



Time Series Analysis and Forecasting

Chapter 7: Cointegration and VECM



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Outline

- Motivation
- Spurious Regression
- Cointegration Concept
- Engle-Granger Method
- Johansen Method
- VECM Estimation
- Practical Considerations
- Real-World Examples
- Case Study: Interest Rates
- Summary
- Quiz

Why Cointegration Matters

The Challenge

- ▣ Many economic/financial time series are **non-stationary** ($I(1)$)
- ▣ GDP, stock prices, exchange rates, interest rates all have unit roots
- ▣ Standard regression with $I(1)$ variables \Rightarrow **spurious results**
- ▣ Differencing removes non-stationarity but loses **long-run information**

The Solution: Cointegration

Some non-stationary series share a **common stochastic trend**—they move together in the long run.

Nobel Prize 2003

Granger & Engle received the Nobel Prize for “methods for analyzing economic time series with common trends.”

Real-World Applications

Finance

- ▣ **Pairs Trading:** Cointegrated stocks
- ▣ **Term Structure:** Interest rates
- ▣ **Spot-Futures:** Arbitrage

Macroeconomics

- ▣ **Consumption & Income**
- ▣ **Money & Prices**
- ▣ **PPP:** Exchange rates

Policy Analysis

- ▣ **Fiscal:** Spending & taxes
- ▣ **Monetary:** Rate pass-through
- ▣ **Labor:** Wages & productivity

The Spurious Regression Problem

Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk: $Y_t = \alpha + \beta X_t + u_t$ where Y_t and X_t are independent $I(1)$ processes.

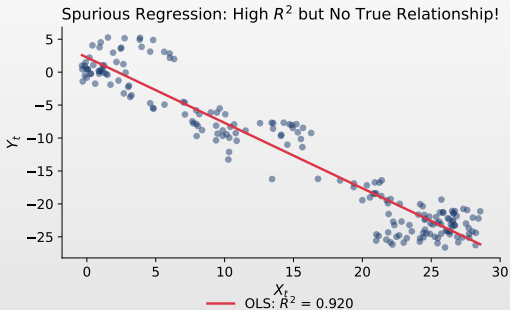
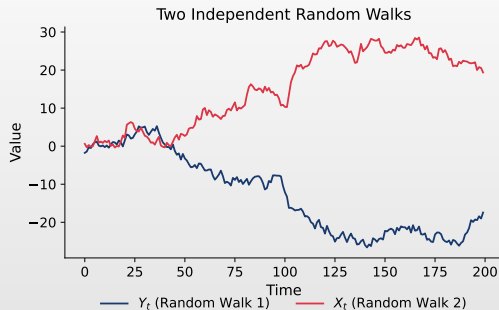
Symptoms of Spurious Regression

- ▣ High R^2 (often > 0.9) even though variables are **unrelated**
- ▣ Highly significant t -statistics (reject $H_0 : \beta = 0$)
- ▣ Very low Durbin-Watson statistic ($DW \approx 0$)
- ▣ Residuals are non-stationary (have unit root)

Rule of Thumb

If $R^2 > DW$, suspect spurious regression!

Spurious Regression: Visual Example



Warning: Two completely independent random walks show high correlation ($R^2 > 0.8$) purely by chance! This is why we need cointegration analysis.

Definition of Cointegration

Definition 1 (Cointegration (Engle & Granger, 1987))

Variables $Y_{1t}, Y_{2t}, \dots, Y_{kt}$ are **cointegrated of order (d, b)** , written $CI(d, b)$, if:

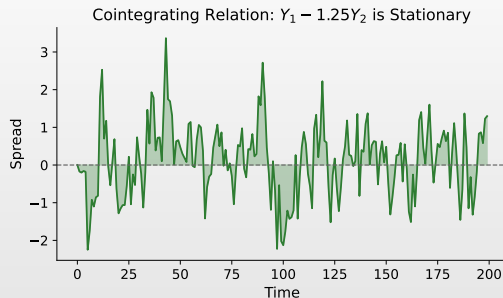
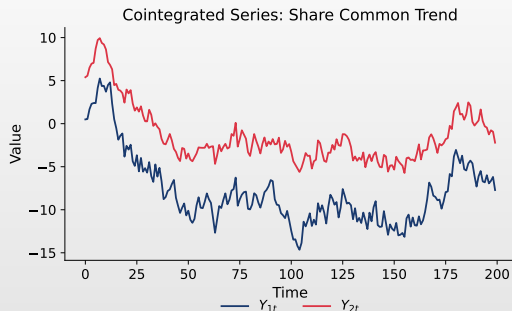
1. All variables are integrated of order d : $Y_{it} \sim I(d)$
2. There exists a linear combination $\beta'Y_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$ that is integrated of order $(d - b)$, where $b > 0$

Most Common Case: $CI(1, 1)$

- ▣ Variables are $I(1)$ (have unit roots)
- ▣ Linear combination is $I(0)$ (stationary)
- ▣ Vector $\beta = (\beta_1, \dots, \beta_k)'$ is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized: $\beta_1 = 1$.

Cointegration: Visual Example



Key insight: Both series are $I(1)$ and trend together, but their linear combination (spread) is stationary—this is cointegration!

Intuition: Common Stochastic Trends

Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**: $Y_{1t} = \gamma_1 \tau_t + S_{1t}$, $Y_{2t} = \gamma_2 \tau_t + S_{2t}$ where τ_t is a common random walk and S_{it} are stationary.

Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

Economic Interpretation

- Cointegration = **long-run equilibrium relationship**
- Variables may deviate in the short run, but are “pulled back”
- The cointegrating vector defines the equilibrium

Cointegrating Rank

How Many Cointegrating Relationships?

For k variables that are $I(1)$:

- Maximum possible cointegrating relationships: $r = k - 1$
- If $r = 0$: No cointegration (variables drift apart)
- If $r = k$: All variables are $I(0)$ (contradiction)

Example: 3 Variables

- $r = 0$: No cointegration
- $r = 1$: One cointegrating relationship
- $r = 2$: Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends = $k - r$

Engle-Granger Two-Step Method

Step 1: Estimate Cointegrating Regression

Run OLS: $Y_t = \alpha + \beta X_t + e_t$. Save residuals: $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

Step 2: Test Residuals for Stationarity

Test if \hat{e}_t is $I(0)$ using ADF: $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$

- ▣ $H_0: \rho = 0$ (unit root \Rightarrow no cointegration)
- ▣ $H_1: \rho < 0$ (stationary \Rightarrow cointegration)

Important

Use **Engle-Granger critical values**, not standard ADF! (More negative because residuals are estimated)

Engle-Granger Critical Values

Critical Values for Cointegration Test

Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

MacKinnon (1991), $T = 100$

Limitations of Engle-Granger

- Only tests for **one** cointegrating relationship
- Results depend on choice of dependent variable
- Small sample bias; cannot test hypotheses on cointegrating vector

Johansen Cointegration Test

Advantages over Engle-Granger

- ▣ Tests for **multiple** cointegrating relationships
- ▣ Maximum likelihood estimation (more efficient)
- ▣ Can test restrictions on cointegrating vectors
- ▣ Does not require choosing a dependent variable

Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...

VECM Representation

Vector Error Correction Model

$$\Delta Y_t = c + \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

- $\Pi = \sum_i A_i - I$ (long-run impact); Γ_j (short-run dynamics)

Key Insight: Rank of Π

The **rank of Π** determines cointegration:

- $\text{rank}(\Pi) = 0$: No cointegration (VAR in differences)
- $\text{rank}(\Pi) = k$: All variables are $I(0)$ (VAR in levels)
- $0 < \text{rank}(\Pi) = r < k$: r cointegrating vectors

Decomposition of Π

When $\text{rank}(\Pi) = r < k$

$\Pi = \alpha\beta'$ where β ($k \times r$) = **cointegrating vectors**, α ($k \times r$) = **adjustment coefficients**

Interpretation

- ▣ $\beta'Y_{t-1}$ = deviations from equilibrium (error correction terms)
- ▣ α = speed of adjustment; rows show each variable's response

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta'Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

Johansen Test Statistics

Two Test Statistics

Based on eigenvalues $\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_k$ of a certain matrix:

Trace Test:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^k \ln(1 - \hat{\lambda}_i)$$

Tests $H_0: \text{rank} \leq r$ vs $H_1: \text{rank} > r$

Maximum Eigenvalue Test:

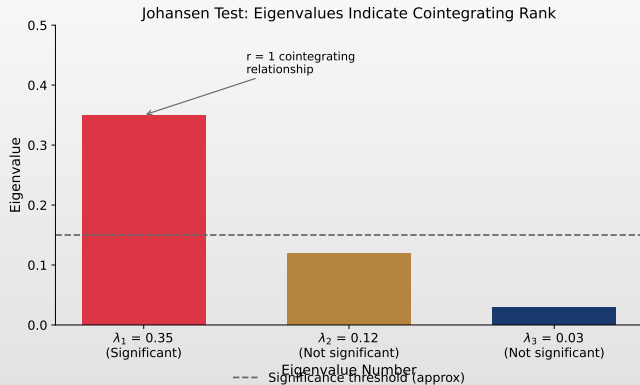
$$\lambda_{\text{max}}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

Tests $H_0: \text{rank} = r$ vs $H_1: \text{rank} = r+1$

Critical values from Johansen & Juselius (1990), depend on:

- Number of variables k
- Deterministic components (constant, trend)

Johansen Test: Visual Interpretation



Significant eigenvalues (above threshold) indicate cointegrating relationships. First eigenvalue significant $\Rightarrow r = 1$.

Testing Procedure

Sequential Testing (Trace Test)

1. Test $H_0: r = 0$. If rejected \Rightarrow continue
2. Test $H_0: r \leq 1$. If not rejected $\Rightarrow r = 1$
3. Continue until H_0 is not rejected

Deterministic Components

- ☐ No constant, no trend (rarely used)
- ☐ Constant in cointegrating relation only
- ☐ **Constant in both** (most common)
- ☐ Constant + trend in cointegrating relation

VECM Structure

Full VECM Specification

For $k = 2$ variables with $r = 1$ cointegrating relation:

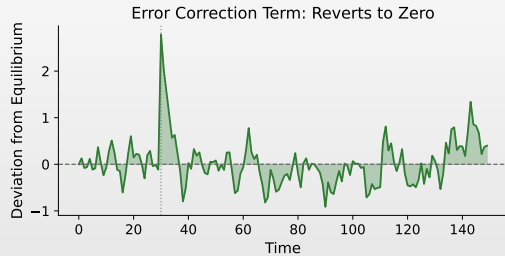
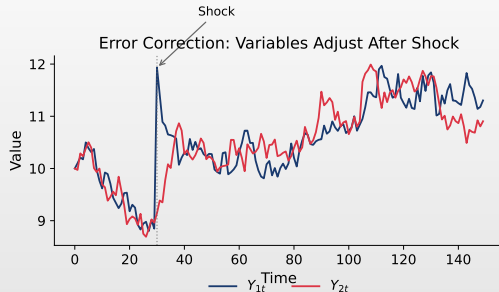
$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$ = error correction term (deviation from equilibrium)
- α_1, α_2 = adjustment speeds (should have opposite signs)
- γ_{ij} = short-run dynamics
- ε_{it} = innovations

Error Correction Mechanism: Visual



Error correction in action: When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment.

Interpreting Adjustment Coefficients

The α Coefficients

If the cointegrating relation is $Y_1 - \beta Y_2 = 0$ (equilibrium):

- ▣ $\alpha_1 < 0$: Y_1 adjusts downward when above equilibrium
- ▣ $\alpha_2 > 0$: Y_2 adjusts upward when Y_1 is above equilibrium

Weak Exogeneity

If $\alpha_i = 0$, variable Y_i does **not** respond to disequilibrium.

- ▣ Y_i is **weakly exogenous** for the long-run parameters
- ▣ The other variable does all the adjusting
- ▣ Can simplify estimation (single-equation approach)

Test weak exogeneity: $H_0 : \alpha_i = 0$ using likelihood ratio test.

VECM vs VAR in Differences

When Variables are Cointegrated

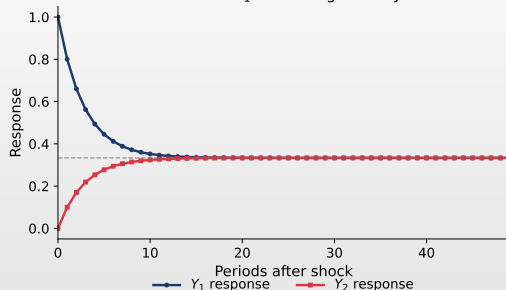
	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

Granger Representation Theorem

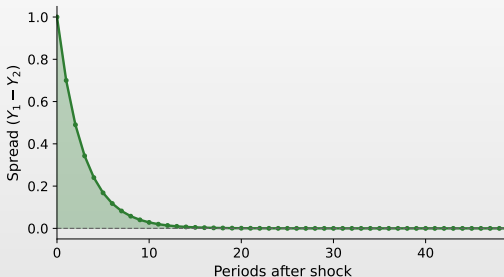
If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

VECM Impulse Response Functions

IRF: Shock to Y_1 in Cointegrated System



Error Correction Term Returns to Zero



IRF interpretation: In a cointegrated system, shocks have **permanent effects** on levels but the system returns to equilibrium. Unlike stationary VAR, effects don't decay to zero—they converge to a new long-run value.

Practical Workflow

Step-by-Step Procedure

1. **Unit Root Tests:** Verify all variables are $I(1)$
 - ▶ ADF, KPSS on levels and first differences
2. **Lag Length Selection:** Choose p for VAR in levels
 - ▶ Use AIC, BIC, or sequential LR tests
3. **Cointegration Test:** Johansen trace/max-eigenvalue tests
 - ▶ Determine cointegrating rank r
4. **Estimate VECM:** If $0 < r < k$
 - ▶ Estimate α , β , Γ_j
5. **Diagnostics:** Check residuals for autocorrelation, normality
6. **Analysis:** IRF, FEVD, hypothesis tests

Common Pitfalls

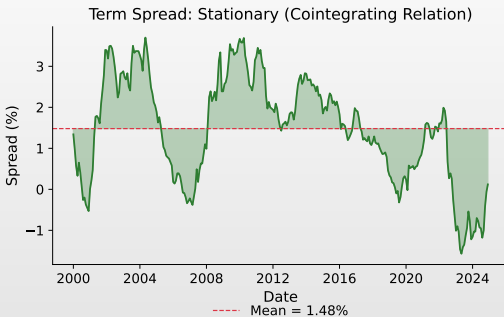
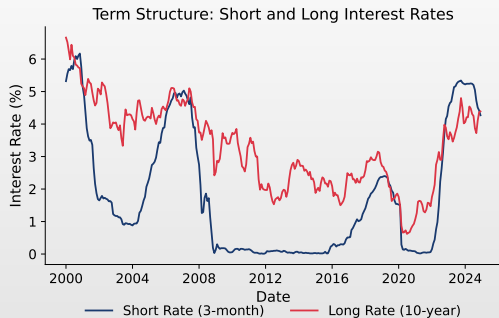
Things to Watch Out For

- ▣ **Structural breaks:** Cause spurious unit roots or cointegration
- ▣ **Near-unit-root:** Tests have low power
- ▣ **Lag selection:** Too many/few lags bias results
- ▣ **Small samples:** Johansen test oversized

Recommendation

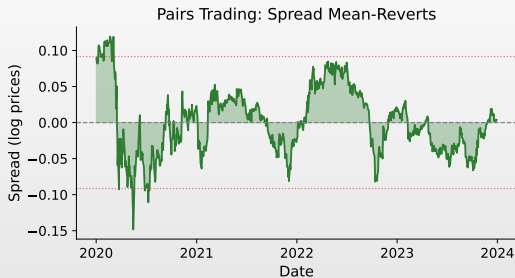
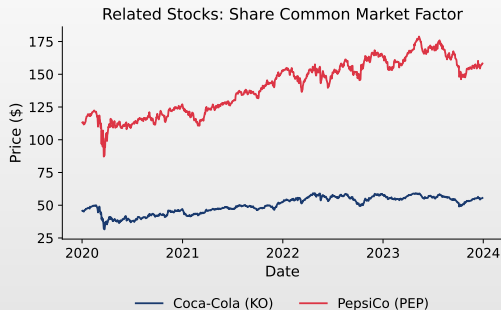
Always check: residual diagnostics, stability of cointegrating relationship, sensitivity to specification

Example 1: Term Structure of Interest Rates



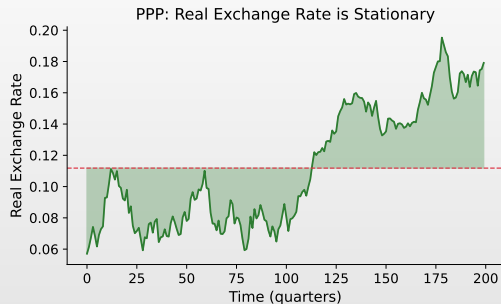
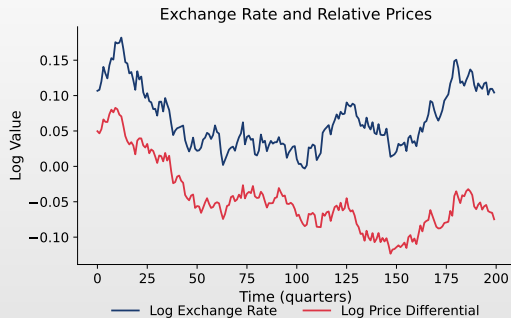
Expectations Hypothesis: Short and long rates share common trend. The spread (term premium) is stationary—evidence of cointegration!

Example 2: Pairs Trading in Finance



Strategy: Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When spread deviates from mean, trade expecting mean reversion. Sell spread when high, buy when low.

Example 3: Purchasing Power Parity (PPP)



PPP Theory: $e_t = p_t - p_t^*$ (log exchange rate equals price differential). Real exchange rate should be stationary in the long run.

Case Study: Cointegration of Interest Rates

Research Question

Are short-term and long-term interest rates cointegrated? Does the expectations hypothesis of the term structure hold?

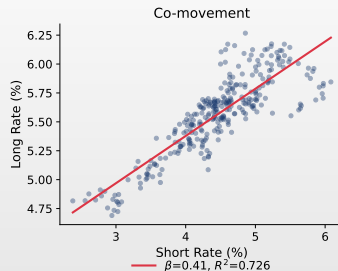
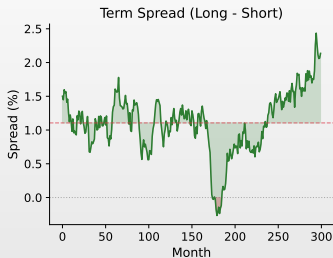
Data

- US Monthly Data (1962-2023)
- 3-Month Treasury Bill Rate
- 10-Year Treasury Bond Yield
- Source: FRED Database

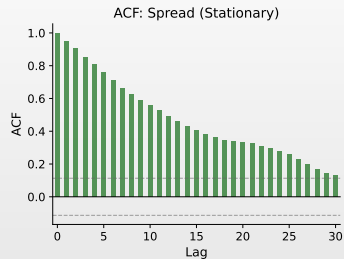
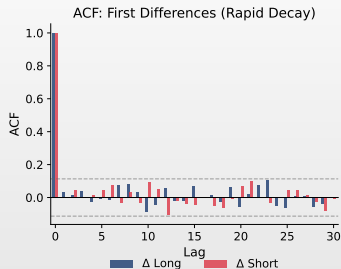
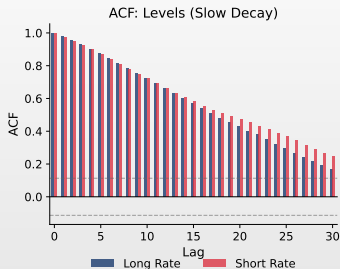
Methodology

- Unit root tests (ADF, PP)
- Engle-Granger cointegration test
- Johansen procedure
- VECM estimation
- Impulse response analysis

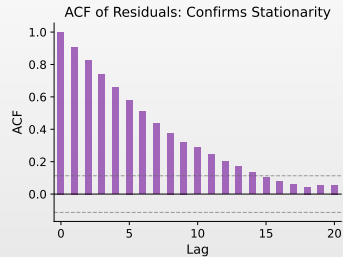
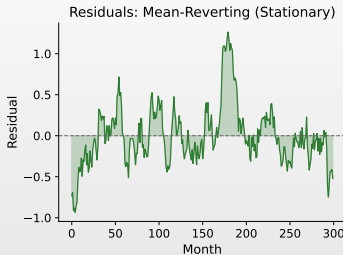
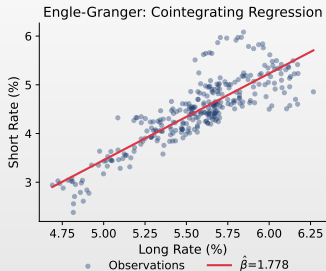
Step 1: Data Visualization



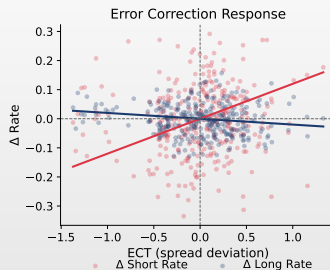
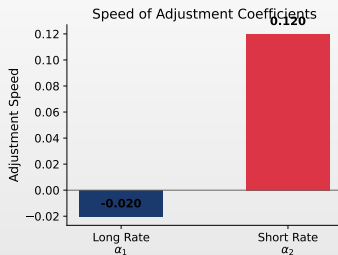
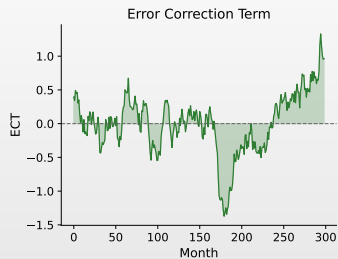
Step 2: Unit Root Tests



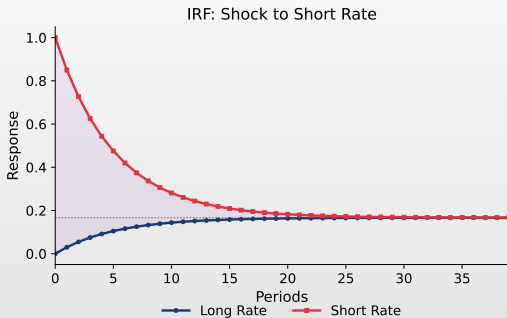
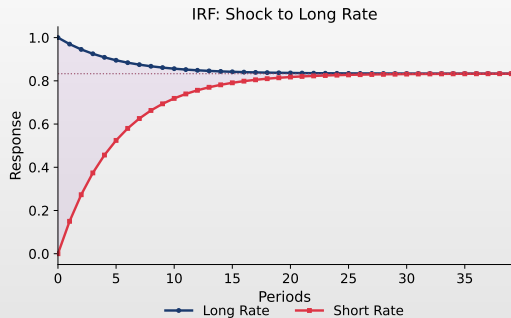
Step 3: Engle-Granger Cointegration Test



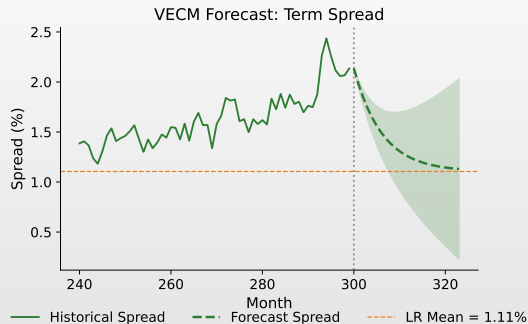
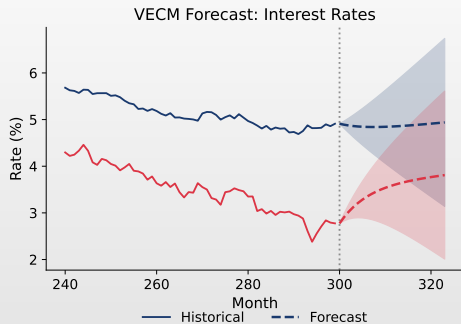
Step 4: VECM Estimation



Step 5: Impulse Response Functions



Step 6: Forecasting



Key Takeaways

Main Concepts

- ▣ **Cointegration:** $I(1)$ variables with stationary linear combination
- ▣ **Spurious regression:** High R^2 with unrelated $I(1)$ variables
- ▣ **VECM:** VAR with error correction for cointegrated systems

Testing Methods

- ▣ **Engle-Granger:** Simple, one vector only
- ▣ **Johansen:** Multiple vectors, MLE-based

Remember

Tests have low power in small samples. Theory should guide specification.

What's Next?

Extensions and Related Topics

- ▣ **Structural VECM:** Identifying structural shocks
- ▣ **Threshold cointegration:** Nonlinear adjustment
- ▣ **Panel cointegration:** Multiple cross-sections
- ▣ **Fractional cointegration:** Long memory
- ▣ **Time-varying cointegration:** Regime changes

Questions?

Quick Quiz

1. What does it mean for two $I(1)$ variables to be cointegrated?
2. What is the “spurious regression” problem?
3. In a VECM, what do the α coefficients represent?
4. What is the main advantage of Johansen over Engle-Granger?
5. If $\alpha_i = 0$ for variable Y_i , what does this imply?

Quiz Answers

1. **Cointegration:** A linear combination of the variables is $I(0)$ (stationary). They share a common stochastic trend.
2. **Spurious regression:** Regressing one $I(1)$ variable on another unrelated $I(1)$ variable gives high R^2 and significant coefficients even though there's no true relationship.
3. **α coefficients:** Speed of adjustment—how quickly each variable responds to deviations from long-run equilibrium.
4. **Johansen advantage:** Can test for multiple cointegrating relationships, uses MLE (more efficient), doesn't require choosing dependent variable.
5. **$\alpha_i = 0$:** Variable Y_i is weakly exogenous—it doesn't respond to disequilibrium. Other variables do all the adjusting.