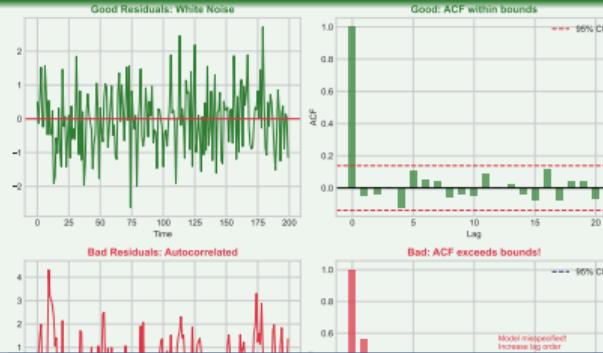
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Answer: B – White noise (no autocorrelation)



Quiz 13: Cointegration and VAR

Question

If variables are $I(1)$ and cointegrated, you should use:

- A) VAR in levels
- B) VAR in first differences
- C) Vector Error Correction Model (VECM)
- D) Univariate ARIMA models

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- A) VAR in levels
- B) VAR in first differences
- C) Vector Error Correction Model (VECM)
- D) Univariate ARIMA models

Answer: C

With cointegration, VAR in differences loses long-run information, while VAR in levels may be inefficient. VECM incorporates both short-run dynamics and long-run equilibrium relationships through the error correction term.

Quiz 14: Instantaneous Causality

Question

Instantaneous causality differs from Granger causality because it tests:

- (A) Lagged relationships only
- (B) Contemporaneous correlation of residuals
- (C) Long-run relationships
- (D) Model stability

Quiz 14: Instantaneous Causality

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- (A) Lagged relationships only
- (B) Contemporaneous correlation of residuals
- (C) Long-run relationships
- (D) Model stability

Answer: B

Instantaneous causality tests whether shocks to X and Y are correlated within the same period (correlation of VAR residuals). Granger causality tests whether lagged values help predict.

True/False Questions

Determine if each statement is True or False:

- ① VAR models treat all variables as endogenous.
- ② Granger causality proves true economic causation.
- ③ A stable VAR always has eigenvalues inside the unit circle.
- ④ FEVD results depend on the ordering of variables.
- ⑤ VAR can be estimated by OLS equation by equation.
- ⑥ Impulse responses eventually die out in a stable VAR.

Answers on next slide...

True/False: Solutions

- ① VAR models treat all variables as endogenous.

TRUE

Each variable is regressed on lags of all variables, including itself.

- ② Granger causality proves true economic causation.

FALSE

It only shows predictive content, not structural causation.

- ③ A stable VAR always has eigenvalues inside the unit circle.

TRUE

Stability condition: all eigenvalues of companion matrix satisfy $|\lambda_i| < 1$.

- ④ FEVD results depend on the ordering of variables.

TRUE

Under Cholesky identification, different orderings give different results.

- ⑤ VAR can be estimated by OLS equation by equation.

TRUE

With same regressors in each equation, OLS = GLS = ML (under normality).

- ⑥ Impulse responses eventually die out in a stable VAR.

TRUE

Stability ensures shocks have transitory effects; IRFs $\rightarrow 0$ as $h \rightarrow \infty$.

Problem 1: Writing VAR Equations

Exercise

Write out the two equations for a bivariate VAR(1) model with variables Y_t (GDP growth) and X_t (inflation).

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Write out the two equations for a bivariate VAR(1) model with variables Y_t (GDP growth) and X_t (inflation).

Solution

$$Y_t = c_1 + a_{11} Y_{t-1} + a_{12} X_{t-1} + \varepsilon_{1t}$$

$$X_t = c_2 + a_{21} Y_{t-1} + a_{22} X_{t-1} + \varepsilon_{2t}$$

Interpretation:

- a_{12} : Effect of past inflation on current GDP growth
- a_{21} : Effect of past GDP growth on current inflation

Problem 2: Parameter Count

Exercise

How many total parameters need to be estimated in a VAR(3) with $K = 4$ variables (including constants)?

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How many total parameters need to be estimated in a VAR(3) with $K = 4$ variables (including constants)?

Solution

Per equation: $1 + K \times p = 1 + 4 \times 3 = 13$ parameters

Total for $K = 4$ equations: $4 \times 13 = 52$ parameters

Plus covariance matrix Σ : $K(K + 1)/2 = 4 \times 5/2 = 10$ unique elements

Grand total: 62 parameters

This is why VARs can be “over-parameterized” with limited data!

