



# Time Series Analysis and Forecasting

## Chapter 4: SARIMA Models



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## Learning Objectives

By the end of this chapter, you will be able to:

1. Identify seasonal patterns in time series data
2. Apply seasonal differencing to remove seasonal unit roots
3. Build and estimate SARIMA models with seasonal components
4. Interpret seasonal ACF/PACF patterns for model identification
5. Evaluate forecasts using rolling window methods for seasonal data
6. Apply the complete methodology on real data (airline passengers)

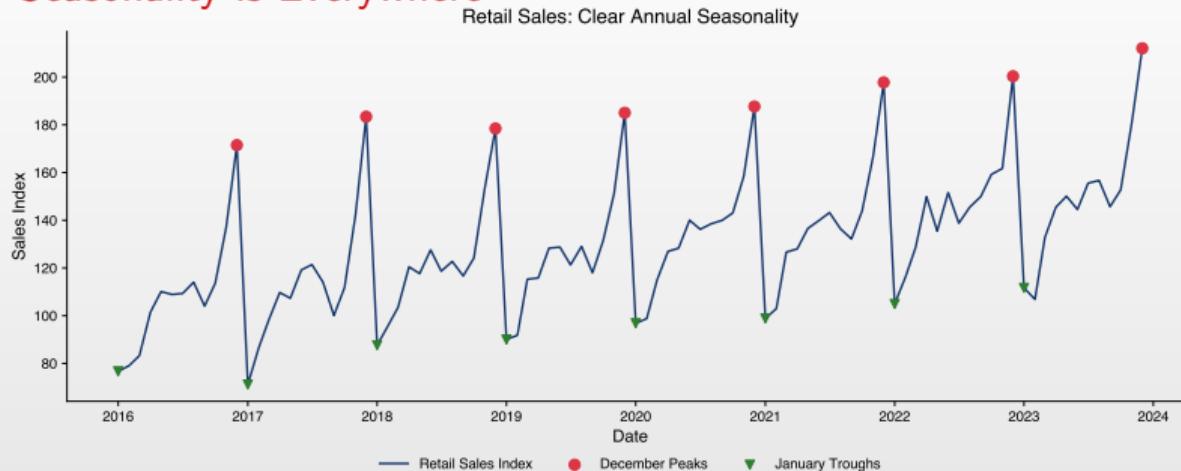


## Outline

- Motivation
- Seasonality in Time Series
- Seasonal Differencing
- The SARIMA Model
- Seasonal ACF and PACF Patterns
- Estimation and Diagnostics
- Forecasting with SARIMA
- Case Study: Airline Passengers
- Practical Aspects
- AI Use Case
- Summary
- Quiz



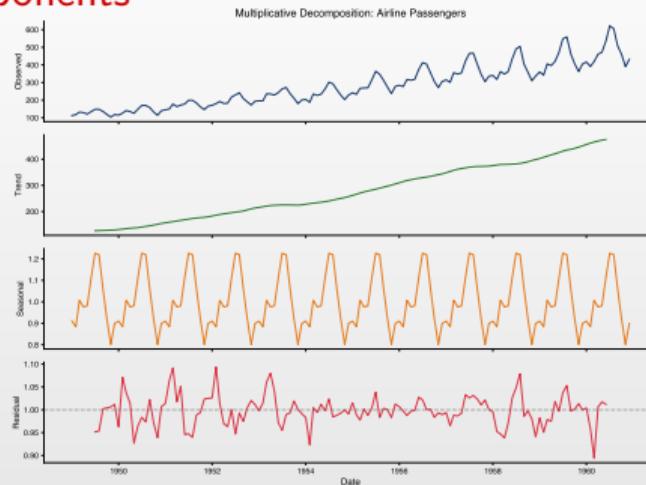
## Why SARIMA? Seasonality Is Everywhere



- Retail sales exhibit clear **annual patterns**: December peaks, January troughs
- Standard ARIMA models cannot capture these **repeating seasonal cycles**
- Ignoring seasonality leads to systematic forecast errors



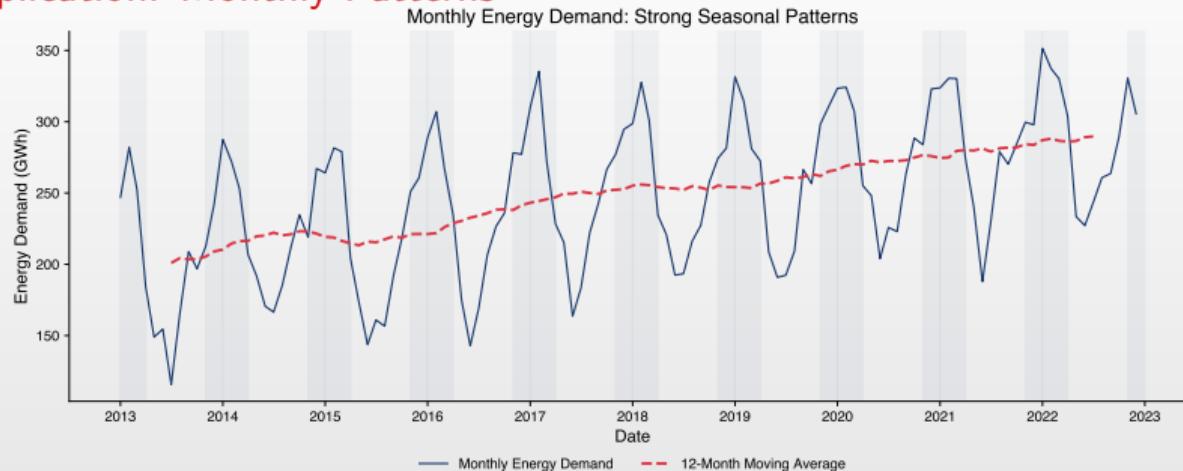
## Understanding Seasonal Components



- ☐ Seasonal time series = **Trend + Seasonal Pattern + Residuals**
- ☐ Decomposition helps visualize each component separately
- ☐ SARIMA models capture both trend dynamics and seasonal behavior



## Real-World Application: Monthly Patterns

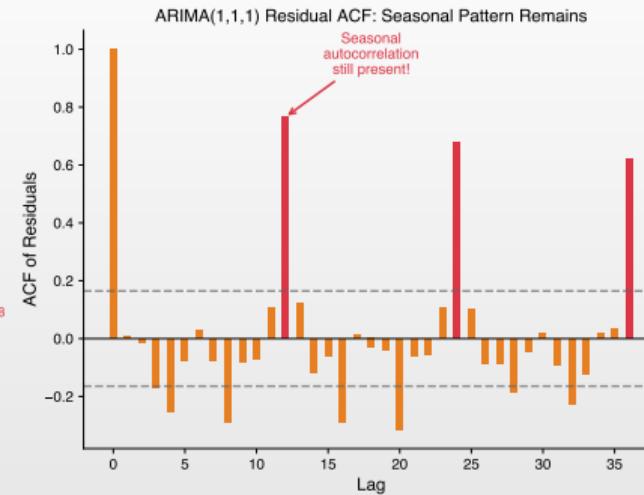
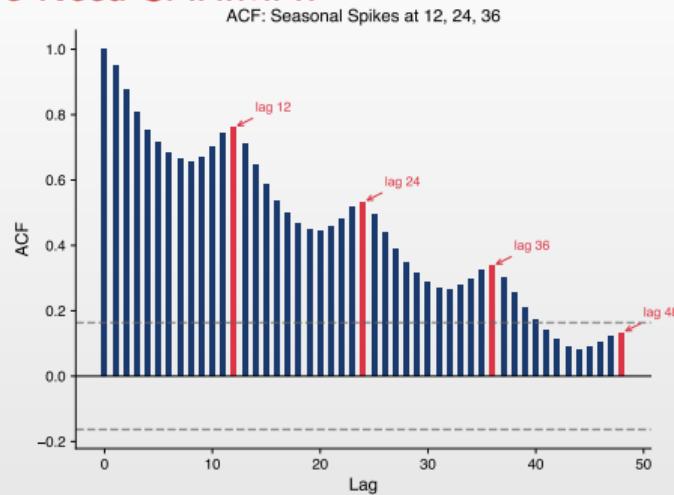


- Energy demand shows strong **monthly seasonality** (heating/cooling cycles)
- Pattern repeats predictably each year with slight variations
- Utility companies use SARIMA forecasts for capacity planning

TSA\_ch4\_motivation\_monthly



## Why Do We Need SARIMA?



- **Left:** Seasonal ACF shows spikes at lags 12, 24, 36... (annual pattern)
- **Right:** ARIMA residuals still show seasonal autocorrelation  $\Rightarrow$  model is incomplete
- **SARIMA adds seasonal AR and MA terms** to capture these patterns



## What We'll Learn Today

### Concepts

- Identifying seasonal patterns
- Seasonal differencing operator
- SARIMA( $p, d, q$ )( $P, D, Q$ ) $_s$  notation
- The famous "Airline Model"
- Model selection for seasonal data

### Skills

- Diagnose seasonality from ACF/PACF
- Determine seasonal period  $s$
- Choose  $(P, D, Q)$  seasonal orders
- Implement SARIMA in Python/R
- Forecast seasonal time series

### Key Insight

- SARIMA = ARIMA applied at **two frequencies**: non-seasonal (short-term) and seasonal (long-term)



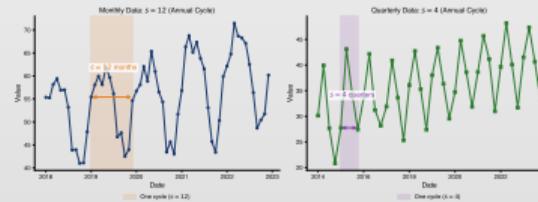
## What is Seasonality?

### Definition 1 (Seasonality)

A time series exhibits seasonality when it shows regular, periodic fluctuations that repeat over a fixed period  $s$  (the seasonal period).

### Common Seasonal Periods

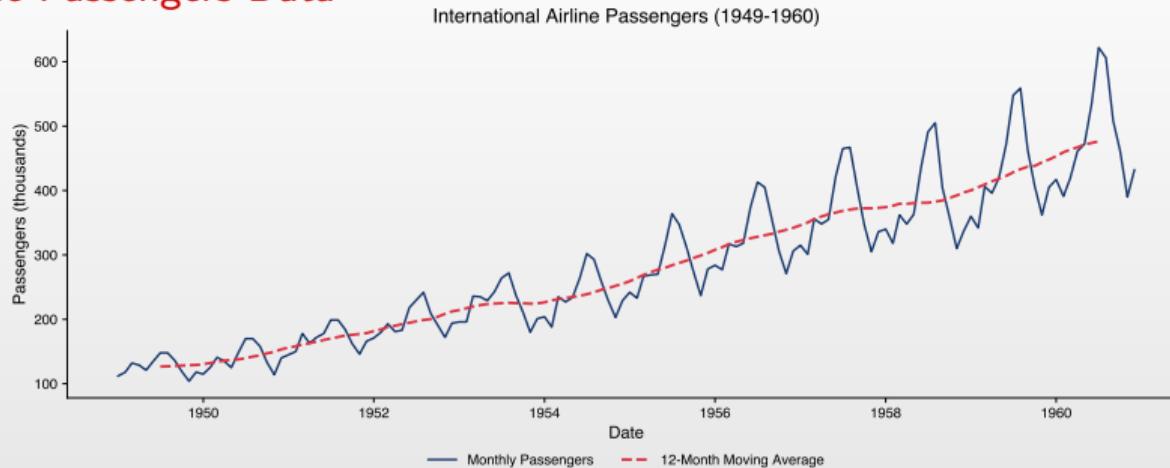
- Monthly data:  $s = 12$  (annual cycle)
- Quarterly data:  $s = 4$  (annual cycle)
- Weekly data:  $s = 52$  (annual) or  $s = 7$  (weekly pattern)
- Daily data:  $s = 7$  (weekly pattern)



Q TSA\_ch4\_seasonality



## Example: Airline Passengers Data



- Monthly international airline passengers (1949–1960)
- Clear **upward trend and growing seasonal amplitude**
- Summer peaks reflect vacation travel patterns



## Examples of Seasonal Data

### Economic Series

- Retail sales (holiday peaks)
- Tourism (summer/winter)
- Agricultural production
- Energy consumption
- Employment (hiring cycles)

### Other Domains

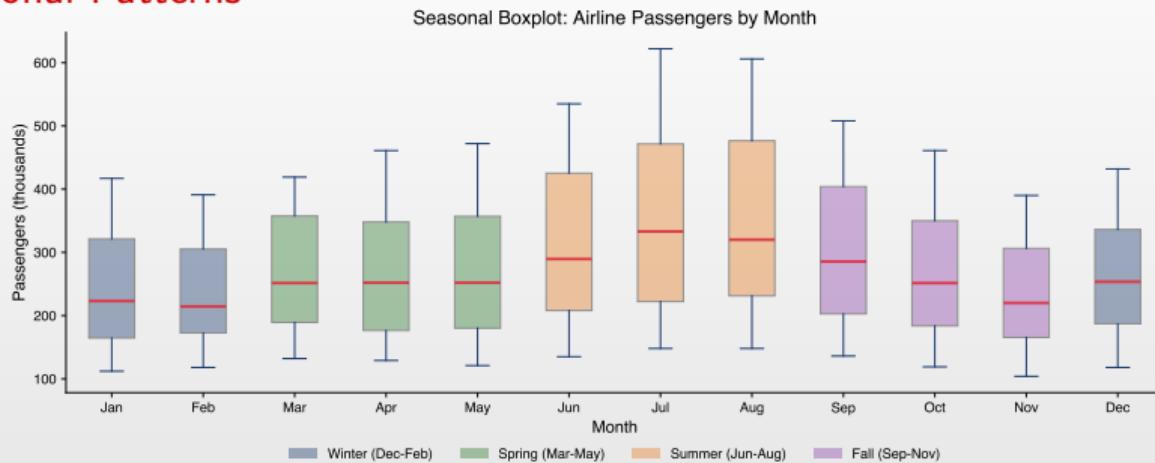
- Weather/temperature
- Website traffic
- Hospital admissions
- Transportation usage
- Electricity demand

### Why It Matters

Ignoring seasonality leads to biased forecasts and invalid inference!



## Visualizing Seasonal Patterns



- Box plot reveals consistent seasonal pattern across years
- July–August: highest passenger counts (summer travel)
- November–February: lowest counts (winter months)

 TSA\_ch4\_seasonal\_boxplot



## Deterministic vs Stochastic Seasonality

### Deterministic Seasonality

- Fixed pattern:**  $Y_t = \sum_{j=1}^s \gamma_j D_{jt} + \varepsilon_t$ 
  - $D_{jt}$  are seasonal dummies
- Pattern constant over time
- Same amplitude every year
- Removed by regression on dummies
- ACF: sharp cutoff at seasonal lags
- Example:** University enrollment peaks every September by the same amount

### Stochastic Seasonality

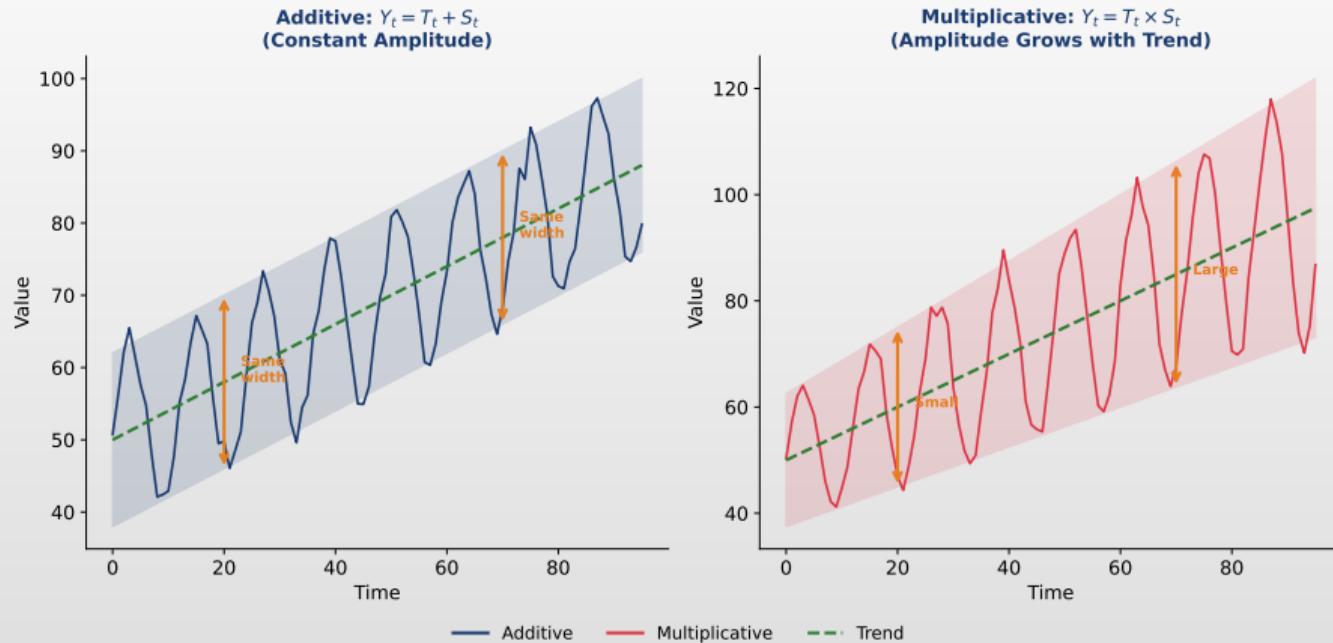
- Evolving pattern:**  $\Delta_s Y_t = Y_t - Y_{t-s}$ 
  - Exhibits dependence structure
- Pattern evolves over time
- Amplitude may grow or shrink
- Requires seasonal differencing
- ACF: slow decay at seasonal lags
- Example:** Retail sales peaks grow larger each December

### How to decide?

- Slow ACF decay at lags  $s, 2s, 3s, \dots \Rightarrow$  stochastic (use  $\Delta_s$ )
- Sharp cutoff  $\Rightarrow$  deterministic (use dummies)
- Use HEGY or Canova-Hansen tests to confirm



## Additive vs Multiplicative Seasonality



## Additive vs Multiplicative Seasonality

$$\text{Additive: } Y_t = T_t + S_t + \varepsilon_t$$

- Seasonal amplitude **constant**
- No transformation needed
- Ex: temperatures, university enrollment

$$\text{Multiplicative: } Y_t = T_t \cdot S_t \cdot \varepsilon_t$$

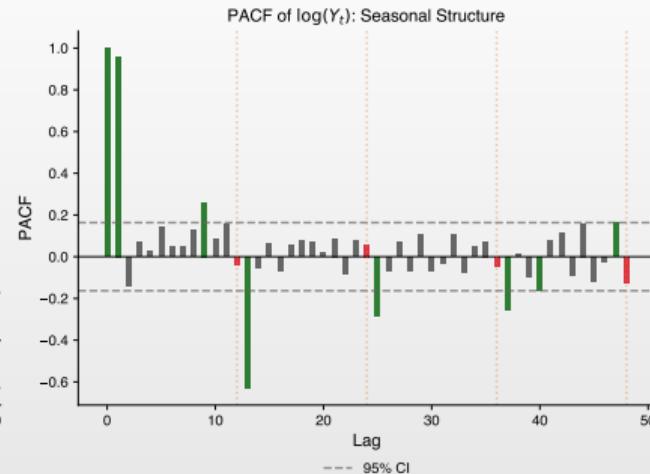
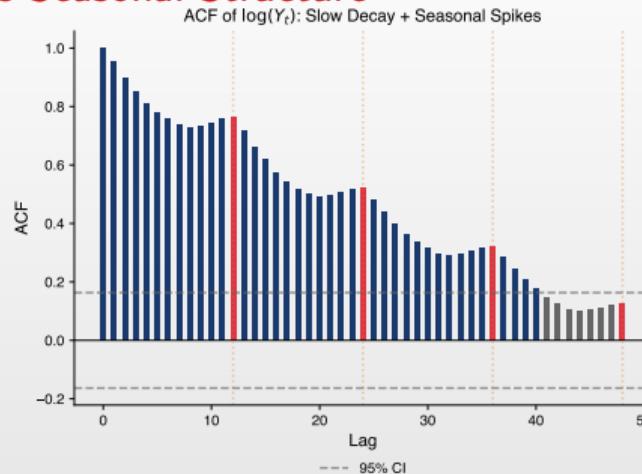
- Amplitude **grows with level**
- Requires log transform (Box-Cox)
- Ex: Airline, retail sales, GDP

### First practical decision

- Amplitude grows with the trend?  $\Rightarrow$  multiplicative  $\Rightarrow$  apply log/Box-Cox *before* differencing



## ACF Reveals Seasonal Structure



- ☐ Slow decay at all lags indicates non-stationarity (trend)
- ☐ Spikes at lags 12, 24, 36 confirm seasonal pattern ( $s = 12$ )
- ☐ ACF at seasonal lags: slow decay  $\Rightarrow$  needs seasonal differencing



## Detecting Seasonality

### Visual Methods

- Time series plot – look for repeating patterns
- Seasonal subseries plot – compare same seasons across years
- ACF plot – spikes at seasonal lags ( $s, 2s, 3s, \dots$ )

### Statistical Tests

- Seasonal unit root tests (HEGY, Canova-Hansen, OCSB<sup>a</sup>)
- F-test for seasonal dummy variables
- Kruskal-Wallis test (non-parametric)

### ACF Signature

- Strong seasonality: ACF shows significant spikes at lags  $s, 2s, 3s, \dots$

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<sup>a</sup>Osborn-Chui-Smith-Birchenhall — the default test in `auto_arima`



## F-Test for Seasonal Dummy Variables: Intuition

### What does this test do?

- Goal:** test whether mean values differ significantly across seasons
- Logic:** if the mean in January  $\neq$  February  $\neq \dots \neq$  December  $\Rightarrow$  seasonality
- Method:** compare a model WITH seasonal dummy variables vs. a model WITHOUT

### Models compared

- Restricted:**  $Y_t = \alpha + \varepsilon_t$     **Unrestricted:**  $Y_t = \alpha + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$
- where  $D_{jt} = 1$  if observation  $t$  is in season  $j$ , 0 otherwise

### Key idea

- If adding seasonal dummy variables **significantly reduces** prediction errors, then seasonality is present



## F-Test for Seasonal Dummy Variables: Formula and Example

### F-statistic formula

- **Formula:**  $F = \frac{(SSR_R - SSR_U)/(s-1)}{SSR_U/(n-s)} \sim F_{s-1, n-s}$ 
  - ▶  $SSR_R$ : sum of squared residuals from the restricted model (no dummies)
  - ▶  $SSR_U$ : sum of squared residuals from the unrestricted model (with dummies)
  - ▶  $s - 1$ : number of restrictions (monthly: 11, quarterly: 3)

### Numerical example (Monthly data, $n=120$ )

- $SSR_R = 15000, SSR_U = 8500, s = 12$
- $F = \frac{(15000 - 8500)/11}{8500/108} = \frac{590.9}{78.7} = 7.51$
- Critical value  $F_{0.05, 11, 108} \approx 1.87$ . Since  $7.51 > 1.87$ : **Reject  $H_0$**   $\Rightarrow$  Seasonality present!



## Kruskal-Wallis Test: Intuition

### What does this test do?

- Non-parametric test:** checks whether observations from different seasons come from the same distribution
- Mechanism:** ranks all observations from smallest to largest
- Check:** whether ranks are uniformly distributed across seasons
- Conclusion:** if one season consistently has higher/lower ranks  $\Rightarrow$  seasonality

### Why use it instead of the F-test?

- No normality assumption** – works with any distribution
- Robust to outliers** – extreme values do not distort results

### Limitation

- Less powerful than the F-test when data ARE normally distributed



## Kruskal-Wallis Test: Formula and Example

### Test statistic

- $H = \frac{12}{N(N+1)} \sum_{j=1}^s \frac{R_j^2}{n_j} - 3(N + 1)$  where  $N$  = total obs.,  $n_j$  = obs. in season  $j$ ,  $R_j$  = rank sum

### Example: Quarterly sales ( $n=20$ , $s=4$ )

- Data ranked 1–20. Rank sums: Q1:  $R_1 = 15$ , Q2:  $R_2 = 35$ , Q3:  $R_3 = 70$ , Q4:  $R_4 = 90$
- $H = \frac{12}{20 \times 21} \left( \frac{15^2}{5} + \frac{35^2}{5} + \frac{70^2}{5} + \frac{90^2}{5} \right) - 3(21) = 19.6$
- Critical value  $\chi^2_{0.05,3} = 7.81$ . Since  $19.6 > 7.81$ : **Reject  $H_0$  ⇒ Seasonality!**

### In Python

- **Implementation:** `scipy.stats.kruskal(q1, q2, q3, q4)`



## HEGY Test: What Problem Does It Solve?

### Key question

- Problem:** given a seasonal series, we need to determine the type of differencing
- Regular differencing** ( $1 - L$ )?  $\Rightarrow$  set  $d = 1$ ; **Seasonal differencing** ( $1 - L^s$ )?  $\Rightarrow$  set  $D = 1$
- HEGY:** tests for both types of unit roots simultaneously!

### Why not just use ADF?

- ADF:** tests only for a regular unit root at frequency zero
- Limitation:** seasonal data may have unit roots at seasonal frequencies that ADF misses!

### HEGY tests multiple frequencies

- Quarterly:** tests at  $0, \pi, \pm\pi/2$
- Monthly:** tests at  $0, \pi, \pm\pi/6, \pm\pi/3, \pm\pi/2, \pm2\pi/3, \pm5\pi/6$



## HEGY Test: Auxiliary Regression (Quarterly)

### HEGY auxiliary regression

- **Quarterly data ( $s = 4$ ):**  $\Delta_4 y_t = \pi_1 z_{1,t-1} + \pi_2 z_{2,t-1} + \pi_3 z_{3,t-2} + \pi_4 z_{4,t-2} + \sum_{j=1}^k \phi_j \Delta_4 y_{t-j} + \varepsilon_t$

### Transformed variables

- $z_{1t}: (1 + L + L^2 + L^3)y_t = y_t + y_{t-1} + y_{t-2} + y_{t-3}$
- $z_{2t}: -(1 - L + L^2 - L^3)y_t = -y_t + y_{t-1} - y_{t-2} + y_{t-3}$
- $z_{3t}: -(1 - L^2)y_t = -y_t + y_{t-2}$
- $z_{4t}: -(L - L^3)y_t = -y_{t-1} + y_{t-3}$

### Hypotheses

- $H_0: \pi_1 = 0:$  unit root at frequency 0
- $H_0: \pi_2 = 0:$  unit root at frequency  $\pi$
- $H_0: \pi_3 = \pi_4 = 0:$  unit root at frequency  $\pm\pi/2$



## HEGY Test: Decision Rules with Examples

HEGY critical values (5%, n=100, with constant)

Test	Statistic	Critical value	If NOT rejected...
$t_1 (\pi_1 = 0)$	t-stat	-2.88	Requires $d = 1$
$t_2 (\pi_2 = 0)$	t-stat	-2.88	Requires $D = 1$
$F_{34} (\pi_3 = \pi_4 = 0)$	F-stat	6.57	Requires $D = 1$

Example: Quarterly GDP

- **HEGY results:**  $t_1 = -1.52$ ,  $t_2 = -4.21$ ,  $F_{34} = 2.15$
- $t_1 = -1.52 > -2.88$ : Cannot reject  $\Rightarrow$  requires  $d = 1$
- $t_2 = -4.21 < -2.88$ : Reject  $\Rightarrow$  no unit root at  $\pi$
- $F_{34} = 2.15 < 6.57$ : Cannot reject  $\Rightarrow$  requires  $D = 1$
- **Conclusion:** Use SARIMA with  $d = 1, D = 1$



## Canova-Hansen Test: The Opposite of HEGY

HEGY vs Canova-Hansen: Different null hypotheses!

	HEGY	Canova-Hansen
$H_0$	Seasonal unit root	No seasonal unit root
$H_1$	No seasonal unit root	Seasonal unit root
Reject $H_0$	Use seasonal dummies	Use differencing ( $1 - L^s$ )
Do not reject	Use differencing ( $1 - L^s$ )	Use seasonal dummies

Why does it matter?

- HEGY: "Prove there is NO unit root" (conservative towards differencing)
- CH: "Prove there IS a unit root" (conservative towards dummies)
- Use **both** tests for robust conclusions!



## Canova-Hansen Test: Formula

### Testing procedure

- **Step 1:** Regress  $y_t$  on seasonal dummies:  $y_t = \sum_{j=1}^s \gamma_j D_{jt} + u_t$
- **Step 2:** Compute partial sums at seasonal frequency  $\lambda_i$ :
  - $S_{it}^{(c)} = \sum_{j=1}^t \hat{u}_j \cos(\lambda_i j)$ ,  $S_{it}^{(s)} = \sum_{j=1}^t \hat{u}_j \sin(\lambda_i j)$

### LM test statistic

- $LM_i = \frac{1}{T^2 \hat{\omega}_i} \left[ \sum_{t=1}^T (S_{it}^{(c)})^2 + \sum_{t=1}^T (S_{it}^{(s)})^2 \right]$
- where  $\hat{\omega}_i$  = consistent estimator of the spectral density at frequency  $\lambda_i$

### Decision

- **Rule:** reject  $H_0$  (stationarity) if  $LM >$  critical value  $\Rightarrow$  seasonal differencing is needed



## Summary: Choosing the Right Seasonality Test

Test	$H_0$	If rejected	Best for
F-test	No seasonality	Seasonality exists	Normal data
Kruskal-Wallis	No difference across seasons	Seasonality exists	Non-normal, outliers
HEGY	Unit root exists	Use dummies	Determining $d, D$
Canova-Hansen	No unit root	Use $(1 - L^s)$	Confirming stability

### Key idea

- F-test / Kruskal-Wallis: “Does seasonality exist?”
- HEGY / Canova-Hansen: “What type?” (deterministic vs stochastic)



## Box-Cox Transformation: Variance Stabilization

### Box-Cox Family of Transformations

- **Formula:** 
$$Y_t^{(\lambda)} = \begin{cases} \frac{Y_t^\lambda - 1}{\lambda} & \text{if } \lambda \neq 0 \\ \ln(Y_t) & \text{if } \lambda = 0 \end{cases}$$
- **Special cases:**  $\lambda = 1$  (no transformation),  $\lambda = 0$  (logarithm),  $\lambda = 0.5$  (square root)

### Automatic Selection of $\lambda$

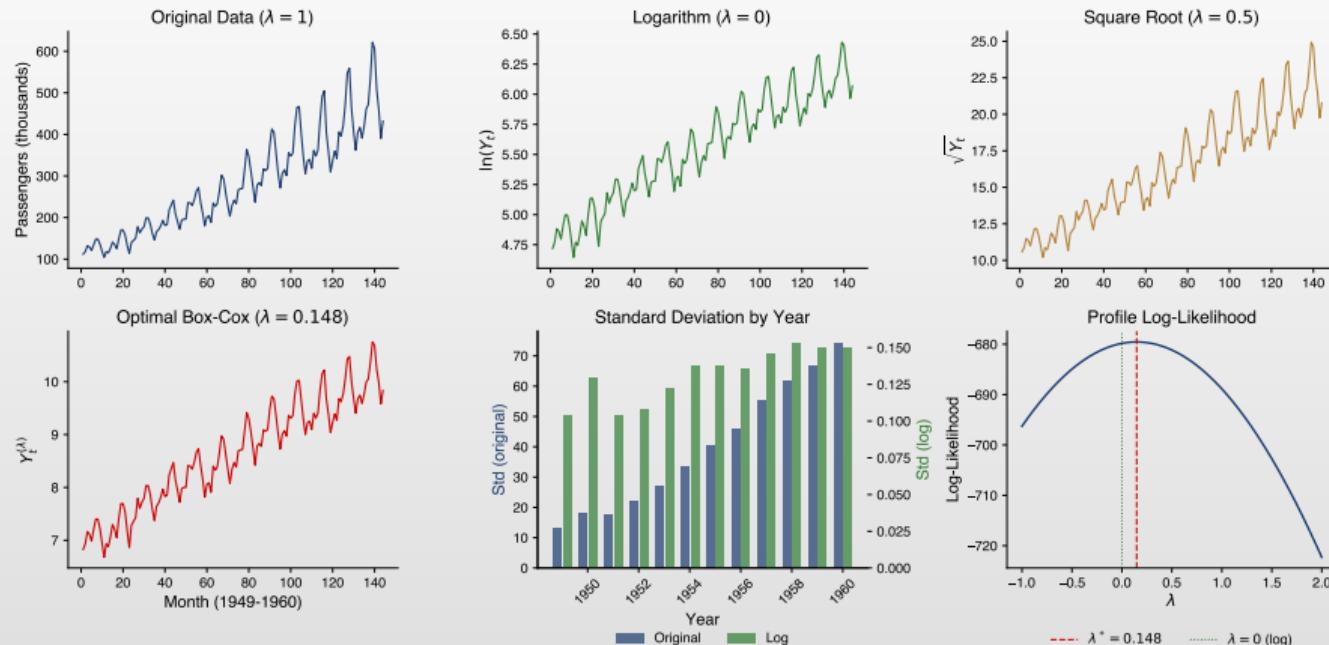
- **Profile likelihood:** maximizes the log-likelihood as a function of  $\lambda$
- **Guerrero method (1993):** minimizes the coefficient of variation of seasonal sub-series
- **Python:** `boxcox(y)` from `scipy.stats` or `BoxCox.lambda_(y)` in R

### Why not just logarithm?

- Log ( $\lambda = 0$ ) assumes variance proportional to level — not always the case
- Box-Cox chooses the optimal transformation based on data, not assumptions



## Box-Cox on the Airline Data: Complete Example



## Box-Cox on the Airline Data: Complete Example

### Result for Airline Passengers

- $\hat{\lambda} = 0.148 \approx 0 \Rightarrow \log$  is nearly optimal
- Standard deviation per year: from increasing (original) to stable (log)

### Bias Correction in Back-Transformation

- On log scale:  $\hat{y}_{T+h}$  is the **median**, not the mean
- Correction:  $\hat{Y}_{T+h} = \exp\left(\hat{y}_{T+h} + \frac{\sigma_h^2}{2}\right)$
- Without correction: systematically under-estimated forecasts!

## STL Decomposition: Modern Alternatives

### STL: Seasonal-Trend Decomposition using Loess (Cleveland et al., 1990)

- Advantages:** time-varying seasonality, robust to outliers, any period  $s$
- Algorithm:** iterative locally weighted regression (loess)

### Key Parameters

- Seasonal window** (`seasonal`): controls how quickly seasonality changes
- Trend window** (`trend`): smoothing of the trend component
- Robustness** (`robust=True`): reduces influence of outliers

### Practical Usage

- STL for exploration and preprocessing; SARIMA for modeling and forecasting
- Python: `STL(y, period=12).fit()` from `statsmodels`



## The Seasonal Difference Operator

### Definition 2 (Seasonal Difference)

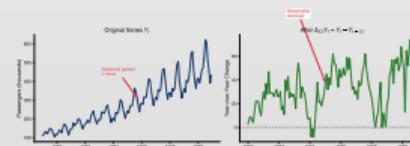
The seasonal difference operator  $\Delta_s$  is defined as:

$$\Delta_s Y_t = (1 - L^s)Y_t = Y_t - Y_{t-s}$$

where  $L^s Y_t = Y_{t-s}$  is the seasonal lag operator.

### Examples

- Monthly data ( $s = 12$ ):  $\Delta_{12} Y_t = Y_t - Y_{t-12}$   
Compares each month to the same month last year
- Quarterly data ( $s = 4$ ):  $\Delta_4 Y_t = Y_t - Y_{t-4}$   
Compares each quarter to the same quarter last year



## Proof: Seasonal Differencing Removes Deterministic Seasonality

**Claim:** If  $Y_t = \mu_t + \varepsilon_t$  where  $\mu_t = \mu_{t-s}$  (periodic mean), then  $\Delta_s Y_t$  removes the seasonal mean.

**Proof:** Let  $Y_t = \mu_t + \varepsilon_t$  where  $\mu_t$  has period  $s$ . Apply seasonal difference:

$$\begin{aligned}\Delta_s Y_t &= Y_t - Y_{t-s} = (\mu_t + \varepsilon_t) - (\mu_{t-s} + \varepsilon_{t-s}) \\ &= \mu_t - \mu_{t-s} + \varepsilon_t - \varepsilon_{t-s} \\ &= 0 + \varepsilon_t - \varepsilon_{t-s} \quad (\text{since } \mu_t = \mu_{t-s})\end{aligned}$$

**Properties of  $\Delta_s Y_t = \varepsilon_t - \varepsilon_{t-s}$ :**

- $\mathbb{E}[\Delta_s Y_t] = 0$  (constant mean)
- $\text{Var}(\Delta_s Y_t) = 2\sigma^2$  (constant variance)
- Autocovariance:  $\gamma(s) = -\sigma^2$ ,  $\gamma(k) = 0$  for  $k \neq 0, s$

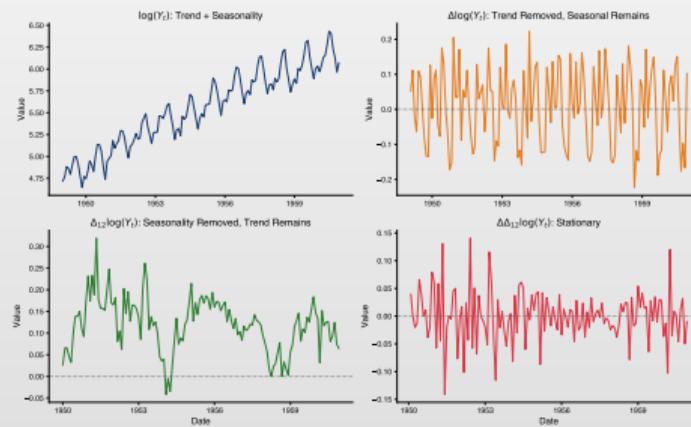
### Result

Seasonal differencing transforms periodic seasonal pattern into MA(1) at seasonal lag.



## Effect of Differencing Operations

- Regular differencing removes trend but seasonal pattern remains
- Seasonal differencing removes seasonality but trend pattern remains
- Both differences needed to achieve stationarity**



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## Combining Regular and Seasonal Differencing

### Full Differencing

For series with both trend and seasonality:

$$\Delta\Delta_s Y_t = (1 - L)(1 - L^s)Y_t$$

### Expansion

$$(1 - L)(1 - L^s)Y_t = Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}. \text{ For monthly: } \Delta\Delta_{12} Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$$

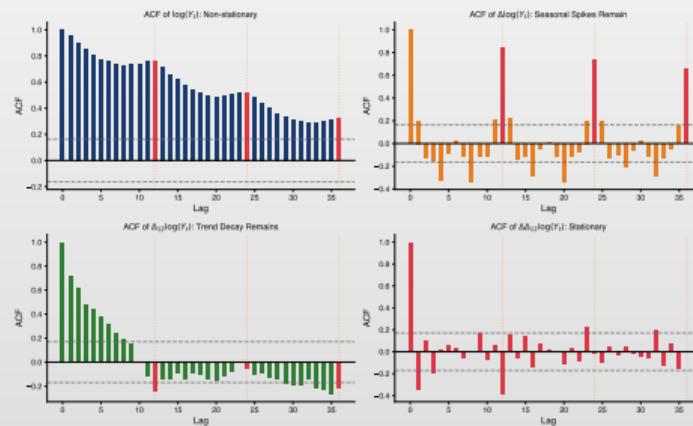
### Order of Differencing

$d$ : regular differences (trend removal);  $D$ : seasonal differences (seasonal trend removal)



## ACF Before and After Differencing

- Original ACF: slow decay indicates non-stationarity
- After  $\Delta$ : seasonal spikes remain at lags 12, 24, 36
- After  $\Delta_{12}$ : trend decay remains at early lags
- After  $\Delta\Delta_{12}$ : ACF cuts off  $\Rightarrow$  stationary



## Seasonal Integration

### Definition 3 (Seasonally Integrated Process)

A series  $Y_t$  is **seasonally integrated** of order  $(d, D)_s$ , written  $Y_t \sim I(d, D)_s$ , if:

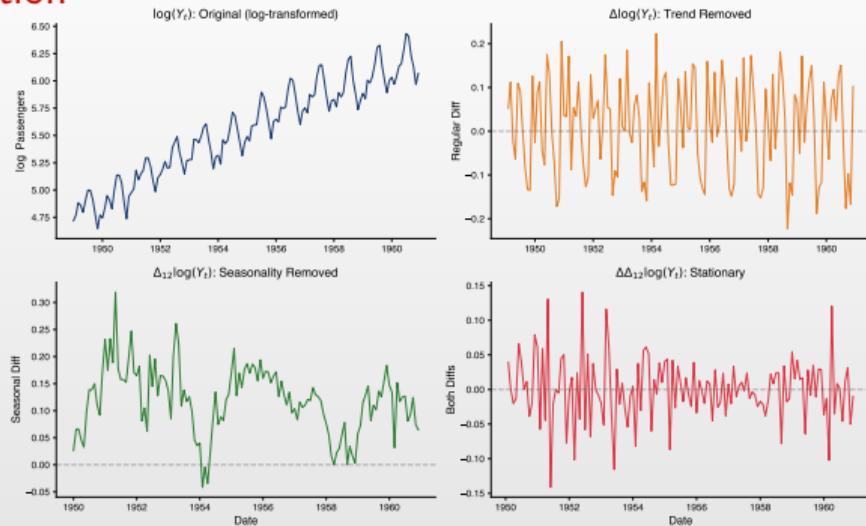
$$(1 - L)^d(1 - L^s)^D Y_t$$

is stationary.

### Common Cases

- $I(1, 0)_{12}$ : Regular unit root only (monthly)
- $I(0, 1)_{12}$ : Seasonal unit root only
- $I(1, 1)_{12}$ :
  - ▶ Both regular and seasonal unit roots

## SARIMA: Visual Illustration



- Original  $\Rightarrow$  regular difference (removes trend)  $\Rightarrow$  seasonal difference (removes seasonality)
- Apply minimum differencing needed to achieve stationarity



## SARIMA Model Definition

Definition 4 (SARIMA( $p, d, q$ )  $\times$  ( $P, D, Q$ ) $_s$ )

The **Seasonal ARIMA** model is:

$$\phi(L)\Phi(L^s)(1 - L)^d(1 - L^s)^D Y_t = c + \theta(L)\Theta(L^s)\varepsilon_t$$

### Components

- $\phi(L) = 1 - \phi_1L - \dots - \phi_pL^p$ : Non-seasonal AR
- $\Phi(L^s) = 1 - \Phi_1L^s - \dots - \Phi_PL^{Ps}$ : Seasonal AR
- $\theta(L) = 1 + \theta_1L + \dots + \theta_qL^q$ : Non-seasonal MA
- $\Theta(L^s) = 1 + \Theta_1L^s + \dots + \Theta_QL^{Qs}$ : Seasonal MA
- $(1 - L)^d$ :
  - ▶ Regular differencing;  $(1 - L^s)^D$ : Seasonal differencing



## Proof: Multiplicative Seasonal Structure

Why multiplicative? Consider  $\text{SARIMA}(1, 0, 0) \times (1, 0, 0)_s$ :

$$(1 - \phi L)(1 - \Phi L^s) Y_t = \varepsilon_t$$

Expand:  $(1 - \phi L)(1 - \Phi L^s) Y_t = Y_t - \phi Y_{t-1} - \Phi Y_{t-s} + \phi \Phi Y_{t-s-1}$

### Interpretation (Monthly, $s = 12$ )

$Y_t$  depends on:  $Y_{t-1}$  (last month),  $Y_{t-12}$  (same month last year),  $Y_{t-13}$  (interaction).

Parsimony: Multiplicative form uses 2 parameters ( $\phi, \Phi$ ); additive would need 3+.



## SARIMA Notation

### Full Specification

SARIMA( $p, d, q$ )  $\times$  ( $P, D, Q$ ) $_s$  has 7 parameters to specify:

Parameter	Meaning
$p$	Non-seasonal AR order
$d$	Non-seasonal differencing order
$q$	Non-seasonal MA order
$P$	Seasonal AR order
$D$	Seasonal differencing order
$Q$	Seasonal MA order
$s$	Seasonal period

### Example

SARIMA(1, 1, 1)  $\times$  (1, 1, 1) $_{12}$ : Monthly data with AR(1), MA(1), seasonal AR(1), seasonal MA(1), and both regular and seasonal differencing.



## Common SARIMA Models

Airline Model:  $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_s$

$$(1 - L)(1 - L^s)Y_t = (1 + \theta L)(1 + \Theta L^s)\varepsilon_t \text{ -- Classic model (Box & Jenkins, 1970)}$$

$\text{SARIMA}(1, 0, 0) \times (1, 0, 0)_s$

$$(1 - \phi L)(1 - \Phi L^s)Y_t = \varepsilon_t \text{ -- Pure seasonal and non-seasonal AR}$$

$\text{SARIMA}(0, 1, 1) \times (0, 1, 0)_s$

$$(1 - L)(1 - L^s)Y_t = (1 + \theta L)\varepsilon_t \text{ -- Random walk + seasonal diff + MA(1)}$$



## ACF/PACF for Seasonal Models

### Key Insight

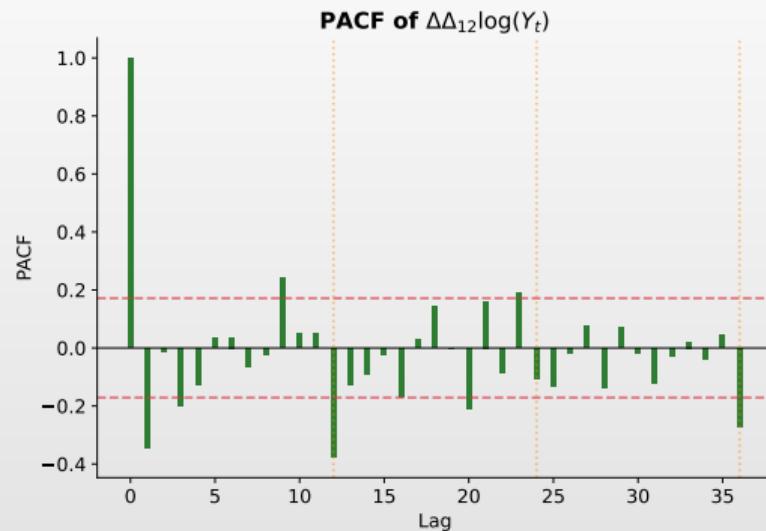
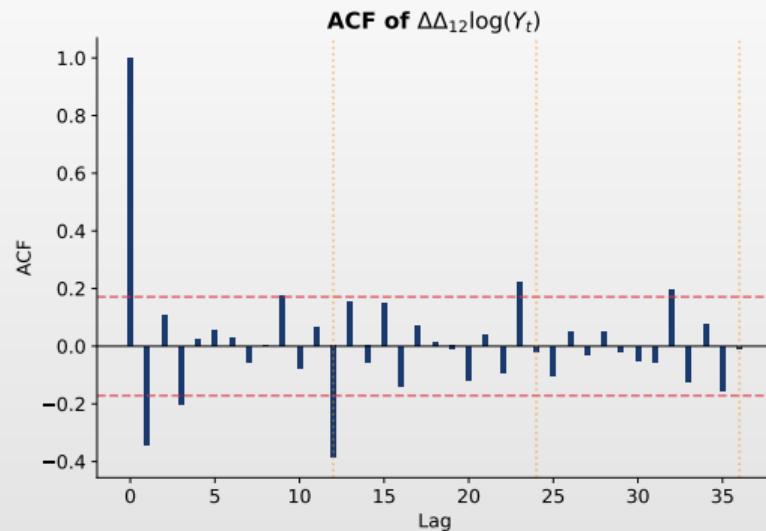
Seasonal models show patterns at both:

- Non-seasonal lags:  $1, 2, 3, \dots$
- Seasonal lags:  $s, 2s, 3s, \dots$

Model	ACF	PACF
SAR( $P$ )	Decays at $s, 2s, \dots$	Cuts off after $Ps$
SMA( $Q$ )	Cuts off after $Qs$	Decays at $s, 2s, \dots$
SARMA	Decays at seasonal lags	Decays at seasonal lags



## Example: Airline Model ACF/PACF



## Example: Airline Model ACF/PACF

ACF:  $\Delta\Delta_{12} \log(Y_t)$

- Spike at lag 1  $\leftarrow$  MA(1),  $\theta$
- Spike at lag 12  $\leftarrow$  SMA(1),  $\Theta$
- Rest  $\approx$  zero

PACF: exponential decay

- Decays at lags 1, 2, 3, ...
- Decays at lags 12, 24, 36
- $\Rightarrow$  MA, not AR

- 
- Conclusion: ACF cuts off  $\Rightarrow$  MA; PACF decays  $\Rightarrow$  not AR. Model:  $(0, 1, 1) \times (0, 1, 1)_{12}$



## Model Identification Guidelines

### Step-by-Step Process

1. Examine ACF for slow decay at seasonal lags  $\Rightarrow$  seasonal differencing
2. After differencing, check ACF/PACF patterns
3. Non-seasonal behavior at lags  $1, 2, \dots, s - 1$
4. Seasonal behavior at lags  $s, 2s, 3s, \dots$

### Practical Tips

- Start with  $d \leq 1$  and  $D \leq 1$
- Usually  $P, Q \leq 2$  is sufficient
- Use information criteria (AIC, BIC) for final selection
- Auto-SARIMA algorithms can help



## Estimation Methods

### Maximum Likelihood Estimation

Standard approach for SARIMA:

- Conditional MLE (conditional on initial values)
- Exact MLE (via Kalman filter)

### Computational Considerations

- More parameters than ARIMA  $\Rightarrow$  more data needed
- Seasonal parameters estimated from lags  $s, 2s, \dots$
- Need sufficient seasonal cycles (at least 3-4 years of monthly data)



## Exact Likelihood: Prediction Error Decomposition

### Why the Kalman Filter?

- **SARIMA:** has the structure of a state-space model
- **Kalman filter:** recursively computes prediction errors  $v_t$  and their variances  $f_t$ , without conditioning on initial values

### Exact Log-Likelihood (Prediction Error Decomposition)

- **Formula:**  $\ell(\theta) = -\frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \left[ \ln(f_t) + \frac{v_t^2}{f_t} \right]$
- $v_t: Y_t - \hat{Y}_{t|t-1}$  (innovation);  $f_t: \text{Var}(v_t)$  (innovation variance)

### Advantages over Conditional MLE

- Does not require choosing initial values
- Each term  $\ln(f_t)$  weights observations differently (variable variance at start)
- Essential for short series where initial values matter
- Implemented by default in `statsmodels.tsa.SARIMAX()` with `method='mle'`



## Stationarity and Invertibility

### Stationarity Conditions

Both non-seasonal and seasonal AR polynomials must have roots outside the unit circle:

- $\phi(z) = 0 \Rightarrow |z| > 1$
- $\Phi(z^s) = 0 \Rightarrow |z| > 1$

### Invertibility Conditions

Both non-seasonal and seasonal MA polynomials must have roots outside the unit circle:

- $\theta(z) = 0 \Rightarrow |z| > 1$
- $\Theta(z^s) = 0 \Rightarrow |z| > 1$



## Diagnostic Checking

### Residual Analysis

After fitting SARIMA, check that residuals are white noise:

1. Plot residuals over time (no patterns)
2. ACF of residuals (no significant spikes)
3. Ljung-Box test at multiple lags including seasonal
4. Normality tests (Q-Q plot, Jarque-Bera)

### Important

Check ACF at **both** non-seasonal and seasonal lags!

Significant ACF at lag 12 suggests inadequate seasonal modeling.



## Model Selection Criteria

### Information Criteria

Compare competing SARIMA models using:

- $AIC = -2 \ln(L) + 2k$
- $BIC = -2 \ln(L) + k \ln(n)$
- $AICc = AIC + \frac{2k(k+1)}{n-k-1}$  (corrected for small samples)

where  $k = p + q + P + Q + 1$  (plus 1 for variance).

### Auto-SARIMA

Python's `pmdarima.auto_arima()` with `seasonal=True` automatically searches for optimal  $(p, d, q) \times (P, D, Q)_s$ .



## Hyndman-Khandakar Algorithm (auto\_arima)

How does automatic selection work? (Hyndman & Khandakar, 2008)

1.  $d$ : successive KPSS tests ( $d = 0, 1, 2$ );  $D$ : OCSB or Canova-Hansen test ( $D = 0, 1$ )
2. **Stepwise search**: starts from initial model, explores neighboring models
3. **Criterion**: AICc (correct for small samples)

### Search Strategy

- Initial model:** SARIMA(2,  $d$ , 2)(1,  $D$ , 1)<sub>s</sub> or SARIMA(0,  $d$ , 0)(0,  $D$ , 0)<sub>s</sub>
- Variations tested:**  $\pm 1$  for each order ( $p, q, P, Q$ ); stops when no neighbor improves AICc
- Complexity:**  $O(20-30)$  models evaluated (vs.  $O(k^4)$  for grid search)

Python: `pm.auto_arima(y, seasonal=True, m=12, stepwise=True, trace=True)`

- Set `stepwise=False` for exhaustive search (slower, sometimes better)



## Point Forecasts

### Forecast Computation

SARIMA forecasts are computed recursively:

- Replace future  $\varepsilon_{T+h}$  with 0
- Replace future  $Y_{T+h}$  with forecasts  $\hat{Y}_{T+h|T}$
- Use known past values  $Y_T, Y_{T-1}, \dots$

### Seasonal Pattern in Forecasts

SARIMA forecasts naturally capture seasonality:

- Short-term: influenced by recent values
- Long-term: revert to seasonal pattern



## Forecast Intervals

### Uncertainty Quantification

$(1 - \alpha)\%$  prediction interval:

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$$

Variance computed from MA( $\infty$ ) representation.

### Key Properties

- Intervals widen with forecast horizon
- For  $I(1, 1)_s$  series: intervals grow without bound
- Seasonal pattern visible in point forecasts
- Uncertainty captures both trend and seasonal variation



## Long-Horizon Forecasts

### Behavior as $h \rightarrow \infty$

- Point forecasts converge to deterministic seasonal pattern
- If drift present: linear trend + seasonal pattern
- Forecast intervals continue to widen

### Practical Implication

- Short-term: SARIMA captures both short-term dynamics and season
- Medium-term: Good seasonal forecasts, growing uncertainty
- Long-term: Mainly reflects seasonal pattern, wide intervals



## The Seasonal Naive Benchmark

### Definition: Seasonal Naive Forecast

- Formula:**  $\hat{Y}_{T+h} = Y_{T+h-s}$  (last observed season)
- Monthly example:** Forecast for March 2025 = value from March 2024
- Interpretation:** "The simplest model that respects seasonality"

### Why is it essential?

- Any SARIMA model **must** outperform the seasonal naive benchmark
- If it doesn't  $\Rightarrow$  the model complexity is not justified
- Surprisingly effective for many series with stable seasonality

### Golden Rule

- Always** report SARIMA performance relative to seasonal naive
- This is the **first thing** a reviewer or manager checks



## The MASE Metric: Proper Evaluation for Seasonal Series

MASE — Mean Absolute Scaled Error (Hyndman & Koehler, 2006)

- **Formula:** 
$$\text{MASE} = \frac{\frac{1}{H} \sum_{h=1}^H |e_{T+h}|}{\frac{1}{T-s} \sum_{t=s+1}^T |Y_t - Y_{t-s}|}$$
- **Numerator:** mean absolute error of the model
- **Denominator:** mean absolute error of seasonal naive (on training data)

### Interpretation

- $\text{MASE} < 1$ : Model is **better** than seasonal naive
- $\text{MASE} = 1$ : Model is **equivalent** to seasonal naive
- $\text{MASE} > 1$ : Model is **worse** — abandon it!

### Why MASE and not MAPE?

- MAPE: undefined for  $Y_t = 0$ ; asymmetric; scale-dependent
- MASE: works with any data; symmetric; comparable across different series



## Forecast Evaluation: Rolling Forecast Origin

### Cross-Validation for Seasonal Time Series

- Principle:** re-estimate model  $\rightarrow$  forecast  $h$  steps  $\rightarrow$  advance 1 step  $\rightarrow$  repeat
- Fixed window:** training on last  $w$  observations (constant size)
- Expanding window:** training from beginning to  $T + i$  (grows)

### Step-by-step procedure

1. Train SARIMA on  $Y_1, \dots, Y_T$ ; forecast  $\hat{Y}_{T+1}, \dots, \hat{Y}_{T+h}$
2. Train SARIMA on  $Y_1, \dots, Y_{T+1}$ ; forecast  $\hat{Y}_{T+2}, \dots, \hat{Y}_{T+h+1}$
3. ... repeat  $N$  times; compute RMSE, MAE, MASE on all  $N$  forecasts

### Important

- Minimum  $N \geq 2s$  origins (2 complete seasonal cycles) for reliable results
- Never “look ahead” — test data is strictly after training data



## SARIMA vs Holt-Winters/ETS: When to Use Which?

### Comparison

Criterion	SARIMA	ETS / Holt-Winters
Approach	Box-Jenkins (ACF/PACF)	Exponential smoothing
Seasonality	Stochastic (differencing)	Additive or multiplicative
Interpretation	AR/MA coefficients	Smoothing weights $\alpha, \beta, \gamma$
Flexibility	Very flexible (7 params.)	Less flexible
Automation	auto_arima	ets() / ExponentialSmoothing

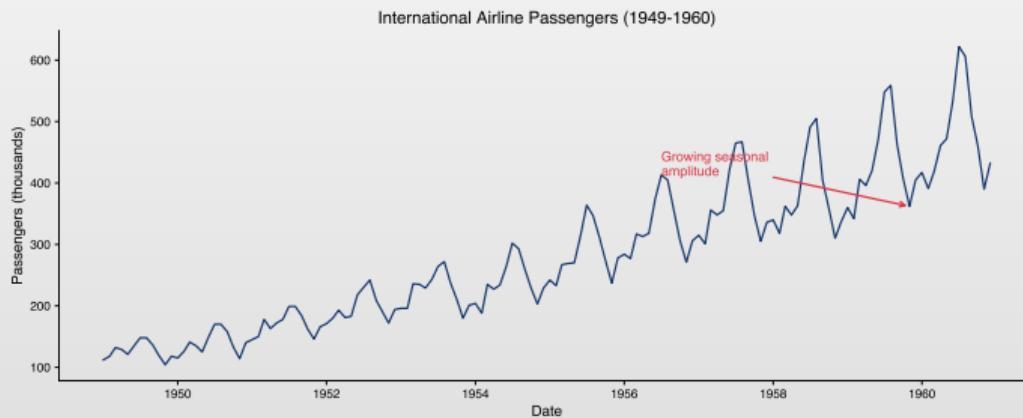
### Practical Selection Guide

- SARIMA preferred:** series with complex autocorrelation, stochastic seasonality, ARMA components
- ETS preferred:** short series, stable seasonality, quick forecasts without diagnostics
- Best:** compare both on out-of-sample data and choose the winner



## Case Study: Airline Passengers Data

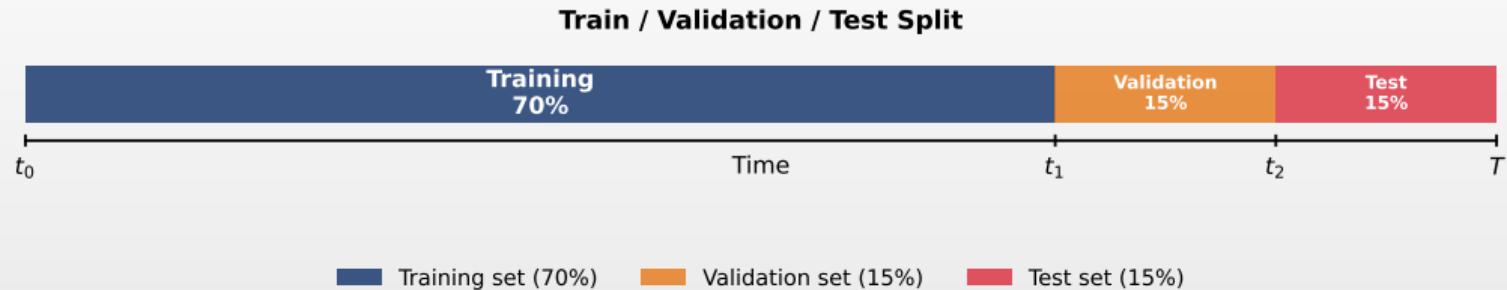
- Classic Box-Jenkins dataset: monthly airline passengers (1949-1960)
- Clear upward trend and increasing seasonal amplitude
- Multiplicative seasonality suggests log transformation



 TSA\_ch4\_case\_raw\_data



## Data Splitting Strategy



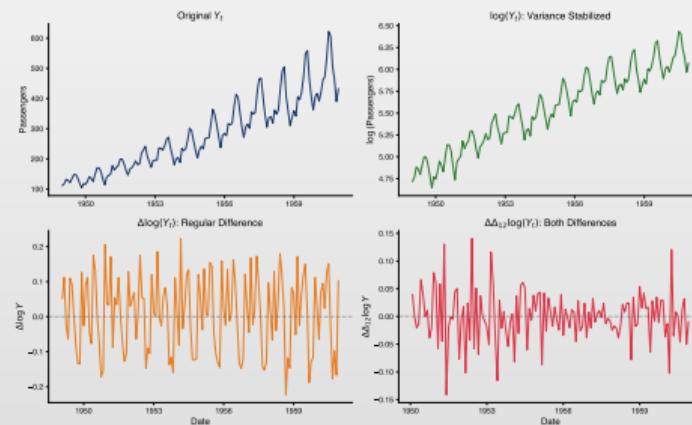
## Data ~~Splitting Strategy~~

- Training set (70%)** — Fit model parameters
  - ▶ Estimate SARIMA coefficients ( $\phi, \theta, \Phi, \Theta$ )
  - ▶ Largest portion ensures reliable parameter estimates
- Validation set (15%)** — Select best model
  - ▶ Compare candidate models (different orders)
  - ▶ Choose model with lowest validation error
- Test set (15%)** — Final evaluation
  - ▶ Unbiased out-of-sample performance; never used during development



## Step 1: Transformations

- Log transform stabilizes variance (multiplicative → additive)
- First difference removes trend; seasonal difference removes seasonality
- Double-differenced series appears stationary

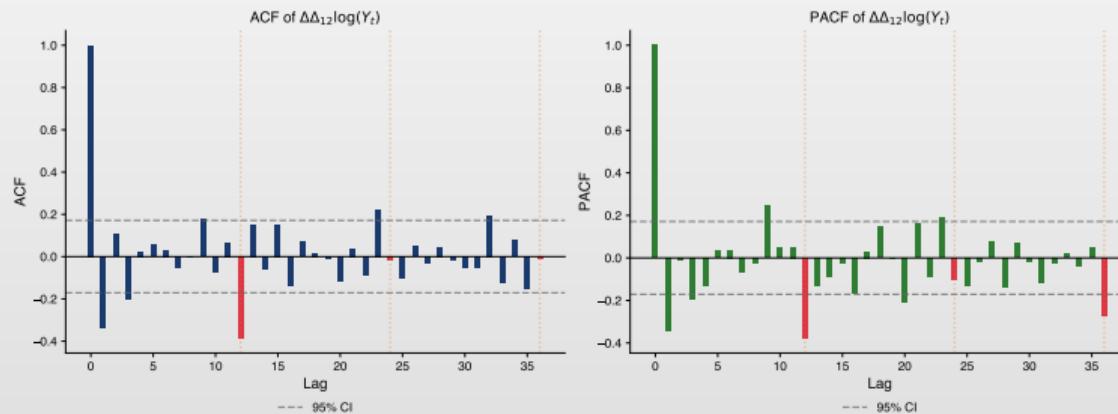


 TSA\_ch4\_case\_transformations



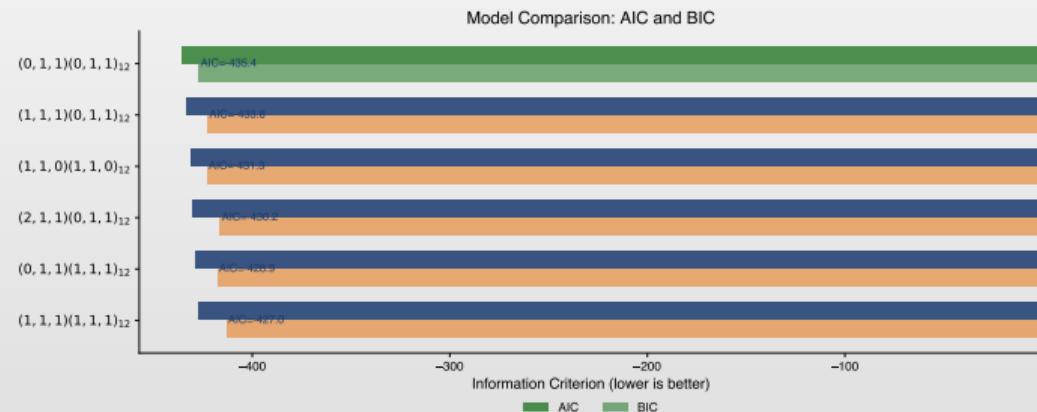
## Step 2: ACF/PACF Analysis

- ACF: Significant spike at lag 1 and lag 12  $\Rightarrow$  MA(1), SMA(1)
- PACF: Exponential decay pattern confirms MA structure
- Suggests SARIMA(0, 1, 1)  $\times$  (0, 1, 1)<sub>12</sub> (airline model)



## Step 3: Model Comparison

- Compare candidate SARIMA models using AIC criterion
- SARIMA(0, 1, 1) × (0, 1, 1)<sub>12</sub> provides best fit (lowest AIC)
- This is the famous “airline model” identified by Box & Jenkins

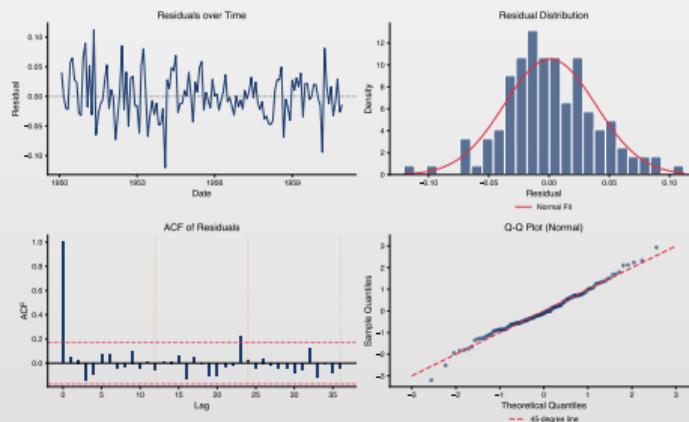


Q TSA\_ch4\_case\_model\_comparison



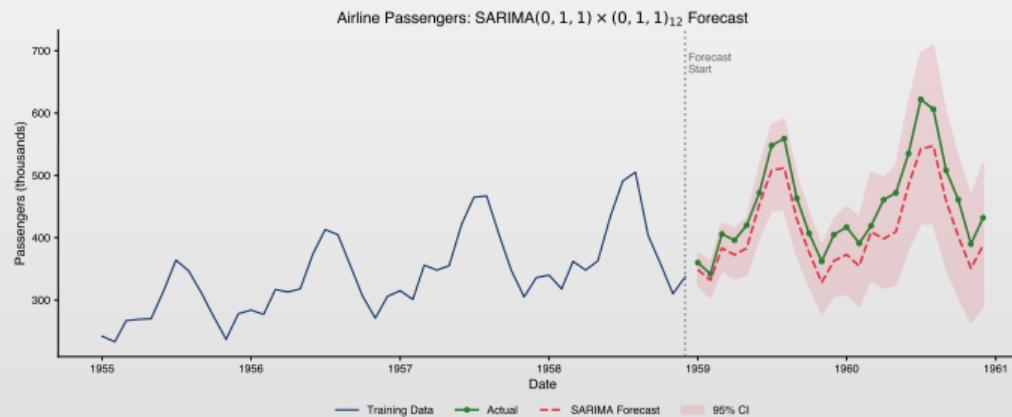
## Step 4: Residual Diagnostics

- Residuals appear random with no remaining autocorrelation
- Q-Q plot shows approximate normality
- Model adequately captures both trend and seasonal structure



## Step 5: Forecasting

- 24-month forecast with 95% confidence interval
- Model captures seasonal pattern and upward trend
- Prediction intervals widen appropriately with forecast horizon



Q TSA\_ch4\_case\_forecast



## Practical Pitfalls in SARIMA Modeling

### 1. Over-differencing

- Symptom:** ACF at lag 1  $\approx -0.5$  (regular) or at lag  $s \approx -0.5$  (seasonal)
- Cause:** applying  $(1 - L)$  or  $(1 - L^s)$  too many times
- Solution:** reduce  $d$  or  $D$  by 1 and re-examine ACF/PACF

### 2. Insufficient Data

- Minimum:** 3–4 complete seasonal cycles (36–48 monthly obs.); **recommended:** 5+ cycles
- Seasonal parameters  $\Phi, \Theta$  are estimated from lags  $s, 2s, 3s, \dots$

### 3. Other Common Pitfalls

- Root cancellation:**  $\phi \approx \theta$  suggests over-parameterization
- Parameters at invertibility boundary:**  $|\theta| \approx 1$  or  $|\Theta| \approx 1$  indicates problems
- Forgetting inverse transformation:** forecasts on log scale must be back-transformed!



## X-13ARIMA-SEATS: Official Seasonal Adjustment

### What is seasonal adjustment?

- Goal:** remove the seasonal component to reveal the true trend
- Users:** Eurostat, US Census Bureau, central banks, national statistical offices
- Example:** "GDP grew 0.3% compared to previous quarter" (seasonally adjusted data)

### X-13ARIMA-SEATS (US Census Bureau)

- Step 1:** Identify and estimate a regARIMA model (SARIMA + calendar effects)
- Step 2:** Extract the seasonal component via SEATS or X-11 filters
- Step 3:**  $Y_t^{\text{adjusted}} = Y_t - \hat{S}_t$  (additive) or  $Y_t^{\text{adjusted}} = Y_t / \hat{S}_t$  (multiplicative)

### Why does it matter for economists?

- Published macroeconomic data is almost always seasonally adjusted
- Misinterpreting unadjusted data can lead to erroneous conclusions



## AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"I have the AirPassengers dataset from statsmodels (monthly data, international airline passengers, 1949–1960, 144 obs.). Identify seasonality, apply Box-Cox transform if needed, estimate a SARIMA model, and forecast 12 months. Give me complete Python code with plots."

**Exercise:**

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it check seasonality with ACF at lags  $s, 2s, 3s$ ? Does it use STL decomposition?
3. Does it apply Box-Cox *before* differencing? Does it justify the choice of  $\lambda$ ?
4. How does it choose orders  $(p, d, q) \times (P, D, Q)_s$ ? Only auto\_arima or also ACF/PACF?
5. Does it evaluate with MASE relative to seasonal naïve? Does it use rolling forecast?

**Warning:** AI-generated code may run without errors and look professional. *That does not mean it is correct.*



## Summary

### What We Learned in This Chapter

- Seasonality in time series
  - ▶ Repetitive patterns at regular intervals; additive vs multiplicative
- Seasonal differencing and Box-Cox transformation
  - ▶  $(1 - L^s)$  removes stochastic seasonality; Box-Cox stabilizes variance
- SARIMA( $p, d, q) \times (P, D, Q)_s$  models
  - ▶ Extend ARIMA with seasonal components; automatic selection via `auto_arima`
- Forecasting and evaluation
  - ▶ Benchmark: MASE relative to seasonal naive; rolling forecast out-of-sample

### Key Idea

- **Parsimony principle:** The Airline Model  $(0, 1, 1) \times (0, 1, 1)_{12}$  with only 2 parameters is remarkably effective for many seasonal economic series.



## What's Next?

### Chapter 5: Volatility Modeling — GARCH

- **Volatility:** conditional variation of financial returns
- **ARCH/GARCH:** models for conditional variance
- **Asymmetric extensions:** GJR-GARCH, EGARCH (leverage effect)
- **VaR:** Value-at-Risk based on GARCH models
- **Case study:** S&P 500 returns volatility

Questions?



## Question 1

### Question

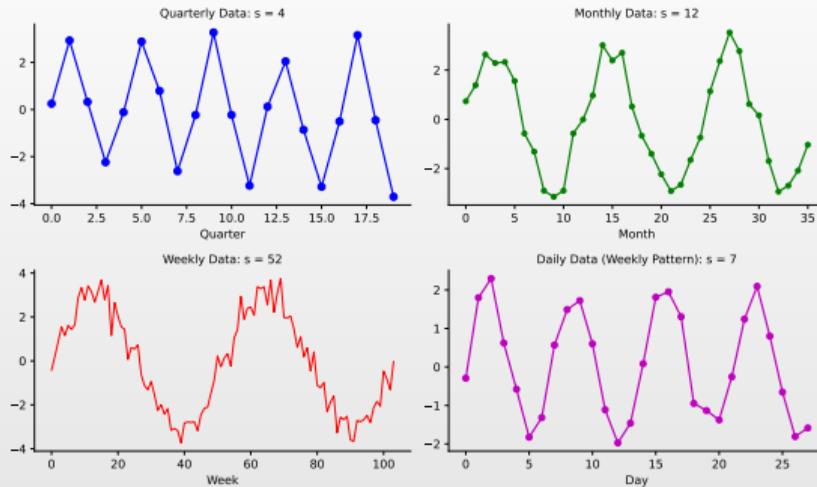
- For monthly data with annual seasonality, what is the seasonal period  $s$ ?

### Answer Choices

- (A)**  $s = 4$
- (B)**  $s = 7$
- (C)**  $s = 12$
- (D)**  $s = 52$



## Question 1: Answer



Answer: (C)

- $s = 12$  (12 months per year). Common periods: Quarterly=4, Monthly=12, Weekly=52, Daily=7, Hourly=24



## Question 2

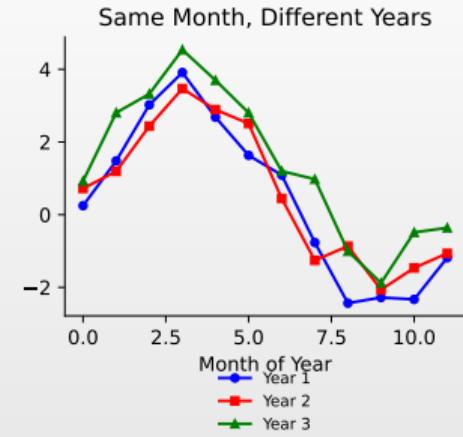
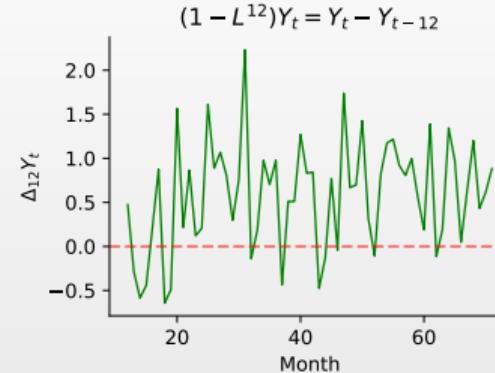
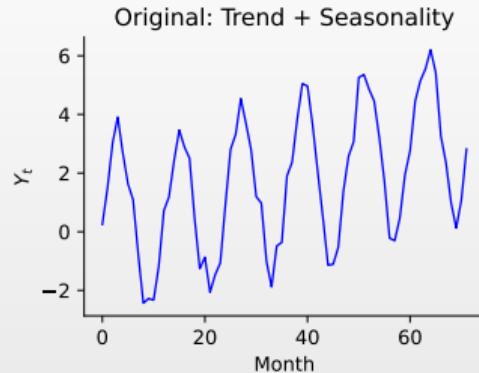
### Question

- What does the seasonal difference operator  $(1 - L^{12})$  do to a monthly series?

### Answer Choices

- (A) Computes  $Y_t - Y_{t-1}$  (month-to-month change)
- (B) Computes  $Y_t - Y_{t-12}$  (year-over-year change)
- (C) Computes the 12-month moving average
- (D) Removes the trend component only

## Question 2: Answer



Answer: (B)

- (1 - L<sup>12</sup>)Y<sub>t</sub> = Y<sub>t</sub> - Y<sub>t-12</sub> removes the seasonal pattern by comparing same months.

Q TSA\_ch4\_quiz2\_seasonal\_diff



## Question 3

### Question

- In SARIMA(1, 1, 1)  $\times$  (1, 1, 1)<sub>12</sub> notation, what does the (1, 1, 1)<sub>12</sub> part represent?

### Answer Choices

- (A) AR(1), differencing once, MA(1) at the regular level
- (B) Seasonal AR(1), seasonal differencing once, seasonal MA(1)
- (C) 12 AR terms, 12 differences, 12 MA terms
- (D) The model has 12 parameters in total

### Question 3: Answer

Answer: (B)

- Seasonal AR(1), seasonal differencing once, seasonal MA(1)

#### SARIMA Notation Breakdown

SARIMA( $p, d, q$ )  $\times$  ( $P, D, Q$ )<sub>s</sub>:

- ( $p, d, q$ )      Non-seasonal: AR( $p$ ),  $d$  differences, MA( $q$ )  
( $P, D, Q$ )<sub>s</sub>    Seasonal: SAR( $P$ ),  $D$  seasonal diffs, SMA( $Q$ )

For  $(1, 1, 1) \times (1, 1, 1)_{12}$ :

- Non-seasonal: AR(1), one regular difference, MA(1)
- Seasonal: SAR(1) at lag 12, one  $\Delta_{12}$ , SMA(1) at lag 12



## Question 4

### Question

- The “Airline Model” is SARIMA(0, 1, 1)  $\times$  (0, 1, 1)<sub>12</sub>. How many parameters need to be estimated (excluding variance)?

### Answer Choices

- (A) 1
- (B) 2
- (C) 4
- (D) 12

## Question 4: Answer

Answer: (B)

- 2 parameters: SARIMA(0, 1, 1)  $\times$  (0, 1, 1)<sub>12</sub>:  $(1 - L)(1 - L^{12})Y_t = (1 + \theta_1 L)(1 + \Theta_1 L^{12})\varepsilon_t$
- Parameters:  $\theta_1$  (non-seasonal MA) and  $\Theta_1$  (seasonal MA), plus  $\sigma^2$ .

### Why “Airline Model”?

Box & Jenkins (1970) used this model to forecast international airline passengers. Remarkably effective for many seasonal economic series!



## Question 5

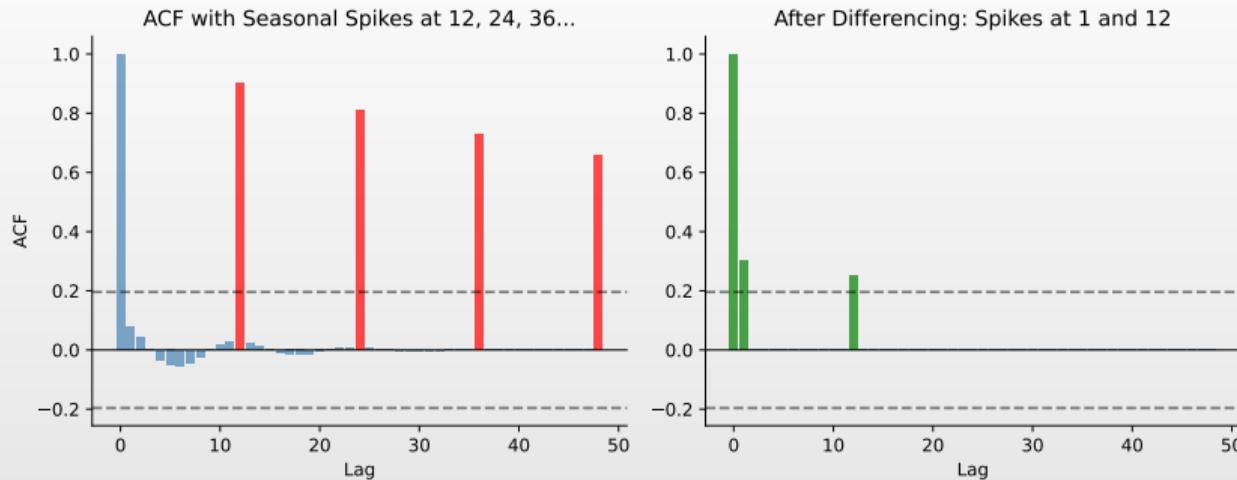
### Question

- You observe significant ACF spikes at lags 12, 24, and 36 in a monthly series. What does this suggest?

### Answer Choices

- (A) The series has a unit root
- (B) The series has annual seasonality that needs seasonal differencing
- (C) The series follows an AR(36) process
- (D) The series is already stationary

## Question 5: Answer



Answer: (B)

- ACF spikes at 12, 24, 36 = stochastic seasonality. Apply  $(1 - L^{12})$  to remove it.

Q TSA\_ch4\_quiz5\_seasonal\_acf



## Question 6

### Question

- After applying  $(1 - L)(1 - L^{12})$  to a monthly series, the ACF shows a significant spike only at lag 1 and lag 12. What SARIMA model is suggested?

### Answer Choices

- (A)** SARIMA(1, 1, 0)  $\times$  (1, 1, 0)<sub>12</sub>
- (B)** SARIMA(0, 1, 1)  $\times$  (0, 1, 1)<sub>12</sub>
- (C)** SARIMA(1, 1, 1)  $\times$  (1, 1, 1)<sub>12</sub>
- (D)** SARIMA(0, 1, 0)  $\times$  (0, 1, 0)<sub>12</sub>

## Question 6: Answer

Answer: (B)

- Model:** SARIMA(0, 1, 1)  $\times$  (0, 1, 1)<sub>12</sub> (The Airline Model)

### ACF/PACF Identification Rules

- Rule:** for MA processes, ACF cuts off after lag  $q$
- ACF spike at lag 1:** MA(1) for non-seasonal part
- ACF spike at lag 12:** SMA(1) for seasonal part
- Combined:** MA(1)  $\times$  SMA(1) = (0,  $d$ , 1)  $\times$  (0,  $D$ , 1)<sub>12</sub>
- With  $d = 1$ ,  $D = 1$ :** (0, 1, 1)  $\times$  (0, 1, 1)<sub>12</sub>



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### Seasonal Decomposition and Diagnostics

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- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.



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### Textbooks and Additional References

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- Brockwell, P.J., & Davis, R.A. (2016). *Introduction to Time Series and Forecasting*, 3rd ed., Springer.
- Hyndman, R.J., & Khandakar, Y. (2008). Automatic Time Series Forecasting: The forecast Package for R, *Journal of Statistical Software*, 27(3), 1–22.

### Online Resources and Code

- Quantlet:** <https://quantlet.com> ↗ Code platform for statistics
- Quantinar:** <https://quantinar.com> ↗ Learning platform for quantitative methods
- GitHub TSA:** [https://github.com/QuantLet/TSA/tree/main/TSA\\_ch4](https://github.com/QuantLet/TSA/tree/main/TSA_ch4) ↗ Python code for this chapter



# Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>

