



Time Series Analysis and Forecasting

Chapter 3: ARIMA Models



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Learning Objectives

By the end of this chapter, you will be able to:

- Understand the concept and implications of non-stationarity
- Apply differencing to achieve stationarity in time series
- Use the Augmented Dickey-Fuller (ADF) test for unit root detection
- Build, estimate, and forecast with ARIMA models



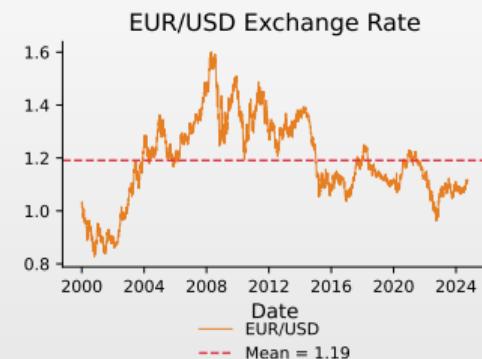
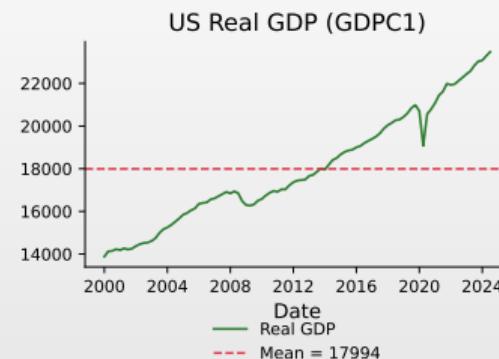
Outline

- Motivation
- Non-Stationarity in Time Series
- Differencing and the Difference Operator
- ARIMA(p,d,q) Models
- Unit Root Tests
- ARIMA Model Identification
- ARIMA Estimation
- Diagnostic Checking
- Forecasting with ARIMA
- Case Study
- Summary
- AI Use Case
- Quiz



Motivating Example: Non-Stationary Data Is Everywhere

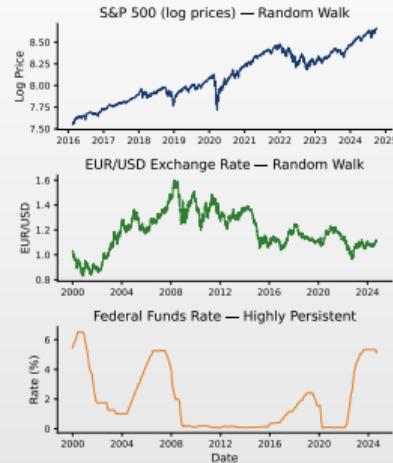
Non-stationary data: sample mean is meaningless



Key Observations

- Stock prices, GDP, exchange rates all exhibit **trends or wandering behavior**
- The sample mean (red line) is meaningless for a random walk
- Standard ARMA models **cannot** handle these series directly

Real-World Applications



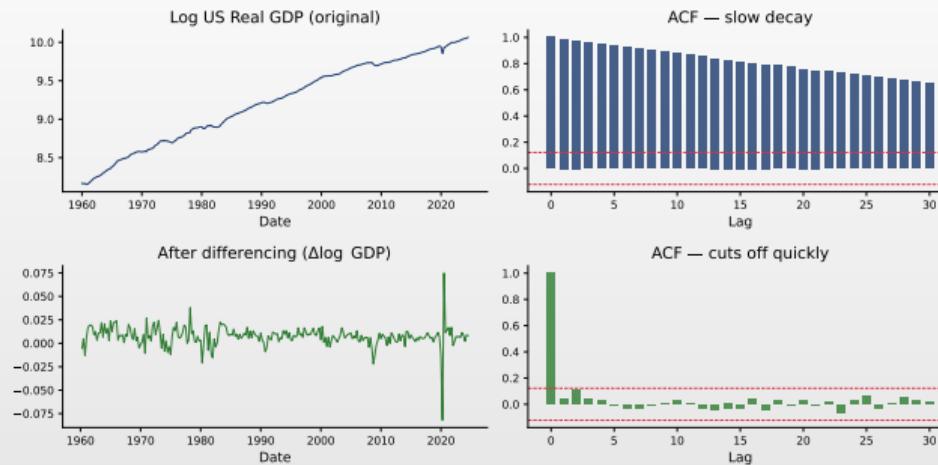
The Challenge

Financial/economic data are typically I(1): stock prices, exchange rates, interest rates.

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The Solution: Differencing



Key Insight

Differencing transforms a non-stationary series into a stationary one: $\Delta Y_t = Y_t - Y_{t-1}$. The ACF changes from slow decay to quick decay!

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What We'll Learn Today

Core Concepts

1. **Non-Stationarity:** Why it matters and how to detect it
2. **Unit Root Tests:** ADF, PP, KPSS tests
3. **Differencing:** The key transformation
4. **ARIMA Models:** Combining differencing with ARMA
5. **Box-Jenkins Methodology:** Identify → Estimate → Diagnose

By the End of This Lecture

You will be able to model and forecast non-stationary time series like stock prices, GDP, and exchange rates using ARIMA models.



Why Non-Stationarity Matters

The Problem

Many economic and financial time series are **non-stationary**:

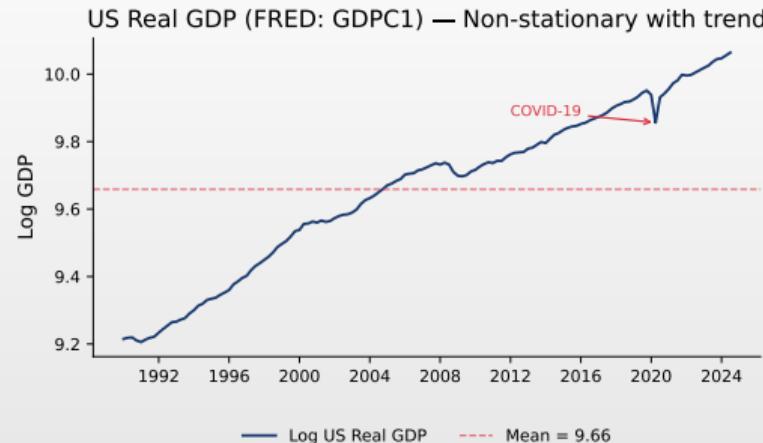
- GDP, stock prices, exchange rates, inflation indices
- They exhibit trends, changing means, or growing variance

Consequences of Non-Stationarity

- Standard ARMA models assume stationarity
- OLS regression with non-stationary data leads to **spurious regression**
- Sample moments (mean, variance, ACF) are not consistent estimators
- Statistical inference becomes invalid



Example: US Real GDP



Key Observations

- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data



Types of Non-Stationarity

Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a deterministic function of time
- Can be removed by **detrending**
- Shocks have temporary effects

Stochastic Trend (Unit Root)

$$Y_t = Y_{t-1} + \varepsilon_t$$

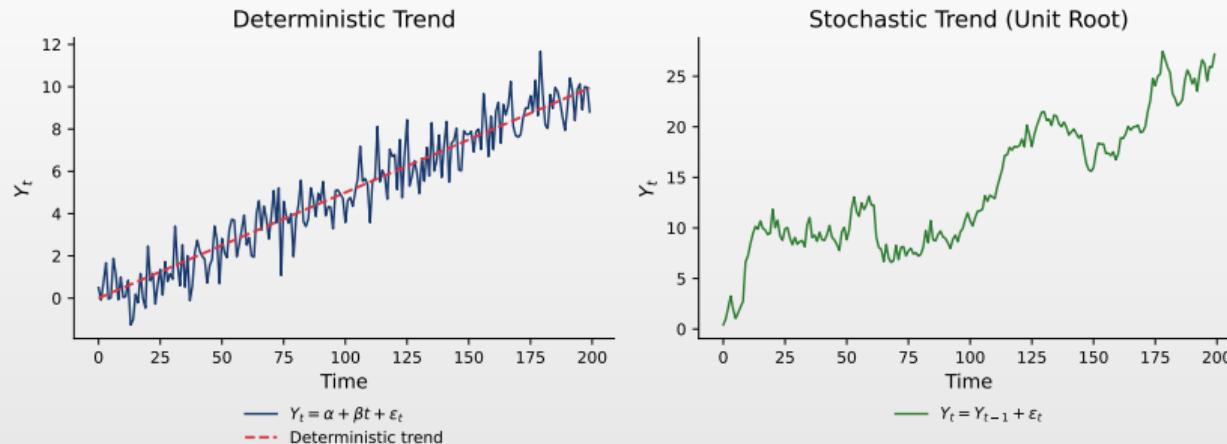
- Random walk process
- Must be removed by **differencing**
- Shocks have permanent effects

Key Distinction

Correct identification is crucial: detrending a unit root → misspecification; differencing trend-stationary → misspecification.



Visualizing the Difference



Key Distinction

- Left:** Deterministic trend – deviations from trend are temporary
- Right:** Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!



The Random Walk Process

Definition 1 (Random Walk)

A **random walk** is defined as:

$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

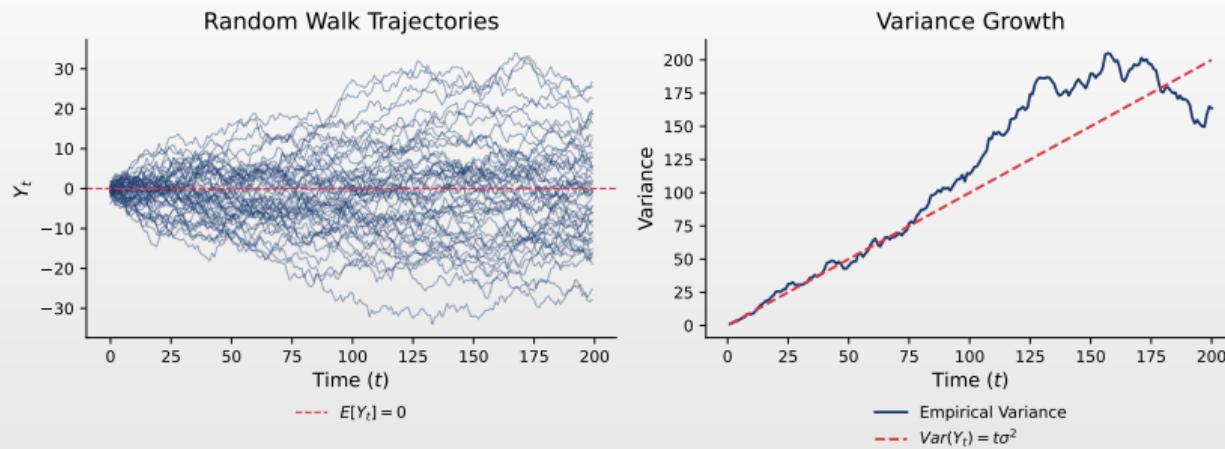
With initial condition $Y_0 = 0$, we have: $Y_t = \sum_{i=1}^t \varepsilon_i$

Properties of Random Walk

- $\mathbb{E}[Y_t] = 0$ (constant mean)
- $\text{Var}(Y_t) = t\sigma^2$ (variance grows with time!)
- $\text{Cov}(Y_t, Y_{t-k}) = (t - k)\sigma^2$ for $k \leq t$
- ACF: $\rho_k = \sqrt{\frac{t-k}{t}} \rightarrow 1$ as $t \rightarrow \infty$



Random Walk: Visual Illustration



Key Properties

Left: Paths wander unpredictably, no mean reversion. **Right:** $\text{Var}(Y_t) = t\sigma^2$ grows linearly \Rightarrow non-stationary.

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Proof: Random Walk Variance

Claim: For $Y_t = Y_{t-1} + \varepsilon_t$ with $Y_0 = 0$: $\text{Var}(Y_t) = t\sigma^2$

Proof: By recursive substitution: $Y_t = \sum_{i=1}^t \varepsilon_i$

Taking variance:

$$\text{Var}(Y_t) = \text{Var}\left(\sum_{i=1}^t \varepsilon_i\right) = \sum_{i=1}^t \text{Var}(\varepsilon_i) + 2 \sum_{i < j} \text{Cov}(\varepsilon_i, \varepsilon_j)$$

Since ε_t independent (white noise): $\text{Var}(Y_t) = \sum_{i=1}^t \sigma^2 = \boxed{t\sigma^2}$

Variance depends on $t \Rightarrow$ non-stationary



Proof: Random Walk Autocovariance

Claim: $\text{Cov}(Y_t, Y_{t-k}) = (t - k)\sigma^2$ for $k \leq t$

Proof: Using $Y_t = \sum_{i=1}^t \varepsilon_i$ and $Y_{t-k} = \sum_{i=1}^{t-k} \varepsilon_i$:

$$\begin{aligned}\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}\left(\sum_{i=1}^t \varepsilon_i, \sum_{j=1}^{t-k} \varepsilon_j\right) \\ &= \sum_{i=1}^t \sum_{j=1}^{t-k} \text{Cov}(\varepsilon_i, \varepsilon_j) = \sum_{i=1}^{t-k} \text{Var}(\varepsilon_i) = \boxed{(t - k)\sigma^2}\end{aligned}$$

Only terms with $i = j$ survive (when $i \leq t - k$).

ACF:

$$\rho(k) = \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} = \frac{(t - k)\sigma^2}{\sqrt{t\sigma^2 \cdot (t - k)\sigma^2}} = \sqrt{\frac{t - k}{t}}$$



Random Walk with Drift

Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

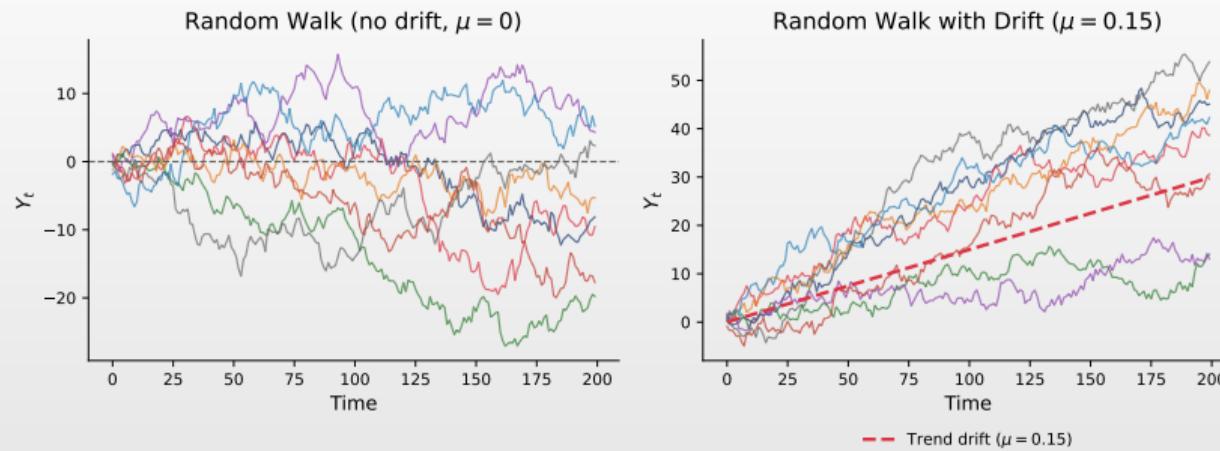
Equivalently: $Y_t = Y_0 + \mu t + \sum_{i=1}^t \varepsilon_i$

Properties

- $\mathbb{E}[Y_t] = Y_0 + \mu t$ (mean grows linearly)
- $\text{Var}(Y_t) = t\sigma^2$ (variance still grows)
- The drift μ creates an upward or downward trend
- Still non-stationary despite having a “trend”



Random Walk with Drift: Visual Illustration



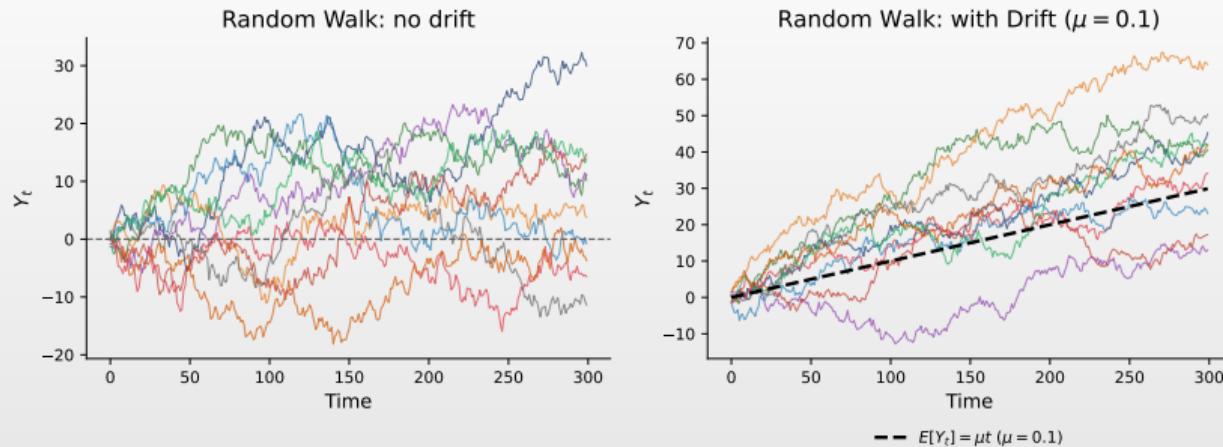
Comparison

Without drift (blue): wanders around zero with no direction. With drift $\mu > 0$ (red): systematic upward trend. Both are non-stationary — drift adds deterministic trend to stochastic wandering.

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Simulating Random Walks

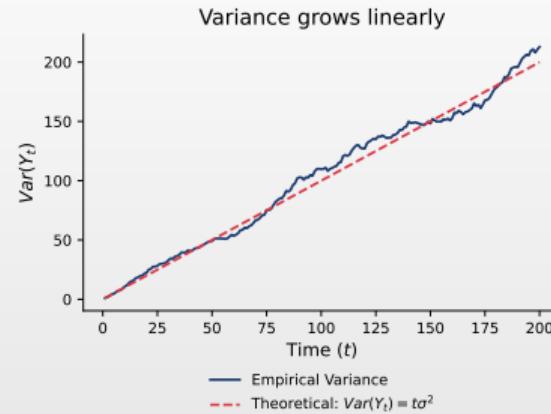
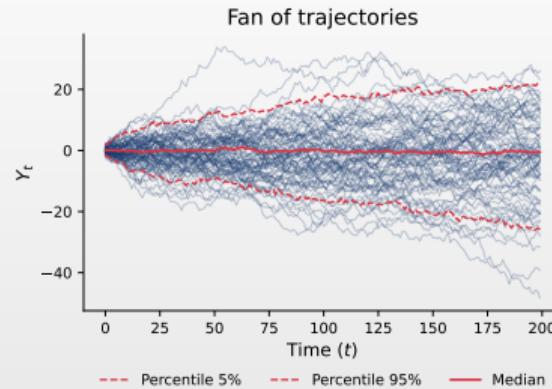


Random Walk Types

- Left:** Pure random walks – no drift, wander unpredictably
- Right:** Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time



Variance Growth: Why Random Walks Are Non-Stationary



Key Observations

- Left: Fan of paths shows uncertainty growing over time
- Right: Variance grows linearly: $\text{Var}(Y_t) = t\sigma^2$
- This violates stationarity (variance should be constant)



Integrated Processes

Definition 3 (Integrated Process of Order d)

A time series $\{Y_t\}$ is **integrated of order d** , written $Y_t \sim I(d)$, if:

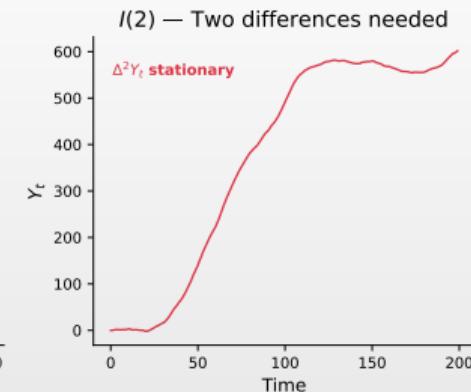
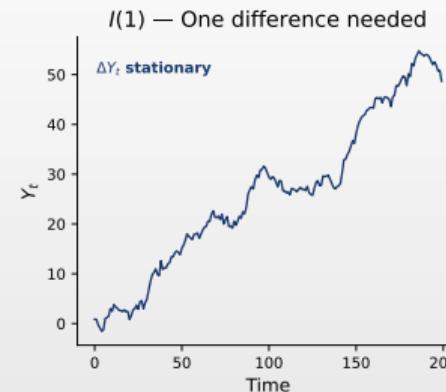
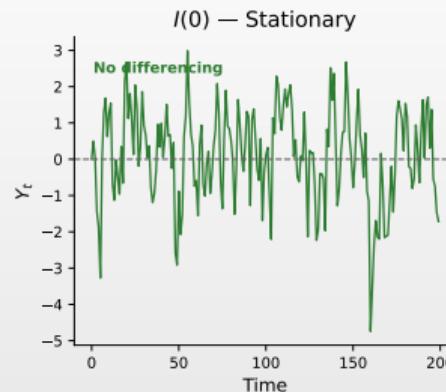
- Y_t is non-stationary
- $(1 - L)^d Y_t = \Delta^d Y_t$ is stationary
- $(1 - L)^{d-1} Y_t$ is still non-stationary

Common Cases

- $I(0)$: Stationary process (e.g., ARMA)
- $I(1)$: First difference is stationary (most common for economic data)
- $I(2)$:
 - ▶ Second difference is stationary (less common)



Integrated Process: Visual Illustration



Order of Integration

- I(0): Stationary \Rightarrow no differencing needed
- I(1): One difference needed (random walk)
- I(2): Two differences needed
- Most economic series are I(0) or I(1)



The Difference Operator

Definition 4 (First Difference)

The **first difference operator** Δ is defined as: $\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$, where L is the lag operator ($LY_t = Y_{t-1}$).

Higher-Order Differences

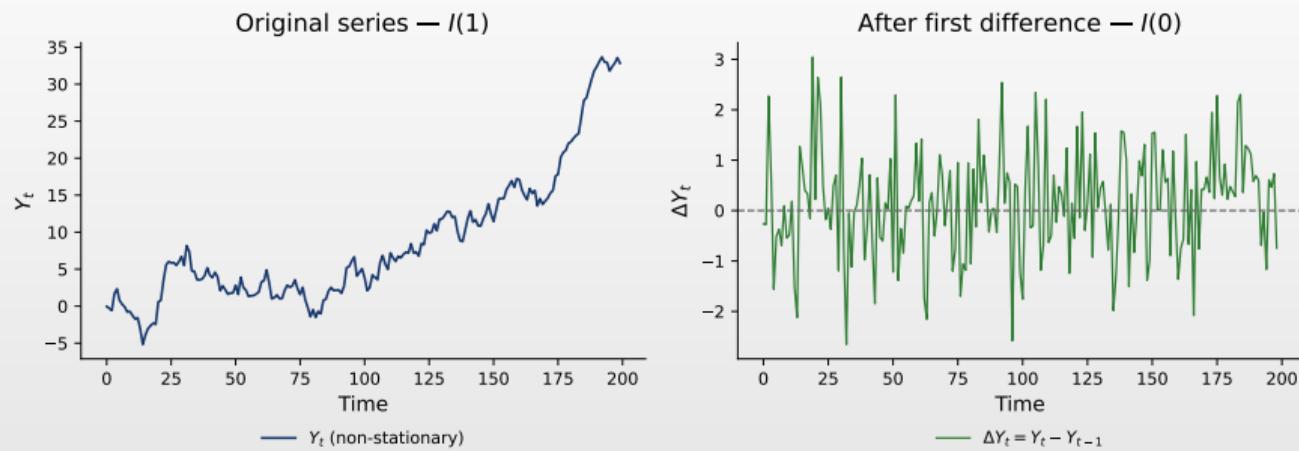
- ◻ Second difference: $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- ◻ $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- ◻ d -th difference: $\Delta^d Y_t = (1 - L)^d Y_t$

Key Result

If $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$ (stationary).



First Difference: Visual Illustration



Observation

- Left: non-stationary series
- Right: after first difference, the series becomes stationary



Example: Differencing a Random Walk

Random Walk to White Noise

Let $Y_t = Y_{t-1} + \varepsilon_t$ (random walk). Taking the first difference:

$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

Interpretation

- A random walk is $I(1)$
- One difference transforms it to $I(0)$
- The “changes” in a random walk are stationary



Proof: Differencing Induces Stationarity

Claim: If $Y_t \sim I(1)$, then $\Delta Y_t = Y_t - Y_{t-1}$ is stationary.

Proof for Random Walk with Drift: $Y_t = \mu + Y_{t-1} + \varepsilon_t$

The first difference is:

$$\Delta Y_t = Y_t - Y_{t-1} = \mu + \varepsilon_t$$

Check stationarity conditions:

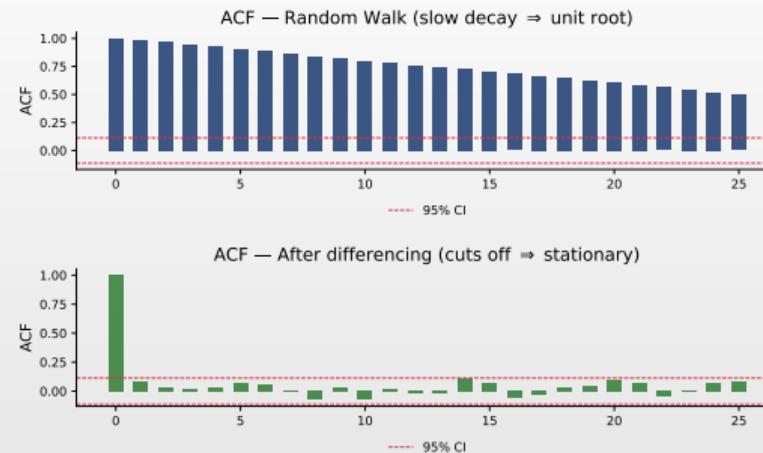
1. **Mean:** $\mathbb{E}[\Delta Y_t] = \mu$ (constant, does not depend on t) ✓
2. **Variance:** $\text{Var}(\Delta Y_t) = \text{Var}(\varepsilon_t) = \sigma^2$ (constant) ✓
3. **Autocovariance:** $\text{Cov}(\Delta Y_t, \Delta Y_{t-k}) = \text{Cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$ for $k \neq 0$ ✓

General Principle

- Differencing removes the “memory” that causes variance to accumulate
- For $I(d)$ processes, d differences are needed



ACF Diagnostic: Detecting Non-Stationarity

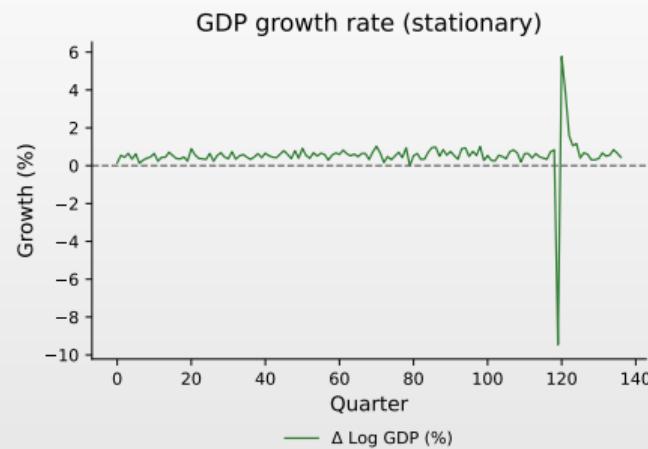
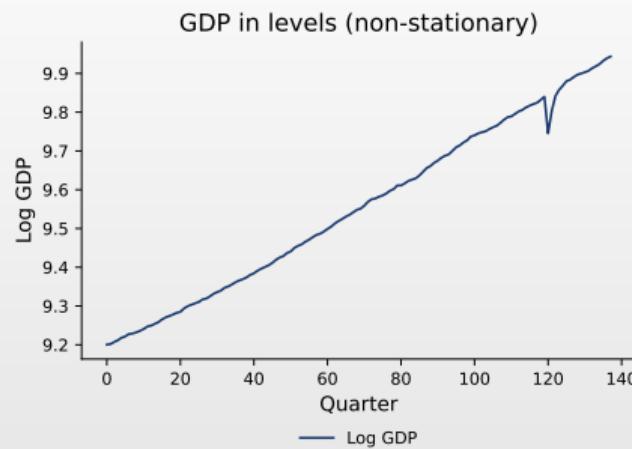


ACF Patterns

- **Top:** Random walk ACF decays very slowly ⇒ unit root
- **Bottom:** After differencing, ACF cuts off ⇒ stationary



Differencing in Practice: GDP Example



Transformation

Left: GDP in levels with clear upward trend (non-stationary). Right: GDP growth rate $\Delta \log(GDP_t)$ fluctuates around constant mean (stationary). One difference removes the stochastic trend.



Overdifferencing

Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
 - ▶ ACF shows spurious patterns
- Inflates variance
 - ▶ Reduces forecast accuracy
- Loses information
 - ▶ Cannot recover original level

Example

If $Y_t \sim I(1)$, then $\Delta Y_t \sim I(0)$. But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with $\theta = 1$ (non-invertible boundary)!



Definition of ARIMA

Definition 5 (ARIMA(p,d,q))

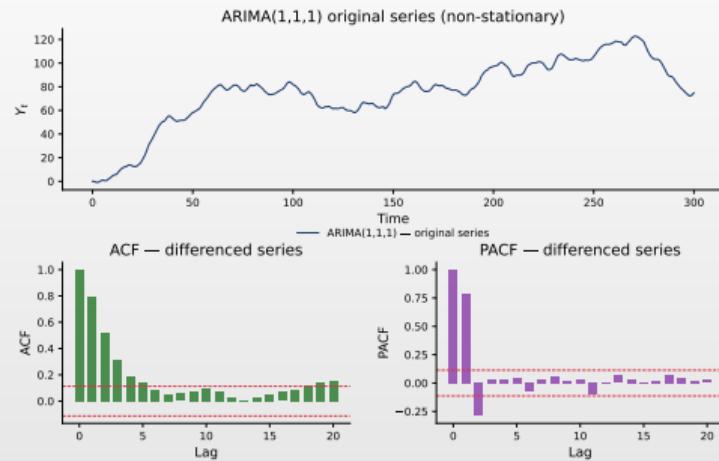
A time series $\{Y_t\}$ follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1 - L)^d Y_t = c + \theta(L)\varepsilon_t$$

where:

- $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$ (AR polynomial)
- $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$ (MA polynomial)
- d is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

ARIMA: Visual Illustration



Interpretation

Top: original ARIMA series (non-stationary). Bottom: after differencing d times, ACF/PACF reveal the AR and MA orders for the stationary component.

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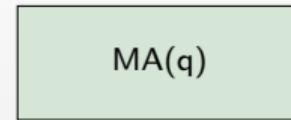
ARIMA Components



Autoregressive
Memory



Integration
Differencing



Moving Average
Shocks

Special Cases

- ARIMA(p,0,q) = ARMA(p,q) – stationary
- ARIMA(0,1,0) = Random walk
- ARIMA(0,1,1) = IMA(1,1) – exponential smoothing
- ARIMA(1,1,0) = ARI(1,1) – differenced AR(1)



ARIMA(1,1,0) Example

ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently: $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$

Interpretation

- The **changes** in Y_t follow an AR(1) process
- If $|\phi_1| < 1$, the changes are stationary
- Y_t itself has a stochastic trend
- Common model for many economic time series



ARIMA(0,1,1) Example

IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently: $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha) \hat{Y}_t$$

where $\alpha = 1 + \theta_1$ (for $-1 < \theta_1 < 0$).



The Role of the Constant in ARIMA

Constant Term in ARIMA(p,d,q)

When $d > 0$, the constant c has a different interpretation: $\phi(L)(1 - L)^d Y_t = c + \theta(L)\varepsilon_t$

Important Implications

- ◻ For $d = 1$: c represents the **drift**
 - ▶ Average change: $\mathbb{E}[\Delta Y_t] = \frac{c}{1-\phi_1-\dots-\phi_p}$
 - ▶ Linear trend in levels
- ◻ For $d = 2$: c affects the **curvature**
 - ▶ Quadratic trend in levels
- ◻ Often $c = 0$ is assumed when $d \geq 1$
 - ▶ No deterministic trend component



Testing for Unit Roots

Why Test?

Before fitting an ARIMA model, we need to determine:

1. Is the series stationary? (Is $d = 0$?)
2. If not, how many differences are needed? (What is d ?)

Common Unit Root Tests

- Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- Phillips-Perron (PP)**
- KPSS** (stationarity test – reversed null hypothesis)



Researcher Spotlight: Dickey & Fuller



David Dickey (*1945)

[W Wikipedia](#)



Wayne Fuller (1931–2022)

[W Wikipedia](#)

Biography

- **David Dickey:** American statistician at NC State University. PhD student of Wayne Fuller at Iowa State
- **Wayne Fuller:** American statistician, professor at Iowa State University
- Together they developed the foundational test for unit roots in time series

Key Contributions

- **Dickey-Fuller test (1979)** — the fundamental unit root test
- **Augmented Dickey-Fuller (ADF)** — extension with lagged differences
- **Critical value tables** — non-standard distributions under the null
- Enabled rigorous testing of integration order for ARIMA modeling



The Dickey-Fuller Test

Setup

Consider the AR(1) model: $Y_t = \phi Y_{t-1} + \varepsilon_t$. Subtract Y_{t-1} : $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$, where $\gamma = \phi - 1$.

Hypotheses

- $H_0: \gamma = 0$ (unit root, $\phi = 1$, non-stationary)
- $H_1: \gamma < 0$ (stationary, $|\phi| < 1$)

Key Issue

Under H_0 , the t -statistic does **not** follow a standard t -distribution! Must use Dickey-Fuller critical values.



Dickey-Fuller Test Variants

Three Specifications

1. **No constant, no trend:** $\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$
2. **With constant (drift):** $\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$
3. **With constant and trend:** $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$

Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference



Augmented Dickey-Fuller (ADF) Test

The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

Definition 6 (ADF Test)

Add lagged differences: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^k \delta_j \Delta Y_{t-j} + \varepsilon_t$

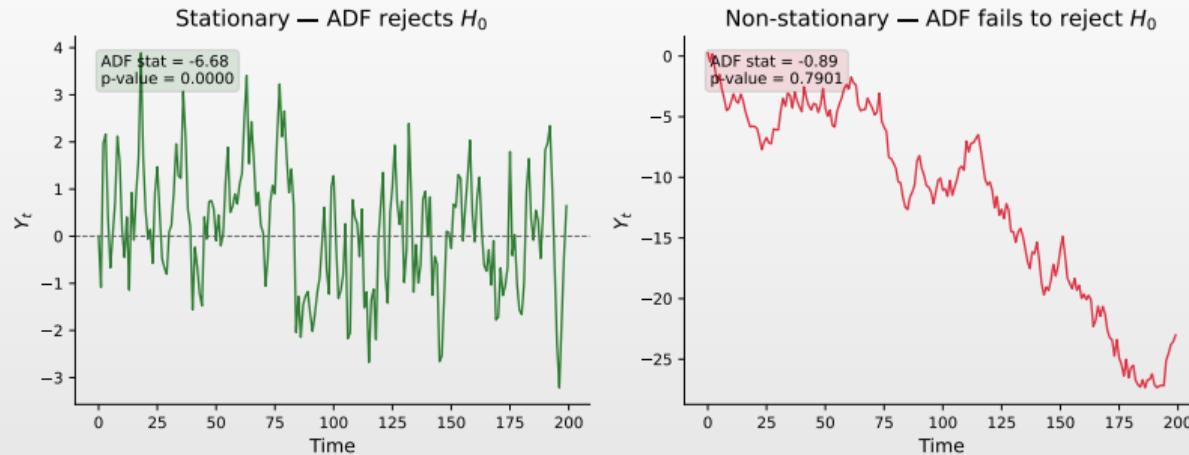
Test $H_0 : \gamma = 0$ using ADF critical values.

Choosing Lag Length k

- Use information criteria (AIC, BIC)
- Start with k_{max} , reduce until last lag significant



ADF Test: Visual Illustration



Observation

- Left: stationary series \Rightarrow ADF rejects unit root
- Right: non-stationary \Rightarrow ADF fails to reject



ADF Test Critical Values

| Model | 1% | 5% | 10% |
|-------------------------|-------|-------|-------|
| No constant, no trend | -2.58 | -1.95 | -1.62 |
| With constant | -3.43 | -2.86 | -2.57 |
| With constant and trend | -3.96 | -3.41 | -3.13 |

Decision Rule

- Test statistic < critical value \Rightarrow Reject H_0 (stationary)
- Test statistic \geq critical value \Rightarrow Fail to reject (unit root)



The Phillips-Perron (PP) Test

Motivation

Like ADF, tests H_0 : Unit root vs H_1 : Stationary, but uses a **non-parametric correction** for serial correlation instead of adding lagged differences.

Test Statistic

The PP test modifies the DF t -statistic:

$$Z_t = t_{\hat{\gamma}} \cdot \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - \hat{\sigma}^2)(se(\hat{\gamma}))}{2\hat{\lambda}^2 \cdot s}$$

where $\hat{\lambda}^2$ is a consistent estimate of the long-run variance using Newey-West.

Advantages over ADF

- Robust to heteroskedasticity and serial correlation
- No need to select lag length (uses bandwidth instead)



The KPSS Test

Reversed Hypotheses

Unlike ADF: H_0 : Stationary vs H_1 : Unit root

KPSS Procedure

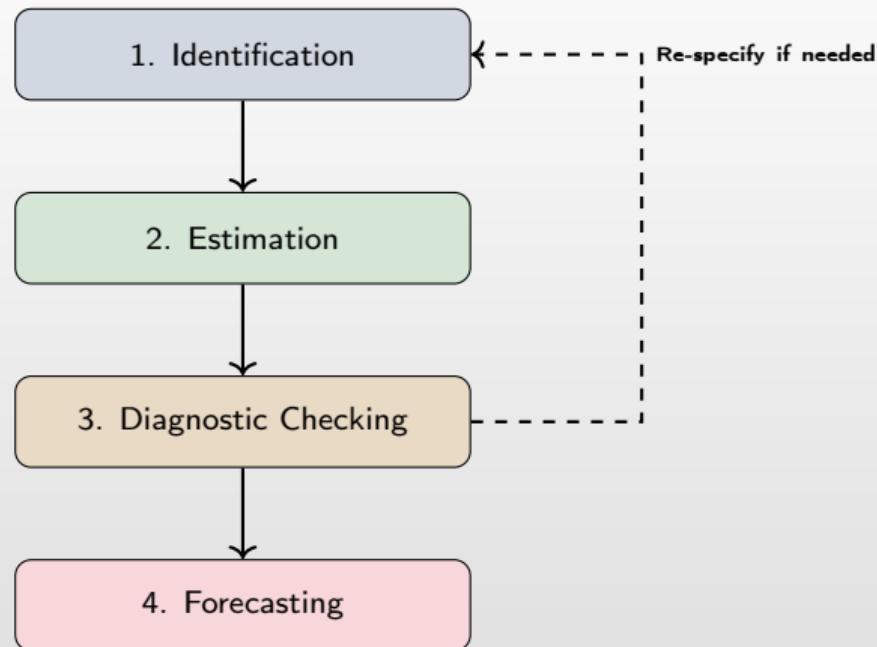
Decompose: $Y_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$. Test whether $\text{Var}(u_t) = 0$.

Complementary Use with ADF

- ADF rejects, KPSS doesn't \Rightarrow Stationary
- ADF doesn't reject, KPSS rejects \Rightarrow Unit root
- Both reject or neither \Rightarrow Inconclusive



The Box-Jenkins Methodology



Step 1: Determining d

Procedure

1. Plot the time series – look for trends, changing variance
2. Examine ACF – slow decay suggests non-stationarity
3. Apply unit root tests (ADF, KPSS)
4. If non-stationary, difference and repeat

Practical Guidelines

- Most economic series: $d = 1$ is sufficient
- Rarely need $d > 2$
- If ACF of ΔY_t still decays slowly, try $d = 2$
- Watch for overdifferencing (ACF with $\rho_1 \approx -0.5$)



Step 2: Determining p and q

After Differencing

Once $W_t = \Delta^d Y_t$ is stationary, use ACF/PACF to identify ARMA(p, q):

| Model | ACF | PACF |
|----------------|------------------------|------------------------|
| AR(p) | Decays exponentially | Cuts off after lag p |
| MA(q) | Cuts off after lag q | Decays exponentially |
| ARMA(p, q) | Decays | Decays |

Information Criteria

When patterns are unclear, compare models using:

$AIC = -2 \ln(L) + 2k; \quad BIC = -2 \ln(L) + k \ln(n)$

Lower is better. BIC penalizes complexity more.



Auto-ARIMA Algorithms

Automated Model Selection

Modern software can automatically select (p, d, q) :

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

How Auto-ARIMA Works

1. Use unit root tests to determine d
2. Fit models for various (p, q) combinations
3. Select model with lowest AIC/BIC
4. Optionally use stepwise search for efficiency

Caution

Automated selection is helpful but not infallible. Always check diagnostics!



Estimation Methods

Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- Assumes $\varepsilon_t \sim N(0, \sigma^2)$
- Maximizes the likelihood function
- Provides consistent, efficient estimators
- Yields standard errors for inference

Conditional vs Exact MLE

- Conditional MLE:** Conditions on initial values
- Exact MLE:** Treats initial values as unknown
- Difference diminishes as sample size grows



Conditional Log-Likelihood

Gaussian Log-Likelihood Function

- ◻ $\ell(\theta, \sigma^2) = -\frac{T}{2} \ln(2\pi) - \frac{T}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^T e_t^2(\theta)$
- ◻ $e_t(\theta) = X_t - \hat{X}_{t|t-1}$ are the **one-step prediction errors**
- ◻ $\theta = (\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q, c)$

Example: ARIMA(1,1,1)

- ◻ Prediction errors: $e_t = \Delta X_t - \phi_1 \Delta X_{t-1} - \theta_1 e_{t-1} - c$
- ◻ Conditional MLE: set $e_0 = 0$, compute e_1, \dots, e_T , maximize ℓ

Estimating σ^2

- ◻ At optimal parameters $\hat{\theta}$: $\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^T e_t^2(\hat{\theta})$



Parameter Constraints

Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- AR stationarity:** Roots of $\phi(z) = 0$ outside unit circle
- MA invertibility:** Roots of $\theta(z) = 0$ outside unit circle

Checking in Practice

Most software reports:

- Estimated coefficients with standard errors
- Roots of AR and MA polynomials
- Warning if near-unit-root detected



Residual Analysis

What to Check

If the model is correct, residuals $\hat{\varepsilon}_t$ should be white noise:

1. Zero mean
2. Constant variance
3. No autocorrelation
4. (Optional) Normality

Diagnostic Tools

- Residual ACF/PACF:** Should show no significant spikes
- Ljung-Box test:** Tests for autocorrelation at multiple lags
- Q-Q plot:** Checks normality assumption
- Residual vs fitted:**
 - ▶ Checks for heteroskedasticity



The Ljung-Box Test

Definition 7 (Ljung-Box Q Statistic)

$$Q(m) = n(n + 2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}. \text{ Under } H_0 \text{ (no autocorrelation): } Q(m) \sim \chi^2(m - p - q)$$

Usage

- Choose $m \approx \ln(n)$ or $m = 10$ for quarterly, $m = 20$ for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if $Q(m)$ exceeds critical value

If Test Fails

Consider adding AR or MA terms, or check for structural breaks.



Point Forecasts

Minimum MSE Forecast

The optimal h -step ahead forecast is the conditional expectation: $\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \dots]$

ARIMA(1,1,1) Forecasting

Model: $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

One-step forecast: $\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1 \hat{\varepsilon}_T$

For $h > 1$: replace unknown ε_{T+j} with 0, unknown Y_{T+j} with $\hat{Y}_{T+j|T}$



Forecast Intervals

Forecast Uncertainty

The h -step forecast error variance: $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$, where ψ_j are MA(∞) coefficients.

Confidence Intervals

Under normality, $(1 - \alpha)\%$ interval: $\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$

Key Property for I(1) Series

For integrated processes, forecast variance grows without bound as $h \rightarrow \infty$. Intervals widen over time!



Long-Run Forecasts for ARIMA

Behavior as $h \rightarrow \infty$

For ARIMA(p,1,q) with drift c :

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with \sqrt{h}

For ARIMA(p,1,q) without drift:

- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

Practical Implication

- ARIMA forecasts are most reliable for short horizons
- Long-term forecasts have very wide uncertainty bands



Rolling Forecasting: Concept

What is Rolling Forecasting?

A technique to evaluate forecast accuracy out-of-sample:

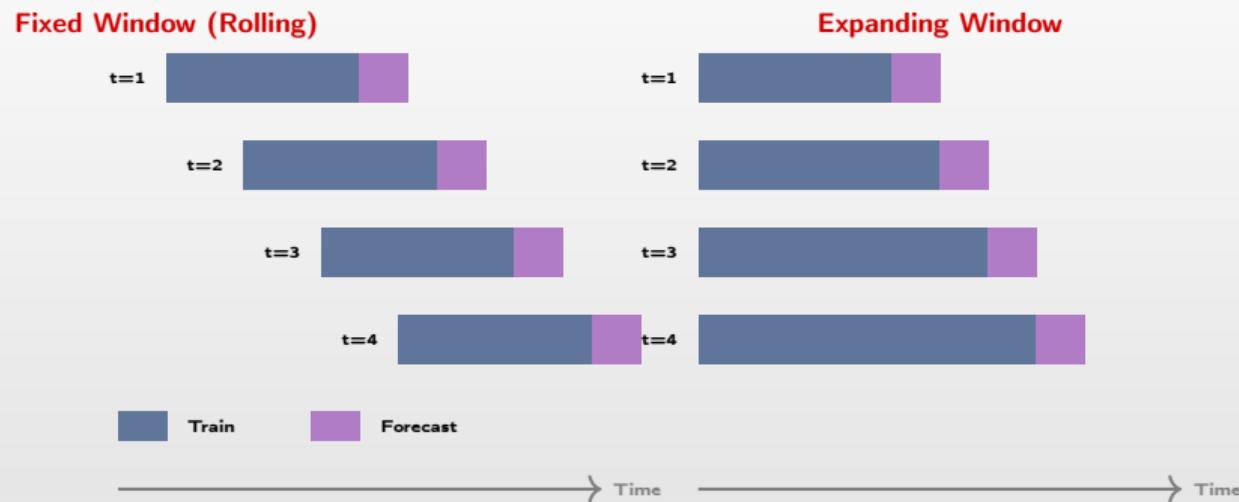
1. Fix a **training window** of size w
2. Estimate model on observations $t = 1, \dots, w$
3. Forecast h steps ahead: $\hat{Y}_{w+h|w}$
4. **Roll** the window forward by one period
5. Repeat until end of sample

Why Rolling Forecasts?

- Mimics real-time forecasting scenario
- Provides multiple forecast errors for evaluation
- Avoids overfitting to full sample



Fixed vs Expanding Window



Comparison

- **Fixed:** Window slides forward, constant size — adapts to regime changes
- **Expanding:** Window grows over time — uses all historical data



1-Step vs Multi-Step Forecasting

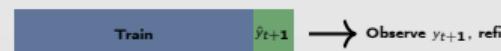
1-Step Ahead (Recursive)

- Forecast only next period
 - ▶ Refit model after each step
 - ▶ Use actual value once revealed
- Most accurate for short horizons

Multi-Step (Direct)

- Forecast multiple periods ahead
 - ▶ No refit between steps
 - ▶ Uses forecasted values as inputs
- Uncertainty compounds over horizon

1-Step Ahead



Multi-Step (h=3)



Rolling Forecast: Step-by-Step Example

Setup: ARIMA(1,1,0) with $\phi_1 = 0.6$

Model: $\Delta Y_t = \phi_1 \Delta Y_{t-1} + \varepsilon_t$ where $\Delta Y_t = Y_t - Y_{t-1}$

Given Data at Time T

$$Y_{T-2} = 100, \quad Y_{T-1} = 103, \quad Y_T = 108 \quad \Rightarrow \quad \Delta Y_{T-1} = 3, \quad \Delta Y_T = 5$$

1-Step Ahead Point Forecast

$$\begin{aligned}\hat{\Delta Y}_{T+1|T} &= \phi_1 \cdot \Delta Y_T = 0.6 \times 5 = 3 \\ \hat{Y}_{T+1|T} &= Y_T + \hat{\Delta Y}_{T+1|T} = 108 + 3 = 111\end{aligned}$$



Multi-Step Point Forecasts

2-Step Ahead Forecast

$$\begin{aligned}\hat{\Delta Y}_{T+2|T} &= \phi_1 \cdot \hat{\Delta Y}_{T+1|T} = 0.6 \times 3 = 1.8 \\ \hat{Y}_{T+2|T} &= \hat{Y}_{T+1|T} + \hat{\Delta Y}_{T+2|T} = 111 + 1.8 = \boxed{112.8}\end{aligned}$$

General Formula for h -Step Forecast (ARIMA(1,1,0))

$$\begin{aligned}\hat{\Delta Y}_{T+h|T} &= \phi_1^h \cdot \Delta Y_T \\ \hat{Y}_{T+h|T} &= Y_T + \Delta Y_T \cdot \frac{\phi_1(1 - \phi_1^h)}{1 - \phi_1}\end{aligned}$$

Numerical: 3-Step Forecast

$$\hat{Y}_{T+3|T} = 108 + 5 \times \frac{0.6(1 - 0.6^3)}{1 - 0.6} = 108 + 5 \times 1.092 = \boxed{113.46}$$



Confidence Intervals: Formulas

Forecast Error Variance

For ARIMA(1,1,0), the h -step forecast error variance:

$$\text{Var}(e_{T+h|T}) = \sigma^2 \left(1 + \sum_{j=1}^{h-1} \psi_j^2 \right)$$

$$\text{where } \psi_j = \phi_1^{j-1} (1 + \phi_1 + \cdots + \phi_1^{j-1}) = \phi_1^{j-1} \cdot \frac{1 - \phi_1^j}{1 - \phi_1}$$

$(1 - \alpha)\%$ Confidence Interval

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(e_{T+h|T})}$$

For 95% CI: $z_{0.025} = 1.96$



Confidence Interval: Numerical Example

Given: $\sigma^2 = 4$, $\phi_1 = 0.6$, $\hat{Y}_{T+1|T} = 111$

1-Step Ahead CI

$$\text{Var}(e_{T+1|T}) = \sigma^2 = 4$$

$$95\% \text{ CI} = 111 \pm 1.96 \times \sqrt{4} = 111 \pm 3.92 = [107.08, 114.92]$$

2-Step Ahead CI (for $\hat{Y}_{T+2|T} = 112.8$)

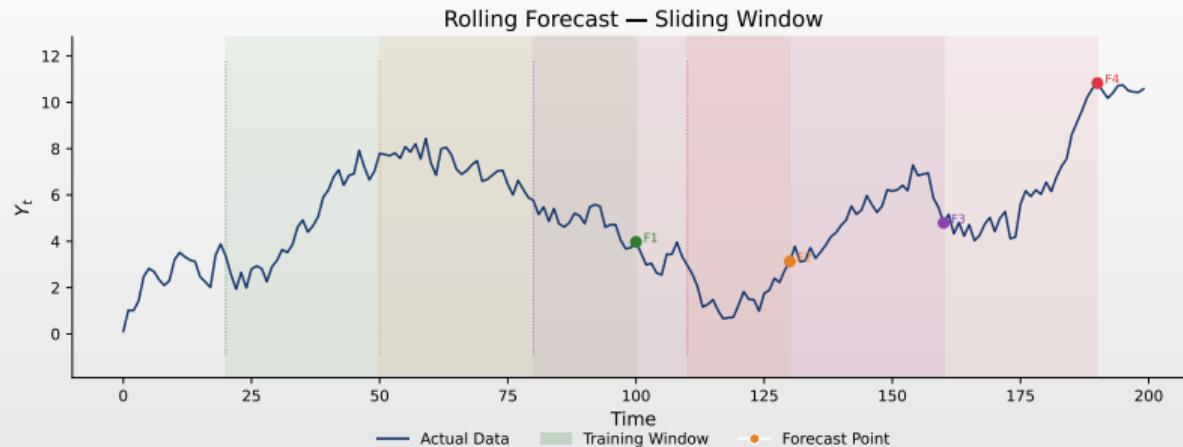
$$\psi_1 = 1 + \phi_1 = 1.6, \quad \text{Var}(e_{T+2|T}) = 4(1 + 1.6^2) = 14.24$$

$$95\% \text{ CI} = 112.8 \pm 1.96 \times \sqrt{14.24} = 112.8 \pm 7.40 = [105.40, 120.20]$$

Note: CI widens as horizon increases!



Rolling Window Illustration



Rolling Procedure

- Each window produces a 1-step ahead forecast
- Compare forecasts to actuals to compute RMSE, MAE
- Rolling window keeps model estimation up-to-date



Case Study: Complete ARIMA Analysis

Objective

- Forecast US Real GDP using the Box-Jenkins methodology

Steps

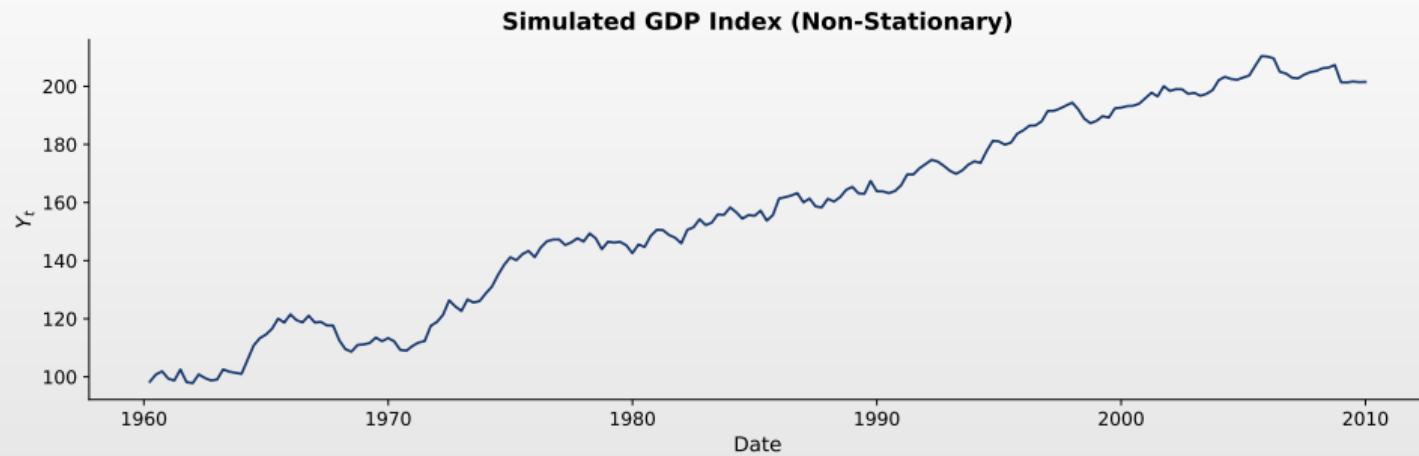
- Step 1:** Visualize data and check stationarity
- Step 2:** Apply unit root tests (ADF, KPSS)
- Step 3:** Difference if needed, identify p and q
- Step 4:** Estimate the ARIMA model
- Step 5:** Model diagnostics
- Step 6:** Generate forecasts with confidence intervals
- Step 7:** Evaluate forecast accuracy

Data

- US Real GDP (FRED: GDPC1), Quarterly, 1990Q1–2024Q2, $n = 138$



Case Study: US Real GDP (FRED)



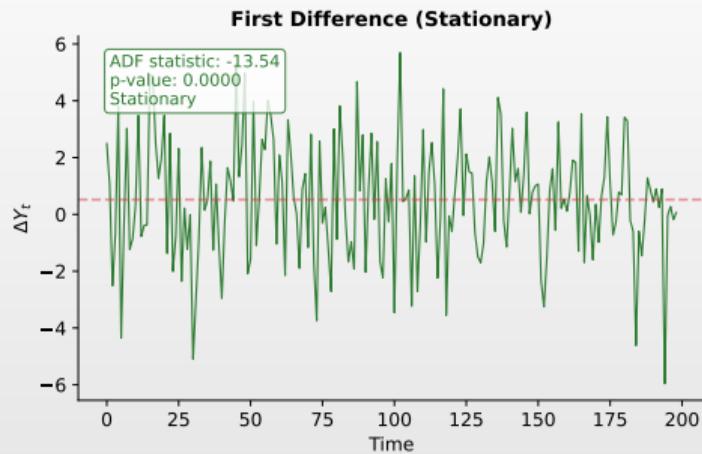
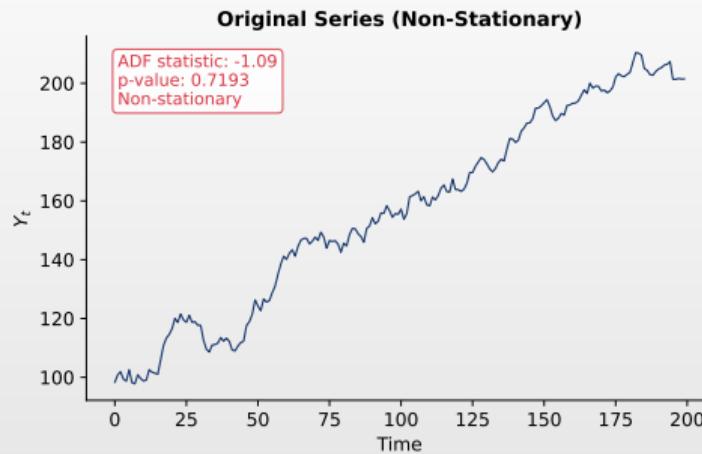
Data: FRED GDPC1 (1960Q1–2024Q3)

Quarterly Real GDP, seasonally adjusted, billions of chained 2017 dollars. Non-stationary series with upward trend \Rightarrow requires differencing.

 **TSA_ch3_case_raw_data**



Step 1: ADF Test for Stationarity



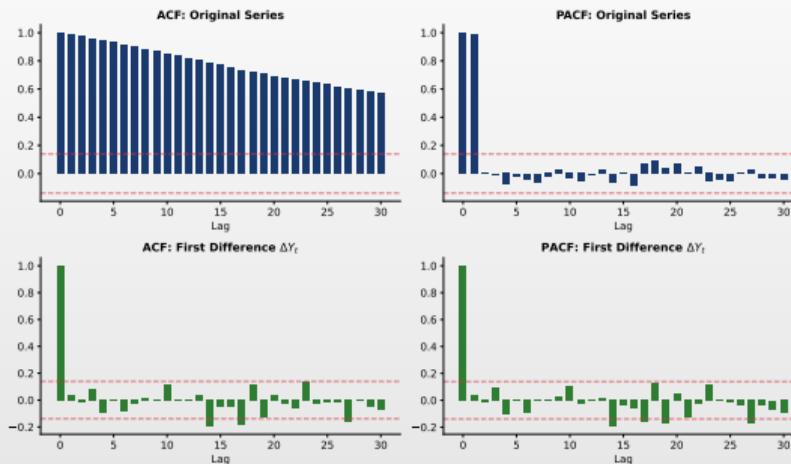
ADF Test Results

Original series: Large p-value \Rightarrow fail to reject H_0 (unit root present). First difference: p-value $< 0.01 \Rightarrow$ reject $H_0 \Rightarrow d = 1$ is sufficient.

TSA_ch3_case_adf_test



Step 2: ACF/PACF Before and After Differencing



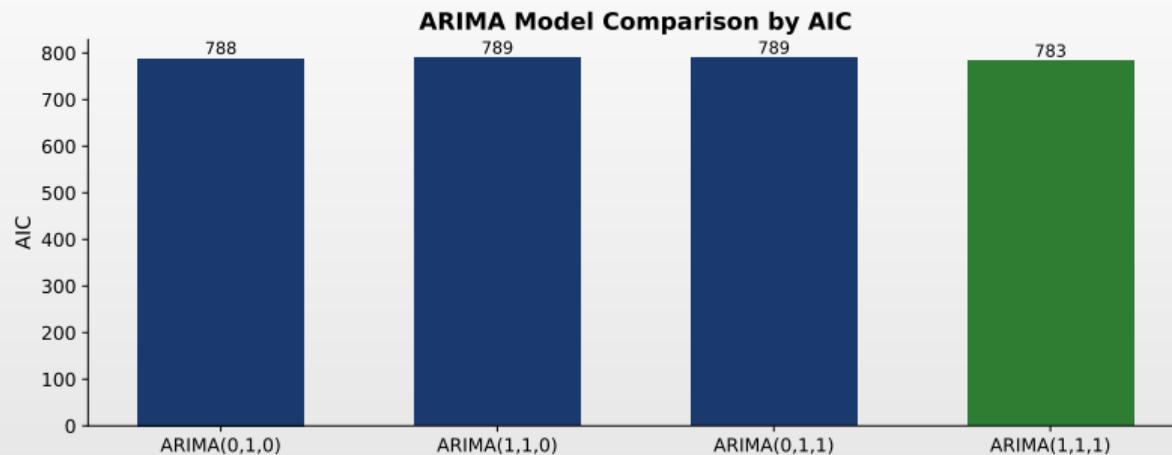
ACF/PACF Analysis

Top: Slow ACF decay (non-stationary) | Bottom: After differencing, low-order ARMA

Q TSA_ch3_case_acf_diff



Step 3: ARIMA Model Comparison



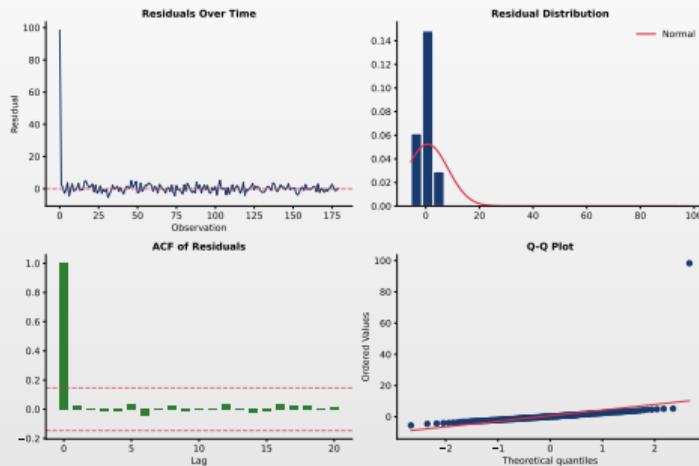
Model Selection

Compare ARIMA(0,1,0), ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1). The model with lowest AIC is selected.

 [TSA_ch3_case_model_comparison](#)



Step 4: Diagnostic Checking

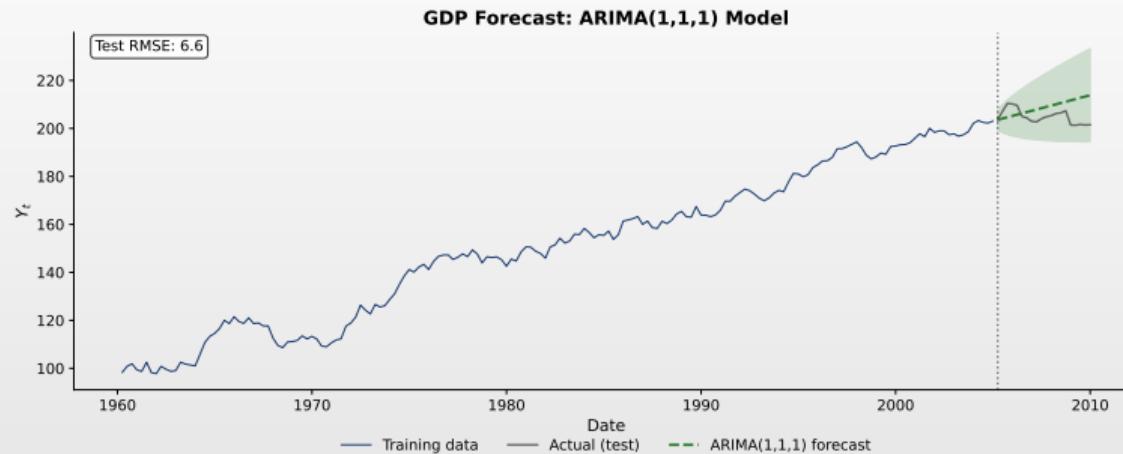


ARIMA(1,1,1) Diagnostics

ACF: no autocorrelation ✓ Q-Q: non-normal (COVID-19 outlier) JB test: $p < 0.001$



Step 5: Out-of-Sample Forecasting



Train/Val/Test Split (70%/15%/15%)

Train 70% (blue): Estimation | Val 15% (green): Tuning | Test 15% (purple): Evaluation with 95% CI

Q TSA_ch3_case_forecast



Step 6: Rolling Forecast with Train/Val/Test



Rolling 1-Step Ahead Forecast (Expanding Window, 95% CI)

Train 70% → Val 15% → Test 15% | Expanding window refits model at each step

Q TSA_ch3_case_rolling_forecast



Summary

What we learned in this chapter

- Non-stationarity in time series
 - ▶ Deterministic vs stochastic trend; consequences for statistical inference
- Differencing and integrated processes
 - ▶ $\Delta Y_t = Y_t - Y_{t-1}$; if $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$
- ARIMA(p, d, q) models and unit root tests
 - ▶ ADF, PP, KPSS; Box-Jenkins: identify → estimate → validate
- Forecasts with confidence intervals
 - ▶ For $I(1)$: CIs widen without bound ($\propto \sqrt{h}$)

Key Insight

- **Difference carefully:** One difference is usually sufficient ($d = 1$). Over-differencing creates artificial autocorrelation.



AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download quarterly US Real GDP from FRED (series GDPC1) for 2000-Q1 to 2024-Q4 (100 observations). Test stationarity, difference if needed, estimate an ARIMA model, and forecast 8 quarters ahead. Give me complete Python code with plots."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it test stationarity with ADF *before* estimating ARIMA? Does it also use KPSS?
3. How does it determine the differencing order d ? Does it check for over-differencing?
4. How does it choose p and q ? ACF/PACF or just auto_arima?
5. Do forecast confidence intervals widen with horizon? (key I(1) property)

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*



What's Next?

Chapter 4: SARIMA Models for Seasonal Data

- **Seasonality:** repetitive patterns at regular intervals
- **Seasonal differencing:** the $(1 - L^s)$ operator
- **SARIMA(p, d, q)(P, D, Q)_s:** seasonal extension of ARIMA
- **Model identification:** seasonal ACF/PACF
- **Case study:** Airline passengers forecast

Questions?



Question 1

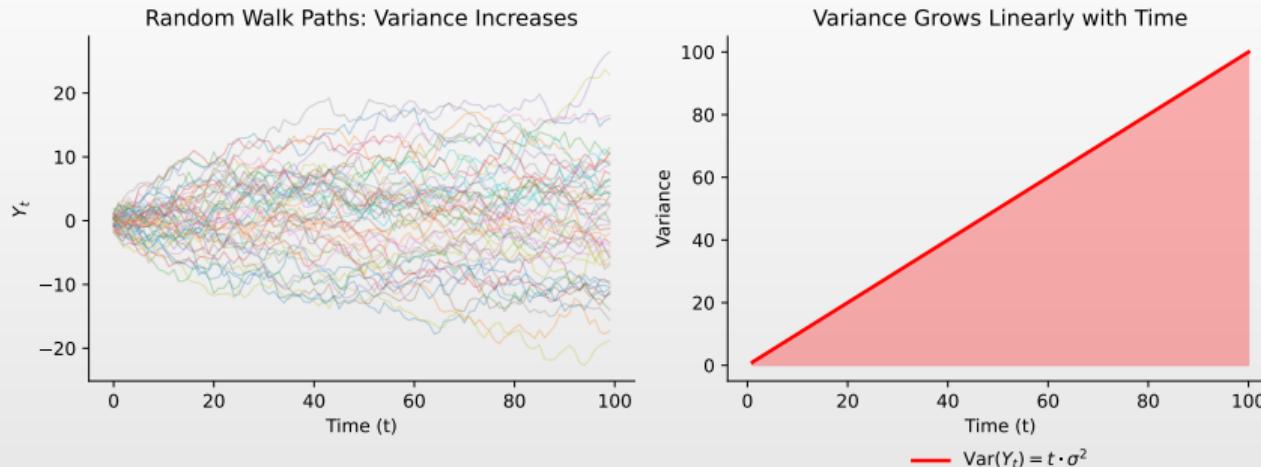
Question

- A time series Y_t follows a random walk: $Y_t = Y_{t-1} + \varepsilon_t$. What is $\text{Var}(Y_t)$?

Answer Choices

- (A) σ^2 (constant)
- (B) $t \cdot \sigma^2$ (grows linearly with time)
- (C) σ^2/t (decreases with time)
- (D) σ^{2t} (grows exponentially)

Question 1: Answer



Answer: (B)

- Random walk variance grows linearly with time — this is why random walks are non-stationary.

Q TSA_ch3_quiz1_rw_variance



Question 2

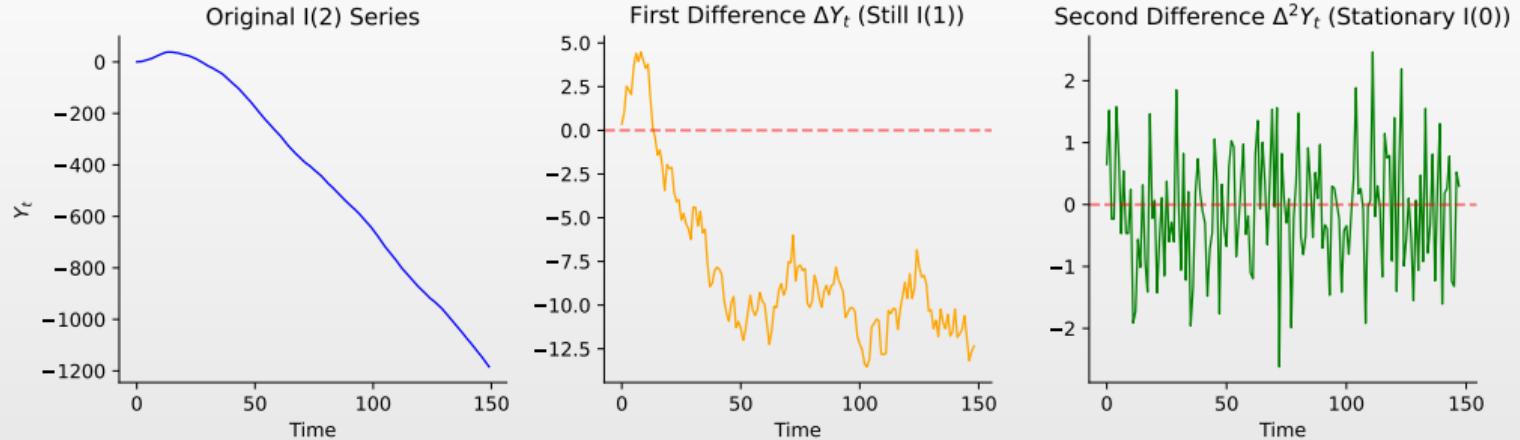
Question

- If a series Y_t is I(2), how many times must you difference it to achieve stationarity?

Answer Choices

- (A)** 0 times (already stationary)
- (B)** 1 time
- (C)** 2 times
- (D)** Cannot be made stationary by differencing

Question 2: Answer



Answer: (C)

- I(d) means “integrated of order d ” — requires d differences for stationarity.

Q TSA_ch3_quiz2_differencing



Question 3

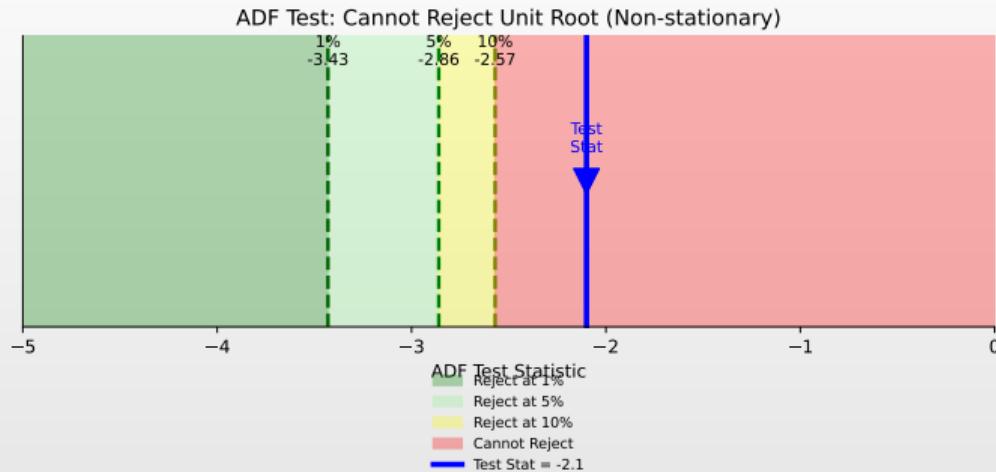
Question

- You run an ADF test and get a test statistic of -2.1 with critical values: -3.43 (1%), -2.86 (5%), -2.57 (10%). What do you conclude?

Answer Choices

- (A)** Reject H_0 : series is stationary at all levels
- (B)** Reject H_0 : series is stationary at 10% level only
- (C)** Fail to reject H_0 : series likely has a unit root
- (D)** The test is inconclusive

Question 3: Answer



Answer: (C)

- Test stat $-2.1 > -2.57$ (10% CV) \Rightarrow Cannot reject at any level. Consider differencing.

Question 4

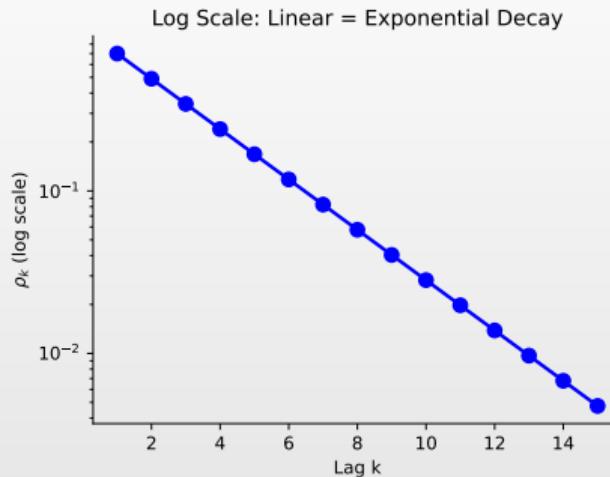
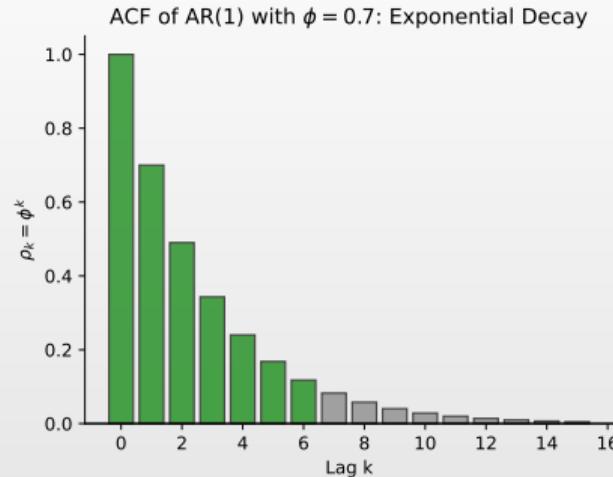
Question

- For an ARIMA(1,1,0) model, what is the ACF pattern of the **differenced** series ΔY_t ?

Answer Choices

- (A) Cuts off after lag 1
- (B) Decays exponentially
- (C) Alternates in sign
- (D) Is zero at all lags

Question 4: Answer



Answer: (B)

- ARIMA(1,1,0) $\Rightarrow \Delta Y_t$ follows AR(1) with ACF $\rho_k = \phi_1^k$ (geometric decay).



Question 5

Question

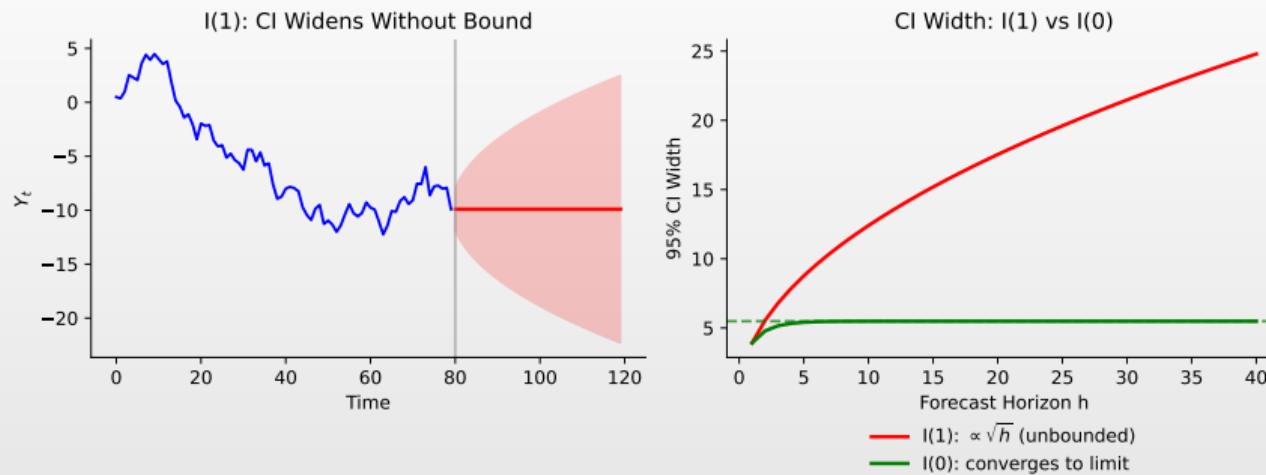
- What happens to ARIMA forecast confidence intervals as the horizon h increases for an $I(1)$ series?

Answer Choices

- (A) They stay constant
- (B) They narrow (more precision)
- (C) They widen without bound
- (D) They widen but converge to a limit



Question 5: Answer



Answer: (C)

- For $I(1)$: CI width $\propto \sqrt{h}$ (unbounded). For $I(0)$: CIs converge to a limit.

Q TSA_ch3_quiz5_forecast_ci



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- Kwiatkowski, D., Phillips, P.C.B., Schmidt, P., & Shin, Y. (1992). Testing the Null Hypothesis of Stationarity, *Journal of Econometrics*, 54(1-3), 159–178.

ARIMA Models and Automatic Selection

- Box, G.E.P., & Jenkins, G.M. (1970). *Time Series Analysis: Forecasting and Control*, Holden-Day.
- Hyndman, R.J., & Khandakar, Y. (2008). Automatic Time Series Forecasting: The forecast Package for R, *Journal of Statistical Software*, 27(3), 1–22.



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- Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.
- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

Online Resources and Code

- **Quantlet:** <https://quantlet.com> ↘ Code repository for statistics
- **Quantinar:** <https://quantinar.com> ↘ Learning platform for quantitative methods
- **GitHub TSA:** https://github.com/QuantLet/TSA/tree/main/TSA_ch3 ↘ Python code for this chapter



Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



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