

Analiza și Prognoza Serilor de Timp

Capitolul 7: Cointegrare & VECM

Relații de Echilibru pe Termen Lung



Cuprins

La finalul acestui capitol, veți fi capabili să:

- ① Înțelegeți conceptul de **cointegrare** și relații de echilibru pe termen lung
- ② Recunoașteți și evitați problema **regresiei false**
- ③ Aplicați metoda **Engle-Granger** în doi pași
- ④ Efectuați testul **Johansen** pentru cointegrare multiplă
- ⑤ Estimați și interpretați modele **VECM**
- ⑥ Analizați viteza de ajustare și vectorii de cointegrare
- ⑦ Implementați analiza de cointegrare în **Python**

De ce contează cointegrarea?

Provocarea

- Multe serii de timp economice/financiare sunt **nestaționare** ($I(1)$)
- PIB, prețuri acțiuni, cursuri valutare, rate ale dobânzii au rădăcini unitare
- Regresia standard cu variabile $I(1)$ ⇒ **rezultate false**
- Diferențierea elimină nestaționaritatea dar pierde **informația pe termen lung**

Soluția: Cointegrarea

Unele serii nestaționare au un **trend stocastic comun**—se mișcă împreună pe termen lung. Această relație pe termen lung poate fi modelată!

Premiul Nobel 2003

Clive Granger a primit Premiul Nobel în Economie (împreună cu Robert Engle) pentru dezvoltarea analizei de cointegrare—“metode pentru analiza seriilor de timp economice cu tendințe comune.”

Finanțe

- **Pairs Trading:** Tranzacționarea spread-ului între acțiuni cointegrate
- **Structura pe Termene:** Rate dobânzi pe termen scurt și lung
- **Spot-Futures:** Relații de arbitraj

Macroeconomie

- **Consum și Venit:** Ipoteza venitului permanent
- **Bani și Prețuri:** Teoria cantitativă a banilor
- **PPP:** Cursuri valutare și niveluri de prețuri

Analiza Politicilor

- **Politica Fiscală:** Cheltuieli guvernamentale și venituri fiscale
- **Politica Monetară:** Transmiterea ratelor dobânzii
- **Piața Muncii:** Salarii și productivitate

The Spurious Regression Problem

Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk:

$$Y_t = \alpha + \beta X_t + u_t$$

where Y_t and X_t are independent $I(1)$ processes.

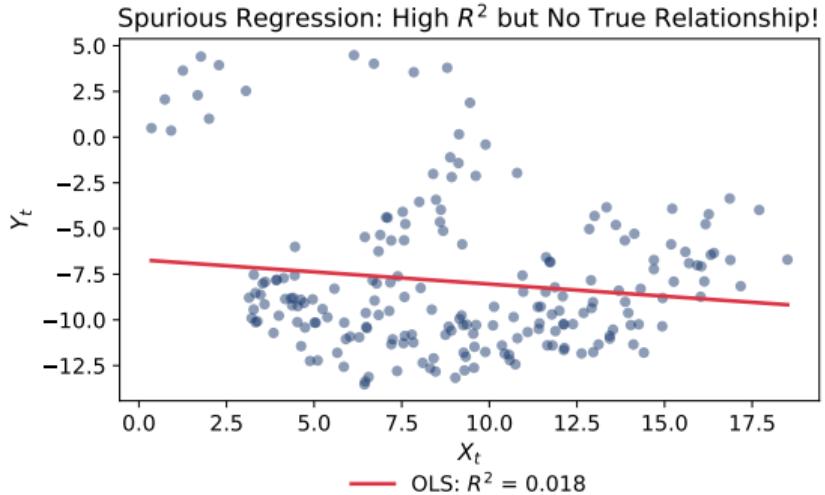
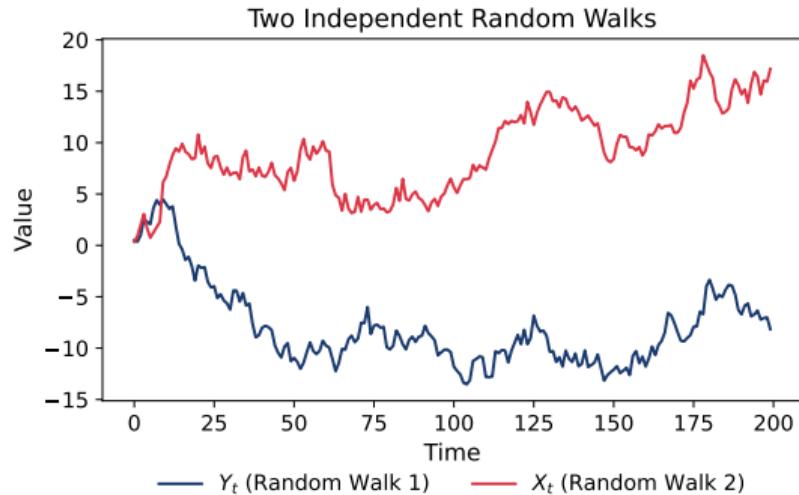
Symptoms of Spurious Regression

