



Time Series Analysis and Forecasting

## Chapter 3: ARIMA Models

Non-Stationary Time Series

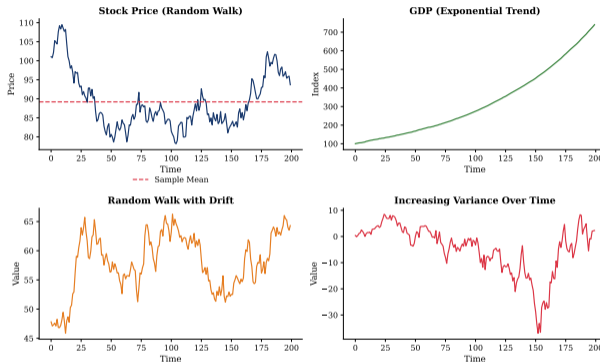


# Outline

- 1 Non-Stationarity in Time Series
- 2 Differencing and the Difference Operator
- 3 ARIMA(p,d,q) Models
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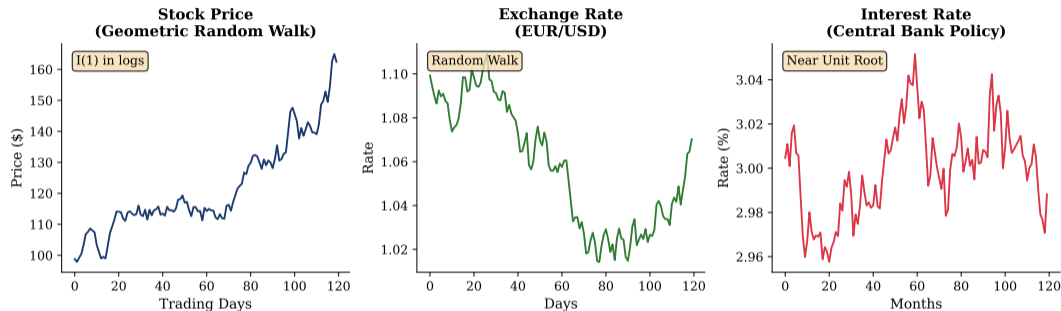
# Motivating Example: Non-Stationary Data Is Everywhere

Examples of Non-Stationary Time Series



- Stock prices, GDP, exchange rates all exhibit **trends** or **wandering behavior**
- The sample mean (red line) is meaningless for a random walk
- Standard ARMA models **cannot** handle these series directly

## Real-World Non-Stationary Series: Why We Need ARIMA



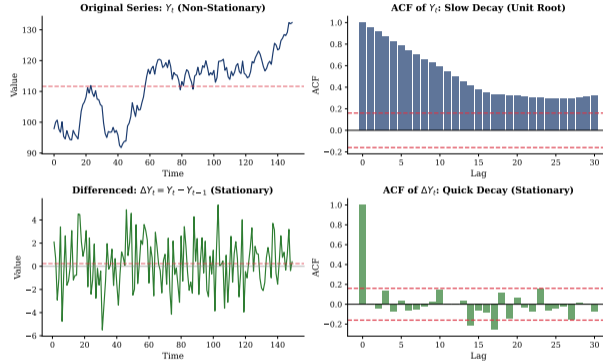
## The Challenge

Financial and economic data are typically **integrated** (I(1) or near unit root):

- Stock prices: random walk in logs
- Exchange rates: random walk
- Interest rates: highly persistent (near unit root)

# The Solution: Differencing

## The Magic of Differencing: Converting Non-Stationary to Stationary



## Key Insight

**Differencing** transforms a non-stationary series into a stationary one:  $\Delta Y_t = Y_t - Y_{t-1}$ . The ACF changes from slow decay to quick decay!

# What We'll Learn Today

## Core Concepts

- ① **Non-Stationarity:** Why it matters and how to detect it
- ② **Unit Root Tests:** ADF, PP, KPSS tests
- ③ **Differencing:** The key transformation
- ④ **ARIMA Models:** Combining differencing with ARMA
- ⑤ **Box-Jenkins Methodology:** Identify → Estimate → Diagnose

## By the End of This Lecture

You will be able to model and forecast non-stationary time series like stock prices, GDP, and exchange rates using ARIMA models.

# Why Non-Stationarity Matters

## The Problem

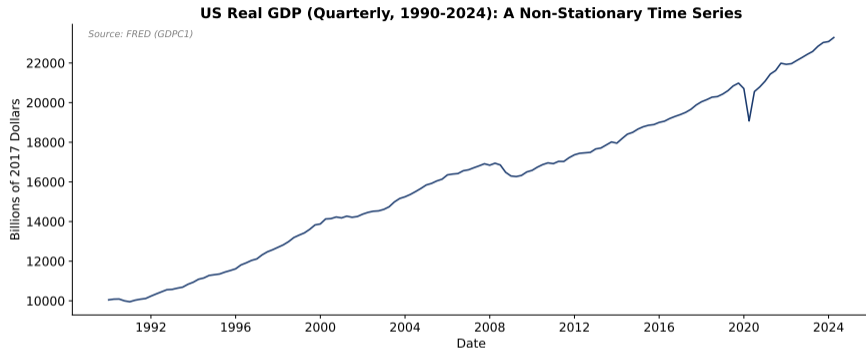
Many economic and financial time series are **non-stationary**:

- GDP, stock prices, exchange rates, inflation indices
- They exhibit trends, changing means, or growing variance

## Consequences of Non-Stationarity

- Standard ARMA models assume stationarity
- OLS regression with non-stationary data leads to **spurious regression**
- Sample moments (mean, variance, ACF) are not consistent estimators
- Statistical inference becomes invalid

## Example: US Real GDP



- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data

# Types of Non-Stationarity

## Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a deterministic function of time
- Can be removed by **detrending**
- Shocks have temporary effects

## Stochastic Trend (Unit Root)

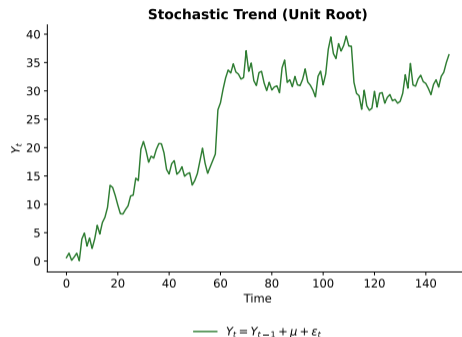
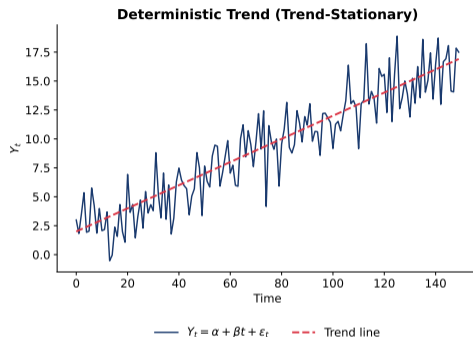
$$Y_t = Y_{t-1} + \varepsilon_t$$

- Random walk process
- Must be removed by **differencing**
- Shocks have permanent effects

## Key Distinction

Correct identification is crucial: detrending a unit root process or differencing a trend-stationary process both lead to misspecification!

# Visualizing the Difference



- **Left:** Deterministic trend – deviations from trend are temporary
- **Right:** Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!

# The Random Walk Process

## Definition 1 (Random Walk)

A **random walk** is defined as:

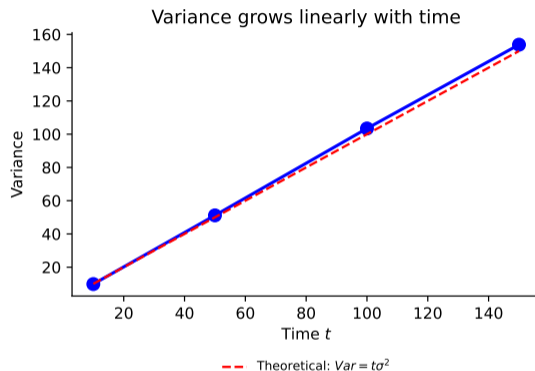
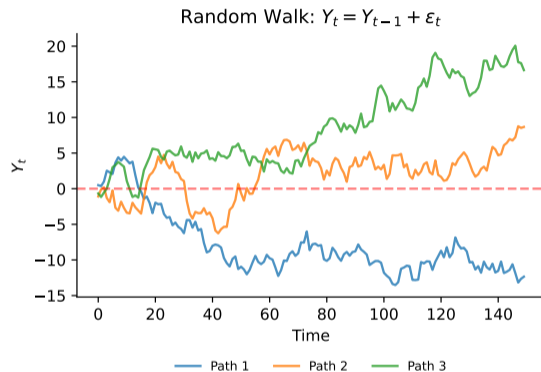
$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

With initial condition  $Y_0 = 0$ , we have:  $Y_t = \sum_{i=1}^t \varepsilon_i$

## Properties of Random Walk

- $\mathbb{E}[Y_t] = 0$  (constant mean)
- $\text{Var}(Y_t) = t\sigma^2$  (variance grows with time!)
- $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$  for  $k \leq t$
- ACF:  $\rho_k = \sqrt{\frac{t-k}{t}} \rightarrow 1$  as  $t \rightarrow \infty$

## Random Walk: Visual Illustration



Left: multiple random walk paths wander unpredictably. Right: variance grows linearly with time.

### Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

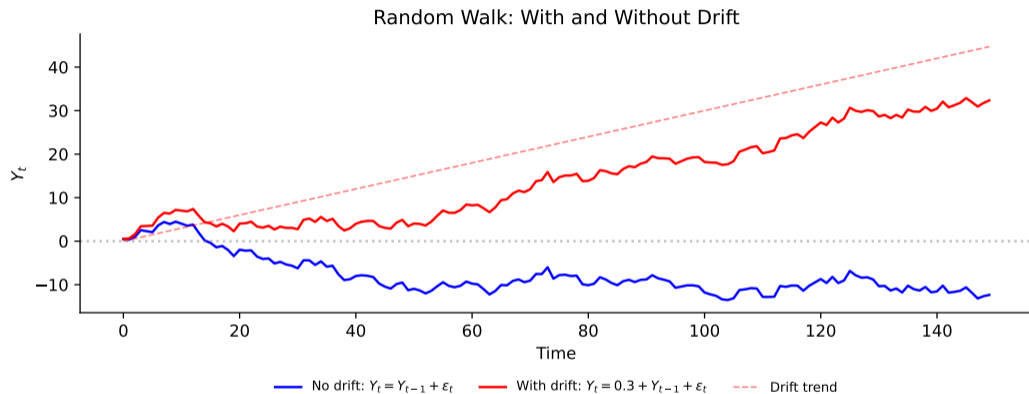
$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

Equivalently:  $Y_t = Y_0 + \mu t + \sum_{i=1}^t \varepsilon_i$

### Properties

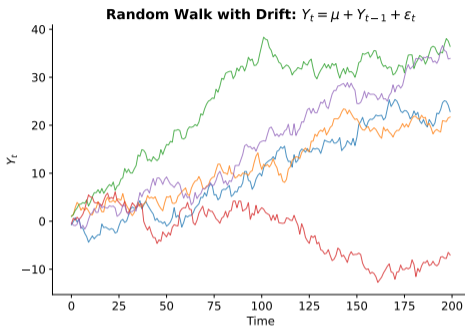
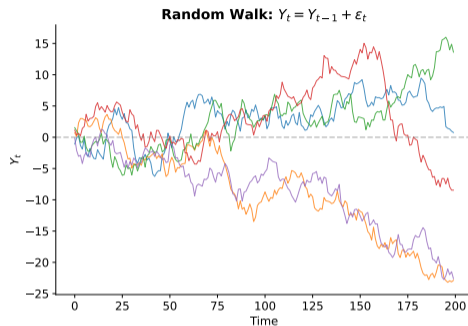
- $\mathbb{E}[Y_t] = Y_0 + \mu t$  (mean grows linearly)
- $\text{Var}(Y_t) = t\sigma^2$  (variance still grows)
- The drift  $\mu$  creates an upward or downward trend
- Still non-stationary despite having a “trend”

## Random Walk with Drift: Visual Illustration



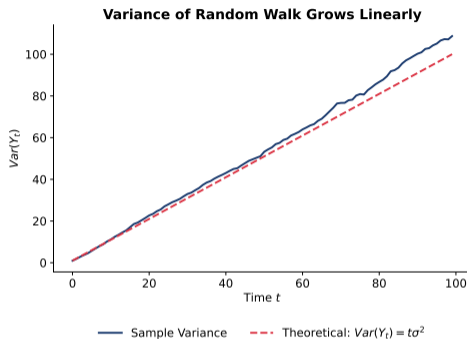
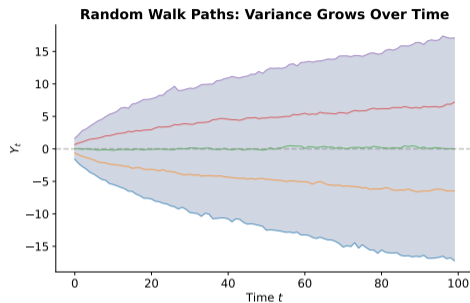
Random walk without drift (blue) wanders around zero. With drift (red), there's a systematic trend.

# Simulating Random Walks



- **Left:** Pure random walks – no drift, wander unpredictably
- **Right:** Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time

# Variance Growth: Why Random Walks Are Non-Stationary



- **Left:** Fan of paths shows uncertainty growing over time
- **Right:** Variance grows linearly:  $\text{Var}(Y_t) = t\sigma^2$
- This violates stationarity (variance should be constant)

## Definition 3 (Integrated Process of Order $d$ )

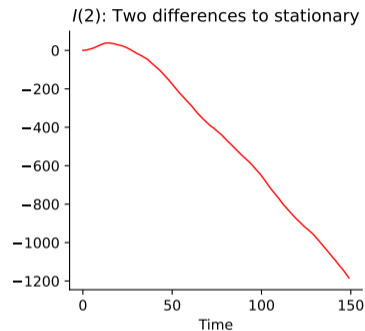
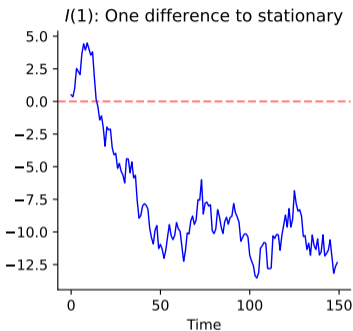
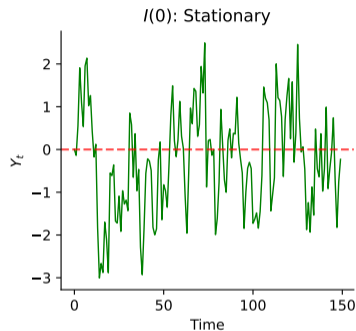
A time series  $\{Y_t\}$  is **integrated of order  $d$** , written  $Y_t \sim I(d)$ , if:

- $Y_t$  is non-stationary
- $(1 - L)^d Y_t = \Delta^d Y_t$  is stationary
- $(1 - L)^{d-1} Y_t$  is still non-stationary

## Common Cases

- $I(0)$ : Stationary process (e.g., ARMA)
- $I(1)$ : First difference is stationary (most common for economic data)
- $I(2)$ : Second difference is stationary (less common)

## Integrated Process: Visual Illustration



$I(0)$ : stationary.  $I(1)$ : one difference needed.  $I(2)$ : two differences needed.

# The Difference Operator

## Definition 4 (First Difference)

The **first difference operator**  $\Delta$  is defined as:  $\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$ , where  $L$  is the lag operator ( $LY_t = Y_{t-1}$ ).

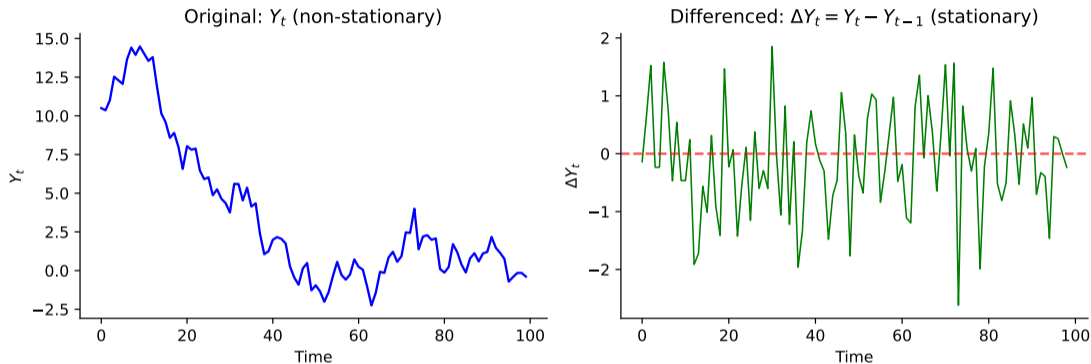
## Higher-Order Differences

- Second difference:  $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- $d$ -th difference:  $\Delta^d Y_t = (1 - L)^d Y_t$

## Key Result

If  $Y_t \sim I(d)$ , then  $\Delta^d Y_t \sim I(0)$  (stationary).

## First Difference: Visual Illustration



Left: non-stationary series. Right: after first difference, the series becomes stationary.

## Example: Differencing a Random Walk

### Random Walk to White Noise

Let  $Y_t = Y_{t-1} + \varepsilon_t$  (random walk). Taking the first difference:

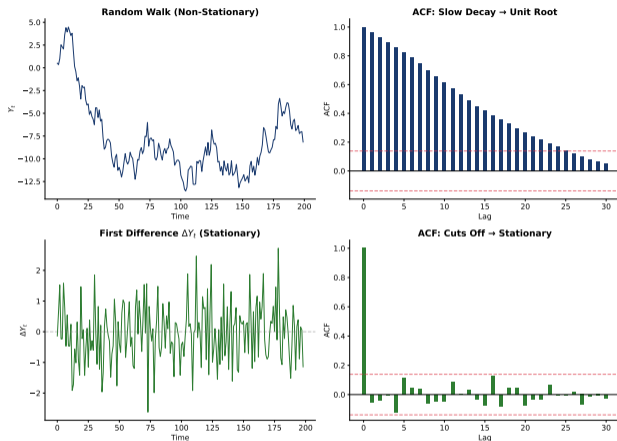
$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

### Interpretation

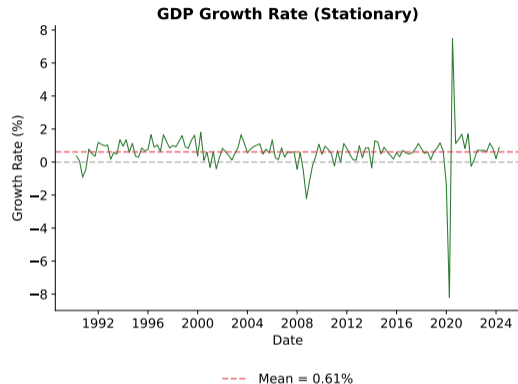
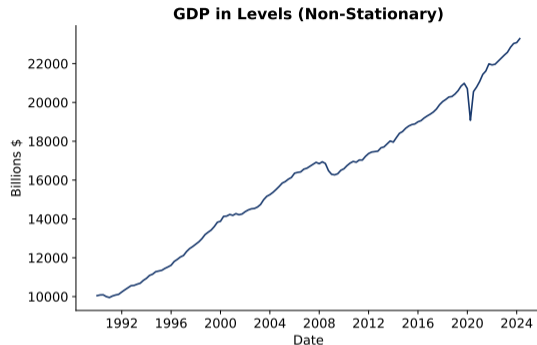
- A random walk is  $I(1)$
- One difference transforms it to  $I(0)$
- The “changes” in a random walk are stationary

# ACF Diagnostic: Detecting Non-Stationarity



- **Top:** Random walk ACF decays very slowly  $\Rightarrow$  unit root
- **Bottom:** After differencing, ACF cuts off  $\Rightarrow$  stationary

# Differencing in Practice: GDP Example



- **Left:** GDP in levels – clear upward trend (non-stationary)
- **Right:** GDP growth rate (log difference) – fluctuates around mean (stationary)
- Differencing removes the trend and achieves stationarity

# Overdifferencing

## Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
- Inflates variance
- Loses information

## Example

If  $Y_t \sim I(1)$ , then  $\Delta Y_t \sim I(0)$ . But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with  $\theta = 1$  (non-invertible boundary)!

## Definition 5 (ARIMA(p,d,q))

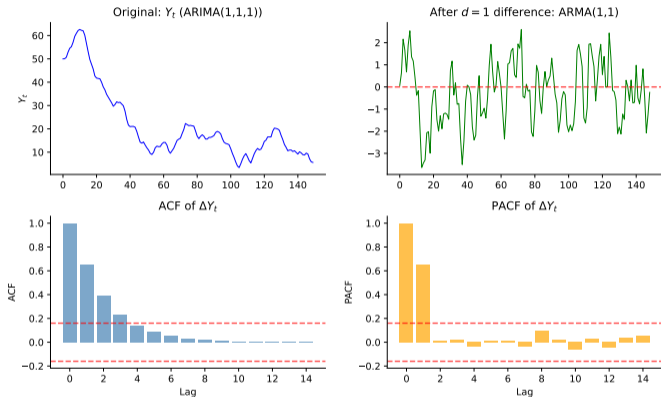
A time series  $\{Y_t\}$  follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$$

where:

- $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$  (AR polynomial)
- $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$  (MA polynomial)
- $d$  is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

# ARIMA: Visual Illustration



Top: original ARIMA series. Bottom left/right: after differencing, ACF/PACF help identify AR and MA orders.

# ARIMA Components

$AR(p)$

Autoregressive  
Memory

$I(d)$

Integration  
Differencing

$MA(q)$

Moving Average  
Shocks

## Special Cases

- $ARIMA(p,0,q) = ARMA(p,q)$  – stationary
- $ARIMA(0,1,0) =$  Random walk
- $ARIMA(0,1,1) = IMA(1,1)$  – exponential smoothing
- $ARIMA(1,1,0) = ARI(1,1)$  – differenced  $AR(1)$

## ARIMA(1,1,0) Example

### ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently:  $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$

### Interpretation

- The **changes** in  $Y_t$  follow an AR(1) process
- If  $|\phi_1| < 1$ , the changes are stationary
- $Y_t$  itself has a stochastic trend
- Common model for many economic time series

## ARIMA(0,1,1) Example

### IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently:  $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

### Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha) \hat{Y}_t$$

where  $\alpha = 1 + \theta_1$  (for  $-1 < \theta_1 < 0$ ).

# The Role of the Constant in ARIMA

## Constant Term in ARIMA(p,d,q)

When  $d > 0$ , the constant  $c$  has a different interpretation:  $\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$

## Important Implications

- For  $d = 1$ :  $c$  represents the **drift** (average change):  $\mathbb{E}[\Delta Y_t] = \frac{c}{1-\phi_1-\dots-\phi_p}$
- For  $d = 2$ :  $c$  affects the **curvature** of the trend
- Often  $c = 0$  is assumed when  $d \geq 1$

# Testing for Unit Roots

## Why Test?

Before fitting an ARIMA model, we need to determine:

- 1 Is the series stationary? (Is  $d = 0$ ?)
- 2 If not, how many differences are needed? (What is  $d$ ?)

## Common Unit Root Tests

- Dickey-Fuller (DF) and Augmented Dickey-Fuller (ADF)
- Phillips-Perron (PP)
- KPSS (stationarity test – reversed null hypothesis)

# The Dickey-Fuller Test

## Setup

Consider the AR(1) model:  $Y_t = \phi Y_{t-1} + \varepsilon_t$ . Subtract  $Y_{t-1}$ :  $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$ , where  $\gamma = \phi - 1$ .

## Hypotheses

- $H_0$ :  $\gamma = 0$  (unit root,  $\phi = 1$ , non-stationary)
- $H_1$ :  $\gamma < 0$  (stationary,  $|\phi| < 1$ )

## Key Issue

Under  $H_0$ , the  $t$ -statistic does **not** follow a standard  $t$ -distribution! Must use Dickey-Fuller critical values.

## Three Specifications

- ① **No constant, no trend:**  $\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$
- ② **With constant (drift):**  $\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$
- ③ **With constant and trend:**  $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$

## Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference

# Augmented Dickey-Fuller (ADF) Test

## The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

## Definition 6 (ADF Test)

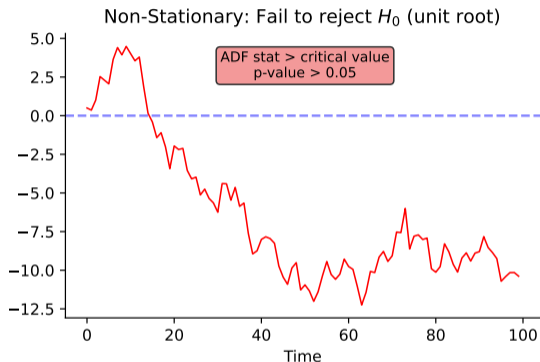
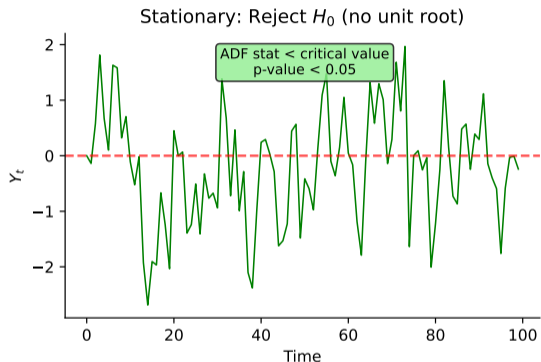
Add lagged differences:  $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^k \delta_j \Delta Y_{t-j} + \varepsilon_t$

Test  $H_0 : \gamma = 0$  using ADF critical values.

## Choosing Lag Length $k$

- Use information criteria (AIC, BIC)
- Start with  $k_{max}$ , reduce until last lag significant

## ADF Test: Visual Illustration



Left: stationary series – ADF rejects unit root. Right: non-stationary – ADF fails to reject.

## ADF Test Critical Values

Model	1%	5%	10%
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

### Decision Rule

- Test statistic  $<$  critical value  $\Rightarrow$  Reject  $H_0$  (stationary)
- Test statistic  $\geq$  critical value  $\Rightarrow$  Fail to reject (unit root)

# The Phillips-Perron (PP) Test

## Motivation

Like ADF, tests  $H_0$ : Unit root vs  $H_1$ : Stationary, but uses a **non-parametric correction** for serial correlation instead of adding lagged differences.

## Test Statistic

The PP test modifies the DF  $t$ -statistic:

$$Z_t = t_{\hat{\gamma}} \cdot \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - \hat{\sigma}^2)(se(\hat{\gamma}))}{2\hat{\lambda}^2 \cdot s}$$

where  $\hat{\lambda}^2$  is a consistent estimate of the long-run variance using Newey-West.

## Advantages over ADF

- Robust to heteroskedasticity and serial correlation
- No need to select lag length (uses bandwidth instead)

# The KPSS Test

## Reversed Hypotheses

Unlike ADF:  $H_0$ : Stationary vs  $H_1$ : Unit root

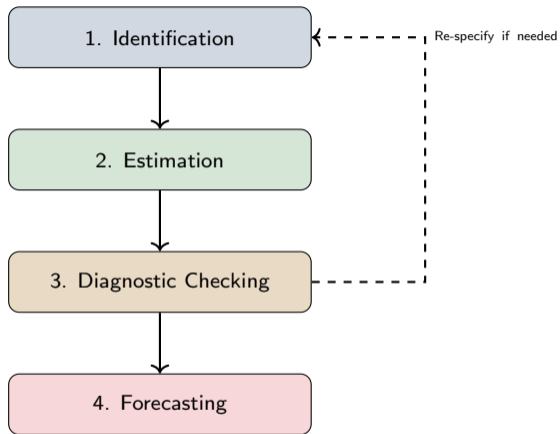
## KPSS Procedure

Decompose:  $Y_t = \xi t + r_t + \varepsilon_t$  where  $r_t = r_{t-1} + u_t$ . Test whether  $\text{Var}(u_t) = 0$ .

## Complementary Use with ADF

- ADF rejects, KPSS doesn't  $\Rightarrow$  Stationary
- ADF doesn't reject, KPSS rejects  $\Rightarrow$  Unit root
- Both reject or neither  $\Rightarrow$  Inconclusive

# The Box-Jenkins Methodology



## Step 1: Determining $d$

### Procedure

- 1 Plot the time series – look for trends, changing variance
- 2 Examine ACF – slow decay suggests non-stationarity
- 3 Apply unit root tests (ADF, KPSS)
- 4 If non-stationary, difference and repeat

### Practical Guidelines

- Most economic series:  $d = 1$  is sufficient
- Rarely need  $d > 2$
- If ACF of  $\Delta Y_t$  still decays slowly, try  $d = 2$
- Watch for overdifferencing (ACF with  $\rho_1 \approx -0.5$ )

## Step 2: Determining $p$ and $q$

### After Differencing

Once  $W_t = \Delta^d Y_t$  is stationary, use ACF/PACF to identify ARMA( $p, q$ ):

Model	ACF	PACF
AR( $p$ )	Decays exponentially	Cuts off after lag $p$
MA( $q$ )	Cuts off after lag $q$	Decays exponentially
ARMA( $p, q$ )	Decays	Decays

### Information Criteria

When patterns are unclear, compare models using:

- $AIC = -2 \ln(L) + 2k$ ;     $BIC = -2 \ln(L) + k \ln(n)$

Lower is better. BIC penalizes complexity more.

## Automated Model Selection

Modern software can automatically select  $(p, d, q)$ :

- Python: `pmdarima.auto.arima()`
- R: `forecast::auto.arima()`

## How Auto-ARIMA Works

- 1 Use unit root tests to determine  $d$
- 2 Fit models for various  $(p, q)$  combinations
- 3 Select model with lowest AIC/BIC
- 4 Optionally use stepwise search for efficiency

## Caution

Automated selection is helpful but not infallible. Always check diagnostics!

## Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- Assumes  $\varepsilon_t \sim N(0, \sigma^2)$
- Maximizes the likelihood function
- Provides consistent, efficient estimators
- Yields standard errors for inference

## Conditional vs Exact MLE

- **Conditional MLE:** Conditions on initial values
- **Exact MLE:** Treats initial values as unknown
- Difference diminishes as sample size grows

### Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- **AR stationarity:** Roots of  $\phi(z) = 0$  outside unit circle
- **MA invertibility:** Roots of  $\theta(z) = 0$  outside unit circle

### Checking in Practice

Most software reports:

- Estimated coefficients with standard errors
- Roots of AR and MA polynomials
- Warning if near-unit-root detected

## What to Check

If the model is correct, residuals  $\hat{\varepsilon}_t$  should be white noise:

- 1 Zero mean
- 2 Constant variance
- 3 No autocorrelation
- 4 (Optional) Normality

## Diagnostic Tools

- **Residual ACF/PACF:** Should show no significant spikes
- **Ljung-Box test:** Tests for autocorrelation at multiple lags
- **Q-Q plot:** Checks normality assumption
- **Residual vs fitted:** Checks for heteroskedasticity

# The Ljung-Box Test

## Definition 7 (Ljung-Box Q Statistic)

$Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}$ . Under  $H_0$  (no autocorrelation):  $Q(m) \sim \chi^2(m - p - q)$

## Usage

- Choose  $m \approx \ln(n)$  or  $m = 10$  for quarterly,  $m = 20$  for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if  $Q(m)$  exceeds critical value

## If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

### Minimum MSE Forecast

The optimal  $h$ -step ahead forecast is the conditional expectation:  $\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \dots]$

### ARIMA(1,1,1) Forecasting

Model:  $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

One-step forecast:  $\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1 \hat{\varepsilon}_T$

For  $h > 1$ : replace unknown  $\varepsilon_{T+j}$  with 0, unknown  $Y_{T+j}$  with  $\hat{Y}_{T+j|T}$

# Forecast Intervals

## Forecast Uncertainty

The  $h$ -step forecast error variance:  $\text{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$ , where  $\psi_j$  are  $\text{MA}(\infty)$  coefficients.

## Confidence Intervals

Under normality,  $(1 - \alpha)\%$  interval:  $\hat{Y}_{T+h|T} \pm z_{\alpha/2} \sqrt{\text{Var}(e_{T+h})}$

## Key Property for I(1) Series

For integrated processes, forecast variance grows without bound as  $h \rightarrow \infty$ . Intervals widen over time!

# Long-Run Forecasts for ARIMA

## Behavior as $h \rightarrow \infty$

For ARIMA(p,1,q) with drift  $c$ :

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with  $\sqrt{h}$

For ARIMA(p,1,q) without drift:

- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

## Practical Implication

ARIMA forecasts are most reliable for short horizons. Long-term forecasts have very wide uncertainty bands.

## What is Rolling Forecasting?

A technique to evaluate forecast accuracy out-of-sample:

- 1 Fix a **training window** of size  $w$
- 2 Estimate model on observations  $t = 1, \dots, w$
- 3 Forecast  $h$  steps ahead:  $\hat{Y}_{w+h|w}$
- 4 **Roll** the window forward by one period
- 5 Repeat until end of sample

## Why Rolling Forecasts?

- Mimics real-time forecasting scenario
- Provides multiple forecast errors for evaluation
- Avoids overfitting to full sample

## Rolling Forecast: Step-by-Step Example

Setup: ARIMA(1,1,0) with  $\phi_1 = 0.6$

Model:  $\Delta Y_t = \phi_1 \Delta Y_{t-1} + \varepsilon_t$  where  $\Delta Y_t = Y_t - Y_{t-1}$

Given Data at Time  $T$

$Y_{T-2} = 100, \quad Y_{T-1} = 103, \quad Y_T = 108 \quad \Rightarrow \quad \Delta Y_{T-1} = 3, \quad \Delta Y_T = 5$

1-Step Ahead Point Forecast

$$\begin{aligned}\hat{\Delta Y}_{T+1|T} &= \phi_1 \cdot \Delta Y_T = 0.6 \times 5 = 3 \\ \hat{Y}_{T+1|T} &= Y_T + \hat{\Delta Y}_{T+1|T} = 108 + 3 = \boxed{111}\end{aligned}$$

# Multi-Step Point Forecasts

## 2-Step Ahead Forecast

$$\begin{aligned}\Delta \hat{Y}_{T+2|T} &= \phi_1 \cdot \Delta \hat{Y}_{T+1|T} = 0.6 \times 3 = 1.8 \\ \hat{Y}_{T+2|T} &= \hat{Y}_{T+1|T} + \Delta \hat{Y}_{T+2|T} = 111 + 1.8 = \boxed{112.8}\end{aligned}$$

## General Formula for $h$ -Step Forecast (ARIMA(1,1,0))

$$\begin{aligned}\Delta \hat{Y}_{T+h|T} &= \phi_1^h \cdot \Delta Y_T \\ \hat{Y}_{T+h|T} &= Y_T + \Delta Y_T \cdot \frac{\phi_1(1 - \phi_1^h)}{1 - \phi_1}\end{aligned}$$

## Numerical: 3-Step Forecast

$$\hat{Y}_{T+3|T} = 108 + 5 \times \frac{0.6(1-0.6^3)}{1-0.6} = 108 + 5 \times 1.092 = \boxed{113.46}$$

### Forecast Error Variance

For ARIMA(1,1,0), the  $h$ -step forecast error variance:

$$\text{Var}(e_{T+h|T}) = \sigma^2 \left( 1 + \sum_{j=1}^{h-1} \psi_j^2 \right)$$

where  $\psi_j = \phi_1^{j-1}(1 + \phi_1 + \dots + \phi_1^{j-1}) = \phi_1^{j-1} \cdot \frac{1-\phi_1^j}{1-\phi_1}$

### $(1 - \alpha)\%$ Confidence Interval

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(e_{T+h|T})}$$

For 95% CI:  $z_{0.025} = 1.96$

## Confidence Interval: Numerical Example

Given:  $\sigma^2 = 4$ ,  $\phi_1 = 0.6$ ,  $\hat{Y}_{T+1|T} = 111$

### 1-Step Ahead CI

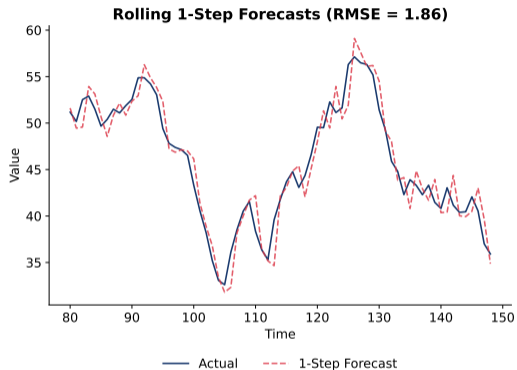
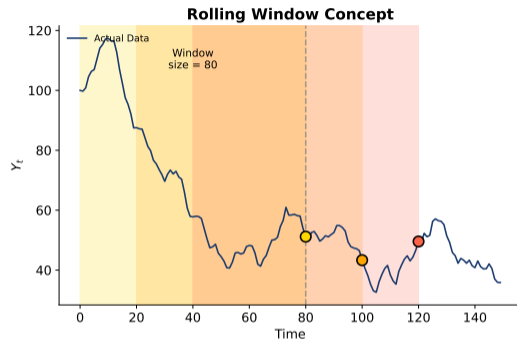
$$\begin{aligned}\text{Var}(e_{T+1|T}) &= \sigma^2 = 4 \\ 95\% \text{ CI} &= 111 \pm 1.96 \times \sqrt{4} = 111 \pm 3.92 = [107.08, 114.92]\end{aligned}$$

### 2-Step Ahead CI (for $\hat{Y}_{T+2|T} = 112.8$ )

$$\begin{aligned}\psi_1 &= 1 + \phi_1 = 1.6, \quad \text{Var}(e_{T+2|T}) = 4(1 + 1.6^2) = 14.24 \\ 95\% \text{ CI} &= 112.8 \pm 1.96 \times \sqrt{14.24} = 112.8 \pm 7.40 = [105.40, 120.20]\end{aligned}$$

**Note:** CI widens as horizon increases!

# Rolling Window Illustration



- Each window produces a 1-step ahead forecast
- Compare forecasts to actuals to compute RMSE, MAE
- Rolling window keeps model estimation up-to-date

## Implementation

```
from statsmodels.tsa.arima.model import ARIMA

window_size = 100
forecasts, actuals = [], []

for t in range(window_size, len(y) - 1):
    train = y[:t]                # Rolling window
    model = ARIMA(train, order=(1,1,0)).fit()
    forecast = model.forecast(steps=1)[0]
    forecasts.append(forecast)
    actuals.append(y[t])

rmse = np.sqrt(np.mean((np.array(forecasts) - np.array(actuals))**2))
```

# Case Study: Complete ARIMA Analysis

## Objective

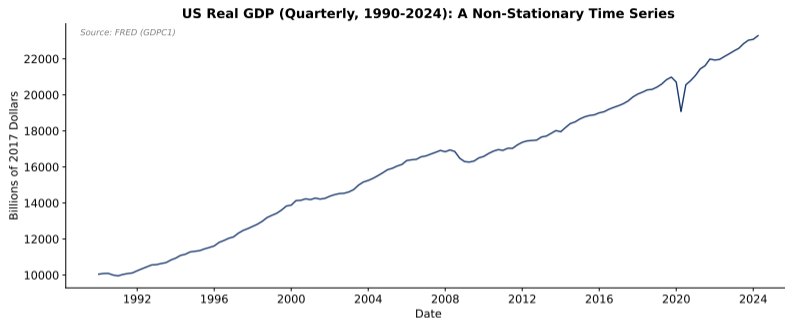
Forecast US Real GDP using the Box-Jenkins methodology

- ➊ **Step 1:** Visualize data and check for stationarity
- ➋ **Step 2:** Apply unit root tests (ADF, KPSS)
- ➌ **Step 3:** Difference if needed, identify  $p$  and  $q$
- ➍ **Step 4:** Estimate ARIMA model
- ➎ **Step 5:** Diagnostic checking
- ➏ **Step 6:** Generate forecasts with confidence intervals
- ➐ **Step 7:** Evaluate forecast accuracy

## Data

US Real GDP (FRED: GDPC1), Quarterly, 1990Q1–2024Q2,  $n = 138$  observations

## Step 1: Initial Data Analysis



### Observations

- Clear upward trend  $\Rightarrow$  non-constant mean
- Variance appears relatively stable (after log transform)
- Notable dip in 2020 (COVID-19 pandemic)
- **Conclusion:** Series is non-stationary, needs differencing

## Step 2: Unit Root Testing

### ADF Test on Log GDP Levels

- Test statistic:  $-0.91$
- Critical values:  $-3.48$  (1%),  $-2.88$  (5%),  $-2.58$  (10%)
- p-value:  $0.79$
- **Result:** Cannot reject  $H_0 \Rightarrow$  **Unit root present**

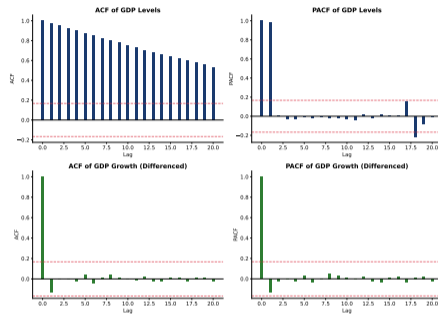
### ADF Test on First Difference (Growth Rate)

- Test statistic:  $-13.24$
- p-value:  $< 0.001$
- **Result:** Reject  $H_0$  at 1%  $\Rightarrow$  **Stationary after differencing**

### Conclusion

GDP is  $I(1) \Rightarrow$  Use  $d = 1$  in ARIMA model

## Step 3: Model Identification via ACF/PACF



### Analysis of Differenced Series

- ACF: Significant spike at lag 1, then cuts off  $\Rightarrow$  suggests MA(1)
- PACF: Significant spike at lag 1, decays  $\Rightarrow$  suggests AR(1)
- **Candidate models:** ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1)

## Step 4: Model Estimation

### Comparing Models using Information Criteria

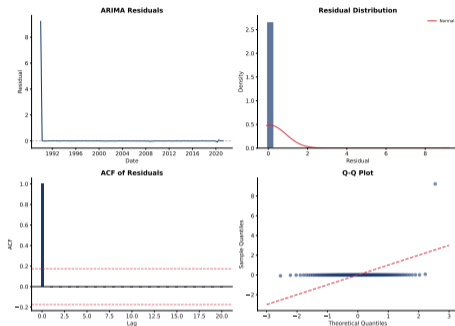
Model	AIC	BIC	Log-Lik
ARIMA(1,1,0)	-725.2	-719.5	364.6
ARIMA(0,1,1)	-724.8	-719.2	364.4
<b>ARIMA(1,1,1)</b>	<b>-747.0</b>	<b>-738.5</b>	<b>376.5</b>

### Selected Model: ARIMA(1,1,1)

$$(1 - 0.35L)(1 - L)Y_t = (1 + 0.58L)\varepsilon_t, \quad \hat{\sigma}^2 = 0.000156$$

- $\hat{\phi}_1 = 0.35$  (SE = 0.09), significant at 1%
- $\hat{\theta}_1 = 0.58$  (SE = 0.08), significant at 1%

## Step 5: Diagnostic Checking



### Residual Analysis

- Ljung-Box test:  $Q(10) = 5.8$ ,  $p\text{-value} = 0.83 \Rightarrow$  No autocorrelation
- Jarque-Bera test:  $JB = 156.4$ ,  $p\text{-value} < 0.001 \Rightarrow$  Non-normal (COVID outlier)
- **Conclusion:** Model passes autocorrelation checks; outliers expected

## Step 6: Forecasting with Confidence Intervals

### Last Observed Values (Log GDP)

$$Y_T = 9.973 \text{ (2024Q2)}, \quad Y_{T-1} = 9.956 \text{ (2024Q1)} \\ \Delta Y_T = 0.017, \quad \hat{\varepsilon}_T = 0.004$$

### 1-Step Ahead Forecast (2024Q3)

$$\Delta \hat{Y}_{T+1} = \hat{\phi}_1 \Delta Y_T + \hat{\theta}_1 \hat{\varepsilon}_T = 0.35(0.017) + 0.58(0.004) = 0.0083$$

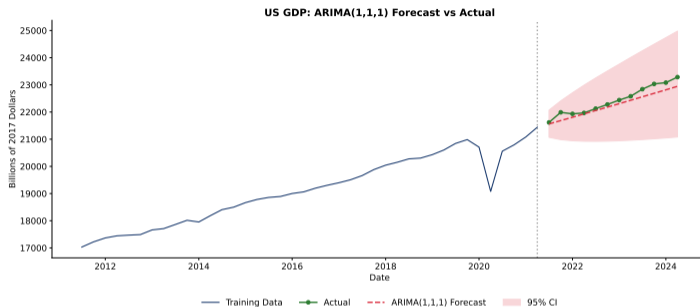
$$\hat{Y}_{T+1} = 9.973 + 0.0083 = \boxed{9.981}$$

### 95% Confidence Interval

$$CI = 9.981 \pm 1.96 \times \sqrt{0.000156} = [9.957, 10.006]$$

In levels: GDP forecast = \$21,652B, CI = [\$21,142B, \$22,175B]

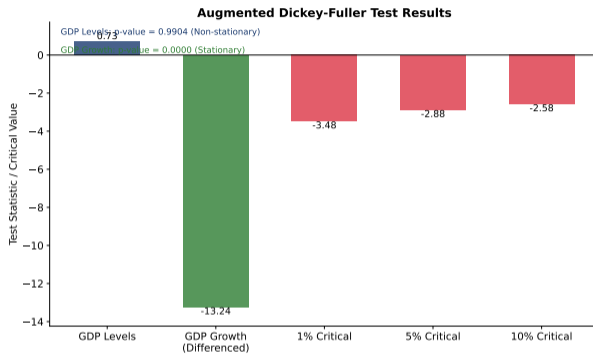
## Step 7: Forecast Evaluation



### Out-of-Sample Performance (Last 12 Quarters)

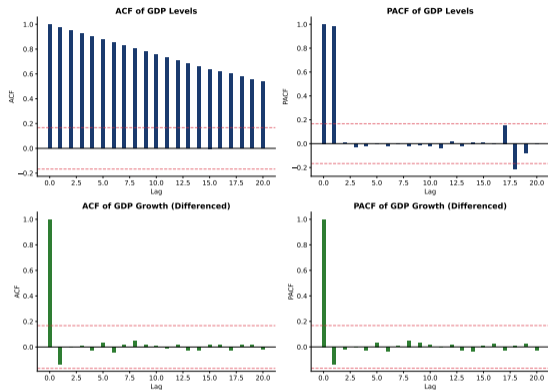
- RMSE = 0.0486 (log scale)  $\approx$  4.86% error
- MAE = 0.0430 (log scale)  $\approx$  4.30% error
- Direction accuracy = 91% (correctly predicted growth/decline)

# Unit Root Test Results



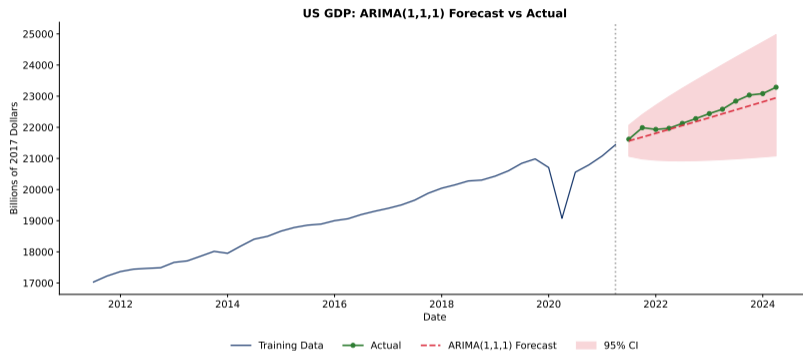
- GDP in levels: Cannot reject unit root (non-stationary)
- GDP growth: Reject unit root at 1% level (stationary)

## ACF/PACF: Levels vs Differenced



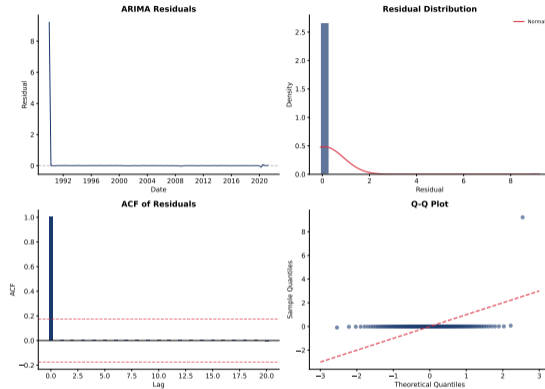
- **Top:** Slow ACF decay in levels suggests non-stationarity
- **Bottom:** After differencing, ACF/PACF help identify  $p$  and  $q$

# ARIMA Forecasting: Actual vs Predicted



- ARIMA(1,1,1) captures the trend dynamics
- Confidence intervals widen with forecast horizon

# Model Diagnostics



- Residuals appear random; ACF within bounds
- Q-Q plot shows approximate normality

## Auto-ARIMA Example

```
# Automatic model selection
model = pm.auto_arima(y, start_p=0, start_q=0,
                      max_p=3, max_q=3, d=None,
                      seasonal=False, trace=True)
print(model.summary())
```

# Key Takeaways

## Main Points

- ① **Non-stationarity** is common in economic data – must be addressed
- ② **Differencing** transforms  $I(d)$  to  $I(0)$
- ③ **ARIMA(p,d,q)** combines differencing with ARMA modeling
- ④ **Unit root tests** (ADF, KPSS) help determine  $d$
- ⑤ **Box-Jenkins methodology**: Identify → Estimate → Diagnose
- ⑥ **Forecasts** for  $I(1)$  series have growing uncertainty

## Next Steps

Chapter 4 will extend ARIMA to handle seasonality: SARIMA models.

## Quiz Question 1

### Question

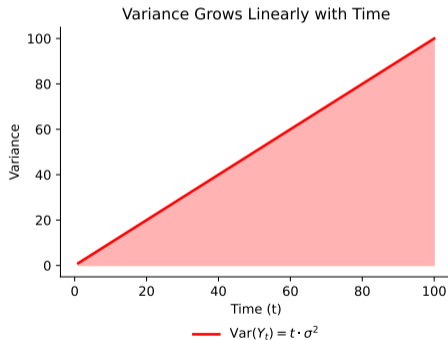
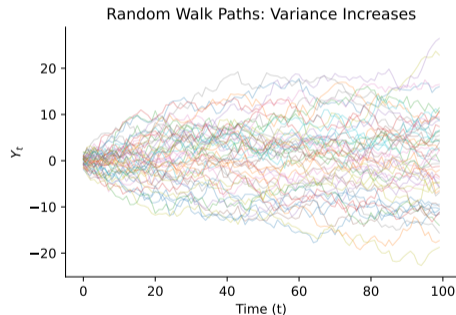
A time series  $Y_t$  follows a random walk:  $Y_t = Y_{t-1} + \varepsilon_t$ . What is  $\text{Var}(Y_t)$ ?

- ☐ A  $\sigma^2$  (constant)
- ☐ B  $t \cdot \sigma^2$  (grows linearly with time)
- ☐ C  $\sigma^2/t$  (decreases with time)
- ☐ D  $\sigma^{2t}$  (grows exponentially)

## Quiz Question 1: Answer

Correct Answer: (B)  $\text{Var}(Y_t) = t \cdot \sigma^2$

Random walk variance grows linearly with time — this is why random walks are non-stationary.



## Quiz Question 2

### Question

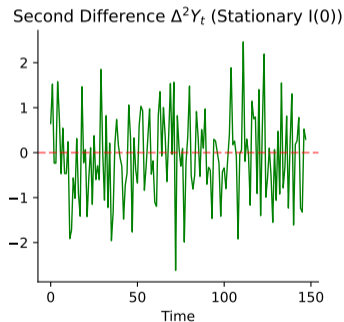
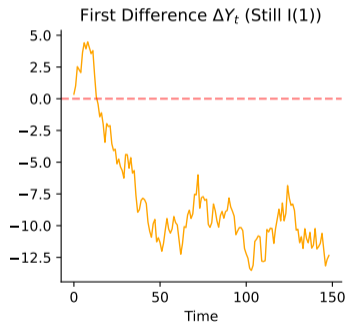
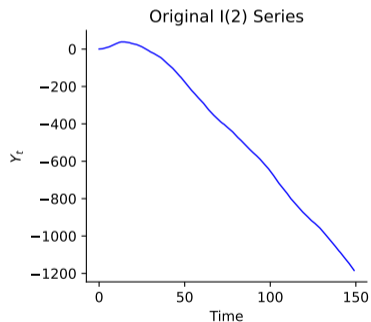
If a series  $Y_t$  is  $I(2)$ , how many times must you difference it to achieve stationarity?

- ☐ A 0 times (already stationary)
- ☐ B 1 time
- ☐ C 2 times
- ☐ D Cannot be made stationary by differencing

## Quiz Question 2: Answer

Correct Answer: (C) 2 times

$I(d)$  means “integrated of order  $d$ ” — requires  $d$  differences for stationarity.



## Quiz Question 3

### Question

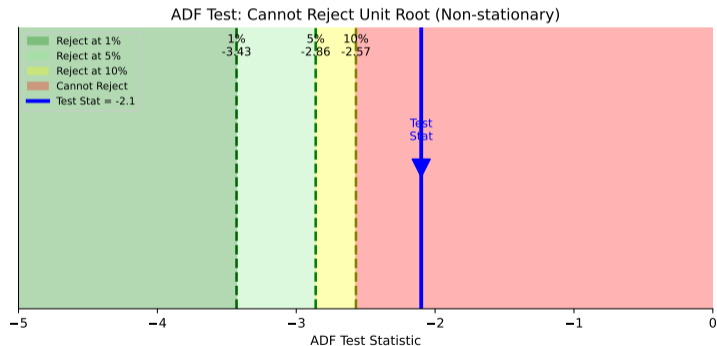
You run an ADF test and get a test statistic of  $-2.1$  with critical values:  $-3.43$  (1%),  $-2.86$  (5%),  $-2.57$  (10%). What do you conclude?

- ☐ A Reject  $H_0$ : series is stationary at all levels
- ☐ B Reject  $H_0$ : series is stationary at 10% level only
- ☐ C Fail to reject  $H_0$ : series likely has a unit root
- ☐ D The test is inconclusive

## Quiz Question 3: Answer

Correct Answer: (C) Fail to reject  $H_0$ : series has unit root

Test stat  $-2.1 > -2.57$  (10% CV)  $\Rightarrow$  Cannot reject at any level. Consider differencing.



## Quiz Question 4

### Question

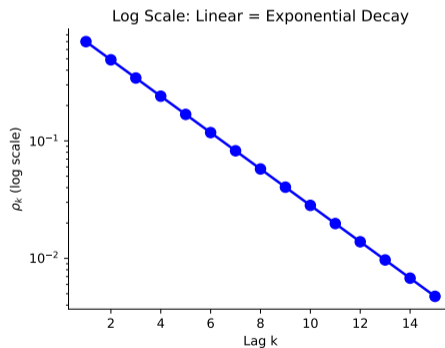
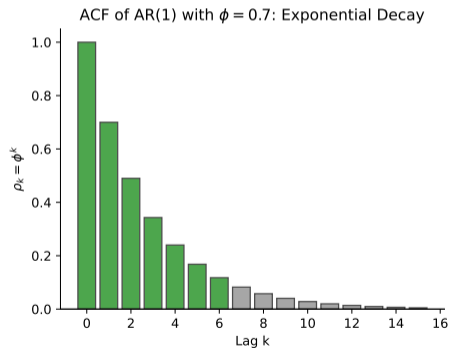
For an ARIMA(1,1,0) model, what is the ACF pattern of the **differenced** series  $\Delta Y_t$ ?

- ☐ A Cuts off after lag 1
- ☐ B Decays exponentially
- ☐ C Alternates in sign
- ☐ D Is zero at all lags

## Quiz Question 4: Answer

Correct Answer: (B) Decays exponentially

ARIMA(1,1,0)  $\Rightarrow \Delta Y_t$  follows AR(1) with ACF  $\rho_k = \phi_1^k$  (geometric decay).



## Quiz Question 5

### Question

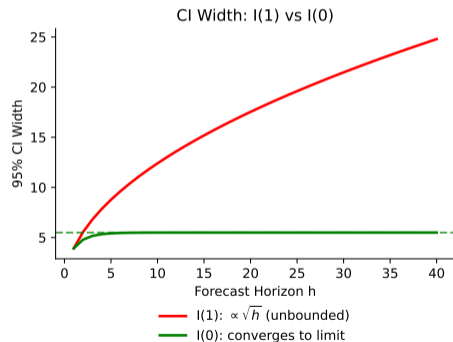
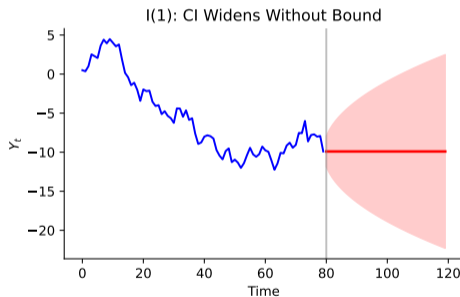
What happens to ARIMA forecast confidence intervals as the horizon  $h$  increases for an  $I(1)$  series?

- ☐ A They stay constant
- ☐ B They narrow (more precision)
- ☐ C They widen without bound
- ☐ D They widen but converge to a limit





## Quiz Question 5: Answer

Correct Answer: (C) They widen without bound

For  $I(1)$ : CI width  $\propto \sqrt{h}$  (unbounded). For  $I(0)$ : CIs converge to a limit.



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