Chapter 5: VAR & Granger Causality

Seminar

Seminar Outline

1. Review Quiz
2. True/False Questions
3. Practice Problems
4. Worked Examples
5. Real Data Analysis
6. Discussion Topics
7. Exercises for Self-Study

Quiz 1: VAR Definition

Question

In a VAR(2) model with 3 variables, how many coefficient matrices $\mathbf{A}_i$ are there?

- A) 2
- B) 3
- C) 6
- D) 9

Answer on next slide...

Quiz 1: Answer

Answer: A – 2 coefficient matrices

VAR(p) model: $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$

VAR(2) with $K = 3$:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \\ Y_{3t} \end{pmatrix} = \mathbf{c} + \underbrace{\mathbf{A}_1}_{3\times 3} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{pmatrix} + \underbrace{\mathbf{A}_2}_{3\times 3} \begin{pmatrix} Y_{1,t-2} \\ Y_{2,t-2} \\ Y_{3,t-2} \end{pmatrix} + \boldsymbol{\varepsilon}_t$$

Key: p = number of lags = number of matrices

Quiz 2: Number of Parameters

Question

A VAR(2) with $K = 3$ variables (including constants) has how many parameters to estimate per equation?

A) 3
B) 6
C) 7
D) 9

Answer on next slide...

Quiz 2: Answer

Answer: C – 7 parameters per equation

VAR(p) Parameter Count: The Curse of Dimensionality

Parameters per equation: $1 + K \times p$

Total parameters: $K(1 + Kp) + K(K + 1)/2$

(coefficients + covariance matrix)

Model	Coefficients	Total
K=2, p=1	2(1+2×1) = 6	**+ 3 = 9**
K=3, p=2	3(1+3×2) = 21	**+ 6 = 27**
K=5, p=4	5(1+5×4) = 105	**+ 15 = 120**
K=10, p=4	10(1+10×4) = 410	**+ 55 = 465**

Warning: Parameters grow as $K^2 \times p$ — need lots of data!

Formula: Per equation $= 1 + K \times p = 1 + 3 \times 2 = 7$. **Total**: $K(1 + Kp) = 3(1 + 6) = 21$ parameters

Quiz 3: Granger Causality

Question

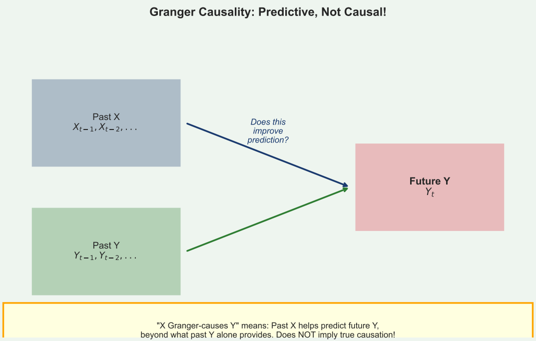
"X Granger-causes Y" means:

- A) X is the economic cause of Y
- B) Past X helps predict future Y
- C) X and Y are contemporaneously correlated
- D) X always increases when Y increases

Answer on next slide...

Quiz 3: Answer

Answer: B – Past X helps predict future Y



Key: Predictive relationship, NOT true causation!

Quiz 4: Granger Causality Test

Question

To test if Y_2 Granger-causes Y_1 in a VAR(p), we test:

- A) All coefficients in the Y_1 equation equal zero
- B) Coefficients on lagged Y_2 in the Y_1 equation equal zero
- C) Coefficients on lagged Y_1 in the Y_2 equation equal zero
- D) The error covariance equals zero

Answer on next slide...

