



# Time Series Analysis and Forecasting

## Chapter 6: VAR and Granger Causality



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## Outline

### Foundations

- ▣ Motivation
- ▣ Introduction to multivariate time series
- ▣ Vector Autoregression (VAR)
- ▣ Granger Causality
- ▣ Impulse Response Functions
- ▣ Forecast Error Variance Decomposition

### Applications

- ▣ VAR Diagnostics
- ▣ VAR Forecasting
- ▣ Practical Example
- ▣ Case Study: GDP and Unemployment
- ▣ Summary and Quiz

## Learning Objectives

By the end of this chapter, you will be able to:

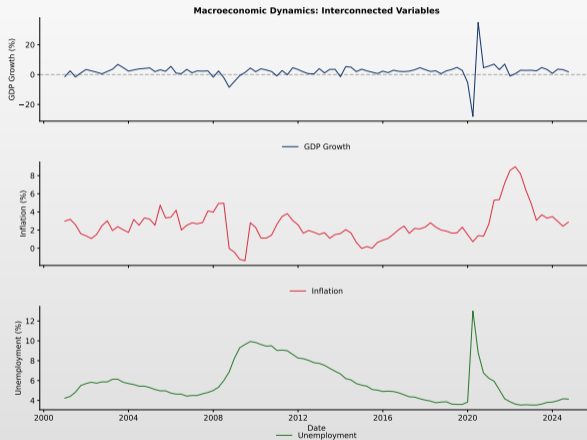
1. Understand the **motivation** for multivariate time series analysis
2. Specify and estimate **VAR(p)** models
3. Apply **Granger causality** tests
4. Interpret **Impulse Response Functions (IRFs)**
5. Perform **Forecast Error Variance Decomposition (FEVD)**
6. Select the optimal lag order using information criteria
7. Implement VAR analysis in **Python**

## Motivating Example: Macroeconomic Dynamics

### Observations

- Economic variables are **interconnected**: GDP affects unemployment, inflation affects interest rates
- Changes in one variable **propagate** through the system
- Understanding these dynamics requires **multivariate** analysis

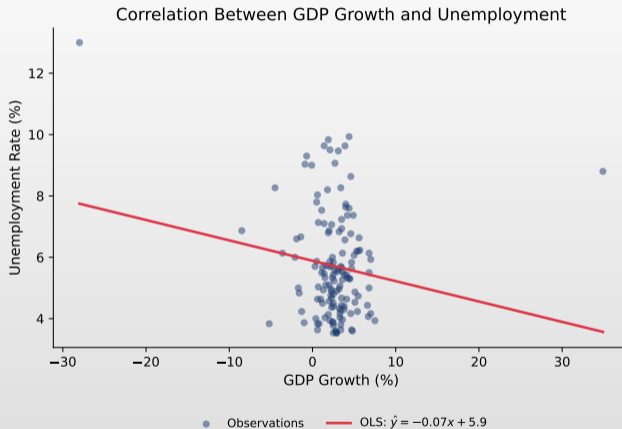
## Motivating Example: Macroeconomic Dynamics



## The Key Insight: Variables Interact

- ▣ **Okun's Law:** Higher GDP growth  $\Rightarrow$  lower unemployment
- ▣ **Taylor Rule:** Higher inflation  $\Rightarrow$  higher interest rates
- ▣ **Phillips Curve:** Unemployment-inflation tradeoff

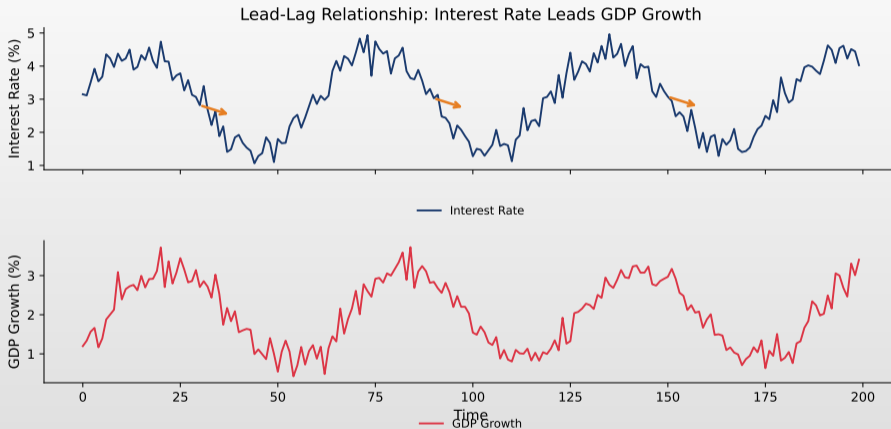
## The Key Insight: Variables Interact



## Lead-Lag Relationships

- Some variables **lead** others: stock market predicts economic activity
- Cross-correlation reveals the **timing** of relationships
- Peak correlation at lag 4: stock market leads unemployment by  $\sim 4$  months

## Lead-Lag Relationships



## Why Univariate Models Are Not Enough

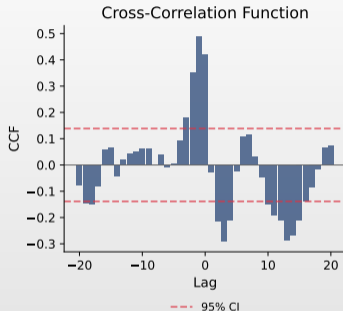
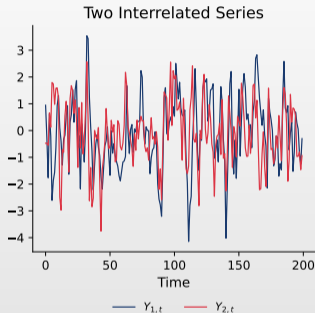
### The Problem

ARIMA models each variable **in isolation**—ignoring valuable information from other variables!

### The Solution

**VAR models** capture the joint dynamics and feedback effects between multiple time series.

## Why Univariate Models Are Not Enough



### Univariate AR(1)

$$Y_{1,t} = \phi_1 Y_{1,t-1} + \varepsilon_t$$

→ Ignores  $Y_2$

### VAR(1)

$$\mathbf{Y}_t = \mathbf{A}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t$$

→ Captures all cross-dynamics

## What We'll Learn Today

### Core Concepts

1. **VAR Models:** How to model multiple time series jointly
2. **Granger Causality:** Does  $X$  help predict  $Y$ ?
3. **Impulse Response Functions:** How do shocks propagate?
4. **Variance Decomposition:** What drives each variable?

### Recurring Example: GDP Growth and Unemployment

- ▣  $Y_{1t}$ : **GDP Growth**    and     $Y_{2t}$ : **Unemployment Rate** (*Okun's Law*)
- ▣ Central question: Does GDP cause unemployment, or vice versa, or both?

### Applications

- ▣ Macroeconomic policy    •    Financial markets    •    Business cycle    •    Risk management

## Multivariate Time Series Notation

### Vector of Variables

Let  $Y_t = (Y_{1t}, Y_{2t}, \dots, Y_{Kt})'$  be a  $K \times 1$  vector of time series.

Example with  $K = 2$ :

$$Y_t = \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \text{GDP growth}_t \\ \text{Inflation}_t \end{pmatrix}$$

### Key Questions

1. Does  $Y_1$  help predict  $Y_2$ ? (Granger causality)
2. How do shocks to  $Y_1$  affect  $Y_2$ ? (Impulse responses)
3. What proportion of  $Y_2$ 's variance is due to  $Y_1$ ? (Variance decomposition)

## Multivariate Stationarity

### Definition: Weak Stationarity

A  $K$ -dimensional time series  $Y_t$  is **weakly stationary** if:

1.  $\mathbb{E}[Y_t] = \mu$  (constant mean vector)
2.  $\text{Cov}(Y_t, Y_{t-h}) = \Gamma(h)$  depends only on  $h$ , not  $t$

### Autocovariance Matrix

$$\Gamma(h) = \mathbb{E}[(Y_t - \mu)(Y_{t-h} - \mu)'] = \begin{pmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{pmatrix}$$

Note:  $\Gamma(-h) = \Gamma(h)'$  (transpose, not equal!)

## Cross-Covariance Properties

### Cross-Covariance Function

For variables  $Y_{it}$  and  $Y_{jt}$ :

$$\gamma_{ij}(h) = \text{Cov}(Y_{it}, Y_{j,t-h}) = \mathbb{E}[(Y_{it} - \mu_i)(Y_{j,t-h} - \mu_j)]$$

### Key Difference from Univariate Case

- In general:  $\gamma_{ij}(h) \neq \gamma_{ij}(-h)$
- But:  $\gamma_{ij}(h) = \gamma_{ji}(-h)$
- The cross-covariance matrix is **not symmetric** for  $h \neq 0$

### Example

If  $Y_1$  leads  $Y_2$ :  $\gamma_{12}(h) > 0$  for  $h > 0$  but  $\gamma_{12}(h) \approx 0$  for  $h < 0$

## Correlation Matrix Function

### Definition

The **autocorrelation matrix** at lag  $h$ :

$$R(h) = D^{-1}\Gamma(h)D^{-1}$$

where  $D = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_K)$  and  $\sigma_i = \sqrt{\gamma_{ii}(0)}$

### For Bivariate Case

$$R(h) = \begin{pmatrix} \rho_{11}(h) & \rho_{12}(h) \\ \rho_{21}(h) & \rho_{22}(h) \end{pmatrix}$$

where  $\rho_{ij}(h) = \frac{\gamma_{ij}(h)}{\sigma_i \sigma_j}$

Diagonal elements: usual ACFs; Off-diagonal: cross-correlations

## Researcher Spotlight: Sims & Granger



Christopher Sims (\*1942)

Nobel Prize 2011



Clive Granger (1934–2009)

Nobel Prize 2003



### Biography

- **Christopher Sims:** American econometrician at Princeton. Nobel Prize (2011) “for empirical research on cause and effect in the macroeconomy”
- **Clive Granger:** British-American economist at UC San Diego. Nobel Prize (2003) “for methods of analyzing economic time series with common trends (cointegration)”

### Key Contributions

- **VAR models** (Sims, 1980) — vector autoregression for macroeconomics
- **Granger causality** (Granger, 1969) — predictive causality concept
- **Impulse response functions** and structural VAR identification
- **Cointegration** (Granger, 1981) — long-run equilibrium relationships

## The VAR(p) Model

### Definition

A **VAR(p)** model for  $K$  variables:

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

where:

- ▣  $Y_t$ :  $K \times 1$  vector of endogenous variables
- ▣  $c$ :  $K \times 1$  vector of constants
- ▣  $A_i$ :  $K \times K$  coefficient matrices
- ▣  $\varepsilon_t$ :  $K \times 1$  vector of error terms with  $\mathbb{E}[\varepsilon_t] = 0$ ,  $\mathbb{E}[\varepsilon_t \varepsilon_t'] = \Sigma$

## VAR(1) with Two Variables

### Bivariate VAR(1)

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

### Equation by Equation

$$Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$$

$$Y_{2t} = c_2 + a_{21} Y_{1,t-1} + a_{22} Y_{2,t-1} + \varepsilon_{2t}$$

**Key insight:** Each equation includes lags of **all** variables!

## Numerical Example: VAR(1)

### Specific VAR(1) Model

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} + \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

### Interpretation of Coefficients

- $a_{11} = 0.7$ : A 1-unit increase in  $Y_1$  at  $t - 1$  increases  $Y_1$  at  $t$  by 0.7
- $a_{12} = 0.2$ : A 1-unit increase in  $Y_2$  at  $t - 1$  increases  $Y_1$  at  $t$  by 0.2
- $a_{21} = -0.1$ : A 1-unit increase in  $Y_1$  at  $t - 1$  **decreases**  $Y_2$  at  $t$  by 0.1
- $a_{22} = 0.6$ :
  - A 1-unit increase in  $Y_2$  at  $t - 1$  increases  $Y_2$  at  $t$  by 0.6

## VAR(2): Higher Order Dynamics

### VAR(2) Specification

$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \epsilon_t$ . For  $K = 2$ :  $2 + 2 \times 4 + 2 \times 4 = 18$  parameters!

