Time Series Analysis and Forecasting

Chapter 7: Cointegration & VECM

Seminar

Seminar Outline

Quiz 1: Cointegration Definition

Question

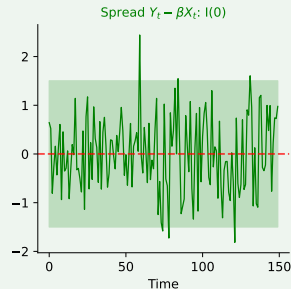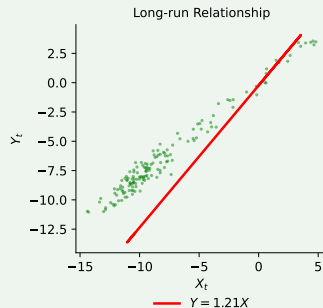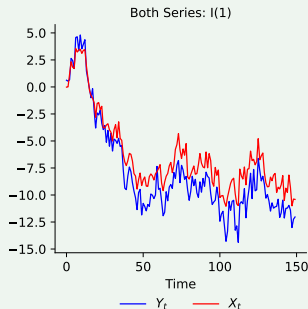Two I(1) variables X_t and Y_t are cointegrated if:

- They are both stationary
- Their sum is I(2)
- A linear combination of them is I(0)
- They have the same mean

Answer on next slide...

Answer: C – A linear combination is I(0)

Key: $Y_t - \beta X_t \sim I(0)$ means they share a common stochastic trend. The linear combination (spread) is stationary even though both series are non-stationary.

Quiz 2: Spurious Regression

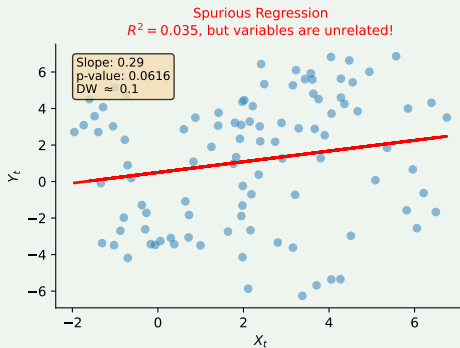### Question

When regressing one independent random walk on another, you typically get:

- Low R^2 and insignificant coefficients
- High R^2 and significant coefficients (spurious!)
- Zero coefficients
- Undefined results

Answer on next slide...

Answer: B – High R^2 and significant coefficients (spurious!)

Two INDEPENDENT Random Walks

- Y_t (Random Walk 1)
- X_t (Random Walk 2)

Spurious Regression
$R^2 = 0.035$, but variables are unrelated!

Slope: 0.29
p-value: 0.0616
DW ≈ 0.1
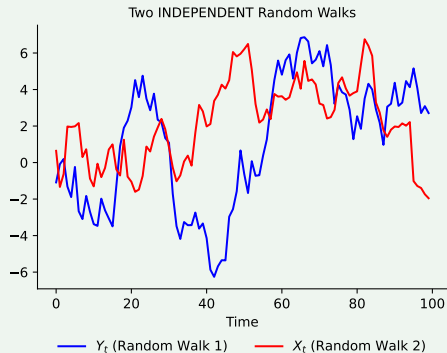
Granger-Newbold (1974): Regressing unrelated I(1) series gives misleading results. Rule of thumb: If $R^2 > DW$, suspect spurious regression!

Quiz 3: Engle-Granger Test

Question

In the Engle-Granger two-step method, what do you test in step 2?

- Whether the original variables are stationary
- Whether the regression residuals have a unit root
- Whether the coefficients are significant
- Whether the R^2 is high enough

Answer on next slide...

Quiz 3: Answer

Answer: B – Whether residuals have unit root

Step 1: Run OLS: $Y_t = \alpha + \beta X_t + e_t$, save residuals $\hat{e}_t$

Step 2: ADF test on residuals: $\Delta \hat{e}_t = \rho \hat{e}_{t-1} + \ldots$

- H_0: $\rho = 0$ (unit root $\Rightarrow$ no cointegration)
- H_1: $\rho < 0$ (stationary $\Rightarrow$ cointegration!)

Important: Use Engle-Granger critical values, not standard ADF!

