Time Series Analysis and Forecasting

Chapter 9: Prophet and TBATS

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Learning Objectives

By the end of this chapter, you will be able to:

- Handle time series with multiple seasonal patterns
- Use Facebook Prophet for flexible forecasting with holidays
- Apply TBATS models for complex seasonality
- Compare and select between modern forecasting methods

Outline

The Problem: Complex Seasonal Patterns

Real-World Examples

- **Hourly electricity demand**: Daily + Weekly + Annual patterns
- **Website traffic**: Daily + Weekly + Holiday effects
- **Retail sales**: Weekly + Monthly + Annual + Holiday effects
- **Call center volume**:
  - Hourly + Daily + Weekly patterns

SARIMA Limitation

Standard SARIMA$(p, d, q)(P, D, Q)_s$ handles only **one** seasonal period s.

For hourly data with daily AND weekly patterns, we need $s_1 = 24$ and $s_2 = 168$.

Solutions for Multiple Seasonalities

Traditional Approaches

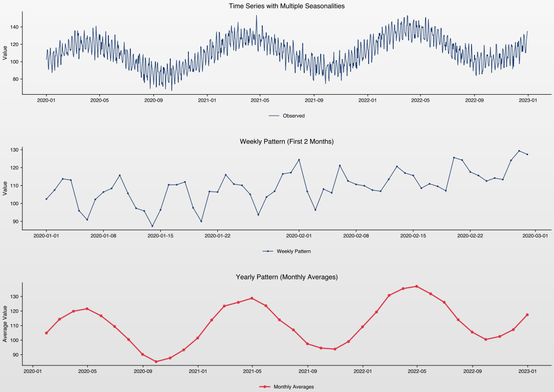
- **Fourier terms**: Add sin/cos regressors
- **Dummy variables**: Many parameters
- **Nested models**: Complex specification

Modern Approaches

- **TBATS**: Automatic, handles many periods
- **Prophet**: Flexible, interpretable
- **Neural methods**:
  - Deep learning

Method	Max Seasonalities	Interpretable
SARIMA	1	Yes
Fourier + ARIMA	Multiple	Moderate
TBATS	Multiple	Moderate
Prophet	Multiple	Yes

Example: Hourly Data with Multiple Seasonalities



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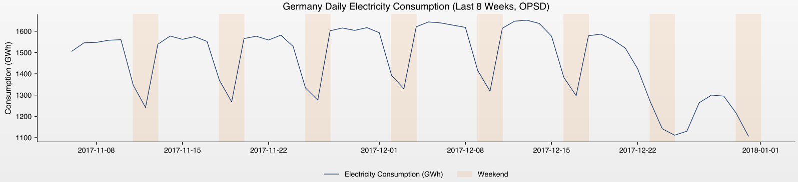
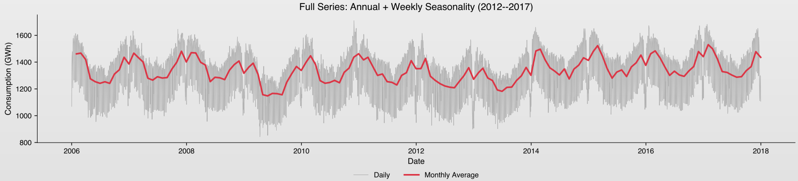
Real Example: Electricity Demand





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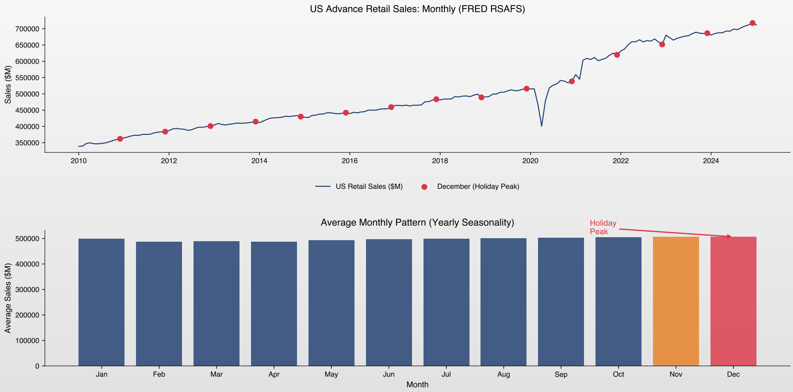
Real Example: Retail Sales with Holidays



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TBATS: What Does It Stand For?

