



# Time Series Analysis and Forecasting

## Chapter 7: Cointegration and VECM



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## Learning Objectives

By the end of this chapter, you will be able to:

- Understand the problem of spurious regression with non-stationary data
- Test for cointegration using Engle-Granger and Johansen methods
- Estimate Vector Error Correction Models (VECM)
- Interpret error correction mechanisms and adjustment speeds

## Outline

- Motivation
- Spurious Regression
- Cointegration Concept
- Engle-Granger Method
- Johansen Method
- VECM Estimation
- Practical Considerations
- Real-World Examples
- Case Study: Interest Rates
- AI Use Case
- Summary
- Quiz

## Why Cointegration Matters

### The Challenge

- ▣ Many economic/financial time series are **non-stationary** ( $I(1)$ )
- ▣ GDP, stock prices, exchange rates, interest rates all have unit roots
- ▣ Standard regression with  $I(1)$  variables  $\Rightarrow$  **spurious results**
- ▣ Differencing removes non-stationarity but loses **long-run information**

### The Solution: Cointegration

Some non-stationary series share a **common stochastic trend**—they move together in the long run.

### Nobel Prize 2003

Granger & Engle received the Nobel Prize for “methods for analyzing economic time series with common trends.”

## Real-World Applications

### Finance

- ▣ **Pairs Trading:** Cointegrated stocks
- ▣ **Term Structure:** Interest rates
- ▣ **Spot-Futures:** Arbitrage

### Macroeconomics

- ▣ **Consumption & Income**
- ▣ **Money & Prices**
- ▣ **PPP:** Exchange rates

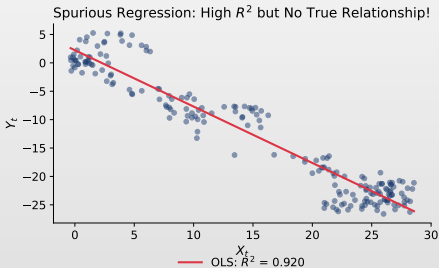
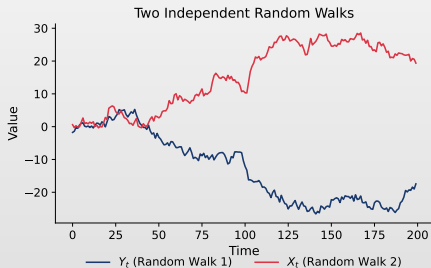
### Policy Analysis

- ▣ **Fiscal:** Spending & taxes
- ▣ **Monetary:** Rate pass-through
- ▣ **Labor:** Wages & productivity

## Spurious Regression: Visual Example

### Warning

- ▣ **Result:** Two completely independent random walks show high correlation ( $R^2 > 0.8$ ) purely by chance! This is why we need cointegration analysis



## The Spurious Regression Problem

### Granger & Newbold (1974)

Regressing one random walk on another **independent** random walk:  $Y_t = \alpha + \beta X_t + u_t$  where  $Y_t$  and  $X_t$  are independent  $I(1)$  processes.

### Symptoms of Spurious Regression

- ▣ High  $R^2$  (often  $> 0.9$ ) even though variables are **unrelated**
- ▣ Highly significant  $t$ -statistics (reject  $H_0 : \beta = 0$ )
- ▣ Very low Durbin-Watson statistic ( $DW \approx 0$ )
- ▣ Residuals are non-stationary (have unit root)

### Rule of Thumb

If  $R^2 > DW$ , suspect spurious regression!

## Spurious Correlations in the Real World

### Data Mining Can Produce Meaningless Correlations

With enough variables and long time series, purely coincidental patterns emerge:

- Distance between Neptune and Uranus  $\leftrightarrow$  SAP SE stock price (2002–2023)
- GMO corn use in South Dakota  $\leftrightarrow$  Google searches for “i cant even” (2004–2023)
- Two and a Half Men* season ratings  $\leftrightarrow$  Jet fuel used in Serbia (2006–2015)
- “Its Wednesday my dudes” meme popularity  $\leftrightarrow$  Boeing stock price (2006–2023)

### Lesson

High correlation  $\neq$  causation. Non-stationary series with common trends produce high  $R^2$  by construction. Always test for stationarity and cointegration before interpreting regression results!

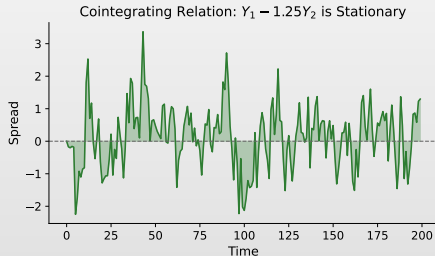
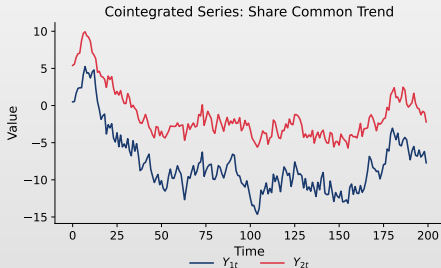
🌐 Explore more examples: [tylervigen.com/spurious-correlations](https://tylervigen.com/spurious-correlations)



## Cointegration: Visual Example

### Key Insight

- **Cointegration:** Both series are  $I(1)$  and trend together, but their linear combination (spread) is stationary — this is cointegration!



## Definition of Cointegration

### Definition 1 (Cointegration (Engle & Granger, 1987))

Variables  $Y_{1t}, Y_{2t}, \dots, Y_{kt}$  are **cointegrated of order**  $(d, b)$ , written  $CI(d, b)$ , if:

1. All variables are integrated of order  $d$ :  $Y_{it} \sim I(d)$
2. There exists a linear combination  $\beta'Y_t = \beta_1 Y_{1t} + \dots + \beta_k Y_{kt}$  that is integrated of order  $(d - b)$ , where  $b > 0$

### Most Common Case: $CI(1, 1)$

- ▣ Variables are  $I(1)$  (have unit roots)
- ▣ Linear combination is  $I(0)$  (stationary)
- ▣ Vector  $\beta = (\beta_1, \dots, \beta_k)'$  is the **cointegrating vector**

The cointegrating vector is unique only up to scalar multiplication. Usually normalized:  $\beta_1 = 1$ .

## Intuition: Common Stochastic Trends

### Why Does Cointegration Occur?

Cointegrated variables share **common stochastic trends**:  $Y_{1t} = \gamma_1 \tau_t + S_{1t}$ ,  $Y_{2t} = \gamma_2 \tau_t + S_{2t}$  where  $\tau_t$  is a common random walk and  $S_{it}$  are stationary.

### Linear Combination Eliminates the Trend

$$\gamma_2 Y_{1t} - \gamma_1 Y_{2t} = \gamma_2 S_{1t} - \gamma_1 S_{2t} \sim I(0)$$

### Economic Interpretation

- Cointegration = **long-run equilibrium relationship**
- Variables may deviate in the short run, but are “pulled back”
- The cointegrating vector defines the equilibrium

## Cointegrating Rank

### How Many Cointegrating Relationships?

For  $k$  variables that are  $I(1)$ :

- ▣ Maximum possible cointegrating relationships:  $r = k - 1$
- ▣ If  $r = 0$ : No cointegration (variables drift apart)
- ▣ If  $r = k$ : All variables are  $I(0)$  (contradiction)

### Example: 3 Variables

- ▣  $r = 0$ : No cointegration
- ▣  $r = 1$ : One cointegrating relationship
- ▣  $r = 2$ : Two cointegrating relationships (only 1 common trend)

The number of common stochastic trends =  $k - r$

## Engle-Granger Two-Step Method

### Step 1: Estimate Cointegrating Regression

Run OLS:  $Y_t = \alpha + \beta X_t + e_t$ . Save residuals:  $\hat{e}_t = Y_t - \hat{\alpha} - \hat{\beta} X_t$

### Step 2: Test Residuals for Stationarity

Test if  $\hat{e}_t$  is  $I(0)$  using ADF:  $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \sum_{j=1}^p \gamma_j \Delta \hat{e}_{t-j} + v_t$

- $H_0: \rho = 0$  (unit root  $\Rightarrow$  no cointegration)
- $H_1: \rho < 0$  (stationary  $\Rightarrow$  cointegration)

### Important

Use Engle-Granger critical values, not standard ADF! (More negative because residuals are estimated)

## Engle-Granger Critical Values

### Critical Values for Cointegration Test

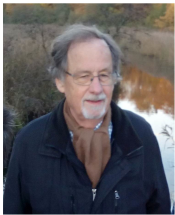
Variables	1%	5%	10%
2	-3.90	-3.34	-3.04
3	-4.29	-3.74	-3.45
4	-4.64	-4.10	-3.81
5	-4.96	-4.42	-4.13

MacKinnon (1991),  $T = 100$

### Limitations of Engle-Granger

- ▣ Only tests for **one** cointegrating relationship
- ▣ Results depend on choice of dependent variable
- ▣ Small sample bias; cannot test hypotheses on cointegrating vector

## Researcher Spotlight: Søren Johansen



\*1939

 Wikipedia

### Biography

- Danish statistician and econometrician, Professor Emeritus at University of Copenhagen
- Known for his rigorous mathematical approach to econometrics
- Fellow of the Econometric Society; recipient of numerous honors in statistical science

### Key Contributions

- **Johansen cointegration test** (1988, 1991) — maximum likelihood approach to testing for multiple cointegrating vectors
- **Trace and maximum eigenvalue** statistics for determining cointegration rank
- **VECM estimation** — linking cointegration with error correction models
- Standard framework for multivariate cointegration analysis in economics and finance

## Johansen Cointegration Test

### Advantages over Engle-Granger

- ▣ Tests for **multiple** cointegrating relationships
- ▣ Maximum likelihood estimation (more efficient)
- ▣ Can test restrictions on cointegrating vectors
- ▣ Does not require choosing a dependent variable

### Starting Point: VAR in Levels

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \varepsilon_t$$

Rewrite in **Vector Error Correction** form...



## VECM Representation

### Vector Error Correction Model

$$\Delta Y_t = c + \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \varepsilon_t$$

- ▣  $\Pi = \sum_i A_i - I$  (long-run impact);  $\Gamma_j$  (short-run dynamics)

### Key Insight: Rank of $\Pi$

The **rank of  $\Pi$**  determines cointegration:

- ▣  $\text{rank}(\Pi) = 0$ : No cointegration (VAR in differences)
- ▣  $\text{rank}(\Pi) = k$ : All variables are  $I(0)$  (VAR in levels)
- ▣  $0 < \text{rank}(\Pi) = r < k$ :  $r$  cointegrating vectors

## Derivation: From VAR to VECM

### Starting Point: VAR(p) in Levels

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \epsilon_t$$

### Step 1: Subtract $Y_{t-1}$ from Both Sides

$$Y_t - Y_{t-1} = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} - Y_{t-1} + \epsilon_t$$

$$\Delta Y_t = (A_1 - I) Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + \epsilon_t$$

### Goal

Rewrite so that all terms are either in levels ( $Y_{t-1}$ ) or differences ( $\Delta Y_{t-j}$ ).

## Derivation: From VAR to VECM (cont.)

### Step 2: Add and Subtract Terms Strategically

Add  $A_2 Y_{t-1}$  and subtract  $A_2 Y_{t-1}$ :  $\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} - A_2(Y_{t-1} - Y_{t-2}) + A_3 Y_{t-3} + \dots + \epsilon_t$   
Continue adding  $A_3 Y_{t-1}$ , etc., until all lagged **levels** are collected in one term.

### Step 3: General Pattern

After algebraic manipulation:  $\Delta Y_t = \Pi Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$

### The Key Matrices

$$\Pi = \sum_{i=1}^p A_i - I = -(I - A_1 - A_2 - \dots - A_p)$$

$$\Gamma_j = - \sum_{i=j+1}^p A_i \text{ for } j = 1, \dots, p-1$$

## Derivation: Verifying the $\Gamma_j$ Formula

### Example: VAR(2)

Starting from:  $Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$

Subtract  $Y_{t-1}$ :

$$\Delta Y_t = (A_1 - I)Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$$

Add and subtract  $A_2 Y_{t-1}$ :

$$\Delta Y_t = (A_1 + A_2 - I)Y_{t-1} + A_2(Y_{t-2} - Y_{t-1}) + \varepsilon_t$$

$$\Delta Y_t = \underbrace{(A_1 + A_2 - I)}_{\Pi} Y_{t-1} - \underbrace{A_2}_{\Gamma_1} \Delta Y_{t-1} + \varepsilon_t$$

### Verification

For VAR(2):  $\Pi = A_1 + A_2 - I$  and  $\Gamma_1 = -A_2$

Using our formula:  $\Gamma_1 = -\sum_{i=2}^2 A_i = -A_2 \quad \checkmark$

## Economic Interpretation of Error Correction

### The VECM with Cointegration

When  $\text{rank}(\Pi) = r$ , we write  $\Pi = \alpha\beta'$ :  $\Delta Y_t = \alpha \underbrace{(\beta' Y_{t-1})}_{\text{equilibrium error}} + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$

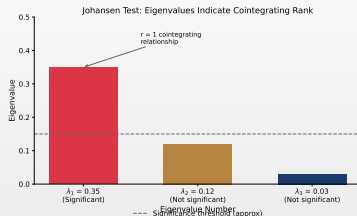
### Economic Interpretation

- ▣  $\beta' Y_{t-1} = \text{equilibrium error}$ : deviation from long-run relationship
- ▣  $\alpha = \text{adjustment speeds}$ : how fast variables correct deviations
- ▣  $\Gamma_j = \text{short-run dynamics}$ : transitory effects

### Error Correction Mechanism

If  $\beta' Y_{t-1} > 0$  (above equilibrium) and  $\alpha_i < 0$ , then  $\Delta Y_{it}$  decreases. **The system self-corrects toward equilibrium!**

## Johansen Test Statistics



When  $\text{rank}(\Pi) = r < k$

$\Pi = \alpha\beta'$  where  $\beta$  ( $k \times r$ ) = cointegrating vectors,  $\alpha$  ( $k \times r$ ) = adjustment coefficients

### Interpretation

- ▣  $\beta'Y_{t-1}$  = deviations from equilibrium (error correction terms)
- ▣  $\alpha$  = speed of adjustment; rows show each variable's response

$$\text{VECM: } \Delta Y_t = c + \alpha(\beta'Y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta Y_{t-j} + \epsilon_t$$

## Testing Procedure

### Sequential Testing (Trace Test)

1. Test  $H_0: r = 0$ . If rejected  $\Rightarrow$  continue
2. Test  $H_0: r \leq 1$ . If not rejected  $\Rightarrow r = 1$
3. Continue until  $H_0$  is not rejected

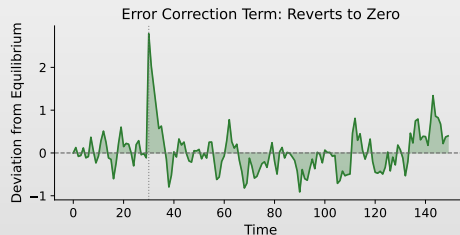
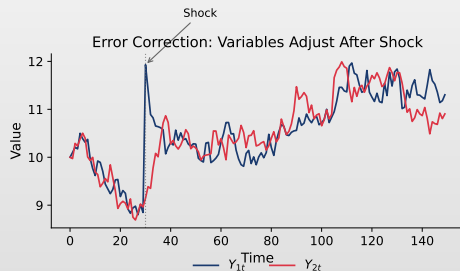
### Deterministic Components

- No constant, no trend (rarely used)
- Constant in cointegrating relation only
- **Constant in both** (most common)
- Constant + trend in cointegrating relation

## Error Correction Mechanism: Visual

### Interpretation

- **Error correction:** When series deviate from equilibrium (shaded regions), the adjustment mechanism pulls them back. Positive deviations lead to downward adjustment, negative deviations lead to upward adjustment





## VECM Structure

### Full VECM Specification

For  $k = 2$  variables with  $r = 1$  cointegrating relation:

$$\Delta Y_{1t} = c_1 + \alpha_1(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{11}\Delta Y_{1,t-1} + \gamma_{12}\Delta Y_{2,t-1} + \varepsilon_{1t}$$

$$\Delta Y_{2t} = c_2 + \alpha_2(Y_{1,t-1} - \beta Y_{2,t-1}) + \gamma_{21}\Delta Y_{1,t-1} + \gamma_{22}\Delta Y_{2,t-1} + \varepsilon_{2t}$$

### Components

- $(Y_{1,t-1} - \beta Y_{2,t-1})$  = error correction term (deviation from equilibrium)
- $\alpha_1, \alpha_2$  = adjustment speeds (should have opposite signs)
- $\gamma_{ij}$  = short-run dynamics
- $\varepsilon_{it}$  = innovations

## Interpreting Adjustment Coefficients

### The $\alpha$ Coefficients

If the cointegrating relation is  $Y_1 - \beta Y_2 = 0$  (equilibrium):

- ▣  $\alpha_1 < 0$ :  $Y_1$  adjusts downward when above equilibrium
- ▣  $\alpha_2 > 0$ :  $Y_2$  adjusts upward when  $Y_1$  is above equilibrium

### Weak Exogeneity

If  $\alpha_i = 0$ , variable  $Y_i$  does **not** respond to disequilibrium.

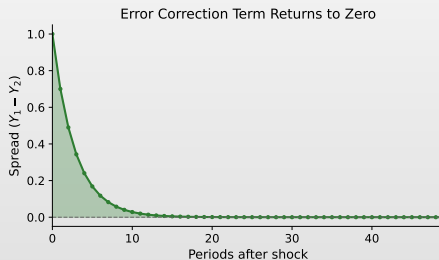
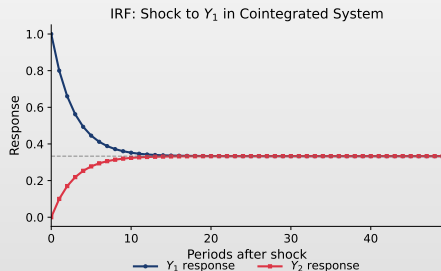
- ▣  $Y_i$  is **weakly exogenous** for the long-run parameters
- ▣ The other variable does all the adjusting
- ▣ Can simplify estimation (single-equation approach)

Test weak exogeneity:  $H_0 : \alpha_i = 0$  using likelihood ratio test.

## VECM Impulse Response Functions

### IRF Interpretation

- **Permanent effects:** In a cointegrated system, shocks have permanent effects on levels, but the system returns to equilibrium — they converge to a new long-run value



## VECM vs VAR in Differences

### When Variables are Cointegrated

	VAR in Differences	VECM
Long-run info	Lost	Preserved
Short-run dynamics	Yes	Yes
Error correction	No	Yes
Forecasting	Poor (long-run)	Better
IRF interpretation	Short-run only	Both

### Granger Representation Theorem

If variables are cointegrated, there **must** exist an error correction representation. Ignoring cointegration = model misspecification!

## Practical Workflow

### Step-by-Step Procedure

1. **Unit Root Tests:** Verify all variables are  $I(1)$ 
  - ▶ ADF, KPSS on levels and first differences
2. **Lag Length Selection:** Choose  $p$  for VAR in levels
  - ▶ Use AIC, BIC, or sequential LR tests
3. **Cointegration Test:** Johansen trace/max-eigenvalue tests
  - ▶ Determine cointegrating rank  $r$
4. **Estimate VECM:** If  $0 < r < k$ 
  - ▶ Estimate  $\alpha$ ,  $\beta$ ,  $\Gamma_j$
5. **Diagnostics:** Check residuals for autocorrelation, normality
6. **Analysis:** IRF, FEVD, hypothesis tests

## Common Pitfalls

### Things to Watch Out For

- ▣ **Structural breaks:** Cause spurious unit roots or cointegration
- ▣ **Near-unit-root:** Tests have low power
- ▣ **Lag selection:** Too many/few lags bias results
- ▣ **Small samples:** Johansen test oversized

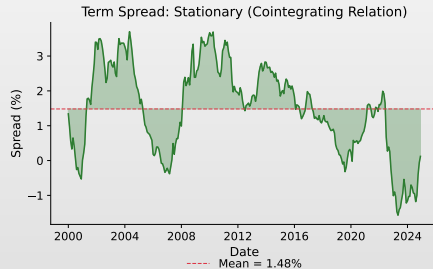
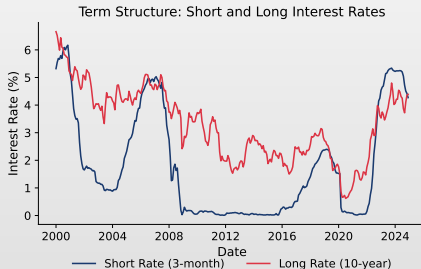
### Recommendation

Always check: residual diagnostics, stability of cointegrating relationship, sensitivity to specification

## Example 1: Term Structure of Interest Rates

### Expectations Hypothesis

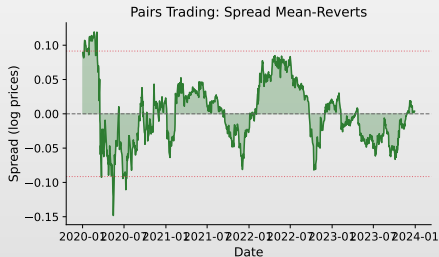
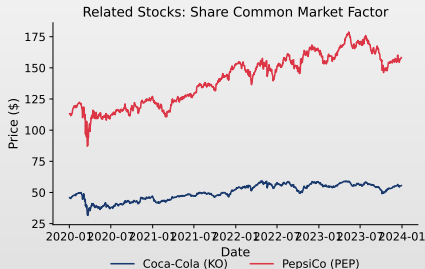
- **Conclusion:** Short and long rates share a common trend. The spread (term premium) is stationary — evidence of cointegration!



## Example 2: Pairs Trading in Finance

### Strategy

- ▣ **Pairs trading:** Find cointegrated stock pairs (e.g., Coca-Cola & Pepsi). When the spread deviates from the mean, trade expecting mean reversion





## Interest Rates: Economic Theory

### Expectations Hypothesis of the Term Structure

- ▣ **Formula:** Long-term rate as average of expected future rates
  - ▶  $R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{term premium}$
- ▣ **Implication:** If the term premium is constant,  $r_t$  and  $R_t$  are cointegrated
  - ▶ Cointegrating vector:  $(1, -1)$

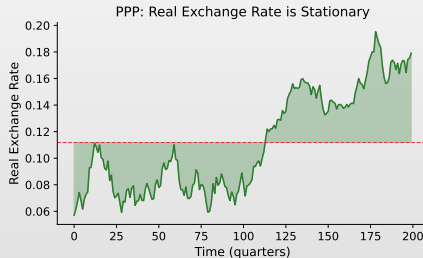
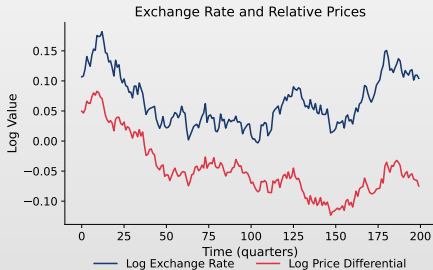
### Empirical Results

- ▣ **Unit root tests:** Both rates are  $I(1)$ 
  - ▶ One cointegrating relationship (Johansen test)
- ▣ **Cointegrating vector:**  $\approx (1, -1)$ , the spread is stationary
  - ▶ The short rate adjusts to disequilibrium (the long rate is weakly exogenous)

## Example 3: Purchasing Power Parity (PPP)

### PPP Theory

- Formula:  $e_t = p_t - p_t^*$  (log exchange rate equals price differential). The real exchange rate should be stationary in the long run



## VECM Results for Interest Rates

### Typical Results

- ▣ **Integration:** Both rates are  $I(1)$ , one cointegrating relationship identified
  - ▶ Cointegrating vector close to  $(1, -1)$ : the spread is stationary
- ▣ **Adjustment:** The short rate adjusts to the long rate
  - ▶ The long rate does not adjust (weakly exogenous)

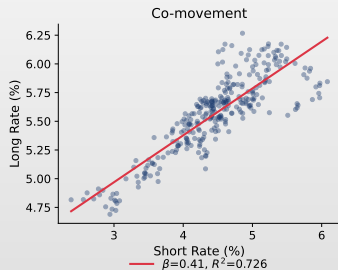
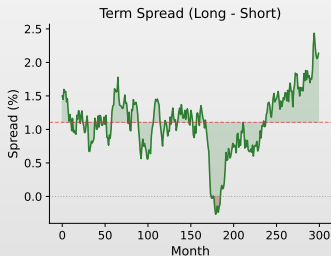
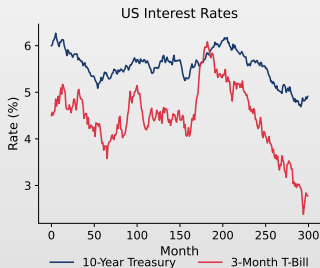
### VECM Equations (Stylized)

- ▣ **Estimated system:**
  - ▶  $\Delta r_t = 0.02 - 0.15(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{1t}$
  - ▶  $\Delta R_t = 0.01 - 0.02(r_{t-1} - R_{t-1}) + \text{lags} + \varepsilon_{2t}$
- ▣ **Interpretation:** The short rate adjusts faster ( $\alpha_1 = -0.15$ )
  - ▶ The long rate is nearly weakly exogenous ( $\alpha_2 \approx 0$ )

## Case Study: Cointegration of Interest Rates

### Data

- US Interest Rates: Long-term (10 years) and short-term (3 months)
- Observation: Both series are  $I(1)$ , but the spread appears stationary

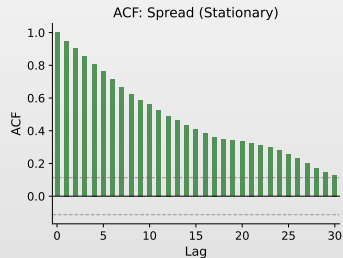
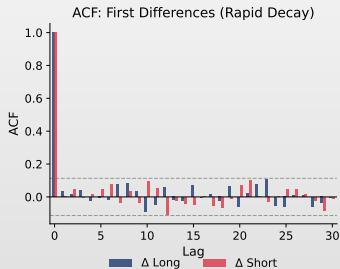
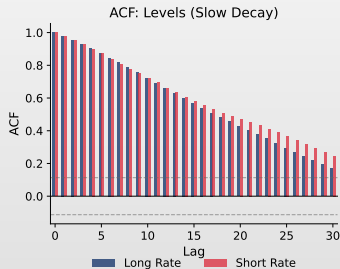


TSA\_ch7\_case\_raw\_data

## Step 1: Unit Root Tests

### Results

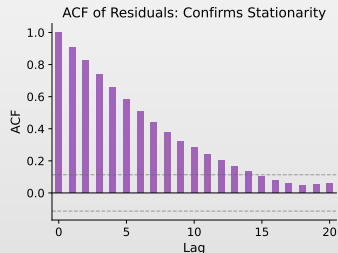
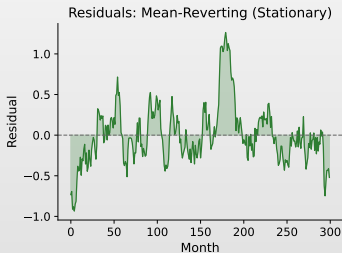
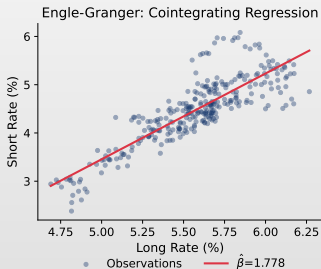
- ACF levels: Slow decay — non-stationarity; after differencing: rapid decay —  $I(1)$
- ACF spread: Stationary — possible cointegration!



## Step 2: Engle-Granger Cointegration Test

### Results

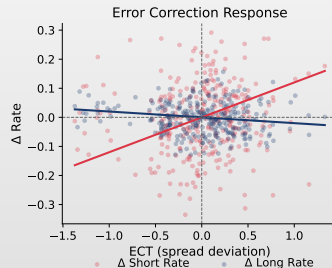
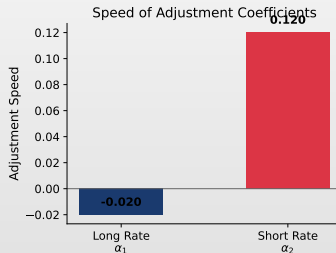
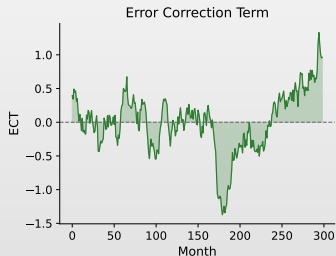
- Engle-Granger regression: Short rate =  $\alpha + \beta \times \text{Long rate} + \varepsilon_t$
- Conclusion: The series are cointegrated — a long-run equilibrium relationship exists



## Step 3: VECM Estimation

### Model

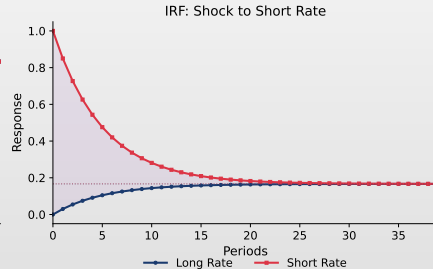
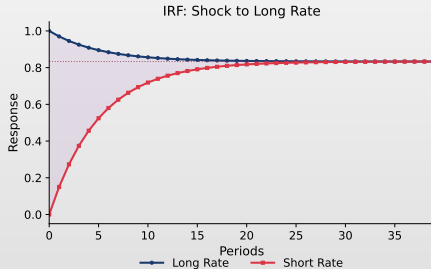
- ▣ **VECM(2)**: Cointegration rank = 1
- ▣ **Adjustment**: The  $\alpha$  coefficients indicate the speed of return to equilibrium



## Step 4: Impulse Response Functions

### Interpretation

- ▣ **Permanent effects:** Shocks to the long rate persistently affect both rates
- ▣ **Cointegration:** Effects do not converge to zero — characteristic of cointegrated series

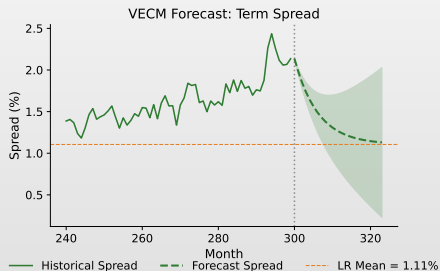
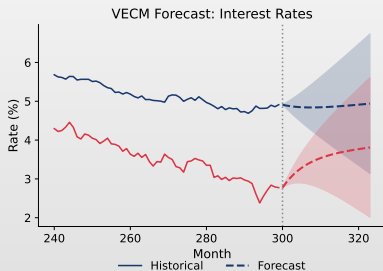




## Step 5: VECM Forecast

### Forecast

- ▣ **Horizon:** 24 months for both rates simultaneously
- ▣ **Advantage:** VECM maintains the cointegrating relationship in the forecast



## AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download daily prices for gold and silver using yfinance. Test for cointegration and estimate a VECM model. Analyze the speed of adjustment parameters. Give me complete Python code."

**Exercise:**

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it verify that each series is  $I(1)$  before testing for cointegration?
3. Does it use both Engle-Granger and Johansen tests? What are the trade-offs?
4. How does it determine the cointegration rank? Trace vs max-eigenvalue statistics?
5. Does it correctly interpret the  $\alpha$  (adjustment) coefficients?

**Warning:** AI-generated code may run without errors and look professional. *That does not mean it is correct.*

## Key Takeaways

### Main Concepts

- ▣ **Cointegration:**  $I(1)$  variables with stationary linear combination
- ▣ **Spurious regression:** High  $R^2$  with unrelated  $I(1)$  variables
- ▣ **VECM:** VAR with error correction for cointegrated systems

### Testing Methods

- ▣ **Engle-Granger:** Simple, one vector only
- ▣ **Johansen:** Multiple vectors, MLE-based

### Remember

Tests have low power in small samples. Theory should guide specification.

## What's Next?

### Extensions and Related Topics

- ▣ **Structural VECM:** Identifying structural shocks
- ▣ **Threshold cointegration:** Nonlinear adjustment
- ▣ **Panel cointegration:** Multiple cross-sections
- ▣ **Fractional cointegration:** Long memory
- ▣ **Time-varying cointegration:** Regime changes

- ▣ **Questions?**

## Key Formulas – Summary

### Cointegration

- ▣ **Definition:**  $Y_t - \beta X_t = u_t \sim I(0)$
- ▣ **Interpretation:** Long-run equilibrium

### Engle-Granger Test

- ▣ **Step 1:**  $Y_t = \alpha + \beta X_t + u_t$
- ▣ **Step 2:** ADF test on  $\hat{u}_t$
- ▣ **Note:** Special critical values

### Cointegration Rank

- ▣ **Rank  $r$ :**  $0 \leq r \leq K - 1$  relationships

### VECM Model

- ▣ **Equation:**  $\Delta Y_t = \Pi Y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta Y_{t-i} + \varepsilon_t$
- ▣ **Factorization:**  $\Pi = \alpha \beta'$

### Interpretation of $\alpha$ and $\beta$

- ▣  $\beta$ : Cointegrating vectors
- ▣  $\alpha$ : Speed of adjustment

### Johansen Test

- ▣ **Trace:**  $\lambda_{trace} = -T \sum_{i=r+1}^K \ln(1 - \hat{\lambda}_i)$
- ▣ **Max-Eigen:**  $\lambda_{max} = -T \ln(1 - \hat{\lambda}_{r+1})$

## Question 1

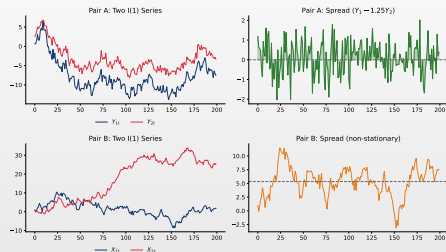
### Question

□ Analyze the two pairs of  $I(1)$  series below. Which pair is cointegrated?

### Answer choices

- (A) Pair A, because the series have the same trend
- (B) Pair B, because the series are uncorrelated
- (C) Pair A, because their spread is stationary
- (D) Both pairs are cointegrated

## Question 1: Answer



Correct answer: (C) Pair A – stationary spread

- Cointegration = stationary linear combination, not just correlation
- Pair B's spread is non-stationary → not cointegrated

## Question 2

### Question

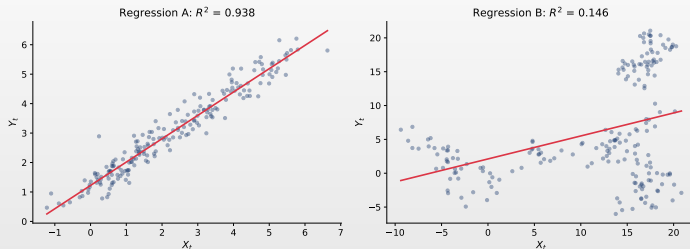
- ☐ Both regressions below have high  $R^2$ . How can you distinguish a spurious regression from a genuine one?

### Answer choices

- (A) Cannot distinguish – both have high  $R^2$
- (B) Test the residuals: stationary residuals = genuine cointegration
- (C) Check the significance of the  $\beta$  coefficient
- (D) Compare  $R^2$  values: higher = more real relationship



## Question 2: Answer



Correct answer: (B) Test the stationarity of residuals

- ▣ Engle-Granger test: if OLS residuals are stationary (ADF), the relationship is genuine
- ▣ High  $R^2$  does NOT imply a real relationship between  $I(1)$  variables!

### Question 3

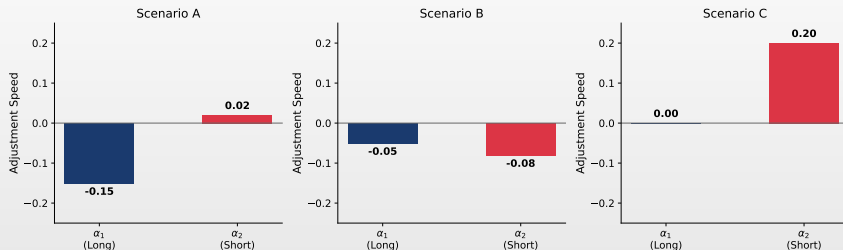
#### Question

- ☐ In which scenario is the long rate weakly exogenous (does not adjust to disequilibrium)?

#### Answer choices

- (A) Scenario A:  $\alpha_1 = -0.15$ ,  $\alpha_2 = 0.02$
- (B) Scenario B:  $\alpha_1 = -0.05$ ,  $\alpha_2 = -0.08$
- (C) Scenario C:  $\alpha_1 = 0.00$ ,  $\alpha_2 = 0.20$
- (D) No scenario – both variables must adjust

## Question 3: Answer



Correct answer: (C) Scenario C –  $\alpha_1 = 0$

- $\alpha_1 = 0$ : the long rate does not respond to disequilibrium (weakly exogenous)
- All adjustment is done by the short rate ( $\alpha_2 = 0.20$ )

## Question 4

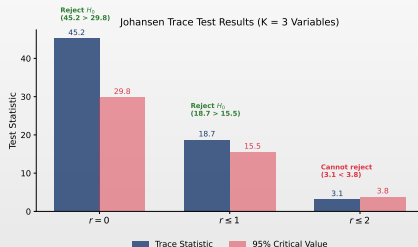
### Question

□ Given the Johansen Trace test results for  $K = 3$  variables, what is the cointegrating rank?

### Answer choices

- (A)  $r = 0$  (no cointegrating relationships)
- (B)  $r = 1$  (one cointegrating relationship)
- (C)  $r = 2$  (two cointegrating relationships)
- (D)  $r = 3$  (fully stationary system)

## Question 4: Answer



Correct answer: (C)  $r = 2$  cointegrating relationships

- Reject  $H_0 : r = 0$  (45.2 > 29.8) and  $H_0 : r \leq 1$  (18.7 > 15.5)
- Cannot reject  $H_0 : r \leq 2$  (3.1 < 3.8)  $\rightarrow$  rank is  $r = 2$

## Question 5

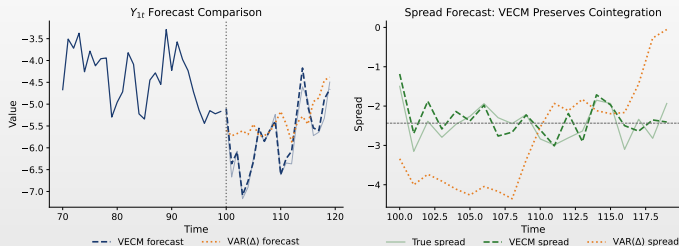
### Question

- ☐ What is the main advantage of VECM over VAR in differences for forecasting?

### Answer choices

- (A) VECM has fewer parameters to estimate
- (B) VECM preserves the cointegrating relationship in long-run forecasts
- (C) VAR in differences cannot produce forecasts
- (D) No advantage – both are equivalent

## Question 5: Answer



Correct answer: (B) VECM preserves cointegration

- VAR(Δ) loses the level relationship → spread diverges
- VECM incorporates long-run equilibrium → forecast stays coherent

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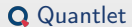
### Online Resources and Code

- ▣ **Quantlet**: <https://quantlet.com> → Code repository for statistics
- ▣ **Quantinar**: <https://quantinar.com> → Learning platform for quantitative methods
- ▣ **GitHub TSA**: <https://github.com/QuantLet/TSA> → Python code for this course

# Thank You!

## Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>



Quantlet



Quantinar