



Time Series Analysis and Forecasting

Chapter 9: Prophet and TBATS



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Learning Objectives

By the end of this chapter, you will be able to:

- ▣ Handle time series with multiple seasonal patterns
- ▣ Use Facebook Prophet for flexible forecasting with holidays
- ▣ Apply TBATS models for complex seasonality
- ▣ Compare and select between modern forecasting methods

Outline

The Problem: Complex Seasonal Patterns

Real-World Examples

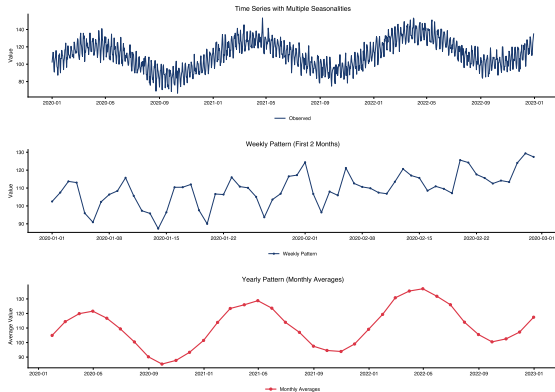
- ▣ **Hourly electricity demand:** Daily + Weekly + Annual patterns
- ▣ **Website traffic:** Daily + Weekly + Holiday effects
- ▣ **Retail sales:** Weekly + Monthly + Annual + Holiday effects
- ▣ **Call center volume:**
 - ▶ Hourly + Daily + Weekly patterns

SARIMA Limitation

Standard $\text{SARIMA}(p, d, q)(P, D, Q)_s$ handles only **one** seasonal period s .

For hourly data with daily AND weekly patterns, we need $s_1 = 24$ and $s_2 = 168$.

Example: Hourly Data with Multiple Seasonalities



Solutions for Multiple Seasonalities

Traditional Approaches

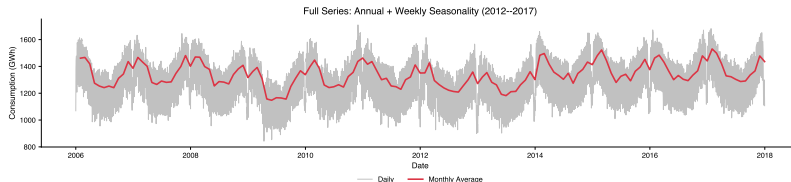
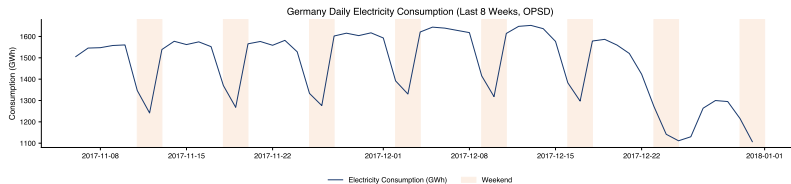
- ▣ **Fourier terms:** Add sin/cos regressors
- ▣ **Dummy variables:** Many parameters
- ▣ **Nested models:** Complex specification

Modern Approaches

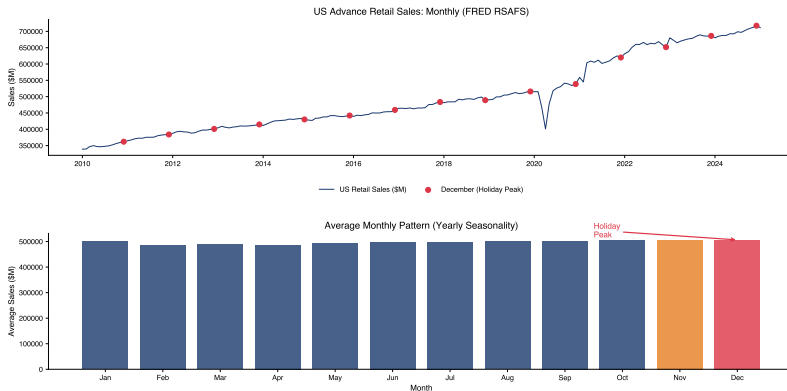
- ▣ **TBATS:** Automatic, handles many periods
- ▣ **Prophet:** Flexible, interpretable
- ▣ **Neural methods:**
 - ▶ Deep learning