Quiz 4: Answer

Answer: B – Coefficients on lagged Y_2 in Y_1 equation $= 0$

Null hypothesis: $H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$

Test statistic: Wald or F-test with p restrictions

Interpretation:

- Reject H_0: Y_2 Granger-causes Y_1
- Don't reject: No evidence of predictive relationship

Note: Test $Y_1 \to Y_2$ separately (different coefficients in Y_2 equation)

Quiz 5: VAR Stability

Question

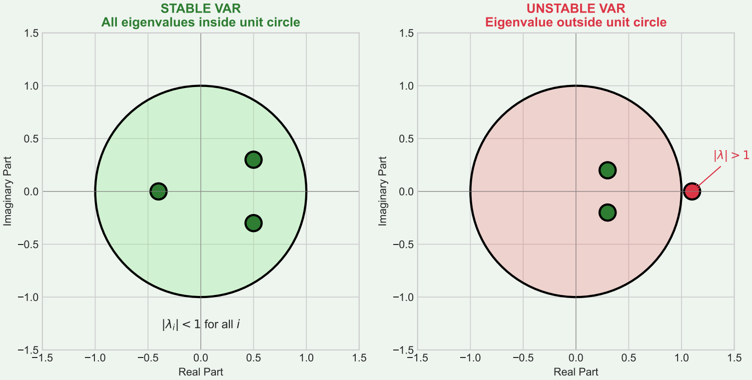
A VAR(1) model is stable (stationary) if:

- A) All diagonal elements of $\mathbf{A}_1$ are less than 1
- B) The determinant of $\mathbf{A}_1$ is less than 1
- C) All eigenvalues of $\mathbf{A}_1$ are less than 1 in absolute value
- D) The trace of $\mathbf{A}_1$ equals zero

Answer on next slide...

Quiz 5: Answer

Answer: C – All eigenvalues of $\mathbf{A}_1$ inside unit circle



Stable: All $|\lambda_i| < 1$ (inside unit circle) $\Rightarrow$ shocks die out over time

Quiz 6: Impulse Response Functions

Question

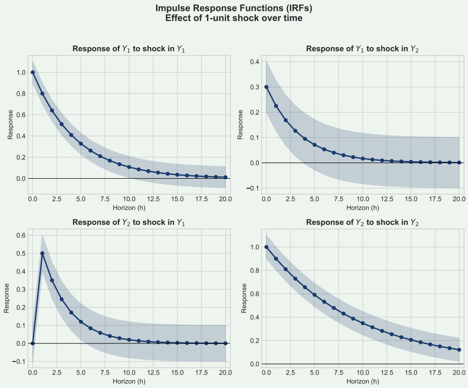
An impulse response function shows:

- A) The correlation between two variables
- B) The effect of a shock to one variable on all variables over time
- C) The forecast accuracy of the model
- D) The p-values of coefficient tests

Answer on next slide...

Quiz 6: Answer

Answer: B – Effect of shock on all variables over time



$\mathbf{IRF}_{ij}(h)$: Response of variable i at horizon h to shock in variable j

Quiz 7: Lag Order Selection

Question

Which criterion typically selects the most parsimonious VAR model?

- A) AIC (Akaike Information Criterion)
- B) BIC (Bayesian Information Criterion)
- C) FPE (Final Prediction Error)
- D) Adjusted R^2

Answer on next slide...

Quiz 7: Answer

Answer: B – BIC (Bayesian Information Criterion)

Penalty comparison (for k parameters, n observations):

- AIC: $-2 \ln L + 2k$
- BIC: $-2 \ln L + k \ln n$

Since $\ln n > 2$ for $n > 8$, BIC penalizes complexity more heavily

Practical guidance:

- Forecasting: AIC may perform better
- Inference/parsimony: BIC preferred
- Large samples: BIC consistent, AIC tends to overfit

Quiz 8: Granger Causality Interpretation

Question

"X Granger-causes Y" means:

- A) X is the true cause of Y
- B) Past values of X help predict Y beyond Y's own past
- C) X and Y are correlated
- D) Y depends only on X

Answer on next slide...

Quiz 8: Answer

Answer: B

Granger causality is about **predictive** content, not true causation. X Granger-causes Y if lagged X terms are jointly significant in the equation for Y, after controlling for lagged Y.