- High R^2 (often > 0.9) even though variables are **unrelated!**
- Highly significant t -statistics (reject $H_0 : \beta = 0$)
- Very low Durbin-Watson statistic ($DW \approx 0$)
- Residuals are non-stationary (have unit root)

Rule of Thumb (Granger)

If $R^2 > DW$, suspect spurious regression!

Spurious Regression: Visual Example



Warning: Two completely independent random walks show high correlation ($R^2 > 0.8$) purely by chance! This is why we need cointegration analysis.

Definition of Cointegration

Definiție 1 (Cointegration (Engle & Granger, 1987))

Variables $Y_{1t}, Y_{2t}, \dots, Y_{kt}$ are **cointegrated of order** (d, b) , written $CI(d, b)$, if:

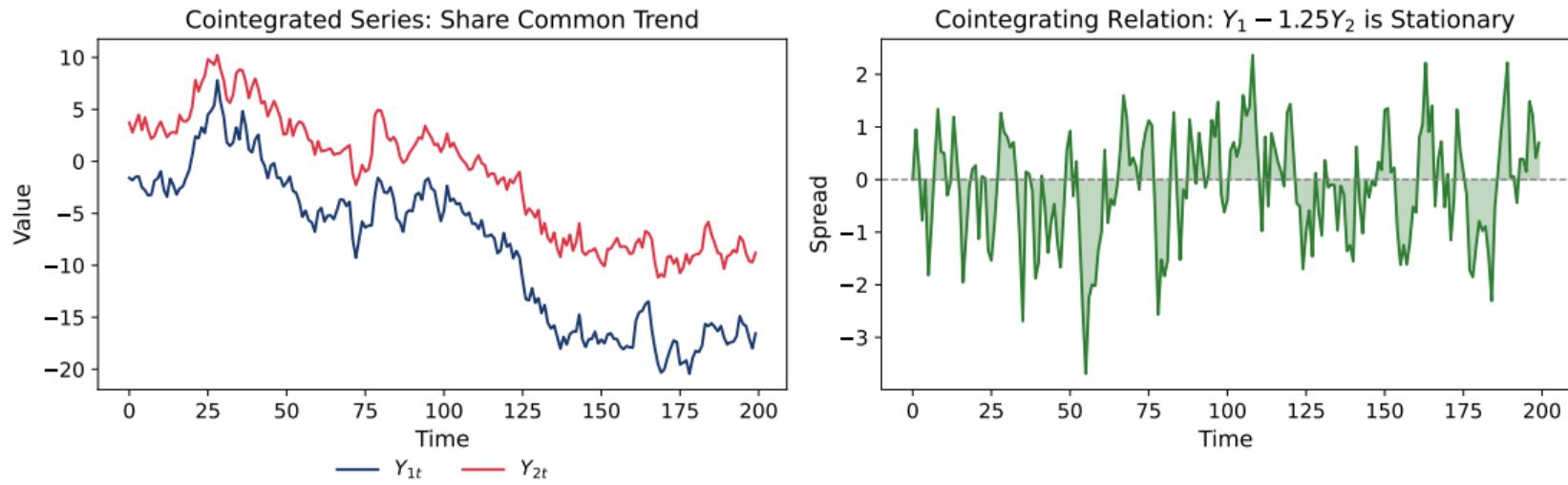
- ① All variables are integrated of order d : $Y_{it} \sim I(d)$
- ② There exists a linear combination $\beta'Y_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$ that is integrated of order $(d - b)$, where $b > 0$

Most Common Case: $CI(1, 1)$

- Variables are $I(1)$ (have unit roots)
- Linear combination is $I(0)$ (stationary)
- Vector $\beta = (\beta_1, \dots, \beta_k)'$ is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized: $\beta_1 = 1$.

Cointegration: Visual Example



Key insight: Both series are $I(1)$ and trend together, but their linear combination (spread) is stationary—this is cointegration!

Intuition: Common Stochastic Trends

Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**:

$$Y_{1t} = \gamma_1 \tau_t + S_{1t}, \quad Y_{2t} = \gamma_2 \tau_t + S_{2t}$$

where τ_t is a common random walk and S_{it} are stationary components.

Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

Economic Interpretation

- Cointegration represents a **long-run equilibrium relationship**
- Variables may deviate in the short run
- But they are “pulled back” to equilibrium over time
- The cointegrating vector defines the equilibrium

How Many Cointegrating Relationships?

For k variables that are $I(1)$:

- Maximum possible cointegrating relationships: $r = k - 1$
- If $r = 0$: No cointegration (variables drift apart)
- If $r = k$: All variables are $I(0)$ (contradiction)

Example: 3 Variables

- $r = 0$: No cointegration
- $r = 1$: One cointegrating relationship
- $r = 2$: Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends = $k - r$

Engle-Granger Two-Step Method

Step 1: Estimate Cointegrating Regression

Run OLS regression (assuming Y_t is the dependent variable):

$$Y_t = \alpha + \beta X_t + e_t$$

Save the residuals: $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

Step 2: Test Residuals for Stationarity

Test if \hat{e}_t is $I(0)$ using ADF test:

$$\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$$

- $H_0: \rho = 0$ (residuals have unit root \Rightarrow no cointegration)
- $H_1: \rho < 0$ (residuals are stationary \Rightarrow cointegration)

Important

Engle-Granger Critical Values

Critical Values for Cointegration Test

Number of Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

Based on MacKinnon (1991) response surface estimates, $T = 100$

Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on which variable is chosen as dependent
- Small sample bias in estimated cointegrating vector
- Cannot test hypotheses on the cointegrating vector

Advantages over Engle-Granger

- Tests for **multiple** cointegrating relationships
- Maximum likelihood estimation (more efficient)
- Can test restrictions on cointegrating vectors
- Does not require choosing a dependent variable

Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \dots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...

Vector Error Correction Model

$$\Delta \mathbf{Y}_t = \mathbf{c} + \boldsymbol{\Pi} \mathbf{Y}_{t-1} + \sum_{j=1}^{p-1} \boldsymbol{\Gamma}_j \Delta \mathbf{Y}_{t-j} + \boldsymbol{\varepsilon}_t$$

where:

- $\boldsymbol{\Pi} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_p - \mathbf{I}$ (long-run impact matrix)
- $\boldsymbol{\Gamma}_j = -(\mathbf{A}_{j+1} + \cdots + \mathbf{A}_p)$ (short-run dynamics)

Key Insight: Rank of $\boldsymbol{\Pi}$

The **rank of $\boldsymbol{\Pi}$** determines cointegration:

- $\text{rank}(\boldsymbol{\Pi}) = 0$: No cointegration (VAR in differences)
- $\text{rank}(\boldsymbol{\Pi}) = k$: All variables are $I(0)$ (VAR in levels)
- $0 < \text{rank}(\boldsymbol{\Pi}) = r < k$: Cointegration with r cointegrating vectors

Decomposition of Π

When $\text{rank}(\Pi) = r < k$

The matrix Π can be decomposed as:

$$\Pi = \alpha\beta'$$

where:

- β is $k \times r$ matrix of **cointegrating vectors**
- α is $k \times r$ matrix of **adjustment coefficients**

Interpretation

- $\beta'Y_{t-1}$ = deviations from long-run equilibrium (error correction terms)
- α = speed of adjustment to equilibrium
- Each row of α shows how each variable responds to disequilibrium

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta' Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$$

Two Test Statistics

Based on eigenvalues $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$ of a certain matrix:

Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests $H_0: \text{rank} \leq r$ vs $H_1: \text{rank} > r$

Maximum Eigenvalue Test:

