



Time Series Analysis and Forecasting

# Chapter 7: Cointegration & VECM

Seminar



## Seminar Outline

Review Quiz

True/False Questions

Practice Problems

Worked Examples

Discussion Topics

Exercises for Self-Study

## Quiz 1: Cointegration Definition

### Question

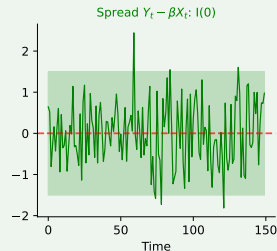
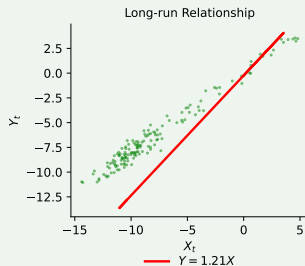
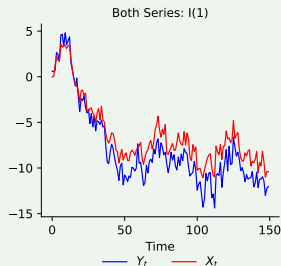
Two  $I(1)$  variables  $X_t$  and  $Y_t$  are cointegrated if:

- A) They are both stationary
- B) Their sum is  $I(2)$
- C) A linear combination of them is  $I(0)$
- D) They have the same mean

*Answer on next slide...*

## Quiz 1: Answer

Answer: C – A linear combination is  $I(0)$



**Key:**  $Y_t - \beta X_t \sim I(0)$  means they share a common stochastic trend. The linear combination (spread) is stationary even though both series are non-stationary.

## Quiz 2: Spurious Regression

### Question

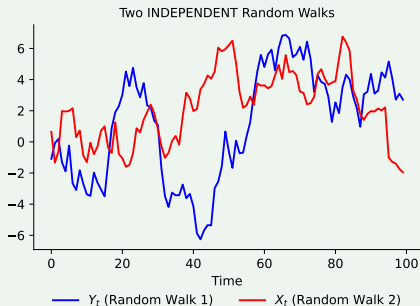
When regressing one independent random walk on another, you typically get:

- A) Low  $R^2$  and insignificant coefficients
- B) High  $R^2$  and significant coefficients (spurious!)
- C) Zero coefficients
- D) Undefined results

*Answer on next slide...*

## Quiz 2: Answer

Answer: B – High  $R^2$  and significant coefficients (spurious!)



**Granger-Newbold (1974):** Regressing unrelated  $I(1)$  series gives misleading results. Rule of thumb: If  $R^2 > DW$ , suspect spurious regression! 🌐 Real-world examples: [tylervigen.com/spurious-correlations](http://tylervigen.com/spurious-correlations)

## Quiz 3: Engle-Granger Test

### Question

In the Engle-Granger two-step method, what do you test in step 2?

- A) Whether the original variables are stationary
- B) Whether the regression residuals have a unit root
- C) Whether the coefficients are significant
- D) Whether the  $R^2$  is high enough

*Answer on next slide...*

### Quiz 3: Answer

Answer: B – Whether residuals have unit root

**Step 1:** Run OLS:  $Y_t = \alpha + \beta X_t + e_t$ , save residuals  $\hat{e}_t$

**Step 2:** ADF test on residuals:  $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \dots$

☐  $H_0: \rho = 0$  (unit root  $\Rightarrow$  no cointegration)

☐  $H_1: \rho < 0$  (stationary  $\Rightarrow$  cointegration!)

**Important:** Use Engle-Granger critical values, not standard ADF!



## Quiz 4: Johansen Test Advantage

### Question

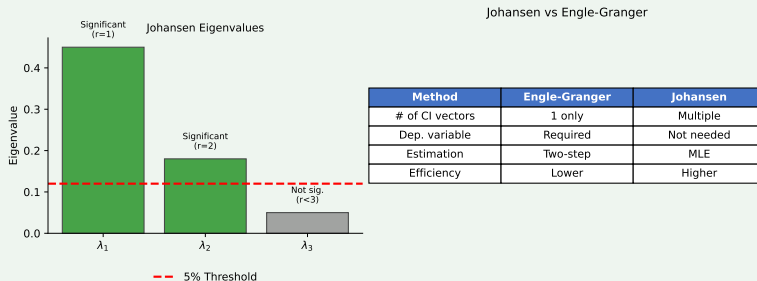
The main advantage of Johansen over Engle-Granger is:

- A) It's simpler to compute
- B) It can detect multiple cointegrating relationships
- C) It doesn't require data
- D) It always finds cointegration

*Answer on next slide...*

## Quiz 4: Answer

Answer: B – Can detect multiple cointegrating relationships



**Johansen advantages:** Tests for  $r = 0, 1, \dots, k - 1$  cointegrating vectors; MLE (more efficient); no need to choose dependent variable.

