Time Series Analysis and Forecasting

Chapter 1: Stochastic Processes and Stationarity

Daniel Traian PELE

Academia de Studii Economice din București
IDA Institute Digital Assets
Blockchain Research Center
AI4EFin Artificial Intelligence for Energy Finance
Academia Română, Institutul de Prognoză Economică
MSCA Digital Finance

Learning Objectives

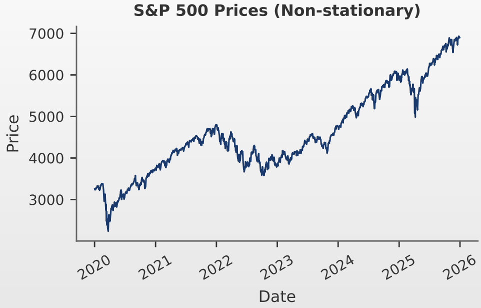
By the end of this chapter, you will be able to:

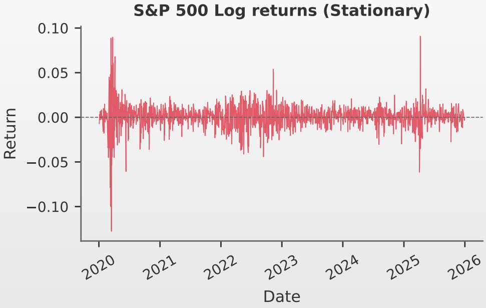
1. **Define** stochastic processes and understand their properties
2. **Distinguish** between strict and weak (covariance) stationarity
3. **Identify** white noise and random walk processes
4. **Compute** and interpret ACF and PACF
5. **Apply** the lag operator and differencing
6. **Conduct** stationarity tests (ADF, KPSS)
7. **Analyze** financial time series data
8. **Distinguish** between unit root and trend-stationary processes

Chapter Outline

- Motivation
- Stochastic Processes
- Stationarity
- Lag Operator and Differencing
- White Noise and Random Walk
- Autocorrelation Functions
- Testing for Stationarity
- Financial Data Application
- Case Study: Stationarity Testing
- AI Use Case
- Summary
- Quiz

Examples: stationary vs. non-stationary series





Observations

- **Prices** (left) are non-stationary: trend, the mean changes over time
- **Returns** (right) are stationary: mean ≈ 0, approximately constant variance
- Log returns: $r_t = \ln P_t - \ln P_{t-1} \succ$ non-stationary $\rightarrow$ stationary

TSA_ch1_stationarity

Stochastic process: definition

Definition 1 (Stochastic Process)

- A **stochastic process** is a collection of random variables indexed by time
  - $\{X_t(\omega) : t \in \mathcal{T}, \omega \in \Omega\}$
  - Ω is the sample space of possible outcomes

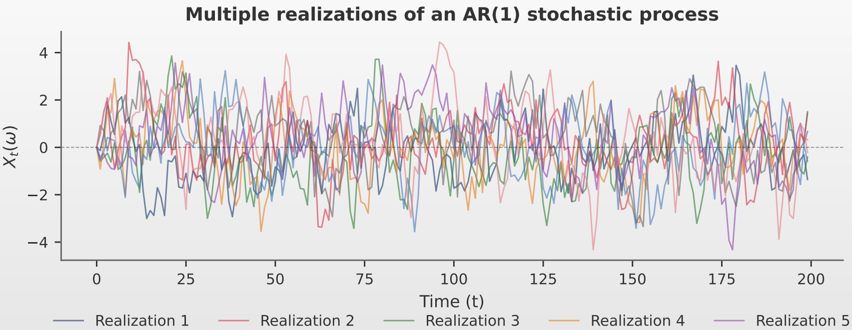
Two Perspectives

- **Fixed** ω: A *realization* $\{X_t(\omega)\}_{t\in\mathcal{T}}$
- **Fixed** t: A *random variable* X_t

Key Insight

- A time series we observe is **one realization** of the underlying stochastic process

Stochastic process: visual illustration



Interpretation

- Each line is a **different realization** from the same underlying stochastic process
- We observe only **one realization**, yet aim to understand the properties of the process

TSA_ch1_stationarity

Moments of a stochastic process

The First Two Moments Characterize the Process

- **Mean Function**: $\mu_t = \mathbb{E}[X_t]$
- **Autocovariance (ACVF)**: $\gamma(t, s) = \text{Cov}(X_t, X_s)$
  - $\gamma(t, s) = \mathbb{E}[(X_t - \mu_t)(X_s - \mu_s)]$
- **Autocorrelation (ACF)**:
  - $\rho(t, s) = \gamma(t, s)/\sqrt{\text{Var}(X_t) \cdot \text{Var}(X_s)}$

ACF Properties

- **Range**: $\rho(t, s) \in [-1, 1]$
- **Normalization**: $\rho(t, t) = 1$ (perfect correlation with itself)

Key Point

- **General**: μ_t and $\gamma(t, s)$ may depend on t
- **Stationary**: Removes this dependence

Why stationarity matters

Without Stationarity

- Mean, variance change over time
  - Estimates are inconsistent
- Past may not predict the future
- Standard methods fail
- Spurious correlations

With Stationarity

- Statistical properties constant
  - Ergodicity justified
- Can estimate from a single realization
- Valid inference possible
- Models are meaningful

Key Principle

- Most time series models (ARMA, ARIMA, etc.) require stationarity
- Non-stationary series must be transformed (e.g., differencing) before modeling

Strict stationarity

Definition 2 (Strict (Strong) Stationarity)

- A process $\{X_t\}$ is **strictly stationary** if for all k, all $t_1, \ldots, t_k$, and all h:
  - $(X_{t_1}, \ldots, X_{t_k}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_k+h})$
- **Notation**: $X \stackrel{d}{=} Y$ means *equality in distribution*
  - $P(X \leq x) = P(Y \leq x)$

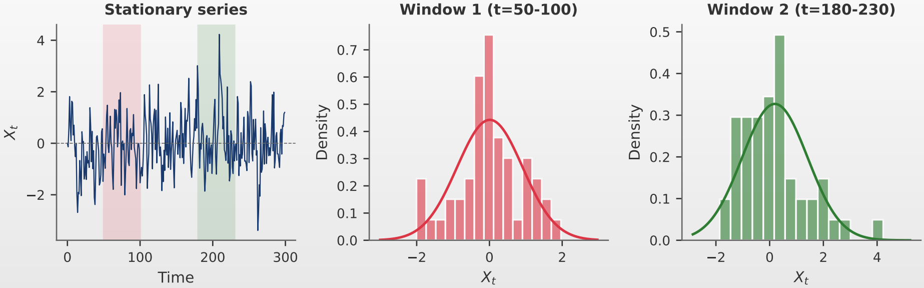
Implications

- **Identical distributions**: $F_{X_t}(x)$ does not depend on t
  - $\mathbb{E}[X_t] = \mu$ (constant mean, if it exists)
  - $\text{Var}(X_t) = \sigma^2$ (constant variance, if it exists)
- **Lag dependence**: Joint distributions depend only on lag

Note

- Strict stationarity is a strong condition, often impossible to verify in practice

