Time Series Analysis and Forecasting

Chapter 5: Volatility Models

ARCH, GARCH, EGARCH, TGARCH

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Learning Objectives

By the end of this chapter, you will be able to:

1. Understand **volatility clustering** and stylized facts of financial returns
2. Estimate and interpret **ARCH** and **GARCH** models
3. Apply asymmetric models (**EGARCH**, **GJR-GARCH**) for the leverage effect
4. Perform model diagnostics and selection
5. Forecast volatility and calculate **Value at Risk (VaR)**

Practical Skills

- Python implementation with the `arch` package
- Interpretation of parameters and volatility persistence
- VaR calculation for risk management

Why Model Volatility?

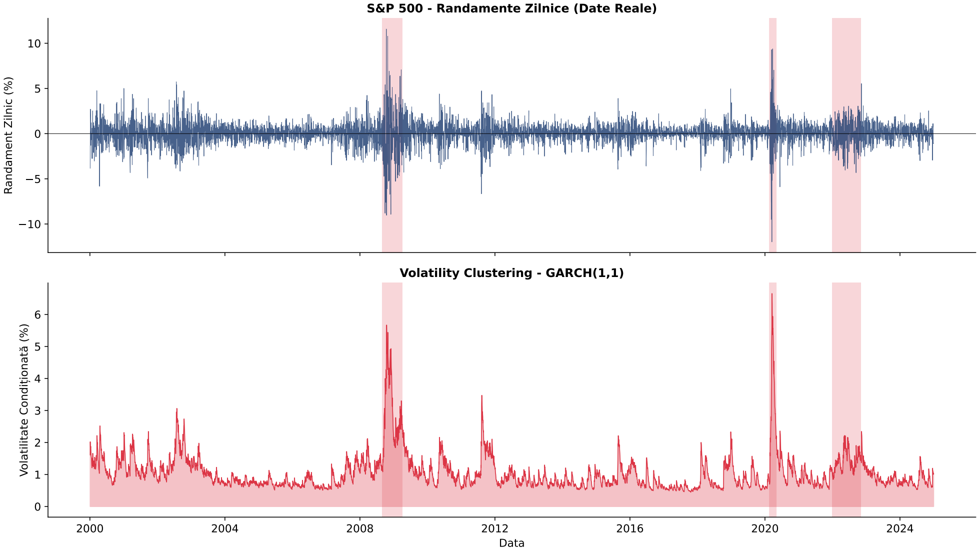
Empirical Observations in Financial Series

- Financial returns exhibit **volatility clustering** — periods of high volatility tend to be followed by periods of high volatility
- The distribution of returns has **fat tails** (leptokurtosis)
- Return correlation is nearly zero, but correlation of squares is significant
- Volatility responds **asymmetrically** to shocks (leverage effect)

Limitation of ARIMA Models

ARIMA models assume **constant variance** (homoskedasticity), which is not realistic for financial series!

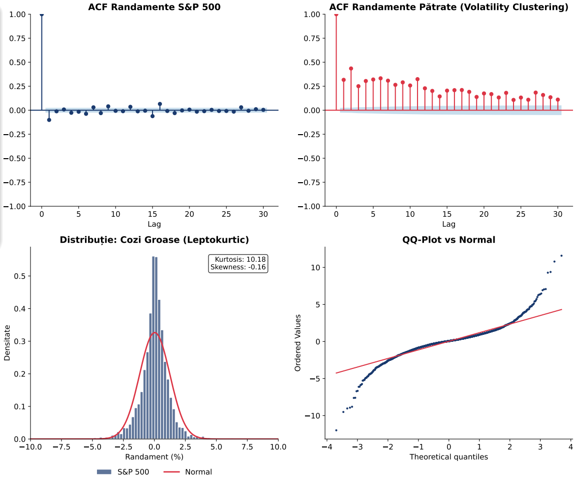
Volatility Clustering



Stylized Facts of Financial Returns

Observed Properties

1. **Absence of autocorrelation** in returns
2. **Significant autocorrelation** in r_t^2 and $|r_t|$
3. **Fat tails** (kurtosis > 3)
4. **Leverage effect** — negative correlation between returns and volatility
5. **Volatility clustering**



Conditional Heteroskedasticity

Definition 1 (Conditional Variance)

Let $\{r_t\}$ be a return series. The **conditional variance** at time t is:

$$\sigma_t^2 = \text{Var}(r_t|\mathcal{F}_{t-1}) = \mathbb{E}[(r_t - \mu_t)^2|\mathcal{F}_{t-1}]$$

where $\mathcal{F}_{t-1}$ represents the information available up to time $t-1$.

General Model

$$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0, 1)$$

where:

- μ_t = conditional mean (can be modeled as ARMA)
- σ_t^2 = conditional variance (modeled as ARCH/GARCH)
- z_t = standardized innovations (Normal, Student-t, GED)

The ARCH(q) Model — Engle (1982)

Definition 2 (ARCH(q))

The **Autoregressive Conditional Heteroskedasticity** model of order q:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1)$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

Stationarity Restrictions

- $\omega > 0$ (positive base variance)
- $\alpha_i \geq 0$ for $i = 1, \ldots, q$ (non-negativity)
- $\sum_{i=1}^{q} \alpha_i < 1$ (stationarity)

Remark 1

Robert Engle received the **Nobel Prize in Economics** in 2003 for developing the ARCH model!

Properties of the ARCH(1) Model

ARCH(1): $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$

- **Unconditional variance**: $\mathbb{E}[\varepsilon_t^2] = \dfrac{\omega}{1-\alpha_1}$ (if $\alpha_1 < 1$)
- **Kurtosis**: $\kappa = 3 \cdot \dfrac{1-\alpha_1^2}{1-3\alpha_1^2}$ (if $\alpha_1^2 < 1/3$)
- Kurtosis > 3 for $\alpha_1 > 0 \Rightarrow$ **fat tails**!

Numerical Example

If $\omega = 0.0001$ and $\alpha_1 = 0.3$:

- Unconditional variance: $\sigma^2 = \frac{0.0001}{1-0.3} = 0.000143$
- Kurtosis: $\kappa = 3 \cdot \frac{1-0.09}{1-0.27} = 3.74 > 3$

Testing for ARCH Effects

Engle's Test for ARCH Effects

Procedure:

1. Estimate the mean model and obtain residuals $\hat{\varepsilon}_t$
2. Calculate $\hat{\varepsilon}_t^2$
3. Regress $\hat{\varepsilon}_t^2$ on its lags:
$$\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \cdots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$$
4. Calculate the statistic $LM = T \cdot R^2 \sim \chi^2(q)$

Hypotheses

- H_0: No ARCH effects ($\alpha_1 = \cdots = \alpha_q = 0$)
- H_1: ARCH effects present (at least one $\alpha_i \neq 0$)

Limitations of the ARCH Model

Practical Problems

1. **High order** — many lags are usually needed (large q)
2. **Many parameters** — estimation difficulties
3. **Non-negativity constraints** — difficult to impose for large q
4. **Does not capture persistence** — observed volatility is very persistent

The Solution

The GARCH Model — introduces lags of conditional variance to capture persistence with fewer parameters!

The GARCH(p,q) Model — Bollerslev (1986)

Definition 3 (GARCH(p,q))

The **Generalized ARCH** model:

$$\varepsilon_t = \sigma_t z_t, \quad z_t \sim \text{i.i.d.}(0,1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

Interpretation

- ω = base level of volatility
- α_i = reaction to recent shocks (news coefficients)
- β_j = volatility persistence (memory)
- $\alpha + \beta$ = total persistence

The GARCH(1,1) Model

The Most Popular Volatility Model

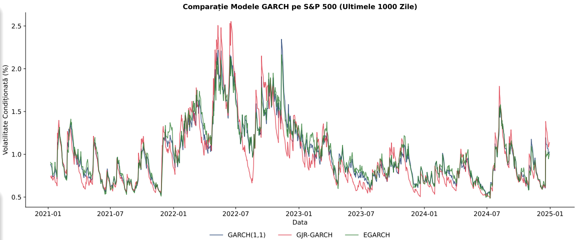
$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

Restrictions

- $\omega > 0$
- $\alpha \geq 0$, $\beta \geq 0$
- $\alpha + \beta < 1$ (stationarity)

Properties

- Unconditional variance: $\bar{\sigma}^2 = \dfrac{\omega}{1 - \alpha - \beta}$
- Half-life: $HL = \dfrac{\ln(0.5)}{\ln(\alpha + \beta)}$



GARCH(1,1) as ARMA for ε_t^2

ARMA(1,1) Representation

Define $\nu_t = \varepsilon_t^2 - \sigma_t^2$ (variance shock). Then:

$$\varepsilon_t^2 = \omega + (\alpha + \beta)\varepsilon_{t-1}^2 + \nu_t - \beta\nu_{t-1}$$

This is an **ARMA(1,1)** for ε_t^2!

Implications

- ACF of ε_t^2 decays exponentially (like ARMA)
- Persistence is given by $\alpha + \beta$
- PACF can help identify the order

Estimation of GARCH Models

Maximum Likelihood Estimation (MLE)

Log-likelihood function (normal distribution):

$$\ell(\theta) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln(\sigma_t^2) + \frac{\varepsilon_t^2}{\sigma_t^2}\right]$$

Alternative Distributions for z_t

- **Student-t**: captures fat tails

$$f(z;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{(\nu-2)\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{z^2}{\nu-2}\right)^{-\frac{\nu+1}{2}}$$

- **GED (Generalized Error Distribution)**: flexibility for kurtosis
- **Skewed Student-t**: asymmetry and fat tails

Typical Values for GARCH(1,1)

Series	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\alpha + \beta}$
S&P 500 daily	0.05–0.10	0.85–0.95	0.95–0.99
EUR/USD daily	0.03–0.08	0.90–0.95	0.95–0.99
Bitcoin daily	0.10–0.20	0.75–0.85	0.90–0.98
Bonds	0.02–0.05	0.90–0.97	0.95–0.99

Observations

- $\alpha + \beta$ close to 1 $\Rightarrow$ **very persistent volatility**
- Small α, large β $\Rightarrow$ slow reaction to shocks, long memory
- Bitcoin: larger α $\Rightarrow$ faster reaction to news

IGARCH — Integrated GARCH

Definition 4 (IGARCH(1,1))

When $\alpha + \beta = 1$:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2$$

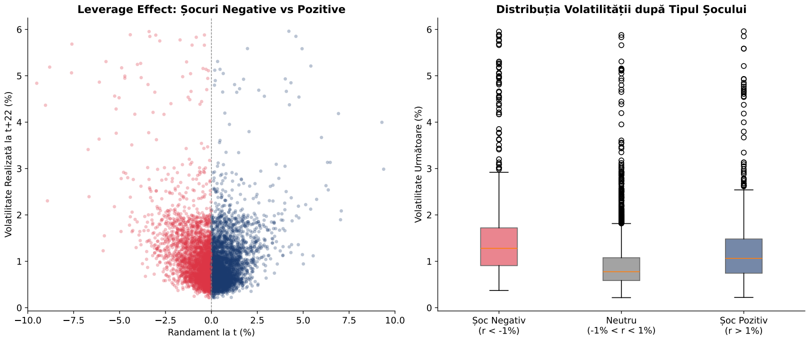
Properties

- Unconditional variance does not exist (infinite)
- Shocks have **permanent** effect on volatility
- Used for series with extreme persistence
- Useful for **RiskMetrics** (J.P. Morgan): $\alpha = 0.06$, $\beta = 0.94$

Remark 2

IGARCH is analogous to a unit root in variance!

Leverage Effect



Definition

Leverage effect: Negative shocks (price declines) tend to increase volatility **more** than positive shocks of the same magnitude.

Problem with Standard GARCH

GARCH(p,q) depends on ε^2_{t-1}, so it treats positive and negative shocks **symmetrically**!

The EGARCH Model — Nelson (1991)

Definition 5 (EGARCH(1,1))

Exponential GARCH:

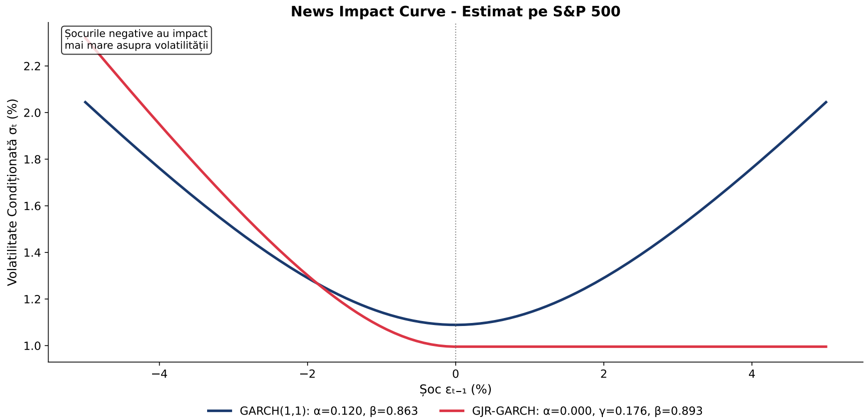
$$\ln(\sigma_t^2) = \omega + \alpha\left(|z_{t-1}| - \mathbb{E}[|z_{t-1}|]\right) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

where $z_t = \varepsilon_t/\sigma_t$.

EGARCH Advantages

- **No non-negativity constraints required** — models $\ln(\sigma_t^2)$
- **Captures leverage effect** through parameter γ
  - $\gamma < 0$: negative shocks $\Rightarrow$ higher volatility
  - $\gamma = 0$: symmetric effect (like GARCH)
- Persistence is given by β

News Impact Curve — EGARCH



Interpretation

News Impact Curve: shows how future volatility σ^2_{t+1} depends on the current shock ε_t, holding σ^2_t constant.

The GJR-GARCH (TGARCH) Model

Definition 6 (GJR-GARCH(1,1))

Glosten, Jagannathan & Runkle (1993):

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 \cdot I_{t-1} + \beta\sigma_{t-1}^2$$

where $I_{t-1} = \begin{cases} 1 & \text{if } \varepsilon_{t-1} < 0 \\ 0 & \text{otherwise} \end{cases}$

Interpretation

- Positive shocks ($\varepsilon_{t-1} > 0$): impact $= \alpha$
- Negative shocks ($\varepsilon_{t-1} < 0$): impact $= \alpha + \gamma$
- Leverage effect present if $\gamma > 0$

Stationarity

$\alpha + \gamma/2 + \beta < 1$

TGARCH — Threshold GARCH

Definition 7 (TGARCH(1,1))

Zakoian (1994) — models standard deviation:

$$\sigma_t = \omega + \alpha^+ \varepsilon_{t-1}^+ + \alpha^- \varepsilon_{t-1}^- + \beta \sigma_{t-1}$$

where $\varepsilon_t^+ = \max(\varepsilon_t, 0)$ and $\varepsilon_t^- = \max(-\varepsilon_t, 0)$.

Comparison of Asymmetric Models

Model	Specification	Leverage
GARCH	σ_t^2	No
EGARCH	$\ln(\sigma_t^2)$	Yes ($\gamma < 0$)
GJR-GARCH	σ_t^2 with indicator	Yes ($\gamma > 0$)
TGARCH	σ_t	Yes ($\alpha^- > \alpha^+$)

Order Selection

Information Criteria

- **AIC** $= -2\ell + 2k$
- **BIC** $= -2\ell + k\ln(T)$
- **HQIC** $= -2\ell + 2k\ln(\ln(T))$

where ℓ = maximized log-likelihood, k = number of parameters.

Practical Recommendations

- GARCH(1,1) is sufficient in **90% of cases**
- Check if asymmetric model significantly improves fit
- Choose innovation distribution that minimizes AIC/BIC

GARCH Model Diagnostics

Standardized Residuals

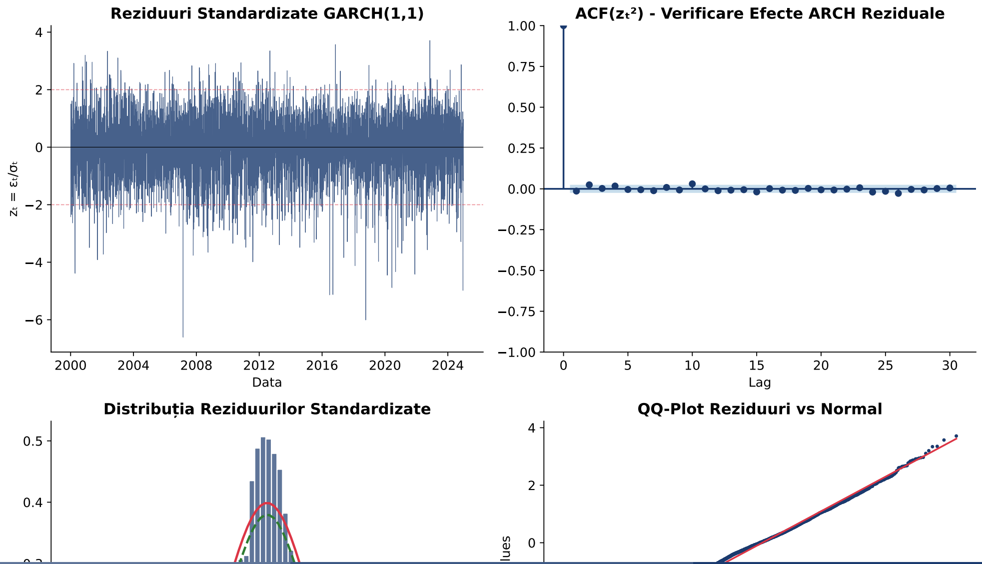
$$\hat{z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t}$$

If the model is correctly specified, $\hat{z}_t$ should be i.i.d.(0,1).

Diagnostic Checks

1. **Ljung-Box on** $\hat{z}_t$: check absence of autocorrelation in mean
2. **Ljung-Box on** $\hat{z}_t^2$: check absence of residual ARCH effects
3. **ARCH-LM test on** $\hat{z}_t$: confirm absence of heteroskedasticity
4. **Histogram + QQ-plot**: verify assumed distribution

Diagnostic Example



Forecasting with GARCH(1,1)

One-Step-Ahead Forecast

$$\hat{\sigma}^2_{T+1} = \omega + \alpha\varepsilon^2_T + \beta\sigma^2_T$$

Multi-Step Forecast

For $h > 1$:

$$\mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2 + (\alpha + \beta)^{h-1}(\sigma^2_{T+1} - \bar{\sigma}^2)$$

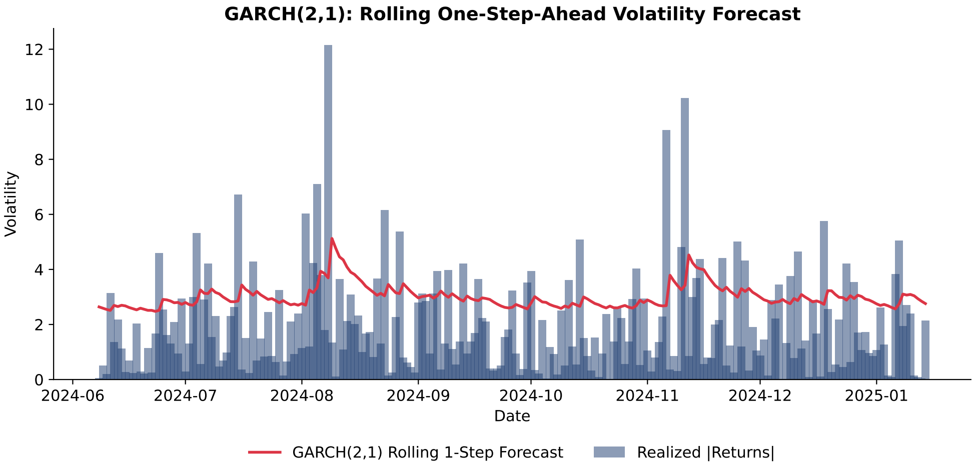
where $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$ = unconditional variance.

Convergence

$$\lim_{h\to\infty} \mathbb{E}_T[\sigma^2_{T+h}] = \bar{\sigma}^2$$

Forecast converges to unconditional variance!

Volatility Forecast — Visualization



- Forecast converges exponentially to $\bar{\sigma}^2$
- Convergence speed depends on $\alpha + \beta$

Applications of Volatility Forecasting

Value at Risk (VaR)

$$\text{VaR}_\alpha = -\mu_{T+1} + z_\alpha \cdot \sigma_{T+1}$$

The probability of losing more than VaR is α (e.g., 1%, 5%).

Expected Shortfall

$$\text{ES}_\alpha = \mathbb{E}[-r_{T+1} | r_{T+1} < -\text{VaR}_\alpha]$$

Other Applications

- Option pricing
- Dynamic hedging
- Portfolio allocation
- Stress testing
- Scenario analysis

Value at Risk — Numerical Example

VaR Calculation for a Portfolio

Data: Portfolio of **1,000,000 EUR**, forecasted volatility $\hat{\sigma}_{T+1} = 1.5\%$

VaR with Normal Distribution

Level	z_α	**VaR (%)**	**VaR (EUR)**
95% (1 day)	1.645	2.47%	24,675
99% (1 day)	2.326	3.49%	34,890
99% (10 days)	$2.326 \cdot \sqrt{10}$	11.03%	110,314

Scaling for Longer Periods

$$\text{VaR}_{h \text{ days}} = \text{VaR}_{1 \text{ day}} \cdot \sqrt{h}$$

This rule assumes i.i.d. returns — an approximation for short horizons.

Value at Risk — Student-t Distribution

Why Student-t?

- The normal distribution **underestimates** tail risk
- Financial returns have **fat tails** (kurtosis > 3)
- Student-t with ν degrees of freedom better captures extremes

VaR 99% (1 day) Comparison for $\sigma = 1.5\%$, Portfolio = 1M EUR

Distribution	**Quantile**	**VaR (EUR)**
Normal	2.326	34,890
Student-t ($\nu = 10$)	2.764	41,460
Student-t ($\nu = 6$)	3.143	47,145
Student-t ($\nu = 4$)	3.747	56,205

Observation

With $\nu = 6$ (typical for stocks), VaR is **35% higher** than normal!

VaR — Complete Example with GARCH

VaR Calculation Procedure

1. Estimate GARCH(1,1) model with Student-t distribution
2. Obtain volatility forecast: $\hat{\sigma}_{T+1}$
3. Calculate VaR: $\text{VaR}_\alpha = t_\alpha(\nu) \cdot \hat{\sigma}_{T+1} \cdot \sqrt{\frac{\nu-2}{\nu}}$

Example: S&P 500

- Estimated parameters: $\alpha = 0.088$, $\beta = 0.900$, $\nu = 6.4$
- Forecasted volatility: $\hat{\sigma}_{T+1} = 1.2\%$
- Portfolio: 10,000,000 EUR

VaR 99% (1 day):

$$\text{VaR} = 3.05 \times 0.012 \times 10,000,000 = \mathbf{366{,}000\ EUR}$$

GARCH Estimation in Python — arch

Installation and Import

```
pip install arch

import numpy as np
import pandas as pd
from arch import arch_model
from arch.univariate import GARCH, EGARCH, ConstantMean
```

GARCH(1,1) Estimation

```
# Assume returns = return series (%)
model = arch_model(returns, vol='Garch', p=1, q=1,
                   dist='normal')
results = model.fit(disp='off')
print(results.summary())
```

Asymmetric Models in Python

EGARCH

```
model_egarch = arch_model(returns, vol='EGARCH', p=1, q=1)
res_egarch = model_egarch.fit(disp='off')
```

GJR-GARCH

```
model_gjr = arch_model(returns, vol='Garch', p=1, o=1, q=1)
res_gjr = model_gjr.fit(disp='off')
```

Alternative Distributions

```
# Student-t
model_t = arch_model(returns, vol='Garch', p=1, q=1,
                     dist='t')
# Skewed Student-t
model_skewt = arch_model(returns, vol='Garch', p=1, q=1,
                         dist='skewt')
```

Forecasting and Diagnostics

Volatility Forecast

```
# Forecast 10 steps ahead
forecasts = results.forecast(horizon=10)
volatility_forecast = np.sqrt(forecasts.variance.values[-1, :])
```

Standardized Residuals

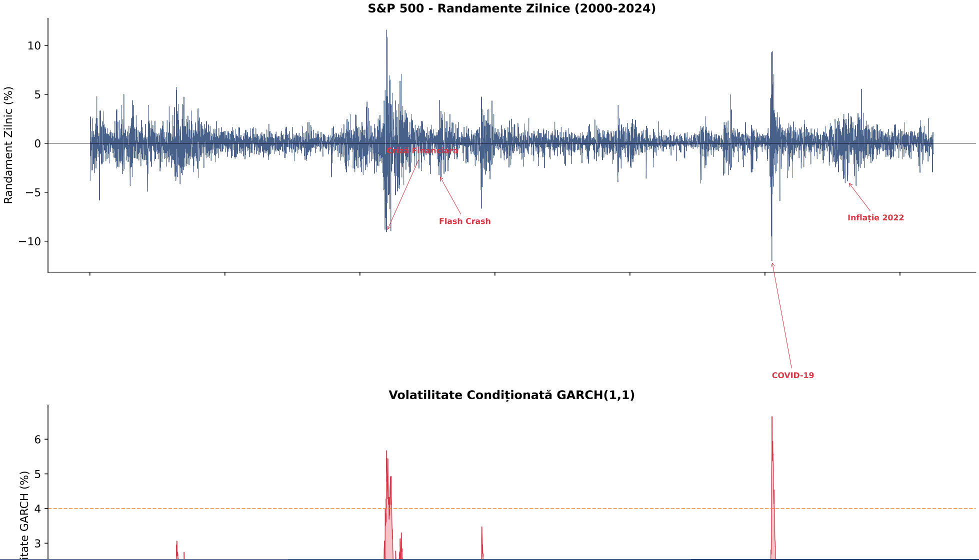
```
std_resid = results.std_resid

# Ljung-Box Test
from statsmodels.stats.diagnostic import acorr_ljungbox
lb_test = acorr_ljungbox(std_resid**2, lags=10)
```

VaR Calculation

```
sigma_forecast = np.sqrt(forecasts.variance.values[-1, 0])
VaR_95 = 1.645 * sigma_forecast  # for alpha = 5%
```

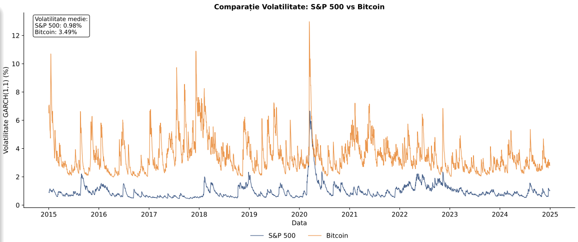
S&P 500 Volatility Analysis



GARCH(1,1) Estimation — S&P 500

Estimation Results

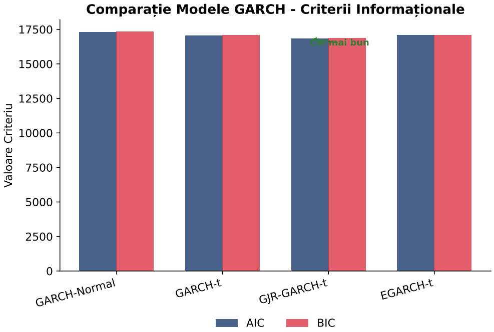
Parameter	Value
ω	0.0108
α	0.0883
β	0.9002
$\alpha + \beta$	0.9885
ν (Student-t df)	6.42

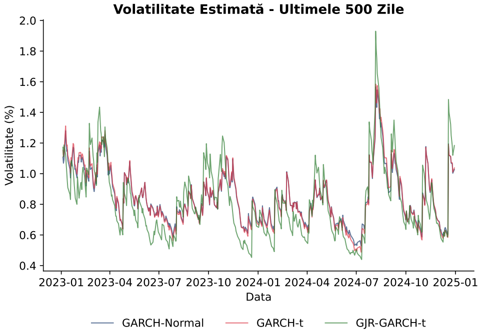


Interpretation

- Very persistent volatility
- Half-life $\approx$ 60 days
- Fat tails (Student-t)

GARCH vs EGARCH Comparison — S&P 500





Leverage Effect Confirmed

EGARCH: $\gamma = -0.12$ (significantly negative) $\Rightarrow$ negative shocks amplify volatility more than positive ones

Key Formulas

Volatility Models

- **ARCH(q):** $\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2$
- **GARCH(1,1):** $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$
- **EGARCH:** $\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$
- **GJR-GARCH:** $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 I_{t-1} + \beta\sigma_{t-1}^2$

Properties and Measures

- **Unconditional variance:** $\bar{\sigma}^2 = \dfrac{\omega}{1-\alpha-\beta}$
- **Half-life:** $HL = \dfrac{\ln(0.5)}{\ln(\alpha+\beta)}$
- **VaR:** $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{T+1}$
- **Stationarity:** $\alpha + \beta < 1$

ARCH-LM Test

$LM = T \cdot R^2 \sim \chi^2(q)$ where R^2 comes from regressing $\hat{\varepsilon}_t^2$ on its lags

Summary — Chapter 5: Volatility Models

Key Concepts

- **ARCH(q)**: conditional variance depends on past squared errors
- **GARCH(p,q)**: adds variance lags for persistence
- **EGARCH**: allows leverage effect, no positivity constraints
- **GJR-GARCH/TGARCH**: captures asymmetry with indicator variables

Applications

- Risk measurement and forecasting (VaR, ES)
- Derivative pricing
- Portfolio management

Practical Tip

Start with GARCH(1,1), check for leverage effect, choose innovation distribution that minimizes AIC/BIC!

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