



Seminar: Volatility Models

# ARCH, GARCH, EGARCH, TGARCH

Review Quiz and Practice Exercises



# Seminar Outline

1 Review Quiz

2 True/False Questions

3 Practice Problems

4 Python Workflow

5 Summary

## Question 1

What is “volatility clustering”?

- A Volatility is constant over time
- B Periods of high volatility tend to be followed by periods of high volatility
- C Returns are correlated over time
- D The return distribution is normal

*Think about the behavior of financial markets during crisis periods...*

## Answer to Question 1

Correct Answer: (B)

Periods of high volatility tend to be followed by periods of high volatility

### Explanation

- **Volatility clustering** is a stylized fact observed in financial time series
- “Turbulent” periods (with large movements) tend to persist
- “Calm” periods (with small movements) also tend to persist
- This implies that conditional variance  $\sigma_t^2$  is **predictable**
- GARCH models capture exactly this phenomenon!

## Question 2

In the GARCH(1,1) model:  $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$

What does the parameter  $\alpha$  represent?

- A Volatility persistence
- B Baseline volatility level
- C Reaction to recent shocks (news coefficient)
- D Unconditional variance

## Answer to Question 2

Correct Answer: (C)

Reaction to recent shocks (news coefficient)

Interpretation of GARCH(1,1) Parameters

- $\omega$  = baseline (floor) volatility level
- $\alpha$  = **reaction** to squared innovations ("news")
- $\beta$  = volatility **persistence** (memory)
- $\alpha + \beta$  = total persistence

A large  $\alpha$  means volatility reacts strongly to recent shocks.

## Question 3

What is the stationarity condition for GARCH(1,1)?

- A  $\omega > 0$
- B  $\alpha + \beta = 1$
- C  $\alpha + \beta < 1$
- D  $\alpha > \beta$

## Answer to Question 3

Correct Answer: (C)

$$\alpha + \beta < 1$$

### Complete Conditions

For GARCH(1,1) stationarity:

- $\omega > 0$  (ensures positive variance)
- $\alpha \geq 0, \beta \geq 0$  (non-negativity)
- $\alpha + \beta < 1$  (**strict stationarity**)

If  $\alpha + \beta = 1 \Rightarrow$  IGARCH (shocks have permanent effect)

## Question 4

What is the formula for unconditional variance in GARCH(1,1)?

- A  $\bar{\sigma}^2 = \omega$
- B  $\bar{\sigma}^2 = \frac{\omega}{1 - \alpha}$
- C  $\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$
- D  $\bar{\sigma}^2 = \frac{\omega}{\alpha + \beta}$

## Answer to Question 4

Correct Answer: (C)

$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$$

### Derivation

Taking unconditional expectation of GARCH(1,1):

$$\mathbb{E}[\sigma_t^2] = \omega + \alpha\mathbb{E}[\varepsilon_{t-1}^2] + \beta\mathbb{E}[\sigma_{t-1}^2]$$

$$\bar{\sigma}^2 = \omega + \alpha\bar{\sigma}^2 + \beta\bar{\sigma}^2$$

$$\bar{\sigma}^2(1 - \alpha - \beta) = \omega$$

$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$$

## Question 5

What is the “leverage effect”?

- A Positive shocks increase volatility more than negative shocks
- B Negative shocks increase volatility more than positive shocks
- C Volatility is independent of shock sign
- D Returns are asymmetric

## Answer to Question 5

Correct Answer: (B)

Negative shocks increase volatility more than positive shocks

### Explanation

- Empirically observed in stock markets
- When prices fall, firm leverage increases (debt becomes larger relative to equity)
- This makes the firm riskier  $\Rightarrow$  higher volatility
- Standard GARCH **cannot** capture this effect (depends on  $\varepsilon^2$ )
- Solutions: **EGARCH, GJR-GARCH, TGARCH**

## Question 6

In the EGARCH model, a negative  $\gamma$  parameter indicates:

- A Absence of leverage effect
- B Presence of leverage effect
- C Constant volatility
- D Non-stationary model

## Answer to Question 6

Correct Answer: (B)

Presence of leverage effect

EGARCH(1,1)

$$\ln(\sigma_t^2) = \omega + \alpha(|z_{t-1}| - \mathbb{E}[|z|]) + \gamma z_{t-1} + \beta \ln(\sigma_{t-1}^2)$$

- $\gamma < 0$ : negative shock ( $z < 0$ )  $\Rightarrow$  increases  $\ln(\sigma_t^2)$
- $\gamma > 0$ : inverse effect (less common)
- $\gamma = 0$ : symmetric effect (like GARCH)

## Question 7

What is the main advantage of EGARCH over GARCH?