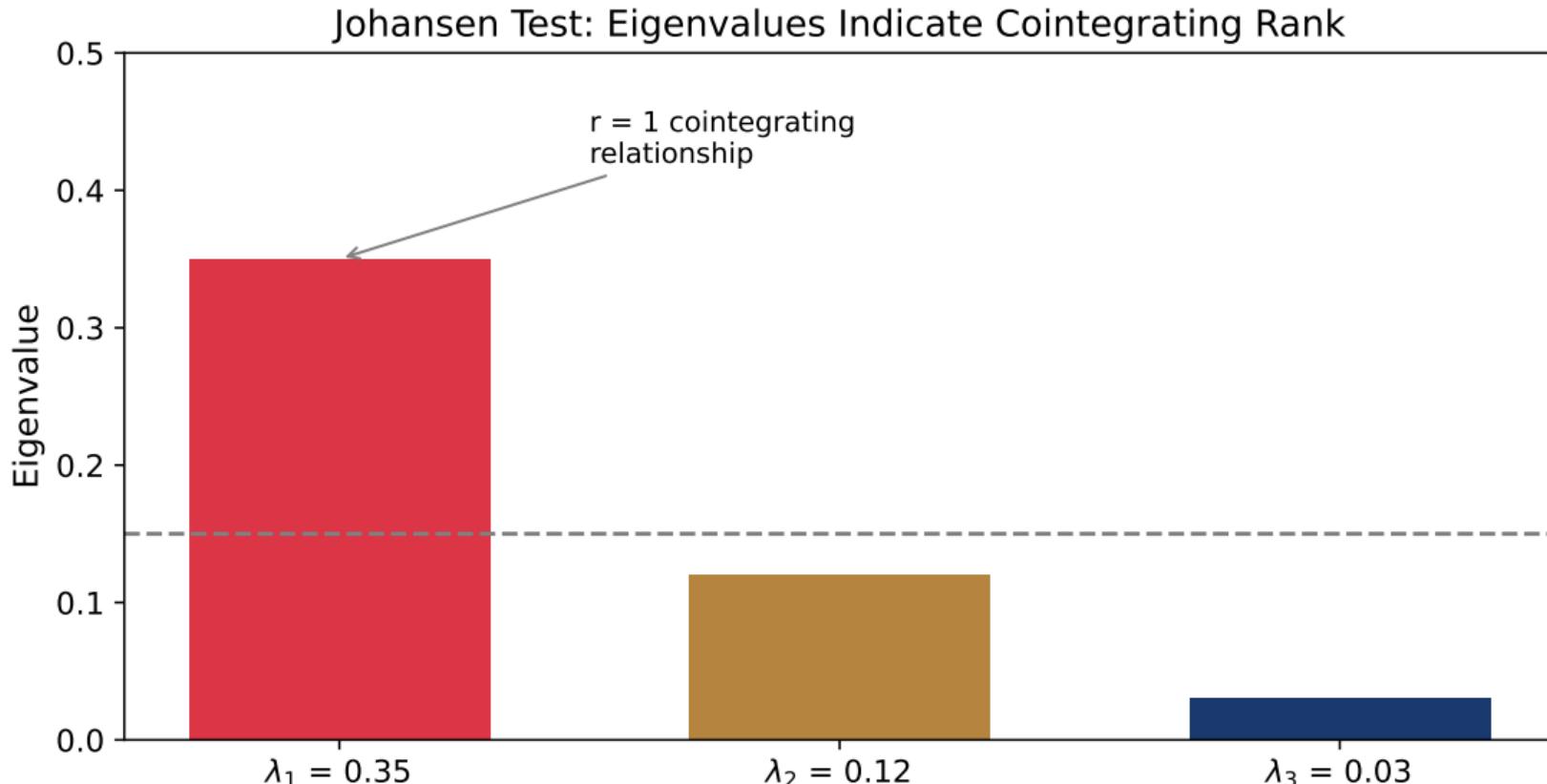
$$\lambda_{\max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

Tests $H_0: \text{rank} = r$ vs $H_1: \text{rank} = r + 1$

Critical values from Johansen & Juselius (1990), depend on:

- Number of variables k
- Deterministic components (constant, trend)

Johansen Test: Visual Interpretation



Sequential Testing (Trace Test)

- ① Test $H_0: r = 0$ vs $H_1: r > 0$
 - If not rejected: No cointegration. Stop.
 - If rejected: At least one cointegrating vector. Continue.
- ② Test $H_0: r \leq 1$ vs $H_1: r > 1$
 - If not rejected: $r = 1$. Stop.
 - If rejected: At least two cointegrating vectors. Continue.
- ③ Continue until H_0 is not rejected...

Deterministic Components

Choose specification carefully:

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- Constant in both (most common)
- Constant + trend in cointegrating relation
- Constant + trend in both

Full VECM Specification

For $k = 2$ variables with $r = 1$ cointegrating relation:

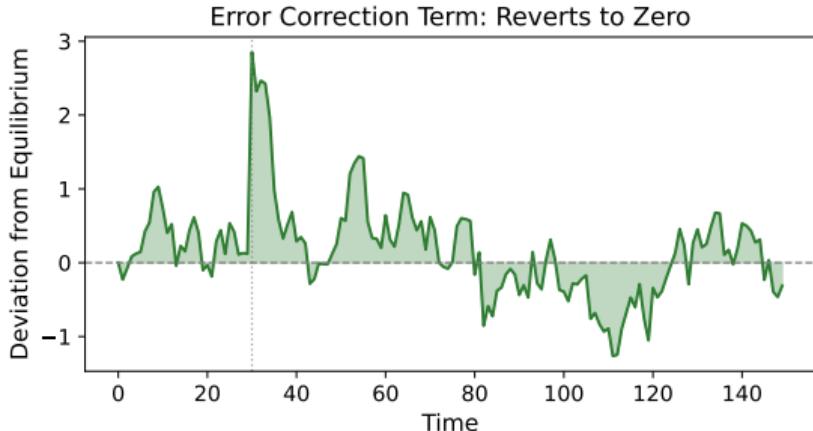
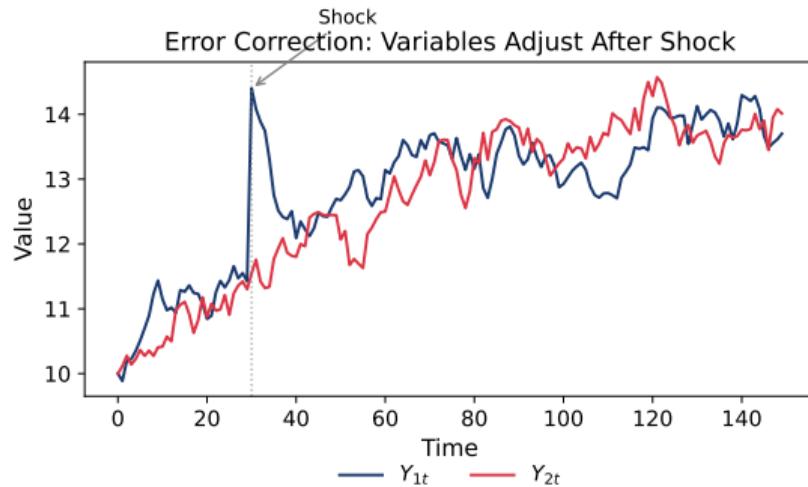
$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$ = error correction term (deviation from equilibrium)
- α_1, α_2 = adjustment speeds (should have opposite signs)
- γ_{ij} = short-run dynamics
- ε_{it} = innovations

Error Correction Mechanism: Visual



Error correction in action: When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment.

Interpreting Adjustment Coefficients

The α Coefficients

If the cointegrating relation is $Y_1 - \beta Y_2 = 0$ (equilibrium):

- $\alpha_1 < 0$: Y_1 adjusts downward when above equilibrium
- $\alpha_2 > 0$: Y_2 adjusts upward when Y_1 is above equilibrium

Weak Exogeneity

If $\alpha_i = 0$, variable Y_i does **not** respond to disequilibrium.

- Y_i is **weakly exogenous** for the long-run parameters
- The other variable does all the adjusting
- Can simplify estimation (single-equation approach)

Test weak exogeneity: $H_0 : \alpha_i = 0$ using likelihood ratio test.

