Time Series Analysis and Forecasting

Chapter 2: ARMA Models

Daniel Traian PELE

Bucharest University of Economic Studies
IDA Institute Digital Assets
Blockchain Research Center
AI4EFin Artificial Intelligence for Energy Finance
Romanian Academy, Institute for Economic Forecasting
MSCA Digital Finance

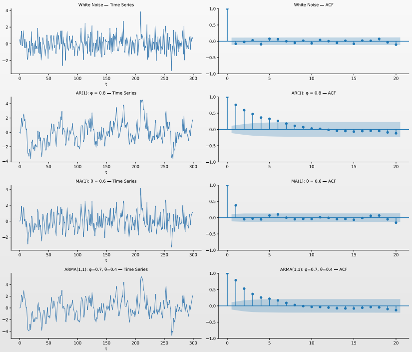
Learning Objectives

By the end of this chapter, you will be able to:

1. **Define** and simulate AR(p), MA(q), and ARMA(p, q) processes
2. **Verify** stationarity and invertibility conditions
3. **Identify** orders p and q through ACF/PACF analysis
4. **Estimate** parameters via Yule-Walker, MLE, and information criteria (AIC, BIC)
5. **Diagnose** the model through residual analysis and the Ljung-Box test
6. **Forecast** using ARMA models with confidence intervals
7. **Apply** the Box-Jenkins methodology to real data (sunspots)

Outline

Why ARMA Models?



- **AR processes**: Current value depends on past values $\Rightarrow$ mean-reverting behavior
- **MA processes**: Current value depends on past shocks $\Rightarrow$ short memory
- **ARMA**: Combines both mechanisms for flexible modeling

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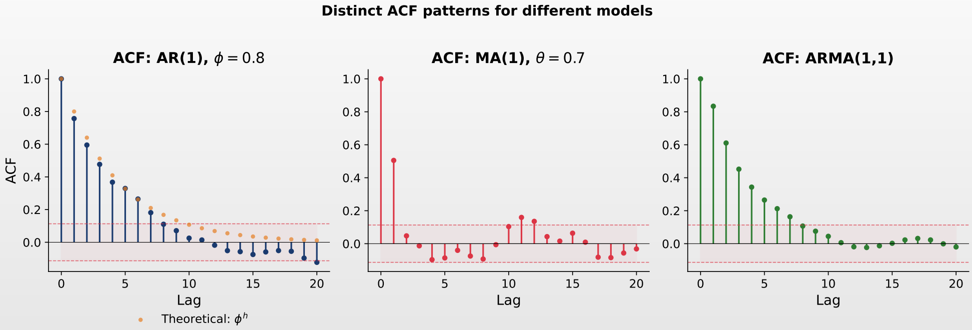
Model Identification Through ACF Patterns



ACF Reflects Model Structure

- **Distinct patterns**: AR: exponential decay; MA: sharp cutoff; ARMA: mixed decay
- **Identification**: Visual analysis of ACF/PACF guides the selection of orders p and q

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Recap: Stationarity

From Chapter 1

- A process $\{X_t\}$ is **weakly stationary** if:
  1. $\mathbb{E}[X_t] = \mu$ (constant mean)
  2. $\text{Var}(X_t) = \sigma^2 < \infty$ (constant, finite variance)
  3. $\text{Cov}(X_t, X_{t+h}) = \gamma(h)$ (covariance depends only on lag h)

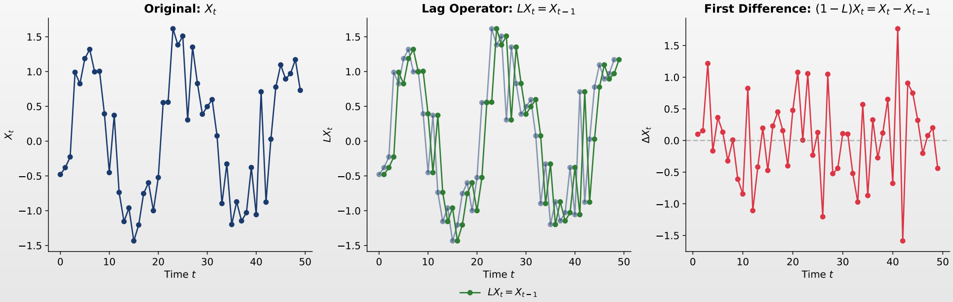
Why Stationarity Matters for ARMA

- ARMA models assume stationarity
  - Parameters remain stable over time
  - Autocorrelation structure is maintained
- Non-stationary data $\Rightarrow$ difference first (ARIMA, Ch. 3)

Chapter Objective

- Parametric models for stationary series $\Rightarrow$ combining dependence on past observations (AR) with the influence of random shocks (MA)

The Lag Operator: Visual Illustration



Role of the Lag Operator

- **Notation foundation**: Enables compact writing of difference equations
- **Utility**: Facilitates algebraic manipulation of ARMA models

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The Lag Operator (Backshift Operator)

Definition 1 (Lag Operator)

- The **lag operator** L (or backshift operator B) shifts a time series back by one period: $LX_t = X_{t-1}$

Properties

- $L^k X_t = X_{t-k}$ (shift back by k periods)
- $L^0 X_t = X_t$ (identity)
- $(1 - L)X_t = X_t - X_{t-1} = \Delta X_t$ (first difference)
- $(1 - L)^d X_t = \Delta^d X_t$ (difference of order d)

Lag Polynomials

- **AR polynomial**: $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$
- **MA polynomial**: $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$

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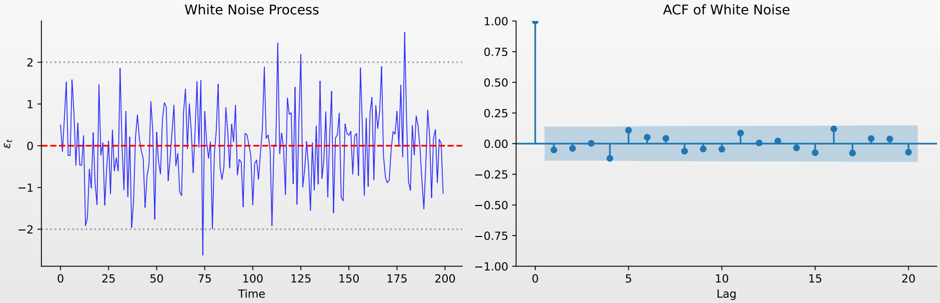
White Noise: Visual Illustration



Key Characteristics

- **Left**: Random fluctuations, no patterns, constant variance
- **Right**: ACF only a spike at lag 0; others within significance bounds $\Rightarrow$ no linear dependence

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The White Noise Process

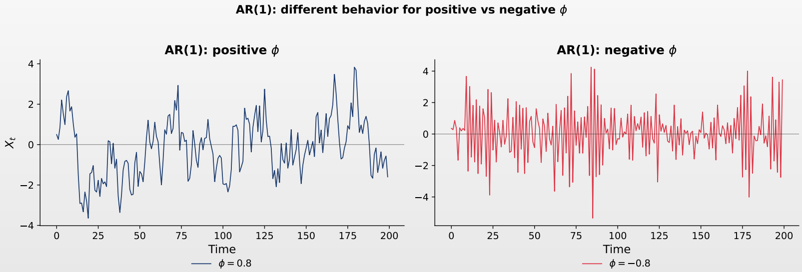
Definition 2 (White Noise)

- A process $\{\varepsilon_t\}$ is **white noise**, denoted $\varepsilon_t \sim WN(0, \sigma^2)$, if:
  1. $\mathbb{E}[\varepsilon_t] = 0$ for all t
  2. $\text{Var}(\varepsilon_t) = \sigma^2$ for all t
  3. $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ for all $t \neq s$

Properties

- **Building block**: White noise underlies all ARMA models
- **ACF**: $\rho(0) = 1$, $\rho(h) = 0$ for $h \neq 0$; PACF: same pattern
- **Gaussian white noise**: $\varepsilon_t \sim N(0, \sigma^2)$ i.i.d.
- **Unpredictable**: White noise is *not* predictable $\Rightarrow$ it is purely random

AR(1): Visual Illustration



Visual Interpretation

- **Positive** ϕ: Persistent fluctuations, gradual mean reversion
- **Negative** ϕ: Oscillating behavior, alternating around the mean
- Larger $|\phi| \Rightarrow$ greater persistence, slower reversion

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The AR(1) Model: Definition

Definition 3 (AR(1) Process)

- An **autoregressive process of order 1** is: $X_t = c + \phi X_{t-1} + \varepsilon_t$
- $\varepsilon_t \sim WN(0, \sigma^2)$ and $|\phi| < 1$ for stationarity

Interpretation

- c: constant (intercept)
- ϕ: autoregressive coefficient
  - Measures the persistence of the series
- ε_t: innovation (shock)

Lag Operator Notation

- $(1 - \phi L)X_t = c + \varepsilon_t$
- $\phi(L)X_t = c + \varepsilon_t$
- $\phi(L) = 1 - \phi L$

AR(1) Stationarity Condition

Necessary and Sufficient Condition: $|\phi| < 1$

- The root of the characteristic equation must lie outside the unit circle

- Shocks diminish over time
  - Process reverts to the mean
  - Finite, stable variance

Non-stationary ($|\phi| \geq 1$)

- $|\phi| = 1$: random walk
  - Unit root, variance $\to \infty$
- $|\phi| > 1$: explosive process

Characteristic Equation

- $\phi(z) = 1 - \phi z = 0 \implies z = 1/\phi$
- Stationarity $\Leftrightarrow$ root outside the unit circle ($|z| > 1$)

AR(1) Properties

Stationary AR(1) with $|\phi| < 1$

- Moment properties:

Mean: $\mu = \mathbb{E}[X_t] = \frac{c}{1-\phi}$

Variance: $\gamma(0) = \mathrm{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$

Autocovariance: $\gamma(h) = \frac{\phi^h \sigma^2}{1-\phi^2}$

Autocorrelation (ACF): $\rho(h) = \phi^h$

Key Observation

- **AR(1) signature**: ACF decays exponentially with factor ϕ
  - $\phi > 0$: monotone decay towards zero
  - $\phi < 0$: damped oscillations (alternating signs)

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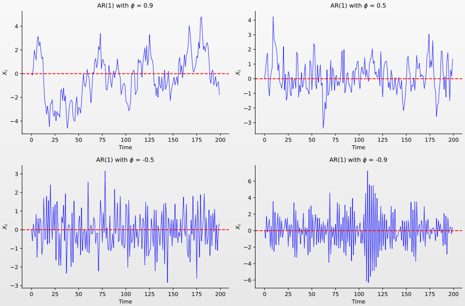
AR(1) Simulations: Effect of ϕ



Interpretation

- Different values of ϕ produce distinct behaviors: larger $|\phi| \Rightarrow$ more persistence
- Positive ϕ creates smooth trajectories; negative ϕ creates oscillations
- As $|\phi| \to 1$, the process approaches non-stationarity

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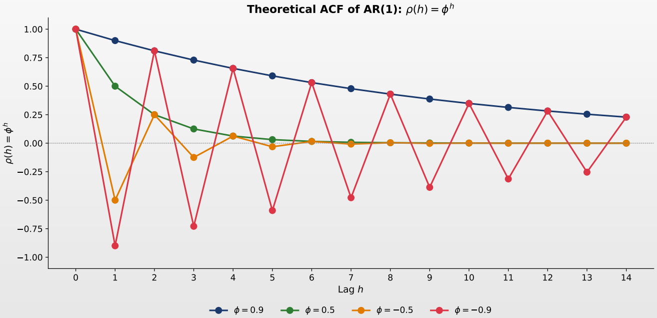
Theoretical AR(1) ACF



ACF Pattern

- **Formula**: $\rho(h) = \phi^h \Rightarrow$ exponential decay
- $\phi > 0$: monotone decay; $\phi < 0$: alternating signs

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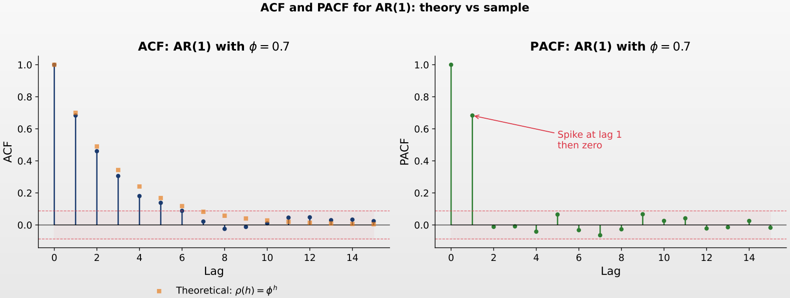
AR(1) ACF and PACF: Theory vs Sample