## Quiz 5: Rank of $\Pi$

### Question

In a VECM with  $k = 3$  variables, if  $\text{rank}(\Pi) = 2$ , this means:

- A) No cointegration
- B) One cointegrating relationship
- C) Two cointegrating relationships
- D) All variables are stationary

*Answer on next slide...*

## Quiz 5: Answer

Answer: C – Two cointegrating relationships

**Rank interpretation** for  $k$  variables:

- ☐  $\text{rank}(\mathbf{\Pi}) = 0$ : No cointegration (use VAR in differences)
- ☐  $0 < \text{rank}(\mathbf{\Pi}) = r < k$ :  $r$  cointegrating vectors (use VECM)
- ☐  $\text{rank}(\mathbf{\Pi}) = k$ : All variables are  $I(0)$  (use VAR in levels)

**With  $k = 3$  and  $r = 2$ :**

- ☐ Two equilibrium relationships
- ☐ Only  $k - r = 1$  common stochastic trend

## Quiz 6: VECM Structure

### Question

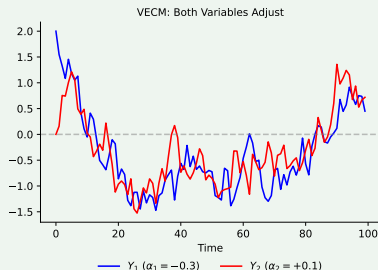
In the VECM equation  $\Delta Y_t = c + \alpha\beta'Y_{t-1} + \dots$ , what does  $\alpha$  represent?

- A) The cointegrating vectors
- B) The adjustment (loading) coefficients
- C) The short-run dynamics
- D) The error variance

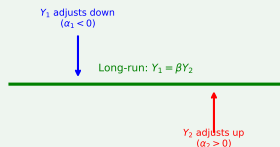
*Answer on next slide...*

## Quiz 6: Answer

Answer: B – The adjustment (loading) coefficients



Error Correction Mechanism



$$\Pi = \alpha\beta'$$

- $\beta$  = cointegrating vectors (define equilibrium)
- $\alpha$  = adjustment speeds (how fast each variable corrects)

## Quiz 7: Error Correction Term

### Question

If  $Y_t - \beta X_t$  is the cointegrating relation and this term is positive, what happens?

- A)  $Y$  is above equilibrium;  $Y$  should decrease (if  $\alpha < 0$ )
- B)  $Y$  is below equilibrium;  $Y$  should increase
- C) Nothing, error correction doesn't affect levels
- D) Both variables increase

*Answer on next slide...*

## Quiz 7: Answer

Answer: A –  $Y$  above equilibrium; decreases if  $\alpha < 0$

**Error correction mechanism:**

$$\Delta Y_t = \alpha(Y_{t-1} - \beta X_{t-1}) + \dots$$

- ▣ If  $Y_{t-1} - \beta X_{t-1} > 0$ :  $Y$  is “too high”
- ▣ With  $\alpha < 0$ :  $\Delta Y_t < 0$  ( $Y$  decreases toward equilibrium)
- ▣ This is the “error correction” pulling  $Y$  back

**Sign convention:**  $\alpha$  should be negative for the dependent variable to move back toward equilibrium.



## Quiz 8: Weak Exogeneity

### Question

If  $\alpha_2 = 0$  in a bivariate VECM, this means:

- A) There is no cointegration
- B) Variable 2 does not adjust to disequilibrium (weakly exogenous)
- C) Variable 1 does not adjust
- D) Both variables are stationary

*Answer on next slide...*

## Quiz 8: Answer

**Answer:** B – Variable 2 is weakly exogenous

**Weak exogeneity:** Variable doesn't respond to disequilibrium.

**Example: Interest rates**

- Long rate ( $R_t$ ) often weakly exogenous ( $\alpha_R \approx 0$ )
- Short rate ( $r_t$ ) adjusts to spread ( $\alpha_r < 0$ )
- Interpretation: Central bank adjusts short rate to maintain term structure

**Implication:** Can estimate single equation for the adjusting variable.

## Quiz 9: Trace Test

### Question

The Johansen trace test with  $H_0 : r \leq 1$  vs  $H_1 : r > 1$  tests whether:

- A) There is exactly one cointegrating vector
- B) There are at most one cointegrating vectors
- C) There are more than one cointegrating vectors
- D) All eigenvalues are zero

*Answer on next slide...*

## Quiz 9: Answer

Answer: B/C –  $H_0$ : at most 1;  $H_1$ : more than 1

**Sequential testing procedure:**

1. Test  $H_0 : r = 0$  vs  $H_1 : r > 0$
2. If rejected, test  $H_0 : r \leq 1$  vs  $H_1 : r > 1$
3. Continue until fail to reject...

**Trace statistic:**

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Reject  $H_0$  if trace statistic  $>$  critical value.

## Quiz 10: VECM vs VAR in Differences

### Question

If variables are cointegrated, using VAR in first differences instead of VECM:

- A) Gives identical results
- B) Is more efficient
- C) Loses long-run information (misspecified)
- D) Is the preferred approach

*Answer on next slide...*

## Quiz 10: Answer

Answer: C – Loses long-run information

**Granger Representation Theorem:** If cointegrated, VECM representation exists and should be used.

	VAR( $\Delta$ )	VECM
Long-run equilibrium	Lost	Preserved
Error correction	No	Yes
Forecasts (long-run)	Poor	Better

**Bottom line:** Differencing removes the long-run relationship that cointegration represents!

## True/False Questions

Determine if each statement is True or False:

1. Cointegration requires all variables to be  $I(1)$ .
2. The cointegrating vector is unique.
3. Spurious regression has low Durbin-Watson statistic.
4. In VECM, both  $\alpha$  coefficients must be non-zero.
5. Johansen test requires choosing a dependent variable.
6. The number of common trends  $= k - r$ .

*Answers on next slide...*

## True/False: Solutions

1. Cointegration requires all variables to be  $I(1)$ .

Standard  $CI(1,1)$  case: all variables  $I(1)$ , linear combination  $I(0)$ .

TRUE

2. The cointegrating vector is unique.

Unique only up to scalar multiplication. Usually normalized ( $\beta_1 = 1$ ).

FALSE

3. Spurious regression has low Durbin-Watson statistic.

$DW \approx 0$  indicates highly autocorrelated residuals (non-stationary).

TRUE

4. In VECM, both  $\alpha$  coefficients must be non-zero.

One can be zero (weak exogeneity). At least one must be non-zero.

FALSE

5. Johansen test requires choosing a dependent variable.

That's Engle-Granger. Johansen treats all variables symmetrically.

FALSE

6. The number of common trends =  $k - r$ .

$k$  variables,  $r$  cointegrating relations  $\Rightarrow k - r$  common stochastic trends.

TRUE



## Problem 1: Cointegration Identification

### Exercise

You have quarterly data on consumption ( $C_t$ ) and income ( $Y_t$ ). ADF tests show both are  $I(1)$ . The regression  $C_t = 0.85Y_t + e_t$  gives residuals with ADF statistic  $= -3.92$ . The 5% Engle-Granger critical value for 2 variables is  $-3.34$ .

Are  $C_t$  and  $Y_t$  cointegrated?

*Answer on next slide...*

## Problem 1: Solution

**Solution:** Yes, they are cointegrated

**Test:**  $H_0$ : No cointegration (residuals have unit root)

**ADF statistic:**  $-3.92$

**Critical value (5%):**  $-3.34$

Since  $-3.92 < -3.34$ , we **reject**  $H_0$  at 5% level.

**Conclusion:** Residuals are stationary  $\Rightarrow$  Cointegration exists!

**Interpretation:** Consumption and income share a common trend. The cointegrating vector is approximately  $(1, -0.85)$ , consistent with permanent income hypothesis.

## Problem 2: VECM Interpretation

### Exercise

A VECM for short rate ( $r_t$ ) and long rate ( $R_t$ ) gives:

$$\Delta r_t = 0.01 - 0.25(r_{t-1} - R_{t-1}) + \dots$$

$$\Delta R_t = 0.005 - 0.02(r_{t-1} - R_{t-1}) + \dots$$

Interpret the adjustment coefficients.

*Answer on next slide...*

## Problem 2: Solution

### Solution

**Error correction term:**  $(r_{t-1} - R_{t-1}) = \text{spread}$

**Short rate** ( $\alpha_r = -0.25$ ):

- When spread is positive (short > long), short rate decreases
- 25% of disequilibrium corrected per period
- Short rate actively adjusts

**Long rate** ( $\alpha_R = -0.02$ ):

- Very small adjustment coefficient
- Long rate is nearly weakly exogenous
- Mostly driven by expectations, not error correction

**Economic interpretation:** Central bank (short rate) adjusts to maintain yield curve.

## Problem 3: Johansen Test Results

### Exercise

Johansen trace test for 3 variables gives:

$H_0$	Trace Stat	5% CV
$r = 0$	45.2	29.8
$r \leq 1$	18.1	15.5
$r \leq 2$	3.2	3.8

What is the cointegrating rank?

*Answer on next slide...*

## Problem 3: Solution

**Solution: Rank = 2**

**Sequential testing:**

1.  $H_0 : r = 0$ :  $45.2 > 29.8 \Rightarrow$  **Reject** (at least 1)
2.  $H_0 : r \leq 1$ :  $18.1 > 15.5 \Rightarrow$  **Reject** (at least 2)
3.  $H_0 : r \leq 2$ :  $3.2 < 3.8 \Rightarrow$  **Fail to reject**

**Conclusion:**  $r = 2$  cointegrating relationships

**Implications:**

- Two equilibrium relationships among 3 variables
- Only  $3 - 2 = 1$  common stochastic trend
- Use VECM with 2 error correction terms

## Problem 4: Testing Weak Exogeneity

### Exercise

In a VECM for prices ( $P$ ) and exchange rate ( $E$ ), you estimate  $\alpha_P = -0.15$  (s.e. = 0.04) and  $\alpha_E = 0.02$  (s.e. = 0.03).

Test whether the exchange rate is weakly exogenous at 5%.

*Answer on next slide...*

## Problem 4: Solution

**Solution:** Exchange rate is weakly exogenous

**Test:**  $H_0 : \alpha_E = 0$  (weak exogeneity)

**t-statistic:**  $t = \frac{0.02}{0.03} = 0.67$

**Critical value (5%, two-tailed):**  $\pm 1.96$

Since  $|0.67| < 1.96$ , **fail to reject  $H_0$ .**

**Conclusion:** Exchange rate does not respond to PPP disequilibrium.

**Implication:** Prices do all the adjusting to restore PPP equilibrium. Can estimate single-equation model for prices.



## Example: Term Structure of Interest Rates

### Economic Theory

Expectations hypothesis:  $R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{premium}$

If premium is constant  $\Rightarrow$  spread  $(R_t - r_t)$  should be stationary.

### Typical Findings

- Both rates are  $I(1)$  (confirmed by ADF)
- Johansen test:  $r = 1$  cointegrating vector
- Cointegrating vector  $\approx (1, -1)$ : spread is stationary
- Short rate adjusts ( $\alpha_r < 0$ ), long rate weakly exogenous

### Policy Implication

Central bank controls short rate; long rate driven by expectations.

## Example: Purchasing Power Parity

### PPP Theory

$e_t = p_t - p_t^*$  (log exchange rate = price differential)

Real exchange rate:  $q_t = e_t - p_t + p_t^*$  should be stationary (long-run PPP)

### Empirical Challenges

- Unit root tests:  $e_t, p_t, p_t^*$  all  $I(1)$
- Cointegration tests: Mixed results depending on sample
- Half-life of PPP deviations: 3-5 years (slow adjustment)
- Weak exogeneity: Exchange rate often doesn't adjust

### PPP Puzzle

Real exchange rate is highly persistent—slow mean reversion is hard to explain with standard models.

## Example: Pairs Trading Strategy

### Idea

Find cointegrated stocks  $\Rightarrow$  trade the stationary spread

### Implementation Steps

1. **Identify pairs:** Test cointegration (e.g., Coca-Cola & Pepsi)
2. **Estimate spread:**  $z_t = P_A - \beta P_B$
3. **Trading rules:**
  - ▶  $z_t > \mu + 2\sigma$ : Sell A, Buy B (spread too wide)
  - ▶  $z_t < \mu - 2\sigma$ : Buy A, Sell B (spread too narrow)
  - ▶ Exit when  $z_t \approx \mu$

### Risks

Cointegration can break down; spread may not revert; transaction costs.

## Python Cointegration Analysis: Key Functions

### Essential Libraries

```
from statsmodels.tsa.stattools import coint, adfuller  
from statsmodels.tsa.vector_ar.vecm import coint_johansen, VECM
```

### Workflow

1. Unit root tests: `adfuller(series)`
2. Engle-Granger: `coint(y, x)` returns test stat & p-value
3. Johansen: `coint_johansen(data, det_order, k_ar_diff)`
4. Fit VECM: `model = VECM(data, k_ar_diff=2, coint_rank=1)`
5. Results: `results = model.fit()`

### Note

Complete working examples are provided in the Jupyter notebooks.

## Discussion: Cointegration vs Correlation

### Key Question

Two series are highly correlated. Are they cointegrated?

Answer: Not necessarily!

- ▣ **Correlation:** Measures co-movement (can be spurious for  $I(1)$ )
- ▣ **Cointegration:** Requires stationary linear combination

### Example

Two independent random walks can have correlation  $> 0.9$  purely by chance (spurious correlation). But they're NOT cointegrated—their spread is also  $I(1)$ .

**Cointegration** implies a meaningful long-run equilibrium relationship.

## Discussion: Choosing Deterministic Components

### Key Question

Johansen test has 5 cases for deterministics. Which to choose?

### Guidelines

1. **No constant, no trend:** Rarely used (requires mean-zero data)
2. **Constant in CE only:** Level series, no drift
3. **Constant unrestricted:** Most common for economic data
4. **Trend in CE:** Series have deterministic trends
5. **Trend unrestricted:** Trending differences (uncommon)

### Practical Advice

Start with Case 3 (constant unrestricted). Check sensitivity to specification. Use economic reasoning: do levels have trends?

## Take-Home Exercises

1. **Theoretical:** Show that if  $Y_t$  and  $X_t$  are both random walks with the same innovation, they are cointegrated.
2. **Computation:** Given VECM estimates:

$$\Delta Y_t = 0.5 - 0.3(Y_{t-1} - 2X_{t-1}) + 0.2\Delta Y_{t-1}$$

$$\Delta X_t = 0.1 + 0.1(Y_{t-1} - 2X_{t-1}) + 0.4\Delta X_{t-1}$$

- ▶ What is the cointegrating vector?
  - ▶ Which variable adjusts more quickly?
  - ▶ What is the long-run equilibrium relationship?
3. **Applied:** Download 10-year and 3-month Treasury rates:
    - ▶ Test for unit roots; Test for cointegration (Engle-Granger and Johansen)
    - ▶ Estimate VECM; Interpret adjustment coefficients
  4. **Critical Thinking:** Why might PPP hold in the long run but not short run?

## Exercise Solutions Hints

### Hints

1. If  $Y_t = Y_{t-1} + \varepsilon_t$  and  $X_t = X_{t-1} + \varepsilon_t$  (same shock), then  $Y_t - X_t = Y_0 - X_0$  is constant (stationary).
2. From the VECM:
  - ▶ Cointegrating vector:  $(1, -2)$  (normalized on  $Y$ )
  - ▶  $Y$  adjusts faster:  $|\alpha_Y| = 0.3 > |\alpha_X| = 0.1$
  - ▶ Long-run:  $Y = 2X$  (when EC term = 0)
3. For interest rates:
  - ▶ Both typically  $I(1)$ ; spread usually stationary
  - ▶ Expect one cointegrating vector with  $(1, -1)$
  - ▶ Short rate typically adjusts; long rate often weakly exogenous
4. PPP deviations: Transportation costs, non-traded goods, sticky prices, tariffs, market segmentation all slow adjustment but don't prevent long-run convergence.



## Key Takeaways from This Seminar

### Main Points

1. **Cointegration:**  $I(1)$  variables with stationary linear combination
2. **Spurious regression:** High  $R^2$  without cointegration is meaningless
3. **Engle-Granger:** Simple, but only one cointegrating vector
4. **Johansen:** Multiple vectors, MLE, more powerful

### VECM Insights

- ▣  $\beta$  defines equilibrium;  $\alpha$  determines adjustment speed
- ▣ Weak exogeneity ( $\alpha = 0$ ): Variable doesn't respond to disequilibrium
- ▣ Always use VECM (not VAR in differences) when cointegrated

### Remember

Cointegration is about **long-run equilibrium**, not just correlation!