Time Series Analysis and Forecasting

# Chapter 6: VAR Models & Granger Causality

Multivariate Time Series

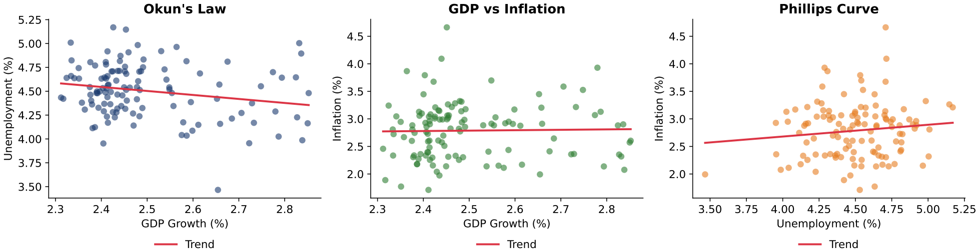
# Lecture Outline

1. Introduction to Multivariate Time Series
2. Vector Autoregression (VAR)
3. Granger Causality
4. Impulse Response Functions
5. Forecast Error Variance Decomposition
6. VAR Diagnostics
7. VAR Forecasting
8. Practical Example
9. Summary
10. Quiz

# Motivating Example: Macroeconomic Dynamics

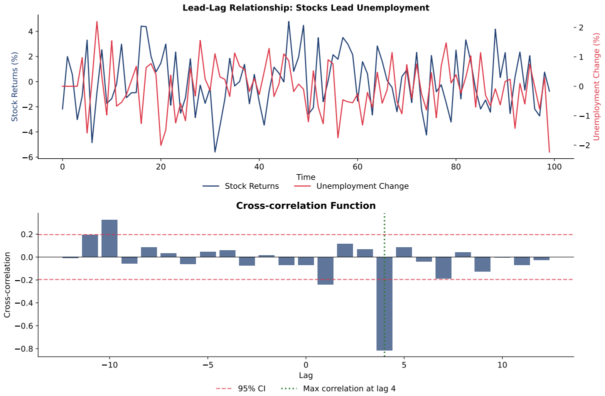

- Economic variables are **interconnected**: GDP affects unemployment, inflation affects interest rates
- Changes in one variable **propagate** through the system
- Understanding these dynamics requires **multivariate** analysis

# The Key Insight: Variables Interact



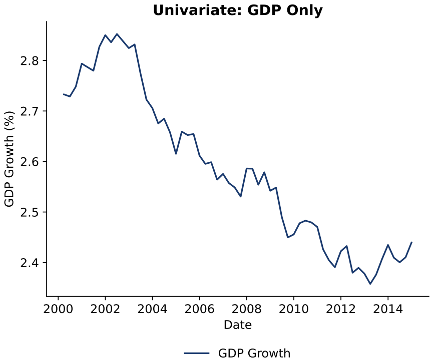

- **Okun's Law**: Higher GDP growth $\Rightarrow$ lower unemployment
- **Taylor Rule**: Higher inflation $\Rightarrow$ higher interest rates
- **Phillips Curve**: Unemployment-inflation tradeoff

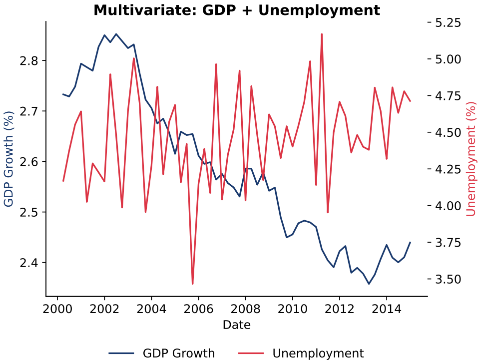
# Lead-Lag Relationships




- Some variables **lead** others: stock market predicts economic activity
- Cross-correlation reveals the **timing** of relationships
- Peak correlation at lag 4: stock market leads unemployment by $\sim$4 months

# Why Univariate Models Are Not Enough




## The Problem

ARIMA models each variable **in isolation**—ignoring valuable information from other variables!

## The Solution

**VAR models** capture the joint dynamics and feedback effects between multiple time series.

# What We'll Learn Today

## Core Concepts

1. **VAR Models**: How to model multiple time series jointly
2. **Granger Causality**: Does $X$ help predict $Y$?
3. **Impulse Response Functions**: How do shocks propagate?
4. **Variance Decomposition**: What drives each variable?

## Applications

- Macroeconomic policy analysis (monetary policy effects)
- Financial market dynamics (stock-bond relationships)
- Business cycle analysis (leading indicators)
- Risk management (volatility transmission)

# Why Multivariate Analysis?

## Limitations of Univariate Models

- ARIMA models each variable **in isolation**
- Ignores potential **interactions** between variables
- Cannot capture **feedback effects**

## Economic Examples of Interdependence

- GDP and unemployment (Okun's law)
- Interest rates and inflation (Taylor rule)
- Stock prices and trading volume
- Exchange rates and trade balance

# Multivariate Time Series Notation

## Vector of Variables

Let $\mathbf{Y}_t = (Y_{1t}, Y_{2t}, \ldots, Y_{Kt})'$ be a $K \times 1$ vector of time series.

Example with $K = 2$:

$$\mathbf{Y}_t = \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \text{GDP growth}_t \\ \text{Inflation}_t \end{pmatrix}$$

## Key Questions

1. Does $Y_1$ help predict $Y_2$? (Granger causality)
2. How do shocks to $Y_1$ affect $Y_2$? (Impulse responses)
3. What proportion of $Y_2$'s variance is due to $Y_1$? (Variance decomposition)

# Multivariate Stationarity

## Definition: Weak Stationarity

A $K$-dimensional time series $\mathbf{Y}_t$ is **weakly stationary** if:

1. $\mathbb{E}[\mathbf{Y}_t] = \boldsymbol{\mu}$ (constant mean vector)
2. $\text{Cov}(\mathbf{Y}_t, \mathbf{Y}_{t-h}) = \boldsymbol{\Gamma}(h)$ depends only on $h$, not $t$

## Autocovariance Matrix

$$\boldsymbol{\Gamma}(h) = \mathbb{E}[(\mathbf{Y}_t - \boldsymbol{\mu})(\mathbf{Y}_{t-h} - \boldsymbol{\mu})'] = \begin{pmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{pmatrix}$$

Note: $\boldsymbol{\Gamma}(-h) = \boldsymbol{\Gamma}(h)'$ (transpose, not equal!)

# Cross-Covariance Properties

## Cross-Covariance Function

For variables $Y_{it}$ and $Y_{jt}$:

$$\gamma_{ij}(h) = \text{Cov}(Y_{it}, Y_{j,t-h}) = \mathbb{E}[(Y_{it} - \mu_i)(Y_{j,t-h} - \mu_j)]$$

## Key Difference from Univariate Case

- In general: $\gamma_{ij}(h) \neq \gamma_{ij}(-h)$
- But: $\gamma_{ij}(h) = \gamma_{ji}(-h)$
- The cross-covariance matrix is **not symmetric** for $h \neq 0$

## Example

If $Y_1$ leads $Y_2$: $\gamma_{12}(h) > 0$ for $h > 0$ but $\gamma_{12}(h) \approx 0$ for $h < 0$

# Correlation Matrix Function

## Definition

The **autocorrelation matrix** at lag $h$:

$$\mathbf{R}(h) = \mathbf{D}^{-1}\mathbf{\Gamma}(h)\mathbf{D}^{-1}$$

where $\mathbf{D} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_K)$ and $\sigma_i = \sqrt{\gamma_{ii}(0)}$

## For Bivariate Case

$$\mathbf{R}(h) = \begin{pmatrix} \rho_{11}(h) & \rho_{12}(h) \\ \rho_{21}(h) & \rho_{22}(h) \end{pmatrix}$$

where $\rho_{ij}(h) = \frac{\gamma_{ij}(h)}{\sigma_i \sigma_j}$

Diagonal elements: usual ACFs; Off-diagonal: cross-correlations

# The VAR(p) Model

## Definition

A **VAR(p)** model for $K$ variables:

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \varepsilon_t$$

where:

- $\mathbf{Y}_t$: $K \times 1$ vector of endogenous variables
- $\mathbf{c}$: $K \times 1$ vector of constants
- $\mathbf{A}_i$: $K \times K$ coefficient matrices
- $\varepsilon_t$: $K \times 1$ vector of error terms with $\mathbb{E}[\varepsilon_t] = \mathbf{0}$, $\mathbb{E}[\varepsilon_t\varepsilon_t'] = \mathbf{\Sigma}$

# VAR(1) with Two Variables

## Bivariate VAR(1)

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

## Equation by Equation

$$Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$$
$$Y_{2t} = c_2 + a_{21} Y_{1,t-1} + a_{22} Y_{2,t-1} + \varepsilon_{2t}$$

**Key insight**: Each equation includes lags of **all** variables!

# Numerical Example: VAR(1)

## Specific VAR(1) Model

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} + \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

## Interpretation of Coefficients

- $a_{11} = 0.7$: A 1-unit increase in $Y_1$ at $t-1$ increases $Y_1$ at $t$ by 0.7
- $a_{12} = 0.2$: A 1-unit increase in $Y_2$ at $t-1$ increases $Y_1$ at $t$ by 0.2
- $a_{21} = -0.1$: A 1-unit increase in $Y_1$ at $t-1$ **decreases** $Y_2$ at $t$ by 0.1
- $a_{22} = 0.6$: A 1-unit increase in $Y_2$ at $t-1$ increases $Y_2$ at $t$ by 0.6

# VAR(2): Higher Order Dynamics

## VAR(2) Specification

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \varepsilon_t$$

For $K = 2$, the full model has $2 + 2 \times 4 + 2 \times 4 = 18$ parameters!

## Written Out

$$Y_{1t} = c_1 + a_{11}^{(1)} Y_{1,t-1} + a_{12}^{(1)} Y_{2,t-1} + a_{11}^{(2)} Y_{1,t-2} + a_{12}^{(2)} Y_{2,t-2} + \varepsilon_{1t}$$
$$Y_{2t} = c_2 + a_{21}^{(1)} Y_{1,t-1} + a_{22}^{(1)} Y_{2,t-1} + a_{21}^{(2)} Y_{1,t-2} + a_{22}^{(2)} Y_{2,t-2} + \varepsilon_{2t}$$

## Curse of Dimensionality

VAR($p$) with $K$ variables has $K + pK^2$ parameters. With $K = 5$, $p = 4$: $5 + 4 \times 25 = 105$ parameters!

# The Companion Form

## Converting VAR(p) to VAR(1)

Any VAR($p$) can be written as a VAR(1) in **companion form**:

$$\boldsymbol{\xi}_t = \mathbf{A}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t$$

## For VAR(2)

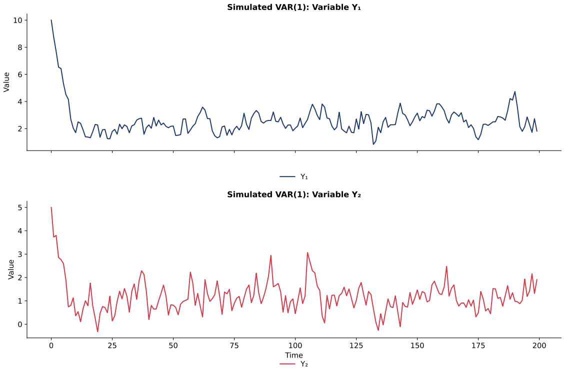
$$\underbrace{\begin{pmatrix} \mathbf{Y}_t \\ \mathbf{Y}_{t-1} \end{pmatrix}}_{\boldsymbol{\xi}_t} = \underbrace{\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I}_K & \mathbf{0} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \mathbf{Y}_{t-1} \\ \mathbf{Y}_{t-2} \end{pmatrix}}_{\boldsymbol{\xi}_{t-1}} + \underbrace{\begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \end{pmatrix}}_{\mathbf{v}_t}$$

The companion matrix $\mathbf{A}$ is $Kp \times Kp$.

## Why Useful?

Stationarity, forecasting, and IRFs are easier to analyze in companion form.

# Simulated VAR Process




- Two simulated series from a bivariate VAR(1) process showing interdependence
- Each variable responds to both its own past and the other variable's past
- Notice how the series co-move due to cross-equation dynamics

# Stationarity of VAR

## Stability Condition

VAR(p) is **stable** (stationary) if all roots of:

$$\det(\mathbf{I}_K - \mathbf{A}_1 z - \mathbf{A}_2 z^2 - \cdots - \mathbf{A}_p z^p) = 0$$

lie **outside** the unit circle (i.e., $|z| > 1$).

## For VAR(1)

The model is stable if all **eigenvalues** of $\mathbf{A}_1$ are less than 1 in absolute value.

Example: For $\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$, eigenvalues are $\lambda_1 = 0.6$ and $\lambda_2 = 0.2$.

Both $< 1 \Rightarrow$ stable!

## Computing Eigenvalues: Example

**For $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$**

Characteristic polynomial: $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det\begin{pmatrix} 0.7-\lambda & 0.2 \\ -0.1 & 0.6-\lambda \end{pmatrix} = (0.7-\lambda)(0.6-\lambda) + 0.02 = 0$$

$$\lambda^2 - 1.3\lambda + 0.44 = 0$$

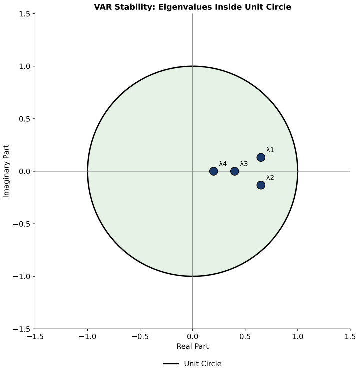
**Solution**

Using quadratic formula:

$$\lambda = \frac{1.3 \pm \sqrt{1.69 - 1.76}}{2} = \frac{1.3 \pm \sqrt{-0.07}}{2} = 0.65 \pm 0.132i$$

$|\lambda| = \sqrt{0.65^2 + 0.132^2} = \sqrt{0.44} = 0.663 < 1 \quad \checkmark$ Stable!

# Stability Condition: Visual Interpretation




- Eigenvalues of the companion matrix must lie inside the unit circle
- Complex eigenvalues come in conjugate pairs
- If any eigenvalue is outside the circle, the VAR is explosive (non-stationary)

# Mean of a Stationary VAR

## Unconditional Mean

For a stationary VAR(1): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t$
Taking expectations:

$$\mathbb{E}[\mathbf{Y}_t] = \mathbf{c} + \mathbf{A}\mathbb{E}[\mathbf{Y}_{t-1}]$$

Since $\mathbb{E}[\mathbf{Y}_t] = \mathbb{E}[\mathbf{Y}_{t-1}] = \boldsymbol{\mu}$ (stationarity):

$$\boldsymbol{\mu} = \mathbf{c} + \mathbf{A}\boldsymbol{\mu} \quad \Rightarrow \quad \boldsymbol{\mu} = (\mathbf{I}_K - \mathbf{A})^{-1}\mathbf{c}$$

## Example

If $\mathbf{c} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$:

$$\boldsymbol{\mu} = \begin{pmatrix} 0.3 & -0.2 \\ 0.1 & 0.4 \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2.3 \\ 1.0 \end{pmatrix}$$

# Covariance Structure of VAR(1)

## Variance-Covariance Matrix $\boldsymbol{\Gamma}(0)$

For VAR(1), the variance satisfies the **discrete Lyapunov equation**:

$$\boldsymbol{\Gamma}(0) = \mathbf{A}\boldsymbol{\Gamma}(0)\mathbf{A}' + \boldsymbol{\Sigma}$$

## Autocovariance at Lag $h$

$$\boldsymbol{\Gamma}(h) = \mathbf{A}^h\boldsymbol{\Gamma}(0), \quad h \geq 0$$

This shows that autocovariances decay geometrically with the eigenvalues of $\mathbf{A}$.

## Solving the Lyapunov Equation

Can solve by vectorization:

$$\text{vec}(\boldsymbol{\Gamma}(0)) = (\mathbf{I}_{K^2} - \mathbf{A} \otimes \mathbf{A})^{-1}\text{vec}(\boldsymbol{\Sigma})$$

where $\otimes$ denotes the Kronecker product.

# Estimation of VAR

## OLS Estimation

Each equation can be estimated by **OLS separately**:

$$\hat{\mathbf{A}} = \left(\sum_{t=1}^{T} \mathbf{Y}_{t-1}\mathbf{Y}_{t-1}'\right)^{-1} \left(\sum_{t=1}^{T} \mathbf{Y}_{t-1}\mathbf{Y}_t'\right)$$

This is efficient because all equations have the **same regressors**.

## Covariance Matrix

$$\hat{\Sigma} = \frac{1}{T - Kp - 1} \sum_{t=1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_t'$$

The errors $\varepsilon_{1t}$ and $\varepsilon_{2t}$ may be **contemporaneously correlated**.

# Lag Order Selection

## Information Criteria

Choose $p$ that minimizes:

$$\text{AIC}(p) = \ln|\hat{\Sigma}_p| + \frac{2pK^2}{T}$$

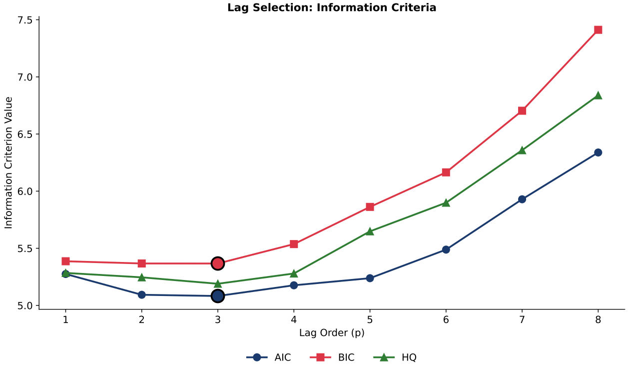
$$\text{BIC}(p) = \ln|\hat{\Sigma}_p| + \frac{pK^2 \ln T}{T}$$

$$\text{HQ}(p) = \ln|\hat{\Sigma}_p| + \frac{2pK^2 \ln\ln T}{T}$$

## Guidelines

- AIC tends to select **larger** models (better for forecasting)
- BIC tends to select **smaller** models (consistent selection)
- Start with maximum $p_{max}$ based on data frequency (e.g., 4 for quarterly, 12 for monthly)

# Lag Selection: Example




- Information criteria values for different lag orders
- AIC and BIC may suggest different optimal lags
- Lower values indicate better model fit (penalized by complexity)

# Restricted VAR Models

## Why Restrict?

Full VAR models can be **overparameterized**:

- Many coefficients may be insignificant
- Poor forecasting performance
- Loss of degrees of freedom

## Common Restrictions

- **Zero restrictions**: Set small coefficients to zero
- **Block exogeneity**: Some variables don't affect others
- **Lag exclusion**: Exclude certain lags

## Testing Restrictions

Use likelihood ratio test: $LR = T(\ln|\hat{\Sigma}_R| - \ln|\hat{\Sigma}_U|) \sim \chi^2_r$
where $r$ = number of restrictions

# What is Granger Causality?

## Clive Granger (1969, Nobel Prize 2003)

"$X$ **Granger-causes** $Y$" if past values of $X$ help predict $Y$, **beyond** what past values of $Y$ alone can predict.

## Important Distinction

**Granger causality $\neq$ True causality**

- Granger causality is about **predictive content**
- Does NOT imply economic/structural causation
- "$X$ Granger-causes $Y$" means: $X$ contains useful information for forecasting $Y$

# Formal Definition

## Granger Causality

$X$ **does not Granger-cause** $Y$ if:

$$\mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \ldots, X_{t-1}, X_{t-2}, \ldots] = \mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \ldots]$$

In other words: adding $X$'s history does not improve the prediction of $Y$.

## In the VAR Context

For VAR(1): $Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$

$Y_2$ does **not** Granger-cause $Y_1$ if $a_{12} = 0$.

For VAR(p): $Y_2$ does not Granger-cause $Y_1$ if $a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$.

# Testing for Granger Causality

## Hypothesis Test

$H_0$: $Y_2$ does **not** Granger-cause $Y_1$

$$H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$$

$H_1$: At least one $a_{12}^{(i)} \neq 0$ (Granger causality exists)

## Test Statistic: Wald Test

$$F = \frac{(RSS_R - RSS_U)/p}{RSS_U/(T - 2p - 1)} \sim F_{p,T-2p-1}$$

where:

- $RSS_R$: Residual sum of squares from restricted model (without $Y_2$ lags)
- $RSS_U$: Residual sum of squares from unrestricted model (full VAR)

# Types of Granger Causality


Unidirectional: $X \to Y$

Bidirectional: $X \leftrightarrow Y$


Unidirectional: $Y \to X$

No causality

## Economic Examples

- Money $\to$ Output? (monetarist view)
- Stock prices $\leftrightarrow$ Trading volume (bidirectional)
- Weather $\to$ Crop yields (unidirectional, obvious)

# Cross-Correlation Function

## Definition 1 (Cross-Correlation Function)

The **cross-correlation** between $X_t$ and $Y_t$ at lag $k$ is:

$$\rho_{XY}(k) = \frac{\gamma_{XY}(k)}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X_t, Y_{t+k})}{\sqrt{\text{Var}(X_t)\text{Var}(Y_t)}}$$

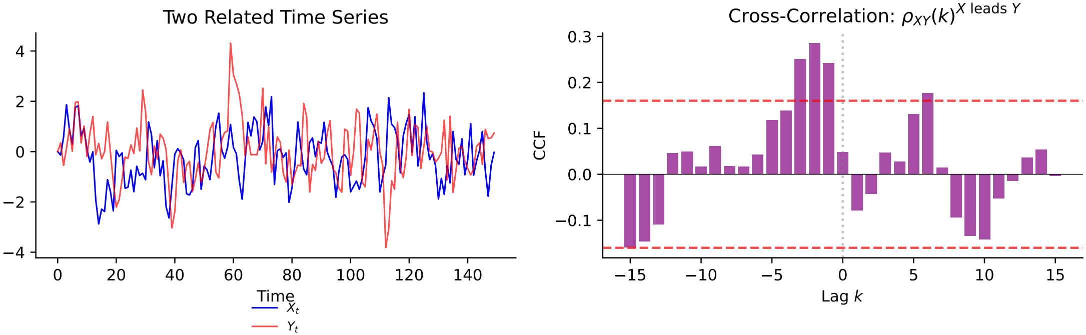
## Interpretation

- $\rho_{XY}(k) > 0$ at $k > 0$: $X$ is positively correlated with future $Y$ (X may lead Y)
- $\rho_{XY}(k) > 0$ at $k < 0$: $X$ is positively correlated with past $Y$ (Y may lead X)

## Note

Unlike ACF, cross-correlation is **not symmetric**: $\rho_{XY}(k) \neq \rho_{XY}(-k)$ in general.

# Cross-Correlation: Visual Illustration




Left: two related series. Right: CCF reveals that $X$ leads $Y$ (significant correlations at positive lags).

# Granger Causality: Practical Considerations

## Common Pitfalls

1. **Omitted variables**: A third variable $Z$ may cause both $X$ and $Y$
2. **Non-stationarity**: Test requires stationary data (or cointegration)
3. **Lag selection**: Results can be sensitive to $p$
4. **Sample size**: Need sufficient observations

## Best Practices

- Test for unit roots first
- Use multiple lag selection criteria
- Check robustness to different lag lengths
- Report results for both directions

# Granger Causality Test: Numerical Example

## Testing: Does Money Growth Granger-cause Output?

**Unrestricted model** (VAR with 2 lags):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \beta_1 \Delta M_{t-1} + \beta_2 \Delta M_{t-2} + \varepsilon_t$$

**Restricted model** ($H_0$: $\beta_1 = \beta_2 = 0$):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \varepsilon_t$$

## Test Computation

With $T = 100$, $RSS_U = 45.2$, $RSS_R = 52.8$:

$$F = \frac{(52.8 - 45.2)/2}{45.2/(100-5)} = \frac{3.8}{0.476} = 7.98$$

$F_{0.05}(2, 95) = 3.09 \Rightarrow$ **Reject** $H_0$: Money Granger-causes output!

# The Toda-Yamamoto Procedure

## Problem with Non-Stationary Data

Standard Granger test has **non-standard distributions** when:

- Variables have unit roots
- Variables are cointegrated

## Toda-Yamamoto Solution (1995)

1. Determine maximum order of integration $d_{max}$
2. Estimate VAR($p + d_{max}$) in **levels**
3. Test restrictions on first $p$ lags only
4. Extra $d_{max}$ lags are **not** tested (just for correct distribution)

## Advantage

Wald test has asymptotic $\chi^2$ distribution regardless of cointegration!

# Instantaneous Causality

## Definition

$X$ **instantaneously causes** $Y$ if:

$$\mathbb{E}[Y_t|\Omega_{t-1}, X_t] \neq \mathbb{E}[Y_t|\Omega_{t-1}]$$

where $\Omega_{t-1}$ contains all past information.

## Testing in VAR

Test whether $\sigma_{12} \neq 0$ in the covariance matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

If $\sigma_{12} = 0$: no instantaneous causality

## Interpretation

Instantaneous causality often reflects **common shocks** or **data aggregation**, not true contemporaneous effects.

# Granger Causality in Multiple Systems

## Block Exogeneity Test

In a VAR with $K > 2$ variables, test whether a **group** of variables Granger-causes another group.

Example: Do financial variables (interest rates, stock prices) Granger-cause real variables (GDP, unemployment)?

## Test Statistic

$$\chi^2 = T \cdot K_1 \cdot p \cdot \left( \ln |\hat{\Sigma}_R| - \ln |\hat{\Sigma}_U| \right) \sim \chi^2_{K_1 \cdot K_2 \cdot p}$$

where $K_1$ = number of “caused” variables, $K_2$ = number of “causing” variables

# What are Impulse Response Functions?

## Definition

An **Impulse Response Function (IRF)** traces the effect of a one-time shock to one variable on the current and future values of all variables.