TBATS Components

- T **Trigonometric** seasonality using Fourier terms
- B **Box-Cox** transformation for variance stabilization
- A **ARMA** errors for remaining autocorrelation
- T **Trend** component (possibly damped)
- S **Seasonal** components (multiple allowed)

Key Innovation: Trigonometric Seasonality

$$s_t^{(i)} = \sum_{j=1}^{k_i} \left[s_j^{(i)} \cos\left(\frac{2\pi jt}{m_i}\right) + s_j^{*(i)} \sin\left(\frac{2\pi jt}{m_i}\right) \right]$$

m_i = seasonal period, k_i = number of harmonics

Box-Cox Transformation

Definition 1 (Box-Cox Transformation)

The Box-Cox transformation with parameter ω is defined as:

$$y_t^{(\omega)} = \begin{cases} \dfrac{y_t^{\omega} - 1}{\omega} & \text{if } \omega \neq 0 \\ \ln(y_t) & \text{if } \omega = 0 \end{cases}$$

Purpose

- **Variance stabilization**: Makes variance constant over time
- **Normalization**: Reduces skewness in the data
- Common values: $\omega = 0$ (log), $\omega = 0.5$ (square root), $\omega = 1$ (no transform)

TBATS Model Structure

State Space Representation

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^{T} s_{t-m_i}^{(i)} + d_t \tag{1}$$

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t, \quad b_t = \phi b_{t-1} + \beta d_t \tag{2}$$

- $y_t^{(\omega)}$: Box-Cox transformed observation
- ℓ_t: local level (smoothed mean)
- b_t: trend with damping $\phi \in (0, 1)$
- $s_t^{(i)}$: i-th seasonal component
- d_t: ARMA(p, q) error process
- α, β: smoothing parameters

TBATS: Trigonometric Seasonality State Evolution

Definition 2 (Trigonometric State-Space Recursion)

For each seasonal component with period m_i and k_i harmonics, define states:

$$\begin{pmatrix} s_{j,t}^{(i)} \\ s_{j,t}^{*(i)} \end{pmatrix} = \begin{pmatrix} \cos(\lambda_j) & \sin(\lambda_j) \\ -\sin(\lambda_j) & \cos(\lambda_j) \end{pmatrix} \begin{pmatrix} s_{j,t-1}^{(i)} \\ s_{j,t-1}^{*(i)} \end{pmatrix} + \begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \end{pmatrix} d_t$$

where $\lambda_j = \frac{2\pi j}{m_i}$ is the j-th harmonic frequency.

Interpretation

- The rotation matrix preserves the periodic structure
- Total seasonal: $s_t^{(i)} = \sum_{j=1}^{k_i} s_{j,t}^{(i)}$
- Parameters: $2k_i$ states per seasonal period

TBATS: Choosing the Number of Harmonics

Why Fourier/Trigonometric Terms?