Quiz 4: Johansen Test Advantage

Question
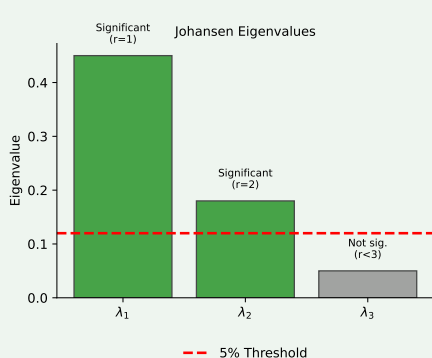
The main advantage of Johansen over Engle-Granger is:

- It's simpler to compute
- It can detect multiple cointegrating relationships
- It doesn't require data
- It always finds cointegration

Answer on next slide...

Answer: B – Can detect multiple cointegrating relationships

Johansen vs Engle-Granger

Johansen Eigenvalues

Method	Engle-Granger	Johansen
# of CI vectors	1 only	Multiple
Dep. variable	Required	Not needed
Estimation	Two-step	MLE
Efficiency	Lower	Higher

-- -- 5% Threshold

Johansen advantages:

- Tests for $r = 0, 1, 2, \ldots, k - 1$ cointegrating vectors
- Maximum likelihood (more efficient)
- No need to choose dependent variable

Question

In a VECM with $k = 3$ variables, if rank($\mathbf{\Pi}$) = 2, this means:

- No cointegration
- One cointegrating relationship
- Two cointegrating relationships
- All variables are stationary

Answer on next slide...

Quiz 5: Answer

Answer: C – Two cointegrating relationships

Rank interpretation for k variables:

- rank$(\mathbf{\Pi}) = 0$: No cointegration (use VAR in differences)
- $0 < \text{rank}(\mathbf{\Pi}) = r < k$: r cointegrating vectors (use VECM)
- rank$(\mathbf{\Pi}) = k$: All variables are I(0) (use VAR in levels)

With $k = 3$ and $r = 2$:

- Two equilibrium relationships
- Only $k - r = 1$ common stochastic trend

Question

In the VECM equation $\Delta\mathbf{Y}_t = \mathbf{c} + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{Y}_{t-1} + \ldots$, what does $\boldsymbol{\alpha}$ represent?

- The cointegrating vectors
- The adjustment (loading) coefficients
- The short-run dynamics
- The error variance

Answer on next slide...

Answer: B – The adjustment (loading) coefficients

VECM: Both Variables Adjust

Error Correction Mechanism

Y_1 adjusts down $(\alpha_1 < 0)$

Long-run: $Y_1 = \beta Y_2$

Y_2 adjusts up $(\alpha_2 > 0)$

— Y_1 ($\alpha_1 = -0.3$)   — Y_2 ($\alpha_2 = +0.1$)

$\mathbf{\Pi} = \boldsymbol{\alpha\beta}'$:
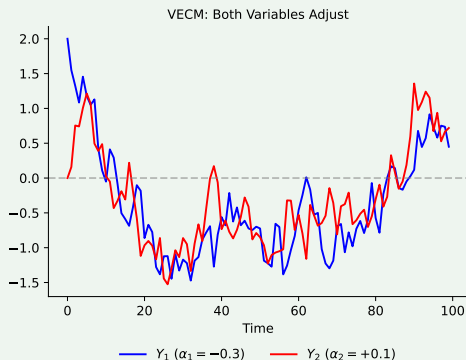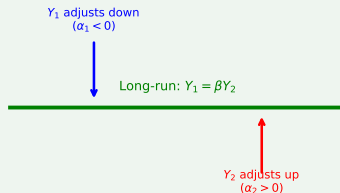
- $\boldsymbol{\beta}$ = cointegrating vectors (define equilibrium)
- $\boldsymbol{\alpha}$ = adjustment speeds (how fast each variable corrects)

Question

If $Y_t - \beta X_t$ is the cointegrating relation and this term is positive, what happens?

- Y is above equilibrium; Y should decrease (if $\alpha < 0$)
- Y is below equilibrium; Y should increase
- Nothing, error correction doesn't affect levels
- Both variables increase

Answer on next slide...

Quiz 7: Answer

Error correction mechanism:

$$\Delta Y_t = \alpha(Y_{t-1} - \beta X_{t-1}) + \ldots$$

- If $Y_{t-1} - \beta X_{t-1} > 0$: Y is "too high"
- With $\alpha < 0$: $\Delta Y_t < 0$ (Y decreases toward equilibrium)
- This is the "error correction" pulling Y back

Sign convention: α should be negative for the dependent variable to move back toward equilibrium.