## Question IRFs Answer

"If there is an unexpected 1-unit shock to $Y_1$ today, what happens to $Y_1$ and $Y_2$ over the next $h$ periods?"

## MA($\infty$) Representation

A stable VAR(p) can be written as:

$$\mathbf{Y}_t = \boldsymbol{\mu} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{\varepsilon}_{t-i}$$

The matrices $\boldsymbol{\Phi}_i$ are the **impulse responses** at horizon $i$.

# Computing IRFs for VAR(1)

## For VAR(1): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t$

The impulse response matrices are:

$$\mathbf{\Phi}_0 = \mathbf{I}_K, \quad \mathbf{\Phi}_1 = \mathbf{A}, \quad \mathbf{\Phi}_2 = \mathbf{A}^2, \quad \ldots, \quad \mathbf{\Phi}_h = \mathbf{A}^h$$

## Interpretation

$[\mathbf{\Phi}_h]_{ij}$ = Effect on $Y_i$ at time $t+h$ of a unit shock to $Y_j$ at time $t$

For stable VAR: $\mathbf{\Phi}_h \to \mathbf{0}$ as $h \to \infty$ (shocks die out)

# Computing IRFs for General VAR(p)

## Recursive Formula for VAR(p)

For $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \varepsilon_t$:

$$\boldsymbol{\Phi}_h = \sum_{j=1}^{\min(h,p)} \mathbf{A}_j\boldsymbol{\Phi}_{h-j}, \quad h = 1, 2, 3, \ldots$$

with $\boldsymbol{\Phi}_0 = \mathbf{I}_K$ and $\boldsymbol{\Phi}_h = \mathbf{0}$ for $h < 0$.

## Example: VAR(2) IRFs

- $\boldsymbol{\Phi}_0 = \mathbf{I}_K$
- $\boldsymbol{\Phi}_1 = \mathbf{A}_1\boldsymbol{\Phi}_0 = \mathbf{A}_1$
- $\boldsymbol{\Phi}_2 = \mathbf{A}_1\boldsymbol{\Phi}_1 + \mathbf{A}_2\boldsymbol{\Phi}_0 = \mathbf{A}_1^2 + \mathbf{A}_2$
- $\boldsymbol{\Phi}_3 = \mathbf{A}_1\boldsymbol{\Phi}_2 + \mathbf{A}_2\boldsymbol{\Phi}_1 = \mathbf{A}_1(\mathbf{A}_1^2 + \mathbf{A}_2) + \mathbf{A}_2\mathbf{A}_1$

# Orthogonalized IRFs

## Problem: Correlated Errors

If $\mathbf{\Sigma}$ is not diagonal, shocks $\varepsilon_{1t}$ and $\varepsilon_{2t}$ are correlated.
A shock to "$Y_1$" also involves a shock to "$Y_2$".

## Solution: Cholesky Decomposition

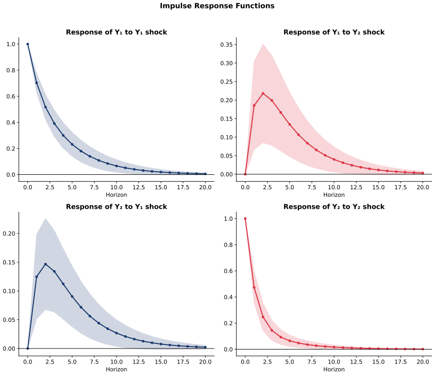
Factor $\mathbf{\Sigma} = \mathbf{PP}'$ where $\mathbf{P}$ is lower triangular.
Define orthogonalized shocks: $\mathbf{u}_t = \mathbf{P}^{-1}\varepsilon_t$ with $\mathbb{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}$

Orthogonalized IRFs: $\mathbf{\Theta}_h = \mathbf{\Phi}_h\mathbf{P}$

## Ordering Matters!

Cholesky assumes variables ordered from "most exogenous" to "most endogenous". Results depend on this ordering.

# Impulse Response Functions: Example




- IRFs show how each variable responds to a one-unit shock over time
- Shaded regions represent confidence intervals (uncertainty in estimates)
- For stable VAR models, responses converge to zero as the horizon increases

# IRF Numerical Example

**For $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$**

$$\boldsymbol{\Phi}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{\Phi}_1 = \mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$$

$$\boldsymbol{\Phi}_2 = \mathbf{A}^2 = \begin{pmatrix} 0.47 & 0.26 \\ -0.13 & 0.34 \end{pmatrix}$$

**Interpretation**

- $[\boldsymbol{\Phi}_2]_{12} = 0.26$: A unit shock to $Y_2$ increases $Y_1$ by 0.26 after 2 periods
- $[\boldsymbol{\Phi}_2]_{21} = -0.13$: A unit shock to $Y_1$ **decreases** $Y_2$ by 0.13 after 2 periods

# Cumulative Impulse Responses

## Definition

The **cumulative IRF** up to horizon $H$:

$$\mathbf{\Psi}_H = \sum_{h=0}^{H} \mathbf{\Phi}_h$$

Measures the **total accumulated effect** of a shock.

## Long-Run Multiplier

For stable VAR: $\mathbf{\Psi}_\infty = (\mathbf{I}_K - \mathbf{A}_1 - \mathbf{A}_2 - \cdots - \mathbf{A}_p)^{-1}$

This gives the **permanent effect** of a one-time shock.

## When to Use

Cumulative IRFs are useful when interested in total impact (e.g., cumulative GDP loss from a shock).

# Confidence Intervals for IRFs

## Sources of Uncertainty

IRFs are functions of estimated parameters $\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_p$, so they have **sampling uncertainty**.

## Methods for Confidence Bands

1. **Asymptotic**: Use delta method to derive standard errors
2. **Monte Carlo**: Simulate from asymptotic distribution of $\hat{\mathbf{A}}$
3. **Bootstrap**: Resample residuals and re-estimate VAR

## Bootstrap Procedure

1. Estimate VAR, save residuals $\{\hat{\varepsilon}_t\}$
2. Draw with replacement to create $\{\hat{\varepsilon}_t^*\}$
3. Generate bootstrap sample using estimated VAR
4. Re-estimate and compute IRFs
5. Repeat $B$ times; use percentiles for CIs

# Structural VAR (SVAR)

## Motivation

Standard VAR shocks $\varepsilon_t$ are **reduced-form** innovations—linear combinations of structural shocks.

We want to identify economically meaningful **structural shocks**.

## Structural Form

$$\mathbf{B}_0\mathbf{Y}_t = \mathbf{\Gamma}_0 + \mathbf{B}_1\mathbf{Y}_{t-1} + \cdots + \mathbf{B}_p\mathbf{Y}_{t-p} + \mathbf{u}_t$$

where $\mathbf{u}_t$ are **structural shocks** with $\mathbb{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}_K$

## Relationship to Reduced Form

$\varepsilon_t = \mathbf{B}_0^{-1}\mathbf{u}_t \quad \Rightarrow \quad \mathbf{\Sigma} = \mathbf{B}_0^{-1}(\mathbf{B}_0^{-1})'$

# Identification in SVAR

## The Identification Problem

$\boldsymbol{\Sigma}$ has $K(K+1)/2$ unique elements, but $\mathbf{B}_0^{-1}$ has $K^2$ elements.

Need $K(K-1)/2$ additional restrictions!

## Common Identification Schemes

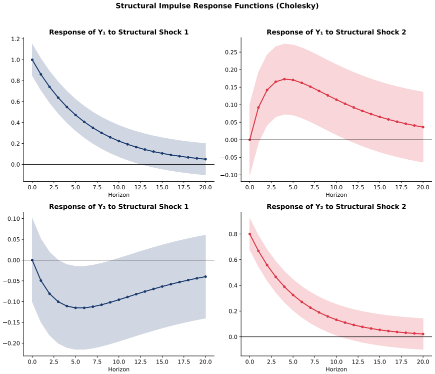
1. **Short-run restrictions**: Zero impact effects (Cholesky)
2. **Long-run restrictions**: Zero long-run effects (Blanchard-Quah)
3. **Sign restrictions**: Inequality constraints on IRFs
4. **External instruments**: Use outside information

## Example: Cholesky (Recursive) Ordering

For $K=2$: $\mathbf{B}_0^{-1} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$

Variable 1 doesn't respond to shock 2 contemporaneously.

# Structural IRF Example




- Structural IRFs based on Cholesky identification
- Order of variables affects interpretation of shocks
- First variable responds only to own shocks contemporaneously

# Variance Decomposition

## Question

What proportion of the forecast error variance of $Y_i$ at horizon $h$ is due to shocks to $Y_j$?

## FEVD Formula

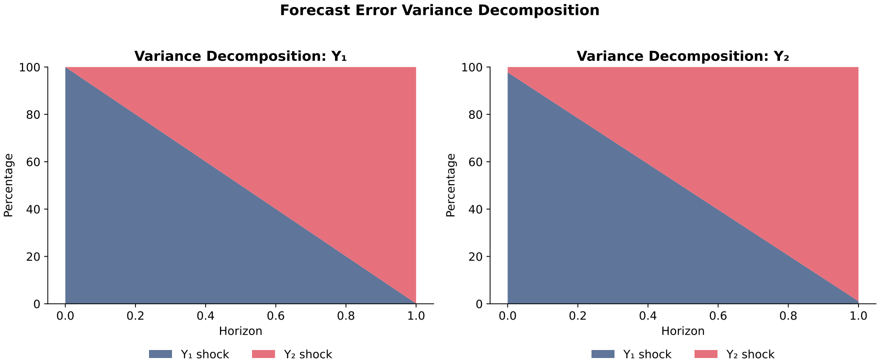
$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\boldsymbol{\Theta}_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^{K} [\boldsymbol{\Theta}_s]_{ik}^2}$$

This gives the **percentage** of $Y_i$'s $h$-step forecast variance explained by shocks to $Y_j$.

## Properties

- $0 \leq \text{FEVD}_{ij}(h) \leq 1$
- $\sum_{j=1}^{K} \text{FEVD}_{ij}(h) = 1$ (sums to 100%)
- At $h = 1$: Own shocks dominate (by construction with Cholesky)

# FEVD: Example




- FEVD shows the proportion of forecast variance attributable to each shock
- At short horizons, own shocks dominate; cross-variable effects grow over time
- Useful for understanding the relative importance of different shocks in the system

# FEVD: Numerical Example

## Computing FEVD for Bivariate VAR

Using orthogonalized IRFs $\mathbf{\Theta}_h$, FEVD at horizon $H$:

$$\text{FEVD}_{11}(H) = \frac{\sum_{h=0}^{H-1} \theta_{11}^2(h)}{\sum_{h=0}^{H-1} [\theta_{11}^2(h) + \theta_{12}^2(h)]}$$

## Example Calculation

| $h$ | $\theta_{11}(h)$ | $\theta_{12}(h)$ | $\theta_{11}^2(h)$ | $\theta_{12}^2(h)$ |
|---|---|---|---|---|
| 0 | 1.00 | 0.00 | 1.00 | 0.00 |
| 1 | 0.70 | 0.20 | 0.49 | 0.04 |
| 2 | 0.47 | 0.26 | 0.22 | 0.07 |

$\text{FEVD}_{11}(3) = \frac{1.00+0.49+0.22}{1.00+0.49+0.22+0.00+0.04+0.07} = \frac{1.71}{1.82} = 94\%$

# Historical Decomposition

## Definition

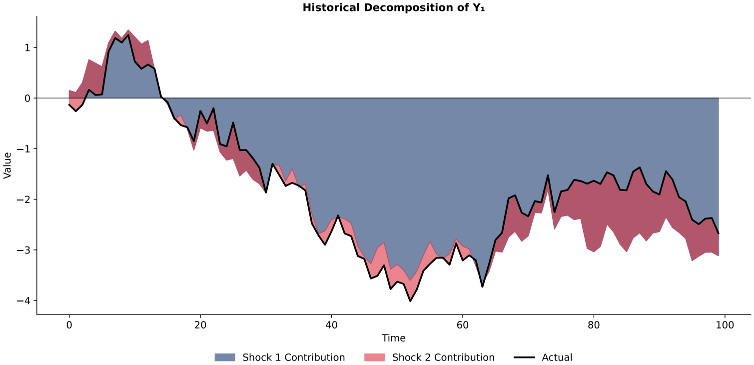
**Historical decomposition** breaks down each observed value into contributions from each structural shock:

$$Y_{it} - \bar{Y}_i = \sum_{j=1}^{K} \sum_{s=0}^{t-1} \theta_{ij}(s) \cdot u_{j,t-s}$$

## Application

- "How much of the 2008 GDP decline was due to financial shocks vs. oil shocks?"
- Attributes historical movements to specific identified shocks
- Useful for policy analysis and narrative interpretation

# Historical Decomposition: Example




- Each color represents the contribution of a different structural shock
- Stacked contributions sum to the actual observed deviation from mean
- Helps identify which shocks drove historical episodes

# Residual Diagnostics

## What to Check

After estimating VAR, verify that residuals $\hat{\varepsilon}_t$ behave like white noise:

1. No serial correlation
2. Constant variance (homoskedasticity)
3. Normality (for inference)

## Why It Matters

- Autocorrelated residuals $\Rightarrow$ inefficient estimates
- Heteroskedasticity $\Rightarrow$ invalid standard errors
- Non-normality $\Rightarrow$ inference may be unreliable

# Testing for Serial Correlation

## Portmanteau Test (Ljung-Box)

$$Q_h = T(T+2)\sum_{j=1}^{h} \frac{1}{T-j} \text{tr}(\hat{\mathbf{C}}_j' \hat{\mathbf{C}}_0^{-1} \hat{\mathbf{C}}_j \hat{\mathbf{C}}_0^{-1})$$

where $\hat{\mathbf{C}}_j = \frac{1}{T}\sum_{t=j+1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_{t-j}'$
Under $H_0$ (no autocorrelation): $Q_h \sim \chi^2_{K^2(h-p)}$

## Breusch-Godfrey LM Test

1. Regress $\hat{\varepsilon}_t$ on $\hat{\varepsilon}_{t-1}, \ldots, \hat{\varepsilon}_{t-h}$ and original regressors
2. $LM = T \cdot R^2 \sim \chi^2_{K^2 h}$ under $H_0$

## If Rejected

Consider increasing lag order $p$ or adding additional variables.

# Testing for Heteroskedasticity

## ARCH-LM Test

Test for autoregressive conditional heteroskedasticity in residuals:

$$\hat{\varepsilon}_{it}^2 = \alpha_0 + \alpha_1 \hat{\varepsilon}_{i,t-1}^2 + \cdots + \alpha_q \hat{\varepsilon}_{i,t-q}^2 + v_t$$

$H_0$: $\alpha_1 = \cdots = \alpha_q = 0$ (homoskedasticity)
$LM = TR^2 \sim \chi_q^2$

## Multivariate Version

Test all equations jointly using:

$$\text{vech}(\hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t') = \mathbf{c} + \sum_{j=1}^{q} \mathbf{B}_j \text{vech}(\hat{\boldsymbol{\varepsilon}}_{t-j} \hat{\boldsymbol{\varepsilon}}_{t-j}') + \mathbf{v}_t$$

# Normality Testing

## Jarque-Bera Test (Univariate)

$$JB = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right) \sim \chi^2_2$$

where $S$ = skewness, $K$ = kurtosis

## Multivariate Normality (Doornik-Hansen)

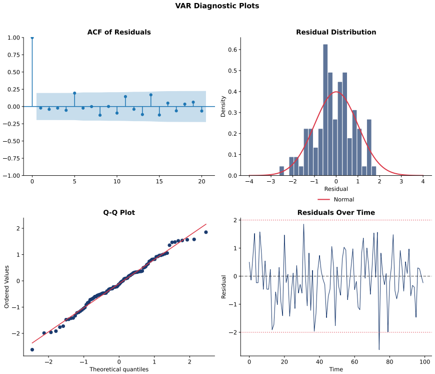
Transform residuals and test joint skewness and kurtosis:

$$DH = s_1'(\boldsymbol{\Omega}^{-1/2})'(\boldsymbol{\Omega}^{-1/2})s_1 + s_2'(\boldsymbol{\Omega}^{-1/2})'(\boldsymbol{\Omega}^{-1/2})s_2 \sim \chi^2_{2K}$$

## Note

Normality is often rejected in financial data. Consider robust standard errors if non-normality is severe.

# Diagnostic Summary Plot




- Residual ACF should show no significant autocorrelation
- Histogram should approximate normal distribution
- Q-Q plot should follow 45-degree line

# Point Forecasts from VAR

## Iterative Forecasting

For VAR(1): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t$

**1-step forecast**: $\hat{\mathbf{Y}}_{T+1|T} = \mathbf{c} + \mathbf{A}\mathbf{Y}_T$

**2-step forecast**: $\hat{\mathbf{Y}}_{T+2|T} = \mathbf{c} + \mathbf{A}\hat{\mathbf{Y}}_{T+1|T}$

***h*-step forecast**: $\hat{\mathbf{Y}}_{T+h|T} = \mathbf{c} + \mathbf{A}\hat{\mathbf{Y}}_{T+h-1|T}$

## Direct Formula

$$\hat{\mathbf{Y}}_{T+h|T} = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{h-1})\mathbf{c} + \mathbf{A}^h\mathbf{Y}_T$$

For stable VAR: converges to $\boldsymbol{\mu} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c}$ as $h \to \infty$

# Forecast Error and MSE

## $h$-Step Forecast Error

$$\mathbf{e}_{T+h|T} = \mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T} = \sum_{j=0}^{h-1} \mathbf{A}^j \boldsymbol{\varepsilon}_{T+h-j}$$

## Mean Squared Error Matrix

$$\text{MSE}(\hat{\mathbf{Y}}_{T+h|T}) = \mathbb{E}[\mathbf{e}_{T+h|T}\mathbf{e}'_{T+h|T}] = \sum_{j=0}^{h-1} \mathbf{A}^j \boldsymbol{\Sigma} (\mathbf{A}^j)'$$

## Key Insight

- MSE increases with horizon $h$
- For stable VAR: MSE converges to unconditional variance $\boldsymbol{\Gamma}(0)$
- Long-horizon forecasts $\to$ unconditional mean with uncertainty $= \boldsymbol{\Gamma}(0)$

# Forecast Confidence Intervals

## Constructing Intervals

For normally distributed errors, $(1-\alpha)$ confidence interval:

$$\hat{Y}_{i,T+h|T} \pm z_{\alpha/2}\sqrt{[\mathrm{MSE}(\hat{\mathbf{Y}}_{T+h|T})]_{ii}}$$

## Joint Confidence Regions

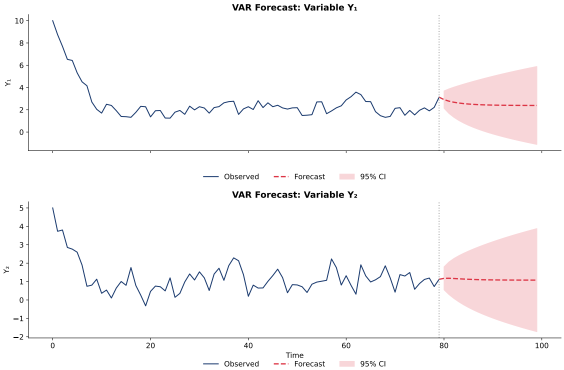
For multiple variables, use ellipsoids:

$$(\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T})'[\mathrm{MSE}(\hat{\mathbf{Y}}_{T+h|T})]^{-1}(\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T}) \leq \chi^2_{K,\alpha}$$

## Note

These assume known parameters. Bootstrap methods account for parameter uncertainty.

# VAR Forecasts: Example




- Point forecasts shown as solid line beyond observed data
- Confidence bands widen as forecast horizon increases
- Forecasts converge to unconditional mean for long horizons

# Forecast Evaluation

## Out-of-Sample Evaluation

Split data: estimation sample (1 to $T_1$) and test sample ($T_1 + 1$ to $T$). Compute forecast errors: $e_{t+h} = Y_{t+h} - \hat{Y}_{t+h|t}$

## Common Metrics

- **RMSE**: $\sqrt{\frac{1}{n}\sum e_{t+h}^2}$ **MAE**: $\frac{1}{n}\sum |e_{t+h}|$ **MAPE**: $\frac{100}{n}\sum \left|\frac{e_{t+h}}{Y_{t+h}}\right|$

## Diebold-Mariano Test

Test whether VAR forecasts are significantly better than alternative: $DM = \frac{\bar{d}}{\sqrt{\hat{\sigma}_d^2/n}} \sim N(0,1)$ where $d_t = L(e_{1t}) - L(e_{2t})$ is the loss differential.

# Example: GDP and Unemployment

## Okun's Law

There is a negative relationship between GDP growth and unemployment:

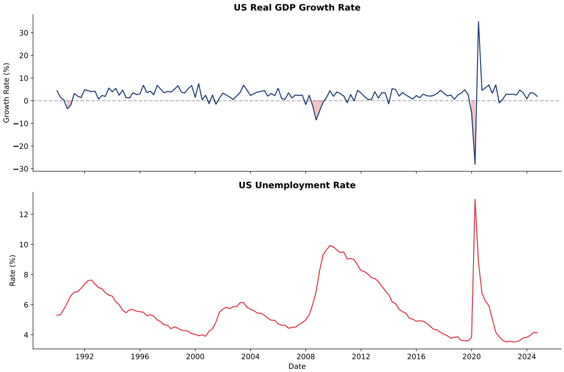
$$\Delta U_t \approx -\beta(\Delta Y_t - \bar{g})$$

where $\bar{g}$ is trend GDP growth and $\beta \approx 0.4$.

## VAR Analysis Questions

1. Does GDP growth Granger-cause unemployment changes?
2. Does unemployment Granger-cause GDP growth?
3. How do shocks propagate between variables?

# GDP and Unemployment: Data

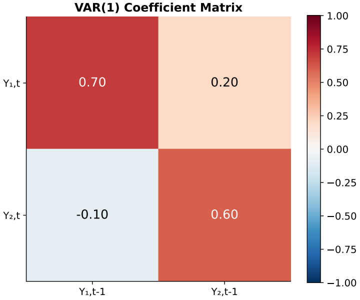


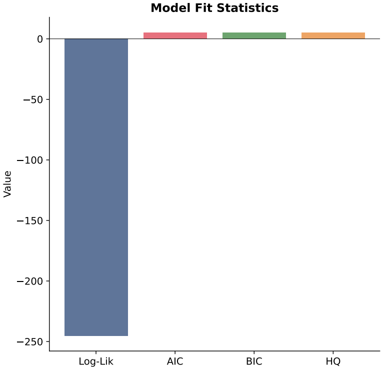

- GDP growth and unemployment rate show clear negative correlation (Okun's Law)
- Both series exhibit cyclical patterns related to business cycle fluctuations
- This bivariate system is ideal for VAR analysis and Granger causality testing

# VAR Workflow

1. **Data preparation**
   - Check for stationarity (unit root tests)
   - Transform if necessary (differences, logs)
2. **Lag selection**
   - Use AIC, BIC, HQ criteria
   - Check residual autocorrelation
3. **Estimation**
   - OLS equation by equation
   - Check stability (eigenvalues)
4. **Analysis**
   - Granger causality tests
   - Impulse response functions
   - Variance decomposition
5. **Forecasting**

# Estimated VAR Results




- Estimated coefficients with standard errors and t-statistics
- Information criteria values for model comparison
- Model diagnostics summary (residual tests)

# Granger Causality Results

## Test Results: GDP and Unemployment

| Null Hypothesis | F-statistic | df | p-value | Decision |
|---|---|---|---|---|
| GDP $\not\rightarrow$ Unemployment | 8.42 | (2, 95) | 0.0004 | Reject |
| Unemployment $\not\rightarrow$ GDP | 2.15 | (2, 95) | 0.1220 | Fail to Reject |

## Interpretation

- GDP growth Granger-causes unemployment (consistent with Okun's Law)
- Unemployment does not significantly Granger-cause GDP
- Evidence of **unidirectional** causality: GDP $\rightarrow$ Unemployment

# Python Implementation

## VAR in Python (statsmodels)

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests

# Fit VAR model
model = VAR(data)
results = model.fit(maxlags=4, ic='aic')

# Granger causality test
granger_test = grangercausalitytests(data[['Y1', 'Y2']],
                                      maxlag=4)
# Impulse response functions
irf = results.irf(periods=20)
irf.plot()

# Variance decomposition
fevd = results.fevd(periods=20)
fevd.plot()
```

# R Implementation

## VAR in R (vars package)

```
library(vars)

# Select optimal lag order
lag_select <- VARselect(data, lag.max = 8)
print(lag_select$selection)

# Fit VAR model
var_model <- VAR(data, p = 2, type = "const")
summary(var_model)

# Granger causality test
causality(var_model, cause = "GDP")

# Impulse response functions
irf_results <- irf(var_model, n.ahead = 20, boot = TRUE)
plot(irf_results)

# Forecast error variance decomposition
fevd_results <- fevd(var_model, n.ahead = 20)
plot(fevd_results)
```

# Example: Monetary Policy Analysis

## Three-Variable VAR

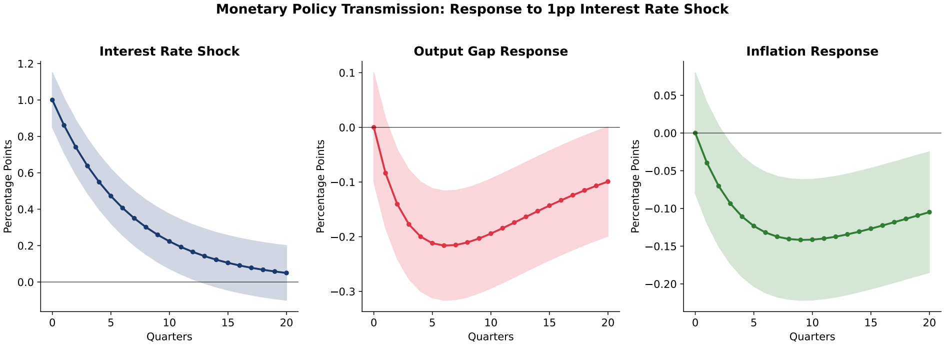
Study the monetary transmission mechanism with:

- $Y_1$: Output gap (GDP deviation from trend)
- $Y_2$: Inflation rate
- $Y_3$: Interest rate (policy instrument)

## Key Questions

1. How does an interest rate shock affect output and inflation?
2. How long until the maximum effect is felt?
3. What fraction of output variance is due to monetary shocks?

# Monetary Policy VAR: IRFs




- Contractionary monetary policy shock (interest rate increase)
- Output decreases with peak effect after 4-6 quarters ("long and variable lags")
- Inflation responds more slowly, decreasing after output

# Key Takeaways

## VAR Models

- Model **multiple** time series jointly
- Each variable depends on its own lags AND lags of other variables
- Estimated by OLS equation by equation; requires stationarity

## Granger Causality

- Tests whether $X$ helps predict $Y$ beyond $Y$'s own history
- **Not** the same as true causality; F-test on coefficient restrictions

## IRF and FEVD

- IRF: How shocks propagate through the system
- FEVD: What proportion of variance is due to each shock
- Both depend on variable ordering (Cholesky decomposition)

# VAR Model Selection Checklist

## Before Estimation

- ☐ Test for unit roots in each variable
- ☐ Transform to stationary if needed (differences, logs)
- ☐ Check for outliers and structural breaks

## Model Specification

- ☐ Select lag order using AIC/BIC
- ☐ Estimate VAR by OLS
- ☐ Check stability (eigenvalues inside unit circle)

## Post-Estimation

- ☐ Test residuals for autocorrelation
- ☐ Test for ARCH effects
- ☐ Test for normality
- ☐ Compute IRFs, FEVDs, Granger tests

# Common Mistakes to Avoid

## Pitfalls in VAR Analysis

1. **Ignoring non-stationarity**: Always test for unit roots first
2. **Overfitting**: Too many lags $\Rightarrow$ poor forecasts
3. **Wrong ordering**: Cholesky results depend on variable order
4. **Confusing correlation with causation**: Granger causality $\neq$ true causality
5. **Ignoring parameter uncertainty**: Use bootstrap CIs for IRFs
6. **Short samples**: VAR requires many observations ($T > 50$)

## What's Next?

### Topics for Further Study

- **Cointegration**: Long-run relationships between non-stationary variables
- **VECM**: Error correction models for cointegrated systems
- **Structural VAR**: Imposing economic theory restrictions
- **Panel VAR**: VAR for panel data
- **Bayesian VAR**: Shrinkage priors for high-dimensional systems

Questions?

# Quiz Question 1

## Question

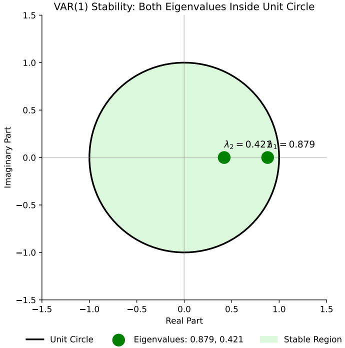
For a VAR(1) model with coefficient matrix $\mathbf{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.5 \end{pmatrix}$, is the model stable?

- A Yes, because all diagonal elements are less than 1
- B Yes, because all eigenvalues are inside the unit circle
- C No, because the sum of coefficients exceeds 1
- D Cannot be determined without knowing $\boldsymbol{\Sigma}$

# Quiz Question 1: Answer

## Correct Answer: (B) Eigenvalues inside unit circle

$\lambda_1 = 0.879$, $\lambda_2 = 0.421$ — both $|\lambda| < 1 \Rightarrow$ Stable!



# Quiz Question 2

**Question**

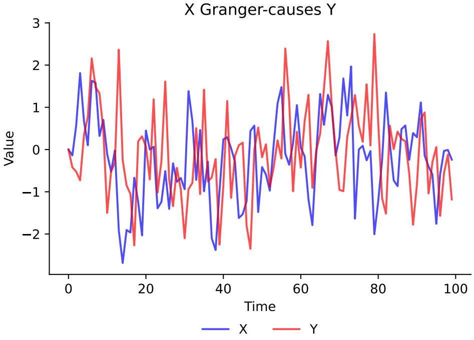
If $X$ Granger-causes $Y$ at the 5% significance level, which of the following statements is TRUE?

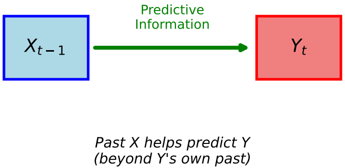
- A. $X$ is the economic cause of $Y$
- B. Past values of $X$ contain useful information for predicting $Y$
- C. $Y$ cannot Granger-cause $X$
- D. The correlation between $X$ and $Y$ is positive

# Quiz Question 2: Answer

**Correct Answer: (B) Predictive information**

Granger causality = predictive content, not true economic causation. Past X helps predict Y.



# Quiz Question 3

## Question

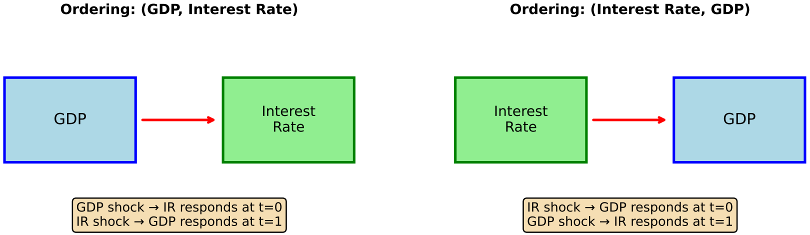
In a VAR with Cholesky-identified IRFs, what does the ordering of variables determine?

- A. The magnitude of the impulse responses
- B. The speed at which shocks die out
- C. Which variables can respond contemporaneously to which shocks
- D. The number of lags in the VAR

# Quiz Question 3: Answer

**Correct Answer: (C) Contemporaneous responses**

Ordering determines which variables respond immediately to which shocks.



# Quiz Question 4

## Question

For a bivariate VAR(1), how many parameters need to be estimated (excluding the error covariance matrix)?

- A 4
- B 6
- C 8
- D 10

# Quiz Question 4: Answer

## Correct Answer: (B) 6 parameters

## Detailed Count

VAR(1) with $K = 2$ variables:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\text{2 params}} + \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\text{4 params}} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

- Constant vector $\mathbf{c}$: $K = 2$ parameters
- Coefficient matrix $\mathbf{A}$: $K^2 = 4$ parameters
- Total: $K + K^2 = 2 + 4 = 6$ parameters

## General Formula

VAR($p$) with $K$ variables: $K + pK^2$ parameters (excluding $\mathbf{\Sigma}$)

## Quiz Question 5

**Question**

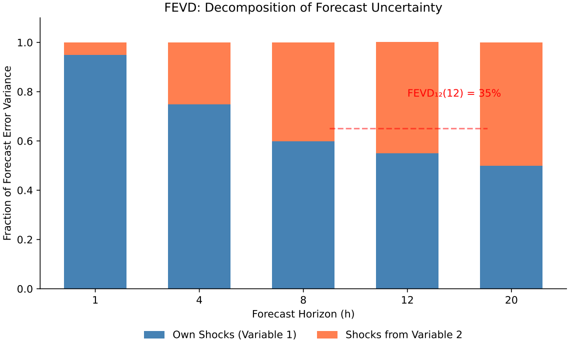
What does $\text{FEVD}_{12}(h) = 0.35$ mean?

- A. 35% of variable 1's total variance is explained by variable 2
- B. 35% of variable 1's $h$-step forecast error variance is due to shocks to variable 2
- C. The correlation between variables 1 and 2 at lag $h$ is 0.35
- D. Variable 2 explains 35% of the impulse response of variable 1

# Quiz Question 5: Answer

**Correct Answer: (B) Forecast error variance decomposition**

35% of variable 1's *h*-step forecast error variance is due to shocks from variable 2.



# Case Study: VAR Analysis of GDP and Inflation

## Research Question

Is there a dynamic relationship between real GDP growth and inflation? Does one variable Granger-cause the other?

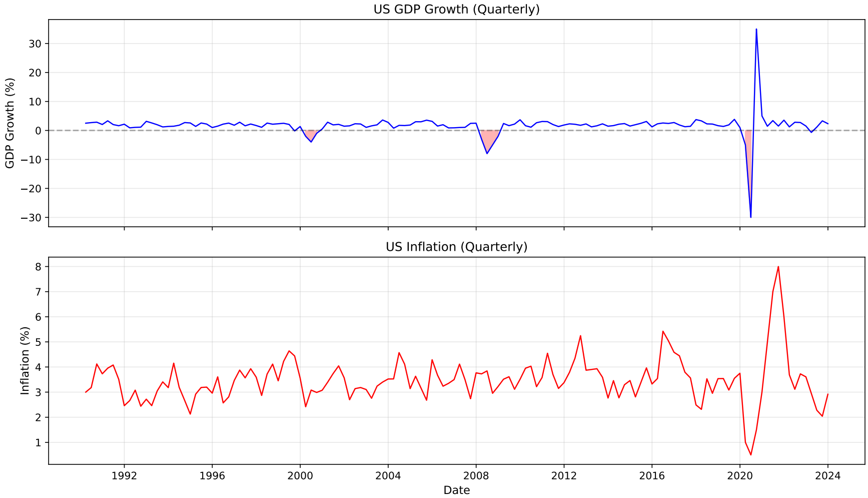
**Data**

- US Quarterly Data (1960-2023)
- Real GDP Growth Rate
- CPI Inflation Rate
- Source: FRED Database

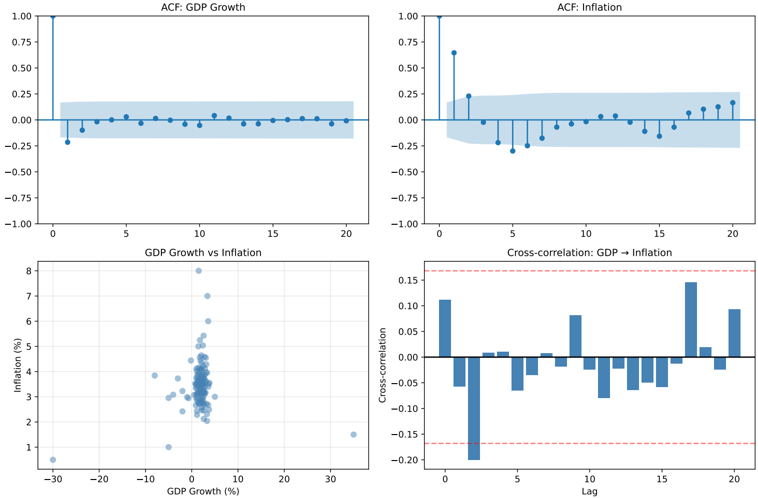
**Methodology**

- Stationarity tests (ADF)
- Lag order selection
- VAR estimation
- Granger causality tests
- Impulse response analysis

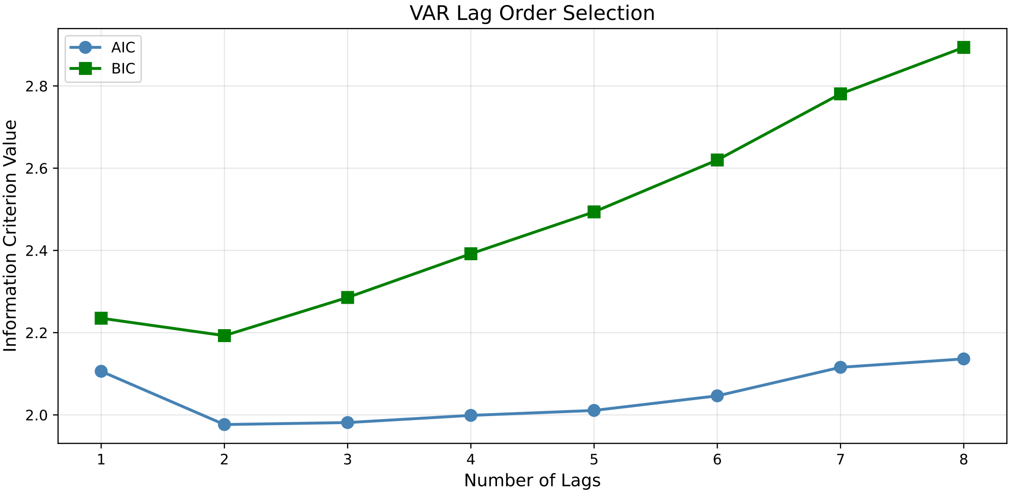
# Step 1: Data Visualization



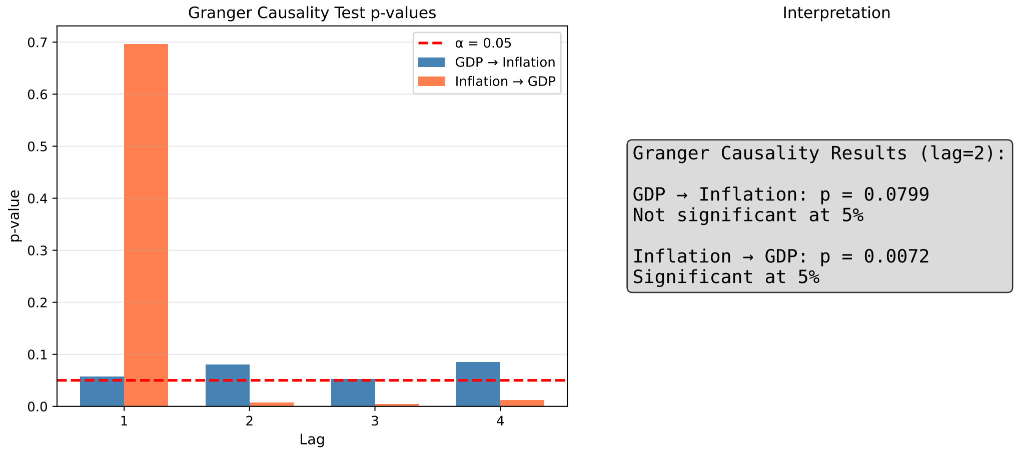
# Step 2: Stationarity Tests



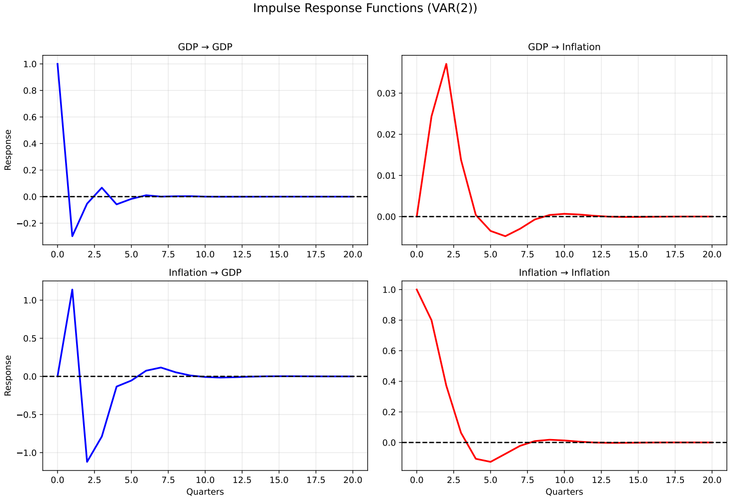
# Step 3: Lag Selection and VAR Estimation



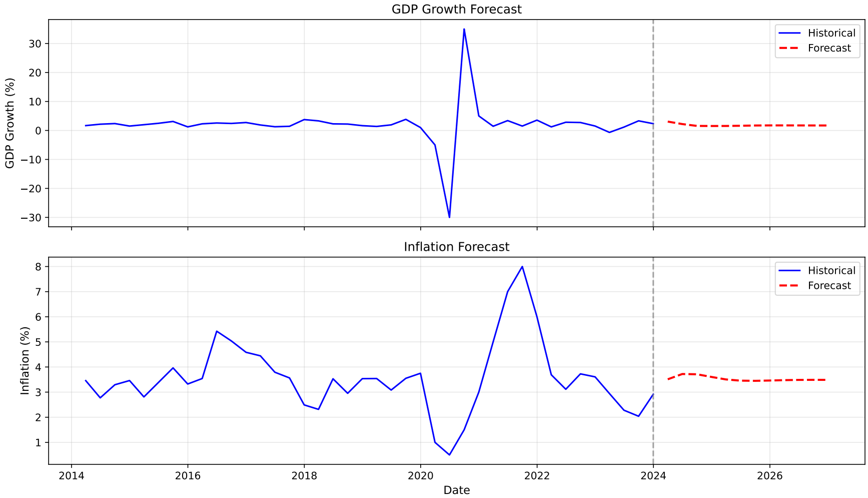
# Step 4: Granger Causality Tests



# Step 5: Impulse Response Functions



# Step 6: Forecasting



# References


- Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*. Springer.
- Hamilton, J.D. (1994). *Time Series Analysis*. Princeton University Press.
- Sims, C.A. (1980). Macroeconomics and Reality. *Econometrica*, 48(1), 1-48.
- Granger, C.W.J. (1969). Investigating Causal Relations by Econometric Models. *Econometrica*, 37(3), 424-438.
- Toda, H.Y. & Yamamoto, T. (1995). Statistical Inference in Vector Autoregressions with Possibly Integrated Processes. *Journal of Econometrics*, 66(1-2), 225-250.