



# Chapter 3: ARIMA Models

Non-Stationary Time Series



# Outline

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## The Problem

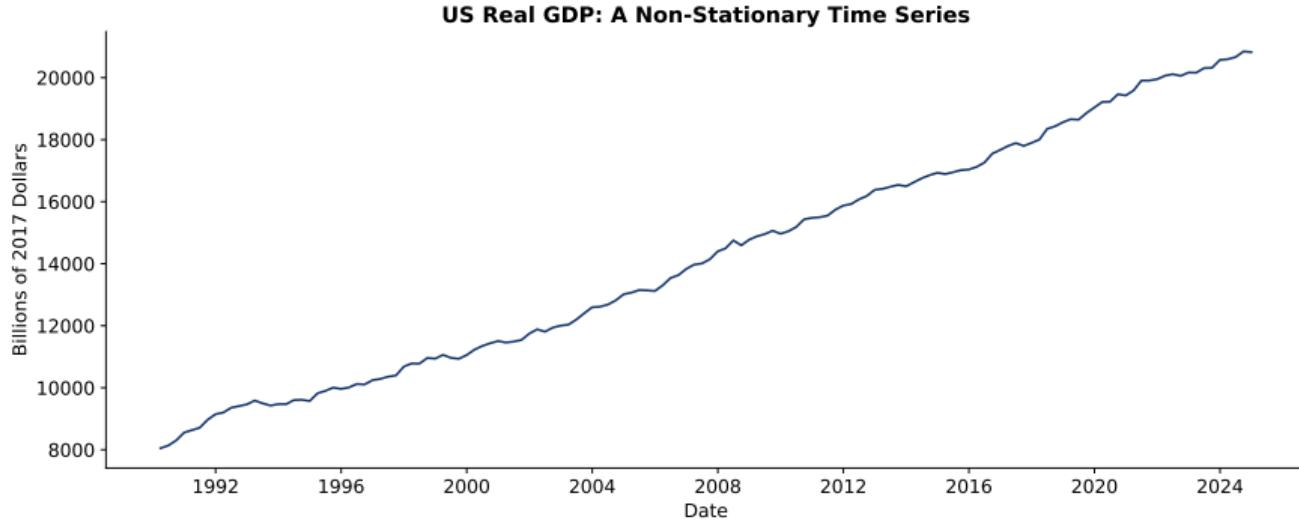
Many economic and financial time series are **non-stationary**:

- GDP, stock prices, exchange rates, inflation indices
- They exhibit trends, changing means, or growing variance

## Consequences of Non-Stationarity

- Standard ARMA models assume stationarity
- OLS regression with non-stationary data leads to **spurious regression**
- Sample moments (mean, variance, ACF) are not consistent estimators
- Statistical inference becomes invalid

## Example: US Real GDP



- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data

# Types of Non-Stationarity

## Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a deterministic function of time
- Can be removed by **detrending**
- Shocks have temporary effects

## Stochastic Trend (Unit Root)

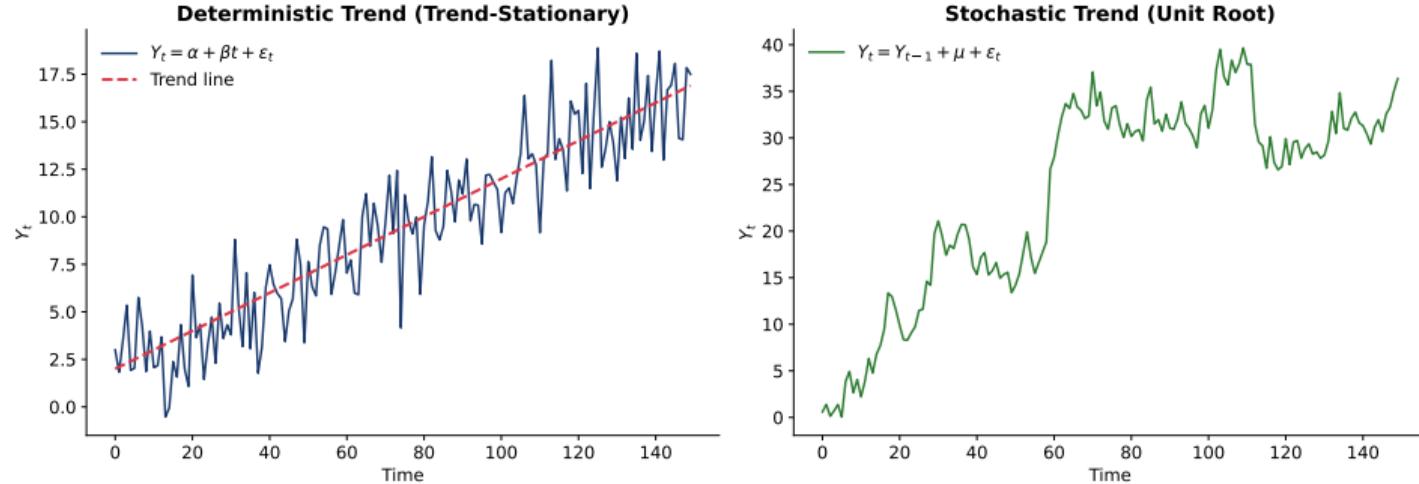
$$Y_t = Y_{t-1} + \varepsilon_t$$

- Random walk process
- Must be removed by **differencing**
- Shocks have permanent effects

## Key Distinction

Correct identification is crucial: detrending a unit root process or differencing a trend-stationary process both lead to misspecification!

# Visualizing the Difference



- **Left:** Deterministic trend – deviations from trend are temporary
- **Right:** Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!

# The Random Walk Process

## Definition 1 (Random Walk)

A **random walk** is defined as:

$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

With initial condition  $Y_0 = 0$ , we have:  $Y_t = \sum_{i=1}^t \varepsilon_i$

## Properties of Random Walk

- $\mathbb{E}[Y_t] = 0$  (constant mean)
- $\text{Var}(Y_t) = t\sigma^2$  (variance grows with time!)
- $\text{Cov}(Y_t, Y_{t-k}) = (t - k)\sigma^2$  for  $k \leq t$
- ACF:  $\rho_k = \sqrt{\frac{t-k}{t}} \rightarrow 1$  as  $t \rightarrow \infty$

## Random Walk with Drift

### Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

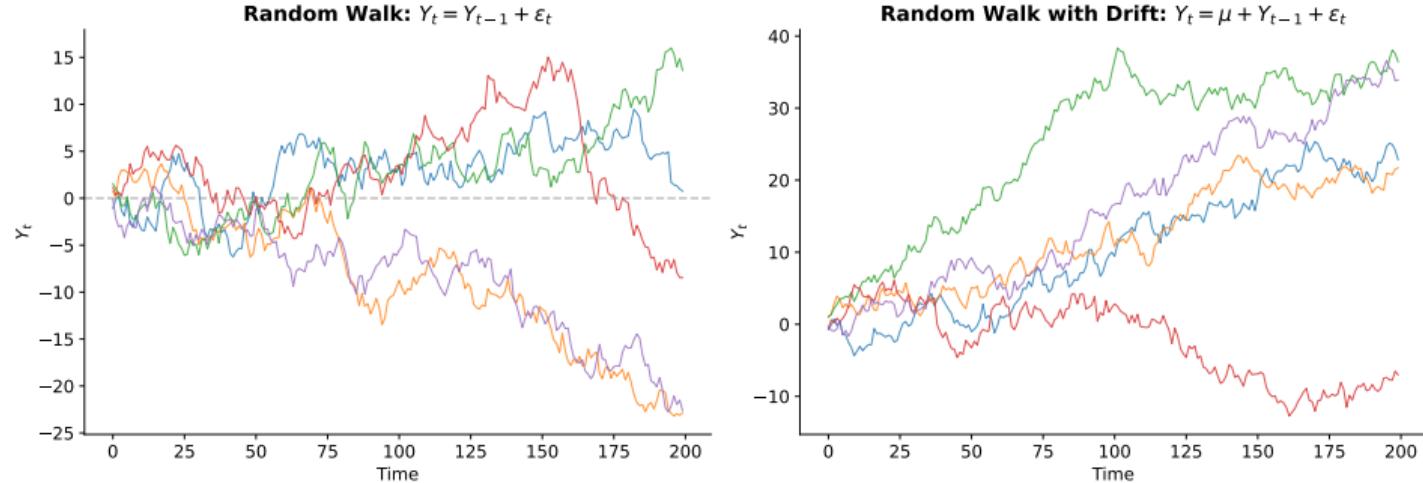
$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

Equivalently:  $Y_t = Y_0 + \mu t + \sum_{i=1}^t \varepsilon_i$

### Properties

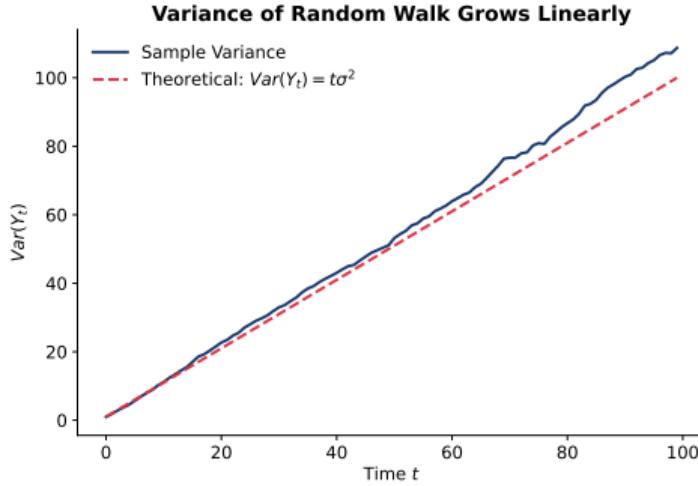
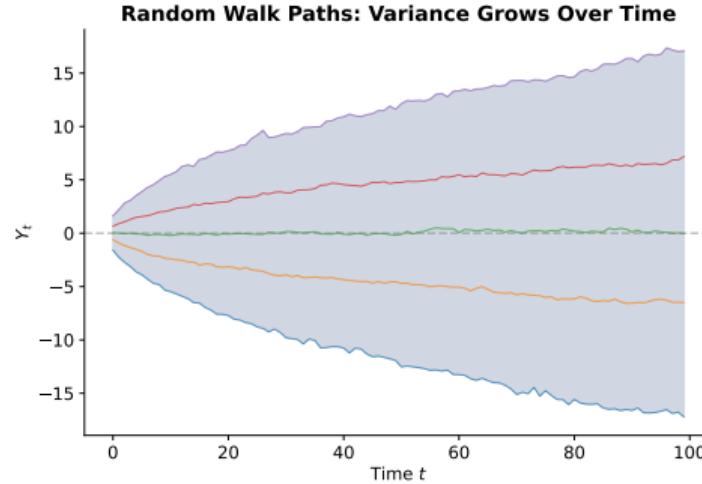
- $\mathbb{E}[Y_t] = Y_0 + \mu t$  (mean grows linearly)
- $\text{Var}(Y_t) = t\sigma^2$  (variance still grows)
- The drift  $\mu$  creates an upward or downward trend
- Still non-stationary despite having a “trend”

# Simulating Random Walks



- **Left:** Pure random walks – no drift, wander unpredictably
- **Right:** Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time

# Variance Growth: Why Random Walks Are Non-Stationary



- Left: Fan of paths shows uncertainty growing over time
- Right: Variance grows linearly:  $\text{Var}(Y_t) = t\sigma^2$
- This violates stationarity (variance should be constant)

## Definition 3 (Integrated Process of Order $d$ )

A time series  $\{Y_t\}$  is **integrated of order  $d$** , written  $Y_t \sim I(d)$ , if:

- $Y_t$  is non-stationary
- $(1 - L)^d Y_t = \Delta^d Y_t$  is stationary
- $(1 - L)^{d-1} Y_t$  is still non-stationary

## Common Cases

- $I(0)$ : Stationary process (e.g., ARMA)
- $I(1)$ : First difference is stationary (most common for economic data)
- $I(2)$ : Second difference is stationary (less common)

# The Difference Operator

## Definition 4 (First Difference)

The **first difference operator**  $\Delta$  is defined as:  $\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$ , where  $L$  is the lag operator ( $LY_t = Y_{t-1}$ ).

## Higher-Order Differences

- Second difference:  $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- $d$ -th difference:  $\Delta^d Y_t = (1 - L)^d Y_t$

## Key Result

If  $Y_t \sim I(d)$ , then  $\Delta^d Y_t \sim I(0)$  (stationary).

## Example: Differencing a Random Walk

### Random Walk to White Noise

Let  $Y_t = Y_{t-1} + \varepsilon_t$  (random walk). Taking the first difference:

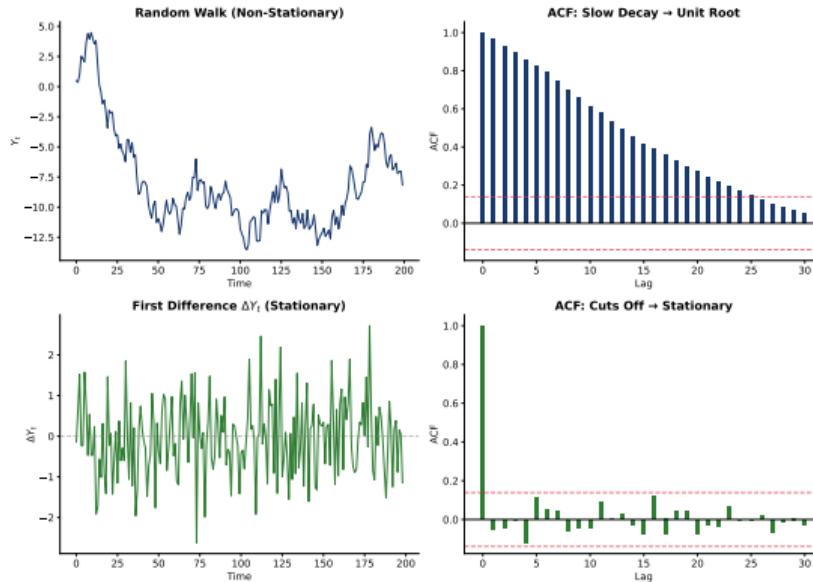
$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

### Interpretation

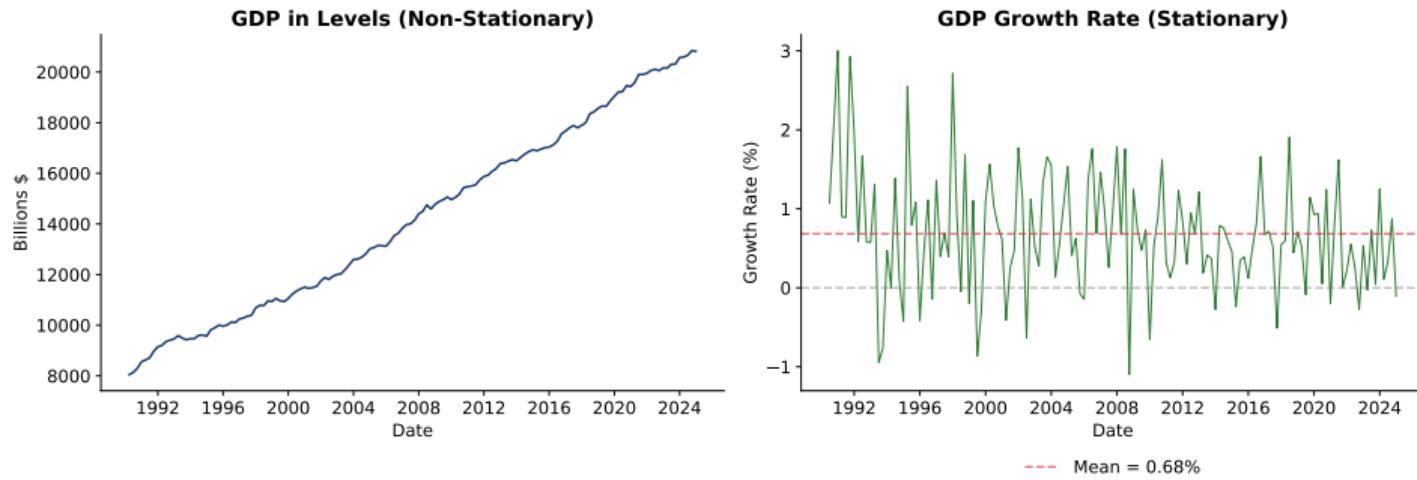
- A random walk is  $I(1)$
- One difference transforms it to  $I(0)$
- The “changes” in a random walk are stationary

# ACF Diagnostic: Detecting Non-Stationarity



- **Top:** Random walk ACF decays very slowly  $\Rightarrow$  unit root
- **Bottom:** After differencing, ACF cuts off  $\Rightarrow$  stationary

## Differencing in Practice: GDP Example



- **Left:** GDP in levels – clear upward trend (non-stationary)
- **Right:** GDP growth rate (log difference) – fluctuates around mean (stationary)
- Differencing removes the trend and achieves stationarity

# Overdifferencing

## Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
- Inflates variance
- Loses information

## Example

If  $Y_t \sim I(1)$ , then  $\Delta Y_t \sim I(0)$ . But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with  $\theta = 1$  (non-invertible boundary)!

## Definition of ARIMA

### Definition 5 (ARIMA(p,d,q))

A time series  $\{Y_t\}$  follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1 - L)^d Y_t = c + \theta(L)\varepsilon_t$$

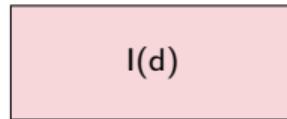
where:

- $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$  (AR polynomial)
- $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$  (MA polynomial)
- $d$  is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

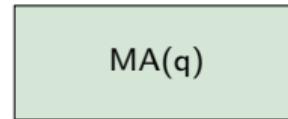
# ARIMA Components



AR(p)



I(d)



MA(q)

Autoregressive  
Memory

Integration  
Differencing

Moving Average  
Shocks

## Special Cases

- ARIMA( $p, 0, q$ ) = ARMA( $p, q$ ) – stationary
- ARIMA( $0, 1, 0$ ) = Random walk
- ARIMA( $0, 1, 1$ ) = IMA( $1, 1$ ) – exponential smoothing
- ARIMA( $1, 1, 0$ ) = ARI( $1, 1$ ) – differenced AR(1)

# ARIMA(1,1,0) Example

## ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently:  $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$

## Interpretation

- The **changes** in  $Y_t$  follow an AR(1) process
- If  $|\phi_1| < 1$ , the changes are stationary
- $Y_t$  itself has a stochastic trend
- Common model for many economic time series

# ARIMA(0,1,1) Example

## IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently:  $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

## Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha) \hat{Y}_t$$

where  $\alpha = 1 + \theta_1$  (for  $-1 < \theta_1 < 0$ ).

## Constant Term in ARIMA(p,d,q)

When  $d > 0$ , the constant  $c$  has a different interpretation:  $\phi(L)(1 - L)^d Y_t = c + \theta(L)\varepsilon_t$

## Important Implications

- For  $d = 1$ :  $c$  represents the **drift** (average change):  $\mathbb{E}[\Delta Y_t] = \frac{c}{1 - \phi_1 - \dots - \phi_p}$
- For  $d = 2$ :  $c$  affects the **curvature** of the trend
- Often  $c = 0$  is assumed when  $d \geq 1$

# Testing for Unit Roots

## Why Test?

Before fitting an ARIMA model, we need to determine:

- ① Is the series stationary? (Is  $d = 0$ ?)
- ② If not, how many differences are needed? (What is  $d$ ?)

## Common Unit Root Tests

- **Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- **Phillips-Perron (PP)**
- **KPSS** (stationarity test – reversed null hypothesis)

# The Dickey-Fuller Test

## Setup

Consider the AR(1) model:  $Y_t = \phi Y_{t-1} + \varepsilon_t$ . Subtract  $Y_{t-1}$ :  $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$ , where  $\gamma = \phi - 1$ .

## Hypotheses

- $H_0: \gamma = 0$  (unit root,  $\phi = 1$ , non-stationary)
- $H_1: \gamma < 0$  (stationary,  $|\phi| < 1$ )

## Key Issue

Under  $H_0$ , the  $t$ -statistic does **not** follow a standard  $t$ -distribution! Must use Dickey-Fuller critical values.

## Three Specifications

- ① **No constant, no trend:**  $\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$
- ② **With constant (drift):**  $\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$
- ③ **With constant and trend:**  $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$

## Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference

# Augmented Dickey-Fuller (ADF) Test

## The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

## Definition 6 (ADF Test)

Add lagged differences:  $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^k \delta_j \Delta Y_{t-j} + \varepsilon_t$

Test  $H_0 : \gamma = 0$  using ADF critical values.

## Choosing Lag Length $k$

- Use information criteria (AIC, BIC)
- Start with  $k_{max}$ , reduce until last lag significant

## ADF Test Critical Values

Model	1%	5%	10%
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

### Decision Rule

- Test statistic  $<$  critical value  $\Rightarrow$  Reject  $H_0$  (stationary)
- Test statistic  $\geq$  critical value  $\Rightarrow$  Fail to reject (unit root)

# The KPSS Test

## Reversed Hypotheses

Unlike ADF:  $H_0$ : Stationary vs  $H_1$ : Unit root

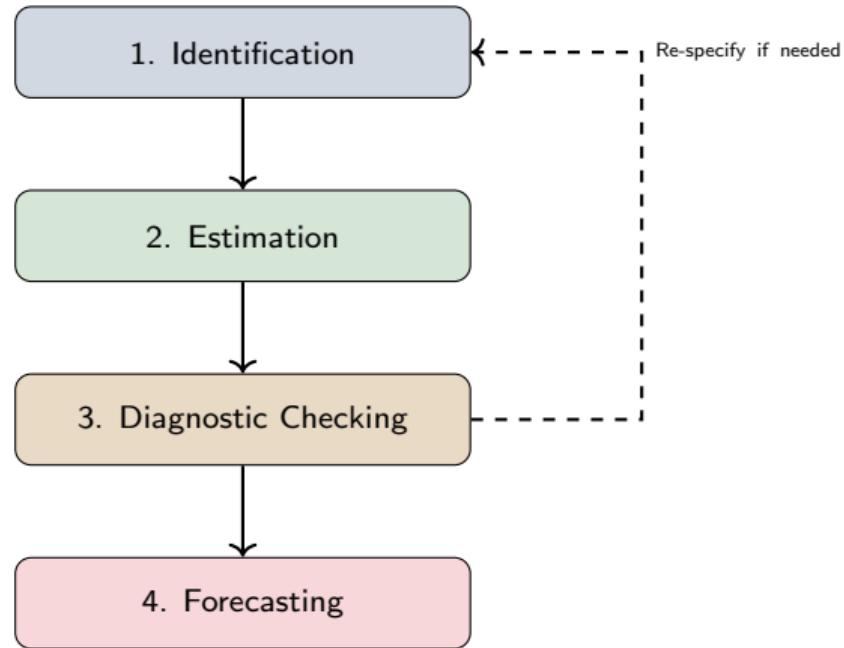
## KPSS Procedure

Decompose:  $Y_t = \xi t + r_t + \varepsilon_t$  where  $r_t = r_{t-1} + u_t$ . Test whether  $\text{Var}(u_t) = 0$ .

## Complementary Use with ADF

- ADF rejects, KPSS doesn't  $\Rightarrow$  Stationary
- ADF doesn't reject, KPSS rejects  $\Rightarrow$  Unit root
- Both reject or neither  $\Rightarrow$  Inconclusive

# The Box-Jenkins Methodology



## Step 1: Determining $d$

### Procedure

- ① Plot the time series – look for trends, changing variance
- ② Examine ACF – slow decay suggests non-stationarity
- ③ Apply unit root tests (ADF, KPSS)
- ④ If non-stationary, difference and repeat

### Practical Guidelines

- Most economic series:  $d = 1$  is sufficient
- Rarely need  $d > 2$
- If ACF of  $\Delta Y_t$  still decays slowly, try  $d = 2$
- Watch for overdifferencing (ACF with  $\rho_1 \approx -0.5$ )

## Step 2: Determining $p$ and $q$

### After Differencing

Once  $W_t = \Delta^d Y_t$  is stationary, use ACF/PACF to identify ARMA( $p, q$ ):

Model	ACF	PACF
AR( $p$ )	Decays exponentially	Cuts off after lag $p$
MA( $q$ )	Cuts off after lag $q$	Decays exponentially
ARMA( $p, q$ )	Decays	Decays

### Information Criteria

When patterns are unclear, compare models using:

- $AIC = -2 \ln(L) + 2k$ ;    $BIC = -2 \ln(L) + k \ln(n)$

Lower is better. BIC penalizes complexity more.

# Auto-ARIMA Algorithms

## Automated Model Selection

Modern software can automatically select  $(p, d, q)$ :

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

## How Auto-ARIMA Works

- ① Use unit root tests to determine  $d$
- ② Fit models for various  $(p, q)$  combinations
- ③ Select model with lowest AIC/BIC
- ④ Optionally use stepwise search for efficiency

## Caution

Automated selection is helpful but not infallible. Always check diagnostics!

## Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- Assumes  $\varepsilon_t \sim N(0, \sigma^2)$
- Maximizes the likelihood function
- Provides consistent, efficient estimators
- Yields standard errors for inference

## Conditional vs Exact MLE

- **Conditional MLE:** Conditions on initial values
- **Exact MLE:** Treats initial values as unknown
- Difference diminishes as sample size grows

## Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- **AR stationarity:** Roots of  $\phi(z) = 0$  outside unit circle
- **MA invertibility:** Roots of  $\theta(z) = 0$  outside unit circle

## Checking in Practice

Most software reports:

- Estimated coefficients with standard errors
- Roots of AR and MA polynomials
- Warning if near-unit-root detected

# Residual Analysis

## What to Check

If the model is correct, residuals  $\hat{\varepsilon}_t$  should be white noise:

- ① Zero mean
- ② Constant variance
- ③ No autocorrelation
- ④ (Optional) Normality

## Diagnostic Tools

- **Residual ACF/PACF:** Should show no significant spikes
- **Ljung-Box test:** Tests for autocorrelation at multiple lags
- **Q-Q plot:** Checks normality assumption
- **Residual vs fitted:** Checks for heteroskedasticity

# The Ljung-Box Test

## Definition 7 (Ljung-Box Q Statistic)

$$Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}. \text{ Under } H_0 \text{ (no autocorrelation): } Q(m) \sim \chi^2(m-p-q)$$

## Usage

- Choose  $m \approx \ln(n)$  or  $m = 10$  for quarterly,  $m = 20$  for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if  $Q(m)$  exceeds critical value

## If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

## Point Forecasts

### Minimum MSE Forecast

The optimal  $h$ -step ahead forecast is the conditional expectation:  $\hat{Y}_{T+h|\tau} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \dots]$

### ARIMA(1,1,1) Forecasting

Model:  $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

One-step forecast:  $\hat{Y}_{T+1|\tau} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1 \hat{\varepsilon}_T$

For  $h > 1$ : replace unknown  $\varepsilon_{T+j}$  with 0, unknown  $Y_{T+j}$  with  $\hat{Y}_{T+j|\tau}$

## Forecast Intervals

### Forecast Uncertainty

The  $h$ -step forecast error variance:  $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$ , where  $\psi_j$  are MA( $\infty$ ) coefficients.

### Confidence Intervals

Under normality,  $(1 - \alpha)\%$  interval:  $\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$

### Key Property for I(1) Series

For integrated processes, forecast variance grows without bound as  $h \rightarrow \infty$ . Intervals widen over time!

## Behavior as $h \rightarrow \infty$

For ARIMA(p,1,q) with drift  $c$ :

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with  $\sqrt{h}$

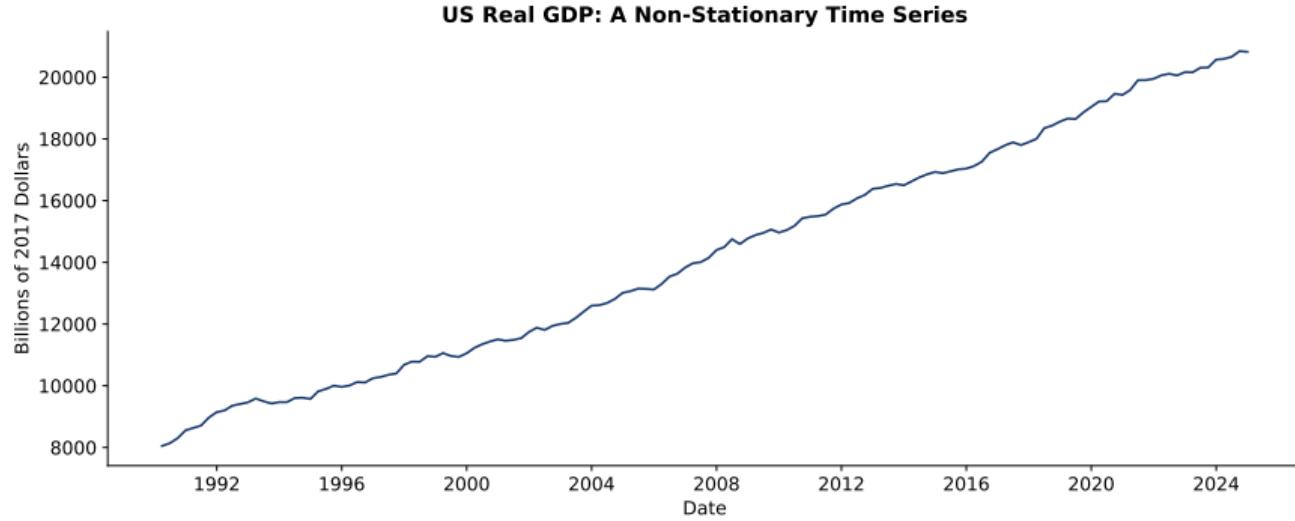
For ARIMA(p,1,q) without drift:

- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

## Practical Implication

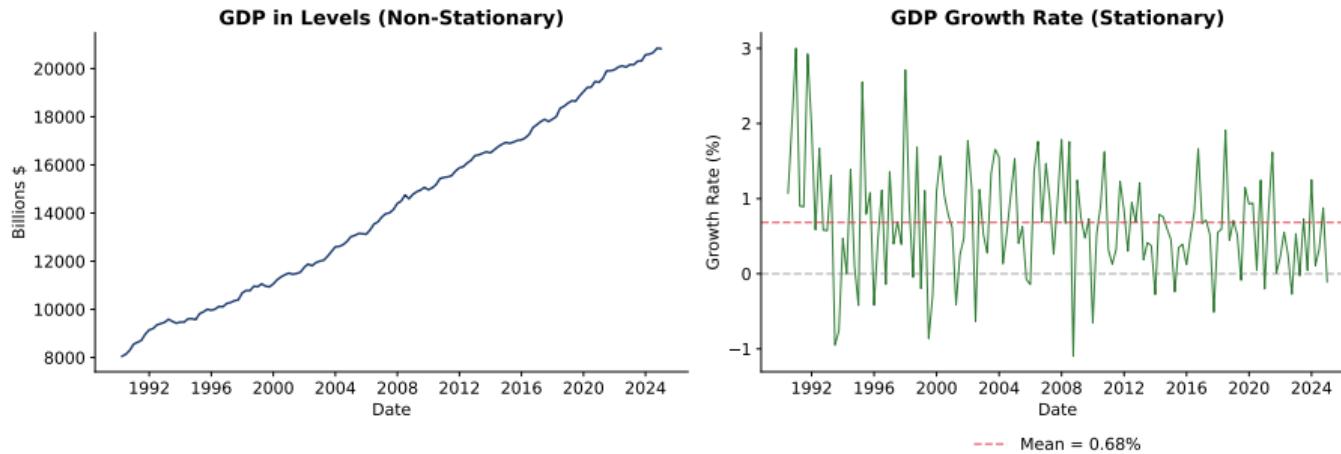
ARIMA forecasts are most reliable for short horizons. Long-term forecasts have very wide uncertainty bands.

## US Real GDP: A Non-Stationary Series



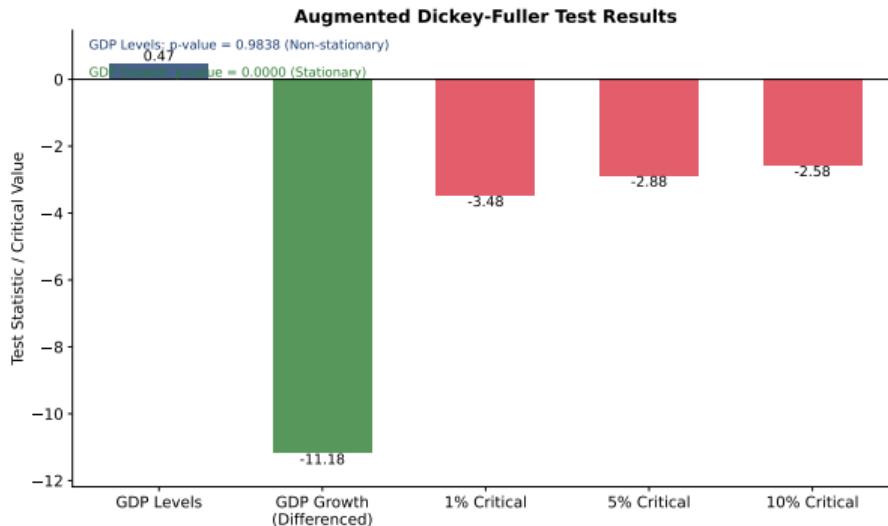
- Clear upward trend – non-stationary in levels
- Needs differencing before ARMA modeling

## Effect of Differencing



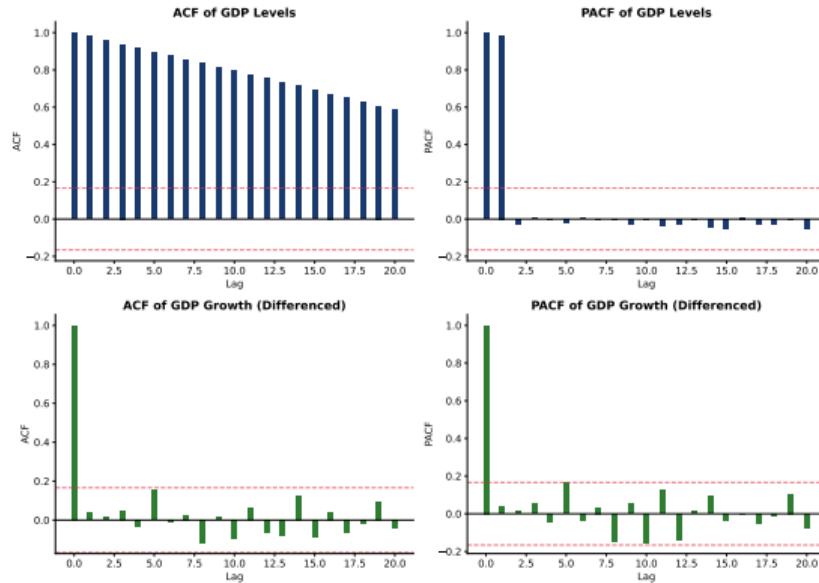
- **Left:** GDP in levels – non-stationary (clear trend)
- **Right:** GDP growth rate – stationary (fluctuates around mean)

# Unit Root Test Results



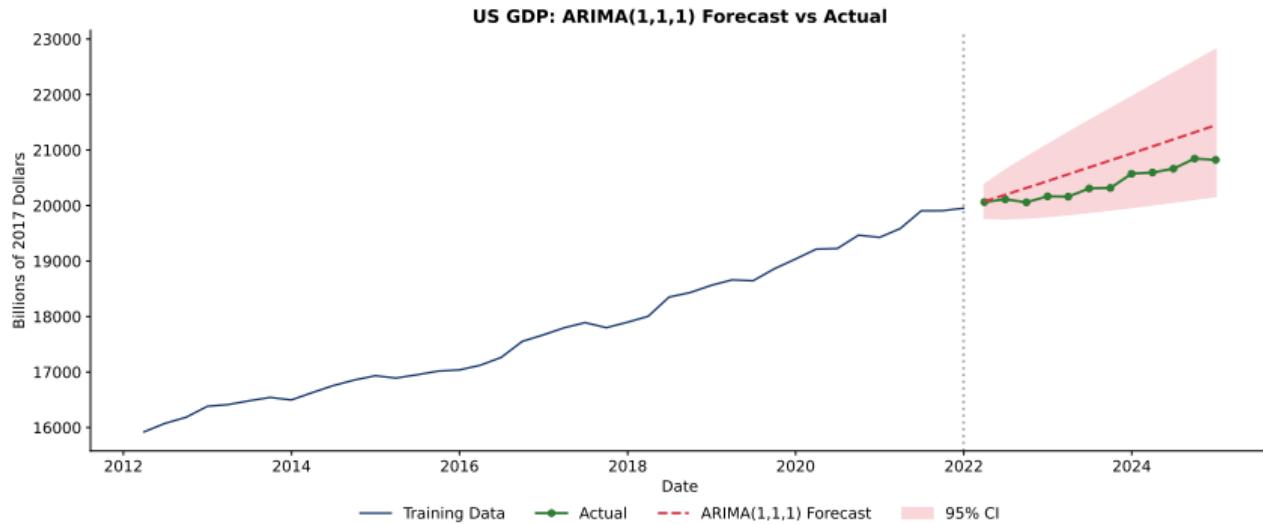
- GDP in levels: Cannot reject unit root (non-stationary)
- GDP growth: Reject unit root at 1% level (stationary)

# ACF/PACF: Levels vs Differenced



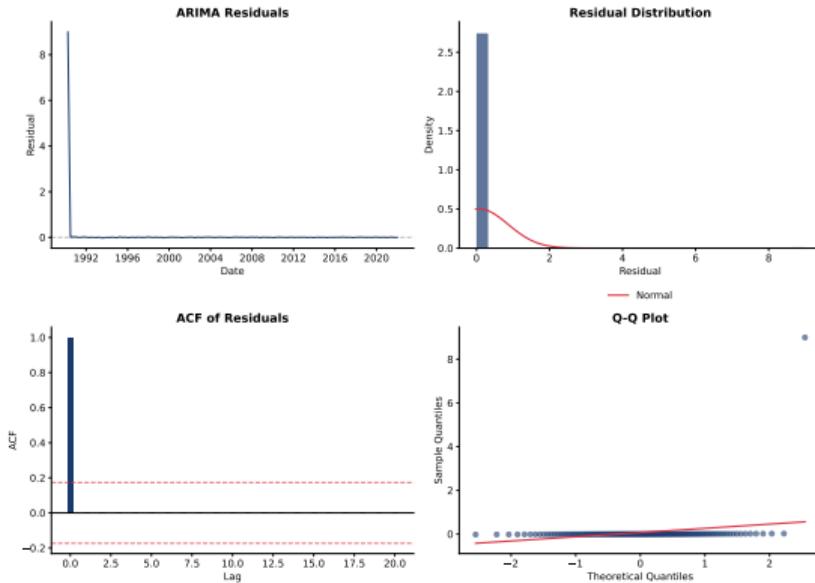
- **Top:** Slow ACF decay in levels suggests non-stationarity
- **Bottom:** After differencing, ACF/PACF help identify  $p$  and  $q$

## ARIMA Forecasting: Actual vs Predicted



- ARIMA(1,1,1) captures the trend dynamics
- Confidence intervals widen with forecast horizon

# Model Diagnostics



- Residuals appear random; ACF within bounds
- Q-Q plot shows approximate normality

## Auto-ARIMA Example

```
# Automatic model selection
model = pm.auto_arima(y, start_p=0, start_q=0,
                      max_p=3, max_q=3, d=None,
                      seasonal=False, trace=True)
print(model.summary())
```

## Key Takeaways

### Main Points

- ① Non-stationarity is common in economic data – must be addressed
- ② Differencing transforms  $I(d)$  to  $I(0)$
- ③ ARIMA( $p,d,q$ ) combines differencing with ARMA modeling
- ④ Unit root tests (ADF, KPSS) help determine  $d$
- ⑤ Box-Jenkins methodology: Identify → Estimate → Diagnose
- ⑥ Forecasts for  $I(1)$  series have growing uncertainty

### Next Steps

Chapter 4 will extend ARIMA to handle seasonality: SARIMA models.

## References

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