Chapter 2: ARMA Models

Seminar

Seminar Outline

Quiz 1: Lag Operator

Question

What is the result of applying $(1 - L)^2$ to X_t?

- A. $X_t - X_{t-1}$
- B. $X_t - 2X_{t-1} + X_{t-2}$
- C. $X_t + X_{t-1} + X_{t-2}$
- D. $X_t - X_{t-2}$

Answer on next slide...

Quiz 1: Solution

Answer: B – $X_t - 2X_{t-1} + X_{t-2}$

Explanation:

$$\begin{aligned}(1-L)^2 X_t &= (1 - 2L + L^2)X_t \\ &= X_t - 2LX_t + L^2 X_t \\ &= X_t - 2X_{t-1} + X_{t-2}\end{aligned}$$

This is the **second difference** of X_t.
Note: $(1-L)$ is the first difference operator, $(1-L)^2$ is the second difference.

Quiz 2: AR(1) Stationarity

Question

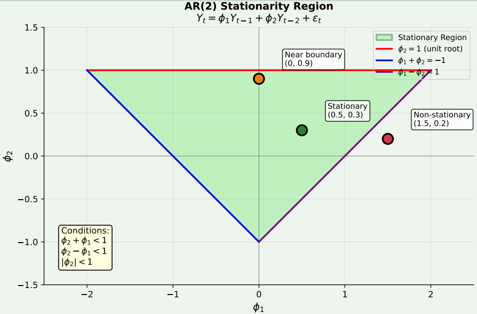
For which value of ϕ is the AR(1) process $X_t = 0.5 + \phi X_{t-1} + \varepsilon_t$ stationary?

A. $\phi = 1.2$
B. $\phi = 1.0$
C. $\phi = -0.8$
D. $\phi = -1.5$

Answer on next slide...

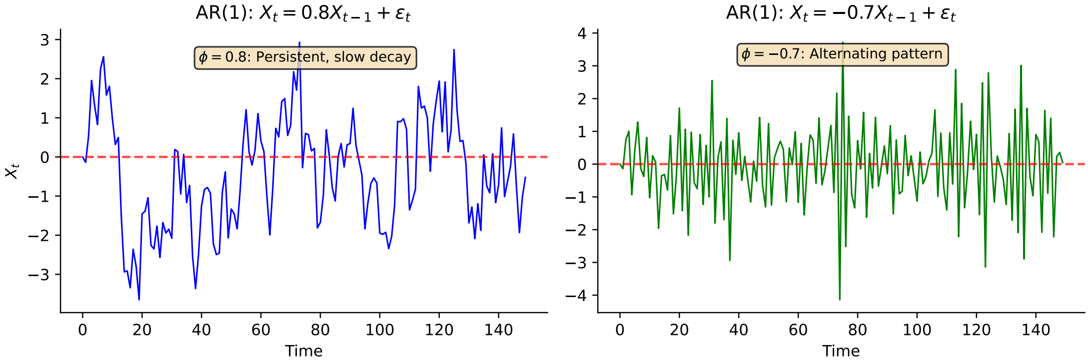
Quiz 2: Solution

Answer: C – $\phi = -0.8$ (Stationary)



AR(1) stationarity: $|\phi| < 1$ (root outside unit circle). Only C satisfies: $|-0.8| = 0.8 < 1$

Visual: AR(1) Process Behavior



Positive ϕ: persistent, smooth patterns. Negative ϕ: oscillating behavior around mean.

Quiz 3: ACF Pattern

Question

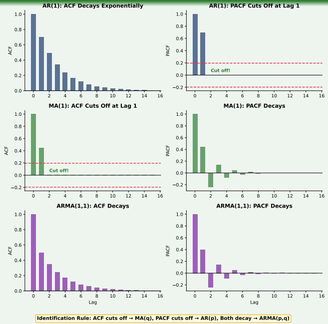
You observe the following ACF pattern: significant spike at lag 1, then all other lags are within confidence bands. The PACF shows gradual decay. What model is suggested?

A. AR(1)
B. MA(1)
C. ARMA(1,1)
D. White noise

Answer on next slide...

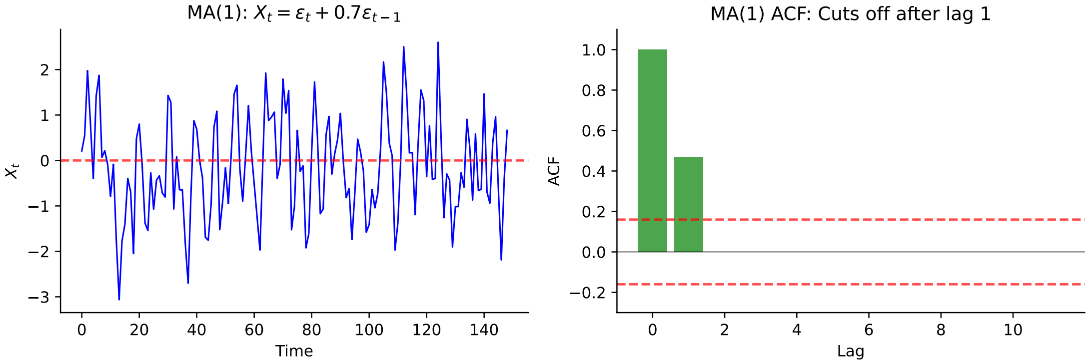
Quiz 3: Solution

Answer: B – MA(1)



Pattern: ACF cuts off after lag 1 $\Rightarrow$ MA(1); PACF decays $\Rightarrow$ confirms MA structure (not AR)

Visual: MA(1) Process and ACF



MA(1) process (left). Key signature: ACF cuts off sharply after lag 1 (right).

Quiz 4: MA Invertibility

Question

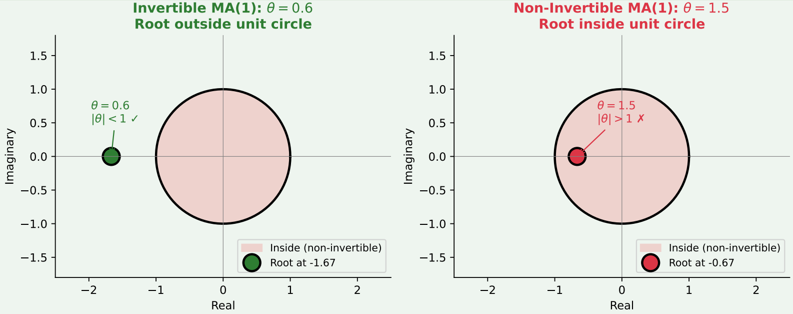
For the MA(1) process $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$, is the process invertible?

- A. Yes, because MA processes are always invertible
- B. Yes, because $1.5 > 0$
- C. No, because $|\theta| = 1.5 > 1$
- D. No, because MA processes are never invertible

Answer on next slide...

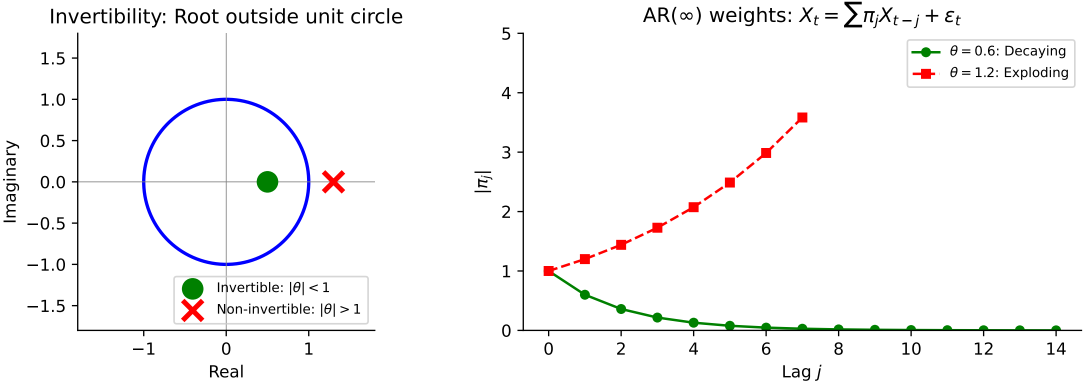
Quiz 4: Solution

Answer: C – Not invertible ($|\theta| = 1.5 > 1$)



MA invertibility: Root $z = -1/\theta$ must be outside unit circle $\Leftrightarrow |\theta| < 1$. Here $z = -0.67$ is inside!