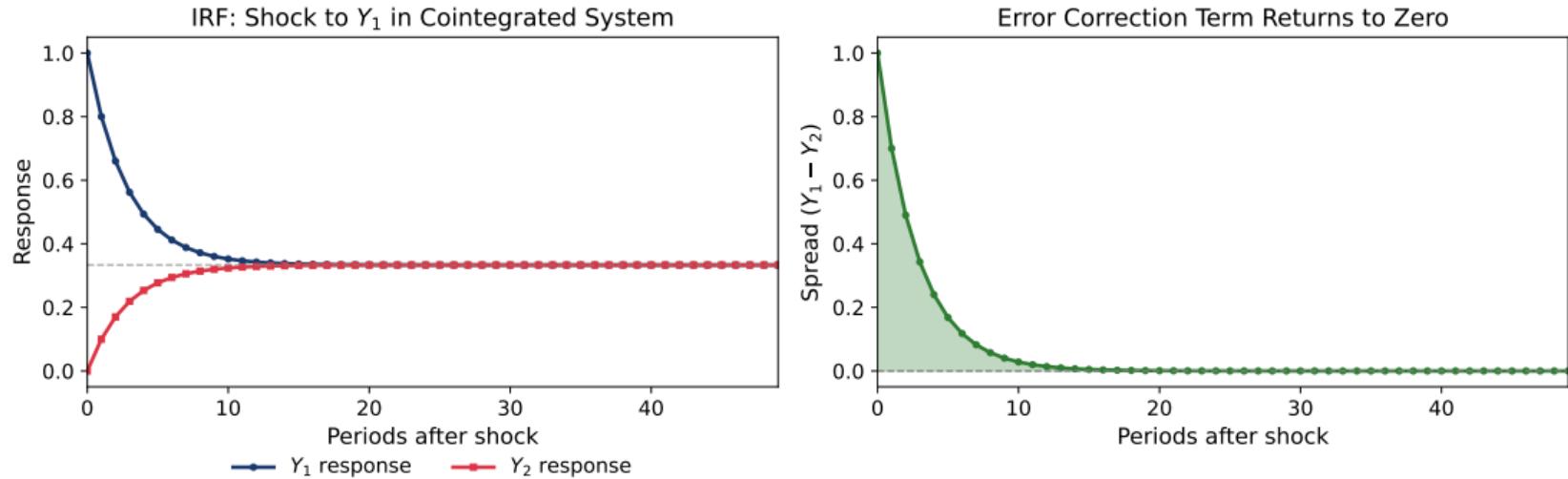
When Variables are Cointegrated

	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

VECM Impulse Response Functions



IRF interpretation: In a cointegrated system, shocks have **permanent effects** on levels but the system returns to equilibrium. Unlike stationary VAR, effects don't decay to zero—they converge to a new long-run value.

Step-by-Step Procedure

- 1 Unit Root Tests:** Verify all variables are $I(1)$
 - ADF, KPSS on levels and first differences
- 2 Lag Length Selection:** Choose p for VAR in levels
 - Use AIC, BIC, or sequential LR tests
- 3 Cointegration Test:** Johansen trace/max-eigenvalue tests
 - Determine cointegrating rank r
- 4 Estimate VECM:** If $0 < r < k$
 - Estimate α, β, Γ_j
- 5 Diagnostics:** Check residuals for autocorrelation, normality
- 6 Analysis:** IRF, FEVD, hypothesis tests

Common Pitfalls

Things to Watch Out For

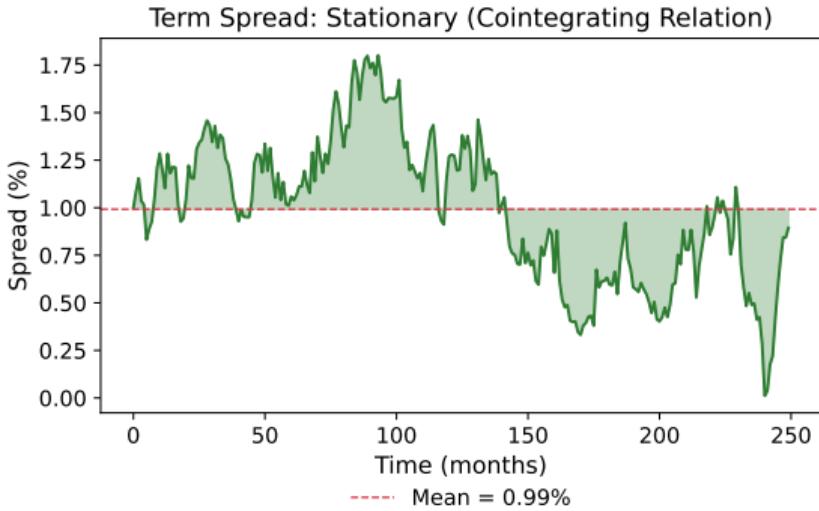
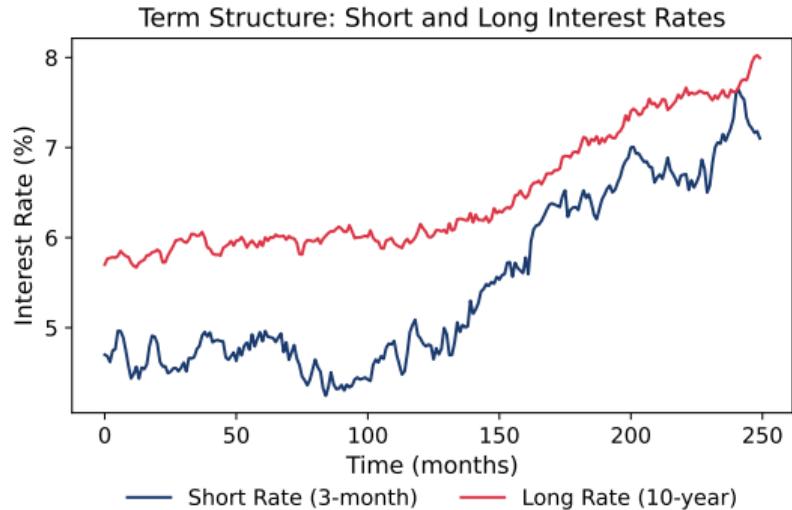
- **Structural breaks:** Can cause spurious unit roots or cointegration
- **Near-unit-root processes:** Tests have low power
- **Too many lags:** Over-parameterization, loss of efficiency
- **Too few lags:** Residual autocorrelation, biased estimates
- **Wrong deterministic specification:** Affects critical values
- **Small samples:** Johansen test oversized in small samples

Recommendation

Always check:

- Residual diagnostics (Portmanteau test, normality)
- Stability of estimated cointegrating relationship over time
- Sensitivity to lag length and deterministic specification

Example 1: Term Structure of Interest Rates



Expectations Hypothesis: Short and long rates share common trend. The spread (term premium) is stationary—evidence of cointegration!

Expectations Hypothesis of Term Structure

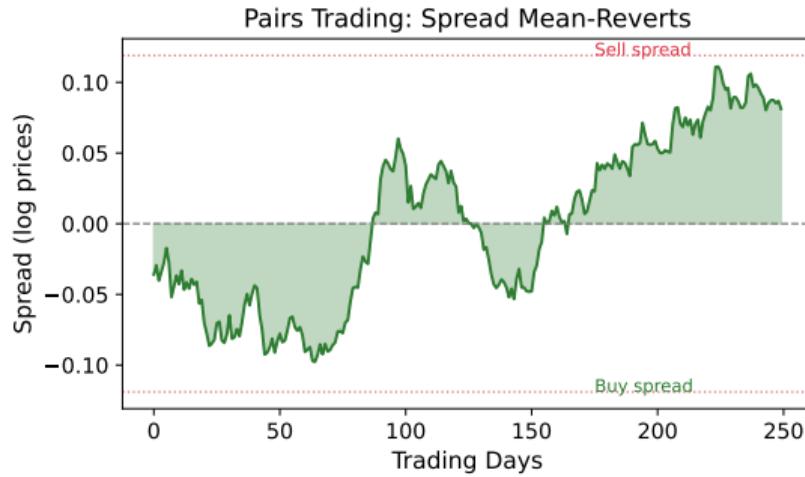
$$R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{term premium}$$

If term premium is constant, short rate r_t and long rate R_t should be cointegrated with vector $(1, -1)$.

Empirical Findings

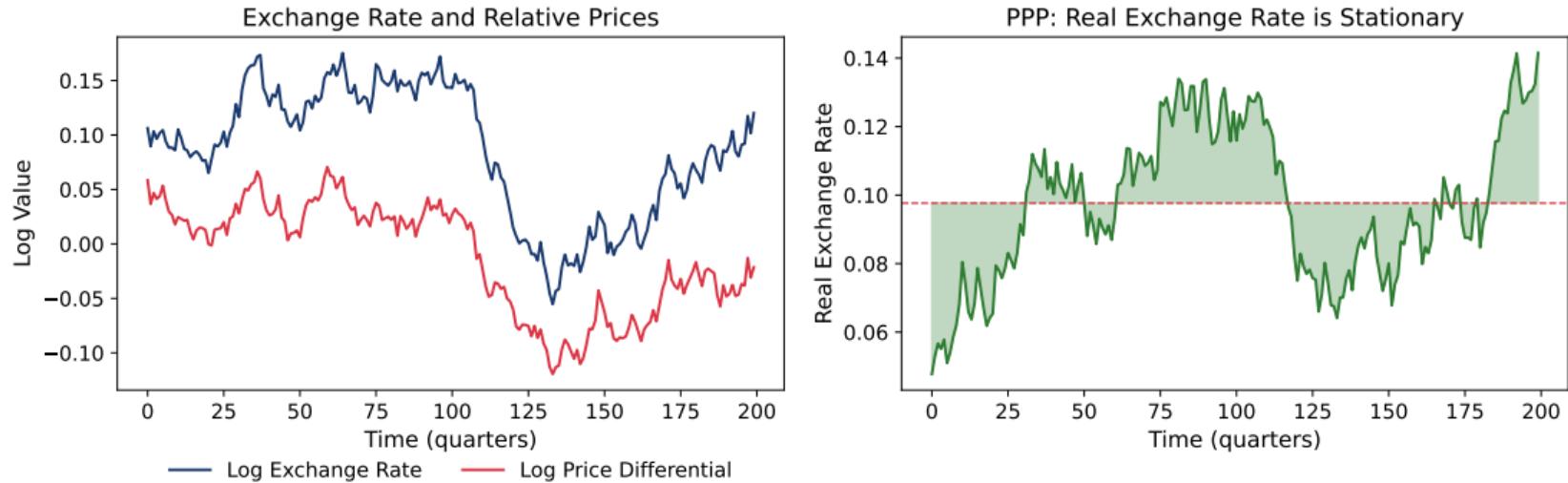
- ① Both rates are $I(1)$ (unit root tests)
- ② One cointegrating relationship (Johansen test)
- ③ Cointegrating vector $\approx (1, -1)$: spread is stationary
- ④ Short rate adjusts to disequilibrium (long rate is weakly exogenous)

Example 2: Pairs Trading in Finance



Strategy: Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When spread deviates from mean, trade expecting mean reversion. Sell spread when high, buy when low.