Interpretation

- **ACF**: Exponential decay with factor ϕ; formula: $\rho(h) = \phi^h$
- **PACF**: A single spike at lag 1, then cuts off $\Rightarrow$ identifies AR(1)
- Sample estimates fluctuate around theoretical values

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Proof: AR(1) Mean

Claim

- For AR(1): $X_t = c + \phi X_{t-1} + \varepsilon_t$, the mean is $\mu = \frac{c}{1-\phi}$

Proof

- Take expectations of both sides: $\mathbb{E}[X_t] = c + \phi \mathbb{E}[X_{t-1}] + \mathbb{E}[\varepsilon_t]$
- By stationarity, $\mathbb{E}[X_t] = \mathbb{E}[X_{t-1}] = \mu$, and $\mathbb{E}[\varepsilon_t] = 0$: $\mu = c + \phi\mu$
- Solving: $\mu - \phi\mu = c \implies \mu(1-\phi) = c \implies \boxed{\mu = \dfrac{c}{1-\phi}}$

Requirement

- **Necessary condition**: $\phi \neq 1$ for the mean to be defined
  - If $\phi = 1$ (unit root), the mean is undefined
  - The process becomes a random walk (non-stationarity)

Proof: AR(1) Variance

Claim

- $\text{Var}(X_t) = \frac{\sigma^2}{1-\phi^2}$

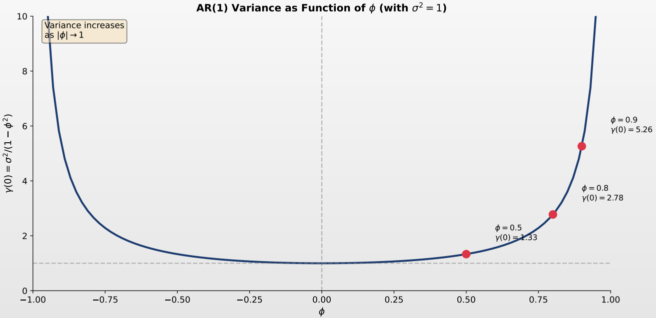
Proof

- Assume $c = 0$. Take the variance of $X_t = \phi X_{t-1} + \varepsilon_t$:
- $\text{Var}(X_t) = \phi^2 \text{Var}(X_{t-1}) + \text{Var}(\varepsilon_t) + 2\phi \underbrace{\text{Cov}(X_{t-1}, \varepsilon_t)}_{=0}$
- By stationarity, $\text{Var}(X_t) = \text{Var}(X_{t-1}) = \gamma(0)$:
- $\gamma(0) = \phi^2\gamma(0) + \sigma^2 \implies \gamma(0)(1-\phi^2) = \sigma^2 \implies \boxed{\gamma(0) = \frac{\sigma^2}{1-\phi^2}}$

Note

- Requires $|\phi| < 1$ for positive variance. When $|\phi| \to 1$, variance $\to \infty$

AR(1) Variance as a Function of ϕ



Observations

- As $|\phi| \to 1$, the variance explodes $\Rightarrow$ non-stationarity
- For $\phi = 0$: $\gamma(0) = \sigma^2$ (white noise); variance increases monotonically with $|\phi|$

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Proof: AR(1) Autocorrelation Function

Claim: $\rho(h) = \phi^h$ for $h \geq 0$

- Find the autocovariance $\gamma(h) = \text{Cov}(X_t, X_{t-h})$

Proof

- Multiply $X_t = \phi X_{t-1} + \varepsilon_t$ by X_{t-h} and take expectations:
- $\mathbb{E}[X_t X_{t-h}] = \phi \mathbb{E}[X_{t-1} X_{t-h}] + \mathbb{E}[\varepsilon_t X_{t-h}]$
- For $h \geq 1$: $\mathbb{E}[\varepsilon_t X_{t-h}] = 0 \Rightarrow \gamma(h) = \phi\gamma(h-1)$
- Recursive relation from $\gamma(0)$: $\gamma(1) = \phi\gamma(0)$, $\gamma(2) = \phi^2\gamma(0)$, $\ldots$ $\boxed{\gamma(h) = \phi^h\gamma(0)}$
- ACF: $\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\phi^h\gamma(0)}{\gamma(0)} = \boxed{\phi^h}$

Proof: AR(1) Stationarity Condition

Claim

- AR(1) is stationary if and only if $|\phi| < 1$

Proof

- Recursive substitution: $X_t = \phi X_{t-1} + \varepsilon_t = \phi(\phi X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \cdots$
- After n steps: $X_t = \phi^n X_{t-n} + \sum_{j=0}^{n-1} \phi^j \varepsilon_{t-j}$
- If $|\phi| < 1$: $\phi^n \to 0$ as $n \to \infty$, so $X_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$
- Finite variance: $\text{Var}(X_t) = \sigma^2 \sum_{j=0}^{\infty} \phi^{2j} = \frac{\sigma^2}{1-\phi^2} < \infty$ (geometric series)

Conclusion

- Converges $\iff |\phi| < 1$. For $|\phi| \geq 1$, the term $\phi^n X_{t-n}$ does not vanish $\Rightarrow$ infinite variance

The Partial Autocorrelation Function (PACF)

Definition 4 (PACF)

- The **partial autocorrelation** of order k, denoted π_k, measures the correlation between X_t and X_{t-k} **after removing** the linear effects of the intermediate variables $X_{t-1}, \ldots, X_{t-k+1}$

Formal Definition

- $\pi_1 = \rho(1)$
- For $k \geq 2$: π_k is the last coefficient in:
  $X_t = \alpha_1 X_{t-1} + \cdots + \alpha_k X_{t-k} + e_t$
- $\pi_k = \alpha_k$ (coefficient of X_{t-k})

Computation via Yule-Walker

- Solve the Yule-Walker equations of order k
- $\pi_k =$ last element of the solution vector

Utility

- **Identification**: PACF determines the order p of an AR model
  - PACF cuts off after lag p

Durbin-Levinson Algorithm for PACF

Durbin-Levinson Recursion

- Computes PACF (π_k) recursively, without inverting the Toeplitz matrix:

1. **Initialize**: $\pi_1 = \hat{\rho}(1)$, $v_1 = \hat{\gamma}(0)(1 - \pi_1^2)$
2. **Recursion** ($k = 2, 3, \ldots$):
$$\pi_k = \frac{\hat{\rho}(k) - \sum_{j=1}^{k-1} \phi_{k-1,j}\, \hat{\rho}(k-j)}{1 - \sum_{j=1}^{k-1} \phi_{k-1,j}\, \hat{\rho}(j)}$$
3. **Update**: $\phi_{k,j} = \phi_{k-1,j} - \pi_k\, \phi_{k-1,k-j}$ for $j = 1, \ldots, k-1$; $\phi_{k,k} = \pi_k$
4. **Prediction variance**: $v_k = v_{k-1}(1 - \pi_k^2)$

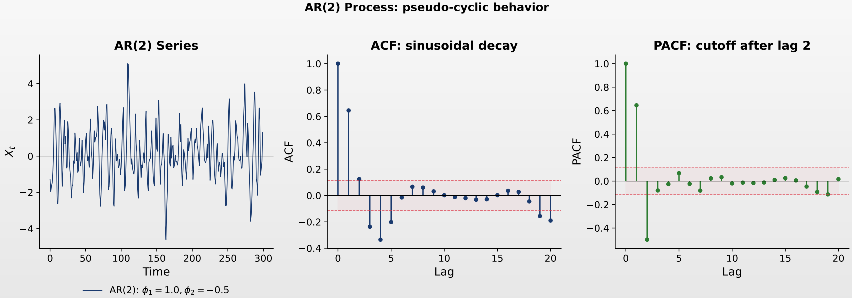
Complexity

- $O(k^2)$ vs $O(k^3)$ (direct inversion)
- Exploits Toeplitz structure of $\boldsymbol{\Gamma}_k$
- Guarantees $v_k > 0$ if stationary

AR(p) Identification

- $\pi_k = 0$ for $k > p \Rightarrow$ order is p
- CI: $|\pi_k| > 1.96/\sqrt{T} \Rightarrow$ significant
- Equivalent to t-test on last OLS coeff.

AR(p): Visual Illustration



Observations

- AR(2) can exhibit pseudo-cyclic behavior (complex roots); damped sinusoidal ACF
- PACF cuts off after lag 2 $\Rightarrow$ key identification pattern

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The AR(p) Model: General Form

Definition 5 (AR(p) Process)

- An **autoregressive process of order p** is: $X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \varepsilon_t$
- **Lag operator**: $\phi(L)X_t = c + \varepsilon_t$, where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$

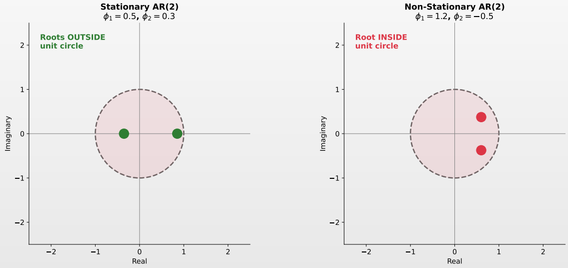
Stationarity Condition

- All roots of $\phi(z) = 0$ must lie **outside** the unit circle
- Equivalently: all roots have modulus > 1

PACF Pattern

- PACF cuts off after lag p
- ACF decays (exponentially or with damped oscillations)

AR(2) Stationarity: Unit Circle Visualization



Characteristic Polynomial and Unit Circle Condition

- **Characteristic polynomial** of AR(p): $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$
- All roots of $\phi(z) = 0$ must lie **outside** the unit circle ($|z| > 1$)
- Roots on the circle: non-stationary; roots inside: explosive process

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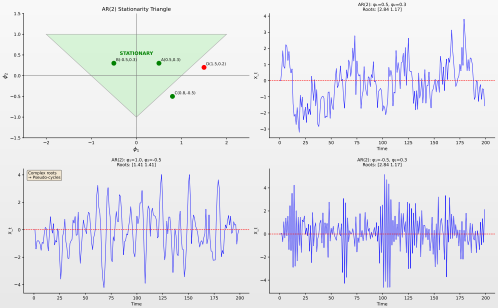
The AR(2) Stationarity Triangle



Stationarity Region

- The triangular region defines the stationary AR(2) parameter combinations
- **Boundaries**: $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$ and $|\phi_2| < 1$
- Points outside the region $\Rightarrow$ non-stationary or explosive processes

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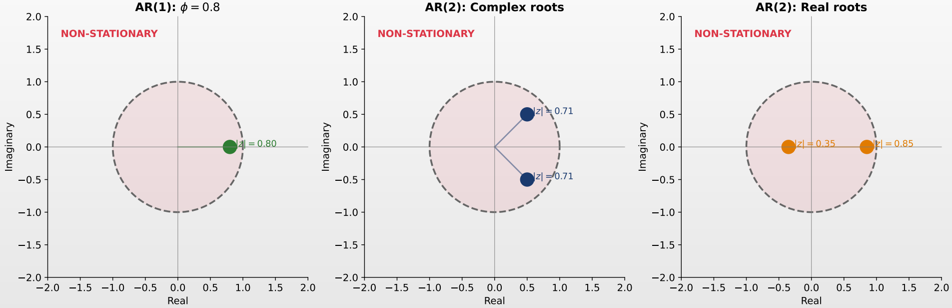
Characteristic Polynomial Roots



Types of Roots

- **Real roots**: exponential decay in ACF
- **Complex roots**: damped oscillations (pseudo-cycles)
- All roots must lie outside the unit circle

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The AR(2) Model

Definition 6 (AR(2) Process)

- $X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$

Stationarity Conditions

- $\phi_1 + \phi_2 < 1; \quad \phi_2 - \phi_1 < 1; \quad |\phi_2| < 1$

ACF Behavior

- **Real roots**: mixture of two exponential decays
- **Complex roots**: damped sinusoidal pattern (pseudo-cycles)
- **PACF**: Cuts off after lag 2 ($\pi_k = 0$ for $k > 2$)