Problem 3: Granger Causality Interpretation

Exercise

A Granger causality test yields:

- H_0 : Money does not Granger-cause GDP. $p\text{-value} = 0.02$
- H_0 : GDP does not Granger-cause Money. $p\text{-value} = 0.35$

Interpret these results.

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- H_0 : GDP does not Granger-cause Money. $p\text{-value} = 0.35$

Interpret these results.

Solution

- Reject H_0 at 5%: Money **Granger-causes** GDP
- Fail to reject H_0 : GDP does **not** Granger-cause Money

Conclusion: Unidirectional causality: Money \rightarrow GDP

Interpretation: Past money supply helps predict GDP growth. This is consistent with monetarist views, but remember: Granger causality \neq structural causality!

Problem 4: Stability Check

Exercise

For VAR(1) with $\mathbf{A}_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.1 & 0.5 \end{pmatrix}$, check stability.

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Solution

Find eigenvalues: $\det(\mathbf{A}_1 - \lambda \mathbf{I}) = 0$

$$(0.7 - \lambda)(0.5 - \lambda) - (0.2)(0.1) = 0$$

$$\lambda^2 - 1.2\lambda + 0.33 = 0$$

$$\lambda = \frac{1.2 \pm \sqrt{1.44 - 1.32}}{2} = \frac{1.2 \pm 0.346}{2}$$

$$\lambda_1 = 0.773, \quad \lambda_2 = 0.427$$

Both $|\lambda_i| < 1 \Rightarrow \mathbf{Stable!}$

Problem 5: IRF Computation

Exercise

For VAR(1) with $\mathbf{A} = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$, compute Φ_2 (response at $h = 2$).

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Solution

$$\begin{aligned}\Phi_2 &= \mathbf{A}^2 = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix} \\ &= \begin{pmatrix} 0.25 + 0 & 0.10 + 0.12 \\ 0 + 0 & 0 + 0.36 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.22 \\ 0 & 0.36 \end{pmatrix}\end{aligned}$$

Interpretation: A unit shock to Y_2 at t increases Y_1 by 0.22 at $t + 2$.

Example: Stock Returns and Trading Volume

Scenario

You have daily data on stock returns (R_t) and trading volume (V_t). You want to test for Granger causality in both directions.

Typical Findings in Finance Literature

- Returns often Granger-cause volume (price changes trigger trading)
- Volume sometimes Granger-causes returns (volume as leading indicator)
- Results: Often **bidirectional** causality $R \leftrightarrow V$

Practical Issue

Stock returns are typically stationary, but volume may need transformation (log or difference).

Example: Interest Rates and Inflation

Taylor Rule Context

Central banks set interest rates (i_t) in response to inflation (π_t):

$$i_t = r^* + \pi^* + 1.5(\pi_t - \pi^*) + 0.5(y_t - y^*)$$

VAR Analysis

- Does inflation Granger-cause interest rates? (Should, if central bank reacts)
- Do interest rates Granger-cause inflation? (Monetary policy transmission)

Expected: Bidirectional causality with:

- Quick response: $\pi \rightarrow i$ (policy reaction)
- Delayed response: $i \rightarrow \pi$ (policy takes effect)

Python VAR Analysis: Key Functions

Essential Libraries

```
from statsmodels.tsa.api import VAR  
from statsmodels.tsa.stattools import grangercausalitytests
```

