Chapter 3: ARIMA Models

Seminar

Seminar Outline

Quiz 1: Integration Order

Question

A time series Y_t requires two differences to become stationary. What is its order of integration?

A) $I(0)$
B) $I(1)$
C) $I(2)$
D) Cannot be determined

Quiz 1: Integration Order

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- A) $I(0)$
- B) $I(1)$
- C) $I(2)$
- D) Cannot be determined

Answer: C

By definition, $Y_t \sim I(d)$ means d differences are needed to achieve stationarity. Since two differences are required, $Y_t \sim I(2)$.

Quiz 2: Random Walk Properties

Question

For a random walk $Y_t = Y_{t-1} + \varepsilon_t$ with $\text{Var}(\varepsilon_t) = \sigma^2$, what is $\text{Var}(Y_t)$?

- A) σ^2
- B) $t \cdot \sigma^2$
- C) σ^2/t
- D) $\sigma^2/(1-\phi^2)$

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- B) $t \cdot \sigma^2$
- C) σ^2/t
- D) $\sigma^2/(1-\phi^2)$

Answer: B

Since $Y_t = \sum_{i=1}^{t} \varepsilon_i$ (assuming $Y_0 = 0$), we have $\text{Var}(Y_t) = t \cdot \sigma^2$. The variance grows linearly with time – a key feature of non-stationarity.

Quiz 3: ADF Test Hypotheses

Question

In the Augmented Dickey-Fuller test, what is the null hypothesis?

- A) The series is stationary
- B) The series has a unit root
- C) The series has no autocorrelation
- D) The series is normally distributed

Quiz 3: ADF Test Hypotheses

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- D) The series is normally distributed

Answer: B

The ADF test has H_0: unit root (non-stationary) vs H_1: stationary. We reject H_0 if the test statistic is sufficiently negative (below critical value).

Quiz 4: ARIMA Notation

Question

What does ARIMA(2,1,1) mean?

- A) AR(2) on differenced data with MA(1) errors
- B) AR(1) with 2 differences and MA(1)
- C) MA(2) with 1 difference and AR(1)
- D) 2 lags, 1 trend, 1 seasonal component

Quiz 4: ARIMA Notation

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- B) AR(1) with 2 differences and MA(1)
- C) MA(2) with 1 difference and AR(1)
- D) 2 lags, 1 trend, 1 seasonal component

Answer: A

ARIMA(p,d,q) means: p=AR order, d=differencing order, q=MA order. So ARIMA(2,1,1) has AR(2) and MA(1) components applied to the first-differenced series.

Quiz 5: Difference Operator

Question

What is $(1 - L)^2 Y_t$ expanded?

- A) $Y_t - Y_{t-1}$
- B) $Y_t - 2Y_{t-1} + Y_{t-2}$
- C) $Y_t + 2Y_{t-1} + Y_{t-2}$
- D) $Y_t - Y_{t-2}$

Quiz 5: Difference Operator

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- B) $Y_t - 2Y_{t-1} + Y_{t-2}$
- C) $Y_t + 2Y_{t-1} + Y_{t-2}$
- D) $Y_t - Y_{t-2}$

Answer: B

$(1 - L)^2 = 1 - 2L + L^2$, so $(1 - L)^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$. This is the second difference of Y_t.

Quiz 6: KPSS vs ADF

Question

How does the KPSS test differ from the ADF test?

A) KPSS tests for seasonality, ADF tests for trends
B) KPSS has stationarity as null, ADF has unit root as null
C) KPSS is more powerful than ADF
D) There is no difference

Quiz 6: KPSS vs ADF

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- B) KPSS has stationarity as null, ADF has unit root as null
- C) KPSS is more powerful than ADF
- D) There is no difference

Answer: B

The key difference is reversed hypotheses. KPSS: H_0 = stationary. ADF: H_0 = unit root. Using both tests together provides stronger evidence.

Quiz 7: Overdifferencing

Question

If $Y_t \sim I(1)$ and we compute $\Delta^2 Y_t$, what happens?

- A) We get a better stationary series
- B) We introduce artificial negative autocorrelation
- C) The variance decreases
- D) Nothing changes

Quiz 7: Overdifferencing

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- A) We get a better stationary series
- B) We introduce artificial negative autocorrelation
- C) The variance decreases
- D) Nothing changes

Answer: B

Overdifferencing creates an MA component at the invertibility boundary. If $\Delta Y_t = \varepsilon_t$, then $\Delta^2 Y_t = \varepsilon_t - \varepsilon_{t-1}$, which is MA(1) with $\theta = -1$ (non-invertible).

Quiz 8: Forecast Variance

Question

For an ARIMA(0,1,0) model (random walk), how does forecast variance behave as horizon h increases?

- A) Stays constant
- B) Decreases to zero
- C) Grows linearly with h
- D) Converges to a finite limit

Quiz 8: Forecast Variance

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- A) Stays constant
- B) Decreases to zero
- C) Grows linearly with h
- D) Converges to a finite limit

Answer: C

For a random walk, $\text{Var}(Y_{T+h} - \hat{Y}_{T+h|T}) = h\sigma^2$. The forecast uncertainty grows without bound – a key characteristic of I(1) processes.

Quiz 9: Unit Root Test Power

Question

The ADF test has low power when:

- A) Sample size is very large
- B) The true root is close to but not equal to 1
- C) The series has no trend
- D) The series is clearly stationary

Quiz 9: Unit Root Test Power

Question

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- A) Sample size is very large
- B) The true root is close to but not equal to 1
- C) The series has no trend
- D) The series is clearly stationary

Answer: B

Near-unit-root processes (e.g., AR(1) with $\phi = 0.95$) are difficult to distinguish from true unit roots. The ADF test often fails to reject H_0 even when the process is stationary but highly persistent. Consider using Phillips-Perron or panel unit root tests.

Quiz 10: ARIMA Model Selection

Question

After differencing once, the ACF shows a spike at lag 1 only, and PACF decays. The appropriate model is:

- A) ARIMA(1,1,0)
- B) ARIMA(0,1,1)
- C) ARIMA(1,1,1)
- D) ARIMA(0,2,1)

Quiz 10: ARIMA Model Selection

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- C) ARIMA(1,1,1)
- D) ARIMA(0,2,1)

Answer: B

ACF cutoff at lag 1 + PACF decay = MA(1) signature. Combined with $d = 1$ differencing, this gives ARIMA(0,1,1). This is the IMA(1,1) model, common for economic data.

Quiz 11: Trend Stationarity vs Difference Stationarity

Question

A trend-stationary process is made stationary by:

- A) Taking first differences
- B) Removing the deterministic trend via regression
- C) Taking second differences
- D) Applying seasonal adjustment

Quiz 11: Trend Stationarity vs Difference Stationarity

Question

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- A) Taking first differences
- B) Removing the deterministic trend via regression
- C) Taking second differences
- D) Applying seasonal adjustment

Answer: B

Trend-stationary: $Y_t = \alpha + \beta t + \varepsilon_t$ (deterministic trend). Detrend by regression.
Difference-stationary: $Y_t = Y_{t-1} + \varepsilon_t$ (stochastic trend). Use differencing.
Misidentifying the type leads to incorrect inference!

Quiz 12: ARIMA Invertibility

Question

ARIMA(0,1,1) with $\theta_1 = 1.2$ is:

- A) Stationary and invertible
- B) Non-stationary but invertible
- C) Non-stationary and non-invertible
- D) Stationary but non-invertible

Quiz 12: ARIMA Invertibility

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- C) Non-stationary and non-invertible
- D) Stationary but non-invertible

Answer: C

With $d = 1$, the model is non-stationary (has unit root). With $|\theta_1| = 1.2 > 1$, the MA part is non-invertible. Invertibility requires MA roots outside unit circle, i.e., $|\theta| < 1$.

Quiz 13: Spurious Regression

Question

Regressing one random walk on another independent random walk typically shows:

- A) No significant relationship
- B) High R^2 and significant t-statistics (spuriously)
- C) Negative correlation
- D) Perfect multicollinearity

Quiz 13: Spurious Regression

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- A) No significant relationship
- B) High R^2 and significant t-statistics (spuriously)
- C) Negative correlation
- D) Perfect multicollinearity

Answer: B

Spurious regression: Two independent I(1) series often show "significant" relationships due to common stochastic trends. High R^2, significant coefficients, but residuals are non-stationary. Solution: Use cointegration analysis or work with differenced data.

Quiz 14: Long-Run Forecast

Question

The long-run forecast from ARIMA(1,1,0) with $\phi_1 = 0.7$ converges to:

- A) Zero
- B) The unconditional mean
- C) A linear trend extrapolation
- D) The last observed value

Quiz 14: Long-Run Forecast

Question

The long-run forecast from ARIMA(1,1,0) with $\phi_1 = 0.7$ converges to:

- A) Zero
- B) The unconditional mean
- C) A linear trend extrapolation
- D) The last observed value

Answer: C

For I(1) models, long-run forecasts follow a linear trend based on the estimated drift. Unlike stationary models (which revert to mean), integrated models project forward along the current trajectory. The slope depends on any constant term in the model.

True/False Questions

Determine if each statement is True or False:

1. An I(2) process requires two differences to become stationary.
2. The ADF test always includes a constant term.
3. ARIMA(0,1,0) is another name for a random walk.
4. Differencing a stationary series makes it "more stationary."
5. The KPSS test has stationarity as the null hypothesis.
6. ARIMA models can only capture linear patterns.

Answers on next slide...

True/False: Solutions

1. An I(2) process requires two differences to become stationary. **TRUE**
   I(d) means d differences needed. I(2) = two unit roots.
2. The ADF test always includes a constant term. **FALSE**
   You choose: no constant, constant only, or constant + trend.
3. ARIMA(0,1,0) is another name for a random walk. **TRUE**
   $(1 - L)Y_t = \varepsilon_t \Rightarrow Y_t = Y_{t-1} + \varepsilon_t$.
4. Differencing a stationary series makes it "more stationary." **FALSE**
   Over-differencing creates non-invertible MA; hurts model performance.
5. The KPSS test has stationarity as the null hypothesis. **TRUE**
   KPSS: H_0 = stationary. Opposite of ADF.
6. ARIMA models can only capture linear patterns. **TRUE**
   ARIMA is linear in parameters. Nonlinear patterns need GARCH, neural nets, etc.

Problem 1: Unit Root Testing

Exercise

You have quarterly GDP data for 80 quarters. The ADF test (with constant and trend) gives a test statistic of -2.85. The 5% critical value is -3.41.

1. What is your conclusion about stationarity?
2. What would you do next?

Problem 1: Unit Root Testing

Exercise

You have quarterly GDP data for 80 quarters. The ADF test (with constant and trend) gives a test statistic of -2.85. The 5% critical value is -3.41.

1. What is your conclusion about stationarity?
2. What would you do next?

Solution

1. Since $-2.85 > -3.41$, we **fail to reject** H_0. The data appears to have a unit root (non-stationary).
2. Take the first difference ΔY_t and repeat the ADF test on the differenced series to confirm it is now stationary.

Problem 2: Model Identification

Exercise

After differencing a time series once, the ACF shows:

- Significant spike at lag 1 ($\rho_1 = 0.4$)
- All other lags insignificant

The PACF shows gradual decay.
What ARIMA model is suggested?

Problem 2: Model Identification

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- Significant spike at lag 1 ($\rho_1 = 0.4$)
- All other lags insignificant

The PACF shows gradual decay.
What ARIMA model is suggested?

Solution

- ACF cuts off after lag 1 $\Rightarrow$ MA(1) component
- PACF decays $\Rightarrow$ Confirms MA structure
- Since we differenced once: $d = 1$

Suggested model: ARIMA(0,1,1) or IMA(1,1)

Problem 3: ARIMA Equation

Exercise

Write out the full equation for ARIMA(1,1,1):

$$(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$$

Expand this completely in terms of Y_t, Y_{t-1}, Y_{t-2}, etc.

Problem 3: ARIMA Equation

Exercise

Write out the full equation for ARIMA(1,1,1):

$$(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$$

Expand this completely in terms of Y_t, Y_{t-1}, Y_{t-2}, etc.

Solution

Expanding $(1 - \phi_1 L)(1 - L) = 1 - L - \phi_1 L + \phi_1 L^2 = 1 - (1 + \phi_1)L + \phi_1 L^2$:

$$Y_t - (1 + \phi_1)Y_{t-1} + \phi_1 Y_{t-2} = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Or equivalently:

$$Y_t = c + (1 + \phi_1)Y_{t-1} - \phi_1 Y_{t-2} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Problem 4: Forecast Calculation

Exercise

Given ARIMA(0,1,1): $\Delta Y_t = \varepsilon_t + 0.3\varepsilon_{t-1}$
At time T: $Y_T = 100$, $\hat{\varepsilon}_T = 2$, $\sigma^2 = 4$
Calculate:

1. $\hat{Y}_{T+1|T}$ (one-step forecast)
2. $\hat{Y}_{T+2|T}$ (two-step forecast)

Problem 4: Forecast Calculation

Exercise

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At time T: $Y_T = 100$, $\hat{\varepsilon}_T = 2$, $\sigma^2 = 4$
Calculate:

1. $\hat{Y}_{T+1|T}$ (one-step forecast)
2. $\hat{Y}_{T+2|T}$ (two-step forecast)

Solution

1. $\hat{Y}_{T+1|T} = Y_T + 0.3\hat{\varepsilon}_T = 100 + 0.3(2) = \mathbf{100.6}$
2. $\hat{Y}_{T+2|T} = \hat{Y}_{T+1|T} + 0.3 \cdot 0 = 100.6 + 0 = \mathbf{100.6}$
   (Future shocks $\varepsilon_{T+1}, \varepsilon_{T+2}$ are forecast as 0)

Problem 5: Confidence Intervals

Exercise

Continuing from Problem 4, calculate the 95% forecast intervals for $\hat{Y}_{T+1|T}$ and $\hat{Y}_{T+2|T}$.
Recall: $\sigma^2 = 4$, $\theta_1 = 0.3$

Problem 5: Confidence Intervals

Exercise

Continuing from Problem 4, calculate the 95% forecast intervals for $\hat{Y}_{T+1|T}$ and $\hat{Y}_{T+2|T}$.
Recall: $\sigma^2 = 4$, $\theta_1 = 0.3$

Solution

For IMA(1,1), the MA(∞) weights are $\psi_0 = 1$, $\psi_j = 1 + \theta_1$ for $j \geq 1$.
1-step: $\text{Var}(e_{T+1}) = \sigma^2\psi_0^2 = 4$, so $SE = 2$
$100.6 \pm 1.96(2) = [\mathbf{96.68}, \mathbf{104.52}]$
2-step: $\text{Var}(e_{T+2}) = \sigma^2(\psi_0^2 + \psi_1^2) = 4(1 + 1.3^2) = 10.76$, $SE = 3.28$
$100.6 \pm 1.96(3.28) = [\mathbf{94.17}, \mathbf{107.03}]$

Example: Testing for Unit Root in Stock Prices

Scenario

You have daily closing prices for a stock over 500 days. You want to determine if prices follow a random walk.

Step-by-step Approach

1. **Visual inspection**: Plot prices – likely shows trend
2. **ADF test on prices**: Expect to fail to reject H_0 (unit root)
3. **Take log returns**: $r_t = \ln(P_t/P_{t-1}) = \Delta \ln(P_t)$
4. **ADF test on returns**: Should reject H_0 (stationary)
5. **Conclusion**: Log prices are $I(1)$, returns are $I(0)$

Example: Box-Jenkins for Inflation Data

Scenario

Monthly inflation rates for 10 years. Build an ARIMA model.

Workflow

1. **Plot & test**: ADF suggests borderline – try both $d = 0$ and $d = 1$
2. **If** $d = 0$: Fit ARMA models, compare AIC
3. **If** $d = 1$: Examine ACF/PACF of ΔY_t
   - ACF: spike at lag 1, then cuts off
   - PACF: decays
   - $\Rightarrow$ Try ARIMA(0,1,1)
4. **Estimate**: Fit ARIMA(0,1,1), check coefficients
5. **Diagnose**: Ljung-Box on residuals (want $p > 0.05$)
6. **Compare**: AIC of ARIMA(0,1,1) vs ARMA(1,1) on levels

Example: Interpreting Python Output

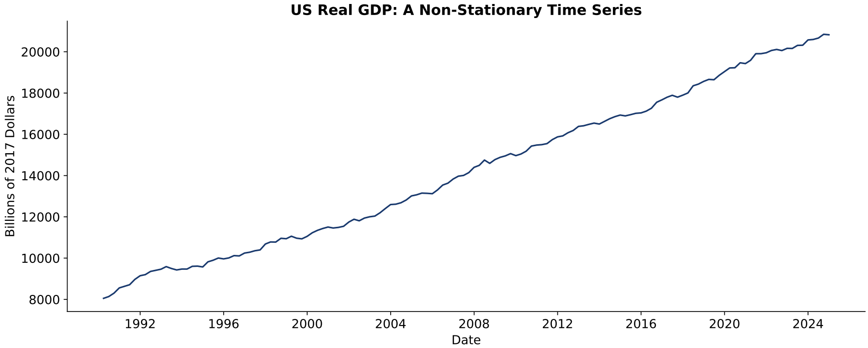
statsmodels ARIMA Output

```
                            ARIMA Model Results
==============================================================
Dep. Variable:               D.y   No. Observations:    99
Model:                ARIMA(1,1,1)    AIC                    285.32
                                      BIC                    295.63
==============================================================
                 coef    std err       z      P>|z|
--------------------------------------------------------------
const          0.0521     0.048     1.085    0.278
ar.L1          0.4532     0.102     4.443    0.000
ma.L1         -0.2891     0.118    -2.450    0.014
sigma2         1.2340     0.176     7.011    0.000
```

Interpretation

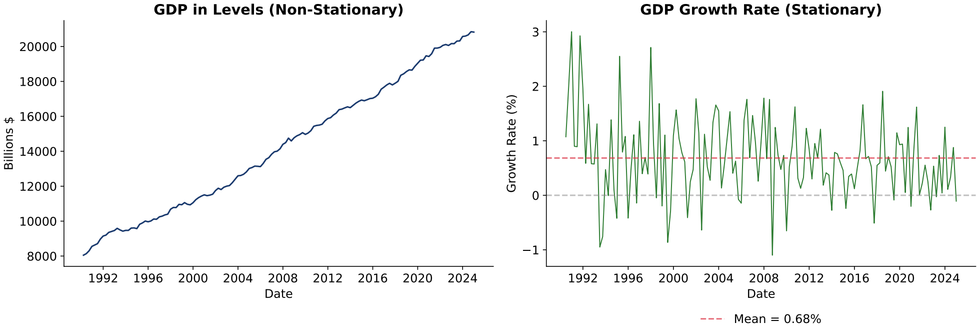
- AR coefficient (0.45) is significant, MA coefficient (-0.29) is significant
- Constant (0.052) not significant – could set $c = 0$
- Check: $|\phi_1| < 1$ (stationary), $|\theta_1| < 1$ (invertible) – OK!

Case Study: US Real GDP (1990–2024)



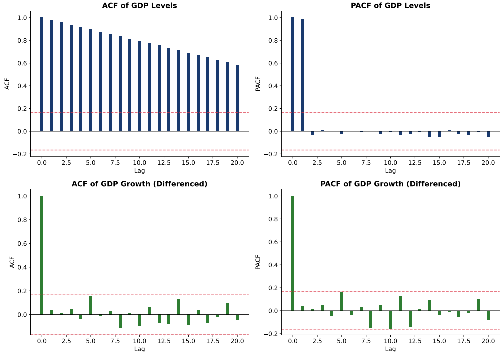
- US Real GDP in billions of 2017 dollars (quarterly data)
- Clear **upward trend** – typical of macroeconomic series
- Notable drops during recessions (2008-2009, 2020)
- Non-stationary: needs differencing before ARIMA modeling

Stationarity Through Differencing



- **Left**: GDP in levels – clear upward trend (non-stationary)
- **Right**: GDP growth rate $= \Delta \log(Y_t) \times 100$ – stationary
- First differencing of log GDP removes the stochastic trend
- Growth rate fluctuates around a constant mean ($\approx 0.6\%$ quarterly)

ACF/PACF: Levels vs Differenced



- **Top row**: ACF/PACF of GDP levels – slow decay indicates non-stationarity
- **Bottom row**: ACF/PACF of GDP growth – mostly within confidence bands
- Pattern suggests low-order ARIMA model is appropriate

ARIMA Estimation Results: US GDP Growth

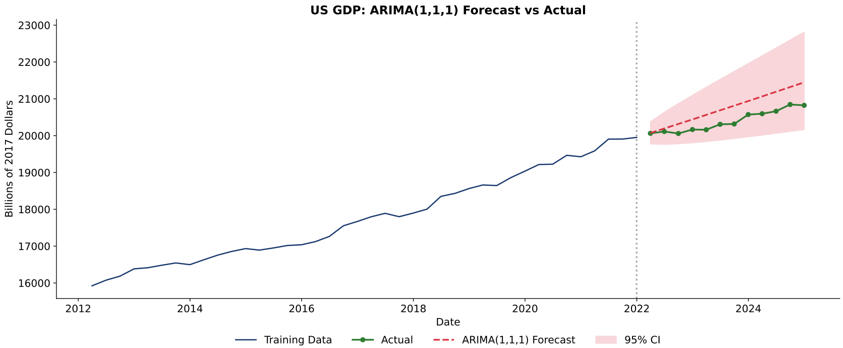
Model: ARIMA$(1,1,1)$ on log(GDP)

Parameter	Estimate	Std. Error	z-stat	p-value
ϕ_1 (AR.L1)	0.312	0.185	1.69	0.091
θ_1 (MA.L1)	-0.087	0.203	-0.43	0.668
σ^2	0.00012	–	–	–

Interpretation

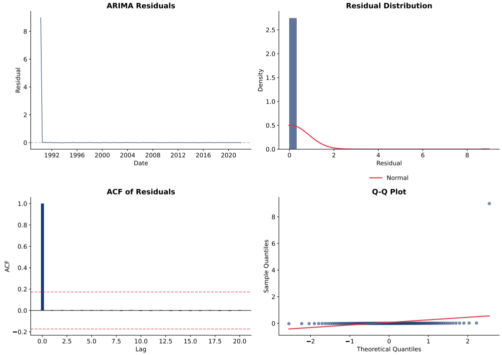
- Low-order ARIMA captures GDP dynamics reasonably well
- AR(1) coefficient positive – GDP growth shows persistence
- Alternative: simpler random walk (ARIMA(0,1,0)) often competitive

Forecast: ARIMA vs Actual



- Blue: historical training data; Green: actual test data
- Red dashed: ARIMA forecasts with 95% confidence interval
- Forecasts capture the general trend direction
- Confidence intervals widen as forecast horizon increases

Model Diagnostics: Residual Analysis



- Residuals show no systematic patterns over time
- Distribution approximately normal (histogram and Q-Q plot)
- ACF of residuals within bounds – no significant autocorrelation remaining
- Model adequately captures the data generating process

Discussion: Deterministic vs Stochastic Trends

Key Question

Why is it important to distinguish between deterministic and stochastic trends?

Discussion Points

- **Wrong treatment consequences**:
  - Detrending a unit root $\Rightarrow$ spurious stationarity
  - Differencing a trend-stationary $\Rightarrow$ overdifferencing
- **Economic interpretation**:
  - Deterministic trend: shocks are temporary
  - Stochastic trend: shocks have permanent effects
- **Policy implications**:
  - Does a recession permanently lower GDP, or does the economy return to trend?

Discussion: Model Selection Criteria

Key Question

When should you use AIC vs BIC for ARIMA model selection?

Considerations

- **AIC**: Minimizes prediction error, may overfit
  - Better for forecasting
  - Tends to select larger models
- **BIC**: Consistent model selection, more parsimonious
  - Better for identifying "true" model
  - Penalizes complexity more heavily
- **Practical advice**: Report both, prefer BIC if they disagree substantially

Discussion: Limitations of ARIMA

Key Question

What are the main limitations of ARIMA models?

Discussion Points

- **Linearity**: Cannot capture nonlinear dynamics
- **Constant variance**: Assumes homoskedasticity (no GARCH effects)
- **No structural breaks**: Parameters assumed constant
- **Univariate**: Ignores relationships with other variables
- **Symmetric**: Treats positive and negative shocks equally
- **Long-horizon forecasts**: Uncertainty grows rapidly

Extensions

These limitations motivate GARCH (volatility), VAR (multivariate), regime-switching models, etc.

Key Points from Today's Seminar

What We Covered

1. **Integration and differencing**: $I(d)$ processes require d differences
2. **Unit root testing**: ADF tests H_0: unit root; KPSS tests H_0: stationary
3. **ARIMA(p,d,q)**: Combines ARMA with differencing
4. **Model identification**: Use ACF/PACF patterns and information criteria
5. **Forecasting**: Point forecasts and growing confidence intervals

Next Seminar

Hands-on Python exercises with real economic data:

- Unit root testing with `statsmodels`
- Auto-ARIMA with `pmdarima`
- Forecasting and model diagnostics