



# Time Series Analysis and Forecasting

Chapter 0: Fundamentals



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## Learning Objectives

By the end of this chapter, you will be able to:

1. Define time series and distinguish them from cross-sectional and panel data
2. Decompose time series into trend-cycle, seasonality, and residual components
3. Apply exponential smoothing methods (SES, Holt, Holt-Winters, ETS)
4. Evaluate forecasts using MAE, RMSE, MAPE, sMAPE
5. Implement train/validation/test splitting and cross-validation
6. Model seasonality using dummy variables or Fourier terms
7. Remove trend and seasonality through appropriate methods
8. Distinguish between deterministic and stochastic trends



## Data Sources and Software Tools

### Data Sources

- **Yahoo Finance**
  - ▶ Stock prices, cryptocurrencies, exchange rates
- **FRED (Federal Reserve)**
  - ▶ GDP, unemployment, interest rates
- **Eurostat / INS / BNR**
  - ▶ European and Romanian economic data
- **Classic datasets**
  - ▶ AirPassengers, Sunspots, CO<sub>2</sub>

### Python

- `yfinance` — Yahoo Finance data
- `pandas_datareader` — FRED, Eurostat
- `statsmodels` — statistical models
- `pandas` — data manipulation
- `matplotlib` — visualization

### R

- `quantmod` — financial data
- `tseries` — time series tests
- `forecast` — forecasting models
- `fredr` — FRED API access



## Chapter Outline

- Motivation
- What Is a Time Series?
- Time Series Decomposition
- Exponential Smoothing Methods
- Forecast Evaluation
- Seasonality Modeling
- Handling Trend and Seasonality
- AI Use Case
- Summary



## Time Series Are Everywhere

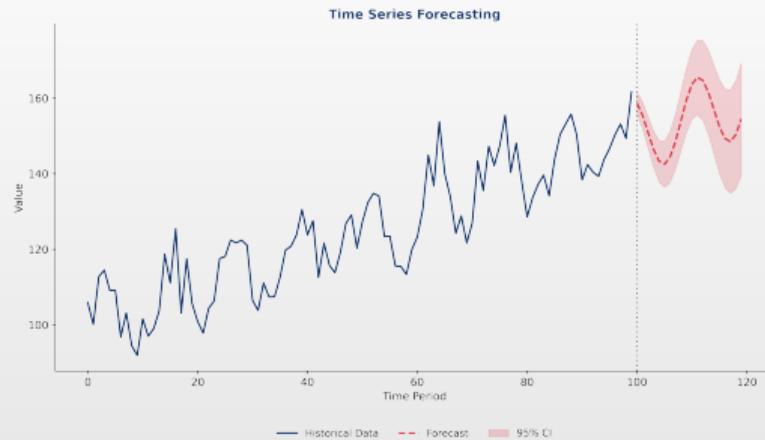


- **Finance:** Stock prices, exchange rates, volumes
- **Economics:** GDP, unemployment, inflation rates
- **Business:** Sales, website traffic, demand
- **Science:** Temperature, pollution, vital signs

Q TSA\_ch0\_real\_data



## Why Do We Study Time Series?



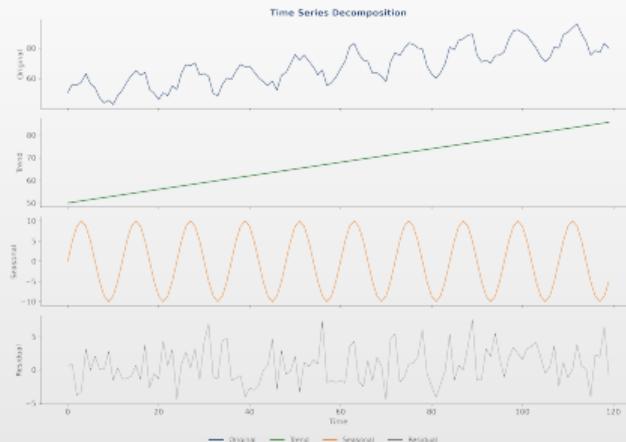
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Main objective: forecasting

- We use historical patterns to predict future values → essential for business planning, risk management, and policy decisions



## Understanding Time Series Structure



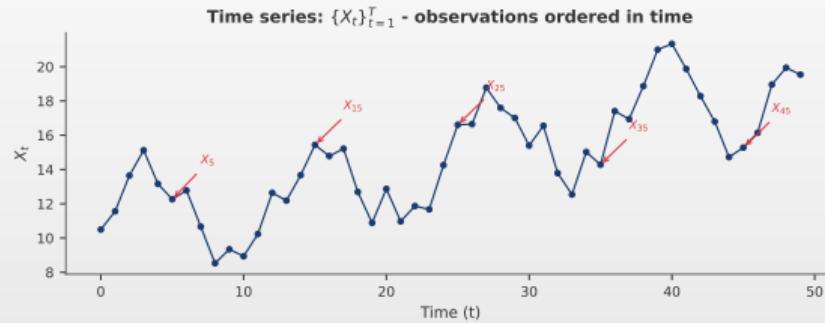
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### Decomposition

- Any time series can be decomposed into: **trend-cycle + seasonality + noise**



## Time Series: Conceptual Illustration



Q TSA\_ch0\_definition

### Fundamental Elements

- Formal notation:**  $X_t$  = value at time  $t$ ,  $t \in \{1, 2, \dots, T\}$
- Autocorrelation:**  $\rho_k = \text{Corr}(X_t, X_{t-k})$  — measures temporal dependence

## Definition of a Time Series

### Definition 1 (Time Series)

- **Time series:** a sequence of observations  $\{X_t\}$  indexed by time:  $\{X_t : t \in \mathcal{T}\}$  where  $\mathcal{T}$  is a set of indices representing time points

### Key Characteristics

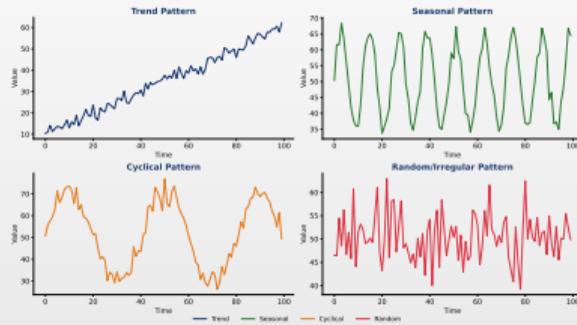
- **Ordered:** natural temporal order
- **Dependent:** consecutive observations are correlated
- **Discrete/Continuous:**  $t = 1, 2, 3, \dots$

### Notation

- $X_t$ : observation at time  $t$
- $\{X_t\}_{t=1}^T$ : series with  $T$  observations



## Common Patterns in Time Series



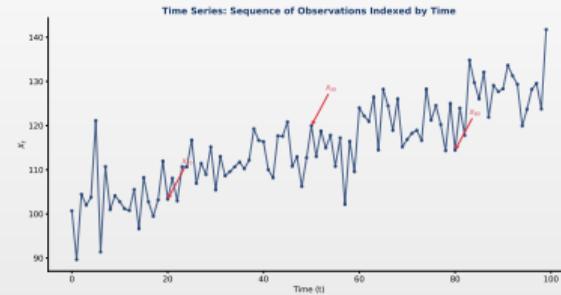
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### Types of Patterns

- Trend:** long-term increase or decrease
- Seasonal:** regular periodic patterns
- Cyclical:** medium-term fluctuations (2–10 years)
- Random:** unpredictable fluctuations



