Chapter 3: ARIMA Models

Non-Stationary Time Series

Outline

Why Non-Stationarity Matters

The Problem

Many economic and financial time series are **non-stationary**:

- GDP, stock prices, exchange rates, inflation indices
- They exhibit trends, changing means, or growing variance

Consequences of Non-Stationarity

- Standard ARMA models assume stationarity
- OLS regression with non-stationary data leads to **spurious regression**
- Sample moments (mean, variance, ACF) are not consistent estimators
- Statistical inference becomes invalid

Types of Non-Stationarity

Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a deterministic function of time
- Can be removed by **detrending**
- Shocks have temporary effects

Stochastic Trend (Unit Root)

$$Y_t = Y_{t-1} + \varepsilon_t$$

- Random walk process
- Must be removed by **differencing**
- Shocks have permanent effects

Key Distinction

Correct identification is crucial: detrending a unit root process or differencing a trend-stationary process both lead to misspecification!

The Random Walk Process

Definition 1 (Random Walk)

A **random walk** is defined as:

$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

With initial condition $Y_0 = 0$, we have:

$$Y_t = \sum_{i=1}^{t} \varepsilon_i$$

Properties of Random Walk

- $\mathbb{E}[Y_t] = 0$ (constant mean)
- $\text{Var}(Y_t) = t\sigma^2$ (variance grows with time!)
- $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$ for $k \leq t$
- ACF: $\rho_k = \sqrt{\frac{t-k}{t}} \to 1$ as $t \to \infty$

Random Walk with Drift

Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

Equivalently:

$$Y_t = Y_0 + \mu t + \sum_{i=1}^{t} \varepsilon_i$$

Properties

- $\mathbb{E}[Y_t] = Y_0 + \mu t$ (mean grows linearly)
- $\text{Var}(Y_t) = t\sigma^2$ (variance still grows)
- The drift μ creates an upward or downward trend
- Still non-stationary despite having a "trend"

Integrated Processes

Definition 3 (Integrated Process of Order d)

A time series $\{Y_t\}$ is **integrated of order** d, written $Y_t \sim I(d)$, if:

- Y_t is non-stationary
- $(1-L)^d Y_t = \Delta^d Y_t$ is stationary
- $(1-L)^{d-1} Y_t$ is still non-stationary

Common Cases

- $I(0)$: Stationary process (e.g., ARMA)
- $I(1)$: First difference is stationary (most common for economic data)
- $I(2)$: Second difference is stationary (less common)

The Difference Operator

Definition 4 (First Difference)

The **first difference operator** Δ is defined as:

$$\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$$

where L is the lag operator ($LY_t = Y_{t-1}$).

Higher-Order Differences

- Second difference: $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- d-th difference: $\Delta^d Y_t = (1 - L)^d Y_t$

Key Result

If $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$ (stationary).

Example: Differencing a Random Walk

Random Walk to White Noise

Let $Y_t = Y_{t-1} + \varepsilon_t$ (random walk).

Taking the first difference:

$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

Interpretation

- A random walk is $I(1)$
- One difference transforms it to $I(0)$
- The “changes” in a random walk are stationary

Overdifferencing

Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
- Inflates variance
- Loses information

Example

If $Y_t \sim I(1)$, then $\Delta Y_t \sim I(0)$.
But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with $\theta = 1$ (non-invertible boundary)!

Definition of ARIMA

Definition 5 (ARIMA(p,d,q))

A time series $\{Y_t\}$ follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$$

where:

- $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$ (AR polynomial)
- $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$ (MA polynomial)
- d is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

ARIMA Components

AR(p)	I(d)	MA(q)
Autoregressive Memory	Integration Differencing	Moving Average Shocks

Special Cases

- ARIMA(p,0,q) = ARMA(p,q) – stationary
- ARIMA(0,1,0) = Random walk
- ARIMA(0,1,1) = IMA(1,1) – exponential smoothing
- ARIMA(1,1,0) = ARI(1,1) – differenced AR(1)

ARIMA(1,1,0) Example

ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently:

$$(1 - \phi_1 L)(1 - L) Y_t = c + \varepsilon_t$$

Interpretation

- The **changes** in Y_t follow an AR(1) process
- If $|\phi_1| < 1$, the changes are stationary
- Y_t itself has a stochastic trend
- Common model for many economic time series

ARIMA(0,1,1) Example

IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently:

$$(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$$

Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha)\hat{Y}_t$$

where $\alpha = 1 + \theta_1$ (for $-1 < \theta_1 < 0$).

The Role of the Constant in ARIMA

Constant Term in ARIMA(p,d,q)

When $d > 0$, the constant c has a different interpretation:

$$\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$$

Important Implications

- For $d = 1$: c represents the **drift** (average change)

$$\mathbb{E}[\Delta Y_t] = \frac{c}{1 - \phi_1 - \cdots - \phi_p}$$

- For $d = 2$: c affects the **curvature** of the trend
- Often $c = 0$ is assumed when $d \geq 1$

Testing for Unit Roots

Why Test?

Before fitting an ARIMA model, we need to determine:

1. Is the series stationary? (Is $d = 0$?)
2. If not, how many differences are needed? (What is d?)

Common Unit Root Tests

- **Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- **Phillips-Perron (PP)**
- **KPSS** (stationarity test – reversed null hypothesis)

The Dickey-Fuller Test

Setup

Consider the AR(1) model: $Y_t = \phi Y_{t-1} + \varepsilon_t$
Subtract Y_{t-1} from both sides:

$$\Delta Y_t = (\phi - 1) Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$$

where $\gamma = \phi - 1$.

Hypotheses

- H_0: $\gamma = 0$ (unit root, $\phi = 1$, non-stationary)
- H_1: $\gamma < 0$ (stationary, $|\phi| < 1$)

Key Issue

Under H_0, the t-statistic does **not** follow a standard t-distribution! Must use Dickey-Fuller critical values.

Dickey-Fuller Test Variants

Three Specifications

1. **No constant, no trend**:
$$\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$$
2. **With constant (drift)**:
$$\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$$
3. **With constant and trend**:
$$\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$$

Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference

Augmented Dickey-Fuller (ADF) Test

The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

Definition 6 (ADF Test)

Add lagged differences: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^{k} \delta_j \Delta Y_{t-j} + \varepsilon_t$
Test $H_0 : \gamma = 0$ using ADF critical values.

Choosing Lag Length k

- Use information criteria (AIC, BIC)
- Start with k_{max}, reduce until last lag significant

ADF Test Critical Values

Model	**1%**	**5%**	**10%**
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

Decision Rule

- Test statistic $<$ critical value $\Rightarrow$ Reject H_0 (stationary)
- Test statistic $\geq$ critical value $\Rightarrow$ Fail to reject (unit root)

The KPSS Test

Reversed Hypotheses

Unlike ADF: H_0: Stationary vs H_1: Unit root

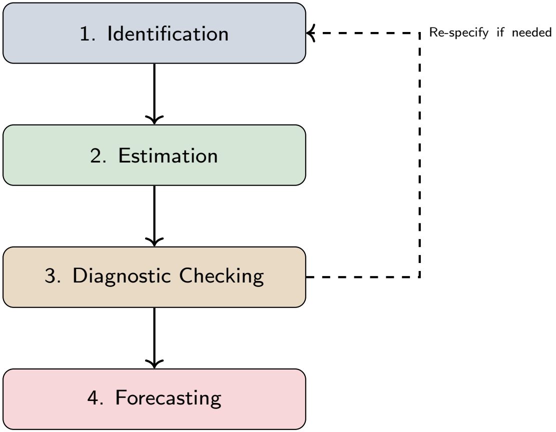
KPSS Procedure

Decompose: $Y_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$. Test whether $\text{Var}(u_t) = 0$.

Complementary Use with ADF

- ADF rejects, KPSS doesn't $\Rightarrow$ Stationary
- ADF doesn't reject, KPSS rejects $\Rightarrow$ Unit root
- Both reject or neither $\Rightarrow$ Inconclusive

The Box-Jenkins Methodology



Step 1: Determining d

Procedure

1. Plot the time series – look for trends, changing variance
2. Examine ACF – slow decay suggests non-stationarity
3. Apply unit root tests (ADF, KPSS)
4. If non-stationary, difference and repeat

Practical Guidelines

- Most economic series: $d = 1$ is sufficient
- Rarely need $d > 2$
- If ACF of ΔY_t still decays slowly, try $d = 2$
- Watch for overdifferencing (ACF with $\rho_1 \approx -0.5$)

Step 2: Determining p and q

After Differencing

Once $W_t = \Delta^d Y_t$ is stationary, use ACF/PACF to identify ARMA(p,q):

Model	**ACF**	**PACF**
AR(p)	Decays exponentially	Cuts off after lag p
MA(q)	Cuts off after lag q	Decays exponentially
ARMA(p,q)	Decays	Decays

Information Criteria

When patterns are unclear, compare models using:

- $\text{AIC} = -2\ln(L) + 2k$
- $\text{BIC} = -2\ln(L) + k\ln(n)$

Lower is better. BIC penalizes complexity more.

Auto-ARIMA Algorithms

Automated Model Selection

Modern software can automatically select (p, d, q):

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

How Auto-ARIMA Works

1. Use unit root tests to determine d
2. Fit models for various (p, q) combinations
3. Select model with lowest AIC/BIC
4. Optionally use stepwise search for efficiency

Caution

Automated selection is helpful but not infallible. Always check diagnostics!

Estimation Methods

Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- Assumes $\varepsilon_t \sim N(0, \sigma^2)$
- Maximizes the likelihood function
- Provides consistent, efficient estimators
- Yields standard errors for inference

Conditional vs Exact MLE

- **Conditional MLE**: Conditions on initial values
- **Exact MLE**: Treats initial values as unknown
- Difference diminishes as sample size grows

Parameter Constraints

Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- **AR stationarity**: Roots of $\phi(z) = 0$ outside unit circle
- **MA invertibility**: Roots of $\theta(z) = 0$ outside unit circle

Checking in Practice

Most software reports:

- Estimated coefficients with standard errors
- Roots of AR and MA polynomials
- Warning if near-unit-root detected

Residual Analysis

What to Check

If the model is correct, residuals $\hat{\varepsilon}_t$ should be white noise:

1. Zero mean
2. Constant variance
3. No autocorrelation
4. (Optional) Normality

Diagnostic Tools

- **Residual ACF/PACF**: Should show no significant spikes
- **Ljung-Box test**: Tests for autocorrelation at multiple lags
- **Q-Q plot**: Checks normality assumption
- **Residual vs fitted**: Checks for heteroskedasticity

The Ljung-Box Test

Definition 7 (Ljung-Box Q Statistic)

$$Q(m) = n(n+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{n-k}$$

Under H_0 (no autocorrelation): $Q(m) \sim \chi^2(m-p-q)$

Usage

- Choose $m \approx \ln(n)$ or $m = 10$ for quarterly, $m = 20$ for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if $Q(m)$ exceeds critical value

If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

Point Forecasts

Minimum MSE Forecast

The optimal h-step ahead forecast is the conditional expectation:

$$\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \ldots]$$

ARIMA(1,1,1) Forecasting

Model: $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$
One-step forecast:

$$\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1\hat{\varepsilon}_T$$

For $h > 1$: replace unknown ε_{T+j} with 0, unknown Y_{T+j} with $\hat{Y}_{T+j|T}$

Forecast Intervals

Forecast Uncertainty

The h-step forecast error variance:

$$\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$$

where ψ_j are MA(∞) coefficients.

Confidence Intervals

Under normality, $(1-\alpha)\%$ interval:

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$$

Key Property for I(1) Series

For integrated processes, forecast variance grows without bound as $h \to \infty$. Intervals widen over time!

Long-Run Forecasts for ARIMA

Behavior as $h \to \infty$

For ARIMA(p,1,q) with drift c:

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with $\sqrt{h}$

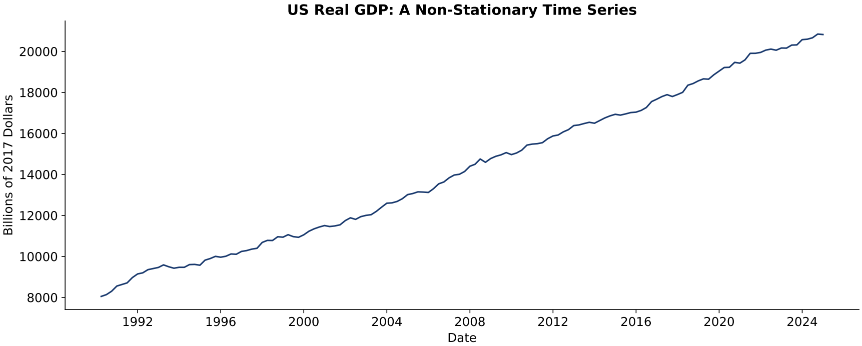
For ARIMA(p,1,q) without drift:

- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

Practical Implication

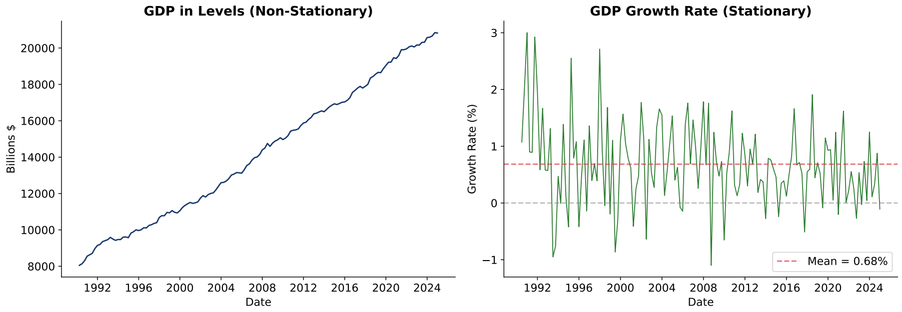
ARIMA forecasts are most reliable for short horizons. Long-term forecasts have very wide uncertainty bands.

US Real GDP: A Non-Stationary Series



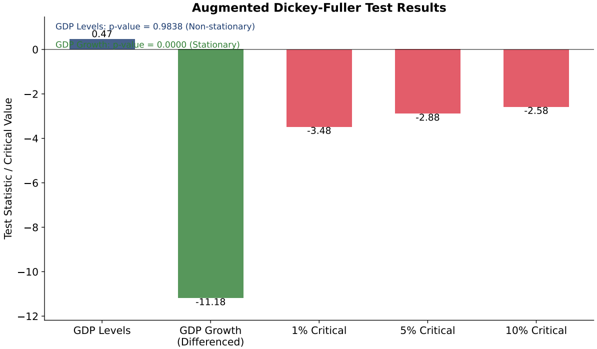
- Clear upward trend – non-stationary in levels
- Needs differencing before ARMA modeling

Effect of Differencing



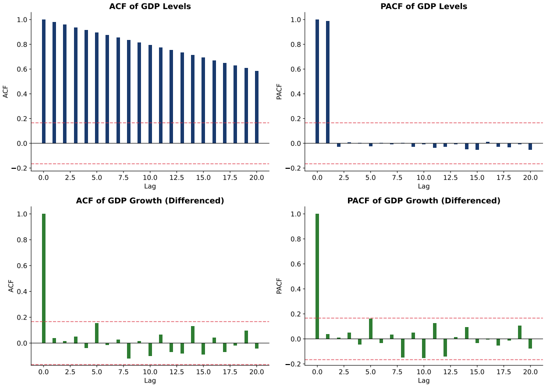
- **Left**: GDP in levels – non-stationary (clear trend)
- **Right**: GDP growth rate – stationary (fluctuates around mean)

Unit Root Test Results



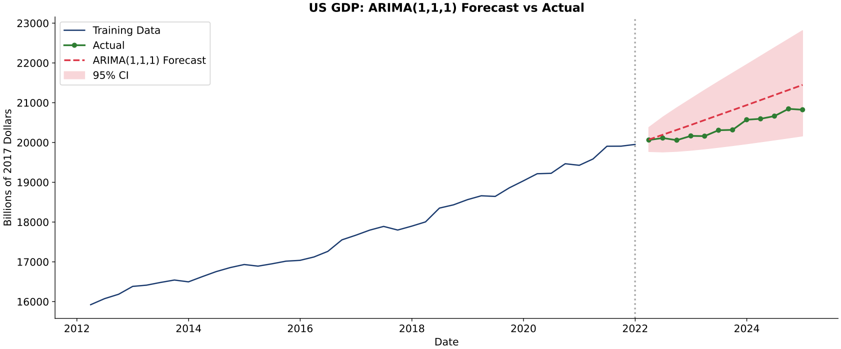
- GDP in levels: Cannot reject unit root (non-stationary)
- GDP growth: Reject unit root at 1% level (stationary)

ACF/PACF: Levels vs Differenced



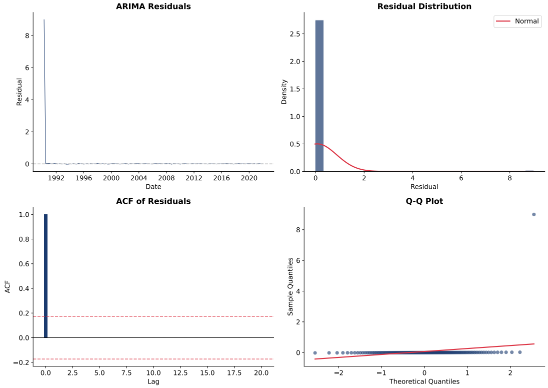
- **Top**: Slow ACF decay in levels suggests non-stationarity
- **Bottom**: After differencing, ACF/PACF help identify p and q

ARIMA Forecasting: Actual vs Predicted



- ARIMA(1,1,1) captures the trend dynamics
- Confidence intervals widen with forecast horizon

Model Diagnostics



- Residuals appear random; ACF within bounds
- Q-Q plot shows approximate normality

Python Implementation

Key Libraries

```
import pandas as pd
import numpy as np
from statsmodels.tsa.arima.model import ARIMA
from statsmodels.tsa.stattools import adfuller, kpss
import pmdarima as pm
```

Auto-ARIMA Example

```
# Automatic model selection
model = pm.auto_arima(y, start_p=0, start_q=0,
                      max_p=3, max_q=3, d=None,
                      seasonal=False, trace=True)
print(model.summary())
```

Key Takeaways

Main Points

1. **Non-stationarity** is common in economic data – must be addressed
2. **Differencing** transforms I(d) to I(0)
3. **ARIMA(p,d,q)** combines differencing with ARMA modeling
4. **Unit root tests** (ADF, KPSS) help determine d
5. **Box-Jenkins methodology**: Identify $\rightarrow$ Estimate $\rightarrow$ Diagnose
6. **Forecasts** for I(1) series have growing uncertainty

Next Steps

Chapter 4 will extend ARIMA to handle seasonality: SARIMA models.

References

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