



# Time Series Analysis and Forecasting

## Chapter 1: Stochastic Processes and Stationarity



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## Learning Objectives

By the end of this chapter, you will be able to:

1. Define stochastic processes and understand their properties
2. Distinguish between strict and weak (covariance) stationarity
3. Identify white noise and random walk processes
4. Compute and interpret ACF and PACF
5. Apply the lag operator and differencing
6. Conduct stationarity tests (ADF, KPSS)
7. Analyze financial time series data
8. Distinguish between unit root and trend-stationary processes

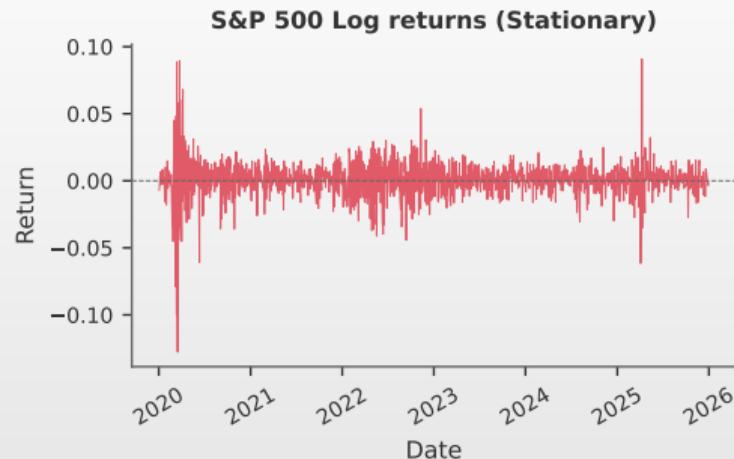
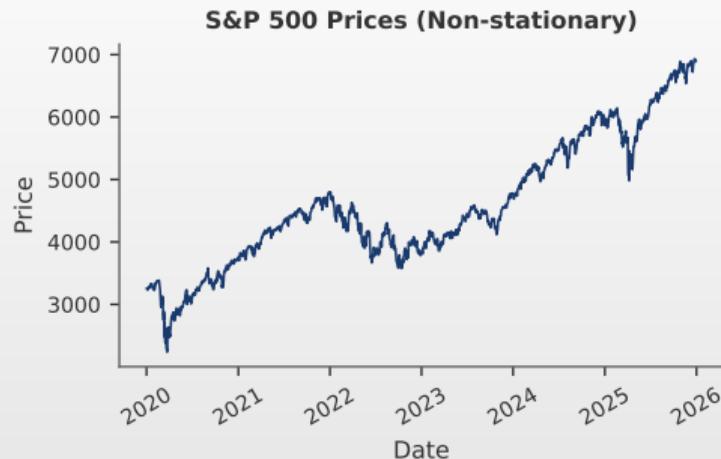


## Outline

- Motivation
- Stochastic Processes
- Stationarity
- Lag Operator and Differencing
- White Noise and Random Walk
- Autocorrelation Functions
- Testing for Stationarity
- Financial Data Application
- Case Study: Stationarity Testing
- AI Use Case
- Summary
- Quiz
- References



## Examples: stationary vs. non-stationary series

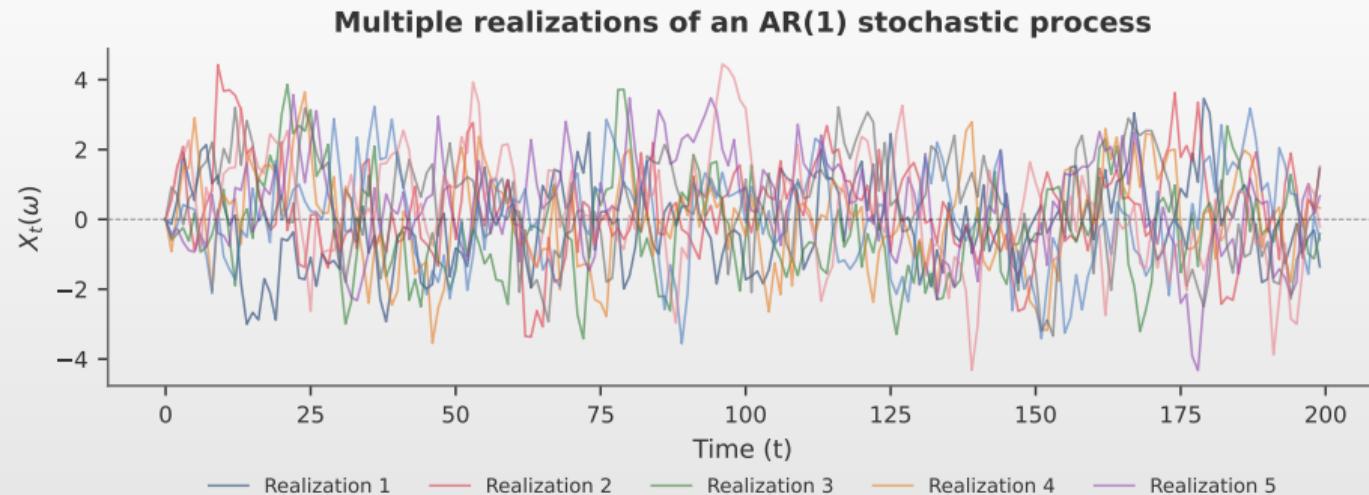


### Observations

- Prices (left) are non-stationary: trend, the mean changes over time
- Returns (right) are stationary: mean  $\approx 0$ , approximately constant variance
- Log returns:  $r_t = \ln P_t - \ln P_{t-1} \succ$  non-stationary  $\rightarrow$  stationary



## Stochastic process: visual illustration



### Interpretation

- Each line is a **different realization** from the same underlying stochastic process
- We observe only **one realization**, yet aim to understand the properties of the process



## Stochastic process: definition

### Definition 1 (Stochastic Process)

- ◻ A **stochastic process** is a collection of random variables indexed by time
  - ▶  $\{X_t(\omega) : t \in \mathcal{T}, \omega \in \Omega\}$
  - ▶  $\Omega$  is the sample space of possible outcomes

### Two Perspectives

- ◻ Fixed  $\omega$ : A *realization*  $\{X_t(\omega)\}_{t \in \mathcal{T}}$
- ◻ Fixed  $t$ : A *random variable*  $X_t$

### Key Insight

- ◻ A time series we observe is **one realization** of the underlying stochastic process

## Moments of a stochastic process

### The First Two Moments Characterize the Process

- **Mean Function:**  $\mu_t = \mathbb{E}[X_t]$
- **Autocovariance (ACVF):**  $\gamma(t, s) = \text{Cov}(X_t, X_s)$ 
  - ▶  $\gamma(t, s) = \mathbb{E}[(X_t - \mu_t)(X_s - \mu_s)]$
- **Autocorrelation (ACF):**
  - ▶  $\rho(t, s) = \gamma(t, s) / \sqrt{\text{Var}(X_t) \cdot \text{Var}(X_s)}$

### ACF Properties

- **Range:**  $\rho(t, s) \in [-1, 1]$
- **Normalization:**  $\rho(t, t) = 1$  (perfect correlation with itself)

### Key Point

- **General:**  $\mu_t$  and  $\gamma(t, s)$  may depend on  $t$
- **Stationary:** Removes this dependence



## Why stationarity matters

### Without Stationarity

- ◻ Mean, variance change over time
  - ▶ Estimates are inconsistent
- ◻ Past may not predict the future
- ◻ Standard methods fail
- ◻ Spurious correlations

### With Stationarity

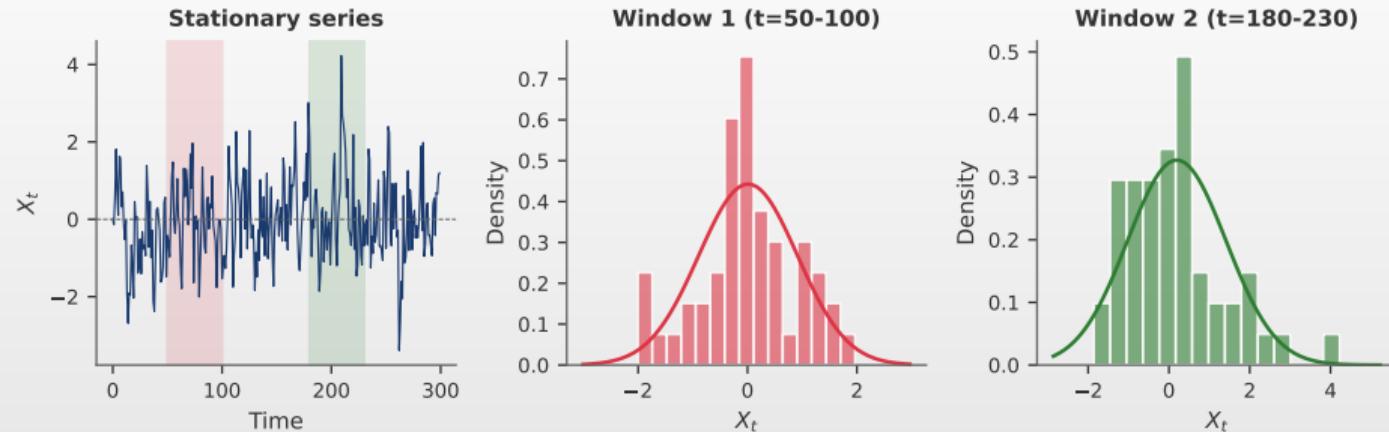
- ◻ Statistical properties constant
  - ▶ Ergodicity justified
- ◻ Can estimate from a single realization
- ◻ Valid inference possible
- ◻ Models are meaningful

### Key Principle

- ◻ Most time series models (ARMA, ARIMA, etc.) require stationarity
- ◻ Non-stationary series must be transformed (e.g., differencing) before modeling



## Strict stationarity: visual illustration



### Interpretation

- Time translation does not change the joint distribution of the variables
- Any two time windows have the same statistical properties
- In practice: we only check the first moments (weak stationarity)



## Strict stationarity

### Definition 2 (Strict (Strong) Stationarity)

- ◻ A process  $\{X_t\}$  is **strictly stationary** if for all  $k$ , all  $t_1, \dots, t_k$ , and all  $h$ :
  - ▶  $(X_{t_1}, \dots, X_{t_k}) \stackrel{d}{=} (X_{t_1+h}, \dots, X_{t_k+h})$
- ◻ **Notation:**  $X \stackrel{d}{=} Y$  means *equality in distribution*
  - ▶  $P(X \leq x) = P(Y \leq x)$

### Implications

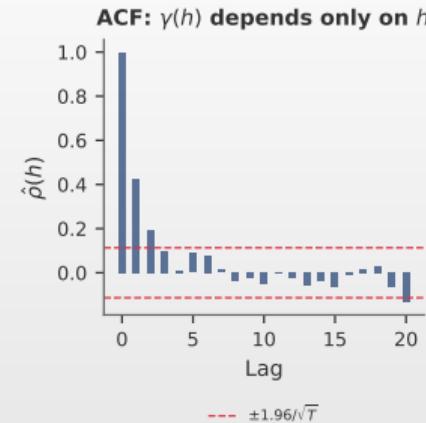
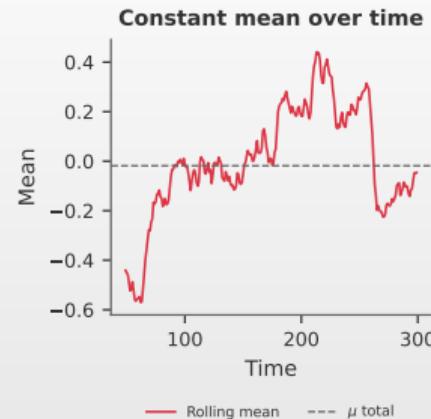
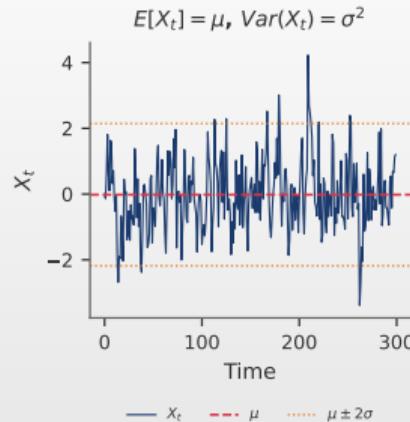
- ◻ **Identical distributions:**  $F_{X_t}(x)$  does not depend on  $t$ 
  - ▶  $\mathbb{E}[X_t] = \mu$  (constant mean, if it exists)
  - ▶  $\text{Var}(X_t) = \sigma^2$  (constant variance, if it exists)
- ◻ **Lag dependence:** Joint distributions depend only on lag

### Note

- ◻ Strict stationarity is a strong condition, often impossible to verify in practice



## Weak stationarity: visual illustration



### The Three Conditions

- $\mathbb{E}[X_t] = \mu$  constant  $\succ$  mean does not depend on time
- $\text{Var}(X_t) = \sigma^2$  constant  $\succ$  variance does not depend on time
- $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$   $\succ$  autocovariance depends only on lag  $h$



## Weak (covariance) stationarity

### Definition 3 (Weak Stationarity)

- ◻ A process  $\{X_t\}$  is **weakly stationary** (or covariance stationary) if:
  - ▶  $\mathbb{E}[X_t^2] < \infty$  for all  $t$  — finite second-order moments
  - ▶  $\mathbb{E}[X_t] = \mu$  for all  $t$  — constant mean
  - ▶  $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$  — covariance depends only on lag  $h$ , not on  $t$

### Key Properties

- ◻ **Autocovariance:**  $\gamma(h) = \text{Cov}(X_t, X_{t+h}) = \mathbb{E}[(X_t - \mu)(X_{t+h} - \mu)]$
- ◻ **Autocorrelation:**  $\rho(h) = \gamma(h)/\gamma(0) = \text{Cov}(X_t, X_{t+h})/\text{Var}(X_t)$
- ◻ **Note:**  $\rho(0) = 1$ ,  $|\rho(h)| \leq 1$ ,  $\rho(h) = \rho(-h)$  (symmetry)



## Relationship between strict and weak stationarity

### Theorem 1 (Fundamental Implication)

If  $\{X_t\}$  is **strictly stationary** and  $\mathbb{E}[X_t^2] < \infty$ , then  $\{X_t\}$  is also **weakly stationary**.

### Proof.

- Let  $t_1, t_2$  be arbitrary and  $h$  any time shift
- From joint distribution invariance:  $(X_{t_1}, X_{t_2}) \stackrel{d}{=} (X_{t_1+h}, X_{t_2+h})$
- $\mathbb{E}[X_{t_1}] = \mathbb{E}[X_{t_1+h}] = \mu$  (constant mean)
- $\text{Cov}(X_{t_1}, X_{t_2}) = \text{Cov}(X_{t_1+h}, X_{t_2+h})$
- Thus autocovariance depends only on the difference  $t_2 - t_1 = h$ , not on  $t_1$



### Warning: The Converse is NOT True!

- There exist weakly stationary processes that are **not** strictly stationary



## Example: AR(1) is weakly stationary

### Model

◻  $X_t = \phi X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2), \quad |\phi| < 1$

### Verification of the three conditions

1. **Constant mean:**  $\mathbb{E}[X_t] = \phi \mathbb{E}[X_{t-1}] + 0 = \phi \mathbb{E}[X_t] \Rightarrow \mathbb{E}[X_t] = 0$

2. **Constant variance:**  $\text{Var}(X_t) = \phi^2 \text{Var}(X_{t-1}) + \sigma^2 \Rightarrow \text{Var}(X_t) = \frac{\sigma^2}{1 - \phi^2}$

3. **Autocovariance depends only on lag:**  $\gamma(h) = \phi^{|h|} \cdot \frac{\sigma^2}{1 - \phi^2}, \quad \rho(h) = \phi^{|h|}$

### Numerical example: $\phi = 0.8, \sigma^2 = 1$

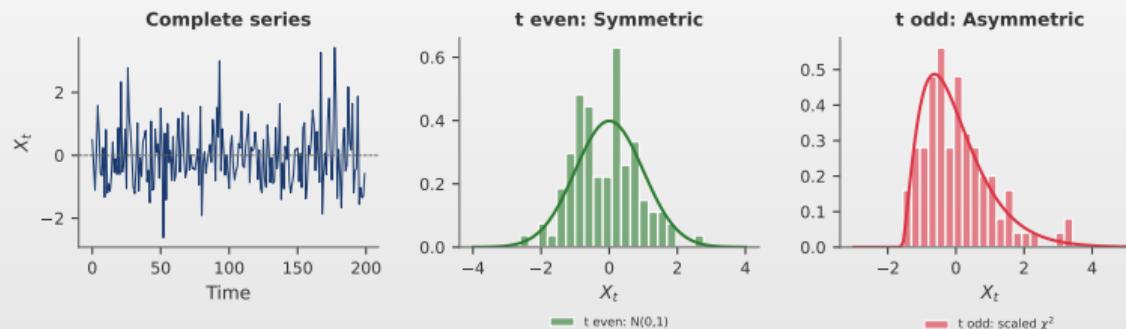
◻  $\mathbb{E}[X_t] = 0, \quad \text{Var}(X_t) = \frac{1}{1 - 0.64} = 2.78, \quad \rho(1) = 0.8, \quad \rho(2) = 0.64, \quad \rho(5) = 0.33$



## Counterexample: weakly stationary but NOT strictly stationary

### Construction

- Let  $\{X_t\}$  be **independent** random variables with:  $t$  even:  $X_t \sim N(0, 1)$ ;  $t$  odd:  $X_t \sim \frac{\chi^2(5)-5}{\sqrt{10}}$



Weakly stationary ✓

- $\mathbb{E}[X_t] = 0$ ,  $\text{Var}(X_t) = 1$ ,  $\text{Cov}(X_t, X_{t+h}) = 0$

NOT strictly stationary ✗

- Skewness differs (0 vs  $> 0$ )  $\succ X_1 \stackrel{d}{\neq} X_2$

Q TSA\_ch1\_stationarity



## Properties of the autocovariance function

### Proposition 1

For a weakly stationary process, the ACVF  $\gamma(h)$  satisfies:

- **Symmetry:**  $\gamma(h) = \gamma(-h)$
- **Maximum at zero:**  $|\gamma(h)| \leq \gamma(0) = \text{Var}(X_t)$
- **Non-negative definiteness:**  $\sum_{i,j} a_i a_j \gamma(i - j) \geq 0$  for any  $a_1, \dots, a_n$

### Proof (property 3)

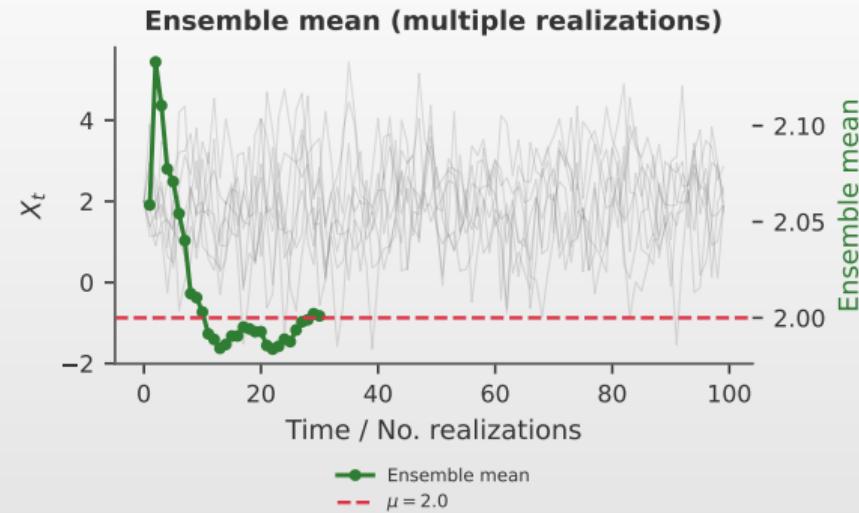
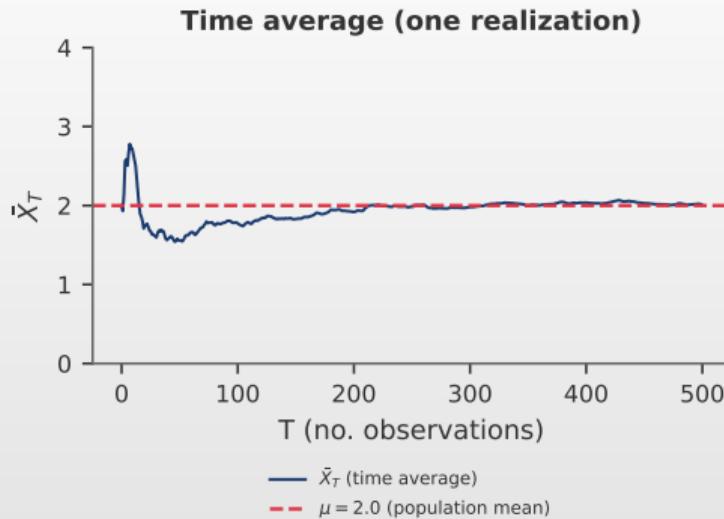
- $\text{Var}(\sum_{i=1}^n a_i X_{t+i}) = \sum_{i,j} a_i a_j \gamma(i - j) \geq 0$  (variance  $\geq 0$ )

### Implication

- Not every function can be a valid autocovariance function



## Ergodicity: visual illustration



- Time average (single realization) and ensemble average (multiple realizations) both converge to  $\mu$
- Ergodicity guarantees that we can estimate  $\mu$  from a single sufficiently long time series



## Ergodicity: the foundation of inference from data

### Definition 4 (Ergodicity for Mean)

- A stationary process  $\{X_t\}$  is **ergodic for the mean** if:
  - ▶  $\bar{X}_T = \frac{1}{T} \sum_{t=1}^T X_t \xrightarrow{P} \mathbb{E}[X_t] = \mu$  as  $T \rightarrow \infty$

### Why does ergodicity matter?

- **Problem:** We have only **one realization** of the stochastic process
- **Solution:** Ergodicity allows estimating  $\mu$  from  $\bar{X}_T$ 
  - ▶ The time average converges to the population mean
  - ▶ Without ergodicity, statistical inference is not possible!

### Theorem 2 (Sufficient Condition)

If  $\sum_{h=0}^{\infty} |\gamma(h)| < \infty$  (absolutely summable autocovariances), the process is ergodic.



## The Wold decomposition theorem

### Theorem 3 (Wold, 1938)

Any **covariance stationary** process  $\{X_t\}$  can be written as:  $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \eta_t$

- ◻  $\varepsilon_t \sim WN(0, \sigma^2) \succ$  white noise
  - ▶  $\psi_0 = 1, \sum \psi_j^2 < \infty$
- ◻  $\eta_t \succ$  deterministic component (perfectly predictable)

### Significance of the Wold Theorem

- ◻ **Decomposition:** Any stationary process =  $MA(\infty) +$  deterministic component
  - ▶ Theoretically justifies  $MA(q)$  and  $ARMA(p, q)$  models
  - ▶ Coefficients  $\psi_j$  measure the impact of past shocks



## Proof of the Wold theorem (sketch)

Proof sketch.

1. **Hilbert space of the past:** Define  $\mathcal{H}_t = \overline{\text{sp}}\{X_s : s \leq t\}$  — the closed linear span of past and present values, with inner product  $\langle X, Y \rangle = \text{Cov}(X, Y)$ .
2. **Innovation:** Define  $\varepsilon_t = X_t - \hat{X}_t$ , where  $\hat{X}_t = \text{Proj}_{\mathcal{H}_{t-1}}(X_t)$  is the orthogonal projection. By construction,  $\varepsilon_t \perp \mathcal{H}_{t-1}$ , so  $\varepsilon_t \perp \varepsilon_s$  for  $t \neq s \Rightarrow \{\varepsilon_t\}$  is white noise.
3. **Iterative representation:** Applying the projection recursively:

$$X_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots + \eta_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \eta_t$$

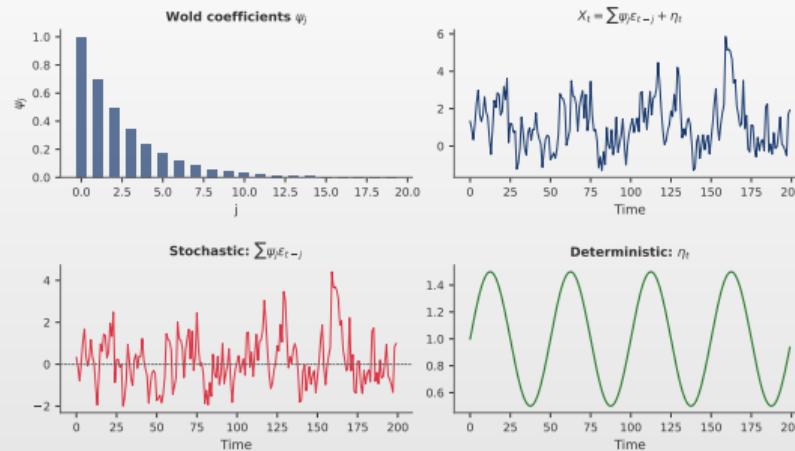
where  $\psi_j$  arise from successive projections, and  $\eta_t \in \mathcal{H}_{-\infty} = \bigcap_t \mathcal{H}_t$ .

4. **Convergence:**  $\sum_{j=0}^{\infty} \psi_j^2 < \infty$  because  $\text{Var}(X_t) < \infty$  (stationarity).
5. **Deterministic component:**  $\eta_t \in \mathcal{H}_{-\infty} \Rightarrow \eta_t$  is perfectly predictable from the infinite past.

□



## The Wold theorem: visual illustration

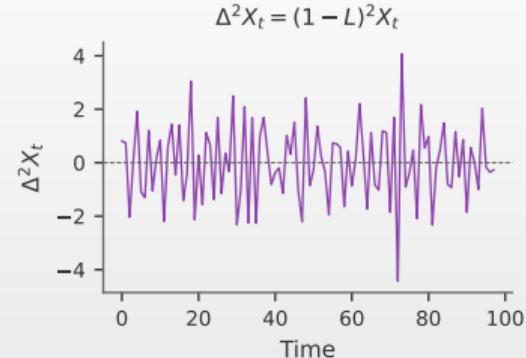
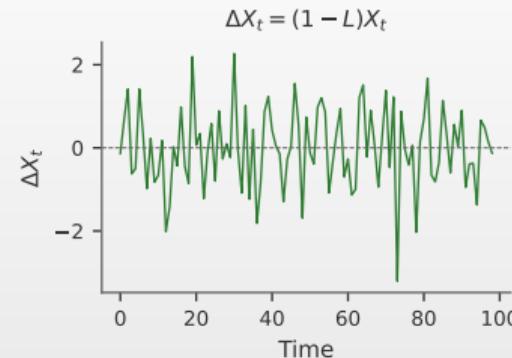
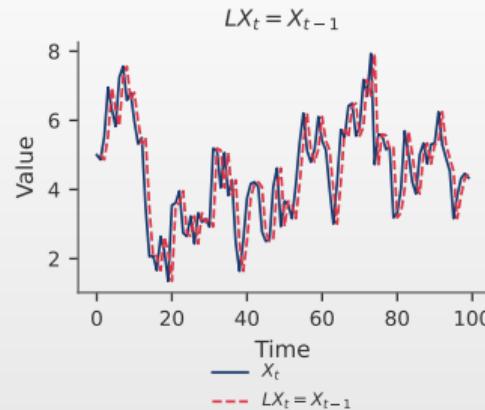


### Interpretation

- $X_t$  decomposes into a **stochastic** component ( $MA(\infty)$ ) and a **deterministic** component ( $\eta_t$ )
- Coefficients  $\psi_j$  decay  $\succ$  recent shocks have greater impact than distant ones



## Lag operator: visual illustration



## Properties

- $LX_t = X_{t-1}$   $\succ$  the lag operator shifts the series back by one period
- $L^k X_t = X_{t-k}$   $\succ$  shift by  $k$  periods;  $L^0 = I$  (identity)
- Difference operator:**  $\Delta = (1 - L)$ , so  $\Delta X_t = X_t - X_{t-1}$



## The lag operator

### Definition 5 (Lag Operator)

- The **lag operator** (or backshift operator)  $L$  is defined by:  $LX_t = X_{t-1}$

### Properties

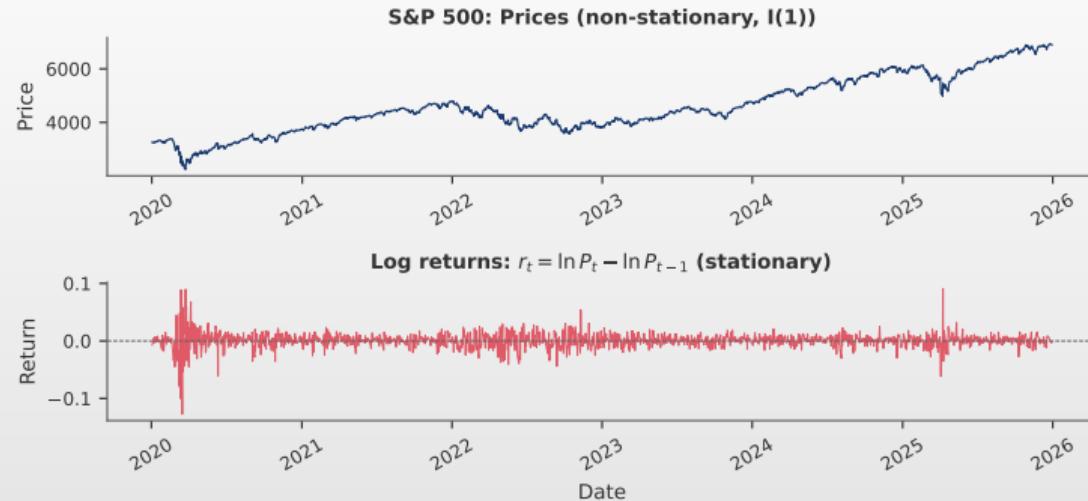
- **Powers:**  $L^k X_t = X_{t-k}$  (lag by  $k$  periods)
  - ▶ Compact notation for models
- **Identity:**  $L^0 = I$
- **Polynomial:**  $(1 - \phi L)X_t = X_t - \phi X_{t-1}$

### Examples

- **First difference:**  $(1 - L)X_t = X_t - X_{t-1}$
- **Second difference:**  $(1 - L)^2 X_t = \Delta^2 X_t$
- **Seasonal:**  $(1 - L^{12})X_t$



## Effect of differencing: S&P 500



### Interpretation

- Top:** S&P 500 prices  $\succ$  clear trend, non-stationary ( $I(1)$ )
- Bottom:** Log returns  $r_t = \ln P_t - \ln P_{t-1} \succ$  fluctuates around mean  $\approx 0$ , stationary



## Differencing

### Why Do We Difference?

- **First Difference:**  $\Delta X_t = X_t - X_{t-1} = (1 - L)X_t$ 
  - ▶ Removes trend and unit root
  - ▶ Random walk:  $\Delta X_t = \varepsilon_t$

### Definition 6 (Integrated Process of Order $d$ )

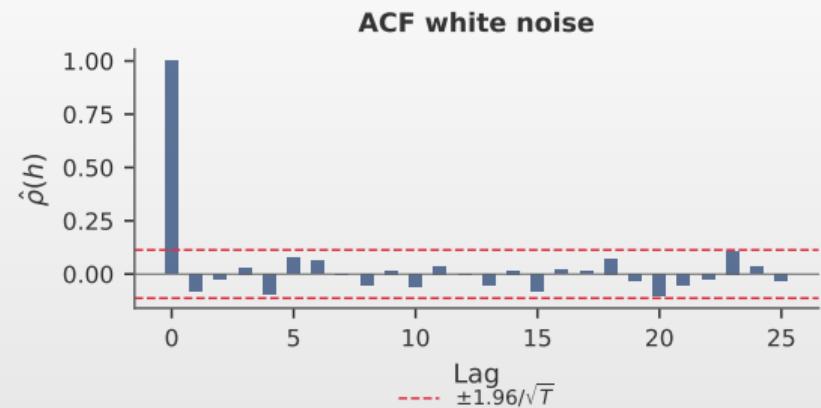
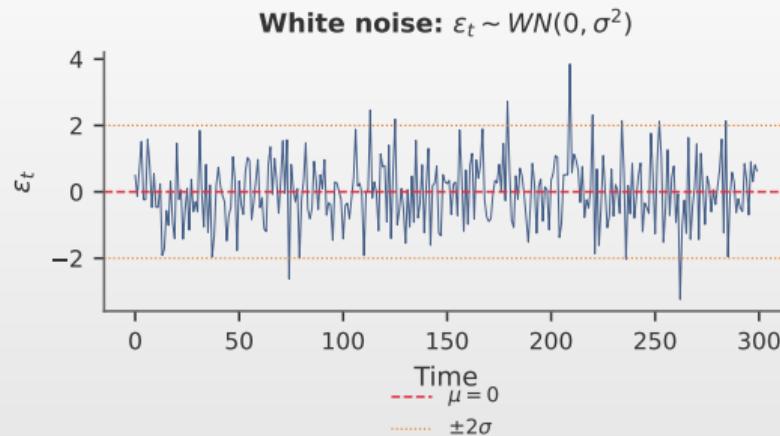
- A process  $\{X_t\}$  is **integrated of order  $d$** , denoted  $X_t \sim I(d)$ , if:
  - ▶  $\Delta^d X_t = (1 - L)^d X_t$  is stationary ( $I(0)$  process)
  - ▶  $\Delta^{d-1} X_t$  is **not** stationary

### Examples

- $I(0)$ : Stationary process (white noise, stationary AR)
- $I(1)$ : Random walk  $\succ \Delta X_t = \varepsilon_t$  is stationary
- $I(2)$ : Requires two differences for stationarity



## White noise: visual illustration



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## White noise process

### Definition 7 (White Noise)

- A process  $\{\varepsilon_t\}$  is **white noise**, denoted  $\varepsilon_t \sim WN(0, \sigma^2)$ , if:
  - ▶  $\mathbb{E}[\varepsilon_t] = 0$  for all  $t$  (zero mean)
  - ▶  $\text{Var}(\varepsilon_t) = \sigma^2$  for all  $t$  (constant variance)
  - ▶  $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$  for  $t \neq s$  (uncorrelated)

### ACF of White Noise

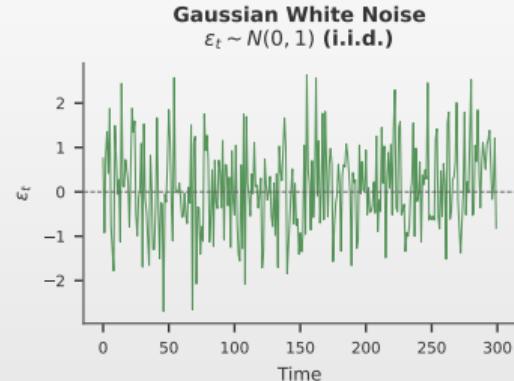
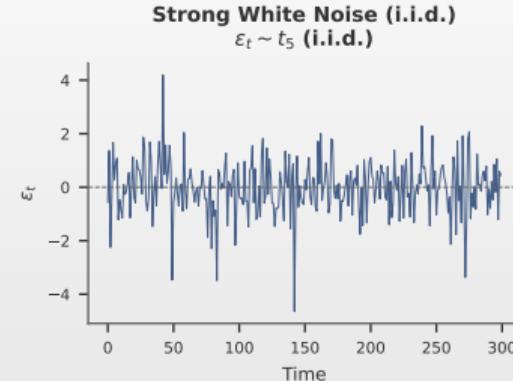
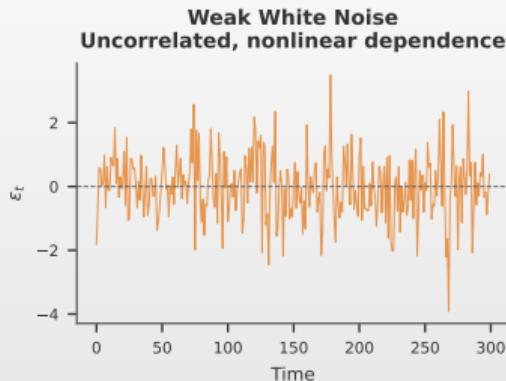
- By definition:  $\gamma(0) = \sigma^2$  and  $\gamma(h) = 0$  for  $h \neq 0$ ;  $\rho(h) = \begin{cases} 1 & h = 0 \\ 0 & h \neq 0 \end{cases}$

### Types of white noise (in order of increasing restrictions)

- **Weak:** uncorrelated, but nonlinear dependencies may exist
- **Strong:**  $\varepsilon_t$  are *independent* and identically distributed (i.i.d.)
- **Gaussian:**  $\varepsilon_t \stackrel{iid}{\sim} N(0, \sigma^2)$ 
  - ▶ Uncorrelated  $\Rightarrow$  independent



## The three types of white noise

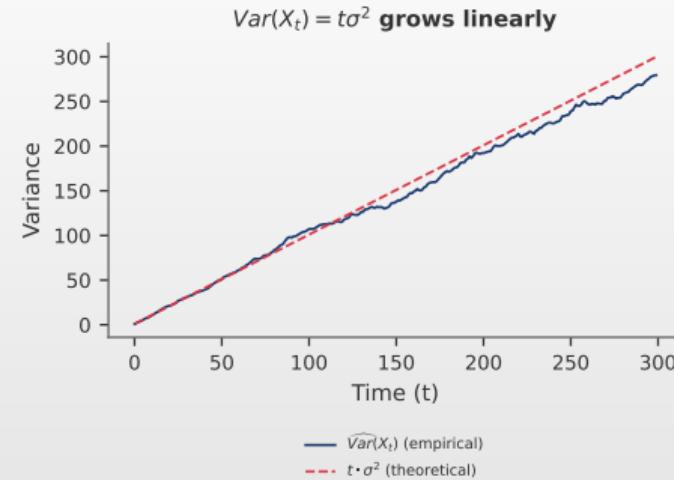
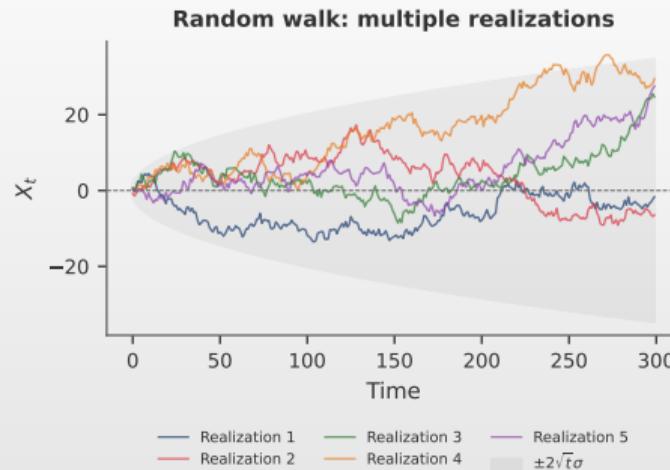


Inclusion relationship: Gaussian  $\subset$  Strong (i.i.d.)  $\subset$  Weak (uncorrelated)

- **Weak:**  $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ , but nonlinear dependencies may exist (e.g. GARCH)
- **Strong:**  $\varepsilon_t$  are i.i.d. — any distribution (e.g. Student- $t$ )
- **Gaussian:**  $\varepsilon_t \stackrel{iid}{\sim} N(0, \sigma^2)$  — uncorrelated  $\Leftrightarrow$  independent



## Random walk: visualization



### Observations

- Each shock has a **permanent effect**;  $\text{Var}(X_t) = t\sigma^2$  grows linearly with time
- **Solution** — differencing transforms into white noise,  $\Delta X_t = \varepsilon_t$



## Random walk process

### Definition 8 (Random Walk)

$X_t = X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2), \quad X_0 = 0 \quad \Rightarrow \text{Explicit form: } X_t = \sum_{i=1}^t \varepsilon_i$

### Proposition 2 (Properties)

- ◻  $\mathbb{E}[X_t] = 0$
- ◻  $\text{Var}(X_t) = t\sigma^2$  (grows with time!)
- ◻  $\text{Cov}(X_t, X_s) = \min(t, s) \cdot \sigma^2$

### Proofs.

- ◻  $\mathbb{E}[X_t] = \mathbb{E}\left[\sum_{i=1}^t \varepsilon_i\right] = 0$
- ◻  $\text{Var}(X_t) = \text{Var}\left(\sum_{i=1}^t \varepsilon_i\right) = \sum_{i=1}^t \text{Var}(\varepsilon_i) = t\sigma^2 \quad (\text{independence})$
- ◻  $\text{Cov}(X_t, X_s) = \min(t, s) \sigma^2 \quad (\text{for } s \leq t)$

□

## Non-Stationary!

$\text{Var}(X_t) = t\sigma^2$  depends on  $t \succ$  random walk is **not stationary**



## Random walk with drift

### Definition 9 (Random Walk with Drift)

$X_t = c + X_{t-1} + \varepsilon_t$ ,  $c \neq 0$  is the **drift**  $\Rightarrow$  **Explicit form:**  $X_t = ct + \sum_{i=1}^t \varepsilon_i$

### Proposition 3 (Properties)

- $\mathbb{E}[X_t] = ct$  (linear trend)
- $\text{Var}(X_t) = t\sigma^2$  (grows with time)

### Differencing

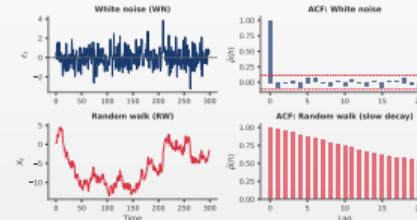
$\Delta X_t = c + \varepsilon_t$  — constant plus white noise  $\succ$  the differenced series is stationary

### Practical Importance

- Nominal GDP, stock prices  $\succ$  often modeled as RW with drift
- The ADF test includes variants: without constant, with constant, with constant and trend



## White noise vs random walk: comparison



Q [TSA\\_ch1\\_random\\_walk](#)

### White Noise

- ☐ Stationary;  $\text{Var} = \sigma^2$  (const.);  $\text{ACF} = 0, h \neq 0$ ; no memory

### Random Walk

- ☐ Non-stationary;  $\text{Var} = t\sigma^2$  (grows);  $\text{ACF} \approx 1$  (slow); permanent shocks

### Link

- ☐  $\Delta X_t = \varepsilon_t$



## Trend-stationary vs. difference-stationary

### Trend-Stationary (TS)

- ◻ **Model:**  $Y_t = \alpha + \beta t + \varepsilon_t$ 
  - ▶ **Deterministic trend**
  - ▶ Deviations from the trend are temporary
- ◻ **Solution:** regression on  $t$ , extract residuals
- ◻ **Effect:** Shocks do NOT have a permanent effect

### Difference-Stationary (DS)

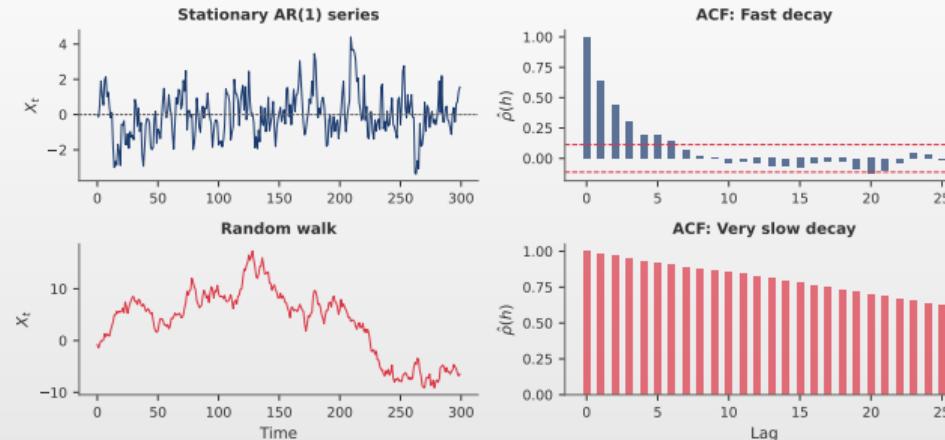
- ◻ **Model:**  $Y_t = c + Y_{t-1} + \varepsilon_t$ 
  - ▶ **Stochastic trend**
  - ▶ Deviations from the trend are permanent
- ◻ **Solution:** differencing ( $\Delta Y_t$ )
- ◻ **Effect:** Shocks HAVE a permanent effect

### Why does the distinction matter?

- ◻ **Differencing a TS process:** introduces an artificial unit root in the MA part
- ◻ **Regression on a DS process:** produces residuals that are still non-stationary
- ◻ **Solution:** ADF and KPSS tests help distinguish between the two



## ACF comparison: stationary vs random walk

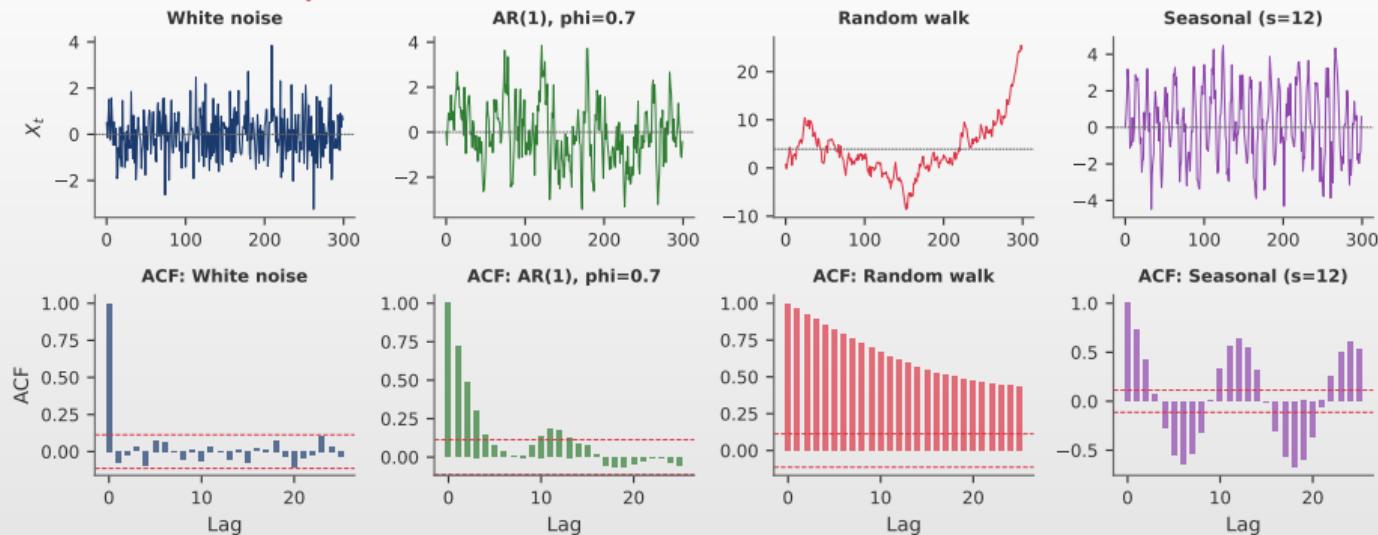


### Interpretation

- **Stationary:** ACF decays rapidly (exponentially or oscillating) toward zero
- **Random walk:** ACF decays very slowly, stays close to 1
- **Rule of thumb:** Slow ACF decay  $\succ$  suspect unit root  $\succ$  ADF test



## ACF patterns for different processes



### Interpretation

- White noise:  $\text{ACF} = 0$ ; Stationary: decays fast; Non-stationary: decays slowly
- Seasonal: Spikes at seasonal lags (12, 24 for monthly data)



## Sample autocorrelation function

### Sample ACF at Lag $h$

$\hat{\rho}(h) = \frac{\sum_{t=1}^{T-h} (x_t - \bar{x})(x_{t+h} - \bar{x})}{\sum_{t=1}^T (x_t - \bar{x})^2}$

► Properties:  $\hat{\rho}(0) = 1$ ,  $|\hat{\rho}(h)| \leq 1$

### Theorem 4 (Bartlett, 1946)

Under  $H_0$ : white noise, for large  $T$ :  $\hat{\rho}(h) \approx N(0, 1/T)$

### 95% Confidence Interval

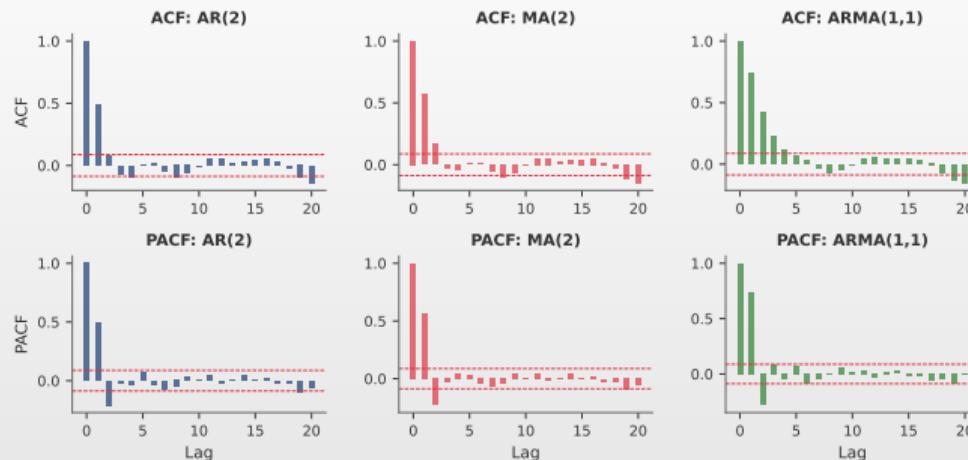
$\pm 1.96/\sqrt{T}$  (the bands in ACF plots)

### Caution

- Bartlett's formula is valid **only under  $H_0$ : white noise**
- For AR/MA, the asymptotic variance differs



## ACF and PACF patterns



### Identification Rules

- **AR( $p$ ):** ACF decays exponentially, PACF cuts off after lag  $p$
- **MA( $q$ ):** ACF cuts off after lag  $q$ , PACF decays exponentially
- **ARMA( $p, q$ ):** Both decay exponentially  $\succ$  identification requires information criteria



## Partial autocorrelation function (PACF)

### Definition 10 (Partial Autocorrelation)

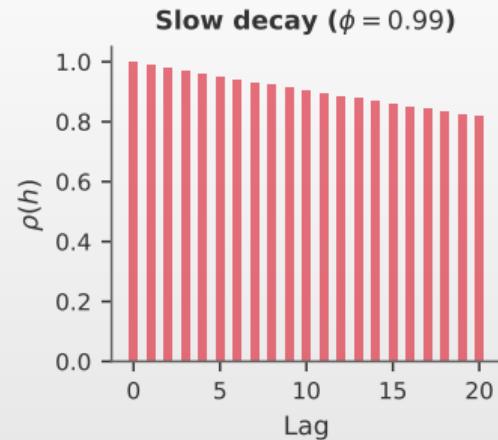
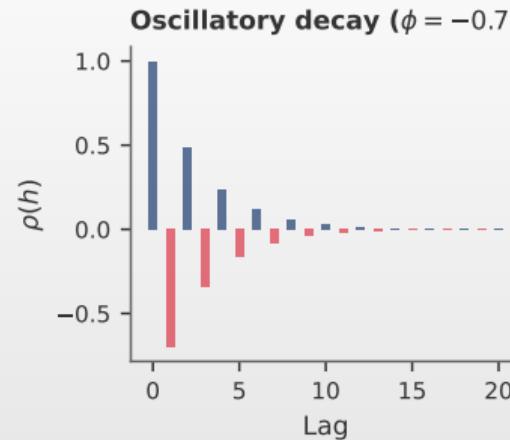
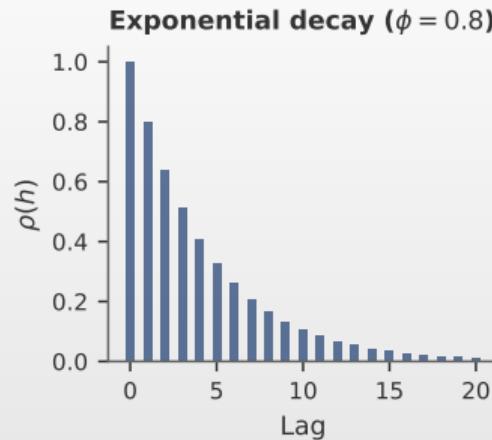
- PACF at lag  $h$ , denoted  $\phi_{hh}$ : the last coefficient in the regression:
  - ▶  $X_t = \phi_{h1}X_{t-1} + \phi_{h2}X_{t-2} + \cdots + \phi_{hh}X_{t-h} + e_t$
- Alternatively:
  - ▶  $\phi_{hh} = \text{Corr}(X_t - \hat{X}_t^{(h-1)}, X_{t-h} - \hat{X}_{t-h}^{(h-1)})$
- Interpretation: Direct dependence at lag  $h$ 
  - ▶ Removes the effect of intermediate lags

### Key Application: Model Order Identification

- AR( $p$ ): PACF cuts off after lag  $p$ 
  - ▶ ACF decays exponentially or oscillates
- MA( $q$ ): ACF cuts off after lag  $q$ 
  - ▶ PACF decays exponentially or oscillates



## ACF decay patterns



### Interpretation

- Exponential decay:** Persistent positive dependence (AR with  $\phi > 0$ )
- Oscillating decay:** Alternating dependence (AR with  $\phi < 0$ )
- The decay rate indicates the strength of the process memory



## Augmented Dickey-Fuller (ADF) test

### ADF Model

$$\Delta X_t = \alpha + \gamma X_{t-1} + \sum_{i=1}^p \delta_i \Delta X_{t-i} + \varepsilon_t, \quad \gamma = \rho - 1, \quad H_0 : \gamma = 0 \Leftrightarrow \rho = 1$$

### Hypotheses

- $H_0: \gamma = 0$  (unit root)
- $H_1: \gamma < 0$  (stationary)

### Test Statistic

- $\tau_{ADF} = \hat{\gamma} / SE(\hat{\gamma})$
- $\hat{\gamma}$  = OLS coefficient of  $X_{t-1}$
- $SE(\hat{\gamma})$  from the OLS regression

### Decision Rule

- $\tau_{ADF} <$  critical value  $\succ$  Reject  $H_0 \succ$  Stationary
- $\tau_{ADF} \geq$  critical value  $\succ$  Non-stationary (unit root)
- Critical values follow the Dickey-Fuller distribution (**not t-Student!**)



## KPSS test

### Model

- $X_t = \xi t + r_t + \varepsilon_t$  where  $r_t = r_{t-1} + u_t$

### Hypotheses (opposite of ADF)

- $H_0: \sigma_u^2 = 0$  (stationary)
- $H_1: \sigma_u^2 > 0$  (unit root)

### Test Statistic

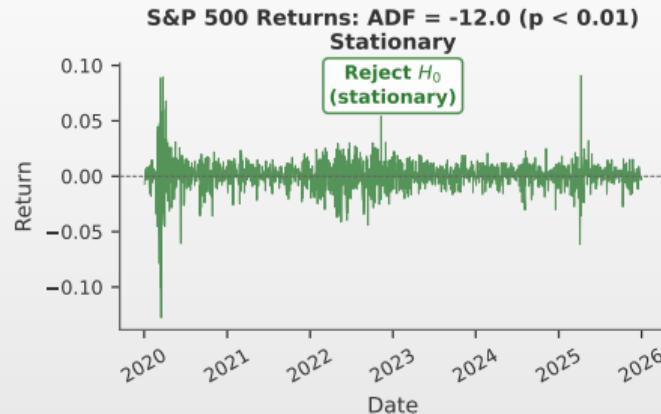
- $LM = \frac{\sum_{t=1}^T S_t^2}{T^2 \hat{\sigma}_{LR}^2}$
- $S_t = \sum_{i=1}^t \hat{e}_i, \quad \hat{\sigma}_{LR}^2 = \text{long-run variance}$

### Decision Rule

- $LM > \text{critical value} \succ \text{Reject } H_0 \succ \text{Non-stationary}$
- $LM \leq \text{critical value} \succ \text{Stationary}$



## ADF test: visualization with S&P 500



Q TSA\_ch1\_unit\_root\_tests

### Interpreting the ADF Test

- Hypothesis:  $H_0$ : Unit root
  - ▶ Critical values:  $-3.43$  (1%),  $-2.86$  (5%),  $-2.57$  (10%)
  - ▶  $\tau <$  critical value  $\succ$  reject  $H_0 \succ$  stationary series
- S&P 500: Prices non-stationary; Returns stationary



## Using ADF and KPSS together

### Confirmatory Testing

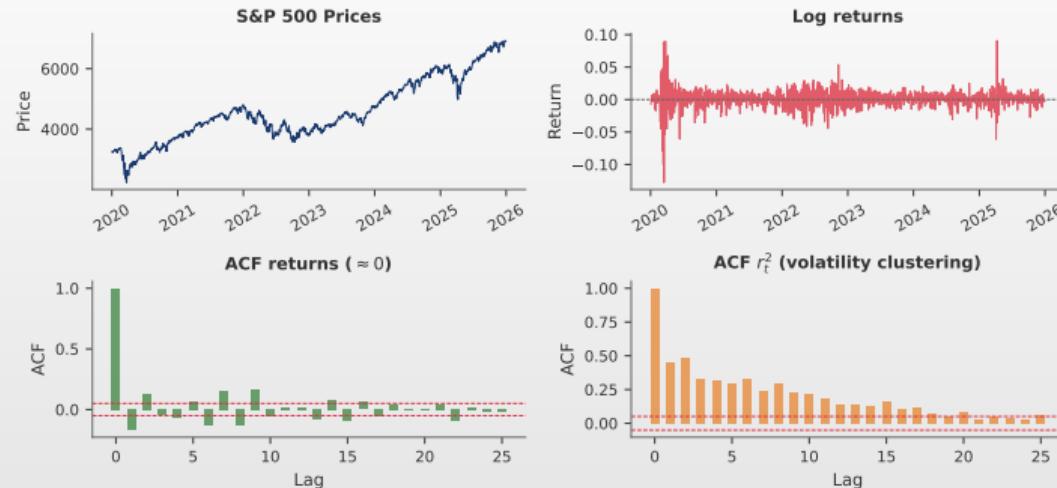
- ADF rejects  $H_0$  + KPSS fails to reject: **Stationary**
- ADF fails to reject + KPSS rejects  $H_0$ : **Unit Root**
- Both reject or both fail to reject: **Inconclusive**
  - ▶ Additional tests required (PP, DF-GLS)

### Workflow

- Step 1:** ADF test ( $H_0$ : unit root)
- Step 2:** KPSS test ( $H_0$ : stationary)
- Step 3:** Concordant results  $\succ$  OK
  - ▶ Otherwise: PP, DF-GLS tests



## S&P 500 analysis: overview

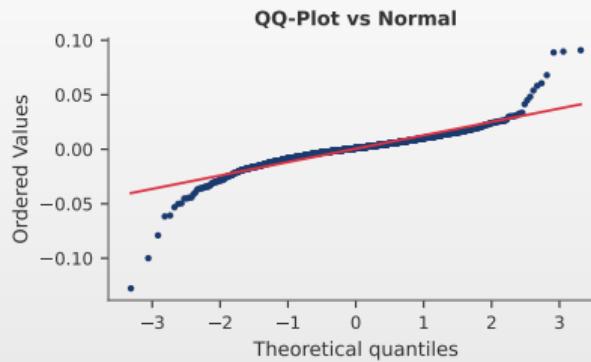
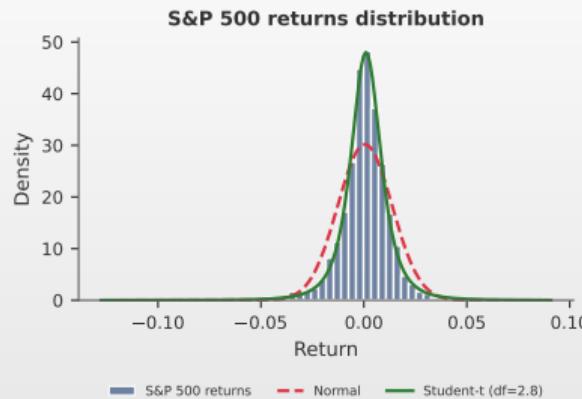


### Observations

- **Prices:** Upward trend, non-stationary; **Returns:** Mean  $\approx 0$ , stationary
- **ACF returns:**  $\approx 0$  (efficient); **ACF  $r_t^2$ :** Significant (volatility clustering)



## Stylized facts of financial returns



### Observed Properties

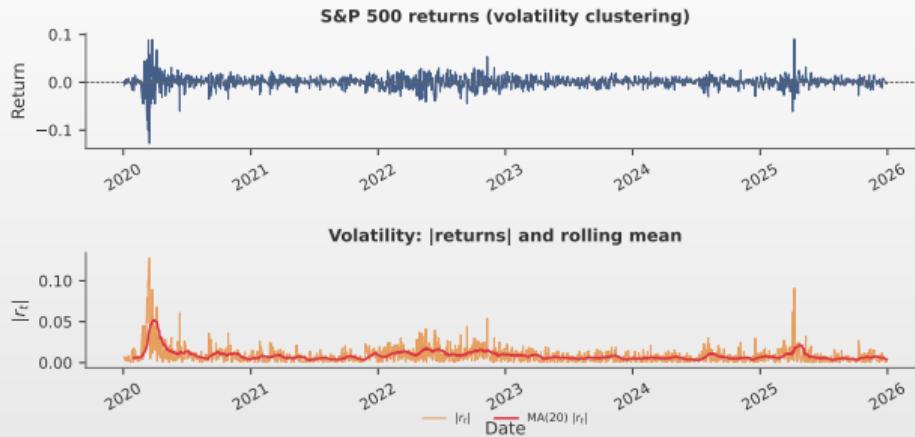
- Negative skewness (left tail)
- Excess kurtosis ( $\gg 3$ )
- Heavy tails (fat tails)

### Implications

- Normal distribution inadequate
- Extreme events more likely
- Student-t or GED required



## Volatility clustering

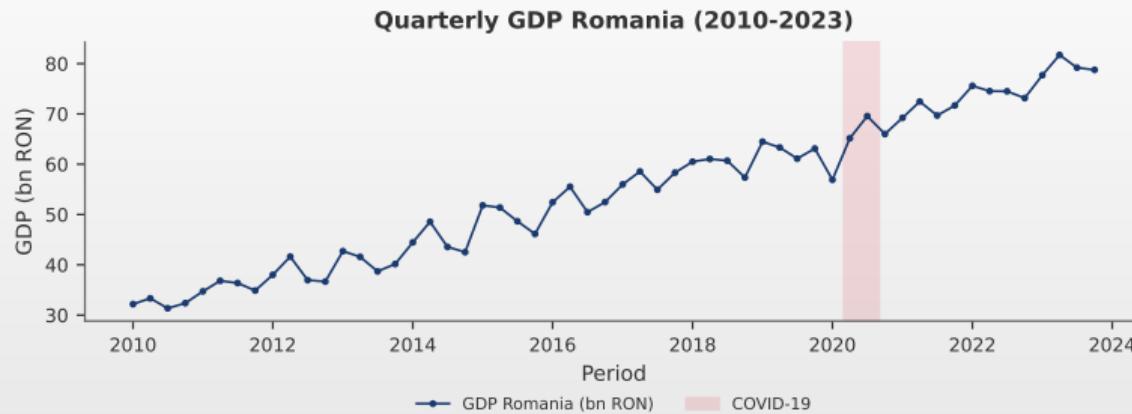


### Observations

- Large returns (in absolute value) followed by large returns
- Calm periods followed by calm periods
- Time-varying volatility**  $\succ$  ARCH/GARCH models (Ch. 5)



## Case study: Romanian quarterly GDP



 TSA\_ch1\_case\_gdp

### Initial Analysis

- Data:** Romanian quarterly GDP 2010–2023 (56 obs., INS/Eurostat)
- Observations:** Upward trend, possibly seasonal
  - ▶ COVID-19 structural shock visible
- Hypothesis:** Non-stationary series  $\succ$  test with ADF and KPSS



## Stationarity testing: ADF and KPSS

### ADF Test

- ◻ Hypothesis:  $H_0$ : Unit root
- ◻ Result: ADF stat.:  $-1.23$ 
  - ▶ Critical value:  $-2.89$
  - ▶ Fail to reject  $H_0$

### KPSS Test

- ◻ Hypothesis:  $H_0$ : Stationary
- ◻ Result: KPSS stat.:  $0.89$ 
  - ▶ Critical value:  $0.46$
  - ▶ Reject  $H_0$

### Conclusion: Both Tests Agree

- ◻ The GDP series is **non-stationary** ↗ requires differencing



## Differencing: transformation to stationarity

### After Differencing

- **Tests:** Both confirm stationarity
  - ▶ ADF:  $-4.56$  ( $p < 0.01$ )
  - ▶ KPSS:  $0.21$  ( $p > 0.10$ )

### Conclusion

- **GDP level:** non-stationary
- $\Delta GDP$ : stationary
  - ▶ Use  $\Delta GDP_t$  for modeling

### Final Result

- GDP requires one differencing to become stationary



## AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Using yfinance, download daily EUR/RON exchange rate data (EURRON=X) from 2020-01-01 to 2024-12-31 (approx. 1,250 observations). Test whether the series is stationary using ADF and KPSS tests. Fit an appropriate model and forecast the exchange rate for the next 5 trading days. Tell me if the forecast is reliable."

**Exercise:**

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Download real EUR/RON data and reproduce the analysis. Do the results match?
3. Is the ADF test correctly specified (trend, lags)? What changes if you modify the options?
4. Compare the AI model's forecast against a naïve benchmark ( $\hat{X}_{t+1} = X_t$ ).
5. If the series is a random walk, does fitting an ARMA model make sense?

**Warning:** Low RMSE and significant coefficients *do not guarantee* a useful forecast.



## Key takeaways

### Summary

- **Stochastic process:** collection of random variables indexed by time
- **Weak stationarity:** constant mean, variance, autocovariance
- **White noise:**  $\varepsilon_t \sim WN(0, \sigma^2)$ 
  - ▶ Stationary, ACF = 0 for  $h \neq 0$
- **Random walk:**  $X_t = X_{t-1} + \varepsilon_t$ 
  - ▶ Non-stationary,  $\text{Var}(X_t) = t\sigma^2$
- **ACF/PACF:** key tools for identifying structure
- **Differencing:** transforms non-stationary series into stationary ones
- **Unit root tests:**
  - ▶ ADF ( $H_0$ : unit root) vs KPSS ( $H_0$ : stationary)



## Important formulas

### Weak Stationarity

- **Constant moments:**
  - ▶  $\mathbb{E}[X_t] = \mu$  (constant mean)
  - ▶  $\text{Var}(X_t) = \sigma^2$  (constant variance)
- **Autocovariance:**  $\gamma(h) = \text{Cov}(X_t, X_{t+h})$
- **Autocorrelation:**  $\rho(h) = \gamma(h)/\gamma(0)$

### Lag Operator

- **Lag:**  $LX_t = X_{t-1}$
- **Difference:**  $\Delta X_t = (1 - L)X_t$

### White Noise (WN)

- **Model:**  $\varepsilon_t \sim WN(0, \sigma^2)$
- **ACF:**  $\rho(h) = 0$  for  $h \neq 0$

### Random Walk (RW)

- **Model:**  $X_t = X_{t-1} + \varepsilon_t$
- **Variance:**  $\text{Var}(X_t) = t\sigma^2$  (grows!)

## Next chapter preview

### Chapter 2: ARMA Models

- **AR( $p$ ):** Autoregressive Models
- **MA( $q$ ):** Moving Average Models
- **ARMA( $p, q$ ):** Combined Models
- **Identification:** Using ACF/PACF

### What We Will Learn

- **Estimation:** Model parameters
- **Diagnostics:** Model validation
- **Forecasting:** Confidence intervals
- **Selection:** AIC, BIC



## Question 1

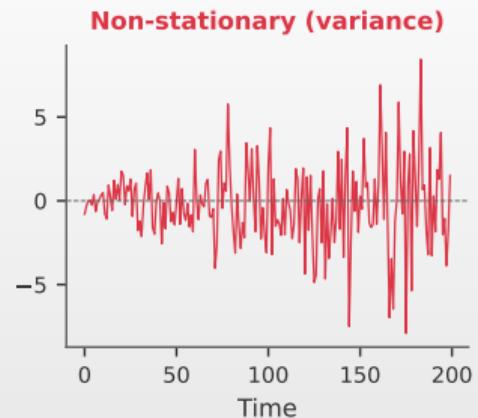
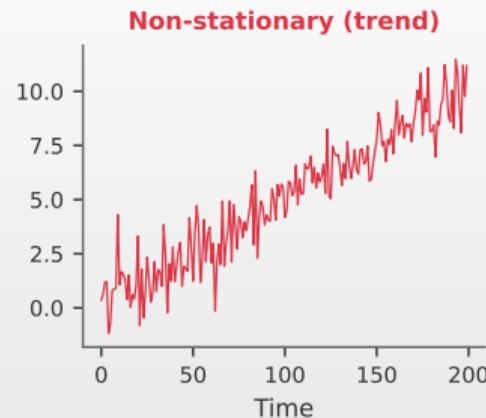
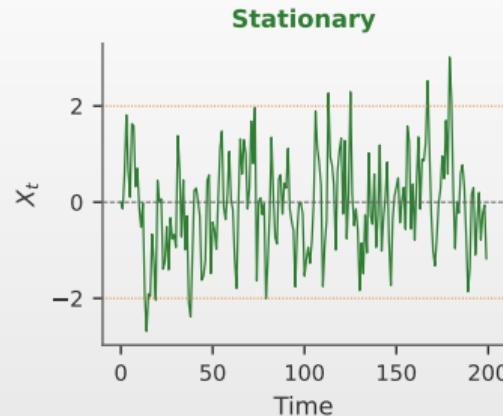
### Question

- What are the three conditions for weak (covariance) stationarity?

### Answer Choices

- (A) Zero mean, infinite variance, time-dependent covariance
- (B) Constant mean, constant variance, autocovariance depends only on lag
- (C) Normal distribution, independence, unit variance
- (D) Linear trend, constant seasonality, white residuals

## Question 1: Answer



Answer: (B)

- $\mathbb{E}[X_t] = \mu, \text{Var}(X_t) = \sigma^2, \gamma(t, s) = \gamma(|t - s|)$

Q TSA\_ch1\_stationarity



## Question 2

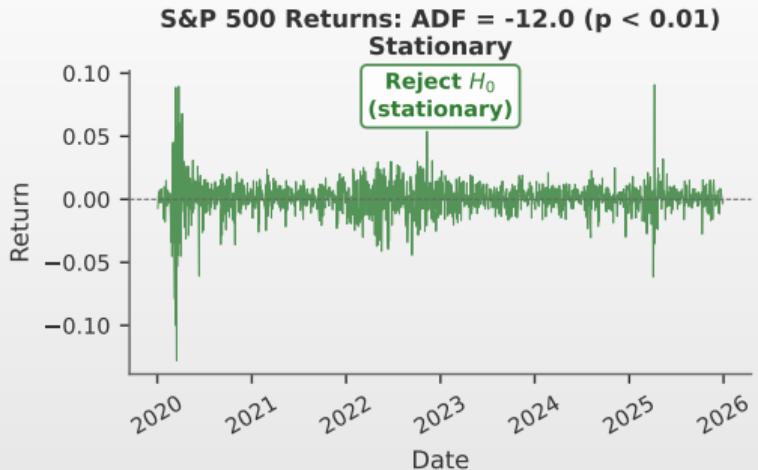
### Question

- What is the null hypothesis ( $H_0$ ) of the ADF (Augmented Dickey-Fuller) test?

### Answer Choices

- (A)** The series is stationary
- (B)** The series has a unit root (is non-stationary)
- (C)** The series has no autocorrelation
- (D)** The series has a normal distribution

## Question 2: Answer



Answer: (B)

- $H_0$ : unit root;  $\tau <$  critical value  $\succ$  stationary



## Question 3

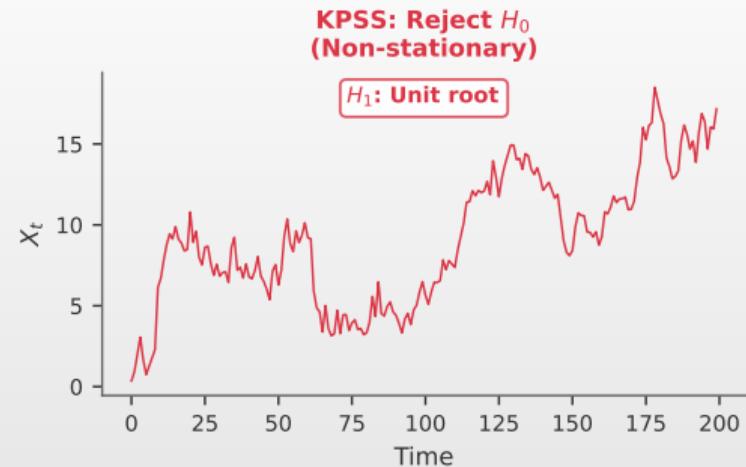
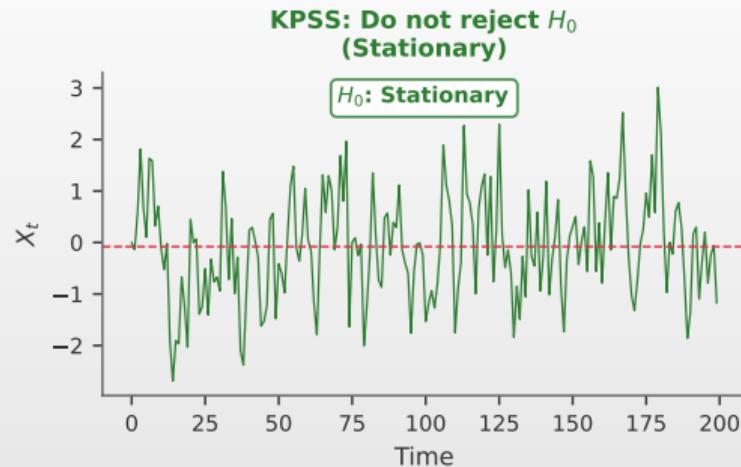
### Question

- What is the null hypothesis ( $H_0$ ) of the KPSS test?

### Answer Choices

- (A)** The series has a unit root (non-stationary)
- (B)** The series is stationary
- (C)** The series is a random walk
- (D)** The series has a deterministic trend

### Question 3: Answer



Answer: (B)

- KPSS:  $H_0$  stationary (opposite of ADF). Use both tests!



## Question 4

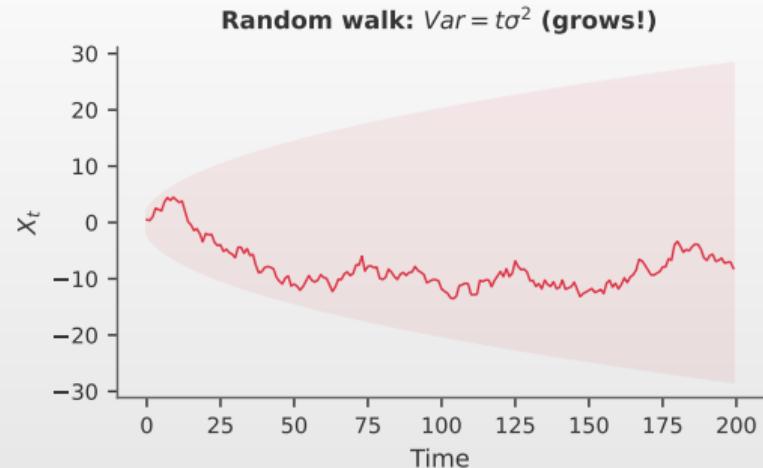
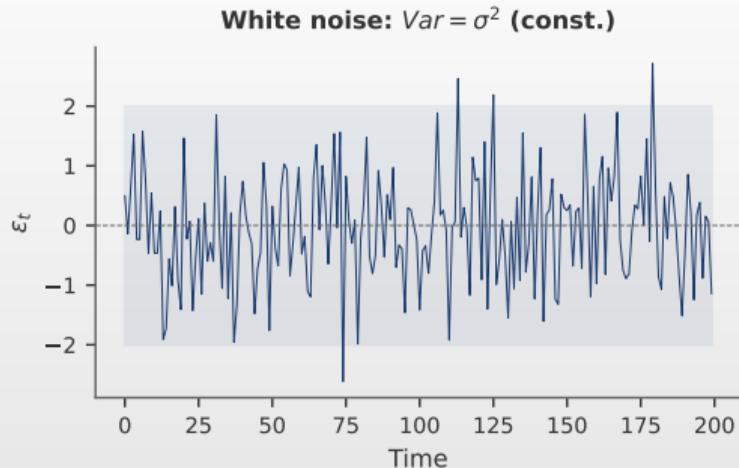
### Question

- What is the key property of the variance of a random walk  $X_t = X_{t-1} + \varepsilon_t$ ?

### Answer Choices

- (A)** Variance is constant:  $\text{Var}(X_t) = \sigma^2$
- (B)** Variance grows linearly with time:  $\text{Var}(X_t) = t\sigma^2$
- (C)** Variance decreases with time
- (D)** Variance is zero

## Question 4: Answer



Answer: (B)

- $\text{Var}(X_t) = t\sigma^2$  grows linearly  $\succ$  non-stationary



## Question 5

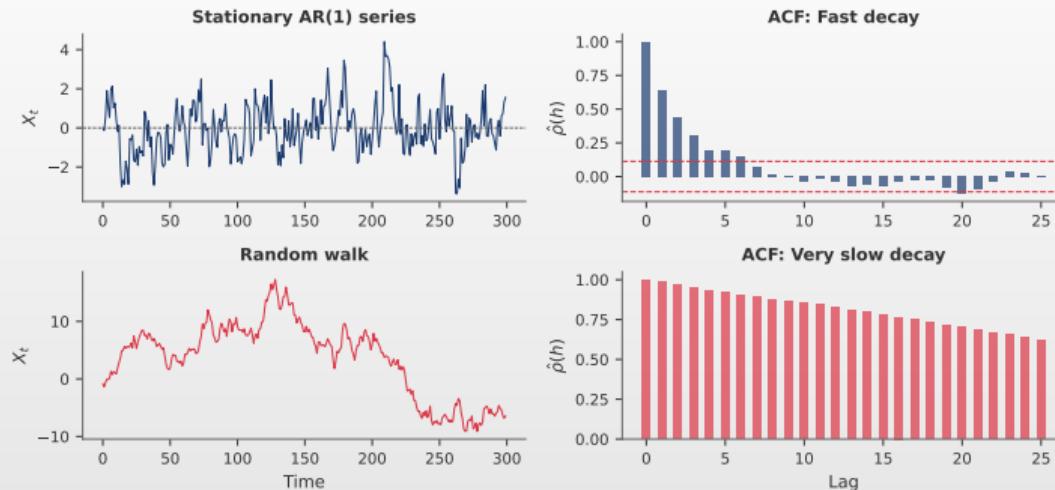
### Question

- What does the ACF of a random walk (non-stationary series with unit root) look like?

### Answer Choices

- (A)** All values are zero after lag 0
- (B)** Decays exponentially fast
- (C)** Decays very slowly (high persistence)
- (D)** Oscillates between positive and negative

## Question 5: Answer



Answer: (C)

- ACF  $\approx 1$  for many lags, slow decay  $\succ$  ADF test

Q TSA\_ch1\_random\_walk



## Question 6

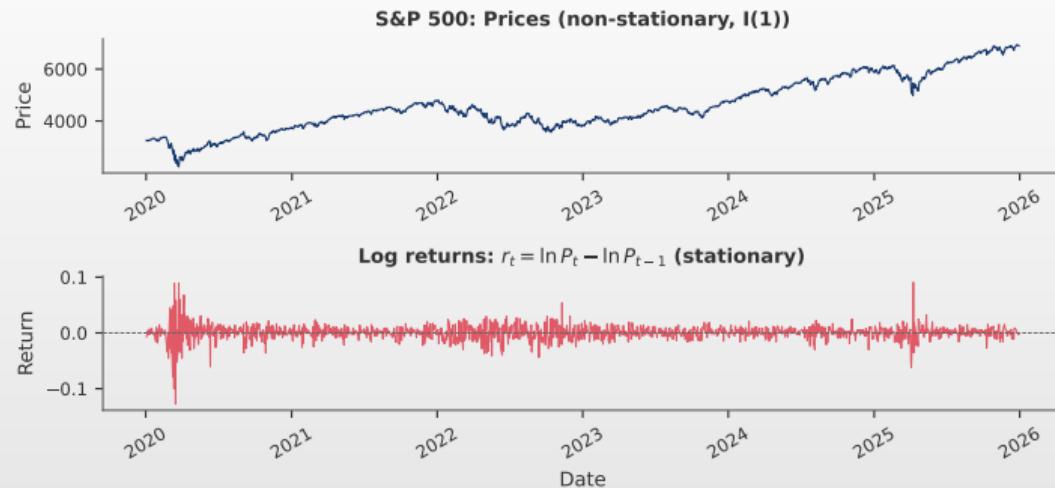
### Question

- How do we obtain stationary returns from a financial price series  $P_t$ ?

### Answer Choices

- (A)** Simple differencing:  $\Delta P_t = P_t - P_{t-1}$
- (B)** Log then differencing:  $r_t = \ln P_t - \ln P_{t-1}$
- (C)** Log only:  $\ln P_t$
- (D)** Standardization:  $(P_t - \bar{P})/s_P$

## Question 6: Answer



Answer: (B)

- Log returns:  $r_t = \ln P_t - \ln P_{t-1}$
- First  $\ln$  (stabilizes variance), then  $\Delta$  (removes trend)  $\succ$  stationary series



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- Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.

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- Kwiatkowski, D., et al. (1992). Testing the Null Hypothesis of Stationarity, *Journal of Econometrics*, 54(1–3), 159–178.

### Online Resources and Code

- **Quantlet:** <https://quantlet.com> ↗ Code repository for statistics
- **Quantinar:** <https://quantinar.com> ↗ Quantitative methods learning platform
- **GitHub TSA:** [https://github.com/QuantLet/TSA/tree/main/TSA\\_ch1](https://github.com/QuantLet/TSA/tree/main/TSA_ch1) ↗ Python code for this chapter



# Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>

