



# Chapter 5: VAR Models & Granger Causality

Multivariate Time Series



# Lecture Outline

- 1 Introduction to Multivariate Time Series
- 2 Vector Autoregression (VAR)
- 3 Granger Causality
- 4 Impulse Response Functions
- 5 Forecast Error Variance Decomposition
- 6 Practical Example
- 7 Summary

# Why Multivariate Analysis?

## Limitations of Univariate Models

- ARIMA models each variable **in isolation**
- Ignores potential **interactions** between variables
- Cannot capture **feedback effects**

## Economic Examples of Interdependence

- GDP and unemployment (Okun's law)
- Interest rates and inflation (Taylor rule)
- Stock prices and trading volume
- Exchange rates and trade balance

## Vector of Variables

Let  $\mathbf{Y}_t = (Y_{1t}, Y_{2t}, \dots, Y_{Kt})'$  be a  $K \times 1$  vector of time series.

Example with  $K = 2$ :

$$\mathbf{Y}_t = \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \text{GDP growth}_t \\ \text{Inflation}_t \end{pmatrix}$$

## Key Questions

- ① Does  $Y_1$  help predict  $Y_2$ ? (Granger causality)
- ② How do shocks to  $Y_1$  affect  $Y_2$ ? (Impulse responses)
- ③ What proportion of  $Y_2$ 's variance is due to  $Y_1$ ? (Variance decomposition)

# The VAR(p) Model

## Definition

A **VAR(p)** model for  $K$  variables:

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$$

where:

- $\mathbf{Y}_t$ :  $K \times 1$  vector of endogenous variables
- $\mathbf{c}$ :  $K \times 1$  vector of constants
- $\mathbf{A}_i$ :  $K \times K$  coefficient matrices
- $\boldsymbol{\varepsilon}_t$ :  $K \times 1$  vector of error terms with  $\mathbb{E}[\boldsymbol{\varepsilon}_t] = \mathbf{0}$ ,  $\mathbb{E}[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}'_t] = \Sigma$

# VAR(1) with Two Variables

## Bivariate VAR(1)

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

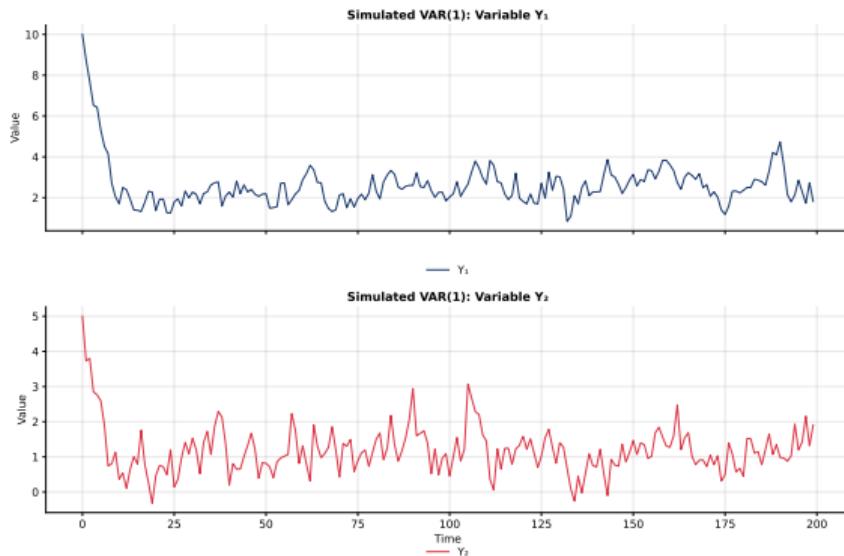
## Equation by Equation

$$Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$$

$$Y_{2t} = c_2 + a_{21} Y_{1,t-1} + a_{22} Y_{2,t-1} + \varepsilon_{2t}$$

**Key insight:** Each equation includes lags of **all** variables!

## Simulated VAR Process



- Two simulated series from a bivariate VAR(1) process showing interdependence
- Each variable responds to both its own past and the other variable's past
- Notice how the series co-move due to cross-equation dynamics

## Stationarity of VAR

### Stability Condition

VAR( $p$ ) is **stable** (stationary) if all roots of:

$$\det(\mathbf{I}_K - \mathbf{A}_1 z - \mathbf{A}_2 z^2 - \cdots - \mathbf{A}_p z^p) = 0$$

lie **outside** the unit circle (i.e.,  $|z| > 1$ ).

### For VAR(1)

The model is stable if all **eigenvalues** of  $\mathbf{A}_1$  are less than 1 in absolute value.

Example: For  $\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$ , eigenvalues are  $\lambda_1 = 0.6$  and  $\lambda_2 = 0.2$ .  
Both  $< 1 \Rightarrow$  stable!

## Estimation of VAR

### OLS Estimation

Each equation can be estimated by **OLS separately**:

$$\hat{\mathbf{A}} = \left( \sum_{t=1}^T \mathbf{Y}_{t-1} \mathbf{Y}'_{t-1} \right)^{-1} \left( \sum_{t=1}^T \mathbf{Y}_{t-1} \mathbf{Y}'_t \right)$$

This is efficient because all equations have the **same regressors**.

### Covariance Matrix

$$\hat{\Sigma} = \frac{1}{T - Kp - 1} \sum_{t=1}^T \hat{\epsilon}_t \hat{\epsilon}'_t$$

The errors  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  may be **contemporaneously correlated**.

## Information Criteria

Choose  $p$  that minimizes:

$$AIC(p) = \ln |\hat{\Sigma}_p| + \frac{2pK^2}{T}$$

$$BIC(p) = \ln |\hat{\Sigma}_p| + \frac{pK^2 \ln T}{T}$$

$$HQ(p) = \ln |\hat{\Sigma}_p| + \frac{2pK^2 \ln \ln T}{T}$$

## Guidelines

- AIC tends to select **larger** models (better for forecasting)
- BIC tends to select **smaller** models (consistent selection)
- Start with maximum  $p_{max}$  based on data frequency (e.g., 4 for quarterly, 12 for monthly)

# What is Granger Causality?

Clive Granger (1969, Nobel Prize 2003)

" $X$  Granger-causes  $Y$ " if past values of  $X$  help predict  $Y$ , **beyond** what past values of  $Y$  alone can predict.

## Important Distinction

**Granger causality  $\neq$  True causality**

- Granger causality is about **predictive content**
- Does NOT imply economic/structural causation
- " $X$  Granger-causes  $Y$ " means:  $X$  contains useful information for forecasting  $Y$

## Formal Definition

### Granger Causality

**$X$  does not Granger-cause  $Y$  if:**

$$\mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \dots, X_{t-1}, X_{t-2}, \dots] = \mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \dots]$$

In other words: adding  $X$ 's history does not improve the prediction of  $Y$ .

### In the VAR Context

For VAR(1):  $Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$

$Y_2$  does **not** Granger-cause  $Y_1$  if  $a_{12} = 0$ .

For VAR( $p$ ):  $Y_2$  does not Granger-cause  $Y_1$  if  $a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$ .

# Testing for Granger Causality

## Hypothesis Test

$H_0$ :  $Y_2$  does **not** Granger-cause  $Y_1$

$$H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$$

$H_1$ : At least one  $a_{12}^{(i)} \neq 0$  (Granger causality exists)

## Test Statistic: Wald Test

$$F = \frac{(RSS_R - RSS_U)/p}{RSS_U/(T - 2p - 1)} \sim F_{p, T-2p-1}$$

where:

- $RSS_R$ : Residual sum of squares from restricted model (without  $Y_2$  lags)
- $RSS_U$ : Residual sum of squares from unrestricted model (full VAR)

