Time Series Analysis and Forecasting

Chapter 0: Fundamentals

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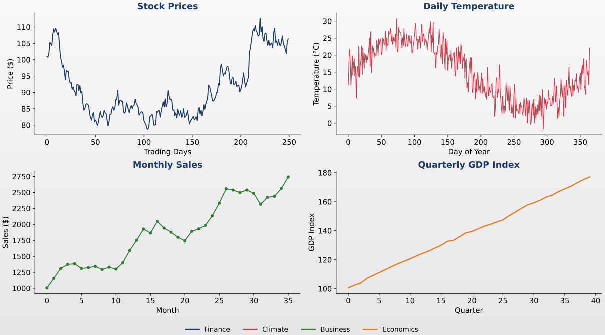
Learning Objectives

By the end of this chapter, you will be able to:

1. **Define** time series and distinguish from cross-sectional and panel data
2. **Decompose** time series into trend-cycle, seasonal, and residual components
3. **Apply** exponential smoothing methods (SES, Holt, Holt-Winters, ETS)
4. **Evaluate** forecasts using MAE, RMSE, MAPE, sMAPE
5. **Implement** train/validation/test splits and cross-validation
6. **Model** seasonality using dummy variables or Fourier terms
7. **Remove** trend and seasonality through appropriate methods
8. **Distinguish** between deterministic and stochastic trends

Chapter Outline

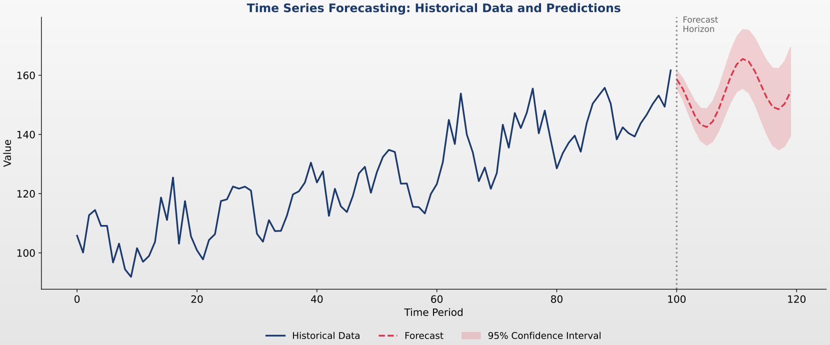
Time Series Are Everywhere



- **Finance**: Stock prices, exchange rates, trading volumes
- **Economics**: GDP, unemployment, inflation rates
- **Business**: Sales, website traffic, customer demand
- **Science**: Temperature, pollution levels, patient vitals

TSA_ch1_motivation

Why Study Time Series?

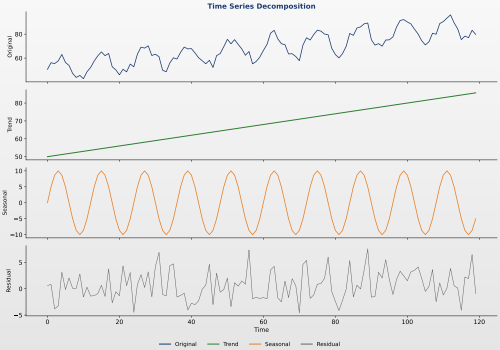


Key Goal: Forecasting

Use historical patterns to predict future values — critical for business planning, risk management, and policy decisions.

TSA_ch1_forecast

Understanding Time Series Structure



Decomposition

Every time series can be decomposed into interpretable components: trend-cycle, seasonality, and noise.

TSA_ch1_components

Definition of a Time Series

Definition 1 (Time Series)

A **time series** is a sequence of observations $\{X_t\}$ indexed by time:

$$\{X_t : t \in \mathcal{T}\}$$

where $\mathcal{T}$ is an index set representing time points.

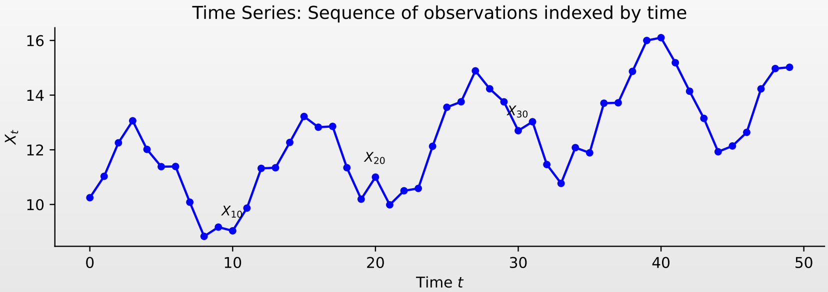
Key Characteristics

- **Ordered**: Natural temporal ordering
- **Dependent**: Consecutive observations correlated
- **Discrete/Continuous**: $t = 1, 2, 3, \ldots$

Notation

- X_t = observation at time t
- $\{X_t\}_{t=1}^{T}$ = series with T observations

Time Series: Visual Illustration

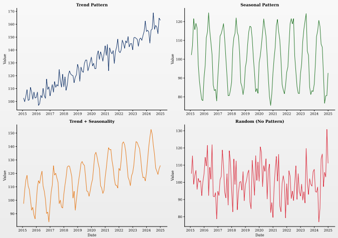


Interpretation

Each point X_t represents an observation at time t. The sequence is ordered and consecutive observations are typically correlated.

TSA_ch1_def_timeseries

Common Time Series Patterns

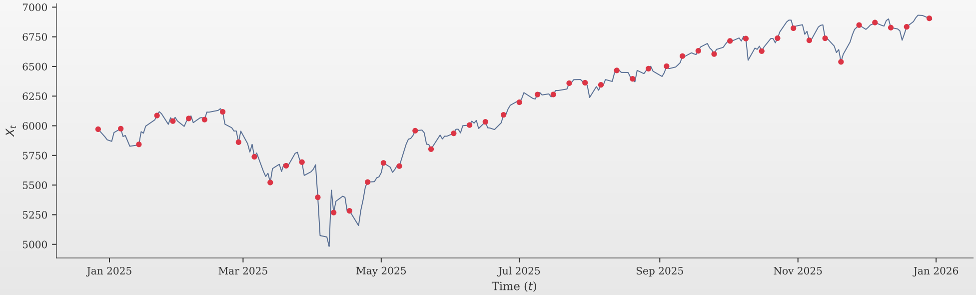


Pattern Types

- **Trend**: Long-term increase or decrease in the data
- **Seasonal**: Regular periodic patterns (e.g., monthly, quarterly)
- **Cyclical**: Medium-term fluctuations (business cycles, 2–10 years)
- **Random**: No systematic pattern – unpredictable fluctuations