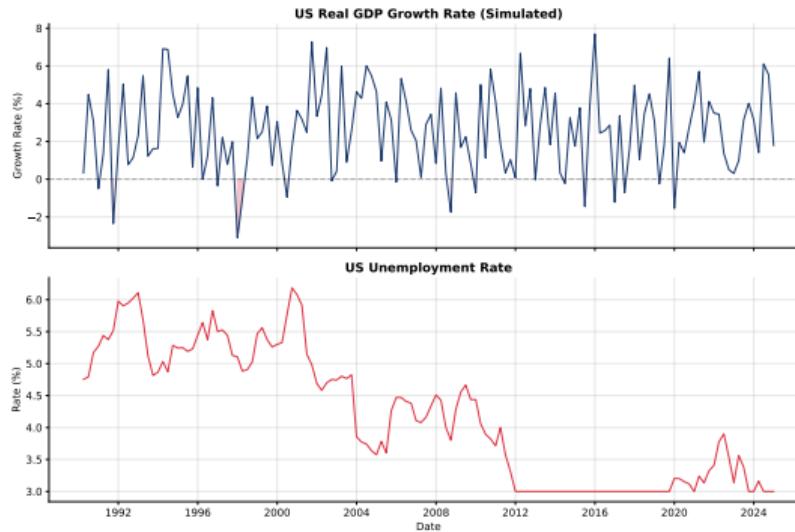
Workflow

- ① Create DataFrame: `data = pd.DataFrame({'gdp': ... , 'unemp': ...})`
- ② Fit VAR: `model = VAR(data); results = model.fit(maxlags=8, ic='aic')`
- ③ Get IRF: `irf = results.irf(periods=20)`
- ④ Get FEVD: `fevd = results.fevd(periods=20)`
- ⑤ Granger tests: `grangercausalitytests(data[['y', 'x']], maxlag=4)`

Note

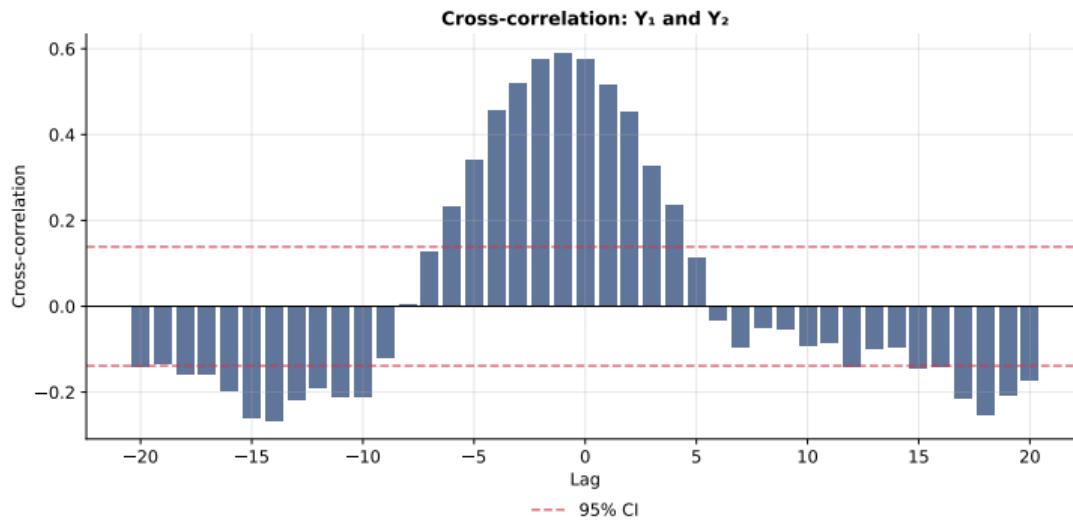
Complete working examples are provided in the Jupyter notebooks.

Case Study: GDP and Unemployment



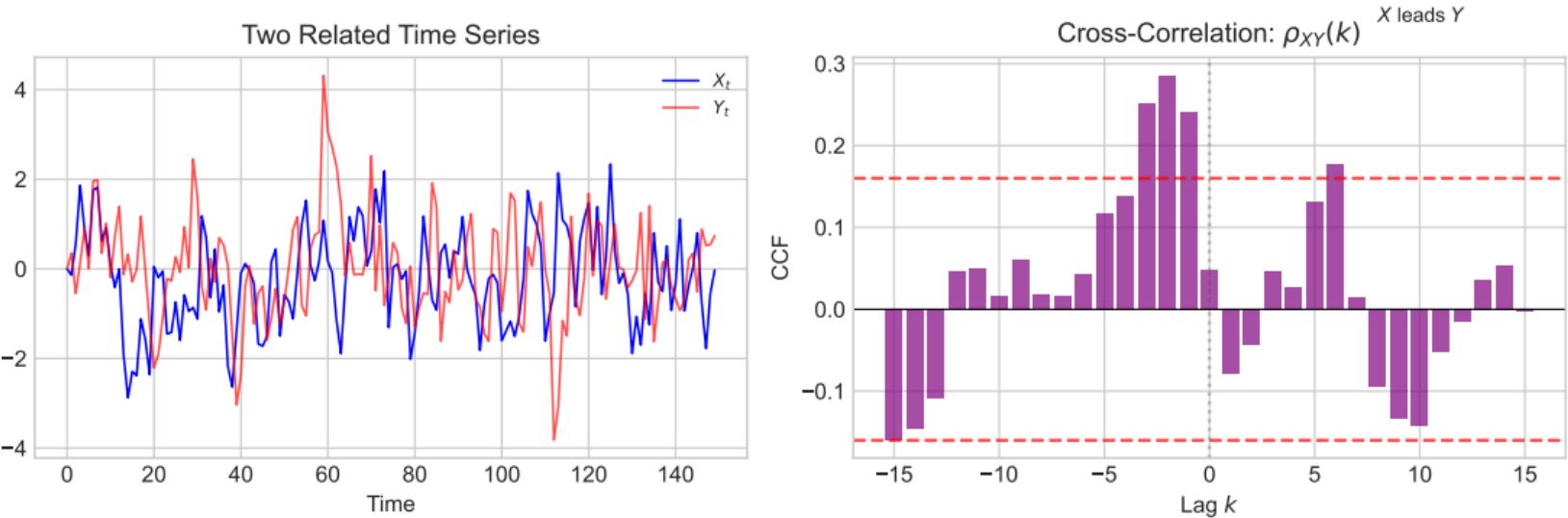
- **Top:** US Real GDP growth rate (quarterly)
- **Bottom:** US Unemployment rate
- Clear negative relationship (Okun's Law)
- VAR model can capture dynamic interactions between these variables