Visual: Invertibility Concept



Left: invertibility requires roots outside unit circle. Right: AR(∞) weights decay only when $|\theta| < 1$.

Quiz 5: ARMA Representation

Question

The compact form $\phi(L)X_t = \theta(L)\varepsilon_t$ represents which model?

A. Pure AR model
B. Pure MA model
C. ARMA model
D. None of the above

Answer on next slide...

Quiz 5: Solution

Answer: C – ARMA model

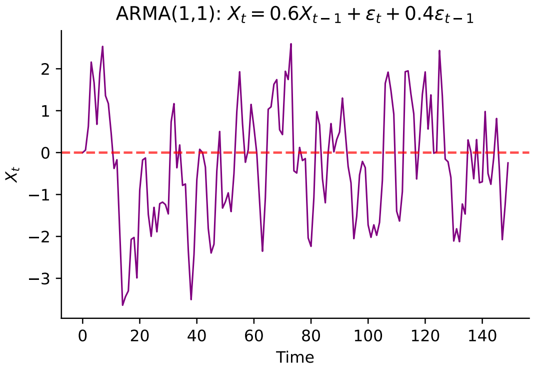
- $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ is the AR polynomial
- $\theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$ is the MA polynomial

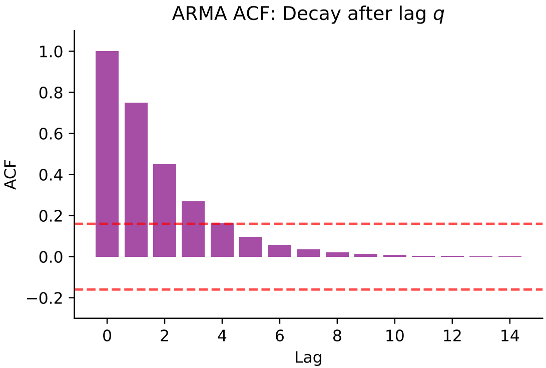
The equation $\phi(L)X_t = \theta(L)\varepsilon_t$ expands to:

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

This is the general **ARMA(p,q)** model.
Special cases: $\theta(L) = 1$ (no MA): Pure AR; $\phi(L) = 1$ (no AR): Pure MA

Visual: ARMA Process





ARMA(1,1) combines AR and MA components. ACF shows decay after initial lag.

Quiz 6: Information Criteria

Question

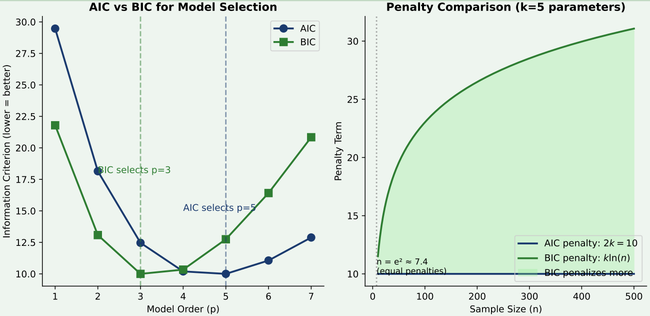
When comparing ARMA(1,1) and ARMA(2,1) using BIC, which statement is correct?

- A. Lower BIC always means better forecasts
- B. BIC penalizes complexity less than AIC
- C. The model with lower BIC is preferred
- D. BIC can only compare models with same number of parameters

Answer on next slide...

