



Time Series Analysis and Forecasting

Chapter 4: SARIMA Models



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Learning Objectives

By the end of this chapter, you will be able to:

1. **Identify** seasonal patterns in time series data
2. **Apply** seasonal differencing to remove seasonal unit roots
3. **Build** and estimate SARIMA models with seasonal components
4. **Interpret** seasonal ACF/PACF patterns for model identification
5. **Evaluate** forecasts using rolling window methods for seasonal data
6. **Apply** the complete methodology on real data (airline passengers)

Outline

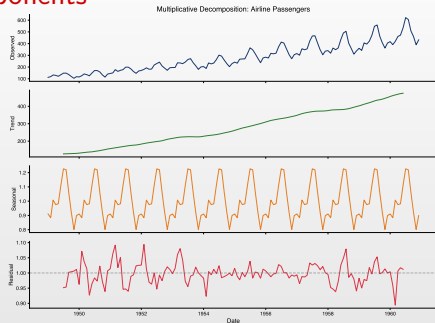
- Motivation
- Seasonality in Time Series
- Seasonal Differencing
- The SARIMA Model
- Seasonal ACF and PACF Patterns
- Estimation and Diagnostics
- Forecasting with SARIMA
- Case Study: Airline Passengers
- Practical Aspects
- AI Use Case
- Summary
- Quiz

Why SARIMA? Seasonality Is Everywhere



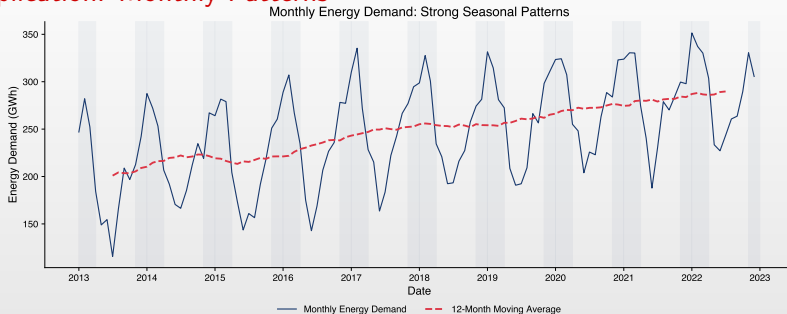
- Retail sales exhibit clear **annual patterns**: December peaks, January troughs
- Standard ARIMA models cannot capture these **repeating seasonal cycles**
- Ignoring seasonality leads to systematic forecast errors

Understanding Seasonal Components



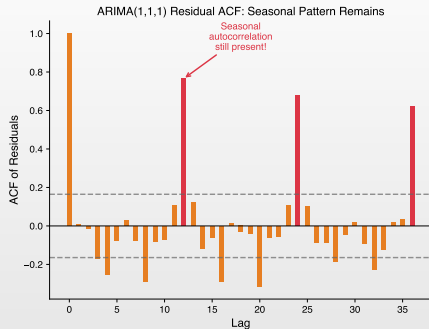
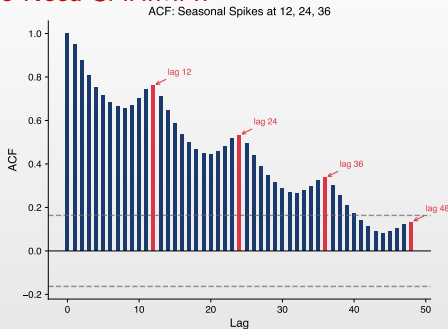
- ▣ Seasonal time series = **Trend** + **Seasonal Pattern** + **Residuals**
- ▣ Decomposition helps visualize each component separately
- ▣ SARIMA models capture both trend dynamics and seasonal behavior

Real-World Application: Monthly Patterns



- Energy demand shows strong **monthly seasonality** (heating/cooling cycles)
- Pattern repeats predictably each year with slight variations
- Utility companies use SARIMA forecasts for capacity planning

Why Do We Need SARIMA?



- Left: Seasonal ACF shows spikes at lags 12, 24, 36... (annual pattern)
- Right: ARIMA residuals still show seasonal autocorrelation \Rightarrow model is incomplete
- SARIMA adds **seasonal AR and MA terms** to capture these patterns

What is Seasonality?

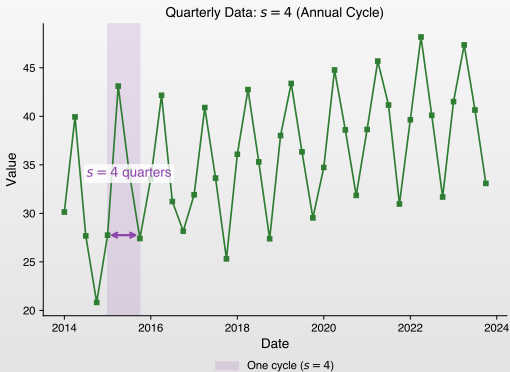
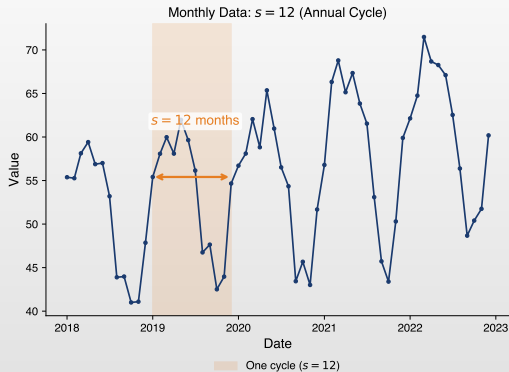
Definition 1 (Seasonality)

A time series exhibits seasonality when it shows regular, periodic fluctuations that repeat over a fixed period s (the seasonal period).

Common Seasonal Periods

- Monthly data: $s = 12$ (annual cycle)
- Quarterly data: $s = 4$ (annual cycle)
- Weekly data: $s = 52$ (annual) or $s = 7$ (weekly pattern)
- Daily data: $s = 7$ (weekly pattern)

What is Seasonality?



Example: Airline Passengers Data



- Monthly international airline passengers (1949–1960)
- Clear **upward trend** and **growing seasonal amplitude**
- Summer peaks reflect vacation travel patterns

Examples of Seasonal Data

Economic Series

- ▣ Retail sales (holiday peaks)
- ▣ Tourism (summer/winter)
- ▣ Agricultural production
- ▣ Energy consumption
- ▣ Employment (hiring cycles)

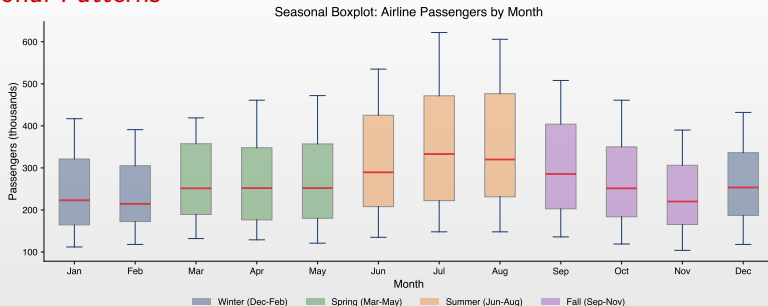
Other Domains

- ▣ Weather/temperature
- ▣ Website traffic
- ▣ Hospital admissions
- ▣ Transportation usage
- ▣ Electricity demand

Why It Matters

Ignoring seasonality leads to biased forecasts and invalid inference!

Visualizing Seasonal Patterns



- Box plot reveals consistent seasonal pattern across years
- July–August: highest passenger counts (summer travel)
- November–February: lowest counts (winter months)

Deterministic vs Stochastic Seasonality

Deterministic Seasonality

- ▣ **Fixed pattern:** $Y_t = \sum_{j=1}^s \gamma_j D_{jt} + \varepsilon_t$
 - ▶ D_{jt} are seasonal dummies
- ▣ Pattern constant over time
- ▣ Same amplitude every year
- ▣ Removed by regression on dummies
- ▣ ACF: sharp cutoff at seasonal lags
- ▣ **Example:** University enrollment peaks every September by the same amount

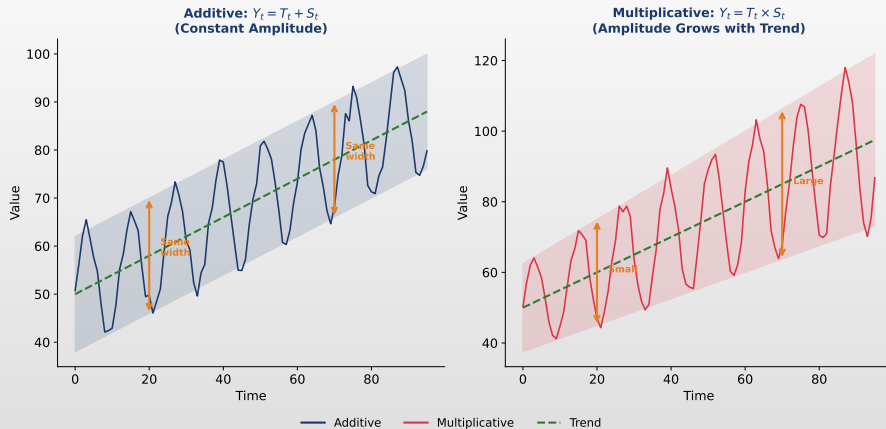
Stochastic Seasonality

- ▣ **Evolving pattern:** $\Delta_s Y_t = Y_t - Y_{t-s}$
 - ▶ Exhibits dependence structure
- ▣ Pattern evolves over time
- ▣ Amplitude may grow or shrink
- ▣ Requires seasonal differencing
- ▣ ACF: slow decay at seasonal lags
- ▣ **Example:** Retail sales peaks grow larger each December

How to decide?

- ▣ Slow ACF decay at lags $s, 2s, 3s, \dots \Rightarrow$ stochastic (use Δ_s)
- ▣ Sharp cutoff \Rightarrow deterministic (use dummies)
- ▣ Use HEGY or Canova-Hansen tests to confirm

Additive vs Multiplicative Seasonality



Additive vs Multiplicative Seasonality

Additive: $Y_t = T_t + S_t + \varepsilon_t$

- ▣ Seasonal amplitude **constant**
- ▣ No transformation needed
- ▣ Ex: temperatures, university enrollment

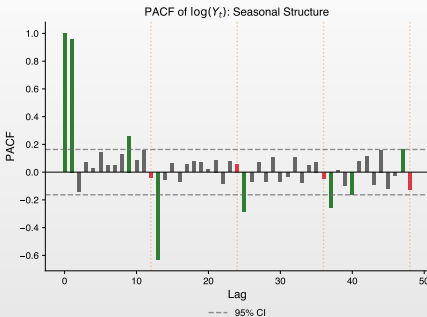
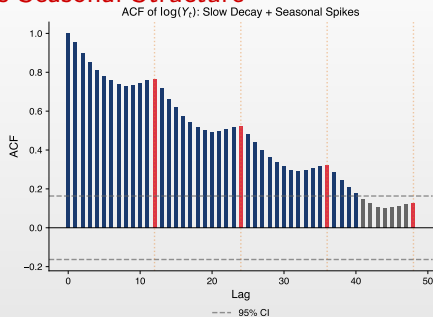
Multiplicative: $Y_t = T_t \cdot S_t \cdot \varepsilon_t$

- ▣ Amplitude **grows with level**
- ▣ Requires log transform (Box-Cox)
- ▣ Ex: Airline, retail sales, GDP

First practical decision

- ▣ Amplitude grows with the trend? \Rightarrow multiplicative \Rightarrow apply log/Box-Cox *before* differencing

ACF Reveals Seasonal Structure



- **Slow decay** at all lags indicates non-stationarity (trend)
- **Spikes at lags 12, 24, 36** confirm seasonal pattern ($s = 12$)
- ACF at seasonal lags: slow decay \Rightarrow needs seasonal differencing

Detecting Seasonality

Visual Methods

- ▣ Time series plot – look for repeating patterns
- ▣ Seasonal subseries plot – compare same seasons across years
- ▣ ACF plot – spikes at seasonal lags ($s, 2s, 3s, \dots$)

Statistical Tests

- ▣ Seasonal unit root tests (HEGY, Canova-Hansen, OCSB^a)
- ▣ F-test for seasonal dummy variables
- ▣ Kruskal-Wallis test (non-parametric)

ACF Signature

- ▣ Strong seasonality: ACF shows significant spikes at lags $s, 2s, 3s, \dots$

^aOsborn-Chui-Smith-Birchenhall — the default test in `auto_arima`

F-Test for Seasonal Dummy Variables: Intuition

What does this test do?

- ▣ **Goal:** test whether mean values differ significantly across seasons
- ▣ **Logic:** if the mean in January \neq February $\neq \dots \neq$ December \Rightarrow seasonality
- ▣ **Method:** compare a model WITH seasonal dummy variables vs. a model WITHOUT

Models compared

- ▣ **Restricted:** $Y_t = \alpha + \varepsilon_t$ **Unrestricted:** $Y_t = \alpha + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$
- ▣ where $D_{jt} = 1$ if observation t is in season j , 0 otherwise

Key idea

- ▣ If adding seasonal dummy variables **significantly reduces** prediction errors, then seasonality is present

F-Test for Seasonal Dummy Variables: Formula and Example

F-statistic formula

- **Formula:** $F = \frac{(SSR_R - SSR_U)/(s-1)}{SSR_U/(n-s)} \sim F_{s-1, n-s}$
 - ▶ SSR_R : sum of squared residuals from the restricted model (no dummies)
 - ▶ SSR_U : sum of squared residuals from the unrestricted model (with dummies)
 - ▶ $s - 1$: number of restrictions (monthly: 11, quarterly: 3)

Numerical example (Monthly data, $n=120$)

- $SSR_R = 15000, SSR_U = 8500, s = 12$
- $F = \frac{(15000 - 8500)/11}{8500/108} = \frac{590.9}{78.7} = 7.51$
- Critical value $F_{0.05, 11, 108} \approx 1.87$. Since $7.51 > 1.87$: **Reject $H_0 \Rightarrow$ Seasonality present!**

Kruskal-Wallis Test: Intuition

What does this test do?

- ▣ **Non-parametric test:** checks whether observations from different seasons come from the same distribution
 - ▶ No assumptions needed about the shape of the data distribution
- ▣ **Mechanism:** ranks all observations from smallest to largest
 - ▶ Replaces original values with ranks $(1, 2, \dots, N)$
- ▣ **Check:** whether ranks are uniformly distributed across seasons
 - ▶ Computes sum of ranks per season and compares with expected distribution
- ▣ **Conclusion:** if one season consistently has higher/lower ranks \Rightarrow seasonality
 - ▶ Example: if summer always has high ranks \Rightarrow seasonal energy consumption

Why use it instead of the F-test?

- ▣ **No normality assumption** – works with any distribution
 - ▶ Ideal for financial data with heavy tails or skewed distributions
- ▣ **Robust to outliers** – extreme values do not distort results
 - ▶ Ranks bound the influence of any single observation

Kruskal-Wallis Test: Formula and Example

Test statistic

$$\square H = \frac{12}{N(N+1)} \sum_{j=1}^s \frac{R_j^2}{n_j} - 3(N+1) \quad \text{where } N = \text{total obs.}, n_j = \text{obs. in season } j, R_j = \text{rank sum}$$

Example: Quarterly sales (n=20, s=4)

- Data ranked 1–20. Rank sums: Q1: $R_1 = 15$, Q2: $R_2 = 35$, Q3: $R_3 = 70$, Q4: $R_4 = 90$
- $H = \frac{12}{20 \times 21} \left(\frac{15^2}{5} + \frac{35^2}{5} + \frac{70^2}{5} + \frac{90^2}{5} \right) - 3(21) = 19.6$
- Critical value $\chi_{0.05,3}^2 = 7.81$. Since $19.6 > 7.81$: **Reject $H_0 \Rightarrow$ Seasonality!**

In Python

- **Implementation:** `scipy.stats.kruskal(q1, q2, q3, q4)`

HEGY Test: What Problem Does It Solve?

Key question

- **Problem:** given a seasonal series, we need to determine the type of differencing
 - ▶ Wrong choice leads to over-differencing or under-differencing
- **Regular differencing** $(1 - L)^d \Rightarrow$ set $d = 1$; **Seasonal differencing** $(1 - L^s)^D \Rightarrow$ set $D = 1$
- **HEGY:** tests for both types of unit roots simultaneously!
 - ▶ Proposed by Hylleberg, Engle, Granger and Yoo (1990)

Why not just use ADF?

- **ADF:** tests only for a regular unit root at frequency zero
 - ▶ Completely ignores the seasonal components of the series
- **Limitation:** seasonal data may have unit roots at seasonal frequencies that ADF misses!

HEGY tests multiple frequencies

- **Quarterly:** tests at $0, \pi, \pm\pi/2$
 - ▶ 3 frequencies \Rightarrow one regular root + two seasonal roots
- **Monthly:** tests at $0, \pi, \pm\pi/6, \pm\pi/3, \pm\pi/2, \pm2\pi/3, \pm5\pi/6$
 - ▶ 6 frequencies \Rightarrow one regular root + five seasonal roots

HEGY Test: Auxiliary Regression (Quarterly)

HEGY auxiliary regression

- **Quarterly data** ($s = 4$): $\Delta_4 y_t = \pi_1 z_{1,t-1} + \pi_2 z_{2,t-1} + \pi_3 z_{3,t-2} + \pi_4 z_{4,t-2} + \sum_{j=1}^k \phi_j \Delta_4 y_{t-j} + \varepsilon_t$
 - Lags k chosen by AIC/BIC to eliminate residual autocorrelation

Transformed variables

- z_{1t} : $(1 + L + L^2 + L^3)y_t = y_t + y_{t-1} + y_{t-2} + y_{t-3}$
 - Captures the zero-frequency component (trend)
- z_{2t} : $-(1 - L + L^2 - L^3)y_t = -y_t + y_{t-1} - y_{t-2} + y_{t-3}$
 - Captures the frequency- π component (semi-annual alternation)
- z_{3t}, z_{4t} : capture the frequency $\pm\pi/2$ components (annual cycle)

Hypotheses

- $H_0 : \pi_1 = 0$: unit root at frequency 0 \Rightarrow requires $d = 1$
- $H_0 : \pi_2 = 0$: unit root at frequency $\pi \Rightarrow$ requires $D = 1$
- $H_0 : \pi_3 = \pi_4 = 0$: unit root at frequency $\pm\pi/2 \Rightarrow$ requires $D = 1$

HEGY Test: Decision Rules with Examples

HEGY critical values (5%, $n=100$, with constant)

Test	Statistic	Critical value	If NOT rejected...
t_1 ($\pi_1 = 0$)	t-stat	-2.88	Requires $d = 1$
t_2 ($\pi_2 = 0$)	t-stat	-2.88	Requires $D = 1$
F_{34} ($\pi_3 = \pi_4 = 0$)	F-stat	6.57	Requires $D = 1$

Example: Quarterly GDP

- ▣ **HEGY results:** $t_1 = -1.52$, $t_2 = -4.21$, $F_{34} = 2.15$
- ▣ $t_1 = -1.52 > -2.88$: Cannot reject \Rightarrow **requires** $d = 1$
- ▣ $t_2 = -4.21 < -2.88$: Reject \Rightarrow no unit root at π
- ▣ $F_{34} = 2.15 < 6.57$: Cannot reject \Rightarrow **requires** $D = 1$
- ▣ **Conclusion:** Use SARIMA with $d = 1, D = 1$

Canova-Hansen Test: The Opposite of HEGY

HEGY vs Canova-Hansen: Different null hypotheses!

	HEGY	Canova-Hansen
H_0	Seasonal unit root	No seasonal unit root
H_1	No seasonal unit root	Seasonal unit root
Reject H_0	Use seasonal dummies	Use differencing $(1 - L^5)$
Do not reject	Use differencing $(1 - L^5)$	Use seasonal dummies

Why does it matter?

- HEGY: “Prove there is NO unit root” (conservative towards differencing)
- CH: “Prove there IS a unit root” (conservative towards dummies)
- Use **both** tests for robust conclusions!

Canova-Hansen Test: Formula

Testing procedure

- ▣ **Step 1:** Regress y_t on seasonal dummies: $y_t = \sum_{j=1}^s \gamma_j D_{jt} + u_t$
- ▣ **Step 2:** Compute partial sums at seasonal frequency λ_i :
 - ▶ $S_{it}^{(c)} = \sum_{j=1}^t \hat{u}_j \cos(\lambda_i j), \quad S_{it}^{(s)} = \sum_{j=1}^t \hat{u}_j \sin(\lambda_i j)$

LM test statistic

- ▣ $LM_i = \frac{1}{T^2 \hat{\omega}_i} \left[\sum_{t=1}^T (S_{it}^{(c)})^2 + \sum_{t=1}^T (S_{it}^{(s)})^2 \right]$
- ▣ where $\hat{\omega}_i$ = consistent estimator of the spectral density at frequency λ_i

Decision

- ▣ **Rule:** reject H_0 (stationarity) if $LM > \text{critical value} \Rightarrow$ seasonal differencing is needed

Summary: Choosing the Right Seasonality Test

Test	H_0	If rejected	Best for
F-test	No seasonality	Seasonality exists	Normal data
Kruskal-Wallis	No difference across seasons	Seasonality exists	Non-normal, outliers
HEGY	Unit root exists	Use dummies	Determining d , D
Canova-Hansen	No unit root	Use $(1 - L^s)$	Confirming stability

Key idea

- ▣ **F-test / Kruskal-Wallis:** “*Does seasonality exist?*”
- ▣ **HEGY / Canova-Hansen:** “*What type?*” (deterministic vs stochastic)

Box-Cox Transformation: Variance Stabilization

Box-Cox Family of Transformations

- **Formula:** $Y_t^{(\lambda)} = \begin{cases} \frac{Y_t^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln(Y_t) & \text{if } \lambda = 0 \end{cases}$
- **Special cases:** $\lambda = 1$ (no transformation), $\lambda = 0$ (logarithm), $\lambda = 0.5$ (square root)
 - ▶ $\lambda < 1$ compresses large values; $\lambda > 1$ amplifies them

Automatic Selection of λ

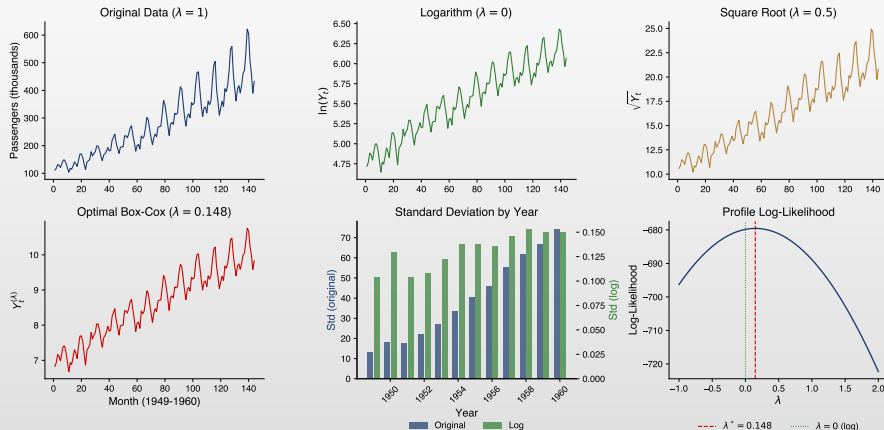
- **Profile likelihood:** maximizes the log-likelihood as a function of λ
 - ▶ Most commonly used method; provides confidence interval for λ
- **Guerrero method (1993):** minimizes the coefficient of variation of seasonal sub-series
 - ▶ Preferred for seasonal data with varying amplitude
- **Python:** `boxcox(y)` from `scipy.stats` or `BoxCox.lambda_(y)` in R

Why not just logarithm?