Strict stationarity: visual illustration



Interpretation

- Time translation does not change the joint distribution of the variables
- Any two time windows have the same statistical properties
- In practice: we only check the first moments (weak stationarity)

TSA_ch1_stationarity

Weak (covariance) stationarity

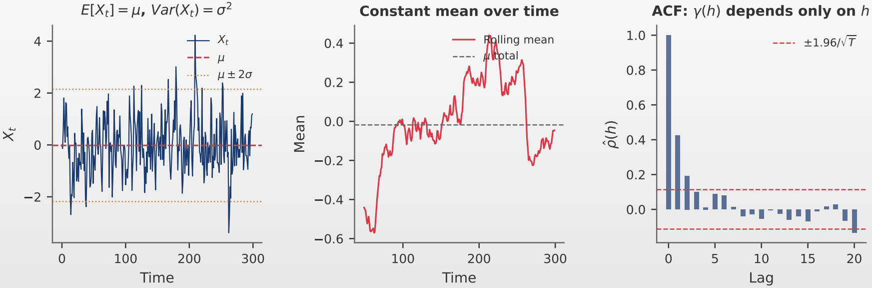
Definition 3 (Weak Stationarity)

- A process $\{X_t\}$ is **weakly stationary** (or covariance stationary) if:
  - $\mathbb{E}[X_t^2] < \infty$ for all t — finite second-order moments
  - $\mathbb{E}[X_t] = \mu$ for all t — constant mean
  - $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ — covariance depends only on lag h, not on t

Key Properties

- **Autocovariance**: $\gamma(h) = \text{Cov}(X_t, X_{t+h}) = \mathbb{E}[(X_t - \mu)(X_{t+h} - \mu)]$
- **Autocorrelation**: $\rho(h) = \gamma(h)/\gamma(0) = \text{Cov}(X_t, X_{t+h})/\text{Var}(X_t)$
- **Note**: $\rho(0) = 1$, $|\rho(h)| \leq 1$, $\rho(h) = \rho(-h)$ (symmetry)

Weak stationarity: visual illustration



The Three Conditions

- $\mathbb{E}[X_t] = \mu$ constant $\succ$ mean does not depend on time
- $\text{Var}(X_t) = \sigma^2$ constant $\succ$ variance does not depend on time
- $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ $\succ$ autocovariance depends only on lag h

TSA_ch1_stationarity

Relationship between strict and weak stationarity

Theorem 1 (Fundamental Implication)

If $\{X_t\}$ is **strictly stationary** and $\mathbb{E}[X_t^2] < \infty$, then $\{X_t\}$ is also **weakly stationary**.

Proof.

- Let t_1, t_2 be arbitrary and h any time shift
- From joint distribution invariance: $(X_{t_1}, X_{t_2}) \stackrel{d}{=} (X_{t_1+h}, X_{t_2+h})$
- $\mathbb{E}[X_{t_1}] = \mathbb{E}[X_{t_1+h}] = \mu$ (constant mean)
- $\text{Cov}(X_{t_1}, X_{t_2}) = \text{Cov}(X_{t_1+h}, X_{t_2+h})$
- Thus autocovariance depends only on the difference $t_2 - t_1 = h$, not on t_1

$\square$

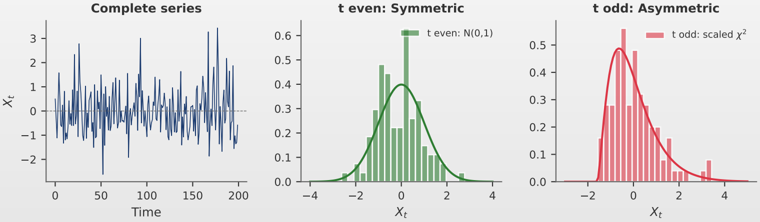
Warning: The Converse is NOT True!

- There exist weakly stationary processes that are **not** strictly stationary

Counterexample: weakly stationary but NOT strictly stationary

Construction

- Let $\{X_t\}$ be **independent** random variables with: t even: $X_t \sim N(0,1)$; t odd: $X_t \sim \frac{\chi^2(5)-5}{\sqrt{10}}$



Weakly stationary ✓

- $\mathbb{E}[X_t] = 0$, $\mathrm{Var}(X_t) = 1$, $\mathrm{Cov}(X_t, X_{t+h}) = 0$

NOT strictly stationary ×

- Skewness differs (0 vs > 0) $\succ X_1 \overset{d}{\neq} X_2$

TSA_ch1_stationarity

Properties of the autocovariance function

Proposition 1

For a weakly stationary process, the ACVF $\gamma(h)$ satisfies:

- **Symmetry:** $\gamma(h) = \gamma(-h)$
- **Maximum at zero:** $|\gamma(h)| \leq \gamma(0) = \mathrm{Var}(X_t)$
- **Non-negative definiteness:** $\sum_{i,j} a_i a_j \gamma(i-j) \geq 0$ for any $a_1, \ldots, a_n$

Proof (property 3)

- $\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_{t+i}\right) = \sum_{i,j} a_i a_j \gamma(i-j) \geq 0 \quad$ (variance ≥ 0)

Implication

- Not every function can be a valid autocovariance function

Ergodicity: the foundation of inference from data

Definition 4 (Ergodicity for Mean)

- A stationary process $\{X_t\}$ is **ergodic for the mean** if:
  - $\bar{X}_T = \frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow{P} \mathbb{E}[X_t] = \mu$ as $T \to \infty$

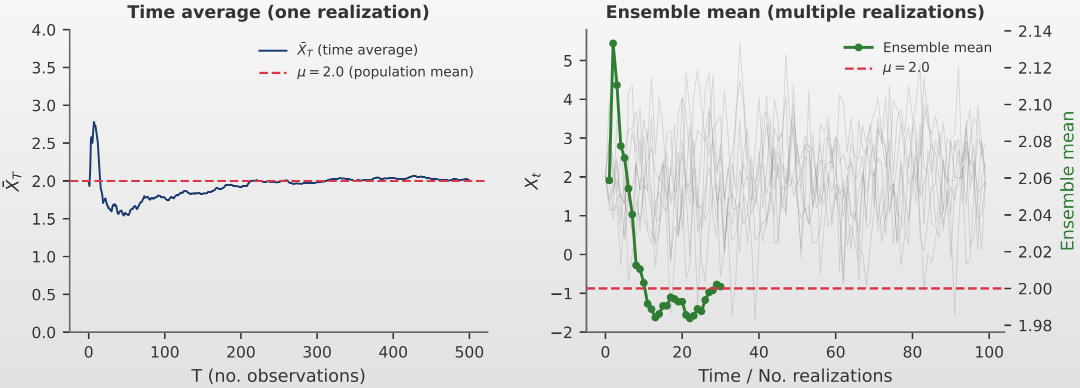
Why does ergodicity matter?

- **Problem**: We have only **one realization** of the stochastic process
- **Solution**: Ergodicity allows estimating μ from $\bar{X}_T$
  - The time average converges to the population mean
  - Without ergodicity, statistical inference is not possible!

Theorem 2 (Sufficient Condition)

If $\sum_{h=0}^{\infty} |\gamma(h)| < \infty$ (absolutely summable autocovariances), the process is ergodic.

Ergodicity: visual illustration



- **Time average** (single realization) and **ensemble average** (multiple realizations) both converge to μ
- Ergodicity guarantees that we can estimate μ from a **single sufficiently long time series**

TSA_ch1_stationarity

The Wold decomposition theorem

Theorem 3 (Wold, 1938)

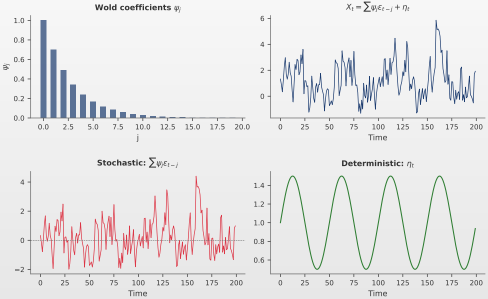
Any **covariance stationary** process $\{X_t\}$ can be written as: $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \eta_t$

- $\varepsilon_t \sim WN(0, \sigma^2) \succ$ white noise
  - $\psi_0 = 1$, $\sum \psi_j^2 < \infty$
- $\eta_t \succ$ deterministic component (perfectly predictable)

Significance of the Wold Theorem

- **Decomposition**: Any stationary process = **MA(∞)** + deterministic component
  - Theoretically justifies MA(q) and ARMA(p, q) models
  - Coefficients ψ_j measure the impact of past shocks

The Wold theorem: visual illustration



Interpretation

- X_t decomposes into a **stochastic** component (MA(∞)) and a **deterministic** component (η_t)
- Coefficients ψ_j decay $\succ$ recent shocks have greater impact than distant ones

TSA_ch1_operators

The lag operator

Definition 5 (Lag Operator)

- The **lag operator** (or backshift operator) L is defined by: $LX_t = X_{t-1}$

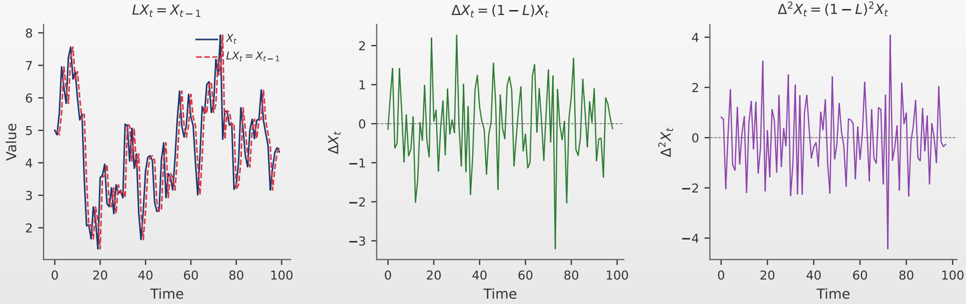
Properties

- **Powers**: $L^k X_t = X_{t-k}$ (lag by k periods)
  - Compact notation for models
- **Identity**: $L^0 = I$
- **Polynomial**: $(1 - \phi L)X_t = X_t - \phi X_{t-1}$

Examples

- **First difference**: $(1 - L)X_t = X_t - X_{t-1}$
- **Second difference**: $(1 - L)^2 X_t = \Delta^2 X_t$
- **Seasonal**: $(1 - L^{12})X_t$

Lag operator: visual illustration



Properties

- $LX_t = X_{t-1}$ ≻ the lag operator shifts the series back by one period
- $L^k X_t = X_{t-k}$ ≻ shift by k periods; $L^0 = I$ (identity)
- **Difference operator**: $\Delta = (1-L)$, so $\Delta X_t = X_t - X_{t-1}$

TSA_ch1_operators

Differencing

Why Do We Difference?

- **First Difference**: $\Delta X_t = X_t - X_{t-1} = (1 - L)X_t$
  - Removes trend and unit root
  - Random walk: $\Delta X_t = \varepsilon_t$

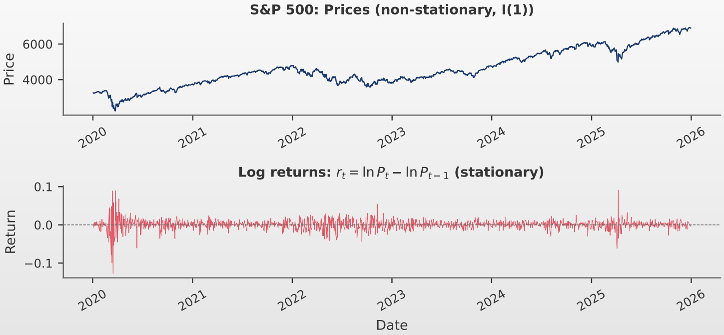
Definition 6 (Integrated Process of Order d)

- A process $\{X_t\}$ is **integrated of order** d, denoted $X_t \sim I(d)$, if:
  - $\Delta^d X_t = (1 - L)^d X_t$ is stationary ($I(0)$ process)
  - $\Delta^{d-1} X_t$ is **not** stationary

Examples

- $I(0)$: Stationary process (white noise, stationary AR)
- $I(1)$: Random walk $\succ$ $\Delta X_t = \varepsilon_t$ is stationary
- $I(2)$: Requires two differences for stationarity

Effect of differencing: S&P 500



Interpretation

- **Top**: S&P 500 prices $\succ$ clear trend, non-stationary ($I(1)$)
- **Bottom**: Log returns $r_t = \ln P_t - \ln P_{t-1}$ $\succ$ fluctuates around mean ≈ 0, stationary

TSA_ch1_operators

White noise process

Definition 7 (White Noise)

- A process $\{\varepsilon_t\}$ is **white noise**, denoted $\varepsilon_t \sim WN(0, \sigma^2)$, if:
  - $\mathbb{E}[\varepsilon_t] = 0$ for all t (zero mean)
  - $\mathrm{Var}(\varepsilon_t) = \sigma^2$ for all t (constant variance)
  - $\mathrm{Cov}(\varepsilon_t, \varepsilon_s) = 0$ for $t \neq s$ (uncorrelated)

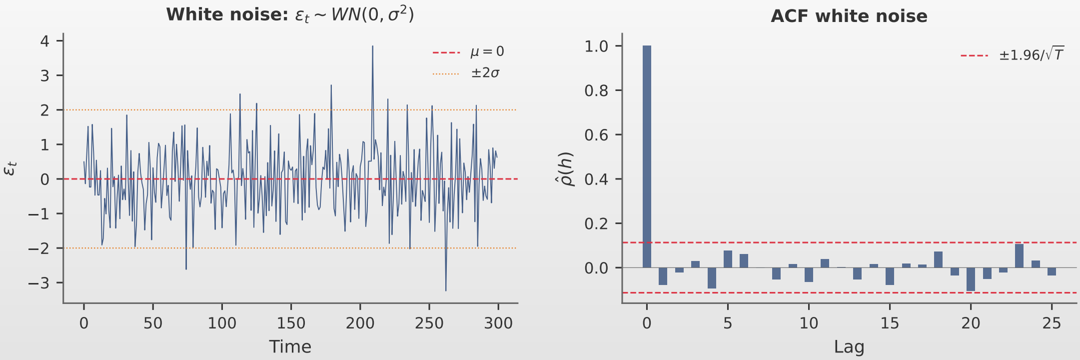
ACF of White Noise

- By definition: $\gamma(0) = \sigma^2$ and $\gamma(h) = 0$ for $h \neq 0$; $\rho(h) = \begin{cases} 1 & h = 0 \\ 0 & h \neq 0 \end{cases}$

Types of white noise (in order of increasing restrictions)

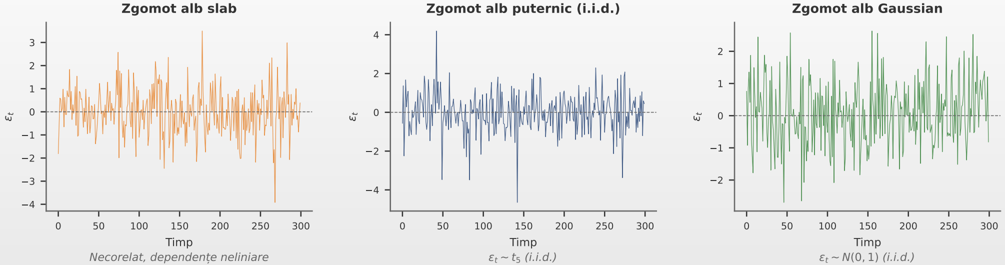
- **Weak**: uncorrelated, but nonlinear dependencies may exist
- **Strong**: ε_t are *independent* and identically distributed (i.i.d.)
- **Gaussian**: $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$
  - Uncorrelated $\Rightarrow$ independent

White noise: visual illustration



TSA_ch1_white_noise

The three types of white noise



Inclusion relationship: Gaussian $\subset$ Strong (i.i.d.) $\subset$ Weak (uncorrelated)

- **Weak**: $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$, but nonlinear dependencies may exist (e.g. GARCH)
- **Strong**: ε_t are i.i.d. — any distribution (e.g. Student-t)
- **Gaussian**: $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$ — uncorrelated $\Leftrightarrow$ independent

TSA_ch1_white_noise

Random walk process

Definition 8 (Random Walk)

$X_t = X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2), \; X_0 = 0 \quad \Rightarrow$ **Explicit form**: $X_t = \sum_{i=1}^{t} \varepsilon_i$

Proposition 2 (Properties)

- $\mathbb{E}[X_t] = 0$
- $\mathrm{Var}(X_t) = t\sigma^2$ (grows with time!)
- $\mathrm{Cov}(X_t, X_s) = \min(t, s) \cdot \sigma^2$

Proofs.

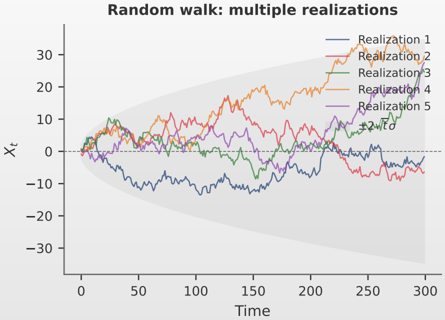
- $\mathbb{E}[X_t] = \mathbb{E}\left[\sum_{i=1}^{t} \varepsilon_i\right] = 0$
- $\mathrm{Var}(X_t) = \mathrm{Var}\left(\sum_{i=1}^{t} \varepsilon_i\right) = \sum_{i=1}^{t} \mathrm{Var}(\varepsilon_i) = t\sigma^2$ (independence)
- $\mathrm{Cov}(X_t, X_s) = \min(t, s)\,\sigma^2$ (for $s \leq t$)

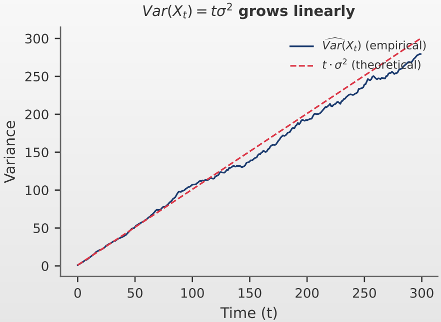
$\square$

Non-Stationary!

$\mathrm{Var}(X_t) = t\sigma^2$ depends on t $\succ$ random walk is **not stationary**

Random walk: visualization





Observations

- Each shock has a **permanent effect**; $\text{Var}(X_t) = t\sigma^2$ grows linearly with time
- **Solution** — differencing transforms into white noise, $\Delta X_t = \varepsilon_t$

TSA_ch1_random_walk

Random walk with drift

Definition 9 (Random Walk with Drift)

$X_t = c + X_{t-1} + \varepsilon_t$, $c \neq 0$ is the **drift** $\Rightarrow$ **Explicit form**: $X_t = ct + \sum_{i=1}^{t} \varepsilon_i$

Proposition 3 (Properties)

- $\mathbb{E}[X_t] = ct$ (linear trend)
- $\text{Var}(X_t) = t\sigma^2$ (grows with time)

Differencing

$\Delta X_t = c + \varepsilon_t$ — constant plus white noise $\succ$ the differenced series is stationary

Practical Importance

- Nominal GDP, stock prices $\succ$ often modeled as RW with drift
- The ADF test includes variants: without constant, with constant, with constant and trend

Trend-stationary vs. difference-stationary

Trend-Stationary (TS)

- **Model**: $Y_t = \alpha + \beta t + \varepsilon_t$
  - **Deterministic** trend
  - Deviations from the trend are temporary
- **Solution**: regression on t, extract residuals
- **Effect**: Shocks do NOT have a permanent effect

Difference-Stationary (DS)

- **Model**: $Y_t = c + Y_{t-1} + \varepsilon_t$
  - **Stochastic** trend
  - Deviations from the trend are permanent
- **Solution**: differencing (ΔY_t)
- **Effect**: Shocks HAVE a permanent effect

Why does the distinction matter?

- **Differencing a TS process**: introduces an artificial unit root in the MA part
- **Regression on a DS process**: produces residuals that are **still non-stationary**
- **Solution**: ADF and KPSS tests help distinguish between the two

White noise vs random walk: comparison

White Noise

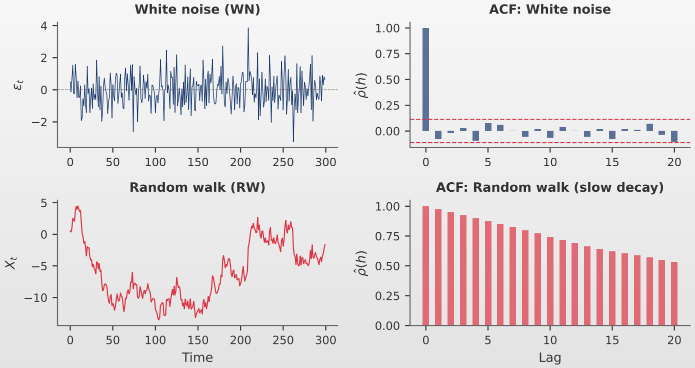
- Stationary
- $\text{Var} = \sigma^2$ (const.)
- $\text{ACF} = 0,\ h \neq 0$
- No memory

Random Walk

- Non-stationary
- $\text{Var} = t\sigma^2$ (grows)
- $\text{ACF} \approx 1$ (slow)
- Permanent shocks

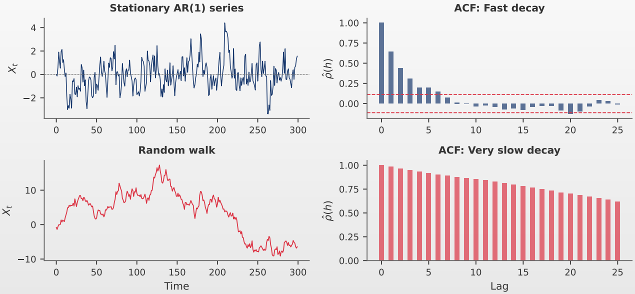
Link

- $\Delta X_t = \varepsilon_t$



TSA_ch1_random_walk

ACF comparison: stationary vs random walk



Interpretation

- **Stationary**: ACF decays rapidly (exponentially or oscillating) toward zero
- **Random walk**: ACF decays very slowly, stays close to 1
- **Rule of thumb**: Slow ACF decay $\succ$ suspect unit root $\succ$ ADF test

TSA_ch1_random_walk

Sample autocorrelation function

Sample ACF at Lag h

- $\hat{\rho}(h) = \frac{\sum_{t=1}^{T-h}(x_t - \bar{x})(x_{t+h} - \bar{x})}{\sum_{t=1}^{T}(x_t - \bar{x})^2}$
  - Properties: $\hat{\rho}(0) = 1$, $|\hat{\rho}(h)| \leq 1$

Theorem 4 (Bartlett, 1946)

Under H_0: white noise, for large T: $\hat{\rho}(h) \approx N(0, 1/T)$

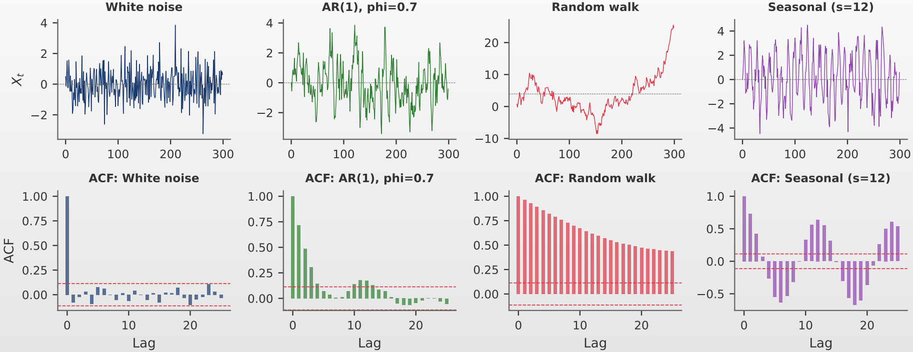
95% Confidence Interval

- $\pm 1.96/\sqrt{T}$ (the bands in ACF plots)

Caution

- Bartlett's formula is valid **only under H_0: white noise**
- For AR/MA, the asymptotic variance differs

ACF patterns for different processes



Interpretation

- **White noise**: ACF = 0; **Stationary**: decays fast; **Non-stationary**: decays slowly
- **Seasonal**: Spikes at seasonal lags (12, 24 for monthly data)

TSA_ch1_acf_patterns

Partial autocorrelation function (PACF)

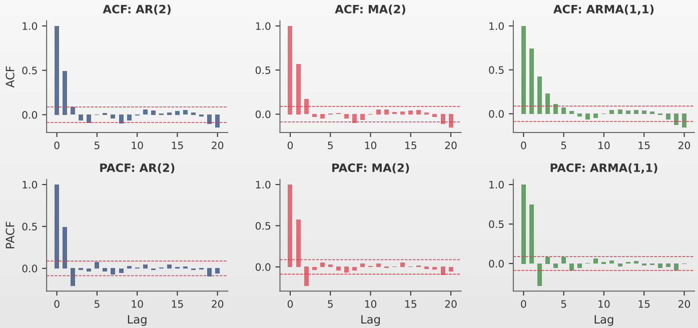
Definition 10 (Partial Autocorrelation)

- **PACF** at lag h, denoted ϕ_{hh}: the last coefficient in the regression:
  - $X_t = \phi_{h1}X_{t-1} + \phi_{h2}X_{t-2} + \cdots + \phi_{hh}X_{t-h} + e_t$
- **Alternatively**:
  - $\phi_{hh} = \text{Corr}(X_t - \hat{X}_t^{(h-1)}, X_{t-h} - \hat{X}_{t-h}^{(h-1)})$
- **Interpretation**: *Direct* dependence at lag h
  - Removes the effect of intermediate lags

Key Application: Model Order Identification

- **AR(p)**: PACF **cuts off** after lag p
  - ACF decays exponentially or oscillates
- **MA(q)**: ACF **cuts off** after lag q
  - PACF decays exponentially or oscillates

ACF and PACF patterns

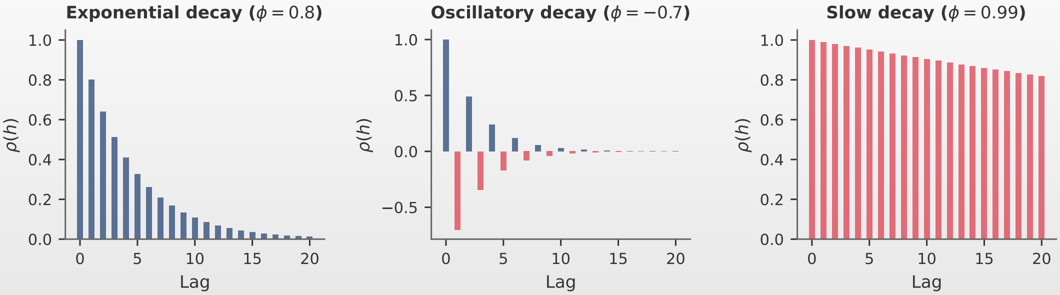


Identification Rules

- **AR(p)**: ACF decays exponentially, PACF cuts off after lag p
- **MA(q)**: ACF cuts off after lag q, PACF decays exponentially
- **ARMA(p, q)**: Both decay exponentially $\succ$ identification requires information criteria

TSA_ch1_acf_patterns

ACF decay patterns



Interpretation

- **Exponential decay**: Persistent positive dependence (AR with $\phi > 0$)
- **Oscillating decay**: Alternating dependence (AR with $\phi < 0$)
- The decay rate indicates the strength of the process memory

TSA_ch1_acf_patterns

Augmented Dickey-Fuller (ADF) test

ADF Model

$\Delta X_t = \alpha + \gamma X_{t-1} + \sum_{i=1}^{p} \delta_i \Delta X_{t-i} + \varepsilon_t, \quad \gamma = \rho - 1, \quad H_0 : \gamma = 0 \Leftrightarrow \rho = 1$

Hypotheses

- H_0: $\gamma = 0$ (unit root)
- H_1: $\gamma < 0$ (stationary)

Test Statistic

- $\tau_{ADF} = \hat{\gamma} \,/\, SE(\hat{\gamma})$
- $\hat{\gamma}$ = OLS coefficient of X_{t-1}
- $SE(\hat{\gamma}) = \hat{\sigma}_\varepsilon / \sqrt{\sum X_{t-1}^2}$

Decision Rule

- $\tau_{ADF} <$ critical value $\succ$ Reject H_0 $\succ$ Stationary
- $\tau_{ADF} \geq$ critical value $\succ$ Non-stationary (unit root)
- Critical values follow the Dickey-Fuller distribution (**not** t-Student!)

KPSS test

Model

- $X_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$

Hypotheses (opposite of ADF)

- H_0: $\sigma_u^2 = 0$ (stationary)
- H_1: $\sigma_u^2 > 0$ (unit root)

Test Statistic

- $LM = \frac{\sum_{t=1}^{T} S_t^2}{T^2 \hat{\sigma}^2}$
- where $S_t = \sum_{i=1}^{t} \hat{e}_i$

Decision Rule

- $LM >$ critical value $\succ$ Reject H_0 $\succ$ Non-stationary
- $LM \leq$ critical value $\succ$ Stationary

Using ADF and KPSS together

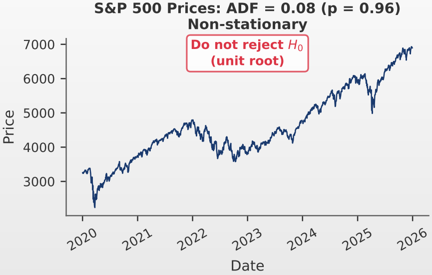
Confirmatory Testing

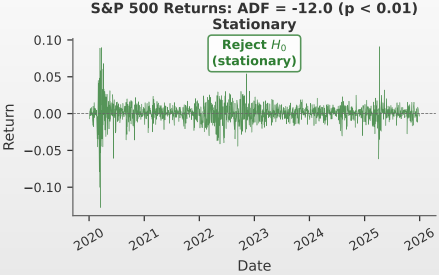
- **ADF rejects H_0 + KPSS fails to reject**: Stationary
- **ADF fails to reject + KPSS rejects H_0**: Unit Root
- **Both reject or both fail to reject**: Inconclusive
  - Additional tests required (PP, DF-GLS)

Workflow

- **Step 1**: ADF test (H_0: unit root)
- **Step 2**: KPSS test (H_0: stationary)
- **Step 3**: Concordant results $\succ$ OK
  - Otherwise: PP, DF-GLS tests

ADF test: visualization with S&P 500



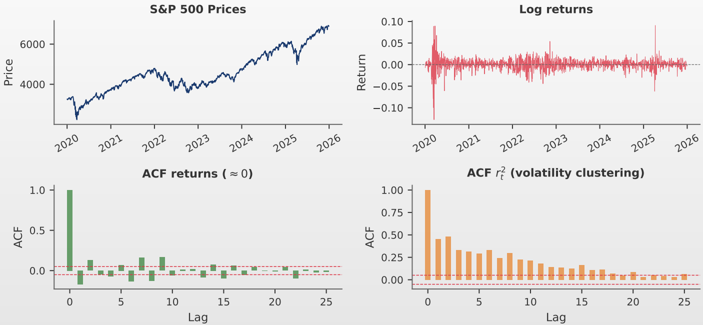


TSA_ch1_unit_root_tests

Interpreting the ADF Test

- **Hypothesis**: H_0: Unit root
  - Critical values: -3.43 (1%), -2.86 (5%), -2.57 (10%)
  - $\tau <$ critical value $\succ$ reject H_0 $\succ$ stationary series
- **S&P 500**: Prices non-stationary; Returns stationary

S&P 500 analysis: overview

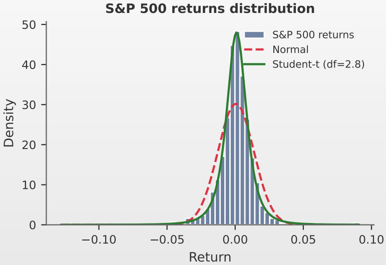


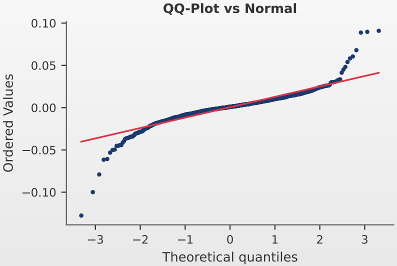
Observations

- **Prices**: Upward trend, non-stationary; **Returns**: Mean ≈ 0, stationary
- **ACF returns**: ≈ 0 (efficient); **ACF** r_t^2: Significant (volatility clustering)

TSA_ch1_ts_basics

Stylized facts of financial returns





Observed Properties

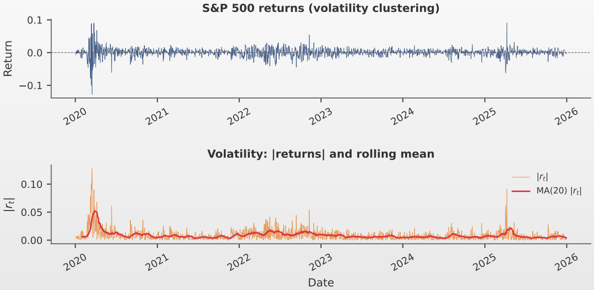
- Negative skewness (left tail)
- Excess kurtosis ($\gg 3$)
- Heavy tails (fat tails)

Implications

- Normal distribution inadequate
- Extreme events more likely
- Student-t or GED required

TSA_ch1_ts_basics

Volatility clustering

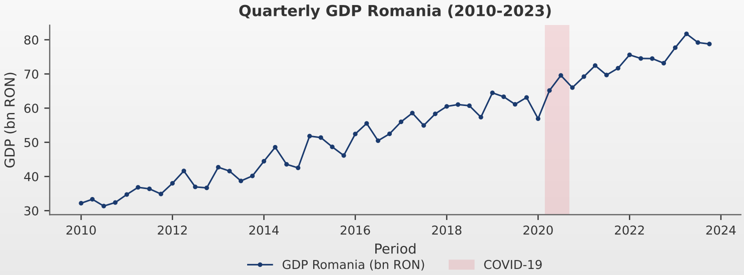


Observations

- Large returns (in absolute value) followed by large returns
- Calm periods followed by calm periods
- **Time-varying volatility** $\succ$ ARCH/GARCH models (Ch. 5)

TSA_ch1_ts_basics

Case study: Romanian quarterly GDP



TSA_ch1_case_gdp

Initial Analysis

- **Data**: Romanian quarterly GDP 2010–2023 (56 obs., INS/Eurostat)
- **Observations**: Upward trend, possibly seasonal
  - COVID-19 structural shock visible
- **Hypothesis**: Non-stationary series ≻ test with ADF and KPSS

Stationarity testing: ADF and KPSS

ADF Test

- **Hypothesis**: H_0: Unit root
- **Result**: ADF stat.: -1.23
  - Critical value: -2.89
  - Fail to reject H_0

KPSS Test

- **Hypothesis**: H_0: Stationary
- **Result**: KPSS stat.: 0.89
  - Critical value: 0.46
  - Reject H_0

Conclusion: Both Tests Agree

- The GDP series is **non-stationary** $\succ$ requires differencing

Differencing: transformation to stationarity

After Differencing

- **Tests**: Both confirm stationarity
  - ADF: -4.56 ($p < 0.01$)
  - KPSS: 0.21 ($p > 0.10$)

Conclusion

- **GDP level**: non-stationary
- **ΔGDP**: stationary
  - Use ΔGDP_t for modeling

Final Result

- GDP requires one differencing to become stationary

Experiment: ChatGPT vs Fundamentals

Prompt → Response

You: "I have daily EUR/RON exchange rate data for the last 5 years. Can you forecast next week?"

ChatGPT: "ADF: $p = 0.67 \to$ non-stationary. After differencing: $p < 0.01 \to$ stationary. Fitted ARMA(2,1). Coefficient $\phi_1 = 0.03$ significant ($p = 0.02$). RMSE $= 0.0043$."

Three errors a trained analyst catches immediately:

1. **Fitted noise**: EUR/RON $\approx$ random walk $\succ$ $\Delta X_t = \varepsilon_t$ is **white noise**
   ACF(k) $\approx 0\ \forall k \geq 1$ $\succ$ no model can beat the naïve forecast $\hat{X}_{t+1} = X_t$
2. **Spurious significance**: with $T = 5000$, ACF confidence band $= \pm\frac{1.96}{\sqrt{T}} = \pm 0.028$
   $\hat{\rho}_1 = 0.03$ barely outside the band $\succ$ fitting this is **overfitting noise**
3. **ADF misspecified**: series has drift but ADF ran without trend regressor
   Wrong specification $\succ$ reduced power $\succ$ unreliable conclusion

Discussion: If prices follow a random walk, can *any* AI model predict them?

Quantitative Audit: Verifying the AI

Reproduce the AI's claims with actual EUR/RON data (2019–2024, $T \approx 1300$)

AI claim	Verification	Verdict
ADF $p = 0.67$ on levels	`adfuller(x, regression='ct')`	Plausible (depends on lags)
$\hat{\rho}_1 = 0.03$ on Δx_t	Band: $\pm 1.96/\sqrt{1300} = \pm 0.054$	Inside the band!
ARMA RMSE $= 0.0043$	Naïve RMSE on same period	Naïve ≈ 0.0044

The decisive test: Diebold–Mariano

- H_0: ARMA and naïve have **equal forecast accuracy**
- Typical result: $p > 0.5 \succ$ cannot reject H_0
- The ARMA model adds **nothing** over $\hat{X}_{t+1|t} = X_t$

Lesson

Always benchmark against the **naïve forecast**. A model with low RMSE is worthless if the random walk achieves the same.

Key takeaways

Summary

- **Stochastic process**: collection of random variables indexed by time
- **Weak stationarity**: constant mean, variance, autocovariance
- **White noise**: $\varepsilon_t \sim WN(0, \sigma^2)$
  - Stationary, ACF $= 0$ for $h \neq 0$
- **Random walk**: $X_t = X_{t-1} + \varepsilon_t$
  - Non-stationary, $\text{Var}(X_t) = t\sigma^2$
- **ACF/PACF**: key tools for identifying structure
- **Differencing**: transforms non-stationary series into stationary ones
- **Unit root tests**:
  - ADF (H_0: unit root) vs KPSS (H_0: stationary)

Important formulas

Weak Stationarity

- **Constant moments**:
  - $\mathbb{E}[X_t] = \mu$ (constant mean)
  - $\text{Var}(X_t) = \sigma^2$ (constant variance)
- **Autocovariance**: $\gamma(h) = \text{Cov}(X_t, X_{t+h})$
- **Autocorrelation**: $\rho(h) = \gamma(h)/\gamma(0)$

Lag Operator

- **Lag**: $LX_t = X_{t-1}$
- **Difference**: $\Delta X_t = (1 - L)X_t$

White Noise (WN)

- **Model**: $\varepsilon_t \sim WN(0, \sigma^2)$
- **ACF**: $\rho(h) = 0$ for $h \neq 0$

Random Walk (RW)

- **Model**: $X_t = X_{t-1} + \varepsilon_t$
- **Variance**: $\text{Var}(X_t) = t\sigma^2$ (grows!)

Next chapter preview

Chapter 2: ARMA Models

- **AR(p)**: Autoregressive Models
- **MA(q)**: Moving Average Models
- **ARMA(p, q)**: Combined Models
- **Identification**: Using ACF/PACF

What We Will Learn

- **Estimation**: Model parameters
- **Diagnostics**: Model validation
- **Forecasting**: Confidence intervals
- **Selection**: AIC, BIC

Question 1

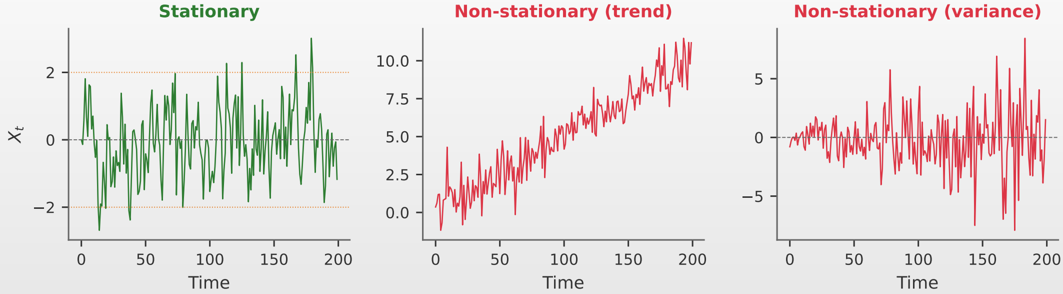
Question

- What are the three conditions for weak (covariance) stationarity?

Answer Choices

(A) Zero mean, infinite variance, time-dependent covariance
(B) Constant mean, constant variance, autocovariance depends only on lag
(C) Normal distribution, independence, unit variance
(D) Linear trend, constant seasonality, white residuals

Question 1: Answer



Answer: (B)

$\boxdot$ $\mathbb{E}[X_t] = \mu$, $\text{Var}(X_t) = \sigma^2$, $\gamma(t, s) = \gamma(|t - s|)$

TSA_ch1_stationarity

Question 2

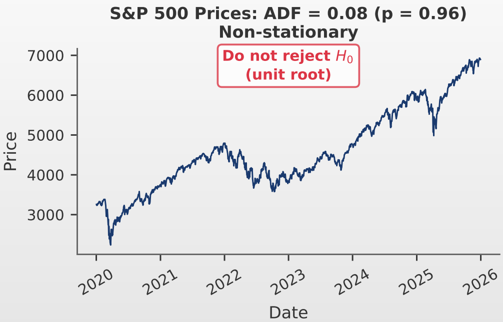
Question

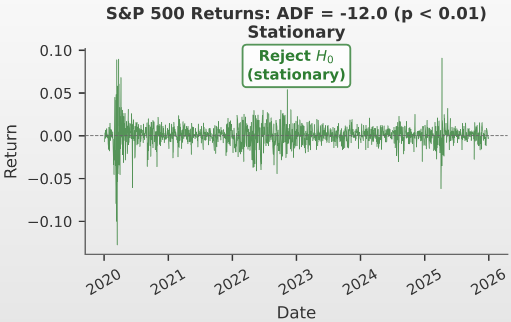
- What is the null hypothesis (H_0) of the ADF (Augmented Dickey-Fuller) test?

Answer Choices

(A) The series is stationary
(B) The series has a unit root (is non-stationary)
(C) The series has no autocorrelation
(D) The series has a normal distribution

Question 2: Answer





Answer: (B)

- H_0: unit root; $\tau <$ critical value $\succ$ stationary

TSA_ch1_unit_root_tests

Question 3

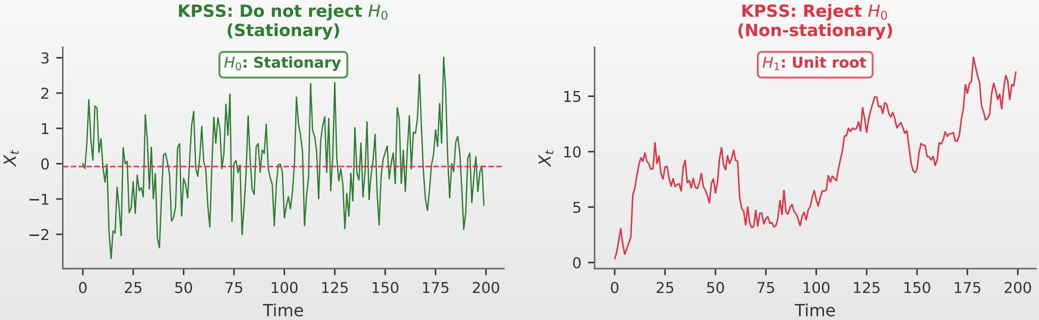
Question

- What is the null hypothesis (H_0) of the KPSS test?

Answer Choices

(A) The series has a unit root (non-stationary)
(B) The series is stationary
(C) The series is a random walk
(D) The series has a deterministic trend

Question 3: Answer



Answer: (B)

- KPSS: H_0 stationary (opposite of ADF). Use both tests!

TSA_ch1_unit_root_tests

Question 4

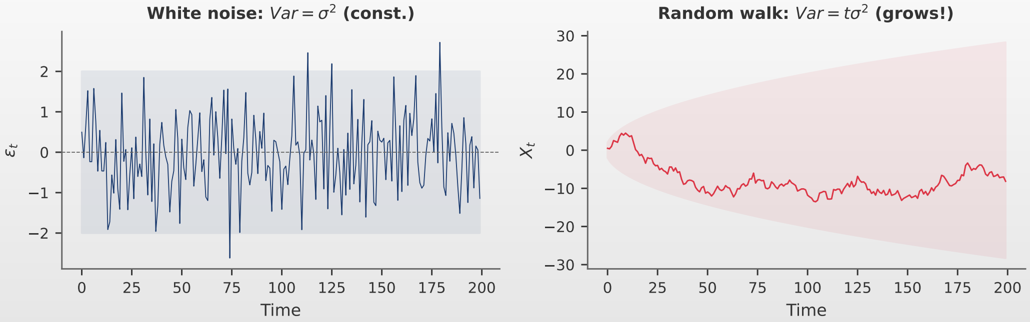
Question

- What is the key property of the variance of a random walk $X_t = X_{t-1} + \varepsilon_t$?

Answer Choices

(A) Variance is constant: $\text{Var}(X_t) = \sigma^2$
(B) Variance grows linearly with time: $\text{Var}(X_t) = t\sigma^2$
(C) Variance decreases with time
(D) Variance is zero

Question 4: Answer



Answer: (B)

- $\text{Var}(X_t) = t\sigma^2$ grows linearly $\succ$ non-stationary

TSA_ch1_random_walk

Question 5

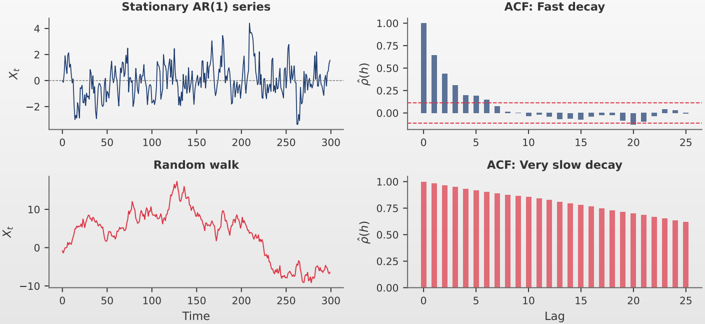
Question

- What does the ACF of a random walk (non-stationary series with unit root) look like?

Answer Choices

(A) All values are zero after lag 0
(B) Decays exponentially fast
(C) Decays very slowly (high persistence)
(D) Oscillates between positive and negative

Question 5: Answer



Answer: (C)

- ACF $\approx$ 1 for many lags, slow decay $\succ$ ADF test

TSA_ch1_random_walk

Question 6

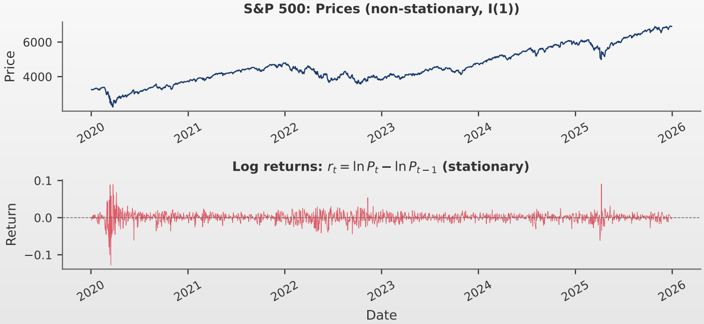
Question

- How do we obtain stationary returns from a financial price series P_t?

Answer Choices

(A) Simple differencing: $\Delta P_t = P_t - P_{t-1}$
(B) Log then differencing: $r_t = \ln P_t - \ln P_{t-1}$
(C) Log only: $\ln P_t$
(D) Standardization: $(P_t - \bar{P})/s_P$

Question 6: Answer



Answer: (B)

- Log returns: $r_t = \ln P_t - \ln P_{t-1}$
- First ln (stabilizes variance), then Δ (removes trend) $\succ$ stationary series

TSA_ch1_operators

References

Core Textbooks

- Hyndman & Athanasopoulos (2021). *Forecasting*, OTexts
- Shumway & Stoffer (2017). *Time Series Analysis*, Springer
- Hamilton (1994). *Time Series Analysis*, Princeton

Classic References

- Wold (1938). *Analysis of Stationary Time Series*
- Bartlett (1946). "Sampling Properties", *JRSS*

Online Resources

- **Quantlet**: `https://quantlet.com` ≻ statistics code
- **Quantinar**: `https://quantinar.com` ≻ tutorials
- **GitHub TSA_ch1**: `https://github.com/QuantLet/TSA/tree/main/TSA_ch1`

Data sources and software

Data Used

- **S&P 500**: Yahoo Finance
  - Prices, returns
- **Romanian GDP**: INS/Eurostat
  - Quarterly data
- **Exchange rates**: BNR

Software

- **Python**: statsmodels, pandas, matplotlib, scipy
- **R**: forecast, tseries, urca
- **Data**: Yahoo Finance, FRED, Eurostat

Thank You!

Questions?

Course materials available at: `https://danpele.github.io/Time-Series-Analysis/`

Quantlet Quantinar