Quiz 6: Solution

Answer: C – Lower BIC is preferred



AIC: $-2\ln(\hat{L}) + 2k$ **BIC**: $-2\ln(\hat{L}) + k\ln(n)$ BIC penalizes complexity more $\Rightarrow$ more parsimonious models

Quiz 7: Ljung-Box Test

Question

After fitting an ARMA(2,1) model, you run the Ljung-Box test on residuals and get p-value $= 0.02$. What do you conclude?

A. The model is adequate
B. Residuals are white noise
C. There is significant autocorrelation in residuals
D. The model has too many parameters

Answer on next slide...

Quiz 7: Solution

Answer: C – There is significant autocorrelation in residuals

The Ljung-Box test has:

- H_0: Residuals are white noise (no autocorrelation)
- H_1: Residuals have significant autocorrelation

With p-value $= 0.02 < 0.05$:

- We **reject** H_0
- Conclusion: residuals are **not** white noise
- The model is **inadequate** — significant structure remains

Next step: Try a different model (e.g., increase p or q)

Quiz 8: Forecasting

Question

For an AR(1) model with $\phi = 0.6$ and mean $\mu = 10$, what happens to forecasts as horizon $h \to \infty$?

- A. Forecasts grow without bound
- B. Forecasts converge to 0
- C. Forecasts converge to $\mu = 10$
- D. Forecasts oscillate forever

Answer on next slide...

Quiz 8: Solution

Answer: C – Forecasts converge to $\mu = 10$

For AR(1), the *h*-step ahead forecast is:

$$\hat{X}_{n+h|n} = \mu + \phi^h (X_n - \mu)$$

Since $|\phi| = 0.6 < 1$:

$$\lim_{h \to \infty} \phi^h = 0$$

Therefore:

$$\lim_{h \to \infty} \hat{X}_{n+h|n} = \mu + 0 \cdot (X_n - \mu) = \mu = 10$$

Key insight: Long-run forecasts from stationary ARMA models always converge to the unconditional mean.

Quiz 9: AR(2) Roots

Question

An AR(2) process has characteristic roots $z_1 = 0.8$ and $z_2 = -0.5$. Is it stationary?

A. Yes, because both roots are inside the unit circle
B. No, because one root is negative
C. No, because roots must be outside the unit circle
D. Cannot determine without more information

Answer on next slide...

Quiz 9: Solution

Answer: C – Roots must be outside the unit circle

For AR stationarity, roots of $\phi(z) = 0$ must lie **outside** the unit circle, i.e., $|z| > 1$.
Here: $|z_1| = 0.8 < 1$ and $|z_2| = 0.5 < 1$ – both **inside** unit circle.
$\rightarrow$ **Non-stationary** (actually explosive)
Note: Equivalent condition: coefficients ϕ_1, ϕ_2 must satisfy stationarity triangle.

Quiz 10: MA(q) Properties

Question

For an MA(2) process, the ACF:

A. Decays exponentially
B. Cuts off after lag 2
C. Cuts off after lag 1
D. Never cuts off

Answer on next slide...

Quiz 10: Solution

Answer: B – Cuts off after lag 2

For MA(q), the ACF is exactly zero for lags $> q$.

- MA(1): ACF cuts off after lag 1
- MA(2): ACF cuts off after lag 2
- MA(q): ACF cuts off after lag q

This is the key identification feature: ACF cutoff $\Rightarrow$ MA order.
Meanwhile, PACF of MA processes decays (doesn't cut off).

Quiz 11: ARMA Parsimony

Question

Why might ARMA(1,1) be preferred over AR(5) even if both fit equally well?

A. ARMA models are always better
B. Fewer parameters reduce overfitting risk
C. AR models cannot capture trends
D. MA components are more stable

Answer on next slide...

Quiz 11: Solution

Answer: B – Fewer parameters reduce overfitting risk

Parsimony principle: prefer simpler models.

- ARMA(1,1): 2 parameters (ϕ_1, θ_1)
- AR(5): 5 parameters ($\phi_1, \ldots, \phi_5$)

Fewer parameters means:

- Less risk of overfitting
- Better out-of-sample forecasts
- More interpretable model

BIC penalizes complexity more than AIC, often selecting more parsimonious models.

Quiz 12: Residual Diagnostics

Question

After fitting an ARMA model, the residual ACF shows a significant spike at lag 5. This suggests:

- A. The model is adequate
- B. The model may need higher order terms
- C. Residuals are white noise
- D. The data is non-stationary

Answer on next slide...

Quiz 12: Solution

Answer: B – The model may need higher order terms

Good residuals should be white noise with no significant ACF.
A significant spike at lag 5 indicates remaining autocorrelation structure not captured by the model.
Actions:

- Consider adding AR or MA terms
- Check if AR(5) or MA(5) component helps
- Re-run Ljung-Box test after modification

Quiz 13: Wold Decomposition

Question

The Wold decomposition theorem states that any stationary process can be written as:

- A. A finite AR process
- B. A finite MA process
- C. An infinite MA process plus a deterministic component
- D. An ARIMA process

Answer on next slide...

Quiz 13: Solution

Answer: C – An infinite MA process plus a deterministic component

Wold's theorem: Any stationary process can be written as:

$$X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \eta_t$$

where η_t is deterministic and $\sum \psi_j^2 < \infty$.
Implication: MA(∞) is the most general representation. ARMA models are parsimonious approximations to this infinite MA.

Quiz 14: Unit Root vs Trend Stationary

Question

How do you make a unit root process stationary?

A. Subtract a linear trend
B. Take first differences
C. Apply moving average
D. Use seasonal adjustment

Answer on next slide...

Quiz 14: Solution

Answer: B – Take first differences

- **Unit root** (stochastic trend): Use **differencing**
- **Trend stationary** (deterministic trend): Use **detrending** (regression)

For random walk $X_t = X_{t-1} + \varepsilon_t$:

$$\Delta X_t = X_t - X_{t-1} = \varepsilon_t$$

which is stationary white noise.

True/False Questions

Question

Determine if each statement is True or False:

1. An AR(2) process can exhibit pseudo-cyclical behavior.
2. MA processes require a stationarity condition.
3. The PACF of an AR(p) process cuts off after lag p.
4. If AIC selects ARMA(2,1) and BIC selects ARMA(1,1), they cannot both be correct.
5. Forecast confidence intervals narrow as the forecast horizon increases.
6. The Yule-Walker equations can be used to estimate MA parameters.

Answer on next slide...

True/False: Solutions

Answers

1. **TRUE**: AR(2) with complex roots shows damped oscillations
2. **FALSE**: MA processes are always stationary; they need *invertibility* condition
3. **TRUE**: This is the key identification feature of AR(p)
4. **FALSE**: Both can be "correct" — they optimize different criteria (AIC favors fit, BIC favors parsimony)
5. **FALSE**: Confidence intervals *widen* as horizon increases (more uncertainty)
6. **FALSE**: Yule-Walker is for AR models only; MA uses MLE

Exercise 1: AR(1) Properties

Problem: Consider the AR(1) process:

$$X_t = 2 + 0.7X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, 9)$$

Calculate:

1. The mean μ
2. The variance $\gamma(0)$
3. The autocovariance $\gamma(1)$ and $\gamma(2)$
4. The autocorrelation $\rho(1)$ and $\rho(2)$

Exercise 1: Solution

Given: $c = 2$, $\phi = 0.7$, $\sigma^2 = 9$

1. **Mean:**

$$\mu = \frac{c}{1-\phi} = \frac{2}{1-0.7} = \frac{2}{0.3} = 6.67$$

2. **Variance:**

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2} = \frac{9}{1-0.49} = \frac{9}{0.51} = 17.65$$

3. **Autocovariance:**

$$\gamma(1) = \phi \cdot \gamma(0) = 0.7 \times 17.65 = 12.35$$

$$\gamma(2) = \phi^2 \cdot \gamma(0) = 0.49 \times 17.65 = 8.65$$

4. **Autocorrelation:**

$$\rho(1) = \phi = 0.7, \quad \rho(2) = \phi^2 = 0.49$$

Exercise 2: MA(1) Properties

Problem: Consider the MA(1) process:

$$X_t = 5 + \varepsilon_t - 0.4\varepsilon_{t-1}, \quad \varepsilon_t \sim WN(0, 4)$$

Calculate:

1. The mean μ
2. The variance $\gamma(0)$
3. The autocovariance $\gamma(1)$
4. The autocorrelation $\rho(1)$
5. Is this process invertible?

Exercise 2: Solution

Given: $\mu = 5$, $\theta = -0.4$, $\sigma^2 = 4$

1. **Mean:**
$$\mathbb{E}[X_t] = \mu = 5$$
2. **Variance:**
$$\gamma(0) = \sigma^2(1 + \theta^2) = 4(1 + 0.16) = 4 \times 1.16 = 4.64$$
3. **Autocovariance at lag 1:**
$$\gamma(1) = \theta\sigma^2 = -0.4 \times 4 = -1.6$$
4. **Autocorrelation:**
$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{-1.6}{4.64} = -0.345$$
5. **Invertibility:** $|\theta| = 0.4 < 1 \rightarrow$ **Yes, invertible**

Exercise 3: Characteristic Roots

Problem: Consider the AR(2) process:

$$X_t = 0.5X_{t-1} + 0.3X_{t-2} + \varepsilon_t$$

1. Write the characteristic equation
2. Find the characteristic roots
3. Is this process stationary?

Exercise 3: Solution

1. Characteristic equation:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 1 - 0.5z - 0.3z^2 = 0$$

Or: $0.3z^2 + 0.5z - 1 = 0$

2. Roots (using quadratic formula):

$$z = \frac{-0.5 \pm \sqrt{0.25 + 1.2}}{0.6} = \frac{-0.5 \pm 1.204}{0.6}$$

$$z_1 = \frac{0.704}{0.6} = 1.17, \quad z_2 = \frac{-1.704}{0.6} = -2.84$$

3. Stationarity check:
Both roots have $|z| > 1$: $|z_1| = 1.17 > 1$ and $|z_2| = 2.84 > 1$
$\rightarrow$ **Stationary** (roots outside unit circle)

Exercise 4: Forecasting

Problem: You have fit an AR(1) model:

$$X_t = 3 + 0.8X_{t-1} + \varepsilon_t, \quad \sigma^2 = 4$$

Given $X_{100} = 20$, calculate:

1. The 1-step ahead forecast $\hat{X}_{101|100}$
2. The 2-step ahead forecast $\hat{X}_{102|100}$
3. The long-run forecast $\hat{X}_{100+h|100}$ as $h \to \infty$
4. The 95% confidence interval for $\hat{X}_{101|100}$

Exercise 4: Solution

Given: $c = 3$, $\phi = 0.8$, $\sigma^2 = 4$, $X_{100} = 20$
Mean: $\mu = \frac{3}{1-0.8} = 15$

1. **One-step forecast:**

$$\hat{X}_{101|100} = c + \phi X_{100} = 3 + 0.8 \times 20 = 19$$

2. **Two-step forecast:**

$$\hat{X}_{102|100} = c + \phi \hat{X}_{101|100} = 3 + 0.8 \times 19 = 18.2$$

3. **Long-run forecast:**

$$\lim_{h \to \infty} \hat{X}_{100+h|100} = \mu = 15$$

4. **95% CI for 1-step:**

$$\text{MSFE}(1) = \sigma^2 = 4, \quad \sqrt{\text{MSFE}(1)} = 2$$

$$CI : 19 \pm 1.96 \times 2 = [15.08, 22.92]$$

Python Exercise 1: Simulate and Fit AR(1)

Task:

1. Simulate 500 observations from AR(1) with $\phi = 0.7$
2. Plot the series and ACF/PACF
3. Fit an AR(1) model and check if $\hat{\phi} \approx 0.7$
4. Examine residual diagnostics

Hint code:

```
np.random.seed(42)
n = 500
phi = 0.7
x = np.zeros(n)
for t in range(1, n):
  x[t] = phi * x[t-1] + np.random.randn()
```

Python Exercise 2: Model Selection

Task:

1. Load a real time series (e.g., stock returns)
2. Check stationarity using ADF test
3. Compare AIC/BIC for ARMA(1,0), ARMA(0,1), ARMA(1,1), ARMA(2,1)
4. Select the best model
5. Generate forecasts with confidence intervals

Key functions:

- `adfuller()` for stationarity test
- `ARIMA(data, order=(p,0,q)).fit()` for fitting
- `results.aic`, `results.bic` for criteria
- `results.get_forecast(h)` for predictions

Python Exercise 3: Diagnostic Checking

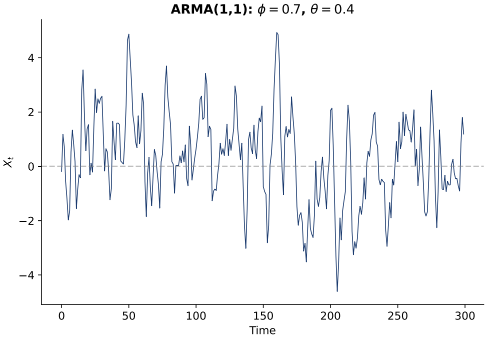
Task: After fitting a model, perform complete diagnostics:

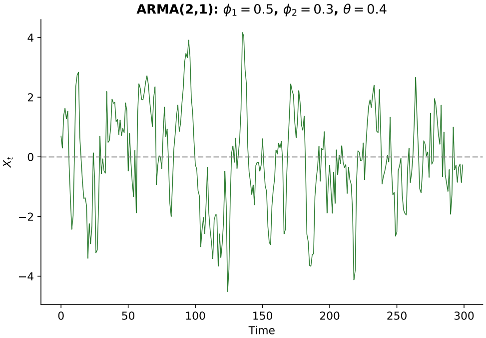
1. Plot residuals over time
2. Plot ACF of residuals
3. Create Q-Q plot
4. Run Ljung-Box test
5. Check if AR/MA roots are outside unit circle

Key functions:

- `results.resid` for residuals
- `plot_acf(resid)` for ACF plot
- `stats.probplot(resid)` for Q-Q plot
- `acorr_ljungbox(resid)` for portmanteau test
- `results.arroots`, `results.maroots` for roots

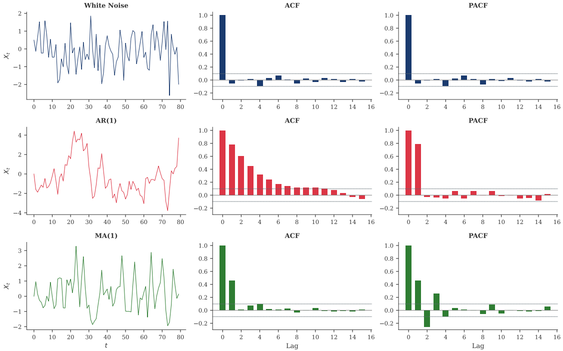
Case Study: Industrial Production Index





- US Industrial Production: monthly data, already stationary (growth rates)
- Shows typical ARMA patterns: mean-reverting with short-term dependence
- Volatility clustering visible – ARMA captures the conditional mean
- Suitable for ARMA modeling without differencing

ACF/PACF Pattern Recognition



- ACF shows gradual decay – suggests AR component
- PACF cuts off after lag 2 – suggests AR(2) might be appropriate
- Some significant lags in ACF beyond lag 2 – MA terms may help
- Pattern consistent with ARMA(2,1) or similar low-order models

ARMA Estimation Results

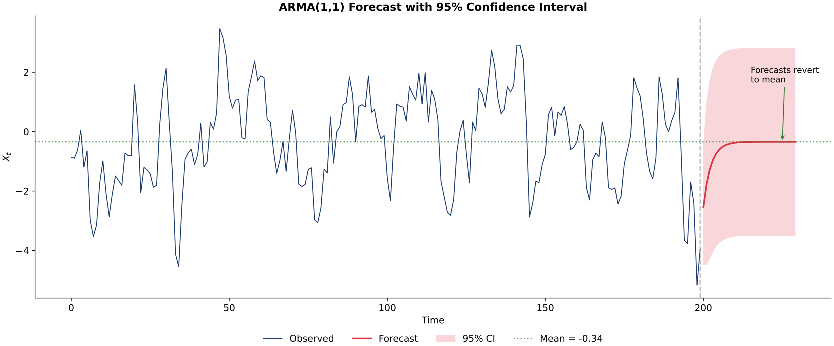
Model: ARMA(2,1) for Industrial Production Growth

Parameter	Estimate	Std. Error	z-stat	p-value
c (const)	0.156	0.048	3.25	0.001
ϕ_1 (AR.L1)	0.423	0.089	4.75	< 0.001
ϕ_2 (AR.L2)	0.187	0.072	2.60	0.009
θ_1 (MA.L1)	-0.156	0.091	-1.71	0.087

Model Selection

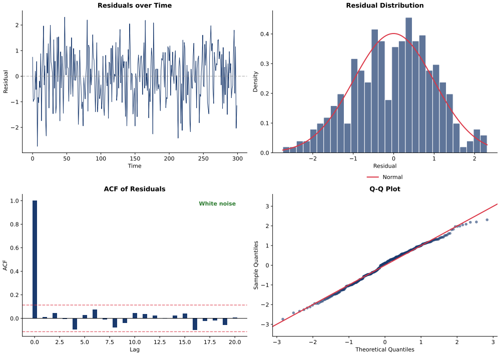
AIC: -412.5, BIC: -398.2. Model passes stationarity and invertibility checks.

Forecast Performance



- ARMA forecasts mean-revert toward unconditional mean
- Short-term forecasts capture recent dynamics
- Confidence intervals expand with forecast horizon
- Comparison with naive forecast shows ARMA improvement

Model Diagnostics



- Residuals appear random with no systematic patterns
- ACF of residuals within confidence bounds
- Q-Q plot shows approximate normality
- Ljung-Box test: $p > 0.05$ – no significant autocorrelation in residuals

Discussion 1: Model Selection

Scenario: You're modeling monthly inflation rates. After checking stationarity (passed), you find:

- ACF: significant at lags 1, 2, 3, then decays
- PACF: significant at lags 1, 2, then cuts off
- AIC selects ARMA(2,3)
- BIC selects ARMA(2,0) = AR(2)

Questions:

1. What does the ACF/PACF pattern suggest?
2. Why do AIC and BIC disagree?
3. Which model would you choose and why?
4. What additional checks would you perform?

Discussion 2: Forecast Evaluation

Scenario: You fit an ARMA(1,1) model to daily stock returns. The in-sample fit looks good (Ljung-Box p-value $= 0.45$), but out-of-sample RMSE is worse than a simple random walk forecast.

Questions:

1. Is this surprising? Why or why not?
2. What does this tell us about stock return predictability?
3. Should you conclude the ARMA model is useless?
4. What alternatives might you consider?

Hint: Think about the Efficient Market Hypothesis and what ARMA captures vs. what it doesn't (e.g., volatility clustering).

Discussion 3: Real-World Application

Scenario: A central bank economist asks you to forecast quarterly GDP growth for policy planning.

Questions:

1. What preliminary analysis would you do before fitting ARMA?
2. GDP is often non-stationary — how would you handle this?
3. Would you use AIC or BIC for model selection? Why?
4. How would you communicate forecast uncertainty to policymakers?
5. What limitations of ARMA models should you mention?

Key Takeaways from Today's Seminar

1. **AR models:** Current value depends on past values
   - Stationarity: $|\phi| < 1$ for AR(1)
   - PACF cuts off at lag p
2. **MA models:** Current value depends on past shocks
   - Always stationary; invertibility: $|\theta| < 1$ for MA(1)
   - ACF cuts off at lag q
3. **Model selection:** Use ACF/PACF patterns + information criteria
4. **Diagnostics:** Residuals must be white noise (Ljung-Box test)
5. **Forecasting:** Point forecasts converge to mean; uncertainty grows

Next Seminar: ARIMA and Seasonal Models