### Written Out

$$Y_{1t} = c_1 + a_{11}^{(1)} Y_{1,t-1} + a_{12}^{(1)} Y_{2,t-1} + a_{11}^{(2)} Y_{1,t-2} + a_{12}^{(2)} Y_{2,t-2} + \epsilon_{1t}$$

$$Y_{2t} = c_2 + a_{21}^{(1)} Y_{1,t-1} + a_{22}^{(1)} Y_{2,t-1} + a_{21}^{(2)} Y_{1,t-2} + a_{22}^{(2)} Y_{2,t-2} + \epsilon_{2t}$$

### Curse of Dimensionality

VAR( $p$ ) with  $K$  variables has  $K + pK^2$  parameters. With  $K = 5$ ,  $p = 4$ :  $5 + 4 \times 25 = 105$  parameters!

## The Companion Form

### Converting VAR(p) to VAR(1)

Any VAR( $p$ ) can be written as a VAR(1) in **companion form**:  $\xi_t = A\xi_{t-1} + v_t$

### For VAR(2)

$$\underbrace{\begin{pmatrix} Y_t \\ Y_{t-1} \end{pmatrix}}_{\xi_t} = \underbrace{\begin{pmatrix} A_1 & A_2 \\ I_K & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} Y_{t-1} \\ Y_{t-2} \end{pmatrix}}_{\xi_{t-1}} + \underbrace{\begin{pmatrix} \varepsilon_t \\ 0 \end{pmatrix}}_{v_t}$$

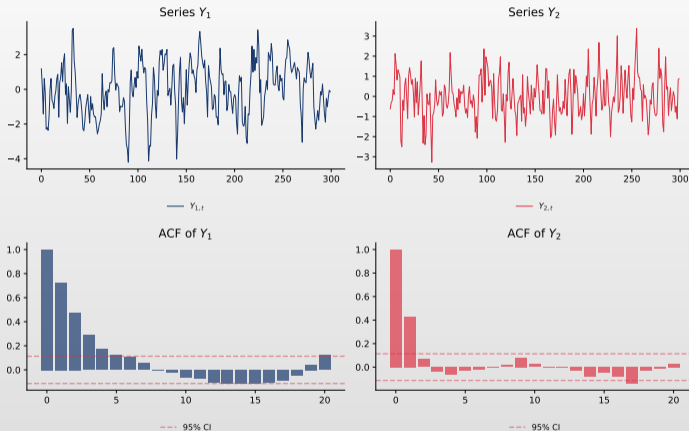
### Why Useful?

Stationarity, forecasting, and IRFs are easier in companion form. Matrix  $A$  is  $Kp \times Kp$ .

## VAR Process: GDP and Unemployment (FRED)

- ▣ **Data:** US GDP Growth (GDPC1) and Unemployment Rate (UNRATE) from FRED
- ▣ Each variable responds to both its own past and the other variable's past
- ▣ Classic example of macroeconomic interdependence (Okun's Law)

## VAR Process: GDP and Unemployment (FRED)



## Stationarity of VAR

### Stability Condition

VAR(p) is **stable** (stationary) if all roots of:

$$\det(I_K - A_1 z - A_2 z^2 - \dots - A_p z^p) = 0$$

lie **outside** the unit circle (i.e.,  $|z| > 1$ ).

### For VAR(1)

The model is stable if all **eigenvalues** of  $A_1$  are less than 1 in absolute value.

Example: For  $A_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$ , eigenvalues are  $\lambda_1 = 0.6$  and  $\lambda_2 = 0.2$ .

Both  $< 1 \Rightarrow$  stable!

## Computing Eigenvalues: Example

$$\text{For } A = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$$

Characteristic polynomial:  $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 0.7 - \lambda & 0.2 \\ -0.1 & 0.6 - \lambda \end{pmatrix} = (0.7 - \lambda)(0.6 - \lambda) + 0.02 = 0$$

$$\lambda^2 - 1.3\lambda + 0.44 = 0$$

### Solution

Using quadratic formula:

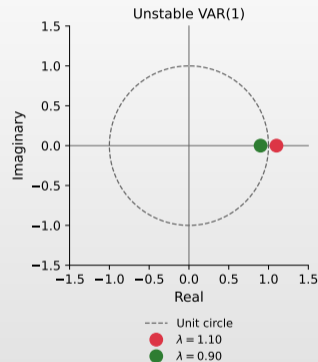
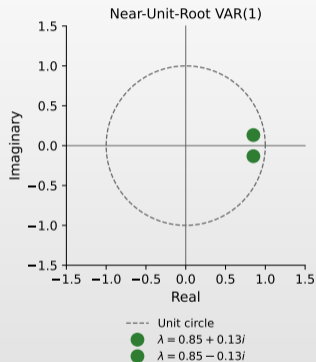
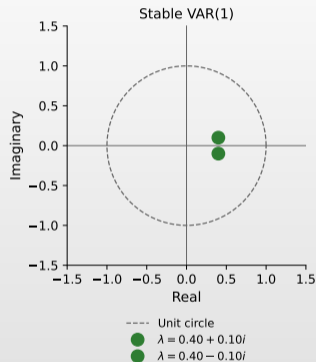
$$\lambda = \frac{1.3 \pm \sqrt{1.69 - 1.76}}{2} = \frac{1.3 \pm \sqrt{-0.07}}{2} = 0.65 \pm 0.132i$$

$$|\lambda| = \sqrt{0.65^2 + 0.132^2} = \sqrt{0.44} = 0.663 < 1 \quad \checkmark \text{ Stable!}$$

## Stability Condition: Visual Interpretation

- Eigenvalues of the companion matrix must lie inside the unit circle
- Complex eigenvalues come in conjugate pairs
- If any eigenvalue is outside the circle, the VAR is explosive (non-stationary)

## Stability Condition: Visual Interpretation



## Mean of a Stationary VAR

### Unconditional Mean

For stationary VAR(1):  $Y_t = c + AY_{t-1} + \varepsilon_t$ . Since  $\mathbb{E}[Y_t] = \mathbb{E}[Y_{t-1}] = \mu$ :

$$\mu = c + A\mu \quad \Rightarrow \quad \mu = (I_K - A)^{-1}c$$

### Example

$$\text{If } c = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}: \mu = \begin{pmatrix} 0.3 & -0.2 \\ 0.1 & 0.4 \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2.3 \\ 1.0 \end{pmatrix}$$

## Covariance Structure of VAR(1)

### Variance-Covariance Matrix $\Gamma(0)$

For VAR(1), the variance satisfies the **discrete Lyapunov equation**:

$$\Gamma(0) = A\Gamma(0)A' + \Sigma$$

### Autocovariance at Lag $h$

$$\Gamma(h) = A^h\Gamma(0), \quad h \geq 0$$

This shows that autocovariances decay geometrically with the eigenvalues of  $A$ .

### Solving the Lyapunov Equation

Can solve by vectorization:

$$\text{vec}(\Gamma(0)) = (I_{K^2} - A \otimes A)^{-1} \text{vec}(\Sigma)$$

where  $\otimes$  denotes the Kronecker product.

## Estimation of VAR

### OLS Estimation

Each equation can be estimated by **OLS separately**:

$$\hat{A} = \left( \sum_{t=1}^T Y_{t-1} Y'_{t-1} \right)^{-1} \left( \sum_{t=1}^T Y_{t-1} Y'_t \right)$$

This is efficient because all equations have the **same regressors**.

### Covariance Matrix

$$\hat{\Sigma} = \frac{1}{T - Kp - 1} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}'_t$$

The errors  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  may be **contemporaneously correlated**.

## Lag Order Selection

### Information Criteria

Choose  $p$  that minimizes:

$$\text{AIC}(p) = \ln |\hat{\Sigma}_p| + \frac{2pK^2}{T} \quad \text{BIC}(p) = \ln |\hat{\Sigma}_p| + \frac{pK^2 \ln T}{T}$$

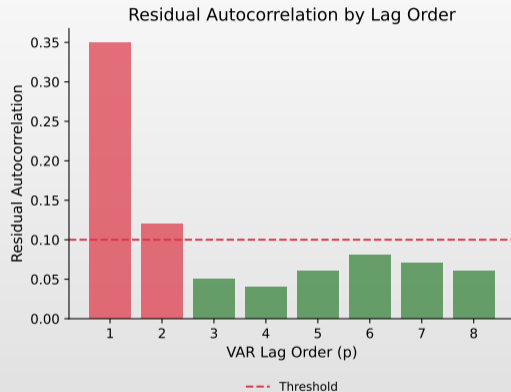
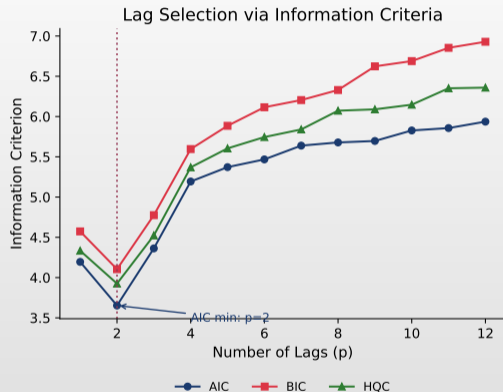
### Guidelines

- ▣ AIC: larger models (better forecasting); BIC: smaller models (consistent)
- ▣ Start with  $p_{\max}$  based on frequency (4 quarterly, 12 monthly)

## Lag Selection: Example

- ▣ Information criteria values for different lag orders
- ▣ AIC and BIC may suggest different optimal lags
- ▣ Lower values indicate better model fit (penalized by complexity)

## Lag Selection: Example



## Restricted VAR Models

### Why Restrict?

Full VAR models can be **overparameterized**:

- ▣ Many coefficients may be insignificant
- ▣ Poor forecasting performance
- ▣ Loss of degrees of freedom

### Common Restrictions

- ▣ **Zero restrictions**: Set small coefficients to zero
- ▣ **Block exogeneity**: Some variables don't affect others
- ▣ **Lag exclusion**: Exclude certain lags

### Testing Restrictions

Use likelihood ratio test:  $LR = T(\ln |\hat{\Sigma}_R| - \ln |\hat{\Sigma}_U|) \sim \chi_r^2$   
where  $r$  = number of restrictions

## What is Granger Causality?

Clive Granger (1969, Nobel Prize 2003)

“X **Granger-causes** Y” if past values of X help predict Y, **beyond** what past values of Y alone can predict.