## Types of Granger Causality



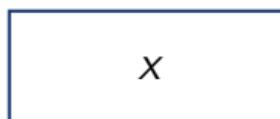
Unidirectional:  $X \rightarrow Y$



Bidirectional:  $X \leftrightarrow Y$



Unidirectional:  $Y \rightarrow X$

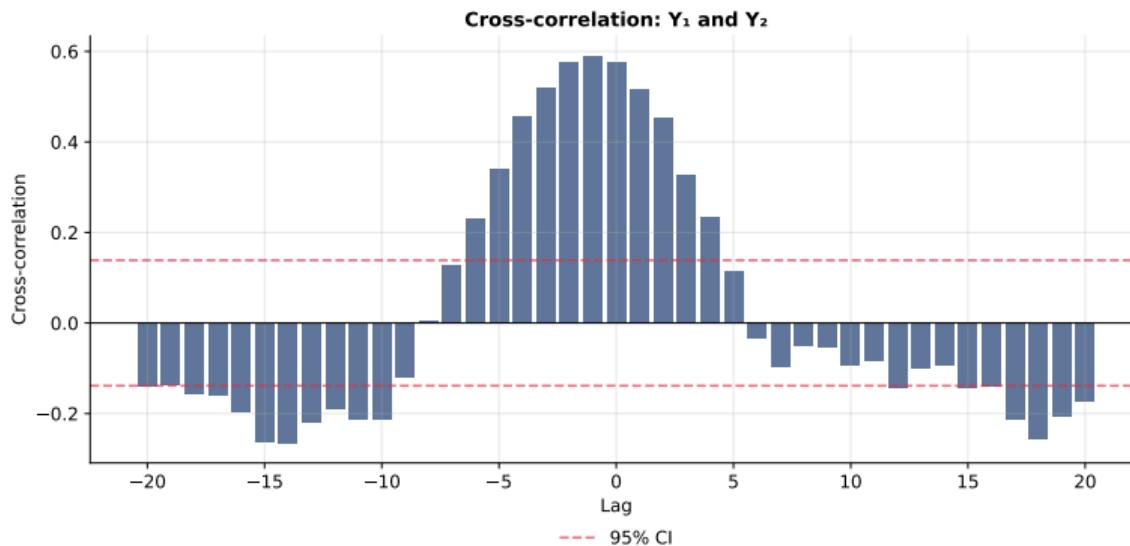


No causality

### Economic Examples

- Money  $\rightarrow$  Output? (monetarist view)
- Stock prices  $\leftrightarrow$  Trading volume (bidirectional)
- Weather  $\rightarrow$  Crop yields (unidirectional, obvious)

## Cross-Correlation Analysis



- Cross-correlation function measures linear dependence at different lags
- Significant correlations at negative lags suggest  $X$  leads  $Y$ ; positive lags suggest  $Y$  leads  $X$
- Useful for preliminary analysis before formal Granger causality testing

## Common Pitfalls

- ① **Omitted variables:** A third variable  $Z$  may cause both  $X$  and  $Y$
- ② **Non-stationarity:** Test requires stationary data (or cointegration)
- ③ **Lag selection:** Results can be sensitive to  $p$
- ④ **Sample size:** Need sufficient observations

## Best Practices

- Test for unit roots first
- Use multiple lag selection criteria
- Check robustness to different lag lengths
- Report results for both directions

# What are Impulse Response Functions?

## Definition

An **Impulse Response Function (IRF)** traces the effect of a one-time shock to one variable on the current and future values of all variables.

## Question IRFs Answer

"If there is an unexpected 1-unit shock to  $Y_1$  today, what happens to  $Y_1$  and  $Y_2$  over the next  $h$  periods?"

## MA( $\infty$ ) Representation

A stable VAR(p) can be written as:

$$\mathbf{Y}_t = \boldsymbol{\mu} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{\varepsilon}_{t-i}$$

The matrices  $\boldsymbol{\Phi}_i$  are the **impulse responses** at horizon  $i$ .

## Computing IRFs for VAR(1)

For VAR(1):  $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t$

The impulse response matrices are:

$$\Phi_0 = \mathbf{I}_K, \quad \Phi_1 = \mathbf{A}, \quad \Phi_2 = \mathbf{A}^2, \quad \dots, \quad \Phi_h = \mathbf{A}^h$$

### Interpretation

$[\Phi_h]_{ij}$  = Effect on  $Y_i$  at time  $t+h$  of a unit shock to  $Y_j$  at time  $t$

For stable VAR:  $\Phi_h \rightarrow \mathbf{0}$  as  $h \rightarrow \infty$  (shocks die out)

## Orthogonalized IRFs

### Problem: Correlated Errors

If  $\Sigma$  is not diagonal, shocks  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  are correlated.

A shock to "Y<sub>1</sub>" also involves a shock to "Y<sub>2</sub>".

### Solution: Cholesky Decomposition

Factor  $\Sigma = \mathbf{P}\mathbf{P}'$  where  $\mathbf{P}$  is lower triangular.

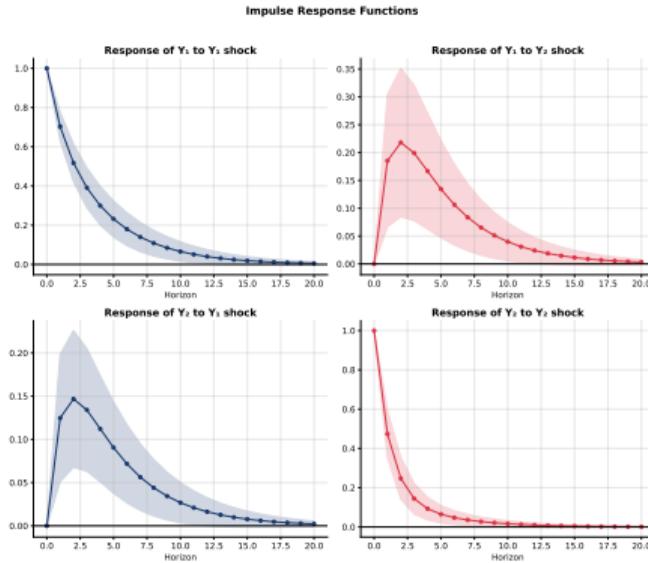
Define orthogonalized shocks:  $\mathbf{u}_t = \mathbf{P}^{-1}\varepsilon_t$  with  $\mathbb{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}$

Orthogonalized IRFs:  $\Theta_h = \Phi_h \mathbf{P}$

### Ordering Matters!

Cholesky assumes variables ordered from "most exogenous" to "most endogenous". Results depend on this ordering.

# Impulse Response Functions: Example



- IRFs show how each variable responds to a one-unit shock over time
- Shaded regions represent confidence intervals (uncertainty in estimates)
- For stable VAR models, responses converge to zero as the horizon increases

## Variance Decomposition

### Question

What proportion of the forecast error variance of  $Y_i$  at horizon  $h$  is due to shocks to  $Y_j$ ?

### FEVD Formula

$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\Theta_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^K [\Theta_s]_{ik}^2}$$

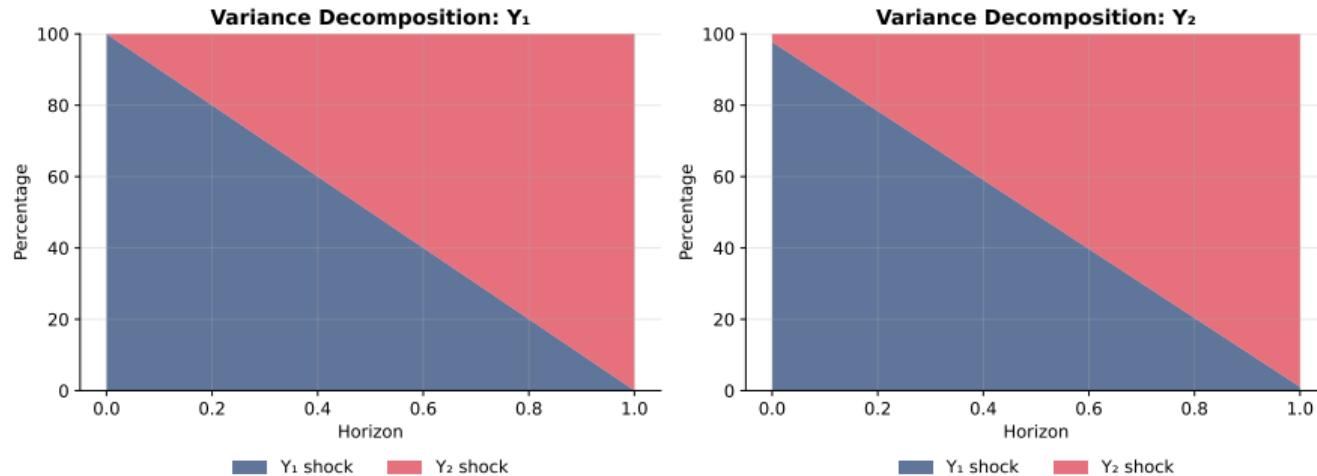
This gives the **percentage** of  $Y_i$ 's  $h$ -step forecast variance explained by shocks to  $Y_j$ .

### Properties

- $0 \leq \text{FEVD}_{ij}(h) \leq 1$
- $\sum_{j=1}^K \text{FEVD}_{ij}(h) = 1$  (sums to 100%)
- At  $h = 1$ : Own shocks dominate (by construction with Cholesky)

## FEVD: Example

Forecast Error Variance Decomposition



- FEVD shows the proportion of forecast variance attributable to each shock
- At short horizons, own shocks dominate; cross-variable effects grow over time
- Useful for understanding the relative importance of different shocks in the system

## Example: GDP and Unemployment

### Okun's Law

There is a negative relationship between GDP growth and unemployment:

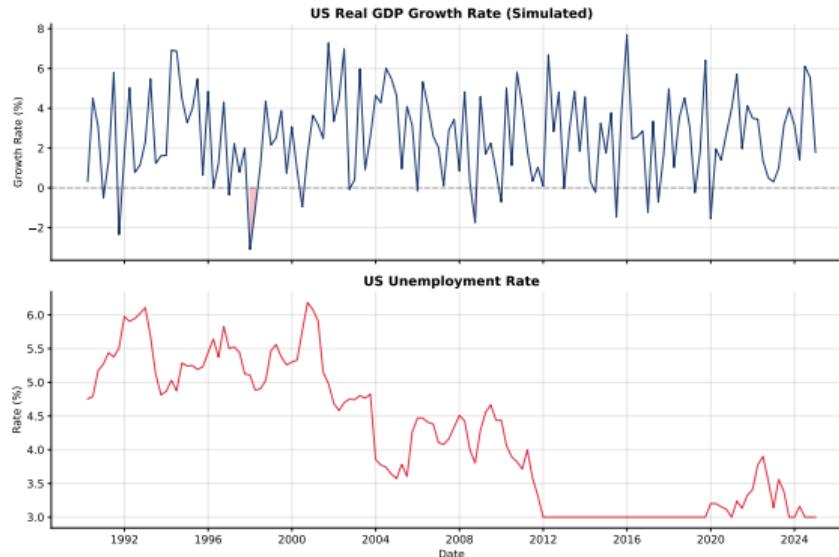
$$\Delta U_t \approx -\beta(\Delta Y_t - \bar{g})$$

where  $\bar{g}$  is trend GDP growth and  $\beta \approx 0.4$ .

### VAR Analysis Questions

- ① Does GDP growth Granger-cause unemployment changes?
- ② Does unemployment Granger-cause GDP growth?
- ③ How do shocks propagate between variables?

## GDP and Unemployment: Data



- GDP growth and unemployment rate show clear negative correlation (Okun's Law)
- Both series exhibit cyclical patterns related to business cycle fluctuations
- This bivariate system is ideal for VAR analysis and Granger causality testing

## ① Data preparation

- Check for stationarity (unit root tests)
- Transform if necessary (differences, logs)

## ② Lag selection

- Use AIC, BIC, HQ criteria
- Check residual autocorrelation

## ③ Estimation

- OLS equation by equation
- Check stability (eigenvalues)

## ④ Analysis

- Granger causality tests
- Impulse response functions
- Variance decomposition

## ⑤ Forecasting

## VAR in Python (statsmodels)

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests

# Fit VAR model
model = VAR(data)
results = model.fit(maxlags=4, ic='aic')

# Granger causality test
granger_test = grangercausalitytests(data[['Y1', 'Y2']],
                                      maxlag=4)

# Impulse response functions
irf = results.irf(periods=20)
irf.plot()

# Variance decomposition
fevd = results.fevd(periods=20)
fevd.plot()
```

## Key Takeaways

### VAR Models

- Model **multiple** time series jointly
- Each variable depends on its own lags AND lags of other variables
- Estimated by OLS equation by equation
- Requires stationarity (or differencing)

### Granger Causality

- Tests whether  $X$  helps predict  $Y$  beyond  $Y$ 's own history
- **Not** the same as true causality
- F-test on coefficient restrictions

### IRF and FEVD

- IRF: How shocks propagate through the system
- FEVD: What proportion of variance is due to each shock
- Both depend on variable ordering (Cholesky)

## Topics for Further Study

- **Cointegration:** Long-run relationships between non-stationary variables
- **VECM:** Error correction models for cointegrated systems
- **Structural VAR:** Imposing economic theory restrictions
- **Panel VAR:** VAR for panel data

Questions?