## Practical Example: Real Financial Data



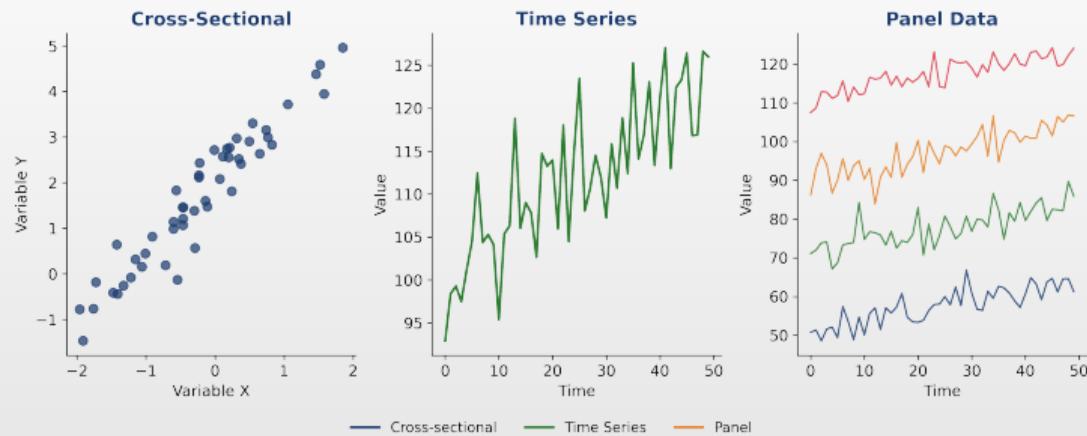
Q [TSA\\_ch0\\_definition](#)

### S&P 500 (2024)

- Daily frequency:**  $\approx 252$  trading days/year
- Observed characteristics:** upward trend, volatility clustering, persistence (momentum)



## Data Types: Comparison



Data Type	Units ( $N$ )	Time ( $T$ )	Example
Cross-sectional	Many	1	Survey of 1000 households
Time series	1	Many	Daily S&P 500 prices
Panel	Many	Many	GDP of 50 countries, 20 years

Q TSA\_cho\_definition



## Examples of Time Series Data



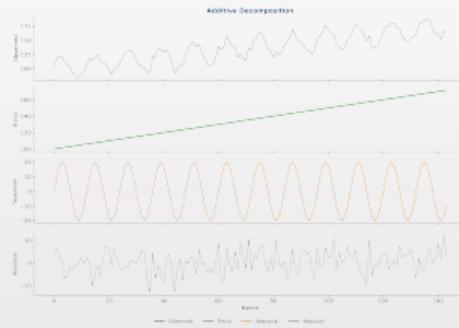
Q TSA\_ch0\_real\_data

### Real financial data

- Source: Yahoo Finance (2019–2025), normalized to base 100
- Bitcoin: most volatile; Gold: most stable



## Time Series Decomposition: Visual Example



Q TSA\_ch0\_decomposition

### Components Explained

- Original:** observed series
- Trend-Cycle:** long-term movement
- Seasonal:** periodic pattern
- Residual:** random noise



## Why Do We Decompose a Time Series?

### Objectives

- ◻ Understanding underlying patterns
- ◻ Removing seasonality for modeling
- ◻ Identifying trend direction
- ◻ Isolating irregular fluctuations
- ◻ Improving forecast accuracy

### Components

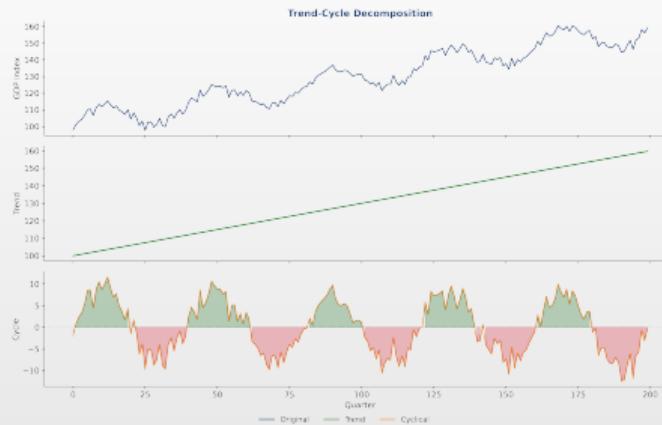
- ◻  $T_t$ : Trend-Cycle
  - ▶ Long-term movement
- ◻  $S_t$ : Seasonal
  - ▶ Regular periodic pattern
- ◻  $\varepsilon_t$ : Residual
  - ▶ Random noise

### Classical decomposition models

- ◻ **Additive:**  $X_t = T_t + S_t + \varepsilon_t$ 
  - ▶ Constant seasonal amplitude
- ◻ **Multiplicative:**  $X_t = T_t \times S_t \times \varepsilon_t$ 
  - ▶ Seasonal amplitude grows with the level



## The Cyclical Component



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### Characteristics

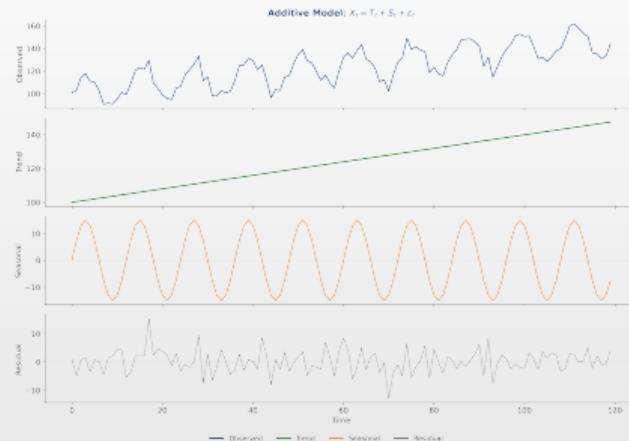
- **Duration:** medium-term fluctuations (2–10 years)
- **Aperiodic:** no fixed period (vs seasonality)
- **Origin:** reflects business cycles

### In Practice

- **Combination:** cycle combined with trend
- **Difficulty:** hard to identify in short series
- **Solution:** usually absorbed into trend-cycle



## Additive Decomposition: Visualization



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### Interpretation

- **Decomposition:** Original = Trend + Seasonal + Residual
- **Property:** constant seasonal amplitude, does not depend on the level



## The Additive Decomposition Model

### Model

- **Equation:**  $X_t = T_t + S_t + \varepsilon_t$ 
  - ▶ Components are added together to form the observed series

### When to Use

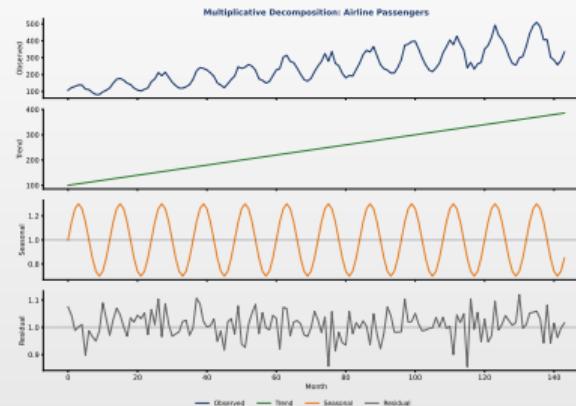
- **Constant seasonal fluctuations**
  - ▶ Amplitude does not depend on the level
- **Stable series variance**
  - ▶ Measures dispersion around the mean
  - ▶ Estimator:  $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$