### Important Distinction

**Granger causality  $\neq$  True causality**

- Granger causality is about **predictive content**
- Does NOT imply economic/structural causation
- “X Granger-causes Y” means:
  - ▶ X contains useful information for forecasting Y

## Formal Definition

### Granger Causality

$X$  does not Granger-cause  $Y$  if:

$$\mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \dots, X_{t-1}, X_{t-2}, \dots] = \mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \dots]$$

In other words: adding  $X$ 's history does not improve the prediction of  $Y$ .

### In the VAR Context

For VAR(1):  $Y_{1t} = c_1 + a_{11}Y_{1,t-1} + a_{12}Y_{2,t-1} + \varepsilon_{1t}$

$Y_2$  does not Granger-cause  $Y_1$  if  $a_{12} = 0$ .

For VAR(p):  $Y_2$  does not Granger-cause  $Y_1$  if  $a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$ .

## Testing for Granger Causality

### Hypothesis Test

$H_0$ :  $Y_2$  does **not** Granger-cause  $Y_1$

$$H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \dots = a_{12}^{(p)} = 0$$

$H_1$ : At least one  $a_{12}^{(i)} \neq 0$  (Granger causality exists)

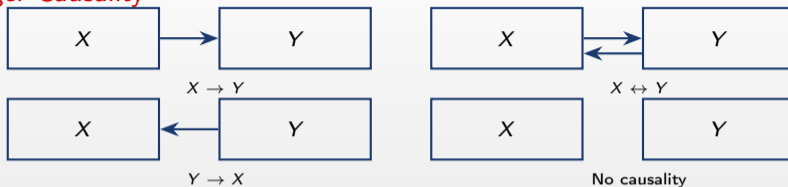
### Test Statistic: Wald Test

$$F = \frac{(RSS_R - RSS_U)/p}{RSS_U/(T - 2p - 1)} \sim F_{p, T-2p-1}$$

where:

- ▣  $RSS_R$ : Residual sum of squares from restricted model (without  $Y_2$  lags)
- ▣  $RSS_U$ : Residual sum of squares from unrestricted model (full VAR)

## Types of Granger Causality



### Economic Examples

- Money → Output? (monetarist); Stock prices ↔ Volume (bidirectional)

## Cross-Correlation Function

### Definition 1 (Cross-Correlation Function)

The **cross-correlation** between  $X_t$  and  $Y_t$  at lag  $k$  is:

$$\rho_{XY}(k) = \frac{\gamma_{XY}(k)}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X_t, Y_{t+k})}{\sqrt{\text{Var}(X_t)\text{Var}(Y_t)}}$$

### Interpretation

- ▣  $\rho_{XY}(k) > 0$  at  $k > 0$ :  $X$  is positively correlated with future  $Y$  ( $X$  may lead  $Y$ )
- ▣  $\rho_{XY}(k) > 0$  at  $k < 0$ :  $X$  is positively correlated with past  $Y$  ( $Y$  may lead  $X$ )

### Note

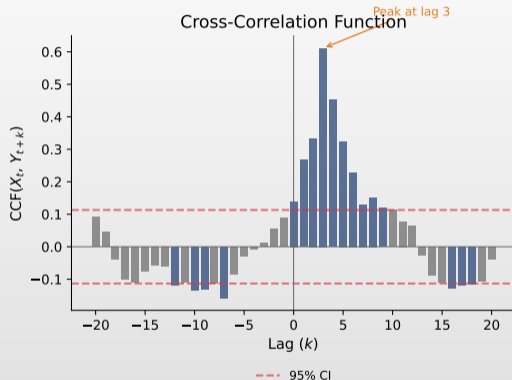
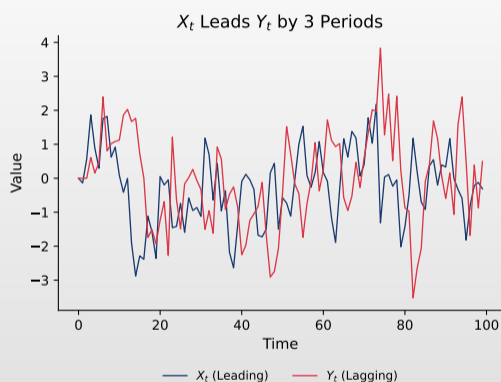
Unlike ACF, cross-correlation is **not symmetric**:  $\rho_{XY}(k) \neq \rho_{XY}(-k)$  in general.

## Cross-Correlation: Visual Illustration

### Interpretation

- Left: two related series; Right: CCF reveals that  $X$  leads  $Y$  (significant correlations at positive lags)

## Cross-Correlation: Visual Illustration



## Granger Causality: Practical Considerations

### Common Pitfalls

1. **Omitted variables:** A third variable  $Z$  may cause both  $X$  and  $Y$
2. **Non-stationarity:** Test requires stationary data (or cointegration)
3. **Lag selection:** Results can be sensitive to  $p$
4. **Sample size:** Need sufficient observations

### Best Practices

- Test for unit roots first
- Use multiple lag selection criteria
- Check robustness to different lag lengths
- Report results for both directions

## Granger Causality Test: Numerical Example

### Testing: Does Money Growth Granger-cause Output?

**Unrestricted model** (VAR with 2 lags):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \beta_1 \Delta M_{t-1} + \beta_2 \Delta M_{t-2} + \varepsilon_t$$

**Restricted model** ( $H_0: \beta_1 = \beta_2 = 0$ ):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \varepsilon_t$$

### Test Computation

With  $T = 100$ ,  $RSS_U = 45.2$ ,  $RSS_R = 52.8$ :

$$F = \frac{(52.8 - 45.2)/2}{45.2/(100 - 5)} = \frac{3.8}{0.476} = 7.98$$

$F_{0.05}(2, 95) = 3.09 \Rightarrow$  **Reject**  $H_0$ : Money Granger-causes output!

## The Toda-Yamamoto Procedure

### Problem with Non-Stationary Data

Standard Granger test has **non-standard distributions** when:

- Variables have unit roots
- Variables are cointegrated

### Toda-Yamamoto Solution (1995)

1. Determine maximum order of integration  $d_{max}$
2. Estimate  $\text{VAR}(p + d_{max})$  in **levels**
3. Test restrictions on first  $p$  lags only
4. Extra  $d_{max}$  lags are **not** tested (just for correct distribution)

### Advantage

Wald test has asymptotic  $\chi^2$  distribution regardless of cointegration!

## Instantaneous Causality

### Definition

$X$  **instantaneously causes**  $Y$  if  $\mathbb{E}[Y_t | \Omega_{t-1}, X_t] \neq \mathbb{E}[Y_t | \Omega_{t-1}]$ , where  $\Omega_{t-1}$  contains all past information.

### Testing in VAR

Test whether  $\sigma_{12} \neq 0$  in the covariance matrix:  $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$ . If  $\sigma_{12} = 0$ : no instantaneous causality.

### Interpretation

Instantaneous causality often reflects **common shocks** or **data aggregation**, not true contemporaneous effects.

## Granger Causality in Multiple Systems

### Block Exogeneity Test

In a VAR with  $K > 2$  variables, test whether a **group** of variables Granger-causes another group.

Example: Do financial variables (interest rates, stock prices) Granger-cause real variables (GDP, unemployment)?

### Test Statistic

$$\chi^2 = T \cdot K_1 \cdot p \cdot \left( \ln |\hat{\Sigma}_R| - \ln |\hat{\Sigma}_U| \right) \sim \chi_{K_1 \cdot K_2 \cdot p}^2$$

where  $K_1$  = number of “caused” variables,  $K_2$  = number of “causing” variables

## What are Impulse Response Functions?

### Definition

An **Impulse Response Function (IRF)** traces the effect of a one-time shock to one variable on the current and future values of all variables.

### Question IRFs Answer

"If there is an unexpected 1-unit shock to  $Y_1$  today, what happens to  $Y_1$  and  $Y_2$  over the next  $h$  periods?"

### MA( $\infty$ ) Representation

A stable VAR(p) can be written as:

$$Y_t = \mu + \sum_{i=0}^{\infty} \Phi_i \epsilon_{t-i}$$

The matrices  $\Phi_i$  are the **impulse responses** at horizon  $i$ .

## Computing IRFs for VAR(1)

For VAR(1):  $Y_t = c + AY_{t-1} + \varepsilon_t$

The impulse response matrices are:

$$\Phi_0 = I_K, \quad \Phi_1 = A, \quad \Phi_2 = A^2, \quad \dots, \quad \Phi_h = A^h$$

### Interpretation

$[\Phi_h]_{ij}$  = Effect on  $Y_i$  at time  $t + h$  of a unit shock to  $Y_j$  at time  $t$

For stable VAR:  $\Phi_h \rightarrow 0$  as  $h \rightarrow \infty$  (shocks die out)

## Computing IRFs for General VAR(p)

### Recursive Formula for VAR(p)

For  $Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$ :

$$\Phi_h = \sum_{j=1}^{\min(h,p)} A_j \Phi_{h-j}, \quad h = 1, 2, 3, \dots$$

with  $\Phi_0 = I_K$  and  $\Phi_h = 0$  for  $h < 0$ .

### Example: VAR(2) IRFs

- ▣  $\Phi_0 = I_K$
- ▣  $\Phi_1 = A_1 \Phi_0 = A_1$
- ▣  $\Phi_2 = A_1 \Phi_1 + A_2 \Phi_0 = A_1^2 + A_2$
- ▣  $\Phi_3 = A_1 \Phi_2 + A_2 \Phi_1 = A_1(A_1^2 + A_2) + A_2 A_1$

## Orthogonalized IRFs

### Problem: Correlated Errors

If  $\Sigma$  is not diagonal, shocks  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  are correlated.  
A shock to “ $Y_1$ ” also involves a shock to “ $Y_2$ ”.

