Time Series Analysis and Forecasting

Seminar 2: ARMA Models

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Seminar Outline

Today's Activities:

1. **Review Quiz** — Checking understanding of ARMA concepts
2. **True/False Questions** — Conceptual checks
3. **Practice Problems** — Practice with AR/MA
4. **Worked Examples** — Fitting and diagnostics
5. **Discussion Topics** — Practical applications
6. **AI-Assisted Exercises** — Human vs. AI modeling

Quiz 1: Lag Operator

Question

What is the result of applying $(1 - L)^2$ to X_t?

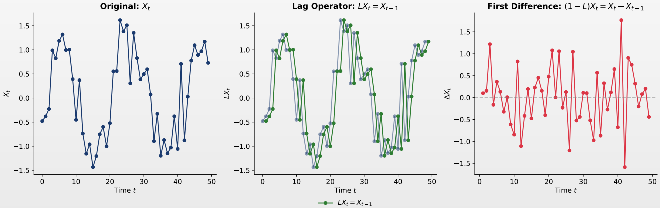
A. $X_t - X_{t-1}$

B. $X_t - 2X_{t-1} + X_{t-2}$

C. $X_t + X_{t-1} + X_{t-2}$

D. $X_t - X_{t-2}$

Quiz 1: Answer



Answer: B

$X_t - 2X_{t-1} + X_{t-2}$

Explanation: $(1-L)^2 X_t = (1 - 2L + L^2)X_t = X_t - 2X_{t-1} + X_{t-2}$ — the **second difference** of X_t.

TSA_ch2_lag_operator

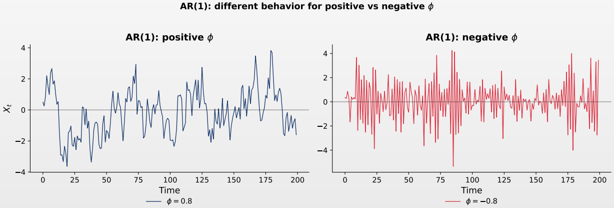
Quiz 2: AR(1) Stationarity

Question

For which value of ϕ is the AR(1) process $X_t = 0.5 + \phi X_{t-1} + \varepsilon_t$ stationary?

A. $\phi = 1.2$
B. $\phi = 1.0$
C. $\phi = -0.8$
D. $\phi = -1.5$

Quiz 2: Answer



Answer: C

$\phi = -0.8$ (Stationary)

AR(1) stationarity condition: $|\phi| < 1$. Checking each option:

- A: $|1.2| = 1.2 > 1$ × B: $|1.0| = 1$ (unit root) × C: $|-0.8| = 0.8 < 1$ ✓ D: $|-1.5| = 1.5 > 1$ ×

TSA_ch2_ar1

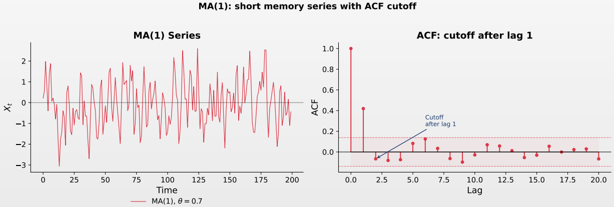
Quiz 3: ACF Pattern

Question

You observe the following ACF pattern: significant spike at lag 1, then all lags within confidence bands. PACF shows gradual decay. What model is suggested?

A. AR(1)
B. MA(1)
C. ARMA(1,1)
D. White noise

Quiz 3: Answer



Answer: B

MA(1)

Key identification rule:

- ACF cuts off after lag $q \Rightarrow$ MA(q); PACF cuts off after lag $p \Rightarrow$ AR(p)

Here: ACF cuts off at lag 1, PACF decays $\Rightarrow$ **MA(1)**

TSA_ch2_ma1

Quiz 4: MA Invertibility

Question

For the MA(1) process $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$, is the process invertible?

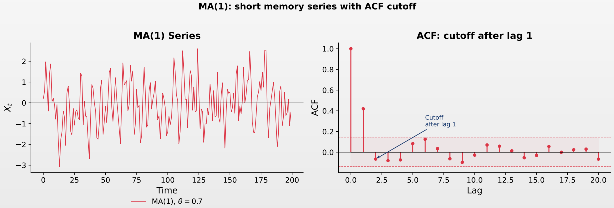
A. Yes, because MA processes are always invertible

B. Yes, because $1.5 > 0$

C. No, because $|\theta| = 1.5 > 1$

D. No, because MA processes are never invertible

Quiz 4: Answer



Answer: C

Not invertible ($|\theta| = 1.5 > 1$)

MA(1) invertibility: Requires $|\theta| < 1$. The root of $\theta(z) = 1 + \theta z = 0$ must be outside the unit circle. Here: $z = -1/1.5 = -0.67$ is **inside**! $\Rightarrow$ **Not invertible**

TSA_ch2_ma1

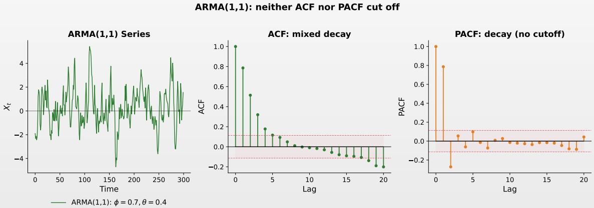
Quiz 5: ARMA Representation

Question

The compact form $\phi(L)X_t = \theta(L)\varepsilon_t$ represents which model?

A. Pure AR model
B. Pure MA model
C. ARMA model
D. None of the above

Quiz 5: Answer



Answer: C

ARMA model

Lag polynomial notation:

- $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p; \quad \theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$
- Special cases: $\theta(L) = 1 \Rightarrow$ Pure AR; $\quad \phi(L) = 1 \Rightarrow$ Pure MA

TSA_ch2_arma

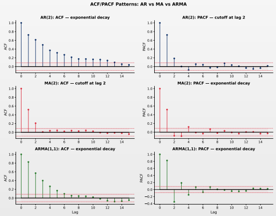
Quiz 6: Information Criteria

Question

When comparing ARMA(1,1) and ARMA(2,1) using BIC, which statement is correct?

A. Lower BIC always means better forecasts
B. BIC penalizes complexity less than AIC
C. The model with lower BIC is preferred
D. BIC can only compare models with the same number of parameters

Quiz 6: Answer



Answer: C

The model with lower BIC is preferred

Information Criteria: $\text{AIC} = -2\ln(\hat{L}) + 2k; \quad \text{BIC} = -2\ln(\hat{L}) + k\ln(n)$

$\hat{L}$ = maximum of the likelihood function, k = number of parameters, n = sample size

BIC penalizes complexity **more** than AIC (for $n > 7$) $\Rightarrow$ BIC favors simpler models.

TSA_ch2_model_selection

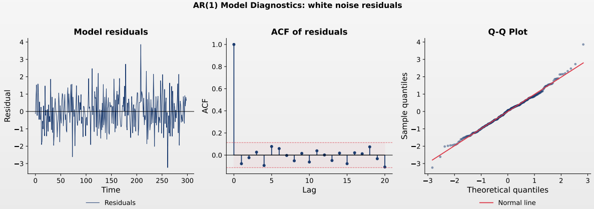
Quiz 7: Ljung-Box Test

Question

After fitting an ARMA(2,1) model, you run the Ljung-Box test on residuals and get p-value $= 0.02$. What do you conclude?

A. The model is adequate
B. Residuals are white noise
C. There is significant autocorrelation in residuals
D. The model has too many parameters

Quiz 7: Answer



Answer: C

Significant autocorrelation in residuals

Ljung-Box test: H_0: Residuals are white noise; H_1: Autocorrelation present.
p-value $= 0.02 < 0.05 \Rightarrow$ **Reject H_0**. The model is **inadequate** — try other orders.

TSA_ch2_diagnostics

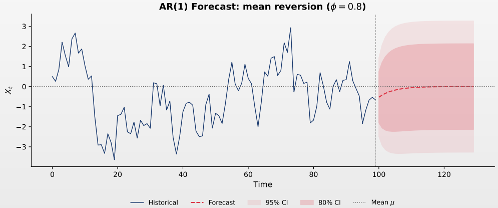
Quiz 8: Forecasting

Question

For an AR(1) model with $\phi = 0.6$ and mean $\mu = 10$, what happens to forecasts as horizon $h \to \infty$?

A. Forecasts grow without bound
B. Forecasts converge to 0
C. Forecasts converge to $\mu = 10$
D. Forecasts oscillate forever

Quiz 8: Answer



Answer: C

Forecasts converge to $\mu = 10$

AR(1) forecast formula: $\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu)$
Since $|\phi| = 0.6 < 1$: $\lim_{h\to\infty} \phi^h = 0 \Rightarrow$ Forecasts converge to μ. **Mean reversion!**

TSA_ch2_forecasting

Quiz 9: AR(2) Roots

Question

An AR(2) process has characteristic roots $z_1 = 0.8$ and $z_2 = -0.5$. Is it stationary?

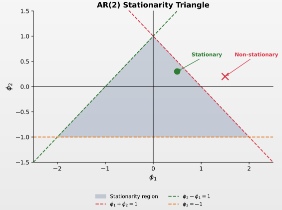
A. Yes, because both roots are inside the unit circle

B. No, because one root is negative

C. No, because roots must be outside the unit circle

D. Cannot determine without more information

Quiz 9: Answer



Answer: C

Roots must be outside the unit circle

Stationarity condition: The roots of $\phi(z) = 0$ must be **outside** the unit circle ($|z| > 1$).
Here: $|z_1| = 0.8 < 1$ $\times$; $|z_2| = 0.5 < 1$ $\times$. Both inside $\Rightarrow$ **Non-stationary**

TSA_ch2_ar2

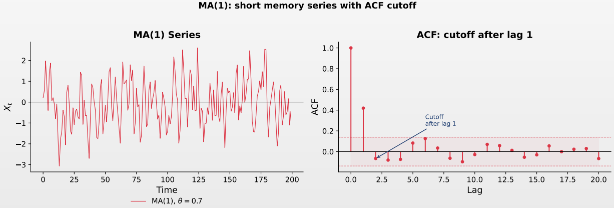
Quiz 10: MA(q) Properties

Question

For an MA(2) process, the ACF:

A. Decays exponentially
B. Cuts off after lag 2
C. Cuts off after lag 1
D. Never cuts off

Quiz 10: Answer



Answer: B

Cuts off after lag 2

ACF property for MA(q): $\rho(h) = 0$ for $h > q$

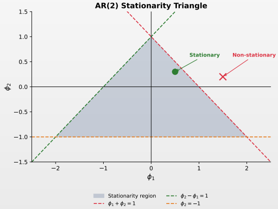
- MA(1): cuts off after lag 1; MA(2): cuts off after lag 2; MA(q): cuts off after lag q — key identification feature!

TSA_ch2_ma1

True or False? — Questions

Statement	T/F?
1. An AR(2) process can exhibit pseudo-cyclical behavior.	?
2. MA processes require a stationarity condition.	?
3. The PACF of an AR(p) process cuts off after lag p.	?
4. If AIC selects ARMA(2,1) and BIC selects ARMA(1,1), they cannot both be correct.	?
5. Confidence intervals narrow as the horizon increases.	?
6. The Yule-Walker equations can be used to estimate MA parameters.	?

True or False? — Answers



1. **TRUE**: AR(2) with complex roots $\Rightarrow$ damped oscillations
2. **FALSE**: MA processes are always stationary; they need *invertibility*
3. **TRUE**: Key identification feature of AR(p)
4. **FALSE**: Both are "correct" for their criteria (AIC: estimation, BIC: parsimony)
5. **FALSE**: CIs *widen* with horizon (more uncertainty)
6. **FALSE**: Yule-Walker is for AR; MA uses MLE

TSA_ch2_ar2

Exercise 1: AR(1) Properties

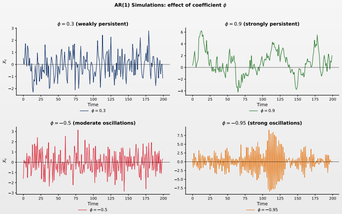
Problem: Consider the AR(1) process:

$$X_t = 2 + 0.7X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, 9)$$

Calculate:

1. The mean μ
2. The variance $\gamma(0)$
3. The autocovariance $\gamma(1)$ and $\gamma(2)$
4. The autocorrelation $\rho(1)$ and $\rho(2)$

Exercise 1: Solution



Given: $c = 2$, $\phi = 0.7$, $\sigma^2 = 9$

1. **Mean:** $\mu = \frac{c}{1-\phi} = \frac{2}{0.3} = \mathbf{6.67}$ 2. **Variance:** $\gamma(0) = \frac{\sigma^2}{1-\phi^2} = \frac{9}{0.51} = \mathbf{17.65}$
3. **Autocovariance:** $\gamma(1) = \phi \cdot \gamma(0) = 0.7 \times 17.65 = \mathbf{12.35}$; $\gamma(2) = \phi^2 \cdot \gamma(0) = 0.49 \times 17.65 = \mathbf{8.65}$
4. **Autocorrelation:** $\rho(1) = \phi = \mathbf{0.7}$, $\rho(2) = \phi^2 = \mathbf{0.49}$

TSA_ch2_ex1_ar1

Exercise 2: MA(1) Properties

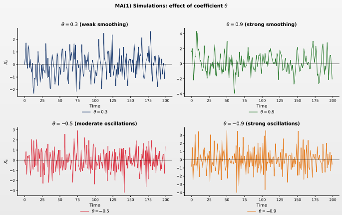
Problem: Consider the MA(1) process:

$$X_t = 5 + \varepsilon_t - 0.4\varepsilon_{t-1}, \quad \varepsilon_t \sim WN(0, 4)$$

Calculate:

1. The mean μ
2. The variance $\gamma(0)$
3. The autocovariance $\gamma(1)$
4. The autocorrelation $\rho(1)$
5. Is this process invertible?

Exercise 2: Solution



Given: $\mu = 5$, $\theta = -0.4$, $\sigma^2 = 4$

1. Mean: $\mathbb{E}[X_t] = \mu = \mathbf{5}$ **2. Variance:** $\gamma(0) = \sigma^2(1+\theta^2) = 4(1.16) = \mathbf{4.64}$

3. Autocovariance: $\gamma(1) = \theta\sigma^2 = -0.4 \times 4 =$ **-1.6** **4. Autocorrelation:** $\rho(1) = \frac{-1.6}{4.64} =$ **-0.345**

5. Invertibility: $|\theta| = 0.4 < 1 \Rightarrow$ **Yes**

TSA_ch2_ex2_ma1

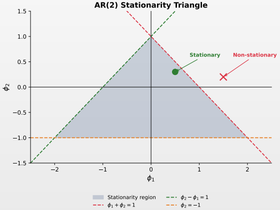
Exercise 3: Characteristic Roots

Problem: Consider the AR(2) process:

$$X_t = 0.5X_{t-1} + 0.3X_{t-2} + \varepsilon_t$$

1. Write the characteristic equation
2. Find the characteristic roots
3. Is this process stationary?

Exercise 3: Solution



1. **Characteristic equation:** $\phi(z) = 1 - 0.5z - 0.3z^2 = 0$, i.e., $0.3z^2 + 0.5z - 1 = 0$
2. **Roots (quadratic formula):** $z = \frac{-0.5 \pm \sqrt{0.25+1.2}}{0.6} \Rightarrow z_1 = \mathbf{1.17}$, $z_2 = \mathbf{-2.84}$
3. **Stationarity check:** $|z_1| = 1.17 > 1$ ✓; $|z_2| = 2.84 > 1$ ✓. Both outside the unit circle $\Rightarrow$ **Stationary**

TSA_ch2_ex3_roots

Exercise 4: Forecasting

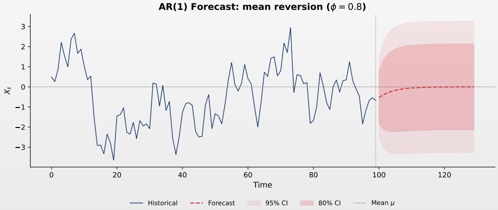
Problem: You have fit an AR(1) model:

$$X_t = 3 + 0.8X_{t-1} + \varepsilon_t, \quad \sigma^2 = 4$$

Given $X_{100} = 20$, calculate:

1. The 1-step ahead forecast $\hat{X}_{101|100}$
2. The 2-step ahead forecast $\hat{X}_{102|100}$
3. The long-run forecast as $h \to \infty$
4. The 95% confidence interval for $\hat{X}_{101|100}$

Exercise 4: Solution



Given: $c = 3$, $\phi = 0.8$, $\sigma^2 = 4$, $X_{100} = 20$. **Mean:** $\mu = \frac{3}{1-0.8} = \mathbf{15}$

1. One-step: $\hat{X}_{101|100} = 3 + 0.8 \times 20 = \mathbf{19}$ **2. Two-step:** $\hat{X}_{102|100} = 3 + 0.8 \times 19 = \mathbf{18.2}$

3. Long-run: $\lim_{h\to\infty} \hat{X}_{100+h|100} = \mu = \mathbf{15}$ **4. 95% CI:** $19 \pm 1.96 \times 2 = \mathbf{[15.08, 22.92]}$

TSA_ch2_ex4_forecast

Python Exercise 1: AR(1) Simulation and Fitting

Task:

1. Simulate 300 observations from an AR(1) with $\phi = 0.6$
2. Plot the series and ACF/PACF
3. Fit an AR(1) model and compare $\hat{\phi}$ vs actual ϕ
4. Examine residual diagnostics

Key code:

```
from statsmodels.tsa.arima.model import ARIMA
model = ARIMA(x, order=(1, 0, 0)).fit()
print(model.summary())
```

TSA_ch2_python_simulate

Python Exercise 2: Model Selection

Task:

1. Load a time series and check stationarity (ADF test)
2. Compare AIC/BIC for AR(1), MA(1), ARMA(1,1), ARMA(2,1)
3. Select the best model
4. Generate forecasts with confidence intervals

Key functions:

- `adfuller(x)` for stationarity test
- `model.aic`, `model.bic` for criteria
- `model.get_forecast(h)` for predictions

TSA_ch2_python_selection

Python Exercise 3: Diagnostic Checking

Task: After fitting a model, perform complete diagnostics:

1. Plot residuals over time
2. Plot the ACF of residuals
3. Create a Q-Q plot for normality
4. Run the Ljung-Box test

Key functions:

- `model.resid` for residuals
- `plot_acf(resid)` for ACF plot
- `stats.probplot(resid)` for Q-Q plot
- `acorr_ljungbox(resid, lags=[10])` for test

TSA_ch2_python_diagnostics

Discussion 1: Model Selection

Scenario: You are modeling monthly inflation rates. After checking stationarity:

- ACF: significant at lags 1, 2, 3, then decays
- PACF: significant at lags 1, 2, then cuts off
- AIC selects ARMA(2,3)
- BIC selects AR(2)

Questions:

1. What does the ACF/PACF pattern suggest?
2. Why do AIC and BIC disagree?
3. Which model would you choose and why?
4. What additional checks would you perform?

Discussion 2: Forecast Evaluation

Scenario: You fit an ARMA(1,1) model to daily stock returns. The in-sample fit looks good (Ljung-Box p = 0.45), but out-of-sample RMSE is worse than a random walk.

Questions:

1. Is this surprising? Why or why not?
2. What does this tell us about return predictability?
3. Should you conclude that the ARMA model is useless?
4. What alternatives might you consider?

Hint: Think about the Efficient Market Hypothesis and what ARMA captures vs. volatility clustering.

AI in ARMA Modeling

Context

AI tools can fit ARMA models and generate diagnostics automatically. The critical skill is **evaluating whether the methodology is correct**.

Key questions to ask about any AI-generated ARMA analysis:

1. Did it check stationarity **before** fitting?
2. Is the model order justified by ACF/PACF?
3. Are residuals white noise (Ljung-Box test)?
4. Are the roots inside the unit circle?
5. Is the forecast horizon reasonable for the model?

AI Exercise 1: Critique an AI Analysis

Scenario

You asked an AI: "Fit the best model to this sunspot data." It returned:

- Fitted ARMA(4,3) with AIC = 2415.3
- No stationarity test performed
- Ljung-Box p-value = 0.03 (reported as "acceptable")
- 50-year forecast with tight confidence intervals

Your critique:

1. Is ARMA(4,3) over-parameterized? What would BIC suggest?
2. Why is Ljung-Box p = 0.03 **not** acceptable at 5% level?
3. Are 50-year forecasts reliable for ARMA models? Why/why not?
4. What is the correct Box-Jenkins methodology that was skipped?

AI Exercise 2: Prompt Refinement for ARMA

Task

Iteratively improve prompts for fitting an AR model to sunspot data.

Round 1 (vague): *"Fit a time series model to sunspots"*

- What did the AI produce? What's missing?

Round 2 (better): *"Test stationarity with ADF, examine ACF/PACF, fit AR(p) using BIC, check residuals with Ljung-Box"*

- Did the AI follow the Box-Jenkins methodology?

Round 3 (expert): *"Follow Box-Jenkins: (1) plot & test stationarity, (2) identify order from ACF/PACF, (3) estimate AR(2), (4) Ljung-Box on residuals, (5) forecast 20 steps with 95% CI"*

- Compare results across all three rounds

AI Exercise 3: Model Selection Competition

Task

Download monthly unemployment data from `statsmodels.datasets`.

Your approach (manual):

- ACF/PACF analysis → candidate models
- Compare AIC/BIC across AR(1), AR(2), MA(1), ARMA(1,1)
- Residual diagnostics for selected model
- Rolling 1-step forecast on last 20 observations

AI approach:

- Ask AI to "find the best ARMA model and forecast"

Compare:

- Which model did each select? Do they agree?
- Compare out-of-sample RMSE
- Did the AI use proper rolling forecasts or just multi-step?
- **Submit:** 1-page reflection on AI strengths and weaknesses

Key Formulas Summary

Concept	Formula
AR(1) mean	$\mu = c/(1-\phi)$
AR(1) variance	$\gamma(0) = \sigma^2/(1-\phi^2)$
AR(1) ACF	$\rho(h) = \phi^h$
AR(1) stationarity	$\|\phi\| < 1$
MA(1) variance	$\gamma(0) = \sigma^2(1+\theta^2)$
MA(1) ACF	$\rho(1) = \theta/(1+\theta^2)$, $\rho(h) = 0$ for $h > 1$
MA(1) invertibility	$\|\theta\| < 1$
AR(1) forecast	$\hat{X}_{n+h\|n} = \mu + \phi^h(X_n - \mu)$
Forecast CI	$\hat{X} \pm z_{\alpha/2} \times \sqrt{\text{MSFE}(h)}$
AIC	$-2\ln(\hat{L}) + 2k$
BIC	$-2\ln(\hat{L}) + k\ln(n)$

Notation: $\hat{L}$ = maximum of the likelihood function, k = no. of parameters, n = sample size, c = constant, σ^2 = white noise variance

Questions?

Good luck with the exercises!

Next Seminar: ARIMA and Seasonal Models

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Online resources and code

- **Quantlet**: `https://quantlet.com` — Code repository for statistics
- **Quantinar**: `https://quantinar.com` — Learning platform for quantitative methods
- **GitHub TSA**: `https://github.com/QuantLet/TSA/tree/main/TSA_ch2` — Python code for this seminar