



Time Series Analysis and Forecasting

Chapter 3: ARIMA Models

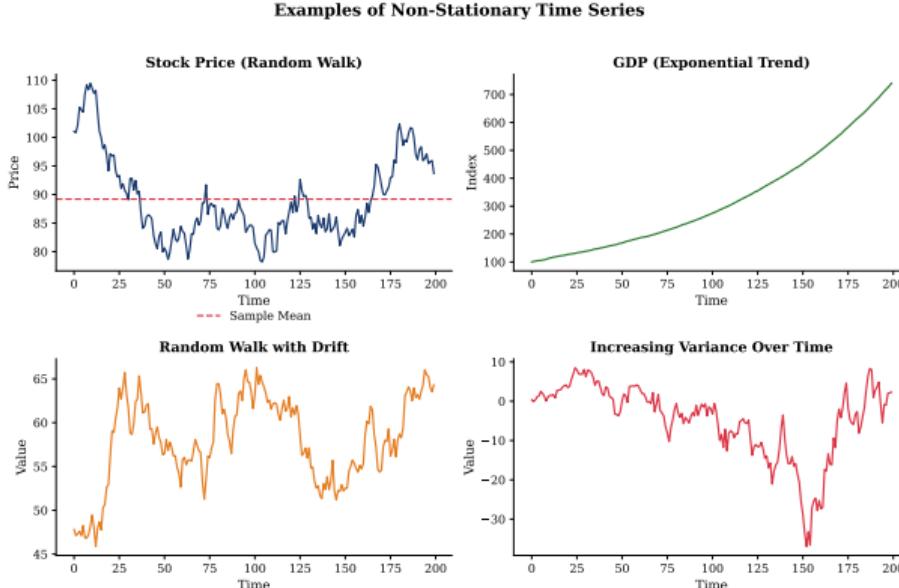
Non-Stationary Time Series



Outline

- 1 Non-Stationarity in Time Series
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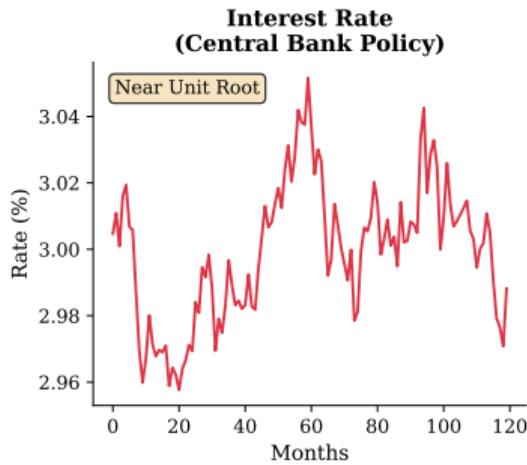
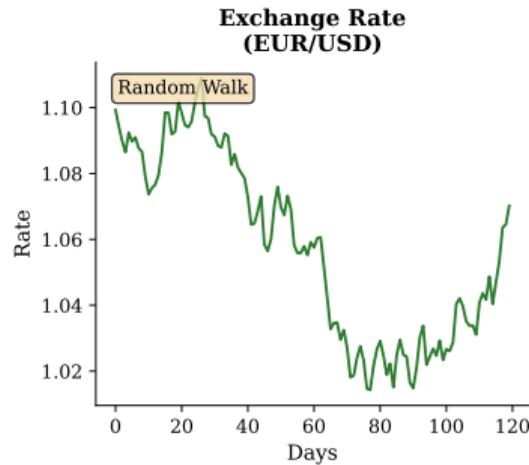
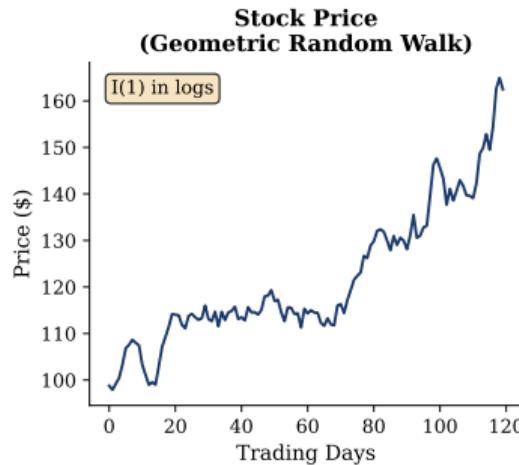
Motivating Example: Non-Stationary Data Is Everywhere



- Stock prices, GDP, exchange rates all exhibit **trends or wandering behavior**
- The sample mean (red line) is meaningless for a random walk
- Standard ARMA models **cannot** handle these series directly

Real-World Applications

Real-World Non-Stationary Series: Why We Need ARIMA

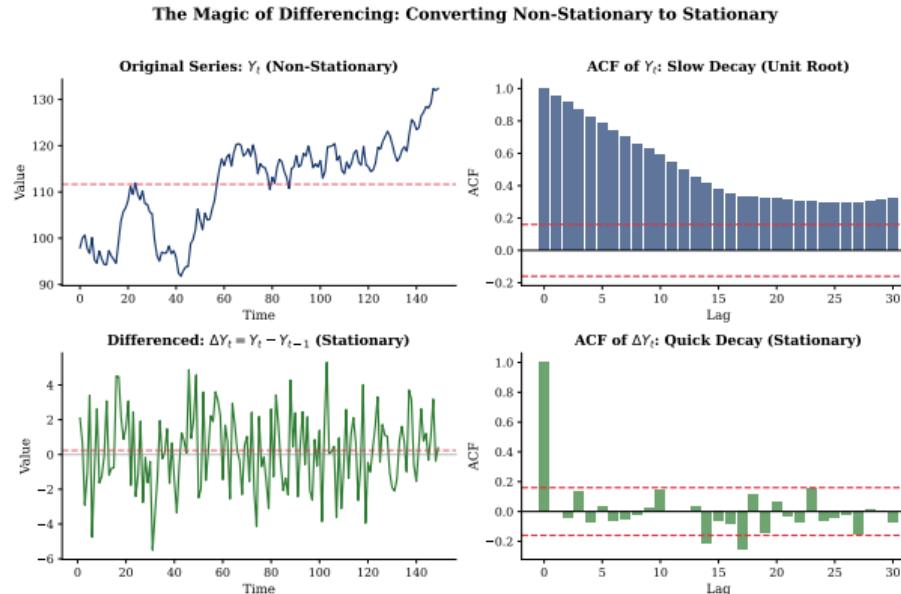


The Challenge

Financial and economic data are typically **integrated** ($I(1)$ or near unit root):

- Stock prices: random walk in logs
- Exchange rates: random walk
- Interest rates: highly persistent (near unit root)

The Solution: Differencing



Key Insight

Differencing transforms a non-stationary series into a stationary one: $\Delta Y_t = Y_t - Y_{t-1}$. The ACF changes from slow decay to quick decay!

What We'll Learn Today

Core Concepts

- ① **Non-Stationarity:** Why it matters and how to detect it
- ② **Unit Root Tests:** ADF, PP, KPSS tests
- ③ **Differencing:** The key transformation
- ④ **ARIMA Models:** Combining differencing with ARMA
- ⑤ **Box-Jenkins Methodology:** Identify → Estimate → Diagnose

By the End of This Lecture

You will be able to model and forecast non-stationary time series like stock prices, GDP, and exchange rates using ARIMA models.

Why Non-Stationarity Matters

The Problem

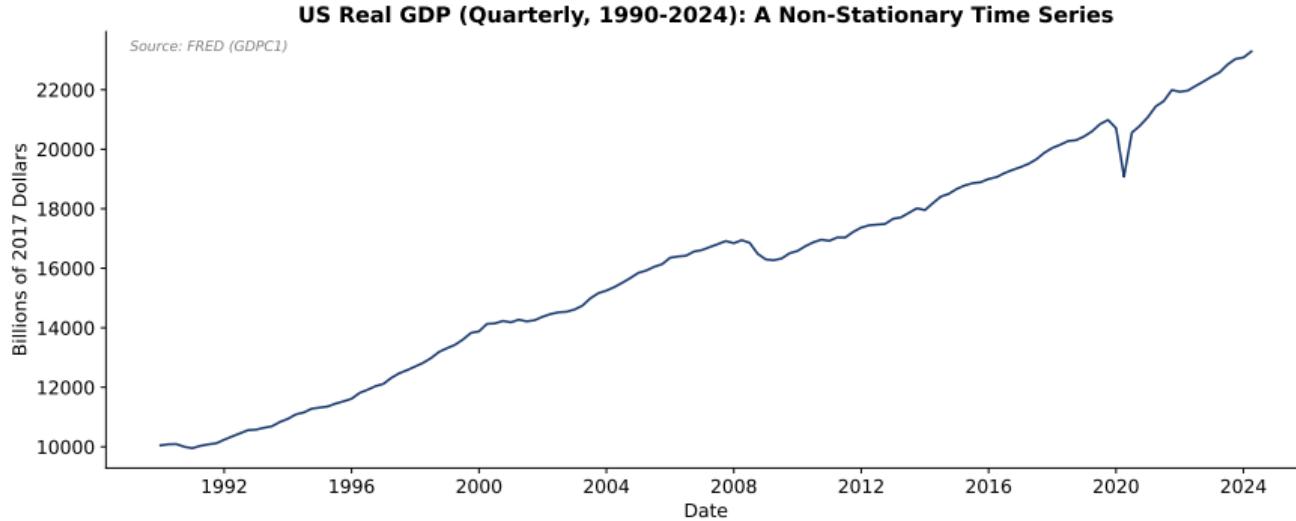
Many economic and financial time series are **non-stationary**:

- GDP, stock prices, exchange rates, inflation indices
- They exhibit trends, changing means, or growing variance

Consequences of Non-Stationarity

- Standard ARMA models assume stationarity
- OLS regression with non-stationary data leads to **spurious regression**
- Sample moments (mean, variance, ACF) are not consistent estimators
- Statistical inference becomes invalid

Example: US Real GDP



- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data

Types of Non-Stationarity

Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a deterministic function of time
- Can be removed by **detrending**
- Shocks have temporary effects

Stochastic Trend (Unit Root)

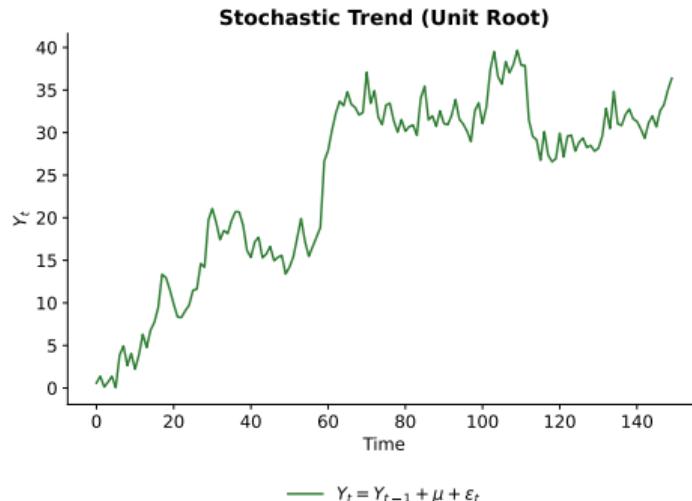
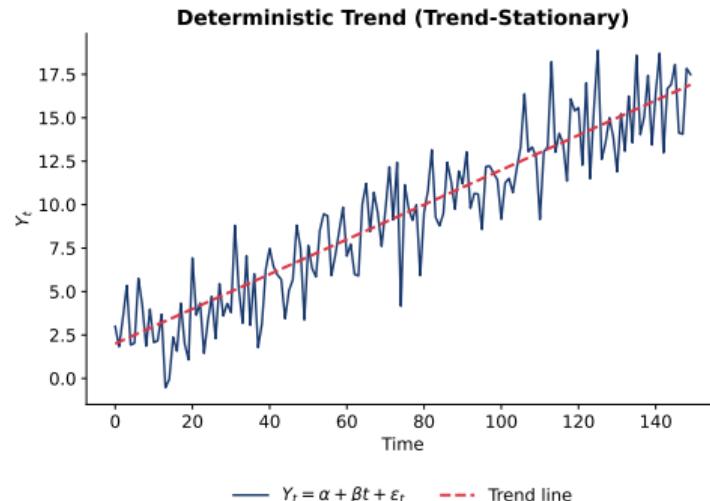
$$Y_t = Y_{t-1} + \varepsilon_t$$

- Random walk process
- Must be removed by **differencing**
- Shocks have permanent effects

Key Distinction

Correct identification is crucial: detrending a unit root process or differencing a trend-stationary process both lead to misspecification!

Visualizing the Difference



- **Left:** Deterministic trend – deviations from trend are temporary
- **Right:** Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!

The Random Walk Process

Definition 1 (Random Walk)

A **random walk** is defined as:

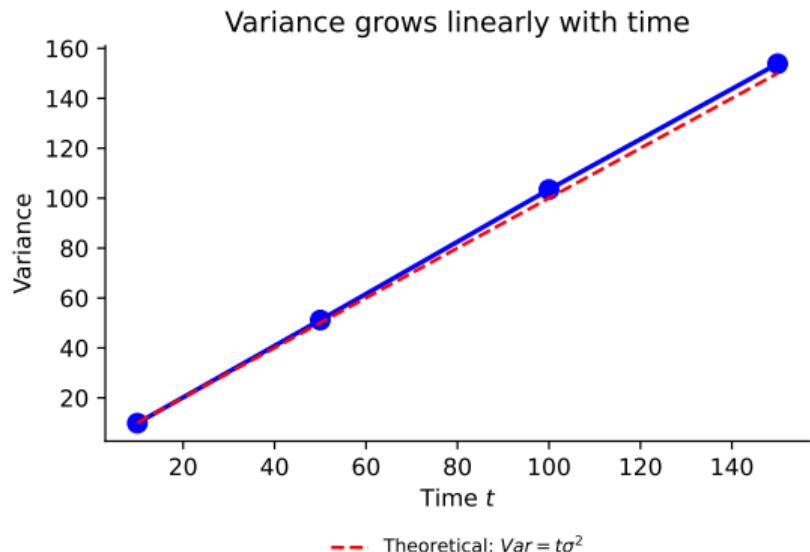
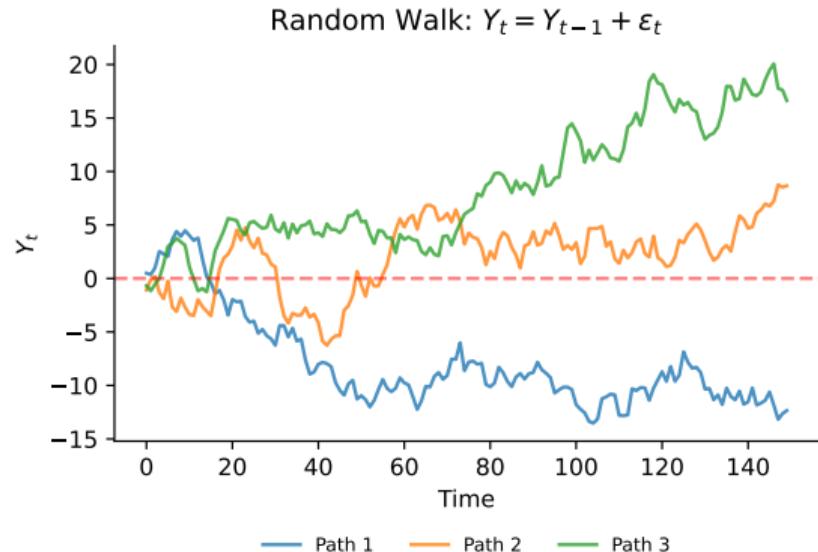
$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

With initial condition $Y_0 = 0$, we have: $Y_t = \sum_{i=1}^t \varepsilon_i$

Properties of Random Walk

- $\mathbb{E}[Y_t] = 0$ (constant mean)
- $\text{Var}(Y_t) = t\sigma^2$ (variance grows with time!)
- $\text{Cov}(Y_t, Y_{t-k}) = (t - k)\sigma^2$ for $k \leq t$
- ACF: $\rho_k = \sqrt{\frac{t-k}{t}} \rightarrow 1$ as $t \rightarrow \infty$

Random Walk: Visual Illustration



Left: multiple random walk paths wander unpredictably. Right: variance grows linearly with time.

Random Walk with Drift

Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

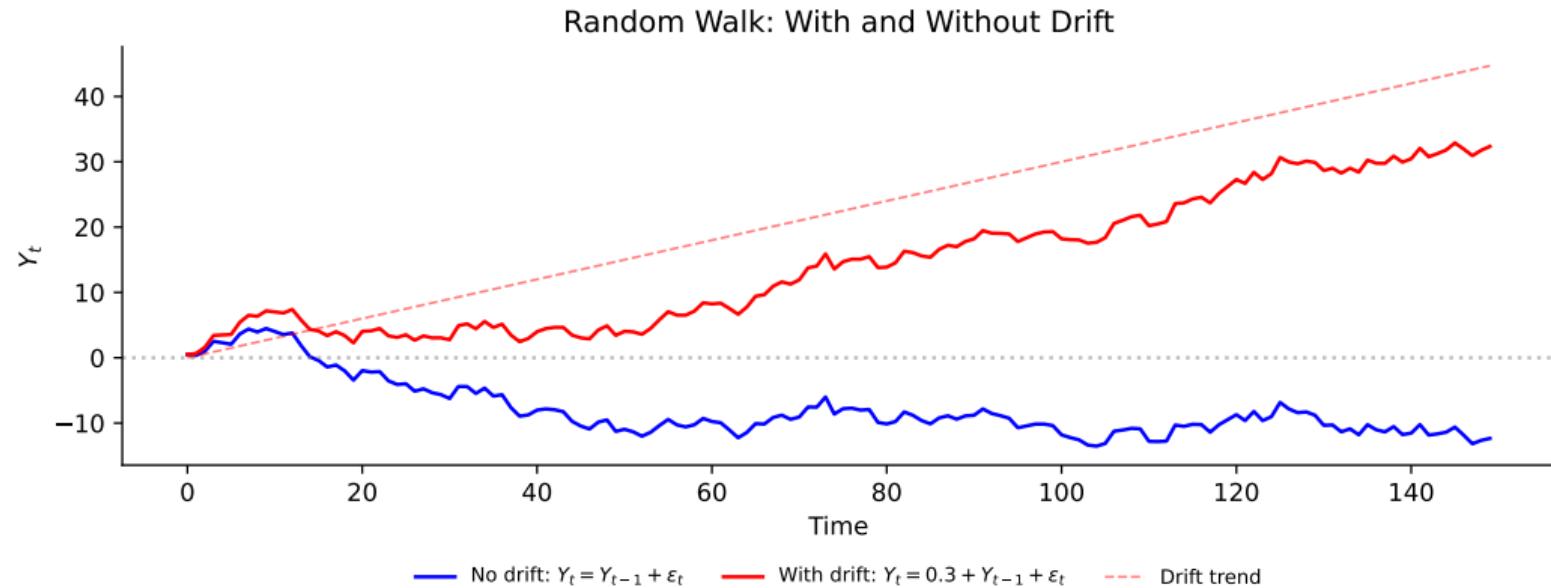
$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

Equivalently: $Y_t = Y_0 + \mu t + \sum_{i=1}^t \varepsilon_i$

Properties

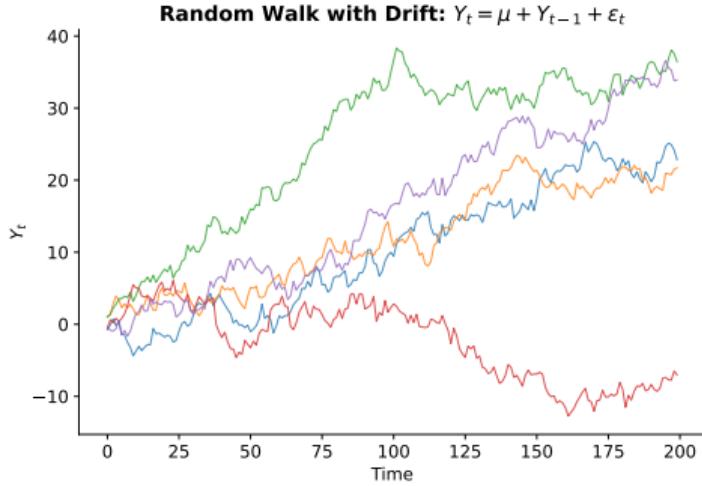
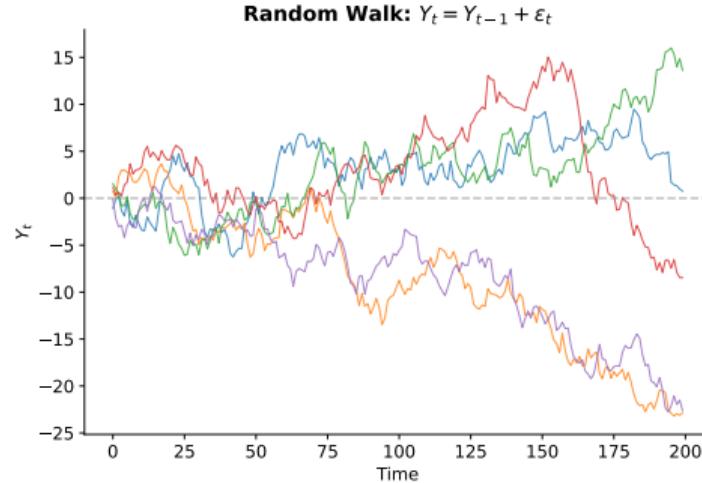
- $\mathbb{E}[Y_t] = Y_0 + \mu t$ (mean grows linearly)
- $\text{Var}(Y_t) = t\sigma^2$ (variance still grows)
- The drift μ creates an upward or downward trend
- Still non-stationary despite having a “trend”

Random Walk with Drift: Visual Illustration



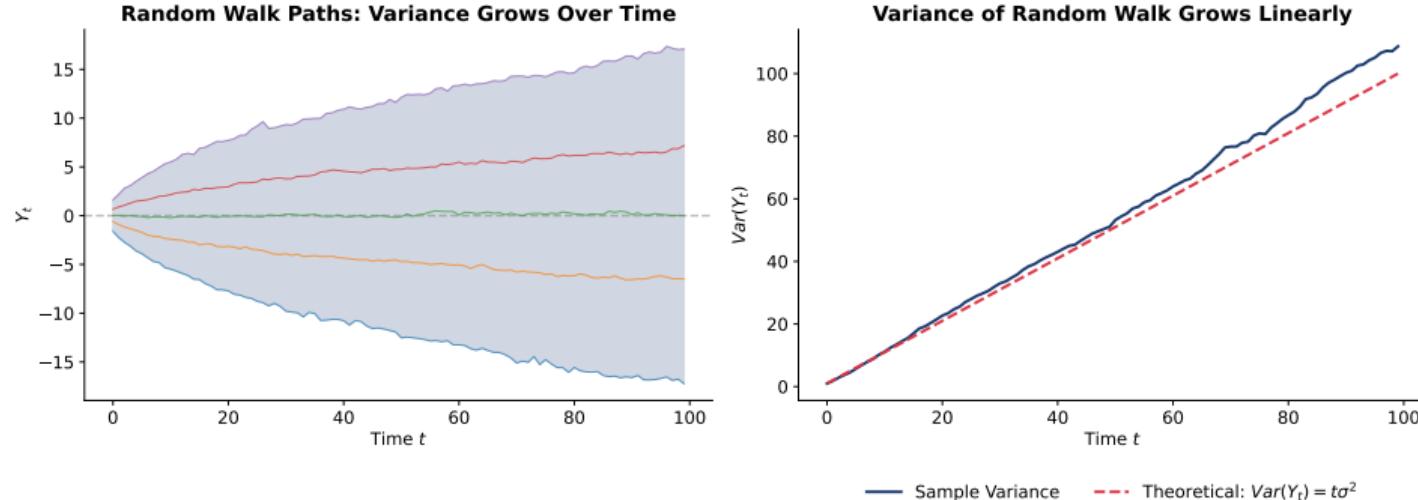
Random walk without drift (blue) wanders around zero. With drift (red), there's a systematic trend.

Simulating Random Walks



- **Left:** Pure random walks – no drift, wander unpredictably
- **Right:** Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time

Variance Growth: Why Random Walks Are Non-Stationary



- Left: Fan of paths shows uncertainty growing over time
- Right: Variance grows linearly: $\text{Var}(Y_t) = t\sigma^2$
- This violates stationarity (variance should be constant)

Definition 3 (Integrated Process of Order d)

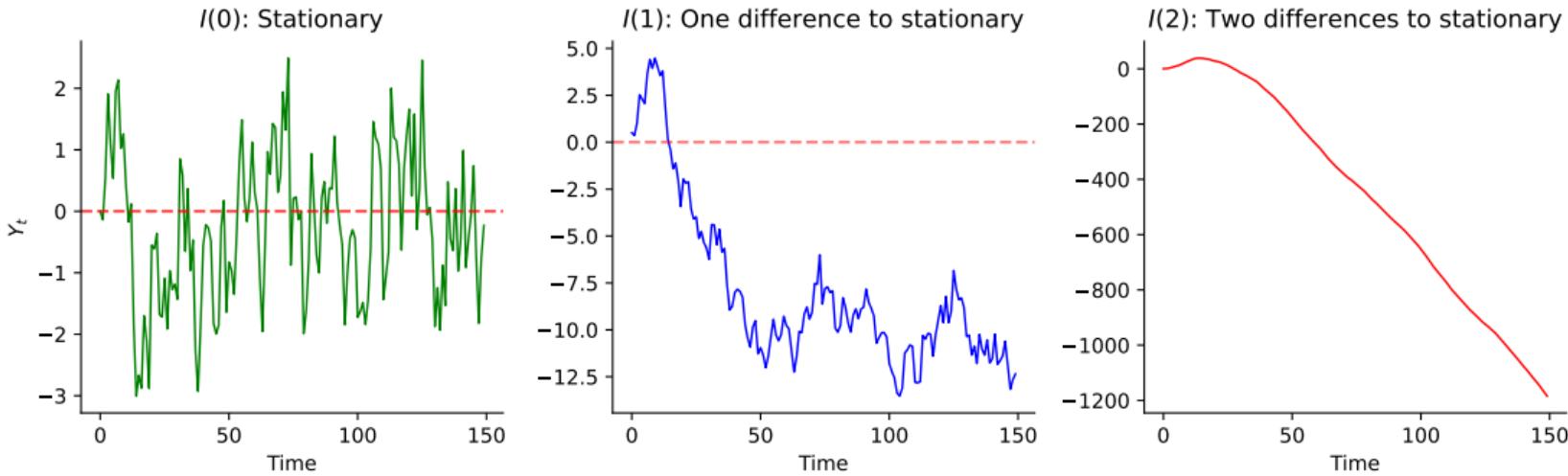
A time series $\{Y_t\}$ is **integrated of order d** , written $Y_t \sim I(d)$, if:

- Y_t is non-stationary
- $(1 - L)^d Y_t = \Delta^d Y_t$ is stationary
- $(1 - L)^{d-1} Y_t$ is still non-stationary

Common Cases

- $I(0)$: Stationary process (e.g., ARMA)
- $I(1)$: First difference is stationary (most common for economic data)
- $I(2)$: Second difference is stationary (less common)

Integrated Process: Visual Illustration



$I(0)$: stationary. $I(1)$: one difference needed. $I(2)$: two differences needed.

The Difference Operator

Definition 4 (First Difference)

The **first difference operator** Δ is defined as: $\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$, where L is the lag operator ($LY_t = Y_{t-1}$).

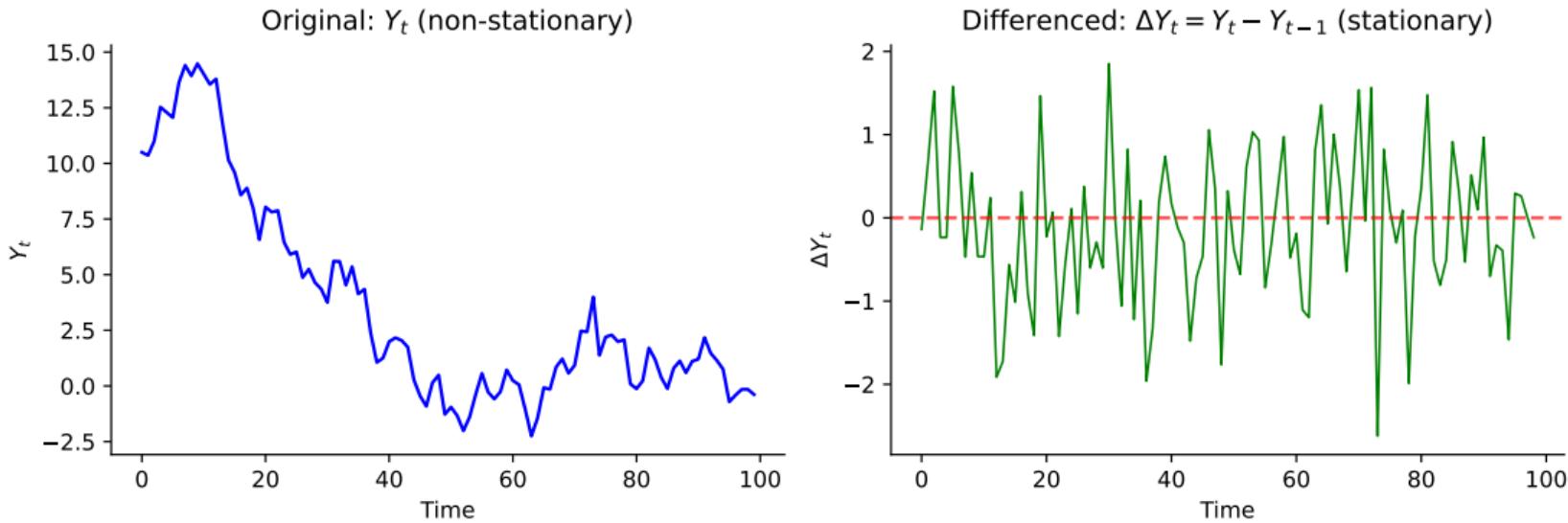
Higher-Order Differences

- Second difference: $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- d -th difference: $\Delta^d Y_t = (1 - L)^d Y_t$

Key Result

If $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$ (stationary).

First Difference: Visual Illustration



Left: non-stationary series. Right: after first difference, the series becomes stationary.

Example: Differencing a Random Walk

Random Walk to White Noise

Let $Y_t = Y_{t-1} + \varepsilon_t$ (random walk). Taking the first difference:

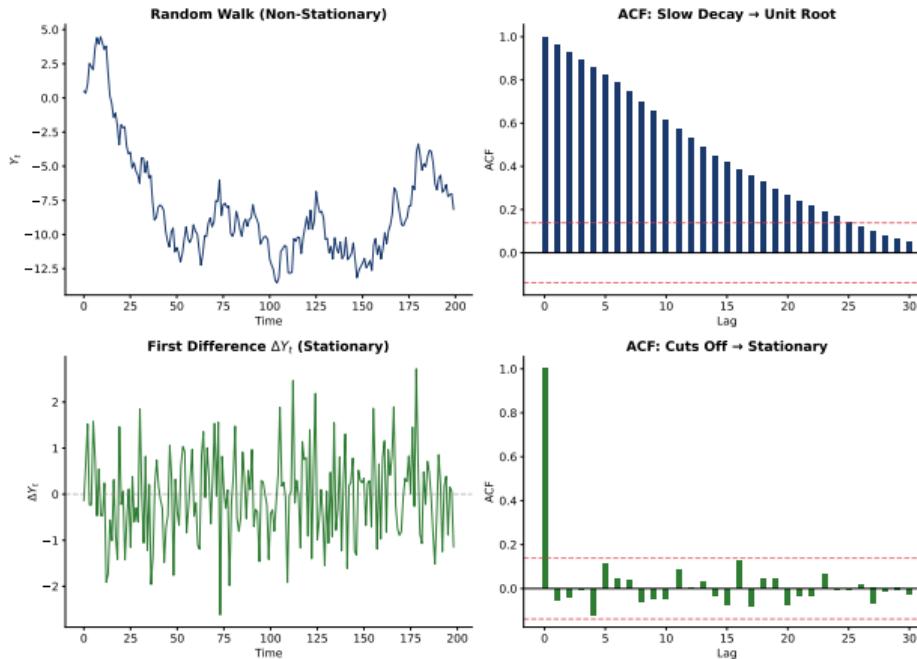
$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

Interpretation

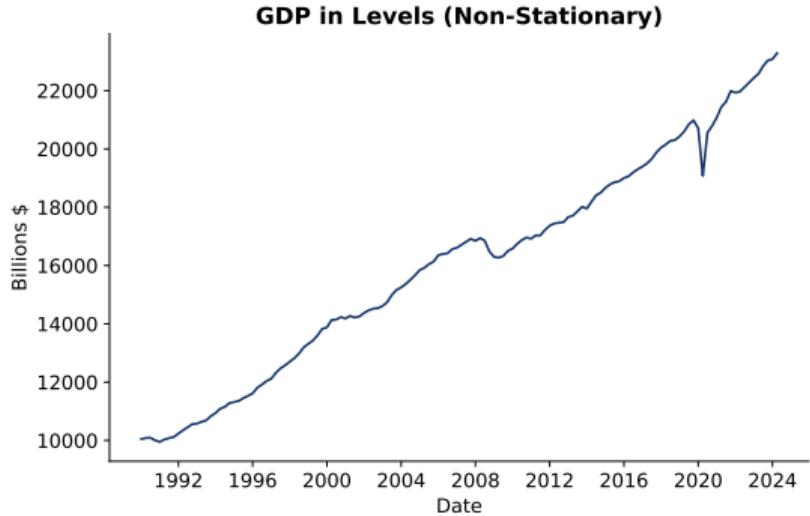
- A random walk is $I(1)$
- One difference transforms it to $I(0)$
- The “changes” in a random walk are stationary

ACF Diagnostic: Detecting Non-Stationarity



- **Top:** Random walk ACF decays very slowly \Rightarrow unit root
- **Bottom:** After differencing, ACF cuts off \Rightarrow stationary

Differencing in Practice: GDP Example



- **Left:** GDP in levels – clear upward trend (non-stationary)
- **Right:** GDP growth rate (log difference) – fluctuates around mean (stationary)
- Differencing removes the trend and achieves stationarity

Overdifferencing

Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
- Inflates variance
- Loses information

Example

If $Y_t \sim I(1)$, then $\Delta Y_t \sim I(0)$. But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with $\theta = 1$ (non-invertible boundary)!

Definition of ARIMA

Definition 5 (ARIMA(p,d,q))

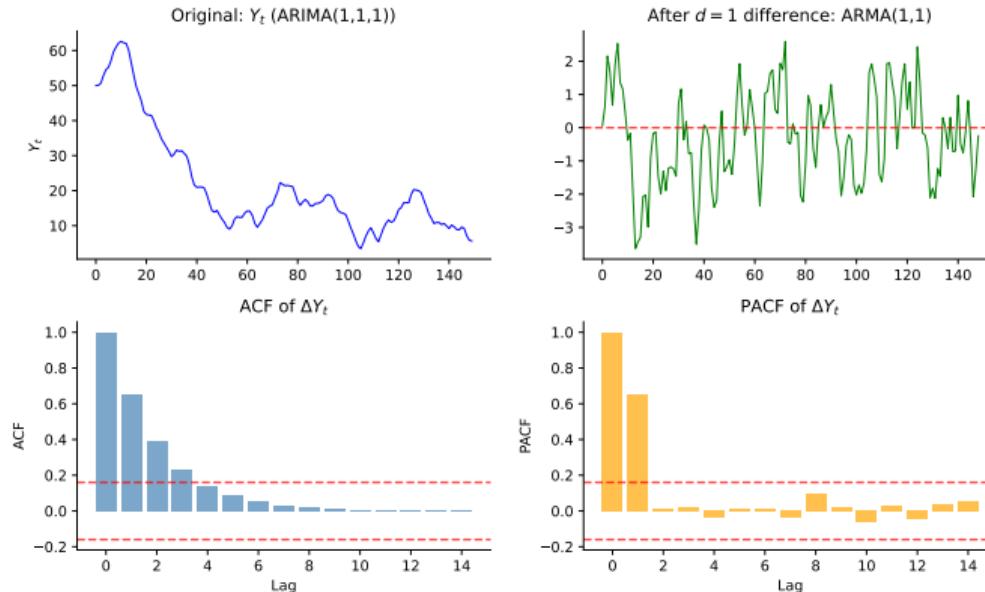
A time series $\{Y_t\}$ follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1 - L)^d Y_t = c + \theta(L)\varepsilon_t$$

where:

- $\phi(L) = 1 - \phi_1L - \phi_2L^2 - \cdots - \phi_pL^p$ (AR polynomial)
- $\theta(L) = 1 + \theta_1L + \theta_2L^2 + \cdots + \theta_qL^q$ (MA polynomial)
- d is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

ARIMA: Visual Illustration

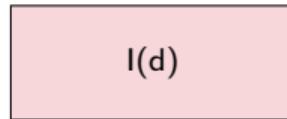


Top: original ARIMA series. Bottom left/right: after differencing, ACF/PACF help identify AR and MA orders.

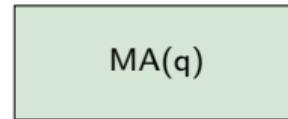
ARIMA Components



Autoregressive
Memory



Integration
Differencing



Moving Average
Shocks

Special Cases

- ARIMA($p, 0, q$) = ARMA(p, q) – stationary
- ARIMA($0, 1, 0$) = Random walk
- ARIMA($0, 1, 1$) = IMA($1, 1$) – exponential smoothing
- ARIMA($1, 1, 0$) = ARI($1, 1$) – differenced AR(1)

ARIMA(1,1,0) Example

ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently: $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$

Interpretation

- The **changes** in Y_t follow an AR(1) process
- If $|\phi_1| < 1$, the changes are stationary
- Y_t itself has a stochastic trend
- Common model for many economic time series

IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently: $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha) \hat{Y}_t$$

where $\alpha = 1 + \theta_1$ (for $-1 < \theta_1 < 0$).

The Role of the Constant in ARIMA

Constant Term in ARIMA(p,d,q)

When $d > 0$, the constant c has a different interpretation: $\phi(L)(1 - L)^d Y_t = c + \theta(L)\varepsilon_t$

Important Implications

- For $d = 1$: c represents the **drift** (average change): $\mathbb{E}[\Delta Y_t] = \frac{c}{1-\phi_1-\cdots-\phi_p}$
- For $d = 2$: c affects the **curvature** of the trend
- Often $c = 0$ is assumed when $d \geq 1$

Testing for Unit Roots

Why Test?

Before fitting an ARIMA model, we need to determine:

- ① Is the series stationary? (Is $d = 0$?)
- ② If not, how many differences are needed? (What is d ?)

Common Unit Root Tests

- **Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- **Phillips-Perron (PP)**
- **KPSS** (stationarity test – reversed null hypothesis)

The Dickey-Fuller Test

Setup

Consider the AR(1) model: $Y_t = \phi Y_{t-1} + \varepsilon_t$. Subtract Y_{t-1} : $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$, where $\gamma = \phi - 1$.

Hypotheses

- $H_0: \gamma = 0$ (unit root, $\phi = 1$, non-stationary)
- $H_1: \gamma < 0$ (stationary, $|\phi| < 1$)

Key Issue

Under H_0 , the t -statistic does **not** follow a standard t -distribution! Must use Dickey-Fuller critical values.

Dickey-Fuller Test Variants

Three Specifications

- ① **No constant, no trend:** $\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$
- ② **With constant (drift):** $\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$
- ③ **With constant and trend:** $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$

Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference

Augmented Dickey-Fuller (ADF) Test

The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

Definition 6 (ADF Test)

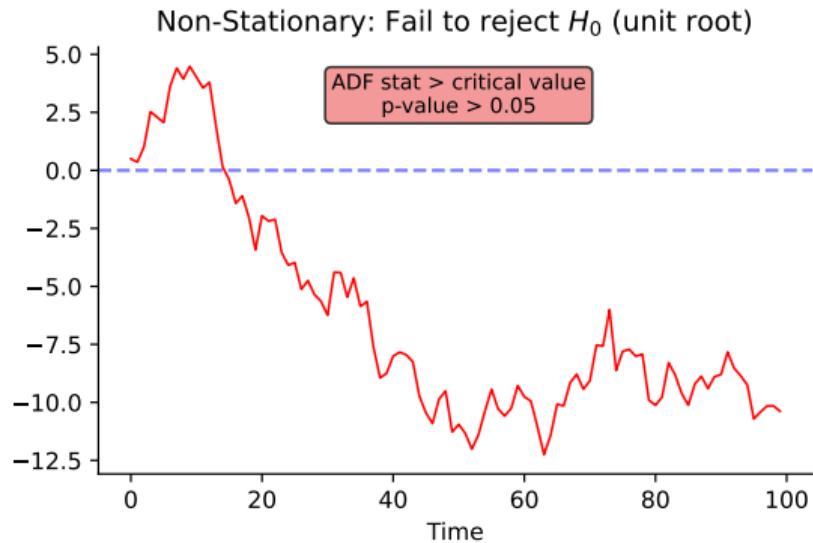
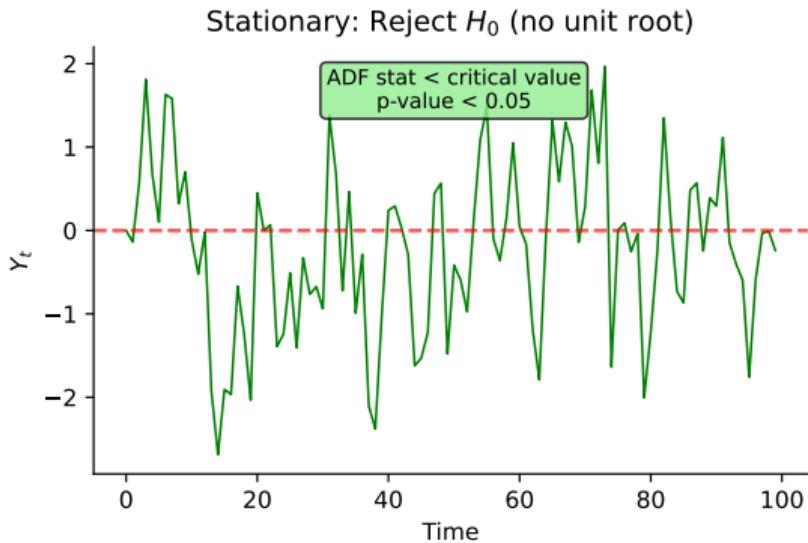
Add lagged differences: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^k \delta_j \Delta Y_{t-j} + \varepsilon_t$

Test $H_0 : \gamma = 0$ using ADF critical values.

Choosing Lag Length k

- Use information criteria (AIC, BIC)
- Start with k_{max} , reduce until last lag significant

ADF Test: Visual Illustration



Left: stationary series – ADF rejects unit root. Right: non-stationary – ADF fails to reject.

ADF Test Critical Values

Model	1%	5%	10%
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

Decision Rule

- Test statistic $<$ critical value \Rightarrow Reject H_0 (stationary)
- Test statistic \geq critical value \Rightarrow Fail to reject (unit root)

The Phillips-Perron (PP) Test

Motivation

Like ADF, tests H_0 : Unit root vs H_1 : Stationary, but uses a **non-parametric correction** for serial correlation instead of adding lagged differences.

Test Statistic

The PP test modifies the DF t -statistic:

$$Z_t = t_{\hat{\gamma}} \cdot \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - \hat{\sigma}^2)(se(\hat{\gamma}))}{2\hat{\lambda}^2 \cdot s}$$

where $\hat{\lambda}^2$ is a consistent estimate of the long-run variance using Newey-West.

Advantages over ADF

- Robust to heteroskedasticity and serial correlation
- No need to select lag length (uses bandwidth instead)

Reversed Hypotheses

Unlike ADF: H_0 : Stationary vs H_1 : Unit root

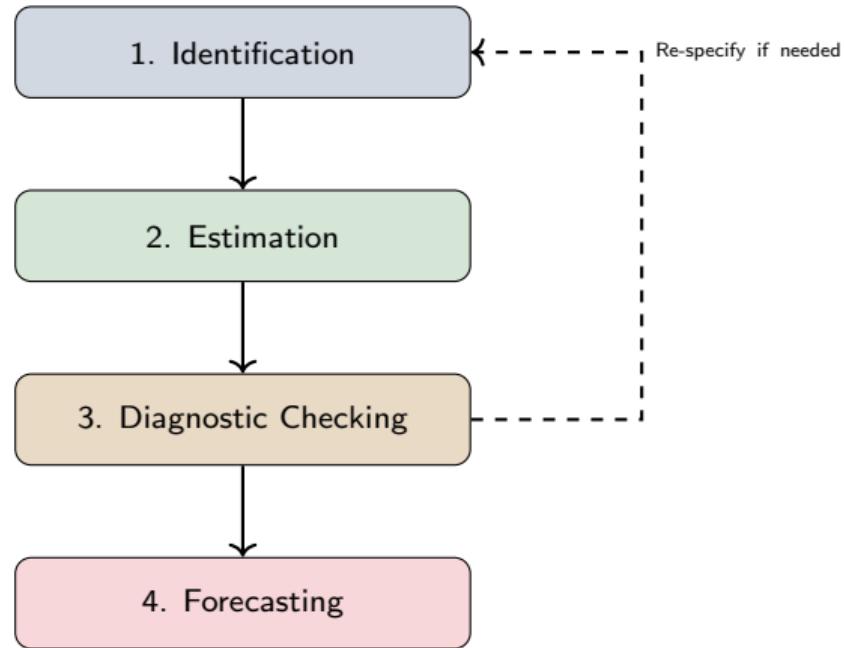
KPSS Procedure

Decompose: $Y_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$. Test whether $\text{Var}(u_t) = 0$.

Complementary Use with ADF

- ADF rejects, KPSS doesn't \Rightarrow Stationary
- ADF doesn't reject, KPSS rejects \Rightarrow Unit root
- Both reject or neither \Rightarrow Inconclusive

The Box-Jenkins Methodology



Step 1: Determining d

Procedure

- ① Plot the time series – look for trends, changing variance
- ② Examine ACF – slow decay suggests non-stationarity
- ③ Apply unit root tests (ADF, KPSS)
- ④ If non-stationary, difference and repeat

Practical Guidelines

- Most economic series: $d = 1$ is sufficient
- Rarely need $d > 2$
- If ACF of ΔY_t still decays slowly, try $d = 2$
- Watch for overdifferencing (ACF with $\rho_1 \approx -0.5$)

Step 2: Determining p and q

After Differencing

Once $W_t = \Delta^d Y_t$ is stationary, use ACF/PACF to identify ARMA(p, q):

Model	ACF	PACF
AR(p)	Decays exponentially	Cuts off after lag p
MA(q)	Cuts off after lag q	Decays exponentially
ARMA(p, q)	Decays	Decays

Information Criteria

When patterns are unclear, compare models using:

- $AIC = -2 \ln(L) + 2k$; $BIC = -2 \ln(L) + k \ln(n)$

Lower is better. BIC penalizes complexity more.

Auto-ARIMA Algorithms

Automated Model Selection

Modern software can automatically select (p, d, q) :

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

How Auto-ARIMA Works

- ① Use unit root tests to determine d
- ② Fit models for various (p, q) combinations
- ③ Select model with lowest AIC/BIC
- ④ Optionally use stepwise search for efficiency

Caution

Automated selection is helpful but not infallible. Always check diagnostics!

Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- Assumes $\varepsilon_t \sim N(0, \sigma^2)$
- Maximizes the likelihood function
- Provides consistent, efficient estimators
- Yields standard errors for inference

Conditional vs Exact MLE

- **Conditional MLE:** Conditions on initial values
- **Exact MLE:** Treats initial values as unknown
- Difference diminishes as sample size grows

Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- **AR stationarity:** Roots of $\phi(z) = 0$ outside unit circle
- **MA invertibility:** Roots of $\theta(z) = 0$ outside unit circle

Checking in Practice

Most software reports:

- Estimated coefficients with standard errors
- Roots of AR and MA polynomials
- Warning if near-unit-root detected

Residual Analysis

What to Check

If the model is correct, residuals $\hat{\varepsilon}_t$ should be white noise:

- ① Zero mean
- ② Constant variance
- ③ No autocorrelation
- ④ (Optional) Normality

Diagnostic Tools

- **Residual ACF/PACF:** Should show no significant spikes
- **Ljung-Box test:** Tests for autocorrelation at multiple lags
- **Q-Q plot:** Checks normality assumption
- **Residual vs fitted:** Checks for heteroskedasticity

The Ljung-Box Test

Definition 7 (Ljung-Box Q Statistic)

$$Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}. \text{ Under } H_0 \text{ (no autocorrelation): } Q(m) \sim \chi^2(m-p-q)$$

Usage

- Choose $m \approx \ln(n)$ or $m = 10$ for quarterly, $m = 20$ for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if $Q(m)$ exceeds critical value

If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

Minimum MSE Forecast

The optimal h -step ahead forecast is the conditional expectation: $\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \dots]$

ARIMA(1,1,1) Forecasting

Model: $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

One-step forecast: $\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1 \hat{\varepsilon}_T$

For $h > 1$: replace unknown ε_{T+j} with 0, unknown Y_{T+j} with $\hat{Y}_{T+j|T}$

Forecast Intervals

Forecast Uncertainty

The h -step forecast error variance: $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$, where ψ_j are MA(∞) coefficients.

Confidence Intervals

Under normality, $(1 - \alpha)\%$ interval: $\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$

Key Property for I(1) Series

For integrated processes, forecast variance grows without bound as $h \rightarrow \infty$. Intervals widen over time!

Long-Run Forecasts for ARIMA

Behavior as $h \rightarrow \infty$

For ARIMA(p,1,q) with drift c:

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with \sqrt{h}

For ARIMA(p,1,q) without drift:

- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

Practical Implication

ARIMA forecasts are most reliable for short horizons. Long-term forecasts have very wide uncertainty bands.

Rolling Forecasting: Concept

What is Rolling Forecasting?

A technique to evaluate forecast accuracy out-of-sample:

- ① Fix a **training window** of size w
- ② Estimate model on observations $t = 1, \dots, w$
- ③ Forecast h steps ahead: $\hat{Y}_{w+h|w}$
- ④ **Roll** the window forward by one period
- ⑤ Repeat until end of sample

Why Rolling Forecasts?

- Mimics real-time forecasting scenario
- Provides multiple forecast errors for evaluation
- Avoids overfitting to full sample

Rolling Forecast: Step-by-Step Example

Setup: ARIMA(1,1,0) with $\phi_1 = 0.6$

Model: $\Delta Y_t = \phi_1 \Delta Y_{t-1} + \varepsilon_t$ where $\Delta Y_t = Y_t - Y_{t-1}$

Given Data at Time T

$$Y_{T-2} = 100, \quad Y_{T-1} = 103, \quad Y_T = 108 \quad \Rightarrow \quad \Delta Y_{T-1} = 3, \quad \Delta Y_T = 5$$

1-Step Ahead Point Forecast

$$\begin{aligned}\hat{\Delta Y}_{T+1|T} &= \phi_1 \cdot \Delta Y_T = 0.6 \times 5 = 3 \\ \hat{Y}_{T+1|T} &= Y_T + \hat{\Delta Y}_{T+1|T} = 108 + 3 = \boxed{111}\end{aligned}$$

2-Step Ahead Forecast

$$\begin{aligned}\hat{\Delta Y}_{T+2|T} &= \phi_1 \cdot \hat{\Delta Y}_{T+1|T} = 0.6 \times 3 = 1.8 \\ \hat{Y}_{T+2|T} &= \hat{Y}_{T+1|T} + \hat{\Delta Y}_{T+2|T} = 111 + 1.8 = \boxed{112.8}\end{aligned}$$

General Formula for h -Step Forecast (ARIMA(1,1,0))

$$\begin{aligned}\hat{\Delta Y}_{T+h|T} &= \phi_1^h \cdot \Delta Y_T \\ \hat{Y}_{T+h|T} &= Y_T + \Delta Y_T \cdot \frac{\phi_1(1 - \phi_1^h)}{1 - \phi_1}\end{aligned}$$

Numerical: 3-Step Forecast

$$\hat{Y}_{T+3|T} = 108 + 5 \times \frac{0.6(1 - 0.6^3)}{1 - 0.6} = 108 + 5 \times 1.092 = \boxed{113.46}$$

Confidence Intervals: Formulas

Forecast Error Variance

For ARIMA(1,1,0), the h -step forecast error variance:

$$\text{Var}(e_{T+h|T}) = \sigma^2 \left(1 + \sum_{j=1}^{h-1} \psi_j^2 \right)$$

where $\psi_j = \phi_1^{j-1} (1 + \phi_1 + \dots + \phi_1^{j-1}) = \phi_1^{j-1} \cdot \frac{1 - \phi_1^j}{1 - \phi_1}$

$(1 - \alpha)\%$ Confidence Interval

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(e_{T+h|T})}$$

For 95% CI: $z_{0.025} = 1.96$

Confidence Interval: Numerical Example

Given: $\sigma^2 = 4$, $\phi_1 = 0.6$, $\hat{Y}_{T+1|T} = 111$

1-Step Ahead CI

$$\text{Var}(e_{T+1|T}) = \sigma^2 = 4$$

$$95\% \text{ CI} = 111 \pm 1.96 \times \sqrt{4} = 111 \pm 3.92 = [107.08, 114.92]$$

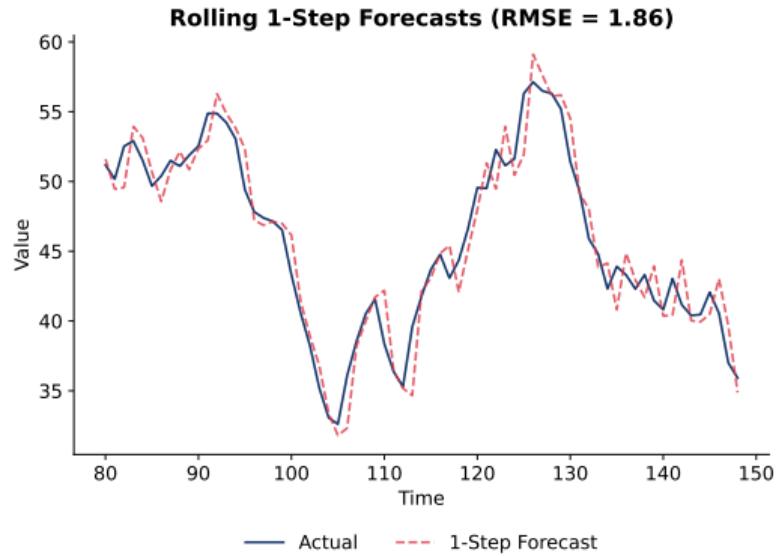
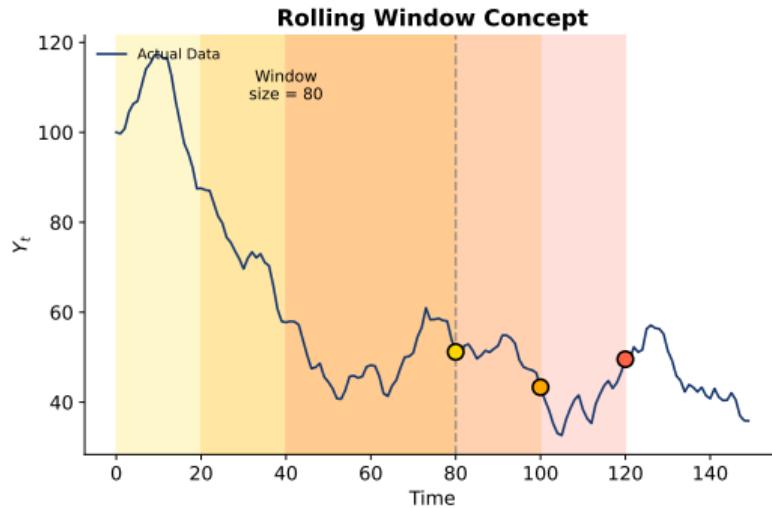
2-Step Ahead CI (for $\hat{Y}_{T+2|T} = 112.8$)

$$\psi_1 = 1 + \phi_1 = 1.6, \quad \text{Var}(e_{T+2|T}) = 4(1 + 1.6^2) = 14.24$$

$$95\% \text{ CI} = 112.8 \pm 1.96 \times \sqrt{14.24} = 112.8 \pm 7.40 = [105.40, 120.20]$$

Note: CI widens as horizon increases!

Rolling Window Illustration



- Each window produces a 1-step ahead forecast
- Compare forecasts to actuals to compute RMSE, MAE
- Rolling window keeps model estimation up-to-date

Rolling Forecast: Python Code

Implementation

```
from statsmodels.tsa.arima.model import ARIMA

window_size = 100
forecasts, actuals = [], []

for t in range(window_size, len(y) - 1):
    train = y[:t]                      # Rolling window
    model = ARIMA(train, order=(1,1,0)).fit()
    forecast = model.forecast(steps=1)[0]
    forecasts.append(forecast)
    actuals.append(y[t])

rmse = np.sqrt(np.mean((np.array(forecasts) - np.array(actuals))**2))
```

Case Study: Complete ARIMA Analysis

Objective

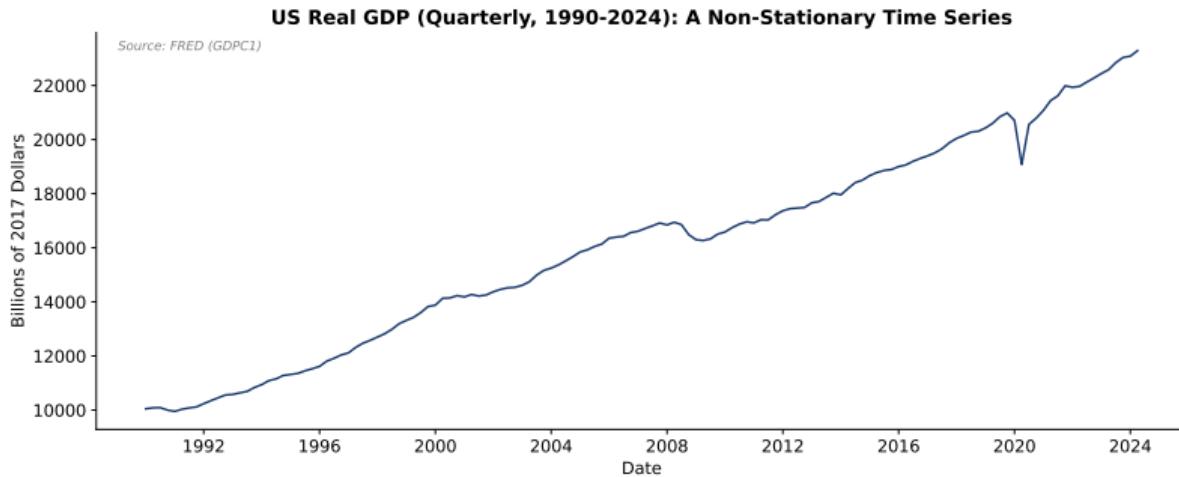
Forecast US Real GDP using the Box-Jenkins methodology

- ① **Step 1:** Visualize data and check for stationarity
- ② **Step 2:** Apply unit root tests (ADF, KPSS)
- ③ **Step 3:** Difference if needed, identify p and q
- ④ **Step 4:** Estimate ARIMA model
- ⑤ **Step 5:** Diagnostic checking
- ⑥ **Step 6:** Generate forecasts with confidence intervals
- ⑦ **Step 7:** Evaluate forecast accuracy

Data

US Real GDP (FRED: GDPC1), Quarterly, 1990Q1–2024Q2, $n = 138$ observations

Step 1: Initial Data Analysis



Observations

- Clear upward trend \Rightarrow non-constant mean
- Variance appears relatively stable (after log transform)
- Notable dip in 2020 (COVID-19 pandemic)
- **Conclusion:** Series is non-stationary, needs differencing

Step 2: Unit Root Testing

ADF Test on Log GDP Levels

- Test statistic: -0.91
- Critical values: -3.48 (1%), -2.88 (5%), -2.58 (10%)
- p-value: 0.79
- **Result:** Cannot reject $H_0 \Rightarrow$ **Unit root present**

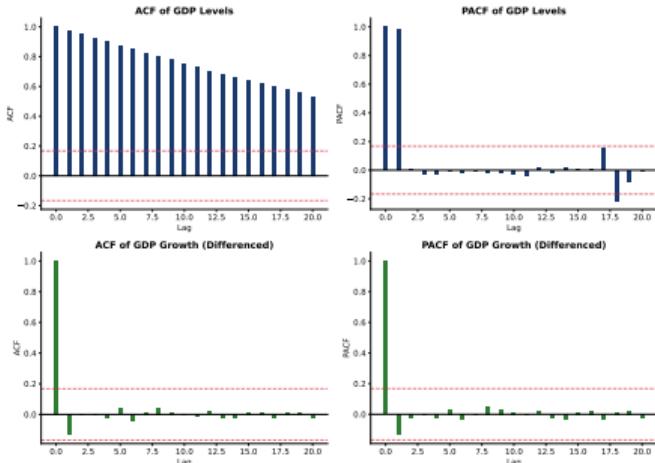
ADF Test on First Difference (Growth Rate)

- Test statistic: -13.24
- p-value: < 0.001
- **Result:** Reject H_0 at 1% \Rightarrow **Stationary after differencing**

Conclusion

GDP is $I(1) \Rightarrow$ Use $d = 1$ in ARIMA model

Step 3: Model Identification via ACF/PACF



Analysis of Differenced Series

- ACF: Significant spike at lag 1, then cuts off \Rightarrow suggests MA(1)
- PACF: Significant spike at lag 1, decays \Rightarrow suggests AR(1)
- Candidate models: ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1)

Step 4: Model Estimation

Comparing Models using Information Criteria

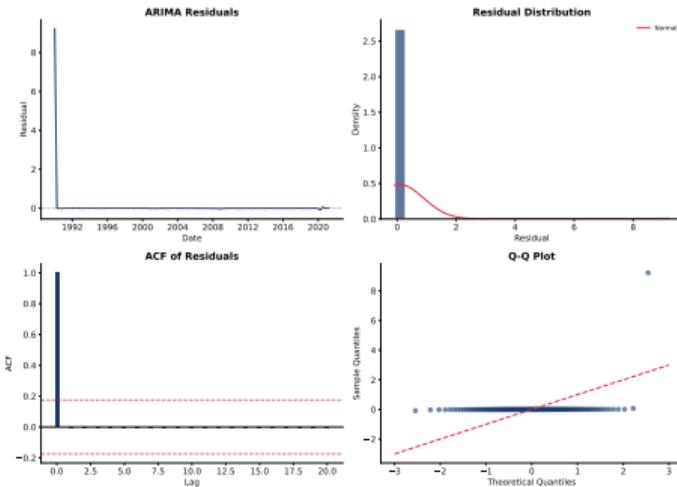
Model	AIC	BIC	Log-Lik
ARIMA(1,1,0)	-725.2	-719.5	364.6
ARIMA(0,1,1)	-724.8	-719.2	364.4
ARIMA(1,1,1)	-747.0	-738.5	376.5

Selected Model: ARIMA(1,1,1)

$$(1 - 0.35L)(1 - L)Y_t = (1 + 0.58L)\varepsilon_t, \quad \hat{\sigma}^2 = 0.000156$$

- $\hat{\phi}_1 = 0.35$ (SE = 0.09), significant at 1%
- $\hat{\theta}_1 = 0.58$ (SE = 0.08), significant at 1%

Step 5: Diagnostic Checking



Residual Analysis

- Ljung-Box test: $Q(10) = 5.8$, p-value = 0.83 \Rightarrow No autocorrelation
- Jarque-Bera test: $JB = 156.4$, p-value < 0.001 \Rightarrow Non-normal (COVID outlier)
- Conclusion: Model passes autocorrelation checks; outliers expected

Step 6: Forecasting with Confidence Intervals

Last Observed Values (Log GDP)

$$Y_T = 9.973 \text{ (2024Q2)}, \quad Y_{T-1} = 9.956 \text{ (2024Q1)} \\ \Delta Y_T = 0.017, \quad \hat{\varepsilon}_T = 0.004$$

1-Step Ahead Forecast (2024Q3)

$$\hat{\Delta}Y_{T+1} = \hat{\phi}_1\Delta Y_T + \hat{\theta}_1\hat{\varepsilon}_T = 0.35(0.017) + 0.58(0.004) = 0.0083$$

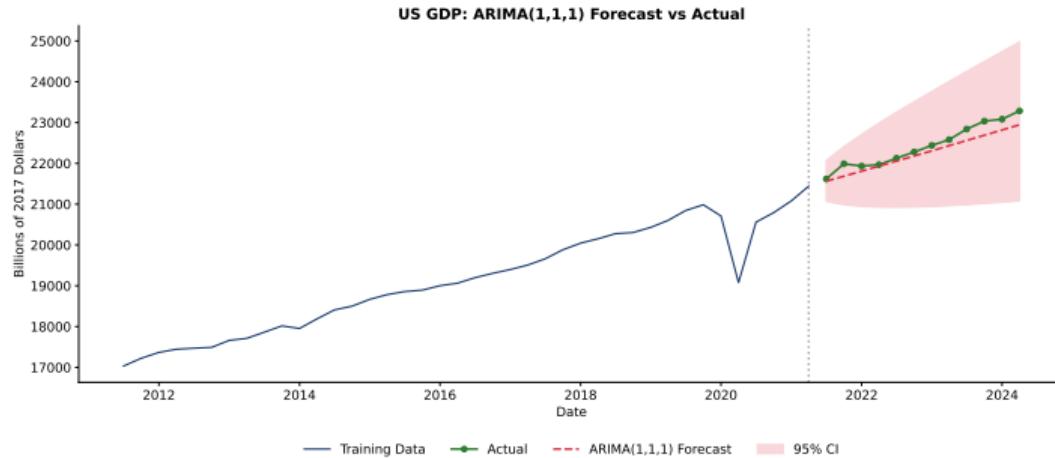
$$\hat{Y}_{T+1} = 9.973 + 0.0083 = \boxed{9.981}$$

95% Confidence Interval

$$CI = 9.981 \pm 1.96 \times \sqrt{0.000156} = [9.957, 10.006]$$

In levels: GDP forecast = \$21,652B, CI = [\$21,142B, \$22,175B]

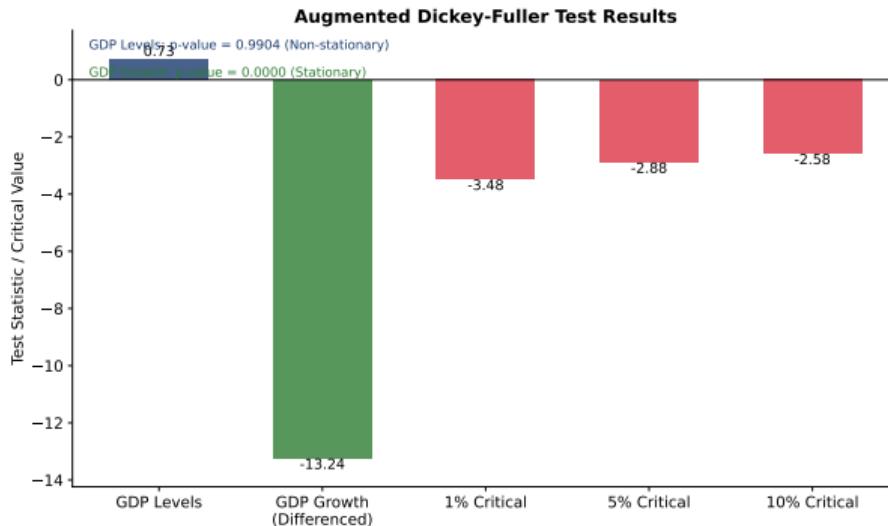
Step 7: Forecast Evaluation



Out-of-Sample Performance (Last 12 Quarters)

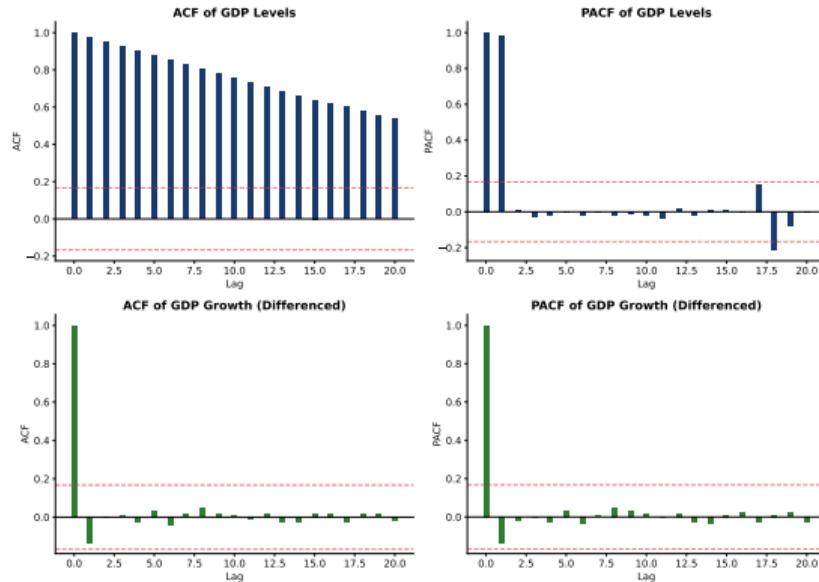
- RMSE = 0.0486 (log scale) $\approx 4.86\%$ error
- MAE = 0.0430 (log scale) $\approx 4.30\%$ error
- Direction accuracy = 91% (correctly predicted growth/decline)

Unit Root Test Results



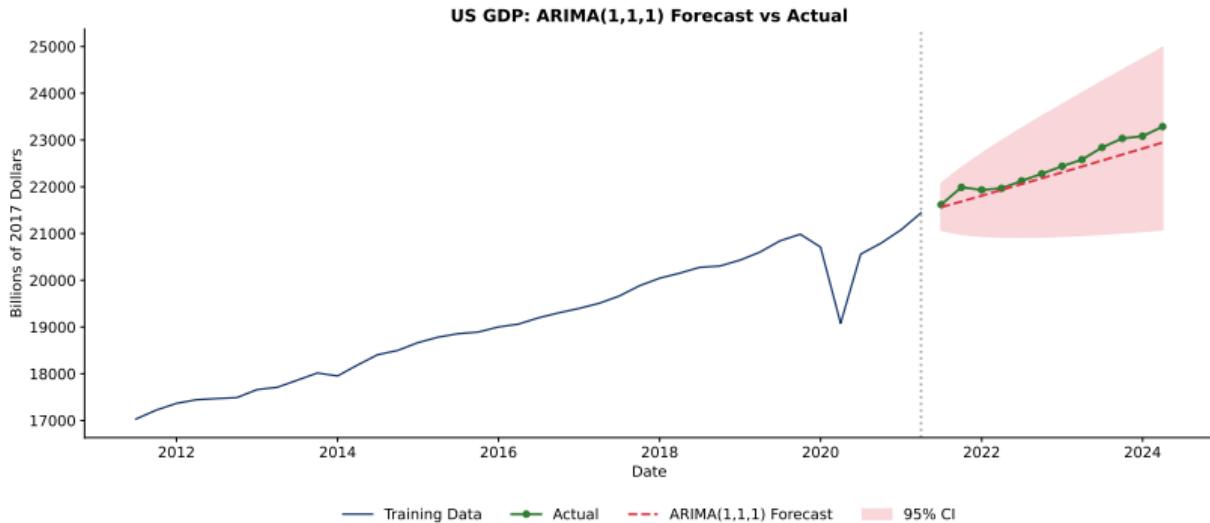
- GDP in levels: Cannot reject unit root (non-stationary)
- GDP growth: Reject unit root at 1% level (stationary)

ACF/PACF: Levels vs Differenced



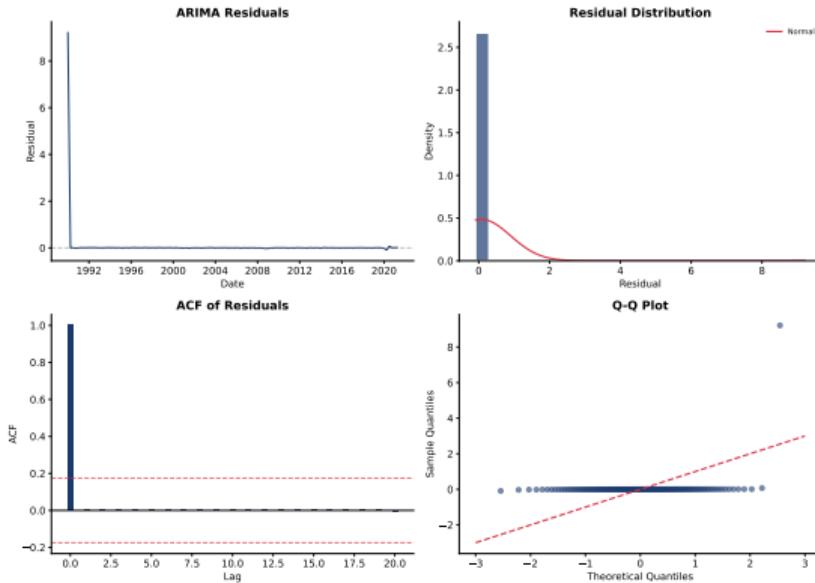
- **Top:** Slow ACF decay in levels suggests non-stationarity
- **Bottom:** After differencing, ACF/PACF help identify p and q

ARIMA Forecasting: Actual vs Predicted



- ARIMA(1,1,1) captures the trend dynamics
- Confidence intervals widen with forecast horizon

Model Diagnostics



- Residuals appear random; ACF within bounds
- Q-Q plot shows approximate normality

Auto-ARIMA Example

```
# Automatic model selection
model = pm.auto_arima(y, start_p=0, start_q=0,
                      max_p=3, max_q=3, d=None,
                      seasonal=False, trace=True)
print(model.summary())
```

Key Takeaways

Main Points

- ① Non-stationarity is common in economic data – must be addressed
- ② Differencing transforms $I(d)$ to $I(0)$
- ③ ARIMA(p,d,q) combines differencing with ARMA modeling
- ④ Unit root tests (ADF, KPSS) help determine d
- ⑤ Box-Jenkins methodology: Identify → Estimate → Diagnose
- ⑥ Forecasts for $I(1)$ series have growing uncertainty

Next Steps

Chapter 4 will extend ARIMA to handle seasonality: SARIMA models.

Quiz Question 1

Question

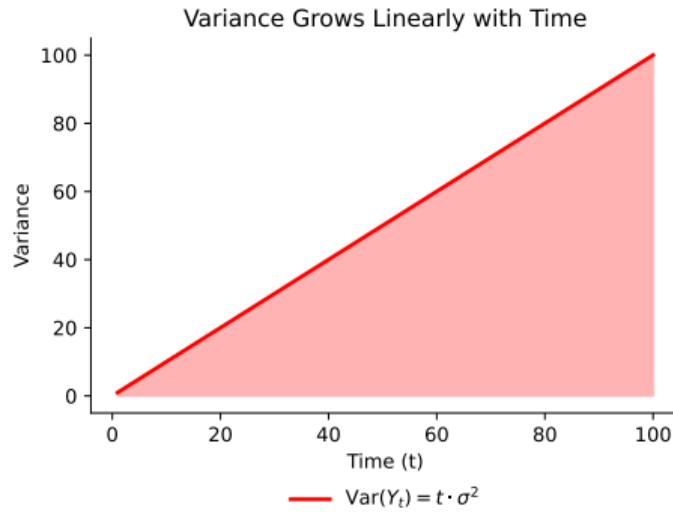
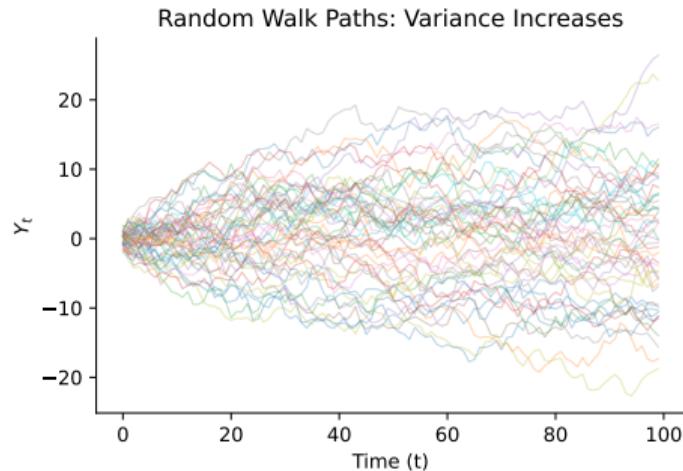
A time series Y_t follows a random walk: $Y_t = Y_{t-1} + \varepsilon_t$. What is $\text{Var}(Y_t)$?

- A σ^2 (constant)
- B $t \cdot \sigma^2$ (grows linearly with time)
- C σ^2/t (decreases with time)
- D σ^{2t} (grows exponentially)

Quiz Question 1: Answer

Correct Answer: (B) $\text{Var}(Y_t) = t \cdot \sigma^2$

Random walk variance grows linearly with time — this is why random walks are non-stationary.



Quiz Question 2

Question

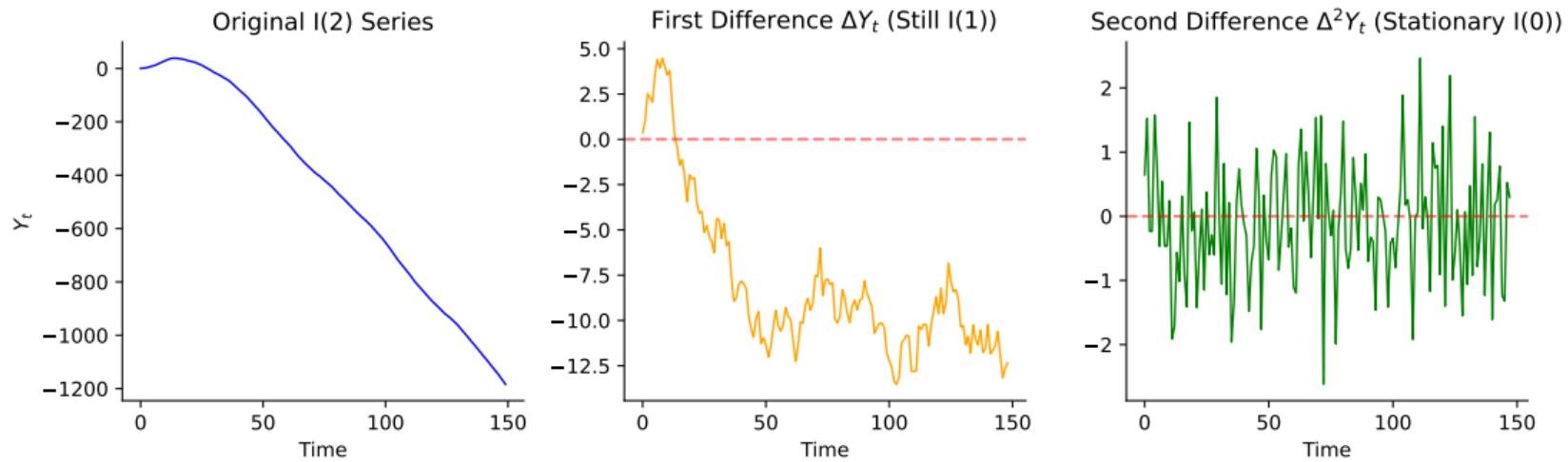
If a series Y_t is $I(2)$, how many times must you difference it to achieve stationarity?

- A 0 times (already stationary)
- B 1 time
- C 2 times
- D Cannot be made stationary by differencing

Quiz Question 2: Answer

Correct Answer: (C) 2 times

$I(d)$ means “integrated of order d ” — requires d differences for stationarity.



Quiz Question 3

Question

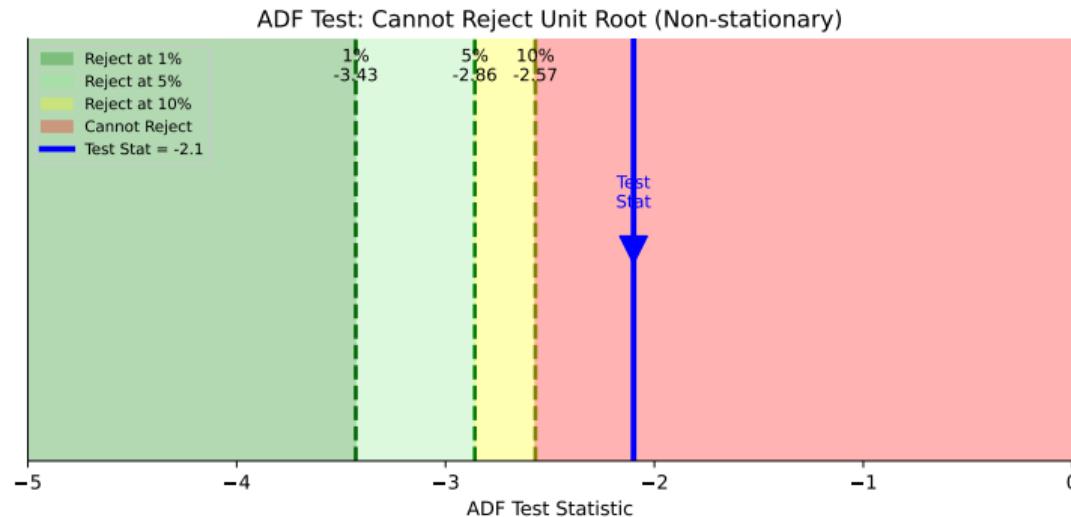
You run an ADF test and get a test statistic of -2.1 with critical values: -3.43 (1%), -2.86 (5%), -2.57 (10%). What do you conclude?

- A Reject H_0 : series is stationary at all levels
- B Reject H_0 : series is stationary at 10% level only
- C Fail to reject H_0 : series likely has a unit root
- D The test is inconclusive

Quiz Question 3: Answer

Correct Answer: (C) Fail to reject H_0 : series has unit root

Test stat $-2.1 > -2.57$ (10% CV) \Rightarrow Cannot reject at any level. Consider differencing.



Quiz Question 4

Question

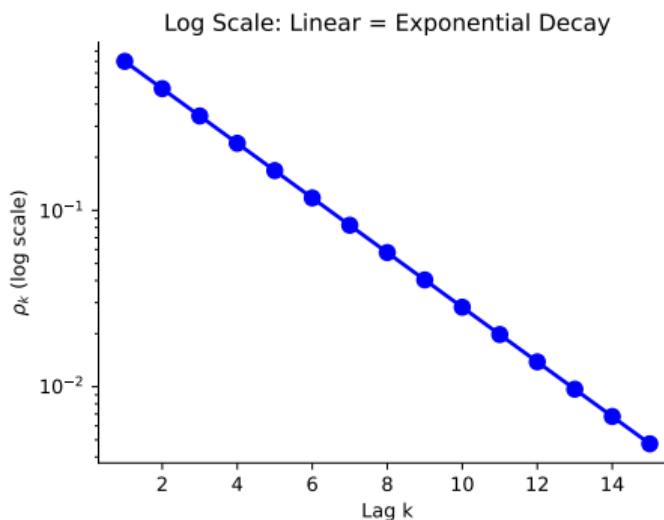
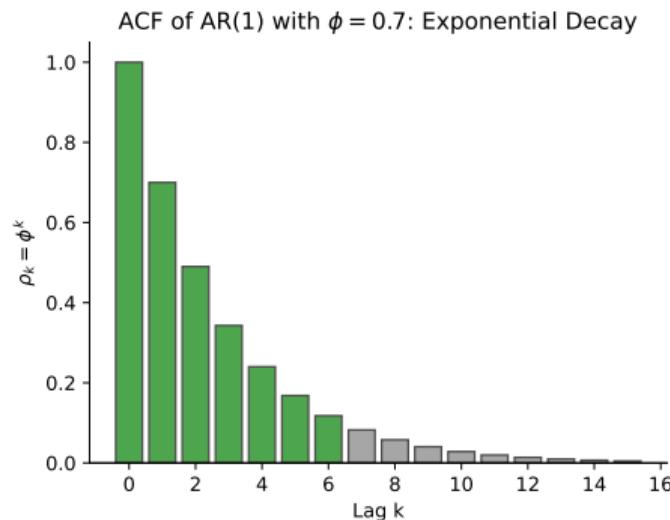
For an ARIMA(1,1,0) model, what is the ACF pattern of the **differenced** series ΔY_t ?

- A Cuts off after lag 1
- B Decays exponentially
- C Alternates in sign
- D Is zero at all lags

Quiz Question 4: Answer

Correct Answer: (B) Decays exponentially

$\text{ARIMA}(1,1,0) \Rightarrow \Delta Y_t$ follows AR(1) with ACF $\rho_k = \phi^k$ (geometric decay).



Quiz Question 5

Question

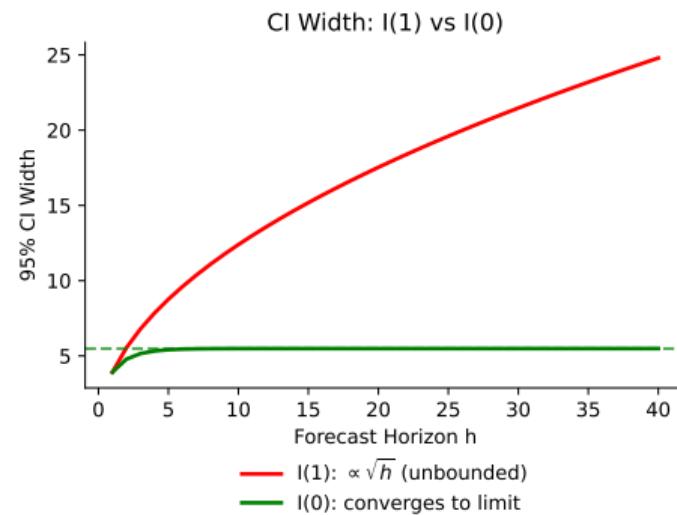
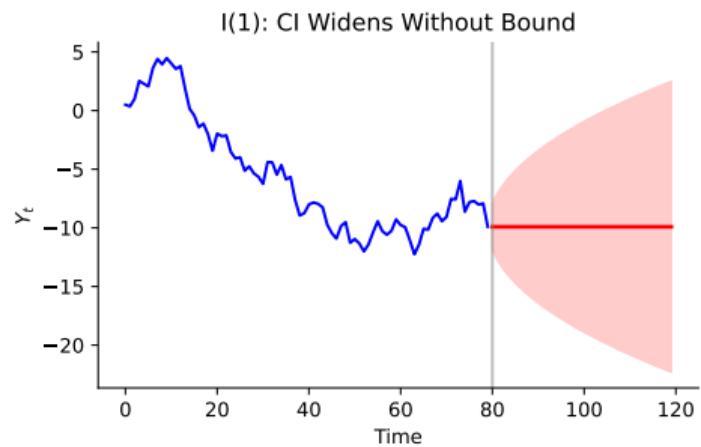
What happens to ARIMA forecast confidence intervals as the horizon h increases for an $I(1)$ series?

- A They stay constant
- B They narrow (more precision)
- C They widen without bound
- D They widen but converge to a limit

Quiz Question 5: Answer

Correct Answer: (C) They widen without bound

For $I(1)$: CI width $\propto \sqrt{h}$ (unbounded). For $I(0)$: CIs converge to a limit.



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