Method	Max Seasonalities	Interpretable
SARIMA	1	Yes
Fourier + ARIMA	Multiple	Moderate
TBATS	Multiple	Moderate
Prophet	Multiple	Yes

Real Example: Electricity Demand



Real Example: Retail Sales with Holidays



Researcher Spotlight: Rob J. Hyndman



*1967

 Wikipedia

Biography

- ▣ Australian statistician, Professor at Monash University
- ▣ One of the most influential researchers in time series forecasting
- ▣ Creator of the widely-used `forecast` package for R
- ▣ Editor-in-Chief of the *International Journal of Forecasting* (2005–2018)

Key Contributions

- ▣ **TBATS model** (2011) — trigonometric Box-Cox ARMA with multiple seasonal periods
- ▣ **ETS framework** — exponential smoothing state space models with automatic selection
- ▣ **forecast package** for R — the standard toolkit for time series forecasting
- ▣ **Hierarchical forecasting** and forecast reconciliation methods

TBATS: What Does It Stand For?

TBATS Components

- T** **Trigonometric** seasonality using Fourier terms
- B** **Box-Cox** transformation for variance stabilization
- A** **ARMA** errors for remaining autocorrelation
- T** **Trend** component (possibly damped)
- S** **Seasonal** components (multiple allowed)

Key Innovation: Trigonometric Seasonality

$$s_t^{(i)} = \sum_{j=1}^{k_i} \left[s_j^{(i)} \cos\left(\frac{2\pi jt}{m_i}\right) + s_j^{*(i)} \sin\left(\frac{2\pi jt}{m_i}\right) \right]$$

m_i = seasonal period, k_i = number of harmonics

Box-Cox Transformation

Definition 1 (Box-Cox Transformation)

The Box-Cox transformation with parameter ω is defined as:

$$y_t^{(\omega)} = \begin{cases} \frac{y_t^\omega - 1}{\omega} & \text{if } \omega \neq 0 \\ \ln(y_t) & \text{if } \omega = 0 \end{cases}$$

Purpose

- ▣ **Variance stabilization:** Makes variance constant over time
- ▣ **Normalization:** Reduces skewness in the data
- ▣ Common values: $\omega = 0$ (log), $\omega = 0.5$ (square root), $\omega = 1$ (no transform)

TBATS Model Structure

State Space Representation

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^T s_{t-m_i}^{(i)} + d_t \quad (1)$$

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t, \quad b_t = \phi b_{t-1} + \beta d_t \quad (2)$$

- $y_t^{(\omega)}$: Box-Cox transformed observation
- ℓ_t : local level (smoothed mean)
- b_t : trend with damping $\phi \in (0, 1)$
- $s_t^{(i)}$: i -th seasonal component
- d_t : $\text{ARMA}(p, q)$ error process
- α, β : smoothing parameters

TBATS: Trigonometric Seasonality State Evolution

Definition 2 (Trigonometric State-Space Recursion)

For each seasonal component with period m_i and k_i harmonics, define states:

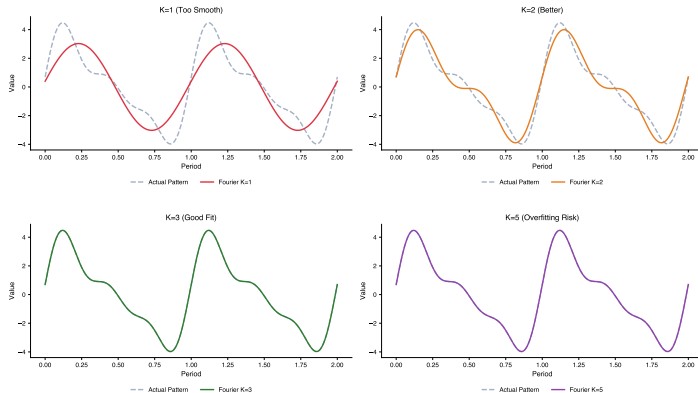
$$\begin{pmatrix} s_{j,t}^{(i)} \\ s_{j,t}^{*(i)} \end{pmatrix} = \begin{pmatrix} \cos(\lambda_j) & \sin(\lambda_j) \\ -\sin(\lambda_j) & \cos(\lambda_j) \end{pmatrix} \begin{pmatrix} s_{j,t-1}^{(i)} \\ s_{j,t-1}^{*(i)} \end{pmatrix} + \begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \end{pmatrix} d_t$$

where $\lambda_j = \frac{2\pi j}{m_i}$ is the j -th harmonic frequency.

Interpretation

- The rotation matrix preserves the periodic structure
- Total seasonal: $s_t^{(i)} = \sum_{j=1}^{k_i} s_{j,t}^{(i)}$
- Parameters: $2k_i$ states per seasonal period

Fourier Approximation of Seasonality



TBATS: Choosing the Number of Harmonics

Why Fourier/Trigonometric Terms?

1. **Parsimonious:** $2k$ parameters vs m dummy variables
2. **Smooth:** Captures smooth seasonal patterns naturally
3. **Flexible:** Number of harmonics k controls complexity
4. **Non-integer periods:** Can handle $s = 365.25$ for daily data

Low k (few harmonics)

- Smooth pattern
- Fewer parameters
- May miss sharp peaks

High k (many harmonics)

- Can capture any pattern
- More parameters ($2k$ total)
- Maximum useful: $k \leq \lfloor m/2 \rfloor$

TBATS: Key Features

Automatic Model Selection

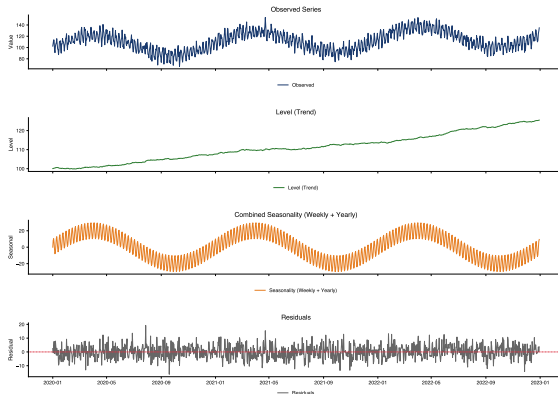
TBATS automatically determines:

- Box-Cox parameter ω for variance stabilization
- Number of harmonics k_i for each seasonal period
- ARMA orders (p, q) for residual autocorrelation
- Damped vs non-damped trend specification

BATS vs TBATS

- **BATS**: Traditional seasonal states (dummy variables)
- **TBATS**: Trigonometric (Fourier) seasonal representation
- TBATS more parsimonious for long seasonal periods

TBATS Decomposition Example



TBATS: Advantages and Limitations

Advantages

- ▣ Handles **multiple** seasonal periods
- ▣ **Automatic** model selection
- ▣ Handles **non-integer** periods (365.25)
- ▣ **Box-Cox** for heteroskedasticity
- ▣ Good for **high-frequency** data

Limitations

- ▣ **Computationally intensive**
- ▣ No **external regressors**
- ▣ Less **interpretable** than Prophet
- ▣ Can be **slow** for very long series
- ▣ Requires **sufficient data** per season

Prophet: Overview

What is Prophet?

Forecasting procedure developed by Facebook (Meta) in 2017 for **business time series**:

- ▣ Strong seasonal effects (daily, weekly, yearly)
- ▣ Holiday effects and trend changes (changepoints)
- ▣ Handles missing data and outliers

Key Philosophy: “Analyst-in-the-loop”

Designed for analysts with domain knowledge but without time series expertise.

Prophet Model Structure

Decomposition Approach

Prophet uses an **additive decomposition**:

$$y(t) = g(t) + s(t) + h(t) + \varepsilon_t$$

$g(t)$: Trend

- ▣ Linear or logistic
- ▣ Automatic changepoints
- ▣ Growth saturation

$s(t)$: Seasonality

- ▣ Fourier series
- ▣ Multiple periods
- ▣ Custom seasonality

$h(t)$: Holidays

- ▣ Country holidays
- ▣ Custom events
- ▣ Window effects

Prophet: Seasonality Component

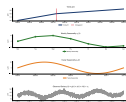
Fourier Series Representation

$$s(t) = \sum_{n=1}^N \left[a_n \cos\left(\frac{2\pi nt}{P}\right) + b_n \sin\left(\frac{2\pi nt}{P}\right) \right]$$

Default Settings

Seasonality	Period	Fourier Order
Yearly	365.25 days	10
Weekly	7 days	3
Daily	1 day	4

Higher N = more flexibility but risk of overfitting



Prophet: Trend Component

Linear Trend with Changepoints

$$g(t) = (k + \mathbf{a}(t)^T \boldsymbol{\delta}) \cdot t + (m + \mathbf{a}(t)^T \boldsymbol{\gamma})$$

- ▣ k : base growth rate (slope)
- ▣ $\boldsymbol{\delta} = (\delta_1, \dots, \delta_S)$: slope changes at S changepoints
- ▣ $\mathbf{a}(t) \in \{0, 1\}^S$: indicator if changepoint s is active at time t

Logistic Growth

For saturating trends:

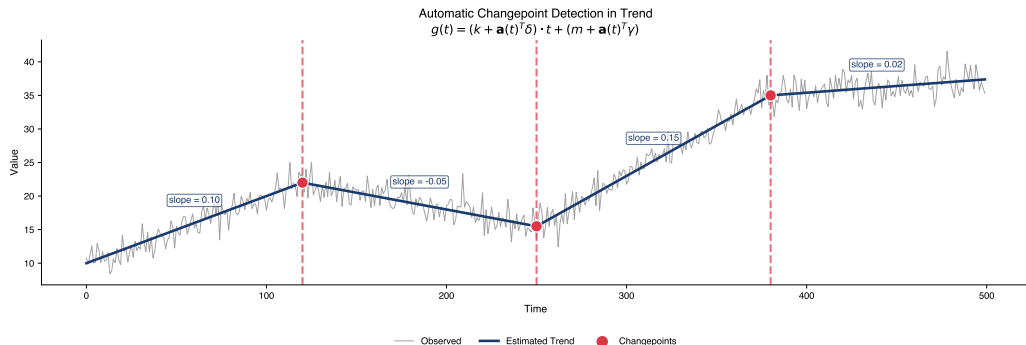
$$g(t) = \frac{C(t)}{1 + e^{-(k + \mathbf{a}(t)^T \boldsymbol{\delta})(t - m - \mathbf{a}(t)^T \boldsymbol{\gamma})}}$$

$C(t)$ = time-varying carrying capacity

Continuity Constraint

The offset $\gamma_j = -s_j \cdot \delta_j$ ensures $g(t)$ is continuous at each changepoint s_j .

Trend Changepoint Detection



Prophet: Holiday Effects

Holiday Model

$$h(t) = Z(t) \cdot \kappa$$

where $Z(t)$ is an indicator matrix for holidays and κ are holiday effects.

Built-in Features

- ▣ 60+ countries supported
- ▣ Custom holiday definitions
- ▣ Window effects (before/after)

Holiday Types

- ▣ National holidays
- ▣ Religious observances
- ▣ Business events

Prophet: Customization Options

Seasonality Customization

- ▣ Add custom seasonal periods (monthly, quarterly)
- ▣ Control Fourier order for each seasonality
- ▣ Enable/disable default seasonalities

External Regressors

Prophet supports adding external variables:

- ▣ Weather data, promotions, special events
- ▣ Binary or continuous regressors
- ▣ Automatic regularization

Prophet: Uncertainty Quantification

Bayesian Framework

Prophet uses a **Laplace prior** on changepoint magnitudes:

$$\delta_j \sim \text{Laplace}(0, \tau), \quad \tau = \text{changepoint_prior_scale}$$

Smaller τ = sparser, smaller changepoints (more regularization).

Sources of Uncertainty

1. **Trend**: Future changepoints
2. **Seasonality**: Coefficient variance
3. **Observation**: Residual noise σ^2

Prediction Intervals

- ▣ MAP estimation for point forecasts
- ▣ Monte Carlo sampling for intervals
- ▣ Default: 80% credible interval

Prophet: Tuning Parameters

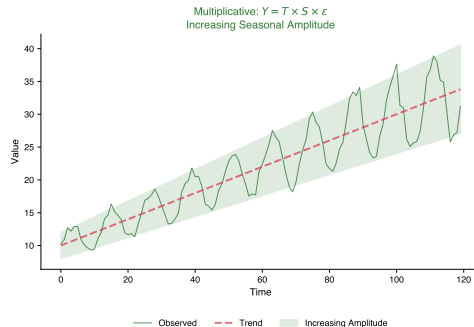
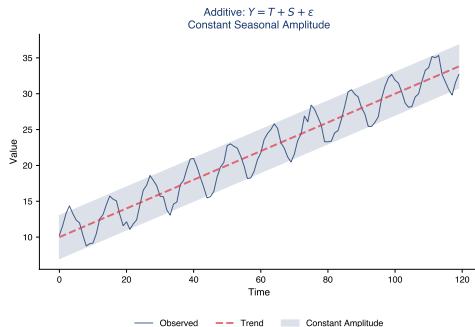
Key Parameters

Parameter	Effect
<code>changepoint_prior_scale</code>	Trend flexibility (default: 0.05)
<code>seasonality_prior_scale</code>	Seasonality flexibility (default: 10)
<code>holidays_prior_scale</code>	Holiday effect size (default: 10)
<code>seasonality_mode</code>	'additive' or 'multiplicative'
<code>changepoint_range</code>	Portion of history for changepoints

Practical Tips

- ▣ **Overfitting trend?** Decrease `changepoint_prior_scale`
- ▣ **Underfitting seasonality?** Increase `seasonality_prior_scale`
- ▣ **Seasonal amplitude varies?** Use `seasonality_mode='multiplicative'`

Additive vs Multiplicative Seasonality



 TSA_ch9_additive_vs_multiplicative

Prophet: Advantages and Limitations

Advantages

- **Easy to use:** Minimal tuning needed
- **Interpretable:** Clear decomposition
- **Handles missing data** well
- **Holiday effects** built-in
- **Multiple seasonalities**
- **External regressors** supported
- **Fast** fitting

Limitations

- **Not ARIMA-based:** No autocorrelation modeling
- **Daily data focus:** Less suited for very high frequency
- **Trend assumptions:** Linear/logistic may not fit
- **No built-in CV:** Must implement manually
- **Overfitting risk** with many seasonalities

TBATS vs Prophet: Head-to-Head

Feature	TBATS	Prophet
Multiple seasonalities	Yes (automatic)	Yes (manual or auto)
Holiday effects	No	Yes (built-in)
External regressors	No	Yes
Trend changepoints	No (smooth)	Yes (automatic)
Missing data	Interpolation needed	Handles natively
Interpretability	Moderate	High
Computation speed	Slow	Fast
High-frequency data	Good	Moderate
Non-integer periods	Yes (e.g., 365.25)	Yes
Uncertainty intervals	Yes	Yes

When to Use Each Model

Use TBATS when:

- ▣ High-frequency data
- ▣ Multiple seasonal periods
- ▣ No external regressors
- ▣ Automatic selection preferred

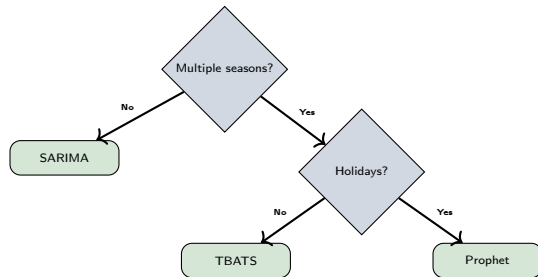
Use Prophet when:

- ▣ Business forecasting
- ▣ Holiday effects important
- ▣ Trend has changepoints
- ▣ External regressors available

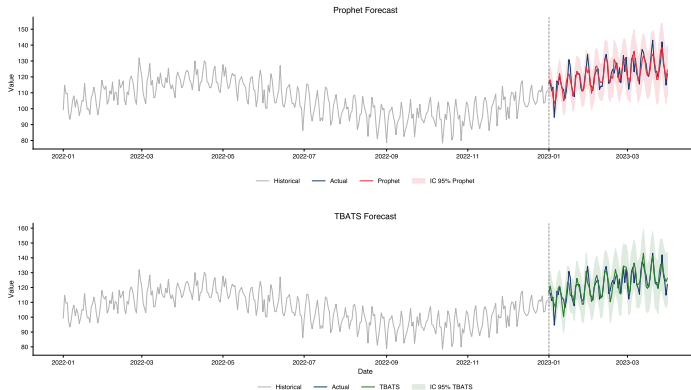
General Guideline

Prophet: business applications with daily data
TBATS: technical applications with high-frequency data

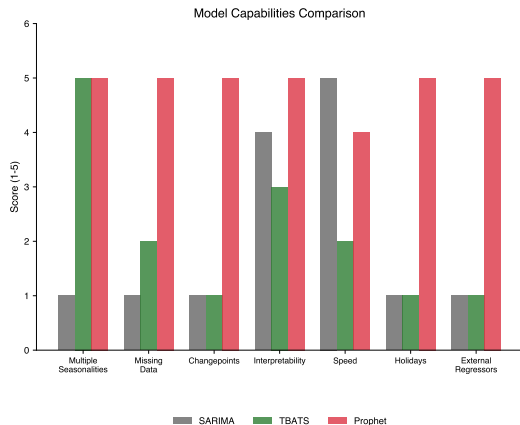
Decision Flowchart



Prophet vs TBATS: Forecast Comparison



Model Selection Guide



When to Use Each Model

SARIMA

- Single seasonality
- Regular data
- Statistical inference
- Short-term forecast

TBATS

- High frequency (hourly)
- Non-integer periods
- Automatic selection
- No external regressors

Prophet

- Business forecasting
- Holiday effects
- Missing data
- Changepoints trend
- External regressors



Evaluation Metrics

Definition 3 (Forecast Accuracy Metrics)

Let y_t denote actual values, \hat{y}_t forecasts, and n the forecast horizon:

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_{t=1}^n (y_t - \hat{y}_t)^2} \quad (\text{penalizes large errors})$$

$$\text{MAE} = \frac{1}{n} \sum_{t=1}^n |y_t - \hat{y}_t| \quad (\text{robust to outliers})$$

$$\text{MAPE} = \frac{100\%}{n} \sum_{t=1}^n \left| \frac{y_t - \hat{y}_t}{y_t} \right| \quad (\text{scale-free})$$

Coverage

For prediction intervals $[\hat{y}_t^L, \hat{y}_t^U]$, coverage rate is the proportion of actual values falling within the interval. Target: match the nominal level (e.g., 80%).

Case Study: Energy Demand Forecasting

Problem

Forecast hourly electricity demand with:

- ▣ **Daily pattern:** Peak at noon and evening
- ▣ **Weekly pattern:** Lower on weekends
- ▣ **Annual pattern:** Higher in summer (AC) and winter (heating)
- ▣ **Holiday effects:** Lower demand on holidays

Approach

1. Try TBATS with periods [24, 168, 8766]
2. Try Prophet with daily, weekly, yearly seasonality + holidays
3. Compare using cross-validation

Case Study: Results

Model Comparison

Model	MAPE	RMSE	Coverage
SARIMA (daily only)	8.5%	450 MW	75%
TBATS	4.2%	220 MW	82%
Prophet	4.8%	250 MW	85%
Prophet + holidays	3.9%	200 MW	88%

Key Finding

Multiple seasonality models significantly outperform single-seasonality SARIMA.

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"I have 3 years of hourly energy consumption data. Use Facebook Prophet to forecast the next week. Include holidays and special events. Give me complete Python code."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does Prophet automatically detect multiple seasonalities (daily, weekly)?
3. How are holidays specified? Country-specific or custom events?
4. Does it use cross-validation with cutoffs (performance _metrics)?
5. Would TBATS be more appropriate for this frequency? Why or why not?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Question 1

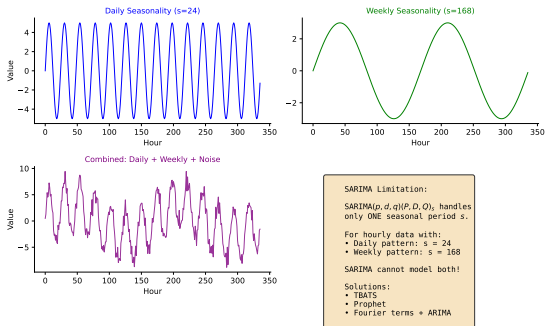
Question

- ☐ Why can't standard $\text{SARIMA}(p, d, q)(P, D, Q)_s$ model hourly electricity data with both daily and weekly patterns?

Answer Choices

- (A) SARIMA can only handle one seasonal period s at a time
- (B) SARIMA requires normally distributed errors for multiple seasonalities
- (C) SARIMA can handle multiple seasonalities but requires more data
- (D) SARIMA only works with monthly or quarterly data

Question 1: Answer



Answer: (A)

- ☐ SARIMA handles only **one** seasonal period s . You cannot set $s = 24$ (daily) and $s = 168$ (weekly) simultaneously in a single SARIMA model.

Question 2

Question

□ What does each letter in TBATS represent?

Answer Choices

- (A) Trend, Bayes, Autoregressive, Time, Stationarity
- (B) Trigonometric seasonality, Box-Cox, ARMA errors, Trend, Seasonal components
- (C) Taylor, Box-Cox, ARIMA, Transformation, Smoothing
- (D) Trigonometric, Bayesian, ARMA, Trend, Spectral analysis

Question 2: Answer

TBATS: What Does It Stand For?

T	Trigonometric	Fourier terms for seasonality $\sum [a_n \cos(\frac{2\pi nt}{m}) + b_n \sin(\frac{2\pi nt}{m})]$
B	Box-Cox	Variance stabilization $y^{(\omega)} = (y^\omega - 1)/\omega$
A	ARMA	Error autocorrelation $\phi(L)d_t = \theta(L)\varepsilon_t$
T	Trend	Level + slope (possibly damped) $\ell_t = \ell_{t-1} + \phi b_{t-1}$
S	Seasonal	Multiple seasonal periods m_1, m_2, \dots, m_T

Answer: (B)

- ☐ Trigonometric seasonality, **Box-Cox** transformation, **ARMA** errors, **Trend**, **Seasonal** components.

Question 3

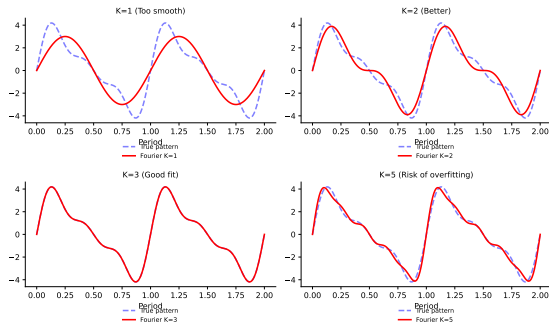
Question

□ What happens when we increase the number of Fourier harmonics K ?

Answer Choices

- (A) The model becomes simpler and more robust
- (B) The model captures more complex seasonal patterns but risks overfitting
- (C) The forecast horizon increases proportionally
- (D) The seasonal period s changes automatically

Question 3: Answer



Answer: (B)

- Higher K captures more complex seasonal patterns but increases the risk of overfitting. The maximum is $K \leq s/2$.

Question 4

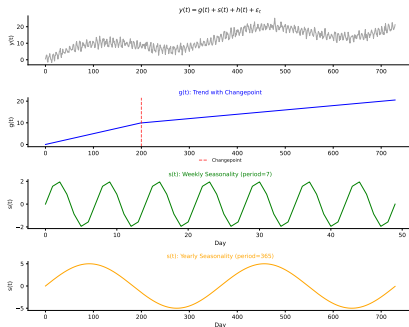
Question

□ What are the main components in Prophet's model $y(t) = g(t) + s(t) + h(t) + \varepsilon_t$?

Answer Choices

- (A) $g(t)$ = GARCH volatility, $s(t)$ = stationarity test, $h(t)$ = heteroskedasticity
- (B) $g(t)$ = growth (trend with changepoints), $s(t)$ = seasonality, $h(t)$ = holiday effects
- (C) $g(t)$ = Gaussian noise, $s(t)$ = smoothing, $h(t)$ = harmonic terms
- (D) $g(t)$ = gradient, $s(t)$ = spectral density, $h(t)$ = Hurst exponent

Question 4: Answer



Answer: (B)

- $g(t)$ = trend with changepoints, $s(t)$ = seasonality (Fourier terms), $h(t)$ = holiday effects, ε_t = error term.

Question 5

Question

□ What key features does Prophet have that TBATS lacks?

Answer Choices

- (A) Trigonometric seasonality and Box-Cox transformation
- (B) Automatic parameter selection and exponential smoothing
- (C) Holiday effects, external regressors, trend changepoints, and native missing data handling
- (D) State-space formulation and ARMA error modeling

Question 5: Answer

TBATS vs Prophet: Head-to-Head Comparison

Feature	TBATS	Prophet
Multiple seasonalities	Yes (automatic)	Yes (manual/auto)
Holiday effects	No	Yes (built-in)
External regressors	No	Yes
Trend changepoints	No (smooth)	Yes (automatic)
Missing data	Needs interpolation	Handles natively
Interpretability	Moderate	High
Computation speed	Slow	Fast
High-frequency data	Good	Moderate
Non-integer periods	Yes (e.g., 365.25)	Yes
Best for	Technical/high-freq	Business/daily

Answer: (C)

- ☐ Prophet offers holiday effects, external regressors, trend changepoints, and native missing data handling—features not available in TBATS.

Key Takeaways

What We Learned

- TBATS handles multiple seasonalities with Fourier terms and Box-Cox transformation
- Prophet provides interpretable decomposition with trend changepoints and holiday effects
- Both methods scale better than SARIMA for high-frequency and complex seasonal data

Important

Choose Prophet for business forecasting with holidays and interpretability needs. Use TBATS for automatic modeling of high-frequency data. Always validate with time series cross-validation—never standard k-fold!

Questions?

Questions?

Next Steps:

- Practice with the Jupyter notebook
- Try Prophet on your own data
- Explore NeuralProphet for deep learning extension

Bibliography I

Prophet

- ▣ Taylor, S.J., & Letham, B. (2018). Forecasting at Scale, *The American Statistician*, 72(1), 37–45.
- ▣ Harvey, A.C. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge University Press.

TBATS and Exponential Smoothing

- ▣ De Livera, A.M., Hyndman, R.J., & Snyder, R.D. (2011). Forecasting Time Series with Complex Seasonal Patterns Using Exponential Smoothing, *JASA*, 106(496), 1513–1527.
- ▣ Hyndman, R.J., Koehler, A.B., Ord, J.K., & Snyder, R.D. (2008). *Forecasting with Exponential Smoothing: The State Space Approach*, Springer.
- ▣ Taylor, J.W. (2003). Short-term Electricity Demand Forecasting Using Double Seasonal Exponential Smoothing, *Journal of the Operational Research Society*, 54(8), 799–805.

Bibliography II

Forecasting Comparisons and Competitions

- ▣ Makridakis, S., Spiliotis, E., & Assimakopoulos, V. (2020). The M4 Competition, *International Journal of Forecasting*, 36(1), 54–74.
- ▣ Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.
- ▣ Petropoulos, F., et al. (2022). Forecasting: Theory and Practice, *International Journal of Forecasting*, 38(3), 845–1054.

Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> → Code repository for statistics
- ▣ **Quantinar**: <https://quantinar.com> → Learning platform for quantitative methods
- ▣ **GitHub TSA**: <https://github.com/QuantLet/TSA> → Python code for this course

Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar