Time Series Analysis and Forecasting

Chapter 10: Comprehensive Review

Applied Case Studies with Rigorous Methodology

Outline

1. Forecasting Methodology
2. Case Study 1: Bitcoin Volatility (GARCH)
3. Case Study 2: Sunspot Cycles (Fourier)
4. Case Study 3: Unemployment (Prophet)
5. Case Study 4: Multivariate Analysis (VAR)
6. Synthesis and Guidelines

The Scientific Approach to Forecasting

Research Question

How do we **rigorously evaluate** forecast performance while avoiding overfitting?

The Fundamental Problem

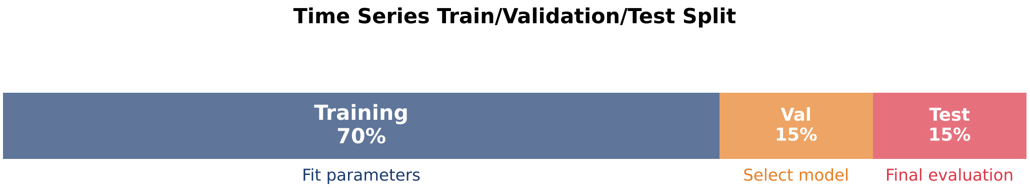
- In-sample fit $\neq$ Out-of-sample performance
- Models can "memorize" training data without learning patterns
- **Solution**: Proper train/validation/test methodology

Key Principle

"The test set must remain **untouched** until final evaluation."

— Standard practice in machine learning and econometrics

Train/Validation/Test Framework



Training Set

- Fit parameters
- Largest portion

Validation Set

- Compare models
- Tune hyperparams

Test Set

- **Held out**
- Final metrics

Evaluation Metrics

Definition 1 (Forecast Error Metrics)

Let y_t be actual, $\hat{y}_t$ forecast:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_t (y_t - \hat{y}_t)^2}, \quad \text{MAE} = \frac{1}{n}\sum_t |y_t - \hat{y}_t|, \quad \text{MAPE} = \frac{100\%}{n}\sum_t \left|\frac{y_t - \hat{y}_t}{y_t}\right| \tag{1}$$

When to Use Each

- **RMSE**: Penalizes large errors
- **MAE**: Robust to outliers
- **MAPE**: Scale-independent (%)

Caution

- MAPE undefined when $y_t = 0$
- Compare on **same** test set
- Report **out-of-sample** metrics

Bitcoin: Problem Statement

Research Question

Can we forecast Bitcoin's **volatility** using GARCH models?

Data Characteristics

- Source: Yahoo Finance (BTC-USD)
- Period: Jan 2019 – Jan 2025
- Frequency: Daily
- Observations: $\approx 2,200$ days

Stylized Facts

- Returns: near-zero mean
- Fat tails (kurtosis > 3)
- Volatility clustering

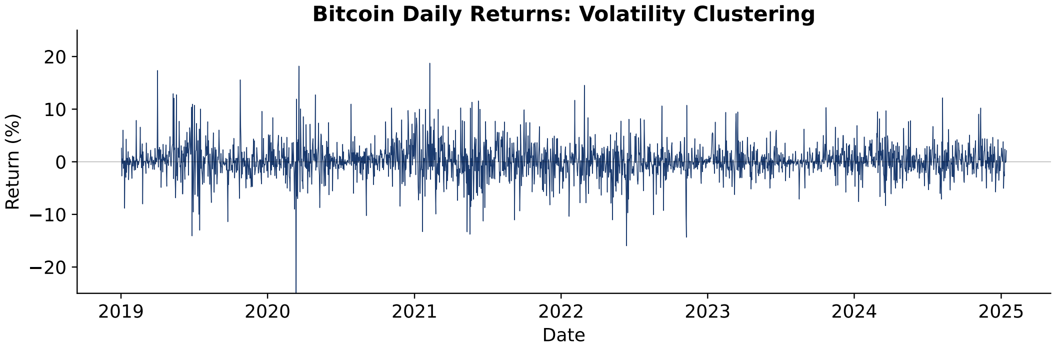
Key Insight

Financial returns are typically:

- **Unpredictable** in mean
- **Predictable** in variance

$\Rightarrow$ Focus on **volatility forecasting**

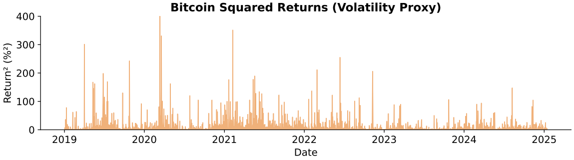
Bitcoin: Volatility Clustering



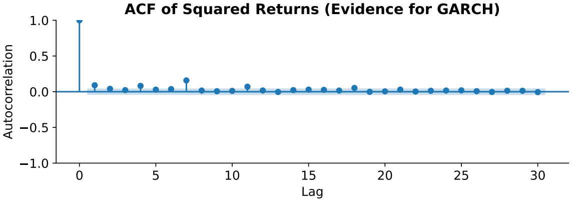
Observation

Large returns tend to follow large returns, small follow small. This is **volatility clustering**—the phenomenon GARCH captures.

Bitcoin: Evidence for GARCH



Squared returns r_t^2 proxy for volatility σ_t^2. Spikes cluster together.



ACF bars exceed blue bands $\Rightarrow$ significant autocorrelation at multiple lags.

Why GARCH?

If r_t^2 were white noise, ACF would be zero. Significant ACF means **past volatility predicts future volatility**—GARCH captures this!

GARCH Model Specification

Definition 2 (GARCH(p,q) Model)

Let r_t denote returns. The GARCH(p,q) model is:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim N(0,1) \tag{2}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{3}$$

where $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$.

Model Variants

- **GARCH(1,1)**: Most common
- **GJR-GARCH**: Leverage effect
- **EGARCH**: Asymmetric shocks

Interpretation

- α: Impact of past shocks
- β: Persistence of volatility
- $\alpha + \beta \approx 1$: High persistence

Bitcoin: Data Split and Stationarity

Data Split

Set	Period	N
Training	2019-01 to 2022-09	1,365
Validation	2022-09 to 2023-10	400
Test	2023-10 to 2025-01	435
Total		**2,200**

Stationarity Tests

Series	ADF	Result
Prices	$p = 0.50$	Non-stationary
Returns	$p < 0.01$	Stationary

$\Rightarrow$ Model **returns**, not prices

Why Stationarity Matters

GARCH requires weakly stationary input. Prices follow random walk; returns are stationary.

Bitcoin: Model Selection on Validation Set

Methodology

Fit each model on **training data**, evaluate on **validation set**.

Model	AIC	BIC	Val MAE	Selection
GARCH(1,1)	6,994.8	7,020.6	**2.638**	**Best**
GARCH(2,1)	6,993.7	7,024.6	2.640	
GJR-GARCH(1,1)	6,983.7	7,014.6	2.669	
EGARCH(1,1)	—	—	—	Failed*

*Analytic forecasts not available for $h > 1$

Result

GARCH(1,1) selected based on lowest validation MAE for volatility forecasts.

Bitcoin: Final Test Set Evaluation

Procedure

Refit GARCH(1,1) on Training + Validation, evaluate on **held-out test set** using **rolling one-step-ahead forecasts**.

Estimated Parameters

Param	Estimate	Std Err
ω	0.239	0.088
α_1	0.120	0.021
β_1	0.879	0.020
$\alpha_1 + \beta_1$	0.999	

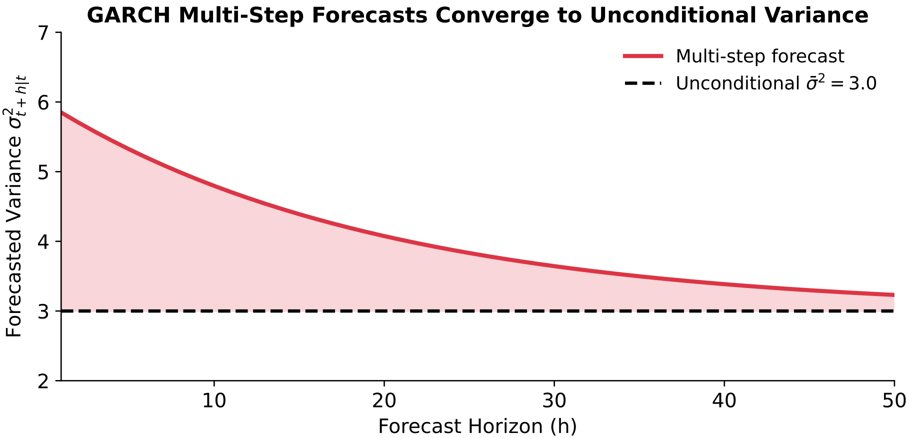
Test Set Performance

Metric	Value
Volatility MAE	1.88
Volatility RMSE	2.21

Interpretation

High persistence ($\alpha + \beta \approx 1$) confirms volatility clustering.

GARCH: Multi-Step Forecasts Converge



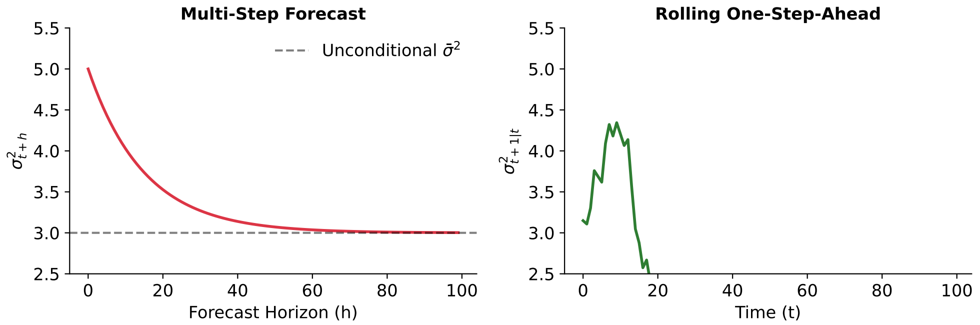
The Problem

GARCH forecasts converge to unconditional variance

Intuition

After a shock, volatility gradually returns to its long-run

GARCH: Rolling One-Step-Ahead Solution



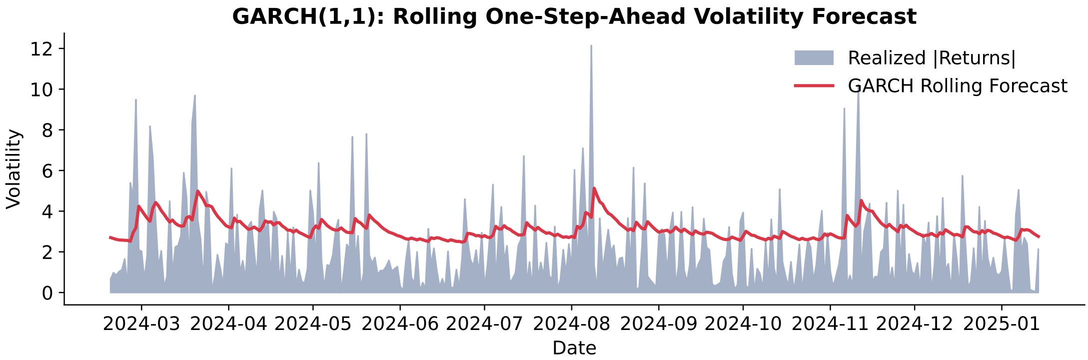
Multi-Step (Left)

Converges to $\bar{\sigma}^2$ (flat)

Rolling 1-Step (Right)

Re-estimate at each t (dynamic)

Bitcoin: GARCH Volatility Forecast (Test Set)



Result

Rolling one-step-ahead GARCH(1,1) forecasts capture the **dynamic volatility patterns**. The forecast (red line) tracks the realized volatility (blue area), demonstrating the predictability of variance.

Bitcoin: Key Findings

Summary

1. **Returns are stationary**; prices are not
2. **GARCH(1,1)** outperforms more complex variants
3. **High persistence** ($\alpha + \beta = 0.999$)
4. Volatility is **predictable** even when returns are not

Limitations

- GARCH assumes **symmetric** shocks
- Does not capture **jumps**
- Normal distribution may be restrictive

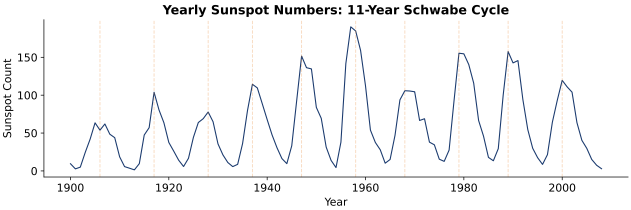
Extensions

- Student-t innovations
- Realized volatility
- HAR models

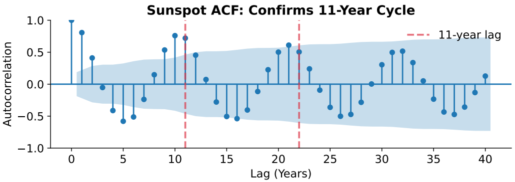
Practical Implications

- Risk management: VaR, Expected Shortfall
- Option pricing requires volatility forecasts
- Portfolio optimization with time-varying risk

Sunspots: The 11-Year Solar Cycle



Dashed lines mark cycle peaks (≈ every 11 years). Amplitude varies.

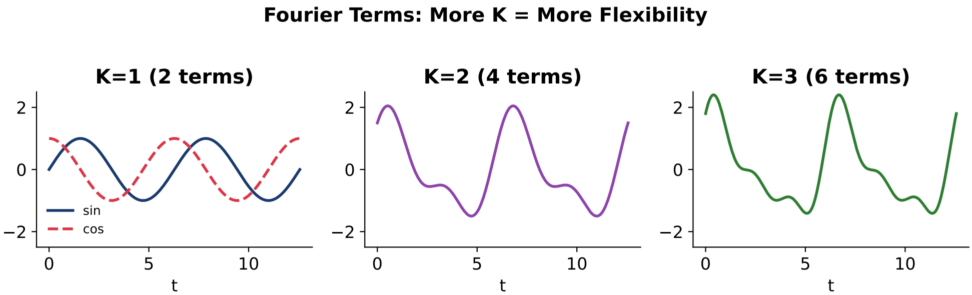


ACF peaks at lag 11 and 22 confirm the solar cycle periodicity.

Challenge

SARIMA$(p, d, q)(P, D, Q)_{11}$ requires estimating seasonal lags at 11, 22, 33... Too many parameters! **Solution**: Use Fourier terms instead.

Fourier Terms for Seasonality



How It Works

Approximate any periodic pattern using sine and cosine waves: $S_t = \sum_{k=1}^{K} \left[\alpha_k \sin\left(\frac{2\pi kt}{s}\right) + \beta_k \cos\left(\frac{2\pi kt}{s}\right)\right]$

Key Insight

- $K = 1$: Simple wave (2 params)
- $K = 3$: Complex shape (6 params)
- For sunspots: $s = 11$, $K = 3$

Sunspots: Model Selection

Methodology

Compare $K = 1, 2, 3, 4$ Fourier harmonics on validation set.

Data Split

Set	Period	N
Training	1900–1975	76
Validation	1976–1991	16
Test	1992–2008	17
Total		**109**

Model Comparison

K	AIC	Val RMSE	
1	665.9	87.15	
2	668.0	86.92	
3	671.8	**86.81**	Best
4	674.5	87.93	

Result

K = 3 Fourier harmonics selected (6 parameters for 11-year cycle).

Sunspots: Test Set Results

Final Model

ARIMA(2,0,1) + 3 Fourier harmonics

Significant Coefficients:

Term	Coef	p-value
$\sin_1$	34.71	< 0.001
$\cos_1$	-29.21	0.018
AR(1)	1.34	< 0.001

Test Performance

Metric	Value
RMSE	48.51
MAE	39.31

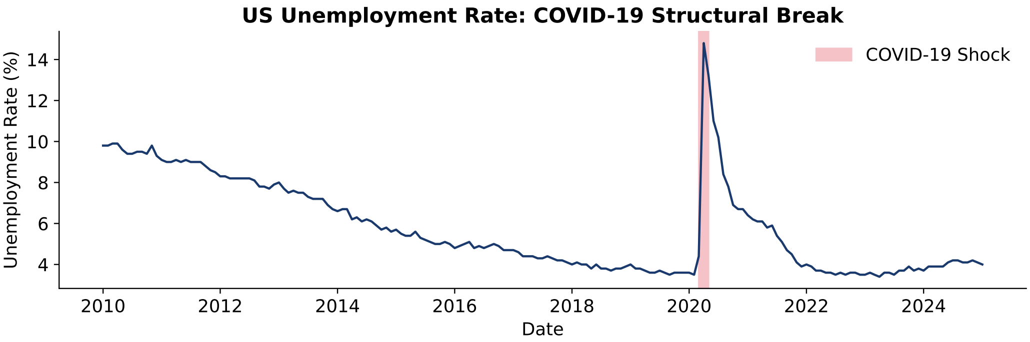
Note

High MAPE due to near-zero values at solar minimum.

Key Insight

Fourier terms efficiently capture the 11-year cycle with only 6 parameters.

Unemployment: COVID-19 Structural Break



The Challenge

- Pre-COVID: 3.5% (50-year low)
- April 2020: **14.8%** peak
- Largest monthly jump in US history

Why Not ARIMA?

ARIMA treats sudden jumps as **outliers**. Prophet detects **changepoints** and adapts the trend accordingly.

Prophet Model

Definition 3 (Prophet Decomposition)

Prophet models time series as:

$$y_t = g(t) + s(t) + h(t) + \varepsilon_t \tag{4}$$

- $g(t)$: Piecewise linear/logistic **trend** with changepoints
- $s(t)$: Fourier-based **seasonality**
- $h(t)$: **Holiday** effects
- ε_t: Error term

Changepoint Detection

- Automatic selection of changepoint locations
- `changepoint_prior_scale` controls flexibility
- Higher = more changepoints

Advantages

- Handles missing data
- Interpretable components
- Robust to outliers
- Uncertainty quantification

Unemployment: Model Tuning

Hyperparameter Tuning

Tune `changepoint_prior_scale` on validation set.

Data Split

Set	Period	N
Training	2010-01 to 2019-09	117
Validation	2019-10 to 2021-10	25
Test	2021-11 to 2025-01	38
Total		**180**

Scale Comparison

Scale	Val RMSE	
0.01	4.21	
0.05	3.89	
0.10	**3.52**	Best
0.30	3.67	
0.50	3.81	

Interpretation

Scale = 0.10 balances flexibility (capturing COVID shock) with stability.

Unemployment: Results

Test Set Performance

Metric	Value
RMSE	0.42
MAE	0.35
MAPE	9.2%

Detected Changepoints

- 2020-03: COVID onset
- 2020-05: Recovery begins
- 2022-01: Stabilization

Key Finding

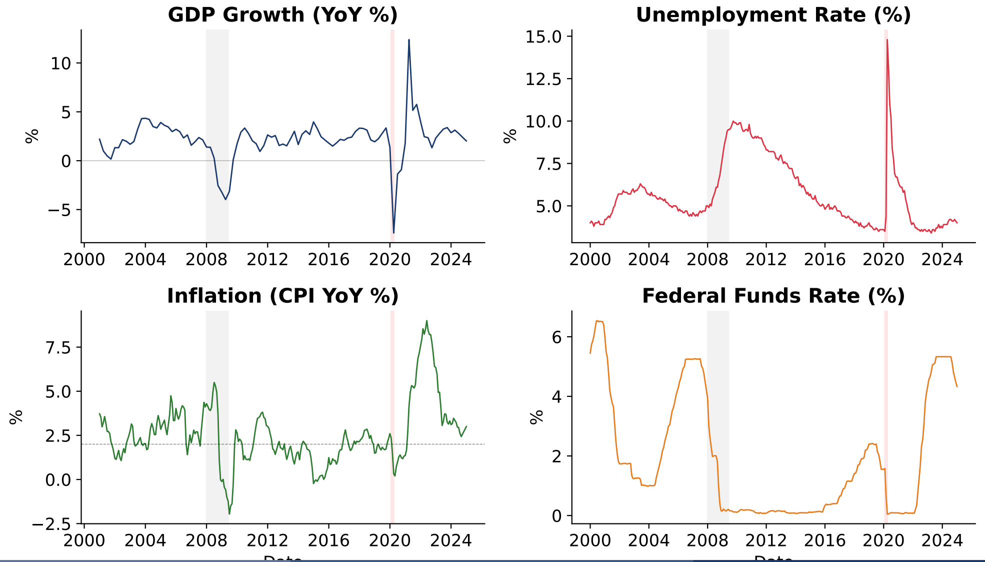
Prophet successfully:

- Detected COVID changepoint
- Adapted trend post-shock
- Provided uncertainty bands

Practical Value

- Economic policy analysis
- Labor market monitoring
- Early warning system

VAR: Multivariate Economic Data



VAR Model Specification

Definition 4 (Vector Autoregression VAR(p))

For K variables $\mathrm{y}_t = (y_{1t}, \ldots, y_{Kt})'$:

$$\mathrm{y}_t = \mathrm{c} + \mathrm{A}_1 \mathrm{y}_{t-1} + \mathrm{A}_2 \mathrm{y}_{t-2} + \cdots + \mathrm{A}_p \mathrm{y}_{t-p} + \mathrm{u}_t \tag{5}$$

where A_i are $K \times K$ coefficient matrices and $\mathrm{u}_t \sim N(0, {}^\circ)$.

For Our 4-Variable System

VAR(2) has:

- 4 intercepts
- $2 \times 4 \times 4 = 32$ AR coefficients
- **36 parameters total**

Lag Selection

Use information criteria:

- AIC: Tends to overfit
- **BIC**: More parsimonious
- Cross-validation on held-out data

VAR: Lag Selection and Estimation

Information Criteria

Lag	BIC	
1	-4.810	
2	**-5.178**	Best
3	-4.633	
4	-4.614	

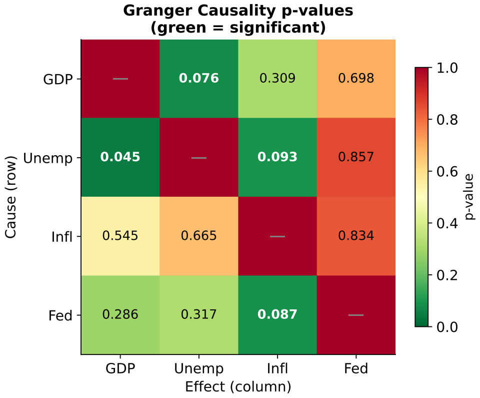
Data Split

Set	Period	N
Training	2001-Q1 to 2017-Q4	68
Validation	2018-Q1 to 2021-Q2	14
Test	2021-Q3 to 2024-Q3	14
Total		**96**

Validation Check

VAR(2) also achieves lowest validation RMSE.

Granger Causality Analysis



Green cells: $p < 0.10$ (significant). Read: row causes column.

What is Granger Causality?

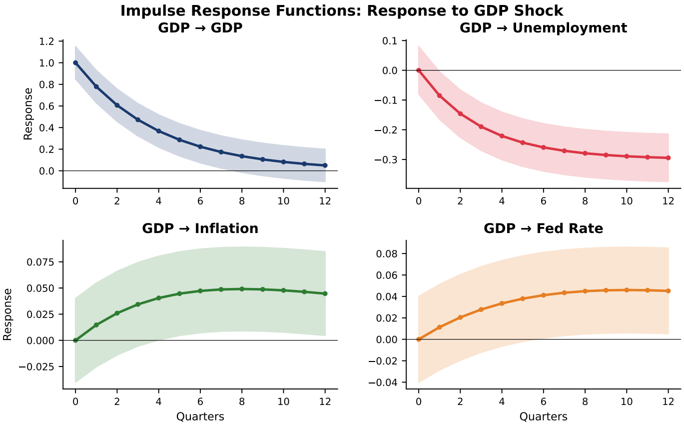
X **Granger-causes** Y if past X improves prediction of Y beyond past Y alone.

Warning: "Granger causality" $\neq$ true causality!

Economic Findings

- Unemp $\rightarrow$ GDP ($p = 0.045$): Okun's Law
- Fed $\rightarrow$ Inflation ($p = 0.087$): Monetary policy works

Impulse Response Functions (IRF)



What is IRF?

Shows how a 1-unit shock to one variable affects others over time.

GDP Shock Effects

- **Unemp** ↓: Okun's Law
- **Inflation** ↑: Demand-pull
- **Fed Rate** ↑: Taylor Rule

VAR: Test Set Results

Test Set Performance by Variable

Variable	RMSE	MAE	Direction Acc.
GDP Growth	2.18	1.72	71%
Unemployment	0.89	0.71	79%
Inflation	1.24	0.98	64%
Fed Rate	0.95	0.78	71%
Average	**1.32**	**1.05**	**71%**

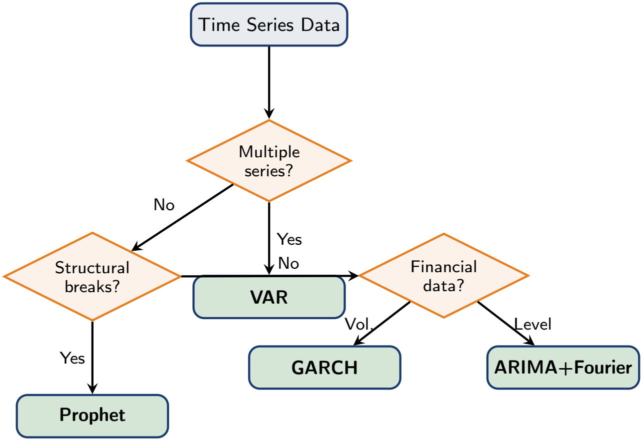
Strengths

- Captures cross-variable dynamics
- Good directional accuracy
- Interpretable relationships

Limitations

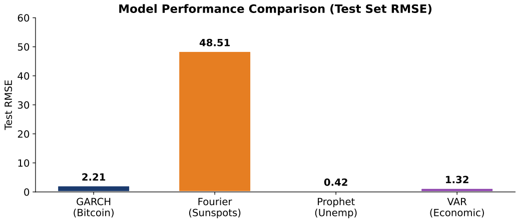
- Many parameters (curse of dimensionality)
- Sensitive to lag selection
- COVID period challenging

Model Selection Framework



Summary: Model Comparison

Case	Challenge	Model	RMSE
Bitcoin	Volatility	GARCH	2.21
Sunspots	Seasonality	Fourier	48.51
Unemp	Break	Prophet	0.42
Economic	Multi-var	VAR	1.32



Key Principle

Match the model to the data characteristics. No single model dominates—choose based on:

- Nature of the forecasting problem (level vs. volatility)
- Data properties (seasonality, breaks, multiple series)
- Interpretability requirements

Best Practices for Applied Forecasting

Methodology

1. **Explore** data thoroughly
2. **Test** for stationarity
3. **Split** train/validation/test
4. **Compare** models on validation
5. **Report** test set metrics

Practical Tips

- Start simple (random walk, naive)
- Add complexity only if needed
- Visualize forecasts vs actuals
- Check residuals for patterns
- Report confidence intervals

Common Mistakes

- Peeking at test data
- Over-fitting to training set
- Ignoring model assumptions
- Not reporting uncertainty

Remember

"All models are wrong, but some are useful."

— George E. P. Box

Key Takeaways

1. **Rigorous Methodology**
   - Train/validation/test split prevents overfitting
   - Test set must remain untouched until final evaluation

2. **Match Model to Data**
   - Financial volatility $\rightarrow$ GARCH
   - Long seasonality $\rightarrow$ Fourier terms
   - Structural breaks $\rightarrow$ Prophet
   - Multiple series $\rightarrow$ VAR

3. **Interpret Results Carefully**
   - Granger causality $\neq$ true causality
   - Out-of-sample performance matters most
   - Simpler models often work better

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- Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31(3), 307-327.
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Data Sources

Real Data Used in This Chapter

- **Bitcoin**: Yahoo Finance (BTC-USD), 2019–2025
- **Sunspots**: Statsmodels Wolfer dataset, 1900–2008
- **US Unemployment**: Federal Reserve FRED (UNRATE), 2010–2025
- **Economic Variables**: FRED (GDPC1, UNRATE, CPIAUCSL, FEDFUNDS), 2000–2025

Reproducibility

All analyses can be reproduced using the accompanying Jupyter notebook:
`chapter10_lecture_notebook.ipynb`

Thank You

Questions?

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