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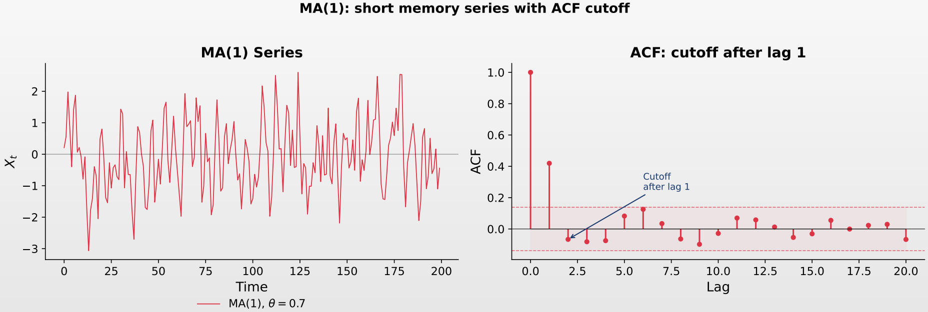
MA(1): Visual Illustration



Visual Interpretation

- **Left panel**: MA(1) series $\Rightarrow$ rapid mean reversion
- **Right panel**: ACF with **cutoff after lag 1**; PACF exponential decay

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The MA(1) Model: Definition

Definition 7 (MA(1) Process)

- A **moving average process of order 1** is: $X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$
- $\varepsilon_t \sim WN(0, \sigma^2)$

Interpretation

- μ: process mean
- θ: MA coefficient
  - Measures the impact of the past shock
- Depends on ε_t and ε_{t-1}

Lag Operator Notation

- $X_t = \mu + \theta(L)\varepsilon_t$
- $\theta(L) = 1 + \theta L$

Key Property

- **Guaranteed stationarity**: MA processes are always stationary
  - Does not depend on the value of θ

MA(1) Properties

MA(1): $X_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$

- **Mean**: $\mathbb{E}[X_t] = \mu$; **Variance**: $\gamma(0) = \sigma^2(1 + \theta^2)$
- **Autocovariance**: $\gamma(1) = \theta\sigma^2$, $\gamma(h) = 0$ $(h > 1)$
- **ACF**: $\rho(1) = \frac{\theta}{1+\theta^2}$, $\rho(h) = 0$ $(h > 1)$

Key Observation

- **MA(1) signature**: ACF cuts off after lag 1
  - $\rho(1) \neq 0$, but $\rho(h) = 0$ for $h > 1$; opposite pattern to AR(1)

TSA_ch2_ma1

Proof: MA(1) Variance and Autocovariance

Starting point: $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ (assuming $\mu = 0$)

- **Variance**:

$$\gamma(0) = \text{Var}(\varepsilon_t + \theta\varepsilon_{t-1}) = \sigma^2 + \theta^2\sigma^2 + 0 = \boxed{\sigma^2(1+\theta^2)}$$

Autocovariance at lag 1

- $\gamma(1) = \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-1} + \theta\varepsilon_{t-2})$
- $= \text{Cov}(\varepsilon_t, \varepsilon_{t-1}) + \theta\text{Cov}(\varepsilon_t, \varepsilon_{t-2}) + \theta\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-1}) + \theta^2\text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-2})$
- $= 0 + 0 + \theta\sigma^2 + 0 = \boxed{\theta\sigma^2}$

Autocovariance at lag $h \geq 2$

- No common ε terms $\Rightarrow \gamma(h) = 0$

Proof: Maximum ACF for MA(1)

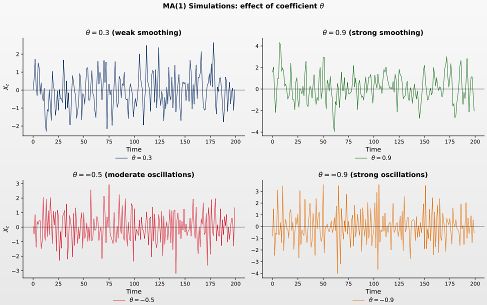
Claim: $|\rho(1)| \leq 0.5$ for any value of θ

- ACF at lag 1: $\rho(1) = \frac{\theta}{1+\theta^2}$
- Differentiate: $\frac{d\rho(1)}{d\theta} = \frac{1-\theta^2}{(1+\theta^2)^2} = 0 \Rightarrow \theta = \pm 1$
- At these values: $\rho(1)\big|_{\theta=1} = \frac{1}{2}$, $\rho(1)\big|_{\theta=-1} = -\frac{1}{2}$

Implication

- **Practical test**: If $|\hat{\rho}(1)| > 0.5$ from data, the process is **not** MA(1)
  - The maximum $|\rho(1)| = 0.5$ is reached at $\theta = \pm 1$
  - Consider AR or ARMA models as alternatives

MA(1) Simulations: Effect of θ



Interpretation

- MA(1) is always stationary regardless of $\theta \Rightarrow$ finite memory of only one lag
- Positive θ smooths the series; negative θ creates faster fluctuations
- Unlike AR(1), MA(1) shocks affect the process for only one period

TSA_ch2_ma1

Proof: ACF for MA(1)

Claim: $\rho(1) = \frac{\theta}{1+\theta^2}$ and $\rho(h) = 0$ for $h > 1$

- MA(1) has non-zero autocorrelation **only** at lag 1

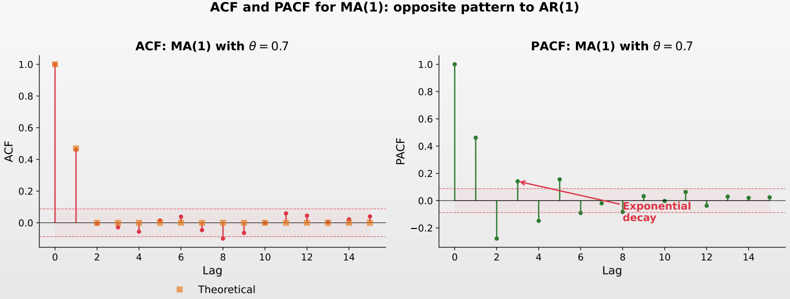
Proof

- Let $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$. Autocorrelation at lag 1:
- $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta\sigma^2}{\sigma^2(1+\theta^2)} = \boxed{\frac{\theta}{1+\theta^2}}$
- For $h > 1$: $\gamma(h) = \text{Cov}(\varepsilon_t + \theta\varepsilon_{t-1},\ \varepsilon_{t-h} + \theta\varepsilon_{t-h-1})$
- The terms $\varepsilon_t, \varepsilon_{t-1}$ do not overlap with $\varepsilon_{t-h}, \varepsilon_{t-h-1}$ when $h > 1$, so $\boxed{\gamma(h) = 0}$

Practical Consequence

- ACF cuts off sharply after lag 1 $\Rightarrow$ distinctive signature of MA(1) processes

MA(1) ACF and PACF: Visual Comparison

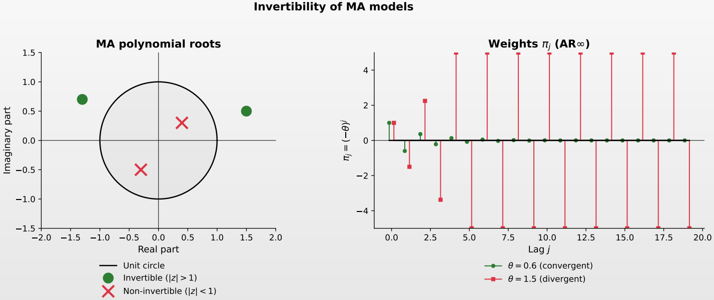


Interpretation

- **ACF**: A single spike at lag 1, then cuts off $\Rightarrow$ key MA(1) signature
- **PACF**: Exponential decay $\Rightarrow$ opposite pattern to AR(1)
- This reversal differentiates MA processes from AR processes

TSA_ch2_ma1

Invertibility: Visual Illustration



Interpretation

- **Left**: invertibility requires roots outside the unit circle
- **Right**: AR(∞) weights decay only when $|\theta| < 1$

Invertibility of MA Models

Definition 8 (Invertibility)

- An MA process is **invertible** if it can be written as an infinite AR process:
- $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$

Invertibility Conditions

- **MA(1)**: Invertible if $|\theta| < 1$
- **MA(q)**: Roots of $\theta(z) = 0$ outside the unit circle

Why Invertibility Matters

- Ensures unique representation (without invertibility, multiple MA models describe the same data)
- Necessary for forecasting and estimation
- **Stationarity** $\Rightarrow$ AR; **Invertibility** $\Rightarrow$ MA

Proof: MA(1) Invertibility

Claim

- MA(1) is invertible if and only if $|\theta| < 1$

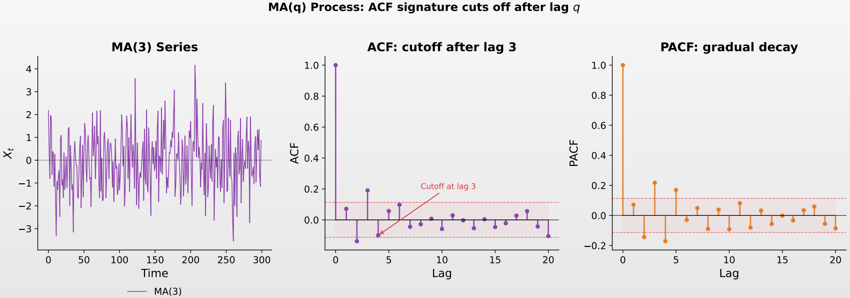
Proof

- From $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$, isolate: $\varepsilon_t = X_t - \theta\varepsilon_{t-1}$
- Recursive back-substitution: $\varepsilon_t = X_t - \theta(X_{t-1} - \theta\varepsilon_{t-2}) = X_t - \theta X_{t-1} + \theta^2\varepsilon_{t-2}$
- Continuing: $\varepsilon_t = \sum_{j=0}^{n}(-\theta)^j X_{t-j} + (-\theta)^{n+1}\varepsilon_{t-n-1}$
- If $|\theta| < 1$: $(-\theta)^{n+1} \to 0$, so $\boxed{\varepsilon_t = \sum_{j=0}^{\infty}(-\theta)^j X_{t-j}}$

Conclusion

- Geometric series converges $\iff$ $|\theta| < 1 \Rightarrow$ MA(1) can be written as AR(∞)

MA(q): Visual Illustration



Observation

- MA(3) process: key signature $\Rightarrow$ ACF cuts off after lag q ($\rho(h) = 0$ for $h > 3$)

TSA_ch2_acf_pacf_patterns

The MA(q) Model: General Form

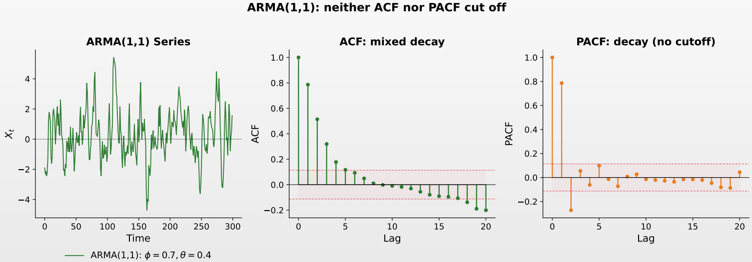
Definition 9 (MA(q) Process)

- A **moving average process of order q**: $X_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q}$
- **Lag operator**: $X_t = \mu + \theta(L)\varepsilon_t$, where $\theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$

Properties

- Always stationary (finite variance)
- ACF cuts off after lag q: $\rho(h) = 0$ for $h > q$; PACF decays gradually
- Invertible if all roots of $\theta(z) = 0$ lie outside the unit circle

ARMA: Visual Illustration



ARMA(1,1) Interpretation

- **Combines** AR persistence with MA shock response
- **ACF pattern**: Decay after the first lag (lags decay geometrically)
- **PACF pattern**: Also decays (no sharp cutoff as in pure AR)
- Neither ACF nor PACF cuts off $\Rightarrow$ key identifier for mixed models

The ARMA(p,q) Model: Definition

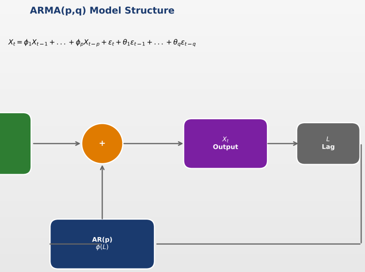
Definition 10 (ARMA(p,q) Process)

- $X_t = c + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$
- **Compact form**: $\phi(L)X_t = c + \theta(L)\varepsilon_t$, where $\mu = \frac{c}{1-\phi_1-\cdots-\phi_p}$

Key Idea

- **Flexibility**: Combines AR and MA components
  - AR captures persistence; MA captures shock response
- **Parsimony**: ARMA(1,1) can be better than AR(5) or MA(5)
  - Fewer parameters, less risk of overfitting

ARMA Model Structure

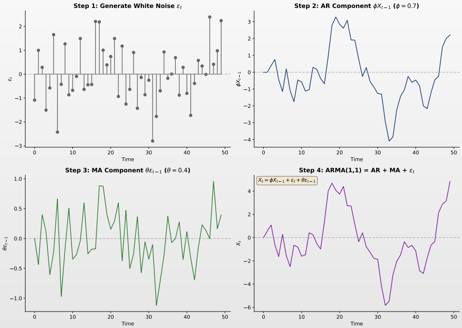


Components

- **AR component**: influence of past values of the series
- **MA component**: impact of past random shocks

TSA_ch2_arma

How ARMA Simulation Works

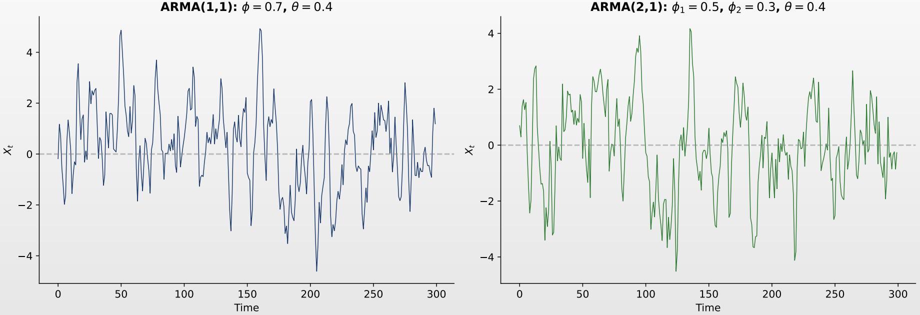


Steps

- Generate white noise, apply the ARMA equation recursively, obtain simulated series

TSA_ch2_arma

ARMA Examples



Observation

- Different combinations of orders (p, q) produce distinct behaviors

TSA_ch2_arma

The ARMA(1,1) Model

Definition 11 (ARMA(1,1) Process)

- $X_t = c + \phi X_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$

Properties (stationarity and invertibility)

- **Mean**: $\mu = \frac{c}{1-\phi}$; **Variance**: $\gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$

ACF

- $\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}$; $\quad \rho(h) = \phi \cdot \rho(h-1)$ for $h \geq 2$
- ACF decays exponentially after lag 1 (starting point depends on ϕ and θ)

TSA_ch2_arma

Proof: ARMA(1,1) Variance

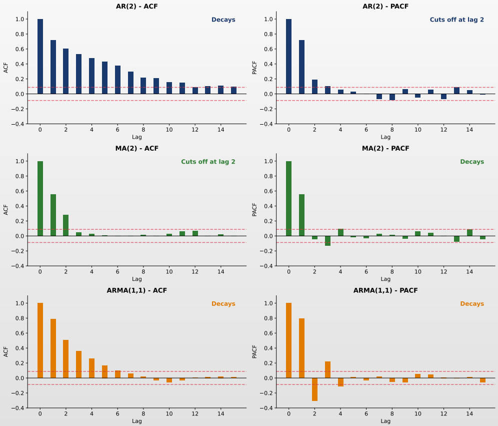
Claim

- $\gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}$

Proof

- Let $Y_t = X_t - \mu$: $Y_t = \phi Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$
- Square: $Y_t^2 = \phi^2 Y_{t-1}^2 + \varepsilon_t^2 + \theta^2\varepsilon_{t-1}^2 + 2\phi Y_{t-1}\varepsilon_t + 2\phi\theta Y_{t-1}\varepsilon_{t-1} + 2\theta\varepsilon_t\varepsilon_{t-1}$
- Take expectations; $\mathbb{E}[\varepsilon_t Y_{t-1}] = 0$, $\mathbb{E}[\varepsilon_t\varepsilon_{t-1}] = 0$:
- $\gamma(0) = \phi^2\gamma(0) + \sigma^2 + \theta^2\sigma^2 + 2\phi\theta\,\mathbb{E}[\varepsilon_{t-1}Y_{t-1}]$
- From $Y_{t-1} = \phi Y_{t-2} + \varepsilon_{t-1} + \theta\varepsilon_{t-2}$: only ε_{t-1}^2 contributes $\Rightarrow \mathbb{E}[\varepsilon_{t-1}Y_{t-1}] = \sigma^2$
- $\gamma(0)(1-\phi^2) = (1+2\phi\theta+\theta^2)\sigma^2 \implies \boxed{\gamma(0) = \frac{(1+2\phi\theta+\theta^2)\sigma^2}{1-\phi^2}}$

ACF/PACF Patterns: AR vs MA vs ARMA



TSA_ch2_acf_pacf_patterns

Proof: ARMA(1,1) ACF at Lag 1

Claim

- $\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}$; $\rho(h) = \phi\,\rho(h-1)$ for $h \geq 2$

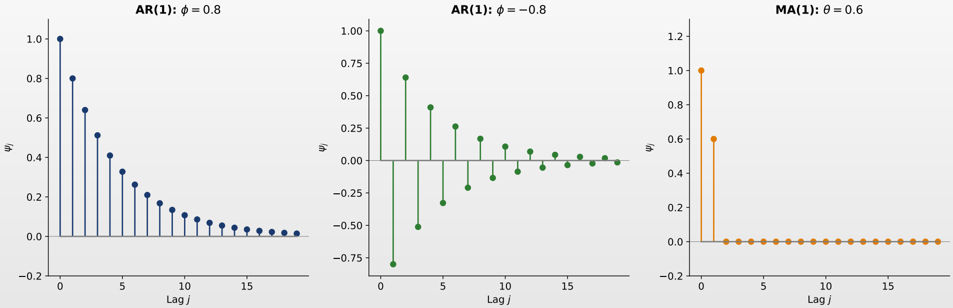
Proof

- Multiply Y_t by Y_{t-1} and take expectations:
- $\gamma(1) = \phi\gamma(0) + \underbrace{\mathbb{E}[\varepsilon_t Y_{t-1}]}_{=0} + \theta\,\underbrace{\mathbb{E}[\varepsilon_{t-1} Y_{t-1}]}_{=\sigma^2} = \phi\gamma(0) + \theta\sigma^2$
- Divide by $\gamma(0)$: $\rho(1) = \phi + \frac{\theta\sigma^2}{\gamma(0)}$. Substitute $\gamma(0)$:
- $\rho(1) = \phi + \frac{\theta(1-\phi^2)}{1+2\phi\theta+\theta^2} = \frac{\phi(1+2\phi\theta+\theta^2)+\theta(1-\phi^2)}{1+2\phi\theta+\theta^2}$
- Numerator: $\phi + \theta + \phi^2\theta + \phi\theta^2 = (\phi+\theta)(1+\phi\theta)$, so $\boxed{\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2}}$

Recursion

- For $h \geq 2$: $\gamma(h) = \phi\gamma(h-1)$, so $\rho(h) = \phi\,\rho(h-1) \Rightarrow$ exponential decay from lag 1

Impulse Response Functions



Shock Propagation

- Shows how a unit shock propagates through the system over time
- **AR**: exponential or oscillating decay; **MA**: effect limited to q periods

TSA_ch2_arma

Stationarity and Invertibility Summary

Conditions for a Valid ARMA(p,q) Model

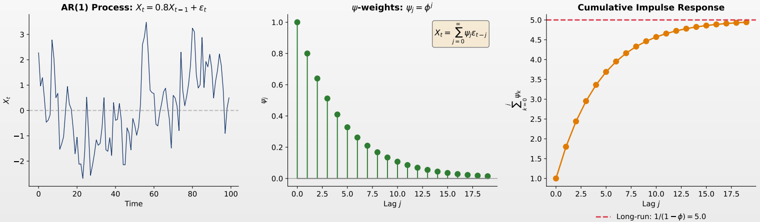
- Requirements summary:

Condition	**Requirement**
Stationarity	Roots of $\phi(z) = 0$ outside the unit circle
Invertibility	Roots of $\theta(z) = 0$ outside the unit circle

Implications

- **Stationarity**: Can be written as MA(∞): $X_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$
- **Invertibility**: Can be written as AR(∞): $X_t = \mu + \sum_{j=1}^{\infty} \pi_j (X_{t-j} - \mu) + \varepsilon_t$
- **Causal representation**: X_t depends only on *past* shocks $\Rightarrow$ necessary for forecasting

Wold's Decomposition Theorem



Wold's Theorem

- Any purely non-deterministic stationary process can be written as MA(∞):
- $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ with $\sum \psi_j^2 < \infty$
- Theoretical justification for ARMA modeling

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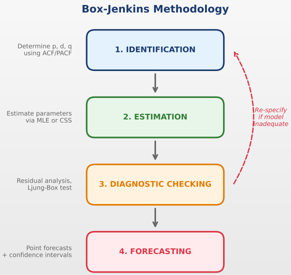
The Box-Jenkins Methodology



Iterative Approach

- Identification $\Rightarrow$ estimation $\Rightarrow$ validation; repeat until residuals are white noise

TSA_ch2_case_study

ACF/PACF Identification Rules

Theoretical Patterns for Stationary Processes

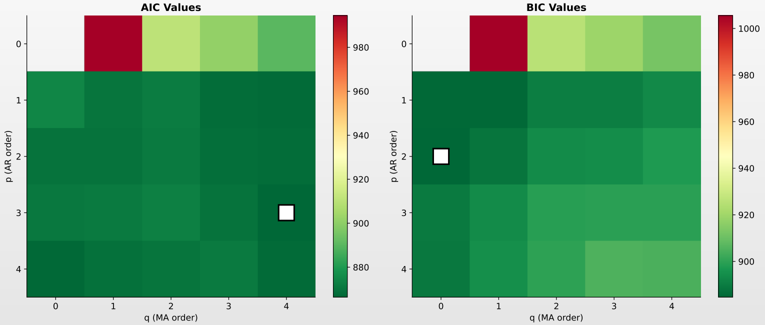
Model	ACF Pattern	PACF Pattern
AR(1)	Exponential decay	Spike at lag 1, then 0
AR(2)	Exp./damped sinusoid	Spikes at lags 1-2, then 0
AR(p)	Gradual decay	Cuts off after lag p
MA(1)	Spike at lag 1, then 0	Exponential decay
MA(2)	Spikes at lags 1-2, then 0	Exp./damped sinusoid
MA(q)	Cuts off after lag q	Gradual decay
ARMA(p,q)	Decays	Decays

Model Identification: ACF/PACF Patterns

Model	ACF Pattern	PACF Pattern
AR(p)	Exponential decay or damped oscillation	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponential decay or damped oscillation
ARMA(p,q)	Exponential decay after lag q-p	Exponential decay after lag p-q

TSA_ch2_acf_pacf_patterns

AIC vs BIC: Model Selection



Interpretation

- White square marks the best model; lower values (green) are better

TSA_ch2_model_selection

Information Criteria

AIC (Akaike)

- $\text{AIC} = -2\ln(\hat{L}) + 2k$
- Moderate penalty
  - Tends to select larger models
  - Optimal for forecasting

BIC (Bayesian)

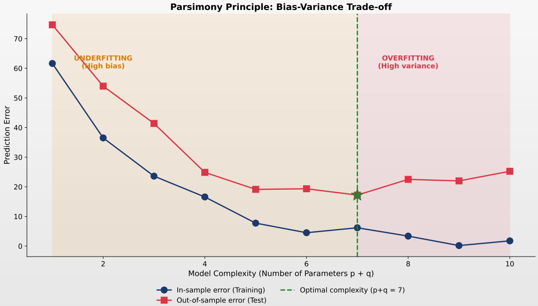
- $\text{BIC} = -2\ln(\hat{L}) + k\ln(n)$
- Stronger penalty
  - Prefers parsimonious models
  - Consistent for identification

where: $\hat{L}$ = maximum of the likelihood function, k = number of estimated parameters, n = sample size

Rules

- Lower values = better model. Compare models on the *same data*

Parsimony Principle: Bias-Variance Trade-off



Bias-Variance Trade-off

- Too simple model $\Rightarrow$ high bias (underfitting)
- Too complex model $\Rightarrow$ high variance (overfitting)
- The optimum lies at the intersection of the two curves

TSA_ch2_model_selection

Automatic Model Selection

Grid Search Approach

- Estimate ARMA(p,q) for $p = 0, \ldots, p_{max}$ and $q = 0, \ldots, q_{max}$
- Select the model with the lowest AIC or BIC; verify with validation tests

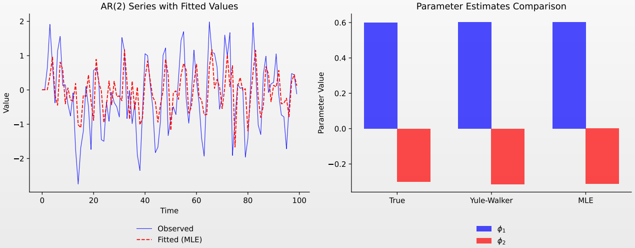
In Python

- `pm.auto_arima()` from the `pmdarima` package
- Automatically tests stationarity, iterates over orders (p, q), returns the best model

Caution

- Automatic selection is not the final answer $\Rightarrow$ verify model validity
- Full Auto-ARIMA (including selection of d) $\Rightarrow$ Chapter 3

Estimation Methods



The Three Main Approaches

- **Yule-Walker**: closed-form, AR only; equates sample autocorrelations with theoretical values
- **MLE**: most efficient and consistent; requires distributional assumption (Gaussian)
- **Conditional Least Squares**: compromise; minimizes sum of squared residuals

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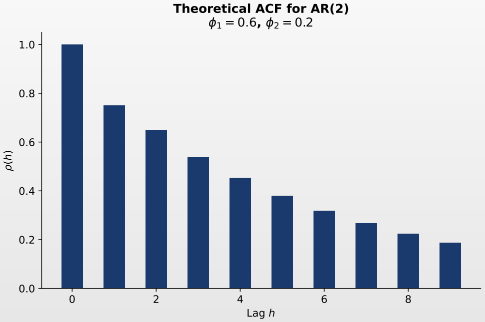
The Yule-Walker Equations for AR(p)



Yule-Walker Equations

$$\rho(1) = \phi_1 + \phi_2\rho(1)$$

$$\rho(2) = \phi_1\rho(1) + \phi_2$$

Matrix form: $R \cdot \phi = \rho$

R = autocorrelation matrix

Solution: $\hat{\phi} = R^{-1}\rho$

Main Idea

- Linear relationship between autocorrelations and AR parameters
- Allows closed-form estimation (no numerical optimization)

TSA_ch2_estimation

The Yule-Walker Equations: Matrix Form

Yule-Walker Equations for AR(p)

- $\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \cdots + \phi_p\rho(k-p), \quad k = 1, 2, \ldots, p$

Matrix Form

- $$\begin{pmatrix} \rho(0) & \rho(1) & \cdots & \rho(p-1) \\ \rho(1) & \rho(0) & \cdots & \rho(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(p-1) & \rho(p-2) & \cdots & \rho(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{pmatrix}$$
- **Estimation**: Replace $\rho(k)$ with $\hat{\rho}(k)$; the Toeplitz matrix is symmetric and positive definite

Numerical Example: Yule-Walker for AR(2)

Sample Data ($T = 100$)

- **Estimated autocorrelations**: $\hat{\rho}(1) = 0.75$, $\hat{\rho}(2) = 0.65$
  - Estimated variance: $\hat{\gamma}(0) = 4.0$

Step 1: Matrix System

- **Yule-Walker**: $R\hat{\phi} = \boldsymbol{\rho}$
  - $\begin{pmatrix} 1 & 0.75 \\ 0.75 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.65 \end{pmatrix}$

Step 2: Solution (Cramer's Rule)

- $\det(R) = 1 - 0.75^2 = 0.4375$
- $\hat{\phi}_1 = \frac{0.75 \times 1 - 0.75 \times 0.65}{0.4375} = \frac{0.2625}{0.4375} = \boxed{0.600}$ $\qquad \hat{\phi}_2 = \frac{0.65 \times 1 - 0.75 \times 0.75}{0.4375} = \frac{0.0875}{0.4375} = \boxed{0.200}$

Step 3: Noise Variance

- $\hat{\sigma}^2 = \hat{\gamma}(0)(1 - \hat{\phi}_1\hat{\rho}(1) - \hat{\phi}_2\hat{\rho}(2)) = 4.0(1 - 0.45 - 0.13) = \boxed{1.68}$

Stationarity check: $\hat{\phi}_1 + \hat{\phi}_2 = 0.8 < 1$ ✓ $\quad |\hat{\phi}_2| = 0.2 < 1$ ✓ $\quad \hat{\phi}_2 - \hat{\phi}_1 = -0.4 > -1$ ✓

Proof: The Yule-Walker Equations

Goal: Derive $\rho(k) = \phi_1\rho(k-1) + \cdots + \phi_p\rho(k-p)$

- Start from AR(p): $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t$
- Multiply by X_{t-k} and take expectations:
- $\mathbb{E}[X_t X_{t-k}] = \phi_1 \mathbb{E}[X_{t-1}X_{t-k}] + \cdots + \phi_p \mathbb{E}[X_{t-p}X_{t-k}] + \mathbb{E}[\varepsilon_t X_{t-k}]$
- For $k \geq 1$: $\mathbb{E}[\varepsilon_t X_{t-k}] = 0 \Rightarrow \gamma(k) = \phi_1\gamma(k-1) + \cdots + \phi_p\gamma(k-p)$
- Dividing by $\gamma(0)$: $\boxed{\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2) + \cdots + \phi_p\rho(k-p)}$

Special Case AR(1)

- $\rho(k) = \phi_1\rho(k-1) = \phi_1^k$ (using $\rho(0) = 1$)

Maximum Likelihood Estimation

ARMA(p,q) Log-Likelihood (Gaussian errors: $\varepsilon_t \sim N(0, \sigma^2)$)

- $\ell(\boldsymbol{\phi}, \boldsymbol{\theta}, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\varepsilon_t^2$
- ε_t are the innovations computed recursively

Estimation Procedure

- Initialization: use method of moments or OLS for starting values
- Optimization: numerical methods (BFGS, Newton-Raphson)
- Iterate until convergence

In Practice

- `statsmodels.tsa.arima.model.ARIMA` $\Rightarrow$ implements exact MLE with automatic initialization

Standard Errors and Inference

Asymptotic Distribution of MLE

- $\hat{\boldsymbol{\theta}} \xrightarrow{d} N\left(\boldsymbol{\theta}_0, \frac{1}{n}\mathrm{I}(\boldsymbol{\theta}_0)^{-1}\right)$, where $\mathrm{I}(\boldsymbol{\theta})$ is the **Fisher information matrix**
- $\mathrm{I}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 \ln L(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\,\partial\boldsymbol{\theta}'}\right] \Rightarrow$ average curvature of the log-likelihood
- Estimated variance-covariance matrix: $\hat{\mathrm{V}} = \frac{1}{n}\hat{\mathrm{I}}^{-1}$

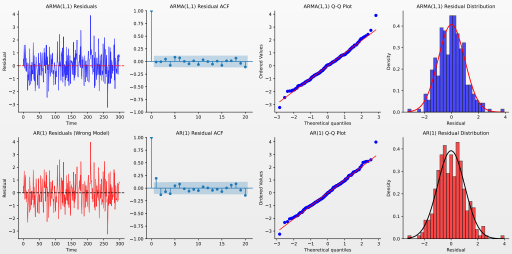
What is the Standard Error (SE)?

- $SE(\hat{\theta}_j) = \sqrt{\hat{\mathrm{V}}_{jj}} = \sqrt{\mathrm{diag}_j\left(\frac{1}{n}\hat{\mathrm{I}}^{-1}\right)} \Rightarrow$ measures estimation uncertainty
- **Example AR(1)**: $SE(\hat{\phi}) \approx \sqrt{(1-\hat{\phi}^2)/n}$; for $\hat{\phi} = 0.8$, $n = 100$: $SE \approx 0.06$
- **Interpretation**: small SE $\Rightarrow$ parameter is estimated with high precision

Testing Parameter Significance

- $H_0 : \theta_j = 0$ Statistic: $z = \frac{\hat{\theta}_j}{SE(\hat{\theta}_j)} \sim N(0,1)$ asymptotically
- Reject if $|z| > 1.96$ at 5% $\Rightarrow$ **CI**: $\hat{\theta}_j \pm 1.96 \cdot SE(\hat{\theta}_j)$

Residual Diagnostics



If the model is correctly specified, residuals must be white noise

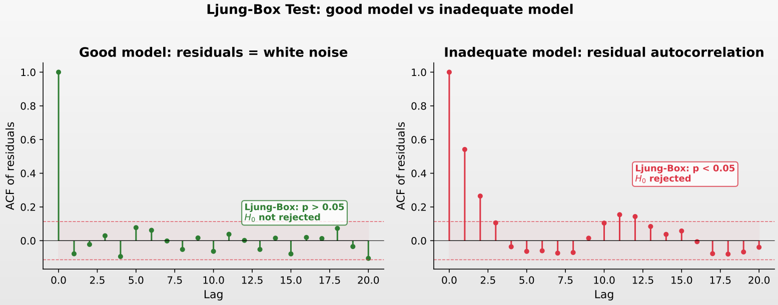
- **Residual plot**: random fluctuations around zero, constant variance
- **Residual ACF**: no significant spikes $\Rightarrow$ white noise
- **Q-Q plot**: points on diagonal $\Rightarrow$ normal; heavy tails $\Rightarrow$ non-normal errors

Decision

- ✓ All checks OK $\Rightarrow$ adequate model   × Not satisfied $\Rightarrow$ return to identification

TSA_ch2_diagnostics

The Ljung-Box Test: Visual Illustration



Interpretation

- **Left**: good model $\Rightarrow$ white noise residuals
- **Right**: inadequate model $\Rightarrow$ residual autocorrelation $\Rightarrow$ re-specification needed

TSA_ch2_diagnostics

The Ljung-Box Test

Definition 12 (Ljung-Box Test)

- Tests whether residuals are independently distributed (no autocorrelation)
- **Statistic**: $Q(m) = n(n+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{n-k}$

Hypotheses and Distribution

- H_0: Residuals are white noise; H_1: Residuals are autocorrelated
- Under H_0, $Q(m) \sim \chi^2(m-p-q)$ approximately

Decision

- p-value $> 0.05 \Rightarrow$ do not reject $H_0 \Rightarrow$ residuals are white noise
- p-value $< 0.05 \Rightarrow$ residual autocorrelation $\Rightarrow$ inadequate model

TSA_ch2_diagnostics

Model Checklist

A Good ARMA Model Should Satisfy

- **Stationarity**: AR roots outside the unit circle (`arroots`)
- **Invertibility**: MA roots outside the unit circle (`maroots`)
- **White noise residuals**: No significant ACF (Ljung-Box test)
- **Normal residuals**: Q-Q plot, Jarque-Bera test
- **No heteroscedasticity**: Constant variance (ARCH test)
- **Simple**: Lowest AIC/BIC among adequate models

If Checks Are Not Satisfied

- Return to identification, try different orders

Point Forecasts

Optimal Forecast: $\hat{X}_{n+h|n} = \mathbb{E}[X_{n+h}|X_n, X_{n-1}, \ldots]$

- The conditional expectation minimizes MSE

AR(1): $X_t = c + \phi X_{t-1} + \varepsilon_t$

- $\hat{X}_{n+1|n} = c + \phi X_n; \quad \hat{X}_{n+h|n} = \mu + \phi^h (X_n - \mu)$
- Forecasts converge to the mean μ as $h \to \infty$ (mean reversion)

MA(1): $X_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

- $\hat{X}_{n+1|n} = \mu + \theta \varepsilon_n; \quad \hat{X}_{n+h|n} = \mu$ for $h > 1$

Forecast Uncertainty

Mean Square Forecast Error (MSFE)

- **Error**: $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- **MSFE**: $\text{MSFE}(h) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$, where ψ_j are the MA(∞) coefficients

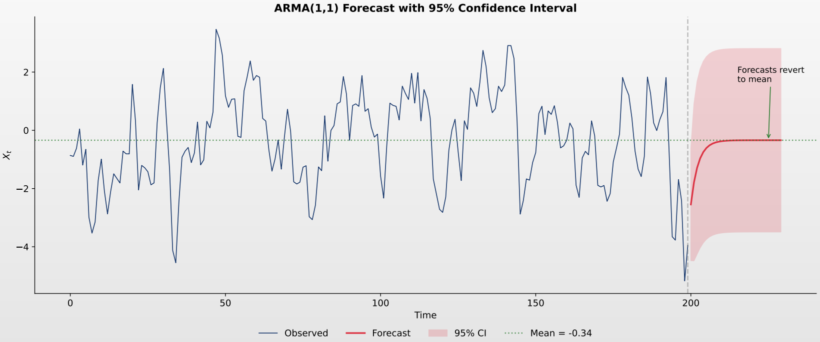
For AR(1): $\psi_j = \phi^j$

- $\text{MSFE}(h) = \sigma^2 \frac{1-\phi^{2h}}{1-\phi^2} \to \frac{\sigma^2}{1-\phi^2} = \text{Var}(X_t)$

Key Observation

- Forecast uncertainty increases with the horizon
- Converges to the unconditional variance $\text{Var}(X_t)$

ARMA Forecast with Confidence Intervals



Observation

- The confidence band widens with the horizon $\Rightarrow$ convergence to the unconditional interval

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Proof: MSFE for AR(1)

Claim

- $\text{MSFE}(h) = \sigma^2 \frac{1-\phi^{2h}}{1-\phi^2}$ and $\text{MSFE}(\infty) = \gamma(0)$

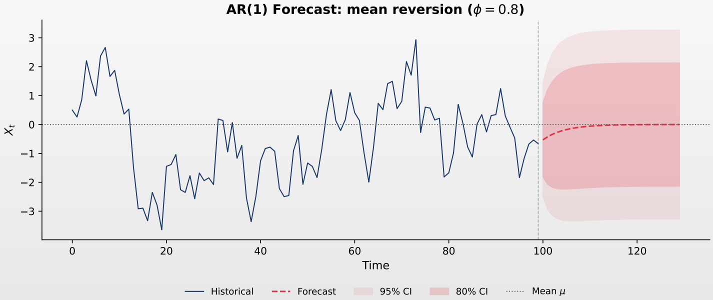
Proof

- Forecast error at horizon h: $e_{n+h|n} = X_{n+h} - \hat{X}_{n+h|n}$
- By recursive substitution: $e_{n+h|n} = \sum_{j=0}^{h-1} \phi^j \varepsilon_{n+h-j}$
- $\text{MSFE}(h) = \mathbb{E}[e^2_{n+h|n}] = \sigma^2 \sum_{j=0}^{h-1} \phi^{2j} = \boxed{\sigma^2 \frac{1-\phi^{2h}}{1-\phi^2}}$
- Limit: $\text{MSFE}(\infty) = \frac{\sigma^2}{1-\phi^2} = \gamma(0) \Rightarrow$ forecast converges to unconditional mean

Interpretation

- At long horizons, we do no better than the unconditional mean: CI $\to 2 \times 1.96\sqrt{\gamma(0)}$

AR(1) Forecast: Mean Reversion

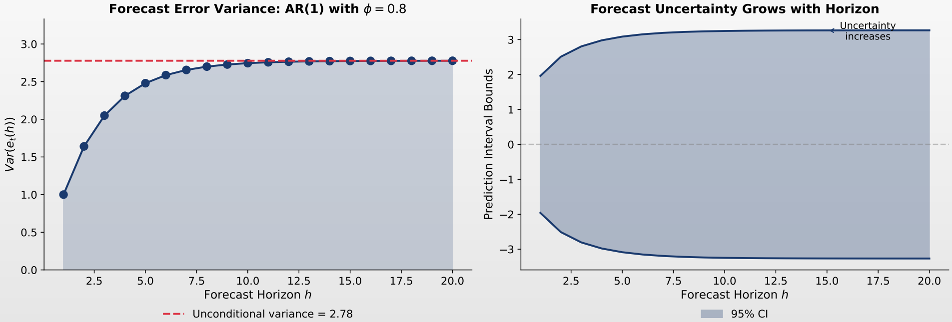


Properties

- Forecasts converge to the unconditional mean μ as the horizon increases
- Larger $|\phi| \Rightarrow$ slower reversion; CIs widen with the horizon

TSA_ch2_forecasting

Forecast Error Variance by Horizon



Observation

- MSFE increases monotonically with horizon $h \Rightarrow$ convergence to $\mathrm{Var}(X_t)$ (predictability limit)

TSA_ch2_forecasting

Confidence Intervals for Forecasts

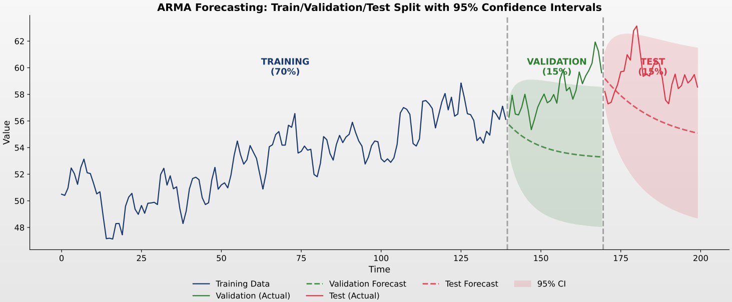
Formulas

- $X_{n+h}|X_n, \ldots \sim N\left(\hat{X}_{n+h|n}, \text{MSFE}(h)\right)$
- **CI** $(1-\alpha)$: $\hat{X}_{n+h|n} \pm z_{\alpha/2} \cdot \sqrt{\text{MSFE}(h)}$, where $z_{\alpha/2} = 1.96$ for 95%

Properties

- Intervals widen as the horizon increases
  - Converge to the unconditional interval: $\mu \pm z_{\alpha/2}\sigma_X$
- Width depends on model parameters
  - Larger AR coefficients $\Rightarrow$ wider intervals
- **Python**: `model.get_forecast(h).conf_int()`

Train/Validation/Test Forecast Example



Best Practice

- Always evaluate forecasts on data not used for estimation (train/validation/test split)

TSA_ch2_forecasting

Forecast Evaluation

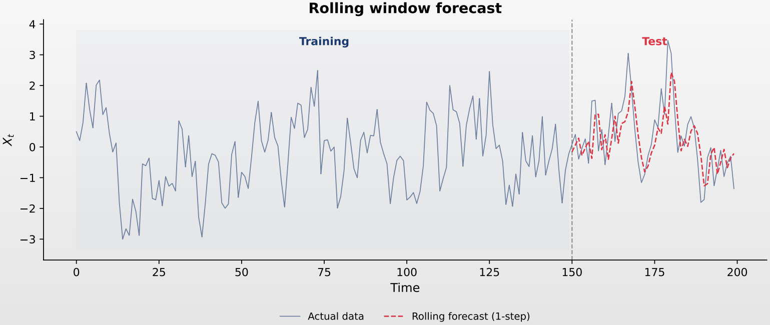
Out-of-Sample Testing

- Split data: training + test
- Generate forecasts on test
- Compare with actual values
- **Rolling window**: re-estimate as new data arrives

Error Metrics

- **MAE** $= \frac{1}{n}\sum |e_t|$
  - Robust to outliers
- **RMSE** $= \sqrt{\frac{1}{n}\sum e_t^2}$
  - Penalizes large errors
- **MAPE** $= \frac{100}{n}\sum \left|\frac{e_t}{X_t}\right|$
  - Percentage-based, interpretable

Rolling Window Forecast

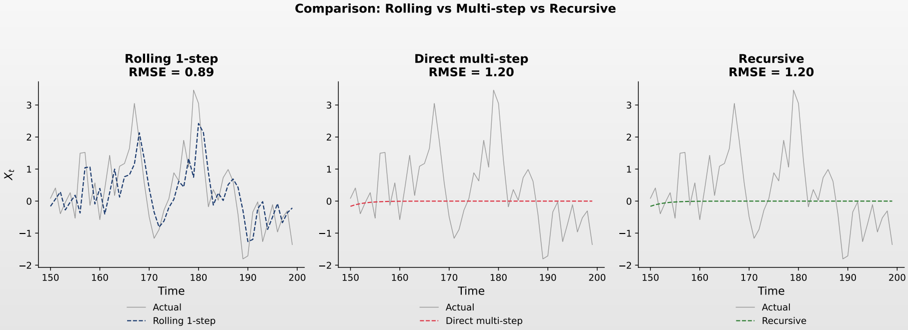


Rolling Forecast Methodology

- **Fixed window** (last w obs.) vs **expanding** (all data); generate 1-step forecast, repeat

TSA_ch2_forecasting

Rolling vs Multi-Step Forecast

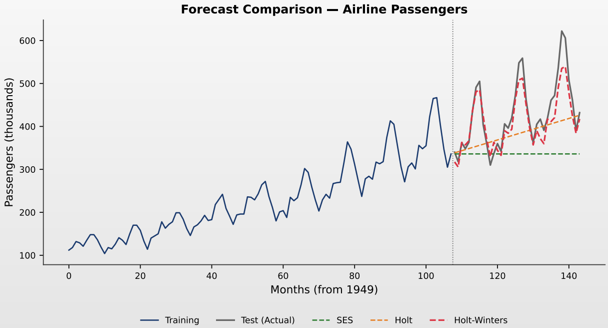


Key Differences

- **Rolling 1-step** (accurate); **Multi-step direct** (separate model/horizon); **Recursive** (error accumulation)

TSA_ch2_forecasting

Real Data Application: Forecast Comparison



Practical Considerations

- Real data: non-stationarity, structural breaks; compare models; use rolling window validation

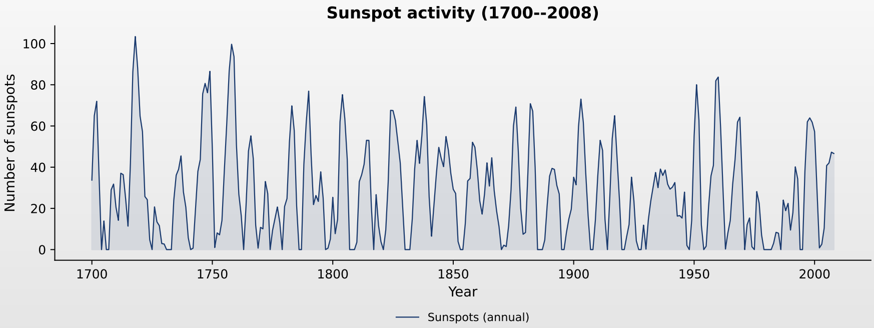
TSA_ch2_forecasting

Workflow Summary

Box-Jenkins Methodology Steps

- **1. Data preparation**: Check for missing values, outliers; transform if necessary
- **2. Stationarity check**: Visual inspection, formal tests (ADF, KPSS); difference if non-stationary
- **3. Model identification**: ACF/PACF patterns; grid search with information criteria
- **4. Estimation and validation**: Estimate model, check significance; residual analysis, Ljung-Box test
- **5. Forecasting**: Point forecasts with confidence intervals; out-of-sample validation

Case Study: Sunspots

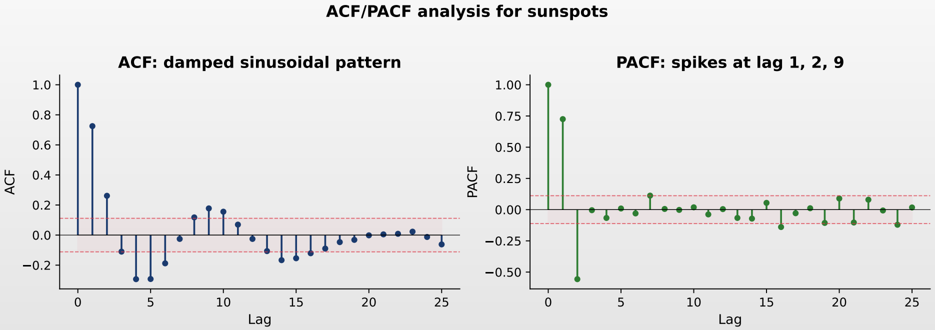


Data Description

- Annual sunspots (1700–2008): stationary series with ∼11-year cycles; Box-Jenkins methodology

TSA_ch2_case_study

Step 1: ACF/PACF Analysis

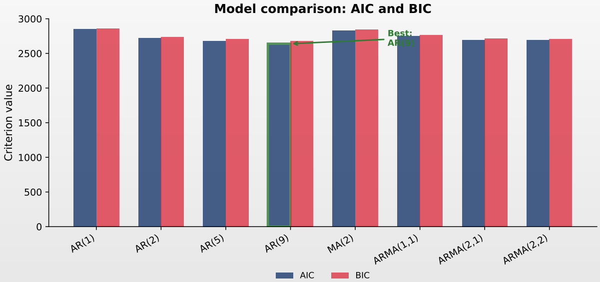


Identification

- Sinusoidal ACF (AR); PACF with spikes at lags 1, 2, 9 ⇒ AR(2) or AR(9); stationary series ($d = 0$)

TSA_ch2_case_study

Step 2: Model Comparison

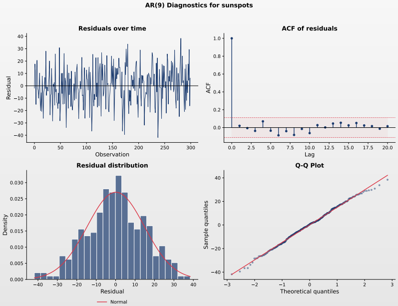


Model Selection

- Compare multiple candidate models using AIC; the **AR(9)** has the lowest AIC, capturing the 11-year solar cycle

TSA_ch2_case_study

Step 3: Model Diagnostics

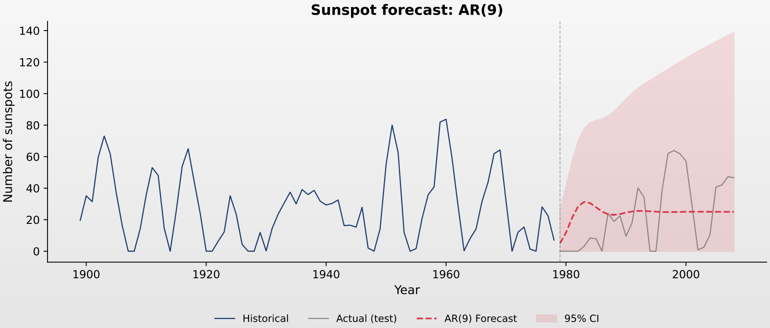


AR(9) Diagnostics

- Residuals: white noise, zero mean, constant variance, ACF without structure, $\approx$ normal

TSA_ch2_case_study

Step 4: Forecasting



Results

- AR(9) captures the cyclicality; 95% CI covers actual values; RMSE $\approx$ 30

TSA_ch2_case_study

Key Takeaways

Chapter Summary

- **AR(p)**: Depends on p past values; stationarity: roots outside the unit circle; PACF cuts off at lag p
- **MA(q)**: Depends on q past shocks; always stationary; ACF cuts off at lag q
- **ARMA(p,q)**: Combines AR and MA; both ACF and PACF decay
- **Box-Jenkins**: Identification $\Rightarrow$ Estimation $\Rightarrow$ Validation $\Rightarrow$ Forecasting
- **Validation**: Residuals must be white noise
- **Forecasts**: Converge to the mean; uncertainty increases with the horizon

Next Chapter Preview

Chapter 3: ARIMA Models for Non-Stationary Data

- Non-stationarity: types, unit root tests (ADF, PP, KPSS)
- Differencing and the difference operator
- ARIMA(p,d,q): integrated models for non-stationary data
- The Auto-ARIMA algorithm: automatic model selection
- Case study: US GDP Forecasting

Reading

- Hyndman & Athanasopoulos, *Forecasting: Principles and Practice*, Ch. 9
- Box, Jenkins, Reinsel & Ljung, *Time Series Analysis*, Ch. 3-4

AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Download monthly US Industrial Production Index from FRED (series INDPRO) for 2010-01 to 2024-12 (180 observations). Compute monthly log-differences (growth rates). Estimate an ARMA model, perform residual diagnostics, and forecast 12 months ahead. Give me complete Python code with plots."

Exercise:

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. Does it verify stationarity *before* estimating ARMA? Justify.
3. How does it choose the orders p and q? Does it use ACF/PACF or AIC/BIC?
4. Are residuals tested correctly? (Ljung-Box, Q-Q plot, heteroscedasticity)
5. Do forecast confidence intervals converge to the unconditional mean?

Warning: AI-generated code may run without errors and look professional. *That does not mean it is correct.*

Question 1

Question

- For which value of ϕ is the AR(1) process $X_t = c + \phi X_{t-1} + \varepsilon_t$ stationary?

Answer Choices

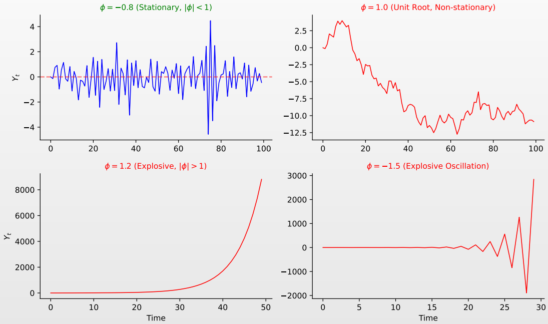
(A) $\phi = 1.2$

(B) $\phi = 1.0$

(C) $\phi = -0.8$

(D) $\phi = -1.5$

Question 1: Answer



Answer: (C)

- AR(1) is stationary if and only if $|\phi| < 1$
- Only $|-0.8| = 0.8 < 1$

TSA_ch2_ar1

Question 2

Question

- You observe: ACF has a spike at lag 1, then cuts off. PACF decays gradually. What model?

Answer Choices

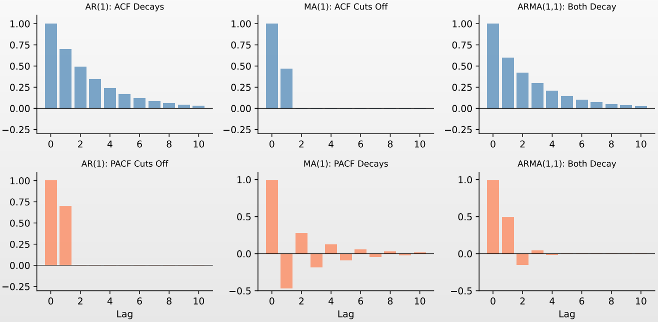
(A) AR(1)

(B) MA(1)

(C) ARMA(1,1)

(D) White noise

Question 2: Answer



Answer: (B)

- ACF cuts off $\Rightarrow$ MA process
- PACF decays $\Rightarrow$ confirms MA(1)

TSA_ch2_ma1

Question 3

Question

- Is the MA(1) $X_t = \varepsilon_t + 1.5\varepsilon_{t-1}$ invertible?

Answer Choices

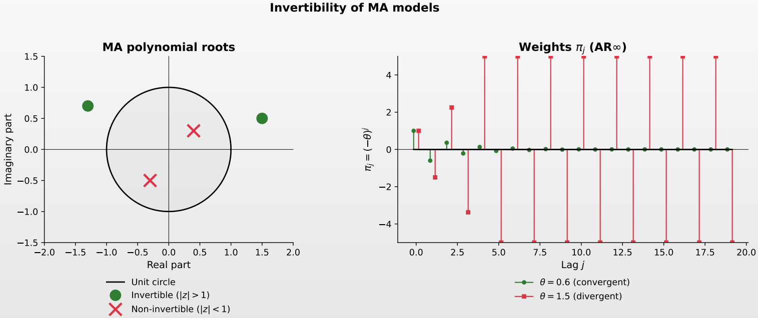
(A) Yes, MA processes are always invertible

(B) Yes, because $1.5 > 0$

(C) No, because $|\theta| = 1.5 > 1$

(D) No, MA processes are never invertible

Question 3: Answer



Answer: (C)

- Invertibility requires $|\theta| < 1$
- Here $|\theta| = 1.5 > 1$, so it is not invertible

Question 4

Question

- The compact form $\phi(L)X_t = \theta(L)\varepsilon_t$ represents which model?

Answer Choices

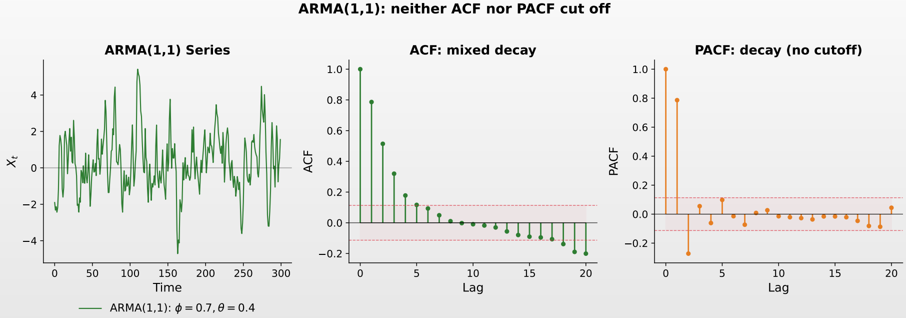
(A) Pure AR model

(B) Pure MA model

(C) ARMA model

(D) None of the above

Question 4: Answer



Answer: (C)

- $\phi(L)$ is the AR polynomial, $\theta(L)$ is the MA polynomial $\Rightarrow$ ARMA(p,q)

TSA_ch2_arma

Question 5

Question

- What is $(1-L)^2 X_t$?

Answer Choices

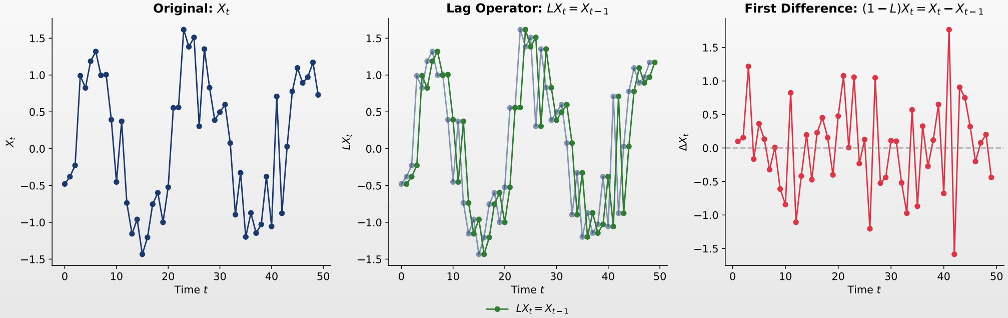
(A) $X_t - X_{t-1}$

(B) $X_t - 2X_{t-1} + X_{t-2}$

(C) $X_t + X_{t-1} + X_{t-2}$

(D) $X_t - X_{t-2}$

Question 5: Answer



Answer: (B)

- $(1-L)^2 = 1 - 2L + L^2$
- $(1-L)^2 X_t = X_t - 2X_{t-1} + X_{t-2}$

TSA_ch2_lag_operator

Question 6

Question

- Comparing ARMA(1,1) vs ARMA(2,1) using BIC, which is correct?

Answer Choices

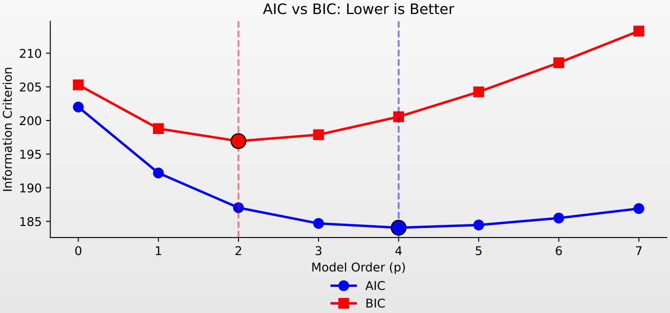
(A) Lower BIC always means better forecasts

(B) BIC penalizes complexity less than AIC

(C) The model with lower BIC is preferred

(D) BIC can only compare models with the same number of parameters

Question 6: Answer



Answer: (C)

- Lower BIC indicates a better balance between estimation quality and complexity
- BIC penalizes complexity *more* than AIC

TSA_ch2_model_selection

Question 7

Question

- After estimating an ARMA model, you run the Ljung-Box test on residuals and obtain p-value = 0.03. What does this mean?

Answer Choices

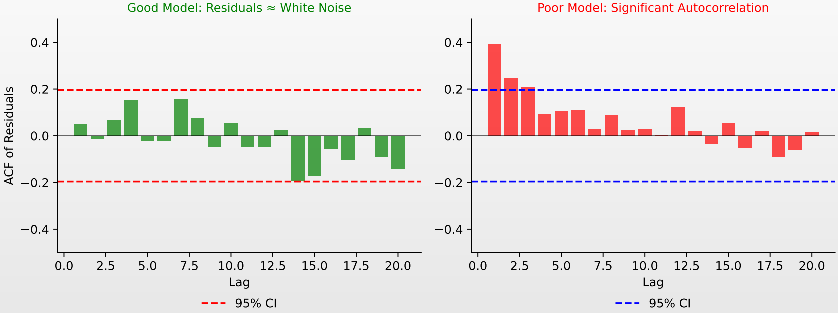
(A) The model is adequate, residuals are white noise

(B) The model is inadequate, residuals have autocorrelation

(C) You need to increase the sample size

(D) The test is inconclusive

Question 7: Answer



Answer: (B)

- p-value < 0.05 rejects H_0 (white noise)
- Indicates remaining residual autocorrelation

TSA_ch2_diagnostics

Question 8

Question

- For a stationary AR(1) model, what happens to forecasts as the horizon $h \to \infty$?

Answer Choices

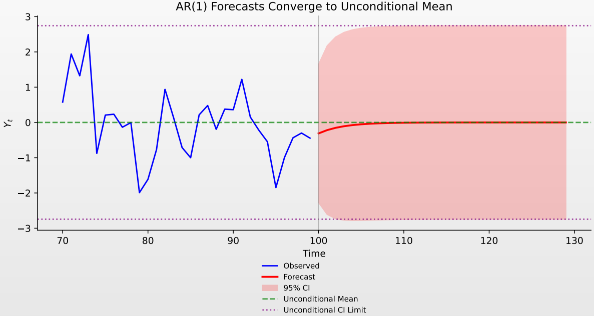
(A) Forecasts increase without bound

(B) Forecasts oscillate indefinitely

(C) Forecasts converge to the unconditional mean μ

(D) Forecasts become more precise

Question 8: Answer



Answer: (C)

- $\hat{X}_{n+h|n} = \mu + \phi^h(X_n - \mu) \to \mu$ as $h \to \infty$ (since $|\phi| < 1$)

TSA_ch2_forecasting

Question 9

Question

- Consider an AR(1) process with $\phi = 0.6$ and $\sigma^2 = 4$. What is $\text{Var}(X_t)$?

Answer Choices

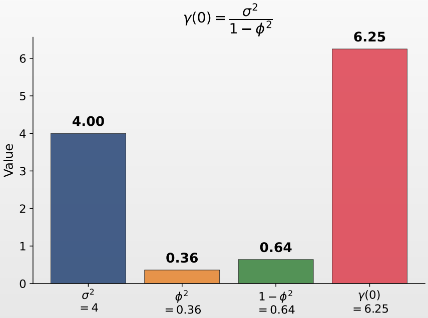
(A) 4.0

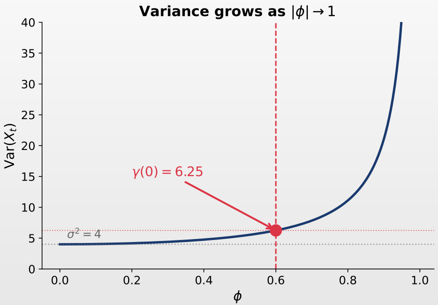
(B) 5.56

(C) 6.25

(D) 10.0

Question 9: Answer





Answer: (C)

- $\mathrm{Var}(X_t) = \frac{\sigma^2}{1-\phi^2} = \frac{4}{1-0.36} = \frac{4}{0.64} = 6.25$
- The process variance exceeds σ^2 due to persistence

TSA_ch2_ar1

Question 10

Question

- Consider an MA(1) process with $\theta = 0.5$. What is $\rho(1)$?

Answer Choices

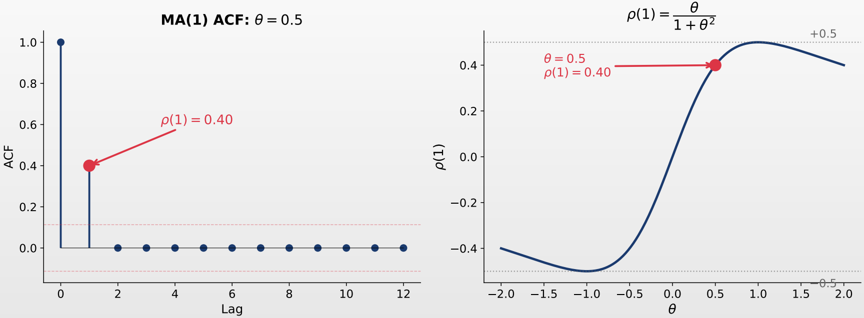
(A) 0.50

(B) 0.40

(C) 0.25

(D) 0.33

Question 10: Answer



Answer: (B)

- $\rho(1) = \frac{\theta}{1+\theta^2} = \frac{0.5}{1+0.25} = \frac{0.5}{1.25} = 0.40$
- Note that $\rho(1) < \theta$ — the autocorrelation is **always** attenuated

TSA_ch2_ma1

Question 11

Question

Which statement about the ACF of an ARMA(1,1) process is **true**?

Answer Choices

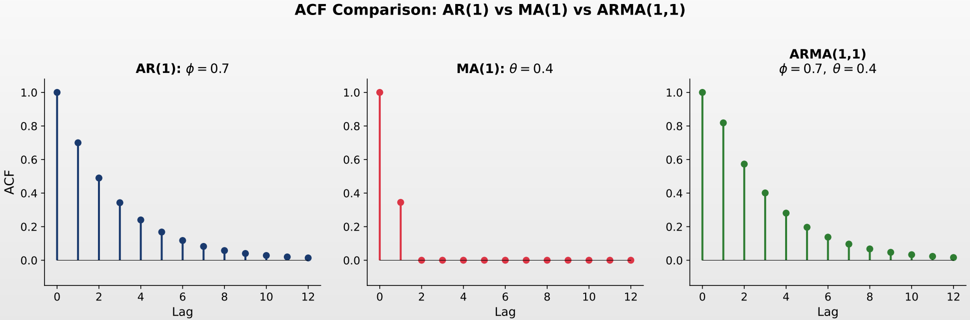
(A) It cuts off after lag 1

(B) Exponential decay starting from lag 1, with $\rho(1) \neq \phi$

(C) It is zero for all lags

(D) It exactly follows the pattern ϕ^h for all $h \geq 0$

Question 11: Answer



Answer: (B)

- $\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+2\phi\theta+\theta^2} \neq \phi$ (the MA component modifies lag 1)
- For $h \geq 2$: $\rho(h) = \phi\,\rho(h-1)$ — exponential decay as in AR(1)

TSA_ch2_arma

Question 12

Question

- An AR(2) process has $\phi_1 = 0.8$ and $\phi_2 = 0.3$. Is it stationary?

Answer Choices

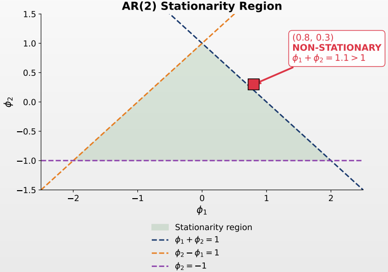
(A) Yes, it is stationary

(B) No, because $\phi_1 + \phi_2 = 1.1 > 1$

(C) Cannot be determined without data

(D) Depends on the value of σ^2

Question 12: Answer



Answer: (B)

- Necessary conditions for AR(2) stationarity: $\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1, \quad |\phi_2| < 1$
- Here $0.8 + 0.3 = 1.1 > 1 \Rightarrow$ the first condition is violated

TSA_ch2_ar2

Question 13

Question

- What does the Wold decomposition theorem guarantee?

Answer Choices

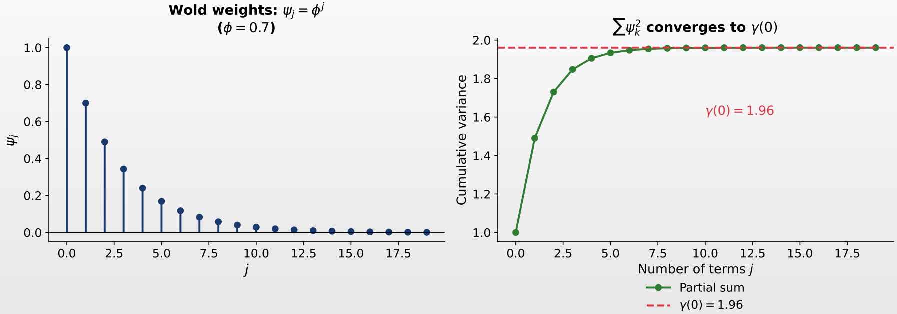
(A) Any time series is an AR process

(B) Any stationary process can be written as MA(∞): $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$

(C) Any process has finite variance

(D) ARMA models are always invertible

Question 13: Answer



Answer: (B)

- Wold's theorem: $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + D_t$, where D_t is the deterministic component
- This justifies ARMA models: they are parsimonious approximations of MA(∞)

Question 14

Question

- AR(1) with $\phi = 0.9$, $\sigma^2 = 1$. What happens to the CI width as $h \to \infty$?

Answer Choices

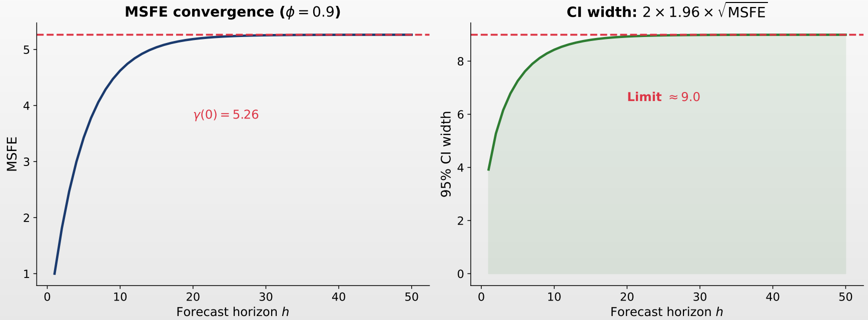
(A) It remains constant

(B) It decreases to zero

(C) It grows toward $2 \times 1.96 \times \sqrt{1/(1-0.81)} \approx 9.0$

(D) It grows to infinity

Question 14: Answer



Answer: (C)

- $\text{MSFE}(\infty) = \frac{\sigma^2}{1-\phi^2} = \frac{1}{1-0.81} = \frac{1}{0.19} \approx 5.26$
- CI width $= 2 \times 1.96\sqrt{5.26} \approx 2 \times 1.96 \times 2.29 \approx 9.0$

Data Sources and Software

Software Packages

- `statsmodels` ⇒ Statistical models for Python, including ARIMA
- `pmdarima` ⇒ Automatic ARIMA selection for Python
- `scipy` ⇒ Optimization and statistical functions
- `numpy`, `pandas` ⇒ Data manipulation
- `matplotlib` ⇒ Visualization

Data and Examples

- Simulated time series for illustrations
- Examples based on Hyndman & Athanasopoulos (2021)

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- Akaike, H. (1974). A New Look at the Statistical Model Identification, *IEEE Transactions on Automatic Control*, 19(6), 716–723.
- Schwarz, G. (1978). Estimating the Dimension of a Model, *The Annals of Statistics*, 6(2), 461–464.

Diagnostics and Validation

- Ljung, G.M., & Box, G.E.P. (1978). On a Measure of Lack of Fit in Time Series Models, *Biometrika*, 65(2), 297–303.
- Brockwell, P.J., & Davis, R.A. (2016). *Introduction to Time Series and Forecasting*, 3rd ed., Springer.

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- Shumway, R.H., & Stoffer, D.S. (2017). *Time Series Analysis and Its Applications*, 4th ed., Springer.
- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

Online Resources and Code

- **Quantlet**: https://quantlet.com – Code platform for quantitative methods
- **Quantinar**: https://quantinar.com – Learning platform for quantitative methods
- **GitHub TSA**: https://github.com/QuantLet/TSA/tree/main/TSA_ch2 – Python code for this chapter

Thank You!

Questions?

Course materials available at: https://danpele.github.io/Time-Series-Analysis/

Quantlet Quantinar