### Solution: Cholesky Decomposition

Factor  $\Sigma = PP'$  where  $P$  is lower triangular.  
Define orthogonalized shocks:  $u_t = P^{-1}\varepsilon_t$  with  $\mathbb{E}[u_t u_t'] = I$

Orthogonalized IRFs:  $\Theta_h = \Phi_h P$

### Ordering Matters!

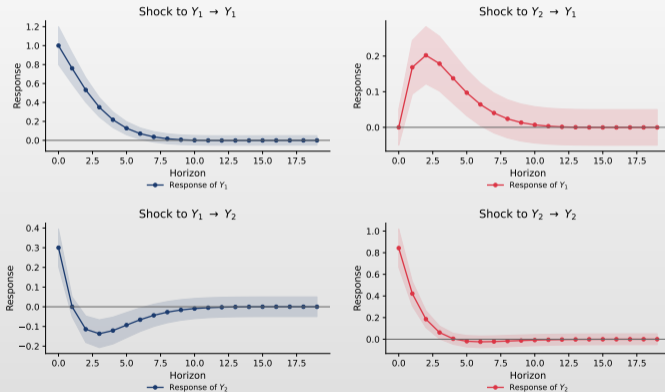
Cholesky assumes variables ordered from “most exogenous” to “most endogenous”. Results depend on this ordering.

## Impulse Response Functions: Example

- ▣ IRFs show how each variable responds to a one-unit shock over time
- ▣ Shaded regions represent confidence intervals (uncertainty in estimates)
- ▣ For stable VAR models, responses converge to zero as the horizon increases

# Impulse Response Functions: Example

Orthogonalized Impulse Response Functions



## IRF Numerical Example

$$\text{For } A = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$$

$$\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Phi_1 = A = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$$

$$\Phi_2 = A^2 = \begin{pmatrix} 0.47 & 0.26 \\ -0.13 & 0.34 \end{pmatrix}$$

## Interpretation

- ▣  $[\Phi_2]_{12} = 0.26$ : A unit shock to  $Y_2$  increases  $Y_1$  by 0.26 after 2 periods
- ▣  $[\Phi_2]_{21} = -0.13$ : A unit shock to  $Y_1$  **decreases**  $Y_2$  by 0.13 after 2 periods

## Cumulative Impulse Responses

### Definition

The **cumulative IRF** up to horizon  $H$ :

$$\Psi_H = \sum_{h=0}^H \Phi_h$$

Measures the **total accumulated effect** of a shock.

### Long-Run Multiplier

For stable VAR:  $\Psi_{\infty} = (I_K - A_1 - A_2 - \dots - A_p)^{-1}$

This gives the **permanent effect** of a one-time shock.

### When to Use

Cumulative IRFs are useful when interested in total impact (e.g., cumulative GDP loss from a shock).

## Confidence Intervals for IRFs

### Sources of Uncertainty

IRFs are functions of estimated parameters  $\hat{A}_1, \dots, \hat{A}_p$ , so they have **sampling uncertainty**.

### Methods for Confidence Bands

1. **Asymptotic:** Delta method for standard errors
2. **Monte Carlo:** Simulate from asymptotic distribution of  $\hat{A}$
3. **Bootstrap:** Resample residuals and re-estimate VAR

### Bootstrap Procedure

1. Estimate VAR, save residuals  $\{\hat{\epsilon}_t\}$
2. Draw with replacement to create  $\{\hat{\epsilon}_t^*\}$
3. Generate bootstrap sample, re-estimate, compute IRFs
4. Repeat  $B$  times; use percentiles for CIs

## Structural VAR (SVAR)

### Motivation

Standard VAR shocks  $\varepsilon_t$  are **reduced-form** innovations—linear combinations of structural shocks.

We want to identify economically meaningful **structural shocks**.

### Structural Form

$$B_0 Y_t = \Gamma_0 + B_1 Y_{t-1} + \cdots + B_p Y_{t-p} + u_t$$

where  $u_t$  are **structural shocks** with  $\mathbb{E}[u_t u_t'] = I_K$

### Relationship to Reduced Form

$$\varepsilon_t = B_0^{-1} u_t \quad \Rightarrow \quad \Sigma = B_0^{-1} (B_0^{-1})'$$

## Identification in SVAR

### The Identification Problem

$\Sigma$  has  $K(K + 1)/2$  unique elements, but  $B_0^{-1}$  has  $K^2$  elements.

Need  $K(K - 1)/2$  additional restrictions!

### Common Identification Schemes

1. **Short-run restrictions:** Zero impact effects (Cholesky)
2. **Long-run restrictions:** Zero long-run effects (Blanchard-Quah)
3. **Sign restrictions:** Inequality constraints on IRFs
4. **External instruments:** Use outside information

### Example: Cholesky (Recursive) Ordering

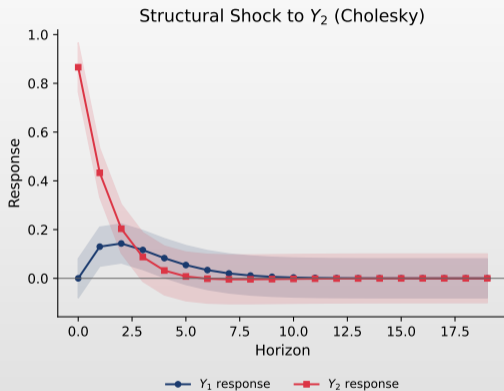
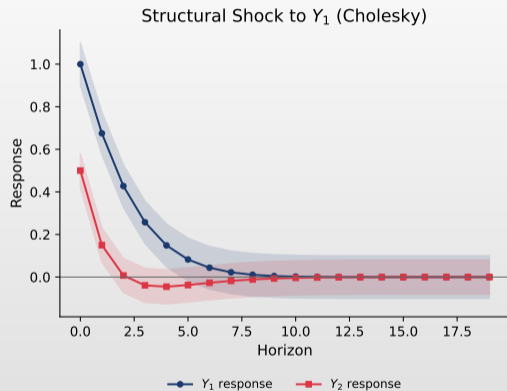
For  $K = 2$ :  $B_0^{-1} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$

Variable 1 doesn't respond to shock 2 contemporaneously.

## Structural IRF Example

- ▣ Structural IRFs based on Cholesky identification
- ▣ Order of variables affects interpretation of shocks
- ▣ First variable responds only to own shocks contemporaneously

## Structural IRF Example



## Variance Decomposition

### Question

What proportion of the forecast error variance of  $Y_i$  at horizon  $h$  is due to shocks to  $Y_j$ ?

### FEVD Formula

$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\Theta_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^K [\Theta_s]_{ik}^2}$$

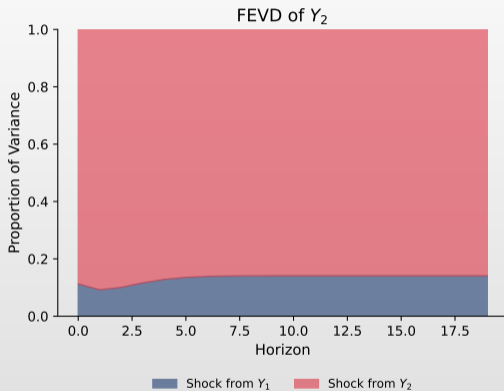
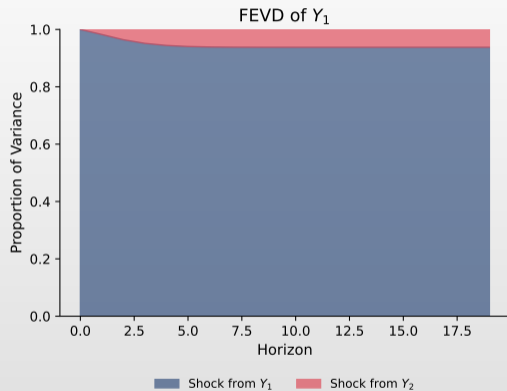
### Properties

- ▣  $0 \leq \text{FEVD}_{ij}(h) \leq 1$  and  $\sum_{j=1}^K \text{FEVD}_{ij}(h) = 1$  (sums to 100%)
- ▣ At  $h = 1$ : own shocks dominate (by Cholesky construction)

## FEVD: Example

- FEVD shows the proportion of forecast variance attributable to each shock
- At short horizons, own shocks dominate; cross-variable effects grow over time
- Useful for understanding the relative importance of different shocks in the system

## FEVD: Example



## FEVD: Numerical Example

## Computing FEVD for Bivariate VAR

Using orthogonalized IRFs  $\Theta_h$ , FEVD at horizon  $H$ :

$$\text{FEVD}_{11}(H) = \frac{\sum_{h=0}^{H-1} \theta_{11}^2(h)}{\sum_{h=0}^{H-1} [\theta_{11}^2(h) + \theta_{12}^2(h)]}$$

## Example Calculation

$h$	$\theta_{11}(h)$	$\theta_{12}(h)$	$\theta_{11}^2(h)$	$\theta_{12}^2(h)$
0	1.00	0.00	1.00	0.00
1	0.70	0.20	0.49	0.04
2	0.47	0.26	0.22	0.07

$$\text{FEVD}_{11}(3) = \frac{1.00+0.49+0.22}{1.00+0.49+0.22+0.00+0.04+0.07} = \frac{1.71}{1.82} = 94\%$$

## Historical Decomposition

### Definition

**Historical decomposition** breaks down each observed value into contributions from each structural shock:

$$Y_{it} - \bar{Y}_i = \sum_{j=1}^K \sum_{s=0}^{t-1} \theta_{ij}(s) \cdot u_{j,t-s}$$

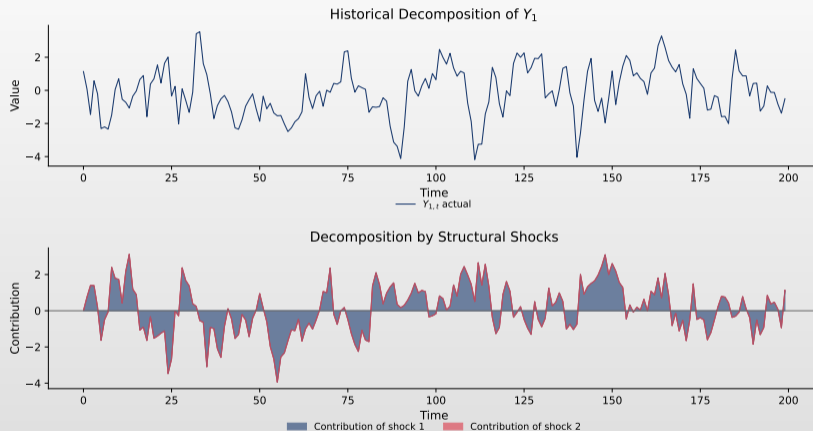
### Application

- “How much of the 2008 GDP decline was due to financial shocks vs. oil shocks?”
- Attributes historical movements to specific identified shocks
- Useful for policy analysis and narrative interpretation

## Historical Decomposition: Example

- Each color represents the contribution of a different structural shock
- Stacked contributions sum to the actual observed deviation from mean
- Helps identify which shocks drove historical episodes

## Historical Decomposition: Example



## Residual Diagnostics

### What to Check

After estimating VAR, verify that residuals  $\hat{\varepsilon}_t$  behave like white noise:

1. No serial correlation
2. Constant variance (homoskedasticity)
3. Normality (for inference)

### Why It Matters

- Autocorrelated residuals  $\Rightarrow$  inefficient estimates
- Heteroskedasticity  $\Rightarrow$  invalid standard errors
- Non-normality  $\Rightarrow$  inference may be unreliable

## Testing for Serial Correlation

### Portmanteau Test (Ljung-Box)

$$Q_h = T(T+2) \sum_{j=1}^h \frac{1}{T-j} \text{tr}(\hat{C}_j' \hat{C}_0^{-1} \hat{C}_j \hat{C}_0^{-1})$$

where  $\hat{C}_j = \frac{1}{T} \sum_{t=j+1}^T \hat{\varepsilon}_t \hat{\varepsilon}_{t-j}'$

Under  $H_0$  (no autocorrelation):  $Q_h \sim \chi_{K^2}^2(h-p)$

### Breusch-Godfrey LM Test

1. Regress  $\hat{\varepsilon}_t$  on  $\hat{\varepsilon}_{t-1}, \dots, \hat{\varepsilon}_{t-h}$  and original regressors
2.  $LM = T \cdot R^2 \sim \chi_{K^2}^2(h)$  under  $H_0$

### If Rejected

Consider increasing lag order  $p$  or adding additional variables.

## Testing for Heteroskedasticity

### ARCH-LM Test

Test for autoregressive conditional heteroskedasticity in residuals:

$$\hat{\varepsilon}_{it}^2 = \alpha_0 + \alpha_1 \hat{\varepsilon}_{i,t-1}^2 + \cdots + \alpha_q \hat{\varepsilon}_{i,t-q}^2 + v_t$$

$H_0: \alpha_1 = \cdots = \alpha_q = 0$  (homoskedasticity)

$$LM = TR^2 \sim \chi_q^2$$

### Multivariate Version

Test all equations jointly using:

$$\text{vech}(\hat{\varepsilon}_t \hat{\varepsilon}_t') = c + \sum_{j=1}^q B_j \text{vech}(\hat{\varepsilon}_{t-j} \hat{\varepsilon}_{t-j}') + v_t$$

## Normality Testing

### Jarque-Bera Test (Univariate)

$$JB = \frac{T}{6} \left( S^2 + \frac{(K-3)^2}{4} \right) \sim \chi^2_2$$

where  $S$  = skewness,  $K$  = kurtosis

### Multivariate Normality (Doornik-Hansen)

Transform residuals and test joint skewness and kurtosis:

$$DH = s_1'(\Omega^{-1/2})'(\Omega^{-1/2})s_1 + s_2'(\Omega^{-1/2})'(\Omega^{-1/2})s_2 \sim \chi^2_{2K}$$

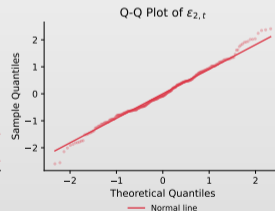
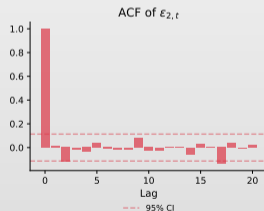
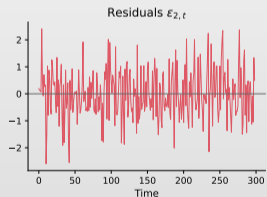
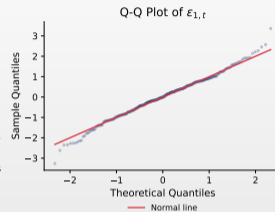
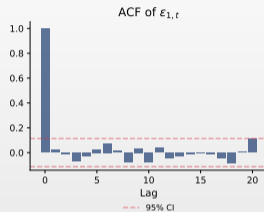
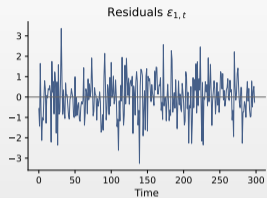
### Note

Normality is often rejected in financial data. Consider robust standard errors if non-normality is severe.

## Diagnostic Summary Plot

- Residual ACF should show no significant autocorrelation
- Histogram should approximate normal distribution
- Q-Q plot should follow 45-degree line

## Diagnostic Summary Plot



## Point Forecasts from VAR

### Iterative Forecasting

For VAR(1):  $Y_t = c + AY_{t-1} + \varepsilon_t$

**1-step forecast:**  $\hat{Y}_{T+1|T} = c + AY_T$

**2-step forecast:**  $\hat{Y}_{T+2|T} = c + A\hat{Y}_{T+1|T}$

**$h$ -step forecast:**  $\hat{Y}_{T+h|T} = c + A\hat{Y}_{T+h-1|T}$

### Direct Formula

$$\hat{Y}_{T+h|T} = (I + A + A^2 + \cdots + A^{h-1})c + A^h Y_T$$

For stable VAR: converges to  $\mu = (I - A)^{-1}c$  as  $h \rightarrow \infty$

## Forecast Error and MSE

### $h$ -Step Forecast Error

$$\mathbf{e}_{T+h|T} = \mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T} = \sum_{j=0}^{h-1} \mathbf{A}^j \boldsymbol{\varepsilon}_{T+h-j}$$

### Mean Squared Error Matrix

$$\text{MSE}(\hat{\mathbf{Y}}_{T+h|T}) = \mathbb{E}[\mathbf{e}_{T+h|T} \mathbf{e}_{T+h|T}' ] = \sum_{j=0}^{h-1} \mathbf{A}^j \boldsymbol{\Sigma} (\mathbf{A}^j)'$$

### Key Insight

- MSE increases with  $h$ ; converges to  $\boldsymbol{\Gamma}(0)$  for stable VAR
- Long-horizon forecasts  $\rightarrow$  unconditional mean

## Forecast Confidence Intervals

### Constructing Intervals

For normally distributed errors,  $(1 - \alpha)$  confidence interval:

$$\hat{Y}_{i,T+h|T} \pm z_{\alpha/2} \sqrt{[\text{MSE}(\hat{Y}_{T+h|T})]_{ii}}$$

### Joint Confidence Regions

For multiple variables, use ellipsoids:

$$(\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T})' [\text{MSE}(\hat{\mathbf{Y}}_{T+h|T})]^{-1} (\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T}) \leq \chi_{K,\alpha}^2$$

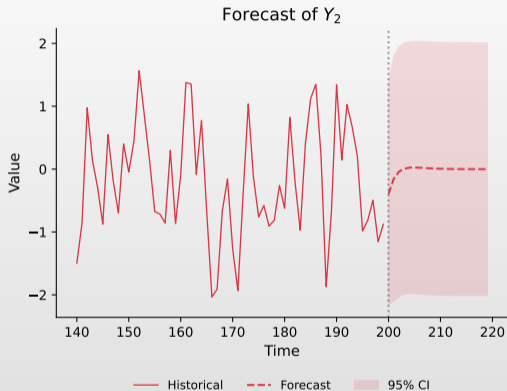
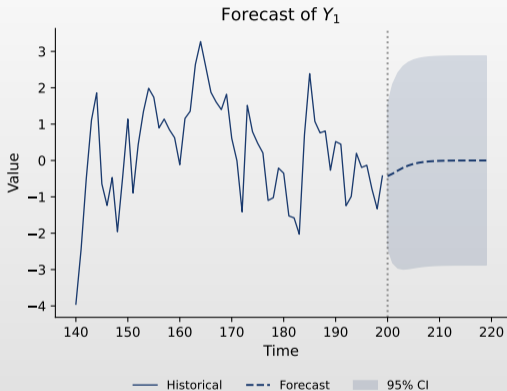
### Note

These assume known parameters. Bootstrap methods account for parameter uncertainty.

## VAR Forecasts: Example

- ▣ Point forecasts shown as solid line beyond observed data
- ▣ Confidence bands widen as forecast horizon increases
- ▣ Forecasts converge to unconditional mean for long horizons

## VAR Forecasts: Example



## Forecast Evaluation

### Out-of-Sample Evaluation

Split data: estimation sample (1 to  $T_1$ ) and test sample ( $T_1 + 1$  to  $T$ ). Compute forecast errors:  $e_{t+h} = Y_{t+h} - \hat{Y}_{t+h|t}$

### Common Metrics

□ **RMSE:**  $\sqrt{\frac{1}{n} \sum e_{t+h}^2}$     **MAE:**  $\frac{1}{n} \sum |e_{t+h}|$     **MAPE:**  $\frac{100}{n} \sum \left| \frac{e_{t+h}}{Y_{t+h}} \right|$

### Diebold-Mariano Test

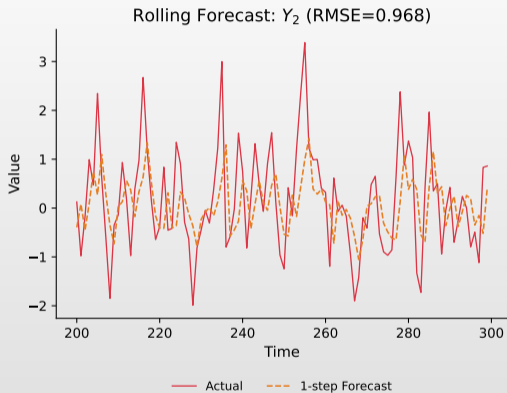
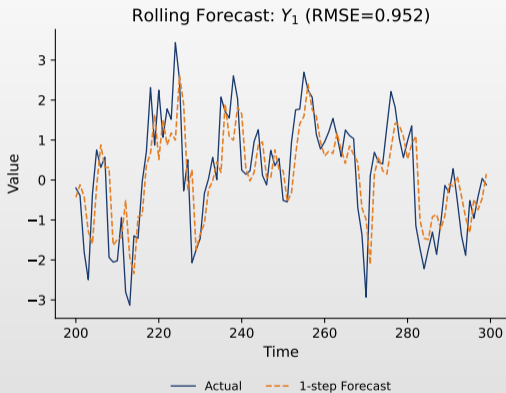
Test whether VAR forecasts are significantly better than alternative:  $DM = \frac{\bar{d}}{\sqrt{\hat{\sigma}_d^2/n}} \sim N(0, 1)$  where  $d_t = L(e_{1t}) - L(e_{2t})$  is the loss differential.

## Out-of-Sample Evaluation: VAR vs. AR

### General Methodology

- Train / Test Split: split data into training + test; estimate on train, evaluate on test
- Why VAR vs. AR? AR ignores other variables; VAR exploits interdependencies
- Metric:  $RMSE = \sqrt{\frac{1}{h} \sum e_i^2}$

## Out-of-Sample Evaluation: VAR vs. AR



## Example: GDP and Unemployment

### Okun's Law

There is a negative relationship between GDP growth and unemployment:

$$\Delta U_t \approx -\beta(\Delta Y_t - \bar{g})$$

where  $\bar{g}$  is trend GDP growth and  $\beta \approx 0.4$ .

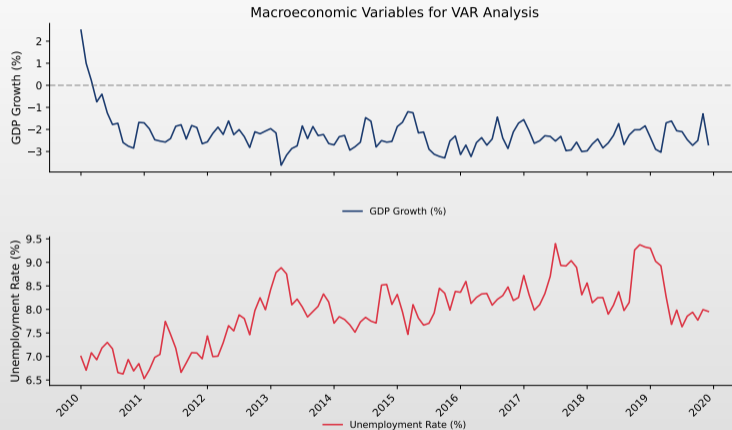
### VAR Analysis Questions

1. Does GDP growth Granger-cause unemployment changes?
2. Does unemployment Granger-cause GDP growth?
3. How do shocks propagate between variables?

## GDP and Unemployment: Data

- GDP growth and unemployment rate show clear negative correlation (Okun's Law)
- Both series exhibit cyclical patterns related to business cycle fluctuations
- This bivariate system is ideal for VAR analysis and Granger causality testing

## GDP and Unemployment: Data



## VAR Workflow

### 1. Data preparation

- ▶ Check for stationarity (unit root tests)
- ▶ Transform if necessary (differences, logs)

### 2. Lag selection

- ▶ Use AIC, BIC, HQ criteria
- ▶ Check residual autocorrelation

### 3. Estimation

- ▶ OLS equation by equation
- ▶ Check stability (eigenvalues)

### 4. Analysis

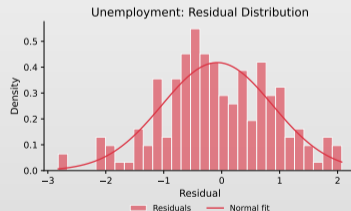
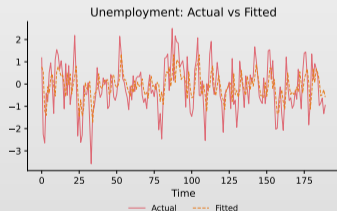
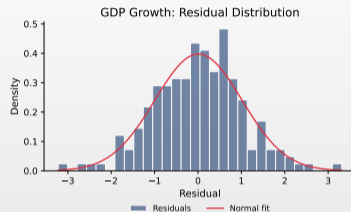
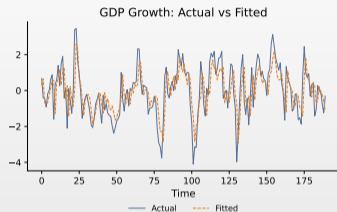
- ▶ Granger causality tests
- ▶ Impulse response functions
- ▶ Variance decomposition

### 5. Forecasting

## Estimated VAR Results

- ▣ Estimated coefficients with standard errors and t-statistics
- ▣ Information criteria values for model comparison
- ▣ Model diagnostics summary (residual tests)

## Estimated VAR Results



## Granger Causality Results

### Test Results: GDP and Unemployment

Null Hypothesis	F-statistic	df	p-value	Decision
GDP $\nrightarrow$ Unemployment	8.42	(2, 95)	0.0004	Reject
Unemployment $\nrightarrow$ GDP	2.15	(2, 95)	0.1220	Fail to Reject

### Interpretation

- GDP growth Granger-causes unemployment (consistent with Okun's Law)
- Unemployment does not significantly Granger-cause GDP
- Evidence of **unidirectional** causality: GDP  $\rightarrow$  Unemployment

## Python Implementation

### VAR in Python (statsmodels)

□ Code:

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests
# Fit the VAR model
model = VAR(data)
results = model.fit(maxlags=4, ic='aic')
# Granger causality test
granger_test = grangercausalitytests(data[['Y1','Y2']], maxlag=4)
# Impulse response functions
irf = results.irf(periods=20)
irf.plot()
# Variance decomposition
fevd = results.fevd(periods=20)
fevd.plot()
```

## R Implementation

### VAR in R (vars package)

□ Code:

```
library(vars)
# Select optimal lag order
lag_select <- VARselect(data, lag.max = 8)
# Fit the VAR model
var_model <- VAR(data, p = 2, type = "const")
# Granger causality test
causality(var_model, cause = "GDP")
# Impulse response functions
irf_results <- irf(var_model, n.ahead=20, boot=TRUE)
plot(irf_results)
# Forecast error variance decomposition
fevd_results <- fevd(var_model, n.ahead = 20)
plot(fevd_results)
```

## Example: Monetary Policy Analysis

### Three-Variable VAR

Study the monetary transmission mechanism with:

- ▣  $Y_1$ : Output gap (GDP deviation from trend)
- ▣  $Y_2$ : Inflation rate
- ▣  $Y_3$ : Interest rate (policy instrument)

### Key Questions

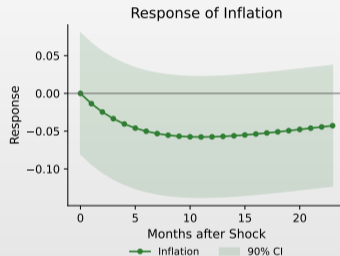
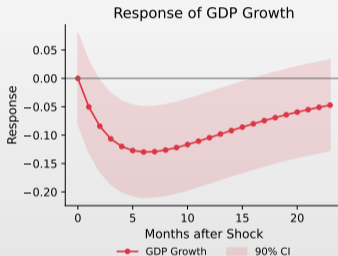
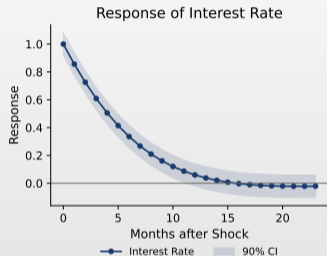
1. How does an interest rate shock affect output and inflation?
2. How long until the maximum effect is felt?
3. What fraction of output variance is due to monetary shocks?

## Monetary Policy VAR: IRFs

- ▣ Contractionary monetary policy shock (interest rate increase)
- ▣ Output decreases with peak effect after 4-6 quarters (“long and variable lags”)
- ▣ Inflation responds more slowly, decreasing after output

## Monetary Policy VAR: IRFs

Monetary Policy Shock (Interest Rate  $\uparrow$ )



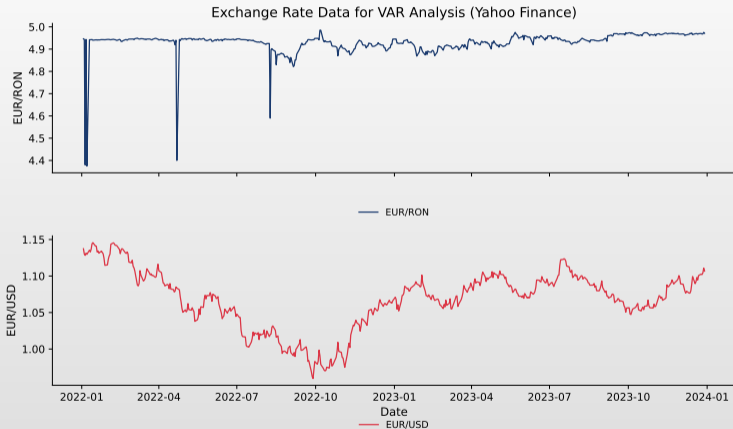
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## Case Study: The Relationship between GDP and Unemployment

### Data

- ▣ Real US data (FRED, 1990–2024): GDP Growth and Unemployment Rate ( $T = 140$  quarters)
- ▣ Visible negative correlation between series (Okun's Law); bidirectional dynamics

## Case Study: The Relationship between GDP and Unemployment

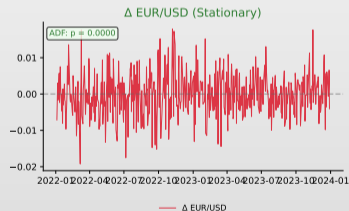
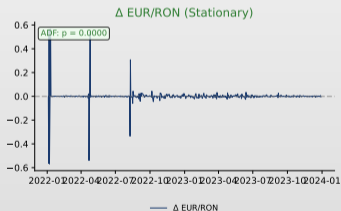
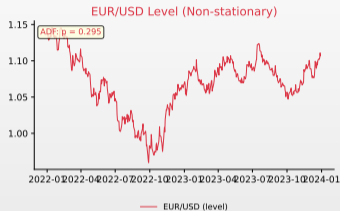
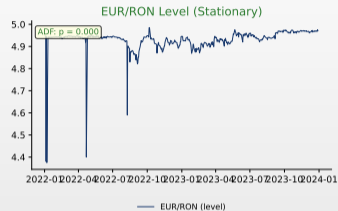


## Step 1: Preliminary Analysis

### Results

- GDP: ACF decays quickly  $\Rightarrow$  stationary; Unemployment: persistent ACF (ADF:  $p = 0.02$ )
- Negative GDP–Unemployment correlation ( $\rho = -0.17$ ); cross-correlation suggests bidirectional relationships

## Step 1: Preliminary Analysis

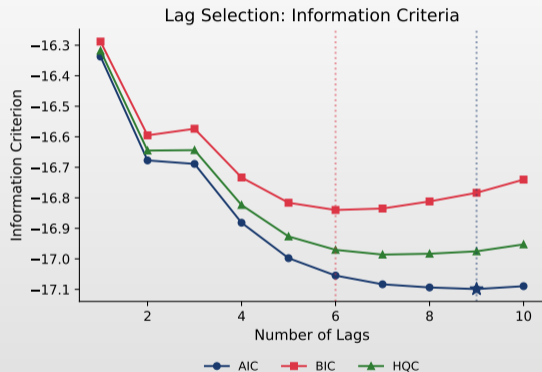


## Step 2: VAR Order Selection

### Results

- AIC and BIC criteria suggest VAR(2); trade-off between complexity and fit

## Step 2: VAR Order Selection



Optimal Lag Summary

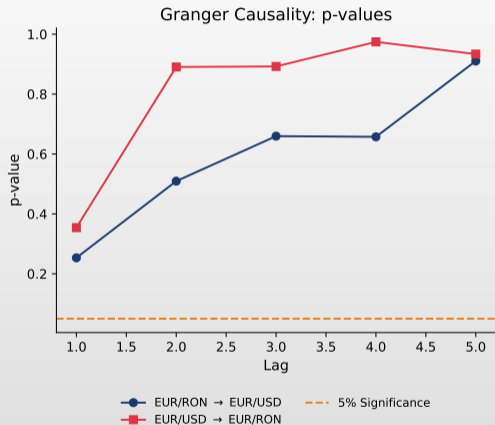
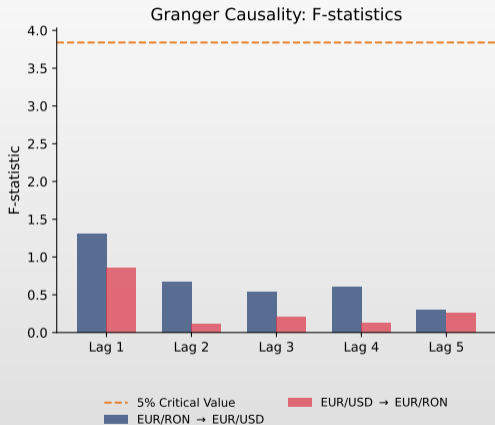
Criterion	Optimal Lag
AIC	$p = 9$
BIC	$p = 6$
HQC	$p = 7$
FPE	$p = 9$

## Step 3: Granger Causality Test

### Results

- GDP  $\Rightarrow$  Unemployment:  $F = 17.35$ ,  $p < 0.001 \Rightarrow$  GDP “Granger-causes” Unemployment
- Unemployment  $\Rightarrow$  GDP:  $F = 38.93$ ,  $p < 0.001 \Rightarrow$  bidirectional causality (Okun’s Law)

## Step 3: Granger Causality Test



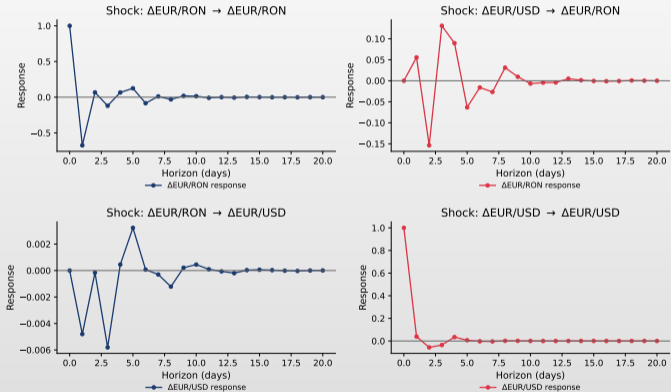
## Step 4: Impulse Response Functions (IRF)

### IRF Results

- GDP shock  $\Rightarrow$  persistent negative effect on unemployment (Okun's Law,  $>20$  quarters)
- Unemployment shock  $\Rightarrow$  short-lived positive effect on GDP (recovery, 2–3 quarters)

## Step 4: Impulse Response Functions (IRF)

Impulse Response Functions (Real VAR)

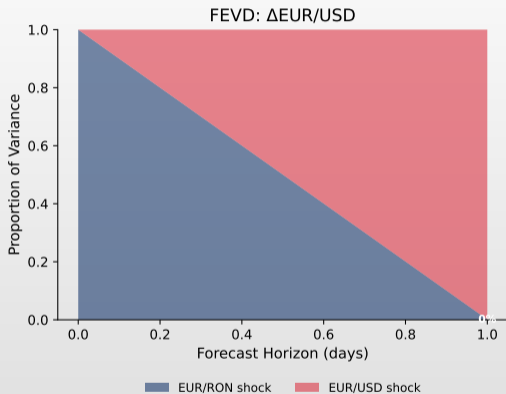
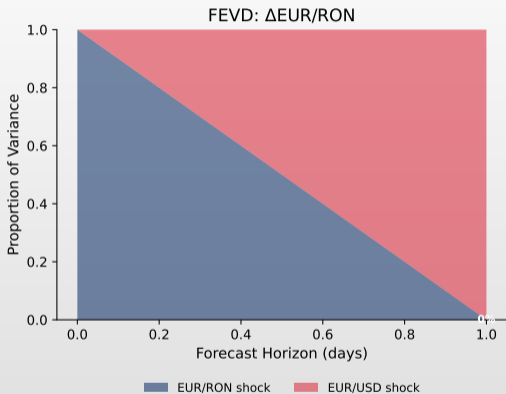


## Step 5: Variance Decomposition (FEVD)

### FEVD Results

- GDP:  $\sim 65\%$  of variance explained by own shocks,  $\sim 35\%$  by Unemployment shocks
- Unemployment: dominated by GDP shocks ( $\sim 65\%$  at  $h = 1$ , increasing to  $\sim 92\%$  at  $h = 20$ )

## Step 5: Variance Decomposition (FEVD)

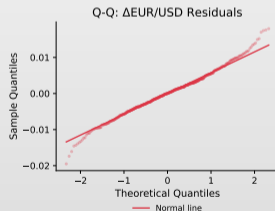
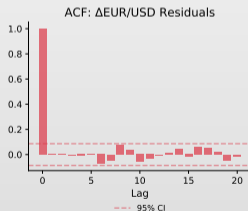
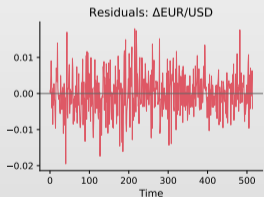
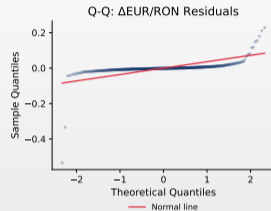
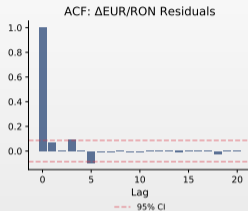
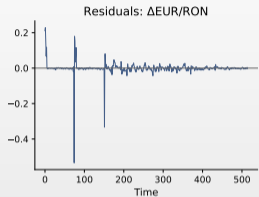


## Step 6: Residual Diagnostics

### Diagnostics

- ▣ Residuals show no significant autocorrelation (ACF within bounds)
- ▣ Significant non-normality (JB rejected)  $\Rightarrow$  due to extreme COVID-19 values

## Step 6: Residual Diagnostics

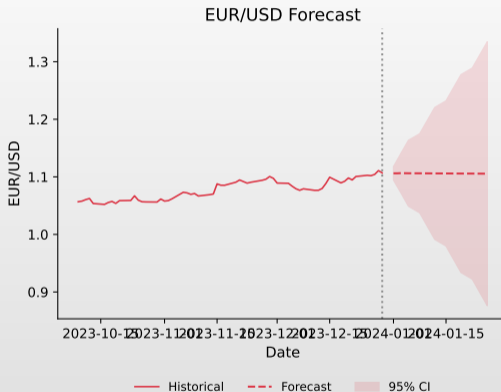
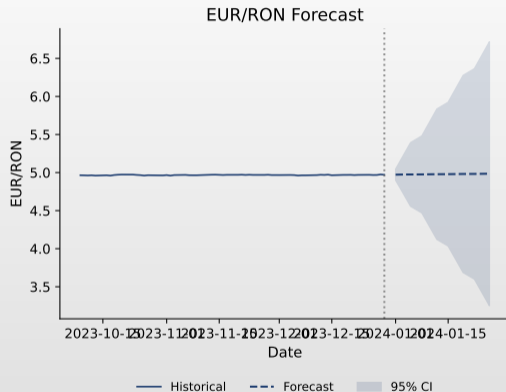


## Step 7: VAR Forecasting

### Forecast Results

- ▣ 12-quarter forecast; VAR captures interdependencies between series
- ▣ Forecasts converge to long-run equilibrium values

## Step 7: VAR Forecasting

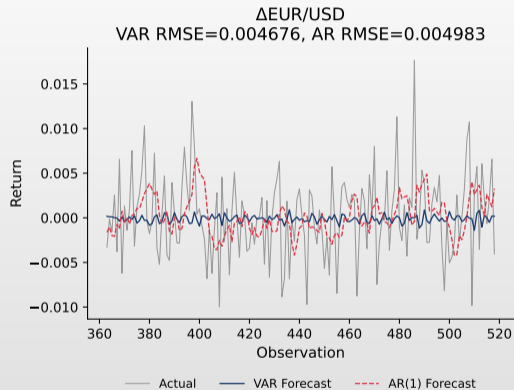
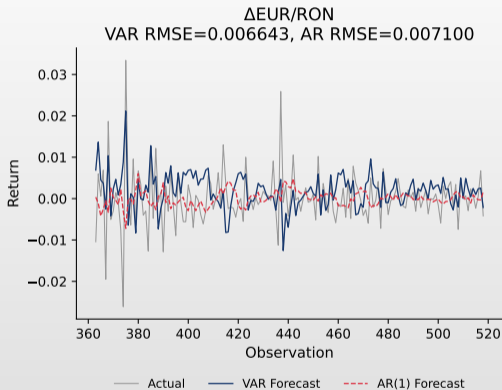


## Step 8: Rolling Forecast – VAR vs AR

### Procedure

- ▣ Recursive Train/Test:  $W = 80$ , re-estimation at each step; AR(2) vs VAR(2) models
- ▣ Expanding RMSE:  $RMSE_t = \sqrt{\frac{1}{t} \sum_{s=1}^t e_s^2}$ ; 95% confidence intervals
- ▣ Mixed results:
  - ▶ Unemployment — VAR −10% RMSE vs AR (information from GDP helps)
  - ▶ GDP — VAR +6% RMSE vs AR (information from unemployment does not help)

## Step 8: Rolling Forecast – VAR vs AR

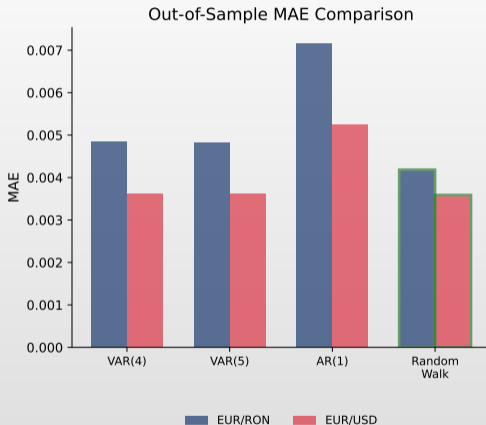
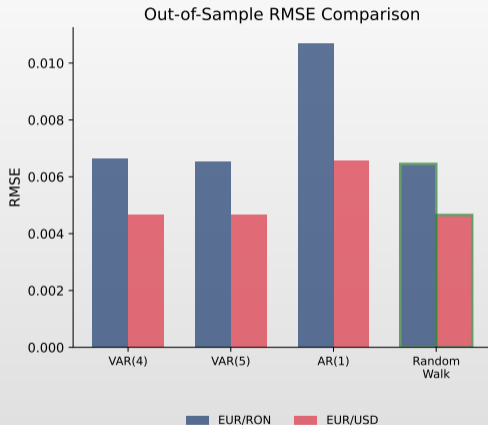


## Step 9: Out-of-Sample Comparison – AR vs VAR

### Methodology

- ▣ Recursive forecast: estimate on  $[1, \dots, t]$ , forecast  $\hat{y}_{t+1|t}$ ; AR(2) vs VAR(2) models
- ▣ Expanding RMSE:  $\text{RMSE}_t = \sqrt{\frac{1}{t} \sum_{s=1}^t e_s^2}$
- ▣ VAR reduces RMSE for Unemployment ( $\sim 10\%$ ), not for GDP

## Step 9: Out-of-Sample Comparison – AR vs VAR



## AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download quarterly GDP and unemployment data for the US. Test for Granger causality, estimate a VAR model, and analyze impulse response functions. Give me complete Python code."

**Exercise:**

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it test stationarity of each variable before estimating the VAR?
3. How does it select the lag order? Does it compare AIC, BIC, HQIC?
4. Are impulse response functions orthogonalized? Does it discuss Cholesky ordering?
5. Does it check the stability condition (eigenvalues inside the unit circle)?

**Warning:** AI-generated code may run without errors and look professional. *That does not mean it is correct.*

## Key Takeaways

### VAR Models

- ▣ Model **multiple** time series jointly
- ▣ Each variable depends on its own lags AND lags of other variables
- ▣ Estimated by OLS equation by equation; requires stationarity

### Granger Causality

- ▣ Tests whether  $X$  helps predict  $Y$  beyond  $Y$ 's own history
- ▣ **Not** the same as true causality; F-test on coefficient restrictions

### IRF and FEVD

- ▣ IRF: How shocks propagate through the system
- ▣ FEVD: What proportion of variance is due to each shock
- ▣ Both depend on variable ordering (Cholesky decomposition)

## VAR Model Selection Checklist

### Before Estimation

- ☐ Test for unit roots
- ☐ Transform if needed
- ☐ Check for breaks

### Model Specification

- ☐ Select lag order (AIC/BIC)
- ☐ Estimate VAR by OLS
- ☐ Check stability

### Post-Estimation

- ☐ Test autocorrelation
- ☐ Test ARCH effects
- ☐ Test normality
- ☐ Compute IRFs, FEVDs

## Common Mistakes to Avoid

### Pitfalls in VAR Analysis

1. **Ignoring non-stationarity:** Always test for unit roots first
2. **Overfitting:** Too many lags  $\Rightarrow$  poor forecasts
3. **Wrong ordering:** Cholesky results depend on variable order
4. **Confusing correlation with causation:** Granger causality  $\neq$  true causality
5. **Ignoring parameter uncertainty:** Use bootstrap CIs for IRFs
6. **Short samples:** VAR requires many observations ( $T > 50$ )

## What's Next?

### Topics for Further Study

- ▣ **Cointegration:** Long-run relationships between non-stationary variables
- ▣ **VECM:** Error correction models for cointegrated systems
- ▣ **Structural VAR:** Imposing economic theory restrictions
- ▣ **Panel VAR:** VAR for panel data
- ▣ **Bayesian VAR:**
  - ▶ Shrinkage priors for high-dimensional systems

Questions?

## Quiz Question 1

### Question

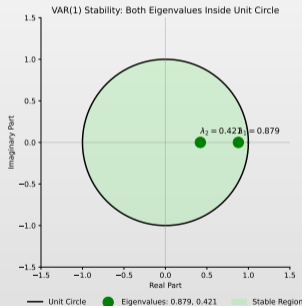
For a VAR(1) model with coefficient matrix  $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.5 \end{pmatrix}$ , is the model stable?

- (A) Yes, because all diagonal elements are less than 1
- (B) Yes, because all eigenvalues are inside the unit circle
- (C) No, because the sum of coefficients exceeds 1
- (D) Cannot be determined without knowing  $\Sigma$

## Quiz Question 1: Answer

Correct Answer: (B) Eigenvalues inside unit circle

$\lambda_1 = 0.879$ ,  $\lambda_2 = 0.421$  — both  $|\lambda| < 1 \Rightarrow$  Stable!



## Quiz Question 2

### Question

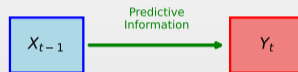
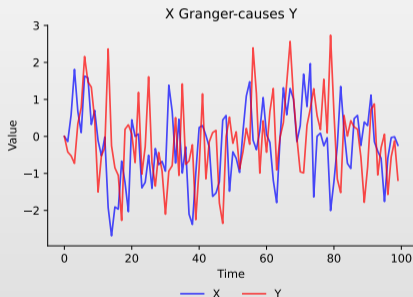
If  $X$  Granger-causes  $Y$  at the 5% significance level, which of the following statements is TRUE?

- (A)  $X$  is the economic cause of  $Y$
- (B) Past values of  $X$  contain useful information for predicting  $Y$
- (C)  $Y$  cannot Granger-cause  $X$
- (D) The correlation between  $X$  and  $Y$  is positive

## Quiz Question 2: Answer

Correct Answer: (B) Predictive information

Granger causality = predictive content, not true economic causation. Past X helps predict Y.



*Past X helps predict Y  
(beyond Y's own past)*

## Quiz Question 3

### Question

In a VAR with Cholesky-identified IRFs, what does the ordering of variables determine?

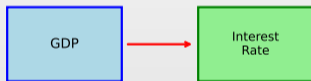
- (A) The magnitude of the impulse responses
- (B) The speed at which shocks die out
- (C) Which variables can respond contemporaneously to which shocks
- (D) The number of lags in the VAR

## Quiz Question 3: Answer

Correct Answer: (C) Contemporaneous responses

Ordering determines which variables respond immediately to which shocks.

**Ordering: (GDP, Interest Rate)**



GDP shock → IR responds at  $t=0$   
IR shock → GDP responds at  $t=1$

**Ordering: (Interest Rate, GDP)**



IR shock → GDP responds at  $t=0$   
GDP shock → IR responds at  $t=1$

## Quiz Question 4

### Question

For a bivariate VAR(1), how many parameters need to be estimated (excluding the error covariance matrix)?

- (A) 4
- (B) 6
- (C) 8
- (D) 10

## Quiz Question 4: Answer

Correct Answer: (B) 6 parameters

### Detailed Count

VAR(1) with  $K = 2$  variables:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{2 \text{ params}} + \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{4 \text{ params}} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

- ▣ Constant vector  $c$ :  $K = 2$  parameters
- ▣ Coefficient matrix  $A$ :  $K^2 = 4$  parameters
- ▣ Total:  $K + K^2 = 2 + 4 = 6$  parameters

### General Formula

VAR( $p$ ) with  $K$  variables:  $K + pK^2$  parameters (excluding  $\Sigma$ )

## Quiz Question 5

### Question

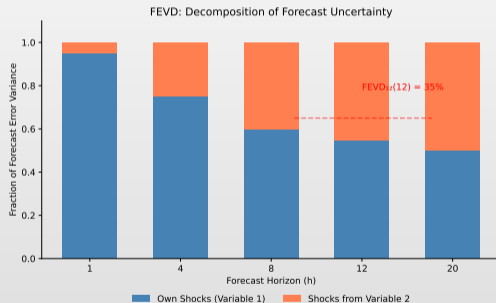
What does  $\text{FEVD}_{12}(h) = 0.35$  mean?

- (A) 35% of variable 1's total variance is explained by variable 2
- (B) 35% of variable 1's  $h$ -step forecast error variance is due to shocks to variable 2
- (C) The correlation between variables 1 and 2 at lag  $h$  is 0.35
- (D) Variable 2 explains 35% of the impulse response of variable 1

## Quiz Question 5: Answer

Correct Answer: (B) Forecast error variance decomposition

35% of variable 1's  $h$ -step forecast error variance is due to shocks from variable 2.



## Key Formulas – Summary

### VAR(p) Model

- $Y_t = c + \sum_{i=1}^p A_i Y_{t-i} + \epsilon_t$
- $\epsilon_t \sim \mathcal{N}(0, \Sigma)$ , i.i.d.

### Granger Causality

- $H_0$ :  $X$  does not Granger-cause  $Y$
- F or Wald test on lag coefficients of  $X$

### Lag Selection

- $AIC = \ln |\hat{\Sigma}| + \frac{2pK^2}{T}$
- $BIC = \ln |\hat{\Sigma}| + \frac{pK^2 \ln T}{T}$

### Impulse Response Functions

- $Y_{t+h} = \sum_{i=0}^{\infty} \Phi_i \epsilon_{t+h-i}$
- $\Phi_i$  = multipliers at horizon  $i$

### FEVD

- $FEVD_{jk}(h) = \frac{\sum_{i=0}^{h-1} (e_j' \Phi_i P e_k)^2}{\sum_{i=0}^{h-1} e_j' \Phi_i \Sigma \Phi_i' e_j}$
- Contribution of shock  $k$  to variance of  $j$

### VAR Stationarity

- All eigenvalues of  $A$  inside the unit circle

## Bibliography I

### Fundamental works on VAR and causality

- ▣ Sims, C.A. (1980). Macroeconomics and Reality, *Econometrica*, 48(1), 1–48.
- ▣ Granger, C.W.J. (1969). Investigating Causal Relations by Econometric Models and Cross-Spectral Methods, *Econometrica*, 37(3), 424–438.
- ▣ Toda, H.Y., & Yamamoto, T. (1995). Statistical Inference in Vector Autoregressions with Possibly Integrated Processes, *Journal of Econometrics*, 66(1-2), 225–250.

### VAR Textbooks

- ▣ Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*, Springer.
- ▣ Kilian, L., & Lütkepohl, H. (2017). *Structural Vector Autoregressive Analysis*, Cambridge University Press.

## Bibliography II

### Impulse response functions and variance decomposition

- ▣ Pesaran, H.H., & Shin, Y. (1998). Generalized Impulse Response Analysis in Linear Multivariate Models, *Economics Letters*, 58(1), 17–29.
- ▣ Hamilton, J.D. (1994). *Time Series Analysis*, Princeton University Press.
- ▣ Tsay, R.S. (2014). *Multivariate Time Series Analysis with R and Financial Applications*, Wiley.

### Online resources and code

- ▣ **Quantlet**: <https://quantlet.com> → Code repository for statistics
- ▣ **Quantinar**: <https://quantinar.com> → Learning platform for quantitative methods
- ▣ **GitHub TSA**: [https://github.com/QuantLet/TSA/tree/main/TSA\\_ch6](https://github.com/QuantLet/TSA/tree/main/TSA_ch6) → Python code for this chapter

# Thank You!

## Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar