



Time Series Analysis and Forecasting

Chapter 5: GARCH and Volatility



Daniel Traian PELE

Bucharest University of Economic Studies

IDA Institute Digital Assets

Blockchain Research Center

AI4EFin Artificial Intelligence for Energy Finance

Romanian Academy, Institute for Economic Forecasting

MSCA Digital Finance

Outline

- Introduction to Volatility Modeling
- The ARCH Model
- The GARCH Model
- Asymmetric GARCH Models
- Model Selection and Diagnostics
- Volatility Forecasting
- Case Study: S&P 500
- Summary



Why Model Volatility?

Empirical Observations in Financial Series

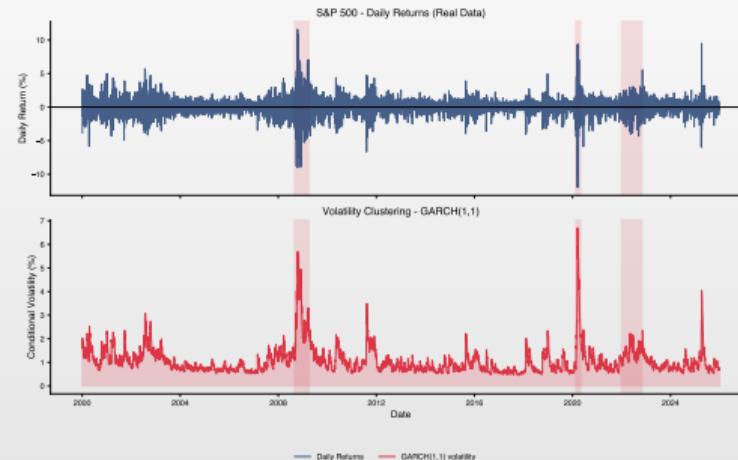
- Financial returns exhibit **volatility clustering** — periods of high volatility tend to be followed by periods of high volatility
- The distribution of returns has **fat tails** (leptokurtosis)
- Return correlation is nearly zero, but correlation of squares is significant
- Volatility responds **asymmetrically** to shocks (leverage effect)

Limitation of ARIMA Models

ARIMA models assume **constant variance** (homoskedasticity), which is not realistic for financial series!



Volatility Clustering



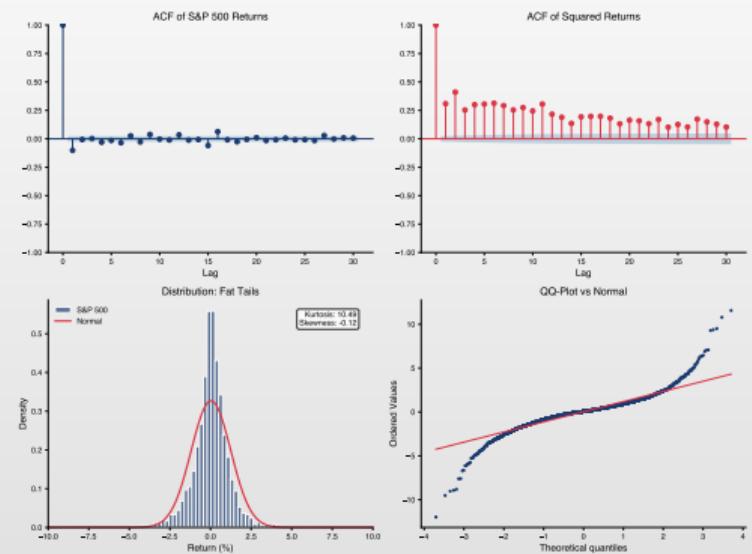
- High volatility periods are followed by high volatility; calm by calm
- This suggests that **conditional variance** is predictable



Stylized Facts of Financial Returns

Observed Properties

1. No autocorrelation in returns
2. Autocorrelation in r_t^2 , $|r_t|$
3. Fat tails (kurtosis > 3)
4. Leverage effect
5. Volatility clustering



Conditional Heteroskedasticity

Definition 1 (Conditional Variance)

For return series $\{r_t\}$, the **conditional variance** at time t is: $\sigma_t^2 = \text{Var}(r_t | \mathcal{F}_{t-1}) = \mathbb{E}[(r_t - \mu_t)^2 | \mathcal{F}_{t-1}]$ where \mathcal{F}_{t-1} is the information available up to time $t-1$.

General Model

$$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0, 1)$$

- ◻ μ_t = conditional mean (ARMA); σ_t^2 = conditional variance (GARCH)
- ◻ z_t = standardized innovations (Normal, Student-t, GED)



The ARCH(q) Model — Engle (1982)

Definition 2 (ARCH(q))

The Autoregressive Conditional Heteroskedasticity model of order q :

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0, 1), \quad \sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2$$

Stationarity Restrictions

- $\omega > 0$ (positive base variance), $\alpha_i \geq 0$ (non-negativity)
- $\sum_{i=1}^q \alpha_i < 1$ (stationarity)

Remark 1

Robert Engle received the **Nobel Prize in Economics** in 2003 for developing the ARCH model!



Properties of the ARCH(1) Model

$$\text{ARCH}(1): \sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$$

- **Unconditional variance:** $\mathbb{E}[\varepsilon_t^2] = \frac{\omega}{1 - \alpha_1}$ (if $\alpha_1 < 1$)
- **Kurtosis:** $\kappa = 3 \cdot \frac{1 - \alpha_1^2}{1 - 3\alpha_1^2}$ (if $\alpha_1^2 < 1/3$)
- Kurtosis > 3 for $\alpha_1 > 0 \Rightarrow$ **fat tails!**

Numerical Example

If $\omega = 0.0001$ and $\alpha_1 = 0.3$:

- Unconditional variance: $\sigma^2 = \frac{0.0001}{1-0.3} = 0.000143$
- Kurtosis: $\kappa = 3 \cdot \frac{1-0.09}{1-0.27} = 3.74 > 3$



Testing for ARCH Effects

Engle's Test for ARCH Effects

Procedure:

1. Estimate the mean model and obtain residuals $\hat{\varepsilon}_t$
2. Calculate $\hat{\varepsilon}_t^2$
3. Regress $\hat{\varepsilon}_t^2$ on its lags:

$$\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \cdots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$$

4. Calculate the statistic $LM = T \cdot R^2 \sim \chi^2(q)$

Hypotheses

- H_0 : No ARCH effects ($\alpha_1 = \cdots = \alpha_q = 0$)
- H_1 : ARCH effects present (at least one $\alpha_i \neq 0$)



Limitations of the ARCH Model

Practical Problems

1. **High order** — many lags are usually needed (large q)
2. **Many parameters** — estimation difficulties
3. **Non-negativity constraints** — difficult to impose for large q
4. **Does not capture persistence** — observed volatility is very persistent

The Solution

The GARCH Model — introduces lags of conditional variance to capture persistence with fewer parameters!



The GARCH(p,q) Model — Bollerslev (1986)

Definition 3 (GARCH(p,q))

The Generalized ARCH model:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0, 1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^p \beta_j \sigma_{t-j}^2$$

Interpretation

- ◻ ω = base level of volatility
- ◻ α_i = reaction to recent shocks (news coefficients)
- ◻ β_j = volatility persistence (memory)
- ◻ $\alpha + \beta$ = total persistence



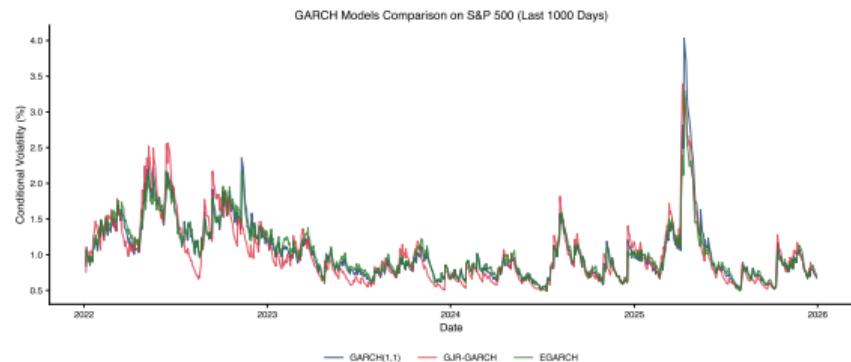
The GARCH(1,1) Model

The Most Popular Volatility Model

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

Restrictions & Properties

- $\omega > 0, \alpha \geq 0, \beta \geq 0$
- $\alpha + \beta < 1$ (stationarity)
- $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- Half-life: $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$



GARCH(1,1) as ARMA for ε_t^2

ARMA(1,1) Representation

Define $\nu_t = \varepsilon_t^2 - \sigma_t^2$ (variance shock). Then:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

This is an ARMA(1,1) for ε_t^2 !

