Time Series Analysis and Forecasting

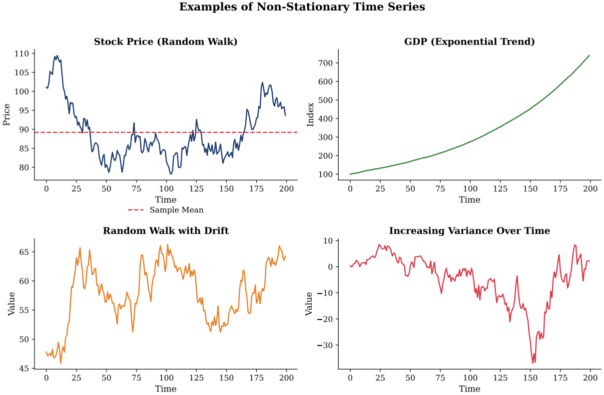
Chapter 3: ARIMA Models

Non-Stationary Time Series

Outline

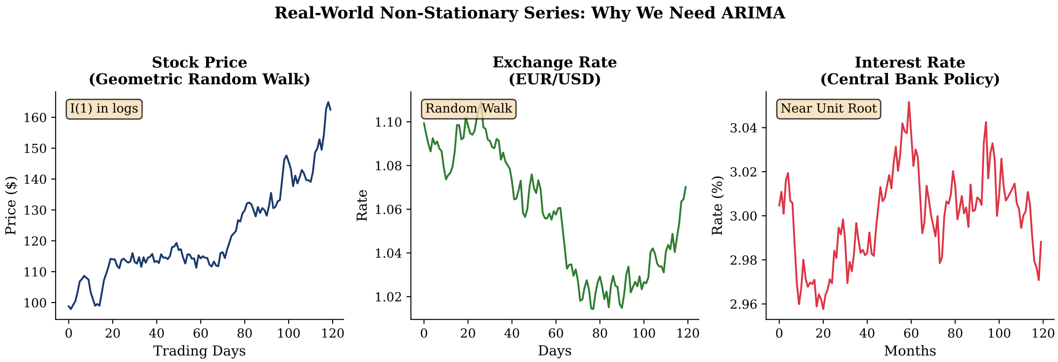
1. Non-Stationarity in Time Series
2. Differencing and the Difference Operator
3. ARIMA(p,d,q) Models
4. Unit Root Tests
5. ARIMA Model Identification
6. ARIMA Estimation
7. Diagnostic Checking
8. Forecasting with ARIMA
9. Case Study: US GDP Forecasting
10. Case Study: Real Data
11. Summary

Motivating Example: Non-Stationary Data Is Everywhere



- Stock prices, GDP, exchange rates all exhibit **trends** or **wandering behavior**
- The sample mean (red line) is meaningless for a random walk
- Standard ARMA models **cannot** handle these series directly

Real-World Applications

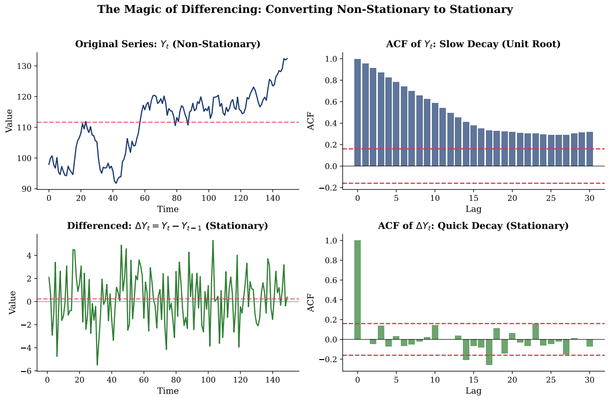


The Challenge

Financial and economic data are typically **integrated** (I(1) or near unit root):

- Stock prices: random walk in logs
- Exchange rates: random walk
- Interest rates: highly persistent (near unit root)

The Solution: Differencing



Key Insight

Differencing transforms a non-stationary series into a stationary one: $\Delta Y_t = Y_t - Y_{t-1}$. The ACF changes from slow decay to quick decay!

What We'll Learn Today

Core Concepts

1. **Non-Stationarity**: Why it matters and how to detect it
2. **Unit Root Tests**: ADF, PP, KPSS tests
3. **Differencing**: The key transformation
4. **ARIMA Models**: Combining differencing with ARMA
5. **Box-Jenkins Methodology**: Identify $\rightarrow$ Estimate $\rightarrow$ Diagnose

By the End of This Lecture

You will be able to model and forecast non-stationary time series like stock prices, GDP, and exchange rates using ARIMA models.

Why Non-Stationarity Matters

The Problem

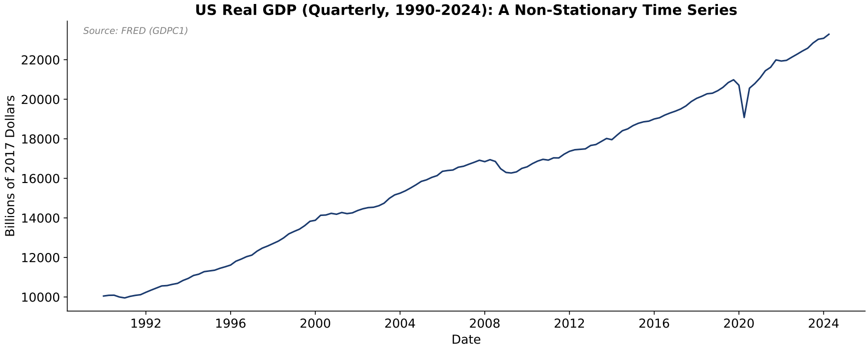
Many economic and financial time series are **non-stationary**:

- GDP, stock prices, exchange rates, inflation indices
- They exhibit trends, changing means, or growing variance

Consequences of Non-Stationarity

- Standard ARMA models assume stationarity
- OLS regression with non-stationary data leads to **spurious regression**
- Sample moments (mean, variance, ACF) are not consistent estimators
- Statistical inference becomes invalid

Example: US Real GDP



- Clear upward **trend** – mean is not constant
- This is a classic example of a **non-stationary** time series
- We cannot apply ARMA models directly to this data

Types of Non-Stationarity

Deterministic Trend

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- Trend is a deterministic function of time
- Can be removed by **detrending**
- Shocks have temporary effects

Stochastic Trend (Unit Root)

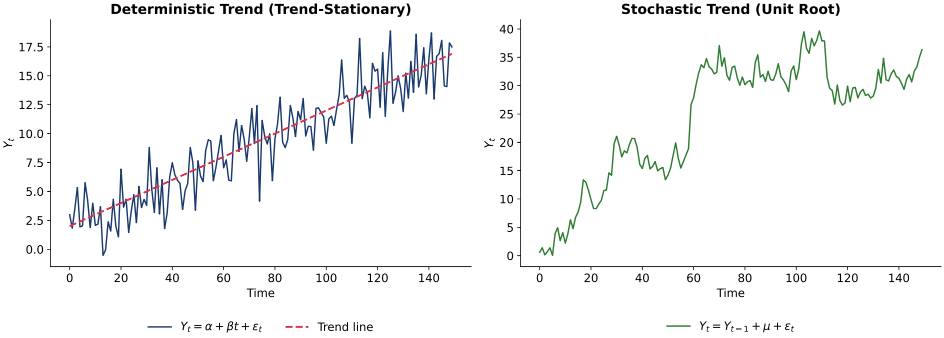
$$Y_t = Y_{t-1} + \varepsilon_t$$

- Random walk process
- Must be removed by **differencing**
- Shocks have permanent effects

Key Distinction

Correct identification is crucial: detrending a unit root process or differencing a trend-stationary process both lead to misspecification!

Visualizing the Difference



- **Left**: Deterministic trend – deviations from trend are temporary
- **Right**: Stochastic trend – shocks accumulate permanently
- Both look similar, but require **different** treatments!

The Random Walk Process

Definition 1 (Random Walk)

A **random walk** is defined as:

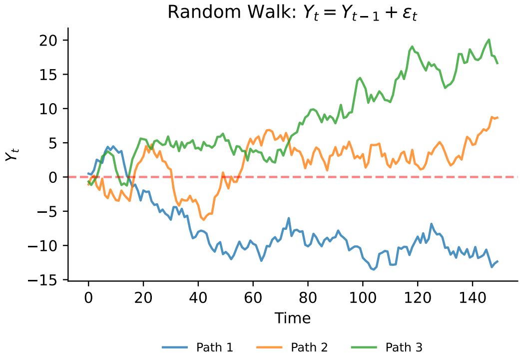
$$Y_t = Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2)$$

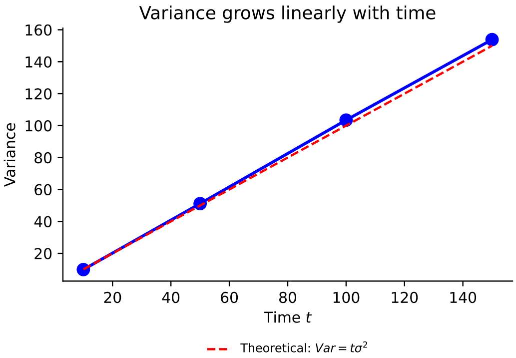
With initial condition $Y_0 = 0$, we have: $Y_t = \sum_{i=1}^{t} \varepsilon_i$

Properties of Random Walk

- $\mathbb{E}[Y_t] = 0$ (constant mean)
- $\text{Var}(Y_t) = t\sigma^2$ (variance grows with time!)
- $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$ for $k \leq t$
- ACF: $\rho_k = \sqrt{\frac{t-k}{t}} \to 1$ as $t \to \infty$

Random Walk: Visual Illustration





Key Properties

Left: Multiple random walk paths wander unpredictably with no mean reversion. Right: Variance $\text{Var}(Y_t) = t\sigma^2$ grows linearly — the defining feature of non-stationarity.

Proof: Random Walk Variance

Claim: For $Y_t = Y_{t-1} + \varepsilon_t$ with $Y_0 = 0$: $\text{Var}(Y_t) = t\sigma^2$

Proof: By recursive substitution:

$$Y_t = Y_{t-1} + \varepsilon_t = Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t = \cdots = \sum_{i=1}^{t} \varepsilon_i$$

Taking variance:

$$\text{Var}(Y_t) = \text{Var}\left(\sum_{i=1}^{t} \varepsilon_i\right) = \sum_{i=1}^{t} \text{Var}(\varepsilon_i) + 2\sum_{i<j} \text{Cov}(\varepsilon_i, \varepsilon_j)$$

Since ε_t are independent (white noise), all covariances are zero:

$$\text{Var}(Y_t) = \sum_{i=1}^{t} \sigma^2 = \boxed{t\sigma^2}$$

Non-Stationarity

Variance depends on $t \Rightarrow$ violates stationarity requirement ($\text{Var}(Y_t) = \gamma(0)$ constant).

Proof: Random Walk Autocovariance

Claim: $\text{Cov}(Y_t, Y_{t-k}) = (t-k)\sigma^2$ for $k \leq t$

Proof: Using $Y_t = \sum_{i=1}^{t} \varepsilon_i$ and $Y_{t-k} = \sum_{i=1}^{t-k} \varepsilon_i$:

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}\left(\sum_{i=1}^{t} \varepsilon_i, \sum_{j=1}^{t-k} \varepsilon_j\right) \\
&= \sum_{i=1}^{t} \sum_{j=1}^{t-k} \text{Cov}(\varepsilon_i, \varepsilon_j) = \sum_{i=1}^{t-k} \text{Var}(\varepsilon_i) = \boxed{(t-k)\sigma^2}
\end{aligned}$$

Only terms with $i = j$ survive (when $i \leq t-k$).

ACF:

$$\rho(k) = \frac{\text{Cov}(Y_t, Y_{t-k})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-k})}} = \frac{(t-k)\sigma^2}{\sqrt{t\sigma^2 \cdot (t-k)\sigma^2}} = \sqrt{\frac{t-k}{t}}$$

Random Walk with Drift

Definition 2 (Random Walk with Drift)

A random walk with drift includes a constant term:

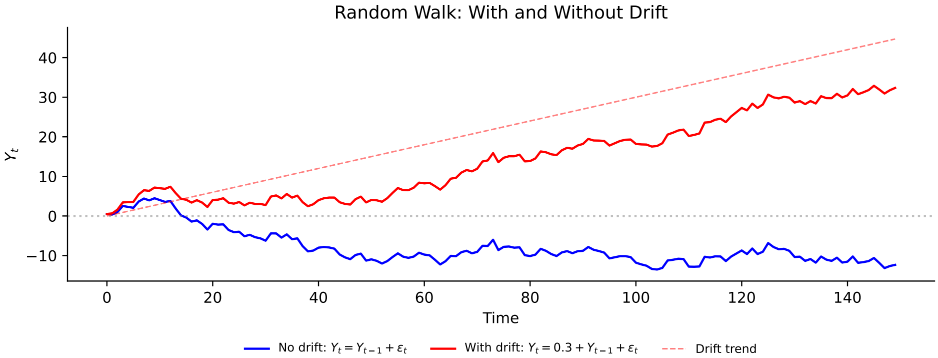
$$Y_t = \mu + Y_{t-1} + \varepsilon_t$$

Equivalently: $Y_t = Y_0 + \mu t + \sum_{i=1}^{t} \varepsilon_i$

Properties

- $\mathbb{E}[Y_t] = Y_0 + \mu t$ (mean grows linearly)
- $\text{Var}(Y_t) = t\sigma^2$ (variance still grows)
- The drift μ creates an upward or downward trend
- Still non-stationary despite having a "trend"

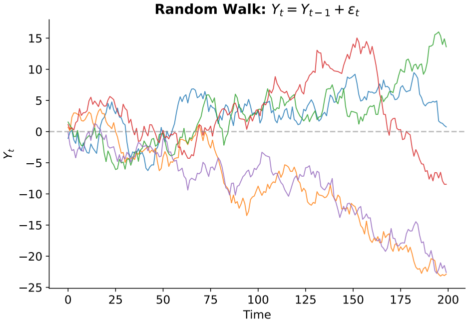
Random Walk with Drift: Visual Illustration

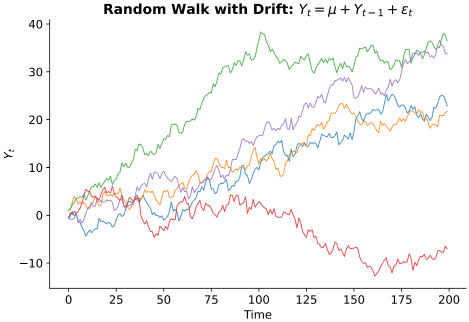


Comparison

Without drift (blue): wanders around zero with no direction. With drift $\mu > 0$ (red): systematic upward trend. Both are non-stationary — drift adds deterministic trend to stochastic wandering.

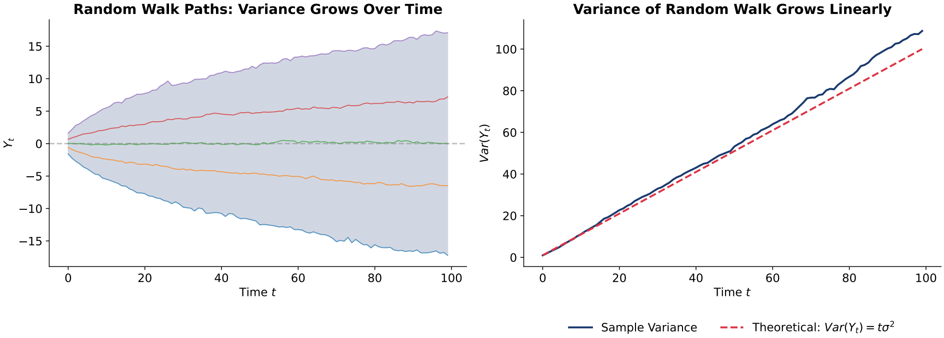
Simulating Random Walks





- **Left**: Pure random walks – no drift, wander unpredictably
- **Right**: Random walks with drift – upward trend on average
- Each path is unique; uncertainty grows over time

Variance Growth: Why Random Walks Are Non-Stationary



- **Left**: Fan of paths shows uncertainty growing over time
- **Right**: Variance grows linearly: $\text{Var}(Y_t) = t\sigma^2$
- This violates stationarity (variance should be constant)

Integrated Processes

Definition 3 (Integrated Process of Order d)

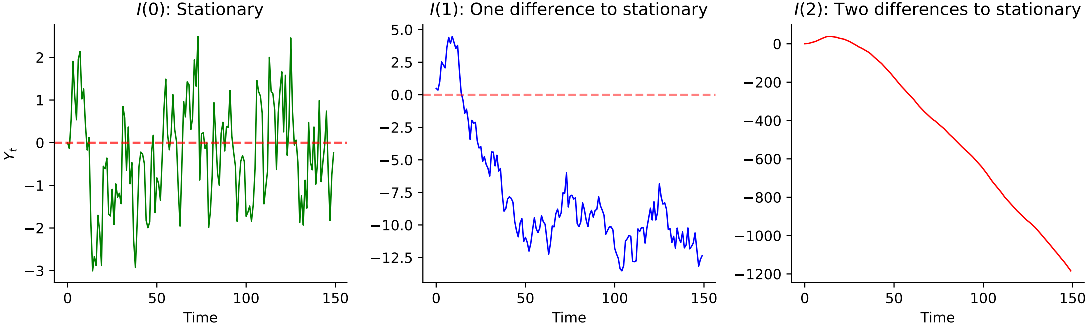
A time series $\{Y_t\}$ is **integrated of order** d, written $Y_t \sim I(d)$, if:

- Y_t is non-stationary
- $(1-L)^d Y_t = \Delta^d Y_t$ is stationary
- $(1-L)^{d-1} Y_t$ is still non-stationary

Common Cases

- $I(0)$: Stationary process (e.g., ARMA)
- $I(1)$: First difference is stationary (most common for economic data)
- $I(2)$: Second difference is stationary (less common)

Integrated Process: Visual Illustration



$I(0)$: stationary. $I(1)$: one difference needed. $I(2)$: two differences needed.

The Difference Operator

Definition 4 (First Difference)

The **first difference operator** Δ is defined as: $\Delta Y_t = Y_t - Y_{t-1} = (1 - L)Y_t$, where L is the lag operator ($LY_t = Y_{t-1}$).

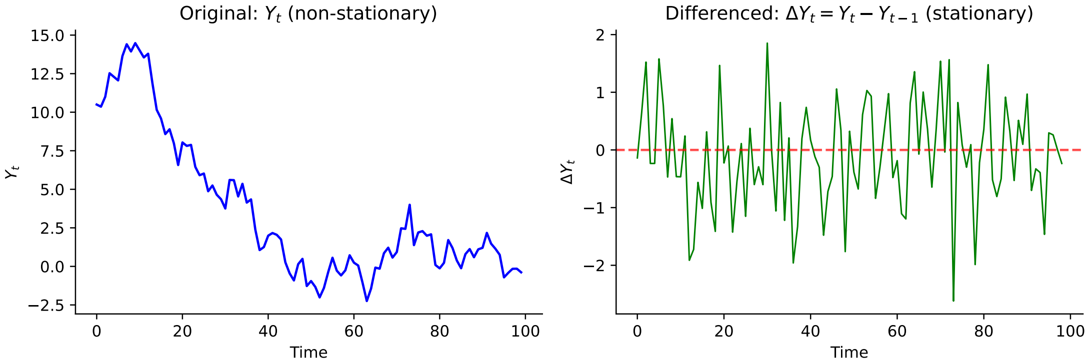
Higher-Order Differences

- Second difference: $\Delta^2 Y_t = \Delta(\Delta Y_t) = (1 - L)^2 Y_t$
- $\Delta^2 Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$
- d-th difference: $\Delta^d Y_t = (1 - L)^d Y_t$

Key Result

If $Y_t \sim I(d)$, then $\Delta^d Y_t \sim I(0)$ (stationary).

First Difference: Visual Illustration



Left: non-stationary series. Right: after first difference, the series becomes stationary.

Example: Differencing a Random Walk

Random Walk to White Noise

Let $Y_t = Y_{t-1} + \varepsilon_t$ (random walk). Taking the first difference:

$$\Delta Y_t = Y_t - Y_{t-1} = \varepsilon_t$$

The first difference is white noise – a stationary process!

Interpretation

- A random walk is $I(1)$
- One difference transforms it to $I(0)$
- The "changes" in a random walk are stationary

Proof: Differencing Induces Stationarity

Claim: If $Y_t \sim I(1)$, then $\Delta Y_t = Y_t - Y_{t-1}$ is stationary.

Proof for Random Walk with Drift: $Y_t = \mu + Y_{t-1} + \varepsilon_t$
The first difference is:

$$\Delta Y_t = Y_t - Y_{t-1} = \mu + \varepsilon_t$$

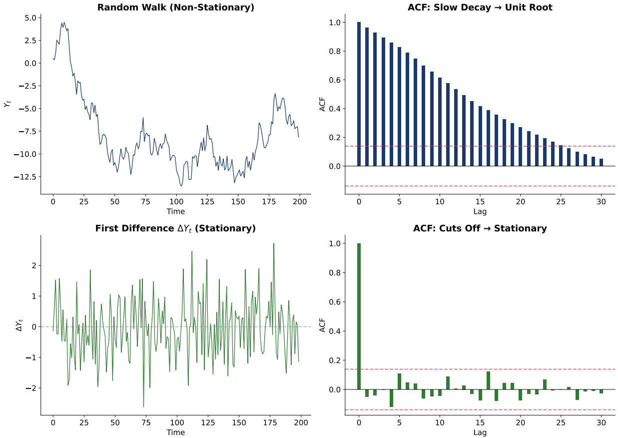
Check stationarity conditions:

1. **Mean:** $\mathbb{E}[\Delta Y_t] = \mu$ (constant, does not depend on t) ✓
2. **Variance:** $\text{Var}(\Delta Y_t) = \text{Var}(\varepsilon_t) = \sigma^2$ (constant) ✓
3. **Autocovariance:** $\text{Cov}(\Delta Y_t, \Delta Y_{t-k}) = \text{Cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$ for $k \neq 0$ ✓

General Principle

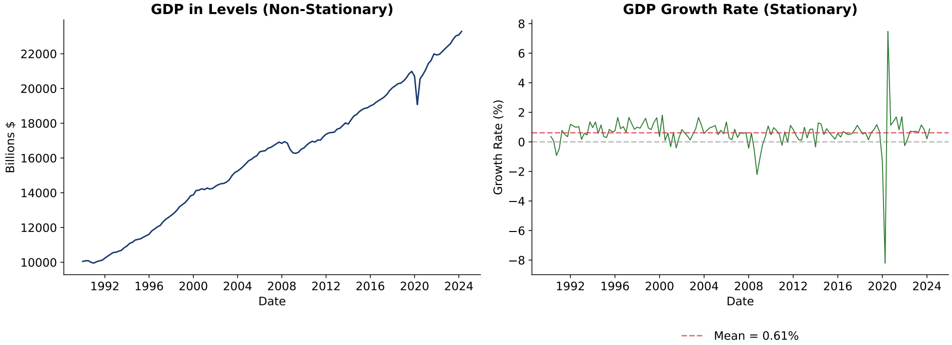
Differencing removes the "memory" that causes variance to accumulate. For $I(d)$ processes, d differences are needed.

ACF Diagnostic: Detecting Non-Stationarity



- **Top**: Random walk ACF decays very slowly $\Rightarrow$ unit root
- **Bottom**: After differencing, ACF cuts off $\Rightarrow$ stationary

Differencing in Practice: GDP Example



Transformation

Left: GDP in levels with clear upward trend (non-stationary). **Right**: GDP growth rate $\Delta \log(GDP_t)$ fluctuates around constant mean (stationary). One difference removes the stochastic trend.

Overdifferencing

Warning: Overdifferencing

Differencing more than necessary introduces problems:

- Creates artificial negative autocorrelation
- Inflates variance
- Loses information

Example

If $Y_t \sim I(1)$, then $\Delta Y_t \sim I(0)$. But if we difference again:

$$\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}$$

This is an MA(1) with $\theta = 1$ (non-invertible boundary)!

Definition of ARIMA

Definition 5 (ARIMA(p,d,q))

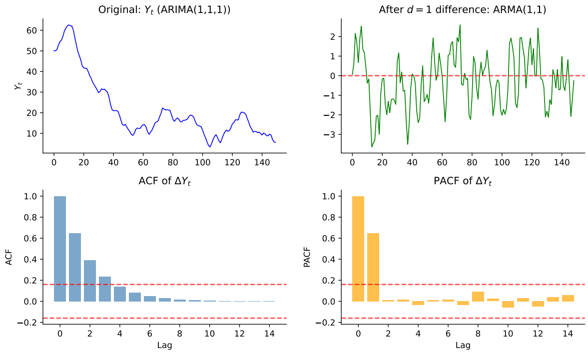
A time series $\{Y_t\}$ follows an **ARIMA(p,d,q)** process if:

$$\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$$

where:

- $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$ (AR polynomial)
- $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$ (MA polynomial)
- d is the order of integration (number of differences)
- $\varepsilon_t \sim WN(0, \sigma^2)$

ARIMA: Visual Illustration



Interpretation

Top: original ARIMA series (non-stationary). Bottom: after differencing d times, ACF/PACF reveal the AR and MA orders for the stationary component.

ARIMA Components

AR(p)	I(d)	MA(q)
Autoregressive Memory	Integration Differencing	Moving Average Shocks

Special Cases

- ARIMA(p,0,q) = ARMA(p,q) – stationary
- ARIMA(0,1,0) = Random walk
- ARIMA(0,1,1) = IMA(1,1) – exponential smoothing
- ARIMA(1,1,0) = ARI(1,1) – differenced AR(1)

ARIMA(1,1,0) Example

ARI(1,1) Model

$$\Delta Y_t = c + \phi_1 \Delta Y_{t-1} + \varepsilon_t$$

Equivalently: $(1 - \phi_1 L)(1 - L)Y_t = c + \varepsilon_t$

Interpretation

- The **changes** in Y_t follow an AR(1) process
- If $|\phi_1| < 1$, the changes are stationary
- Y_t itself has a stochastic trend
- Common model for many economic time series

ARIMA(0,1,1) Example

IMA(1,1) Model

$$\Delta Y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Equivalently: $(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$

Connection to Exponential Smoothing

The IMA(1,1) model is equivalent to **simple exponential smoothing**:

$$\hat{Y}_{t+1} = \alpha Y_t + (1 - \alpha)\hat{Y}_t$$

where $\alpha = 1 + \theta_1$ (for $-1 < \theta_1 < 0$).

The Role of the Constant in ARIMA

Constant Term in ARIMA(p,d,q)

When $d > 0$, the constant c has a different interpretation: $\phi(L)(1-L)^d Y_t = c + \theta(L)\varepsilon_t$

Important Implications

- For $d = 1$: c represents the **drift** (average change): $\mathbb{E}[\Delta Y_t] = \frac{c}{1-\phi_1-\cdots-\phi_p}$
- For $d = 2$: c affects the **curvature** of the trend
- Often $c = 0$ is assumed when $d \geq 1$

Testing for Unit Roots

Why Test?

Before fitting an ARIMA model, we need to determine:

1. Is the series stationary? (Is $d = 0$?)
2. If not, how many differences are needed? (What is d?)

Common Unit Root Tests

- **Dickey-Fuller (DF)** and **Augmented Dickey-Fuller (ADF)**
- **Phillips-Perron (PP)**
- **KPSS** (stationarity test – reversed null hypothesis)

The Dickey-Fuller Test

Setup

Consider the AR(1) model: $Y_t = \phi Y_{t-1} + \varepsilon_t$. Subtract Y_{t-1}: $\Delta Y_t = (\phi - 1)Y_{t-1} + \varepsilon_t = \gamma Y_{t-1} + \varepsilon_t$, where $\gamma = \phi - 1$.

Hypotheses

- H_0: $\gamma = 0$ (unit root, $\phi = 1$, non-stationary)
- H_1: $\gamma < 0$ (stationary, $|\phi| < 1$)

Key Issue

Under H_0, the t-statistic does **not** follow a standard t-distribution! Must use Dickey-Fuller critical values.

Dickey-Fuller Test Variants

Three Specifications

1. **No constant, no trend**: $\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$
2. **With constant (drift)**: $\Delta Y_t = \alpha + \gamma Y_{t-1} + \varepsilon_t$
3. **With constant and trend**: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \varepsilon_t$

Choosing the Right Specification

- Examine the data: does it have a visible trend?
- Including unnecessary terms reduces power
- Excluding necessary terms leads to incorrect inference

Augmented Dickey-Fuller (ADF) Test

The Problem with Simple DF

If AR dynamics beyond AR(1) exist, DF residuals will be autocorrelated.

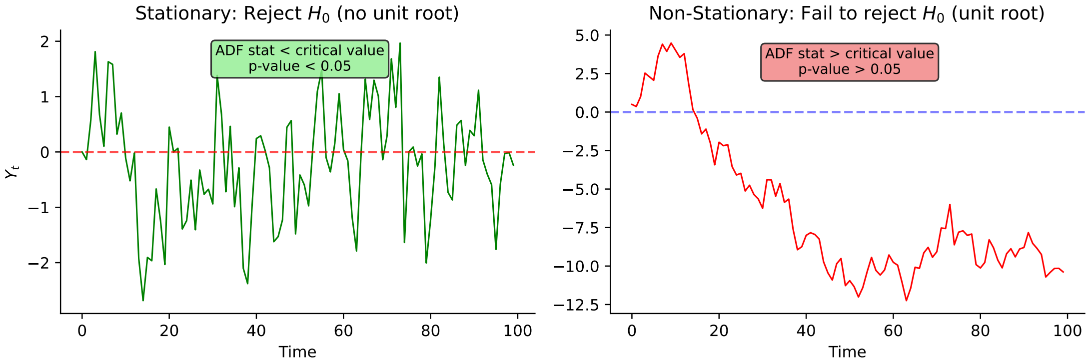
Definition 6 (ADF Test)

Add lagged differences: $\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{j=1}^{k} \delta_j \Delta Y_{t-j} + \varepsilon_t$
Test $H_0 : \gamma = 0$ using ADF critical values.

Choosing Lag Length k

- Use information criteria (AIC, BIC)
- Start with k_{max}, reduce until last lag significant

ADF Test: Visual Illustration



Left: stationary series – ADF rejects unit root. Right: non-stationary – ADF fails to reject.

ADF Test Critical Values

Model	**1%**	**5%**	**10%**
No constant, no trend	-2.58	-1.95	-1.62
With constant	-3.43	-2.86	-2.57
With constant and trend	-3.96	-3.41	-3.13

Decision Rule

- Test statistic $<$ critical value $\Rightarrow$ Reject H_0 (stationary)
- Test statistic $\geq$ critical value $\Rightarrow$ Fail to reject (unit root)

The Phillips-Perron (PP) Test

Motivation

Like ADF, tests H_0: Unit root vs H_1: Stationary, but uses a **non-parametric correction** for serial correlation instead of adding lagged differences.

Test Statistic

The PP test modifies the DF t-statistic:

$$Z_t = t_{\hat{\gamma}} \cdot \sqrt{\frac{\hat{\sigma}^2}{\hat{\lambda}^2}} - \frac{T(\hat{\lambda}^2 - \hat{\sigma}^2)(se(\hat{\gamma}))}{2\hat{\lambda}^2 \cdot s}$$

where $\hat{\lambda}^2$ is a consistent estimate of the long-run variance using Newey-West.

Advantages over ADF

- Robust to heteroskedasticity and serial correlation
- No need to select lag length (uses bandwidth instead)

The KPSS Test

Reversed Hypotheses

Unlike ADF: H_0: Stationary vs H_1: Unit root

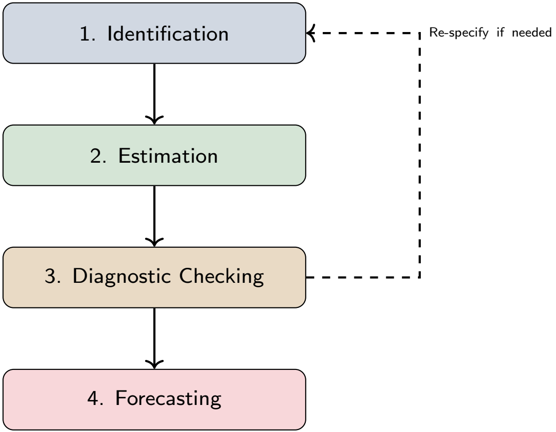
KPSS Procedure

Decompose: $Y_t = \xi t + r_t + \varepsilon_t$ where $r_t = r_{t-1} + u_t$. Test whether $\text{Var}(u_t) = 0$.

Complementary Use with ADF

- ADF rejects, KPSS doesn't $\Rightarrow$ Stationary
- ADF doesn't reject, KPSS rejects $\Rightarrow$ Unit root
- Both reject or neither $\Rightarrow$ Inconclusive

The Box-Jenkins Methodology



Step 1: Determining d

Procedure

1. Plot the time series – look for trends, changing variance
2. Examine ACF – slow decay suggests non-stationarity
3. Apply unit root tests (ADF, KPSS)
4. If non-stationary, difference and repeat

Practical Guidelines

- Most economic series: $d = 1$ is sufficient
- Rarely need $d > 2$
- If ACF of ΔY_t still decays slowly, try $d = 2$
- Watch for overdifferencing (ACF with $\rho_1 \approx -0.5$)

Step 2: Determining p and q

After Differencing

Once $W_t = \Delta^d Y_t$ is stationary, use ACF/PACF to identify ARMA(p,q):

Model	**ACF**	**PACF**
AR(p)	Decays exponentially	Cuts off after lag p
MA(q)	Cuts off after lag q	Decays exponentially
ARMA(p,q)	Decays	Decays

Information Criteria

When patterns are unclear, compare models using:

- AIC $= -2\ln(L) + 2k$; BIC $= -2\ln(L) + k\ln(n)$

Lower is better. BIC penalizes complexity more.

Auto-ARIMA Algorithms

Automated Model Selection

Modern software can automatically select (p, d, q):

- Python: `pmdarima.auto_arima()`
- R: `forecast::auto.arima()`

How Auto-ARIMA Works

1. Use unit root tests to determine d
2. Fit models for various (p, q) combinations
3. Select model with lowest AIC/BIC
4. Optionally use stepwise search for efficiency

Caution

Automated selection is helpful but not infallible. Always check diagnostics!

Estimation Methods

Maximum Likelihood Estimation (MLE)

The standard approach for ARIMA:

- Assumes $\varepsilon_t \sim N(0, \sigma^2)$
- Maximizes the likelihood function
- Provides consistent, efficient estimators
- Yields standard errors for inference

Conditional vs Exact MLE

- **Conditional MLE**: Conditions on initial values
- **Exact MLE**: Treats initial values as unknown
- Difference diminishes as sample size grows

Parameter Constraints

Stationarity and Invertibility

The estimated ARIMA model should satisfy:

- **AR stationarity**: Roots of $\phi(z) = 0$ outside unit circle
- **MA invertibility**: Roots of $\theta(z) = 0$ outside unit circle

Checking in Practice

Most software reports:

- Estimated coefficients with standard errors
- Roots of AR and MA polynomials
- Warning if near-unit-root detected

Residual Analysis

What to Check

If the model is correct, residuals $\hat{\varepsilon}_t$ should be white noise:

1. Zero mean
2. Constant variance
3. No autocorrelation
4. (Optional) Normality

Diagnostic Tools

- **Residual ACF/PACF**: Should show no significant spikes
- **Ljung-Box test**: Tests for autocorrelation at multiple lags
- **Q-Q plot**: Checks normality assumption
- **Residual vs fitted**: Checks for heteroskedasticity

The Ljung-Box Test

Definition 7 (Ljung-Box Q Statistic)

$Q(m) = n(n+2)\sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{n-k}$. Under H_0 (no autocorrelation): $Q(m) \sim \chi^2(m-p-q)$

Usage

- Choose $m \approx \ln(n)$ or $m = 10$ for quarterly, $m = 20$ for monthly
- Degrees of freedom adjusted for estimated parameters
- Reject if $Q(m)$ exceeds critical value

If Test Fails

Consider adding AR or MA terms, or check for structural breaks.

Point Forecasts

Minimum MSE Forecast

The optimal h-step ahead forecast is the conditional expectation: $\hat{Y}_{T+h|T} = \mathbb{E}[Y_{T+h}|Y_T, Y_{T-1}, \ldots]$

ARIMA(1,1,1) Forecasting

Model: $(1 - \phi_1 L)(1 - L)Y_t = c + (1 + \theta_1 L)\varepsilon_t$
One-step forecast: $\hat{Y}_{T+1|T} = c + Y_T + \phi_1(Y_T - Y_{T-1}) + \theta_1\hat{\varepsilon}_T$
For $h > 1$: replace unknown ε_{T+j} with 0, unknown Y_{T+j} with $\hat{Y}_{T+j|T}$

Forecast Intervals

Forecast Uncertainty

The h-step forecast error variance: $\mathrm{Var}(e_{T+h}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2$, where ψ_j are MA(∞) coefficients.

Confidence Intervals

Under normality, $(1-\alpha)\%$ interval: $\hat{Y}_{T+h|T} \pm z_{\alpha/2}\sqrt{\mathrm{Var}(e_{T+h})}$

Key Property for I(1) Series

For integrated processes, forecast variance grows without bound as $h \to \infty$. Intervals widen over time!

Long-Run Forecasts for ARIMA

Behavior as $h \to \infty$

For ARIMA(p,1,q) with drift c:

- Point forecasts: Linear trend with slope = drift
- Forecast intervals: Width grows with $\sqrt{h}$

For ARIMA(p,1,q) without drift:

- Point forecasts: Converge to last level
- Forecast intervals: Still grow unboundedly

Practical Implication

ARIMA forecasts are most reliable for short horizons. Long-term forecasts have very wide uncertainty bands.

Rolling Forecasting: Concept

What is Rolling Forecasting?

A technique to evaluate forecast accuracy out-of-sample:

1. Fix a **training window** of size w
2. Estimate model on observations $t = 1, \ldots, w$
3. Forecast h steps ahead: $\hat{Y}_{w+h|w}$
4. **Roll** the window forward by one period
5. Repeat until end of sample

Why Rolling Forecasts?

- Mimics real-time forecasting scenario
- Provides multiple forecast errors for evaluation
- Avoids overfitting to full sample

Rolling Forecast: Step-by-Step Example

Setup: ARIMA(1,1,0) with $\phi_1 = 0.6$

Model: $\Delta Y_t = \phi_1 \Delta Y_{t-1} + \varepsilon_t$ where $\Delta Y_t = Y_t - Y_{t-1}$

Given Data at Time T

$Y_{T-2} = 100, \quad Y_{T-1} = 103, \quad Y_T = 108 \quad \Rightarrow \quad \Delta Y_{T-1} = 3, \quad \Delta Y_T = 5$

1-Step Ahead Point Forecast

$$\hat{\Delta Y}_{T+1|T} = \phi_1 \cdot \Delta Y_T = 0.6 \times 5 = 3$$
$$\hat{Y}_{T+1|T} = Y_T + \hat{\Delta Y}_{T+1|T} = 108 + 3 = \boxed{111}$$

Multi-Step Point Forecasts

2-Step Ahead Forecast

$$\hat{\Delta Y}_{T+2|T} = \phi_1 \cdot \hat{\Delta Y}_{T+1|T} = 0.6 \times 3 = 1.8$$
$$\hat{Y}_{T+2|T} = \hat{Y}_{T+1|T} + \hat{\Delta Y}_{T+2|T} = 111 + 1.8 = \boxed{112.8}$$

General Formula for *h*-Step Forecast (ARIMA(1,1,0))

$$\hat{\Delta Y}_{T+h|T} = \phi_1^h \cdot \Delta Y_T$$
$$\hat{Y}_{T+h|T} = Y_T + \Delta Y_T \cdot \frac{\phi_1(1-\phi_1^h)}{1-\phi_1}$$

Numerical: 3-Step Forecast

$$\hat{Y}_{T+3|T} = 108 + 5 \times \frac{0.6(1-0.6^3)}{1-0.6} = 108 + 5 \times 1.092 = \boxed{113.46}$$

Confidence Intervals: Formulas

Forecast Error Variance

For ARIMA(1,1,0), the h-step forecast error variance:

$$\text{Var}(e_{T+h|T}) = \sigma^2 \left(1 + \sum_{j=1}^{h-1} \psi_j^2\right)$$

where $\psi_j = \phi_1^{j-1}(1 + \phi_1 + \cdots + \phi_1^{j-1}) = \phi_1^{j-1} \cdot \frac{1-\phi_1^j}{1-\phi_1}$

$(1-\alpha)\%$ Confidence Interval

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2} \cdot \sqrt{\text{Var}(e_{T+h|T})}$$

For 95% CI: $z_{0.025} = 1.96$

Confidence Interval: Numerical Example

Given: $\sigma^2 = 4,\ \phi_1 = 0.6,\ \hat{Y}_{T+1|T} = 111$

1-Step Ahead CI

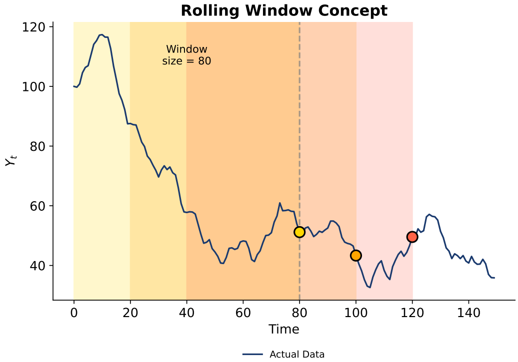
$$\text{Var}(e_{T+1|T}) = \sigma^2 = 4$$
$$95\%\ \text{CI} = 111 \pm 1.96 \times \sqrt{4} = 111 \pm 3.92 = \boxed{[107.08,\ 114.92]}$$

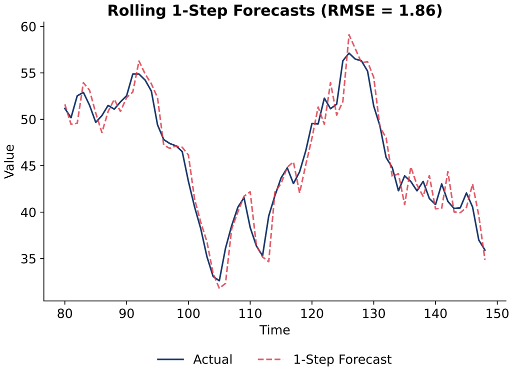
2-Step Ahead CI (for $\hat{Y}_{T+2|T} = 112.8$)

$$\psi_1 = 1 + \phi_1 = 1.6, \quad \text{Var}(e_{T+2|T}) = 4(1 + 1.6^2) = 14.24$$
$$95\%\ \text{CI} = 112.8 \pm 1.96 \times \sqrt{14.24} = 112.8 \pm 7.40 = \boxed{[105.40,\ 120.20]}$$

Note: CI widens as horizon increases!

Rolling Window Illustration





- Each window produces a 1-step ahead forecast
- Compare forecasts to actuals to compute RMSE, MAE
- Rolling window keeps model estimation up-to-date

Rolling Forecast: Python Code

Implementation

```
from statsmodels.tsa.arima.model import ARIMA

window_size = 100
forecasts, actuals = [], []

for t in range(window_size, len(y) - 1):
    train = y[:t]                    # Rolling window
    model = ARIMA(train, order=(1,1,0)).fit()
    forecast = model.forecast(steps=1)[0]
    forecasts.append(forecast)
    actuals.append(y[t])

rmse = np.sqrt(np.mean((np.array(forecasts) - np.array(actuals))**2))
```

Case Study: Complete ARIMA Analysis

Objective

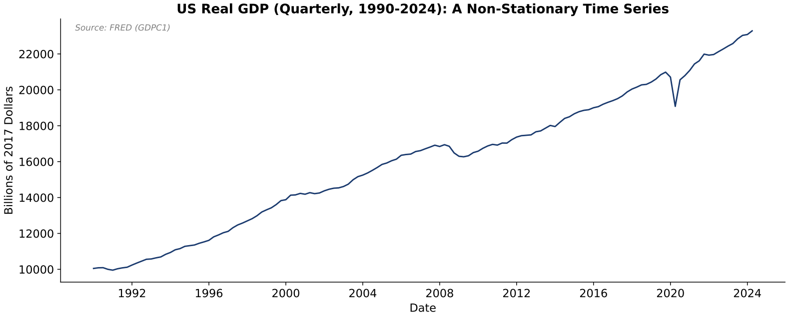
Forecast US Real GDP using the Box-Jenkins methodology

1. **Step 1**: Visualize data and check for stationarity
2. **Step 2**: Apply unit root tests (ADF, KPSS)
3. **Step 3**: Difference if needed, identify p and q
4. **Step 4**: Estimate ARIMA model
5. **Step 5**: Diagnostic checking
6. **Step 6**: Generate forecasts with confidence intervals
7. **Step 7**: Evaluate forecast accuracy

Data

US Real GDP (FRED: GDPC1), Quarterly, 1990Q1–2024Q2, $n = 138$ observations

Step 1: Initial Data Analysis



Observations

- Clear upward trend $\Rightarrow$ non-constant mean
- Variance appears relatively stable (after log transform)
- Notable dip in 2020 (COVID-19 pandemic)
- **Conclusion**: Series is non-stationary, needs differencing

Step 2: Unit Root Testing

ADF Test on Log GDP Levels

- Test statistic: -0.91
- Critical values: -3.48 (1%), -2.88 (5%), -2.58 (10%)
- p-value: 0.79
- **Result**: Cannot reject $H_0 \Rightarrow$ Unit root present

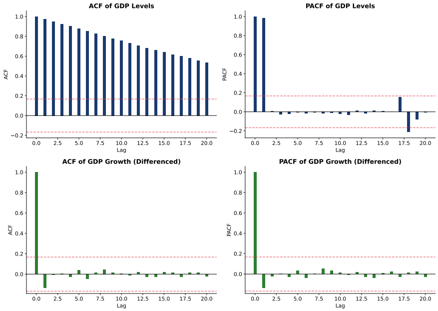
ADF Test on First Difference (Growth Rate)

- Test statistic: -13.24
- p-value: < 0.001
- **Result**: Reject H_0 at 1% $\Rightarrow$ Stationary after differencing

Conclusion

GDP is $I(1) \Rightarrow$ Use $d = 1$ in ARIMA model

Step 3: Model Identification via ACF/PACF



Analysis of Differenced Series

- ACF: Significant spike at lag 1, then cuts off $\Rightarrow$ suggests MA(1)
- PACF: Significant spike at lag 1, decays $\Rightarrow$ suggests AR(1)
- **Candidate models**: ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1)

Step 4: Model Estimation

Comparing Models using Information Criteria

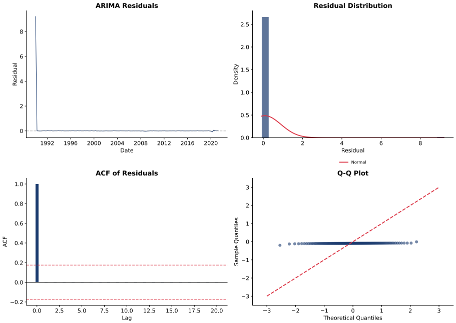
Model	AIC	BIC	Log-Lik
ARIMA(1,1,0)	−725.2	−719.5	364.6
ARIMA(0,1,1)	−724.8	−719.2	364.4
ARIMA(1,1,1)	**−747.0**	**−738.5**	**376.5**

Selected Model: ARIMA(1,1,1)

$$(1 - 0.35L)(1 - L)Y_t = (1 + 0.58L)\varepsilon_t, \quad \hat{\sigma}^2 = 0.000156$$

- $\hat{\phi}_1 = 0.35$ (SE $= 0.09$), significant at 1%
- $\hat{\theta}_1 = 0.58$ (SE $= 0.08$), significant at 1%

Step 5: Diagnostic Checking



Residual Analysis

- Ljung-Box test: $Q(10) = 5.8$, p-value $= 0.83 \Rightarrow$ No autocorrelation
- Jarque-Bera test: $JB = 156.4$, p-value $< 0.001 \Rightarrow$ Non-normal (COVID outlier)
- **Conclusion**: Model passes autocorrelation checks; outliers expected

Step 6: Forecasting with Confidence Intervals

Last Observed Values (Log GDP)

$Y_T = 9.973$ (2024Q2), $\quad Y_{T-1} = 9.956$ (2024Q1)
$\Delta Y_T = 0.017, \quad \hat{\varepsilon}_T = 0.004$

1-Step Ahead Forecast (2024Q3)

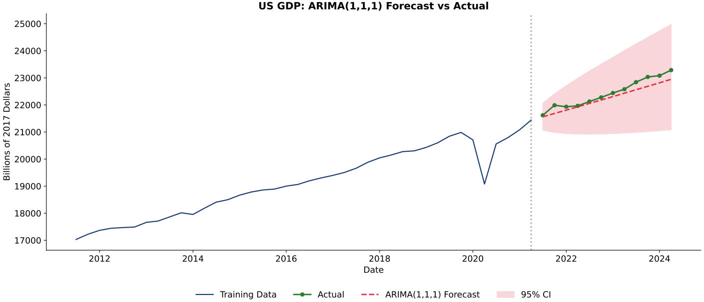
$$\Delta \hat{Y}_{T+1} = \hat{\phi}_1 \Delta Y_T + \hat{\theta}_1 \hat{\varepsilon}_T = 0.35(0.017) + 0.58(0.004) = 0.0083$$
$$\hat{Y}_{T+1} = 9.973 + 0.0083 = \boxed{9.981}$$

95% Confidence Interval

$$\text{CI} = 9.981 \pm 1.96 \times \sqrt{0.000156} = [9.957,\ 10.006]$$

In levels: GDP forecast = \$21,652B, CI = [\$21,142B, \$22,175B]

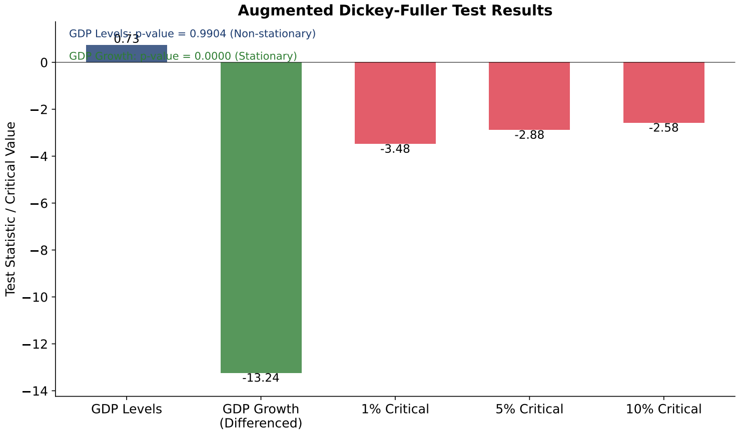
Step 7: Forecast Evaluation



Out-of-Sample Performance (Last 12 Quarters)

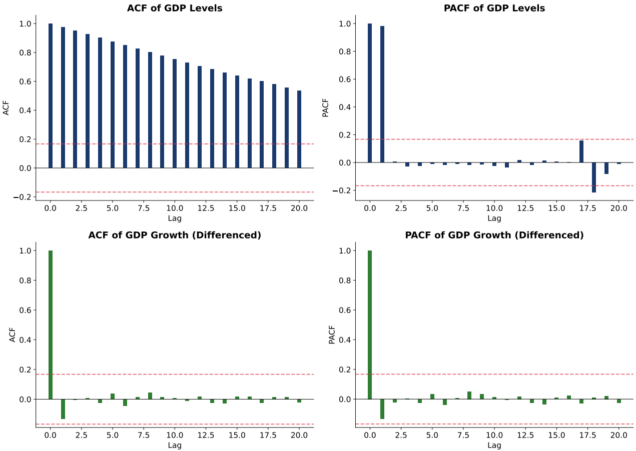
- RMSE $= 0.0486$ (log scale) $\approx 4.86\%$ error
- MAE $= 0.0430$ (log scale) $\approx 4.30\%$ error
- Direction accuracy $= 91\%$ (correctly predicted growth/decline)

Unit Root Test Results



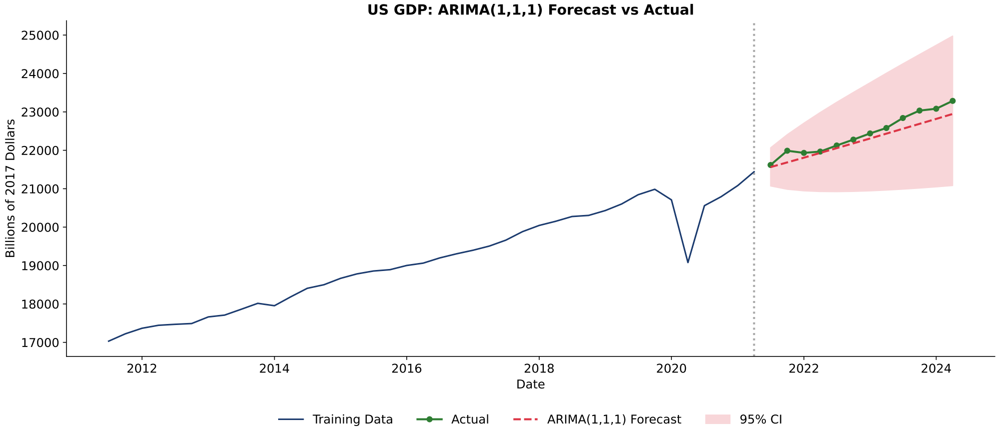
- GDP in levels: Cannot reject unit root (non-stationary)
- GDP growth: Reject unit root at 1% level (stationary)

ACF/PACF: Levels vs Differenced



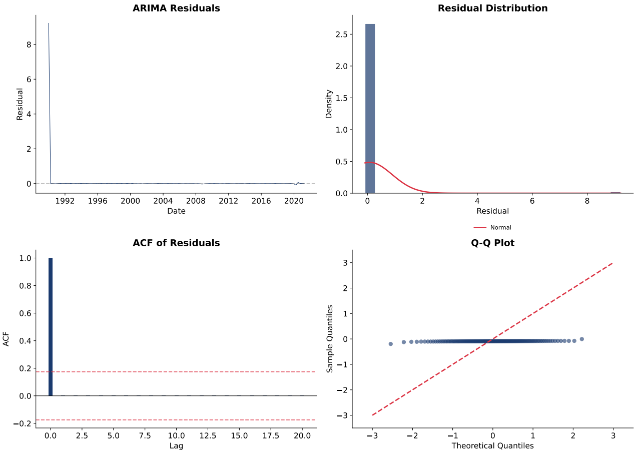
- **Top**: Slow ACF decay in levels suggests non-stationarity
- **Bottom**: After differencing, ACF/PACF help identify p and q

ARIMA Forecasting: Actual vs Predicted



- ARIMA(1,1,1) captures the trend dynamics
- Confidence intervals widen with forecast horizon

Model Diagnostics

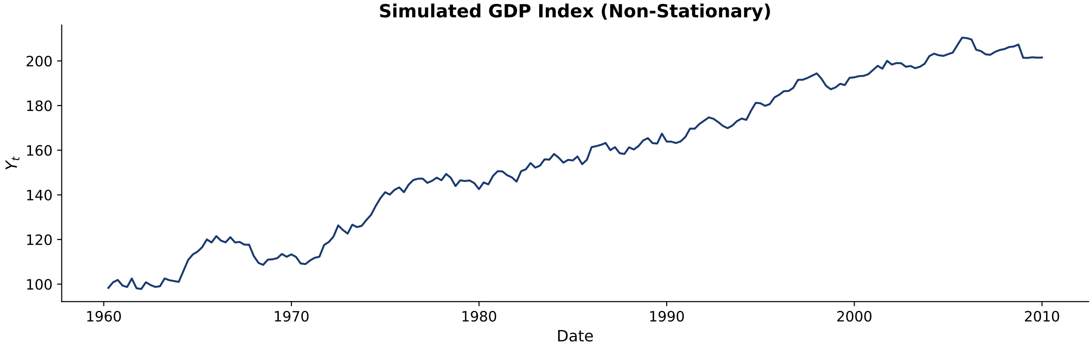


- Residuals appear random; ACF within bounds
- Q-Q plot shows approximate normality

Auto-ARIMA Example

```
# Automatic model selection
model = pm.auto_arima(y, start_p=0, start_q=0,
                      max_p=3, max_q=3, d=None,
                      seasonal=False, trace=True)
print(model.summary())
```

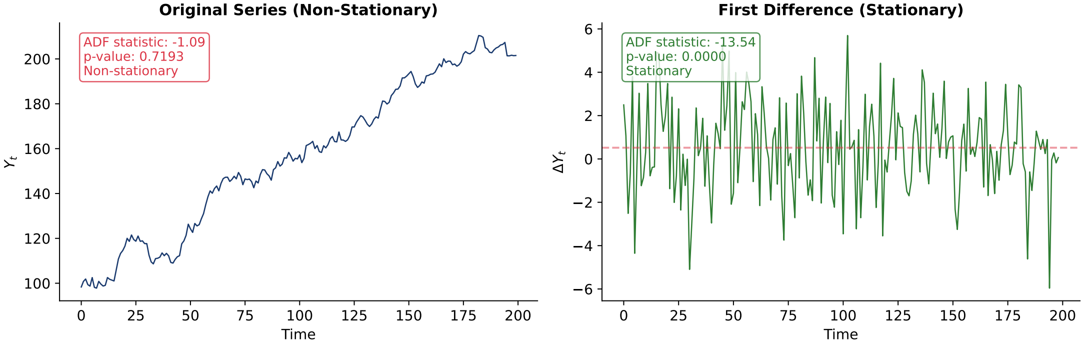
Case Study: GDP Index (Simulated)



Data Description

Simulated quarterly GDP index (1960–2010): Non-stationary series with upward trend. Demonstrates the need for differencing before ARMA modeling.

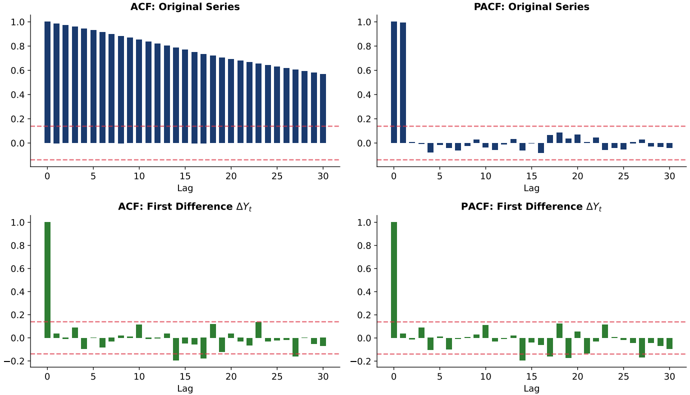
Step 1: ADF Test for Stationarity



ADF Test Results

Original series: Large p-value $\Rightarrow$ fail to reject H_0 (unit root present). **First difference**: p-value $< 0.01 \Rightarrow$ reject $H_0 \Rightarrow d = 1$ is sufficient.

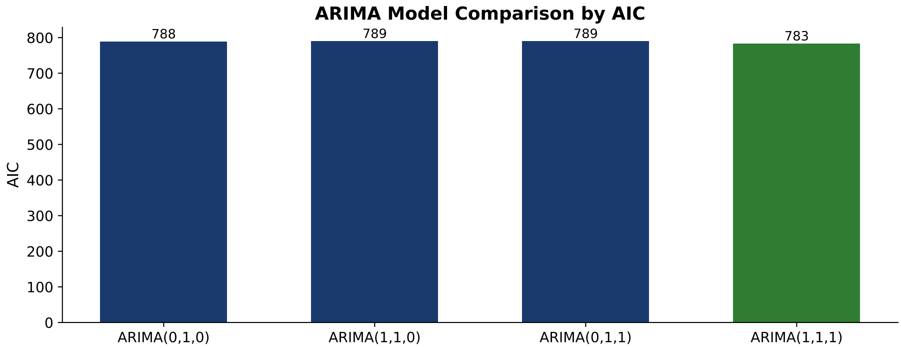
Step 2: ACF/PACF Before and After Differencing



Identification

Top: Slow ACF decay $\Rightarrow$ non-stationarity. **Bottom**: After differencing, ACF and PACF suggest low-order ARMA.

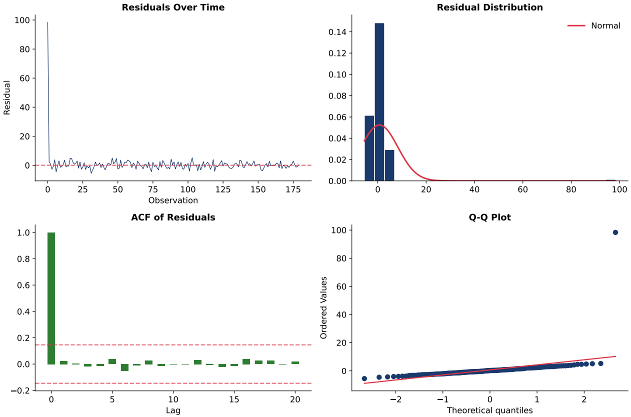
Step 3: ARIMA Model Comparison



Model Selection

Compare ARIMA(0,1,0), ARIMA(1,1,0), ARIMA(0,1,1), ARIMA(1,1,1). The model with lowest AIC is selected.

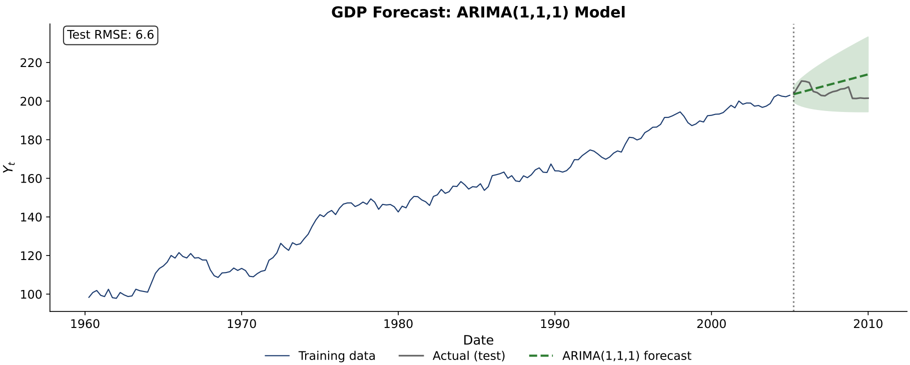
Step 4: Diagnostic Checking



ARIMA(1,1,1) Diagnostics

Residuals are approximately white noise: no significant autocorrelation, nearly normal distribution.

Step 5: Forecasting



Results

- ARIMA(1,1,1) forecasts follow the data trend
- Confidence intervals widen with horizon (characteristic of I(1))
- Uncertainty grows because errors accumulate through differencing

Key Takeaways

Main Points

1. **Non-stationarity** is common in economic data – must be addressed
2. **Differencing** transforms I(d) to I(0)
3. **ARIMA(p,d,q)** combines differencing with ARMA modeling
4. **Unit root tests** (ADF, KPSS) help determine d
5. **Box-Jenkins methodology**: Identify $\rightarrow$ Estimate $\rightarrow$ Diagnose
6. **Forecasts** for I(1) series have growing uncertainty

Next Steps

Chapter 4 will extend ARIMA to handle seasonality: SARIMA models.

Quiz Question 1

Question

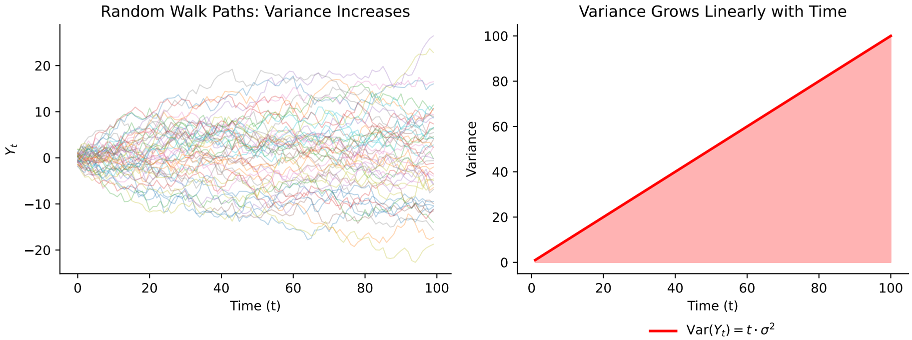
A time series Y_t follows a random walk: $Y_t = Y_{t-1} + \varepsilon_t$. What is $\text{Var}(Y_t)$?

- A σ^2 (constant)
- B $t \cdot \sigma^2$ (grows linearly with time)
- C σ^2/t (decreases with time)
- D σ^{2t} (grows exponentially)

Quiz Question 1: Answer

Correct Answer: (B) $\text{Var}(Y_t) = t \cdot \sigma^2$

Random walk variance grows linearly with time — this is why random walks are non-stationary.



Quiz Question 2

Question

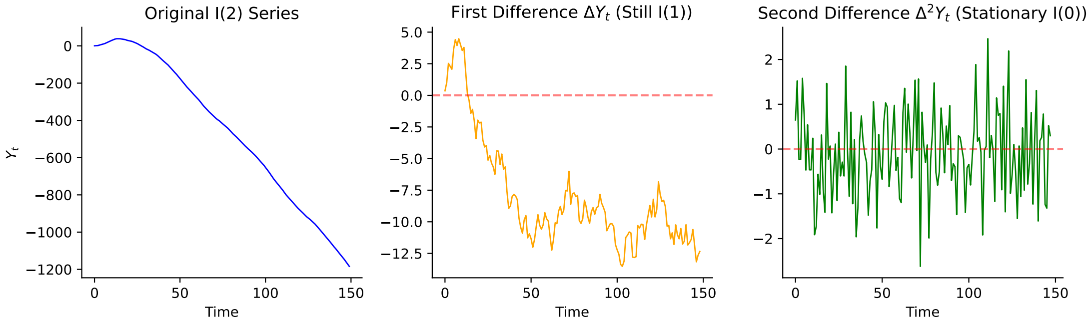
If a series Y_t is I(2), how many times must you difference it to achieve stationarity?

- A. 0 times (already stationary)
- B. 1 time
- C. 2 times
- D. Cannot be made stationary by differencing

Quiz Question 2: Answer

Correct Answer: (C) 2 times

I(d) means "integrated of order d" — requires d differences for stationarity.



Quiz Question 3

Question

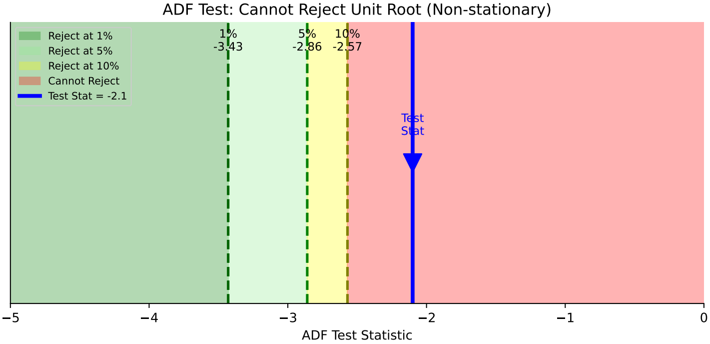
You run an ADF test and get a test statistic of -2.1 with critical values: -3.43 (1%), -2.86 (5%), -2.57 (10%). What do you conclude?

- A. Reject H_0: series is stationary at all levels
- B. Reject H_0: series is stationary at 10% level only
- C. Fail to reject H_0: series likely has a unit root
- D. The test is inconclusive

Quiz Question 3: Answer

Correct Answer: (C) Fail to reject H_0: series has unit root

Test stat $-2.1 > -2.57$ (10% CV) $\Rightarrow$ Cannot reject at any level. Consider differencing.



Quiz Question 4

Question

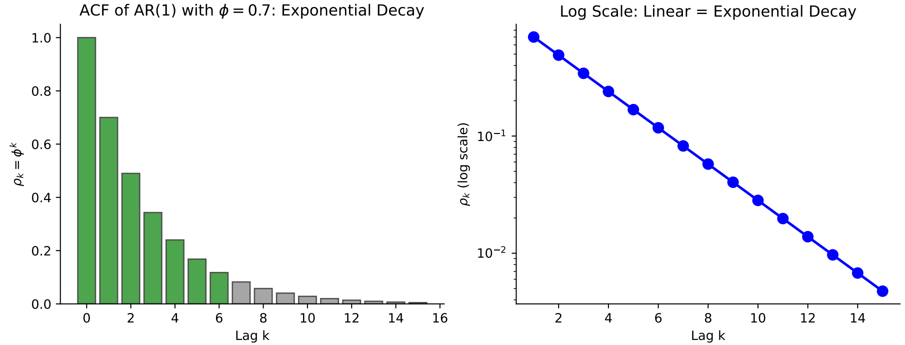
For an ARIMA(1,1,0) model, what is the ACF pattern of the **differenced** series ΔY_t?

- (A) Cuts off after lag 1
- (B) Decays exponentially
- (C) Alternates in sign
- (D) Is zero at all lags

Quiz Question 4: Answer

Correct Answer: (B) Decays exponentially

ARIMA(1,1,0) $\Rightarrow \Delta Y_t$ follows AR(1) with ACF $\rho_k = \phi_1^k$ (geometric decay).



Quiz Question 5

Question

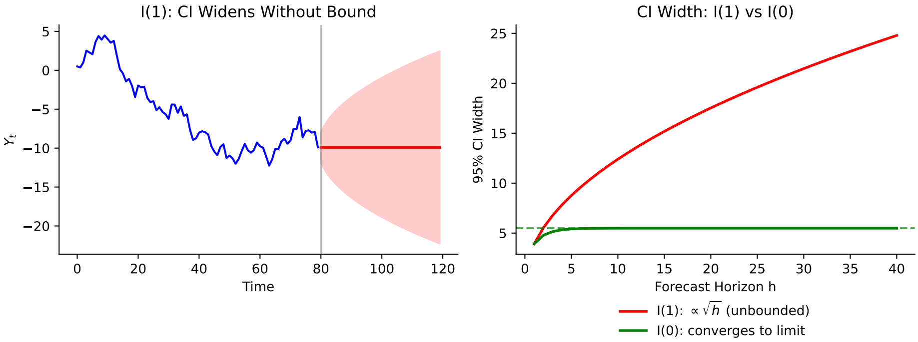
What happens to ARIMA forecast confidence intervals as the horizon h increases for an I(1) series?

- A. They stay constant
- B. They narrow (more precision)
- C. They widen without bound
- D. They widen but converge to a limit

Quiz Question 5: Answer

Correct Answer: (C) They widen without bound

For I(1): CI width $\propto \sqrt{h}$ (unbounded). For I(0): CIs converge to a limit.



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