Quiz 9: Forecast Error Variance Decomposition

Question

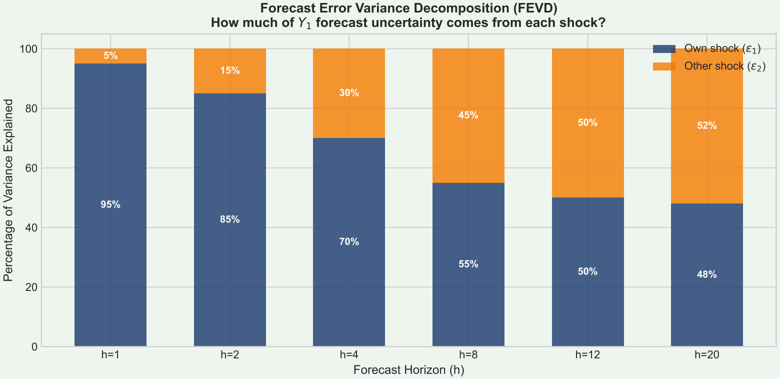
FEVD (Forecast Error Variance Decomposition) tells us:

- A) The correlation between variables
- B) What proportion of forecast error variance comes from each shock
- C) The optimal forecast horizon
- D) Which variables to include in the model

Answer on next slide...

Quiz 9: Answer

Answer: B – Proportion of forecast error variance from each shock



FEVD: Shows how much forecast uncertainty comes from each shock at different horizons

Quiz 10: Structural vs Reduced Form VAR

Question

The difference between structural VAR (SVAR) and reduced-form VAR is:

- A) SVAR has more variables
- B) SVAR allows contemporaneous effects between variables
- C) SVAR uses different estimation methods
- D) There is no difference

Answer on next slide...

Quiz 10: Answer

Answer: B

Reduced-form VAR: shocks are correlated, no contemporaneous effects in equations. SVAR: imposes identifying restrictions to recover structural shocks with economic interpretation (e.g., monetary policy shock).

Quiz 11: Cholesky Decomposition

Question

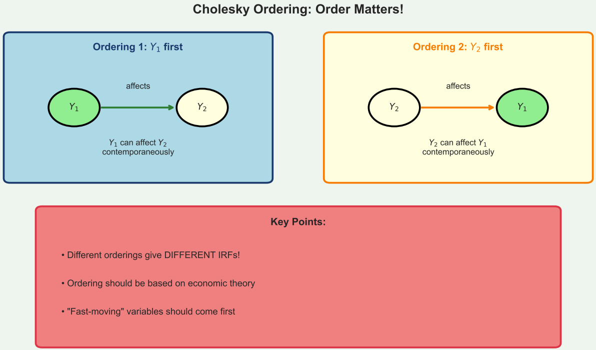
Cholesky ordering in IRF analysis assumes:

- A) All variables are equally important
- B) Variables ordered first affect later variables contemporaneously, not vice versa
- C) Shocks are uncorrelated
- D) No restrictions are needed

Answer on next slide...

Quiz 11: Answer

Answer: B – Variables ordered first affect later ones contemporaneously



Cholesky: Recursive structure. Ordering matters – justify by economic theory (most exogenous first)!

Quiz 12: VAR Residual Diagnostics

Question

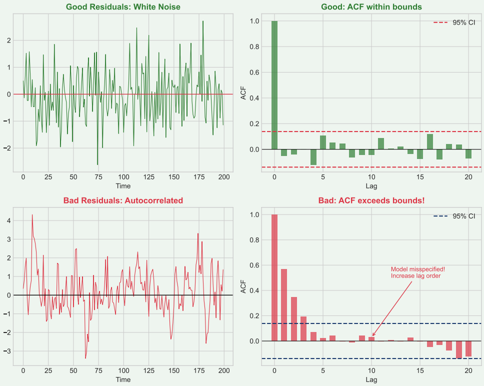
In a well-specified VAR, residuals should be:

- A) Autocorrelated but homoskedastic
- B) White noise (no autocorrelation)
- C) Normally distributed only
- D) Correlated across equations

Answer on next slide...

Quiz 12: Answer

Answer: B – White noise (no autocorrelation)



Diagnostics: Residuals should be white noise. Use Portmanteau/LM test. Cross-equation correlation allowed (Σ_u).

Quiz 13: Cointegration and VAR

Question

If variables are I(1) and cointegrated, you should use:

- A) VAR in levels
- B) VAR in first differences
- C) Vector Error Correction Model (VECM)
- D) Univariate ARIMA models

Answer on next slide...

Quiz 13: Answer

Answer: C

With cointegration, VAR in differences loses long-run information, while VAR in levels may be inefficient. VECM incorporates both short-run dynamics and long-run equilibrium relationships through the error correction term.

Quiz 14: Instantaneous Causality

Question

Instantaneous causality differs from Granger causality because it tests:

- A) Lagged relationships only
- B) Contemporaneous correlation of residuals
- C) Long-run relationships
- D) Model stability

Answer on next slide...

Quiz 14: Answer

Answer: B

Instantaneous causality tests whether shocks to X and Y are correlated within the same period (correlation of VAR residuals). Granger causality tests whether *lagged* values help predict.

True/False Questions

Determine if each statement is True or False:

1. VAR models treat all variables as endogenous.
2. Granger causality proves true economic causation.
3. A stable VAR always has eigenvalues inside the unit circle.
4. FEVD results depend on the ordering of variables.
5. VAR can be estimated by OLS equation by equation.
6. Impulse responses eventually die out in a stable VAR.

Answers on next slide...

True/False: Solutions

1. VAR models treat all variables as endogenous. **TRUE**
   Each variable is regressed on lags of all variables, including itself.
2. Granger causality proves true economic causation. **FALSE**
   It only shows predictive content, not structural causation.
3. A stable VAR always has eigenvalues inside the unit circle. **TRUE**
   Stability condition: all eigenvalues of companion matrix satisfy $|\lambda_i| < 1$.
4. FEVD results depend on the ordering of variables. **TRUE**
   Under Cholesky identification, different orderings give different results.
5. VAR can be estimated by OLS equation by equation. **TRUE**
   With same regressors in each equation, OLS $=$ GLS $=$ ML (under normality).
6. Impulse responses eventually die out in a stable VAR. **TRUE**
   Stability ensures shocks have transitory effects; IRFs $\to 0$ as $h \to \infty$.

Problem 1: Writing VAR Equations

Exercise

Write out the two equations for a bivariate VAR(1) model with variables Y_t (GDP growth) and X_t (inflation).

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Solution

$$Y_t = c_1 + a_{11} Y_{t-1} + a_{12} X_{t-1} + \varepsilon_{1t}$$

$$X_t = c_2 + a_{21} Y_{t-1} + a_{22} X_{t-1} + \varepsilon_{2t}$$

Interpretation:

- a_{12}: Effect of past inflation on current GDP growth
- a_{21}: Effect of past GDP growth on current inflation

Problem 2: Parameter Count

Exercise

How many total parameters need to be estimated in a VAR(3) with $K = 4$ variables (including constants)?

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Solution

Per equation: $1 + K \times p = 1 + 4 \times 3 = 13$ parameters

Total for $K = 4$ equations: $4 \times 13 = \mathbf{52}$ parameters

Plus covariance matrix $\boldsymbol{\Sigma}$: $K(K+1)/2 = 4 \times 5/2 = 10$ unique elements

Grand total: 62 parameters

This is why VARs can be "over-parameterized" with limited data!

Problem 3: Granger Causality Interpretation

Exercise

A Granger causality test yields:

- H_0: Money does not Granger-cause GDP. p-value $= 0.02$
- H_0: GDP does not Granger-cause Money. p-value $= 0.35$

Interpret these results.

Problem 3: Granger Causality Interpretation

Exercise

A Granger causality test yields:

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- H_0: GDP does not Granger-cause Money. p-value $= 0.35$

Interpret these results.

Solution

- **Reject** H_0 at 5%: Money **Granger-causes** GDP
- **Fail to reject** H_0: GDP does **not** Granger-cause Money

Conclusion: Unidirectional causality: Money $\rightarrow$ GDP

Interpretation: Past money supply helps predict GDP growth. This is consistent with monetarist views, but remember: Granger causality $\neq$ structural causality!

Problem 4: Stability Check

Exercise

For VAR(1) with $\mathbf{A}_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.1 & 0.5 \end{pmatrix}$, check stability.

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Solution

Find eigenvalues: $\det(\mathbf{A}_1 - \lambda \mathbf{I}) = 0$

$(0.7 - \lambda)(0.5 - \lambda) - (0.2)(0.1) = 0$

$\lambda^2 - 1.2\lambda + 0.33 = 0$

$\lambda = \frac{1.2 \pm \sqrt{1.44 - 1.32}}{2} = \frac{1.2 \pm 0.346}{2}$

$\lambda_1 = 0.773, \quad \lambda_2 = 0.427$

Both $|\lambda_i| < 1 \Rightarrow$ **Stable!**

Problem 5: IRF Computation

Exercise

For VAR(1) with $\mathbf{A} = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$, compute $\mathbf{\Phi}_2$ (response at $h = 2$).

Problem 5: IRF Computation

Exercise

For VAR(1) with $\mathbf{A} = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$, compute $\mathbf{\Phi}_2$ (response at $h = 2$).

Solution

$$\mathbf{\Phi}_2 = \mathbf{A}^2 = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 0.25 + 0 & 0.10 + 0.12 \\ 0 + 0 & 0 + 0.36 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.22 \\ 0 & 0.36 \end{pmatrix}$$

Interpretation: A unit shock to Y_2 at t increases Y_1 by 0.22 at $t + 2$.

Example: Stock Returns and Trading Volume

Scenario

Daily data on stock returns (R_t) and trading volume (V_t). Test Granger causality both directions.

Typical Findings in Finance Literature

- Returns often Granger-cause volume (price changes trigger trading)
- Volume sometimes Granger-causes returns (volume as leading indicator)
- Results: Often **bidirectional** causality $R \leftrightarrow V$

Practical Issue

Stock returns are typically stationary, but volume may need transformation (log or difference).

Example: Interest Rates and Inflation

Taylor Rule Context

Central banks set interest rates (i_t) in response to inflation (π_t): $i_t = r^* + \pi^* + 1.5(\pi_t - \pi^*) + 0.5(y_t - y^*)$

VAR Analysis Questions

- Does inflation Granger-cause interest rates? (Central bank reaction)
- Do interest rates Granger-cause inflation? (Monetary policy transmission)

Expected Results

Bidirectional causality: Quick $\pi \rightarrow i$ (policy reaction), Delayed $i \rightarrow \pi$ (policy effect)

Python VAR Analysis: Key Functions

Essential Libraries

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests
```

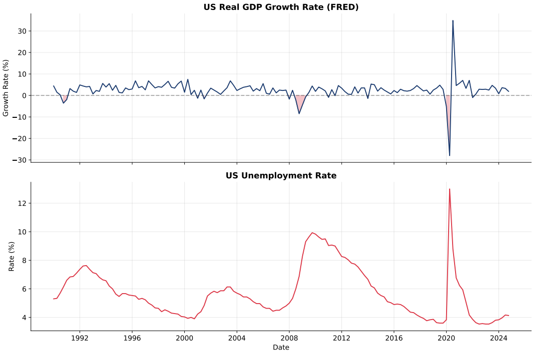
Workflow

1. Create DataFrame: `data = pd.DataFrame({'gdp': ..., 'unemp': ...})`
2. Fit VAR: `model = VAR(data); results = model.fit(maxlags=8, ic='aic')`
3. Get IRF: `irf = results.irf(periods=20)`
4. Get FEVD: `fevd = results.fevd(periods=20)`
5. Granger tests: `grangercausalitytests(data[['y', 'x']], maxlag=4)`

Note

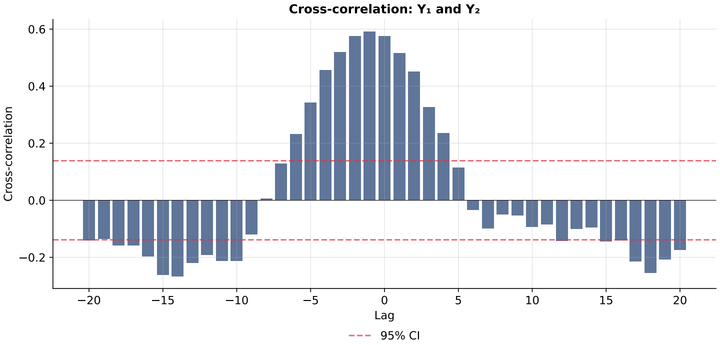
Complete working examples are provided in the Jupyter notebooks.

Case Study: GDP and Unemployment



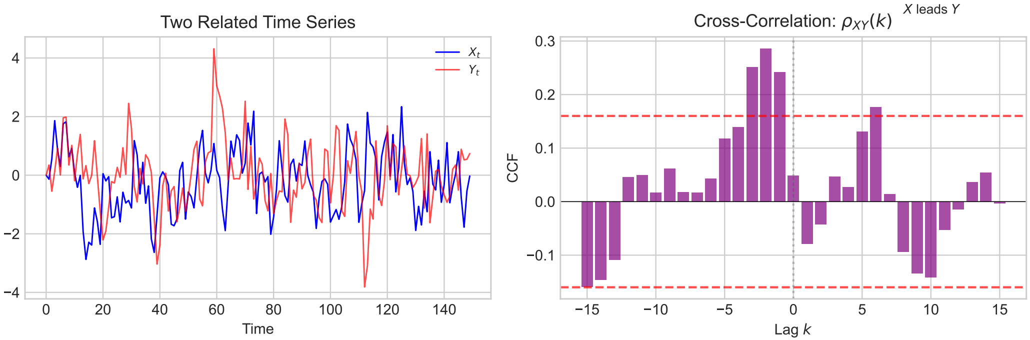
- **Top**: US Real GDP growth rate (quarterly)
- **Bottom**: US Unemployment rate
- Clear negative relationship (Okun's Law)
- VAR model can capture dynamic interactions between these variables

Cross-Correlation Analysis



- Cross-correlation measures lead-lag relationships
- Negative correlation at lag 0: contemporaneous inverse relationship
- Asymmetric pattern suggests unemployment responds to GDP with lag
- Helps inform VAR lag order selection

Visual: Cross-Correlation Function



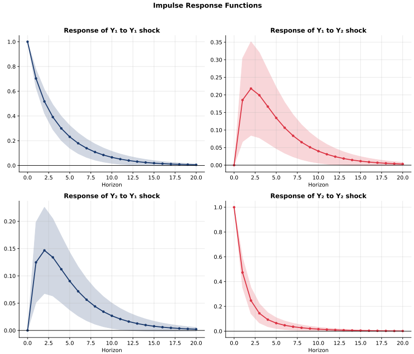
The CCF measures correlation between two series at different lags, revealing lead-lag relationships.

VAR Estimation Results

Model: VAR(2) for GDP Growth and Unemployment

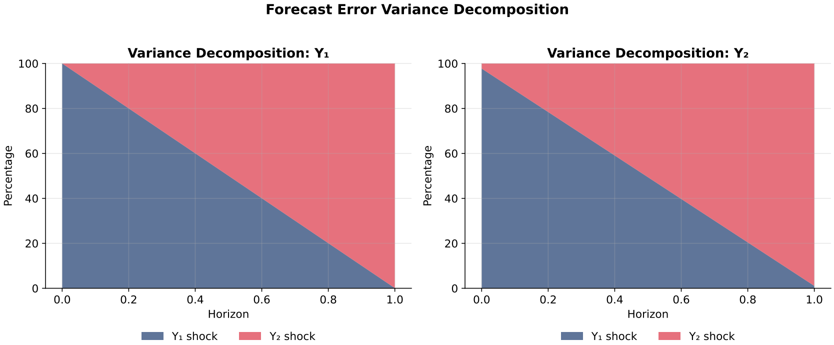
Equation	Variable	Coef.	Std. Error	t-stat
ΔGDP_t	ΔGDP_{t-1}	0.312	0.087	3.59
	ΔGDP_{t-2}	0.145	0.082	1.77
	U_{t-1}	-0.421	0.156	-2.70
	U_{t-2}	0.198	0.148	1.34
U_t	ΔGDP_{t-1}	-0.087	0.032	-2.72
	ΔGDP_{t-2}	-0.045	0.030	-1.50
	U_{t-1}	1.456	0.058	25.1
	U_{t-2}	-0.521	0.055	-9.47

Impulse Response Functions



- IRFs show dynamic response to one-unit shocks
- GDP shock: temporary positive effect on GDP, negative on unemployment
- Unemployment shock: negative effect on GDP, persistent on unemployment
- 95% confidence bands show uncertainty in responses

Forecast Error Variance Decomposition



- FEVD shows proportion of variance explained by each shock
- GDP variance: mostly explained by own shocks, some by unemployment
- Unemployment variance: highly persistent (own shocks dominant)
- Provides insight into relative importance of different shocks

Discussion: Granger Causality vs True Causality

Key Question

If X Granger-causes Y, does that mean X actually causes Y? **NO!**

Why Granger Causality Can Fail

- **Omitted variable bias**: Z might cause both X and Y (e.g., weather $\rightarrow$ ice cream & drownings)
- **Anticipation effects**: Markets anticipate future events (stock prices $\rightarrow$ earnings)
- **Aggregation issues**: Timing of data collection matters

Conclusion

Granger causality is about **prediction**, not **mechanism**. For structural causality, need theory + identification strategy.

Discussion: Variable Ordering in IRFs

Key Question

Why does variable ordering matter for orthogonalized IRFs?

Cholesky Decomposition Assumes

- First variable: Affects all others contemporaneously
- Second variable: Affected by first, affects remaining
- Last variable: Affected by all, affects none contemporaneously

Example: Monetary Policy VAR Ordering

1. Oil prices (exogenous) → 2. GDP (slow) → 3. Inflation → 4. Interest rates (fast)

Rule

Order from "most exogenous" to "most endogenous" — justify with economic theory!

Take-Home Exercises

1. **Theoretical**: Show that a VAR(1) can be written as MA(∞): $\mathbf{Y}_t = \sum_{i=0}^{\infty} \mathbf{A}^i \boldsymbol{\varepsilon}_{t-i}$ when stable.
2. **Computation**: For VAR(1) with $\mathbf{A} = \begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
    - Check stability; Compute IRFs for $h = 0, 1, 2, 3$
    - Plot the response of Y_1 to a shock in Y_2
3. **Applied**: Download US GDP growth and unemployment data:
    - Test stationarity; Estimate VAR (select optimal lag)
    - Test Granger causality; Compute and interpret IRFs
4. **Critical Thinking**: Why might stock prices "Granger-cause" GDP even though GDP is determined by real factors?

Exercise Solutions Hints

Hints

1. Use recursive substitution: $\mathbf{Y}_t = \mathbf{A}\mathbf{Y}_{t-1} + \boldsymbol{\varepsilon}_t = \mathbf{A}(\mathbf{A}\mathbf{Y}_{t-2} + \boldsymbol{\varepsilon}_{t-1}) + \boldsymbol{\varepsilon}_t = \ldots$
2. Eigenvalues of $\begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
   - Characteristic equation: $\lambda^2 - 1.2\lambda + 0.35 = 0$
   - $\lambda_1 \approx 0.85$, $\lambda_2 \approx 0.41$ (both < 1, stable)
3. For GDP/Unemployment:
   - GDP growth is usually I(0), unemployment may be I(1)
   - Use unemployment rate changes if needed
   - Expect GDP growth $\rightarrow$ unemployment (Okun's Law)
4. Stock prices anticipate future economic conditions—they reflect expectations about future GDP, so they "lead" GDP in the data even though causation runs the other way.

Key Takeaways from This Seminar

Main Points

1. VAR models capture **interdependencies** between multiple time series
2. Parameter count grows quickly: $K^2p + K$ per system
3. **Granger causality** tests predictive content, not true causation
4. **IRFs** show dynamic propagation of shocks; ordering matters

Practical Points

- Always check stationarity before estimating VAR
- Use information criteria (AIC/BIC) for lag selection
- Report Granger tests in both directions
- Justify variable ordering with economic theory

Remember

Granger causality is about **prediction**, not **mechanism**!