



# Time Series Analysis and Forecasting

## Chapter 2: ARMA Models



Daniel Traian PELE

Bucharest University of Economic Studies

IDA Institute Digital Assets

Blockchain Research Center

AI4EFin Artificial Intelligence for Energy Finance

Romanian Academy, Institute for Economic Forecasting

MSCA Digital Finance

## Learning Objectives

By the end of this chapter, you will be able to:

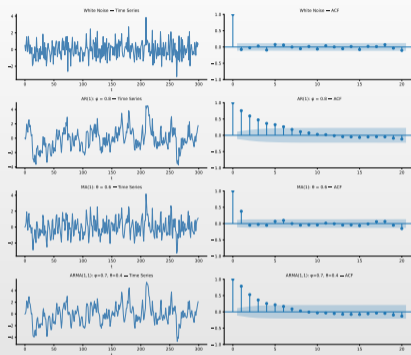
1. **Define** and simulate  $AR(p)$ ,  $MA(q)$ , and  $ARMA(p, q)$  processes
2. **Verify** stationarity and invertibility conditions
3. **Identify** orders  $p$  and  $q$  through ACF/PACF analysis
4. **Estimate** parameters via Yule-Walker, MLE, and information criteria (AIC, BIC)
5. **Diagnose** the model through residual analysis and the Ljung-Box test
6. **Forecast** using ARMA models with confidence intervals
7. **Apply** the Box-Jenkins methodology to real data (sunspots)

# Outline

- Motivation
- Introduction and the Lag Operator
- Autoregressive (AR) Models
- Moving Average (MA) Models
- ARMA Models
- Model Identification
- Parameter Estimation
- Model Diagnostics
- Forecasting with ARMA
- Practical Implementation
- Case Study: Real Data
- Summary
- Quiz



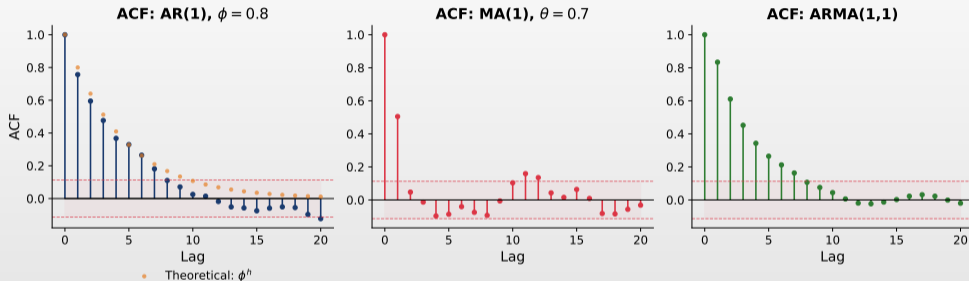
## Why ARMA Models?



- **AR processes:** Current value depends on past values  $\Rightarrow$  mean-reverting behavior
- **MA processes:** Current value depends on past shocks  $\Rightarrow$  short memory
- **ARMA:** Combines both mechanisms for flexible modeling

## Model Identification Through ACF Patterns

Distinct ACF patterns for different models



### ACF Reflects Model Structure

- ▣ **Distinct patterns:** AR: exponential decay; MA: sharp cutoff; ARMA: mixed decay
- ▣ **Identification:** Visual analysis of ACF/PACF guides the selection of orders  $p$  and  $q$

## Recap: Stationarity

### From Chapter 1

- A process  $\{X_t\}$  is **weakly stationary** if:
  1.  $\mathbb{E}[X_t] = \mu$  (constant mean)
  2.  $\text{Var}(X_t) = \sigma^2 < \infty$  (constant, finite variance)
  3.  $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$  (covariance depends only on lag  $h$ )

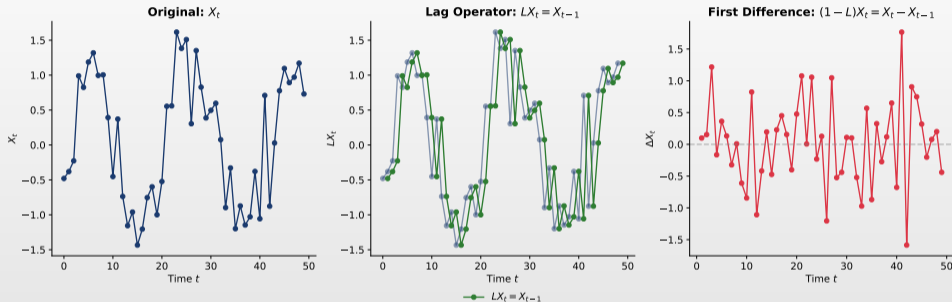
### Why Stationarity Matters for ARMA

- ARMA models assume stationarity
  - ▶ Parameters remain stable over time
  - ▶ Autocorrelation structure is maintained
- Non-stationary data  $\Rightarrow$  difference first (ARIMA, Ch. 3)

### Chapter Objective

- Parametric models for stationary series  $\Rightarrow$  combining dependence on past observations (AR) with the influence of random shocks (MA)

## The Lag Operator: Visual Illustration



### Role of the Lag Operator

- ▣ **Notation foundation:** Enables compact writing of difference equations
- ▣ **Utility:** Facilitates algebraic manipulation of ARMA models

## The Lag Operator (Backshift Operator)

### Definition 1 (Lag Operator)

- The **lag operator**  $L$  (or backshift operator  $B$ ) shifts a time series back by one period:  $LX_t = X_{t-1}$

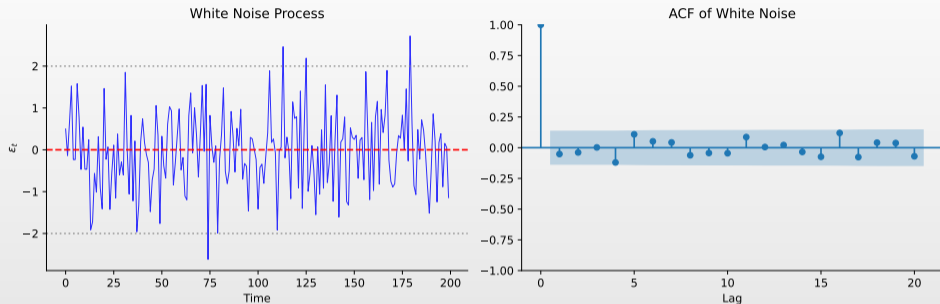
### Properties

- $L^k X_t = X_{t-k}$  (shift back by  $k$  periods)
- $L^0 X_t = X_t$  (identity)
- $(1 - L)X_t = X_t - X_{t-1} = \Delta X_t$  (first difference)
- $(1 - L)^d X_t = \Delta^d X_t$  (difference of order  $d$ )

### Lag Polynomials

- **AR polynomial:**  $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$
- **MA polynomial:**  $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$

## White Noise: Visual Illustration



### Key Characteristics

- **Left:** Random fluctuations, no patterns, constant variance
- **Right:** ACF only a spike at lag 0; others within significance bounds  $\Rightarrow$  no linear dependence

## The White Noise Process

### Definition 2 (White Noise)

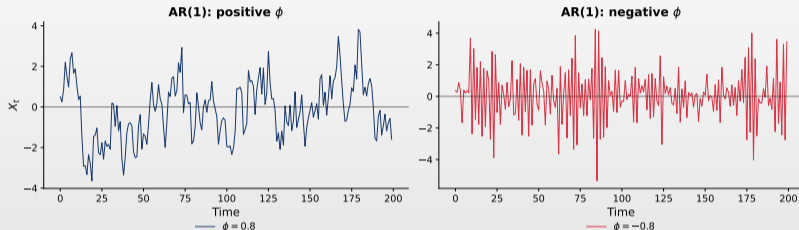
- A process  $\{\varepsilon_t\}$  is **white noise**, denoted  $\varepsilon_t \sim WN(0, \sigma^2)$ , if:
  1.  $\mathbb{E}[\varepsilon_t] = 0$  for all  $t$
  2.  $\text{Var}(\varepsilon_t) = \sigma^2$  for all  $t$
  3.  $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$  for all  $t \neq s$

### Properties

- **Building block:** White noise underlies all ARMA models
- **ACF:**  $\rho(0) = 1$ ,  $\rho(h) = 0$  for  $h \neq 0$ ; PACF: same pattern
- **Gaussian white noise:**  $\varepsilon_t \sim N(0, \sigma^2)$  i.i.d.
- **Unpredictable:** White noise is *not* predictable  $\Rightarrow$  it is purely random

## AR(1): Visual Illustration

AR(1): different behavior for positive vs negative  $\phi$



### Visual Interpretation

- ▣ **Positive  $\phi$ :** Persistent fluctuations, gradual mean reversion
- ▣ **Negative  $\phi$ :** Oscillating behavior, alternating around the mean
- ▣ **Larger  $|\phi|$   $\Rightarrow$  greater persistence, slower reversion**

## The AR(1) Model: Definition

### Definition 3 (AR(1) Process)

- An **autoregressive process of order 1** is:  $X_t = c + \phi X_{t-1} + \varepsilon_t$
- $\varepsilon_t \sim WN(0, \sigma^2)$  and  $|\phi| < 1$  for stationarity

### Interpretation

- $c$ : constant (intercept)
- $\phi$ : autoregressive coefficient
  - ▶ Measures the persistence of the series
- $\varepsilon_t$ : innovation (shock)

### Lag Operator Notation

- $(1 - \phi L)X_t = c + \varepsilon_t$
- $\phi(L)X_t = c + \varepsilon_t$
- $\phi(L) = 1 - \phi L$

## AR(1) Stationarity Condition

Necessary and Sufficient Condition:  $|\phi| < 1$

- The root of the characteristic equation must lie outside the unit circle

- Shocks diminish over time
  - ▶ Process reverts to the mean
  - ▶ Finite, stable variance

Non-stationary ( $|\phi| \geq 1$ )

- $|\phi| = 1$ : random walk
  - ▶ Unit root, variance  $\rightarrow \infty$
- $|\phi| > 1$ : explosive process

### Characteristic Equation

- $\phi(z) = 1 - \phi z = 0 \implies z = 1/\phi$
- Stationarity  $\Leftrightarrow$  root outside the unit circle ( $|z| > 1$ )

## AR(1) Properties

### Stationary AR(1) with $|\phi| < 1$

□ Moment properties:

**Mean:**  $\mu = \mathbb{E}[X_t] = \frac{c}{1-\phi}$

**Variance:**  $\gamma(0) = \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$

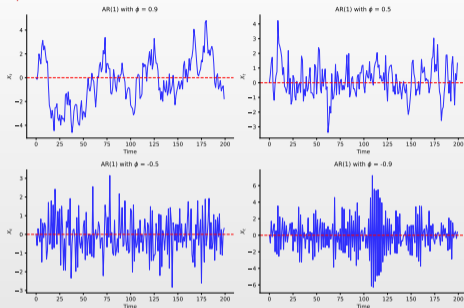
**Autocovariance:**  $\gamma(h) = \frac{\phi^h \sigma^2}{1-\phi^2}$

**Autocorrelation (ACF):**  $\rho(h) = \phi^h$

### Key Observation

□ **AR(1) signature:** ACF decays exponentially with factor  $\phi$

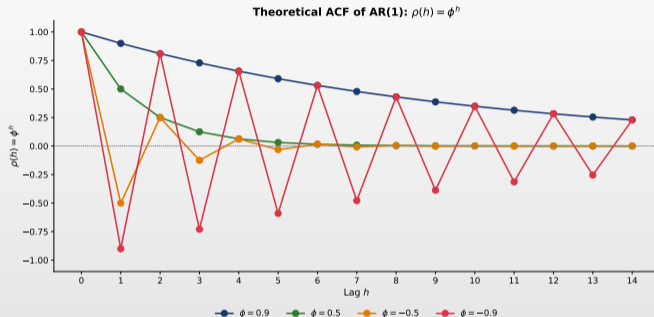
- ▶  $\phi > 0$ : monotone decay towards zero
- ▶  $\phi < 0$ : damped oscillations (alternating signs)

AR(1) Simulations: Effect of  $\phi$ 

## Interpretation

- Different values of  $\phi$  produce distinct behaviors: larger  $|\phi| \Rightarrow$  more persistence
- Positive  $\phi$  creates smooth trajectories; negative  $\phi$  creates oscillations
- As  $|\phi| \rightarrow 1$ , the process approaches non-stationarity

## Theoretical AR(1) ACF



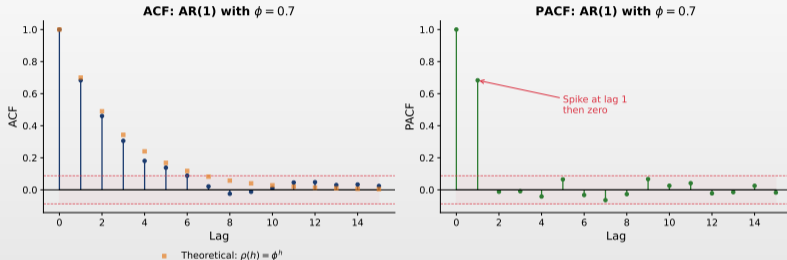
## ACF Pattern

- Formula:  $\rho(h) = \phi^h \Rightarrow$  exponential decay
- $\phi > 0$ : monotone decay;  $\phi < 0$ : alternating signs



## AR(1) ACF and PACF: Theory vs Sample

ACF and PACF for AR(1): theory vs sample



### Interpretation

- ACF: Exponential decay with factor  $\phi$ ; formula:  $\rho(h) = \phi^h$
- PACF: A single spike at lag 1, then cuts off  $\Rightarrow$  identifies AR(1)
- Sample estimates fluctuate around theoretical values

## Proof: AR(1) Mean

### Claim

- For AR(1):  $X_t = c + \phi X_{t-1} + \varepsilon_t$ , the mean is  $\mu = \frac{c}{1-\phi}$

### Proof

- Take expectations of both sides:  $\mathbb{E}[X_t] = c + \phi\mathbb{E}[X_{t-1}] + \mathbb{E}[\varepsilon_t]$
- By stationarity,  $\mathbb{E}[X_t] = \mathbb{E}[X_{t-1}] = \mu$ , and  $\mathbb{E}[\varepsilon_t] = 0$ :  $\mu = c + \phi\mu$
- Solving:  $\mu - \phi\mu = c \implies \mu(1 - \phi) = c \implies \boxed{\mu = \frac{c}{1 - \phi}}$

### Requirement

- **Necessary condition:**  $\phi \neq 1$  for the mean to be defined
  - ▶ If  $\phi = 1$  (unit root), the mean is undefined
  - ▶ The process becomes a random walk (non-stationarity)

## Proof: AR(1) Variance

### Claim

$$\square \text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$$

### Proof

$\square$  Assume  $c = 0$ . Take the variance of  $X_t = \phi X_{t-1} + \varepsilon_t$ :

$$\square \text{Var}(X_t) = \phi^2 \text{Var}(X_{t-1}) + \text{Var}(\varepsilon_t) + 2\phi \underbrace{\text{Cov}(X_{t-1}, \varepsilon_t)}_{=0}$$

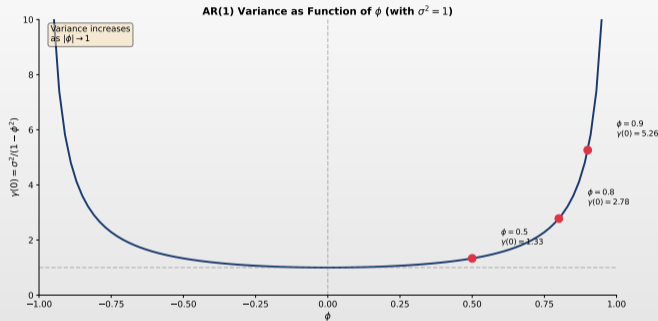
$\square$  By stationarity,  $\text{Var}(X_t) = \text{Var}(X_{t-1}) = \gamma(0)$ :

$$\square \gamma(0) = \phi^2 \gamma(0) + \sigma^2 \implies \gamma(0)(1 - \phi^2) = \sigma^2 \implies \boxed{\gamma(0) = \frac{\sigma^2}{1 - \phi^2}}$$

### Note

$\square$  Requires  $|\phi| < 1$  for positive variance. When  $|\phi| \rightarrow 1$ , variance  $\rightarrow \infty$



AR(1) Variance as a Function of  $\phi$ 

## Observations

- As  $|\phi| \rightarrow 1$ , the variance explodes  $\Rightarrow$  non-stationarity
- For  $\phi = 0$ :  $\gamma(0) = \sigma^2$  (white noise); variance increases monotonically with  $|\phi|$

## Proof: AR(1) Autocorrelation Function

Claim:  $\rho(h) = \phi^h$  for  $h \geq 0$

- Find the autocovariance  $\gamma(h) = \text{Cov}(X_t, X_{t-h})$

### Proof

- Multiply  $X_t = \phi X_{t-1} + \varepsilon_t$  by  $X_{t-h}$  and take expectations:
- $\mathbb{E}[X_t X_{t-h}] = \phi \mathbb{E}[X_{t-1} X_{t-h}] + \mathbb{E}[\varepsilon_t X_{t-h}]$
- For  $h \geq 1$ :  $\mathbb{E}[\varepsilon_t X_{t-h}] = 0 \Rightarrow \gamma(h) = \phi \gamma(h-1)$
- Recursive relation from  $\gamma(0)$ :  $\gamma(1) = \phi \gamma(0)$ ,  $\gamma(2) = \phi^2 \gamma(0)$ , ...  $\gamma(h) = \phi^h \gamma(0)$
- ACF:  $\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\phi^h \gamma(0)}{\gamma(0)} = \boxed{\phi^h}$

## Proof: AR(1) Stationarity Condition

### Claim

- AR(1) is stationary if and only if  $|\phi| < 1$

### Proof

- Recursive substitution:  $X_t = \phi X_{t-1} + \varepsilon_t = \phi(\phi X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \dots$
- After  $n$  steps:  $X_t = \phi^n X_{t-n} + \sum_{j=0}^{n-1} \phi^j \varepsilon_{t-j}$
- If  $|\phi| < 1$ :  $\phi^n \rightarrow 0$  as  $n \rightarrow \infty$ , so  $X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$
- Finite variance:  $\text{Var}(X_t) = \sigma^2 \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma^2}{1-\phi^2} < \infty$  (geometric series)

### Conclusion

- Converges  $\iff |\phi| < 1$ . For  $|\phi| \geq 1$ , the term  $\phi^n X_{t-n}$  does not vanish  $\Rightarrow$  infinite variance

## The Partial Autocorrelation Function (PACF)

### Definition 4 (PACF)

- The **partial autocorrelation** of order  $k$ , denoted  $\pi_k$ , measures the correlation between  $X_t$  and  $X_{t-k}$  **after removing** the linear effects of the intermediate variables  $X_{t-1}, \dots, X_{t-k+1}$

### Formal Definition

- $\pi_1 = \rho(1)$
- For  $k \geq 2$ :  $\pi_k$  is the last coefficient in:  
$$X_t = \alpha_1 X_{t-1} + \dots + \alpha_k X_{t-k} + e_t$$
- $\pi_k = \alpha_k$  (coefficient of  $X_{t-k}$ )

### Computation via Yule-Walker

- Solve the Yule-Walker equations of order  $k$
- $\pi_k =$  last element of the solution vector

### Utility

- **Identification:** PACF determines the order  $p$  of an AR model
  - PACF cuts off after lag  $p$

## Durbin-Levinson Algorithm for PACF

### Durbin-Levinson Recursion

▣ Computes PACF ( $\pi_k$ ) recursively, without inverting the Toeplitz matrix:

1. **Initialize:**  $\pi_1 = \hat{\rho}(1)$ ,  $v_1 = \hat{\gamma}(0)(1 - \pi_1^2)$

2. **Recursion** ( $k = 2, 3, \dots$ ):

$$\pi_k = \frac{\hat{\rho}(k) - \sum_{j=1}^{k-1} \phi_{k-1,j} \hat{\rho}(k-j)}{1 - \sum_{j=1}^{k-1} \phi_{k-1,j} \hat{\rho}(j)}$$

3. **Update:**  $\phi_{k,j} = \phi_{k-1,j} - \pi_k \phi_{k-1,k-j}$  for  $j = 1, \dots, k-1$ ;  $\phi_{k,k} = \pi_k$

4. **Prediction variance:**  $v_k = v_{k-1}(1 - \pi_k^2)$

### Complexity

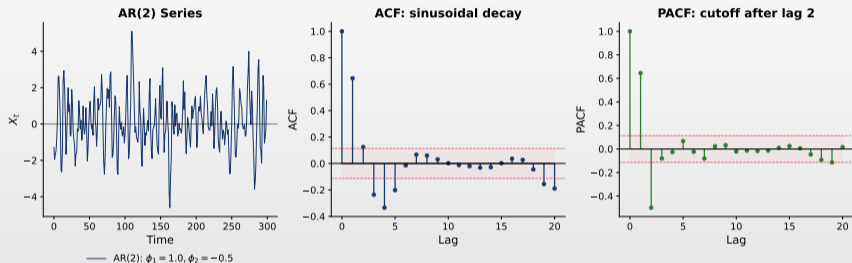
- ▣  $O(k^2)$  vs  $O(k^3)$  (direct inversion)
- ▣ Exploits Toeplitz structure of  $\Gamma_k$
- ▣ Guarantees  $v_k > 0$  if stationary

### AR( $p$ ) Identification

- ▣  $\pi_k = 0$  for  $k > p \Rightarrow$  order is  $p$
- ▣ CI:  $|\pi_k| > 1.96/\sqrt{T} \Rightarrow$  significant
- ▣ Equivalent to  $t$ -test on last OLS coeff.

## AR(p): Visual Illustration

AR(2) Process: pseudo-cyclic behavior



### Observations

- AR(2) can exhibit pseudo-cyclic behavior (complex roots); damped sinusoidal ACF
- PACF cuts off after lag 2  $\Rightarrow$  key identification pattern

## The AR(p) Model: General Form

### Definition 5 (AR(p) Process)

- An **autoregressive process of order p** is:  $X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + \varepsilon_t$
- **Lag operator**:  $\phi(L)X_t = c + \varepsilon_t$ , where  $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$

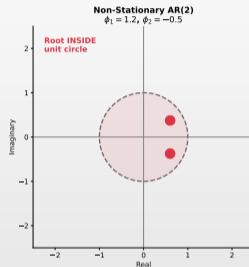
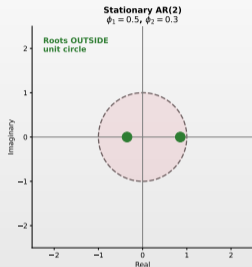
### Stationarity Condition

- All roots of  $\phi(z) = 0$  must lie **outside** the unit circle
- Equivalently: all roots have modulus  $> 1$

### PACF Pattern

- PACF cuts off after lag  $p$
- ACF decays (exponentially or with damped oscillations)

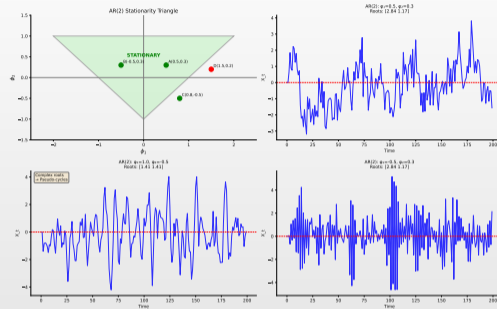
## AR(2) Stationarity: Unit Circle Visualization



### Characteristic Polynomial and Unit Circle Condition

- **Characteristic polynomial** of  $AR(p)$ :  $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$
- All roots of  $\phi(z) = 0$  must lie **outside** the unit circle ( $|z| > 1$ )
- Roots on the circle: non-stationary; roots inside: explosive process

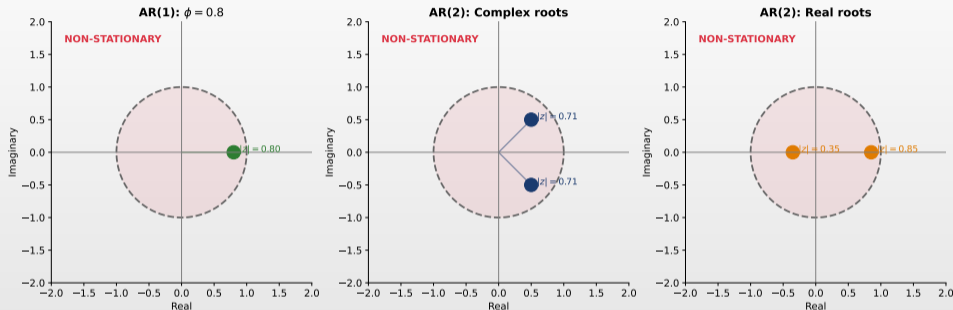
## The AR(2) Stationarity Triangle



### Stationarity Region

- The triangular region defines the stationary AR(2) parameter combinations
- **Boundaries:**  $\phi_1 + \phi_2 < 1$ ,  $\phi_2 - \phi_1 < 1$  and  $|\phi_2| < 1$
- Points outside the region  $\Rightarrow$  non-stationary or explosive processes

