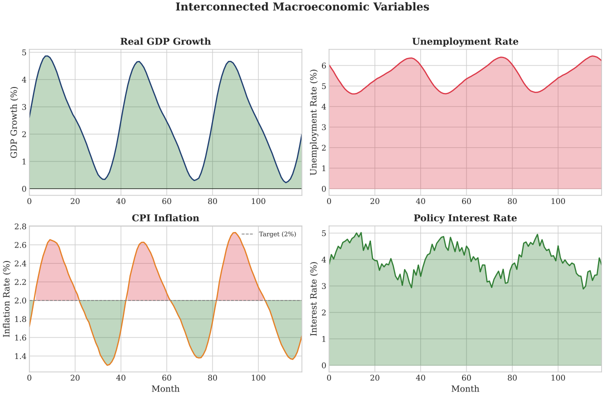
Chapter 5: VAR Models & Granger Causality

Multivariate Time Series

Lecture Outline

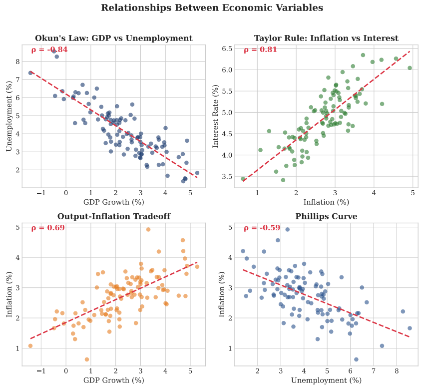
1. Introduction to Multivariate Time Series
2. Vector Autoregression (VAR)
3. Granger Causality
4. Impulse Response Functions
5. Forecast Error Variance Decomposition
6. VAR Diagnostics
7. VAR Forecasting
8. Practical Example
9. Summary
10. Quiz

Motivating Example: Macroeconomic Dynamics



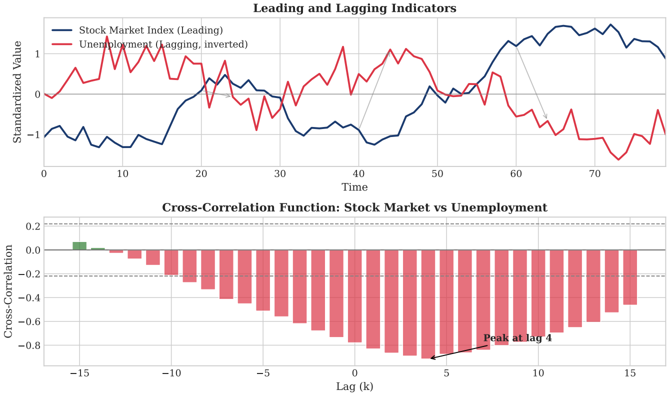
- Economic variables are **interconnected**: GDP affects unemployment, inflation affects interest rates
- Changes in one variable **propagate** through the system
- Understanding these dynamics requires **multivariate** analysis

The Key Insight: Variables Interact



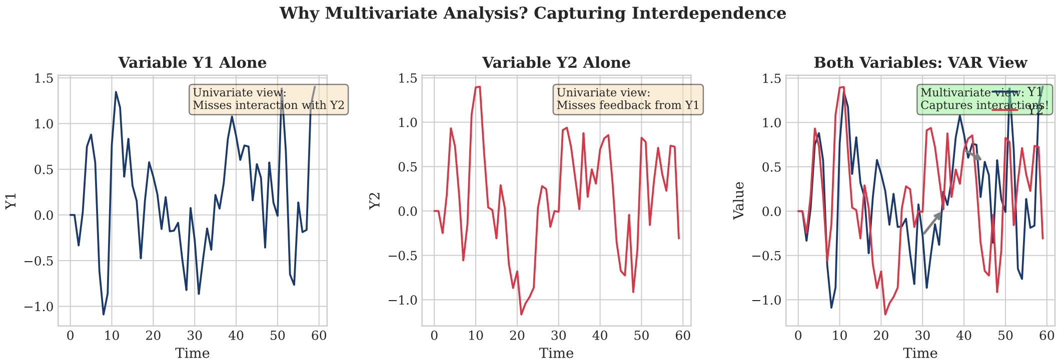
- **Okun's Law**: Higher GDP growth $\Rightarrow$ lower unemployment
- **Taylor Rule**: Higher inflation $\Rightarrow$ higher interest rates
- **Phillips Curve**: Unemployment-inflation tradeoff

Lead-Lag Relationships



- Some variables **lead** others: stock market predicts economic activity
- Cross-correlation reveals the **timing** of relationships
- Peak correlation at lag 4: stock market leads unemployment by ∼4 months

Why Univariate Models Are Not Enough



The Problem

ARIMA models each variable **in isolation**—ignoring valuable information from other variables!

The Solution

VAR models capture the joint dynamics and feedback effects between multiple time series.

What We'll Learn Today

Core Concepts

1. **VAR Models**: How to model multiple time series jointly
2. **Granger Causality**: Does X help predict Y?
3. **Impulse Response Functions**: How do shocks propagate?
4. **Variance Decomposition**: What drives each variable?

Applications

- Macroeconomic policy analysis (monetary policy effects)
- Financial market dynamics (stock-bond relationships)
- Business cycle analysis (leading indicators)
- Risk management (volatility transmission)

Why Multivariate Analysis?

Limitations of Univariate Models

- ARIMA models each variable **in isolation**
- Ignores potential **interactions** between variables
- Cannot capture **feedback effects**

Economic Examples of Interdependence

- GDP and unemployment (Okun's law)
- Interest rates and inflation (Taylor rule)
- Stock prices and trading volume
- Exchange rates and trade balance

Multivariate Time Series Notation

Vector of Variables

Let $\mathbf{Y}_t = (Y_{1t}, Y_{2t}, \ldots, Y_{Kt})'$ be a $K \times 1$ vector of time series.

Example with $K = 2$:

$$\mathbf{Y}_t = \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \text{GDP growth}_t \\ \text{Inflation}_t \end{pmatrix}$$

Key Questions

1. Does Y_1 help predict Y_2? (Granger causality)
2. How do shocks to Y_1 affect Y_2? (Impulse responses)
3. What proportion of Y_2's variance is due to Y_1? (Variance decomposition)

Multivariate Stationarity

Definition: Weak Stationarity

A K-dimensional time series $\mathbf{Y}_t$ is **weakly stationary** if:

1. $\mathbb{E}[\mathbf{Y}_t] = \boldsymbol{\mu}$ (constant mean vector)
2. $\mathrm{Cov}(\mathbf{Y}_t, \mathbf{Y}_{t-h}) = \boldsymbol{\Gamma}(h)$ depends only on h, not t

Autocovariance Matrix

$$\boldsymbol{\Gamma}(h) = \mathbb{E}[(\mathbf{Y}_t - \boldsymbol{\mu})(\mathbf{Y}_{t-h} - \boldsymbol{\mu})'] = \begin{pmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{pmatrix}$$

Note: $\boldsymbol{\Gamma}(-h) = \boldsymbol{\Gamma}(h)'$ (transpose, not equal!)

Cross-Covariance Properties

Cross-Covariance Function

For variables Y_{it} and Y_{jt}:

$$\gamma_{ij}(h) = \text{Cov}(Y_{it}, Y_{j,t-h}) = \mathbb{E}[(Y_{it} - \mu_i)(Y_{j,t-h} - \mu_j)]$$

Key Difference from Univariate Case

- In general: $\gamma_{ij}(h) \neq \gamma_{ij}(-h)$
- But: $\gamma_{ij}(h) = \gamma_{ji}(-h)$
- The cross-covariance matrix is **not symmetric** for $h \neq 0$

Example

If Y_1 leads Y_2: $\gamma_{12}(h) > 0$ for $h > 0$ but $\gamma_{12}(h) \approx 0$ for $h < 0$

Correlation Matrix Function

Definition

The **autocorrelation matrix** at lag h:

$$\mathbf{R}(h) = \mathbf{D}^{-1}\mathbf{\Gamma}(h)\mathbf{D}^{-1}$$

where $\mathbf{D} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_K)$ and $\sigma_i = \sqrt{\gamma_{ii}(0)}$

For Bivariate Case

$$\mathbf{R}(h) = \begin{pmatrix} \rho_{11}(h) & \rho_{12}(h) \\ \rho_{21}(h) & \rho_{22}(h) \end{pmatrix}$$

where $\rho_{ij}(h) = \frac{\gamma_{ij}(h)}{\sigma_i \sigma_j}$

Diagonal elements: usual ACFs; Off-diagonal: cross-correlations

The VAR(p) Model

Definition

A **VAR(p)** model for K variables:

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1 \mathbf{Y}_{t-1} + \mathbf{A}_2 \mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{Y}_{t-p} + \varepsilon_t$$

where:

- $\mathbf{Y}_t$: $K \times 1$ vector of endogenous variables
- $\mathbf{c}$: $K \times 1$ vector of constants
- $\mathbf{A}_i$: $K \times K$ coefficient matrices
- ε_t: $K \times 1$ vector of error terms with $\mathbb{E}[\varepsilon_t] = \mathbf{0}$, $\mathbb{E}[\varepsilon_t \varepsilon_t'] = \mathbf{\Sigma}$

VAR(1) with Two Variables

Bivariate VAR(1)

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Equation by Equation

$$Y_{1t} = c_1 + a_{11} Y_{1,t-1} + a_{12} Y_{2,t-1} + \varepsilon_{1t}$$
$$Y_{2t} = c_2 + a_{21} Y_{1,t-1} + a_{22} Y_{2,t-1} + \varepsilon_{2t}$$

Key insight: Each equation includes lags of **all** variables!

Numerical Example: VAR(1)

Specific VAR(1) Model

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} + \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Interpretation of Coefficients

- $a_{11} = 0.7$: A 1-unit increase in Y_1 at $t-1$ increases Y_1 at t by 0.7
- $a_{12} = 0.2$: A 1-unit increase in Y_2 at $t-1$ increases Y_1 at t by 0.2
- $a_{21} = -0.1$: A 1-unit increase in Y_1 at $t-1$ **decreases** Y_2 at t by 0.1
- $a_{22} = 0.6$: A 1-unit increase in Y_2 at $t-1$ increases Y_2 at t by 0.6

VAR(2): Higher Order Dynamics

VAR(2) Specification

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \varepsilon_t$$

For $K = 2$, the full model has $2 + 2 \times 4 + 2 \times 4 = 18$ parameters!

Written Out

$$Y_{1t} = c_1 + a_{11}^{(1)} Y_{1,t-1} + a_{12}^{(1)} Y_{2,t-1} + a_{11}^{(2)} Y_{1,t-2} + a_{12}^{(2)} Y_{2,t-2} + \varepsilon_{1t}$$
$$Y_{2t} = c_2 + a_{21}^{(1)} Y_{1,t-1} + a_{22}^{(1)} Y_{2,t-1} + a_{21}^{(2)} Y_{1,t-2} + a_{22}^{(2)} Y_{2,t-2} + \varepsilon_{2t}$$

Curse of Dimensionality

VAR(p) with K variables has $K + pK^2$ parameters. With $K = 5$, $p = 4$: $5 + 4 \times 25 = 105$ parameters!

The Companion Form

Converting VAR(p) to VAR(1)

Any VAR(p) can be written as a VAR(1) in **companion form**:

$$\boldsymbol{\xi}_t = \mathbf{A}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t$$

For VAR(2)

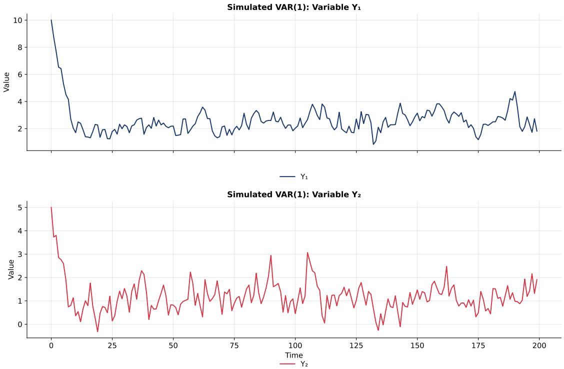
$$\underbrace{\begin{pmatrix} \mathbf{Y}_t \\ \mathbf{Y}_{t-1} \end{pmatrix}}_{\boldsymbol{\xi}_t} = \underbrace{\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I}_K & \mathbf{0} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \mathbf{Y}_{t-1} \\ \mathbf{Y}_{t-2} \end{pmatrix}}_{\boldsymbol{\xi}_{t-1}} + \underbrace{\begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \end{pmatrix}}_{\mathbf{v}_t}$$

The companion matrix $\mathbf{A}$ is $Kp \times Kp$.

Why Useful?

Stationarity, forecasting, and IRFs are easier to analyze in companion form.

Simulated VAR Process



- Two simulated series from a bivariate VAR(1) process showing interdependence
- Each variable responds to both its own past and the other variable's past
- Notice how the series co-move due to cross-equation dynamics

Stationarity of VAR

Stability Condition

VAR(p) is **stable** (stationary) if all roots of:

$$\det(\mathbf{I}_K - \mathbf{A}_1 z - \mathbf{A}_2 z^2 - \cdots - \mathbf{A}_p z^p) = 0$$

lie **outside** the unit circle (i.e., $|z| > 1$).

For VAR(1)

The model is stable if all **eigenvalues** of $\mathbf{A}_1$ are less than 1 in absolute value.

Example: For $\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.2 & 0.3 \end{pmatrix}$, eigenvalues are $\lambda_1 = 0.6$ and $\lambda_2 = 0.2$.
Both $< 1 \Rightarrow$ stable!

Computing Eigenvalues: Example

For $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$

Characteristic polynomial: $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$$\det \begin{pmatrix} 0.7 - \lambda & 0.2 \\ -0.1 & 0.6 - \lambda \end{pmatrix} = (0.7 - \lambda)(0.6 - \lambda) + 0.02 = 0$$

$$\lambda^2 - 1.3\lambda + 0.44 = 0$$

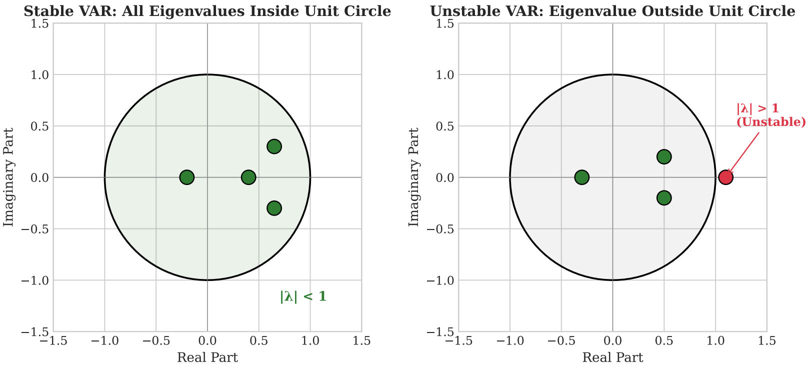
Solution

Using quadratic formula:

$$\lambda = \frac{1.3 \pm \sqrt{1.69 - 1.76}}{2} = \frac{1.3 \pm \sqrt{-0.07}}{2} = 0.65 \pm 0.132i$$

$|\lambda| = \sqrt{0.65^2 + 0.132^2} = \sqrt{0.44} = 0.663 < 1$ ✓ Stable!

Stability Condition: Visual Interpretation



- Eigenvalues of the companion matrix must lie inside the unit circle
- Complex eigenvalues come in conjugate pairs
- If any eigenvalue is outside the circle, the VAR is explosive (non-stationary)

Mean of a Stationary VAR

Unconditional Mean

For a stationary VAR(1): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t$
Taking expectations:

$$\mathbb{E}[\mathbf{Y}_t] = \mathbf{c} + \mathbf{A}\mathbb{E}[\mathbf{Y}_{t-1}]$$

Since $\mathbb{E}[\mathbf{Y}_t] = \mathbb{E}[\mathbf{Y}_{t-1}] = \boldsymbol{\mu}$ (stationarity):

$$\boldsymbol{\mu} = \mathbf{c} + \mathbf{A}\boldsymbol{\mu} \quad \Rightarrow \quad \boldsymbol{\mu} = (\mathbf{I}_K - \mathbf{A})^{-1}\mathbf{c}$$

Example

If $\mathbf{c} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$:

$$\boldsymbol{\mu} = \begin{pmatrix} 0.3 & -0.2 \\ 0.1 & 0.4 \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2.3 \\ 1.0 \end{pmatrix}$$

Covariance Structure of VAR(1)

Variance-Covariance Matrix $\mathbf{\Gamma}(0)$

For VAR(1), the variance satisfies the **discrete Lyapunov equation**:

$$\mathbf{\Gamma}(0) = \mathbf{A}\mathbf{\Gamma}(0)\mathbf{A}' + \mathbf{\Sigma}$$

Autocovariance at Lag h

$$\mathbf{\Gamma}(h) = \mathbf{A}^h\mathbf{\Gamma}(0), \quad h \geq 0$$

This shows that autocovariances decay geometrically with the eigenvalues of $\mathbf{A}$.

Solving the Lyapunov Equation

Can solve by vectorization:

$$\text{vec}(\mathbf{\Gamma}(0)) = (\mathbf{I}_{K^2} - \mathbf{A} \otimes \mathbf{A})^{-1}\text{vec}(\mathbf{\Sigma})$$

where $\otimes$ denotes the Kronecker product.

Estimation of VAR

OLS Estimation

Each equation can be estimated by **OLS separately**:

$$\hat{\mathbf{A}} = \left(\sum_{t=1}^{T} \mathbf{Y}_{t-1}\mathbf{Y}_{t-1}'\right)^{-1} \left(\sum_{t=1}^{T} \mathbf{Y}_{t-1}\mathbf{Y}_t'\right)$$

This is efficient because all equations have the **same regressors**.

Covariance Matrix

$$\hat{\Sigma} = \frac{1}{T - Kp - 1} \sum_{t=1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_t'$$

The errors ε_{1t} and ε_{2t} may be **contemporaneously correlated**.

Lag Order Selection

Information Criteria

Choose p that minimizes:

$$\text{AIC}(p) = \ln|\hat{\Sigma}_p| + \frac{2pK^2}{T}$$

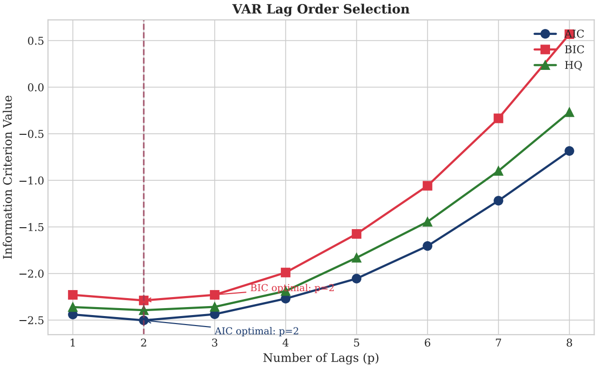
$$\text{BIC}(p) = \ln|\hat{\Sigma}_p| + \frac{pK^2 \ln T}{T}$$

$$\text{HQ}(p) = \ln|\hat{\Sigma}_p| + \frac{2pK^2 \ln\ln T}{T}$$

Guidelines

- AIC tends to select **larger** models (better for forecasting)
- BIC tends to select **smaller** models (consistent selection)
- Start with maximum p_{max} based on data frequency (e.g., 4 for quarterly, 12 for monthly)

Lag Selection: Example



- Information criteria values for different lag orders
- AIC and BIC may suggest different optimal lags
- Lower values indicate better model fit (penalized by complexity)

Restricted VAR Models

Why Restrict?

Full VAR models can be **overparameterized**:

- Many coefficients may be insignificant
- Poor forecasting performance
- Loss of degrees of freedom

Common Restrictions

- **Zero restrictions**: Set small coefficients to zero
- **Block exogeneity**: Some variables don't affect others
- **Lag exclusion**: Exclude certain lags

Testing Restrictions

Use likelihood ratio test: $LR = T(\ln|\hat{\Sigma}_R| - \ln|\hat{\Sigma}_U|) \sim \chi^2_r$
where r = number of restrictions

What is Granger Causality?

Clive Granger (1969, Nobel Prize 2003)

"X **Granger-causes** Y" if past values of X help predict Y, **beyond** what past values of Y alone can predict.

Important Distinction

Granger causality $\neq$ True causality

- Granger causality is about **predictive content**
- Does NOT imply economic/structural causation
- "X Granger-causes Y" means: X contains useful information for forecasting Y

Formal Definition

Granger Causality

X **does not Granger-cause** Y if:

$$\mathbb{E}[Y_t|Y_{t-1}, Y_{t-2}, \ldots, X_{t-1}, X_{t-2}, \ldots] = \mathbb{E}[Y_t|Y_{t-1}, Y_{t-2}, \ldots]$$

In other words: adding X's history does not improve the prediction of Y.

In the VAR Context

For VAR(1): $Y_{1t} = c_1 + a_{11}Y_{1,t-1} + a_{12}Y_{2,t-1} + \varepsilon_{1t}$

Y_2 does **not** Granger-cause Y_1 if $a_{12} = 0$.

For VAR(p): Y_2 does not Granger-cause Y_1 if $a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$.

Testing for Granger Causality

Hypothesis Test

H_0: Y_2 does **not** Granger-cause Y_1

$$H_0 : a_{12}^{(1)} = a_{12}^{(2)} = \cdots = a_{12}^{(p)} = 0$$

H_1: At least one $a_{12}^{(i)} \neq 0$ (Granger causality exists)

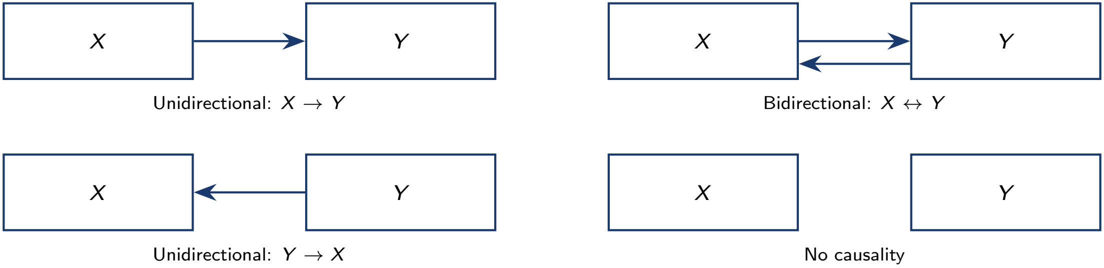
Test Statistic: Wald Test

$$F = \frac{(RSS_R - RSS_U)/p}{RSS_U/(T - 2p - 1)} \sim F_{p,T-2p-1}$$

where:

- RSS_R: Residual sum of squares from restricted model (without Y_2 lags)
- RSS_U: Residual sum of squares from unrestricted model (full VAR)

Types of Granger Causality



Economic Examples

- Money $\to$ Output? (monetarist view)
- Stock prices $\leftrightarrow$ Trading volume (bidirectional)
- Weather $\to$ Crop yields (unidirectional, obvious)

Cross-Correlation Function

Definition 1 (Cross-Correlation Function)

The **cross-correlation** between X_t and Y_t at lag k is:

$$\rho_{XY}(k) = \frac{\gamma_{XY}(k)}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X_t, Y_{t+k})}{\sqrt{\text{Var}(X_t)\text{Var}(Y_t)}}$$

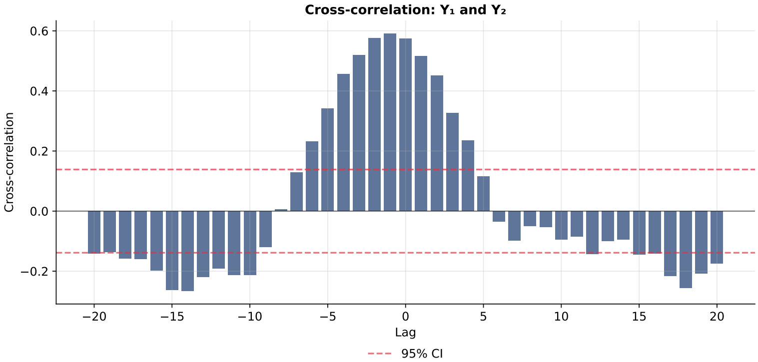
Interpretation

- $\rho_{XY}(k) > 0$ at $k > 0$: X is positively correlated with future Y (X may lead Y)
- $\rho_{XY}(k) > 0$ at $k < 0$: X is positively correlated with past Y (Y may lead X)

Note

Unlike ACF, cross-correlation is **not symmetric**: $\rho_{XY}(k) \neq \rho_{XY}(-k)$ in general.

Cross-Correlation Analysis



- Cross-correlation function measures linear dependence at different lags
- Significant correlations at negative lags suggest X leads Y; positive lags suggest Y leads X
- Useful for preliminary analysis before formal Granger causality testing

Granger Causality: Practical Considerations

Common Pitfalls

1. **Omitted variables**: A third variable Z may cause both X and Y
2. **Non-stationarity**: Test requires stationary data (or cointegration)
3. **Lag selection**: Results can be sensitive to p
4. **Sample size**: Need sufficient observations

Best Practices

- Test for unit roots first
- Use multiple lag selection criteria
- Check robustness to different lag lengths
- Report results for both directions

Granger Causality Test: Numerical Example

Testing: Does Money Growth Granger-cause Output?

Unrestricted model (VAR with 2 lags):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \beta_1 \Delta M_{t-1} + \beta_2 \Delta M_{t-2} + \varepsilon_t$$

Restricted model (H_0: $\beta_1 = \beta_2 = 0$):

$$\Delta Y_t = c + \alpha_1 \Delta Y_{t-1} + \alpha_2 \Delta Y_{t-2} + \varepsilon_t$$

Test Computation

With $T = 100$, $RSS_U = 45.2$, $RSS_R = 52.8$:

$$F = \frac{(52.8 - 45.2)/2}{45.2/(100 - 5)} = \frac{3.8}{0.476} = 7.98$$

$F_{0.05}(2, 95) = 3.09 \Rightarrow$ **Reject** H_0: Money Granger-causes output!

The Toda-Yamamoto Procedure

Problem with Non-Stationary Data

Standard Granger test has **non-standard distributions** when:

- Variables have unit roots
- Variables are cointegrated

Toda-Yamamoto Solution (1995)

1. Determine maximum order of integration d_{max}
2. Estimate VAR($p + d_{max}$) in **levels**
3. Test restrictions on first p lags only
4. Extra d_{max} lags are **not** tested (just for correct distribution)

Advantage

Wald test has asymptotic χ^2 distribution regardless of cointegration!

Instantaneous Causality

Definition

X **instantaneously causes** Y if:

$$\mathbb{E}[Y_t|\Omega_{t-1}, X_t] \neq \mathbb{E}[Y_t|\Omega_{t-1}]$$

where Ω_{t-1} contains all past information.

Testing in VAR

Test whether $\sigma_{12} \neq 0$ in the covariance matrix:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

If $\sigma_{12} = 0$: no instantaneous causality

Interpretation

Instantaneous causality often reflects **common shocks** or **data aggregation**, not true contemporaneous effects.

Granger Causality in Multiple Systems

Block Exogeneity Test

In a VAR with $K > 2$ variables, test whether a **group** of variables Granger-causes another group.

Example: Do financial variables (interest rates, stock prices) Granger-cause real variables (GDP, unemployment)?

Test Statistic

$$\chi^2 = T \cdot K_1 \cdot p \cdot \left(\ln|\hat{\Sigma}_R| - \ln|\hat{\Sigma}_U|\right) \sim \chi^2_{K_1 \cdot K_2 \cdot p}$$

where K_1 = number of "caused" variables, K_2 = number of "causing" variables

What are Impulse Response Functions?

Definition

An **Impulse Response Function (IRF)** traces the effect of a one-time shock to one variable on the current and future values of all variables.

Question IRFs Answer

"If there is an unexpected 1-unit shock to Y_1 today, what happens to Y_1 and Y_2 over the next h periods?"

MA(∞) Representation

A stable VAR(p) can be written as:

$$\mathbf{Y}_t = \boldsymbol{\mu} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}_i \boldsymbol{\varepsilon}_{t-i}$$

The matrices $\boldsymbol{\Phi}_i$ are the **impulse responses** at horizon i.

Computing IRFs for VAR(1)

For VAR(1): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t$

The impulse response matrices are:

$$\mathbf{\Phi}_0 = \mathbf{I}_K, \quad \mathbf{\Phi}_1 = \mathbf{A}, \quad \mathbf{\Phi}_2 = \mathbf{A}^2, \quad \dots, \quad \mathbf{\Phi}_h = \mathbf{A}^h$$

Interpretation

$[\mathbf{\Phi}_h]_{ij}$ = Effect on Y_i at time $t + h$ of a unit shock to Y_j at time t

For stable VAR: $\mathbf{\Phi}_h \to \mathbf{0}$ as $h \to \infty$ (shocks die out)

Computing IRFs for General VAR(p)

Recursive Formula for VAR(p)

For $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}_1\mathbf{Y}_{t-1} + \mathbf{A}_2\mathbf{Y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \varepsilon_t$:

$$\mathbf{\Phi}_h = \sum_{j=1}^{\min(h,p)} \mathbf{A}_j \mathbf{\Phi}_{h-j}, \quad h = 1, 2, 3, \ldots$$

with $\mathbf{\Phi}_0 = \mathbf{I}_K$ and $\mathbf{\Phi}_h = \mathbf{0}$ for $h < 0$.

Example: VAR(2) IRFs

- $\mathbf{\Phi}_0 = \mathbf{I}_K$
- $\mathbf{\Phi}_1 = \mathbf{A}_1\mathbf{\Phi}_0 = \mathbf{A}_1$
- $\mathbf{\Phi}_2 = \mathbf{A}_1\mathbf{\Phi}_1 + \mathbf{A}_2\mathbf{\Phi}_0 = \mathbf{A}_1^2 + \mathbf{A}_2$
- $\mathbf{\Phi}_3 = \mathbf{A}_1\mathbf{\Phi}_2 + \mathbf{A}_2\mathbf{\Phi}_1 = \mathbf{A}_1(\mathbf{A}_1^2 + \mathbf{A}_2) + \mathbf{A}_2\mathbf{A}_1$

Orthogonalized IRFs

Problem: Correlated Errors

If $\mathbf{\Sigma}$ is not diagonal, shocks ε_{1t} and ε_{2t} are correlated.
A shock to "Y_1" also involves a shock to "Y_2".

Solution: Cholesky Decomposition

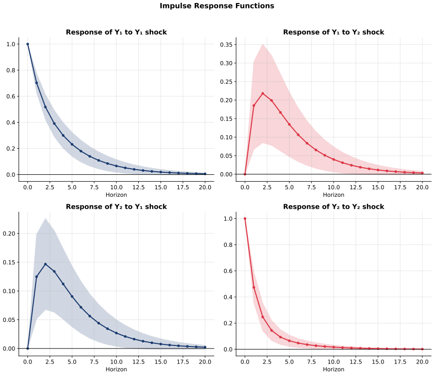
Factor $\mathbf{\Sigma} = \mathbf{P}\mathbf{P}'$ where $\mathbf{P}$ is lower triangular.
Define orthogonalized shocks: $\mathbf{u}_t = \mathbf{P}^{-1}\boldsymbol{\varepsilon}_t$ with $\mathbb{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}$

Orthogonalized IRFs: $\mathbf{\Theta}_h = \mathbf{\Phi}_h\mathbf{P}$

Ordering Matters!

Cholesky assumes variables ordered from "most exogenous" to "most endogenous". Results depend on this ordering.

Impulse Response Functions: Example



- IRFs show how each variable responds to a one-unit shock over time
- Shaded regions represent confidence intervals (uncertainty in estimates)
- For stable VAR models, responses converge to zero as the horizon increases

IRF Numerical Example

For $\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$

$$\mathbf{\Phi}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{\Phi}_1 = \mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ -0.1 & 0.6 \end{pmatrix}$$

$$\mathbf{\Phi}_2 = \mathbf{A}^2 = \begin{pmatrix} 0.47 & 0.26 \\ -0.13 & 0.34 \end{pmatrix}$$

Interpretation

- $[\mathbf{\Phi}_2]_{12} = 0.26$: A unit shock to Y_2 increases Y_1 by 0.26 after 2 periods
- $[\mathbf{\Phi}_2]_{21} = -0.13$: A unit shock to Y_1 **decreases** Y_2 by 0.13 after 2 periods

Cumulative Impulse Responses

Definition

The **cumulative IRF** up to horizon H:

$$\mathbf{\Psi}_H = \sum_{h=0}^{H} \mathbf{\Phi}_h$$

Measures the **total accumulated effect** of a shock.

Long-Run Multiplier

For stable VAR: $\mathbf{\Psi}_\infty = (\mathbf{I}_K - \mathbf{A}_1 - \mathbf{A}_2 - \cdots - \mathbf{A}_p)^{-1}$

This gives the **permanent effect** of a one-time shock.

When to Use

Cumulative IRFs are useful when interested in total impact (e.g., cumulative GDP loss from a shock).

Confidence Intervals for IRFs

Sources of Uncertainty

IRFs are functions of estimated parameters $\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_p$, so they have **sampling uncertainty**.

Methods for Confidence Bands

1. **Asymptotic**: Use delta method to derive standard errors
2. **Monte Carlo**: Simulate from asymptotic distribution of $\hat{\mathbf{A}}$
3. **Bootstrap**: Resample residuals and re-estimate VAR

Bootstrap Procedure

1. Estimate VAR, save residuals $\{\hat{\varepsilon}_t\}$
2. Draw with replacement to create $\{\hat{\varepsilon}_t^*\}$
3. Generate bootstrap sample using estimated VAR
4. Re-estimate and compute IRFs
5. Repeat B times; use percentiles for CIs

Structural VAR (SVAR)

Motivation

Standard VAR shocks ε_t are **reduced-form** innovations—linear combinations of structural shocks.

We want to identify economically meaningful **structural shocks**.

Structural Form

$$\mathbf{B}_0\mathbf{Y}_t = \boldsymbol{\Gamma}_0 + \mathbf{B}_1\mathbf{Y}_{t-1} + \cdots + \mathbf{B}_p\mathbf{Y}_{t-p} + \mathbf{u}_t$$

where $\mathbf{u}_t$ are **structural shocks** with $\mathbb{E}[\mathbf{u}_t\mathbf{u}_t'] = \mathbf{I}_K$

Relationship to Reduced Form

$\varepsilon_t = \mathbf{B}_0^{-1}\mathbf{u}_t \quad \Rightarrow \quad \boldsymbol{\Sigma} = \mathbf{B}_0^{-1}(\mathbf{B}_0^{-1})'$

Identification in SVAR

The Identification Problem

$\boldsymbol{\Sigma}$ has $K(K+1)/2$ unique elements, but $\mathbf{B}_0^{-1}$ has K^2 elements.

Need $K(K-1)/2$ additional restrictions!

Common Identification Schemes

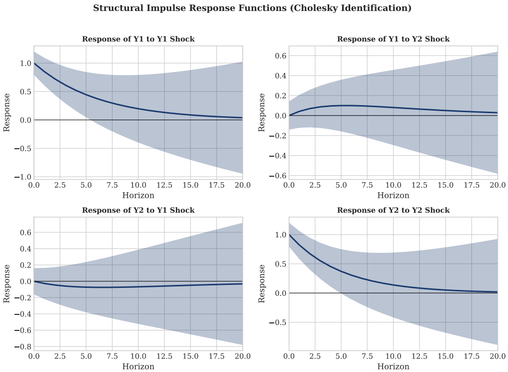
1. **Short-run restrictions**: Zero impact effects (Cholesky)
2. **Long-run restrictions**: Zero long-run effects (Blanchard-Quah)
3. **Sign restrictions**: Inequality constraints on IRFs
4. **External instruments**: Use outside information

Example: Cholesky (Recursive) Ordering

For $K = 2$: $\mathbf{B}_0^{-1} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}$

Variable 1 doesn't respond to shock 2 contemporaneously.

Structural IRF Example



- Structural IRFs based on Cholesky identification
- Order of variables affects interpretation of shocks
- First variable responds only to own shocks contemporaneously

Variance Decomposition

Question

What proportion of the forecast error variance of Y_i at horizon h is due to shocks to Y_j?

FEVD Formula

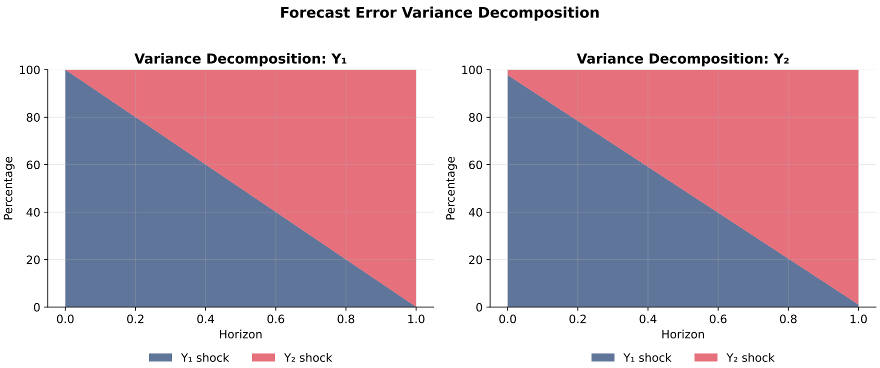
$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\boldsymbol{\Theta}_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^{K} [\boldsymbol{\Theta}_s]_{ik}^2}$$

This gives the **percentage** of Y_i's h-step forecast variance explained by shocks to Y_j.

Properties

- $0 \leq \text{FEVD}_{ij}(h) \leq 1$
- $\sum_{j=1}^{K} \text{FEVD}_{ij}(h) = 1$ (sums to 100%)
- At $h = 1$: Own shocks dominate (by construction with Cholesky)

FEVD: Example



- FEVD shows the proportion of forecast variance attributable to each shock
- At short horizons, own shocks dominate; cross-variable effects grow over time
- Useful for understanding the relative importance of different shocks in the system

FEVD: Numerical Example

Computing FEVD for Bivariate VAR

Using orthogonalized IRFs $\boldsymbol{\Theta}_h$, FEVD at horizon H:

$$\text{FEVD}_{11}(H) = \frac{\sum_{h=0}^{H-1} \theta_{11}^2(h)}{\sum_{h=0}^{H-1}[\theta_{11}^2(h) + \theta_{12}^2(h)]}$$

Example Calculation

h	$\theta_{11}(h)$	$\theta_{12}(h)$	$\theta_{11}^2(h)$	$\theta_{12}^2(h)$
0	1.00	0.00	1.00	0.00
1	0.70	0.20	0.49	0.04
2	0.47	0.26	0.22	0.07

$\text{FEVD}_{11}(3) = \frac{1.00+0.49+0.22}{1.00+0.49+0.22+0.00+0.04+0.07} = \frac{1.71}{1.82} = 94\%$

Historical Decomposition

Definition

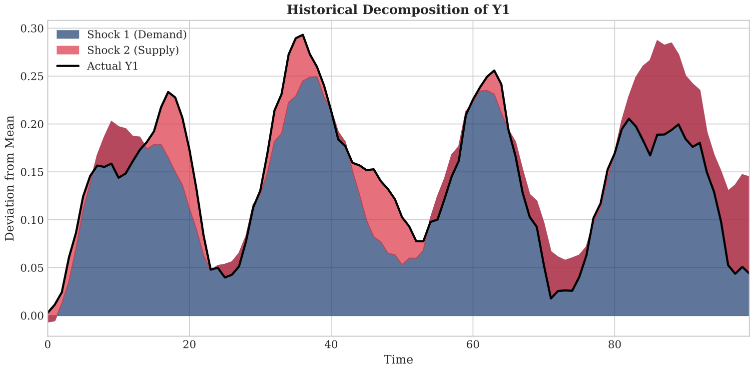
Historical decomposition breaks down each observed value into contributions from each structural shock:

$$Y_{it} - \bar{Y}_i = \sum_{j=1}^{K} \sum_{s=0}^{t-1} \theta_{ij}(s) \cdot u_{j,t-s}$$

Application

- "How much of the 2008 GDP decline was due to financial shocks vs. oil shocks?"
- Attributes historical movements to specific identified shocks
- Useful for policy analysis and narrative interpretation

Historical Decomposition: Example



- Each color represents the contribution of a different structural shock
- Stacked contributions sum to the actual observed deviation from mean
- Helps identify which shocks drove historical episodes

Residual Diagnostics

What to Check

After estimating VAR, verify that residuals $\hat{\varepsilon}_t$ behave like white noise:

1. No serial correlation
2. Constant variance (homoskedasticity)
3. Normality (for inference)

Why It Matters

- Autocorrelated residuals $\Rightarrow$ inefficient estimates
- Heteroskedasticity $\Rightarrow$ invalid standard errors
- Non-normality $\Rightarrow$ inference may be unreliable

Testing for Serial Correlation

Portmanteau Test (Ljung-Box)

$$Q_h = T(T+2)\sum_{j=1}^{h}\frac{1}{T-j}\text{tr}(\hat{\mathbf{C}}_j'\hat{\mathbf{C}}_0^{-1}\hat{\mathbf{C}}_j\hat{\mathbf{C}}_0^{-1})$$

where $\hat{\mathbf{C}}_j = \frac{1}{T}\sum_{t=j+1}^{T}\hat{\varepsilon}_t\hat{\varepsilon}_{t-j}'$
Under H_0 (no autocorrelation): $Q_h \sim \chi^2_{K^2(h-p)}$

Breusch-Godfrey LM Test

1. Regress $\hat{\varepsilon}_t$ on $\hat{\varepsilon}_{t-1}, \ldots, \hat{\varepsilon}_{t-h}$ and original regressors
2. $LM = T \cdot R^2 \sim \chi^2_{K^2 h}$ under H_0

If Rejected

Consider increasing lag order p or adding additional variables.

Testing for Heteroskedasticity

ARCH-LM Test

Test for autoregressive conditional heteroskedasticity in residuals:

$$\hat{\varepsilon}_{it}^2 = \alpha_0 + \alpha_1 \hat{\varepsilon}_{i,t-1}^2 + \cdots + \alpha_q \hat{\varepsilon}_{i,t-q}^2 + v_t$$

H_0: $\alpha_1 = \cdots = \alpha_q = 0$ (homoskedasticity)
$LM = TR^2 \sim \chi_q^2$

Multivariate Version

Test all equations jointly using:

$$\text{vech}(\hat{\boldsymbol{\varepsilon}}_t \hat{\boldsymbol{\varepsilon}}_t') = \mathbf{c} + \sum_{j=1}^{q} \mathbf{B}_j \text{vech}(\hat{\boldsymbol{\varepsilon}}_{t-j} \hat{\boldsymbol{\varepsilon}}_{t-j}') + \mathbf{v}_t$$

Normality Testing

Jarque-Bera Test (Univariate)

$$JB = \frac{T}{6}\left(S^2 + \frac{(K-3)^2}{4}\right) \sim \chi^2_2$$

where S = skewness, K = kurtosis

Multivariate Normality (Doornik-Hansen)

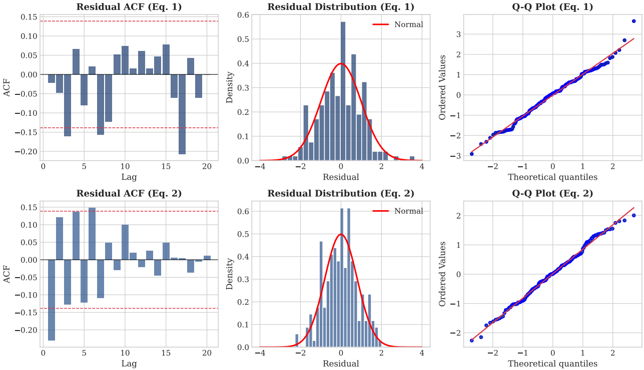
Transform residuals and test joint skewness and kurtosis:

$$DH = s_1'(\boldsymbol{\Omega}^{-1/2})'(\boldsymbol{\Omega}^{-1/2})s_1 + s_2'(\boldsymbol{\Omega}^{-1/2})'(\boldsymbol{\Omega}^{-1/2})s_2 \sim \chi^2_{2K}$$

Note

Normality is often rejected in financial data. Consider robust standard errors if non-normality is severe.

Diagnostic Summary Plot



- Residual ACF should show no significant autocorrelation
- Histogram should approximate normal distribution
- Q-Q plot should follow 45-degree line

Point Forecasts from VAR

Iterative Forecasting

For VAR(1): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A}\mathbf{Y}_{t-1} + \varepsilon_t$

1-step forecast: $\hat{\mathbf{Y}}_{T+1|T} = \mathbf{c} + \mathbf{A}\mathbf{Y}_T$
2-step forecast: $\hat{\mathbf{Y}}_{T+2|T} = \mathbf{c} + \mathbf{A}\hat{\mathbf{Y}}_{T+1|T}$
***h*-step forecast**: $\hat{\mathbf{Y}}_{T+h|T} = \mathbf{c} + \mathbf{A}\hat{\mathbf{Y}}_{T+h-1|T}$

Direct Formula

$$\hat{\mathbf{Y}}_{T+h|T} = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{h-1})\mathbf{c} + \mathbf{A}^h\mathbf{Y}_T$$

For stable VAR: converges to $\boldsymbol{\mu} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c}$ as $h \to \infty$

Forecast Error and MSE

h-Step Forecast Error

$$\mathbf{e}_{T+h|T} = \mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T} = \sum_{j=0}^{h-1} \mathbf{A}^j \varepsilon_{T+h-j}$$

Mean Squared Error Matrix

$$\text{MSE}(\hat{\mathbf{Y}}_{T+h|T}) = \mathbb{E}[\mathbf{e}_{T+h|T}\mathbf{e}'_{T+h|T}] = \sum_{j=0}^{h-1} \mathbf{A}^j \boldsymbol{\Sigma} (\mathbf{A}^j)'$$

Key Insight

- MSE increases with horizon h
- For stable VAR: MSE converges to unconditional variance $\boldsymbol{\Gamma}(0)$
- Long-horizon forecasts $\rightarrow$ unconditional mean with uncertainty $= \boldsymbol{\Gamma}(0)$

Forecast Confidence Intervals

Constructing Intervals

For normally distributed errors, $(1-\alpha)$ confidence interval:

$$\hat{Y}_{i,T+h|T} \pm z_{\alpha/2}\sqrt{[\text{MSE}(\hat{\mathbf{Y}}_{T+h|T})]_{ii}}$$

Joint Confidence Regions

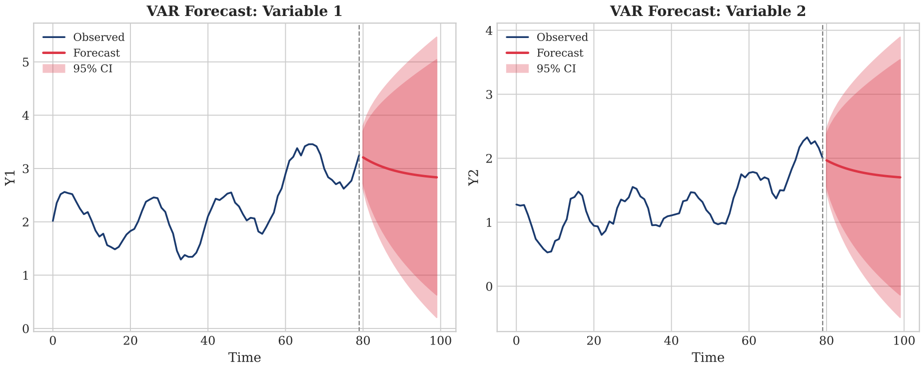
For multiple variables, use ellipsoids:

$$(\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T})'[\text{MSE}(\hat{\mathbf{Y}}_{T+h|T})]^{-1}(\mathbf{Y}_{T+h} - \hat{\mathbf{Y}}_{T+h|T}) \leq \chi^2_{K,\alpha}$$

Note

These assume known parameters. Bootstrap methods account for parameter uncertainty.

VAR Forecasts: Example



- Point forecasts shown as solid line beyond observed data
- Confidence bands widen as forecast horizon increases
- Forecasts converge to unconditional mean for long horizons

Forecast Evaluation

Out-of-Sample Evaluation

Split data: estimation sample (1 to T_1) and test sample ($T_1 + 1$ to T).
Compute forecast errors: $e_{t+h} = Y_{t+h} - \hat{Y}_{t+h|t}$

Common Metrics

- **RMSE**: $\sqrt{\frac{1}{n}\sum e_{t+h}^2}$
- **MAE**: $\frac{1}{n}\sum |e_{t+h}|$
- **MAPE**: $\frac{100}{n}\sum \left|\frac{e_{t+h}}{Y_{t+h}}\right|$

Diebold-Mariano Test

Test whether VAR forecasts are significantly better than alternative:

$$DM = \frac{\bar{d}}{\sqrt{\hat{\sigma}_d^2/n}} \sim N(0,1)$$

where $d_t = L(e_{1t}) - L(e_{2t})$ is the loss differential.

Example: GDP and Unemployment

Okun's Law

There is a negative relationship between GDP growth and unemployment:

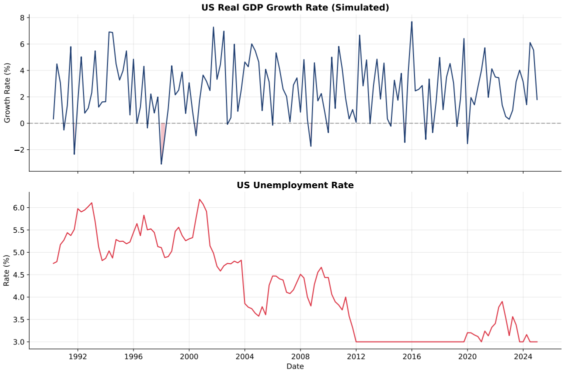
$$\Delta U_t \approx -\beta(\Delta Y_t - \bar{g})$$

where $\bar{g}$ is trend GDP growth and $\beta \approx 0.4$.

VAR Analysis Questions

1. Does GDP growth Granger-cause unemployment changes?
2. Does unemployment Granger-cause GDP growth?
3. How do shocks propagate between variables?

GDP and Unemployment: Data



- GDP growth and unemployment rate show clear negative correlation (Okun's Law)
- Both series exhibit cyclical patterns related to business cycle fluctuations
- This bivariate system is ideal for VAR analysis and Granger causality testing

VAR Workflow

1. **Data preparation**
   - Check for stationarity (unit root tests)
   - Transform if necessary (differences, logs)
2. **Lag selection**
   - Use AIC, BIC, HQ criteria
   - Check residual autocorrelation
3. **Estimation**
   - OLS equation by equation
   - Check stability (eigenvalues)
4. **Analysis**
   - Granger causality tests
   - Impulse response functions
   - Variance decomposition
5. **Forecasting**

Estimated VAR Results

VAR(2) Estimation Results
(Sample: T = 100 observations)

	Y1 Equation		Y2 Equation	
Variable	**Coef.**	**t-stat**	**Coef.**	**t-stat**
Constant	0.523	2.41**	0.318	1.89*
Y1(t-1)	0.712	8.54***	-0.094	-1.42
Y1(t-2)	-0.156	-1.87*	0.067	1.01
Y2(t-1)	0.198	2.76***	0.589	10.32***
Y2(t-2)	-0.043	-0.59	0.124	2.14**
R-squared	0.847		0.791	
Adj. R-sq	0.841		0.783	
F-statistic	131.4***		89.7***	
Residual Tests	Statistic	p-value		
Portmanteau(12)	14.23	0.287		
ARCH-LM(4)	3.56	0.469		
Jarque-Bera	2.18	0.336		

**** p<0.01, ** p<0.05, * p<0.10*

- Estimated coefficients with standard errors and t-statistics
- Information criteria values for model comparison
- Model diagnostics summary (residual tests)

Granger Causality Results

Test Results: GDP and Unemployment

Null Hypothesis	F-statistic	df	p-value	Decision
GDP $\not\to$ Unemployment	8.42	(2, 95)	0.0004	Reject
Unemployment $\not\to$ GDP	2.15	(2, 95)	0.1220	Fail to Reject

Interpretation

- GDP growth Granger-causes unemployment (consistent with Okun's Law)
- Unemployment does not significantly Granger-cause GDP
- Evidence of **unidirectional** causality: GDP $\rightarrow$ Unemployment

Python Implementation

VAR in Python (statsmodels)

```
from statsmodels.tsa.api import VAR
from statsmodels.tsa.stattools import grangercausalitytests

# Fit VAR model
model = VAR(data)
results = model.fit(maxlags=4, ic='aic')

# Granger causality test
granger_test = grangercausalitytests(data[['Y1', 'Y2']],
                                      maxlag=4)
# Impulse response functions
irf = results.irf(periods=20)
irf.plot()

# Variance decomposition
fevd = results.fevd(periods=20)
fevd.plot()
```

R Implementation

VAR in R (vars package)

```r
library(vars)

# Select optimal lag order
lag_select <- VARselect(data, lag.max = 8)
print(lag_select$selection)

# Fit VAR model
var_model <- VAR(data, p = 2, type = "const")
summary(var_model)

# Granger causality test
causality(var_model, cause = "GDP")

# Impulse response functions
irf_results <- irf(var_model, n.ahead = 20, boot = TRUE)
plot(irf_results)

# Forecast error variance decomposition
fevd_results <- fevd(var_model, n.ahead = 20)
plot(fevd_results)
```

Example: Monetary Policy Analysis

Three-Variable VAR

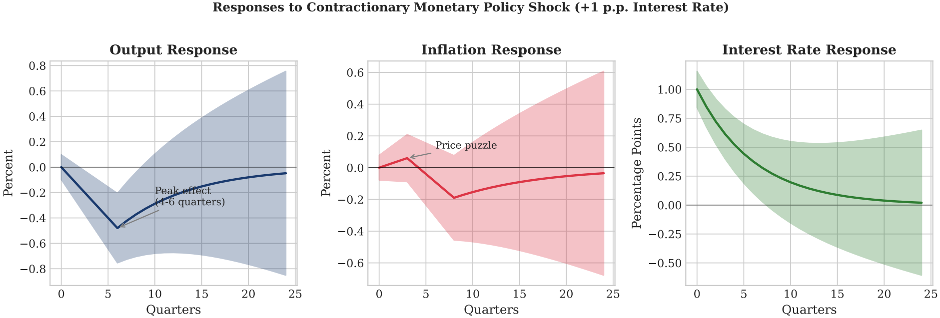
Study the monetary transmission mechanism with:

- Y_1: Output gap (GDP deviation from trend)
- Y_2: Inflation rate
- Y_3: Interest rate (policy instrument)

Key Questions

1. How does an interest rate shock affect output and inflation?
2. How long until the maximum effect is felt?
3. What fraction of output variance is due to monetary shocks?

Monetary Policy VAR: IRFs



- Contractionary monetary policy shock (interest rate increase)
- Output decreases with peak effect after 4-6 quarters ("long and variable lags")
- Inflation responds more slowly, decreasing after output

Key Takeaways

VAR Models

- Model **multiple** time series jointly
- Each variable depends on its own lags AND lags of other variables
- Estimated by OLS equation by equation; requires stationarity

Granger Causality

- Tests whether X helps predict Y beyond Y's own history
- **Not** the same as true causality; F-test on coefficient restrictions

IRF and FEVD

- IRF: How shocks propagate through the system
- FEVD: What proportion of variance is due to each shock
- Both depend on variable ordering (Cholesky decomposition)

VAR Model Selection Checklist

Before Estimation

- ☐ Test for unit roots in each variable
- ☐ Transform to stationary if needed (differences, logs)
- ☐ Check for outliers and structural breaks

Model Specification

- ☐ Select lag order using AIC/BIC
- ☐ Estimate VAR by OLS
- ☐ Check stability (eigenvalues inside unit circle)

Post-Estimation

- ☐ Test residuals for autocorrelation
- ☐ Test for ARCH effects
- ☐ Test for normality
- ☐ Compute IRFs, FEVDs, Granger tests

Common Mistakes to Avoid

Pitfalls in VAR Analysis

1. **Ignoring non-stationarity**: Always test for unit roots first
2. **Overfitting**: Too many lags $\Rightarrow$ poor forecasts
3. **Wrong ordering**: Cholesky results depend on variable order
4. **Confusing correlation with causation**: Granger causality $\neq$ true causality
5. **Ignoring parameter uncertainty**: Use bootstrap CIs for IRFs
6. **Short samples**: VAR requires many observations ($T > 50$)

What's Next?

Topics for Further Study

- **Cointegration**: Long-run relationships between non-stationary variables
- **VECM**: Error correction models for cointegrated systems
- **Structural VAR**: Imposing economic theory restrictions
- **Panel VAR**: VAR for panel data
- **Bayesian VAR**: Shrinkage priors for high-dimensional systems

Questions?

Quiz Question 1

Question

For a VAR(1) model with coefficient matrix $\mathbf{A} = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.5 \end{pmatrix}$, is the model stable?

- A. Yes, because all diagonal elements are less than 1
- B. Yes, because all eigenvalues are inside the unit circle
- C. No, because the sum of coefficients exceeds 1
- D. Cannot be determined without knowing $\mathbf{\Sigma}$

Quiz Question 1: Answer

Correct Answer: (B)

Yes, because all eigenvalues are inside the unit circle.

Proof/Calculation

Eigenvalues of $\mathbf{A}$: $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$$(0.8 - \lambda)(0.5 - \lambda) - 0.03 = \lambda^2 - 1.3\lambda + 0.37 = 0$$

$$\lambda = \frac{1.3 \pm \sqrt{1.69 - 1.48}}{2} = \frac{1.3 \pm 0.458}{2}$$

$$\lambda_1 = 0.879, \quad \lambda_2 = 0.421$$

Both $|\lambda_1| < 1$ and $|\lambda_2| < 1 \Rightarrow$ **Stable!**

Why (A) is Wrong

Diagonal elements alone don't determine stability. Off-diagonal elements affect eigenvalues.

Quiz Question 2

Question

If X Granger-causes Y at the 5% significance level, which of the following statements is TRUE?

- A. X is the economic cause of Y
- B. Past values of X contain useful information for predicting Y
- C. Y cannot Granger-cause X
- D. The correlation between X and Y is positive

Quiz Question 2: Answer

Correct Answer: (B)

Past values of X contain useful information for predicting Y.

Explanation

Granger causality is about **predictive content**, not true economic causation:

$$\mathbb{E}[Y_t | Y_{t-1}, \ldots, X_{t-1}, \ldots] \neq \mathbb{E}[Y_t | Y_{t-1}, \ldots]$$

Adding X's history improves prediction of Y.

Why Other Options are Wrong

- (A): Granger causality $\neq$ economic causality (could be spurious)
- (C): Bidirectional Granger causality is possible ($X \leftrightarrow Y$)
- (D): Direction of correlation is not implied by Granger test

Quiz Question 3

Question

In a VAR with Cholesky-identified IRFs, what does the ordering of variables determine?

- A. The magnitude of the impulse responses
- B. The speed at which shocks die out
- C. Which variables can respond contemporaneously to which shocks
- D. The number of lags in the VAR

Quiz Question 3: Answer

Correct Answer: (C)

Which variables can respond contemporaneously to which shocks.

Illustration with $K = 2$

Cholesky: $\mathbf{P} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & p_{22} \end{pmatrix}$

- Variable 1 ordered first: responds only to shock 1 at $t = 0$
- Variable 2 ordered second: responds to both shocks at $t = 0$

If we reverse order, the contemporaneous restrictions change!

Example

Ordering (GDP, Interest Rate): GDP doesn't respond to monetary shock immediately.
Ordering (Interest Rate, GDP): Interest rate doesn't respond to output shock immediately.

Quiz Question 4

Question

For a bivariate VAR(1), how many parameters need to be estimated (excluding the error covariance matrix)?

- A 4
- B 6
- C 8
- D 10

Quiz Question 4: Answer

Correct Answer: (B) 6 parameters

Detailed Count

VAR(1) with $K = 2$ variables:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\text{2 params}} + \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\text{4 params}} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

- Constant vector $\mathbf{c}$: $K = 2$ parameters
- Coefficient matrix $\mathbf{A}$: $K^2 = 4$ parameters
- Total: $K + K^2 = 2 + 4 = 6$ parameters

General Formula

VAR(p) with K variables: $K + pK^2$ parameters (excluding $\mathbf{\Sigma}$)

Quiz Question 5

Question

What does $\text{FEVD}_{12}(h) = 0.35$ mean?

- (A) 35% of variable 1's total variance is explained by variable 2
- (B) 35% of variable 1's h-step forecast error variance is due to shocks to variable 2
- (C) The correlation between variables 1 and 2 at lag h is 0.35
- (D) Variable 2 explains 35% of the impulse response of variable 1

Quiz Question 5: Answer

Correct Answer: (B)

35% of variable 1's h-step forecast error variance is due to shocks to variable 2.

FEVD Formula Reminder

$$\text{FEVD}_{ij}(h) = \frac{\sum_{s=0}^{h-1} [\boldsymbol{\Theta}_s]_{ij}^2}{\sum_{s=0}^{h-1} \sum_{k=1}^{K} [\boldsymbol{\Theta}_s]_{ik}^2}$$

Measures the proportion of variable i's forecast uncertainty at horizon h that is attributable to shocks originating from variable j.

Interpretation

If we're forecasting GDP h periods ahead, 35% of our forecast uncertainty comes from not knowing future monetary policy shocks.

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