- **Log ($\lambda = 0$)** assumes variance proportional to level — not always the case
 - ▶ **Example:** data with moderate variance may require $\lambda = 0.3$, not $\lambda = 0$

- **Box-Cox** chooses the optimal transformation based on data, not assumptions

Box-Cox on the Airline Data: Complete Example



Box-Cox on the Airline Data: Complete Example

Result for Airline Passengers

- $\hat{\lambda} = 0.148 \approx 0 \Rightarrow \log$ is nearly optimal
- Standard deviation per year: from increasing (original) to stable (log)

Bias Correction in Back-Transformation

- On log scale: \hat{y}_{T+h} is the **median**, not the mean
- Correction: $\hat{Y}_{T+h} = \exp\left(\hat{y}_{T+h} + \frac{\sigma_h^2}{2}\right)$
- Without correction: systematically under-estimated forecasts!

STL Decomposition: Modern Alternatives

STL: Seasonal-Trend Decomposition using Loess (Cleveland et al., 1990)

- ▣ **Advantages:** time-varying seasonality, robust to outliers, any period s
- ▣ **Algorithm:** iterative locally weighted regression (loess)

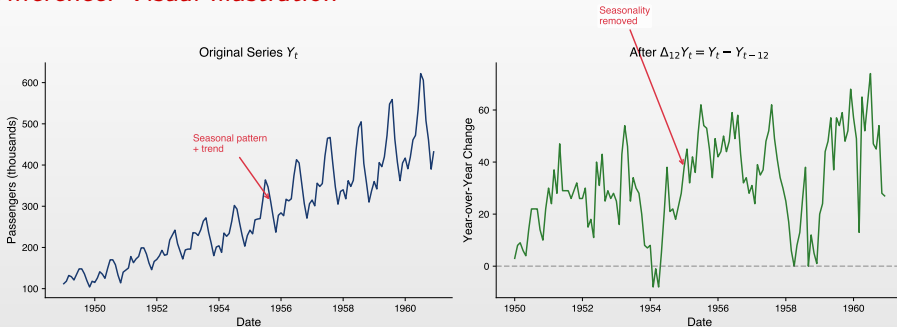
Key Parameters

- ▣ **Seasonal window** (`seasonal`): controls how quickly seasonality changes
- ▣ **Trend window** (`trend`): smoothing of the trend component
- ▣ **Robustness** (`robust=True`): reduces influence of outliers

Practical Usage

- ▣ STL for exploration and preprocessing; SARIMA for modeling and forecasting
- ▣ Python: `STL(y, period=12).fit()` from `statsmodels`

Seasonal Difference: Visual Illustration



- **Left:** Original series with clear seasonal pattern
- **Right:** After $\Delta_{12} = (1 - L^{12})$, the seasonal pattern is removed
 - Year-over-year comparison eliminates seasonal effects

The Seasonal Difference Operator

Definition 2 (Seasonal Difference)

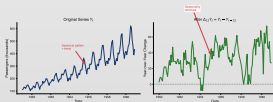
The seasonal difference operator Δ_s is defined as:

$$\Delta_s Y_t = (1 - L^s)Y_t = Y_t - Y_{t-s}$$

where $L^s Y_t = Y_{t-s}$ is the seasonal lag operator.

Examples

- Monthly data ($s = 12$): $\Delta_{12} Y_t = Y_t - Y_{t-12}$
Compares each month to the same month last year
- Quarterly data ($s = 4$): $\Delta_4 Y_t = Y_t - Y_{t-4}$
Compares each quarter to the same quarter last year



Proof: Seasonal Differencing Removes Deterministic Seasonality

Claim: If $Y_t = \mu_t + \varepsilon_t$ where $\mu_t = \mu_{t-s}$ (periodic mean), then $\Delta_s Y_t$ removes the seasonal mean.

- The condition $\mu_t = \mu_{t-s}$ means the mean repeats exactly every s periods

Proof: Let $Y_t = \mu_t + \varepsilon_t$ where μ_t has period s . Apply seasonal difference:

$$\begin{aligned}\Delta_s Y_t &= Y_t - Y_{t-s} = (\mu_t + \varepsilon_t) - (\mu_{t-s} + \varepsilon_{t-s}) \\ &= \mu_t - \mu_{t-s} + \varepsilon_t - \varepsilon_{t-s} \\ &= 0 + \varepsilon_t - \varepsilon_{t-s} \quad (\text{since } \mu_t = \mu_{t-s})\end{aligned}$$

Properties of $\Delta_s Y_t = \varepsilon_t - \varepsilon_{t-s}$:

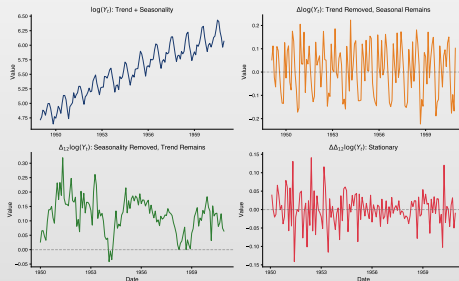
- $\mathbb{E}[\Delta_s Y_t] = 0$ (constant mean); $\text{Var}(\Delta_s Y_t) = 2\sigma^2$ (constant variance)
 - ▶ Both properties are required for second-order stationarity
- Autocovariance: $\gamma(s) = -\sigma^2$, $\gamma(k) = 0$ for $k \neq 0, s$
 - ▶ ACF shows a single spike at lag s , then zero

Result

Seasonal differencing transforms periodic seasonal pattern into MA(1) at seasonal lag.

Effect of Differencing Operations

- Regular differencing removes trend but seasonal pattern remains
- Seasonal differencing removes seasonality but trend pattern remains
- Both differences** needed to achieve stationarity



Combining Regular and Seasonal Differencing

Full Differencing

For series with both trend and seasonality:

$$\Delta\Delta_s Y_t = (1 - L)(1 - L^s)Y_t$$

Expansion

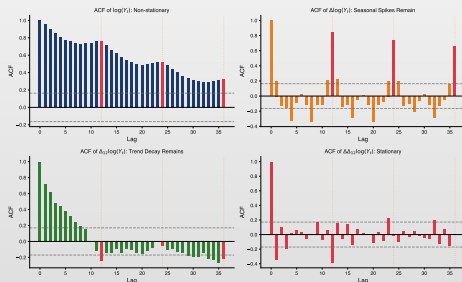
$(1 - L)(1 - L^s)Y_t = Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}$. For monthly: $\Delta\Delta_{12} Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$

Order of Differencing

d : regular differences (trend removal); D : seasonal differences (seasonal trend removal)

ACF Before and After Differencing

- Original ACF: slow decay indicates non-stationarity
- After Δ : seasonal spikes remain at lags 12, 24, 36
- After Δ_{12} : trend decay remains at early lags
- After $\Delta\Delta_{12}$: ACF cuts off \Rightarrow **stationary**



Seasonal Integration

Definition 3 (Seasonally Integrated Process)

A series Y_t is **seasonally integrated** of order $(d, D)_s$, written $Y_t \sim I(d, D)_s$, if:

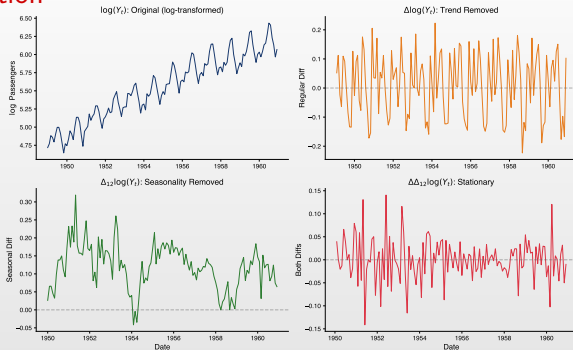
$$(1 - L)^d (1 - L^s)^D Y_t$$

is stationary.

Common Cases

- ▣ $I(1, 0)_{12}$: Regular unit root only (monthly)
- ▣ $I(0, 1)_{12}$: Seasonal unit root only
- ▣ $I(1, 1)_{12}$:
 - ▶ Both regular and seasonal unit roots

SARIMA: Visual Illustration



- Original \Rightarrow regular difference (removes trend) \Rightarrow seasonal difference (removes seasonality)
- Apply minimum differencing needed to achieve stationarity

SARIMA Model Definition

Definition 4 ($\text{SARIMA}(p, d, q) \times (P, D, Q)_s$)

The **Seasonal ARIMA** model is:

$$\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^DY_t = c + \theta(L)\Theta(L^s)\varepsilon_t$$

Components

- ▣ $\phi(L) = 1 - \phi_1L - \dots - \phi_pL^p$: Non-seasonal AR
- ▣ $\Phi(L^s) = 1 - \Phi_1L^s - \dots - \Phi_PL^{Ps}$: Seasonal AR
- ▣ $\theta(L) = 1 + \theta_1L + \dots + \theta_qL^q$: Non-seasonal MA
- ▣ $\Theta(L^s) = 1 + \Theta_1L^s + \dots + \Theta QL^{Qs}$: Seasonal MA
- ▣ $(1-L)^d$:
 - ▶ Regular differencing; $(1-L^s)^D$: Seasonal differencing

Proof: Multiplicative Seasonal Structure

Why multiplicative? Consider $\text{SARIMA}(1, 0, 0) \times (1, 0, 0)_s$:

$$(1 - \phi L)(1 - \Phi L^s)Y_t = \varepsilon_t$$

Expand: $(1 - \phi L)(1 - \Phi L^s)Y_t = Y_t - \phi Y_{t-1} - \Phi Y_{t-s} + \phi\Phi Y_{t-s-1}$

Interpretation (Monthly, $s = 12$)

Y_t depends on: Y_{t-1} (last month), Y_{t-12} (same month last year), Y_{t-13} (interaction).

Parsimony: Multiplicative form uses 2 parameters (ϕ, Φ); additive would need 3+.

SARIMA Notation

Full Specification

$\text{SARIMA}(p, d, q) \times (P, D, Q)_s$ has 7 parameters to specify:

Parameter	Meaning
p	Non-seasonal AR order
d	Non-seasonal differencing order
q	Non-seasonal MA order
P	Seasonal AR order
D	Seasonal differencing order
Q	Seasonal MA order
s	Seasonal period

Example

$\text{SARIMA}(1, 1, 1) \times (1, 1, 1)_{12}$: Monthly data with AR(1), MA(1), seasonal AR(1), seasonal MA(1), and both regular and seasonal differencing.

Common SARIMA Models

Airline Model: $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_s$

$(1 - L)(1 - L^s)Y_t = (1 + \theta L)(1 + \Theta L^s)\varepsilon_t$ - Classic model (Box & Jenkins, 1970)

$\text{SARIMA}(1, 0, 0) \times (1, 0, 0)_s$

$(1 - \phi L)(1 - \Phi L^s)Y_t = \varepsilon_t$ - Pure seasonal and non-seasonal AR

$\text{SARIMA}(0, 1, 1) \times (0, 1, 0)_s$

$(1 - L)(1 - L^s)Y_t = (1 + \theta L)\varepsilon_t$ - Random walk + seasonal diff + MA(1)

ACF/PACF for Seasonal Models

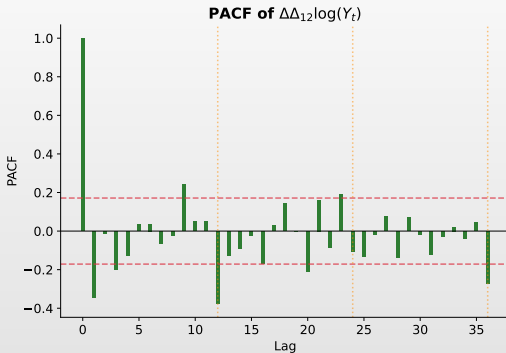
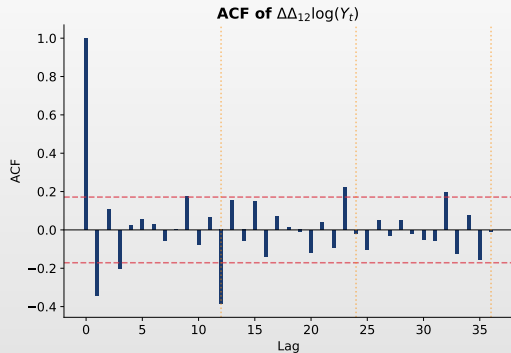
Key Insight

Seasonal models show patterns at both:

- Non-seasonal lags: $1, 2, 3, \dots$
- Seasonal lags: $s, 2s, 3s, \dots$

Model	ACF	PACF
SAR(P)	Decays at $s, 2s, \dots$	Cuts off after P_s
SMA(Q)	Cuts off after Q_s	Decays at $s, 2s, \dots$
SARMA	Decays at seasonal lags	Decays at seasonal lags

Example: Airline Model ACF/PACF



Example: Airline Model ACF/PACF

ACF: $\Delta\Delta_{12} \log(Y_t)$

- ▣ Spike at lag 1 \leftarrow MA(1), θ
- ▣ Spike at lag 12 \leftarrow SMA(1), Θ
- ▣ Rest \approx zero

PACF: exponential decay

- ▣ Decays at lags 1, 2, 3, ...
- ▣ Decays at lags 12, 24, 36
- ▣ \Rightarrow **MA, not AR**

▣ **Conclusion:** ACF cuts off \Rightarrow MA; PACF decays \Rightarrow not AR. Model: $(0, 1, 1) \times (0, 1, 1)_{12}$

Model Identification Guidelines

Step-by-Step Process

1. Examine ACF for slow decay at seasonal lags \Rightarrow seasonal differencing
2. After differencing, check ACF/PACF patterns
3. Non-seasonal behavior at lags $1, 2, \dots, s - 1$
4. Seasonal behavior at lags $s, 2s, 3s, \dots$

Practical Tips

- Start with $d \leq 1$ and $D \leq 1$
- Usually $P, Q \leq 2$ is sufficient
- Use information criteria (AIC, BIC) for final selection
- Auto-SARIMA algorithms can help

Estimation Methods

Maximum Likelihood Estimation

Standard approach for SARIMA:

- ▣ Conditional MLE (conditional on initial values)
- ▣ Exact MLE (via Kalman filter)

Computational Considerations

- ▣ More parameters than ARIMA \Rightarrow more data needed
- ▣ Seasonal parameters estimated from lags $s, 2s, \dots$
- ▣ Need sufficient seasonal cycles (at least 3-4 years of monthly data)

Exact Likelihood: Prediction Error Decomposition

Why the Kalman Filter?

- ▣ **SARIMA**: has the structure of a state-space model
- ▣ **Kalman filter**: recursively computes prediction errors v_t and their variances f_t , without conditioning on initial values

Exact Log-Likelihood (Prediction Error Decomposition)

- ▣ **Formula**: $\ell(\theta) = -\frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \left[\ln(f_t) + \frac{v_t^2}{f_t} \right]$
- ▣ v_t : $Y_t - \hat{Y}_{t|t-1}$ (innovation); f_t : $\text{Var}(v_t)$ (innovation variance)

Advantages over Conditional MLE

- ▣ Does not require choosing initial values
- ▣ Each term $\ln(f_t)$ weights observations differently (variable variance at start)
- ▣ Essential for short series where initial values matter
- ▣ Implemented by default in `statsmodels.tsa.SARIMAX()` with `method='mle'`

Stationarity and Invertibility

Stationarity Conditions

Both non-seasonal and seasonal AR polynomials must have roots outside the unit circle:

- ▣ $\phi(z) = 0 \Rightarrow |z| > 1$
- ▣ $\Phi(z^s) = 0 \Rightarrow |z| > 1$

Invertibility Conditions

Both non-seasonal and seasonal MA polynomials must have roots outside the unit circle:

- ▣ $\theta(z) = 0 \Rightarrow |z| > 1$
- ▣ $\Theta(z^s) = 0 \Rightarrow |z| > 1$

Diagnostic Checking

Residual Analysis

After fitting SARIMA, check that residuals are white noise:

1. Plot residuals over time (no patterns)
2. ACF of residuals (no significant spikes)
3. Ljung-Box test at multiple lags including seasonal
4. Normality tests (Q-Q plot, Jarque-Bera)

Important

Check ACF at **both** non-seasonal and seasonal lags!

Significant ACF at lag 12 suggests inadequate seasonal modeling.

Model Selection Criteria

Information Criteria

Compare competing SARIMA models using:

- ▣ $AIC = -2 \ln(L) + 2k$
- ▣ $BIC = -2 \ln(L) + k \ln(n)$
- ▣ $AICc = AIC + \frac{2k(k+1)}{n-k-1}$ (corrected for small samples)

where $k = p + q + P + Q + 1$ (plus 1 for variance).

Auto-SARIMA

Python's `pmdarima.auto_arima()` with `seasonal=True` automatically searches for optimal $(p, d, q) \times (P, D, Q)_s$.

Hyndman-Khandakar Algorithm (`auto_arima`)

How does automatic selection work? (Hyndman & Khandakar, 2008)

1. d : successive KPSS tests ($d = 0, 1, 2$); D : OCSB or Canova-Hansen test ($D = 0, 1$)
2. **Stepwise search**: starts from initial model, explores neighboring models
3. **Criterion**: AICc (correct for small samples)

Search Strategy

- ▣ **Initial model**: SARIMA($2, d, 2$)($1, D, 1$)_s or SARIMA($0, d, 0$)($0, D, 0$)_s
- ▣ **Variations tested**: ± 1 for each order (p, q, P, Q); stops when no neighbor improves AICc
- ▣ **Complexity**: $O(20-30)$ models evaluated (vs. $O(k^4)$ for grid search)

Python: `pm.auto_arima(y, seasonal=True, m=12, stepwise=True, trace=True)`

- ▣ Set `stepwise=False` for exhaustive search (slower, sometimes better)

Point Forecasts

Forecast Computation

SARIMA forecasts are computed recursively:

- ▣ Replace future ε_{T+h} with 0
- ▣ Replace future Y_{T+h} with forecasts $\hat{Y}_{T+h|T}$
- ▣ Use known past values Y_T, Y_{T-1}, \dots

Seasonal Pattern in Forecasts

SARIMA forecasts naturally capture seasonality:

- ▣ Short-term: influenced by recent values
- ▣ Long-term: revert to seasonal pattern

Forecast Intervals

Uncertainty Quantification

$(1 - \alpha)\%$ prediction interval:

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$$

Variance computed from $\text{MA}(\infty)$ representation.

Key Properties

- ▣ Intervals widen with forecast horizon
- ▣ For $I(1, 1)_s$ series: intervals grow without bound
- ▣ Seasonal pattern visible in point forecasts
- ▣ Uncertainty captures both trend and seasonal variation

Long-Horizon Forecasts

Behavior as $h \rightarrow \infty$

- ▣ Point forecasts converge to deterministic seasonal pattern
- ▣ If drift present: linear trend + seasonal pattern
- ▣ Forecast intervals continue to widen

Practical Implication

- ▣ Short-term: SARIMA captures both short-term dynamics and season
- ▣ Medium-term: Good seasonal forecasts, growing uncertainty
- ▣ Long-term: Mainly reflects seasonal pattern, wide intervals

The Seasonal Naive Benchmark

Definition: Seasonal Naive Forecast

- ▣ **Formula:** $\hat{Y}_{T+h} = Y_{T+h-s}$ (last observed season)
- ▣ **Monthly example:** Forecast for March 2025 = value from March 2024
- ▣ **Interpretation:** “The simplest model that respects seasonality”

Why is it essential?

- ▣ Any SARIMA model **must** outperform the seasonal naive benchmark
- ▣ If it doesn't \Rightarrow the model complexity is not justified
- ▣ Surprisingly effective for many series with stable seasonality

Golden Rule

- ▣ **Always** report SARIMA performance relative to seasonal naive
- ▣ This is the **first thing** a reviewer or manager checks

The MASE Metric: Proper Evaluation for Seasonal Series

MASE — Mean Absolute Scaled Error (Hyndman & Koehler, 2006)

- **Formula:**
$$\text{MASE} = \frac{\frac{1}{H} \sum_{h=1}^H |e_{T+h}|}{\frac{1}{T-s} \sum_{t=s+1}^T |Y_t - Y_{t-s}|}$$
- **Numerator:** mean absolute error of the model
- **Denominator:** mean absolute error of seasonal naive (on training data)

Interpretation

- $\text{MASE} < 1$: Model is **better** than seasonal naive
- $\text{MASE} = 1$: Model is **equivalent** to seasonal naive
- $\text{MASE} > 1$: Model is **worse** — abandon it!

Why MASE and not MAPE?

- MAPE: undefined for $Y_t = 0$; asymmetric; scale-dependent
- MASE: works with any data; symmetric; comparable across different series

Forecast Evaluation: Rolling Forecast Origin

Cross-Validation for Seasonal Time Series

- ▣ **Principle:** re-estimate model \rightarrow forecast h steps \rightarrow advance 1 step \rightarrow repeat
- ▣ **Fixed window:** training on last w observations (constant size)
- ▣ **Expanding window:** training from beginning to $T + i$ (grows)

Step-by-step procedure

1. Train SARIMA on Y_1, \dots, Y_T ; forecast $\hat{Y}_{T+1}, \dots, \hat{Y}_{T+h}$
2. Train SARIMA on Y_1, \dots, Y_{T+1} ; forecast $\hat{Y}_{T+2}, \dots, \hat{Y}_{T+h+1}$
3. ... repeat N times; compute RMSE, MAE, MASE on all N forecasts

Important

- ▣ Minimum $N \geq 2s$ origins (2 complete seasonal cycles) for reliable results
- ▣ Never “look ahead” — test data is strictly after training data

SARIMA vs Holt-Winters/ETS: When to Use Which?

Comparison

Criterion	SARIMA	ETS / Holt-Winters
Approach	Box-Jenkins (ACF/PACF)	Exponential smoothing
Seasonality	Stochastic (differencing)	Additive or multiplicative
Interpretation	AR/MA coefficients	Smoothing weights α, β, γ
Flexibility	Very flexible (7 params.)	Less flexible
Automation	<code>auto_arima</code>	<code>ets()</code> / <code>ExponentialSmoothing</code>

Practical Selection Guide

- ▣ **SARIMA preferred:** series with complex autocorrelation, stochastic seasonality, ARMA components
- ▣ **ETS preferred:** short series, stable seasonality, quick forecasts without diagnostics
- ▣ **Best:** compare both on out-of-sample data and choose the winner

Case Study: Airline Passengers Data

- Classic Box-Jenkins dataset: monthly airline passengers (1949-1960)
- Clear upward trend and increasing seasonal amplitude
- Multiplicative seasonality suggests log transformation



Data Splitting Strategy

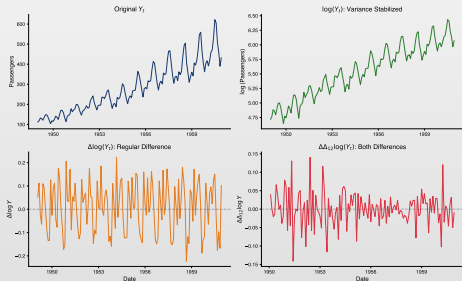
Time Series Train/Validation/Test Split



- ▣ **Training set (70%)** — Fit model parameters; estimate SARIMA coefficients (ϕ , θ , Φ , Θ)
- ▣ **Validation set (15%)** — Select best model; compare candidates, choose lowest validation error
- ▣ **Test set (15%)** — Final evaluation; unbiased out-of-sample performance, never used during development

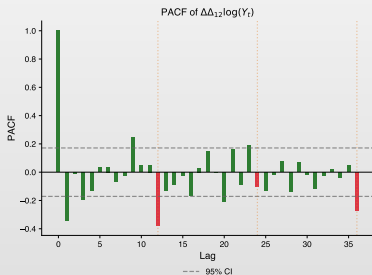
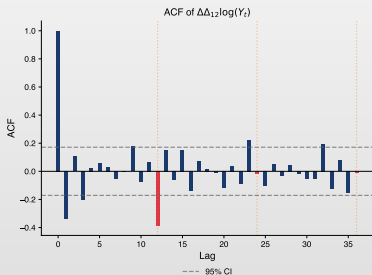
Step 1: Transformations

- Log transform stabilizes variance (multiplicative \Rightarrow additive)
- First difference removes trend; seasonal difference removes seasonality
- Double-differenced series appears stationary



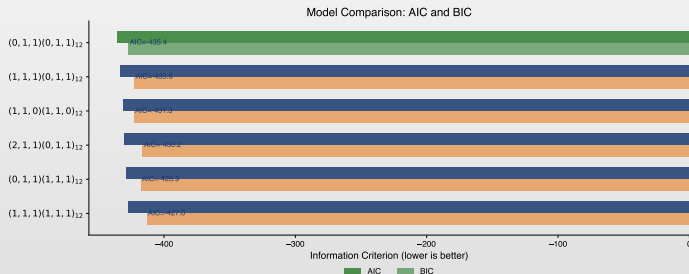
Step 2: ACF/PACF Analysis

- ACF: Significant spike at lag 1 and lag 12 \Rightarrow MA(1), SMA(1)
- PACF: Exponential decay pattern confirms MA structure
- Suggests SARIMA(0, 1, 1) \times (0, 1, 1)₁₂ (airline model)



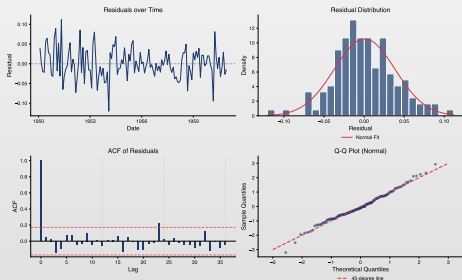
Step 3: Model Comparison

- Compare candidate SARIMA models using AIC criterion
- SARIMA(0, 1, 1) \times (0, 1, 1)₁₂ provides best fit (lowest AIC)
- This is the famous “airline model” identified by Box & Jenkins



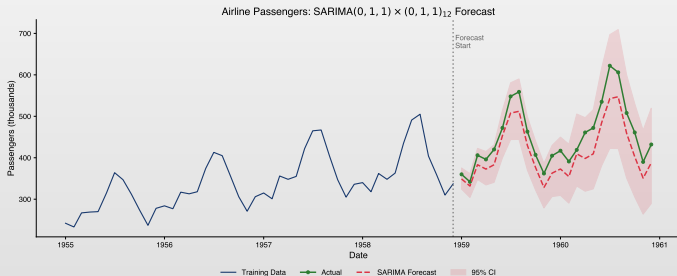
Step 4: Residual Diagnostics

- Residuals appear random with no remaining autocorrelation
- Q-Q plot shows approximate normality
- Model adequately captures both trend and seasonal structure



Step 5: Forecasting

- 24-month forecast with 95% confidence interval
- Model captures seasonal pattern and upward trend
- Prediction intervals widen appropriately with forecast horizon



Practical Pitfalls in SARIMA Modeling

1. Over-differencing

- ▣ **Symptom:** ACF at lag 1 ≈ -0.5 (regular) or at lag $s \approx -0.5$ (seasonal)
- ▣ **Cause:** applying $(1 - L)$ or $(1 - L^s)$ too many times
- ▣ **Solution:** reduce d or D by 1 and re-examine ACF/PACF

2. Insufficient Data

- ▣ **Minimum:** 3–4 complete seasonal cycles (36–48 monthly obs.); **recommended:** 5+ cycles
- ▣ Seasonal parameters Φ, Θ are estimated from lags $s, 2s, 3s, \dots$

3. Other Common Pitfalls

- ▣ **Root cancellation:** $\phi \approx \theta$ suggests over-parameterization
- ▣ **Parameters at invertibility boundary:** $|\theta| \approx 1$ or $|\Theta| \approx 1$ indicates problems
- ▣ **Forgetting inverse transformation:** forecasts on log scale must be back-transformed!

X-13ARIMA-SEATS: Official Seasonal Adjustment

What is seasonal adjustment?

- ▣ **Goal:** remove the seasonal component to reveal the true trend
- ▣ **Users:** Eurostat, US Census Bureau, central banks, national statistical offices
- ▣ **Example:** “GDP grew 0.3% compared to previous quarter” (seasonally adjusted data)

X-13ARIMA-SEATS (US Census Bureau)

- ▣ **Step 1:** Identify and estimate a regARIMA model (SARIMA + calendar effects)
- ▣ **Step 2:** Extract the seasonal component via SEATS or X-11 filters
- ▣ **Step 3:** $Y_t^{adjusted} = Y_t - \hat{S}_t$ (additive) or $Y_t^{adjusted} = Y_t / \hat{S}_t$ (multiplicative)

Why does it matter for economists?

- ▣ Published macroeconomic data is almost always seasonally adjusted
- ▣ Misinterpreting unadjusted data can lead to erroneous conclusions

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"I have the AirPassengers dataset from statsmodels (monthly data, international airline passengers, 1949–1960, 144 obs.). Identify seasonality, apply Box-Cox transform if needed, estimate a SARIMA model, and forecast 12 months. Give me complete Python code with plots."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it check seasonality with ACF at lags $s, 2s, 3s$? Does it use STL decomposition?
3. Does it apply Box-Cox *before* differencing? Does it justify the choice of λ ?
4. How does it choose orders $(p, d, q) \times (P, D, Q)_s$? Only `auto_arima` or also ACF/PACF?
5. Does it evaluate with MASE relative to seasonal naïve? Does it use rolling forecast?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Summary

What We Learned in This Chapter

- Seasonality in time series
 - ▶ Repetitive patterns at regular intervals; additive vs multiplicative
- Seasonal differencing and Box-Cox transformation
 - ▶ $(1 - L^s)$ removes stochastic seasonality; Box-Cox stabilizes variance
- SARIMA(p, d, q) \times (P, D, Q)_s models
 - ▶ Extend ARIMA with seasonal components; automatic selection via `auto_arima`
- Forecasting and evaluation
 - ▶ Benchmark: MASE relative to seasonal naive; rolling forecast out-of-sample

Key Idea

- **Parsimony principle:** The Airline Model $(0, 1, 1) \times (0, 1, 1)_{12}$ with only 2 parameters is remarkably effective for many seasonal economic series.

What's Next?

Chapter 5: Volatility Modeling — GARCH

- ▣ **Volatility:** conditional variation of financial returns
- ▣ **ARCH/GARCH:** models for conditional variance
- ▣ **Asymmetric extensions:** GJR-GARCH, EGARCH (leverage effect)
- ▣ **VaR:** Value-at-Risk based on GARCH models
- ▣ **Case study:** S&P 500 returns volatility

Questions?

Question 1

Question

□ For monthly data with annual seasonality, what is the seasonal period s ?

Answer Choices

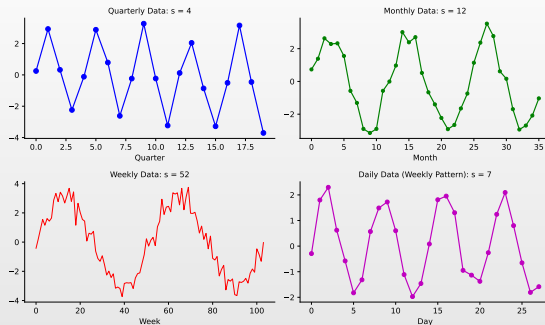
(A) $s = 4$

(B) $s = 7$

(C) $s = 12$

(D) $s = 52$

Question 1: Answer



Answer: (C)

- ☐ $s = 12$ (12 months per year). Common periods: Quarterly=4, Monthly=12, Weekly=52, Daily=7, Hourly=24

Question 2

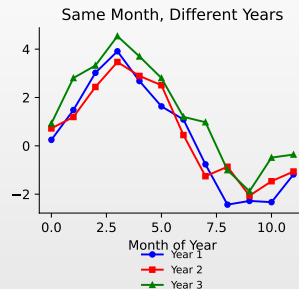
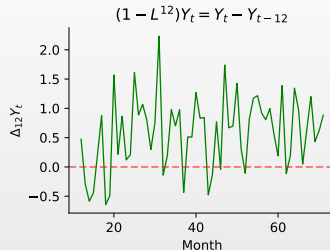
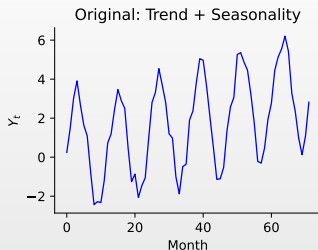
Question

□ What does the seasonal difference operator $(1 - L^{12})$ do to a monthly series?

Answer Choices

- (A) Computes $Y_t - Y_{t-1}$ (month-to-month change)
- (B) Computes $Y_t - Y_{t-12}$ (year-over-year change)
- (C) Computes the 12-month moving average
- (D) Removes the trend component only

Question 2: Answer



Answer: (B)

□ $(1 - L^{12})Y_t = Y_t - Y_{t-12}$ removes the seasonal pattern by comparing same months.

Question 3

Question

□ In $\text{SARIMA}(1, 1, 1) \times (1, 1, 1)_{12}$ notation, what does the $(1, 1, 1)_{12}$ part represent?

Answer Choices

- (A) AR(1), differencing once, MA(1) at the regular level
- (B) Seasonal AR(1), seasonal differencing once, seasonal MA(1)
- (C) 12 AR terms, 12 differences, 12 MA terms
- (D) The model has 12 parameters in total

Question 3: Answer

Answer: (B)

- Seasonal AR(1), seasonal differencing once, seasonal MA(1)

SARIMA Notation Breakdown

$\text{SARIMA}(p, d, q) \times (P, D, Q)_s$:

(p, d, q) Non-seasonal: AR(p), d differences, MA(q)

$(P, D, Q)_s$ Seasonal: SAR(P), D seasonal diffs, SMA(Q)

For $(1, 1, 1) \times (1, 1, 1)_{12}$:

- Non-seasonal: AR(1), one regular difference, MA(1)
- Seasonal: SAR(1) at lag 12, one Δ_{12} , SMA(1) at lag 12

Question 4

Question

- ☐ The “Airline Model” is $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_{12}$. How many parameters need to be estimated (excluding variance)?

Answer Choices

- (A) 1
- (B) 2
- (C) 4
- (D) 12

Question 4: Answer

Answer: (B)

- ▣ 2 parameters: SARIMA(0, 1, 1) \times (0, 1, 1)₁₂: $(1 - L)(1 - L^{12})Y_t = (1 + \theta_1 L)(1 + \Theta_1 L^{12})\varepsilon_t$
- ▣ Parameters: θ_1 (non-seasonal MA) and Θ_1 (seasonal MA), plus σ^2 .

Why “Airline Model”?

Box & Jenkins (1970) used this model to forecast international airline passengers. Remarkably effective for many seasonal economic series!

Question 5

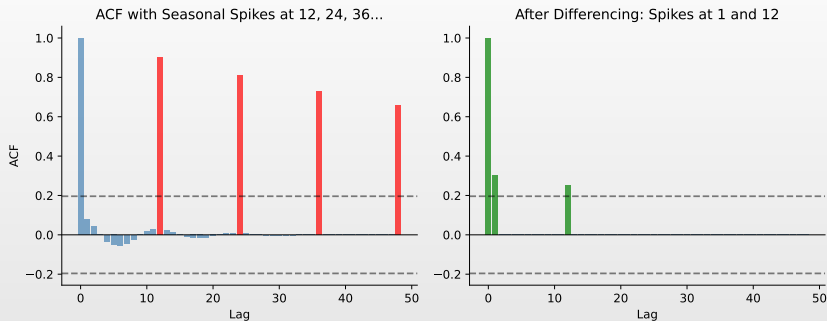
Question

- ☐ You observe significant ACF spikes at lags 12, 24, and 36 in a monthly series. What does this suggest?

Answer Choices

- (A) The series has a unit root
- (B) The series has annual seasonality that needs seasonal differencing
- (C) The series follows an AR(36) process
- (D) The series is already stationary

Question 5: Answer



Answer: (B)

- ☐ ACF spikes at 12, 24, 36 = stochastic seasonality. Apply $(1 - L^{12})$ to remove it.

Question 6

Question

- After applying $(1 - L)(1 - L^{12})$ to a monthly series, the ACF shows a significant spike only at lag 1 and lag 12. What SARIMA model is suggested?

Answer Choices

- (A) $\text{SARIMA}(1, 1, 0) \times (1, 1, 0)_{12}$
- (B) $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_{12}$
- (C) $\text{SARIMA}(1, 1, 1) \times (1, 1, 1)_{12}$
- (D) $\text{SARIMA}(0, 1, 0) \times (0, 1, 0)_{12}$

Question 6: Answer

Answer: (B)

- ▣ **Model:** $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_{12}$ (The Airline Model)

ACF/PACF Identification Rules

- ▣ **Rule:** for MA processes, ACF cuts off after lag q
- ▣ **ACF spike at lag 1:** MA(1) for non-seasonal part
- ▣ **ACF spike at lag 12:** SMA(1) for seasonal part
- ▣ **Combined:** $\text{MA}(1) \times \text{SMA}(1) = (0, d, 1) \times (0, D, 1)_{12}$
- ▣ **With** $d = 1, D = 1$: $(0, 1, 1) \times (0, 1, 1)_{12}$

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- ▣ Brockwell, P.J., & Davis, R.A. (2016). *Introduction to Time Series and Forecasting*, 3rd ed., Springer.
- ▣ Hyndman, R.J., & Khandakar, Y. (2008). Automatic Time Series Forecasting: The forecast Package for R, *Journal of Statistical Software*, 27(3), 1–22.

Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> – Code platform for quantitative methods
- ▣ **Quantinar**: <https://quantinar.com> – Learning platform for quantitative methods
- ▣ **GitHub TSA**: https://github.com/QuantLet/TSA/tree/main/TSA_ch4 – Python code for this chapter

Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar