Chapter 5: VAR Models & Granger Causality

Multivariate Time Series

Lecture Outline

Why Multivariate Analysis?

Limitations of Univariate Models

- ARIMA models each variable **in isolation**
- Ignores potential **interactions** between variables
- Cannot capture **feedback effects**

Economic Examples of Interdependence

- GDP and unemployment (Okun's law)
- Interest rates and inflation (Taylor rule)
- Stock prices and trading volume
- Exchange rates and trade balance

Multivariate Time Series Notation

Vector of Variables

Let $\mathbf{Y}_t = (Y_{1t}, Y_{2t}, \ldots, Y_{Kt})'$ be a $K \times 1$ vector of time series.

Example with $K = 2$:

$$\mathbf{Y}_t = \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \text{GDP growth}_t \\ \text{Inflation}_t \end{pmatrix}$$

Key Questions

1. Does Y_1 help predict Y_2? (Granger causality)
2. How do shocks to Y_1 affect Y_2? (Impulse responses)
3. What proportion of Y_2's variance is due to Y_1? (Variance decomposition)

The VAR(p) Model

Definition

A **VAR(p)** model for K variables:

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{Y}_{t-p} + \varepsilon_t$$

where:

- $\mathbf{Y}_t$: $K \times 1$ vector of endogenous variables
- $\mathbf{c}$: $K \times 1$ vector of constants
- $\mathbf{A}_i$: $K \times K$ coefficient matrices
- ε_t: $K \times 1$ vector of error terms with $\mathbb{E}[\varepsilon_t] = \mathbf{0}$, $\mathbb{E}[\varepsilon_t \varepsilon_t'] = \mathbf{\Sigma}$

VAR(1) with Two Variables

Bivariate VAR(1)

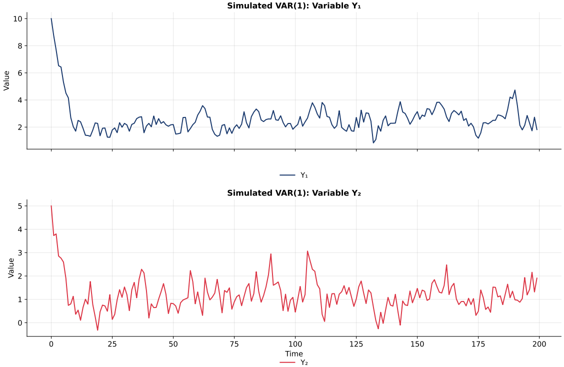
$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Equation by Equation

$$Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$$
$$Y_{2t} = c_2 + a_{21} Y_{1,t-1} + a_{22} Y_{2,t-1} + \varepsilon_{2t}$$

Key insight: Each equation includes lags of **all** variables!

Simulated VAR Process



- Two simulated series from a bivariate VAR(1) process showing interdependence
- Each variable responds to both its own past and the other variable's past
- Notice how the series co-move due to cross-equation dynamics

Stationarity of VAR

Stability Condition

VAR(p) is **stable** (stationary) if all roots of:

$$\det(\mathbf{I}_K - \mathbf{A}_1 z - \mathbf{A}_2 z^2 - \cdots - \mathbf{A}_p z^p) = 0$$

lie **outside** the unit circle (i.e., $|z| > 1$).

For VAR(1)

The model is stable if all **eigenvalues** of $\mathbf{A}_1$ are less than 1 in absolute value.

Example: For $\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$, eigenvalues are $\lambda_1 = 0.6$ and $\lambda_2 = 0.2$.

Both $< 1 \Rightarrow$ stable!

Estimation of VAR

OLS Estimation

Each equation can be estimated by **OLS separately**:

$$\hat{\mathbf{A}} = \left(\sum_{t=1}^{T} \mathbf{Y}_{t-1} \mathbf{Y}_{t-1}' \right)^{-1} \left(\sum_{t=1}^{T} \mathbf{Y}_{t-1} \mathbf{Y}_{t}' \right)$$

This is efficient because all equations have the **same regressors**.

Covariance Matrix

$$\hat{\Sigma} = \frac{1}{T - Kp - 1} \sum_{t=1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_t'$$

The errors ε_{1t} and ε_{2t} may be **contemporaneously correlated**.

Lag Order Selection

Information Criteria

Choose p that minimizes:

$$\text{AIC}(p) = \ln|\hat{\Sigma}_p| + \frac{2pK^2}{T}$$

$$\text{BIC}(p) = \ln|\hat{\Sigma}_p| + \frac{pK^2 \ln T}{T}$$

$$\text{HQ}(p) = \ln|\hat{\Sigma}_p| + \frac{2pK^2 \ln\ln T}{T}$$

Guidelines

- AIC tends to select **larger** models (better for forecasting)
- BIC tends to select **smaller** models (consistent selection)
- Start with maximum p_{max} based on data frequency (e.g., 4 for quarterly, 12 for monthly)

What is Granger Causality?

Clive Granger (1969, Nobel Prize 2003)

"X **Granger-causes** Y" if past values of X help predict Y, **beyond** what past values of Y alone can predict.

Important Distinction

Granger causality $\neq$ True causality

- Granger causality is about **predictive content**
- Does NOT imply economic/structural causation
- "X Granger-causes Y" means: X contains useful information for forecasting Y

Formal Definition

Granger Causality

X **does not Granger-cause** Y if:

$$\mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \ldots, X_{t-1}, X_{t-2}, \ldots] = \mathbb{E}[Y_t | Y_{t-1}, Y_{t-2}, \ldots]$$

In other words: adding X's history does not improve the prediction of Y.

In the VAR Context

For VAR(1): $Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$

Y_2 does **not** Granger-cause Y_1 if $a_{12} = 0$.

For VAR(p): Y_2 does not Granger-cause Y_1 if $a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$.

Testing for Granger Causality

Hypothesis Test

H_0: Y_2 does **not** Granger-cause Y_1

$$H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$$

H_1: At least one $a_{12}^{(i)} \neq 0$ (Granger causality exists)

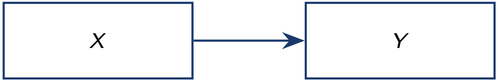
Test Statistic: Wald Test

$$F = \frac{(RSS_R - RSS_U)/p}{RSS_U/(T - 2p - 1)} \sim F_{p, T-2p-1}$$

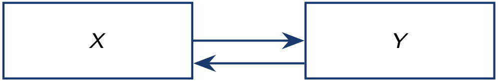
where:

- RSS_R: Residual sum of squares from restricted model (without Y_2 lags)
- RSS_U: Residual sum of squares from unrestricted model (full VAR)

Types of Granger Causality



Unidirectional: $X \rightarrow Y$



Bidirectional: $X \leftrightarrow Y$

Unidirectional: $Y \rightarrow X$

No causality

Economic Examples

- Money $\rightarrow$ Output? (monetarist view)
- Stock prices $\leftrightarrow$ Trading volume (bidirectional)
- Weather $\rightarrow$ Crop yields (unidirectional, obvious)

Cross-Correlation Function

Definition 1 (Cross-Correlation Function)

The **cross-correlation** between X_t and Y_t at lag k is:

$$\rho_{XY}(k) = \frac{\gamma_{XY}(k)}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X_t, Y_{t+k})}{\sqrt{\text{Var}(X_t)\text{Var}(Y_t)}}$$

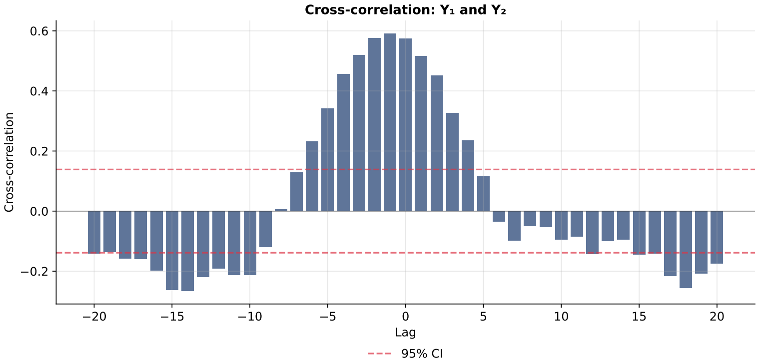
Interpretation

- $\rho_{XY}(k) > 0$ at $k > 0$: X is positively correlated with future Y (X may lead Y)
- $\rho_{XY}(k) > 0$ at $k < 0$: X is positively correlated with past Y (Y may lead X)

Note

Unlike ACF, cross-correlation is **not symmetric**: $\rho_{XY}(k) \neq \rho_{XY}(-k)$ in general.

Cross-Correlation Analysis



- Cross-correlation function measures linear dependence at different lags
- Significant correlations at negative lags suggest X leads Y; positive lags suggest Y leads X
- Useful for preliminary analysis before formal Granger causality testing

Granger Causality: Practical Considerations

Common Pitfalls

1. **Omitted variables**: A third variable Z may cause both X and Y
2. **Non-stationarity**: Test requires stationary data (or cointegration)
3. **Lag selection**: Results can be sensitive to p
4. **Sample size**: Need sufficient observations

Best Practices

- Test for unit roots first
- Use multiple lag selection criteria
- Check robustness to different lag lengths
- Report results for both directions

What are Impulse Response Functions?

Definition

An **Impulse Response Function (IRF)** traces the effect of a one-time shock to one variable on the current and future values of all variables.

Question IRFs Answer

"If there is an unexpected 1-unit shock to Y_1 today, what happens to Y_1 and Y_2 over the next h periods?"

MA(∞) Representation

A stable VAR(p) can be written as:

$$\mathbf{Y}_t = \boldsymbol{\mu} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{\varepsilon}_{t-i}$$

The matrices $\boldsymbol{\Phi}_i$ are the **impulse responses** at horizon i.

Computing IRFs for VAR(1)

For VAR(1): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t$

The impulse response matrices are:

$$\mathbf{\Phi}_0 = \mathbf{I}_K, \quad \mathbf{\Phi}_1 = \mathbf{A}, \quad \mathbf{\Phi}_2 = \mathbf{A}^2, \quad \ldots, \quad \mathbf{\Phi}_h = \mathbf{A}^h$$

Interpretation

$[\mathbf{\Phi}_h]_{ij} =$ Effect on Y_i at time $t + h$ of a unit shock to Y_j at time t

For stable VAR: $\mathbf{\Phi}_h \to \mathbf{0}$ as $h \to \infty$ (shocks die out)

Computing IRFs for General VAR(p)

Recursive Formula for VAR(p)

For $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$:

$$\boldsymbol{\Phi}_h = \sum_{j=1}^{\min(h,p)} \mathbf{A}_j \boldsymbol{\Phi}_{h-j}, \quad h = 1, 2, 3, \ldots$$

with $\boldsymbol{\Phi}_0 = \mathbf{I}_K$ and $\boldsymbol{\Phi}_h = \mathbf{0}$ for $h < 0$.

Example: VAR(2) IRFs

- $\boldsymbol{\Phi}_0 = \mathbf{I}_K$
- $\boldsymbol{\Phi}_1 = \mathbf{A}_1\boldsymbol{\Phi}_0 = \mathbf{A}_1$
- $\boldsymbol{\Phi}_2 = \mathbf{A}_1\boldsymbol{\Phi}_1 + \mathbf{A}_2\boldsymbol{\Phi}_0 = \mathbf{A}_1^2 + \mathbf{A}_2$
- $\boldsymbol{\Phi}_3 = \mathbf{A}_1\boldsymbol{\Phi}_2 + \mathbf{A}_2\boldsymbol{\Phi}_1 = \mathbf{A}_1(\mathbf{A}_1^2 + \mathbf{A}_2) + \mathbf{A}_2\mathbf{A}_1$

Orthogonalized IRFs

Problem: Correlated Errors

If $\boldsymbol{\Sigma}$ is not diagonal, shocks ε_{1t} and ε_{2t} are correlated.
A shock to "Y_1" also involves a shock to "Y_2".

Solution: Cholesky Decomposition

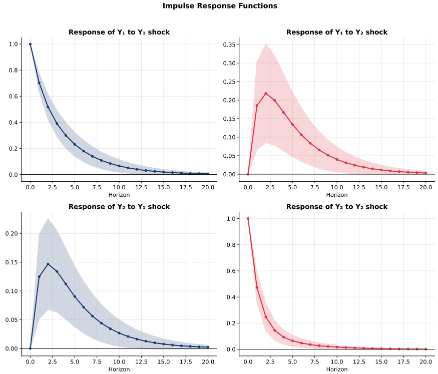
Factor $\boldsymbol{\Sigma} = \mathbf{P}\mathbf{P}'$ where $\mathbf{P}$ is lower triangular.
Define orthogonalized shocks: $\mathbf{u}_t = \mathbf{P}^{-1}\varepsilon_t$ with $\mathbb{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}$

Orthogonalized IRFs: $\boldsymbol{\Theta}_h = \boldsymbol{\Phi}_h\mathbf{P}$

Ordering Matters!

Cholesky assumes variables ordered from "most exogenous" to "most endogenous". Results depend on this ordering.

Impulse Response Functions: Example



- IRFs show how each variable responds to a one-unit shock over time
- Shaded regions represent confidence intervals (uncertainty in estimates)
- For stable VAR models, responses converge to zero as the horizon increases

Variance Decomposition

Question

What proportion of the forecast error variance of Y_i at horizon h is due to shocks to Y_j?

FEVD Formula

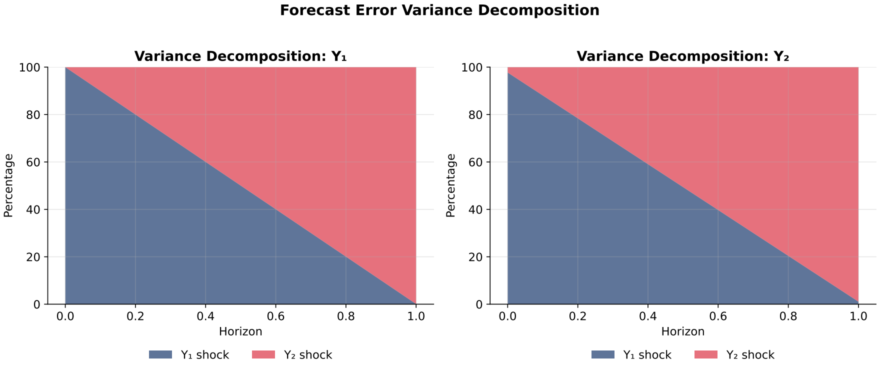
$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\boldsymbol{\Theta}_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^{K} [\boldsymbol{\Theta}_s]_{ik}^2}$$

This gives the **percentage** of Y_i's h-step forecast variance explained by shocks to Y_j.

Properties

- $0 \leq \text{FEVD}_{ij}(h) \leq 1$
- $\sum_{j=1}^{K} \text{FEVD}_{ij}(h) = 1$ (sums to 100%)
- At $h = 1$: Own shocks dominate (by construction with Cholesky)

FEVD: Example



- FEVD shows the proportion of forecast variance attributable to each shock
- At short horizons, own shocks dominate; cross-variable effects grow over time
- Useful for understanding the relative importance of different shocks in the system

Example: GDP and Unemployment

Okun's Law

There is a negative relationship between GDP growth and unemployment:

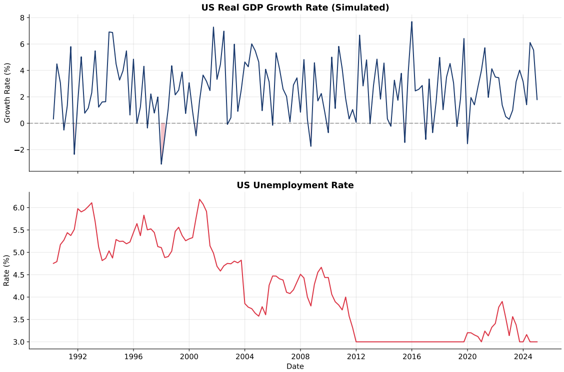
$$\Delta U_t \approx -\beta(\Delta Y_t - \bar{g})$$

where $\bar{g}$ is trend GDP growth and $\beta \approx 0.4$.

VAR Analysis Questions

1. Does GDP growth Granger-cause unemployment changes?
2. Does unemployment Granger-cause GDP growth?
3. How do shocks propagate between variables?

GDP and Unemployment: Data



- GDP growth and unemployment rate show clear negative correlation (Okun's Law)
- Both series exhibit cyclical patterns related to business cycle fluctuations
- This bivariate system is ideal for VAR analysis and Granger causality testing

VAR Workflow

1. **Data preparation**
   - Check for stationarity (unit root tests)
   - Transform if necessary (differences, logs)
2. **Lag selection**
   - Use AIC, BIC, HQ criteria
   - Check residual autocorrelation
3. **Estimation**
   - OLS equation by equation
   - Check stability (eigenvalues)
4. **Analysis**
   - Granger causality tests
   - Impulse response functions
   - Variance decomposition
5. **Forecasting**

Python Implementation

VAR in Python (statsmodels)

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests

# Fit VAR model
model = VAR(data)
results = model.fit(maxlags=4, ic='aic')

# Granger causality test
granger_test = grangercausalitytests(data[['Y1', 'Y2']],
                                      maxlag=4)
# Impulse response functions
irf = results.irf(periods=20)
irf.plot()

# Variance decomposition
fevd = results.fevd(periods=20)
fevd.plot()
```

Key Takeaways

VAR Models

- Model **multiple** time series jointly
- Each variable depends on its own lags AND lags of other variables
- Estimated by OLS equation by equation; requires stationarity

Granger Causality

- Tests whether X helps predict Y beyond Y's own history
- **Not** the same as true causality; F-test on coefficient restrictions

IRF and FEVD

- IRF: How shocks propagate through the system
- FEVD: What proportion of variance is due to each shock
- Both depend on variable ordering (Cholesky decomposition)

What's Next?

Topics for Further Study

- **Cointegration**: Long-run relationships between non-stationary variables
- **VECM**: Error correction models for cointegrated systems
- **Structural VAR**: Imposing economic theory restrictions
- **Panel VAR**: VAR for panel data

Questions?