- A Faster to estimate
- B No non-negativity constraints needed
- C Fewer parameters
- D Easier to interpret

## Answer to Question 7

Correct Answer: (B)

No non-negativity constraints needed

### EGARCH Advantages

- Models  $\ln(\sigma_t^2)$ , not  $\sigma_t^2$
- $\sigma_t^2 = e^{\ln(\sigma_t^2)} > 0$  automatically, regardless of parameter values
- GARCH requires  $\omega > 0, \alpha \geq 0, \beta \geq 0$
- During estimation, these constraints can cause convergence problems

## Question 8

Which test do we use to detect ARCH effects in residuals?

- A Dickey-Fuller test
- B Ljung-Box test on residuals
- C Engle's ARCH-LM test
- D Breusch-Pagan test

## Answer to Question 8

Correct Answer: (C)

Engle's ARCH-LM test

### ARCH-LM Test Procedure

- ① Estimate mean model, obtain residuals  $\hat{\varepsilon}_t$
- ② Compute  $\hat{\varepsilon}_t^2$
- ③ Regress:  $\hat{\varepsilon}_t^2 = \beta_0 + \beta_1 \hat{\varepsilon}_{t-1}^2 + \dots + \beta_q \hat{\varepsilon}_{t-q}^2 + u_t$
- ④ Test statistic:  $LM = T \cdot R^2 \sim \chi^2(q)$  under  $H_0$

$H_0$ : No ARCH effects     $H_1$ : ARCH effects present

## Question 9

For S&P 500, typical values of  $\alpha + \beta$  in GARCH(1,1) are:

- A 0.50 – 0.70
- B 0.70 – 0.85
- C 0.95 – 0.99
- D Greater than 1

## Answer to Question 9

Correct Answer: (C)

0.95 – 0.99

### Highly Persistent Volatility

- Financial time series exhibit very persistent volatility
- $\alpha + \beta \approx 0.98$  for S&P 500
- Half-life:  $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)} \approx 35 - 60$  days
- This means a volatility shock dissipates over several months

Series	$\alpha + \beta$
S&P 500	0.97–0.99
Bitcoin	0.90–0.98
EUR/USD	0.96–0.99

## Question 10

Which distribution is most commonly used for GARCH innovations to capture fat tails?

- A Normal
- B Uniform
- C Student-t
- D Exponential

## Answer to Question 10

Correct Answer: (C)

Student-t

### Innovation Distributions

- **Normal:** standard, but underestimates extreme risk
- **Student-t:** fat tails, parameter  $\nu$  (degrees of freedom)
- **GED:** Generalized Error Distribution, flexible
- **Skewed Student-t:** asymmetry + fat tails

For S&P 500:  $\nu \approx 5 - 8$  (significantly fatter tails than normal)

## True or False?

- ① ARIMA models can capture volatility clustering.
- ② In GARCH(1,1), if  $\alpha + \beta = 1$ , the model is called IGARCH.
- ③ GJR-GARCH uses an indicator variable for negative shocks.
- ④ GARCH volatility forecasts converge to zero in the long run.
- ⑤ EGARCH can have negative parameters without generating negative variance.
- ⑥ Value at Risk (VaR) can be calculated using GARCH volatility forecasts.

## True/False Answers

- ① **FALSE** — ARIMA assumes constant variance; GARCH models volatility.
- ② **TRUE** — IGARCH = Integrated GARCH, volatility has unit root.
- ③ **TRUE** —  $I_{t-1} = 1$  if  $\varepsilon_{t-1} < 0$ , else 0.
- ④ **FALSE** — Converges to unconditional variance  $\bar{\sigma}^2$ , not zero.
- ⑤ **TRUE** — Models  $\ln(\sigma_t^2)$ , so  $\sigma_t^2 = e^{\ln(\sigma_t^2)} > 0$  always.
- ⑥ **TRUE** —  $\text{VaR}_\alpha = z_\alpha \cdot \sigma_{t+1}$  (for zero mean).

## Problem 1: Calculating Unconditional Variance

### Statement

A GARCH(1,1) model has estimated parameters:

- $\omega = 0.000002$
- $\alpha = 0.08$
- $\beta = 0.90$

Calculate:

- a Daily unconditional variance
- b Daily unconditional volatility (as percentage)
- c Annualized volatility (assuming 252 trading days)
- d Volatility half-life

## Solution to Problem 1

### Answers

a)  $\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta} = \frac{0.000002}{1 - 0.08 - 0.90} = \frac{0.000002}{0.02} = 0.0001$

b)  $\bar{\sigma} = \sqrt{0.0001} = 0.01 = 1\% \text{ per day}$

c)  $\sigma_{\text{annual}} = \bar{\sigma} \times \sqrt{252} = 0.01 \times 15.87 = 15.87\% \text{ per year}$

d)  $HL = \frac{\ln(0.5)}{\ln(\alpha + \beta)} = \frac{\ln(0.5)}{\ln(0.98)} = \frac{-0.693}{-0.0202} \approx 34 \text{ days}$

### Interpretation

Volatility of 15.87% per year is typical for a stock index. Half-life of 34 days means a volatility shock is reduced by half after about 7 weeks.

## Problem 2: Volatility Forecast

### Statement

Using the GARCH(1,1) model from Problem 1:

- $\omega = 0.000002$ ,  $\alpha = 0.08$ ,  $\beta = 0.90$
- At time  $T$ :  $\varepsilon_T = -0.03$  (3% drop),  $\sigma_T^2 = 0.0004$

Calculate volatility forecasts for:

- a  $\sigma_{T+1}^2$  (one step ahead)
- b  $\sigma_{T+5}^2$  (five steps ahead)
- c  $\sigma_{T+100}^2$  (one hundred steps ahead)

## Solution to Problem 2

### Answers

a)  $\sigma_{T+1}^2 = \omega + \alpha \varepsilon_T^2 + \beta \sigma_T^2$

$$= 0.000002 + 0.08 \times (0.03)^2 + 0.90 \times 0.0004 = 0.000434$$

Volatility:  $\sqrt{0.000434} = 2.08\%$

b)  $\mathbb{E}_T[\sigma_{T+5}^2] = \bar{\sigma}^2 + (0.98)^4(\sigma_{T+1}^2 - \bar{\sigma}^2)$

$$= 0.0001 + 0.922 \times (0.000434 - 0.0001) = 0.000408$$

Volatility:  $\sqrt{0.000408} = 2.02\%$

c)  $\mathbb{E}_T[\sigma_{T+100}^2] = 0.0001 + (0.98)^{99} \times 0.000334 \approx 0.000145$

Volatility:  $\sqrt{0.000145} = 1.20\%$  (close to  $\bar{\sigma} = 1\%$ )

## Problem 3: Value at Risk

### Statement

A portfolio of 1,000,000 EUR is invested in stocks with returns modeled by GARCH(1,1).

Tomorrow's volatility forecast:  $\sigma_{T+1} = 2\%$  daily.

Assuming normally distributed returns with zero mean, calculate:

- a) VaR at 95% (1 day)
- b) VaR at 99% (1 day)
- c) VaR at 99% (10 days), using "square root of time" rule

Quantiles:  $z_{0.05} = 1.645$ ,  $z_{0.01} = 2.326$

## Solution to Problem 3

### Answers

a) VaR 95% (1 day):

$$\text{VaR}_{95\%} = 1.645 \times 0.02 \times 1,000,000 = 32,900 \text{ EUR}$$

b) VaR 99% (1 day):

$$\text{VaR}_{99\%} = 2.326 \times 0.02 \times 1,000,000 = 46,520 \text{ EUR}$$

c) VaR 99% (10 days):

$$\text{VaR}_{99\%,10d} = \text{VaR}_{99\%,1d} \times \sqrt{10} = 46,520 \times 3.162 = 147,100 \text{ EUR}$$

### Caution

In practice, for Student-t distribution, quantiles are larger (fatter tails)!

## Problem 4: Model Identification

### Statement

Analyze the following estimation results and identify the model:

Parameter	Estimate	Std. Error
$\omega$	0.0000015	0.0000005
$\alpha$	0.0550	0.0120
$\gamma$	0.0850	0.0180
$\beta$	0.9100	0.0150

- a) What model is this?
- b) Is leverage effect present?
- c) What is the impact of negative vs positive shocks?
- d) Is the model stationary?

## Solution to Problem 4

### Answers

- a) GJR-GARCH(1,1,1) — presence of  $\gamma$  parameter (threshold/asymmetry)
- b) Yes, leverage effect present:  $\gamma = 0.085 > 0$  and significant
- c) Impact:
  - Positive shock: impact =  $\alpha = 0.055$
  - Negative shock: impact =  $\alpha + \gamma = 0.055 + 0.085 = 0.140$
  - Negative shocks have **2.5x greater** impact!
- d) Stationarity:  $\alpha + \gamma/2 + \beta = 0.055 + 0.0425 + 0.91 = 1.0075$   
**On the edge!** Almost IGARCH. Very persistent model.

## Step 1: Load and Prepare Data

```
import pandas as pd
import numpy as np
import yfinance as yf
from arch import arch_model
from arch.unitroot import ADF

# Download S&P 500 data
data = yf.download('^GSPC', start='2010-01-01', end='2024-01-01')
returns = 100 * data['Adj Close'].pct_change().dropna()

# Check stationarity
adf = ADF(returns)
print(f'ADF statistic: {adf.stat:.4f}')
print(f'p-value: {adf.pvalue:.4f}')
```

## Step 2: Test for ARCH Effects

```
from statsmodels.stats.diagnostic import het_arch

# ARCH-LM test on residuals
residuals = returns - returns.mean()
lm_stat, lm_pvalue, f_stat, f_pvalue = het_arch(residuals, nlags=10)

print(f'ARCH-LM statistic: {lm_stat:.4f}')
print(f'p-value: {lm_pvalue:.4f}')

if lm_pvalue < 0.05:
    print('=> ARCH effects present! GARCH modeling justified.')
```

## Step 3: Estimate Models

```
# GARCH(1,1) with Student-t distribution
model_garch = arch_model(returns, vol='Garch', p=1, q=1, dist='t')
res_garch = model_garch.fit(disp='off')
print(res_garch.summary())

# GJR-GARCH(1,1,1)
model_gjr = arch_model(returns, vol='Garch', p=1, o=1, q=1, dist='t')
res_gjr = model_gjr.fit(disp='off')

# EGARCH(1,1)
model_egarch = arch_model(returns, vol='EGARCH', p=1, q=1, dist='t')
res_egarch = model_egarch.fit(disp='off')

# Compare AIC
print(f'GARCH AIC: {res_garch.aic:.2f}')
print(f'GJR AIC: {res_gjr.aic:.2f}')
print(f'EGARCH AIC: {res_egarch.aic:.2f}')
```

## Step 4: Diagnostics

```
# Standardized residuals
std_resid = res_gjr.std_resid

# Ljung-Box test on squared residuals
from statsmodels.stats.diagnostic import acorr_ljungbox
lb_test = acorr_ljungbox(std_resid**2, lags=10, return_df=True)
print(lb_test)

# Check for remaining ARCH effects
lm_stat2, lm_pval2, _, _ = het_arch(std_resid, nlags=5)
print(f'ARCH-LM residuals: stat={lm_stat2:.2f}, p={lm_pval2:.4f}')

if lm_pval2 > 0.05:
    print('=> No remaining ARCH effects. Model OK!')
```

## Step 5: Forecast and VaR

```
# Forecast 10 days ahead
forecasts = res_gjr.forecast(horizon=10)
vol_forecast = np.sqrt(forecasts.variance.values[-1, :])

print('Volatility forecast (%):', vol_forecast)

# Value at Risk 99%
portfolio_value = 1_000_000
VaR_99 = 2.326 * vol_forecast[0] / 100 * portfolio_value
print(f'VaR 99% (1 day): {VaR_99:.0f} EUR')

# 10-day VaR
VaR_99_10d = VaR_99 * np.sqrt(10)
print(f'VaR 99% (10 days): {VaR_99_10d:.0f} EUR')
```

## Key Concepts

- **ARCH:** conditional variance depends on past shocks
- **GARCH:** adds persistence through lagged variance
- **EGARCH/GJR:** capture leverage effect (asymmetry)
- **Stationarity:**  $\alpha + \beta < 1$

## Important Formulas

- Unconditional variance:  $\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$
- Half-life:  $HL = \frac{\ln(0.5)}{\ln(\alpha+\beta)}$
- VaR:  $VaR_\alpha = z_\alpha \cdot \sigma \cdot V$

## Practical Tip

Use Student-t distribution to capture fat tails. Verify absence of ARCH effects in residuals!

# Homework Exercises

## Exercise 1

Download daily returns for BET (BVB index) and estimate a GARCH(1,1) model. Compare persistence ( $\alpha + \beta$ ) with S&P 500.

## Exercise 2

For Bitcoin, estimate GARCH, EGARCH, and GJR-GARCH. Is leverage effect present for cryptocurrencies?

## Exercise 3

Calculate daily VaR for a portfolio of 100,000 EUR invested in EUR/USD, using GARCH-forecasted volatility.

## Exercise 4

Compare GARCH(1,1) volatility forecast with realized volatility (sum of squared returns) for a 20-day period.