### Properties

- **Error:**  $\mathbb{E}[\varepsilon_t] = 0$  (zero mean)
- **Seasonal:**  $\sum_{j=1}^s S_j = 0$  (seasonal sum is zero)
- **Units:**  $S_t$  are the same as  $X_t$



## Multiplicative Decomposition: Real Data



### Example

- Box-Jenkins data: monthly passengers (1949–1960). Seasonal amplitude increases with the level

 TSA\_ch0\_decomposition



## The Multiplicative Decomposition Model

### Model

- Equation:  $X_t = T_t \times S_t \times \varepsilon_t \rightarrow$  components are multiplied

### When to Use

- Growing fluctuations: seasonality increases with the level
- Heteroscedasticity: variance increases over time
- Examples: economic/financial data

### Properties

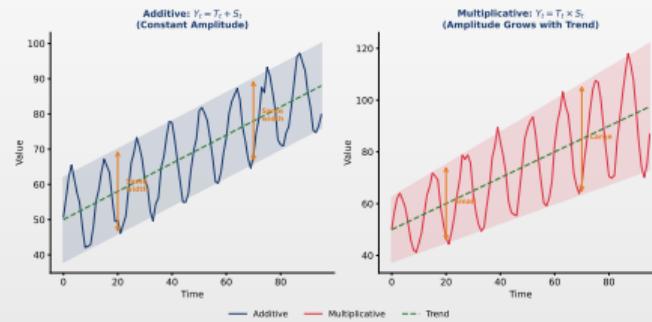
- Error:  $\mathbb{E}[\varepsilon_t] = 1$  (centered at 1)
- Seasonal:  $\frac{1}{s} \sum_{j=1}^s S_j = 1$  (mean is 1)
- Units:  $S_t$  is a dimensionless ratio

### Tip

- Log transformation: multiplicative  $\rightarrow$  additive:  $\log X_t = \log T_t + \log S_t + \log \varepsilon_t$



## Additive vs Multiplicative: Comparison



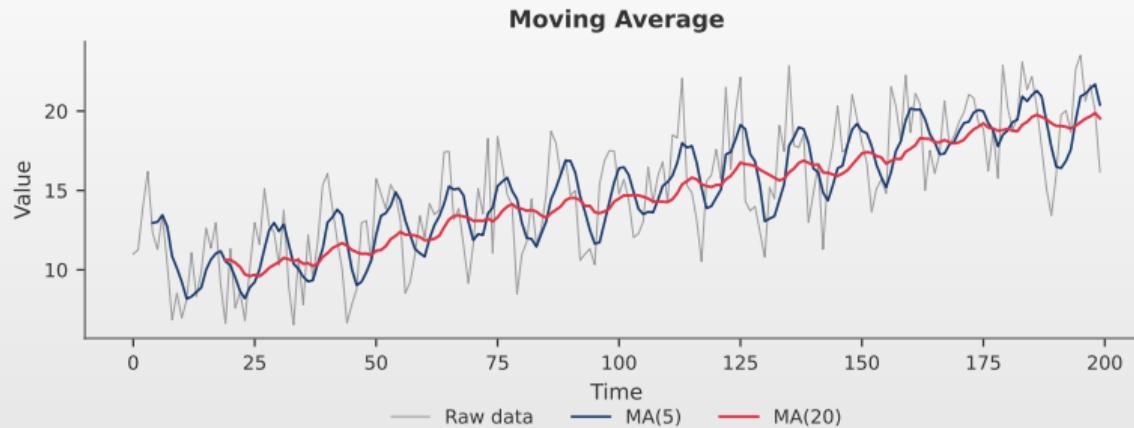
Q TSA\_ch0\_decomposition

### Key Difference

- **Multiplicative:** seasonal component is a *ratio*, centered at value 1
- **Additive:** seasonal component in *absolute units*, centered at value 0



## Centered Moving Average: Visual Illustration



### Interpretation

- Smoothing:** removes short-term fluctuations
- Result:** reveals the underlying trend



## Trend Estimation: Moving Average

### Definition 2 (Centered Moving Average)

- **Centered moving average** of order  $2q + 1$ :

$$\hat{T}_t = \frac{1}{2q+1} \sum_{j=-q}^q X_{t+j} \quad (1)$$

### For Seasonal Data

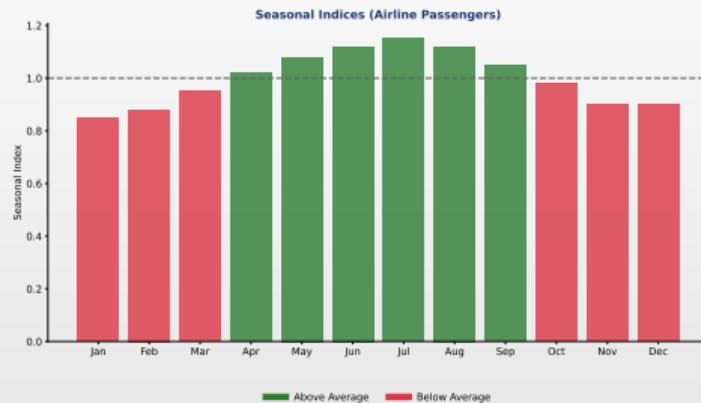
- **Odd period  $s$** 
  - ▶ Use simple average
- **Even period  $s$** 
  - ▶  $2 \times s$  MA with half weights

### Properties

- **Smoothing:** removes seasonal & random components
- **Large window** → smoother estimate
- **Disadvantage:** data loss at endpoints



## Seasonal Indices: Interpretation



### Interpretation

- $S_t > 1$ : above-average activity;  $S_t < 1$ : below average. Travel peak in July–August

TSA\_cho\_seasonal



## Classical Decomposition Algorithm

### Steps for Multiplicative Decomposition

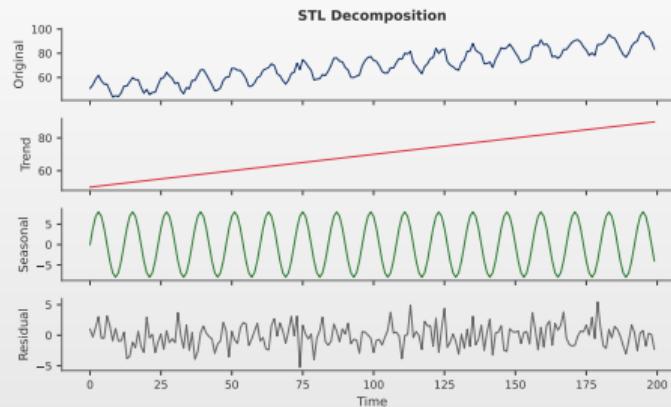
- **Step 1 → Estimate Trend:**  $\hat{T}_t = MA_s(X_t)$ 
  - ▶ Centered moving average of order equal to the seasonal period
- **Step 2 → Detrend:**  $D_t = X_t / \hat{T}_t$
- **Step 3 → Estimate Seasonal:**  $\hat{S}_j = \text{mean}(D_t \text{ for season } j)$
- **Step 4 → Normalize:** scale so that  $\frac{1}{s} \sum_{j=1}^s \hat{S}_j = 1$
- **Step 5 → Compute Residuals:**  $\hat{\varepsilon}_t = X_t / (\hat{T}_t \times \hat{S}_t)$

### Note

- **For additive decomposition:** operations change
  - ▶ Division → subtraction
  - ▶ Multiplication → addition



## STL Decomposition: Visual Illustration



### Key Idea

- ☐ STL (Seasonal-Trend-Loess): separates trend + seasonal + remainder using LOESS regression

 TSA\_ch0\_decomposition



## STL Decomposition: A Modern Approach

### Definition 3 (STL - Seasonal-Trend Decomposition using LOESS)

- **STL:** uses locally weighted regression (LOESS):  $X_t = T_t + S_t + R_t$

#### Advantages

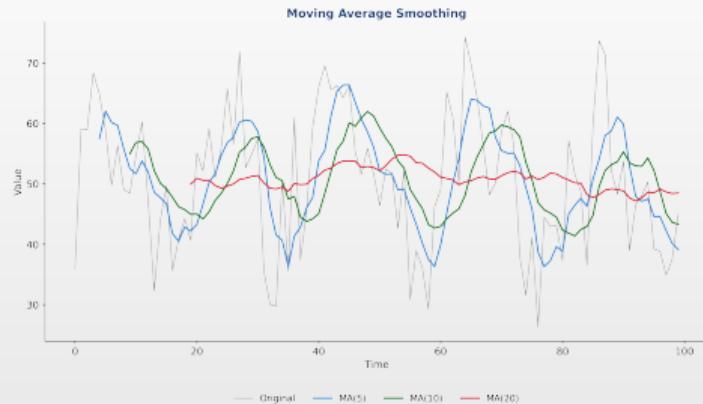
- **Flexibility:** any seasonal period
- **Variability:** seasonality can evolve over time
- **Robustness:** resistant to outliers
- **Smoothing:** smooth trend estimates

#### Key Parameters

- **period:** seasonal period
  - ▶ E.g.: 12 for monthly data, 4 for quarterly
- **seasonal:** smoothing window
- **robust:** reduced weight for outliers



## Moving Average Smoothing



### Window Size Trade-off

- Small window:** reactive but noisy
  - ▶ Captures rapid changes, but amplifies noise
- Large window:** smooth but lagging
  - ▶ Removes noise, but reacts slowly



## Exponential Smoothing: Overview

### Definition

- **Exponential smoothing:** weighted averages of past observations
  - ▶ Weights decrease exponentially over time
  - ▶ Recent observations receive higher weights

### Why Exponential Smoothing?

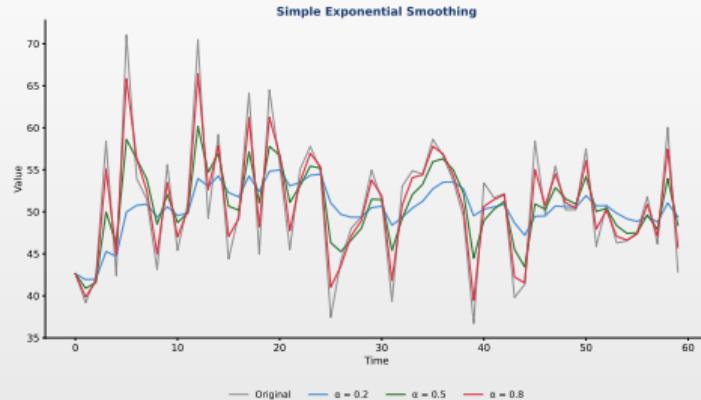
- **Simple:** easy to implement and understand
  - ▶ A single smoothing parameter
- **Adaptive:** higher weights for recent data
- **Versatile:** handles trend and seasonality

### Three Main Methods

- **SES** (Simple Exponential Smoothing): level only
  - ▶ The simplest exponential method
- **Holt**: level + trend
  - ▶ Captures the direction of evolution
- **Holt-Winters**: + seasonality
  - ▶ Complete model with all components



## Simple Exponential Smoothing: Effect of $\alpha$



### Trade-off

- Small  $\alpha \rightarrow$  smooth forecasts**
  - ▶ More weight on distant history
- Large  $\alpha \rightarrow$  tracks the data**
  - ▶ Fast reaction to recent changes



## Simple Exponential Smoothing (SES)

### Model

- **Equation:**  $\hat{X}_{t+1|t} = \alpha X_t + (1 - \alpha)\hat{X}_{t|t-1}$ 
  - ▶  $\alpha \in (0, 1)$  is the smoothing parameter

### How It Works

- **Principle:** weights decrease exponentially
- **Large  $\alpha$** 
  - ▶ Forecast reactive to changes
- **Small  $\alpha$** 
  - ▶ Smoother, more stable forecast

### Level Form

- **Equation:**  $\ell_t = \alpha X_t + (1 - \alpha)\ell_{t-1}$ 
  - ▶  $\ell_t$  = estimated level at time  $t$
  - ▶ Forecast:  $\hat{X}_{t+h|t} = \ell_t$  (constant)



## SES: Step-by-Step Numerical Example

Data: Monthly Sales (thousands EUR)

■ Data:  $X_1 = 100, X_2 = 110, X_3 = 105, X_4 = 115, X_5 = 120$  ( $\alpha = 0.3, \hat{X}_{1|0} = 100$ )

Iterative computation:  $\hat{X}_{t+1|t} = \alpha X_t + (1 - \alpha) \hat{X}_{t|t-1}$

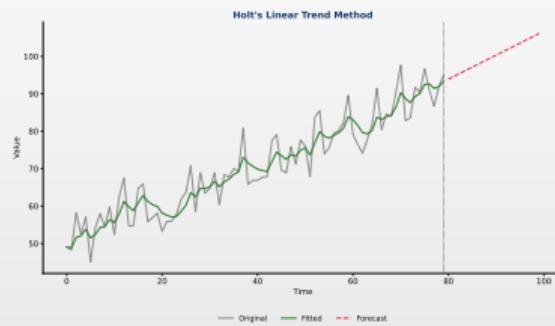
$t$	$X_t$	$\hat{X}_{t t-1}$	$e_t$	Computation $\hat{X}_{t+1 t}$
1	100	100.00	0.00	$0.3 \times 100 + 0.7 \times 100 = 100.00$
2	110	100.00	10.00	$0.3 \times 110 + 0.7 \times 100 = 103.00$
3	105	103.00	2.00	$0.3 \times 105 + 0.7 \times 103 = 103.60$
4	115	103.60	11.40	$0.3 \times 115 + 0.7 \times 103.6 = 107.02$
5	120	107.02	12.98	$0.3 \times 120 + 0.7 \times 107.02 = 110.91$

## Forecast and Evaluation

$\hat{X}_{6|5} = 110.91$  MAE = 7.28 RMSE = 8.97



## Holt's Method: Visualization



 TSA\_ch0\_smoothing

### Interpretation

- ◻ **Holt's method:** captures level and trend, projects them into the forecast horizon
- ◻  $\alpha$ : controls level changes;  $\beta^*$ : controls trend changes



## Holt's Linear Trend Method

### Equations

- Level:**  $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$
- Trend:**  $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$
- Forecast:**  $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t$ 
  - ▶ Extrapolates the linear trend over  $h$  steps

### Parameters

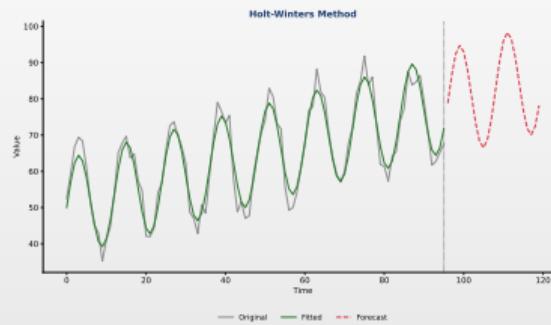
- $\alpha$ : level smoothing
  - ▶ Controls reactivity to level changes
- $\beta^*$ : trend smoothing
  - ▶ Controls reactivity to slope changes

### Components

- $\ell_t$ : estimated level
  - ▶ Local mean of the series
- $b_t$ : estimated trend (slope)
  - ▶ Rate of increase/decrease



## Holt-Winters: Capturing Seasonality



 TSA\_ch0\_smoothing

### Key Feature

- Complete decomposition:** separates level, trend, and seasonal
- Seasonal forecasts:** includes both trend and periodic pattern



## Holt-Winters Seasonal Method

### Equations (Additive Seasonality)

- **Level:**  $\ell_t = \alpha(X_t - S_{t-s}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})$
- **Trend:**  $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$
- **Seasonal:**  $S_t = \gamma(X_t - \ell_t) + (1 - \gamma)S_{t-s}$
- **Forecast:**  $\hat{X}_{t+h|t} = \ell_t + h \cdot b_t + S_{t+h-s(k+1)}$ 
  - ▶ Where  $k = \lfloor (h - 1)/s \rfloor$

### Parameters

- $\alpha$  — level
- $\beta^*$  — trend
- $\gamma$  — seasonal
- $s$  — seasonal period
  - ▶ All in  $(0, 1)$ ; estimated by minimizing error

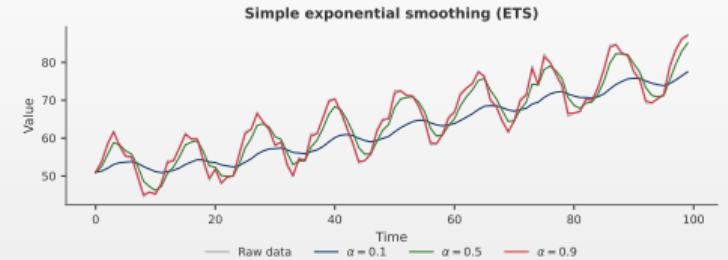


## The ETS Framework: Error-Trend-Seasonality

### Definition 4 (ETS Models)

- **ETS framework:** generalizes exponential smoothing:  
 $\text{ETS}(E, T, S)$

Component	N	A	M
Error (E)	–	Additive	Multiplicative
Trend (T)	None	Additive	Multiplicative
Seasonal (S)	None	Additive	Multiplicative



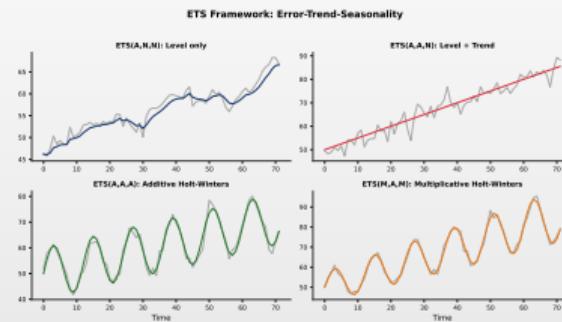
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### Examples

- **ETS(A,N,N):** Simple Exponential Smoothing → level only, no trend or seasonality
- **ETS(A,A,N):** Holt's Linear Method → level + additive trend
- **ETS(A,A,A):** Additive Holt-Winters → level + trend + additive seasonality



## ETS Model Selection



**TSA\_ch0\_smoothing**

### Automatic Selection

- Information criteria:** AIC (Akaike) and BIC (Bayesian)
- Optimal selection:** balance between fit and complexity



## Damped Trend Methods

### Damping Parameter

- **Parameter:**  $\phi \in (0, 1)$ 
  - ▶ Prevents over-projection of the trend
  - ▶ Trend converges to a constant

### Equations

- **Level:**  $\ell_t = \alpha X_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$
- **Trend:**  $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$
- **Forecast:**  $\hat{X}_{t+h|t} = \ell_t + \phi^{\frac{1-\phi^h}{1-\phi}} b_t$

### Key Idea

- **Asymptotic:** as  $h \rightarrow \infty$ , forecast  $\rightarrow$  constant
  - ▶ Prevents unrealistic long-term extrapolation
- **Advantage:** often better for long horizons



## Forecast Evaluation: Visual Example



Q TSA\_ch0\_forecast\_eval

### Observations

- Top:** actual vs. forecast — visual assessment of forecast quality
- Bottom:** residuals — zero mean, constant variance, no pattern



## Forecast Accuracy Metrics

### Forecast Error

- **Definition:**  $e_t = X_t - \hat{X}_t$  (actual minus predicted)
  - ▶ Positive  $\Rightarrow$  underestimates; Negative  $\Rightarrow$  overestimates

### Scale-dependent

- **MAE:**  $\frac{1}{n} \sum |e_t|$
- **MSE:**  $\frac{1}{n} \sum e_t^2$
- **RMSE:**  $\sqrt{\text{MSE}}$

### Scale-independent

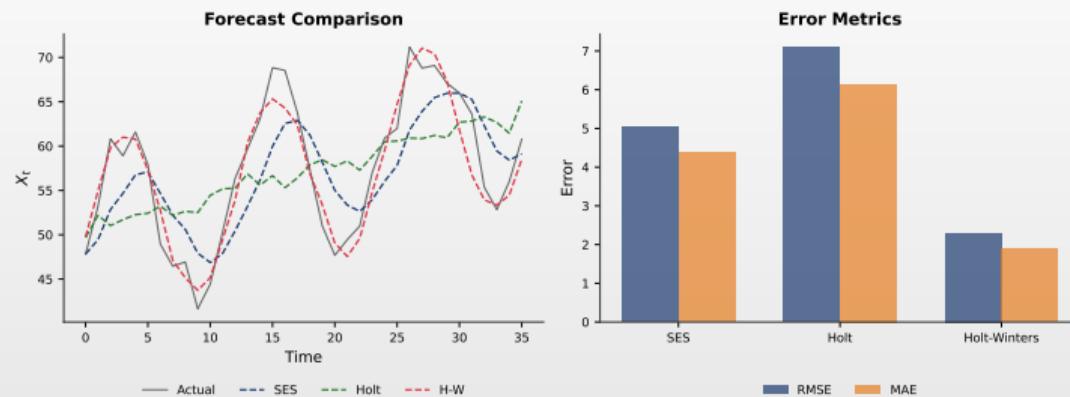
- **MAPE:**  $\frac{100}{n} \sum \left| \frac{e_t}{X_t} \right|$
- **sMAPE:**  $\frac{100}{n} \sum \frac{|e_t|}{(|X_t| + |\hat{X}_t|)/2}$

### What to Use?

- **Same series:** RMSE, MAE  $\rightarrow$  compare models on the same data
- **Across different series:** MAPE, sMAPE  $\rightarrow$  percentage metrics, scale-independent



## Comparing Forecast Methods



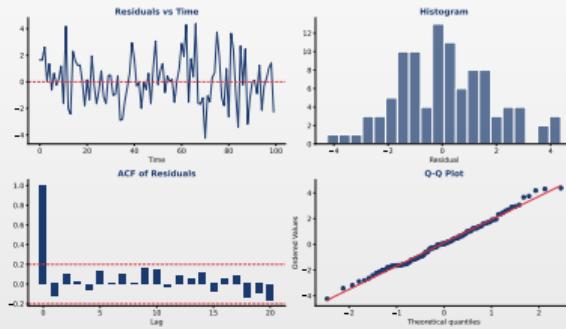
### Interpretation

- Left: SES, Holt, Holt-Winters forecasts. Right: Error metrics. Visual and quantitative comparison

`TSA_ch0_forecast_eval`



## Residual Diagnostics: Visualization



TSA\_ch0\_forecast\_eval

### What to Check

- Time plot:** no systematic patterns
- Histogram:** normality check
- ACF:** no significant autocorrelation
- Q-Q plot:** normality confirmation



## Residual Diagnostics

### Residual Properties

- Zero mean:**  $\mathbb{E}[e_t] = 0$ 
  - ▶ Forecast has no systematic bias
- Uncorrelated:**  $\text{Cov}(e_t, e_{t-k}) = 0$ 
  - ▶ No unexploited information remains
- Constant variance:**  $\text{Var}(e_t) = \sigma^2$
- Normally distributed:** for confidence intervals

### Diagnostic Tests

- Ljung-Box test (autocorrelation):**
  - ▶  $Q = T(T+2) \sum_{k=1}^h \frac{\hat{\rho}_k^2}{T-k} \sim \chi_h^2$
- Jarque-Bera test (normality):**
  - ▶  $JB = \frac{T}{6} \left( S^2 + \frac{(K-3)^2}{4} \right) \sim \chi_2^2$
  - ▶  $S = \text{skewness}, K = \text{kurtosis}$



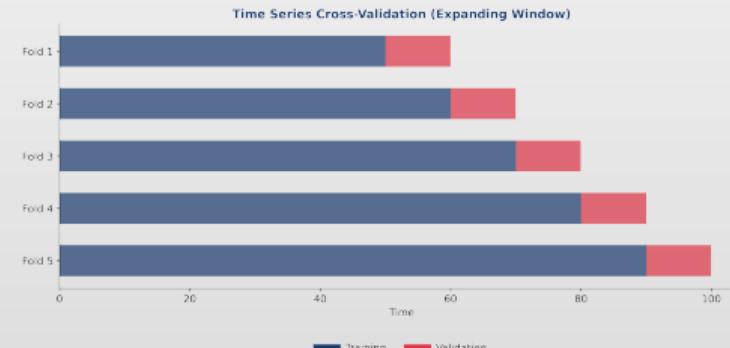
## Cross-Validation for Time Series

### Why Not Standard CV?

- Temporal dependence:** observations are correlated
- Order matters:** chronology must be respected
- Standard k-fold** → data leakage

### CV with Rolling Origin

- Step 1:** train on  $\{X_1, \dots, X_t\}$
- Step 2:** forecast  $\hat{X}_{t+h}$
- Step 3:** increment  $t$ , repeat



## Train / Validation / Test Split

### Training Set

- Fitting model parameters
- Largest portion (60–80%)
- Used for estimation

### Validation Set

- Hyperparameter tuning
- Comparing models
- Selecting the best approach

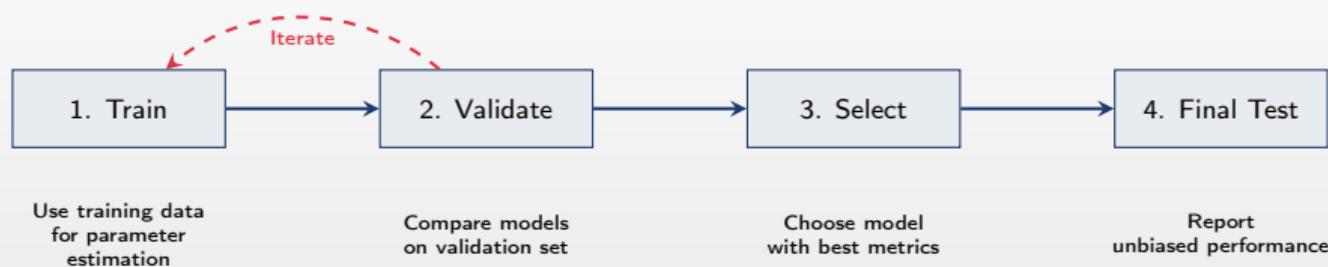
### Test Set

- Final evaluation only
- Never used for tuning
- Unbiased performance



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## Model Development Workflow

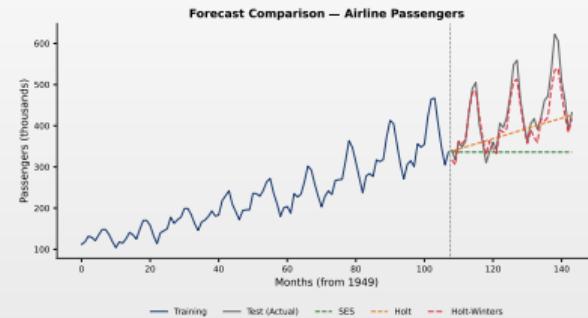


### Critical Rule

- Never use the test set for selection!**
  - ▶ Use it only for final evaluation
- Avoid data leakage**
  - ▶ Overly optimistic performance estimates



## Real Data: Comparing Forecasts



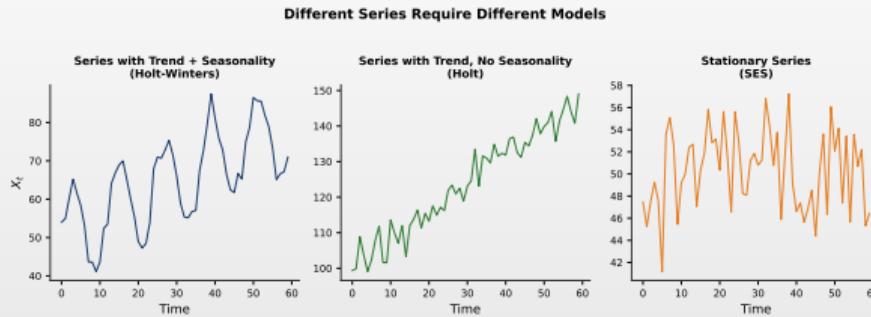
Q TSA\_ch0\_forecast\_eval

### Interpretation

- Data:** airline passengers
- Best:** multiplicative Holt-Winters — ideal for data with growing seasonality



## Forecast Performance Across Different Datasets



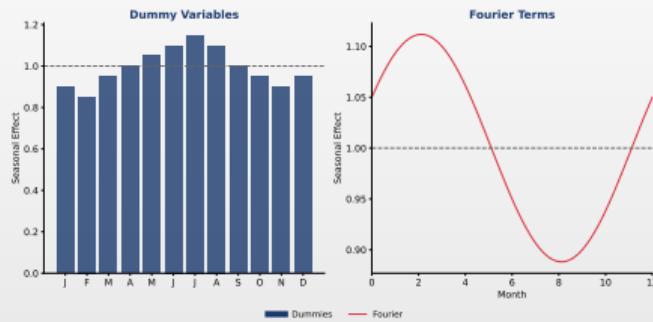
**TSA\_ch0\_forecast\_eval**

### Interpretation

- **Different series:** require different models
- **Seasonal data:** prefer seasonal methods
- **No universal model:** test multiple approaches



## Dummy Variables vs Fourier Terms



Q TSA\_ch0\_seasonal

### Comparison

- ◻ **Dummy variables:** capture any shape, require  $s - 1$  parameters
- ◻ **Fourier terms:** only  $2K$  parameters, smooth sinusoidal patterns



## Modeling Seasonality: Two Approaches

### 1. Dummy Variables

- ◻ **Model:**  $X_t = \mu + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$
- ◻  $D_{jt} = 1$  if  $t$  in season  $j$
- ◻  $s - 1$  parameters
- ◻ Any seasonal pattern

### 2. Fourier Terms

- ◻ **Model:**  
$$X_t = \mu + \sum_{k=1}^K \left[ \alpha_k \sin\left(\frac{2\pi k t}{s}\right) + \beta_k \cos\left(\frac{2\pi k t}{s}\right) \right]$$
- ◻ Sinusoidal functions
- ◻  $2K$  parameters
- ◻ Smooth patterns

### Trade-off

- ◻ **Dummy variables**
  - ▶ Any seasonal pattern, but more parameters
- ◻ **Fourier terms**
  - ▶ Smooth patterns, fewer parameters



## Choosing Between Dummy and Fourier

Criterion	Dummy	Fourier
Parameters (monthly)	11	$2K$ (often 4–6)
Seasonal pattern	Any shape	Smooth/sinusoidal
Interpretation	Direct (monthly effects)	Frequency components
High-frequency seasons	Many parameters	Efficient
Multiple seasonality	Complex	Easy (add terms)

### Recommendations

- Use Dummy**
  - ▶ Irregular patterns, interpretable coefficients
- Use Fourier**
  - ▶ Smooth patterns, high-frequency seasonality
  - ▶ Used in TBATS and Prophet



## Why Do We Remove Trend and Seasonality?

### Reasons for Detrending

- Stationarity requirement
- Focus on fluctuations
- Avoiding spurious regression
- Enabling valid inference

### Reasons for Deseasonalizing

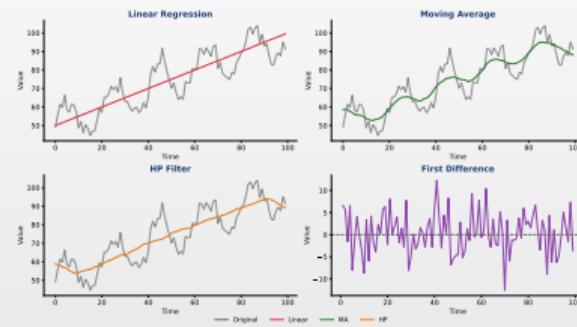
- Revealing the underlying trend
- Cross-season comparisons
- Simplifying modeling
- Focus on the irregular component

### Important

- **We model the transformed series**
  - ▶ With trend and seasonality removed
- **We reverse the transformation**
  - ▶ Bring the forecast back to the original scale



## Detrending Methods: Comparison



[TSA\\_ch0\\_detrending](#)

### Key Idea

- ☐ **Different methods:** produce different residuals
- ☐ **Choose by trend type:** consider the analysis objectives



## Detrending Methods

### Six Common Detrending Approaches

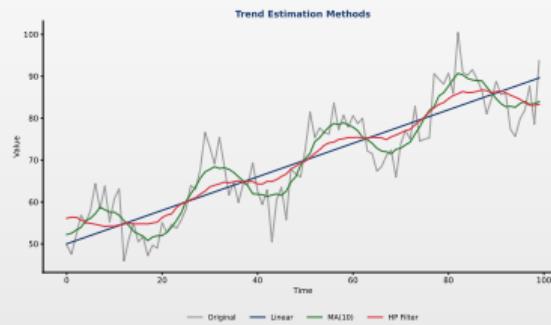
- Differencing:**  $\Delta X_t = X_t - X_{t-1}$ 
  - ▶ Most commonly used, removes stochastic trend
- Linear regression:**  $\hat{T}_t = \hat{\beta}_0 + \hat{\beta}_1 t$
- Polynomial:** higher-order polynomial
- HP filter:** balance between fit and smoothness
- Moving average:**  $\hat{T}_t = MA_q(X_t)$
- LOESS:** local polynomial regression

### The Choice Depends on

- Nature of the trend**
  - ▶ Deterministic vs stochastic
- Purpose of the analysis**
  - ▶ Forecasting vs descriptive analysis



## Trend Estimation: Multiple Approaches



 [TSA\\_ch0\\_detrending](#)

### Method Comparison

- Moving average:** simple but with lag
- Polynomial regression:** flexible, parametric
- HP filter:** macroeconomic standard



## HP Filter: Effect of $\lambda$



 TSA cho\_detrending

### Trade-off

- Small  $\lambda$ :** flexible trend, follows the data closely
- Large  $\lambda$ :** smooth trend, approaches a linear trend



## The Hodrick-Prescott (HP) Filter

### Definition 5 (HP Filter)

- **HP filter:** decomposes  $X_t$  into trend  $\tau_t$  and cycle  $c_t$ :  $X_t = \tau_t + c_t$

$$\min_{\{\tau_t\}} \left\{ \sum_{t=1}^T (X_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right\}$$

### Interpretation

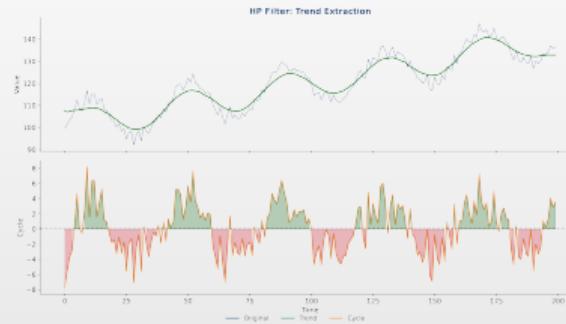
- **First term**
  - ▶ Goodness of fit
- **Second term**
  - ▶ Smoothness penalty
- $\lambda$ 
  - ▶ Controls the balance between fidelity and smoothness

### Standard $\lambda$ Values (Ravn-Uhlig)

- **Annual**
  - ▶  $\lambda = 6.25$
- **Quarterly**
  - ▶  $\lambda = 1600$  (macroeconomic standard)
- **Monthly**
  - ▶  $\lambda = 129600$



## HP Filter: Business Cycle Extraction



 **TSA cho\_detrending**

### Application

- Macroeconomics:** business cycle extraction
- Common series:** GDP, unemployment, inflation



## HP Filter: Limitations

### Known Issues

- Endpoint instability**
  - ▶ Trend estimates unreliable at the beginning and end
- Spurious cycles**
  - ▶ Can create artificial dynamics
- Choice of  $\lambda$** 
  - ▶ Results sensitive to the parameter

### Alternatives

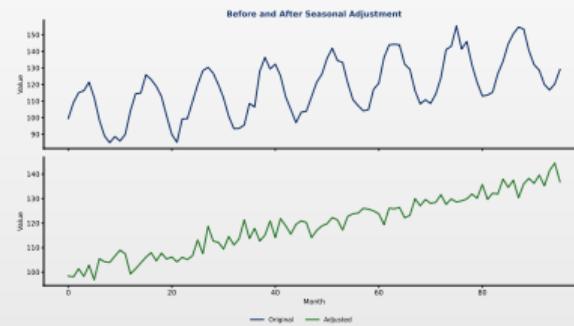
- Band-pass filters:** Baxter-King, Christiano-Fitzgerald
  - ▶ Isolate specific frequencies
- Hamilton filter:** regression-based
- Unobserved components:** state-space models

### Hamilton's Critique (2018)

- "Why You Should Never Use the Hodrick-Prescott Filter"
  - ▶ Suggests using regression on lagged values



## Seasonal Adjustment: Visualization



Q TSA\_ch0\_seasonal

### Result

- Seasonally adjusted series:** reveals the underlying trend, removes periodic fluctuations



## Seasonal Adjustment Methods

### Four Approaches for Seasonal Adjustment

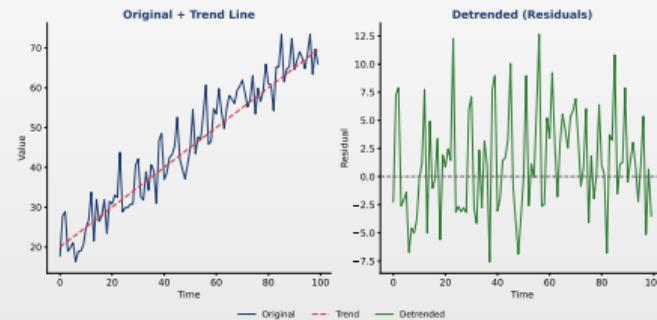
- Seasonal differencing:**  $\Delta_s X_t = X_t - X_{t-s}$ 
  - ▶ Removes periodic pattern, simple to apply
- Division** (multiplicative):  $X_t^{adj} = X_t / \hat{S}_t$
- Subtraction** (additive):  $X_t^{adj} = X_t - \hat{S}_t$
- X-13ARIMA-SEATS**: official US Census Bureau standard
  - ▶ Sophisticated method, used by statistical institutes

### Seasonal Period $s$

- Monthly:  $s = 12$  | Quarterly:  $s = 4$



## Example: Deterministic Trend



 TSA\_ch0\_detrending

### Key

- Method: regression
- Result: stationary residuals, ACF decays rapidly



## Deterministic vs Stochastic Trend

### Deterministic Trend

- Model:**  $X_t = \beta_0 + \beta_1 t + \varepsilon_t$
- Characteristics:**
  - ▶ Trend is a function of time
  - ▶  $\varepsilon_t$  is stationary
- Method:** detrend by regression

### Stochastic Trend

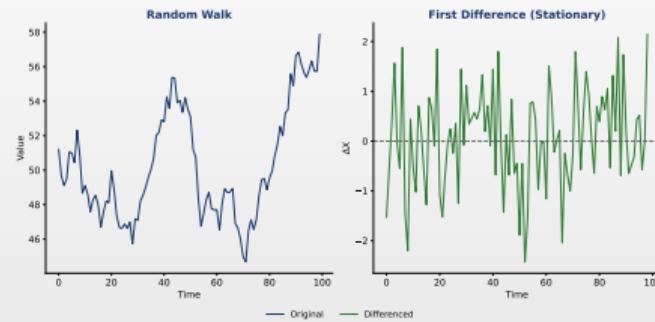
- Model:**  $X_t = X_{t-1} + \varepsilon_t$
- Characteristics:**
  - ▶ Random walk component
  - ▶  $\Delta X_t$  is stationary
- Method:** detrend by differencing

### Wrong Method = Problems

- Differencing a deterministic trend** → over-differencing
  - ▶ Introduces artificial dependence in the series
- Regression on a stochastic trend** → spurious regression
  - ▶ Invalid statistical results



## Example: Stochastic Trend (Random Walk)



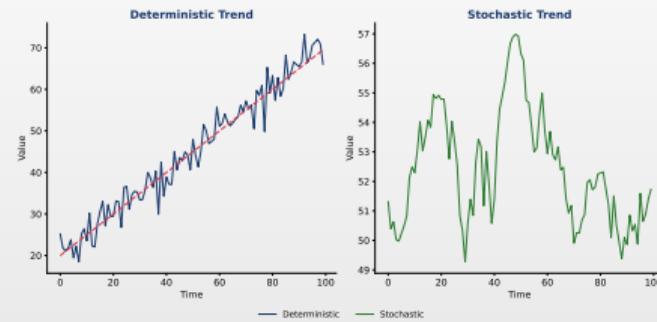
 [TSA\\_ch0\\_detrending](#)

### Key

- Method: differencing
- Result: differences are stationary (white noise)



## Side-by-Side Comparison



[TSA\\_ch0\\_detrending](#)

### Remember

- Deterministic trend:** use regression — trend is a predictable function of time
- Stochastic trend:** use differencing — trend contains a random component



## AI Exercise: Critical Thinking

Prompt to test in ChatGPT / Claude / Copilot

"Using yfinance, download AAPL stock data. I want to see what patterns exist and predict where the price will go next year. Make it look professional with charts."

**Exercise:**

1. Run the prompt in an LLM of your choice and critically analyze the response.
2. What type of decomposition does the model choose? Is it correct? Justify.
3. How does it evaluate forecast quality? Is the metric computed correctly?
4. Check the residuals — do they show unexplained structure?
5. Rewrite the analysis correctly and compare with a seasonal naïve benchmark.

**Warning:** AI-generated code may run without errors and look professional. *That does not mean it is correct.*



## Summary

### What We Learned in This Chapter

- Time Series Definition and Characteristics
  - ▶ Sequence of temporally ordered observations with dependence
- Decomposition (Additive vs Multiplicative)
  - ▶ Components: Trend-Cycle + Seasonal + Residual
- Exponential Smoothing Methods
  - ▶ SES (level), Holt (+ trend), Holt-Winters (+ seasonality), ETS
- Forecast Evaluation and Validation
  - ▶ Metrics: MAE, RMSE, MAPE; Cross-Validation with rolling origin

### Key Idea

- **Understand Before Modeling:**
  - ▶ Visualize and decompose the data first
  - ▶ Choose additive vs multiplicative based on variance behavior



## Quick Quiz

### Test Your Knowledge

1. What is the difference between additive and multiplicative decomposition?
2. When should you use Holt-Winters instead of SES?
3. Why can't we use standard k-fold CV for time series?
4. What does  $\alpha = 0.9$  mean in exponential smoothing?
5. How do you distinguish between a deterministic and stochastic trend?



## Quiz Answers

### Answers

1. **Additive vs multiplicative:** additive when seasonal amplitude is constant; multiplicative when it grows with the level
2. **Holt-Winters:** when data have trend AND seasonality; SES handles only the level
3. **CV:** standard k-fold ignores temporal order → data leakage
4.  $\alpha = 0.9$ : high weight on recent observations, reacts quickly but more volatile
5. **Trend:** deterministic → function of time (regression); stochastic → random walk (differencing)



## What's Next?

### Chapter 1: Stochastic Processes and Stationarity

- Stochastic Processes:** mathematical foundation, random variables indexed by time
- Stationarity:** strict (invariant distribution) vs weak (invariant moments)
- Fundamental Processes:** white noise and random walk → building blocks for ARIMA
- ACF and PACF:** tools for model identification

Questions?



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### Decomposition and Exploratory Analysis

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- Hyndman, R.J., & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*, 3rd ed., OTexts.

## Bibliography II

### Exponential Smoothing and ETS Fundamentals

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- Winters, P.R. (1960). Forecasting Sales by Exponentially Weighted Moving Averages, *Management Science*, 6(3), 324–342.
- Hyndman, R.J., Koehler, A.B., Ord, J.K., & Snyder, R.D. (2008). *Forecasting with Exponential Smoothing: The State Space Approach*, Springer.

### Online Resources and Code

- **Quantlet:** <https://quantlet.com> → Code repository for statistics
- **Quantinar:** <https://quantinar.com> → Learning platform for quantitative methods
- **GitHub TSA\_ch0:** [https://github.com/QuantLet/TSA/tree/main/TSA\\_ch0](https://github.com/QuantLet/TSA/tree/main/TSA_ch0)



# Thank You!

Questions?

Course materials available at: <https://danpele.github.io/Time-Series-Analysis/>