Cross-Correlation Analysis



- Cross-correlation measures lead-lag relationships
- Negative correlation at lag 0: contemporaneous inverse relationship
- Asymmetric pattern suggests unemployment responds to GDP with lag
- Helps inform VAR lag order selection

Visual: Cross-Correlation Function



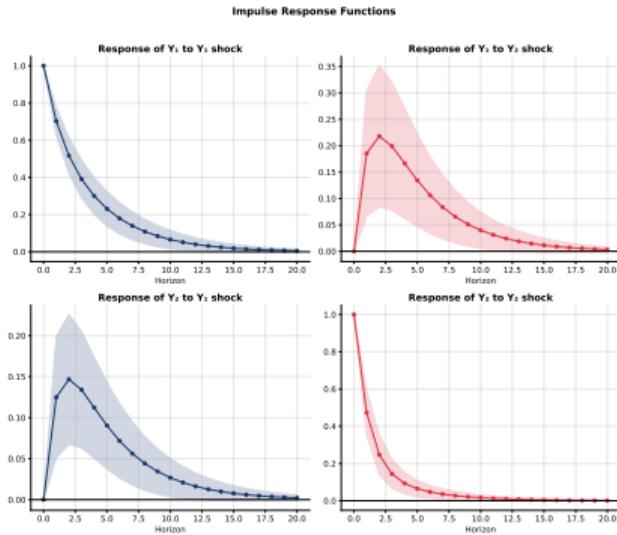
The CCF measures correlation between two series at different lags, revealing lead-lag relationships.

VAR Estimation Results

Model: VAR(2) for GDP Growth and Unemployment

Equation	Variable	Coef.	Std. Error	t-stat
ΔGDP_t	ΔGDP_{t-1}	0.312	0.087	3.59
	ΔGDP_{t-2}	0.145	0.082	1.77
	U_{t-1}	-0.421	0.156	-2.70
	U_{t-2}	0.198	0.148	1.34
U_t	ΔGDP_{t-1}	-0.087	0.032	-2.72
	ΔGDP_{t-2}	-0.045	0.030	-1.50
	U_{t-1}	1.456	0.058	25.1
	U_{t-2}	-0.521	0.055	-9.47

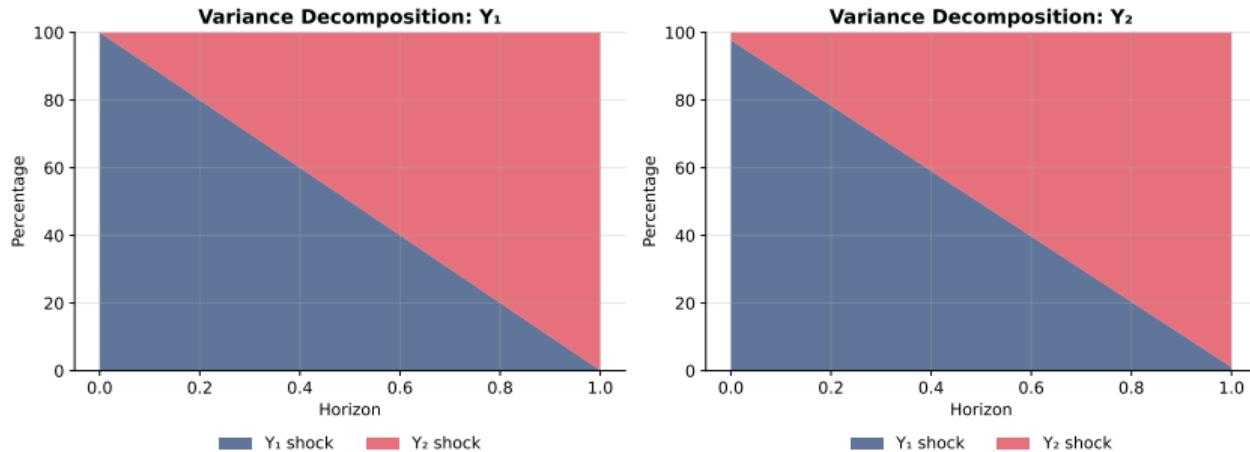
Impulse Response Functions



- IRFs show dynamic response to one-unit shocks
- GDP shock: temporary positive effect on GDP, negative on unemployment
- Unemployment shock: negative effect on GDP, persistent on unemployment
- 95% confidence bands show uncertainty in responses

Forecast Error Variance Decomposition

Forecast Error Variance Decomposition



- FEVD shows proportion of variance explained by each shock
- GDP variance: mostly explained by own shocks, some by unemployment
- Unemployment variance: highly persistent (own shocks dominant)
- Provides insight into relative importance of different shocks

Discussion: Granger Causality vs True Causality

Key Question

If X Granger-causes Y , does that mean X actually causes Y ?

Discussion Points

- **Omitted variable bias:** Z might cause both X and Y
 - Example: Weather affects both ice cream sales and drownings
- **Anticipation effects:** Markets anticipate future events
 - Stock prices “Granger-cause” earnings announcements
- **Aggregation issues:** Timing of data collection matters

Conclusion: Granger causality is about **prediction**, not **mechanism**. For structural causality, need theory + identification strategy.

Discussion: Variable Ordering in IRFs

Key Question

Why does the ordering of variables matter for orthogonalized IRFs?

Explanation

Cholesky decomposition assumes:

- First variable: Affects all others contemporaneously
- Second variable: Affected by first, affects remaining
- Last variable: Affected by all, affects none contemporaneously

Economic reasoning: Order from “most exogenous” to “most endogenous”

Example ordering for monetary policy VAR:

- ① Oil prices (exogenous)
2. GDP (slow)
3. Inflation
4. Interest rates

Take-Home Exercises

- ① **Theoretical:** Show that a VAR(1) can be written as MA(∞): $\mathbf{Y}_t = \sum_{i=0}^{\infty} \mathbf{A}^i \boldsymbol{\epsilon}_{t-i}$ when stable.
- ② **Computation:** For VAR(1) with $\mathbf{A} = \begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
 - Check stability; Compute IRFs for $h = 0, 1, 2, 3$
 - Plot the response of Y_1 to a shock in Y_2
- ③ **Applied:** Download US GDP growth and unemployment data:
 - Test stationarity; Estimate VAR (select optimal lag)
 - Test Granger causality; Compute and interpret IRFs
- ④ **Critical Thinking:** Why might stock prices “Granger-cause” GDP even though GDP is determined by real factors?

Hints

- ① Use recursive substitution: $\mathbf{Y}_t = \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t = \mathbf{A}(\mathbf{A}\mathbf{Y}_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \dots$
- ② Eigenvalues of $\begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
 - Characteristic equation: $\lambda^2 - 1.2\lambda + 0.35 = 0$
 - $\lambda_1 \approx 0.85$, $\lambda_2 \approx 0.41$ (both < 1 , stable)
- ③ For GDP/Unemployment:
 - GDP growth is usually I(0), unemployment may be I(1)
 - Use unemployment rate changes if needed
 - Expect GDP growth \rightarrow unemployment (Okun's Law)
- ④ Stock prices anticipate future economic conditions—they reflect expectations about future GDP, so they “lead” GDP in the data even though causation runs the other way.

Key Takeaways from This Seminar

Main Points

- ① VAR models capture **interdependencies** between multiple time series
- ② Parameter count grows quickly: $K^2p + K$ per system
- ③ **Granger causality** tests predictive content, not true causation
- ④ Test statistic is F-test on coefficient restrictions
- ⑤ **IRFs** show dynamic propagation of shocks
- ⑥ Variable ordering matters for orthogonalized IRFs

Key Practical Points

- Always check stationarity before estimating VAR
- Use information criteria for lag selection
- Report Granger tests in both directions
- Be careful interpreting as “true” causality