## Characteristic Polynomial Roots



### Types of Roots

- Real roots: exponential decay in ACF
- Complex roots: damped oscillations (pseudo-cycles)
- All roots must lie outside the unit circle

## The AR(2) Model

### Definition 6 (AR(2) Process)

$$\square X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

### Stationarity Conditions

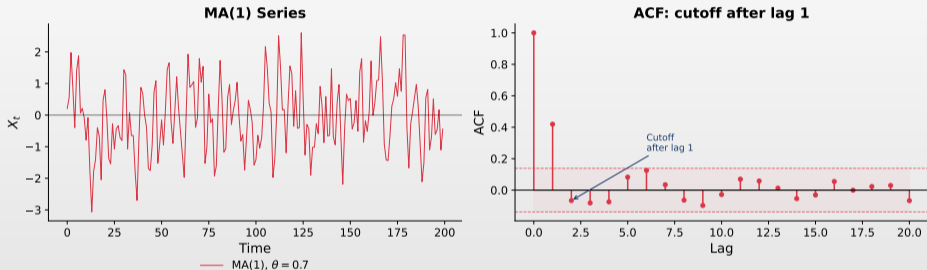
$$\square \phi_1 + \phi_2 < 1; \quad \phi_2 - \phi_1 < 1; \quad |\phi_2| < 1$$

### ACF Behavior

- $\square$  **Real roots:** mixture of two exponential decays
- $\square$  **Complex roots:** damped sinusoidal pattern (pseudo-cycles)
- $\square$  **PACF:** Cuts off after lag 2 ( $\pi_k = 0$  for  $k > 2$ )

## MA(1): Visual Illustration

MA(1): short memory series with ACF cutoff



### Visual Interpretation

- Left panel: MA(1) series  $\Rightarrow$  rapid mean reversion
- Right panel: ACF with **cutoff after lag 1**; PACF exponential decay

## The MA(1) Model: Definition

### Definition 7 (MA(1) Process)

- ▣ A **moving average process of order 1** is:  $X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$
- ▣  $\varepsilon_t \sim WN(0, \sigma^2)$

### Interpretation

- ▣  $\mu$ : process mean
- ▣  $\theta$ : MA coefficient
  - Measures the impact of the past shock
- ▣ Depends on  $\varepsilon_t$  and  $\varepsilon_{t-1}$

### Lag Operator Notation

- ▣  $X_t = \mu + \theta(L)\varepsilon_t$
- ▣  $\theta(L) = 1 + \theta L$

### Key Property

- ▣ **Guaranteed stationarity:** MA processes are always stationary
  - Does not depend on the value of  $\theta$

## MA(1) Properties

$$\text{MA}(1): X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$$

- **Mean:**  $\mathbb{E}[X_t] = \mu$ ;    **Variance:**  $\gamma(0) = \sigma^2(1 + \theta^2)$
- **Autocovariance:**  $\gamma(1) = \theta\sigma^2$ ,  $\gamma(h) = 0$  ( $h > 1$ )
- **ACF:**  $\rho(1) = \frac{\theta}{1+\theta^2}$ ,  $\rho(h) = 0$  ( $h > 1$ )

### Key Observation

- **MA(1) signature:** ACF cuts off after lag 1
  - ▶  $\rho(1) \neq 0$ , but  $\rho(h) = 0$  for  $h > 1$ ; opposite pattern to AR(1)

## Proof: MA(1) Variance and Autocovariance

Starting point:  $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$  (assuming  $\mu = 0$ )

▣ **Variance:**

$$\gamma(0) = \text{Var}(\varepsilon_t + \theta\varepsilon_{t-1}) = \sigma^2 + \theta^2\sigma^2 + 0 = \boxed{\sigma^2(1 + \theta^2)}$$

Autocovariance at lag 1

$$\begin{aligned} \square \gamma(1) &= \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-1} + \theta\varepsilon_{t-2}) \\ \square &= \text{Cov}(\varepsilon_t, \varepsilon_{t-1}) + \theta\text{Cov}(\varepsilon_t, \varepsilon_{t-2}) + \theta\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-1}) + \theta^2\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-2}) \\ \square &= 0 + 0 + \theta\sigma^2 + 0 = \boxed{\theta\sigma^2} \end{aligned}$$

Autocovariance at lag  $h \geq 2$

▣ No common  $\varepsilon$  terms  $\Rightarrow \gamma(h) = 0$

## Proof: Maximum ACF for MA(1)

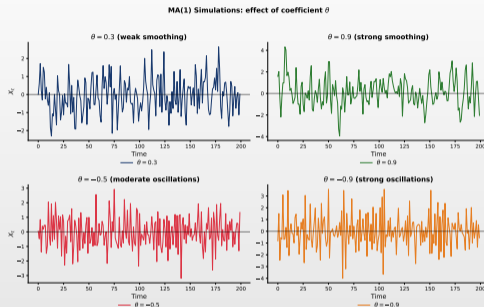
Claim:  $|\rho(1)| \leq 0.5$  for any value of  $\theta$

- ACF at lag 1:  $\rho(1) = \frac{\theta}{1+\theta^2}$
- Differentiate:  $\frac{d\rho(1)}{d\theta} = \frac{1-\theta^2}{(1+\theta^2)^2} = 0 \Rightarrow \theta = \pm 1$
- At these values:  $\rho(1)|_{\theta=1} = \frac{1}{2}$ ,  $\rho(1)|_{\theta=-1} = -\frac{1}{2}$

### Implication

- **Practical test:** If  $|\hat{\rho}(1)| > 0.5$  from data, the process is **not** MA(1)
  - ▶ The maximum  $|\rho(1)| = 0.5$  is reached at  $\theta = \pm 1$
  - ▶ Consider AR or ARMA models as alternatives

## MA(1) Simulations: Effect of $\theta$



### Interpretation

- MA(1) is always stationary regardless of  $\theta \Rightarrow$  finite memory of only one lag
- Positive  $\theta$  smooths the series; negative  $\theta$  creates faster fluctuations
- Unlike AR(1), MA(1) shocks affect the process for only one period

## Proof: ACF for MA(1)

Claim:  $\rho(1) = \frac{\theta}{1+\theta^2}$  and  $\rho(h) = 0$  for  $h > 1$

- MA(1) has non-zero autocorrelation **only** at lag 1

### Proof

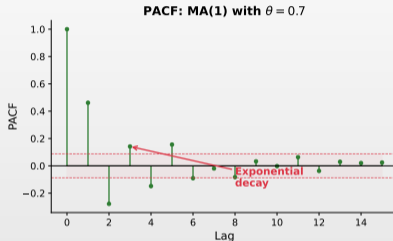
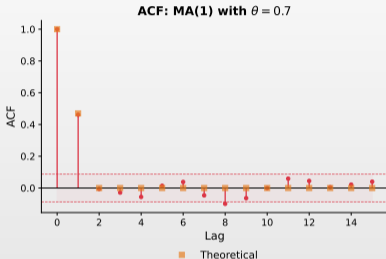
- Let  $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ . Autocorrelation at lag 1:
- $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta\sigma^2}{\sigma^2(1+\theta^2)} = \frac{\theta}{1+\theta^2}$
- For  $h > 1$ :  $\gamma(h) = \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-h} + \theta\varepsilon_{t-h-1})$
- The terms  $\varepsilon_t, \varepsilon_{t-1}$  do not overlap with  $\varepsilon_{t-h}, \varepsilon_{t-h-1}$  when  $h > 1$ , so  $\gamma(h) = 0$

### Practical Consequence

- ACF cuts off sharply after lag 1  $\Rightarrow$  distinctive signature of MA(1) processes

## MA(1) ACF and PACF: Visual Comparison

ACF and PACF for MA(1): opposite pattern to AR(1)

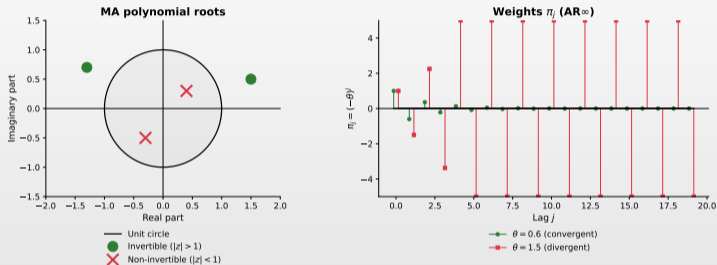


### Interpretation

- ACF: A single spike at lag 1, then cuts off  $\Rightarrow$  key MA(1) signature
- PACF: Exponential decay  $\Rightarrow$  opposite pattern to AR(1)
- This reversal differentiates MA processes from AR processes

## Invertibility: Visual Illustration

Invertibility of MA models



### Interpretation

- **Left:** invertibility requires roots outside the unit circle
- **Right:**  $AR(\infty)$  weights decay only when  $|\theta| < 1$

## Invertibility of MA Models

### Definition 8 (Invertibility)

- An MA process is **invertible** if it can be written as an infinite AR process:
- $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$

### Invertibility Conditions

- **MA(1)**: Invertible if  $|\theta| < 1$
- **MA(q)**: Roots of  $\theta(z) = 0$  outside the unit circle

### Why Invertibility Matters

- Ensures unique representation (without invertibility, multiple MA models describe the same data)
- Necessary for forecasting and estimation
- **Stationarity**  $\Rightarrow$  AR; **Invertibility**  $\Rightarrow$  MA

## Proof: MA(1) Invertibility

### Claim

- MA(1) is invertible if and only if  $|\theta| < 1$

### Proof

- From  $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ , isolate:  $\varepsilon_t = X_t - \theta\varepsilon_{t-1}$
- Recursive back-substitution:  $\varepsilon_t = X_t - \theta(X_{t-1} - \theta\varepsilon_{t-2}) = X_t - \theta X_{t-1} + \theta^2\varepsilon_{t-2}$
- Continuing:  $\varepsilon_t = \sum_{j=0}^n (-\theta)^j X_{t-j} + (-\theta)^{n+1} \varepsilon_{t-n-1}$
- If  $|\theta| < 1$ :  $(-\theta)^{n+1} \rightarrow 0$ , so

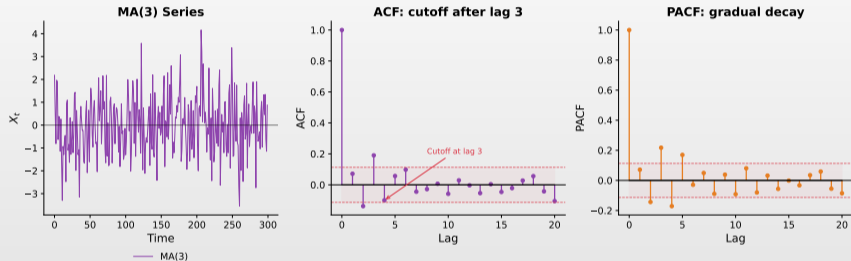
$$\varepsilon_t = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j}$$

### Conclusion

- Geometric series converges  $\iff |\theta| < 1 \Rightarrow$  MA(1) can be written as AR( $\infty$ )

## MA(q): Visual Illustration

MA(q) Process: ACF signature cuts off after lag  $q$



### Observation

- MA(3) process: key signature  $\Rightarrow$  ACF cuts off after lag  $q$  ( $\rho(h) = 0$  for  $h > 3$ )

## The MA(q) Model: General Form

### Definition 9 (MA(q) Process)

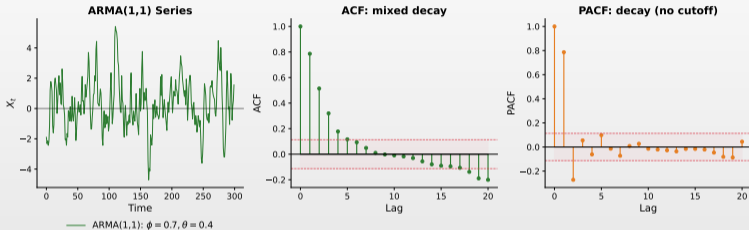
- ▣ A **moving average process of order q**:  $X_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q}$
- ▣ **Lag operator**:  $X_t = \mu + \theta(L)\varepsilon_t$ , where  $\theta(L) = 1 + \theta_1L + \cdots + \theta_qL^q$

### Properties

- ▣ Always stationary (finite variance)
- ▣ ACF cuts off after lag  $q$ :  $\rho(h) = 0$  for  $h > q$ ; PACF decays gradually
- ▣ Invertible if all roots of  $\theta(z) = 0$  lie outside the unit circle

## ARMA: Visual Illustration

ARMA(1,1): neither ACF nor PACF cut off



### ARMA(1,1) Interpretation

- Combines AR persistence with MA shock response
- ACF pattern:** Decay after the first lag (lags decay geometrically)
- PACF pattern:** Also decays (no sharp cutoff as in pure AR)
- Neither ACF nor PACF cuts off  $\Rightarrow$  key identifier for mixed models

## The ARMA(p,q) Model: Definition

### Definition 10 (ARMA(p,q) Process)

- $X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$
- **Compact form:**  $\phi(L)X_t = c + \theta(L)\varepsilon_t$ , where  $\mu = \frac{c}{1 - \phi_1 - \cdots - \phi_p}$

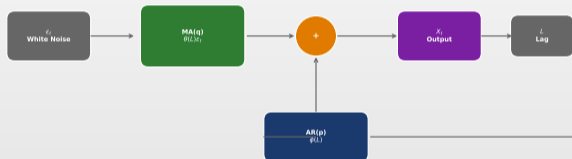
### Key Idea

- **Flexibility:** Combines AR and MA components
  - ▶ AR captures persistence; MA captures shock response
- **Parsimony:** ARMA(1,1) can be better than AR(5) or MA(5)
  - ▶ Fewer parameters, less risk of overfitting

## ARMA Model Structure

ARMA(p,q) Model Structure

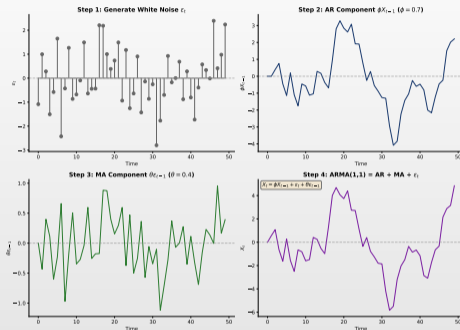
$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$



### Components

- ▣ **AR component:** influence of past values of the series
- ▣ **MA component:** impact of past random shocks

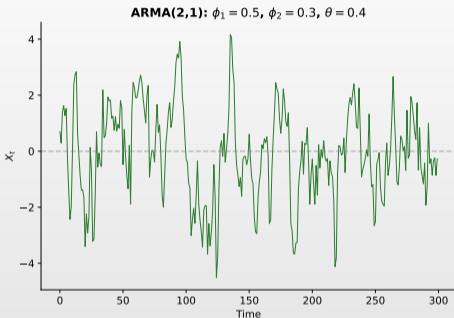
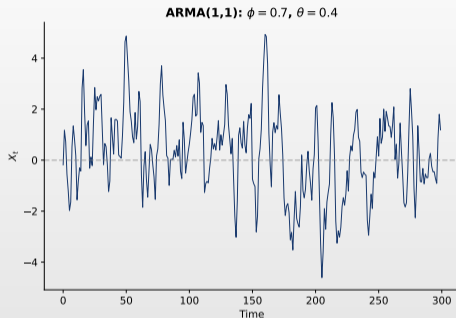
## How ARMA Simulation Works



### Steps

- Generate white noise, apply the ARMA equation recursively, obtain simulated series

## ARMA Examples



### Observation

- Different combinations of orders  $(p, q)$  produce distinct behaviors

## The ARMA(1,1) Model

### Definition 11 (ARMA(1,1) Process)

$$\square X_t = c + \phi X_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

### Properties (stationarity and invertibility)

$$\square \text{Mean: } \mu = \frac{c}{1-\phi}; \text{ Variance: } \gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$$

### ACF

$$\square \rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}; \quad \rho(h) = \phi \cdot \rho(h-1) \text{ for } h \geq 2$$

$\square$  ACF decays exponentially after lag 1 (starting point depends on  $\phi$  and  $\theta$ )

## Proof: ARMA(1,1) Variance

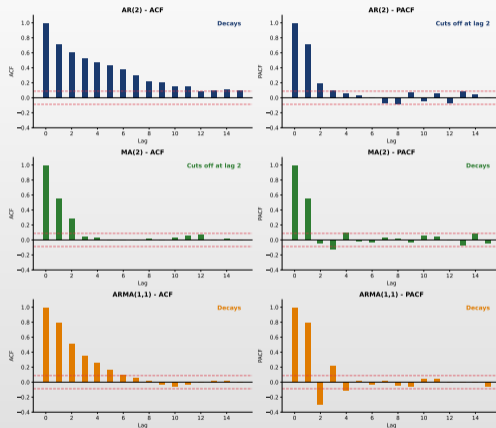
### Claim

$$\square \gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$$

### Proof

- Let  $Y_t = X_t - \mu$ :  $Y_t = \phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$
- Square:  $Y_t^2 = \phi^2 Y_{t-1}^2 + \varepsilon_t^2 + \theta^2 \varepsilon_{t-1}^2 + 2\phi Y_{t-1} \varepsilon_t + 2\phi\theta Y_{t-1} \varepsilon_{t-1} + 2\theta \varepsilon_t \varepsilon_{t-1}$
- Take expectations;  $\mathbb{E}[\varepsilon_t Y_{t-1}] = 0$ ,  $\mathbb{E}[\varepsilon_t \varepsilon_{t-1}] = 0$ :
- $\gamma(0) = \phi^2 \gamma(0) + \sigma^2 + \theta^2 \sigma^2 + 2\phi\theta \mathbb{E}[\varepsilon_{t-1} Y_{t-1}]$
- From  $Y_{t-1} = \phi Y_{t-2} + \varepsilon_{t-1} + \theta \varepsilon_{t-2}$ : only  $\varepsilon_{t-1}^2$  contributes  $\Rightarrow \mathbb{E}[\varepsilon_{t-1} Y_{t-1}] = \sigma^2$
- $\gamma(0)(1 - \phi^2) = (1 + 2\phi\theta + \theta^2)\sigma^2 \implies \boxed{\gamma(0) = \frac{(1 + 2\phi\theta + \theta^2)\sigma^2}{1 - \phi^2}}$

## ACF/PACF Patterns: AR vs MA vs ARMA



## Proof: ARMA(1,1) ACF at Lag 1

### Claim

$$\square \quad \rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}; \quad \rho(h) = \phi \rho(h-1) \text{ for } h \geq 2$$

### Proof

□ Multiply  $Y_t$  by  $Y_{t-1}$  and take expectations:

$$\square \quad \gamma(1) = \phi\gamma(0) + \underbrace{\mathbb{E}[\varepsilon_t Y_{t-1}]}_{=0} + \theta \underbrace{\mathbb{E}[\varepsilon_{t-1} Y_{t-1}]}_{=\sigma^2} = \phi\gamma(0) + \theta\sigma^2$$

□ Divide by  $\gamma(0)$ :  $\rho(1) = \phi + \frac{\theta\sigma^2}{\gamma(0)}$ . Substitute  $\gamma(0)$ :

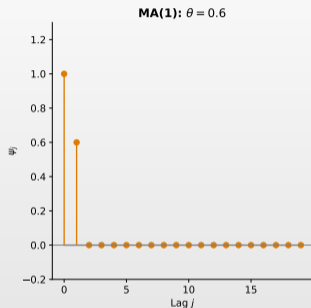
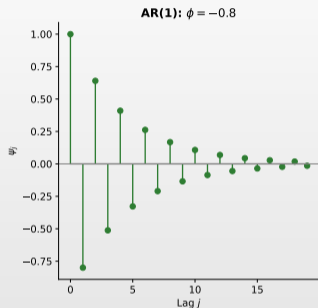
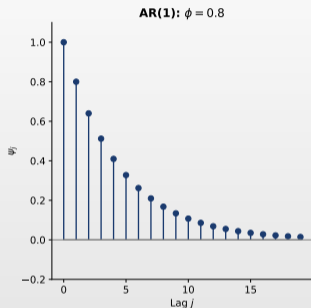
$$\square \quad \rho(1) = \phi + \frac{\theta(1-\phi^2)}{1+2\phi\theta+\theta^2} = \frac{\phi(1+2\phi\theta+\theta^2)+\theta(1-\phi^2)}{1+2\phi\theta+\theta^2}$$

$$\square \quad \text{Numerator: } \phi + \theta + \phi^2\theta + \phi\theta^2 = (\phi + \theta)(1 + \phi\theta), \text{ so } \boxed{\rho(1) = \frac{(1 + \phi\theta)(\phi + \theta)}{1 + 2\phi\theta + \theta^2}}$$

### Recursion

□ For  $h \geq 2$ :  $\gamma(h) = \phi\gamma(h-1)$ , so  $\rho(h) = \phi \rho(h-1) \Rightarrow$  exponential decay from lag 1

## Impulse Response Functions



### Shock Propagation

- ▣ Shows how a unit shock propagates through the system over time
- ▣ **AR**: exponential or oscillating decay; **MA**: effect limited to  $q$  periods

## Stationarity and Invertibility Summary

### Conditions for a Valid ARMA(p,q) Model

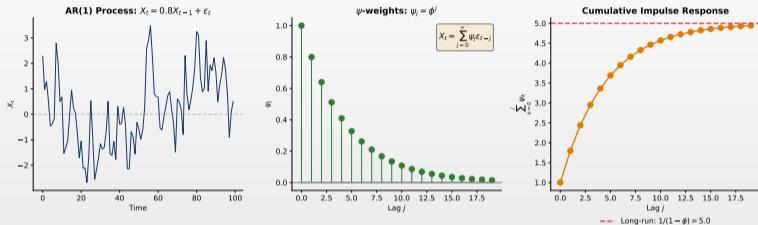
#### ▣ Requirements summary:

Condition	Requirement
Stationarity	Roots of $\phi(z) = 0$ outside the unit circle
Invertibility	Roots of $\theta(z) = 0$ outside the unit circle

### Implications

- ▣ **Stationarity:** Can be written as  $MA(\infty)$ :  $X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- ▣ **Invertibility:** Can be written as  $AR(\infty)$ :  $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$
- ▣ **Causal representation:**  $X_t$  depends only on *past* shocks  $\Rightarrow$  necessary for forecasting

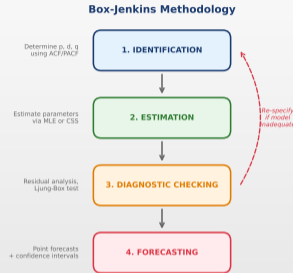
## Wold's Decomposition Theorem



### Wold's Theorem

- Any purely non-deterministic stationary process can be written as  $MA(\infty)$ :
- $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$  with  $\sum \psi_j^2 < \infty$
- Theoretical justification for ARMA modeling

# The Box-Jenkins Methodology



## Iterative Approach

□ Identification  $\Rightarrow$  estimation  $\Rightarrow$  validation; repeat until residuals are white noise

## ACF/PACF Identification Rules

### Theoretical Patterns for Stationary Processes

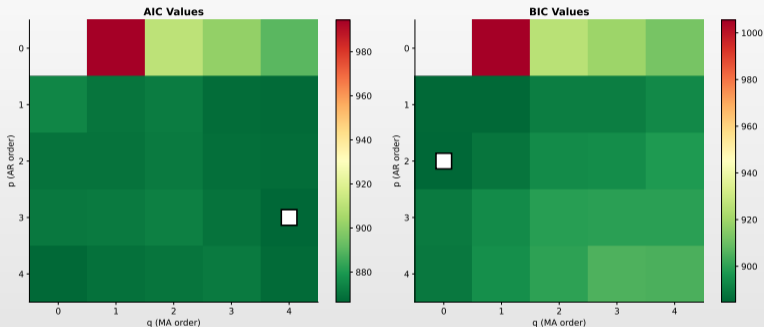
Model	ACF Pattern	PACF Pattern
AR(1)	Exponential decay	Spike at lag 1, then 0
AR(2)	Exp./damped sinusoid	Spikes at lags 1-2, then 0
AR(p)	Gradual decay	Cuts off after lag $p$
MA(1)	Spike at lag 1, then 0	Exponential decay
MA(2)	Spikes at lags 1-2, then 0	Exp./damped sinusoid
MA(q)	Cuts off after lag $q$	Gradual decay
ARMA(p,q)	Decays	Decays

Model Identification: ACF/PACF Patterns

Model	ACF Pattern	PACF Pattern
AR(p)	Exponential decay or damped sinusoid	Cuts off after lag $p$
MA(q)	Cuts off after lag $q$	Exponential decay or damped sinusoid
ARMA(p,q)	Exponential decay after lag $q$	Exponential decay after lag $p$



## AIC vs BIC: Model Selection



### Interpretation

- White square marks the best model; lower values (green) are better

## Information Criteria

### AIC (Akaike)

- ▣  $AIC = -2 \ln(\hat{\mathcal{L}}) + 2k$
- ▣ Moderate penalty
  - ▶ Tends to select larger models
  - ▶ Optimal for forecasting

### BIC (Bayesian)

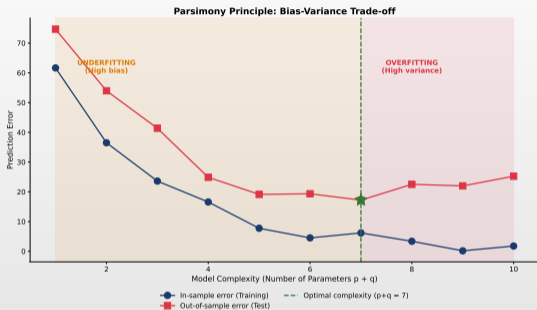
- ▣  $BIC = -2 \ln(\hat{\mathcal{L}}) + k \ln(n)$
- ▣ Stronger penalty
  - ▶ Prefers parsimonious models
  - ▶ Consistent for identification

where:  $\hat{\mathcal{L}}$  = maximum of the likelihood function,  $k$  = number of estimated parameters,  $n$  = sample size

### Rules

- ▣ Lower values = better model. Compare models on the *same data*

## Parsimony Principle: Bias-Variance Trade-off



### Bias-Variance Trade-off

- Too simple model  $\Rightarrow$  high bias (underfitting)
- Too complex model  $\Rightarrow$  high variance (overfitting)
- The optimum lies at the intersection of the two curves

## Automatic Model Selection

### Grid Search Approach

- ▣ Estimate ARMA( $p, q$ ) for  $p = 0, \dots, p_{max}$  and  $q = 0, \dots, q_{max}$
- ▣ Select the model with the lowest AIC or BIC; verify with validation tests

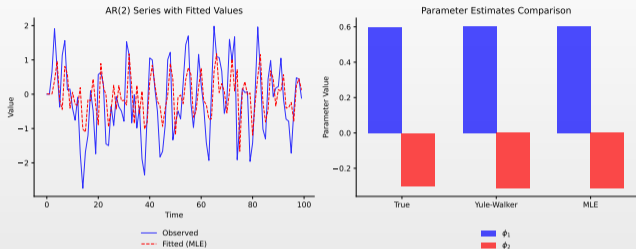
### In Python

- ▣ `pm.auto_arima()` from the `pmdarima` package
- ▣ Automatically tests stationarity, iterates over orders  $(p, q)$ , returns the best model

### Caution

- ▣ Automatic selection is not the final answer  $\Rightarrow$  verify model validity
- ▣ Full Auto-ARIMA (including selection of  $d$ )  $\Rightarrow$  Chapter 3

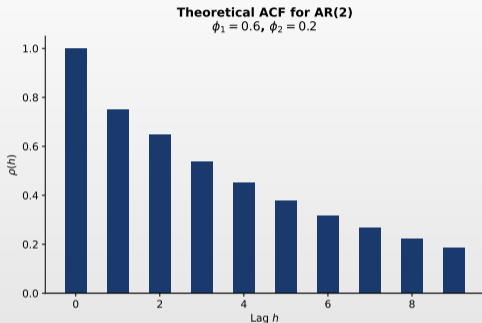
## Estimation Methods



### The Three Main Approaches

- **Yule-Walker**: closed-form, AR only; equates sample autocorrelations with theoretical values
- **MLE**: most efficient and consistent; requires distributional assumption (Gaussian)
- **Conditional Least Squares**: compromise; minimizes sum of squared residuals

## The Yule-Walker Equations for AR(p)



### Yule-Walker Equations

$$\rho(1) = \phi_1 + \phi_2 \rho(1)$$

$$\rho(2) = \phi_1 \rho(1) + \phi_2$$

$$\text{Matrix form: } R \cdot \phi = \rho$$

$R$  = autocorrelation matrix

$$\text{Solution: } \hat{\phi} = R^{-1} \rho$$

### Main Idea

- Linear relationship between autocorrelations and AR parameters
- Allows closed-form estimation (no numerical optimization)

## The Yule-Walker Equations: Matrix Form

### Yule-Walker Equations for AR(p)

$$\square \quad \rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p), \quad k = 1, 2, \dots, p$$

### Matrix Form

$$\square \quad \begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{pmatrix}$$

$\square$  **Estimation:** Replace  $\rho(k)$  with  $\hat{\rho}(k)$ ; the Toeplitz matrix is symmetric and positive definite

## Numerical Example: Yule-Walker for AR(2)

Sample Data ( $T = 100$ )

▣ **Estimated autocorrelations:**  $\hat{\rho}(1) = 0.75$ ,  $\hat{\rho}(2) = 0.65$

▶ Estimated variance:  $\hat{\gamma}(0) = 4.0$

### Step 1: Matrix System

▣ **Yule-Walker:**  $R\hat{\phi} = \rho$

▶ 
$$\begin{pmatrix} 1 & 0.75 \\ 0.75 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.65 \end{pmatrix}$$

### Step 2: Solution (Cramer's Rule)

▣  $\det(R) = 1 - 0.75^2 = 0.4375$

▣  $\hat{\phi}_1 = \frac{0.75 \times 1 - 0.75 \times 0.65}{0.4375} = \frac{0.2625}{0.4375} = \boxed{0.600}$        $\hat{\phi}_2 = \frac{0.65 \times 1 - 0.75 \times 0.75}{0.4375} = \frac{0.0875}{0.4375} = \boxed{0.200}$

### Step 3: Noise Variance

▣  $\hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\phi}_1\hat{\rho}(1) - \hat{\phi}_2\hat{\rho}(2)) = 4.0(1 - 0.45 - 0.13) = \boxed{1.68}$

**Stationarity check:**  $\hat{\phi}_1 + \hat{\phi}_2 = 0.8 < 1 \checkmark$      $|\hat{\phi}_2| = 0.2 < 1 \checkmark$      $\hat{\phi}_2 - \hat{\phi}_1 = -0.4 > -1 \checkmark$

## Proof: The Yule-Walker Equations

Goal: Derive  $\rho(k) = \phi_1\rho(k-1) + \cdots + \phi_p\rho(k-p)$

- Start from AR(p):  $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t$
- Multiply by  $X_{t-k}$  and take expectations:
- $\mathbb{E}[X_t X_{t-k}] = \phi_1 \mathbb{E}[X_{t-1} X_{t-k}] + \cdots + \phi_p \mathbb{E}[X_{t-p} X_{t-k}] + \mathbb{E}[\varepsilon_t X_{t-k}]$
- For  $k \geq 1$ :  $\mathbb{E}[\varepsilon_t X_{t-k}] = 0 \Rightarrow \gamma(k) = \phi_1 \gamma(k-1) + \cdots + \phi_p \gamma(k-p)$
- Dividing by  $\gamma(0)$ :  $\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p)$

### Special Case AR(1)

- $\rho(k) = \phi_1 \rho(k-1) = \phi_1^k$  (using  $\rho(0) = 1$ )

## Maximum Likelihood Estimation

ARMA(p,q) Log-Likelihood (Gaussian errors:  $\varepsilon_t \sim N(0, \sigma^2)$ )

- ▣  $\ell(\phi, \theta, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^n \varepsilon_t^2$
- ▣  $\varepsilon_t$  are the innovations computed recursively

### Estimation Procedure

- ▣ Initialization: use method of moments or OLS for starting values
- ▣ Optimization: numerical methods (BFGS, Newton-Raphson)
- ▣ Iterate until convergence

### In Practice

- ▣ `statsmodels.tsa.arima.model.ARIMA`  $\Rightarrow$  implements exact MLE with automatic initialization

## Standard Errors and Inference

### Asymptotic Distribution of MLE

- ▣  $\hat{\theta} \xrightarrow{d} N(\theta_0, \frac{1}{n}I(\theta_0)^{-1})$ , where  $I(\theta)$  is the **Fisher information matrix**
- ▣  $I(\theta) = -E\left[\frac{\partial^2 \ln L(\theta)}{\partial \theta \partial \theta'}\right] \Rightarrow$  average curvature of the log-likelihood
- ▣ Estimated variance-covariance matrix:  $\hat{V} = \frac{1}{n}\hat{I}^{-1}$

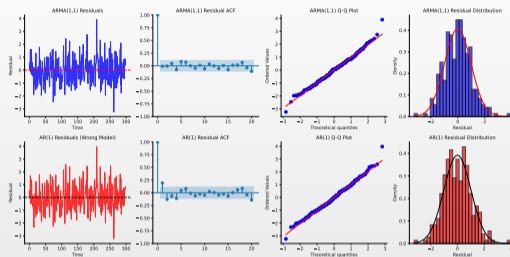
### What is the Standard Error (SE)?

- ▣  $SE(\hat{\theta}_j) = \sqrt{\hat{V}_{jj}} = \sqrt{\text{diag}_j\left(\frac{1}{n}\hat{I}^{-1}\right)} \Rightarrow$  measures estimation uncertainty
- ▣ **Example AR(1):**  $SE(\hat{\phi}) \approx \sqrt{(1 - \hat{\phi}^2)/n}$ ; for  $\hat{\phi} = 0.8$ ,  $n = 100$ :  $SE \approx 0.06$
- ▣ **Interpretation:** small SE  $\Rightarrow$  parameter is estimated with high precision

### Testing Parameter Significance

- ▣  $H_0 : \theta_j = 0$     Statistic:  $z = \frac{\hat{\theta}_j}{SE(\hat{\theta}_j)} \sim N(0, 1)$  asymptotically
- ▣ Reject if  $|z| > 1.96$  at 5%     $\Rightarrow$  **CI:**  $\hat{\theta}_j \pm 1.96 \cdot SE(\hat{\theta}_j)$

## Residual Diagnostics



If the model is correctly specified, residuals must be white noise

- ▣ **Residual plot:** random fluctuations around zero, constant variance
- ▣ **Residual ACF:** no significant spikes  $\Rightarrow$  white noise
- ▣ **Q-Q plot:** points on diagonal  $\Rightarrow$  normal; heavy tails  $\Rightarrow$  non-normal errors

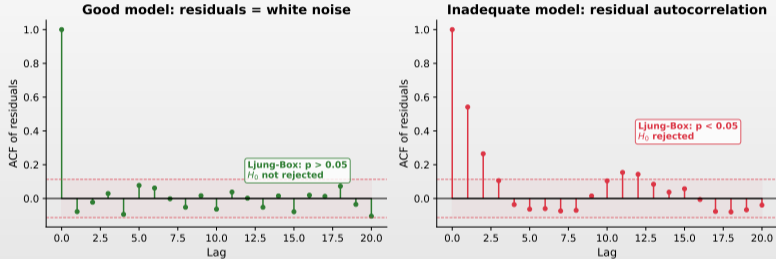
### Decision

- ▣ **✓ All checks OK  $\Rightarrow$  adequate model**
- ✗ Not satisfied  $\Rightarrow$  return to identification**



## The Ljung-Box Test: Visual Illustration

Ljung-Box Test: good model vs inadequate model



### Interpretation

- Left: good model  $\Rightarrow$  white noise residuals
- Right: inadequate model  $\Rightarrow$  residual autocorrelation  $\Rightarrow$  re-specification needed

## The Ljung-Box Test

### Definition 12 (Ljung-Box Test)

- ▣ Tests whether residuals are independently distributed (no autocorrelation)
- ▣ **Statistic:**  $Q(m) = n(n+2) \sum_{k=1}^m \frac{\hat{\rho}_k^2}{n-k}$

### Hypotheses and Distribution

- ▣  $H_0$ : Residuals are white noise;  $H_1$ : Residuals are autocorrelated
- ▣ Under  $H_0$ ,  $Q(m) \sim \chi^2(m-p-q)$  approximately

### Decision

- ▣ **p-value**  $> 0.05 \Rightarrow$  do not reject  $H_0 \Rightarrow$  residuals are white noise
- ▣ **p-value**  $< 0.05 \Rightarrow$  residual autocorrelation  $\Rightarrow$  inadequate model

## Model Checklist

### A Good ARMA Model Should Satisfy

- ▣ **Stationarity:** AR roots outside the unit circle (arroots)
- ▣ **Invertibility:** MA roots outside the unit circle (maroots)
- ▣ **White noise residuals:** No significant ACF (Ljung-Box test)
- ▣ **Normal residuals:** Q-Q plot, Jarque-Bera test
- ▣ **No heteroscedasticity:** Constant variance (ARCH test)
- ▣ **Simple:** Lowest AIC/BIC among adequate models

### If Checks Are Not Satisfied

- ▣ Return to identification, try different orders

## Point Forecasts

Optimal Forecast:  $\hat{X}_{n+h|n} = \mathbb{E}[X_{n+h}|X_n, X_{n-1}, \dots]$

- The conditional expectation minimizes MSE

AR(1):  $X_t = c + \phi X_{t-1} + \varepsilon_t$

- $\hat{X}_{n+1|n} = c + \phi X_n$ ;  $\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu)$
- Forecasts converge to the mean  $\mu$  as  $h \rightarrow \infty$  (mean reversion)

MA(1):  $X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

- $\hat{X}_{n+1|n} = \mu + \theta \varepsilon_n$ ;  $\hat{X}_{n+h|n} = \mu$  for  $h > 1$

## Forecast Uncertainty

### Mean Square Forecast Error (MSFE)

- **Error:**  $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- **MSFE:**  $\text{MSFE}(h) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$ , where  $\psi_j$  are the  $\text{MA}(\infty)$  coefficients

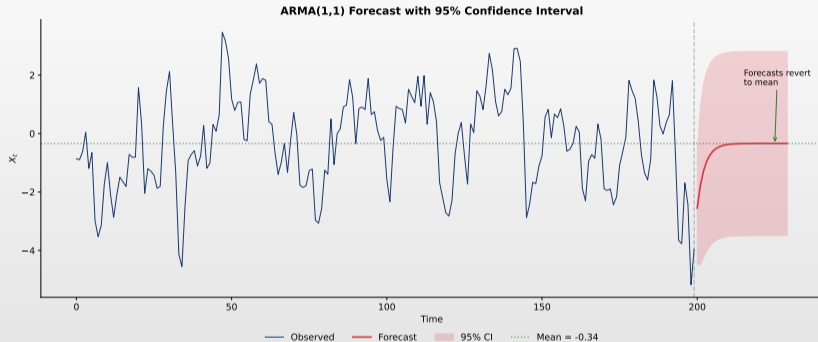
For AR(1):  $\psi_j = \phi^j$

- $\text{MSFE}(h) = \sigma^2 \frac{1-\phi^{2h}}{1-\phi^2} \rightarrow \frac{\sigma^2}{1-\phi^2} = \text{Var}(X_t)$

### Key Observation

- Forecast uncertainty increases with the horizon
- Converges to the unconditional variance  $\text{Var}(X_t)$

## ARMA Forecast with Confidence Intervals



### Observation

- The confidence band widens with the horizon  $\Rightarrow$  convergence to the unconditional interval

## Proof: MSFE for AR(1)

### Claim

$$\square \text{ MSFE}(h) = \sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2} \quad \text{and} \quad \text{MSFE}(\infty) = \gamma(0)$$

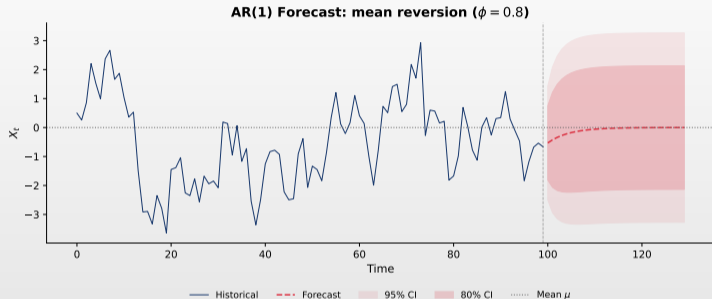
### Proof

- Forecast error at horizon  $h$ :  $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- By recursive substitution:  $e_{n+h|n} = \sum_{j=0}^{h-1} \phi^j \varepsilon_{n+h-j}$
- $\text{MSFE}(h) = \mathbb{E}[e_{n+h|n}^2] = \sigma^2 \sum_{j=0}^{h-1} \phi^{2j} = \boxed{\sigma^2 \frac{1 - \phi^{2h}}{1 - \phi^2}}$
- Limit:  $\text{MSFE}(\infty) = \frac{\sigma^2}{1 - \phi^2} = \gamma(0) \Rightarrow$  forecast converges to unconditional mean

### Interpretation

- At long horizons, we do no better than the unconditional mean:  $\text{CI} \rightarrow 2 \times 1.96 \sqrt{\gamma(0)}$

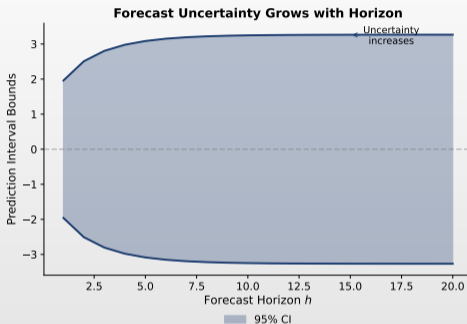
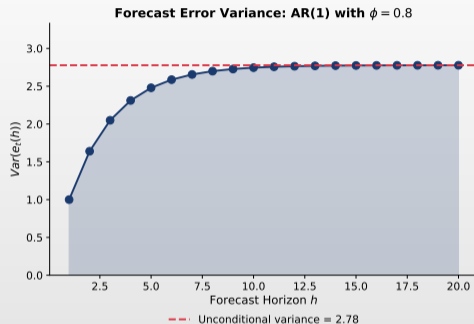
## AR(1) Forecast: Mean Reversion



### Properties

- ▣ Forecasts converge to the unconditional mean  $\mu$  as the horizon increases
- ▣ Larger  $|\phi| \Rightarrow$  slower reversion; CIs widen with the horizon

## Forecast Error Variance by Horizon



### Observation

- MSFE increases monotonically with horizon  $h \Rightarrow$  convergence to  $\text{Var}(X_t)$  (predictability limit)

## Confidence Intervals for Forecasts

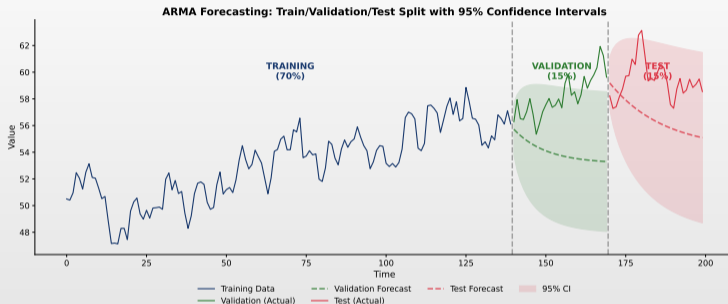
### Formulas

- $X_{n+h}|X_n, \dots \sim N(\hat{X}_{n+h|n}, \text{MSFE}(h))$
- **CI**  $(1 - \alpha)$ :  $\hat{X}_{n+h|n} \pm z_{\alpha/2} \cdot \sqrt{\text{MSFE}(h)}$ , where  $z_{\alpha/2} = 1.96$  for 95%

### Properties

- Intervals widen as the horizon increases
  - ▶ Converge to the unconditional interval:  $\mu \pm z_{\alpha/2}\sigma_X$
- Width depends on model parameters
  - ▶ Larger AR coefficients  $\Rightarrow$  wider intervals
- **Python**: `model.get_forecast(h).conf_int()`

## Train/Validation/Test Forecast Example



### Best Practice

- Always evaluate forecasts on data not used for estimation (train/validation/test split)

## Forecast Evaluation

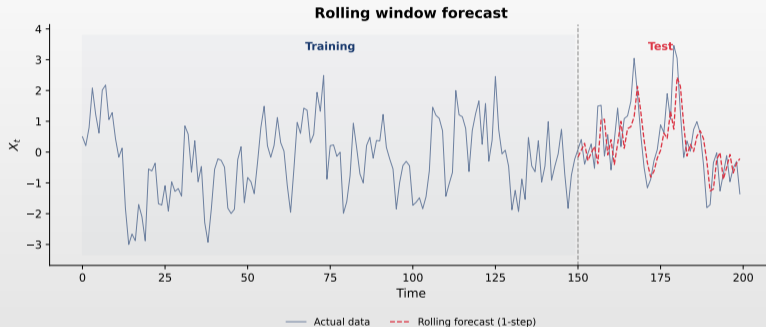
### Out-of-Sample Testing

- Split data: training + test
- Generate forecasts on test
- Compare with actual values
- **Rolling window**: re-estimate as new data arrives

### Error Metrics

- **MAE** =  $\frac{1}{n} \sum |e_t|$ 
  - ▶ Robust to outliers
- **RMSE** =  $\sqrt{\frac{1}{n} \sum e_t^2}$ 
  - ▶ Penalizes large errors
- **MAPE** =  $\frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$ 
  - ▶ Percentage-based, interpretable

## Rolling Window Forecast

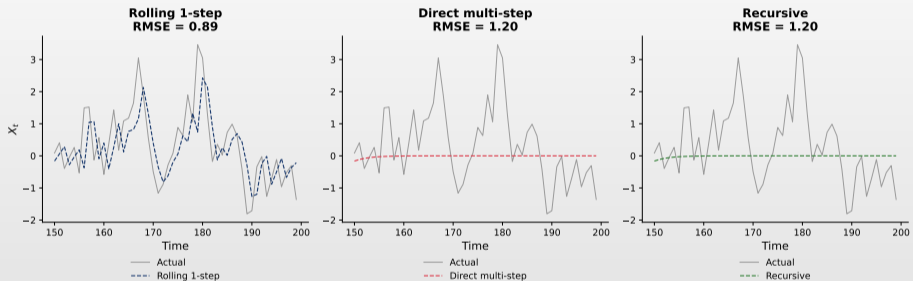


### Rolling Forecast Methodology

- **Fixed window** (last  $w$  obs.) vs **expanding** (all data); generate 1-step forecast, repeat

## Rolling vs Multi-Step Forecast

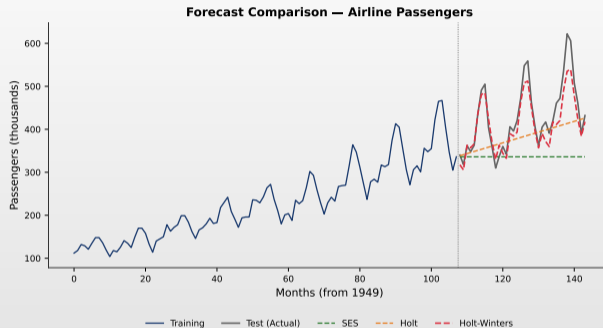
Comparison: Rolling vs Multi-step vs Recursive



### Key Differences

- Rolling 1-step (accurate); Multi-step direct (separate model/horizon); Recursive (error accumulation)

## Real Data Application: Forecast Comparison



### Practical Considerations

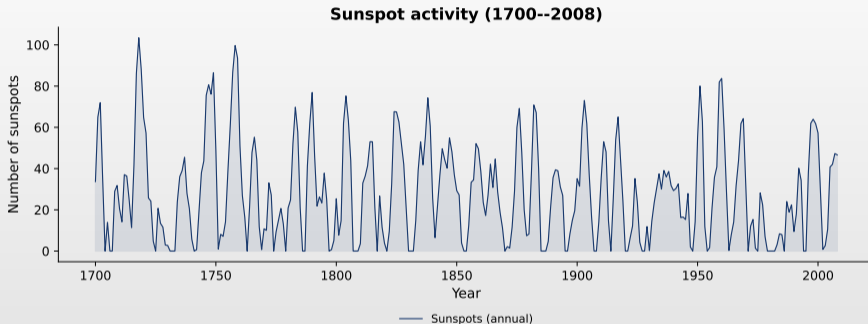
- Real data: non-stationarity, structural breaks; compare models; use rolling window validation

## Workflow Summary

### Box-Jenkins Methodology Steps

- 1. **Data preparation:** Check for missing values, outliers; transform if necessary
- 2. **Stationarity check:** Visual inspection, formal tests (ADF, KPSS); difference if non-stationary
- 3. **Model identification:** ACF/PACF patterns; grid search with information criteria
- 4. **Estimation and validation:** Estimate model, check significance; residual analysis, Ljung-Box test
- 5. **Forecasting:** Point forecasts with confidence intervals; out-of-sample validation

## Case Study: Sunspots

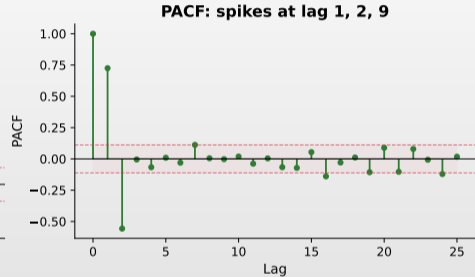
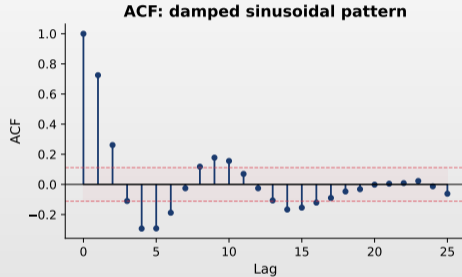


### Data Description

- Annual sunspots (1700–2008): stationary series with  $\sim 11$ -year cycles; Box-Jenkins methodology

## Step 1: ACF/PACF Analysis

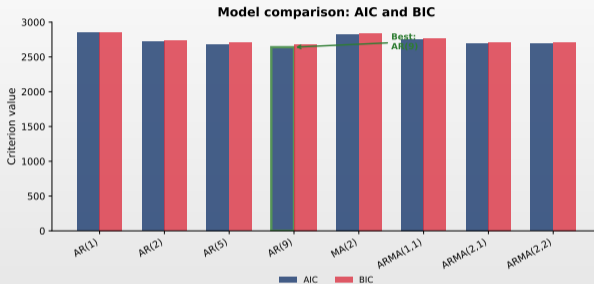
### ACF/PACF analysis for sunspots



### Identification

- Sinusoidal ACF (AR); PACF with spikes at lags 1, 2, 9  $\Rightarrow$  AR(2) or AR(9); stationary series ( $d = 0$ )

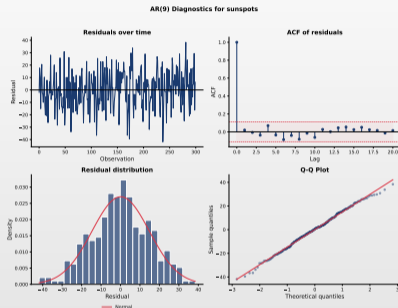
## Step 2: Model Comparison



### Model Selection

- Compare multiple candidate models using AIC; the **AR(9)** has the lowest AIC, capturing the 11-year solar cycle

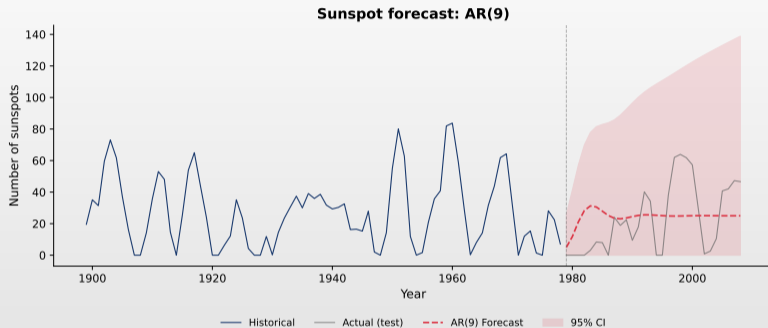
## Step 3: Model Diagnostics



### AR(9) Diagnostics

- Residuals: white noise, zero mean, constant variance, ACF without structure,  $\approx$  normal

## Step 4: Forecasting



### Results

- AR(9) captures the cyclicity; 95% CI covers actual values; RMSE  $\approx 30$

## Key Takeaways

### Chapter Summary

- ▣ **AR( $p$ )**: Depends on  $p$  past values; stationarity: roots outside the unit circle; PACF cuts off at lag  $p$
- ▣ **MA( $q$ )**: Depends on  $q$  past shocks; always stationary; ACF cuts off at lag  $q$
- ▣ **ARMA( $p, q$ )**: Combines AR and MA; both ACF and PACF decay
- ▣ **Box-Jenkins**: Identification  $\Rightarrow$  Estimation  $\Rightarrow$  Validation  $\Rightarrow$  Forecasting
- ▣ **Validation**: Residuals must be white noise
- ▣ **Forecasts**: Converge to the mean; uncertainty increases with the horizon

## Next Chapter Preview

### Chapter 3: ARIMA Models for Non-Stationary Data

- ▣ Non-stationarity: types, unit root tests (ADF, PP, KPSS)
- ▣ Differencing and the difference operator
- ▣ ARIMA(p,d,q): integrated models for non-stationary data
- ▣ The Auto-ARIMA algorithm: automatic model selection
- ▣ Case study: US GDP Forecasting

### Reading

- ▣ Hyndman & Athanasopoulos, *Forecasting: Principles and Practice*, Ch. 9
- ▣ Box, Jenkins, Reinsel & Ljung, *Time Series Analysis*, Ch. 3-4

## AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download monthly US Industrial Production Index from FRED (series INDPRO) for 2010-01 to 2024-12 (180 observations). Compute monthly log-differences (growth rates). Estimate an ARMA model, perform residual diagnostics, and forecast 12 months ahead. Give me complete Python code with plots."

**Exercise:**

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it verify stationarity *before* estimating ARMA? Justify.
3. How does it choose the orders  $p$  and  $q$ ? Does it use ACF/PACF or AIC/BIC?
4. Are residuals tested correctly? (Ljung-Box, Q-Q plot, heteroscedasticity)
5. Do forecast confidence intervals converge to the unconditional mean?

**Warning:** AI-generated code may run without errors and look professional. *That does not mean it is correct.*

## Question 1

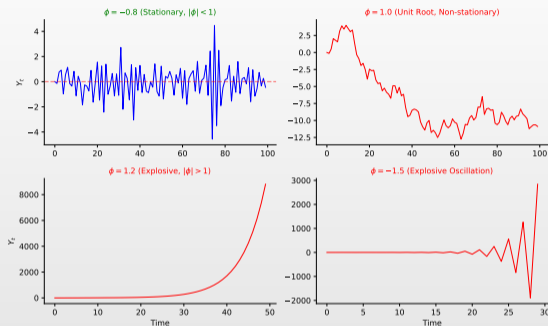
## Question

□ For which value of  $\phi$  is the AR(1) process  $X_t = c + \phi X_{t-1} + \varepsilon_t$  stationary?

## Answer Choices

- (A)  $\phi = 1.2$
- (B)  $\phi = 1.0$
- (C)  $\phi = -0.8$
- (D)  $\phi = -1.5$

## Question 1: Answer



Answer: (C)

- ☐ AR(1) is stationary if and only if  $|\phi| < 1$
- ☐ Only  $|-0.8| = 0.8 < 1$

## Question 2

### Question

☐ You observe: ACF has a spike at lag 1, then cuts off. PACF decays gradually. What model?

### Answer Choices

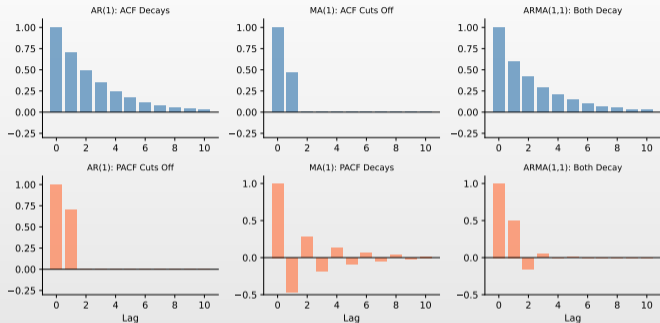
(A) AR(1)

(B) MA(1)

(C) ARMA(1,1)

(D) White noise

## Question 2: Answer



Answer: (B)

- ACF cuts off  $\Rightarrow$  MA process
- PACF decays  $\Rightarrow$  confirms MA(1)

## Question 3

## Question

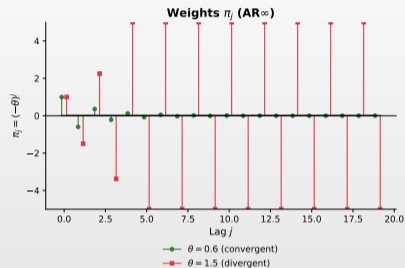
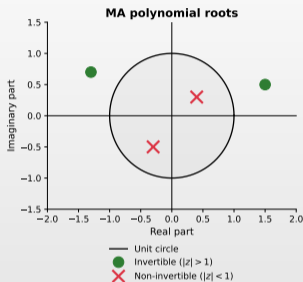
□ Is the MA(1)  $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$  invertible?

## Answer Choices

- (A) Yes, MA processes are always invertible
- (B) Yes, because  $1.5 > 0$
- (C) No, because  $|\theta| = 1.5 > 1$
- (D) No, MA processes are never invertible

## Question 3: Answer

## Invertibility of MA models



Answer: (C)

- Invertibility requires  $|\theta| < 1$
- Here  $|\theta| = 1.5 > 1$ , so it is not invertible

## Question 4

### Question

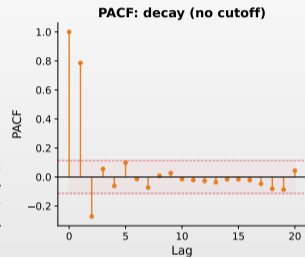
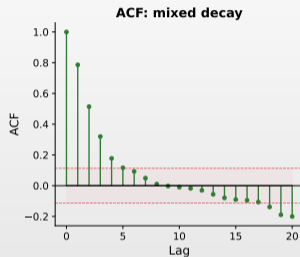
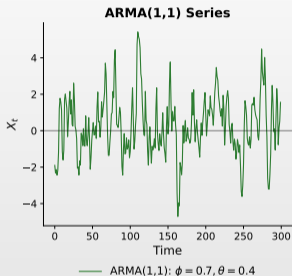
▣ The compact form  $\phi(L)X_t = \theta(L)\varepsilon_t$  represents which model?

### Answer Choices

- (A) Pure AR model
- (B) Pure MA model
- (C) ARMA model
- (D) None of the above

## Question 4: Answer

ARMA(1,1): neither ACF nor PACF cut off



Answer: (C)

□  $\phi(L)$  is the AR polynomial,  $\theta(L)$  is the MA polynomial  $\Rightarrow$  ARMA(p,q)

## Question 5

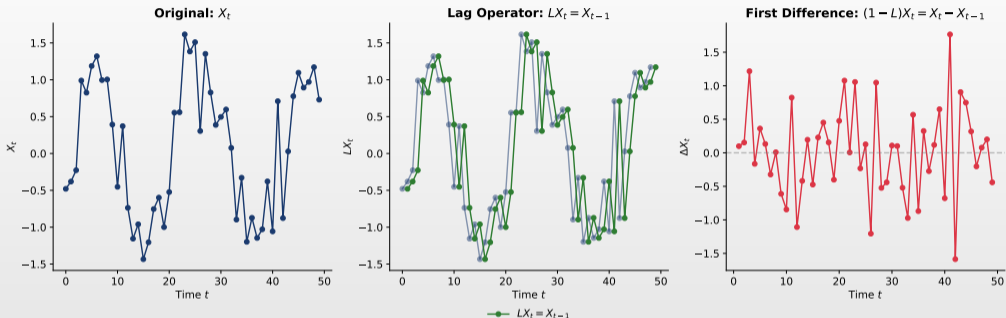
## Question

□ What is  $(1 - L)^2 X_t$ ?

## Answer Choices

- (A)  $X_t - X_{t-1}$
- (B)  $X_t - 2X_{t-1} + X_{t-2}$
- (C)  $X_t + X_{t-1} + X_{t-2}$
- (D)  $X_t - X_{t-2}$

## Question 5: Answer



Answer: (B)

- ☐  $(1 - L)^2 = 1 - 2L + L^2$
- ☐  $(1 - L)^2 X_t = X_t - 2X_{t-1} + X_{t-2}$

## Question 6

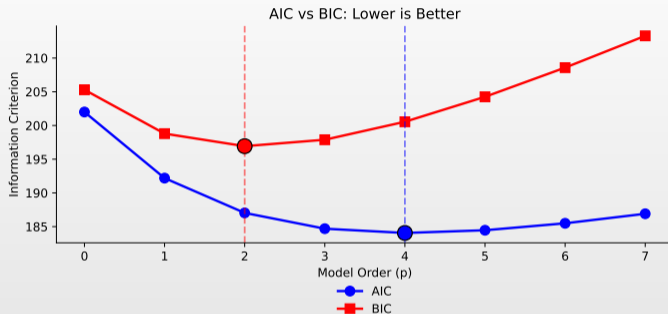
### Question

□ Comparing ARMA(1,1) vs ARMA(2,1) using BIC, which is correct?

### Answer Choices

- (A) Lower BIC always means better forecasts
- (B) BIC penalizes complexity less than AIC
- (C) The model with lower BIC is preferred
- (D) BIC can only compare models with the same number of parameters

## Question 6: Answer



Answer: (C)

- Lower BIC indicates a better balance between estimation quality and complexity
- BIC penalizes complexity *more* than AIC

## Question 7

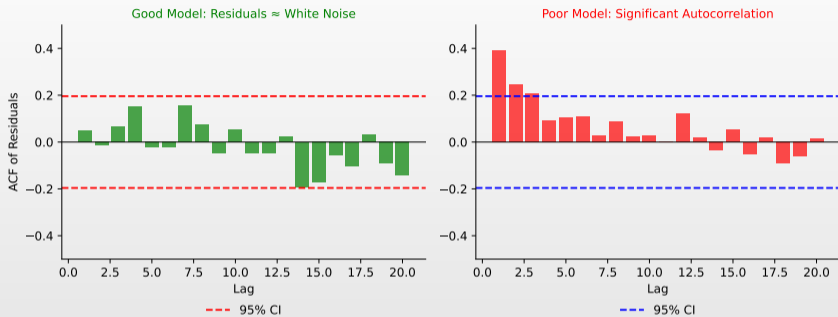
### Question

- ☐ After estimating an ARMA model, you run the Ljung-Box test on residuals and obtain  $p\text{-value} = 0.03$ . What does this mean?

### Answer Choices

- (A) The model is adequate, residuals are white noise
- (B) The model is inadequate, residuals have autocorrelation
- (C) You need to increase the sample size
- (D) The test is inconclusive

## Question 7: Answer



Answer: (B)

- $p\text{-value} < 0.05$  rejects  $H_0$  (white noise)
- Indicates remaining residual autocorrelation

## Question 8

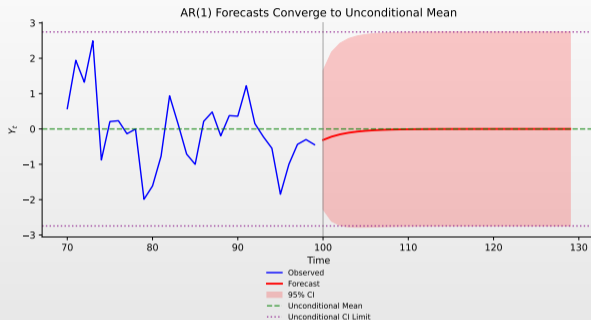
### Question

□ For a stationary AR(1) model, what happens to forecasts as the horizon  $h \rightarrow \infty$ ?

### Answer Choices

- (A) Forecasts increase without bound
- (B) Forecasts oscillate indefinitely
- (C) Forecasts converge to the unconditional mean  $\mu$
- (D) Forecasts become more precise

## Question 8: Answer



Answer: (C)

$$\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu) \rightarrow \mu \text{ as } h \rightarrow \infty \text{ (since } |\phi| < 1)$$

## Question 9

## Question

□ Consider an AR(1) process with  $\phi = 0.6$  and  $\sigma^2 = 4$ . What is  $\text{Var}(X_t)$ ?

## Answer Choices

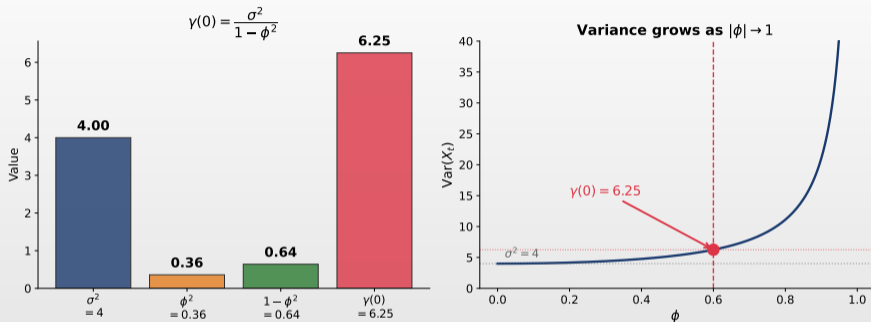
(A) 4.0

(B) 5.56

(C) 6.25

(D) 10.0

## Question 9: Answer



Answer: (C)

- ▣  $\text{Var}(X_t) = \frac{\sigma^2}{1 - \phi^2} = \frac{4}{1 - 0.36} = \frac{4}{0.64} = 6.25$
- ▣ The process variance exceeds  $\sigma^2$  due to persistence

## Question 10

## Question

□ Consider an MA(1) process with  $\theta = 0.5$ . What is  $\rho(1)$ ?

## Answer Choices

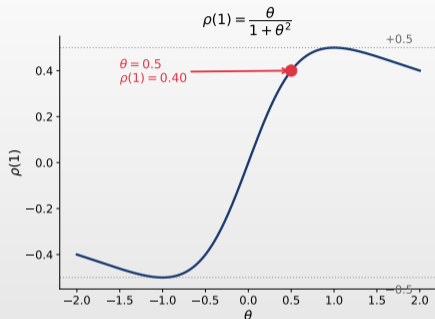
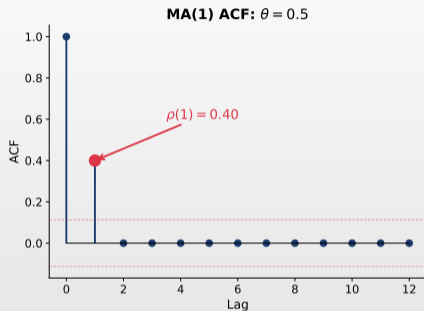
(A) 0.50

(B) 0.40

(C) 0.25

(D) 0.33

## Question 10: Answer



Answer: (B)

- ☐  $\rho(1) = \frac{\theta}{1 + \theta^2} = \frac{0.5}{1 + 0.25} = \frac{0.5}{1.25} = 0.40$
- ☐ Note that  $\rho(1) < \theta$  — the autocorrelation is **always** attenuated

## Question 11

## Question

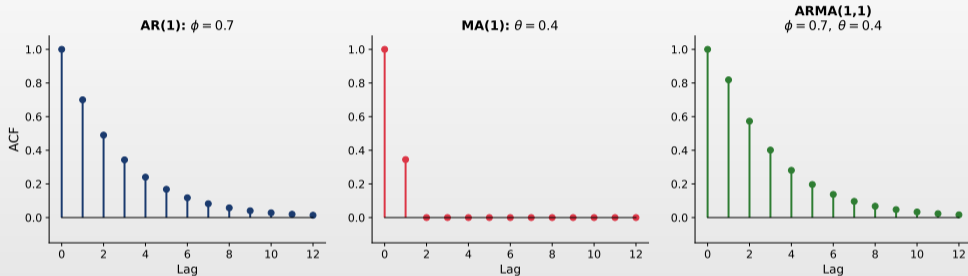
Which statement about the ACF of an ARMA(1,1) process is **true**?

## Answer Choices

- (A) It cuts off after lag 1
- (B) Exponential decay starting from lag 1, with  $\rho(1) \neq \phi$
- (C) It is zero for all lags
- (D) It exactly follows the pattern  $\phi^h$  for all  $h \geq 0$

## Question 11: Answer

## ACF Comparison: AR(1) vs MA(1) vs ARMA(1,1)



Answer: (B)

- ☐  $\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2} \neq \phi$  (the MA component modifies lag 1)
- ☐ For  $h \geq 2$ :  $\rho(h) = \phi \rho(h-1)$  — exponential decay as in AR(1)

## Question 12

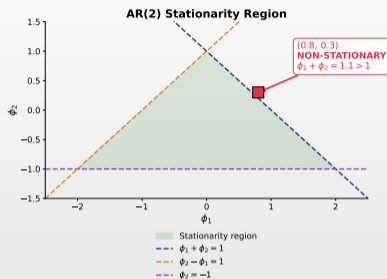
### Question

□ An AR(2) process has  $\phi_1 = 0.8$  and  $\phi_2 = 0.3$ . Is it stationary?

### Answer Choices

- (A) Yes, it is stationary
- (B) No, because  $\phi_1 + \phi_2 = 1.1 > 1$
- (C) Cannot be determined without data
- (D) Depends on the value of  $\sigma^2$

## Question 12: Answer



Answer: (B)

- Necessary conditions for AR(2) stationarity:  $\phi_1 + \phi_2 < 1$ ,  $\phi_2 - \phi_1 < 1$ ,  $|\phi_2| < 1$
- Here  $0.8 + 0.3 = 1.1 > 1 \Rightarrow$  the first condition is violated

## Question 13

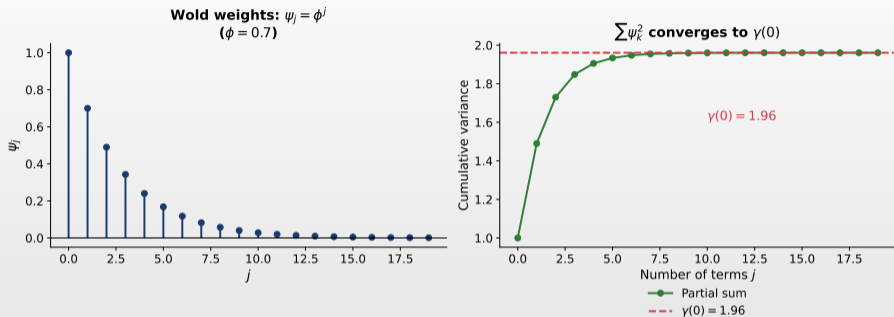
### Question

□ What does the Wold decomposition theorem guarantee?

### Answer Choices

- (A) Any time series is an AR process
- (B) Any stationary process can be written as  $MA(\infty)$ :  $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- (C) Any process has finite variance
- (D) ARMA models are always invertible

## Question 13: Answer



Answer: (B)

- Wold's theorem:  $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + D_t$ , where  $D_t$  is the deterministic component
- This justifies ARMA models: they are parsimonious approximations of  $MA(\infty)$

## Question 14

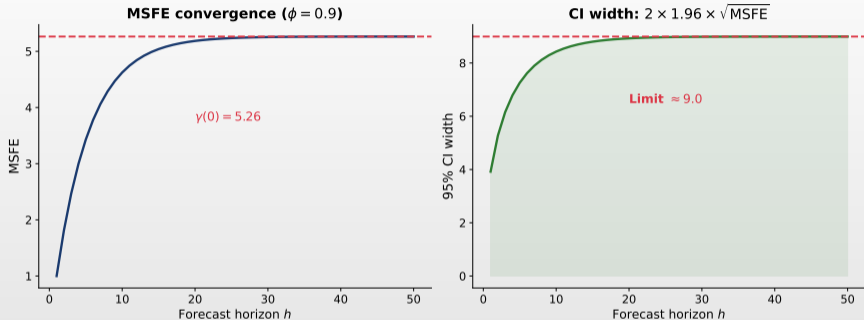
## Question

□ AR(1) with  $\phi = 0.9$ ,  $\sigma^2 = 1$ . What happens to the CI width as  $h \rightarrow \infty$ ?

## Answer Choices

- (A) It remains constant
- (B) It decreases to zero
- (C) It grows toward  $2 \times 1.96 \times \sqrt{1/(1 - 0.81)} \approx 9.0$
- (D) It grows to infinity

## Question 14: Answer



Answer: (C)

- ▣  $\text{MSFE}(\infty) = \frac{\sigma^2}{1-\phi^2} = \frac{1}{1-0.81} = \frac{1}{0.19} \approx 5.26$
- ▣  $\text{CI width} = 2 \times 1.96 \sqrt{5.26} \approx 2 \times 1.96 \times 2.29 \approx 9.0$

## Data Sources and Software

### Software Packages

- ▣ `statsmodels` ⇒ Statistical models for Python, including ARIMA
- ▣ `pmdarima` ⇒ Automatic ARIMA selection for Python
- ▣ `scipy` ⇒ Optimization and statistical functions
- ▣ `numpy`, `pandas` ⇒ Data manipulation
- ▣ `matplotlib` ⇒ Visualization

### Data and Examples

- ▣ Simulated time series for illustrations
- ▣ Examples based on Hyndman & Athanasopoulos (2021)

## Bibliography I

### Fundamental ARMA Works

- Box, G.E.P., & Jenkins, G.M. (1970). *Time Series Analysis: Forecasting and Control*, Holden-Day.
- Akaike, H. (1974). A New Look at the Statistical Model Identification, *IEEE Transactions on Automatic Control*, 19(6), 716–723.
- Schwarz, G. (1978). Estimating the Dimension of a Model, *The Annals of Statistics*, 6(2), 461–464.

### Diagnostics and Validation

- Ljung, G.M., & Box, G.E.P. (1978). On a Measure of Lack of Fit in Time Series Models, *Biometrika*, 65(2), 297–303.
- Brockwell, P.J., & Davis, R.A. (2016). *Introduction to Time Series and Forecasting*, 3rd ed., Springer.

## Bibliography II

### Textbooks and Additional References

- ▣ Hamilton, J.D. (1994). *Time Series Analysis*, Princeton University Press.
- ▣ Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.
- ▣ Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

### Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> – Code platform for quantitative methods
- ▣ **Quantinar**: <https://quantinar.com> – Learning platform for quantitative methods
- ▣ **GitHub TSA**: [https://github.com/QuantLet/TSA/tree/main/TSA\\_ch2](https://github.com/QuantLet/TSA/tree/main/TSA_ch2) – Python code for this chapter

# Thank You!

## Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar