Time Series Analysis and Forecasting

# Chapter 6: VAR & Granger Causality

Seminar

## Seminar Outline

Review Quiz

True/False Questions

Practice Problems

Worked Examples

Real Data Analysis

Discussion Topics

Exercises for Self-Study

## Quiz 1: VAR Definition

### Question

In a VAR(2) model with 3 variables, how many coefficient matrices $A_i$ are there?

A) 2

B) 3

C) 6

D) 9

*Answer on next slide...*

## Quiz 1: Answer

### Answer: A – 2 coefficient matrices

**VAR($p$) model**: $Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$

**VAR(2) with $K = 3$**:
$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \\ Y_{3t} \end{pmatrix} = c + \underbrace{A_1}_{3 \times 3} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{pmatrix} + \underbrace{A_2}_{3 \times 3} \begin{pmatrix} Y_{1,t-2} \\ Y_{2,t-2} \\ Y_{3,t-2} \end{pmatrix} + \varepsilon_t$$

**Key**: $p$ = number of lags = number of matrices

## Quiz 2: Number of Parameters

### Question

A VAR(2) with $K = 3$ variables (including constants) has how many parameters to estimate per equation?

A) 3

B) 6

C) 7

D) 9

*Answer on next slide...*

## Quiz 2: Answer

### Answer: C – 7 parameters per equation

**VAR(p) Parameter Count: The Curse of Dimensionality**

Parameters per equation: $1 + K \times p$

Total parameters: $K(1 + Kp) + K(K + 1)/2$

(coefficients + covariance matrix)

| Model | Coefficients | Total |
|-------|-------------|-------|
| K=2, p=1 | 2(1+2×1) = 6 | + 3 = 9 |
| K=3, p=2 | 3(1+3×2) = 21 | + 6 = 27 |
| K=5, p=4 | 5(1+5×4) = 105 | + 15 = 120 |
| K=10, p=4 | 10(1+10×4) = 410 | + 55 = 465 |

Warning: Parameters grow as $K^2 \times p$ — need lots of data!

**Formula**: Per equation $= 1 + K \times p = 1 + 3 \times 2 = 7$. **Total**: $K(1 + Kp) = 3(1 + 6) = 21$ parameters

## Quiz 3: Granger Causality
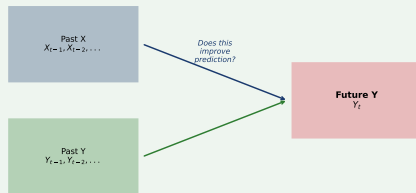
### Question

"$X$ Granger-causes $Y$" means:

A) $X$ is the economic cause of $Y$

B) Past $X$ helps predict future $Y$

C) $X$ and $Y$ are contemporaneously correlated

D) $X$ always increases when $Y$ increases

*Answer on next slide...*

## Quiz 3: Answer

### Answer: B – Past $X$ helps predict future $Y$

**Granger Causality: Predictive, Not Causal!**

Past X
$X_{t-1}, X_{t-2}, \ldots$

*Does this improve prediction?*

Future Y
$Y_t$

Past Y
$Y_{t-1}, Y_{t-2}, \ldots$

"X Granger-causes Y" means: Past X helps predict future Y,
beyond what past Y alone provides. Does NOT imply true causation!

**Key**: Predictive relationship, NOT true causation!

## Quiz 4: Granger Causality Test

### Question

To test if $Y_2$ Granger-causes $Y_1$ in a VAR(p), we test:

