



Time Series Analysis and Forecasting

## Chapter 6: Cointegration & VECM

Long-Run Equilibrium Relationships



# Lecture Outline

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# Why Cointegration?

## The Problem with Non-Stationary Data

- Many economic/financial time series are **non-stationary** ( $I(1)$ )
- Standard regression with  $I(1)$  variables leads to **spurious regression**
- Differencing removes non-stationarity but loses **long-run information**

## Key Insight

Some non-stationary series may share a common stochastic trend—they move together in the long run. This is **cointegration**.

## Economic Examples

- Consumption and Income (permanent income hypothesis)
- Short-term and long-term interest rates (term structure)
- Spot and futures prices (arbitrage relationship)
- Exchange rates and price levels (PPP)

# Spurious Regression Problem

## Granger & Newbold (1974)

Regressing one random walk on another independent random walk:

$$Y_t = \alpha + \beta X_t + u_t$$

where  $Y_t$  and  $X_t$  are independent  $I(1)$  processes.

## Symptoms of Spurious Regression

- High  $R^2$  (often  $> 0.9$ ) even though variables are unrelated
- Highly significant  $t$ -statistics
- Very low Durbin-Watson statistic ( $DW \approx 0$ )
- Residuals are non-stationary (have unit root)

## Rule of Thumb

If  $R^2 > DW$ , suspect spurious regression!

# Definition of Cointegration

## Definition 1 (Cointegration (Engle & Granger, 1987))

Variables  $Y_{1t}, Y_{2t}, \dots, Y_{kt}$  are **cointegrated of order**  $(d, b)$ , written  $CI(d, b)$ , if:

- 1 All variables are integrated of order  $d$ :  $Y_{it} \sim I(d)$
- 2 There exists a linear combination  $\beta' \mathbf{Y}_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$  that is integrated of order  $(d - b)$ , where  $b > 0$

## Most Common Case: $CI(1, 1)$

- Variables are  $I(1)$  (have unit roots)
- Linear combination is  $I(0)$  (stationary)
- Vector  $\beta = (\beta_1, \dots, \beta_k)'$  is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized:  $\beta_1 = 1$ .

## Intuition: Common Stochastic Trends

### Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**:

$$Y_{1t} = \gamma_1 \tau_t + S_{1t}, \quad Y_{2t} = \gamma_2 \tau_t + S_{2t}$$

where  $\tau_t$  is a common random walk and  $S_{it}$  are stationary components.

### Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

### Economic Interpretation

- Cointegration represents a **long-run equilibrium relationship**
- Variables may deviate in the short run
- But they are “pulled back” to equilibrium over time
- The cointegrating vector defines the equilibrium

# Cointegrating Rank

## How Many Cointegrating Relationships?

For  $k$  variables that are  $I(1)$ :

- Maximum possible cointegrating relationships:  $r = k - 1$
- If  $r = 0$ : No cointegration (variables drift apart)
- If  $r = k$ : All variables are  $I(0)$  (contradiction)

## Example: 3 Variables

- $r = 0$ : No cointegration
- $r = 1$ : One cointegrating relationship
- $r = 2$ : Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends =  $k - r$

# Engle-Granger Two-Step Method

## Step 1: Estimate Cointegrating Regression

Run OLS regression (assuming  $Y_t$  is the dependent variable):

$$Y_t = \alpha + \beta X_t + e_t$$

Save the residuals:  $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta}X_t$

## Step 2: Test Residuals for Stationarity

Test if  $\hat{e}_t$  is  $I(0)$  using ADF test:

$$\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$$

- $H_0: \rho = 0$  (residuals have unit root  $\Rightarrow$  no cointegration)
- $H_1: \rho < 0$  (residuals are stationary  $\Rightarrow$  cointegration)

## Important



# Engle-Granger Critical Values

## Critical Values for Cointegration Test

Number of Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

Based on MacKinnon (1991) response surface estimates,  $T = 100$

## Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on which variable is chosen as dependent
- Small sample bias in estimated cointegrating vector
- Cannot test hypotheses on the cointegrating vector

# Johansen Cointegration Test

## Advantages over Engle-Granger

- Tests for **multiple** cointegrating relationships
- Maximum likelihood estimation (more efficient)
- Can test restrictions on cointegrating vectors
- Does not require choosing a dependent variable

## Starting Point: VAR in Levels

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$$

Rewrite in **Vector Error Correction** form...

## Vector Error Correction Model

$$\Delta \mathbf{Y}_t = \mathbf{c} + \mathbf{\Pi} \mathbf{Y}_{t-1} + \sum_{j=1}^{p-1} \mathbf{\Gamma}_j \Delta \mathbf{Y}_{t-j} + \boldsymbol{\varepsilon}_t$$

where:

- $\mathbf{\Pi} = \mathbf{A}_1 + \mathbf{A}_2 + \dots + \mathbf{A}_p - \mathbf{I}$  (long-run impact matrix)
- $\mathbf{\Gamma}_j = -(\mathbf{A}_{j+1} + \dots + \mathbf{A}_p)$  (short-run dynamics)

## Key Insight: Rank of $\mathbf{\Pi}$

The **rank of  $\mathbf{\Pi}$**  determines cointegration:

- $\text{rank}(\mathbf{\Pi}) = 0$ : No cointegration (VAR in differences)
- $\text{rank}(\mathbf{\Pi}) = k$ : All variables are  $I(0)$  (VAR in levels)
- $0 < \text{rank}(\mathbf{\Pi}) = r < k$ : Cointegration with  $r$  cointegrating vectors

## Decomposition of $\Pi$

When  $\text{rank}(\Pi) = r < k$

The matrix  $\Pi$  can be decomposed as:

$$\Pi = \alpha\beta'$$

where:

- $\beta$  is  $k \times r$  matrix of **cointegrating vectors**
- $\alpha$  is  $k \times r$  matrix of **adjustment coefficients**

## Interpretation

- $\beta'Y_{t-1}$  = deviations from long-run equilibrium (error correction terms)
- $\alpha$  = speed of adjustment to equilibrium
- Each row of  $\alpha$  shows how each variable responds to disequilibrium

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta'Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

## Two Test Statistics

Based on eigenvalues  $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$  of a certain matrix:

### Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests  $H_0: \text{rank} \leq r$  vs  $H_1: \text{rank} > r$

### Maximum Eigenvalue Test:

$$\lambda_{\text{max}}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

Tests  $H_0: \text{rank} = r$  vs  $H_1: \text{rank} = r+1$

Critical values from Johansen & Juselius (1990), depend on:

- Number of variables  $k$
- Deterministic components (constant, trend)

## Sequential Testing (Trace Test)

- ① Test  $H_0: r = 0$  vs  $H_1: r > 0$ 
  - If not rejected: No cointegration. Stop.
  - If rejected: At least one cointegrating vector. Continue.
- ② Test  $H_0: r \leq 1$  vs  $H_1: r > 1$ 
  - If not rejected:  $r = 1$ . Stop.
  - If rejected: At least two cointegrating vectors. Continue.
- ③ Continue until  $H_0$  is not rejected...

## Deterministic Components

Choose specification carefully:

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- Constant in both (most common)
- Constant + trend in cointegrating relation
- Constant + trend in both

## Full VECM Specification

For  $k = 2$  variables with  $r = 1$  cointegrating relation:

$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

## Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$  = error correction term (deviation from equilibrium)
- $\alpha_1, \alpha_2$  = adjustment speeds (should have opposite signs)
- $\gamma_{ij}$  = short-run dynamics
- $\varepsilon_{it}$  = innovations

# Interpreting Adjustment Coefficients

## The $\alpha$ Coefficients

If the cointegrating relation is  $Y_1 - \beta Y_2 = 0$  (equilibrium):

- $\alpha_1 < 0$ :  $Y_1$  adjusts downward when above equilibrium
- $\alpha_2 > 0$ :  $Y_2$  adjusts upward when  $Y_1$  is above equilibrium

## Weak Exogeneity

If  $\alpha_i = 0$ , variable  $Y_i$  does **not** respond to disequilibrium.

- $Y_i$  is **weakly exogenous** for the long-run parameters
- The other variable does all the adjusting
- Can simplify estimation (single-equation approach)

Test weak exogeneity:  $H_0 : \alpha_i = 0$  using likelihood ratio test.



### When Variables are Cointegrated

	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

### Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

## Step-by-Step Procedure

- ① **Unit Root Tests:** Verify all variables are  $I(1)$ 
  - ADF, KPSS on levels and first differences
- ② **Lag Length Selection:** Choose  $p$  for VAR in levels
  - Use AIC, BIC, or sequential LR tests
- ③ **Cointegration Test:** Johansen trace/max-eigenvalue tests
  - Determine cointegrating rank  $r$
- ④ **Estimate VECM:** If  $0 < r < k$ 
  - Estimate  $\alpha$ ,  $\beta$ ,  $\Gamma_j$
- ⑤ **Diagnostics:** Check residuals for autocorrelation, normality
- ⑥ **Analysis:** IRF, FEVD, hypothesis tests

## Things to Watch Out For

- **Structural breaks:** Can cause spurious unit roots or cointegration
- **Near-unit-root processes:** Tests have low power
- **Too many lags:** Over-parameterization, loss of efficiency
- **Too few lags:** Residual autocorrelation, biased estimates
- **Wrong deterministic specification:** Affects critical values
- **Small samples:** Johansen test oversized in small samples

## Recommendation

Always check:

- Residual diagnostics (Portmanteau test, normality)
- Stability of estimated cointegrating relationship over time
- Sensitivity to lag length and deterministic specification

## Example: Short and Long Interest Rates

### Economic Theory

Expectations hypothesis of term structure:

$$R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{term premium}$$

If term premium is constant, short rate  $r_t$  and long rate  $R_t$  should be cointegrated with vector  $(1, -1)$ .

### Empirical Approach

- ① Test both rates for unit roots (expect  $I(1)$ )
- ② Test for cointegration (expect  $r = 1$ )
- ③ Estimate VECM
- ④ Test restriction  $\beta = 1$  (spread is stationary)

## Typical Findings

- Both rates are  $I(1)$
- One cointegrating relationship found
- Cointegrating vector close to  $(1, -1)$ : spread is stationary
- Short rate adjusts to long rate (not vice versa)

## VECM Equations (stylized)

$$\Delta r_t = 0.02 - 0.15(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{1t}$$

$$\Delta R_t = 0.01 - 0.02(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{2t}$$

- Short rate adjusts faster ( $\alpha_1 = -0.15$ )
- Long rate nearly weakly exogenous ( $\alpha_2 \approx 0$ )

# Key Takeaways

## Main Concepts

- **Cointegration:**  $I(1)$  variables with stationary linear combination
- **Spurious regression:** High  $R^2$  with unrelated  $I(1)$  variables
- **Error correction:** Adjustment toward long-run equilibrium
- **VECM:** VAR with error correction terms for cointegrated systems

## Testing Methods

- **Engle-Granger:** Simple, but only one cointegrating vector
- **Johansen:** Multiple vectors, more powerful, MLE-based

## Remember

Cointegration tests have low power in small samples. Economic theory should guide specification. Always check diagnostics!

# What's Next?

## Extensions and Related Topics

- **Structural VECM:** Identifying structural shocks
- **Threshold cointegration:** Nonlinear adjustment
- **Panel cointegration:** Multiple cross-sections
- **Fractional cointegration:** Long memory
- **Time-varying cointegration:** Regime changes

Questions?

## Quick Quiz

- 1 What does it mean for two  $I(1)$  variables to be cointegrated?
- 2 What is the “spurious regression” problem?
- 3 In a VECM, what do the  $\alpha$  coefficients represent?
- 4 What is the main advantage of Johansen over Engle-Granger?
- 5 If  $\alpha_i = 0$  for variable  $Y_i$ , what does this imply?



## Quiz Answers

- 1 **Cointegration:** A linear combination of the variables is  $I(0)$  (stationary). They share a common stochastic trend.
- 2 **Spurious regression:** Regressing one  $I(1)$  variable on another unrelated  $I(1)$  variable gives high  $R^2$  and significant coefficients even though there's no true relationship.
- 3  $\alpha$  **coefficients:** Speed of adjustment—how quickly each variable responds to deviations from long-run equilibrium.
- 4 **Johansen advantage:** Can test for multiple cointegrating relationships, uses MLE (more efficient), doesn't require choosing dependent variable.
- 5  $\alpha_i = 0$ : Variable  $Y_i$  is weakly exogenous—it doesn't respond to disequilibrium. Other variables do all the adjusting.