Time Series Analysis and Forecasting

Seminar 6: VAR and Granger Causality

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Seminar Outline

- ☑ **Multiple Choice Quiz** – Knowledge check

- ☑ **True/False** – Conceptual checks

- ☑ **Calculation Exercises** – Applied practice

- ☑ **Worked Examples** – Detailed solutions

- ☑ **Real Data Analysis** – Empirical applications

- ☑ **AI-Assisted Exercise** – Critical thinking

- ☑ **Summary** – Key takeaways

Quiz 1: VAR Definition

Question

In a VAR(2) model with 3 variables, how many coefficient matrices A_i are there?

Answer choices

(A) 2
(B) 3
(C) 6
(D) 9

Answer on next slide...

Quiz 1: Answer

Answer: A – 2 coefficient matrices

VAR(p) model: $Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$

VAR(2) with $K = 3$:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \\ Y_{3t} \end{pmatrix} = c + \underbrace{A_1}_{3 \times 3} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ Y_{3,t-1} \end{pmatrix} + \underbrace{A_2}_{3 \times 3} \begin{pmatrix} Y_{1,t-2} \\ Y_{2,t-2} \\ Y_{3,t-2} \end{pmatrix} + \varepsilon_t$$

Key: p = number of lags = number of matrices

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Quiz 2: Number of Parameters

Question

A VAR(2) with $K = 3$ variables (including constants) has how many parameters to estimate per equation?

Answer choices

(A) 3
(B) 6
(C) 7
(D) 9

Answer on next slide...

Quiz 2: Answer

Answer: C – 7 parameters per equation

VAR(p) Parameter Count: The Curse of Dimensionality

Parameters per equation: $1 + K \times p$

Total parameters: $K(1 + Kp) + K(K + 1)/2$

(coefficients + covariance matrix)

Model	Coefficients	Total
K=2, p=1	2(1+2×1) = 6	+ 3 = 9
K=3, p=2	3(1+3×2) = 21	+ 6 = 27
K=5, p=4	5(1+5×4) = 105	+ 15 = 120
K=10, p=4	10(1+10×4) = 410	+ 55 = 465

Warning: Parameters grow as $K^2 \times p$ — need lots of data!

Formula: Per equation $= 1 + K \times p = 1 + 3 \times 2 = 7$. **Total**: $K(1 + Kp) = 3(1 + 6) = 21$ parameters

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Quiz 3: Granger Causality

Question

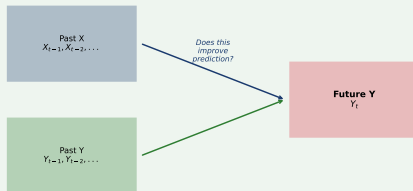"X Granger-causes Y" means:

Answer choices

(A) X is the economic cause of Y

(B) Past X helps predict future Y

(C) X and Y are contemporaneously correlated

(D) X always increases when Y increases

Answer on next slide...

Quiz 3: Answer

Answer: B – Past X helps predict future Y

Granger Causality: Predictive, Not Causal!

Past X
$X_{t-1}, X_{t-2}, \ldots$

Past Y
$Y_{t-1}, Y_{t-2}, \ldots$

Does this improve prediction?

Future Y
Y_t

"X Granger-causes Y" means: Past X helps predict future Y,
beyond what past Y alone provides. Does NOT imply true causation!

Key: Predictive relationship, NOT true causation!

TSA_ch6_case_granger

Quiz 4: Granger Causality Test

Question

To test if Y_2 Granger-causes Y_1 in a VAR(p), we test:

Answer choices

(A) All coefficients in the Y_1 equation equal zero

(B) Coefficients on lagged Y_2 in the Y_1 equation equal zero

(C) Coefficients on lagged Y_1 in the Y_2 equation equal zero

(D) The error covariance equals zero

Answer on next slide...

Quiz 4: Answer

Answer: B – Coefficients on lagged Y_2 in Y_1 equation $= 0$

Null hypothesis: $H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$

Test statistic: Wald or F-test with p restrictions

Interpretation:
- Reject H_0: Y_2 Granger-causes Y_1
- Don't reject: No evidence of predictive relationship

Note: Test $Y_1 \rightarrow Y_2$ separately (different coefficients in Y_2 equation)

TSA_ch6_case_granger

Quiz 5: VAR Stability

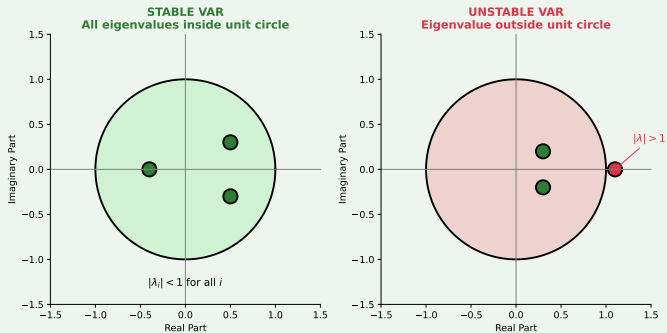### Question

A VAR(1) model is stable (stationary) if:

Answer choices

(A) All diagonal elements of A_1 are less than 1

(B) The determinant of A_1 is less than 1

(C) All eigenvalues of A_1 are less than 1 in absolute value

(D) The trace of A_1 equals zero

Answer on next slide...

Quiz 5: Answer

Answer: C – All eigenvalues of A_1 inside unit circle

Stable: All $|\lambda_i| < 1$ (inside unit circle) $\Rightarrow$ shocks die out over time

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Quiz 6: Impulse Response Functions
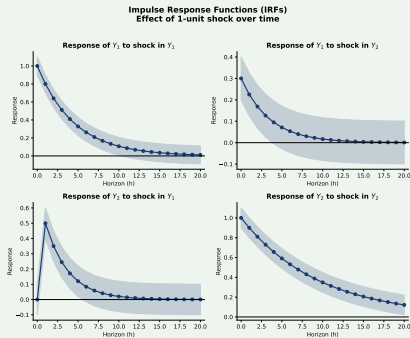
Question

An impulse response function shows:

Answer choices

(A) The correlation between two variables

(B) The effect of a shock to one variable on all variables over time

(C) The forecast accuracy of the model

(D) The p-values of coefficient tests

Answer on next slide...

Quiz 6: Answer

Answer: B – Effect of shock on all variables over time

Impulse Response Functions (IRFs)
Effect of 1-unit shock over time

IRF$_{ij}(h)$: Response of variable i at horizon h to shock in variable j

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Quiz 7: Lag Order Selection

Question

Which criterion typically selects the most parsimonious VAR model?

Answer choices

(A) AIC (Akaike Information Criterion)

(B) BIC (Bayesian Information Criterion)

(C) FPE (Final Prediction Error)

(D) Adjusted R^2

Answer on next slide...

Quiz 7: Answer

Answer: B – BIC (Bayesian Information Criterion)

Penalty comparison (for k parameters, n observations):
- AIC: $-2 \ln L + 2k$
- BIC: $-2 \ln L + k \ln n$

Since $\ln n > 2$ for $n > 8$, BIC penalizes complexity more heavily

Practical guidance:
- Forecasting: AIC may perform better
- Inference/parsimony: BIC preferred
- Large samples: BIC consistent, AIC tends to overfit

TSA_ch6_lag_selection

Quiz 8: Granger Causality – Formal Test

Question

"X Granger-causes Y" means:

Answer choices

(A) X is the true cause of Y

(B) Past values of X help predict Y beyond Y's own past

(C) X and Y are correlated

(D) Y depends only on X

Answer on next slide...

Quiz 8: Answer

Answer: B – Past values of X help predict Y beyond Y's own past

⊡ Granger causality refers to **predictive** content, not true causation

⊡ X Granger-causes Y if lagged X terms are jointly significant in the Y equation, after controlling for lagged Y

Quiz 9: Forecast Error Variance Decomposition

Question
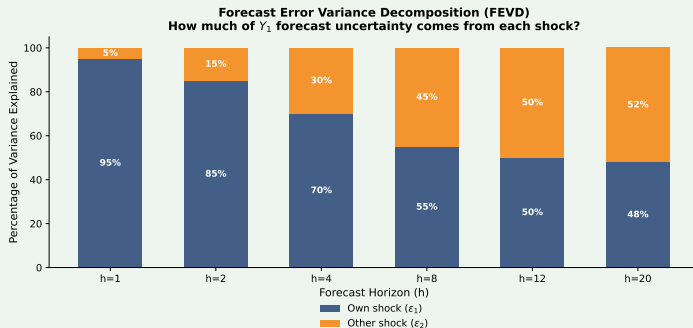
FEVD (Forecast Error Variance Decomposition) tells us:

Answer choices

(A) The correlation between variables

(B) What proportion of forecast error variance comes from each shock

(C) The optimal forecast horizon

(D) Which variables to include in the model

Answer on next slide...

Quiz 9: Answer

Answer: B – Proportion of forecast error variance from each shock

Forecast Error Variance Decomposition (FEVD)
How much of Y_1 forecast uncertainty comes from each shock?

FEVD: Shows how much forecast uncertainty comes from each shock at different horizons

Quiz 10: Structural vs Reduced Form VAR

Question

The difference between structural VAR (SVAR) and reduced-form VAR is:

Answer choices

(A) SVAR has more variables

(B) SVAR allows contemporaneous effects between variables

(C) SVAR uses different estimation methods

(D) There is no difference

Answer on next slide...

Quiz 10: Answer

Answer: B – SVAR allows contemporaneous effects between variables

⊡ Reduced-form VAR: shocks are correlated, no contemporaneous effects in equations

⊡ SVAR: imposes identifying restrictions to recover structural shocks with economic interpretation (e.g., monetary policy shock)

Quiz 11: Cholesky Decomposition
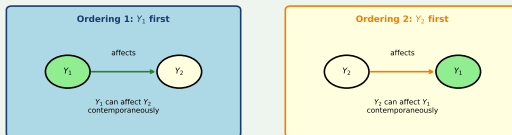
Question

Cholesky ordering in IRF analysis assumes:

Answer choices

(A) All variables are equally important

(B) Variables ordered first affect later variables contemporaneously, not vice versa

(C) Shocks are uncorrelated

(D) No restrictions are needed

Answer on next slide...

Quiz 11: Answer

Answer: B – Variables ordered first affect later ones contemporaneously

Cholesky Ordering: Order Matters!

Ordering 1: Y_1 first
Y_1 — affects → Y_2
Y_1 can affect Y_2 contemporaneously

Ordering 2: Y_2 first
Y_2 — affects → Y_1
Y_2 can affect Y_1 contemporaneously

Key Points:

• Different orderings give DIFFERENT IRFs!

• Ordering should be based on economic theory

• "Fast-moving" variables should come first

Cholesky: Recursive structure. Ordering matters – justify by economic theory (most exogenous first)!

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Quiz 12: VAR Residual Diagnostics

Question

In a well-specified VAR, residuals should be:

Answer choices

(A) Autocorrelated but homoskedastic

(B) White noise (no autocorrelation)

(C) Normally distributed only

(D) Correlated across equations

Answer on next slide...

Quiz 12: Answer

Answer: B – White noise (no autocorrelation)

Diagnostics: Residuals should be white noise. Use Portmanteau/LM test. Cross-equation correlation allowed (Σ_u).

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Quiz 13: Cointegration and VAR

Question

If variables are I(1) and cointegrated, you should use:

Answer choices

(A) VAR in levels

(B) VAR in first differences

(C) Vector Error Correction Model (VECM)

(D) Univariate ARIMA models

Answer on next slide...

Quiz 13: Answer

Answer: C – Vector Error Correction Model (VECM)

⊡ With cointegration, VAR in differences loses long-run information, while VAR in levels may be inefficient

⊡ VECM incorporates both short-run dynamics and long-run equilibrium relationships through the error correction term

Quiz 14: Instantaneous Causality

Question

Instantaneous causality differs from Granger causality because it tests:

Answer choices

(A) Lagged relationships only

(B) Contemporaneous correlation of residuals

(C) Long-run relationships

(D) Model stability

Answer on next slide...

Quiz 14: Answer

Answer: B – Contemporaneous correlation of residuals

⊡ Instantaneous causality tests whether shocks to X and Y are correlated within the same period (correlation of VAR residuals)

⊡ Granger causality tests whether *lagged* values help predict

True or False? — Questions

Statement	T/F?
1. VAR models treat all variables as endogenous.	?
2. Granger causality proves true economic causation.	?
3. A stable VAR always has eigenvalues inside the unit circle.	?
4. FEVD results depend on the ordering of variables.	?
5. VAR can be estimated by OLS equation by equation.	?
6. Impulse responses eventually die out in a stable VAR.	?

True or False? — Answers

Statement	T/F	Explanation
1. VAR models treat all variables as endogenous.	T	Each var regressed on all lags
2. Granger causality proves true economic causation.	F	Only predictive content
3. A stable VAR always has eigenvalues inside the unit circle.	T	$|\lambda_i| < 1$ for all i
4. FEVD results depend on the ordering of variables.	T	Cholesky ordering matters
5. VAR can be estimated by OLS equation by equation.	T	OLS = GLS = ML
6. Impulse responses eventually die out in a stable VAR.	T	IRFs $\to 0$ as $h \to \infty$

Exercise 1: Writing VAR Equations

Problem

⊡ Write out the two equations for a bivariate VAR(1) model with variables Y_t (GDP growth) and X_t (inflation).

Solution

$$Y_t = c_1 + a_{11} Y_{t-1} + a_{12} X_{t-1} + \varepsilon_{1t}$$

$$X_t = c_2 + a_{21} Y_{t-1} + a_{22} X_{t-1} + \varepsilon_{2t}$$

Interpretation:

⊡ a_{12}: Effect of past inflation on current GDP growth
⊡ a_{21}: Effect of past GDP growth on current inflation

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Exercise 2: Parameter Count

Problem

⊡ How many total parameters need to be estimated in a VAR(3) with $K = 4$ variables (including constants)?

Solution

Per equation: $1 + K \times p = 1 + 4 \times 3 = 13$ parameters

Total for $K = 4$ equations: $4 \times 13 = 52$ parameters

Plus covariance matrix $\mathbf{\Sigma}$: $K(K + 1)/2 = 4 \times 5/2 = 10$ unique elements

Grand total: 62 parameters

This is why VARs can be "over-parameterized" with limited data!

Exercise 3: Granger Causality Interpretation

Problem

- ☐ A Granger causality test yields:
- ☐ H_0: Money does not Granger-cause GDP. p-value = 0.02
- ☐ H_0: GDP does not Granger-cause Money. p-value = 0.35
- ☐ Interpret these results.

Solution

- ☐ **Reject** H_0 at 5%: Money **Granger-causes** GDP
- ☐ **Fail to reject** H_0: GDP does **not** Granger-cause Money

Conclusion: Unidirectional causality: Money $\succ$ GDP

Interpretation: Past money supply helps predict GDP growth. This is consistent with monetarist views, but remember: Granger causality $\neq$ structural causality!

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Exercise 4: Stability Check

Problem

□ For VAR(1) with $A_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.1 & 0.5 \end{pmatrix}$, check stability.

Solution

Find eigenvalues: $\det(A_1 - \lambda I) = 0$
$(0.7 - \lambda)(0.5 - \lambda) - (0.2)(0.1) = 0$
$\lambda^2 - 1.2\lambda + 0.33 = 0$
$\lambda = \frac{1.2 \pm \sqrt{1.44 - 1.32}}{2} = \frac{1.2 \pm 0.346}{2}$
$\lambda_1 = 0.773, \quad \lambda_2 = 0.427$

Both $|\lambda_i| < 1 \Rightarrow$ **Stable!**

Q **TSA_ch6_stability_roots**

Exercise 5: IRF Computation

Problem

☐ For VAR(1) with $A = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$, compute Φ_2 (response at $h = 2$).

Solution

$$\Phi_2 = A^2 = \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 & 0.2 \\ 0 & 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 0.25 + 0 & 0.10 + 0.12 \\ 0 + 0 & 0 + 0.36 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.22 \\ 0 & 0.36 \end{pmatrix}$$

Interpretation: A unit shock to Y_2 at t increases Y_1 by 0.22 at $t + 2$.

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Example: Stock Returns and Trading Volume

Scenario

Daily data on stock returns (R_t) and trading volume (V_t). Test Granger causality both directions.

Typical Findings in Finance Literature

⊡ Returns often Granger-cause volume (price changes trigger trading)
⊡ Volume sometimes Granger-causes returns (volume as leading indicator)
⊡ Results: Often **bidirectional** causality $R \leftrightarrow V$

Practical Issue

Stock returns are typically stationary, but volume may need transformation (log or difference).

Example: Interest Rates and Inflation

Taylor Rule Context

Central banks set interest rates (i_t) in response to inflation (π_t): $i_t = r^* + \pi^* + 1.5(\pi_t - \pi^*) + 0.5(y_t - y^*)$

VAR Analysis Questions

⊡ Does inflation Granger-cause interest rates? (Central bank reaction)
⊡ Do interest rates Granger-cause inflation? (Monetary policy transmission)

Expected Results

Bidirectional causality: Quick $\pi \to i$ (policy reaction), Delayed $i \to \pi$ (policy effect)

Python VAR Analysis: Key Functions

Essential Libraries

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests
```
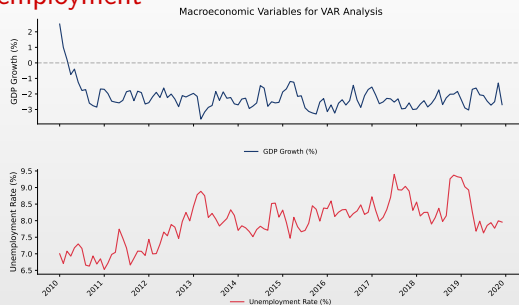
Workflow

1. Create DataFrame: `data = pd.DataFrame({'gdp': ..., 'unemp': ...})`
2. Fit VAR: `model = VAR(data); results = model.fit(maxlags=8, ic='aic')`
3. Get IRF: `irf = results.irf(periods=20)`
4. Get FEVD: `fevd = results.fevd(periods=20)`
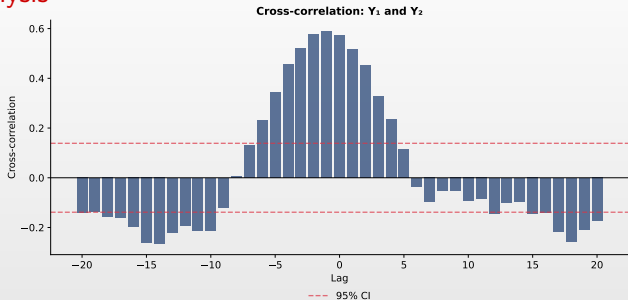5. Granger tests: `grangercausalitytests(data[['y', 'x']], maxlag=4)`

Note

Complete working examples are provided in the Jupyter notebooks.

 TSA_ch6_var_simulation

Case Study: GDP and Unemployment
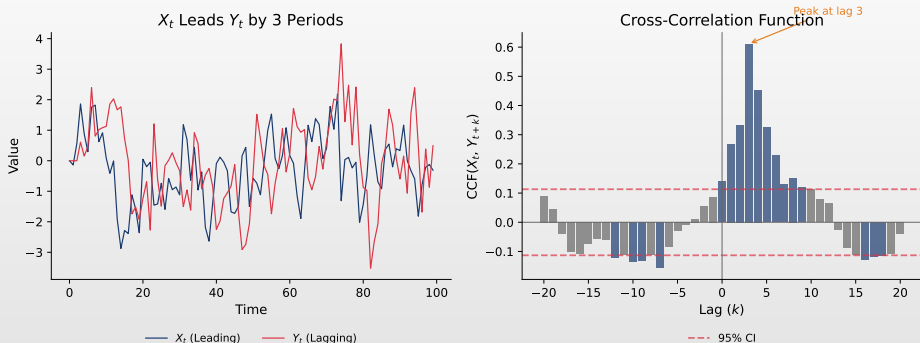

Macroeconomic Variables for VAR Analysis

- ⊡ **Top**: US Real GDP growth rate (quarterly)
- ⊡ **Bottom**: US Unemployment rate
- ⊡ Clear negative relationship (Okun's Law)
- ⊡ VAR model can capture dynamic interactions between these variables

TSA_ch6_gdp_unemployment

Cross-Correlation Analysis

Cross-correlation: Y_1 and Y_2

- ⊡ Cross-correlation measures lead-lag relationships
- ⊡ Negative correlation at lag 0: contemporaneous inverse relationship
- ⊡ Asymmetric pattern suggests unemployment responds to GDP with lag
- ⊡ Helps inform VAR lag order selection

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Visual: Cross-Correlation Function

The CCF measures correlation between two series at different lags, revealing lead-lag relationships.

 TSA_ch6_def_ccf
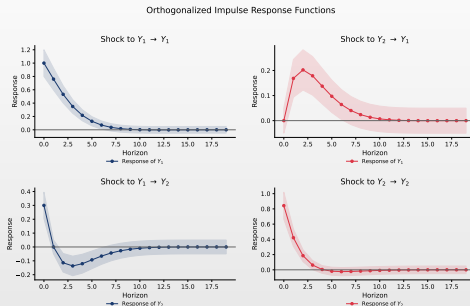
VAR Estimation Results
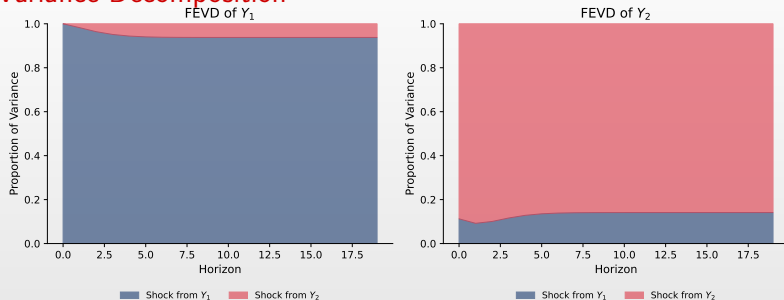
Model: VAR(2) for GDP Growth and Unemployment

Equation	Variable	Coef.	Std. Error	t-stat
ΔGDP_t	ΔGDP_{t-1}	0.312	0.087	3.59
	ΔGDP_{t-2}	0.145	0.082	1.77
	U_{t-1}	-0.421	0.156	-2.70
	U_{t-2}	0.198	0.148	1.34
U_t	ΔGDP_{t-1}	-0.087	0.032	-2.72
	ΔGDP_{t-2}	-0.045	0.030	-1.50
	U_{t-1}	1.456	0.058	25.1
	U_{t-2}	-0.521	0.055	-9.47

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Impulse Response Functions

Orthogonalized Impulse Response Functions

- ⊡ IRFs show dynamic response to one-unit shocks
- ⊡ GDP shock: temporary positive effect on GDP, negative on unemployment
- ⊡ Unemployment shock: negative effect on GDP, persistent on unemployment
- ⊡ 95% confidence bands show uncertainty in responses

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Forecast Error Variance Decomposition

- ⊡ FEVD shows proportion of variance explained by each shock
- ⊡ GDP variance: mostly explained by own shocks, some by unemployment
- ⊡ Unemployment variance: highly persistent (own shocks dominant)
- ⊡ Provides insight into relative importance of different shocks

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Discussion: Granger Causality vs True Causality

Key Question

If X Granger-causes Y, does that mean X actually causes Y? **NO!**

Why Granger Causality Can Fail

- **Omitted variable bias**: Z might cause both X and Y (e.g., weather $\rightarrow$ ice cream & drownings)
- **Anticipation effects**: Markets anticipate future events (stock prices $\rightarrow$ earnings)
- **Aggregation issues**: Timing of data collection matters

Conclusion

Granger causality is about **prediction**, not **mechanism**. For structural causality, need theory + identification strategy.

Discussion: Variable Ordering in IRFs

Key Question

Why does variable ordering matter for orthogonalized IRFs?

Cholesky Decomposition Assumes

- First variable: Affects all others contemporaneously
- Second variable: Affected by first, affects remaining
- Last variable: Affected by all, affects none contemporaneously

Example: Monetary Policy VAR Ordering

1. Oil prices (exogenous) → 2. GDP (slow) → 3. Inflation → 4. Interest rates (fast)

Rule

Order from "most exogenous" to "most endogenous" — justify with economic theory!

Take-Home Exercises

1. **Theoretical**: Show that a VAR(1) can be written as MA(∞): $Y_t = \sum_{i=0}^{\infty} A^i \varepsilon_{t-i}$ when stable.

2. **Computation**: For VAR(1) with $A = \begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
   ▶ Check stability; Compute IRFs for $h = 0, 1, 2, 3$
   ▶ Plot the response of Y_1 to a shock in Y_2

3. **Applied**: Download US GDP growth and unemployment data:
   ▶ Test stationarity; Estimate VAR (select optimal lag)
   ▶ Test Granger causality; Compute and interpret IRFs

4. **Critical Thinking**: Why might stock prices "Granger-cause" GDP even though GDP is determined by real factors?

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Exercise Solutions Hints

Hints

1. Use recursive substitution: $Y_t = AY_{t-1} + \varepsilon_t = A(AY_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \ldots$
2. Eigenvalues of $\begin{pmatrix} 0.8 & -0.1 \\ 0.3 & 0.4 \end{pmatrix}$:
   - Characteristic equation: $\lambda^2 - 1.2\lambda + 0.35 = 0$
   - $\lambda_1 = 0.7$, $\lambda_2 = 0.5$ (both < 1, stable)
3. For GDP/Unemployment:
   - GDP growth is usually I(0), unemployment may be I(1)
   - Use unemployment rate changes if needed
   - Expect GDP growth $\succ$ unemployment (Okun's Law)
4. Stock prices anticipate future economic conditions—they reflect expectations about future GDP, so they "lead" GDP in the data even though causation runs the other way.

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download Romania GDP and inflation data. Estimate a VAR model, test Granger causality and compute impulse response functions. Which variable causes the other?"

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does the AI check stationarity of both series before estimating the VAR?
3. Is the lag order selected via AIC/BIC or chosen arbitrarily?
4. Is the Granger causality interpretation correct? (it does not imply true causation!)
5. Are the IRFs orthogonalized? Does the AI mention the Cholesky decomposition ordering?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Summary: Chapter 6

Key Concepts

1. **VAR models**: capture **interdependencies** between multiple time series
2. **Parameter count**: grows quickly ($K^2 p + K$ per system) — parsimony matters
3. **Granger causality**: tests predictive content, not true causation
4. **IRFs**: show dynamic propagation of shocks; ordering matters
5. **Practical**: check stationarity, use AIC/BIC for lag selection, justify variable ordering

Questions?

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⊡ Makridakis, S., Spiliotis, E., & Assimakopoulos, V. (2020). The M4 Competition, International Journal of Forecasting.
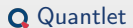
Online Resources and Code

⊡ **Quantlet**: https://quantlet.com — Code repository for statistics
⊡ **Quantinar**: https://quantinar.com — Platform for learning quantitative methods
⊡ **GitHub TSA**: https://github.com/QuantLet/TSA/tree/main/TSA_ch6 — Python code for this seminar

Thank You!

Questions?

Seminar materials are available at: https://danpele.github.io/Time-Series-Analysis/

Q Quantlet   **Q** Quantinar