1. **Parsimonious**: $2k$ parameters vs m dummy variables
2. **Smooth**: Captures smooth seasonal patterns naturally
3. **Flexible**: Number of harmonics k controls complexity
4. **Non-integer periods**: Can handle $s = 365.25$ for daily data

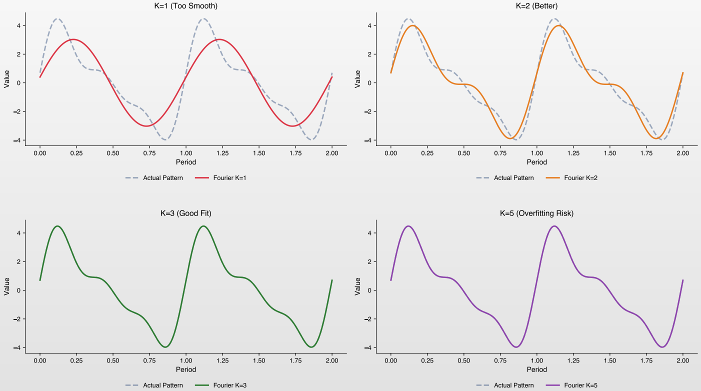
Low k (few harmonics)

- Smooth pattern
- Fewer parameters
- May miss sharp peaks

High k (many harmonics)

- Can capture any pattern
- More parameters ($2k$ total)
- Maximum useful: $k \leq \lfloor m/2 \rfloor$

Fourier Approximation of Seasonality



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TBATS: Key Features

Automatic Model Selection

TBATS automatically determines:

- Box-Cox parameter ω for variance stabilization
- Number of harmonics k_i for each seasonal period
- ARMA orders (p, q) for residual autocorrelation
- Damped vs non-damped trend specification

BATS vs TBATS

- **BATS**: Traditional seasonal states (dummy variables)
- **TBATS**: Trigonometric (Fourier) seasonal representation
- TBATS more parsimonious for long seasonal periods

TBATS: Advantages and Limitations

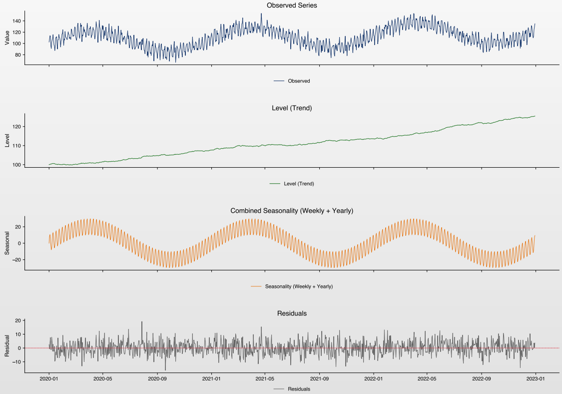
Advantages

- Handles **multiple** seasonal periods
- **Automatic** model selection
- Handles **non-integer** periods (365.25)
- **Box-Cox** for heteroskedasticity
- Good for **high-frequency** data

Limitations

- **Computationally intensive**
- No **external regressors**
- Less **interpretable** than Prophet
- Can be **slow** for very long series
- Requires **sufficient data** per season

TBATS Decomposition Example



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Prophet: Overview

What is Prophet?

Forecasting procedure developed by Facebook (Meta) in 2017 for **business time series**:

- Strong seasonal effects (daily, weekly, yearly)
- Holiday effects and trend changes (changepoints)
- Handles missing data and outliers

Key Philosophy: "Analyst-in-the-loop"

Designed for analysts with domain knowledge but without time series expertise.

Prophet Model Structure

Decomposition Approach

Prophet uses an **additive decomposition**:

$$y(t) = g(t) + s(t) + h(t) + \varepsilon_t$$

$g(t)$: Trend

- Linear or logistic
- Automatic changepoints
- Growth saturation

$s(t)$: Seasonality

- Fourier series
- Multiple periods
- Custom seasonality

$h(t)$: Holidays

- Country holidays
- Custom events
- Window effects

Prophet: Trend Component

Linear Trend with Changepoints

$$g(t) = (k + \boldsymbol{a}(t)^T \boldsymbol{\delta}) \cdot t + (m + \boldsymbol{a}(t)^T \gamma)$$

- k: base growth rate (slope)
- $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_S)$: slope changes at S changepoints
- $\boldsymbol{a}(t) \in \{0, 1\}^S$: indicator if changepoint s is active at time t

Logistic Growth

For saturating trends:

$$g(t) = \frac{C(t)}{1 + e^{-(k + \boldsymbol{a}(t)^T \boldsymbol{\delta})(t - m - \boldsymbol{a}(t)^T \gamma)}}$$

$C(t) =$ time-varying carrying capacity

Continuity Constraint

The offset $\gamma_j = -s_j \cdot \delta_j$ ensures $g(t)$ is continuous at each changepoint s_j.

Prophet: Seasonality Component

Fourier Series Representation

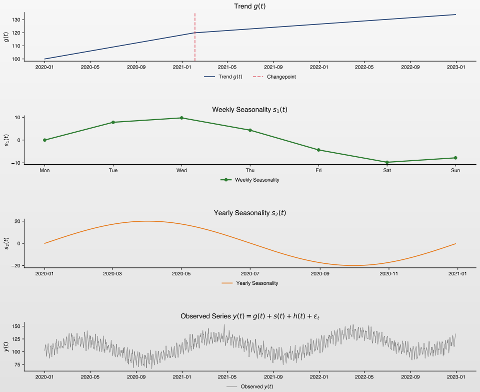
$$s(t) = \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2\pi nt}{P}\right) + b_n \sin\left(\frac{2\pi nt}{P}\right) \right]$$

Default Settings

Seasonality	**Period**	**Fourier Order**
Yearly	365.25 days	10
Weekly	7 days	3
Daily	1 day	4

Higher N = more flexibility but risk of overfitting

Prophet Component Decomposition



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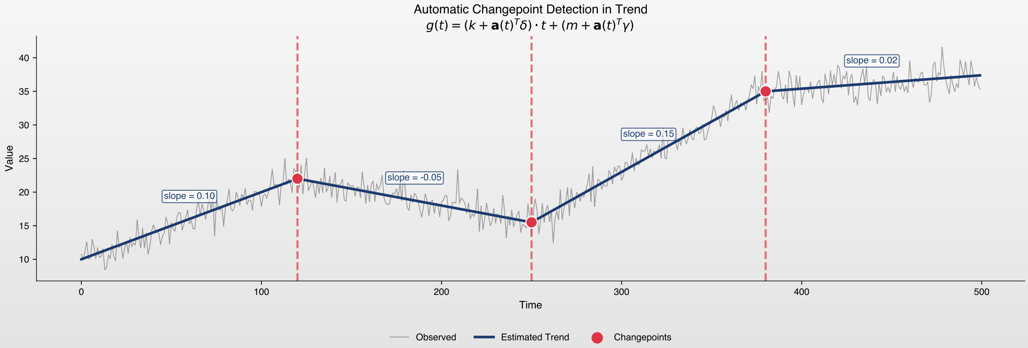
Trend Changepoint Detection



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Prophet: Holiday Effects

Holiday Model

$$h(t) = Z(t) \cdot \boldsymbol{\kappa}$$

where $Z(t)$ is an indicator matrix for holidays and $\boldsymbol{\kappa}$ are holiday effects.

Built-in Features

- 60+ countries supported
- Custom holiday definitions
- Window effects (before/after)

Holiday Types

- National holidays
- Religious observances
- Business events

Prophet: Customization Options

Seasonality Customization

- Add custom seasonal periods (monthly, quarterly)
- Control Fourier order for each seasonality
- Enable/disable default seasonalities

External Regressors

Prophet supports adding external variables:

- Weather data, promotions, special events
- Binary or continuous regressors
- Automatic regularization

Prophet: Uncertainty Quantification

Bayesian Framework

Prophet uses a **Laplace prior** on changepoint magnitudes:

$$\delta_j \sim \text{Laplace}(0, \tau), \quad \tau = \texttt{changepoint\_prior\_scale}$$

Smaller τ = sparser, smaller changepoints (more regularization).

Sources of Uncertainty

1. **Trend**: Future changepoints
2. **Seasonality**: Coefficient variance
3. **Observation**: Residual noise σ^2

Prediction Intervals

- MAP estimation for point forecasts
- Monte Carlo sampling for intervals
- Default: 80% credible interval

Prophet: Tuning Parameters

Key Parameters

Parameter	Effect
`changepoint_prior_scale`	Trend flexibility (default: 0.05)
`seasonality_prior_scale`	Seasonality flexibility (default: 10)
`holidays_prior_scale`	Holiday effect size (default: 10)
`seasonality_mode`	'additive' or 'multiplicative'
`changepoint_range`	Portion of history for changepoints

Practical Tips

- **Overfitting trend?** Decrease `changepoint_prior_scale`
- **Underfitting seasonality?** Increase `seasonality_prior_scale`
- **Seasonal amplitude varies?** Use `seasonality_mode='multiplicative'`

Prophet: Advantages and Limitations

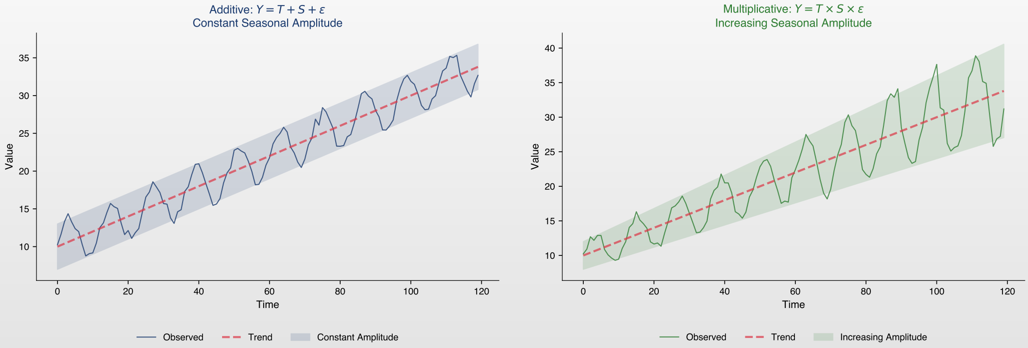
Advantages

- **Easy to use**: Minimal tuning needed
- **Interpretable**: Clear decomposition
- **Handles missing data** well
- **Holiday effects** built-in
- **Multiple seasonalities**
- **External regressors** supported
- **Fast** fitting

Limitations

- **Not ARIMA-based**: No autocorrelation modeling
- **Daily data focus**: Less suited for very high frequency
- **Trend assumptions**: Linear/logistic may not fit
- **No built-in CV**: Must implement manually
- **Overfitting risk** with many seasonalities

Additive vs Multiplicative Seasonality



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TBATS vs Prophet: Head-to-Head

Feature	TBATS	Prophet
Multiple seasonalities	Yes (automatic)	Yes (manual or auto)
Holiday effects	No	Yes (built-in)
External regressors	No	Yes
Trend changepoints	No (smooth)	Yes (automatic)
Missing data	Interpolation needed	Handles natively
Interpretability	Moderate	High
Computation speed	Slow	Fast
High-frequency data	Good	Moderate
Non-integer periods	Yes (e.g., 365.25)	Yes
Uncertainty intervals	Yes	Yes

When to Use Each Model

Use TBATS when:

- High-frequency data
- Multiple seasonal periods
- No external regressors
- Automatic selection preferred

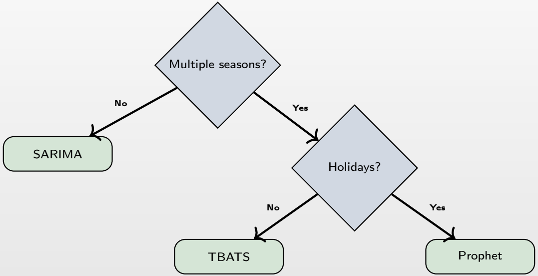
Use Prophet when:

- Business forecasting
- Holiday effects important
- Trend has changepoints
- External regressors available

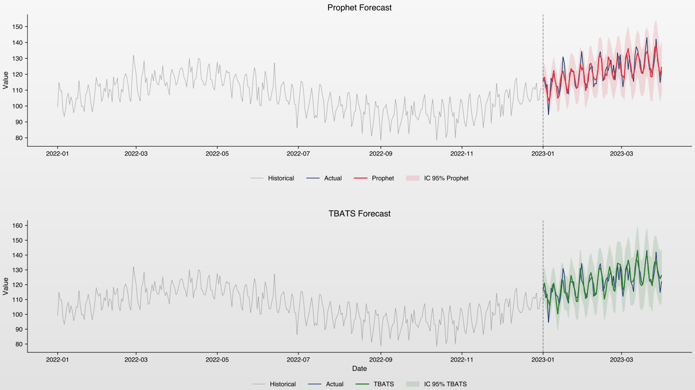
General Guideline

Prophet: business applications with daily data
TBATS: technical applications with high-frequency data

Decision Flowchart



Prophet vs TBATS: Forecast Comparison



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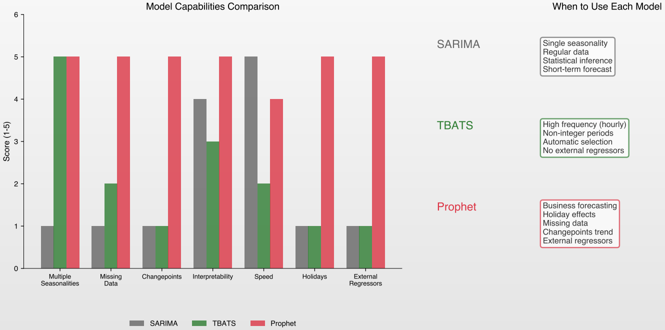
Model Selection Guide



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Evaluation Metrics

Definition 3 (Forecast Accuracy Metrics)

Let y_t denote actual values, $\hat{y}_t$ forecasts, and n the forecast horizon:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)^2} \quad \text{(penalizes large errors)}$$

$$\text{MAE} = \frac{1}{n}\sum_{t=1}^{n}|y_t - \hat{y}_t| \quad \text{(robust to outliers)}$$

$$\text{MAPE} = \frac{100\%}{n}\sum_{t=1}^{n}\left|\frac{y_t - \hat{y}_t}{y_t}\right| \quad \text{(scale-free)}$$

Coverage

For prediction intervals $[\hat{y}_t^L, \hat{y}_t^U]$, coverage rate is the proportion of actual values falling within the interval. Target: match the nominal level (e.g., 80%).

Case Study: Energy Demand Forecasting

Problem

Forecast hourly electricity demand with:

- **Daily pattern**: Peak at noon and evening
- **Weekly pattern**: Lower on weekends
- **Annual pattern**: Higher in summer (AC) and winter (heating)
- **Holiday effects**: Lower demand on holidays

Approach

1. Try TBATS with periods [24, 168, 8766]
2. Try Prophet with daily, weekly, yearly seasonality + holidays
3. Compare using cross-validation

Case Study: Results

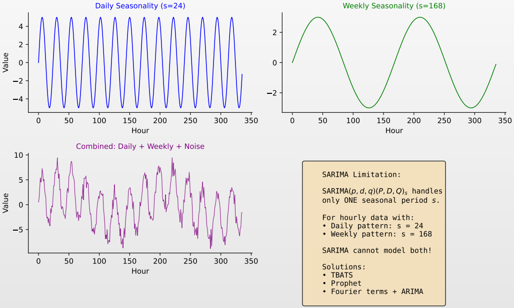
Model Comparison

Model	MAPE	RMSE	Coverage
SARIMA (daily only)	8.5%	450 MW	75%
TBATS	4.2%	220 MW	82%
Prophet	4.8%	250 MW	85%
Prophet + holidays	3.9%	200 MW	88%

Key Finding

Multiple seasonality models significantly outperform single-seasonality SARIMA.

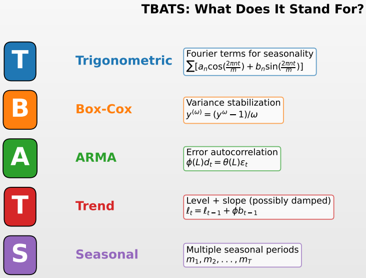
Quiz 1: Multiple Seasonality



Question: Why can't standard SARIMA$(p,d,q)(P,D,Q)_s$ model hourly electricity data with both daily and weekly patterns?

Answer: SARIMA handles only **one** seasonal period s. Cannot set $s = 24$ and $s = 168$ simultaneously.

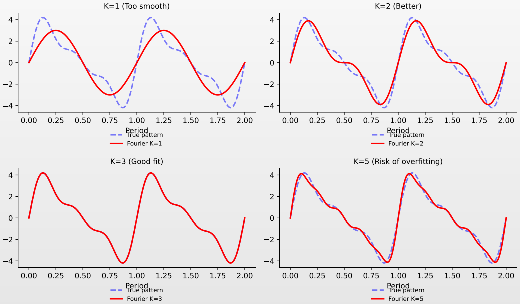
Quiz 2: TBATS Components



Question: What does each letter in TBATS represent?

Answer: **T**rigonometric seasonality, **B**ox-Cox transformation, **A**RMA errors, **T**rend, **S**easonal components.

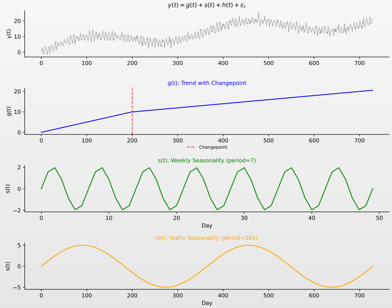
Quiz 3: Fourier Harmonics



Question: What happens when we increase the number of Fourier harmonics K?

Answer: Higher K captures more complex patterns but increases overfitting risk.

Quiz 4: Prophet Decomposition



Question: What are the main components in Prophet's model $y(t) = g(t) + s(t) + h(t) + \varepsilon_t$?

Answer: $g(t)$ = trend with changepoints, $s(t)$ = seasonality, $h(t)$ = holiday effects.

Quiz 5: Model Comparison

TBATS vs Prophet: Head-to-Head Comparison

Feature	TBATS	Prophet
Multiple seasonalities	Yes (automatic)	Yes (manual/auto)
Holiday effects	No	Yes (built-in)
External regressors	No	Yes
Trend changepoints	No (smooth)	Yes (automatic)
Missing data	Needs interpolation	Handles natively
Interpretability	Moderate	High
Computation speed	Slow	Fast
High-frequency data	Good	Moderate
Non-integer periods	Yes (e.g., 365.25)	Yes
Best for	Technical/high-freq	Business/daily

Question: What key features does Prophet have that TBATS lacks?

Answer: Holiday effects, external regressors, trend changepoints, native missing data handling.

Key Takeaways

What We Learned

- TBATS handles multiple seasonalities with Fourier terms and Box-Cox transformation
- Prophet provides interpretable decomposition with trend changepoints and holiday effects
- Both methods scale better than SARIMA for high-frequency and complex seasonal data

Important

Choose Prophet for business forecasting with holidays and interpretability needs. Use TBATS for automatic modeling of high-frequency data. Always validate with time series cross-validation—never standard k-fold!

Questions?

Questions?

Next Steps:

- Practice with the Jupyter notebook
- Try Prophet on your own data
- Explore NeuralProphet for deep learning extension

Online Resources and Code

- **Quantlet**: https://quantlet.com → Code repository for statistics
- **Quantinar**: https://quantinar.com → Learning platform for quantitative methods
- **GitHub TSA_ch9**: https://github.com/QuantLet/TSA/tree/main/TSA_ch9

Thank You!

Questions?

Course materials available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar