



# Chapter 2: ARMA Models

Stationary Time Series



# Outline

- 1 Introduction and Lag Operator
- 2 Autoregressive (AR) Models
- 3 Moving Average (MA) Models
- 4 ARMA Models
- 5 Model Identification
- 6 Parameter Estimation
- 7 Model Diagnostics
- 8 Forecasting with ARMA
- 9 Practical Implementation
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## Recap: Stationarity

**From Chapter 1:** A process  $\{X_t\}$  is **weakly stationary** if:

- ①  $\mathbb{E}[X_t] = \mu$  (constant mean)
- ②  $\text{Var}(X_t) = \sigma^2 < \infty$  (constant, finite variance)
- ③  $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$  (covariance depends only on lag  $h$ )

**Why stationarity matters for ARMA:**

- ARMA models assume the underlying process is stationary
- Non-stationary data must be differenced first (ARIMA)
- Stationarity ensures stable model parameters

**Today:** We build models for stationary time series using past values and past errors.

# The Lag Operator (Backshift Operator)

## Definition 1 (Lag Operator)

The **lag operator**  $L$  (or backshift operator  $B$ ) shifts a time series back by one period:

$$LX_t = X_{t-1}$$

### Properties:

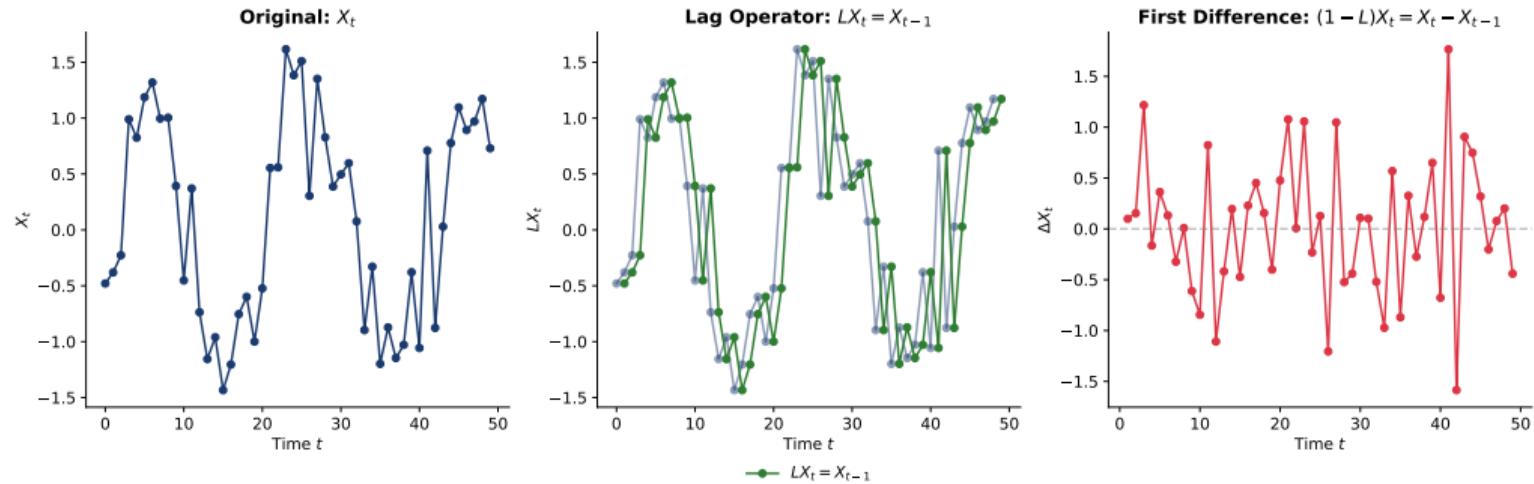
- $L^k X_t = X_{t-k}$  (shift back  $k$  periods)
- $L^0 X_t = X_t$  (identity)
- $(1 - L)X_t = X_t - X_{t-1} = \Delta X_t$  (first difference)
- $(1 - L)^d X_t = \Delta^d X_t$  ( $d$ -th difference)

### Lag Polynomials:

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$$

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$

# Lag Operator: Visual Illustration



**Key insight:** The lag operator is the foundation of ARMA model notation

## Definition 2 (White Noise)

A process  $\{\varepsilon_t\}$  is **white noise**, denoted  $\varepsilon_t \sim WN(0, \sigma^2)$ , if:

- ①  $\mathbb{E}[\varepsilon_t] = 0$  for all  $t$
- ②  $\text{Var}(\varepsilon_t) = \sigma^2$  for all  $t$
- ③  $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$  for all  $t \neq s$

## Properties:

- White noise is the “building block” of ARMA models
- ACF:  $\rho(0) = 1$ ,  $\rho(h) = 0$  for  $h \neq 0$
- PACF: same pattern
- **Gaussian white noise:** additionally  $\varepsilon_t \sim N(0, \sigma^2)$

**Note:** White noise is *not* predictable — it's pure randomness.

## Definition 3 (AR(1) Process)

An **autoregressive process of order 1** is:

$$X_t = c + \phi X_{t-1} + \varepsilon_t$$

where  $\varepsilon_t \sim WN(0, \sigma^2)$  and  $|\phi| < 1$  for stationarity.

### Interpretation:

- $c$ : constant (intercept)
- $\phi$ : autoregressive coefficient — measures persistence
- $\varepsilon_t$ : innovation (unpredictable shock)

### Using lag operator:

$$(1 - \phi L)X_t = c + \varepsilon_t$$

$$\phi(L)X_t = c + \varepsilon_t \quad \text{where } \phi(L) = 1 - \phi L$$

## AR(1) Stationarity Condition

**For AR(1) to be stationary:**  $|\phi| < 1$

**Intuition:**

- If  $|\phi| < 1$ : shocks decay over time  $\rightarrow$  stationary
- If  $|\phi| = 1$ : random walk  $\rightarrow$  non-stationary (unit root)
- If  $|\phi| > 1$ : explosive process  $\rightarrow$  non-stationary

**Characteristic equation:**

$$\phi(z) = 1 - \phi z = 0 \implies z = \frac{1}{\phi}$$

Stationarity requires the root  $z = 1/\phi$  to lie **outside the unit circle**, i.e.,  $|z| > 1$ , which means  $|\phi| < 1$ .

# AR(1) Properties

For a stationary AR(1) with  $|\phi| < 1$ :

**Mean:**

$$\mu = \mathbb{E}[X_t] = \frac{c}{1 - \phi}$$

**Variance:**

$$\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1 - \phi^2}$$

**Autocovariance:**

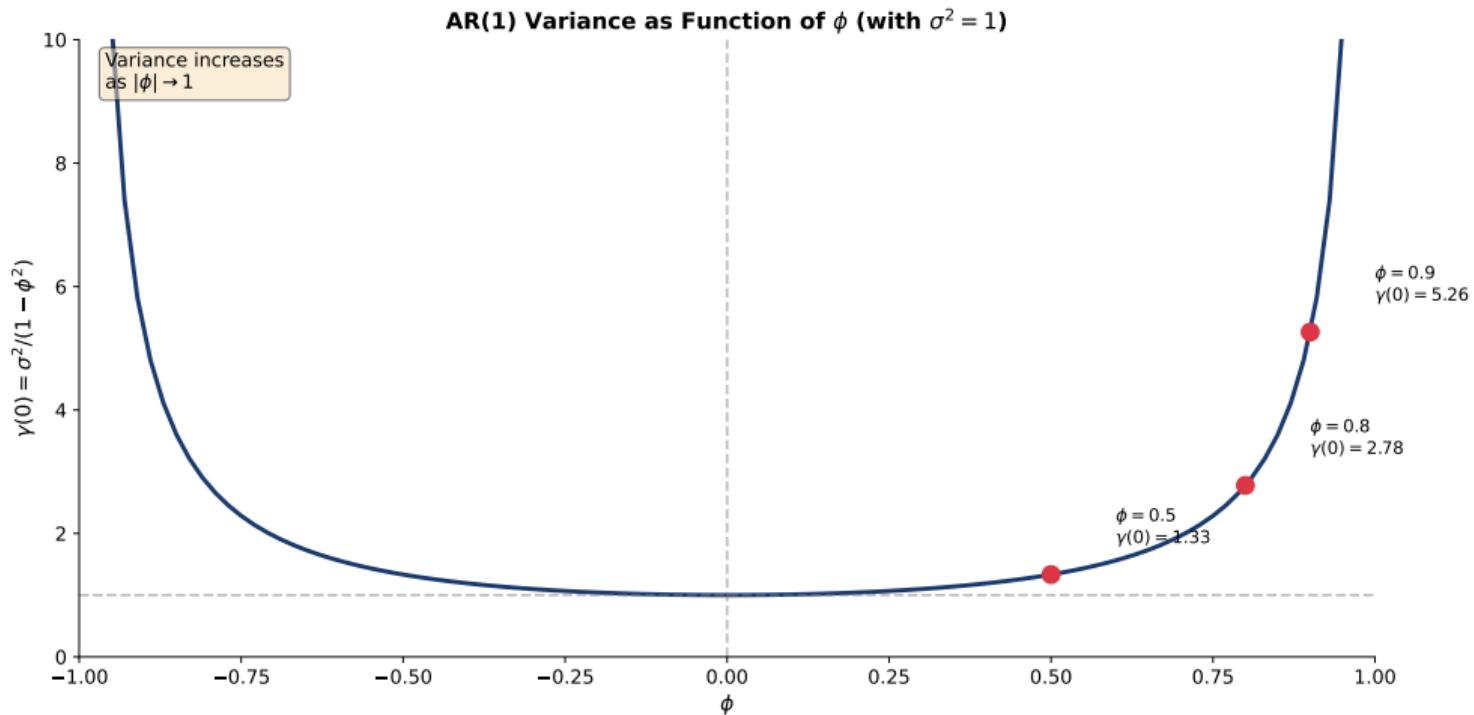
$$\gamma(h) = \phi^h \gamma(0) = \frac{\phi^h \sigma^2}{1 - \phi^2}$$

**Autocorrelation (ACF):**

$$\rho(h) = \phi^h$$

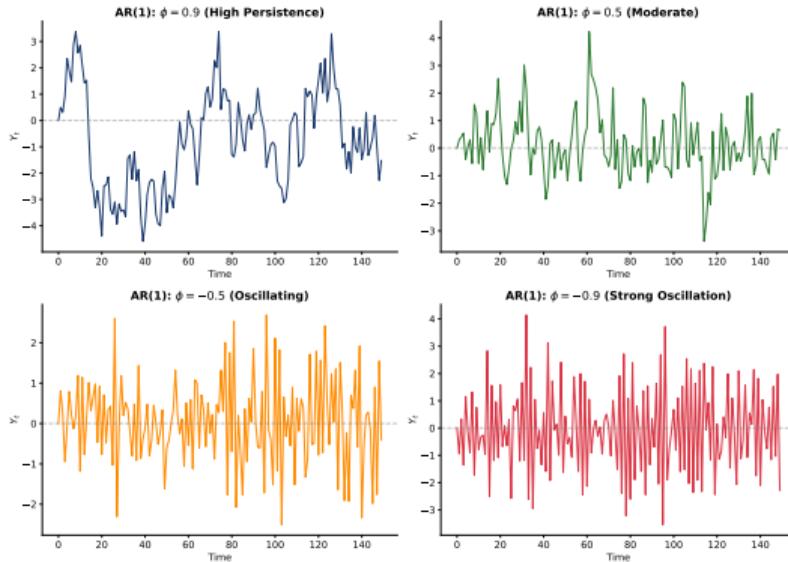
**Key insight:** ACF decays exponentially at rate  $\phi$

## AR(1) Variance as Function of $\phi$



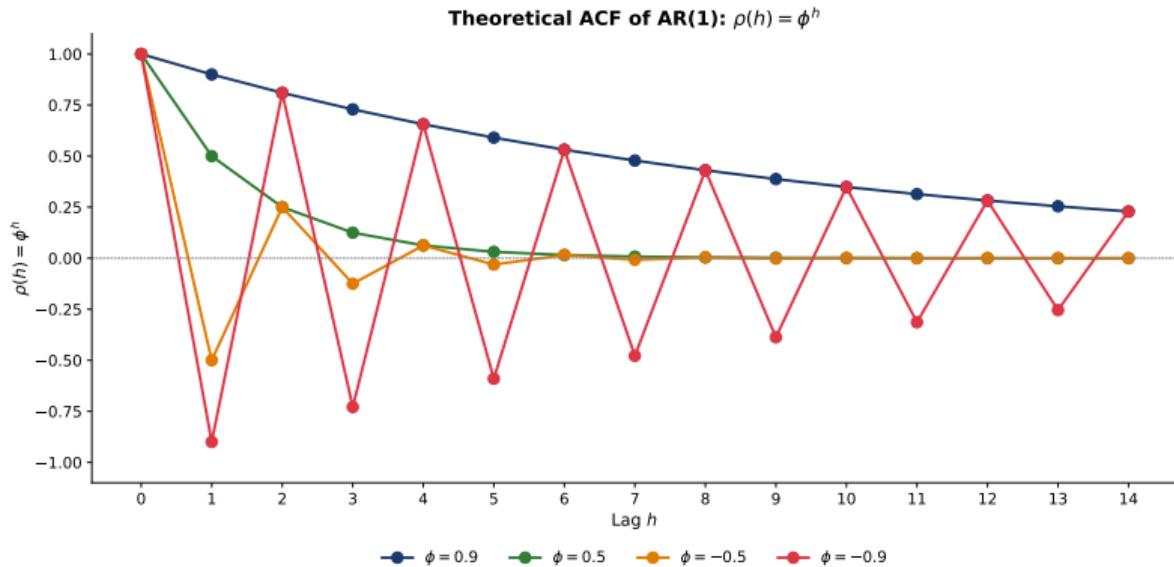
**Key insight:** As  $|\phi| \rightarrow 1$ , variance explodes  $\rightarrow$  non-stationarity

## AR(1) Simulations: Effect of $\phi$



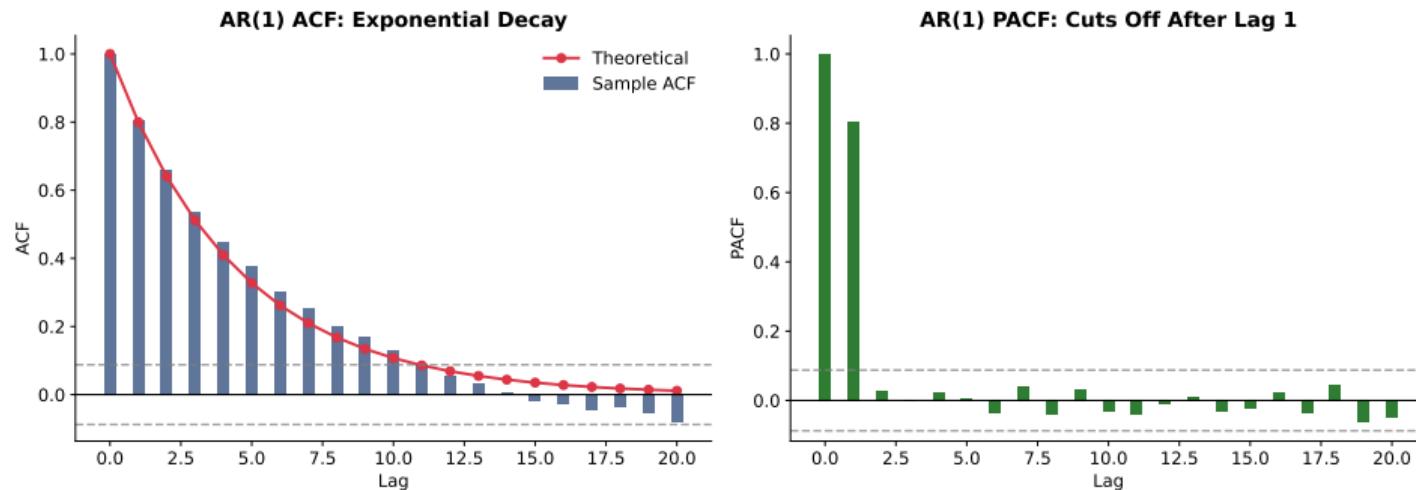
- Different  $\phi$  values produce distinct behavior: higher  $|\phi|$  means more persistence
- Positive  $\phi$  creates smooth, trending patterns; negative  $\phi$  creates oscillations
- As  $|\phi| \rightarrow 1$ , the process becomes more persistent and approaches non-stationarity

## AR(1) Theoretical ACF



**Pattern:**  $\rho(h) = \phi^h$  — exponential decay (or alternating for  $\phi < 0$ )

# AR(1) ACF and PACF: Theory vs Sample



- **ACF:** Exponential decay at rate  $\phi$  – theoretical formula:  $\rho(h) = \phi^h$
- **PACF:** Single spike at lag 1, then cuts off – this identifies AR(1)
- Sample estimates (bars) fluctuate around theoretical values; use confidence bands

## AR(1) ACF and PACF Patterns

### ACF of AR(1):

- Decays exponentially:  $\rho(h) = \phi^h$
- If  $\phi > 0$ : all positive, gradual decay
- If  $\phi < 0$ : alternating signs, decay in magnitude

### PACF of AR(1):

- **Cuts off after lag 1**
- $\pi_1 = \phi$ ,  $\pi_k = 0$  for  $k > 1$

	ACF	PACF
AR(1)	Exponential decay	Cuts off at lag 1

This is the key identification pattern for AR(1)!

## Definition 4 (AR(p) Process)

An autoregressive process of order p is:

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \varepsilon_t$$

Using lag operator:

$$\phi(L)X_t = c + \varepsilon_t$$

where  $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$

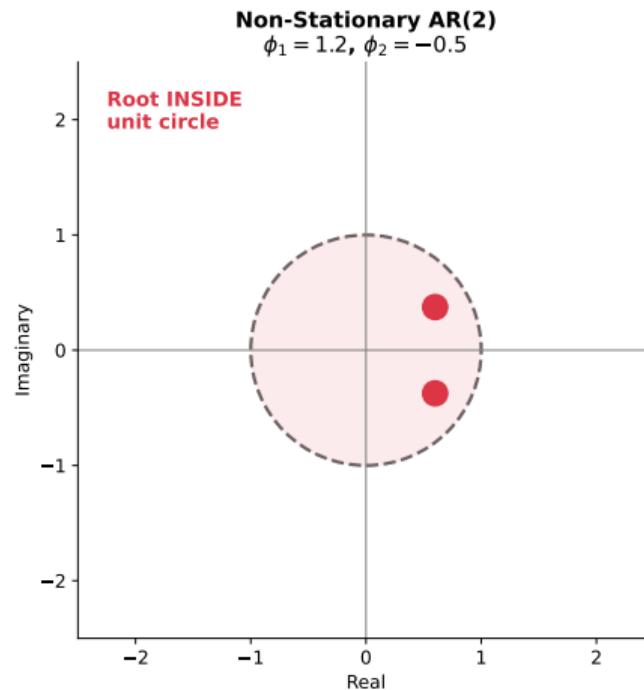
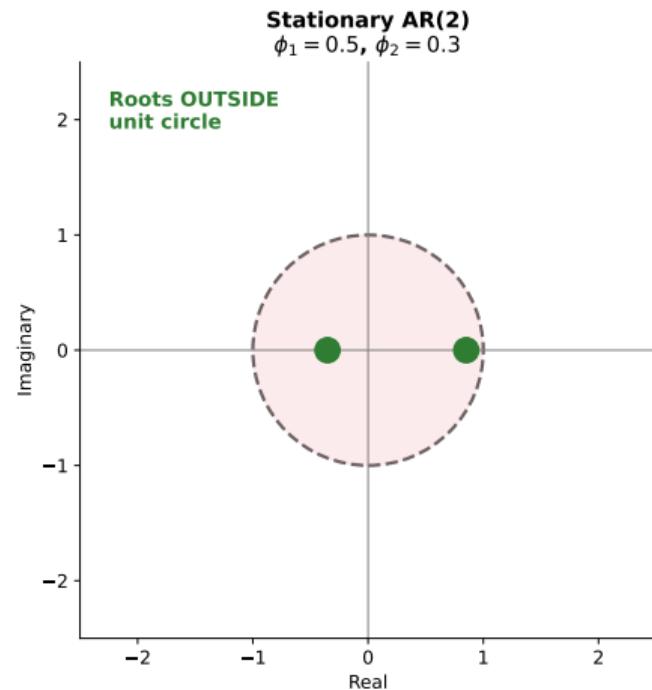
Stationarity condition:

- All roots of  $\phi(z) = 0$  must lie **outside** the unit circle
- Equivalently: all roots have modulus  $> 1$

PACF pattern:

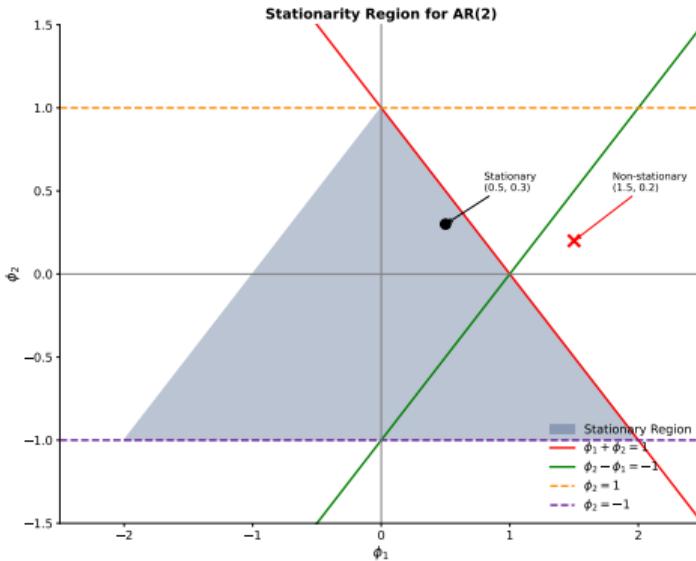
- PACF cuts off after lag  $p$
- ACF decays (exponentially or with damped oscillations)

## AR(2) Stationarity: Unit Circle Visualization



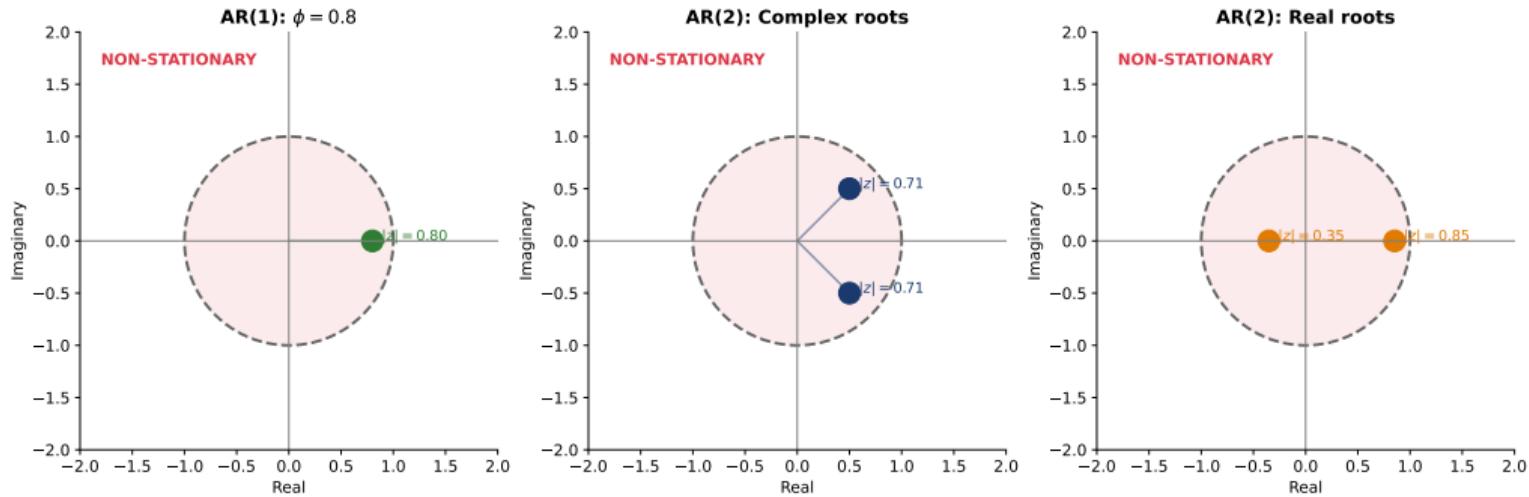
**Rule:** All roots of  $\phi(z) = 0$  must lie **outside** the shaded unit circle

## AR(2) Stationarity Triangle



- The triangular region defines all stationary AR(2) parameter combinations
- Boundaries:  $\phi_1 + \phi_2 < 1$ ,  $\phi_2 - \phi_1 < 1$ , and  $|\phi_2| < 1$
- Points outside this region lead to non-stationary or explosive processes

# Characteristic Polynomial Roots



## Definition 5 (AR(2) Process)

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

**Stationarity conditions for AR(2):**

- ①  $\phi_1 + \phi_2 < 1$
- ②  $\phi_2 - \phi_1 < 1$
- ③  $|\phi_2| < 1$

**ACF behavior depends on roots:**

- **Real roots:** mixture of two exponential decays
- **Complex roots:** damped sinusoidal pattern (pseudo-cycles)

**PACF:** Cuts off after lag 2 ( $\pi_k = 0$  for  $k > 2$ )

## Quiz: AR Stationarity

**Question:** For which value of  $\phi$  is the AR(1) process  $X_t = c + \phi X_{t-1} + \varepsilon_t$  stationary?

- A.  $\phi = 1.2$
- B.  $\phi = 1.0$
- C.  $\phi = -0.8$
- D.  $\phi = -1.5$

## Quiz: AR Stationarity

**Question:** For which value of  $\phi$  is the AR(1) process  $X_t = c + \phi X_{t-1} + \varepsilon_t$  stationary?

- A.  $\phi = 1.2$
- B.  $\phi = 1.0$
- C.  $\phi = -0.8$
- D.  $\phi = -1.5$

**Answer:** C — AR(1) is stationary iff  $|\phi| < 1$ . Only  $|-0.8| = 0.8 < 1$ .

## MA(1) Model: Definition

### Definition 6 (MA(1) Process)

A moving average process of order 1 is:

$$X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$$

where  $\varepsilon_t \sim WN(0, \sigma^2)$ .

#### Interpretation:

- $\mu$ : mean of the process
- $\theta$ : MA coefficient — measures impact of past shock
- Current value depends on current and one past shock

#### Using lag operator:

$$X_t = \mu + \theta(L)\varepsilon_t$$

where  $\theta(L) = 1 + \theta L$

**Key property:** MA processes are **always stationary** for any finite  $\theta$

## MA(1) Properties

For MA(1):  $X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

**Mean:**

$$\mathbb{E}[X_t] = \mu$$

**Variance:**

$$\gamma(0) = \text{Var}(X_t) = \sigma^2(1 + \theta^2)$$

**Autocovariance:**

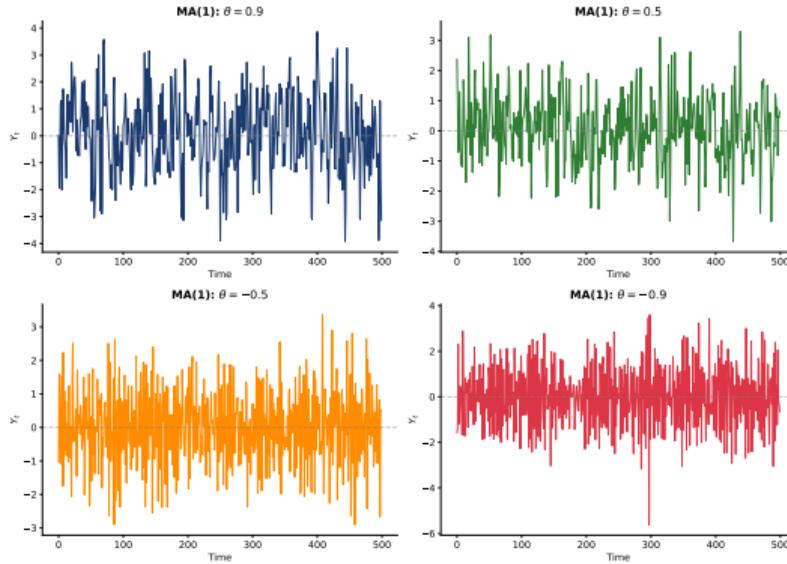
$$\gamma(1) = \theta\sigma^2, \quad \gamma(h) = 0 \text{ for } h > 1$$

**Autocorrelation (ACF):**

$$\rho(1) = \frac{\theta}{1 + \theta^2}, \quad \rho(h) = 0 \text{ for } h > 1$$

**Key insight:** ACF cuts off after lag 1

## MA(1) Simulations: Effect of $\theta$



- MA(1) is always stationary regardless of  $\theta$  – finite memory of only one lag
- Positive  $\theta$  smooths the series; negative  $\theta$  creates more rapid fluctuations
- Unlike AR(1), MA(1) shocks only affect the process for one period

## MA(1) ACF and PACF Patterns

### ACF of MA(1):

- Cuts off after lag 1
- $\rho(1) = \frac{\theta}{1+\theta^2}$ ,  $\rho(h) = 0$  for  $h > 1$
- Note:  $|\rho(1)| \leq 0.5$  always (maximum at  $\theta = \pm 1$ )

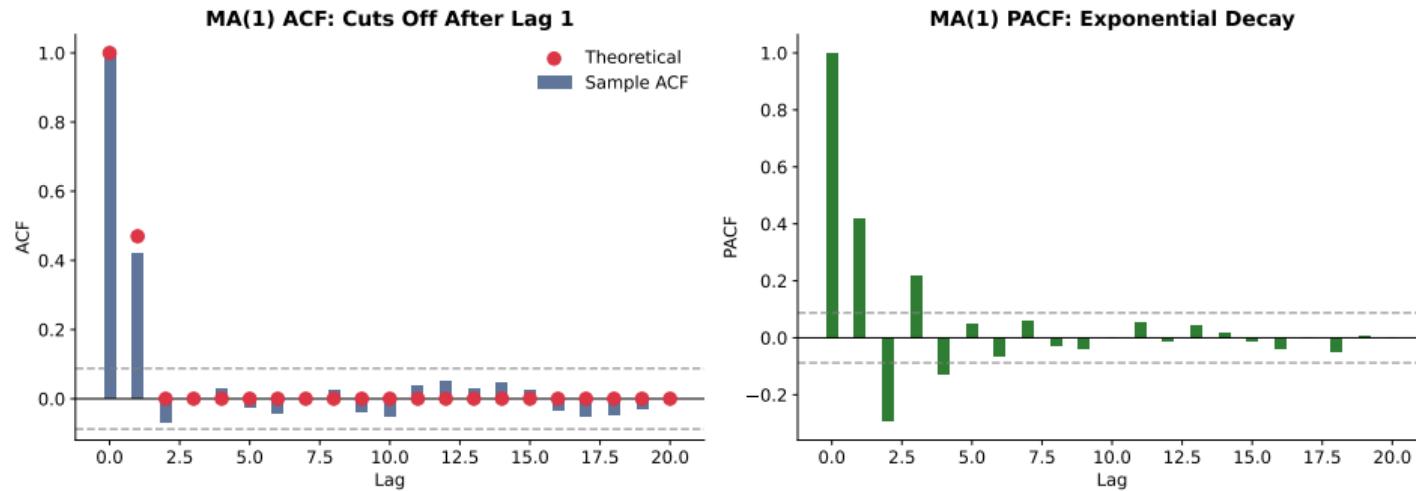
### PACF of MA(1):

- Decays exponentially (or with alternating signs)
- Does *not* cut off

	ACF	PACF
MA(1)	Cuts off at lag 1	Exponential decay

This is the opposite pattern from AR(1)!

# MA(1) ACF and PACF: Visual Comparison



- **ACF:** Single spike at lag 1, then cuts off immediately – key MA(1) signature
- **PACF:** Exponential decay – opposite pattern from AR(1)
- This reversal of ACF/PACF patterns distinguishes MA from AR processes

## Definition 7 (Invertibility)

An MA process is **invertible** if it can be written as an infinite AR process:

$$X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$$

**For MA(1):** Invertible if  $|\theta| < 1$

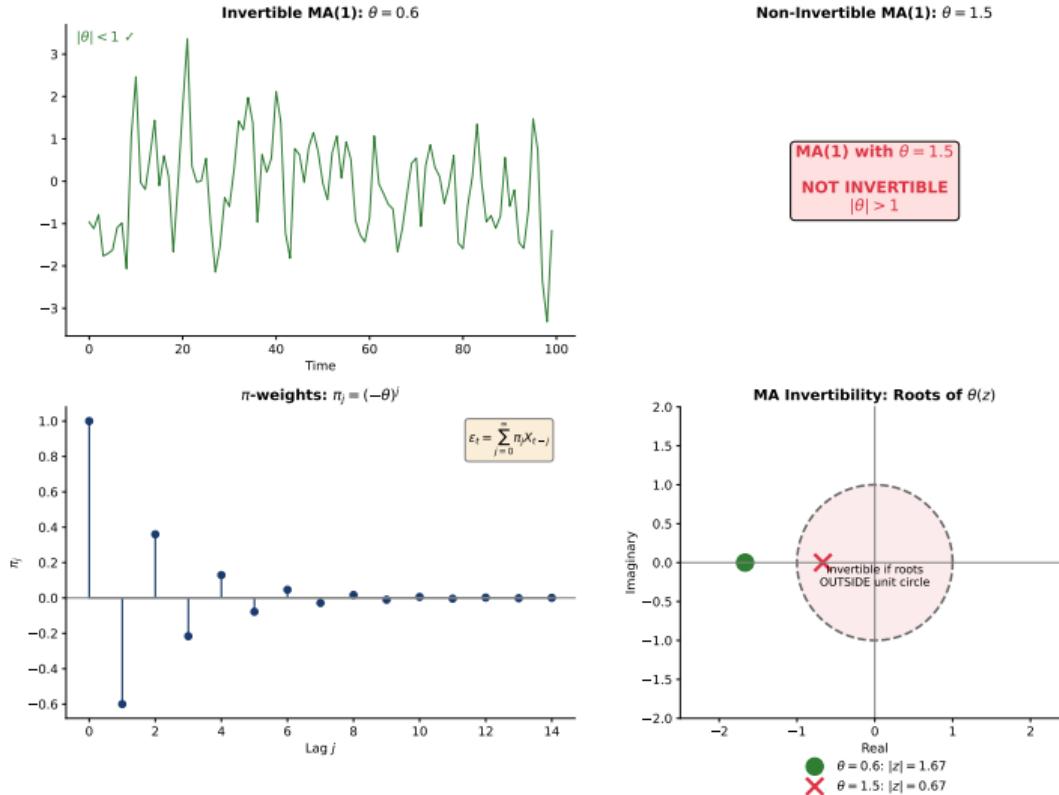
**For MA(q):** All roots of  $\theta(z) = 0$  must lie outside the unit circle

**Why invertibility matters:**

- Ensures unique representation
- Required for forecasting and estimation
- Creates correspondence:  $AR(\infty) \leftrightarrow MA(q)$

**Note:** Stationarity is for AR, Invertibility is for MA

# Invertibility: Visualization



### Definition 8 (MA(q) Process)

A moving average process of order q is:

$$X_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

Using lag operator:

$$X_t = \mu + \theta(L)\varepsilon_t$$

where  $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$

Properties:

- Always stationary (finite variance)
- ACF cuts off after lag  $q$ :  $\rho(h) = 0$  for  $h > q$
- PACF decays gradually
- Invertible if all roots of  $\theta(z) = 0$  lie outside unit circle

## Quiz: ACF/PACF Pattern Recognition

**Question:** You observe: ACF has spike at lag 1, then cuts off. PACF decays gradually. What model?

- A. AR(1)
- B. MA(1)
- C. ARMA(1,1)
- D. White noise

## Quiz: ACF/PACF Pattern Recognition

**Question:** You observe: ACF has spike at lag 1, then cuts off. PACF decays gradually. What model?

- A. AR(1)
- B. MA(1)
- C. ARMA(1,1)
- D. White noise

**Answer:** B — ACF cuts off  $\rightarrow$  MA; PACF decays  $\rightarrow$  confirms MA(1)

## Quiz: MA Invertibility

**Question:** Is MA(1)  $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$  invertible?

- A. Yes, MA processes are always invertible
- B. Yes, because  $1.5 > 0$
- C. No, because  $|\theta| = 1.5 > 1$
- D. No, MA processes are never invertible

## Quiz: MA Invertibility

**Question:** Is MA(1)  $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$  invertible?

- A. Yes, MA processes are always invertible
- B. Yes, because  $1.5 > 0$
- C. No, because  $|\theta| = 1.5 > 1$
- D. No, MA processes are never invertible

**Answer:** C — Invertibility requires  $|\theta| < 1$ . Here  $|\theta| = 1.5 > 1$ .

## Definition 9 (ARMA(p,q) Process)

An autoregressive moving average process of order (p,q) is:

$$X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

**Compact form using lag operators:**

$$\phi(L)(X_t - \mu) = \theta(L)\varepsilon_t$$

or equivalently:

$$\phi(L)X_t = c + \theta(L)\varepsilon_t$$

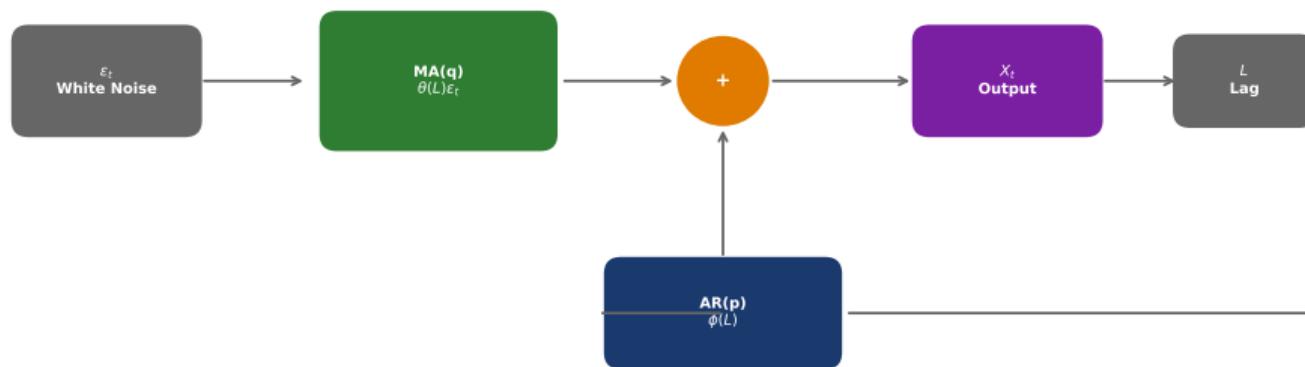
where  $\mu = \frac{c}{1-\phi_1-\cdots-\phi_p}$

**Key idea:** Combines AR and MA components for more flexible modeling

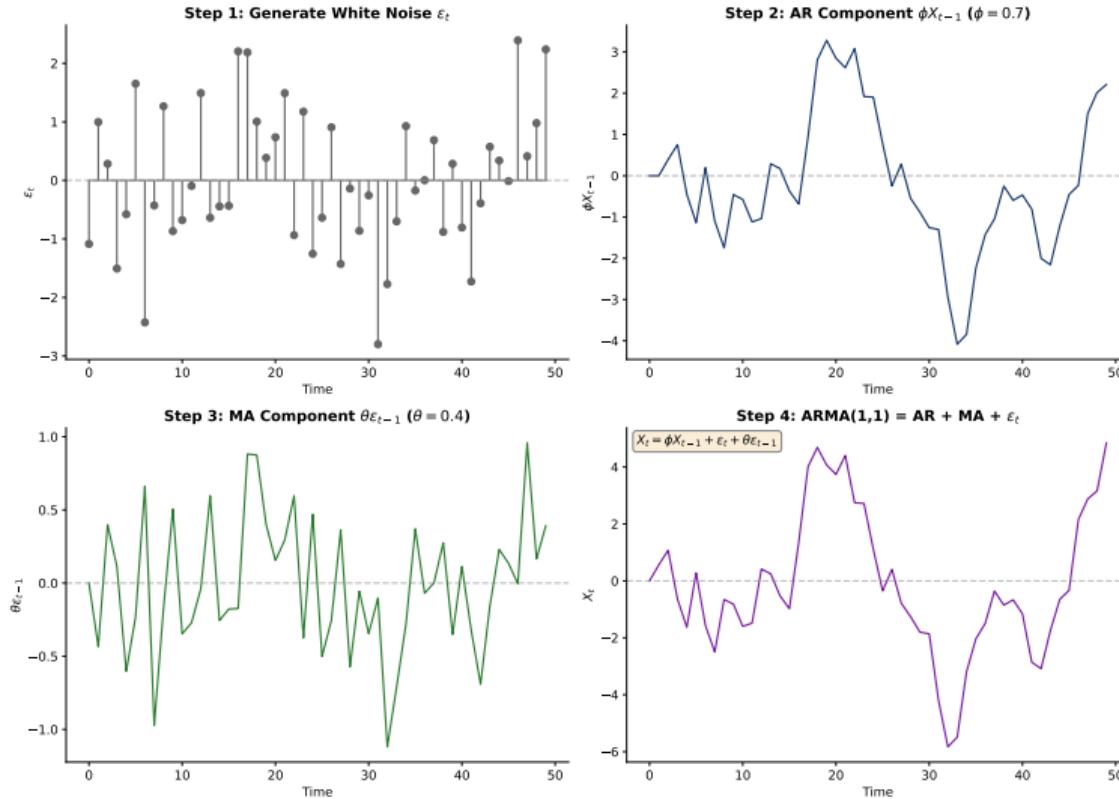
# ARMA Model Structure

## ARMA(p,q) Model Structure

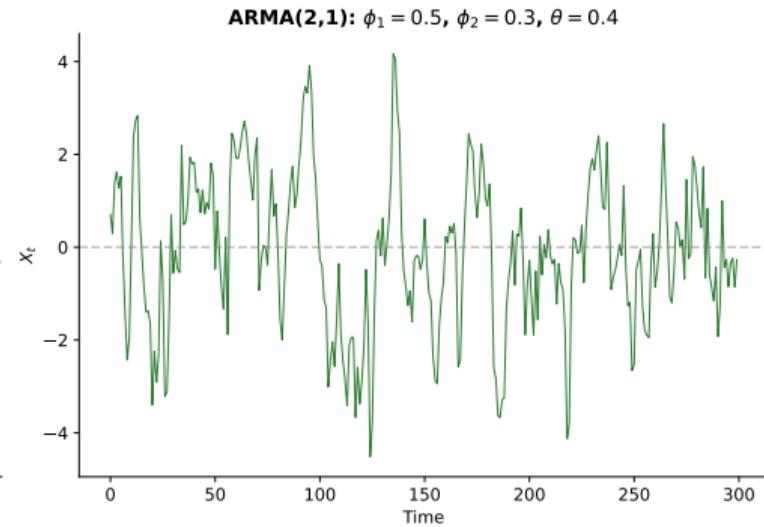
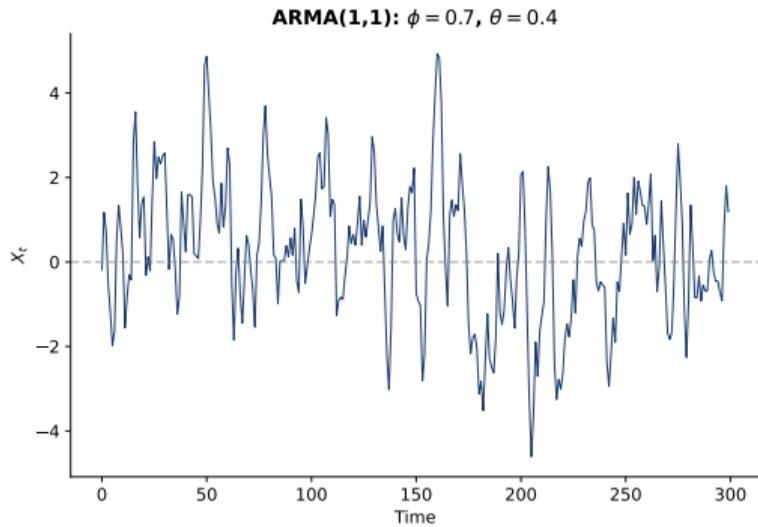
$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$



# How ARMA Simulation Works



# ARMA Examples



## Definition 10 (ARMA(1,1) Process)

$$X_t = c + \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

**Properties (assuming stationarity and invertibility):**

- Mean:  $\mu = \frac{c}{1-\phi}$
- Variance:  $\gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$

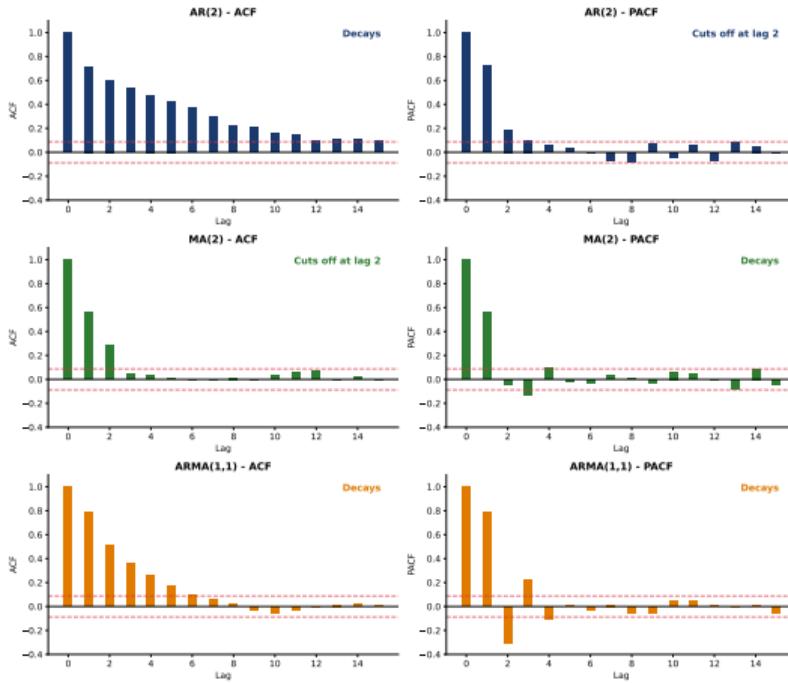
**ACF:**

$$\rho(1) = \frac{(1 + \phi\theta)(\phi + \theta)}{1 + 2\phi\theta + \theta^2}$$

$$\rho(h) = \phi \cdot \rho(h-1) \quad \text{for } h \geq 2$$

**Pattern:** ACF decays exponentially after lag 1 (like AR), but starting point depends on both  $\phi$  and  $\theta$

# ACF/PACF Patterns: AR vs MA vs ARMA



## ARMA ACF and PACF Patterns

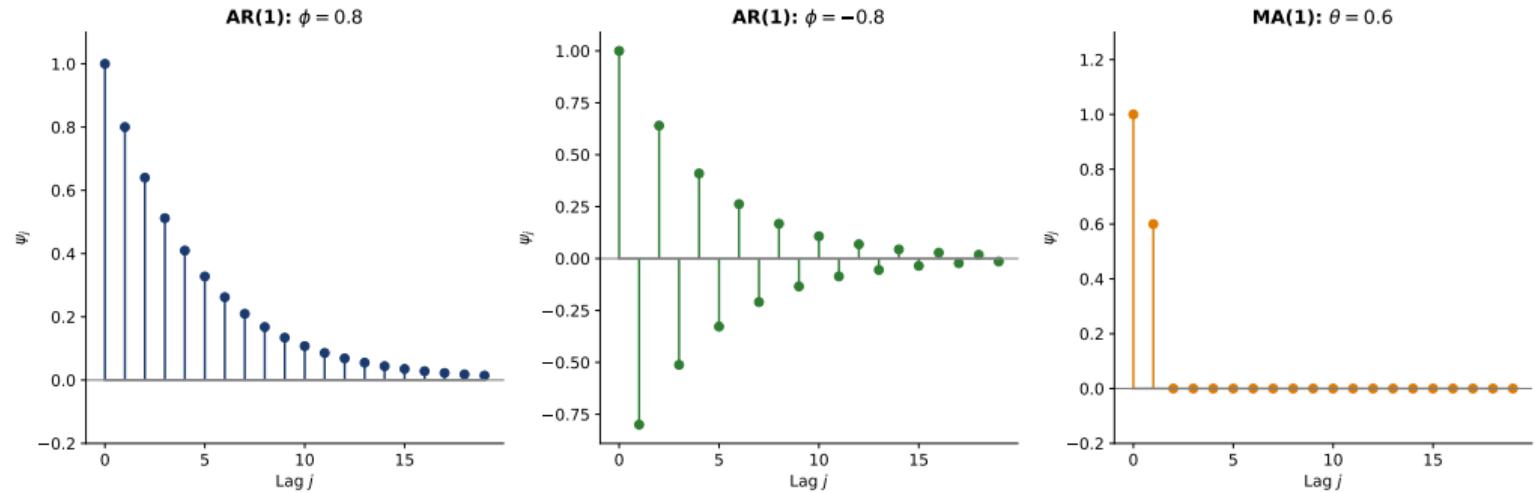
Model	ACF	PACF
AR(p)	Decays (exp./damped)	Cuts off at lag $p$
MA(q)	Cuts off at lag $q$	Decays (exp./damped)
ARMA(p,q)	Decays after lag $q - p$	Decays after lag $p - q$

**Key identification rule:**

- **PACF cuts off** → AR process (order = cutoff lag)
- **ACF cuts off** → MA process (order = cutoff lag)
- **Both decay** → ARMA process

**Caution:** In practice, sample ACF/PACF are noisy; use confidence bands

# Impulse Response Functions



**Interpretation:** Shows how a unit shock propagates through the system over time

## Stationarity and Invertibility Summary

For ARMA(p,q) to be well-behaved:

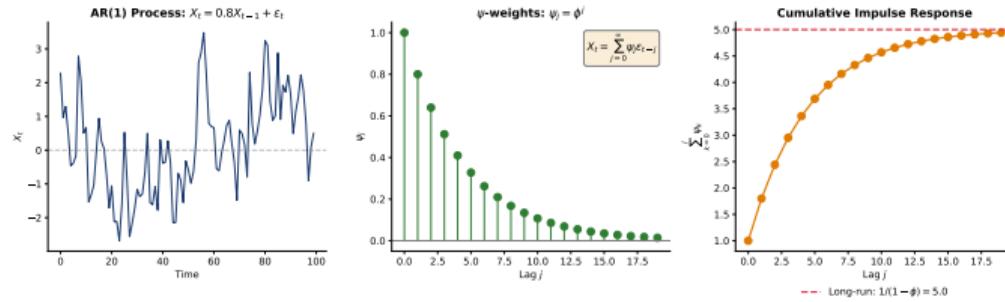
Condition	Requirement
Stationarity	Roots of $\phi(z) = 0$ outside unit circle
Invertibility	Roots of $\theta(z) = 0$ outside unit circle

Implications:

- **Stationarity:** Can write as MA( $\infty$ ):  $X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- **Invertibility:** Can write as AR( $\infty$ ):  $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$

**Causal representation:**  $X_t$  depends only on *past* shocks (not future)

# Wold's Decomposition Theorem



Any stationary process can be written as MA( $\infty$ ):  $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$

## Quiz: ARMA Representation

**Question:** The compact form  $\phi(L)X_t = \theta(L)\varepsilon_t$  represents which model?

- A. Pure AR model
- B. Pure MA model
- C. ARMA model
- D. None of the above

## Quiz: ARMA Representation

**Question:** The compact form  $\phi(L)X_t = \theta(L)\varepsilon_t$  represents which model?

- A. Pure AR model
- B. Pure MA model
- C. ARMA model
- D. None of the above

**Answer: C** —  $\phi(L)$  is AR polynomial,  $\theta(L)$  is MA polynomial  $\rightarrow$  ARMA(p,q)

## Quiz: Lag Operator

**Question:** What is  $(1 - L)^2 X_t$ ?

- A.  $X_t - X_{t-1}$
- B.  $X_t - 2X_{t-1} + X_{t-2}$
- C.  $X_t + X_{t-1} + X_{t-2}$
- D.  $X_t - X_{t-2}$

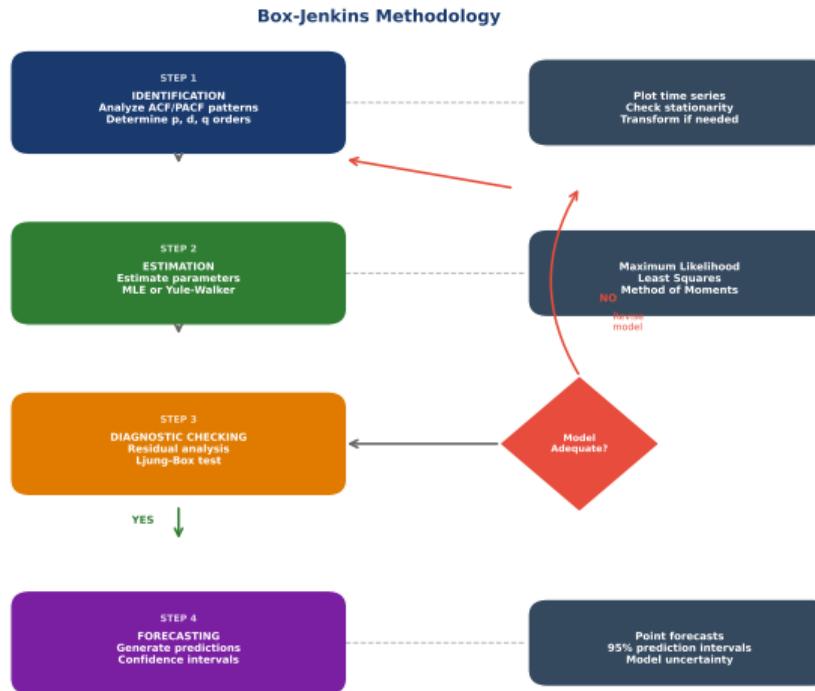
## Quiz: Lag Operator

**Question:** What is  $(1 - L)^2 X_t$ ?

- A.  $X_t - X_{t-1}$
- B.  $X_t - 2X_{t-1} + X_{t-2}$
- C.  $X_t + X_{t-1} + X_{t-2}$
- D.  $X_t - X_{t-2}$

**Answer:** B —  $(1 - L)^2 = 1 - 2L + L^2$ , so  $(1 - L)^2 X_t = X_t - 2X_{t-1} + X_{t-2}$

# The Box-Jenkins Methodology



# Model Identification Summary Table

## Model Identification: ACF/PACF Patterns

Model	ACF Pattern	PACF Pattern
AR(p)	Exponential decay or damped oscillation	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponential decay or damped oscillation
ARMA(p,q)	Exponential decay after lag q-p	Exponential decay after lag p-q

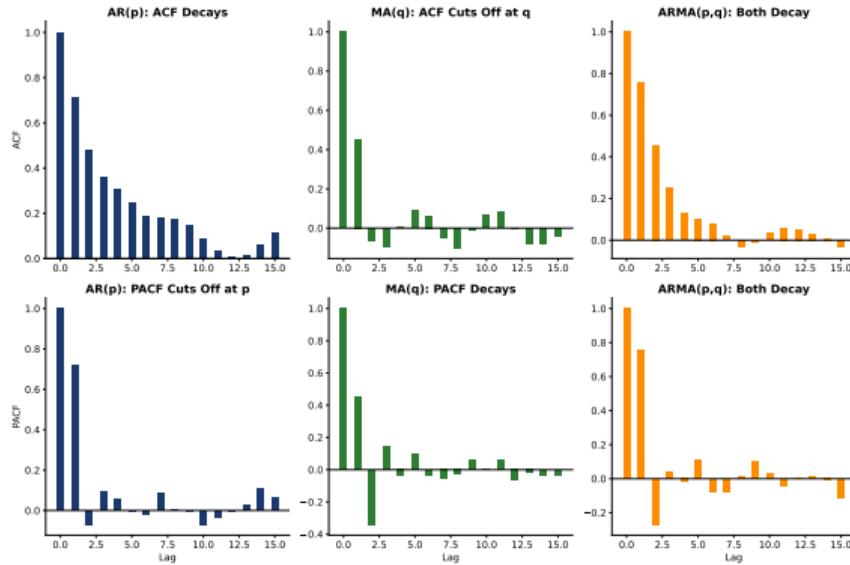
**Practical tip:** Start simple (low  $p, q$ ), increase if diagnostics fail

# ACF/PACF Identification Rules

Theoretical patterns for stationary processes:

Model	ACF Pattern	PACF Pattern
AR(1)	Exponential decay	Spike at lag 1, then 0
AR(2)	Damped exponential/sine	Spikes at lags 1-2, then 0
AR( $p$ )	Decays gradually	Cuts off after lag $p$
MA(1)	Spike at lag 1, then 0	Exponential decay
MA(2)	Spikes at lags 1-2, then 0	Damped exponential/sine
MA( $q$ )	Cuts off after lag $q$	Decays gradually
ARMA( $p,q$ )	Decays	Decays

# ACF/PACF Patterns: Visual Guide



- **AR:** ACF decays, PACF cuts off – use PACF to identify order  $p$
- **MA:** ACF cuts off, PACF decays – use ACF to identify order  $q$
- **ARMA:** Both decay – requires information criteria for model selection

## Information Criteria

**Purpose:** Balance goodness-of-fit against model complexity

**Akaike Information Criterion (AIC):**

$$AIC = -2 \ln(\hat{L}) + 2k$$

**Bayesian Information Criterion (BIC/SBC):**

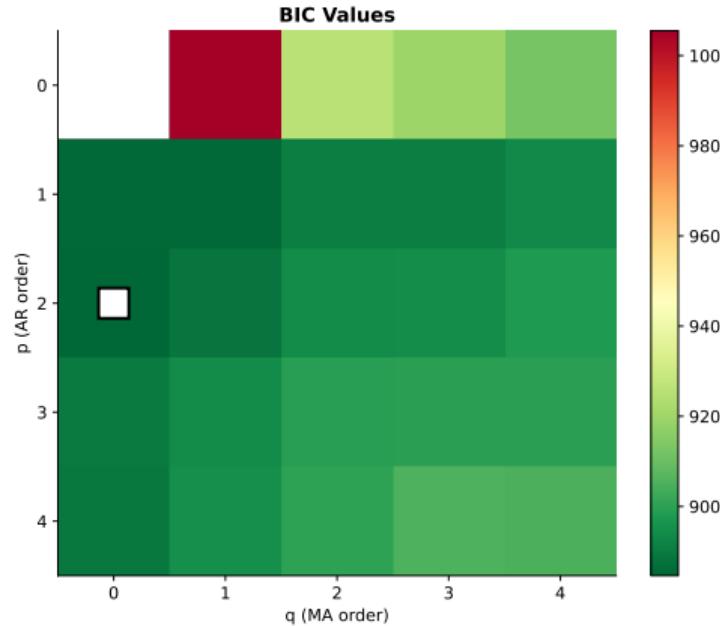
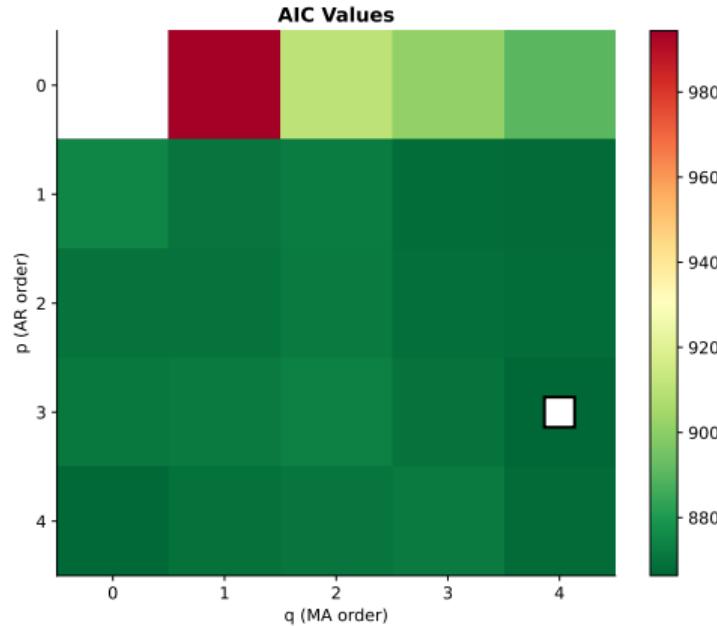
$$BIC = -2 \ln(\hat{L}) + k \ln(n)$$

where  $\hat{L}$  = maximized likelihood,  $k$  = number of parameters,  $n$  = sample size

**Usage:**

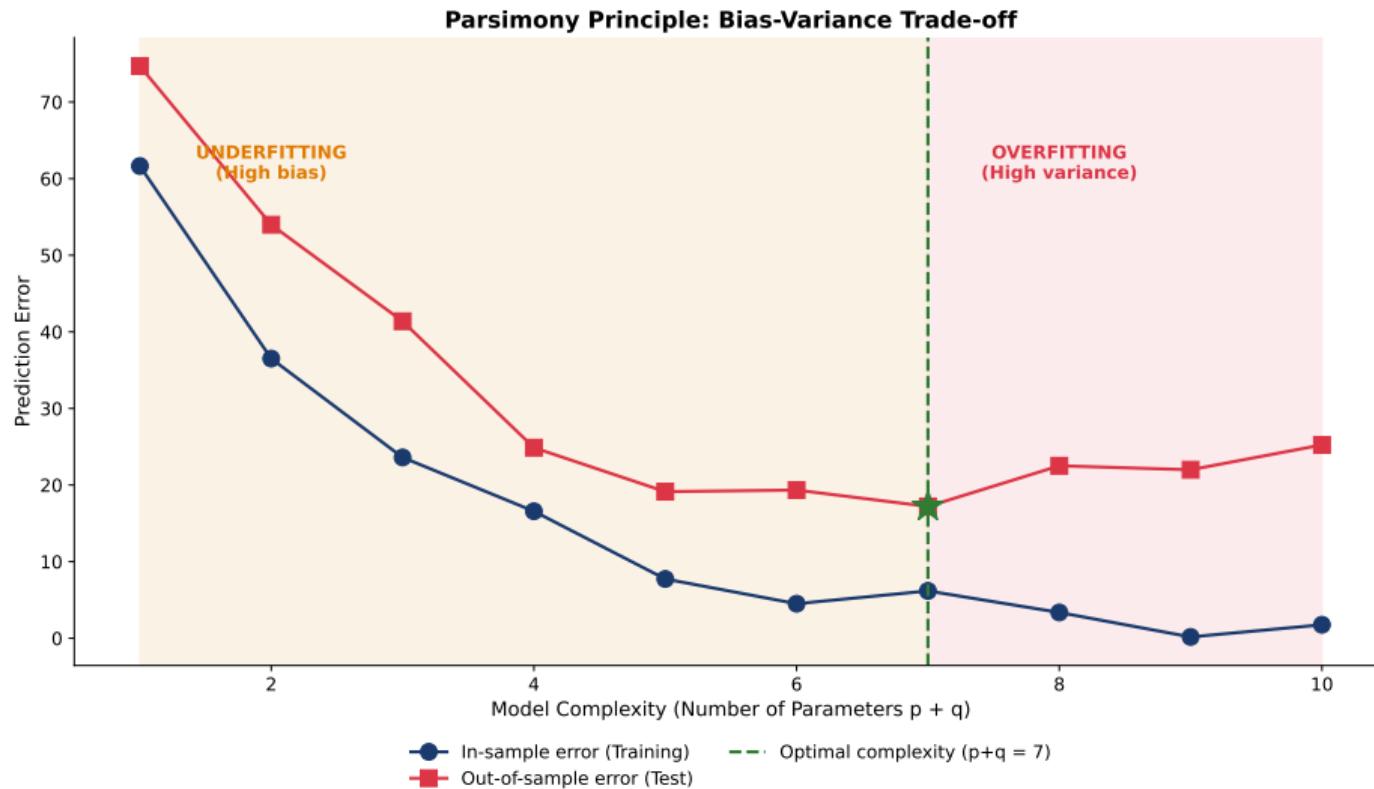
- Lower values are better
- BIC penalizes complexity more strongly than AIC
- AIC tends to choose larger models; BIC more parsimonious
- Compare models fit to the *same data*

## AIC vs BIC: Model Selection



**Note:** White square marks the best model; lower values (green) are better

# Parsimony Principle: Bias-Variance Trade-off



## Grid search approach:

- ① Fit ARMA( $p, q$ ) for  $p = 0, 1, \dots, p_{max}$  and  $q = 0, 1, \dots, q_{max}$
- ② Select model with lowest AIC or BIC
- ③ Verify with diagnostic checks

## In Python (statsmodels):

- `pm.auto_arima()` from `pmdarima` package
- Automatically tests stationarity, searches over orders
- Returns best model by AIC/BIC

## Caution:

- Automatic selection is a starting point, not final answer
- Always check diagnostics
- Consider domain knowledge

## Quiz: Information Criteria

**Question:** Comparing ARMA(1,1) vs ARMA(2,1) using BIC, which is correct?

- A. Lower BIC always means better forecasts
- B. BIC penalizes complexity less than AIC
- C. The model with lower BIC is preferred
- D. BIC can only compare models with same # of parameters

## Quiz: Information Criteria

**Question:** Comparing ARMA(1,1) vs ARMA(2,1) using BIC, which is correct?

- A. Lower BIC always means better forecasts
- B. BIC penalizes complexity less than AIC
- C. The model with lower BIC is preferred
- D. BIC can only compare models with same # of parameters

**Answer:** C — Lower BIC indicates better fit-complexity trade-off. BIC penalizes complexity *more* than AIC.

Three main approaches:

## 1. Method of Moments / Yule-Walker (AR only)

- Match sample autocorrelations to theoretical values
- Simple, closed-form for AR models
- Not efficient for MA components

## 2. Maximum Likelihood Estimation (MLE)

- Most common approach
- Requires distributional assumption (usually Gaussian)
- Efficient and consistent

## 3. Conditional Least Squares

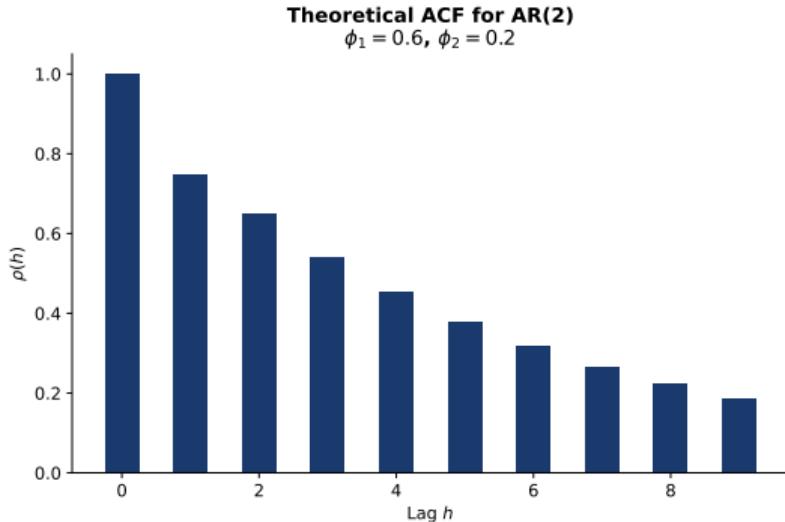
- Minimize sum of squared residuals
- Conditioning on initial observations
- Computationally simpler than exact MLE

## ARMA Parameter Estimation Methods

Yule-Walker	Maximum Likelihood	Conditional LS
<p><b>Pros:</b></p> <ul style="list-style-type: none"><li>+ Simple computation</li><li>+ Closed-form solution</li></ul> <p><b>Cons:</b></p> <ul style="list-style-type: none"><li>- AR only</li><li>- Less efficient</li></ul>	<p><b>Pros:</b></p> <ul style="list-style-type: none"><li>+ Most efficient</li><li>+ Works for ARMA</li></ul> <p><b>Cons:</b></p> <ul style="list-style-type: none"><li>- Iterative</li><li>- Local optima risk</li></ul>	<p><b>Pros:</b></p> <ul style="list-style-type: none"><li>+ Simple to implement</li><li>+ Fast computation</li></ul> <p><b>Cons:</b></p> <ul style="list-style-type: none"><li>- Biased for small n</li><li>- Ignores initial values</li></ul>

Recommendation: Use MLE for final estimation,  
Yule-Walker for initial values

# Yule-Walker Equations for AR(p)



## Yule-Walker Equations

$$\rho(1) = \phi_1 + \phi_2 \rho(1)$$

$$\rho(2) = \phi_1 \rho(1) + \phi_2$$

Matrix form:  $R \cdot \phi = \rho$

$R$  = autocorrelation matrix

Solution:  $\hat{\phi} = R^{-1}\rho$

## Yule-Walker Equations: Matrix Form

For AR(p):  $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t$

**Yule-Walker equations:**

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p)$$

for  $k = 1, 2, \dots, p$

**Matrix form:**

$$\begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{pmatrix}$$

**Estimation:** Replace  $\rho(k)$  with sample autocorrelations  $\hat{\rho}(k)$

# Maximum Likelihood Estimation

Assuming Gaussian errors:  $\varepsilon_t \sim N(0, \sigma^2)$

Log-likelihood for ARMA(p,q):

$$\ell(\phi, \theta, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^n \varepsilon_t^2$$

where  $\varepsilon_t$  are the innovations computed recursively.

Estimation procedure:

- ① Initialize: use method of moments or OLS for starting values
- ② Optimize: numerical methods (e.g., BFGS, Newton-Raphson)
- ③ Iterate until convergence

In practice: Use `statsmodels.tsa.arima.model.ARIMA`

**Asymptotic distribution of MLE:**

$$\hat{\theta} \xrightarrow{d} N\left(\theta_0, \frac{1}{n}\mathbf{I}(\theta_0)^{-1}\right)$$

where  $\mathbf{I}(\theta)$  is the Fisher information matrix.

**Standard errors:** Square root of diagonal of  $\frac{1}{n}\hat{\mathbf{I}}^{-1}$

**Hypothesis testing:**

- $H_0 : \phi_j = 0$  (or  $\theta_j = 0$ )
- Test statistic:  $z = \frac{\hat{\phi}_j}{SE(\hat{\phi}_j)} \sim N(0, 1)$  asymptotically
- Reject if  $|z| > 1.96$  at 5% level

**Confidence interval:**  $\hat{\phi}_j \pm 1.96 \cdot SE(\hat{\phi}_j)$

If model is correctly specified, residuals should be white noise:

## 1. Plot residuals over time

- Should fluctuate around zero
- No obvious patterns or trends
- Constant variance (no heteroskedasticity)

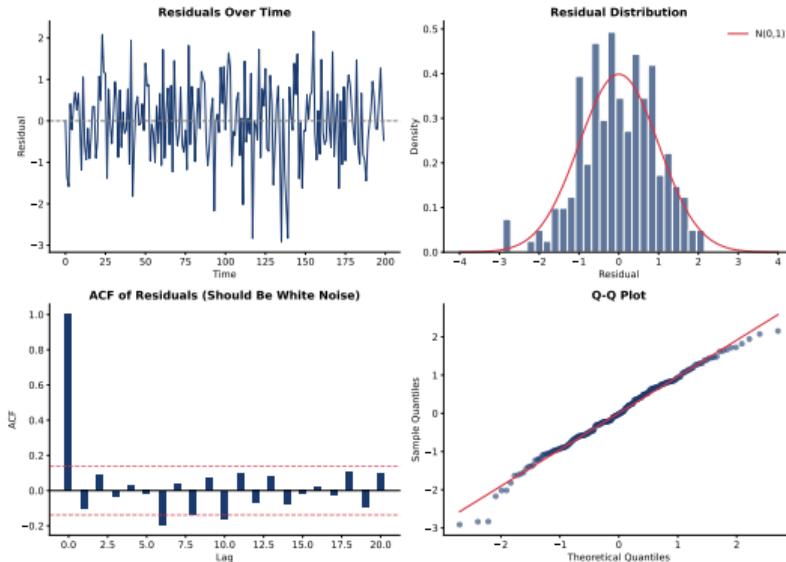
## 2. Check ACF of residuals

- All correlations should be within confidence bands
- No significant spikes → white noise

## 3. Check histogram / Q-Q plot

- Should be approximately normal (if assuming Gaussian)
- Heavy tails suggest non-normal errors

## Residual Diagnostics: Example



- **Residual plot:** Should show random scatter around zero with constant variance
- **ACF of residuals:** No significant spikes indicates white noise (good fit)
- **Q-Q plot:** Points on diagonal line indicate normally distributed residuals

## Definition 11 (Ljung-Box Test)

Tests whether residuals are independently distributed (no autocorrelation).

### Test statistic:

$$Q(m) = n(n + 2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n - k}$$

### Hypotheses:

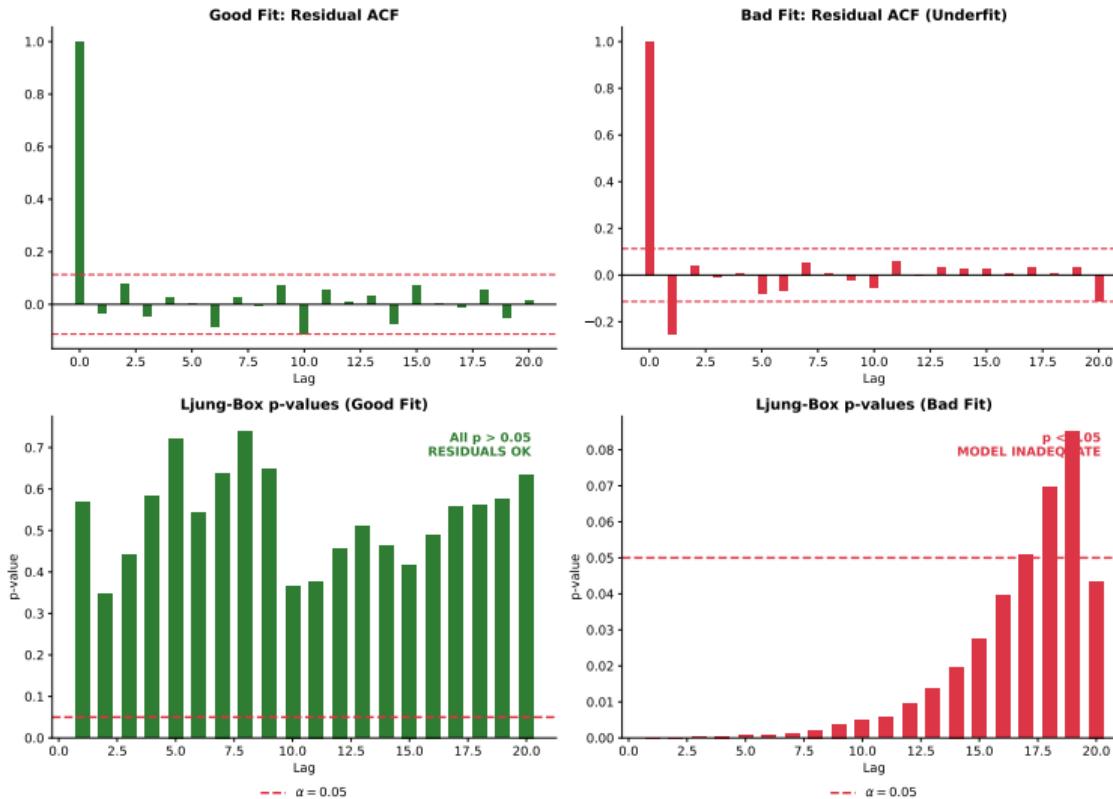
- $H_0$ : Residuals are white noise (no autocorrelation up to lag  $m$ )
- $H_1$ : Residuals are autocorrelated

**Distribution:** Under  $H_0$ ,  $Q(m) \sim \chi^2(m - p - q)$  approximately

### Decision:

- p-value > 0.05 → fail to reject  $H_0$  → residuals look like white noise (good!)
- p-value < 0.05 → significant autocorrelation remains → model inadequate

# Ljung-Box Test: Good vs Bad Model Fit



# Diagnostic Checklist

A good ARMA model should satisfy:

- ① **Stationarity:** AR roots outside unit circle
  - ✓ Check with `arroots`
- ② **Invertibility:** MA roots outside unit circle
  - ✓ Check with `maroots`
- ③ **White noise residuals:** No significant ACF
  - ✓ ACF plot, Ljung-Box test
- ④ **Normal residuals:** (if assumed)
  - ✓ Q-Q plot, Jarque-Bera test
- ⑤ **No heteroskedasticity:** Constant variance
  - ✓ Plot residuals, ARCH test
- ⑥ **Parsimonious:** Lowest AIC/BIC among adequate models

If diagnostics fail: Return to identification, try different orders

## Quiz: Ljung-Box Test

**Question:** After fitting an ARMA model, you run the Ljung-Box test on residuals and get p-value = 0.03. What does this mean?

- A. Model is adequate, residuals are white noise
- B. Model is inadequate, residuals have autocorrelation
- C. Need to increase sample size
- D. Test is inconclusive

## Quiz: Ljung-Box Test

**Question:** After fitting an ARMA model, you run the Ljung-Box test on residuals and get p-value = 0.03. What does this mean?

- A. Model is adequate, residuals are white noise
- B. Model is inadequate, residuals have autocorrelation
- C. Need to increase sample size
- D. Test is inconclusive

**Answer:** B — p-value < 0.05 rejects  $H_0$  (white noise), indicating remaining autocorrelation → model inadequate.

## Point Forecasts

**Optimal forecast:** Conditional expectation minimizes MSE

$$\hat{X}_{n+h|n} = \mathbb{E}[X_{n+h}|X_n, X_{n-1}, \dots]$$

**For AR(1):**  $X_t = c + \phi X_{t-1} + \varepsilon_t$

$$\hat{X}_{n+1|n} = c + \phi X_n$$

$$\hat{X}_{n+2|n} = c + \phi \hat{X}_{n+1|n} = c(1 + \phi) + \phi^2 X_n$$

$$\hat{X}_{n+h|n} = \mu + \phi^h (X_n - \mu)$$

**Key property:** Forecasts converge to mean  $\mu$  as  $h \rightarrow \infty$

**For MA(1):**  $X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

$$\hat{X}_{n+1|n} = \mu + \theta \varepsilon_n$$

$$\hat{X}_{n+h|n} = \mu \quad \text{for } h > 1$$

## Forecast Uncertainty

**Forecast error:**

$$e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$$

**Mean squared forecast error (MSFE):**

$$\text{MSFE}(h) = \mathbb{E}[e_{n+h|n}^2] = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$$

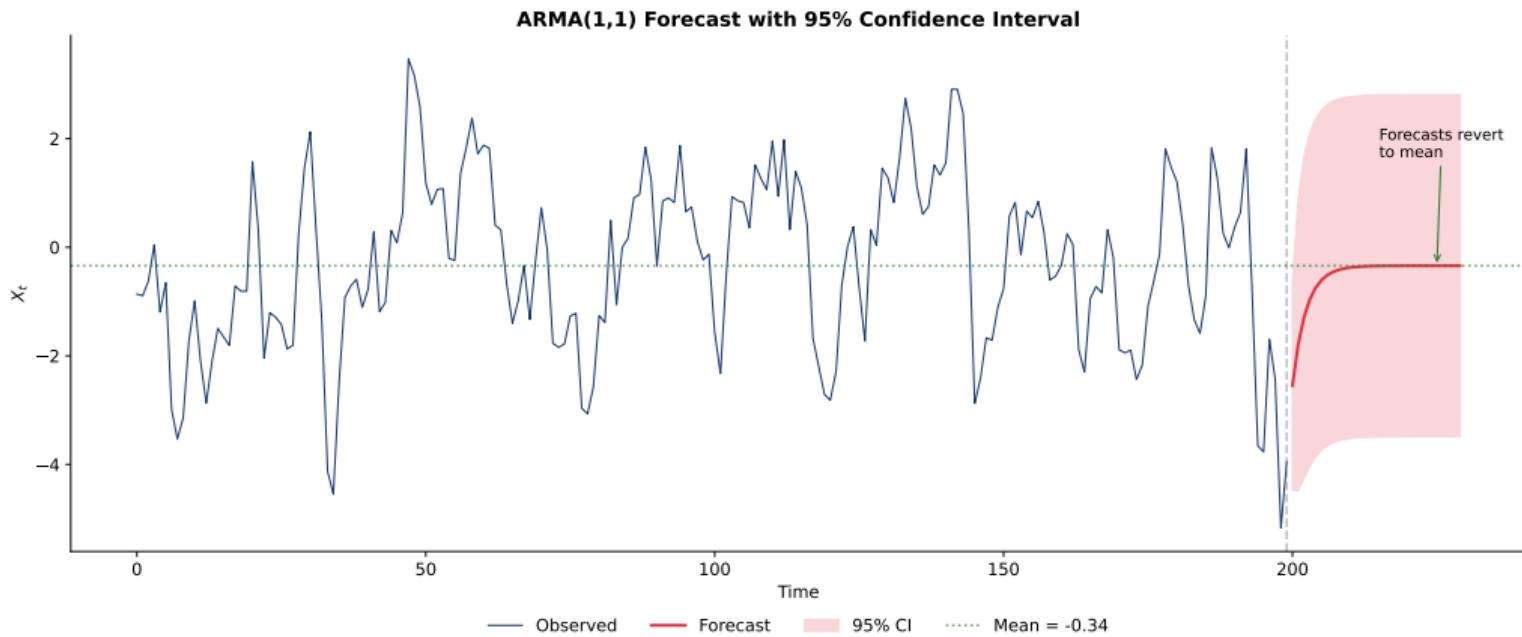
where  $\psi_j$  are the MA( $\infty$ ) coefficients.

**For AR(1):**  $\psi_j = \phi^j$

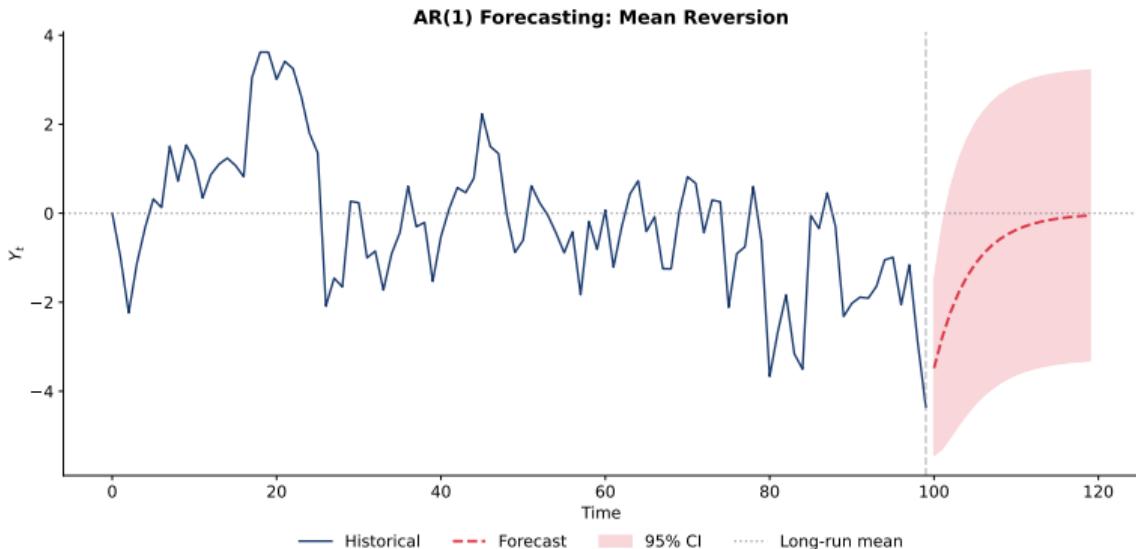
$$\text{MSFE}(h) = \sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2} \rightarrow \frac{\sigma^2}{1 - \phi^2} = \text{Var}(X_t)$$

**Key insight:** Forecast uncertainty increases with horizon, eventually reaching unconditional variance

# ARMA Forecasting with Confidence Intervals

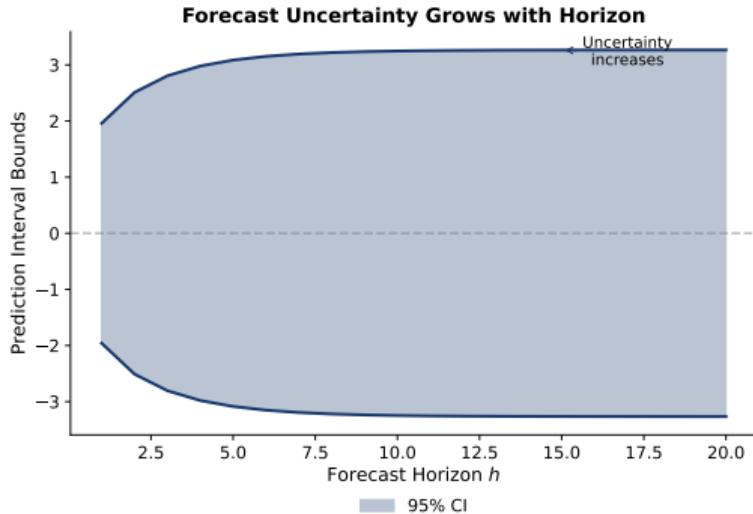
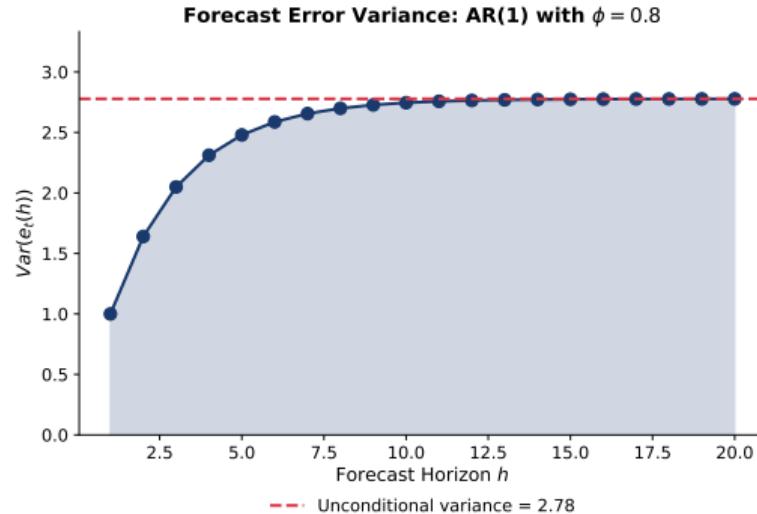


## AR(1) Forecasting: Mean Reversion



- Forecasts converge to the unconditional mean  $\mu$  as horizon increases
- Rate of convergence depends on  $|\phi|$ : higher values mean slower reversion
- Confidence intervals widen with horizon, eventually reaching unconditional variance

# Forecast Error Variance Over Horizon



# Confidence Intervals for Forecasts

**Assuming Gaussian errors:**

$$X_{n+h}|X_n, \dots \sim N\left(\hat{X}_{n+h|n}, \text{MSFE}(h)\right)$$

**$(1 - \alpha)$  confidence interval:**

$$\hat{X}_{n+h|n} \pm z_{\alpha/2} \cdot \sqrt{\text{MSFE}(h)}$$

where  $z_{\alpha/2} = 1.96$  for 95% CI.

**Properties:**

- Intervals widen as horizon increases
- Eventually converge to unconditional interval:  $\mu \pm z_{\alpha/2}\sigma_x$
- Width depends on model parameters (AR coefficients, etc.)

**In Python:** `model.get_forecast(h).conf_int()`

## Out-of-sample testing:

- ① Split data: training set (fit model) and test set (evaluate)
- ② Generate forecasts for test period
- ③ Compare forecasts to actual values

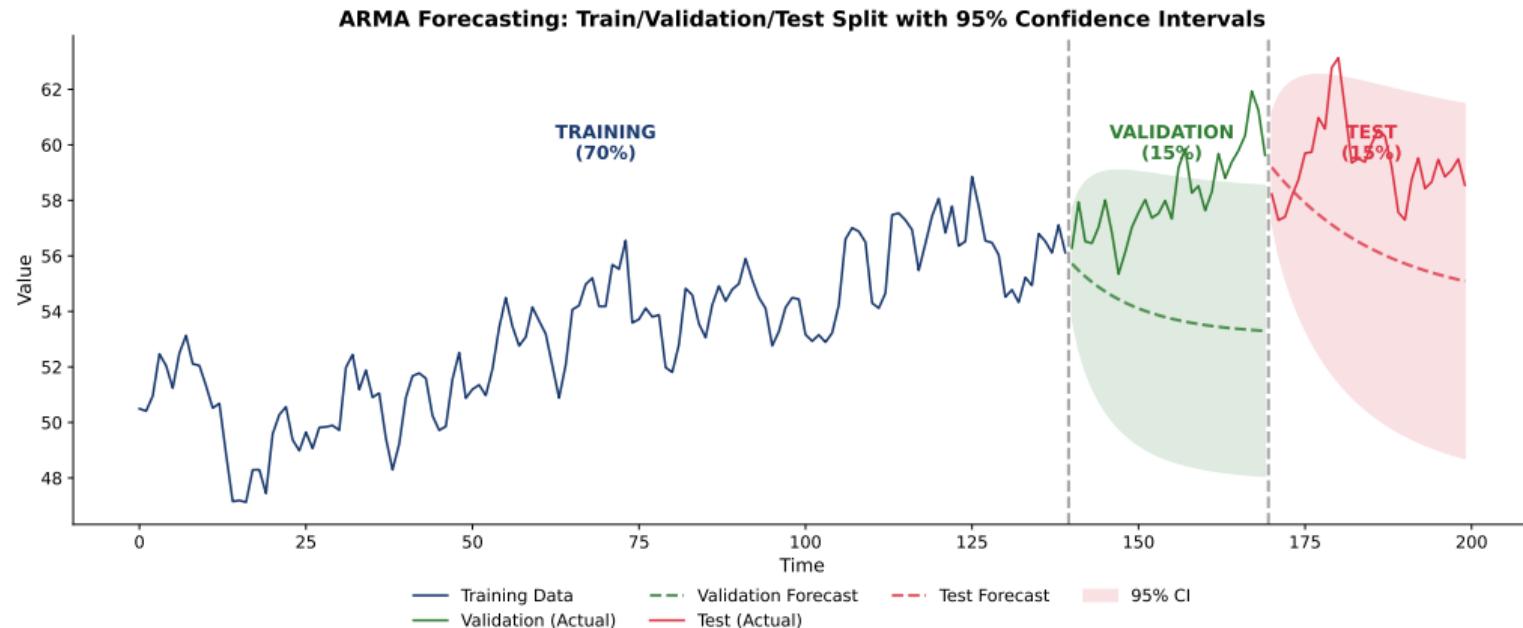
## Metrics (from Chapter 1):

- MAE =  $\frac{1}{n} \sum |e_t|$
- RMSE =  $\sqrt{\frac{1}{n} \sum e_t^2}$
- MAPE =  $\frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$

## Rolling/expanding window:

- Re-estimate model as new data arrives
- More realistic assessment of forecast performance

# Train/Validation/Test Forecasting Example



## Quiz: Forecast Properties

**Question:** For a stationary AR(1) model, what happens to forecasts as horizon  $h \rightarrow \infty$ ?

- A. Forecasts grow without bound
- B. Forecasts oscillate forever
- C. Forecasts converge to the unconditional mean  $\mu$
- D. Forecasts become more accurate

## Quiz: Forecast Properties

**Question:** For a stationary AR(1) model, what happens to forecasts as horizon  $h \rightarrow \infty$ ?

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**Answer: C** —  $\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu) \rightarrow \mu$  as  $h \rightarrow \infty$  (since  $|\phi| < 1$ )

# Python Implementation: Fitting ARMA

## Using statsmodels:

```
from statsmodels.tsa.arima.model import ARIMA

# Fit ARMA(2,1) --- note: ARIMA(p,d,q) with d=0
model = ARIMA(data, order=(2, 0, 1))
results = model.fit()

# Summary
print(results.summary())

# Forecasting
forecast = results.get_forecast(steps=10)
print(forecast.predicted_mean)
print(forecast.conf_int())
```

**Note:** ARIMA with  $d = 0$  is equivalent to ARMA

# Python: Model Selection with pmdarima

## Automatic ARIMA selection:

```
import pmdarima as pm

# Auto ARIMA with AIC criterion
model = pm.auto_arima(data,
                       start_p=0, max_p=5,
                       start_q=0, max_q=5,
                       d=0, # No differencing for stationary data
                       seasonal=False,
                       information_criterion='aic',
                       trace=True)

print(model.summary())
```

**Output:** Best model order and fitted parameters

# Workflow Summary

## ① Data preparation

- Check for missing values, outliers
- Transform if necessary (log, differencing)

## ② Stationarity check

- Visual inspection: time plot, ACF
- Formal tests: ADF, KPSS
- Difference if non-stationary

## ③ Model identification

- ACF/PACF patterns
- Information criteria grid search

## ④ Estimation and diagnostics

- Fit model, check significance
- Residual analysis, Ljung-Box test

## ⑤ Forecasting

- Point forecasts with confidence intervals
- Out-of-sample validation

## Key Takeaways

- ① **AR( $p$ ) models:** Current value depends on  $p$  past values
  - Stationarity: roots of  $\phi(z)$  outside unit circle
  - PACF cuts off at lag  $p$
- ② **MA( $q$ ) models:** Current value depends on  $q$  past shocks
  - Always stationary; invertibility: roots of  $\theta(z)$  outside unit circle
  - ACF cuts off at lag  $q$
- ③ **ARMA( $p,q$ ):** Combines AR and MA for flexible modeling
  - Both ACF and PACF decay
- ④ **Box-Jenkins:** Identify → Estimate → Diagnose → Forecast
- ⑤ **Diagnostics:** Residuals must be white noise
- ⑥ **Forecasts:** Converge to mean; uncertainty increases with horizon

### Chapter 3: ARIMA and Seasonal Models

- ARIMA(p,d,q): Integrated models for non-stationary data
- Seasonal ARIMA: SARIMA(p,d,q)(P,D,Q)<sub>s</sub>
- Seasonal differencing
- Real-world applications with seasonal patterns

#### Reading:

- Hyndman & Athanasopoulos, *Forecasting: Principles and Practice*, Ch. 9
- Box, Jenkins, Reinsel & Ljung, *Time Series Analysis*, Ch. 3-4