Example 3: Purchasing Power Parity (PPP)



PPP Theory: $e_t = p_t - p_t^*$ (log exchange rate equals price differential). Real exchange rate should be stationary in the long run.

Typical Findings

- Both rates are $I(1)$
- One cointegrating relationship found
- Cointegrating vector close to $(1, -1)$: spread is stationary
- Short rate adjusts to long rate (not vice versa)

VECM Equations (stylized)

$$\Delta r_t = 0.02 - 0.15(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{1t}$$

$$\Delta R_t = 0.01 - 0.02(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{2t}$$

- Short rate adjusts faster ($\alpha_1 = -0.15$)
- Long rate nearly weakly exogenous ($\alpha_2 \approx 0$)

Key Takeaways

Main Concepts

- **Cointegration:** $I(1)$ variables with stationary linear combination
- **Spurious regression:** High R^2 with unrelated $I(1)$ variables
- **Error correction:** Adjustment toward long-run equilibrium
- **VECM:** VAR with error correction terms for cointegrated systems

Testing Methods

- **Engle-Granger:** Simple, but only one cointegrating vector
- **Johansen:** Multiple vectors, more powerful, MLE-based

Remember

Cointegration tests have low power in small samples. Economic theory should guide specification. Always check diagnostics!

Extensii și Subiecte Conexe

- **VECM Structural:** Identificarea řocurilor structurale
- **Cointegrare cu prag:** Ajustare neliniară
- **Cointegrare de panel:** Secțiuni transversale multiple
- **Cointegrare fracționară:** Memorie lungă
- **Cointegrare variabilă în timp:** Schimbări de regim

Întrebări?

Formule Cheie – Rezumat

Cointegrare

$$Y_t \sim I(1), X_t \sim I(1)$$

$$Y_t - \beta X_t = u_t \sim I(0)$$

Relație de echilibru pe termen lung

Test Engle-Granger

Pas 1: $Y_t = \alpha + \beta X_t + u_t$

Pas 2: Test ADF pe \hat{u}_t

Valori critice speciale (nu standard ADF)

Rang de Cointegrare

r = numărul de relații de cointegrare

$0 \leq r \leq K - 1$ pentru K variabile $I(1)$

Model VECM

$$\Delta Y_t = \Pi Y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta Y_{t-i} + \epsilon_t$$

$\Pi = \alpha \beta'$ (factorizare)

Interpretare α și β

β : vectori de cointegrare (echilibru)

α : viteza de ajustare

Corecția erorii: $\alpha(\beta' Y_{t-1})$

Test Johansen

Trace: $\lambda_{trace} = -T \sum_{i=r+1}^K \ln(1 - \hat{\lambda}_i)$

Max-Eigen: $\lambda_{max} = -T \ln(1 - \hat{\lambda}_{r+1})$

Quiz Rapid

- ① Ce înseamnă că două variabile $I(1)$ sunt cointegrate?
- ② Care este problema "regresiei false"?
- ③ În VECM, ce reprezintă coeficienții α ?
- ④ Care este avantajul principal al metodei Johansen față de Engle-Granger?
- ⑤ Dacă $\alpha_i = 0$ pentru variabila Y_i , ce implică aceasta?

Răspunsuri Quiz

- ① **Cointegrare:** O combinație liniară a variabilelor este $I(0)$ (staționară). Ele au un trend stocastic comun.
- ② **Regresie falsă:** Regresarea unei variabile $I(1)$ pe alta $I(1)$ necorelată dă R^2 mare și coeficienți semnificativi deși nu există relație reală.
- ③ **Coeficientii α :** Viteza de ajustare—cât de repede răspunde fiecare variabilă la deviații de la echilibrul pe termen lung.
- ④ **Avantajul Johansen:** Poate testa relații multiple de cointegrare, folosește MLE (mai eficient), nu necesită alegerea variabilei dependente.
- ⑤ $\alpha_i = 0$: Variabila Y_i este slab exogenă—nu răspunde la dezechilibru. Alte variabile fac toată ajustarea.