Time Series Analysis and Forecasting

Chapter 3: ARIMA Models

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Learning Objectives

By the end of this chapter, you will be able to:

- Understand the concept and implications of non-stationarity
- Apply differencing to achieve stationarity in time series
- Use the Augmented Dickey-Fuller (ADF) test for unit root detection
- Build, estimate, and forecast with ARIMA models

Data Sources and Software Tools

Data Sources

- **FRED** (Federal Reserve)
  - US Real GDP (GDPC1), interest rates
- **Yahoo Finance**
  - Stock prices, exchange rates
- **Eurostat / World Bank**
  - Macroeconomic data
- **Statsmodels datasets**
  - Sunspots, Nile, Macrodata

Python

- `statsmodels` — ARIMA models
- `pmdarima` — automatic selection
- `pandas-datareader` — FRED download
- `matplotlib` — visualization
- `scipy` — statistical tests

Resources

- `github.com/QuantLet/TSA/TSA_ch3`
- `quantlet.com`

Chapter Outline

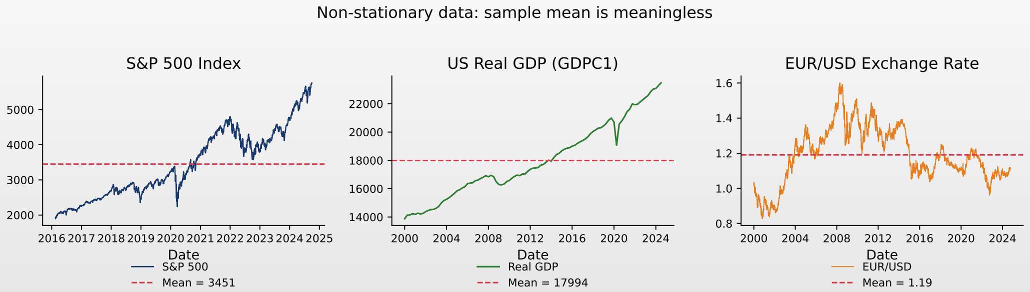
- Motivation
- Non-Stationarity in Time Series
- Differencing and the Difference Operator
- ARIMA(p,d,q) Models
- Unit Root Tests
- ARIMA Model Identification
- ARIMA Estimation
- Diagnostic Checking
- Forecasting with ARIMA
- Case Study
- Summary
- AI Use Case
- Quiz

Motivating Example: Non-Stationary Data Is Everywhere

Key Observations

- Stock prices, GDP, exchange rates all exhibit **trends** or **wandering behavior**
- The sample mean (red line) is meaningless for a random walk
- Standard ARMA models **cannot** handle these series directly

Motivating Example: Non-Stationary Data Is Everywhere



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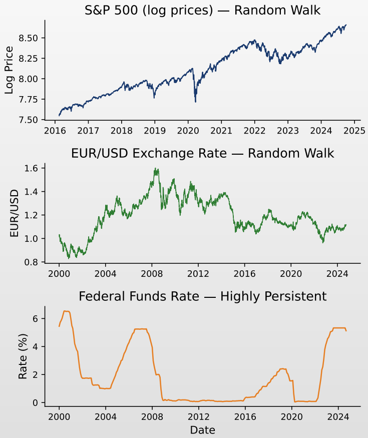
Real-World Applications

The Challenge

Financial/economic data are typically I(1): stock prices, exchange rates, interest rates.

Real-World Applications



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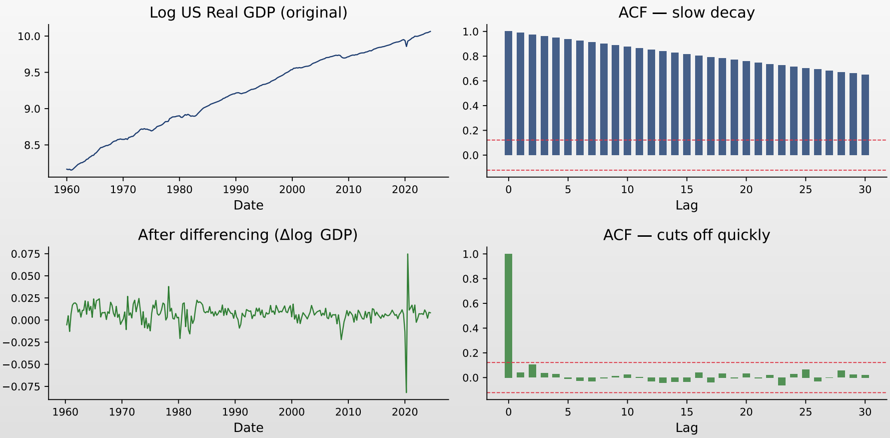
The Solution: Differencing

Key Insight

Differencing transforms a non-stationary series into a stationary one: $\Delta Y_t = Y_t - Y_{t-1}$. The ACF changes from slow decay to quick decay!

The Solution: Differencing



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What We'll Learn Today

Core Concepts

1. **Non-Stationarity**: Why it matters and how to detect it
2. **Unit Root Tests**: ADF, PP, KPSS tests
3. **Differencing**: The key transformation
4. **ARIMA Models**: Combining differencing with ARMA
5. **Box-Jenkins Methodology**: Identify $\rightarrow$ Estimate $\rightarrow$ Diagnose

By the End of This Lecture

You will be able to model and forecast non-stationary time series like stock prices, GDP, and exchange rates using ARIMA models.

Why Non-Stationarity Matters

The Problem

Many economic and financial time series are **non-stationary**:

- GDP, stock prices, exchange rates, inflation indices
- They exhibit trends, changing means, or growing variance

Consequences of Non-Stationarity

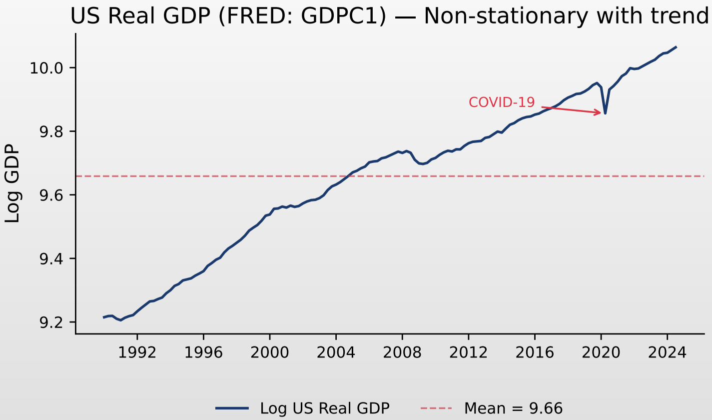
- Standard ARMA models assume stationarity
- OLS regression with non-stationary data leads to **spurious regression**
- Sample moments (mean, variance, ACF) are not consistent estimators
- Statistical inference becomes invalid

Example: US Real GDP

Key Observations

- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data

Example: US Real GDP



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Types of Non-Stationarity

Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a deterministic function of time
- Can be removed by **detrending**
- Shocks have temporary effects

Stochastic Trend (Unit Root)

$$Y_t = Y_{t-1} + \varepsilon_t$$

- Random walk process
- Must be removed by **differencing**
- Shocks have permanent effects

Key Distinction

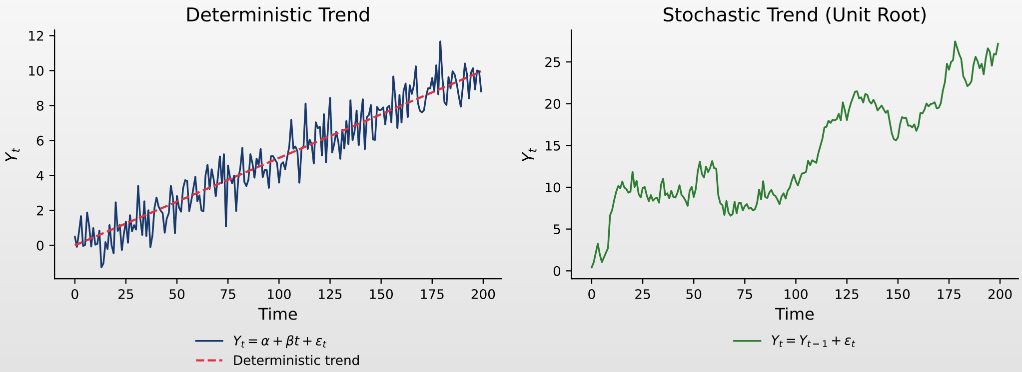
Correct identification is crucial: detrending a unit root $\rightarrow$ misspecification; differencing trend-stationary $\rightarrow$ misspecification.

Visualizing the Difference

Key Distinction

- **Left**: Deterministic trend – deviations from trend are temporary
- **Right**: Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!

Visualizing the Difference



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The Random Walk Process

Definition 1 (Random Walk)

A **random walk** is defined as:

$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

With initial condition $Y_0 = 0$, we have: $Y_t = \sum_{i=1}^{t} \varepsilon_i$

Properties of Random Walk

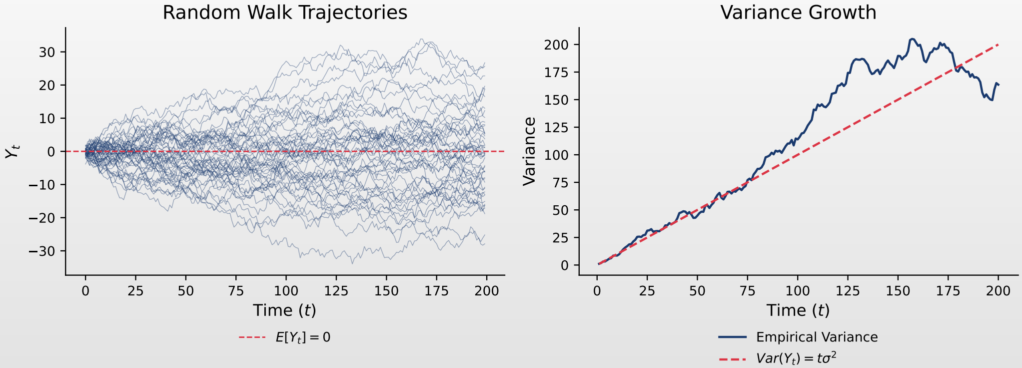
- $\mathbb{E}[Y_t] = 0$ (constant mean)
- $\text{Var}(Y_t) = t\sigma^2$ (variance grows with time!)
- $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$ for $k \leq t$
- ACF: $\rho_k = \sqrt{\frac{t-k}{t}} \to 1$ as $t \to \infty$

Random Walk: Visual Illustration

Key Properties

Left: Paths wander unpredictably, no mean reversion. **Right**: $\text{Var}(Y_t) = t\sigma^2$ grows linearly $\Rightarrow$ non-stationary.

Random Walk: Visual Illustration



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Proof: Random Walk Variance

Claim: For $Y_t = Y_{t-1} + \varepsilon_t$ with $Y_0 = 0$: $\mathrm{Var}(Y_t) = t\sigma^2$

Proof: By recursive substitution: $Y_t = \sum_{i=1}^{t} \varepsilon_i$
Taking variance:

$$\mathrm{Var}(Y_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \varepsilon_i\right) = \sum_{i=1}^{t} \mathrm{Var}(\varepsilon_i) + 2\sum_{i<j} \mathrm{Cov}(\varepsilon_i, \varepsilon_j)$$

Since ε_t independent (white noise): $\mathrm{Var}(Y_t) = \sum_{i=1}^{t} \sigma^2 = \boxed{t\sigma^2}$

Variance depends on $t \Rightarrow$ non-stationary

Proof: Random Walk Autocovariance

Claim: $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$ for $k \le t$

Proof: Using $Y_t = \sum_{i=1}^{t} \varepsilon_i$ and $Y_{t-k} = \sum_{i=1}^{t-k} \varepsilon_i$:

$$\text{Cov}(Y_t, Y_{t-k}) = \text{Cov}\left(\sum_{i=1}^{t} \varepsilon_i, \sum_{j=1}^{t-k} \varepsilon_j\right)$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{t-k} \text{Cov}(\varepsilon_i, \varepsilon_j) = \sum_{i=1}^{t-k} \text{Var}(\varepsilon_i) = \boxed{(t-k)\sigma^2}$$

Only terms with $i = j$ survive (when $i \le t-k$).

ACF:

$$\rho(k) = \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} = \frac{(t-k)\sigma^2}{\sqrt{t\sigma^2 \cdot (t-k)\sigma^2}} = \sqrt{\frac{t-k}{t}}$$

Random Walk with Drift

Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

Equivalently: $Y_t = Y_0 + \mu t + \sum_{i=1}^{t} \varepsilon_i$

Properties

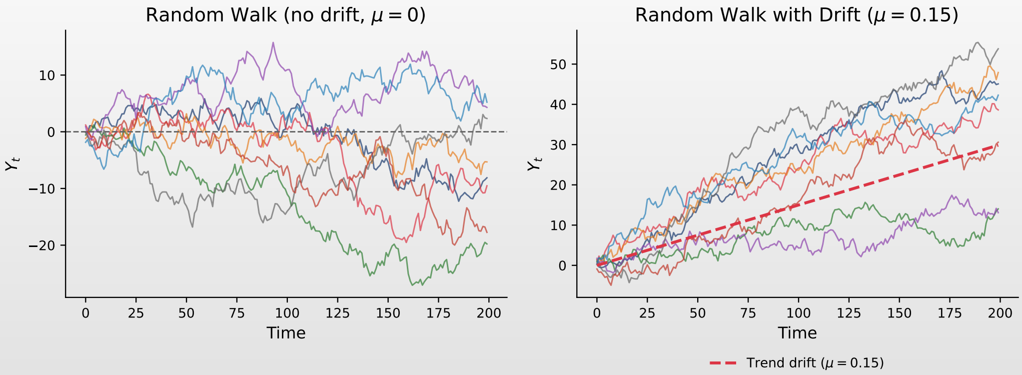
- $\mathbb{E}[Y_t] = Y_0 + \mu t$ (mean grows linearly)
- $\text{Var}(Y_t) = t\sigma^2$ (variance still grows)
- The drift μ creates an upward or downward trend
- Still non-stationary despite having a “trend”

Random Walk with Drift: Visual Illustration

Comparison

Without drift (blue): wanders around zero with no direction. With drift $\mu > 0$ (red): systematic upward trend. Both are non-stationary — drift adds deterministic trend to stochastic wandering.

Random Walk with Drift: Visual Illustration



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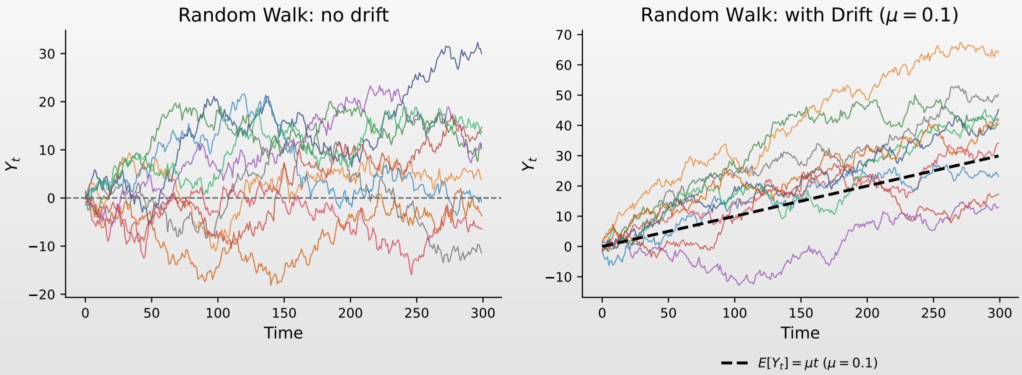
Simulating Random Walks

Random Walk Types

- **Left**: Pure random walks – no drift, wander unpredictably
- **Right**: Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time

Simulating Random Walks



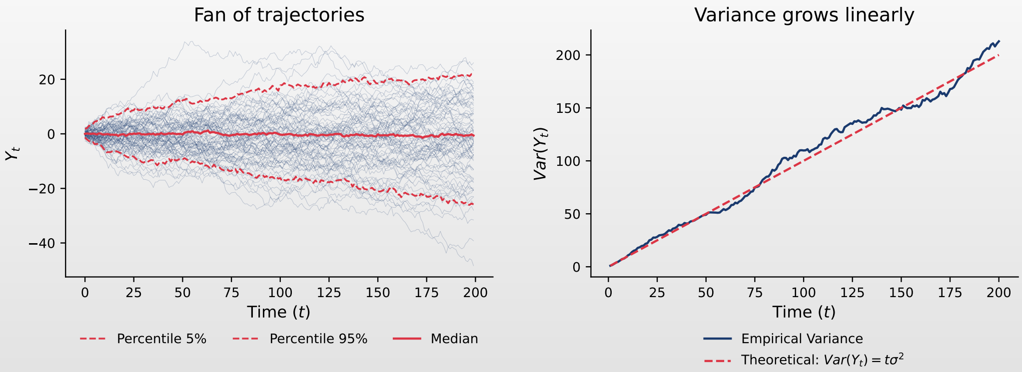
TSA_ch3_random_walk

Variance Growth: Why Random Walks Are Non-Stationary

Key Observations

- **Left**: Fan of paths shows uncertainty growing over time
- **Right**: Variance grows linearly: $\text{Var}(Y_t) = t\sigma^2$
- This violates stationarity (variance should be constant)

Variance Growth: Why Random Walks Are Non-Stationary



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Integrated Processes

Definition 3 (Integrated Process of Order d)

A time series $\{Y_t\}$ is **integrated of order** d, written $Y_t \sim I(d)$, if:

- Y_t is non-stationary
- $(1-L)^d Y_t = \Delta^d Y_t$ is stationary
- $(1-L)^{d-1} Y_t$ is still non-stationary

Common Cases

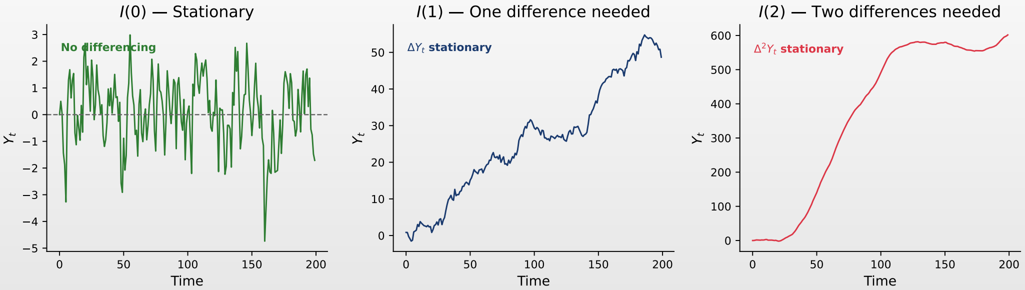
- $I(0)$: Stationary process (e.g., ARMA)
- $I(1)$: First difference is stationary (most common for economic data)
- $I(2)$:
  - Second difference is stationary (less common)

Integrated Process: Visual Illustration

Order of Integration

- $I(0)$: Stationary $\Rightarrow$ no differencing needed
- $I(1)$: One difference needed (random walk)
- $I(2)$: Two differences needed
- Most economic series are $I(0)$ or $I(1)$

Integrated Process: Visual Illustration



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The Difference Operator

Definition 4 (First Difference)

The **first difference operator** Δ is defined as: $\Delta Y_t = Y_t - Y_{t-1} = (1-L)Y_t$, where L is the lag operator ($LY_t = Y_{t-1}$).

Higher-Order Differences

- Second difference: $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1-L)^2 Y_t$
- $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- d-th difference: $\Delta^d Y_t = (1-L)^d Y_t$

Key Result

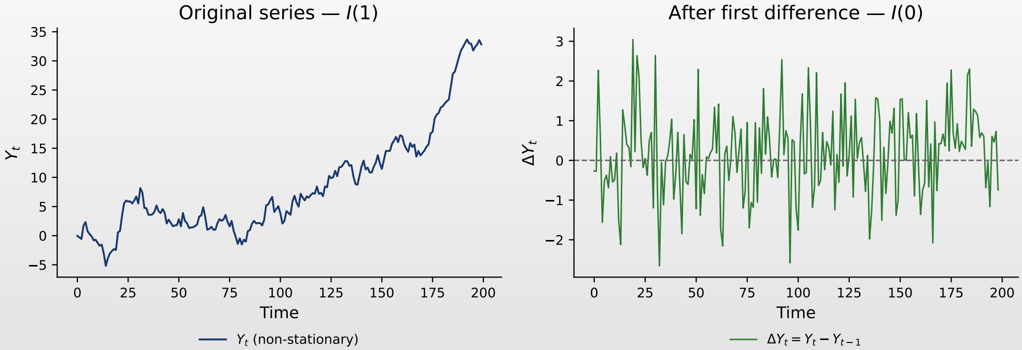
If $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$ (stationary).

First Difference: Visual Illustration

Observation

- **Left**: non-stationary series
- **Right**: after first difference, the series becomes stationary

First Difference: Visual Illustration



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Example: Differencing a Random Walk

Random Walk to White Noise

Let $Y_t = Y_{t-1} + \varepsilon_t$ (random walk). Taking the first difference:

$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

Interpretation

- A random walk is $I(1)$
- One difference transforms it to $I(0)$
- The "changes" in a random walk are stationary

Proof: Differencing Induces Stationarity

Claim: If $Y_t \sim I(1)$, then $\Delta Y_t = Y_t - Y_{t-1}$ is stationary.

Proof for Random Walk with Drift: $Y_t = \mu + Y_{t-1} + \varepsilon_t$
The first difference is:

$$\Delta Y_t = Y_t - Y_{t-1} = \mu + \varepsilon_t$$

Check stationarity conditions:

1. **Mean:** $\mathbb{E}[\Delta Y_t] = \mu$ (constant, does not depend on t) ✓
2. **Variance:** $\text{Var}(\Delta Y_t) = \text{Var}(\varepsilon_t) = \sigma^2$ (constant) ✓
3. **Autocovariance:** $\text{Cov}(\Delta Y_t, \Delta Y_{t-k}) = \text{Cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$ for $k \neq 0$ ✓

General Principle

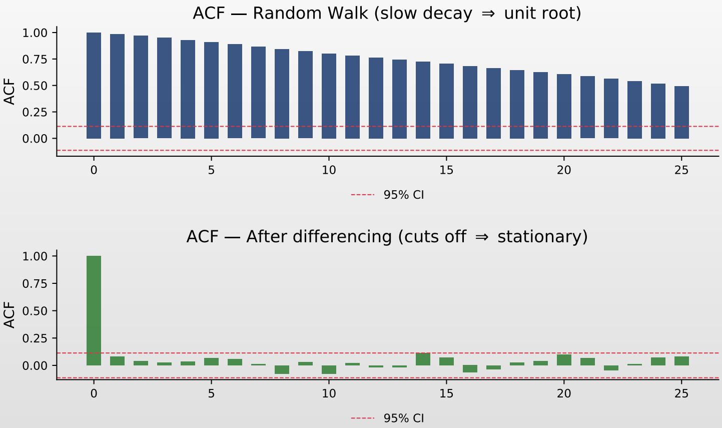
Differencing removes the "memory" that causes variance to accumulate. For $I(d)$ processes, d differences are needed.

ACF Diagnostic: Detecting Non-Stationarity

ACF Patterns

- **Top**: Random walk ACF decays very slowly $\Rightarrow$ unit root
- **Bottom**: After differencing, ACF cuts off $\Rightarrow$ stationary

ACF Diagnostic: Detecting Non-Stationarity



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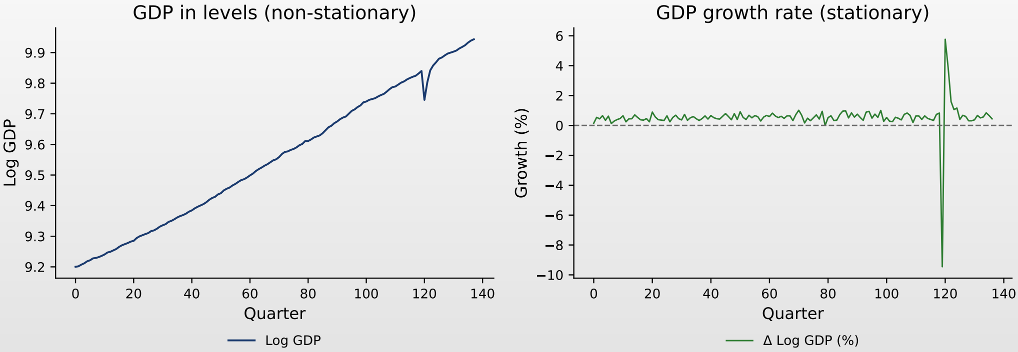
Differencing in Practice: GDP Example

Transformation

Left: GDP in levels with clear upward trend (non-stationary). **Right**: GDP growth rate $\Delta \log(GDP_t)$ fluctuates around constant mean (stationary). One difference removes the stochastic trend.

Differencing in Practice: GDP Example



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Overdifferencing

Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
  - ACF shows spurious patterns
- Inflates variance
  - Reduces forecast accuracy
- Loses information
  - Cannot recover original level

Example

If $Y_t \sim I(1)$, then $\Delta Y_t \sim I(0)$. But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with $\theta = 1$ (non-invertible boundary)!

Definition of ARIMA

Definition 5 (ARIMA(p,d,q))

A time series $\{Y_t\}$ follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$$

where:

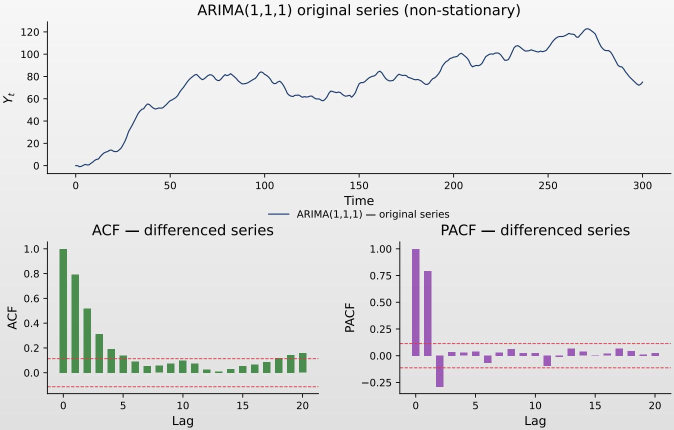
- $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$ (AR polynomial)
- $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$ (MA polynomial)
- d is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

ARIMA: Visual Illustration

Interpretation

Top: original ARIMA series (non-stationary). Bottom: after differencing d times, ACF/PACF reveal the AR and MA orders for the stationary component.

ARIMA: Visual Illustration



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ARIMA Components

AR(p)	I(d)	MA(q)
Autoregressive Memory	Integration Differencing	Moving Average Shocks

Special Cases

- ARIMA(p,0,q) = ARMA(p,q) – stationary
- ARIMA(0,1,0) = Random walk
- ARIMA(0,1,1) = IMA(1,1) – exponential smoothing
- ARIMA(1,1,0) = ARI(1,1) – differenced AR(1)

ARIMA(1,1,0) Example

ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently: $(1 - \phi_1 L)(1 - L) Y_t = c + \varepsilon_t$

Interpretation

- The **changes** in Y_t follow an AR(1) process
- If $|\phi_1| < 1$, the changes are stationary
- Y_t itself has a stochastic trend
- Common model for many economic time series

ARIMA(0,1,1) Example

IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently: $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha)\hat{Y}_t$$

where $\alpha = 1 + \theta_1$ (for $-1 < \theta_1 < 0$).

The Role of the Constant in ARIMA

Constant Term in ARIMA(p,d,q)

When $d > 0$, the constant c has a different interpretation: $\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$

Important Implications

- For $d = 1$: c represents the **drift**
  - Average change: $\mathbb{E}[\Delta Y_t] = \frac{c}{1-\phi_1-\cdots-\phi_p}$
  - Linear trend in levels
- For $d = 2$: c affects the **curvature**
  - Quadratic trend in levels
- Often $c = 0$ is assumed when $d \geq 1$
  - No deterministic trend component

Testing for Unit Roots

Why Test?

Before fitting an ARIMA model, we need to determine:

1. Is the series stationary? (Is $d = 0$?)
2. If not, how many differences are needed? (What is d?)

Common Unit Root Tests

- **Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- **Phillips-Perron (PP)**
- **KPSS** (stationarity test – reversed null hypothesis)

Researcher Spotlight: Dickey & Fuller

David Dickey (*1945)

W Wikipedia

Wayne Fuller (1931–2022)

W Wikipedia

Biography

- **David Dickey**: American statistician at NC State University. PhD student of Wayne Fuller at Iowa State
- **Wayne Fuller**: American statistician, professor at Iowa State University
- Together they developed the foundational test for unit roots in time series

Key Contributions

- **Dickey-Fuller test** (1979) — the fundamental unit root test
- **Augmented Dickey-Fuller (ADF)** — extension with lagged differences
- **Critical value tables** — non-standard distributions under the null
- Enabled rigorous testing of integration order for ARIMA modeling

The Dickey-Fuller Test

Setup

Consider the AR(1) model: $Y_t = \phi Y_{t-1} + \varepsilon_t$. Subtract Y_{t-1}: $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$, where $\gamma = \phi - 1$.

Hypotheses

- H_0: $\gamma = 0$ (unit root, $\phi = 1$, non-stationary)
- H_1: $\gamma < 0$ (stationary, $|\phi| < 1$)

Key Issue

Under H_0, the t-statistic does **not** follow a standard t-distribution! Must use Dickey-Fuller critical values.

Dickey-Fuller Test Variants

Three Specifications

1. **No constant, no trend**: $\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$
2. **With constant (drift)**: $\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$
3. **With constant and trend**: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$

Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference

Augmented Dickey-Fuller (ADF) Test

The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

Definition 6 (ADF Test)

Add lagged differences: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^{k} \delta_j \Delta Y_{t-j} + \varepsilon_t$
Test $H_0 : \gamma = 0$ using ADF critical values.

Choosing Lag Length k

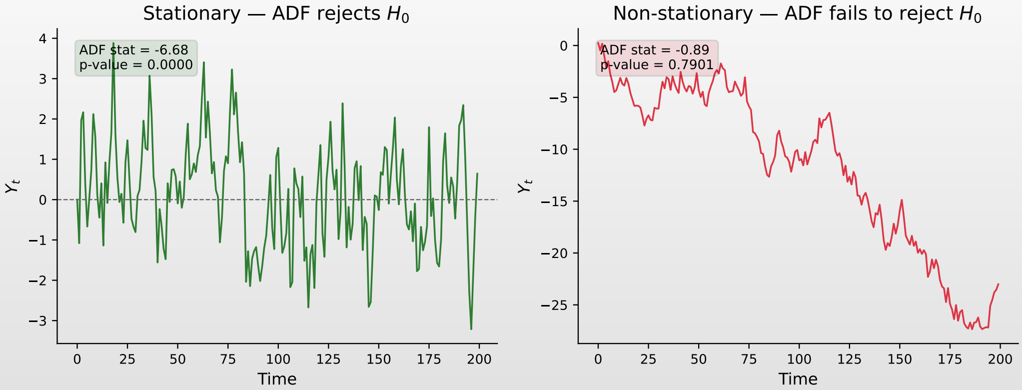
- Use information criteria (AIC, BIC)
- Start with k_{max}, reduce until last lag significant

ADF Test: Visual Illustration

Observation

- **Left**: stationary series $\Rightarrow$ ADF rejects unit root
- **Right**: non-stationary $\Rightarrow$ ADF fails to reject

ADF Test: Visual Illustration



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ADF Test Critical Values

Model	1%	5%	10%
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

Decision Rule

- Test statistic $<$ critical value $\Rightarrow$ Reject H_0 (stationary)
- Test statistic $\geq$ critical value $\Rightarrow$ Fail to reject (unit root)

The Phillips-Perron (PP) Test

Motivation

Like ADF, tests H_0: Unit root vs H_1: Stationary, but uses a **non-parametric correction** for serial correlation instead of adding lagged differences.

Test Statistic

The PP test modifies the DF t-statistic:

$$Z_t = t_{\hat{\gamma}} \cdot \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - \hat{\sigma}^2)(se(\hat{\gamma}))}{2\hat{\lambda}^2 \cdot s}$$

where $\hat{\lambda}^2$ is a consistent estimate of the long-run variance using Newey-West.

Advantages over ADF

- Robust to heteroskedasticity and serial correlation
- No need to select lag length (uses bandwidth instead)

The KPSS Test

Reversed Hypotheses

Unlike ADF: H_0: Stationary vs H_1: Unit root

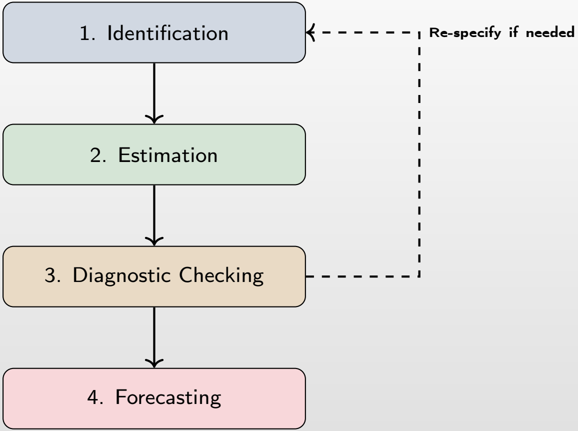
KPSS Procedure

Decompose: $Y_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$. Test whether $\text{Var}(u_t) = 0$.

Complementary Use with ADF

- ADF rejects, KPSS doesn't $\Rightarrow$ Stationary
- ADF doesn't reject, KPSS rejects $\Rightarrow$ Unit root
- Both reject or neither $\Rightarrow$ Inconclusive

The Box-Jenkins Methodology



Step 1: Determining d

Procedure

1. Plot the time series – look for trends, changing variance
2. Examine ACF – slow decay suggests non-stationarity
3. Apply unit root tests (ADF, KPSS)
4. If non-stationary, difference and repeat

Practical Guidelines

- Most economic series: $d = 1$ is sufficient
- Rarely need $d > 2$
- If ACF of ΔY_t still decays slowly, try $d = 2$
- Watch for overdifferencing (ACF with $\rho_1 \approx -0.5$)

Step 2: Determining p and q

After Differencing

Once $W_t = \Delta^d Y_t$ is stationary, use ACF/PACF to identify ARMA(p,q):

Model	ACF	PACF
AR(p)	Decays exponentially	Cuts off after lag p
MA(q)	Cuts off after lag q	Decays exponentially
ARMA(p,q)	Decays	Decays

Information Criteria

When patterns are unclear, compare models using:

- $\text{AIC} = -2\ln(L) + 2k; \quad \text{BIC} = -2\ln(L) + k\ln(n)$

Lower is better. BIC penalizes complexity more.

Auto-ARIMA Algorithms

Automated Model Selection

Modern software can automatically select (p, d, q):

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

How Auto-ARIMA Works

1. Use unit root tests to determine d
2. Fit models for various (p, q) combinations
3. Select model with lowest AIC/BIC
4. Optionally use stepwise search for efficiency

Caution

Automated selection is helpful but not infallible. Always check diagnostics!

Estimation Methods

Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- Assumes $\varepsilon_t \sim N(0, \sigma^2)$
- Maximizes the likelihood function
- Provides consistent, efficient estimators
- Yields standard errors for inference

Conditional vs Exact MLE

- **Conditional MLE**: Conditions on initial values
- **Exact MLE**: Treats initial values as unknown
- Difference diminishes as sample size grows

Conditional Log-Likelihood

Gaussian Log-Likelihood Function

- $\ell(\boldsymbol{\theta}, \sigma^2) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T} e_t^2(\boldsymbol{\theta})$
- $e_t(\boldsymbol{\theta}) = X_t - \hat{X}_{t|t-1}$ are the **one-step prediction errors**
- $\boldsymbol{\theta} = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, c)$

Example: ARIMA(1,1,1)

- Prediction errors: $e_t = \Delta X_t - \phi_1 \Delta X_{t-1} - \theta_1 e_{t-1} - c$
- Conditional MLE: set $e_0 = 0$, compute $e_1, \ldots, e_T$, maximize ℓ

Estimating σ^2

- At optimal parameters $\hat{\boldsymbol{\theta}}$: $\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} e_t^2(\hat{\boldsymbol{\theta}})$

Parameter Constraints

Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- **AR stationarity**: Roots of $\phi(z) = 0$ outside unit circle
- **MA invertibility**: Roots of $\theta(z) = 0$ outside unit circle

Checking in Practice

Most software reports:

- Estimated coefficients with standard errors
- Roots of AR and MA polynomials
- Warning if near-unit-root detected

Residual Analysis

What to Check

If the model is correct, residuals $\hat{\varepsilon}_t$ should be white noise:

1. Zero mean
2. Constant variance
3. No autocorrelation
4. (Optional) Normality

Diagnostic Tools

- **Residual ACF/PACF**: Should show no significant spikes
- **Ljung-Box test**: Tests for autocorrelation at multiple lags
- **Q-Q plot**: Checks normality assumption
- **Residual vs fitted**:
  - Checks for heteroskedasticity

The Ljung-Box Test

Definition 7 (Ljung-Box Q Statistic)

$Q(m) = n(n+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{n-k}$. Under H_0 (no autocorrelation): $Q(m) \sim \chi^2(m-p-q)$

Usage

- Choose $m \approx \ln(n)$ or $m = 10$ for quarterly, $m = 20$ for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if $Q(m)$ exceeds critical value

If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

Point Forecasts

Minimum MSE Forecast

The optimal h-step ahead forecast is the conditional expectation: $\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \ldots]$

ARIMA(1,1,1) Forecasting

Model: $(1-\phi_1 L)(1-L)Y_t = c + (1+\theta_1 L)\varepsilon_t$

One-step forecast: $\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1\hat{\varepsilon}_T$

For $h > 1$: replace unknown ε_{T+j} with 0, unknown Y_{T+j} with $\hat{Y}_{T+j|T}$

Forecast Intervals

Forecast Uncertainty

The h-step forecast error variance: $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$, where ψ_j are MA(∞) coefficients.

Confidence Intervals

Under normality, $(1-\alpha)\%$ interval: $\hat{Y}_{T+h|T} \pm z_{\alpha/2}\sqrt{\text{Var}(e_{T+h})}$

Key Property for I(1) Series

For integrated processes, forecast variance grows without bound as $h \to \infty$. Intervals widen over time!

Long-Run Forecasts for ARIMA

Behavior as $h \to \infty$

For ARIMA(p,1,q) with drift c:

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with $\sqrt{h}$

For ARIMA(p,1,q) without drift:

- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

Practical Implication

ARIMA forecasts are most reliable for short horizons. Long-term forecasts have very wide uncertainty bands.

Rolling Forecasting: Concept

What is Rolling Forecasting?

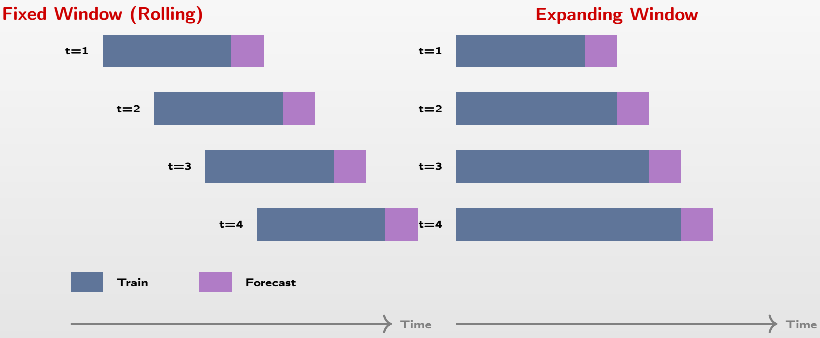
A technique to evaluate forecast accuracy out-of-sample:

1. Fix a **training window** of size w
2. Estimate model on observations $t = 1, \ldots, w$
3. Forecast h steps ahead: $\hat{Y}_{w+h|w}$
4. **Roll** the window forward by one period
5. Repeat until end of sample

Why Rolling Forecasts?

- Mimics real-time forecasting scenario
- Provides multiple forecast errors for evaluation
- Avoids overfitting to full sample

Fixed vs Expanding Window



Comparison

- **Fixed**: Window slides forward, constant size — adapts to regime changes
- **Expanding**: Window grows over time — uses all historical data

1-Step vs Multi-Step Forecasting

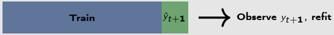
1-Step Ahead (Recursive)

- Forecast only next period
  - Refit model after each step
  - Use actual value once revealed
- Most accurate for short horizons

Multi-Step (Direct)

- Forecast multiple periods ahead
  - No refit between steps
  - Uses forecasted values as inputs
- Uncertainty compounds over horizon

1-Step Ahead



Multi-Step (h=3)

Rolling Forecast: Step-by-Step Example

Setup: ARIMA(1,1,0) with $\phi_1 = 0.6$

Model: $\Delta Y_t = \phi_1 \Delta Y_{t-1} + \varepsilon_t$ where $\Delta Y_t = Y_t - Y_{t-1}$

Given Data at Time T

$Y_{T-2} = 100, \quad Y_{T-1} = 103, \quad Y_T = 108 \quad \Rightarrow \quad \Delta Y_{T-1} = 3, \quad \Delta Y_T = 5$

1-Step Ahead Point Forecast

$$\hat{\Delta Y}_{T+1|T} = \phi_1 \cdot \Delta Y_T = 0.6 \times 5 = 3$$
$$\hat{Y}_{T+1|T} = Y_T + \hat{\Delta Y}_{T+1|T} = 108 + 3 = \boxed{111}$$

Multi-Step Point Forecasts

2-Step Ahead Forecast

$$\Delta\hat{Y}_{T+2|T} = \phi_1 \cdot \Delta\hat{Y}_{T+1|T} = 0.6 \times 3 = 1.8$$
$$\hat{Y}_{T+2|T} = \hat{Y}_{T+1|T} + \Delta\hat{Y}_{T+2|T} = 111 + 1.8 = \boxed{112.8}$$

General Formula for *h*-Step Forecast (ARIMA(1,1,0))

$$\Delta\hat{Y}_{T+h|T} = \phi_1^h \cdot \Delta Y_T$$
$$\hat{Y}_{T+h|T} = Y_T + \Delta Y_T \cdot \frac{\phi_1(1-\phi_1^h)}{1-\phi_1}$$

Numerical: 3-Step Forecast

$$\hat{Y}_{T+3|T} = 108 + 5 \times \frac{0.6(1-0.6^3)}{1-0.6} = 108 + 5 \times 1.092 = \boxed{113.46}$$

Confidence Intervals: Formulas

Forecast Error Variance

For ARIMA(1,1,0), the h-step forecast error variance:

$$\mathrm{Var}(e_{T+h|T}) = \sigma^2 \left(1 + \sum_{j=1}^{h-1} \psi_j^2\right)$$

where $\psi_j = \phi_1^{j-1}(1 + \phi_1 + \cdots + \phi_1^{j-1}) = \phi_1^{j-1} \cdot \frac{1-\phi_1^j}{1-\phi_1}$

$(1-\alpha)$% Confidence Interval

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \cdot \sqrt{\mathrm{Var}(e_{T+h|T})}$$

For 95% CI: $z_{0.025} = 1.96$

Confidence Interval: Numerical Example

Given: $\sigma^2 = 4$, $\phi_1 = 0.6$, $\hat{Y}_{T+1|T} = 111$

1-Step Ahead CI

$$\begin{aligned} \text{Var}(e_{T+1|T}) &= \sigma^2 = 4 \\ 95\%\ \text{CI} &= 111 \pm 1.96 \times \sqrt{4} = 111 \pm 3.92 = \boxed{[107.08,\ 114.92]} \end{aligned}$$

2-Step Ahead CI (for $\hat{Y}_{T+2|T} = 112.8$)

$$\begin{aligned} \psi_1 = 1 + \phi_1 = 1.6, \quad \text{Var}(e_{T+2|T}) &= 4(1 + 1.6^2) = 14.24 \\ 95\%\ \text{CI} = 112.8 \pm 1.96 \times \sqrt{14.24} &= 112.8 \pm 7.40 = \boxed{[105.40,\ 120.20]} \end{aligned}$$

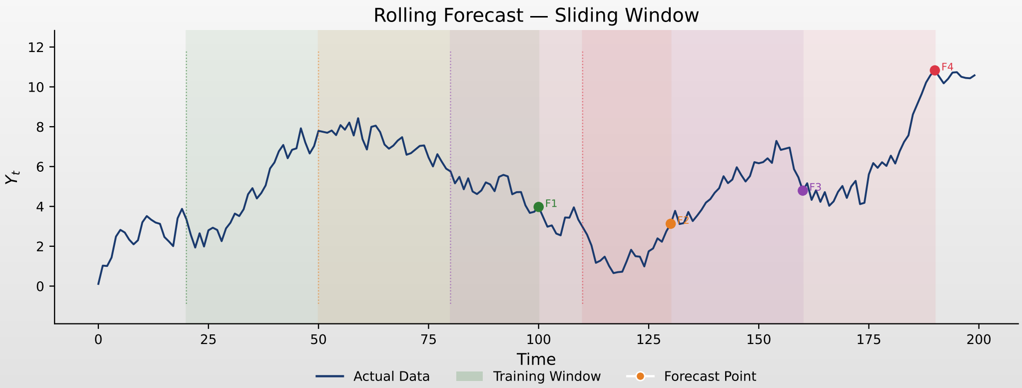
Note: CI widens as horizon increases!

Rolling Window Illustration

Rolling Procedure

- Each window produces a 1-step ahead forecast
- Compare forecasts to actuals to compute RMSE, MAE
- Rolling window keeps model estimation up-to-date

Rolling Window Illustration



TSA_ch3_rolling_forecast

Rolling Forecast: Python Code

Implementation

```
from statsmodels.tsa.arima.model import ARIMA

window_size = 100
forecasts, actuals = [], []

for t in range(window_size, len(y) - 1):
    train = y[:t]                          # Rolling window
    model = ARIMA(train, order=(1,1,0)).fit()
    forecast = model.forecast(steps=1)[0]
    forecasts.append(forecast)
    actuals.append(y[t])

rmse = np.sqrt(np.mean((np.array(forecasts) - np.array(actuals))**2))
```

Case Study: Complete ARIMA Analysis

Objective

- Forecast US Real GDP using the Box-Jenkins methodology

Steps

1. **Step 1**: Visualize data and check stationarity
2. **Step 2**: Apply unit root tests (ADF, KPSS)
3. **Step 3**: Difference if needed, identify p and q
4. **Step 4**: Estimate the ARIMA model
5. **Step 5**: Model diagnostics
6. **Step 6**: Generate forecasts with confidence intervals
7. **Step 7**: Evaluate forecast accuracy

Data

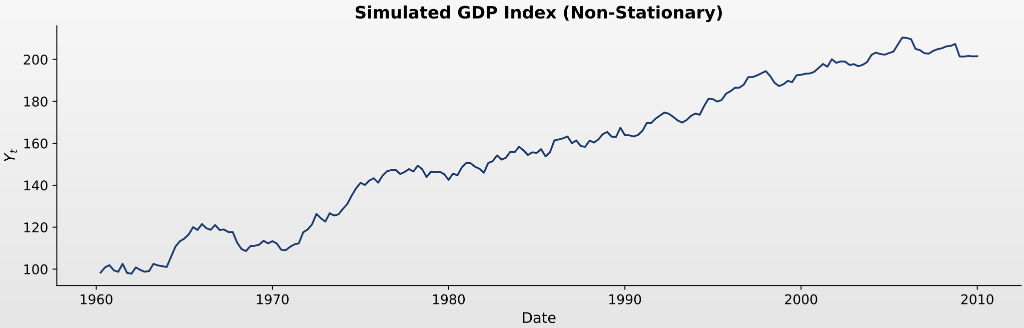
- US Real GDP (FRED: GDPC1), Quarterly, 1990Q1–2024Q2, $n = 138$

Case Study: US Real GDP (FRED)

Data: FRED GDPC1 (1960Q1–2024Q3)

Quarterly Real GDP, seasonally adjusted, billions of chained 2017 dollars. Non-stationary series with upward trend $\Rightarrow$ requires differencing.

Case Study: US Real GDP (FRED)



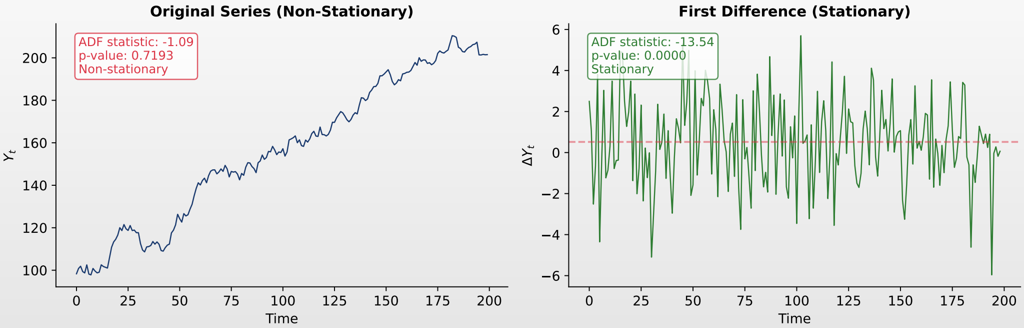
TSA_ch3_case_raw_data

Step 1: ADF Test for Stationarity

ADF Test Results

Original series: Large p-value $\Rightarrow$ fail to reject H_0 (unit root present). **First difference**: p-value $< 0.01 \Rightarrow$ reject $H_0 \Rightarrow d = 1$ is sufficient.

Step 1: ADF Test for Stationarity



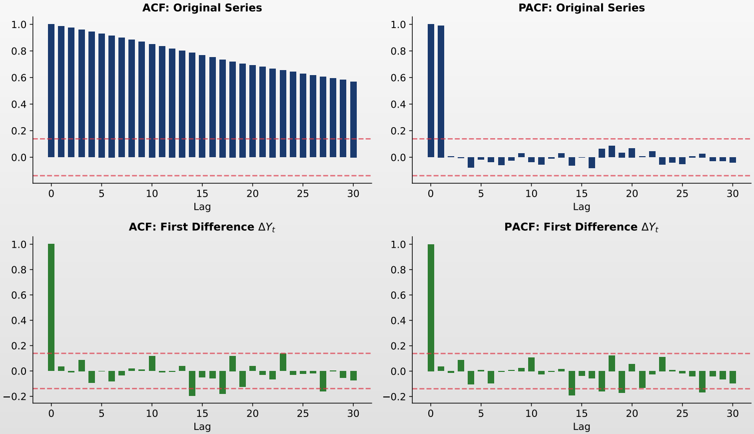
TSA_ch3_case_adf_test

Step 2: ACF/PACF Before and After Differencing

ACF/PACF Analysis

Top: Slow ACF decay (non-stationary) | Bottom: After differencing, low-order ARMA

Step 2: ACF/PACF Before and After Differencing



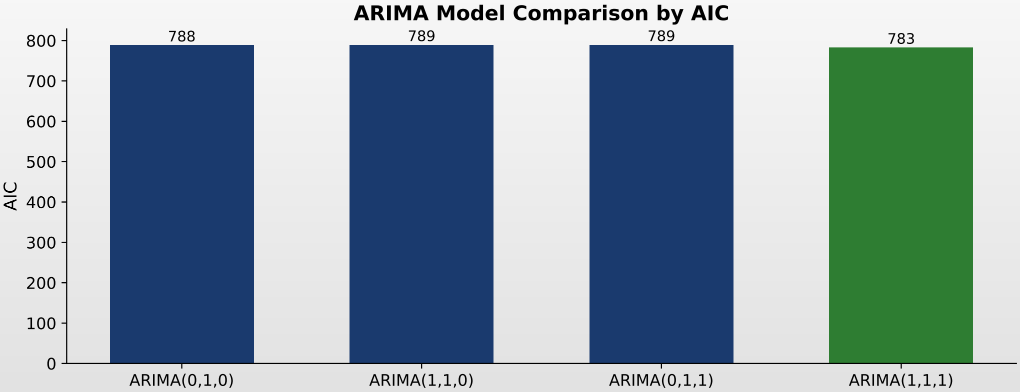
TSA_ch3_case_acf_diff

Step 3: ARIMA Model Comparison

Model Selection

Compare ARIMA(0,1,0), ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1). The model with lowest AIC is selected.

Step 3: ARIMA Model Comparison



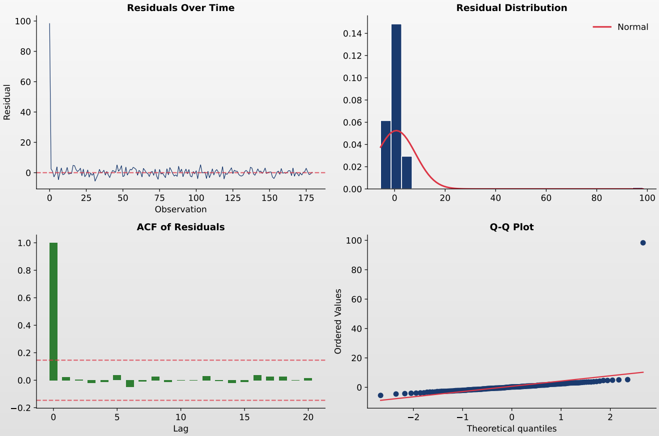
TSA_ch3_case_model_comparison

Step 4: Diagnostic Checking

ARIMA(1,1,1) Diagnostics

ACF: no autocorrelation ✓ Q-Q: **non-normal** (COVID-19 outlier) JB test: $p < 0.001$

Step 4: Diagnostic Checking



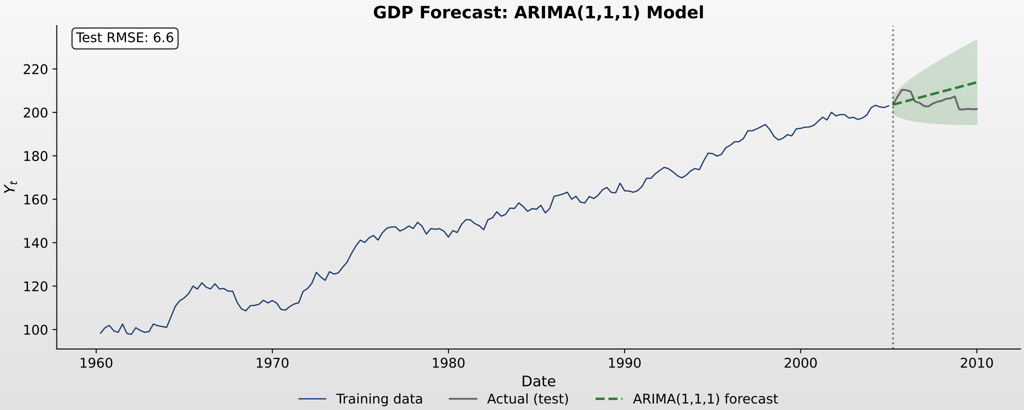
TSA_ch3_case_diagnostics

Step 5: Out-of-Sample Forecasting

Train/Val/Test Split (70%/15%/15%)

Train 70% (blue): Estimation | **Val 15%** (green): Tuning | **Test 15%** (purple): Evaluation with 95% CI

Step 5: Out-of-Sample Forecasting



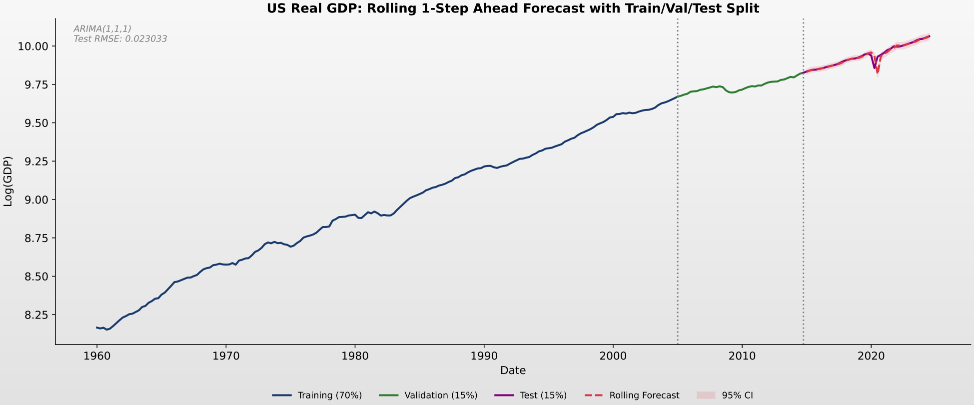
TSA_ch3_case_forecast

Step 6: Rolling Forecast with Train/Val/Test

Rolling 1-Step Ahead Forecast (Expanding Window, 95% CI)

Train 70% $\rightarrow$ Val 15% $\rightarrow$ Test 15% | Expanding window refits model at each step

Step 6: Rolling Forecast with Train/Val/Test



TSA_ch3_case_rolling_forecast

Summary

What we learned in this chapter

- Non-stationarity in time series
  - Deterministic vs stochastic trend; consequences for statistical inference
- Differencing and integrated processes
  - $\Delta Y_t = Y_t - Y_{t-1}$; if $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$
- ARIMA(p, d, q) models and unit root tests
  - ADF, PP, KPSS; Box-Jenkins: identify $\rightarrow$ estimate $\rightarrow$ validate
- Forecasts with confidence intervals
  - For I(1): CIs widen without bound ($\propto \sqrt{h}$)

Key Insight

- **Difference carefully**: One difference is usually sufficient ($d = 1$). Over-differencing creates artificial autocorrelation.

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"I have a quarterly GDP time series (80 observations). Test stationarity, difference if needed, estimate an ARIMA model, and forecast 8 quarters ahead. Give me complete Python code with plots."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it test stationarity with ADF *before* estimating ARIMA? Does it also use KPSS?
3. How does it determine the differencing order d? Does it check for over-differencing?
4. How does it choose p and q? ACF/PACF or just auto_arima?
5. Do forecast confidence intervals widen with horizon? (key I(1) property)

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Quick Quiz

Test your knowledge

1. What types of non-stationarity do you know and how is each treated?
2. Why does random walk variance grow with time?
3. What is the difference between ADF and KPSS tests?
4. What happens if we over-difference a series?
5. Why do ARIMA confidence intervals widen without bound for I(1) series?

Quiz Answers

Answers

1. **Types**: Deterministic trend (regression); stochastic trend/unit root (differencing)
2. **Variance**: $Y_t = \sum_{i=1}^{t} \varepsilon_i \Rightarrow \text{Var}(Y_t) = t\sigma^2$ (shocks accumulate)
3. **ADF vs KPSS**: ADF: H_0 = unit root; KPSS: H_0 = stationary. Used as complements.
4. **Over-differencing**: Creates artificial negative autocorrelation ($\rho_1 \approx -0.5$); non-invertible MA(1)
5. **Unbounded CIs**: $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2 \to \infty$ because ψ_j do not decay

What's Next?

Chapter 4: SARIMA Models for Seasonal Data

- **Seasonality**: repetitive patterns at regular intervals
- **Seasonal differencing**: the $(1 - L^s)$ operator
- **SARIMA$(p, d, q)(P, D, Q)_s$**: seasonal extension of ARIMA
- **Model identification**: seasonal ACF/PACF
- **Case study**: Airline passengers forecast

Questions?

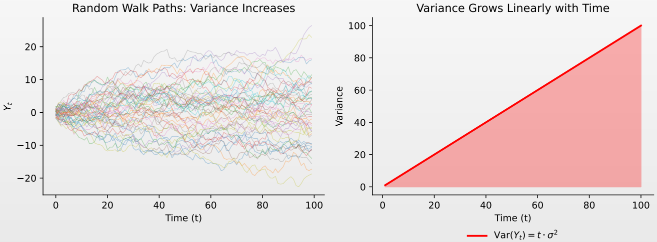
Quiz Question 1

Question

A time series Y_t follows a random walk: $Y_t = Y_{t-1} + \varepsilon_t$. What is $\text{Var}(Y_t)$?

(A) σ^2 (constant)
(B) $t \cdot \sigma^2$ (grows linearly with time)
(C) σ^2/t (decreases with time)
(D) σ^{2t} (grows exponentially)

Quiz Question 1: Answer



Correct Answer: (B) $\mathrm{Var}(Y_t) = t \cdot \sigma^2$

Random walk variance grows linearly with time — this is why random walks are non-stationary.

TSA_ch3_quiz1_rw_variance

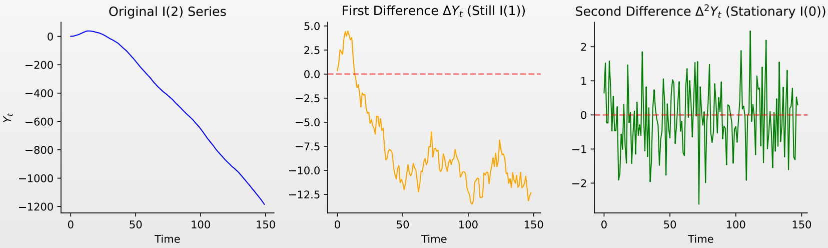
Quiz Question 2

Question

If a series Y_t is I(2), how many times must you difference it to achieve stationarity?

(A) 0 times (already stationary)
(B) 1 time
(C) 2 times
(D) Cannot be made stationary by differencing

Quiz Question 2: Answer



Correct Answer: (C) 2 times

I(d) means "integrated of order d" — requires d differences for stationarity.

TSA_ch3_quiz2_differencing

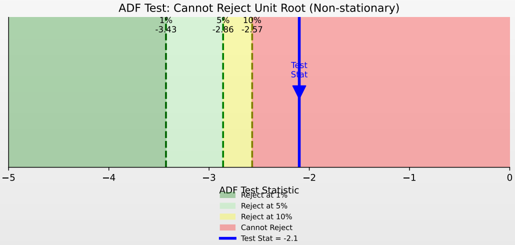
Quiz Question 3

Question

You run an ADF test and get a test statistic of -2.1 with critical values: -3.43 (1%), -2.86 (5%), -2.57 (10%). What do you conclude?

(A) Reject H_0: series is stationary at all levels
(B) Reject H_0: series is stationary at 10% level only
(C) Fail to reject H_0: series likely has a unit root
(D) The test is inconclusive

Quiz Question 3: Answer



Correct Answer: (C) Fail to reject H_0: series has unit root

Test stat $-2.1 > -2.57$ (10% CV) $\Rightarrow$ Cannot reject at any level. Consider differencing.

TSA_ch3_quiz3_adf_test

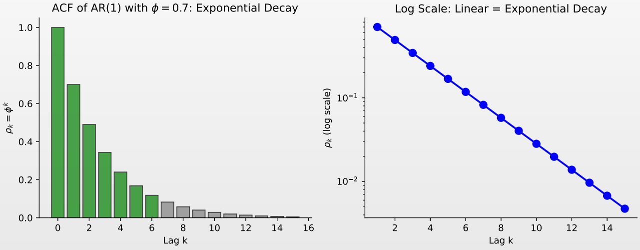
Quiz Question 4

Question

For an ARIMA(1,1,0) model, what is the ACF pattern of the **differenced** series ΔY_t?

(A) Cuts off after lag 1
(B) Decays exponentially
(C) Alternates in sign
(D) Is zero at all lags

Quiz Question 4: Answer



Correct Answer: (B) Decays exponentially

ARIMA(1,1,0) $\Rightarrow \Delta Y_t$ follows AR(1) with ACF $\rho_k = \phi_1^k$ (geometric decay).

TSA_ch3_quiz4_acf_decay

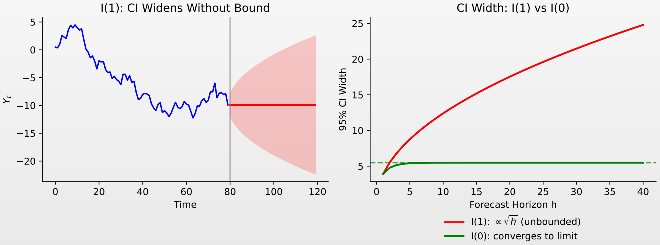
Quiz Question 5

Question

What happens to ARIMA forecast confidence intervals as the horizon h increases for an I(1) series?

(A) They stay constant
(B) They narrow (more precision)
(C) They widen without bound
(D) They widen but converge to a limit

Quiz Question 5: Answer



Correct Answer: (C) They widen without bound

For I(1): CI width $\propto \sqrt{h}$ (unbounded). For I(0): CIs converge to a limit.

TSA_ch3_quiz5_forecast_ci

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- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

Online Resources and Code

- **Quantlet**: https://quantlet.com → Code repository for statistics
- **Quantinar**: https://quantinar.com → Learning platform for quantitative methods
- **GitHub TSA**: https://github.com/QuantLet/TSA/tree/main/TSA_ch3

Thank You!

Questions?

Course materials available at: `https://danpele.github.io/Time-Series-Analysis/`

Quantlet Quantinar