Quiz 8: Weak Exogeneity

Question

If $\alpha_2 = 0$ in a bivariate VECM, this means:

- There is no cointegration
- Variable 2 does not adjust to disequilibrium (weakly exogenous)
- Variable 1 does not adjust
- Both variables are stationary

Answer on next slide...

Quiz 8: Answer

Answer: B – Variable 2 is weakly exogenous

Weak exogeneity: Variable doesn't respond to disequilibrium.

Example: Interest rates
- Long rate (R_t) often weakly exogenous $(\alpha_R \approx 0)$
- Short rate (r_t) adjusts to spread $(\alpha_r < 0)$
- Interpretation: Central bank adjusts short rate to maintain term structure

Implication: Can estimate single equation for the adjusting variable.

Quiz 9: Trace Test

Question

The Johansen trace test with $H_0 : r \leq 1$ vs $H_1 : r > 1$ tests whether:

- There is exactly one cointegrating vector
- There are at most one cointegrating vectors
- There are more than one cointegrating vectors
- All eigenvalues are zero

Answer on next slide...

Quiz 9: Answer

Answer: B/C – H_0: at most 1; H_1: more than 1

Sequential testing procedure:

1. Test $H_0 : r = 0$ vs $H_1 : r > 0$
2. If rejected, test $H_0 : r \leq 1$ vs $H_1 : r > 1$
3. Continue until fail to reject...

Trace statistic:

$$\lambda_{\text{trace}}(r) = -T \sum_{i=r+1}^{k} \ln(1 - \hat{\lambda}_i)$$

Reject H_0 if trace statistic $>$ critical value.

Quiz 10: VECM vs VAR in Differences

Question

If variables are cointegrated, using VAR in first differences instead of VECM:

- Gives identical results
- Is more efficient
- Loses long-run information (misspecified)
- Is the preferred approach

Answer on next slide...

Quiz 10: Answer

Answer: C – Loses long-run information

Granger Representation Theorem: If cointegrated, VECM representation exists and should be used.

	VAR(Δ)	VECM
Long-run equilibrium	Lost	Preserved
Error correction	No	Yes
Forecasts (long-run)	Poor	Better

Bottom line: Differencing removes the long-run relationship that cointegration represents!

True/False Questions

Determine if each statement is True or False:

1. Cointegration requires all variables to be I(1).
2. The cointegrating vector is unique.
3. Spurious regression has low Durbin-Watson statistic.
4. In VECM, both α coefficients must be non-zero.
5. Johansen test requires choosing a dependent variable.
6. The number of common trends $= k - r$.

Answers on next slide...

True/False: Solutions

1. Cointegration requires all variables to be I(1).   **TRUE**

   Standard CI(1,1) case: all variables I(1), linear combination I(0).

2. The cointegrating vector is unique.   **FALSE**

   Unique only up to scalar multiplication. Usually normalized ($\beta_1 = 1$).

3. Spurious regression has low Durbin-Watson statistic.   **TRUE**

   $DW \approx 0$ indicates highly autocorrelated residuals (non-stationary).

4. In VECM, both α coefficients must be non-zero.   **FALSE**

   One can be zero (weak exogeneity). At least one must be non-zero.

5. Johansen test requires choosing a dependent variable.   **FALSE**

   That's Engle-Granger. Johansen treats all variables symmetrically.

6. The number of common trends $= k - r$.   **TRUE**

   k variables, r cointegrating relations $\Rightarrow k - r$ common stochastic trends.

Problem 1: Cointegration Identification

Exercise

You have quarterly data on consumption (C_t) and income (Y_t). ADF tests show both are I(1). The regression $C_t = 0.85 Y_t + e_t$ gives residuals with ADF statistic $= -3.92$. The 5% Engle-Granger critical value for 2 variables is -3.34.

Are C_t and Y_t cointegrated?

Answer on next slide...

Problem 1: Solution

Solution: Yes, they are cointegrated

Test: H_0: No cointegration (residuals have unit root)

ADF statistic: -3.92
Critical value (5%): -3.34

Since $-3.92 < -3.34$, we **reject** H_0 at 5% level.

Conclusion: Residuals are stationary $\Rightarrow$ Cointegration exists!

Interpretation: Consumption and income share a common trend. The cointegrating vector is approximately $(1, -0.85)$, consistent with permanent income hypothesis.

Problem 2: VECM Interpretation

Exercise

A VECM for short rate (r_t) and long rate (R_t) gives:

$$\Delta r_t = 0.01 - 0.25(r_{t-1} - R_{t-1}) + \ldots$$
$$\Delta R_t = 0.005 - 0.02(r_{t-1} - R_{t-1}) + \ldots$$

Interpret the adjustment coefficients.

Answer on next slide...

Problem 2: Solution

Solution

Error correction term: $(r_{t-1} - R_{t-1})$ = spread

Short rate ($\alpha_r = -0.25$):

- When spread is positive (short > long), short rate decreases
- 25% of disequilibrium corrected per period
- Short rate actively adjusts

Long rate ($\alpha_R = -0.02$):

- Very small adjustment coefficient
- Long rate is nearly weakly exogenous
- Mostly driven by expectations, not error correction

Economic interpretation: Central bank (short rate) adjusts to maintain yield curve.

Problem 3: Johansen Test Results

Exercise

Johansen trace test for 3 variables gives:

H_0	Trace Stat	5% CV
$r = 0$	45.2	29.8
$r \leq 1$	18.1	15.5
$r \leq 2$	3.2	3.8

What is the cointegrating rank?

Answer on next slide...

Problem 3: Solution

Solution: Rank = 2

Sequential testing:

1. $H_0 : r = 0$: $45.2 > 29.8 \Rightarrow$ **Reject** (at least 1)
2. $H_0 : r \leq 1$: $18.1 > 15.5 \Rightarrow$ **Reject** (at least 2)
3. $H_0 : r \leq 2$: $3.2 < 3.8 \Rightarrow$ **Fail to reject**

Conclusion: $r = 2$ cointegrating relationships

Implications:

- Two equilibrium relationships among 3 variables
- Only $3 - 2 = 1$ common stochastic trend
- Use VECM with 2 error correction terms

Problem 4: Testing Weak Exogeneity

Exercise

In a VECM for prices (P) and exchange rate (E), you estimate $\alpha_P = -0.15$ (s.e. $= 0.04$) and $\alpha_E = 0.02$ (s.e. $= 0.03$).

Test whether the exchange rate is weakly exogenous at 5%.

Answer on next slide...

Problem 4: Solution

Solution: Exchange rate is weakly exogenous

Test: $H_0 : \alpha_E = 0$ (weak exogeneity)

t-statistic: $t = \frac{0.02}{0.03} = 0.67$
Critical value (5%, two-tailed): ± 1.96

Since $|0.67| < 1.96$, **fail to reject** H_0.

Conclusion: Exchange rate does not respond to PPP disequilibrium.

Implication: Prices do all the adjusting to restore PPP equilibrium. Can estimate single-equation model for prices.

Example: Term Structure of Interest Rates

Economic Theory

Expectations hypothesis: $R_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t[r_{t+i}] + \text{premium}$

If premium is constant $\Rightarrow$ spread $(R_t - r_t)$ should be stationary.

Typical Findings

- Both rates are I(1) (confirmed by ADF)
- Johansen test: $r = 1$ cointegrating vector
- Cointegrating vector $\approx (1, -1)$: spread is stationary
- Short rate adjusts ($\alpha_r < 0$), long rate weakly exogenous

Policy Implication

Central bank controls short rate; long rate driven by expectations.

Example: Purchasing Power Parity

PPP Theory

$e_t = p_t - p_t^*$ (log exchange rate = price differential)

Real exchange rate: $q_t = e_t - p_t + p_t^*$ should be stationary (long-run PPP)

Empirical Challenges

- Unit root tests: e_t, p_t, p_t^* all I(1)
- Cointegration tests: Mixed results depending on sample
- Half-life of PPP deviations: 3-5 years (slow adjustment)
- Weak exogeneity: Exchange rate often doesn't adjust

PPP Puzzle

Real exchange rate is highly persistent—slow mean reversion is hard to explain with standard models.

Example: Pairs Trading Strategy

Idea

Find cointegrated stocks $\Rightarrow$ trade the stationary spread

Implementation Steps

1. **Identify pairs**: Test cointegration (e.g., Coca-Cola & Pepsi)
2. **Estimate spread**: $z_t = P_A - \beta P_B$
3. **Trading rules**:
   - $z_t > \mu + 2\sigma$: Sell A, Buy B (spread too wide)
   - $z_t < \mu - 2\sigma$: Buy A, Sell B (spread too narrow)
   - Exit when $z_t \approx \mu$

Risks

Cointegration can break down; spread may not revert; transaction costs.

Python Cointegration Analysis: Key Functions

Essential Libraries

```
from statsmodels.tsa.stattools import coint, adfuller
from statsmodels.tsa.vector_ar.vecm import coint_johansen, VECM
```

Workflow

1. Unit root tests: `adfuller(series)`
2. Engle-Granger: `coint(y, x)` returns test stat & p-value
3. Johansen: `coint_johansen(data, det_order, k_ar_diff)`
4. Fit VECM: `model = VECM(data, k_ar_diff=2, coint_rank=1)`
5. Results: `results = model.fit()`

Note

Complete working examples are provided in the Jupyter notebooks.

Discussion: Cointegration vs Correlation

Key Question

Two series are highly correlated. Are they cointegrated?

Answer: Not necessarily!

- **Correlation**: Measures co-movement (can be spurious for I(1))
- **Cointegration**: Requires stationary linear combination

Example

Two independent random walks can have correlation > 0.9 purely by chance (spurious correlation). But they're NOT cointegrated—their spread is also I(1).

Cointegration implies a meaningful long-run equilibrium relationship.

Discussion: Choosing Deterministic Components

Key Question

Johansen test has 5 cases for deterministics. Which to choose?

Guidelines

1. **No constant, no trend**: Rarely used (requires mean-zero data)
2. **Constant in CE only**: Level series, no drift
3. **Constant unrestricted**: Most common for economic data
4. **Trend in CE**: Series have deterministic trends
5. **Trend unrestricted**: Trending differences (uncommon)

Practical Advice

Start with Case 3 (constant unrestricted). Check sensitivity to specification. Use economic reasoning: do levels have trends?

Take-Home Exercises

1. **Theoretical**: Show that if Y_t and X_t are both random walks with the same innovation, they are cointegrated.

2. **Computation**: Given VECM estimates:

$$\Delta Y_t = 0.5 - 0.3(Y_{t-1} - 2X_{t-1}) + 0.2\Delta Y_{t-1}$$
$$\Delta X_t = 0.1 + 0.1(Y_{t-1} - 2X_{t-1}) + 0.4\Delta X_{t-1}$$

   - What is the cointegrating vector?
   - Which variable adjusts more quickly?
   - What is the long-run equilibrium relationship?

3. **Applied**: Download 10-year and 3-month Treasury rates:
   - Test for unit roots; Test for cointegration (Engle-Granger and Johansen)
   - Estimate VECM; Interpret adjustment coefficients

4. **Critical Thinking**: Why might PPP hold in the long run but not short run?

Exercise Solutions Hints

Hints

1 If $Y_t = Y_{t-1} + \varepsilon_t$ and $X_t = X_{t-1} + \varepsilon_t$ (same shock), then $Y_t - X_t = Y_0 - X_0$ is constant (stationary).

2 From the VECM:

- Cointegrating vector: $(1, -2)$ (normalized on Y)
- Y adjusts faster: $|\alpha_Y| = 0.3 > |\alpha_X| = 0.1$
- Long-run: $Y = 2X$ (when EC term $= 0$)

3 For interest rates:

- Both typically I(1); spread usually stationary
- Expect one cointegrating vector with $(1, -1)$
- Short rate typically adjusts; long rate often weakly exogenous

4 PPP deviations: Transportation costs, non-traded goods, sticky prices, tariffs, market segmentation all slow adjustment but don't prevent long-run convergence.

Key Takeaways from This Seminar

Main Points

1. **Cointegration**: I(1) variables with stationary linear combination
2. **Spurious regression**: High R^2 without cointegration is meaningless
3. **Engle-Granger**: Simple, but only one cointegrating vector
4. **Johansen**: Multiple vectors, MLE, more powerful

VECM Insights

- β defines equilibrium; α determines adjustment speed
- Weak exogeneity ($\alpha = 0$): Variable doesn't respond to disequilibrium
- Always use VECM (not VAR in differences) when cointegrated

Remember

Cointegration is about **long-run equilibrium**, not just correlation!