TSA_ch1_patterns

Time Series: Visual Definition

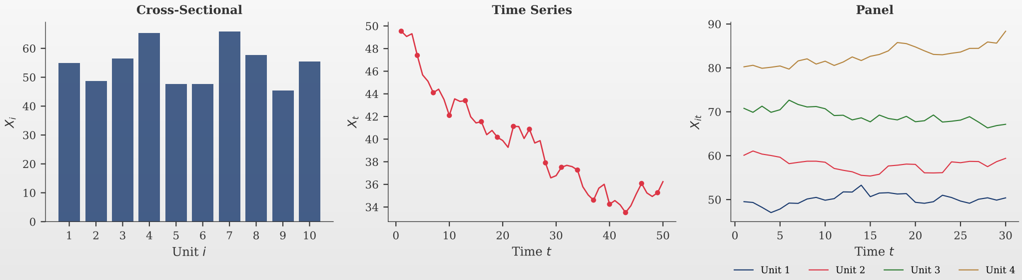


Interpretation

Each point X_t represents a measurement at discrete time t. The temporal ordering creates dependence between observations. Data: S&P 500 (2024).

TSA_ch1_definition

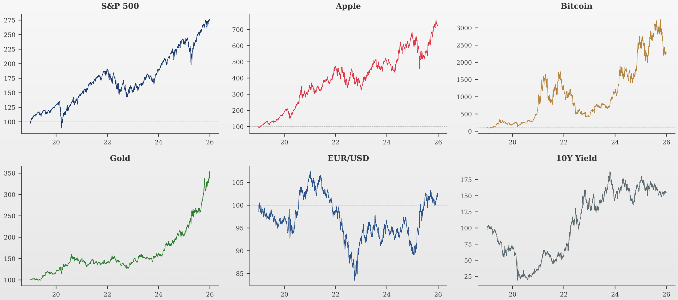
Types of Data: Comparison



Data Type	Units (N)	Time (T)	Example
Cross-sectional	Many	1	Survey of 1000 households
Time series	1	Many	Daily S&P 500 prices
Panel	Many	Many	GDP of 50 countries, 20 years

TSA_ch1_data_types

Examples of Time Series Data



Real Financial Data

Yahoo Finance (2019–2025), normalized to base 100. Notice different volatility patterns: Bitcoin most volatile, Gold most stable.

TSA_ch1_examples

Why Decompose a Time Series?

Decomposition separates a time series into interpretable components:

Goals:

- Understand underlying patterns
- Remove seasonality for modeling
- Identify trend direction
- Isolate irregular fluctuations
- Improve forecasting accuracy

Components:

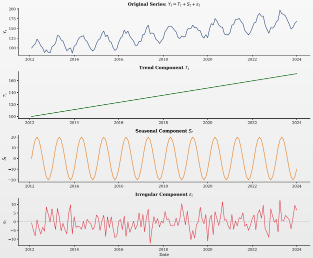
- T_t = **Trend-Cycle**: Long-term movement
- S_t = **Seasonal**: Regular periodic pattern
- ε_t = **Residual**: Random noise

Note: Cyclical component is typically absorbed into T_t

Classical Decomposition Models

- **Additive**: $X_t = T_t + S_t + \varepsilon_t$
- **Multiplicative**: $X_t = T_t \times S_t \times \varepsilon_t$

Time Series Decomposition: Visual Example

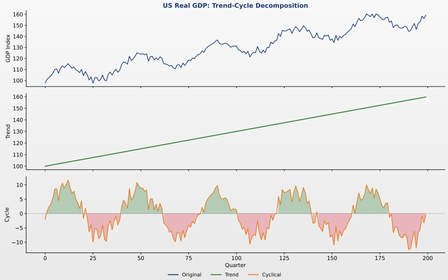


Components Explained

- **Original**: observed series
- **Trend-Cycle**: long-term movement
- **Seasonal**: periodic pattern
- **Residual**: random noise

TSA_ch1_decomposition

The Cyclical Component



Characteristics

- Medium-term fluctuations (2–10 years)
- No fixed period (unlike seasonal)
- Reflects expansions/recessions

In Practice

- Cycle is often combined with trend
- Difficult to identify in short series
- Usually not modeled separately

TSA_ch1_cyclical

Additive Decomposition Model

Model

$$X_t = T_t + S_t + \varepsilon_t \tag{1}$$

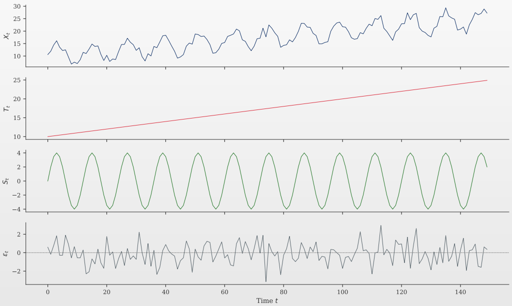
When to Use

- Seasonal fluctuations are **constant** over time
- Variance of the series is **stable**

Properties

- $\mathbb{E}[\varepsilon_t] = 0$ (zero mean)
- $\sum_{j=1}^{s} S_j = 0$ (seasonal sums to zero)
- Units of S_t same as X_t

Additive Decomposition: US Retail Sales (FRED)



Interpretation

Original = Trend + Seasonal + Residual. Seasonal amplitude stays constant. Data: US Retail Sales (RSXFS) from FRED.

TSA_ch0_additive

Multiplicative Decomposition Model

Model

$$X_t = T_t \times S_t \times \varepsilon_t \tag{2}$$

When to Use

- Seasonal fluctuations **grow** with series level
- Variance **increases** over time

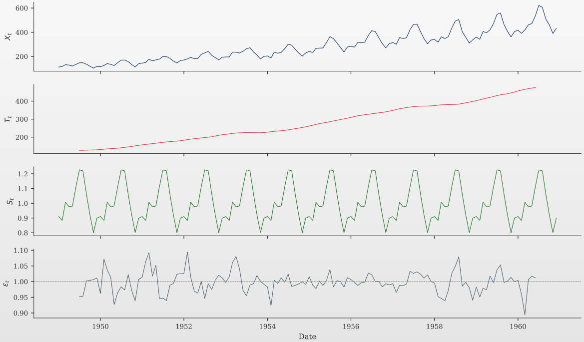
Properties

- $\mathbb{E}[\varepsilon_t] = 1$ (centered at 1)
- $\frac{1}{s} \sum S_j = 1$ (averages to 1)
- S_t is dimensionless ratio

Tip

Log transform converts multiplicative to additive model: $\log X_t = \log T_t + \log S_t + \log \varepsilon_t$

Multiplicative Decomposition: Real Data

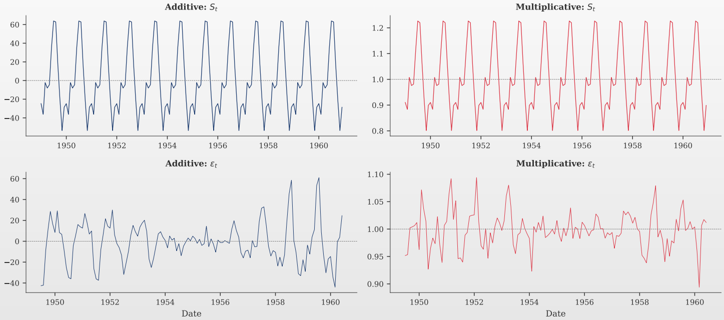


Example

Classic Box-Jenkins airline passengers (1949–1960). Seasonal amplitude grows with level.

TSA_ch0_multiplicative

Additive vs Multiplicative: Comparison



Key Difference

- **Multiplicative**: seasonal is a *ratio* (centered at 1)
- **Additive**: seasonal in *absolute units* (centered at 0)

TSA_ch0_comparison

Trend Estimation: Moving Average

Definition 2 (Centered Moving Average)

The **centered moving average** of order $2q+1$ is:

$$\hat{T}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} X_{t+j} \tag{3}$$

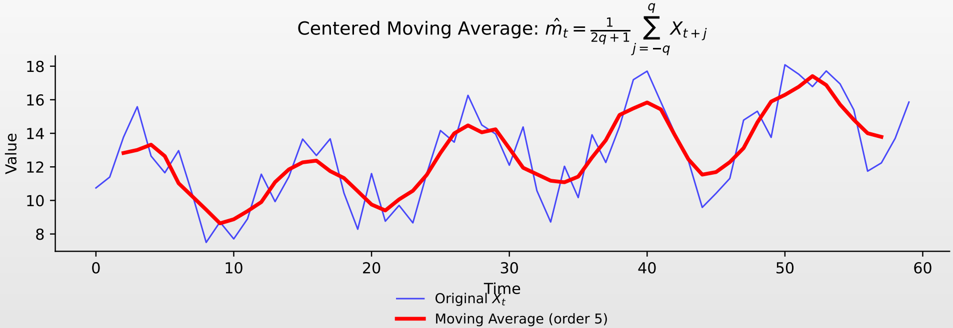
For Seasonal Data

- Period s **odd**: simple average
- Period s **even**: $2 \times s$ MA with half-weights

Properties

- Smooths seasonal & random
- Larger window $\Rightarrow$ smoother
- Trade-off: lose endpoints

Centered Moving Average: Visual Illustration



Interpretation

The moving average smooths out short-term fluctuations, revealing the underlying trend.

TSA_ch0_ma

Classical Decomposition Algorithm

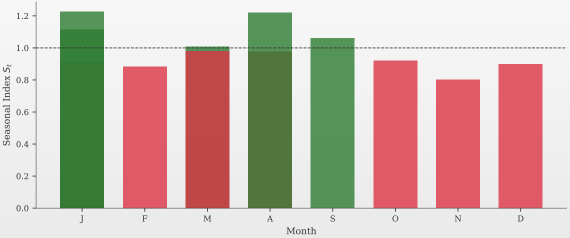
Steps for Multiplicative Decomposition

1. **Estimate Trend**: $\hat{T}_t = MA_s(X_t)$
2. **Detrend**: $D_t = X_t / \hat{T}_t$
3. **Estimate Seasonal**: $\hat{S}_j = \text{mean}(D_t \text{ for season } j)$
4. **Normalize**: Scale so $\frac{1}{s} \sum_{j=1}^{s} \hat{S}_j = 1$
5. **Compute Residuals**: $\hat{\varepsilon}_t = X_t / (\hat{T}_t \times \hat{S}_t)$

Note

For **additive** decomposition: replace division with subtraction and multiplication with addition.

Seasonal Indices: Interpretation



Interpretation

- $S_t > 1$: above-average activity
- $S_t < 1$: below-average activity
- Airline data shows peak travel in July–August

TSA_ch0_seasonal

STL Decomposition: A Modern Approach

Definition 3 (STL - Seasonal-Trend decomposition using LOESS)

STL uses locally weighted regression (LOESS): $X_t = T_t + S_t + R_t$

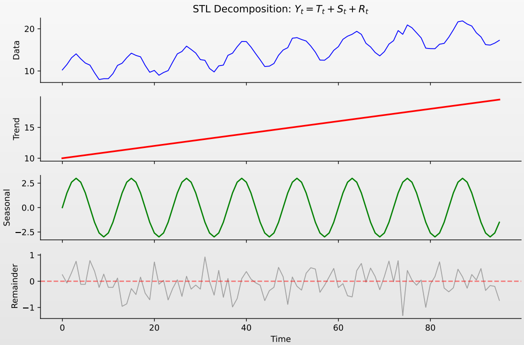
Advantages

- Any seasonal period
- Seasonal can change over time
- Robust to outliers
- Smooth trend estimates

Key Parameters

- `period`: Seasonal period
- `seasonal`: Smoothing window
- `robust`: Downweight outliers

STL Decomposition: Visual Illustration



Key Insight

STL separates the series into trend, seasonal, and remainder using LOESS.

TSA_ch0_stl

Exponential Smoothing: Overview

Definition

Exponential smoothing produces forecasts based on weighted averages of past observations, with weights decaying exponentially.

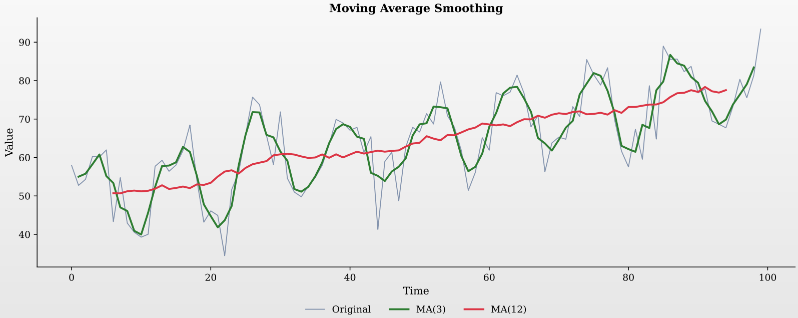
Why Exponential Smoothing?

- Simple yet effective
- Recent obs. get higher weights
- Handles trend & seasonality
- Foundation for ETS models

Three Main Methods

1. **SES**: Level only
2. **Holt**: Level + Trend
3. **Holt-Winters**: + Seasonality

Moving Average Smoothing



Window Size Trade-off

- **Small window**: Responsive but noisy
- **Large window**: Smoother but slower to react

TSA_ch1_moving_average

Simple Exponential Smoothing (SES)

Model

$$\hat{X}_{t+1|t} = \alpha X_t + (1-\alpha)\hat{X}_{t|t-1} \tag{4}$$

where $\alpha \in (0,1)$ is the **smoothing parameter**.

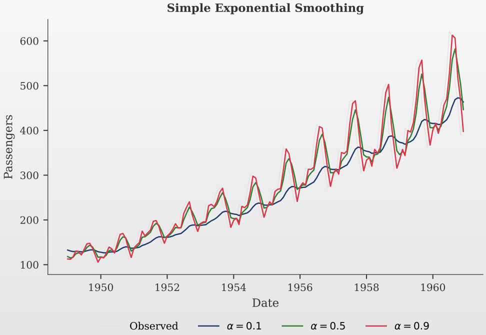
How It Works

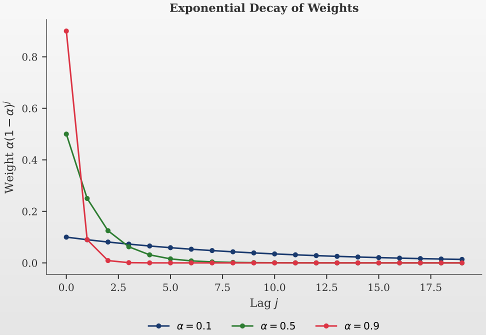
- Weights decay exponentially
- Large α: responsive
- Small α: smoother

Level Form

$$\ell_t = \alpha X_t + (1-\alpha)\ell_{t-1}$$

Simple Exponential Smoothing: Effect of α





Trade-off

Smaller α produces smoother forecasts; larger α follows data more closely.

TSA_ch0_ses

Holt's Linear Trend Method

Equations

- **Level:** $\ell_t = \alpha X_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
- **Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
- **Forecast:** $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t$

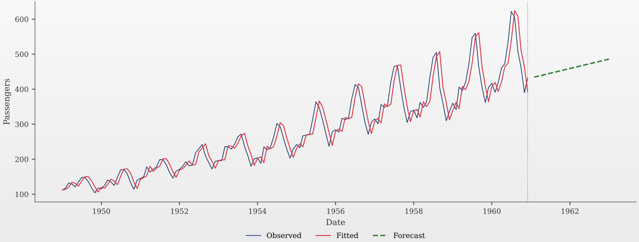
Parameters

- α: Level smoothing
- β^*: Trend smoothing

Components

- ℓ_t: Estimated level
- b_t: Estimated trend (slope)

Holt's Method: Visualization



Interpretation

- Holt's method captures both level and trend
- Projects them into the forecast horizon
- α controls responsiveness to level changes
- β^* controls responsiveness to trend changes

TSA_ch0_holt

Holt-Winters Seasonal Method

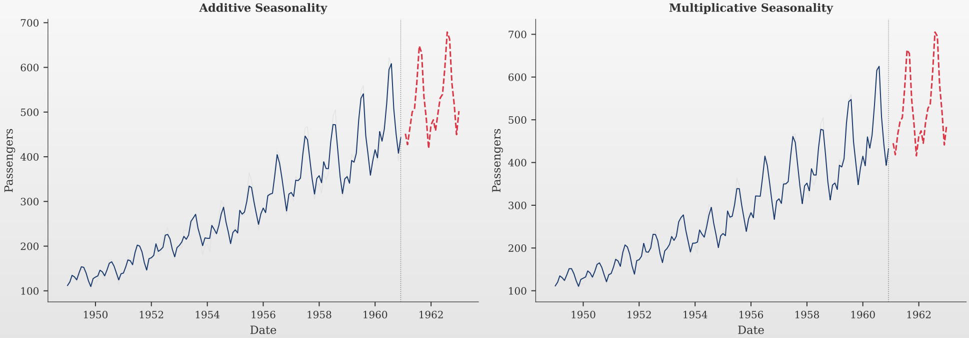
Equations (Additive Seasonality)

- **Level:** $\ell_t = \alpha(X_t - S_{t-s}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$
- **Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$
- **Seasonal:** $S_t = \gamma(X_t - \ell_t) + (1-\gamma)S_{t-s}$
- **Forecast:** $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t + S_{t+h-s(k+1)}$

Parameters

- α: Level smoothing
- β^*: Trend smoothing
- γ: Seasonal smoothing
- s: Seasonal period

Holt-Winters: Capturing Seasonality



Key Feature

Holt-Winters decomposes the series and produces seasonal forecasts with trend.

ETS Framework: Error-Trend-Seasonal

Definition 4 (ETS Models)

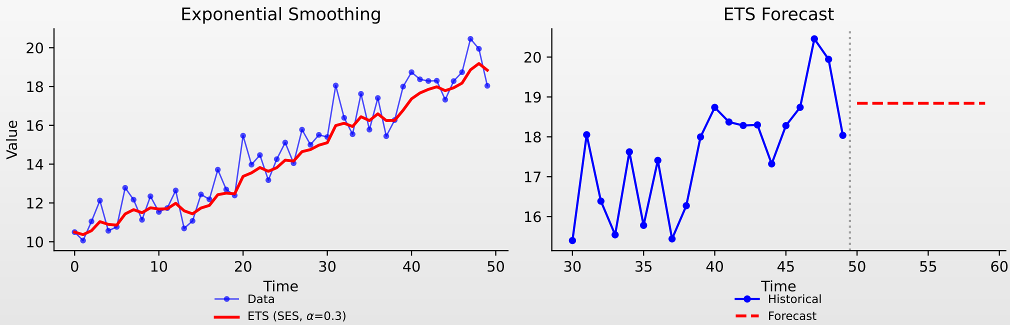
The **ETS framework** generalizes exponential smoothing: ETS(E, T, S)

Component	**N**	**A**	**M**
Error (E)	–	Additive	Multiplicative
Trend (T)	None	Additive	Multiplicative
Seasonal (S)	None	Additive	Multiplicative

Examples

- ETS(A,N,N) = Simple Exponential Smoothing
- ETS(A,A,N) = Holt's Linear Method
- ETS(A,A,A) = Holt-Winters Additive

ETS: Exponential Smoothing Illustration

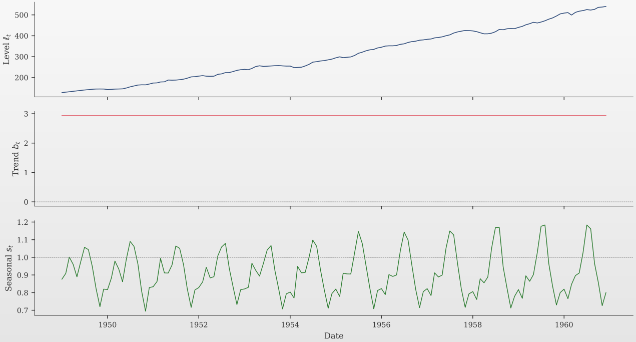


Interpretation

ETS models use exponentially weighted observations for forecasting. Weights decay as observations get older.

TSA_ch0_ets

ETS Model Selection



Interpretation

The ETS framework provides a systematic way to choose the best model using AIC/BIC.

TSA_ch0_ets_select

Damped Trend Methods

Damping Parameter

Introduces $\phi \in (0, 1)$ to prevent over-projection

Equations

- **Level:** $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$
- **Trend:** $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$
- **Forecast:** $\hat{X}_{t+h|t} = \ell_t + \phi \frac{1-\phi^h}{1-\phi} b_t$

Key Insight

- As $h \to \infty$: forecast $\to$ constant
- Prevents unrealistic long-term extrapolation
- Often best for longer horizons

Forecast Accuracy Metrics

Forecast Error

- $e_t = X_t - \hat{X}_t$ (actual minus predicted)

Scale-Dependent

- $\text{MAE} = \frac{1}{n} \sum |e_t|$
- $\text{MSE} = \frac{1}{n} \sum e_t^2$
- $\text{RMSE} = \sqrt{\text{MSE}}$

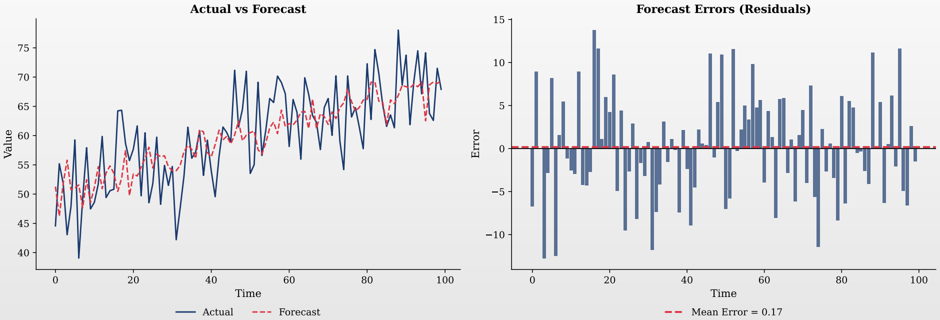
Scale-Independent

- $\text{MAPE} = \frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$
- $\text{sMAPE} = \frac{100}{n} \sum \frac{|e_t|}{(|X_t| + |\hat{X}_t|)/2}$

Which to use?

- Same series: RMSE, MAE
- Compare across series: MAPE, sMAPE

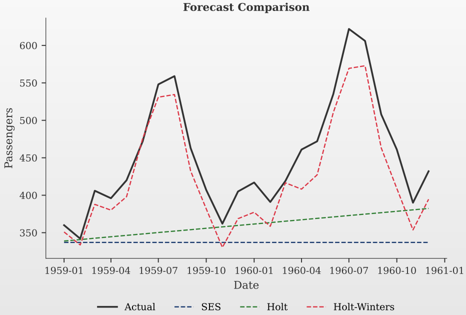
Forecast Evaluation: Visual Example

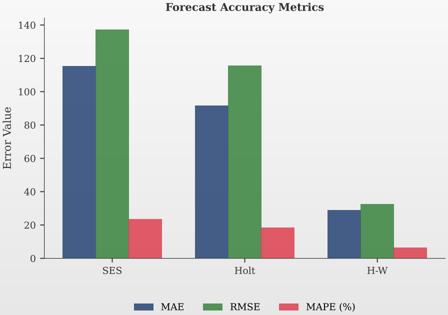


- **Top**: Actual values vs. forecasted values – visual assessment of fit
- **Bottom**: Residuals should be centered around zero with no pattern
- Good forecasts have small, random residuals with constant variance

TSA_ch1_forecast_eval

Comparing Forecast Methods





Interpretation

- **Left:** Comparing SES, Holt, and Holt-Winters forecasts
- **Right:** Error metrics for each method

TSA_ch0_forecast_compare

Residual Diagnostics

Residual Properties

Good forecasts should have residuals that are:

1. **Zero mean**: $\mathbb{E}[e_t] = 0$
2. **Uncorrelated**: $\text{Cov}(e_t, e_{t-k}) = 0$
3. **Constant variance**: $\text{Var}(e_t) = \sigma^2$
4. **Normally distributed**

Diagnostic Tests

Ljung-Box test (autocorrelation):

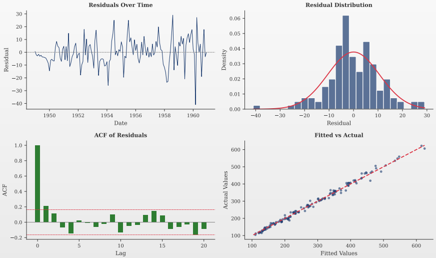
$$Q = T(T+2)\sum_{k=1}^{h}\frac{\hat{\rho}_k^2}{T-k} \sim \chi^2_h$$

Jarque-Bera test (normality):

$$JB = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right) \sim \chi^2_2$$

S = skewness, K = kurtosis

Residual Diagnostics: Visualization



What to Check

- Time plot (no patterns)
- Histogram (normality)
- ACF (no autocorrelation)
- Q-Q plot (normality)

TSA_ch0_residuals

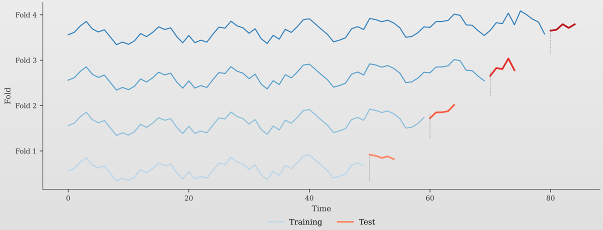
Time Series Cross-Validation

Why Not Standard CV?

- Time series have temporal dependence
- Future data cannot predict the past
- Standard k-fold causes data leakage

Rolling Origin CV

1. Train on $\{X_1, \ldots, X_t\}$
2. Forecast $\hat{X}_{t+h}$
3. Increment t, repeat



TSA_ch0_cv

Train / Validation / Test Split

Three-way split for model development:

Training Set

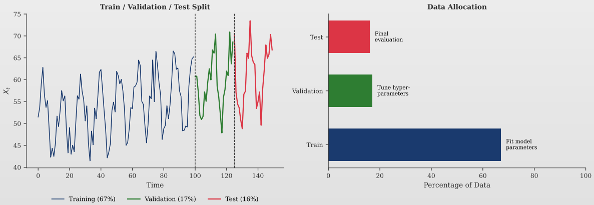
- Fit model parameters
- Largest portion (60–80%)
- Used for estimation

Validation Set

- Tune hyperparameters
- Compare models
- Select best approach

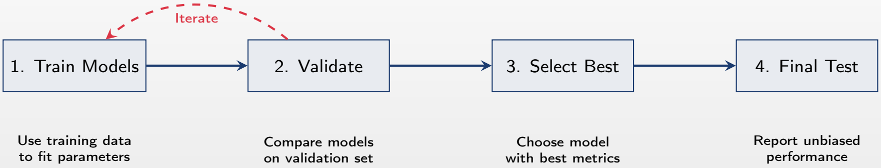
Test Set

- Final evaluation only
- Never used for tuning
- Unbiased performance



TSA_ch0_train_split

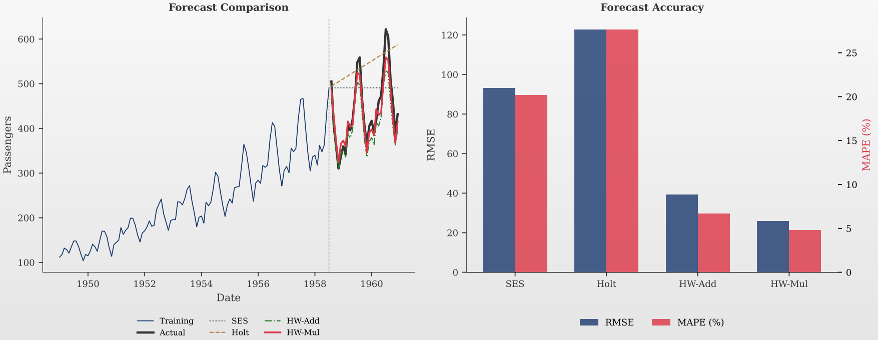
Model Development Workflow



Critical Rule

Never use test set for model selection! This causes *data leakage* and overly optimistic performance estimates.

Real Data: Forecast Comparison

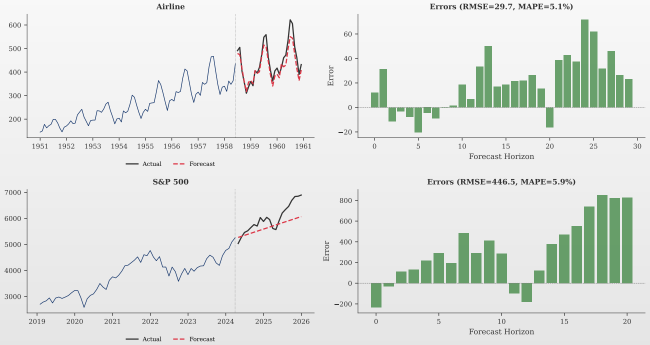


Interpretation

Airline passengers data: Holt-Winters Multiplicative performs best for seasonal data.

TSA_ch0_real_data

Forecast Performance Across Datasets



Interpretation

Different series require different models. Seasonal data needs seasonal methods.

TSA_ch0_multi_series

Modeling Seasonality: Two Approaches

1. Dummy Variables:

$X_t = \mu + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$

- $D_{jt} = 1$ if t in season j
- $s - 1$ parameters
- Any seasonal pattern

2. Fourier Terms:

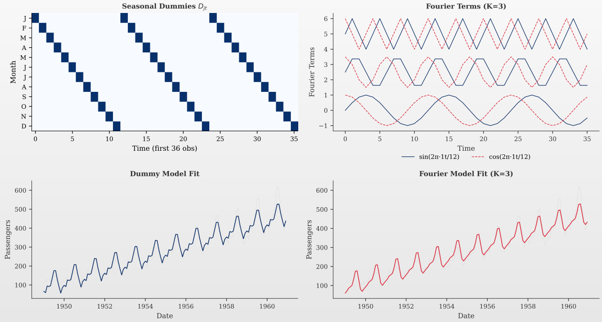
$X_t = \mu + \sum_{k=1}^{K} [\alpha_k \sin(\cdot) + \beta_k \cos(\cdot)]$

- Sinusoidal functions
- $2K$ parameters
- Smooth patterns

Trade-off

- **Dummies**: any pattern, more parameters
- **Fourier**: smooth, fewer parameters

Dummy Variables vs Fourier Terms



Comparison

- **Dummies**: capture any shape but need $s-1$ parameters
- **Fourier**: uses $2K$ parameters for smooth patterns

TSA_ch0_fourier

Choosing Between Dummies and Fourier

Criterion	Dummies	Fourier
Parameters (monthly)	11	$2K$ (often 4–6)
Seasonal pattern	Any shape	Smooth/sinusoidal
Interpretation	Direct (month effects)	Frequency components
High-frequency seasons	Many parameters	Efficient
Multiple seasonality	Complex	Easy (add terms)

Guidelines

- Use **dummies**: irregular patterns, interpretable coefficients
- Use **Fourier**: smooth patterns, high-frequency seasonality, multiple periods
- **Fourier terms** are used in TBATS and Facebook Prophet

Why Remove Trend and Seasonality?

Before modeling, we often need to make series stationary:

Reasons to detrend:

- Stationarity requirement
- Focus on fluctuations
- Avoid spurious regression
- Enable valid inference

Reasons to deseasonalize:

- Reveal underlying trend
- Compare across seasons
- Simplify modeling
- Focus on irregular component

Important

After modeling the detrended/deseasonalized series, we must **reverse the transformation** for forecasting.

Trend Removal Methods

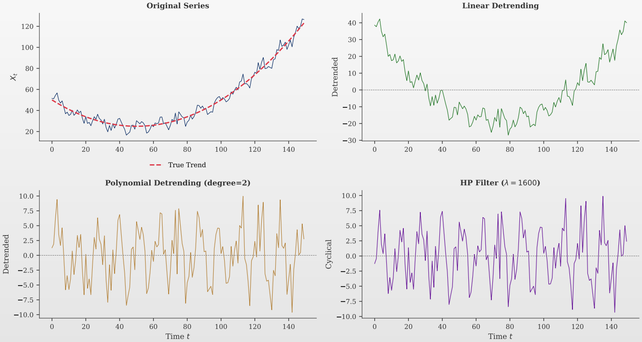
Six Common Detrending Approaches

1. **Differencing**: $\Delta X_t = X_t - X_{t-1}$
2. **Linear regression**: $\hat{T}_t = \hat{\beta}_0 + \hat{\beta}_1 t$
3. **Polynomial**: Higher-order polynomial
4. **HP Filter**: Balance fit vs smoothness
5. **Moving average**: $\hat{T}_t = MA_q(X_t)$
6. **LOESS**: Local polynomial regression

Choice Depends On

- Nature of trend (deterministic vs stochastic)
- Purpose (forecasting vs analysis)

Detrending Methods: Comparison

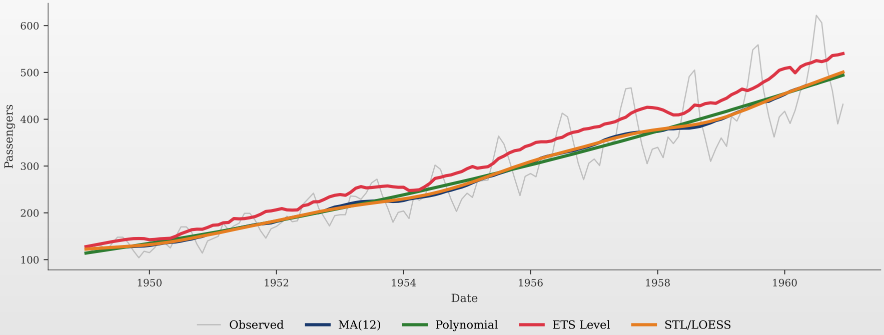


Key Insight

Different methods produce different residuals. Choose based on trend type and analysis goals.

TSA_ch0_detrending

Trend Estimation: Multiple Approaches



Interpretation

Different methods capture trend at varying levels of smoothness.

TSA_ch0_trend

Hodrick-Prescott (HP) Filter

Definition 5 (HP Filter)

The **HP filter** decomposes X_t into trend τ_t and cycle c_t: $X_t = \tau_t + c_t$, by minimizing:

$$\min_{\{\tau_t\}} \left\{ \sum_{t=1}^{T} (X_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right\}$$

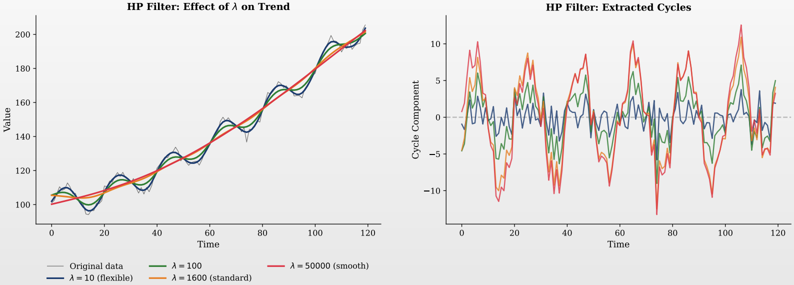
Interpretation

- First term: fit to data
- Second term: smoothness penalty
- λ: trade-off parameter

Standard λ Values

- Annual: $\lambda = 6.25$
- Quarterly: $\lambda = 1600$
- Monthly: $\lambda = 129600$

HP Filter: Effect of λ

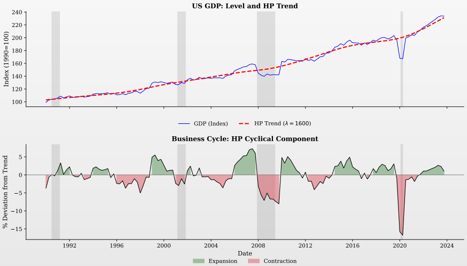


Trade-off

- **Small** λ: Trend follows data closely (more flexible)
- **Large** λ: Trend becomes smoother (approaches linear trend)

TSA_ch1_hp_filter

HP Filter: Business Cycle Extraction



Application

HP filter is widely used in macroeconomics to extract business cycles from GDP and other economic series.

TSA_ch0_hp_cycle

HP Filter: Limitations

Known Issues

- **End-point problem**: Trend estimates unreliable at endpoints
- **Spurious cycles**: Can create artificial dynamics
- λ **choice**: Results sensitive to parameter
- **Non-stationary**: Assumes trend is smooth

Alternatives

- **Band-pass filters**: Baxter-King, Christiano-Fitzgerald
- **Hamilton filter**: Regression-based
- **Unobserved components**: State-space models

Hamilton (2018) Critique

"Why You Should Never Use the Hodrick-Prescott Filter" — suggests using regression on lagged values instead.

Seasonality Removal Methods

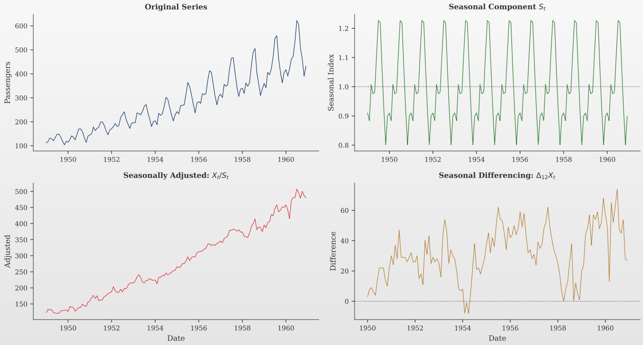
Four Approaches to Remove Seasonality

1. **Seasonal differencing**: $\Delta_s X_t = X_t - X_{t-s}$
2. **Division** (multiplicative): $X_t^{adj} = X_t / \hat{S}_t$
3. **Subtraction** (additive): $X_t^{adj} = X_t - \hat{S}_t$
4. **X-13ARIMA-SEATS**: Government statistical method

Seasonal Period s

- Monthly $\Rightarrow s = 12$
- Quarterly $\Rightarrow s = 4$

Seasonal Adjustment: Visualization



Result

Seasonally adjusted series reveals underlying trend without periodic fluctuations.

TSA_ch0_seasonal_adj

Deterministic vs Stochastic Trend

Deterministic Trend:

$$X_t = \beta_0 + \beta_1 t + \varepsilon_t$$

- Trend is a function of time
- Detrend by regression
- ε_t is stationary

Stochastic Trend:

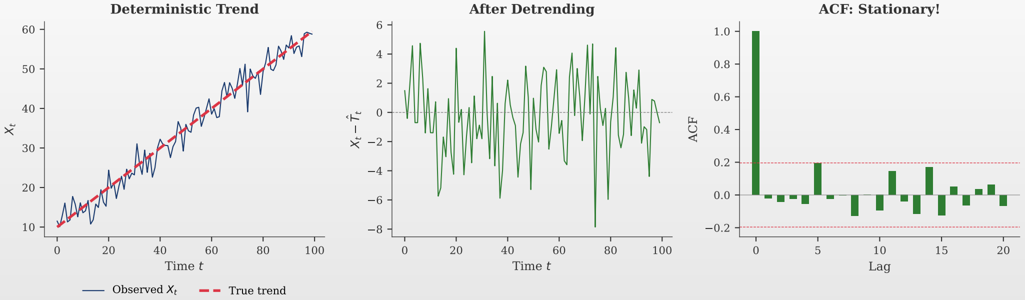
$$X_t = X_{t-1} + \varepsilon_t$$

- Random walk component
- Detrend by differencing
- ΔX_t is stationary

Wrong Method = Problems

- Differencing deterministic trend $\Rightarrow$ over-differencing
- Regression on stochastic trend $\Rightarrow$ spurious regression

Example: Deterministic Trend



Key

Use regression to remove trend → residuals are stationary (ACF decays quickly).

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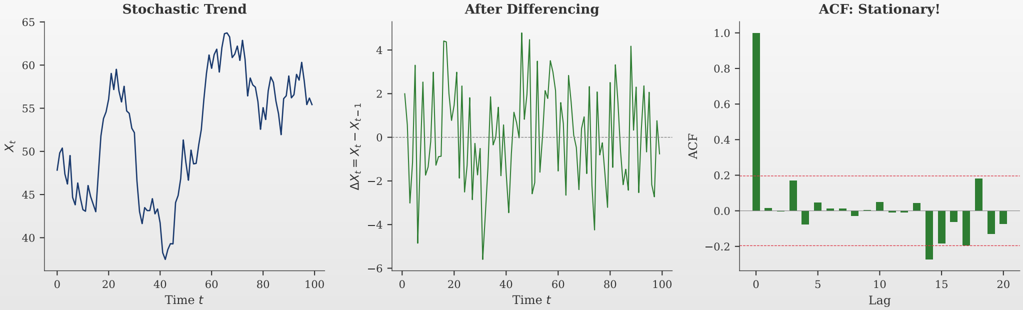
Example: Stochastic Trend (Random Walk)



Key

Use differencing to remove trend → differences are stationary (white noise).

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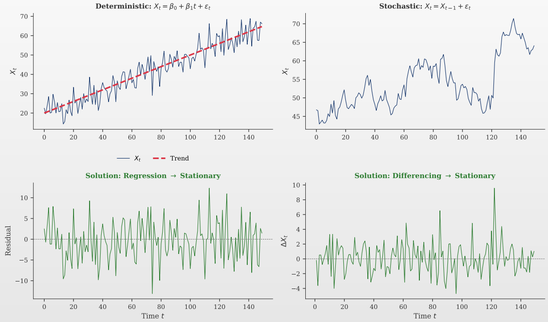
Side-by-Side Comparison



Remember

- Deterministic trend → regression
- Stochastic trend → differencing

TSA_ch0_trend_compare

Summary

What We Learned

- **Time Series Definition**: Sequence of observations indexed by time
- **Decomposition**: Trend-Cycle + Seasonal + Residual components
- **Exponential Smoothing**: SES, Holt, Holt-Winters, ETS framework
- **Forecast Evaluation**: MAE, RMSE, MAPE; train/validation/test splits

Key Takeaway

- **Understand Before Modeling**:
  - Always visualize and decompose your data first
  - Choose additive vs multiplicative based on variance behavior

Quick Quiz

1. What is the difference between additive and multiplicative decomposition?
2. When should you use Holt-Winters instead of simple exponential smoothing?
3. Why can't we use standard k-fold cross-validation for time series?
4. What does $\alpha = 0.9$ mean in exponential smoothing?
5. How do you distinguish between deterministic and stochastic trend?

Quiz Answers

1. **Additive vs Multiplicative**: Additive when seasonal amplitude is constant; multiplicative when it grows with the level.
2. **Holt-Winters**: When data has both trend AND seasonality. SES only handles level.
3. **Time Series CV**: Standard k-fold ignores temporal order — would use future data to predict the past (data leakage).
4. $\alpha = 0.9$: High weight on recent observations, forecast reacts quickly to changes but is more volatile.
5. **Trend type**: Deterministic — predictable function of time (use regression). Stochastic — random walk component (use differencing).

What Comes Next?

Chapter 1: Stochastic Processes and Stationarity

- **Stochastic Processes**: Mathematical foundation for time series
  - Random variables indexed by time
  - Strict vs weak (covariance) stationarity
- **Key Processes**: White noise and random walk
  - Building blocks for ARIMA models
  - Understanding mean reversion vs unit roots
- **ACF and PACF**: Tools for model identification
  - Detecting autocorrelation structure
  - Choosing AR and MA orders

Questions?