A) All coefficients in the $Y_1$ equation equal zero

B) Coefficients on lagged $Y_2$ in the $Y_1$ equation equal zero

C) Coefficients on lagged $Y_1$ in the $Y_2$ equation equal zero

D) The error covariance equals zero

*Answer on next slide...*

### Quiz 4: Answer

Answer: B – Coefficients on lagged $Y_2$ in $Y_1$ equation $= 0$

**Null hypothesis**: $H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$

**Test statistic**: Wald or F-test with $p$ restrictions

**Interpretation**:

⊡ Reject $H_0$: $Y_2$ Granger-causes $Y_1$

⊡ Don't reject: No evidence of predictive relationship

**Note**: Test $Y_1 \rightarrow Y_2$ separately (different coefficients in $Y_2$ equation)
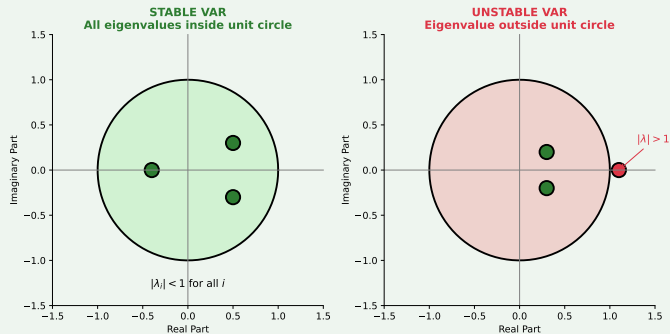
## Quiz 5: VAR Stability

### Question

A VAR(1) model is stable (stationary) if:

A) All diagonal elements of $A_1$ are less than 1

B) The determinant of $A_1$ is less than 1

C) All eigenvalues of $A_1$ are less than 1 in absolute value

D) The trace of $A_1$ equals zero

*Answer on next slide...*

## Quiz 5: Answer

### Answer: C – All eigenvalues of $A_1$ inside unit circle



**STABLE VAR**
**All eigenvalues inside unit circle**

**UNSTABLE VAR**
**Eigenvalue outside unit circle**

$|\lambda_i| < 1$ for all $i$

$|\lambda| > 1$

**Stable**: All $|\lambda_i| < 1$ (inside unit circle) $\Rightarrow$ shocks die out over time
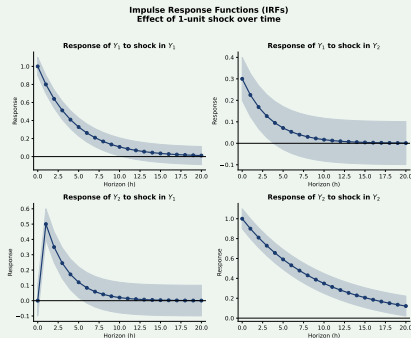
## Quiz 6: Impulse Response Functions

### Question

An impulse response function shows:

A) The correlation between two variables

B) The effect of a shock to one variable on all variables over time

C) The forecast accuracy of the model

D) The p-values of coefficient tests

*Answer on next slide...*

## Quiz 6: Answer

### Answer: B – Effect of shock on all variables over time



**Impulse Response Functions (IRFs)**
**Effect of 1-unit shock over time**

$\text{IRF}_{ij}(h)$: Response of variable $i$ at horizon $h$ to shock in variable $j$

## Quiz 7: Lag Order Selection

### Question

Which criterion typically selects the most parsimonious VAR model?

A) AIC (Akaike Information Criterion)
B) BIC (Bayesian Information Criterion)
C) FPE (Final Prediction Error)
D) Adjusted $R^2$

*Answer on next slide...*

### Quiz 7: Answer

#### Answer: B – BIC (Bayesian Information Criterion)

**Penalty comparison** (for $k$ parameters, $n$ observations):

$\boxdot$ AIC: $-2\ln L + 2k$

$\boxdot$ BIC: $-2\ln L + k\ln n$

Since $\ln n > 2$ for $n > 8$, BIC penalizes complexity more heavily

**Practical guidance**:

$\boxdot$ Forecasting: AIC may perform better

$\boxdot$ Inference/parsimony: BIC preferred

$\boxdot$ Large samples: BIC consistent, AIC tends to overfit

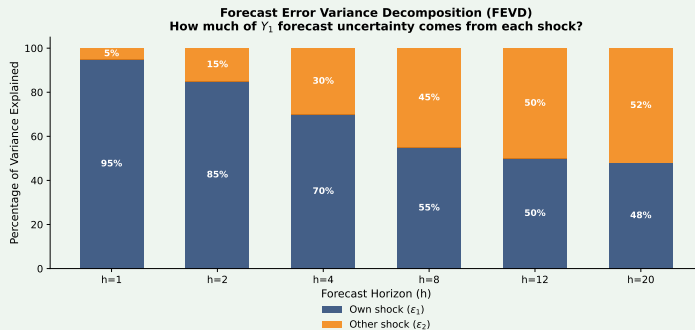## Quiz 8: Forecast Error Variance Decomposition

### Question

FEVD (Forecast Error Variance Decomposition) tells us:

A) The correlation between variables
B) What proportion of forecast error variance comes from each shock
C) The optimal forecast horizon
D) Which variables to include in the model

*Answer on next slide...*

## Quiz 8: Answer

### Answer: B – Proportion of forecast error variance from each shock



**Forecast Error Variance Decomposition (FEVD)**
How much of $Y_1$ forecast uncertainty comes from each shock?

**FEVD**: Shows how much forecast uncertainty comes from each shock at different horizons

## Quiz 9: Structural vs Reduced Form VAR

### Question

The difference between structural VAR (SVAR) and reduced-form VAR is:

A) SVAR has more variables

B) SVAR allows contemporaneous effects between variables

C) SVAR uses different estimation methods

D) There is no difference

*Answer on next slide...*

## Quiz 9: Answer

### Answer: B

Reduced-form VAR: shocks are correlated, no contemporaneous effects in equations. SVAR: imposes identifying restrictions to recover structural shocks with economic interpretation (e.g., monetary policy shock).
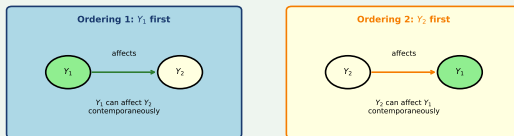
## Quiz 10: Cholesky Decomposition

### Question

Cholesky ordering in IRF analysis assumes:

A) All variables are equally important
B) Variables ordered first affect later variables contemporaneously, not vice versa
C) Shocks are uncorrelated
D) No restrictions are needed

*Answer on next slide...*

## Quiz 10: Answer

**Answer: B – Variables ordered first affect later ones contemporaneously**



**Cholesky**: Recursive structure. Ordering matters – justify by economic theory (most exogenous first)!

## Quiz 11: VAR Residual Diagnostics

### Question

In a well-specified VAR, residuals should be:

A) Autocorrelated but homoskedastic
B) White noise (no autocorrelation)
C) Normally distributed only
D) Correlated across equations

*Answer on next slide...*

## Quiz 11: Answer

### Answer: B – White noise (no autocorrelation)



**Diagnostics**: Residuals should be white noise. Use Portmanteau/LM test. Cross-equation correlation allowed ($\Sigma_u$).

## Quiz 12: Cointegration and VAR

### Question

If variables are I(1) and cointegrated, you should use:

A) VAR in levels
B) VAR in first differences
C) Vector Error Correction Model (VECM)
D) Univariate ARIMA models

*Answer on next slide...*

## Quiz 12: Answer

### Answer: C

With cointegration, VAR in differences loses long-run information, while VAR in levels may be inefficient. VECM incorporates both short-run dynamics and long-run equilibrium relationships through the error correction term.

## Quiz 13: Instantaneous Causality

### Question

Instantaneous causality differs from Granger causality because it tests:

A) Lagged relationships only

B) Contemporaneous correlation of residuals

C) Long-run relationships

D) Model stability

*Answer on next slide...*

## Quiz 13: Answer

### Answer: B

Instantaneous causality tests whether shocks to $X$ and $Y$ are correlated within the same period (correlation of VAR residuals). Granger causality tests whether *lagged* values help predict.

## True/False Questions

Determine if each statement is True or False:

1. VAR models treat all variables as endogenous.
2. Granger causality proves true economic causation.
3. A stable VAR always has eigenvalues inside the unit circle.
4. FEVD results depend on the ordering of variables.
5. VAR can be estimated by OLS equation by equation.
6. Impulse responses eventually die out in a stable VAR.

*Answers on next slide...*

## True/False: Solutions

1. **VAR models treat all variables as endogenous.**                                      **TRUE**
   Each variable is regressed on lags of all variables, including itself.

2. **Granger causality proves true economic causation.**                                **FALSE**
   It only shows predictive content, not structural causation.

3. **A stable VAR always has eigenvalues inside the unit circle.**                        **TRUE**
   Stability condition: all eigenvalues of companion matrix satisfy $|\lambda_i| < 1$.

4. **FEVD results depend on the ordering of variables.**                                   **TRUE**
   Under Cholesky identification, different orderings give different results.

5. **VAR can be estimated by OLS equation by equation.**                                   **TRUE**
   With same regressors in each equation, OLS = GLS = ML (under normality).

6. **Impulse responses eventually die out in a stable VAR.**                               **TRUE**
   Stability ensures shocks have transitory effects; IRFs $\rightarrow 0$ as $h \rightarrow \infty$.

## Problem 1: Writing VAR Equations

### Exercise

Write out the two equations for a bivariate VAR(1) model with variables $Y_t$ (GDP growth) and $X_t$ (inflation).

*Answer on next slide...*

## Problem 1: Solution

### Solution

$$Y_t = c_1 + a_{11}Y_{t-1} + a_{12}X_{t-1} + \varepsilon_{1t}$$

$$X_t = c_2 + a_{21}Y_{t-1} + a_{22}X_{t-1} + \varepsilon_{2t}$$

**Interpretation**:

- $a_{12}$: Effect of past inflation on current GDP growth
- $a_{21}$: Effect of past GDP growth on current inflation

## Problem 2: Parameter Count

### Exercise

How many total parameters need to be estimated in a VAR(3) with $K = 4$ variables (including constants)?

*Answer on next slide...*

## Problem 2: Solution

### Solution

Per equation: $1 + K \times p = 1 + 4 \times 3 = 13$ parameters

Total for $K = 4$ equations: $4 \times 13 = 52$ parameters

Plus covariance matrix $\boldsymbol{\Sigma}$: $K(K+1)/2 = 4 \times 5/2 = 10$ unique elements

**Grand total: 62 parameters**

*This is why VARs can be "over-parameterized" with limited data!*

## Problem 3: Granger Causality Interpretation

### Exercise

A Granger causality test yields:

- $H_0$: Money does not Granger-cause GDP. $p$-value $= 0.02$
- $H_0$: GDP does not Granger-cause Money. $p$-value $= 0.35$

Interpret these results.

*Answer on next slide...*

## Problem 3: Solution

### Solution

⊡ **Reject** $H_0$ at 5%: Money **Granger-causes** GDP
⊡ **Fail to reject** $H_0$: GDP does **not** Granger-cause Money

**Conclusion**: Unidirectional causality: Money $\rightarrow$ GDP

*Interpretation*: Past money supply helps predict GDP growth. This is consistent with monetarist views, but remember: Granger causality $\neq$ structural causality!

## Problem 4: Stability Check

### Exercise

For VAR(1) with $A_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.1 & 0.5 \end{pmatrix}$, check stability.

*Answer on next slide...*

## Problem 4: Solution

### Solution

Find eigenvalues: $\det(A_1 - \lambda I) = 0$

$(0.7 - \lambda)(0.5 - \lambda) - (0.2)(0.1) = 0$

$\lambda^2 - 1.2\lambda + 0.33 = 0$

$\lambda = \frac{1.2 \pm \sqrt{1.44 - 1.32}}{2} = \frac{1.2 \pm 0.346}{2}$

$\lambda_1 = 0.773, \quad \lambda_2 = 0.427$

Both $|\lambda_i| < 1 \Rightarrow$ **Stable!**

## Problem 5: IRF Computation

### Exercise

For VAR(1) with A $= \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$, compute $\Phi_2$ (response at $h = 2$).

*Answer on next slide...*

## Problem 5: Solution

$$\Phi_2 = A^2 = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 0.25 + 0 & 0.10 + 0.12 \\ 0 + 0 & 0 + 0.36 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.22 \\ 0 & 0.36 \end{pmatrix}$$

**Interpretation**: A unit shock to $Y_2$ at $t$ increases $Y_1$ by 0.22 at $t + 2$.

## Example: Stock Returns and Trading Volume

### Scenario

Daily data on stock returns ($R_t$) and trading volume ($V_t$). Test Granger causality both directions.

### Typical Findings in Finance Literature

- Returns often Granger-cause volume (price changes trigger trading)
- Volume sometimes Granger-causes returns (volume as leading indicator)
- Results: Often **bidirectional** causality $R \leftrightarrow V$

### Practical Issue

Stock returns are typically stationary, but volume may need transformation (log or difference).

## Example: Interest Rates and Inflation

### Taylor Rule Context

Central banks set interest rates ($i_t$) in response to inflation ($\pi_t$): $i_t = r^* + \pi^* + 1.5(\pi_t - \pi^*) + 0.5(y_t - y^*)$

### VAR Analysis Questions

- Does inflation Granger-cause interest rates? (Central bank reaction)
- Do interest rates Granger-cause inflation? (Monetary policy transmission)

### Expected Results

Bidirectional causality: Quick $\pi \to i$ (policy reaction), Delayed $i \to \pi$ (policy effect)

## Python VAR Analysis: Key Functions

### Essential Libraries

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests
```
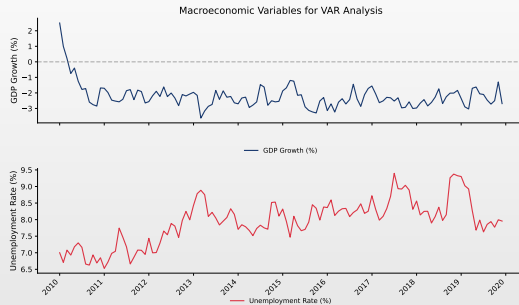
### Workflow

1. Create DataFrame: data = pd.DataFrame({'gdp': ..., 'unemp': ...})
2. Fit VAR: model = VAR(data); results = model.fit(maxlags=8, ic='aic')
3. Get IRF: irf = results.irf(periods=20)
4. Get FEVD: fevd = results.fevd(periods=20)
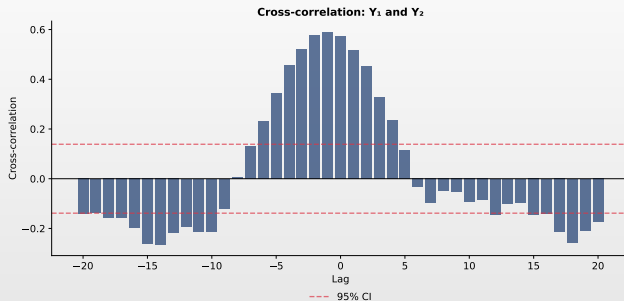5. Granger tests: grangercausalitytests(data[['y', 'x']], maxlag=4)

### Note

Complete working examples are provided in the Jupyter notebooks.

## Case Study: GDP and Unemployment



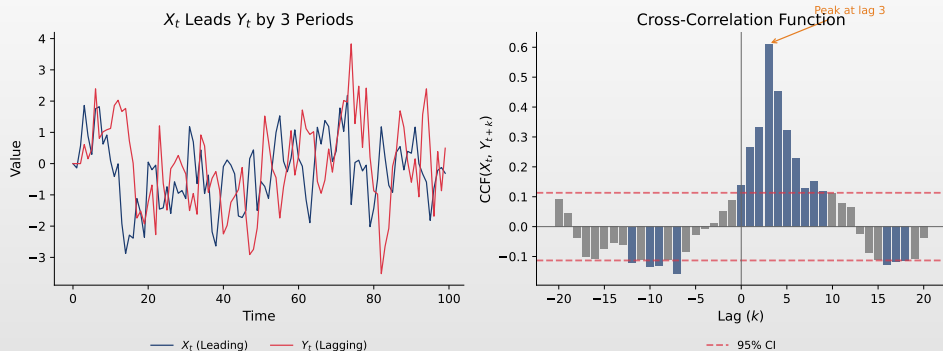Macroeconomic Variables for VAR Analysis

- ⊡ **Top**: US Real GDP growth rate (quarterly)
- ⊡ **Bottom**: US Unemployment rate
- ⊡ Clear negative relationship (Okun's Law)
- ⊡ VAR model can capture dynamic interactions between these variables

## Cross-Correlation Analysis



Cross-correlation: $Y_1$ and $Y_2$

- ⊡ Cross-correlation measures lead-lag relationships
- ⊡ Negative correlation at lag 0: contemporaneous inverse relationship
- ⊡ Asymmetric pattern suggests unemployment responds to GDP with lag
- ⊡ Helps inform VAR lag order selection
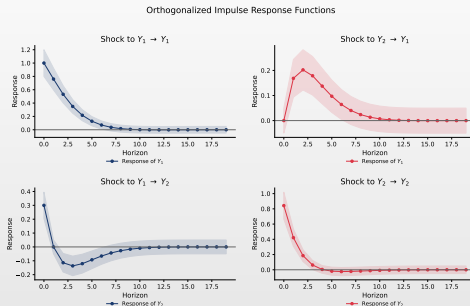
## Visual: Cross-Correlation Function



The CCF measures correlation between two series at different lags, revealing lead-lag relationships.

## VAR Estimation Results
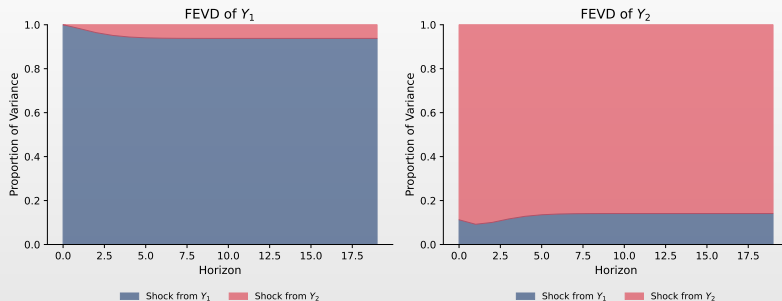
### Model: VAR(2) for GDP Growth and Unemployment

| Equation | Variable | Coef. | Std. Error | t-stat |
|---|---|---|---|---|
| $\Delta GDP_t$ | $\Delta GDP_{t-1}$ | 0.312 | 0.087 | 3.59 |
| | $\Delta GDP_{t-2}$ | 0.145 | 0.082 | 1.77 |
| | $U_{t-1}$ | $-0.421$ | 0.156 | $-2.70$ |
| | $U_{t-2}$ | 0.198 | 0.148 | 1.34 |
| $U_t$ | $\Delta GDP_{t-1}$ | $-0.087$ | 0.032 | $-2.72$ |
| | $\Delta GDP_{t-2}$ | $-0.045$ | 0.030 | $-1.50$ |
| | $U_{t-1}$ | 1.456 | 0.058 | 25.1 |
| | $U_{t-2}$ | $-0.521$ | 0.055 | $-9.47$ |

## Impulse Response Functions



Orthogonalized Impulse Response Functions

- IRFs show dynamic response to one-unit shocks
- GDP shock: temporary positive effect on GDP, negative on unemployment
- Unemployment shock: negative effect on GDP, persistent on unemployment
- 95% confidence bands show uncertainty in responses

# Forecast Error Variance Decomposition



- ☐ FEVD shows proportion of variance explained by each shock
- ☐ GDP variance: mostly explained by own shocks, some by unemployment
- ☐ Unemployment variance: highly persistent (own shocks dominant)
- ☐ Provides insight into relative importance of different shocks

## Discussion: Granger Causality vs True Causality

### Key Question

If $X$ Granger-causes $Y$, does that mean $X$ actually causes $Y$? **NO!**

### Why Granger Causality Can Fail

- **Omitted variable bias**: $Z$ might cause both $X$ and $Y$ (e.g., weather $\rightarrow$ ice cream & drownings)
- **Anticipation effects**: Markets anticipate future events (stock prices $\rightarrow$ earnings)
- **Aggregation issues**: Timing of data collection matters

### Conclusion

Granger causality is about **prediction**, not **mechanism**. For structural causality, need theory + identification strategy.

## Discussion: Variable Ordering in IRFs

### Key Question

Why does variable ordering matter for orthogonalized IRFs?

### Cholesky Decomposition Assumes

- First variable: Affects all others contemporaneously
- Second variable: Affected by first, affects remaining
- Last variable: Affected by all, affects none contemporaneously

### Example: Monetary Policy VAR Ordering

1. Oil prices (exogenous) → 2. GDP (slow) → 3. Inflation → 4. Interest rates (fast)

### Rule

Order from "most exogenous" to "most endogenous" — justify with economic theory!

## Take-Home Exercises

1. **Theoretical**: Show that a VAR(1) can be written as MA($\infty$): $Y_t = \sum_{i=0}^{\infty} A^i \varepsilon_{t-i}$ when stable.

2. **Computation**: For VAR(1) with $A = \begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
   - ▶ Check stability; Compute IRFs for $h = 0, 1, 2, 3$
   - ▶ Plot the response of $Y_1$ to a shock in $Y_2$

3. **Applied**: Download US GDP growth and unemployment data:
   - ▶ Test stationarity; Estimate VAR (select optimal lag)
   - ▶ Test Granger causality; Compute and interpret IRFs

4. **Critical Thinking**: Why might stock prices "Granger-cause" GDP even though GDP is determined by real factors?

## Exercise Solutions Hints

### Hints

1. Use recursive substitution: $Y_t = AY_{t-1} + \varepsilon_t = A(AY_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \ldots$

2. Eigenvalues of $\begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
   - Characteristic equation: $\lambda^2 - 1.2\lambda + 0.35 = 0$
   - $\lambda_1 \approx 0.85$, $\lambda_2 \approx 0.41$ (both $< 1$, stable)

3. For GDP/Unemployment:
   - GDP growth is usually I(0), unemployment may be I(1)
   - Use unemployment rate changes if needed
   - Expect GDP growth $\rightarrow$ unemployment (Okun's Law)

4. Stock prices anticipate future economic conditions—they reflect expectations about future GDP, so they "lead" GDP in the data even though causation runs the other way.

## Key Takeaways from This Seminar

### Main Points

1. VAR models capture **interdependencies** between multiple time series
2. Parameter count grows quickly: $K^2 p + K$ per system
3. **Granger causality** tests predictive content, not true causation
4. **IRFs** show dynamic propagation of shocks; ordering matters

### Practical Points

- ☐ Always check stationarity before estimating VAR
- ☐ Use information criteria (AIC/BIC) for lag selection
- ☐ Report Granger tests in both directions
- ☐ Justify variable ordering with economic theory

### Remember

Granger causality is about **prediction**, not **mechanism**!