Time Series Analysis and Forecasting

Chapter 9: Prophet and TBATS

Daniel Traian PELE

Bucharest University of Economic Studies
IDA Institute Digital Assets
Blockchain Research Center
AI4EFin Artificial Intelligence for Energy Finance
Romanian Academy, Institute for Economic Forecasting
MSCA Digital Finance

Learning Objectives

By the end of this chapter, you will be able to:

- Handle time series with multiple seasonal patterns
- Use Facebook Prophet for flexible forecasting with holidays
- Apply TBATS models for complex seasonality
- Compare and select between modern forecasting methods

Outline

The Problem: Complex Seasonal Patterns

Real-World Examples

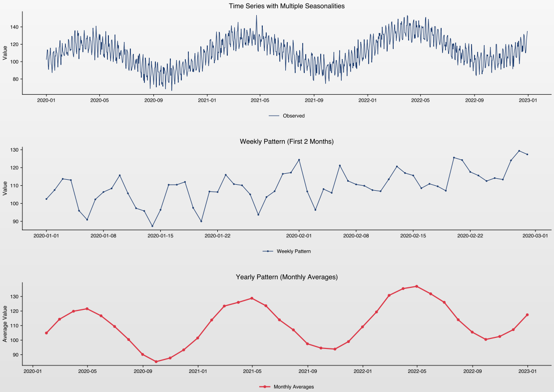
- **Hourly electricity demand**: Daily + Weekly + Annual patterns
- **Website traffic**: Daily + Weekly + Holiday effects
- **Retail sales**: Weekly + Monthly + Annual + Holiday effects
- **Call center volume**:
  - Hourly + Daily + Weekly patterns

SARIMA Limitation

Standard SARIMA$(p, d, q)(P, D, Q)_s$ handles only **one** seasonal period s.

For hourly data with daily AND weekly patterns, we need $s_1 = 24$ and $s_2 = 168$.

Example: Hourly Data with Multiple Seasonalities



TSA_ch9_multiple_seasonality

Solutions for Multiple Seasonalities

Traditional Approaches

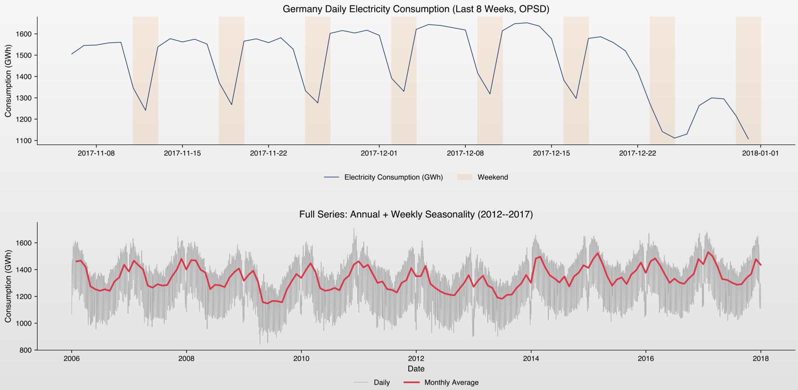
- **Fourier terms**: Add sin/cos regressors
- **Dummy variables**: Many parameters
- **Nested models**: Complex specification

Modern Approaches

- **TBATS**: Automatic, handles many periods
- **Prophet**: Flexible, interpretable
- **Neural methods**:
  - Deep learning

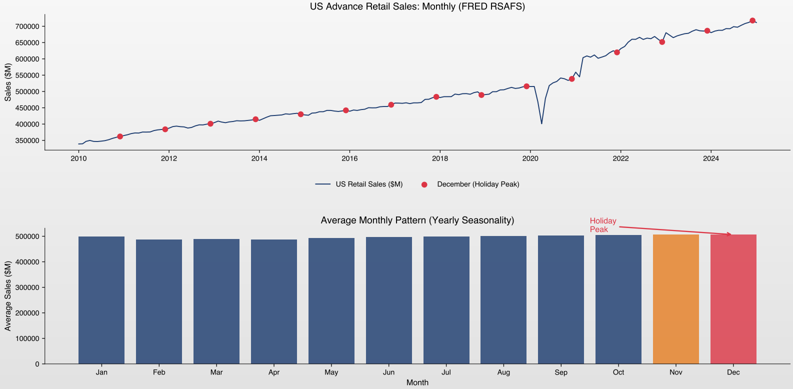
Method	Max Seasonalities	Interpretable
SARIMA	1	Yes
Fourier + ARIMA	Multiple	Moderate
TBATS	Multiple	Moderate
Prophet	Multiple	Yes

Real Example: Electricity Demand



TSA_ch9_electricity_demand

Real Example: Retail Sales with Holidays



TSA_ch9_retail_sales

Researcher Spotlight: Rob J. Hyndman

*1967

W Wikipedia

Biography

- Australian statistician, Professor at Monash University
- One of the most influential researchers in time series forecasting
- Creator of the widely-used `forecast` package for R
- Editor-in-Chief of the *International Journal of Forecasting* (2005–2018)

Key Contributions

- **TBATS model** (2011) — trigonometric Box-Cox ARMA with multiple seasonal periods
- **ETS framework** — exponential smoothing state space models with automatic selection
- **forecast package** for R — the standard toolkit for time series forecasting
- **Hierarchical forecasting** and forecast reconciliation methods

TBATS: What Does It Stand For?

TBATS Components

- T **Trigonometric** seasonality using Fourier terms
- B **Box-Cox** transformation for variance stabilization
- A **ARMA** errors for remaining autocorrelation
- T **Trend** component (possibly damped)
- S **Seasonal** components (multiple allowed)

Key Innovation: Trigonometric Seasonality

$$s_t^{(i)} = \sum_{j=1}^{k_i} \left[s_j^{(i)} \cos\left(\frac{2\pi jt}{m_i}\right) + s_j^{*(i)} \sin\left(\frac{2\pi jt}{m_i}\right) \right]$$

m_i = seasonal period, k_i = number of harmonics

Box-Cox Transformation

Definition 1 (Box-Cox Transformation)

The Box-Cox transformation with parameter ω is defined as:

$$y_t^{(\omega)} = \begin{cases} \dfrac{y_t^{\omega} - 1}{\omega} & \text{if } \omega \neq 0 \\ \ln(y_t) & \text{if } \omega = 0 \end{cases}$$

Purpose

- **Variance stabilization**: Makes variance constant over time
- **Normalization**: Reduces skewness in the data
- Common values: $\omega = 0$ (log), $\omega = 0.5$ (square root), $\omega = 1$ (no transform)

TBATS Model Structure

State Space Representation

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^{T} s_{t-m_i}^{(i)} + d_t \tag{1}$$

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t, \quad b_t = \phi b_{t-1} + \beta d_t \tag{2}$$

- $y_t^{(\omega)}$: Box-Cox transformed observation
- ℓ_t: local level (smoothed mean)
- b_t: trend with damping $\phi \in (0, 1)$
- $s_t^{(i)}$: i-th seasonal component
- d_t: ARMA(p, q) error process
- α, β: smoothing parameters

TBATS: Trigonometric Seasonality State Evolution

Definition 2 (Trigonometric State-Space Recursion)

For each seasonal component with period m_i and k_i harmonics, define states:

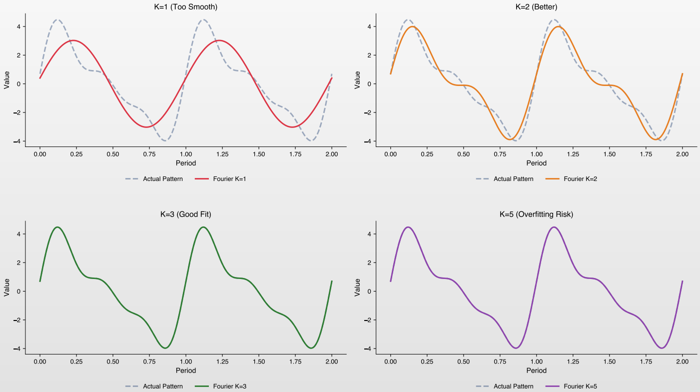
$$\begin{pmatrix} s_{j,t}^{(i)} \\ s_{j,t}^{*(i)} \end{pmatrix} = \begin{pmatrix} \cos(\lambda_j) & \sin(\lambda_j) \\ -\sin(\lambda_j) & \cos(\lambda_j) \end{pmatrix} \begin{pmatrix} s_{j,t-1}^{(i)} \\ s_{j,t-1}^{*(i)} \end{pmatrix} + \begin{pmatrix} \gamma_1^{(i)} \\ \gamma_2^{(i)} \end{pmatrix} d_t$$

where $\lambda_j = \frac{2\pi j}{m_i}$ is the j-th harmonic frequency.

Interpretation

- The rotation matrix preserves the periodic structure
- Total seasonal: $s_t^{(i)} = \sum_{j=1}^{k_i} s_{j,t}^{(i)}$
- Parameters: $2k_i$ states per seasonal period

Fourier Approximation of Seasonality



TSA_ch9_fourier_approximation

TBATS: Choosing the Number of Harmonics

Why Fourier/Trigonometric Terms?

1. **Parsimonious**: $2k$ parameters vs m dummy variables
2. **Smooth**: Captures smooth seasonal patterns naturally
3. **Flexible**: Number of harmonics k controls complexity
4. **Non-integer periods**: Can handle $s = 365.25$ for daily data

Low k (few harmonics)

- Smooth pattern
- Fewer parameters
- May miss sharp peaks

High k (many harmonics)

- Can capture any pattern
- More parameters ($2k$ total)
- Maximum useful: $k \leq \lfloor m/2 \rfloor$

TBATS: Key Features

Automatic Model Selection

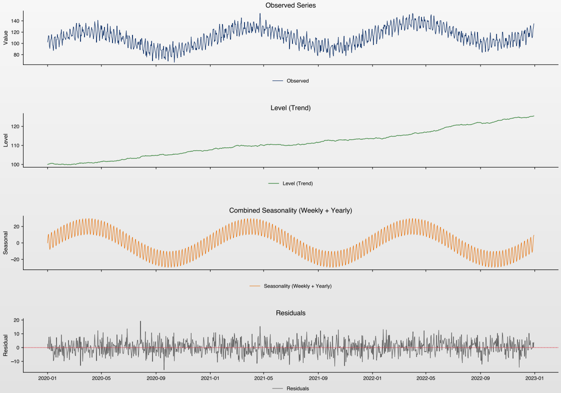
TBATS automatically determines:

- Box-Cox parameter ω for variance stabilization
- Number of harmonics k_i for each seasonal period
- ARMA orders (p, q) for residual autocorrelation
- Damped vs non-damped trend specification

BATS vs TBATS

- **BATS**: Traditional seasonal states (dummy variables)
- **TBATS**: Trigonometric (Fourier) seasonal representation
- TBATS more parsimonious for long seasonal periods

TBATS Decomposition Example



TSA_ch9_tbats_decomposition

TBATS: Advantages and Limitations

Advantages

- Handles **multiple** seasonal periods
- **Automatic** model selection
- Handles **non-integer** periods (365.25)
- **Box-Cox** for heteroskedasticity
- Good for **high-frequency** data

Limitations

- **Computationally intensive**
- No **external regressors**
- Less **interpretable** than Prophet
- Can be **slow** for very long series
- Requires **sufficient data** per season

Prophet: Overview

What is Prophet?

Forecasting procedure developed by Facebook (Meta) in 2017 for **business time series**:

- Strong seasonal effects (daily, weekly, yearly)
- Holiday effects and trend changes (changepoints)
- Handles missing data and outliers

Key Philosophy: "Analyst-in-the-loop"

Designed for analysts with domain knowledge but without time series expertise.

Prophet Model Structure

Decomposition Approach

Prophet uses an **additive decomposition**:

$$y(t) = g(t) + s(t) + h(t) + \varepsilon_t$$

$g(t)$: Trend

- Linear or logistic
- Automatic changepoints
- Growth saturation

$s(t)$: Seasonality

- Fourier series
- Multiple periods
- Custom seasonality

$h(t)$: Holidays

- Country holidays
- Custom events
- Window effects

Prophet: Seasonality Component

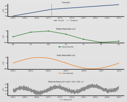
Fourier Series Representation

$$s(t) = \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2\pi nt}{P}\right) + b_n \sin\left(\frac{2\pi nt}{P}\right) \right]$$

Default Settings

Seasonality	Period	Fourier Order
Yearly	365.25 days	10
Weekly	7 days	3
Daily	1 day	4

Higher N = more flexibility but risk of overfitting



TSA_ch9_prophet_components

Prophet: Trend Component

Linear Trend with Changepoints

$$g(t) = (k + \mathbf{a}(t)^T \boldsymbol{\delta}) \cdot t + (m + \mathbf{a}(t)^T \gamma)$$

- k: base growth rate (slope)
- $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_S)$: slope changes at S changepoints
- $\mathbf{a}(t) \in \{0, 1\}^S$: indicator if changepoint s is active at time t

Logistic Growth

For saturating trends:

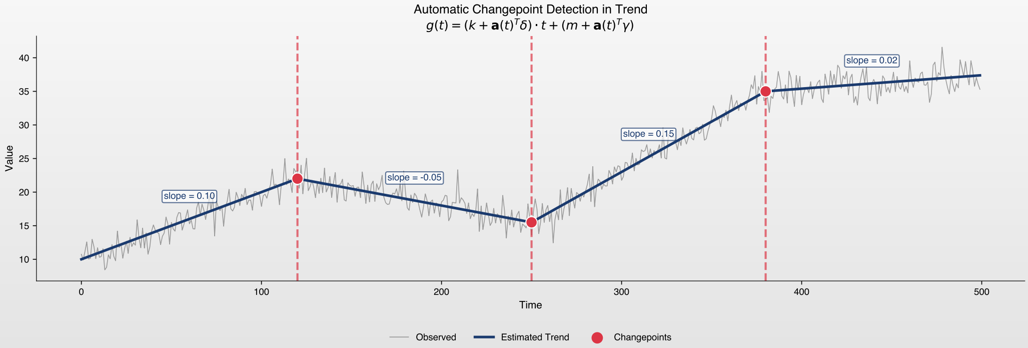
$$g(t) = \frac{C(t)}{1 + e^{-(k+\mathbf{a}(t)^T \boldsymbol{\delta})(t-m-\mathbf{a}(t)^T \gamma)}}$$

$C(t)$ = time-varying carrying capacity

Continuity Constraint

The offset $\gamma_j = -s_j \cdot \delta_j$ ensures $g(t)$ is continuous at each changepoint s_j.

Trend Changepoint Detection



TSA_ch9_changepoint_detection

Prophet: Holiday Effects

Holiday Model

$$h(t) = Z(t) \cdot \boldsymbol{\kappa}$$

where $Z(t)$ is an indicator matrix for holidays and $\boldsymbol{\kappa}$ are holiday effects.

Built-in Features

- 60+ countries supported
- Custom holiday definitions
- Window effects (before/after)

Holiday Types

- National holidays
- Religious observances
- Business events

Prophet: Customization Options

Seasonality Customization

- Add custom seasonal periods (monthly, quarterly)
- Control Fourier order for each seasonality
- Enable/disable default seasonalities

External Regressors

Prophet supports adding external variables:

- Weather data, promotions, special events
- Binary or continuous regressors
- Automatic regularization

Prophet: Uncertainty Quantification

Bayesian Framework

Prophet uses a **Laplace prior** on changepoint magnitudes:

$$\delta_j \sim \text{Laplace}(0, \tau), \quad \tau = \texttt{changepoint\_prior\_scale}$$

Smaller τ = sparser, smaller changepoints (more regularization).

Sources of Uncertainty

1. **Trend**: Future changepoints
2. **Seasonality**: Coefficient variance
3. **Observation**: Residual noise σ^2

Prediction Intervals

- MAP estimation for point forecasts
- Monte Carlo sampling for intervals
- Default: 80% credible interval

Prophet: Tuning Parameters

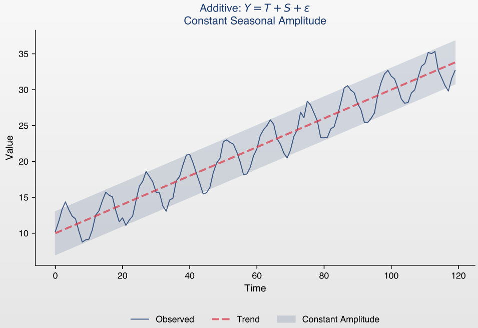
Key Parameters

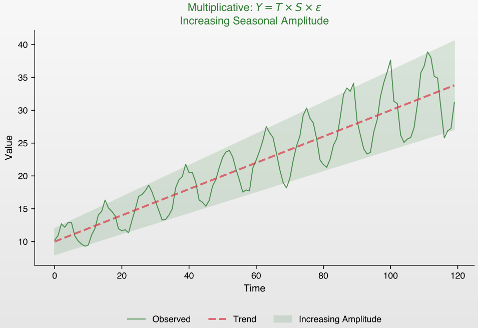
Parameter	Effect
`changepoint_prior_scale`	Trend flexibility (default: 0.05)
`seasonality_prior_scale`	Seasonality flexibility (default: 10)
`holidays_prior_scale`	Holiday effect size (default: 10)
`seasonality_mode`	'additive' or 'multiplicative'
`changepoint_range`	Portion of history for changepoints

Practical Tips

- **Overfitting trend?** Decrease `changepoint_prior_scale`
- **Underfitting seasonality?** Increase `seasonality_prior_scale`
- **Seasonal amplitude varies?** Use `seasonality_mode='multiplicative'`

Additive vs Multiplicative Seasonality





TSA_ch9_additive_vs_multiplicative

Prophet: Advantages and Limitations

Advantages

- **Easy to use**: Minimal tuning needed
- **Interpretable**: Clear decomposition
- **Handles missing data** well
- **Holiday effects** built-in
- **Multiple seasonalities**
- **External regressors** supported
- **Fast** fitting

Limitations

- **Not ARIMA-based**: No autocorrelation modeling
- **Daily data focus**: Less suited for very high frequency
- **Trend assumptions**: Linear/logistic may not fit
- **No built-in CV**: Must implement manually
- **Overfitting risk** with many seasonalities

TBATS vs Prophet: Head-to-Head

Feature	TBATS	Prophet
Multiple seasonalities	Yes (automatic)	Yes (manual or auto)
Holiday effects	No	Yes (built-in)
External regressors	No	Yes
Trend changepoints	No (smooth)	Yes (automatic)
Missing data	Interpolation needed	Handles natively
Interpretability	Moderate	High
Computation speed	Slow	Fast
High-frequency data	Good	Moderate
Non-integer periods	Yes (e.g., 365.25)	Yes
Uncertainty intervals	Yes	Yes

When to Use Each Model

Use TBATS when:

- High-frequency data
- Multiple seasonal periods
- No external regressors
- Automatic selection preferred

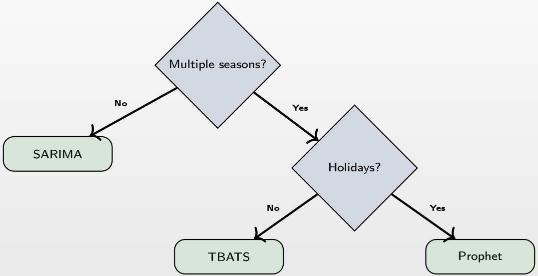
Use Prophet when:

- Business forecasting
- Holiday effects important
- Trend has changepoints
- External regressors available

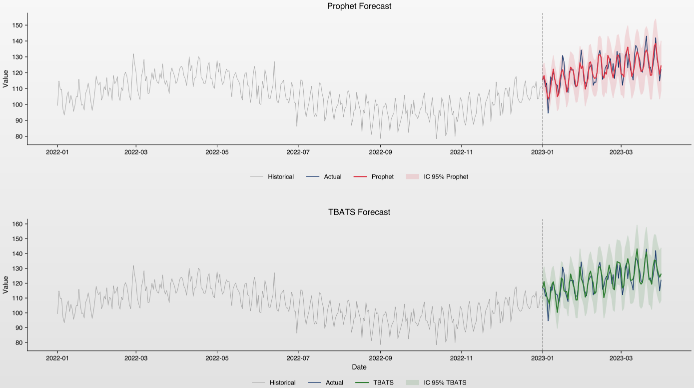
General Guideline

Prophet: business applications with daily data
TBATS: technical applications with high-frequency data

Decision Flowchart

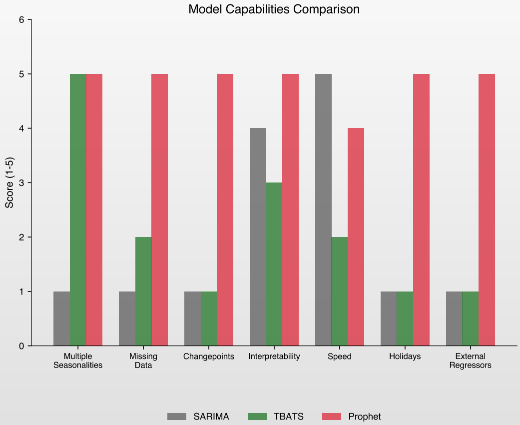


Prophet vs TBATS: Forecast Comparison



TSA_ch9_prophet_vs_tbats

Model Selection Guide



When to Use Each Model

SARIMA
- Single seasonality
- Regular data
- Statistical inference
- Short-term forecast

TBATS
- High frequency (hourly)
- Non-integer periods
- Automatic selection
- No external regressors

Prophet
- Business forecasting
- Holiday effects
- Missing data
- Changepoints trend
- External regressors

TSA_ch9_model_selection_guide

Evaluation Metrics

Definition 3 (Forecast Accuracy Metrics)

Let y_t denote actual values, $\hat{y}_t$ forecasts, and n the forecast horizon:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)^2} \quad \text{(penalizes large errors)}$$

$$\text{MAE} = \frac{1}{n}\sum_{t=1}^{n}|y_t - \hat{y}_t| \quad \text{(robust to outliers)}$$

$$\text{MAPE} = \frac{100\%}{n}\sum_{t=1}^{n}\left|\frac{y_t - \hat{y}_t}{y_t}\right| \quad \text{(scale-free)}$$

Coverage

For prediction intervals $[\hat{y}_t^L, \hat{y}_t^U]$, coverage rate is the proportion of actual values falling within the interval. Target: match the nominal level (e.g., 80%).

Case Study: Energy Demand Forecasting

Problem

Forecast hourly electricity demand with:

- **Daily pattern**: Peak at noon and evening
- **Weekly pattern**: Lower on weekends
- **Annual pattern**: Higher in summer (AC) and winter (heating)
- **Holiday effects**: Lower demand on holidays

Approach

1. Try TBATS with periods [24, 168, 8766]
2. Try Prophet with daily, weekly, yearly seasonality + holidays
3. Compare using cross-validation

Case Study: Results

Model Comparison

Model	MAPE	RMSE	Coverage
SARIMA (daily only)	8.5%	450 MW	75%
TBATS	4.2%	220 MW	82%
Prophet	4.8%	250 MW	85%
Prophet + holidays	3.9%	200 MW	88%

Key Finding

Multiple seasonality models significantly outperform single-seasonality SARIMA.

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download the Peyton Manning Wikipedia pageviews dataset from Prophet (or use daily electricity demand data for 2020-01-01 to 2024-12-31, approx. 1,800 observations). Use Facebook Prophet to forecast the next 30 days. Include US holidays and weekly/yearly seasonality components. Compare with TBATS. Give me complete Python code."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does Prophet automatically detect multiple seasonalities (daily, weekly)?
3. How are holidays specified? Country-specific or custom events?
4. Does it use cross-validation with cutoffs (performance_metrics)?
5. Would TBATS be more appropriate for this frequency? Why or why not?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Question 1

Question

- Why can't standard SARIMA$(p, d, q)(P, D, Q)_s$ model hourly electricity data with both daily and weekly patterns?

Answer Choices

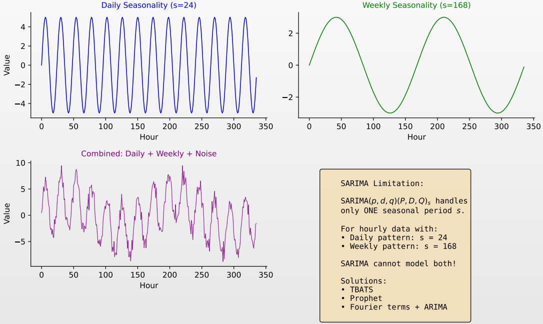
(A) SARIMA can only handle one seasonal period s at a time

(B) SARIMA requires normally distributed errors for multiple seasonalities

(C) SARIMA can handle multiple seasonalities but requires more data

(D) SARIMA only works with monthly or quarterly data

Question 1: Answer



Answer: (A)

- SARIMA handles only **one** seasonal period s. You cannot set $s = 24$ (daily) and $s = 168$ (weekly) simultaneously in a single SARIMA model.

TSA_ch9_quiz1_multiple_seasonality

Question 2

Question

- What does each letter in TBATS represent?

Answer Choices

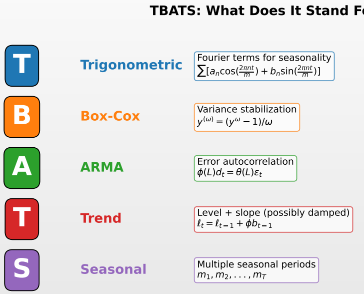
(A) Trend, Bayes, Autoregressive, Time, Stationarity

(B) Trigonometric seasonality, Box-Cox, ARMA errors, Trend, Seasonal components

(C) Taylor, Box-Cox, ARIMA, Transformation, Smoothing

(D) Trigonometric, Bayesian, ARMA, Trend, Spectral analysis

Question 2: Answer



Answer: (B)

- **T**rigonometric seasonality, **B**ox-Cox transformation, **A**RMA errors, **T**rend, **S**easonal components.

TSA_ch9_quiz2_tbats_components

Question 3

Question

- What happens when we increase the number of Fourier harmonics K?

Answer Choices

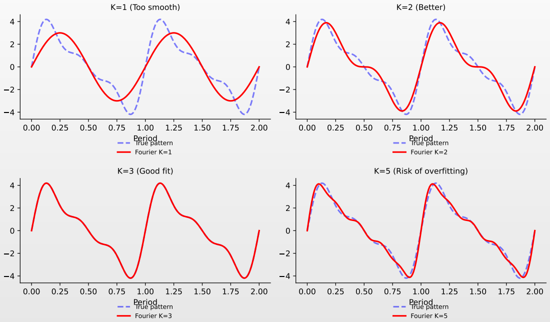
(A) The model becomes simpler and more robust

(B) The model captures more complex seasonal patterns but risks overfitting

(C) The forecast horizon increases proportionally

(D) The seasonal period s changes automatically

Question 3: Answer



Answer: (B)

- Higher K captures more complex seasonal patterns but increases the risk of overfitting. The maximum is $K \leq s/2$.

TSA_ch9_quiz3_fourier_harmonics

Question 4

Question

- What are the main components in Prophet's model $y(t) = g(t) + s(t) + h(t) + \varepsilon_t$?

Answer Choices

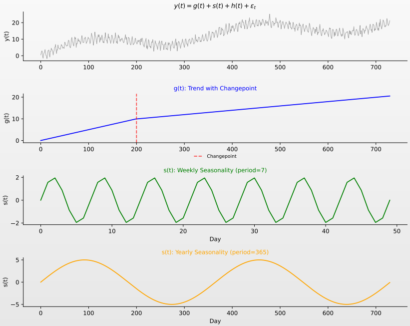
(A) $g(t)$ = GARCH volatility, $s(t)$ = stationarity test, $h(t)$ = heteroskedasticity

(B) $g(t)$ = growth (trend with changepoints), $s(t)$ = seasonality, $h(t)$ = holiday effects

(C) $g(t)$ = Gaussian noise, $s(t)$ = smoothing, $h(t)$ = harmonic terms

(D) $g(t)$ = gradient, $s(t)$ = spectral density, $h(t)$ = Hurst exponent

Question 4: Answer



Answer: (B)

- $g(t)$ = trend with changepoints, $s(t)$ = seasonality (Fourier terms), $h(t)$ = holiday effects, ε_t = error term.

TSA_ch9_quiz4_prophet_decomposition

Question 5

Question

- What key features does Prophet have that TBATS lacks?

Answer Choices

(A) Trigonometric seasonality and Box-Cox transformation

(B) Automatic parameter selection and exponential smoothing

(C) Holiday effects, external regressors, trend changepoints, and native missing data handling

(D) State-space formulation and ARMA error modeling

Question 5: Answer

TBATS vs Prophet: Head-to-Head Comparison

Feature	TBATS	Prophet
Multiple seasonalities	Yes (automatic)	Yes (manual/auto)
Holiday effects	No	Yes (built-in)
External regressors	No	Yes
Trend changepoints	No (smooth)	Yes (automatic)
Missing data	Needs interpolation	Handles natively
Interpretability	Moderate	High
Computation speed	Slow	Fast
High-frequency data	Good	Moderate
Non-integer periods	Yes (e.g., 365.25)	Yes
Best for	Technical/high-freq	Business/daily

Answer: (C)

- Prophet offers holiday effects, external regressors, trend changepoints, and native missing data handling—features not available in TBATS.

TSA_ch9_quiz5_prophet_vs_tbats

Key Takeaways

What We Learned

- TBATS handles multiple seasonalities with Fourier terms and Box-Cox transformation
- Prophet provides interpretable decomposition with trend changepoints and holiday effects
- Both methods scale better than SARIMA for high-frequency and complex seasonal data

Important

Choose Prophet for business forecasting with holidays and interpretability needs. Use TBATS for automatic modeling of high-frequency data. Always validate with time series cross-validation—never standard k-fold!

Questions?

Questions?

Next Steps:

- Practice with the Jupyter notebook
- Try Prophet on your own data
- Explore NeuralProphet for deep learning extension

Bibliography I

Prophet

- Taylor, S.J., & Letham, B. (2018). Forecasting at Scale, *The American Statistician*, 72(1), 37–45.
- Harvey, A.C. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge University Press.

TBATS and Exponential Smoothing

- De Livera, A.M., Hyndman, R.J., & Snyder, R.D. (2011). Forecasting Time Series with Complex Seasonal Patterns Using Exponential Smoothing, *JASA*, 106(496), 1513–1527.
- Hyndman, R.J., Koehler, A.B., Ord, J.K., & Snyder, R.D. (2008). *Forecasting with Exponential Smoothing: The State Space Approach*, Springer.
- Taylor, J.W. (2003). Short-term Electricity Demand Forecasting Using Double Seasonal Exponential Smoothing, *Journal of the Operational Research Society*, 54(8), 799–805.

Bibliography II

Forecasting Comparisons and Competitions

- Makridakis, S., Spiliotis, E., & Assimakopoulos, V. (2020). The M4 Competition, *International Journal of Forecasting*, 36(1), 54–74.
- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.
- Petropoulos, F., et al. (2022). Forecasting: Theory and Practice, *International Journal of Forecasting*, 38(3), 845–1054.

Online Resources and Code

- **Quantlet**: `https://quantlet.com` ≻ Code repository for statistics
- **Quantinar**: `https://quantinar.com` ≻ Learning platform for quantitative methods
- **GitHub TSA**: `https://github.com/QuantLet/TSA/tree/main/TSA_ch9` ≻ Python code for this chapter

Thank You!

Questions?

Course materials available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar