Time Series Analysis and Forecasting

Chapter 5: GARCH and Volatility

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Learning Objectives

By the end of this chapter, you will be able to:

- Understand volatility clustering and its importance in financial data
- Model conditional heteroskedasticity using ARCH and GARCH models
- Estimate GARCH models and interpret their parameters
- Forecast volatility and apply it to risk management

Outline

Why Model Volatility?

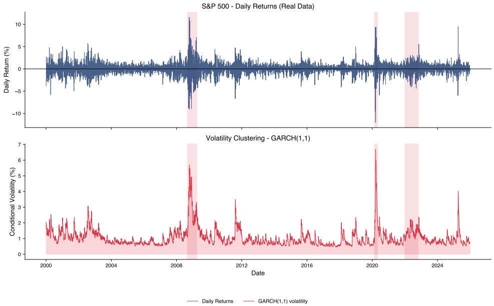
Empirical Observations in Financial Series

- Financial returns exhibit **volatility clustering** — periods of high volatility tend to be followed by periods of high volatility
- The distribution of returns has **fat tails** (leptokurtosis)
- Return correlation is nearly zero, but correlation of squares is significant
- Volatility responds **asymmetrically** to shocks (leverage effect)

Limitation of ARIMA Models

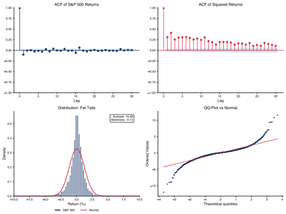
ARIMA models assume **constant variance** (homoskedasticity), which is not realistic for financial series!

Volatility Clustering



- High volatility periods are followed by high volatility; calm by calm
- This suggests that **conditional variance** is predictable

Stylized Facts of Financial Returns



Observed Properties

1. **No autocorrelation** in returns
2. **Autocorrelation** in r_t^2, $|r_t|$
3. **Fat tails** (kurtosis > 3)
4. **Leverage effect**
5. **Volatility clustering**

Conditional Heteroskedasticity

Definition 1 (Conditional Variance)

For return series $\{r_t\}$, the **conditional variance** at time t is: $\sigma_t^2 = \mathrm{Var}(r_t|\mathcal{F}_{t-1}) = \mathbb{E}[(r_t - \mu_t)^2|\mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the information available up to time $t-1$.

General Model

$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1)$

- μ_t = conditional mean (ARMA); σ_t^2 = conditional variance (GARCH)
- z_t = standardized innovations (Normal, Student-t, GED)

The ARCH(q) Model — Engle (1982)

Definition 2 (ARCH(q))

The **Autoregressive Conditional Heteroskedasticity** model of order q:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1), \quad \sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$$

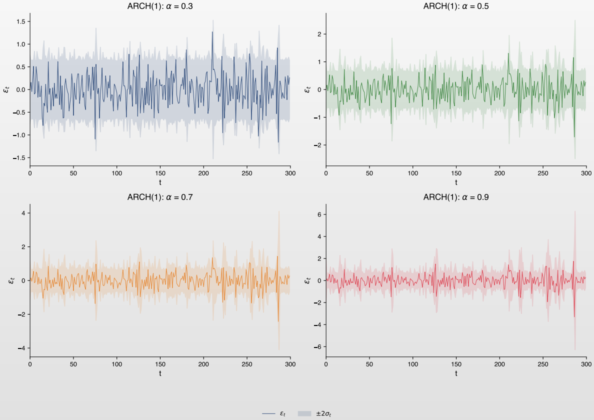
Stationarity Restrictions

- $\omega > 0$ (positive base variance), $\alpha_i \geq 0$ (non-negativity)
- $\sum_{i=1}^{q} \alpha_i < 1$ (stationarity)

Remark 1

Robert Engle received the **Nobel Prize in Economics** in 2003 for developing the ARCH model!

ARCH(1) Simulation: Effect of α Parameter



Higher α means volatility reacts more strongly to recent shocks.

Properties of the ARCH(1) Model

ARCH(1): $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$

- **Unconditional variance**: $\mathbb{E}[\varepsilon_t^2] = \dfrac{\omega}{1-\alpha_1}$ (if $\alpha_1 < 1$)
- **Kurtosis**: $\kappa = 3 \cdot \dfrac{1-\alpha_1^2}{1-3\alpha_1^2}$ (if $\alpha_1^2 < 1/3$)
- Kurtosis > 3 for $\alpha_1 > 0 \Rightarrow$ **fat tails**!

Numerical Example

If $\omega = 0.0001$ and $\alpha_1 = 0.3$:

- Unconditional variance: $\sigma^2 = \frac{0.0001}{1-0.3} = 0.000143$
- Kurtosis: $\kappa = 3 \cdot \frac{1-0.09}{1-0.27} = 3.74 > 3$

Derivation: Unconditional Variance of ARCH(1)

Derivation.

Let $\varepsilon_t = \sigma_t z_t$ with $z_t \sim N(0,1)$ and $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$.

Step 1: Take unconditional expectation:

$$\mathbb{E}[\varepsilon_t^2] = \mathbb{E}[\sigma_t^2 z_t^2] = \mathbb{E}[\sigma_t^2] \cdot \mathbb{E}[z_t^2] = \mathbb{E}[\sigma_t^2]$$

Step 2: Apply expectation to variance equation:

$$\mathbb{E}[\sigma_t^2] = \mathbb{E}[\omega + \alpha_1 \varepsilon_{t-1}^2] = \omega + \alpha_1 \mathbb{E}[\varepsilon_{t-1}^2]$$

Step 3: By stationarity, $\mathbb{E}[\varepsilon_t^2] = \mathbb{E}[\varepsilon_{t-1}^2] = \sigma^2$:

$$\sigma^2 = \omega + \alpha_1 \sigma^2 \quad \Rightarrow \quad \sigma^2(1 - \alpha_1) = \omega$$

Result: $\boxed{\sigma^2 = \frac{\omega}{1 - \alpha_1}}$ (requires $\alpha_1 < 1$ for stationarity) □

Derivation: Kurtosis of ARCH(1)

For $\varepsilon_t = \sigma_t z_t$ with $z_t \sim N(0,1)$:

Step 1: $\mathbb{E}[\varepsilon_t^4] = \mathbb{E}[\sigma_t^4] \cdot \mathbb{E}[z_t^4] = 3\mathbb{E}[\sigma_t^4]$ (since $\mathbb{E}[z^4] = 3$)

Step 2: Using $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$:

$$\mathbb{E}[\sigma_t^4] = \mathbb{E}[(\omega + \alpha_1 \varepsilon_{t-1}^2)^2] = \omega^2 + 2\omega\alpha_1\sigma^2 + \alpha_1^2 \mathbb{E}[\varepsilon_{t-1}^4]$$

Step 3: Solving the recursion yields:

$$\boxed{\kappa = \frac{\mathbb{E}[\varepsilon_t^4]}{(\mathbb{E}[\varepsilon_t^2])^2} = 3 \cdot \frac{1-\alpha_1^2}{1-3\alpha_1^2}}$$

Interpretation

- $\kappa > 3$ for any $\alpha_1 > 0 \Rightarrow$ **fat tails** (leptokurtosis)
- Requires $\alpha_1 < 0.577$ for finite fourth moment
- ARCH naturally generates heavy-tailed distributions!

Testing for ARCH Effects

Engle's Test for ARCH Effects

Procedure:

1. Estimate the mean model and obtain residuals $\hat{\varepsilon}_t$
2. Calculate $\hat{\varepsilon}_t^2$
3. Regress $\hat{\varepsilon}_t^2$ on its lags:
$$\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \cdots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$$
4. Calculate the statistic $LM = T \cdot R^2 \sim \chi^2(q)$

Hypotheses

- H_0: No ARCH effects ($\alpha_1 = \cdots = \alpha_q = 0$)
- H_1: ARCH effects present (at least one $\alpha_i \neq 0$)

Limitations of the ARCH Model

Practical Problems

1. **High order** — many lags are usually needed (large q)
2. **Many parameters** — estimation difficulties
3. **Non-negativity constraints** — difficult to impose for large q
4. **Does not capture persistence** — observed volatility is very persistent

The Solution

The GARCH Model — introduces lags of conditional variance to capture persistence with fewer parameters!

The GARCH(p,q) Model — Bollerslev (1986)

Definition 3 (GARCH(p,q))

The **Generalized ARCH** model:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

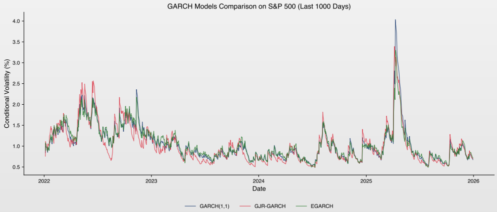
Interpretation

- ω = base level of volatility
- α_i = reaction to recent shocks (news coefficients)
- β_j = volatility persistence (memory)
- $\alpha + \beta$ = total persistence

The GARCH(1,1) Model

The Most Popular Volatility Model

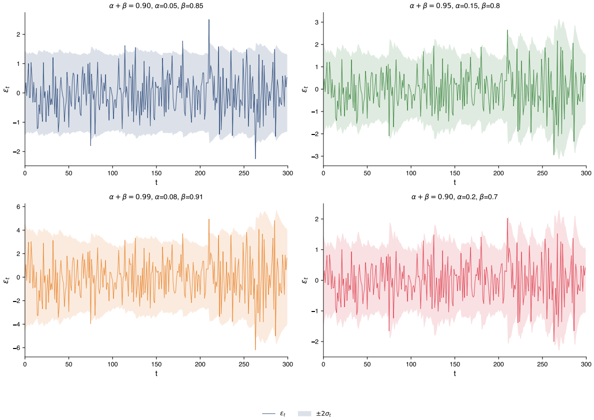
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$



Restrictions & Properties

- $\omega > 0,\ \alpha \geq 0,\ \beta \geq 0;\quad \alpha + \beta < 1$ (stationarity)
- $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$; Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$

GARCH(1,1) Simulation: Persistence Effect



Parameter α controls reaction to shocks, β controls persistence. The sum $\alpha+\beta$ determines mean-reversion speed.

Derivation: Unconditional Variance of GARCH(1,1)

Derivation.

For $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$:

Step 1: Take unconditional expectation: $\mathbb{E}[\sigma_t^2] = \omega + \alpha\mathbb{E}[\varepsilon_{t-1}^2] + \beta\mathbb{E}[\sigma_{t-1}^2]$

Step 2: By stationarity, $\mathbb{E}[\sigma_t^2] = \mathbb{E}[\sigma_{t-1}^2] = \bar{\sigma}^2$ and $\mathbb{E}[\varepsilon_t^2] = \bar{\sigma}^2$: $\bar{\sigma}^2 = \omega + (\alpha + \beta)\bar{\sigma}^2$

Step 3: Solve: $\bar{\sigma}^2(1 - \alpha - \beta) = \omega \Rightarrow \boxed{\bar{\sigma}^2 = \dfrac{\omega}{1 - \alpha - \beta}}$ □

Stationarity Condition

Requires $\alpha + \beta < 1$ for finite unconditional variance.

GARCH(1,1) as ARMA for ε_t^2

ARMA(1,1) Representation

Define $\nu_t = \varepsilon_t^2 - \sigma_t^2$ (variance shock). Then:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

This is an **ARMA(1,1)** for ε_t^2!

Implications

- ACF of ε_t^2 decays exponentially (like ARMA)
- Persistence is given by $\alpha + \beta$
- PACF can help identify the order

Derivation: ARMA Representation of GARCH(1,1)

Derivation.

Step 1: Define variance shock: $\nu_t = \varepsilon_t^2 - \sigma_t^2$

- $\mathbb{E}[\nu_t|\mathcal{F}_{t-1}] = \mathbb{E}[\varepsilon_t^2|\mathcal{F}_{t-1}] - \sigma_t^2 = \sigma_t^2 - \sigma_t^2 = 0$
- ν_t is a martingale difference sequence

Step 2: Substitute $\sigma_t^2 = \varepsilon_t^2 - \nu_t$ into GARCH equation:

$$\varepsilon_t^2 - \nu_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta(\varepsilon_{t-1}^2 - \nu_{t-1})$$

Step 3: Rearrange:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

Result: ARMA(1,1) with AR coefficient $\phi = \alpha + \beta$ and MA coefficient $\theta = -\beta$. □

Volatility Persistence and Half-Life

Persistence

$\alpha + \beta$ measures mean reversion speed:

- ≈ 1: very persistent
- $\ll 1$: quick reversion

Half-Life Formula

$$HL = \frac{-0.693}{\ln(\alpha + \beta)}$$

Example: S&P 500

$\alpha = 0.09$, $\beta = 0.90$
$\alpha + \beta = 0.99$

$$HL = \frac{-0.693}{\ln(0.99)} \approx 69$$

Shock halves in $\sim$69 trading days!

Estimation of GARCH Models

Maximum Likelihood Estimation (MLE)

Log-likelihood (normal): $\ell(\theta) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln(\sigma_t^2) + \frac{\varepsilon_t^2}{\sigma_t^2}\right]$

Alternative Distributions for z_t

- **Student-t**: captures fat tails — most common choice
- **GED**: flexibility for kurtosis
- **Skewed Student-t**: asymmetry and fat tails

Practical Note

Student-t distribution typically provides better fit for financial returns due to fat tails (kurtosis > 3).

Typical Values for GARCH(1,1)

Series	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha + \beta}$
S&P 500 daily	0.05–0.10	0.85–0.95	0.95–0.99
EUR/USD daily	0.03–0.08	0.90–0.95	0.95–0.99
Bitcoin daily	0.10–0.20	0.75–0.85	0.90–0.98
Bonds	0.02–0.05	0.90–0.97	0.95–0.99

Observations

- $\alpha + \beta$ close to 1 $\Rightarrow$ **very persistent volatility**
- Small α, large β $\Rightarrow$ slow reaction to shocks, long memory
- Bitcoin: larger α $\Rightarrow$ faster reaction to news

IGARCH — Integrated GARCH

Definition 4 (IGARCH(1,1))

When $\alpha + \beta = 1$:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2$$

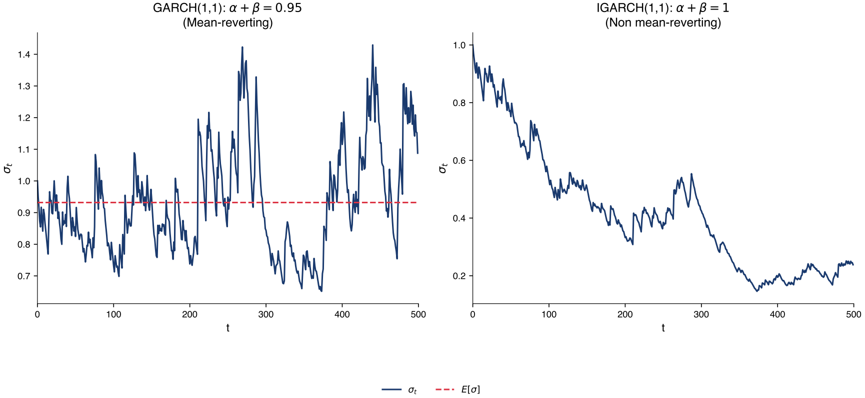
Properties

- Unconditional variance does not exist (infinite)
- Shocks have **permanent** effect on volatility
- Used for series with extreme persistence
- Useful for **RiskMetrics** (J.P. Morgan): $\alpha = 0.06$, $\beta = 0.94$

Remark 2

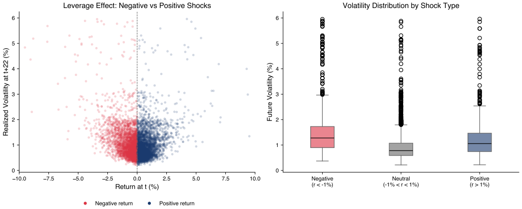
IGARCH is analogous to a unit root in variance!

GARCH vs IGARCH: Persistence Comparison



Standard GARCH reverts to unconditional mean, while IGARCH has no finite mean and shocks persist indefinitely.

Leverage Effect



Definition

Leverage effect: Negative shocks increase volatility **more** than positive shocks of the same magnitude.

Problem with GARCH

Standard GARCH: $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$ — only ε_{t-1}^2 matters, sign is lost! Economic intuition: Bad news $\Rightarrow$ stock price falls $\Rightarrow$ debt/equity ratio rises $\Rightarrow$ volatility increases.

The EGARCH Model — Nelson (1991)

Definition 5 (EGARCH(1,1))

Exponential GARCH:

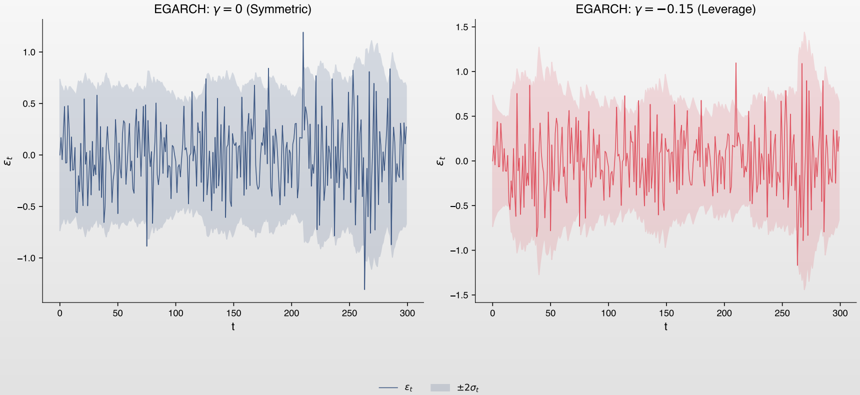
$$\ln(\sigma_t^2) = \omega + \alpha\left(|z_{t-1}| - \mathbb{E}[|z_{t-1}|]\right) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

where $z_t = \varepsilon_t / \sigma_t$.

EGARCH Advantages

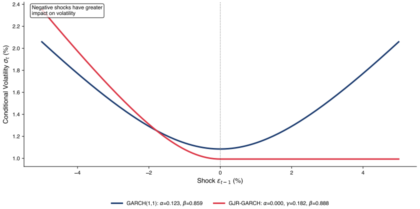
- **No non-negativity constraints required** — models $\ln(\sigma_t^2)$
- **Captures leverage effect** through parameter γ
  - $\gamma < 0$: negative shocks $\Rightarrow$ higher volatility
  - $\gamma = 0$: symmetric effect (like GARCH)
- Persistence is given by β

EGARCH Simulation: Symmetric vs Asymmetric



When $\gamma < 0$, negative shocks (bad news) increase volatility more than positive shocks of the same magnitude.

News Impact Curve — EGARCH



Definition

News Impact Curve: σ^2_{t+1} as function of ε_t, holding σ^2_t constant.

- **GARCH**: Symmetric V-shape (parabola); **EGARCH**: Asymmetric — steeper for negative shocks; **GJR**: Piecewise linear with kink at zero
- The asymmetry captures the leverage effect: bad news has larger impact on volatility than good news.

The GJR-GARCH Model

Definition 6 (GJR-GARCH(1,1))

Glosten, Jagannathan & Runkle (1993): $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 \cdot I_{t-1} + \beta\sigma_{t-1}^2$ where $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$, else 0.

Interpretation

- Positive shocks: impact $= \alpha$; Negative shocks: impact $= \alpha + \gamma$
- Leverage effect present if $\gamma > 0$
- Stationarity: $\alpha + \gamma/2 + \beta < 1$

TGARCH — Threshold GARCH

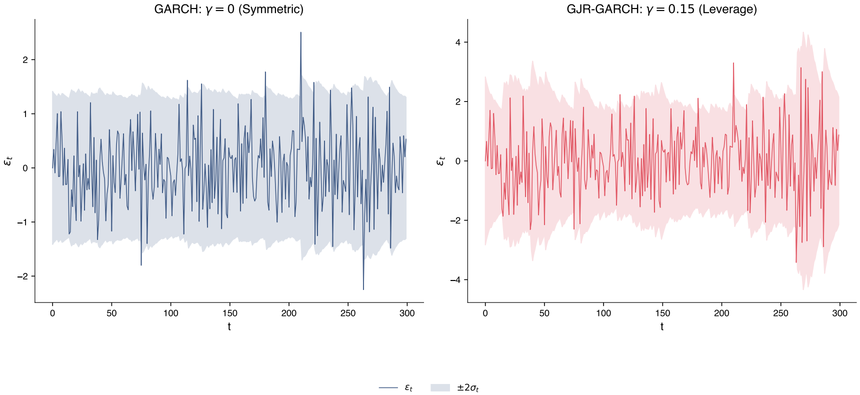
Definition 7 (TGARCH(1,1))

Zakoian (1994) models standard deviation: $\sigma_t = \omega + \alpha^+ \varepsilon_{t-1}^+ + \alpha^- \varepsilon_{t-1}^- + \beta \sigma_{t-1}$

Comparison of Asymmetric Models

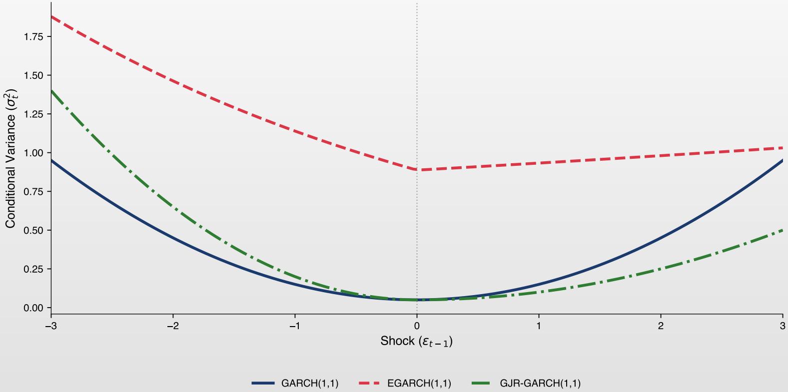
Model	Specification	Leverage
GARCH	σ_t^2	No
EGARCH	$\ln(\sigma_t^2)$	Yes ($\gamma < 0$)
GJR-GARCH	σ_t^2 with indicator	Yes ($\gamma > 0$)
TGARCH	σ_t	Yes ($\alpha^- > \alpha^+$)

GJR-GARCH/TGARCH Simulation



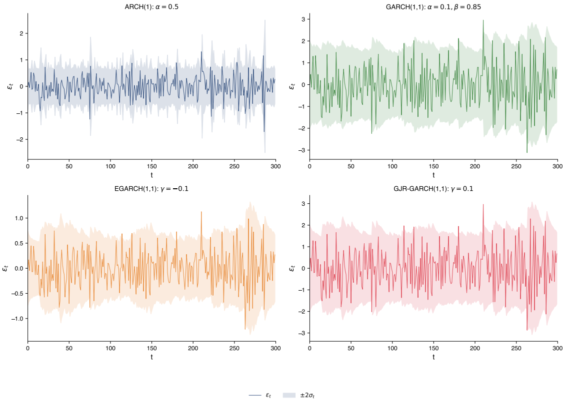
GJR-GARCH adds an indicator term to capture asymmetric response to negative shocks.

News Impact Curves Comparison



Standard GARCH is symmetric, while EGARCH and GJR-GARCH capture asymmetry (leverage effect).

GARCH Family Comparison



All models capture volatility clustering, but differ in how they model asymmetry.

GARCH-M: GARCH-in-Mean Model

Definition 8 (GARCH-M)

The **GARCH-in-Mean** model: $r_t = \mu + \lambda\sigma_t + \varepsilon_t$, where λ is the **risk premium**.

Interpretation

- $\lambda > 0$: Higher risk $\Rightarrow$ higher expected return
- $\lambda = 0$: Reduces to standard GARCH
- $\lambda < 0$: Higher risk $\Rightarrow$ lower return (rare)

Financial Intuition

Investors demand compensation for bearing risk — GARCH-M captures this **risk-return tradeoff**.

GARCH-M: Alternative Specifications

Common Specifications

Risk premium can enter in different forms: (1) $r_t = \mu + \lambda\sigma_t + \varepsilon_t$; (2) $r_t = \mu + \lambda\sigma_t^2 + \varepsilon_t$; (3) $r_t = \mu + \lambda\ln(\sigma_t^2) + \varepsilon_t$

Typical Results for Equity Markets

- Estimated λ often positive but small (0.01–0.10)
- Significance varies across markets and periods
- Variance specification yields larger λ estimates

Remark 3

GARCH-M is used in asset pricing, portfolio optimization, and CAPM testing.

Order Selection

Information Criteria

- **AIC** $= -2\ell + 2k$
- **BIC** $= -2\ell + k\ln(T)$
- **HQIC** $= -2\ell + 2k\ln(\ln(T))$

where ℓ = maximized log-likelihood, k = number of parameters.

Practical Recommendations

- GARCH(1,1) is sufficient in **90% of cases**
- Check if asymmetric model significantly improves fit
- Choose innovation distribution that minimizes AIC/BIC

GARCH Model Diagnostics

Standardized Residuals

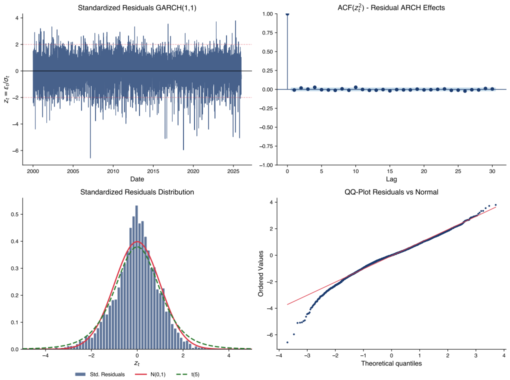
$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$$

If the model is correctly specified, $\hat{z}_t$ should be i.i.d.(0,1).

Diagnostic Checks

1. **Ljung-Box on** $\hat{z}_t$: check absence of autocorrelation in mean
2. **Ljung-Box on** $\hat{z}_t^2$: check absence of residual ARCH effects
3. **ARCH-LM test on** $\hat{z}_t$: confirm absence of heteroskedasticity
4. **Histogram + QQ-plot**: verify assumed distribution

Diagnostic Example



Forecasting with GARCH(1,1)

One-Step-Ahead Forecast

$\hat{\sigma}^2_{T+1} = \omega + \alpha\varepsilon^2_T + \beta\sigma^2_T$

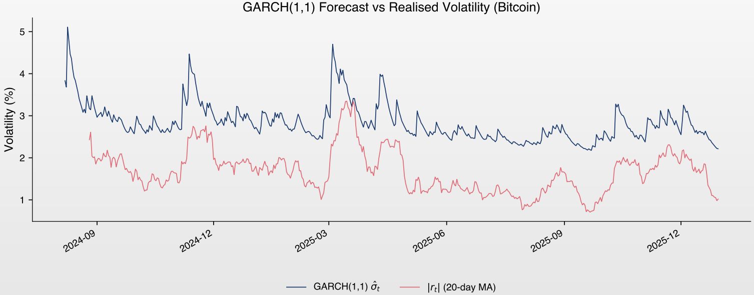
Multi-Step Forecast

For $h > 1$: $\mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2 + (\alpha+\beta)^{h-1}(\sigma^2_{T+1} - \bar{\sigma}^2)$ where $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ = unconditional variance.

Convergence

$\lim_{h\to\infty} \mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2$ — forecast converges to unconditional variance!

Volatility Forecast — Visualization



- Forecast converges exponentially to $\bar{\sigma}^2$; speed depends on $\alpha + \beta$
- The closer $\alpha + \beta$ is to 1, the slower the convergence

Applications of Volatility Forecasting

Value at Risk (VaR)

$\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$
Maximum loss with probability $1 - \alpha$.

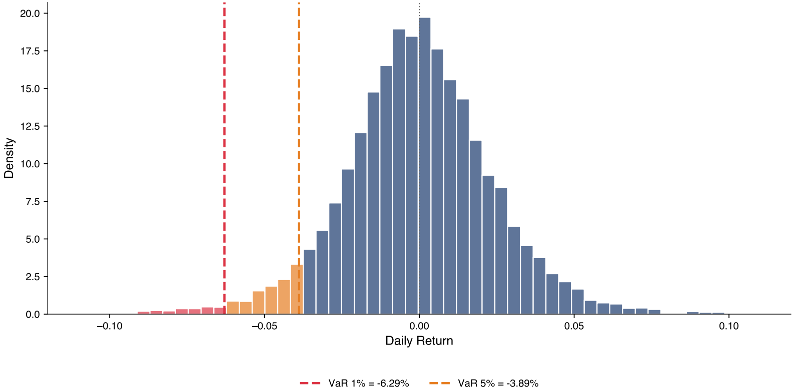
Expected Shortfall (ES)

$\text{ES}_\alpha = \mathbb{E}[-r | r < -\text{VaR}_\alpha]$
Average loss when VaR is exceeded.

Other Applications

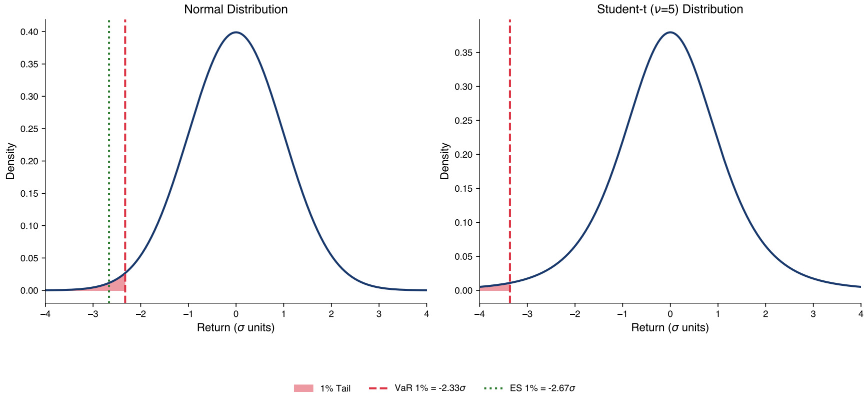
- Option pricing
- Dynamic hedging
- Portfolio allocation
- Stress testing

VaR and ES: Graphical Illustration



VaR 1% = loss exceeded only in 1% of cases. Red area = left tail (extreme losses).

VaR vs Expected Shortfall: Normal vs Student-t



ES (green line) measures average loss when VaR is exceeded. Student-t has heavier tails $\Rightarrow$ larger VaR and ES.

Value at Risk — Numerical Example

VaR Calculation

Portfolio: **1,000,000 EUR**, forecasted volatility $\hat{\sigma}_{T+1} = 1.5\%$

VaR with Normal Distribution

Level	z_α	**VaR (%)**	**VaR (EUR)**
5% (1 day)	1.645	2.47%	24,675
1% (1 day)	2.326	3.49%	34,890

Scaling for Longer Periods

$\text{VaR}_{h\ \text{days}} = \text{VaR}_{1\ \text{day}} \cdot \sqrt{h}$ — assumes i.i.d. returns

Value at Risk — Student-t Distribution

Why Student-t?

Normal distribution **underestimates** tail risk. Student-t with ν degrees of freedom better captures fat tails (kurtosis > 3).

VaR 1% (1 day) Comparison: $\sigma = 1.5\%$, Portfolio = 1M EUR

Distribution	**Quantile**	**VaR (EUR)**
Normal	2.326	34,890
Student-t ($\nu = 6$)	3.143	47,145
Student-t ($\nu = 4$)	3.747	56,205

Observation

With $\nu = 6$ (typical for stocks), VaR is **35% higher** than normal!

VaR — Complete Example with GARCH

VaR Calculation Procedure

1. Estimate GARCH(1,1) model with Student-t distribution
2. Obtain volatility forecast: $\hat{\sigma}_{T+1}$
3. Calculate VaR: $\text{VaR}_\alpha = t_\alpha(\nu) \cdot \hat{\sigma}_{T+1} \cdot \sqrt{\frac{\nu-2}{\nu}}$

Example: S&P 500

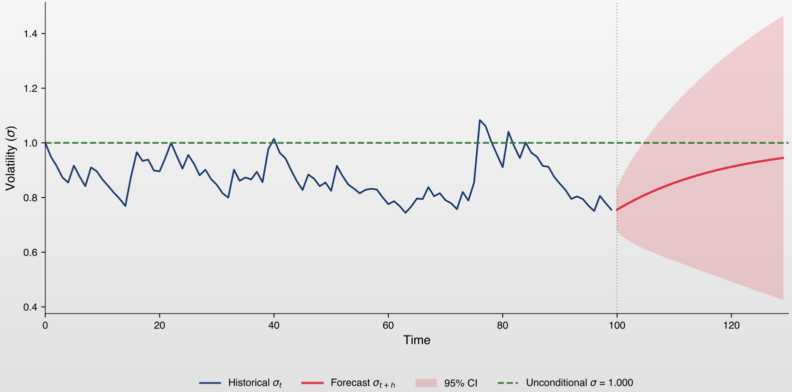
- Estimated parameters: $\alpha = 0.088$, $\beta = 0.900$, $\nu = 6.4$
- Forecasted volatility: $\hat{\sigma}_{T+1} = 1.2\%$
- Portfolio: 10,000,000 EUR

VaR 1% (1 day): VaR $= 3.05 \times 0.012 \times 10{,}000{,}000 =$ **366,000 EUR**

VaR Backtesting: Visual Overview

Backtesting checks if VaR violations match expected rate (e.g., 2.5 violations/year for VaR 1%).

Volatility Forecast with Confidence Intervals



Forecast converges to unconditional volatility $\bar{\sigma}$. Uncertainty increases with forecast horizon.

VaR Backtesting: Kupiec Test

Unconditional Coverage Test

Tests whether the observed violation rate equals the expected rate p (e.g., 1% for VaR 1%).

Let N = number of VaR violations, T = total observations, $\hat{p} = N/T$.

Likelihood Ratio Statistic:

$$LR_{uc} = -2\ln\left[\frac{(1-p)^{T-N}p^N}{(1-\hat{p})^{T-N}\hat{p}^N}\right] \sim \chi^2(1)$$

Hypotheses

- H_0: $\hat{p} = p$ (VaR model is correctly calibrated)
- H_1: $\hat{p} \neq p$ (VaR model under- or over-estimates risk)

VaR Backtesting: Christoffersen Test

Conditional Coverage Test

Tests both **unconditional coverage** and **independence** of violations.
Violations should be independent — no clustering of exceptions!

Test Components

- **Independence test** (LR_{ind}): Tests if violations are serially independent
- **Conditional coverage**: $LR_{cc} = LR_{uc} + LR_{ind} \sim \chi^2(2)$

Interpretation

Reject LR_{uc}: wrong frequency; Reject LR_{ind}: clustered violations; Reject LR_{cc}: model fails

VaR Backtesting: Python Implementation

Kupiec Test Implementation

```
import numpy as np
from scipy import stats
def kupiec_test(violations, T, p=0.01):
    N = np.sum(violations)
    p_hat = N / T
    if N == 0 or N == T:
        return np.nan, np.nan
    LR = -2 * (np.log((1-p)**(T-N) * p**N) -
               np.log((1-p_hat)**(T-N) * p_hat**N))
    return LR, 1 - stats.chi2.cdf(LR, df=1)
```

Usage

```
LR, pval = kupiec_test(violations, T=250, p=0.01)
```

ARMA-GARCH: Joint Mean and Variance Modeling

Why Joint Modeling?

Serial correlation $\Rightarrow$ ARMA for mean; **Volatility clustering** $\Rightarrow$ GARCH for variance.

Definition 9 (ARMA(p,q)-GARCH(r,s))

Mean equation: $r_t = \mu + \sum_{i=1}^{p} \phi_i (r_{t-i} - \mu) + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$

Variance equation: $\varepsilon_t = \sigma_t z_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$

ARMA-GARCH: Model Selection Strategy

Step-by-Step Approach

1. **Identify mean model**: Check ACF/PACF of returns for ARMA structure
2. **Test for ARCH effects**: Apply ARCH-LM test to residuals
3. **Specify variance model**: Usually GARCH(1,1) is sufficient
4. **Joint estimation**: Estimate both equations via MLE
5. **Diagnostic checking**: Standardized residuals should be i.i.d.

Common Specifications

- **Stock returns**: AR(1)-GARCH(1,1) or ARMA(1,1)-GARCH(1,1)
- **Exchange rates**: Often just GARCH(1,1) (no mean dynamics)
- **Interest rates**: AR(1)-EGARCH(1,1) for leverage effects

ARMA-GARCH: Python Implementation

Using the arch Package

```
from arch import arch_model
model = arch_model(returns,
                   mean='ARX',
                   lags=1,
                   vol='Garch',
                   p=1, q=1,
                   dist='t')
result = model.fit(disp='off')
print(result.summary())
```

Parameters

`mean='ARX'`: ARMA mean; `lags=1`: AR(1); `dist='t'`: Student-t

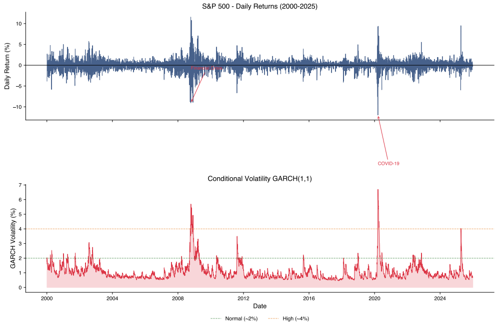
ARMA-GARCH: Complete Example

```
from arch import arch_model
model = arch_model(returns,
                   mean='ARX', lags=[1],
                   vol='EGARCH', p=1, q=1,
                   dist='skewt')
result = model.fit(update_freq=0)
cond_mean = result.conditional_mean
cond_vol = result.conditional_volatility
forecasts = result.forecast(horizon=5)
```

Note

For MA terms, use `mean='HARX'` or pre-filter with statsmodels ARIMA.

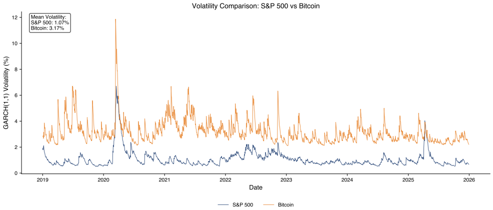
S&P 500 Volatility Analysis



- S&P 500 daily returns (2000–2024) — volatility clustering visible
- Crisis periods: 2008 (financial), 2020 (COVID-19), 2022 (inflation)

GARCH(1,1) Estimation — S&P 500

TSA_ch5_sp500_

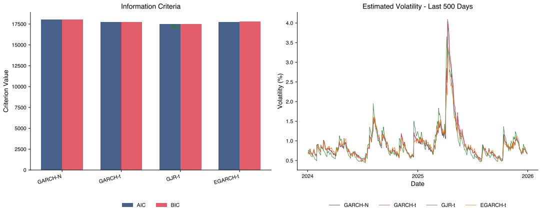


Estimation Results

Parameter	Value
ω	0.0108
α	0.0883
β	0.9002
$\alpha + \beta$	0.9885
ν (df)	6.42

Very persistent; Half-life $\approx$ 60 days

GARCH vs EGARCH Comparison — S&P 500



Leverage Effect Confirmed

EGARCH: $\gamma = -0.12$ (significantly negative) — negative shocks amplify volatility more than positive shocks. Both models capture volatility clustering, but EGARCH better fits crisis periods (2008, 2020).

Key Formulas

Volatility Models

- **ARCH(q):** $\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$
- **GARCH(1,1):** $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$
- **EGARCH:** $\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$
- **GJR-GARCH:** $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 I_{t-1} + \beta\sigma_{t-1}^2$

Properties and Measures

- **Unconditional variance:** $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ **Half-life:** $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$
- **VaR:** $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$ **Stationarity:** $\alpha + \beta < 1$
- **ARCH-LM:** $LM = T \cdot R^2 \sim \chi^2(q)$

Summary — Chapter 5: Volatility Models

Key Concepts

- **ARCH(q)**: conditional variance depends on past squared errors
- **GARCH(p,q)**: adds variance lags for persistence
- **EGARCH/GJR-GARCH**: capture leverage effect (asymmetric response)

Applications

Risk measurement (VaR, ES), derivative pricing, portfolio management

Practical Tip

Start with GARCH(1,1), check for leverage, choose distribution minimizing AIC/BIC!

References

- Engle, R.F. (1982). *Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation*. Econometrica, 50(4), 987-1007.
- Bollerslev, T. (1986). *Generalized Autoregressive Conditional Heteroskedasticity*. Journal of Econometrics, 31(3), 307-327.
- Nelson, D.B. (1991). *Conditional Heteroskedasticity in Asset Returns: A New Approach*. Econometrica, 59(2), 347-370.
- Glosten, L.R., Jagannathan, R., & Runkle, D.E. (1993). *On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks*. The Journal of Finance, 48(5), 1779-1801.
- Tsay, R.S. (2010). *Analysis of Financial Time Series*. 3rd Edition, Wiley.