Implications

- ACF of ε_t^2 decays exponentially (like ARMA)
- Persistence is given by $\alpha + \beta$
- PACF can help identify the order



Estimation of GARCH Models

Maximum Likelihood Estimation (MLE)

Log-likelihood (normal): $\ell(\theta) = -\frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \left[\ln(\sigma_t^2) + \frac{\varepsilon_t^2}{\sigma_t^2} \right]$

Alternative Distributions for z_t

- Student-t**: captures fat tails — most common choice
- GED**: flexibility for kurtosis
- Skewed Student-t**: asymmetry and fat tails

Practical Note

Student-t distribution typically provides better fit for financial returns due to fat tails ($kurtosis > 3$).



Typical Values for GARCH(1,1)

Series	α	β	$\alpha + \beta$
S&P 500 daily	0.05–0.10	0.85–0.95	0.95–0.99
EUR/USD daily	0.03–0.08	0.90–0.95	0.95–0.99
Bitcoin daily	0.10–0.20	0.75–0.85	0.90–0.98
Bonds	0.02–0.05	0.90–0.97	0.95–0.99

Observations

- $\alpha + \beta$ close to 1 \Rightarrow **very persistent volatility**
- Small α , large β \Rightarrow slow reaction to shocks, long memory
- Bitcoin: larger α \Rightarrow faster reaction to news



IGARCH — Integrated GARCH

Definition 4 (IGARCH(1,1))

When $\alpha + \beta = 1$:

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + (1 - \alpha) \sigma_{t-1}^2$$

Properties

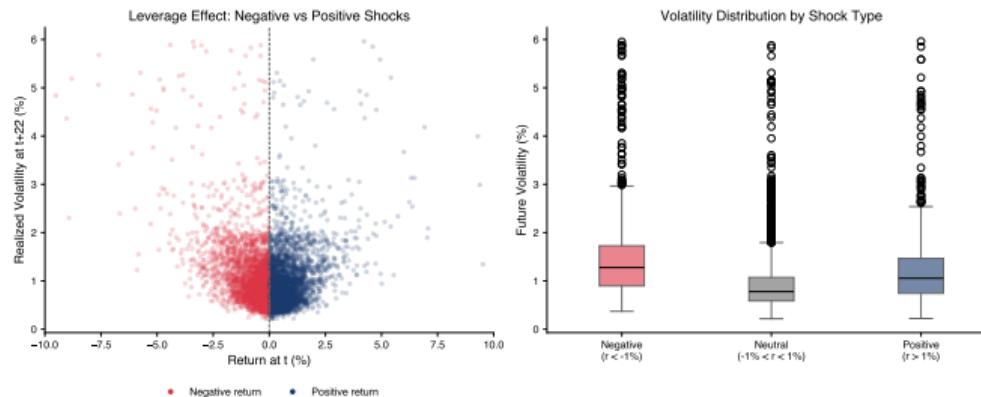
- Unconditional variance does not exist (infinite)
- Shocks have **permanent** effect on volatility
- Used for series with extreme persistence
- Useful for **RiskMetrics** (J.P. Morgan): $\alpha = 0.06$, $\beta = 0.94$

Remark 2

IGARCH is analogous to a unit root in variance!



Leverage Effect



Definition

Leverage effect: Negative shocks increase volatility **more** than positive shocks.

Problem with Standard GARCH

GARCH depends on ε_{t-1}^2 , treating positive and negative shocks **symmetrically**!



The EGARCH Model — Nelson (1991)

Definition 5 (EGARCH(1,1))

Exponential GARCH:

$$\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z_{t-1}|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

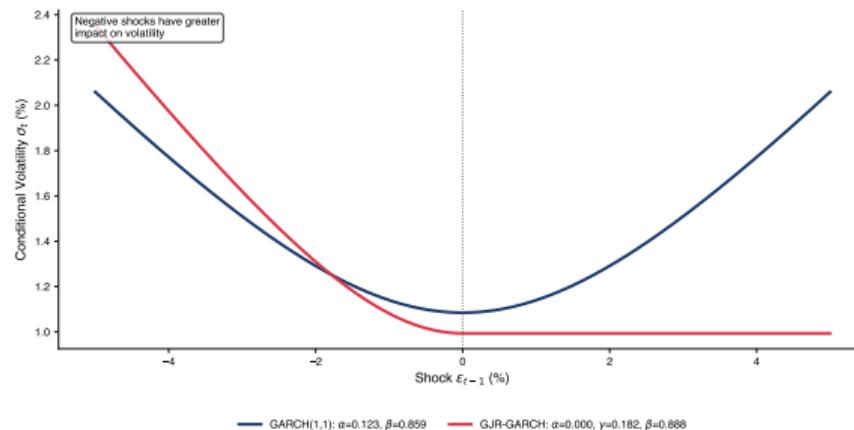
where $z_t = \varepsilon_t / \sigma_t$.

EGARCH Advantages

- No non-negativity constraints required** — models $\ln(\sigma_t^2)$
- Captures leverage effect through parameter γ**
 - ▶ $\gamma < 0$: negative shocks \Rightarrow higher volatility
 - ▶ $\gamma = 0$: symmetric effect (like GARCH)
- Persistence is given by β**



News Impact Curve — EGARCH



Interpretation

News Impact Curve: shows how σ_{t+1}^2 depends on shock ε_t , holding σ_t^2 constant.



The GJR-GARCH Model

Definition 6 (GJR-GARCH(1,1))

Glosten, Jagannathan & Runkle (1993): $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \gamma \varepsilon_{t-1}^2 \cdot I_{t-1} + \beta \sigma_{t-1}^2$ where $I_{t-1} = 1$ if $\varepsilon_{t-1} < 0$, else 0.

Interpretation

- Positive shocks: impact = α ; Negative shocks: impact = $\alpha + \gamma$
- Leverage effect present if $\gamma > 0$
- Stationarity: $\alpha + \gamma/2 + \beta < 1$



TGARCH — Threshold GARCH

Definition 7 (TGARCH(1,1))

Zakoian (1994) models standard deviation: $\sigma_t = \omega + \alpha^+ \varepsilon_{t-1}^+ + \alpha^- \varepsilon_{t-1}^- + \beta \sigma_{t-1}$

Comparison of Asymmetric Models

Model	Specification	Leverage
GARCH	σ_t^2	No
EGARCH	$\ln(\sigma_t^2)$	Yes ($\gamma < 0$)
GJR-GARCH	σ_t^2 with indicator	Yes ($\gamma > 0$)
TGARCH	σ_t	Yes ($\alpha^- > \alpha^+$)



Order Selection

Information Criteria

- AIC** = $-2\ell + 2k$
- BIC** = $-2\ell + k \ln(T)$
- HQIC** = $-2\ell + 2k \ln(\ln(T))$

where ℓ = maximized log-likelihood, k = number of parameters.

Practical Recommendations

- GARCH(1,1) is sufficient in **90% of cases**
- Check if asymmetric model significantly improves fit
- Choose innovation distribution that minimizes AIC/BIC



GARCH Model Diagnostics

Standardized Residuals

$$\hat{z}_t = \frac{\hat{\epsilon}_t}{\hat{\sigma}_t}$$

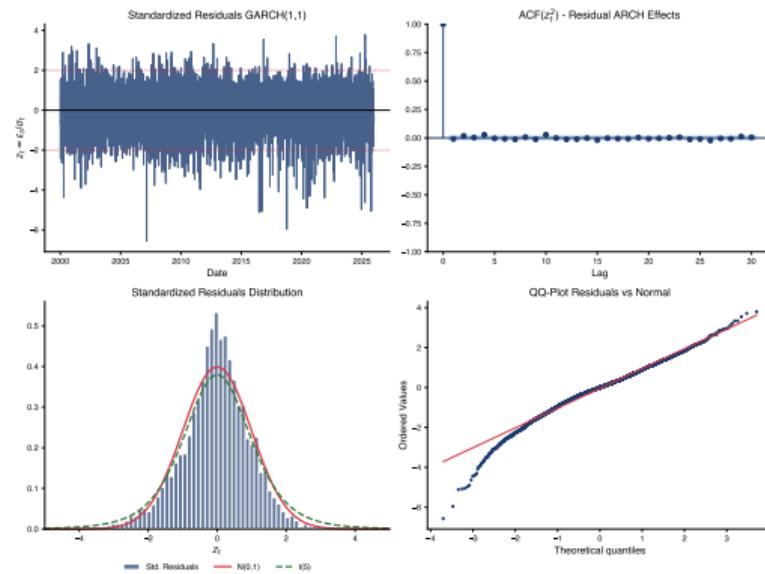
If the model is correctly specified, \hat{z}_t should be i.i.d.(0,1).

Diagnostic Checks

1. **Ljung-Box on \hat{z}_t :** check absence of autocorrelation in mean
2. **Ljung-Box on \hat{z}_t^2 :** check absence of residual ARCH effects
3. **ARCH-LM test on \hat{z}_t :** confirm absence of heteroskedasticity
4. **Histogram + QQ-plot:** verify assumed distribution



Diagnostic Example



Forecasting with GARCH(1,1)

One-Step-Ahead Forecast

$$\hat{\sigma}_{T+1}^2 = \omega + \alpha \varepsilon_T^2 + \beta \sigma_T^2$$

Multi-Step Forecast

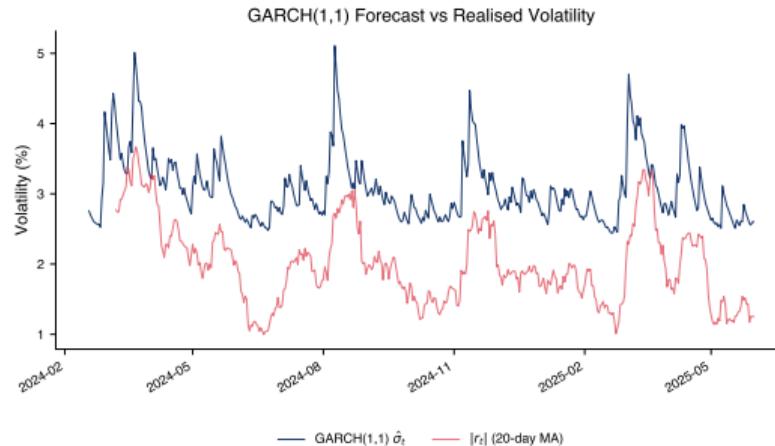
For $h > 1$: $\mathbb{E}_T[\sigma_{T+h}^2] = \bar{\sigma}^2 + (\alpha + \beta)^{h-1}(\sigma_{T+1}^2 - \bar{\sigma}^2)$ where $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ = unconditional variance.

Convergence

$\lim_{h \rightarrow \infty} \mathbb{E}_T[\sigma_{T+h}^2] = \bar{\sigma}^2$ — forecast converges to unconditional variance!



Volatility Forecast — Visualization



- Forecast converges exponentially to $\bar{\sigma}^2$; speed depends on $\alpha + \beta$
- The closer $\alpha + \beta$ is to 1, the slower the convergence

Q TSA_garch_forec



Applications of Volatility Forecasting

Value at Risk (VaR)

$$\text{VaR}_\alpha = -\mu_{T+1} + z_\alpha \cdot \sigma_{T+1}$$

The probability of losing more than VaR is α (e.g., 1%, 5%).

Expected Shortfall

$$\text{ES}_\alpha = \mathbb{E}[-r_{T+1} | r_{T+1} < -\text{VaR}_\alpha]$$

Other Applications

- Option pricing
- Dynamic hedging
- Portfolio allocation
- Stress testing
- Scenario analysis



Value at Risk — Numerical Example

VaR Calculation

Portfolio: 1,000,000 EUR, forecasted volatility $\hat{\sigma}_{T+1} = 1.5\%$

VaR with Normal Distribution

Level	z_α	VaR (%)	VaR (EUR)
95% (1 day)	1.645	2.47%	24,675
99% (1 day)	2.326	3.49%	34,890

Scaling for Longer Periods

$$\text{VaR}_{h \text{ days}} = \text{VaR}_{1 \text{ day}} \cdot \sqrt{h} \text{ — assumes i.i.d. returns}$$



Value at Risk — Student-t Distribution

Why Student-t?

Normal distribution **underestimates** tail risk. Student-t with ν degrees of freedom better captures fat tails (kurtosis > 3).

VaR 99% (1 day) Comparison: $\sigma = 1.5\%$, Portfolio = 1M EUR

Distribution	Quantile	VaR (EUR)
Normal	2.326	34,890
Student-t ($\nu = 6$)	3.143	47,145
Student-t ($\nu = 4$)	3.747	56,205

Observation

With $\nu = 6$ (typical for stocks), VaR is **35% higher** than normal!



VaR — Complete Example with GARCH

VaR Calculation Procedure

1. Estimate GARCH(1,1) model with Student-t distribution
2. Obtain volatility forecast: $\hat{\sigma}_{T+1}$
3. Calculate VaR: $\text{VaR}_\alpha = t_\alpha(\nu) \cdot \hat{\sigma}_{T+1} \cdot \sqrt{\frac{\nu-2}{\nu}}$

Example: S&P 500

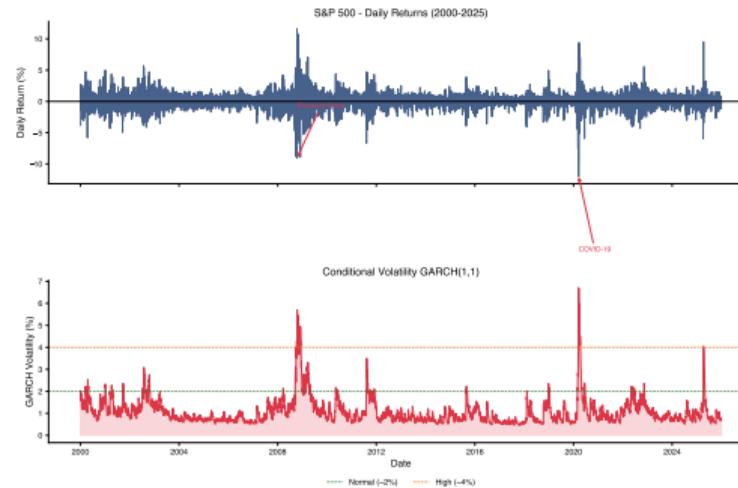
- Estimated parameters: $\alpha = 0.088$, $\beta = 0.900$, $\nu = 6.4$
- Forecasted volatility: $\hat{\sigma}_{T+1} = 1.2\%$
- Portfolio: 10,000,000 EUR

VaR 99% (1 day):

$$\text{VaR} = 3.05 \times 0.012 \times 10,000,000 = \mathbf{366,000 \text{ EUR}}$$



S&P 500 Volatility Analysis



- S&P 500 daily returns (2000–2024) — volatility clustering visible
- Crisis periods: 2008 (financial), 2020 (COVID-19), 2022 (inflation)

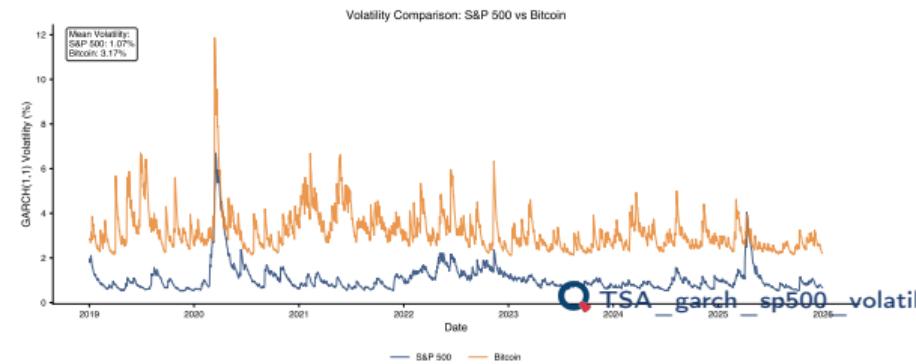


GARCH(1,1) Estimation — S&P 500

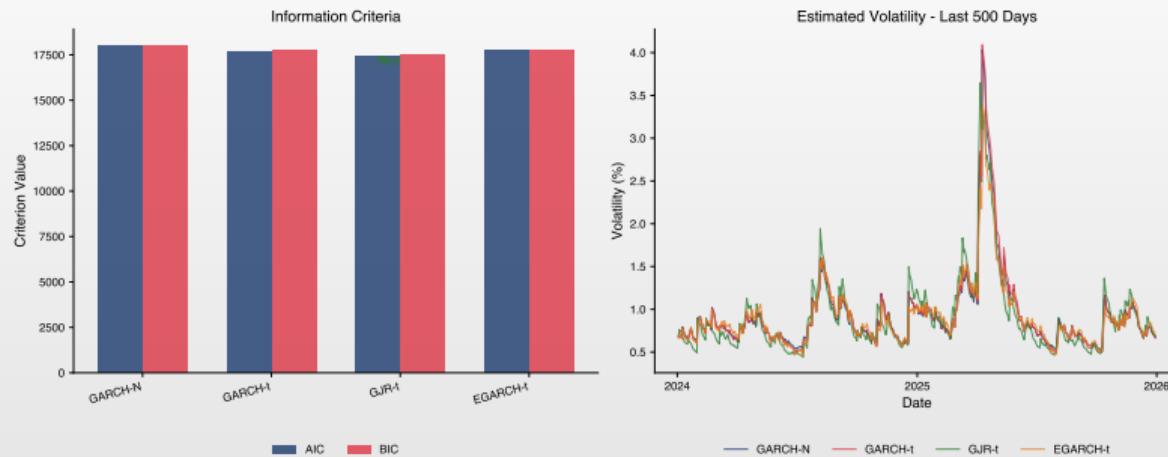
Estimation Results

Parameter	Value
ω	0.0108
α	0.0883
β	0.9002
$\alpha + \beta$	0.9885
ν (df)	6.42

Very persistent; Half-life ≈ 60 days



GARCH vs EGARCH Comparison — S&P 500



Leverage Effect Confirmed

EGARCH: $\gamma = -0.12$ (significantly negative) — negative shocks amplify volatility more

Q [TSA_garch_sp500_comparison.ipynb](#)



Key Formulas

Volatility Models

- **ARCH(q):** $\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2$
- **GARCH(1,1):** $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$
- **EGARCH:** $\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$
- **GJR-GARCH:** $\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \gamma \varepsilon_{t-1}^2 I_{t-1} + \beta \sigma_{t-1}^2$

Properties and Measures

- **Unconditional variance:** $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ **Half-life:** $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$
- **VaR:** $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$ **Stationarity:** $\alpha + \beta < 1$
- **ARCH-LM:** $LM = T \cdot R^2 \sim \chi^2(q)$



Summary — Chapter 5: Volatility Models

Key Concepts

- **ARCH(q)**: conditional variance depends on past squared errors
- **GARCH(p,q)**: adds variance lags for persistence
- **EGARCH/GJR-GARCH**: capture leverage effect (asymmetric response)

Applications

Risk measurement (VaR, ES), derivative pricing, portfolio management

Practical Tip

Start with GARCH(1,1), check for leverage, choose distribution minimizing AIC/BIC!



References

-  Engle, R.F. (1982). *Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation*. *Econometrica*, 50(4), 987-1007.
-  Bollerslev, T. (1986). *Generalized Autoregressive Conditional Heteroskedasticity*. *Journal of Econometrics*, 31(3), 307-327.
-  Nelson, D.B. (1991). *Conditional Heteroskedasticity in Asset Returns: A New Approach*. *Econometrica*, 59(2), 347-370.
-  Glosten, L.R., Jagannathan, R., & Runkle, D.E. (1993). *On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks*. *The Journal of Finance*, 48(5), 1779-1801.
-  Tsay, R.S. (2010). *Analysis of Financial Time Series*. 3rd Edition, Wiley.