Time Series Analysis and Forecasting

Chapter 4: SARIMA Models

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Learning Objectives

By the end of this chapter, you will be able to:

- Identify seasonal patterns in time series data
- Apply seasonal differencing to remove seasonal unit roots
- Build and estimate SARIMA models with seasonal components
- Produce accurate forecasts for seasonal time series

Outline

Motivating Example: Seasonality Is Everywhere

- Retail sales exhibit clear **annual patterns**: December peaks, January troughs
- Standard ARIMA models cannot capture these **repeating seasonal cycles**
- Ignoring seasonality leads to systematic forecast errors

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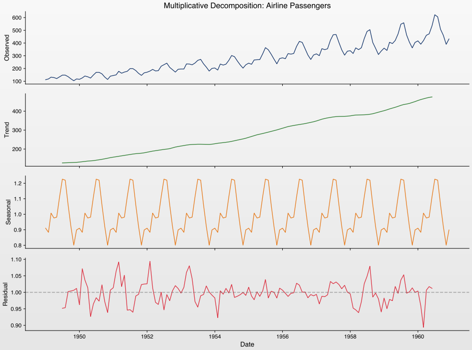
Understanding Seasonal Components



- Seasonal time series = **Trend** + **Seasonal Pattern** + **Residuals**
- Decomposition helps visualize each component separately
- SARIMA models capture both trend dynamics and seasonal behavior

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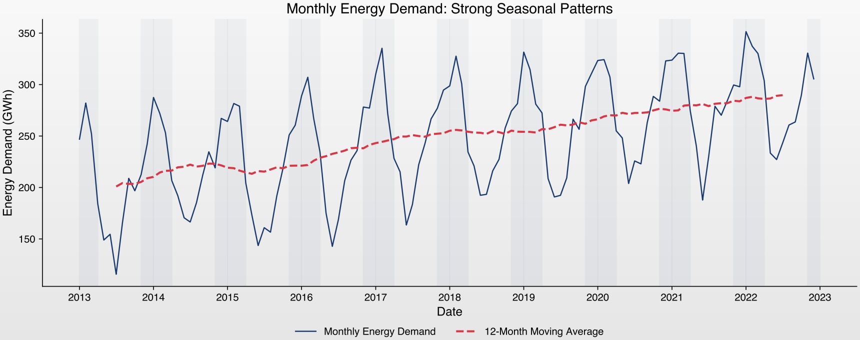
Real-World Application: Monthly Patterns



- Energy demand shows strong **monthly seasonality**
  - Heating cycles in winter, cooling cycles in summer
- Pattern repeats predictably each year with slight weather variations
- Utility companies use SARIMA forecasts for capacity planning

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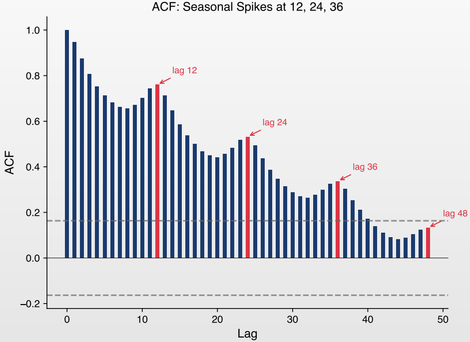
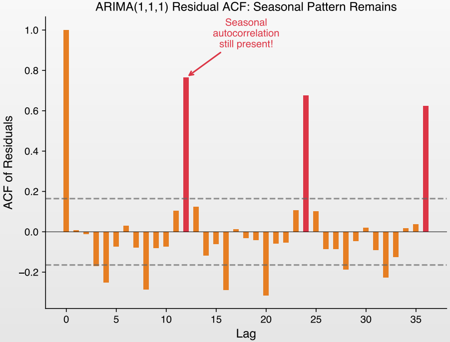
Why Do We Need SARIMA?





- **Left**: Seasonal ACF patterns — spikes at lags 12, 24, 36 reveal annual cycle
- **Right**: ARIMA residuals still show seasonal autocorrelation (incomplete model)
- **SARIMA solution**: Adds seasonal AR/MA terms to capture periodic patterns

TSA_ch4_motivation_why_sarima

What We'll Learn Today

Concepts

- Identifying seasonal patterns
- Seasonal differencing operator
- SARIMA$(p, d, q)(P, D, Q)_s$ notation
- The famous "Airline Model"
- Model selection for seasonal data

Skills

- Diagnose seasonality from ACF/PACF
- Determine seasonal period s
- Choose (P, D, Q) seasonal orders
- Implement SARIMA in Python/R
- Forecast seasonal time series

Key Insight

SARIMA = ARIMA applied at **two frequencies**: the regular (short-term) and seasonal (long-term) levels

What is Seasonality?

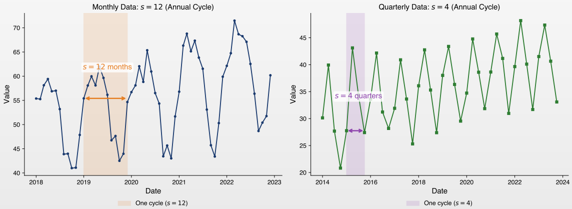
Definition 1 (Seasonality)

A time series exhibits **seasonality** when it shows regular, periodic fluctuations that repeat over a fixed period s (the seasonal period).

Common Seasonal Periods

- Monthly data: $s = 12$ (annual cycle)
- Quarterly data: $s = 4$ (annual cycle)
- Weekly data: $s = 52$ (annual) or $s = 7$ (weekly pattern)
- Daily data: $s = 7$ (weekly pattern)

Seasonality: Visual Illustration



Seasonal Periods

Left: Monthly data with $s = 12$ (annual cycle). Right: Quarterly data with $s = 4$. The pattern repeats every s periods — this regularity is exploited by SARIMA models.

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Examples of Seasonal Data

Economic Series

- Retail sales (holiday peaks)
- Tourism (summer/winter)
- Agricultural production
- Energy consumption
- Employment (hiring cycles)

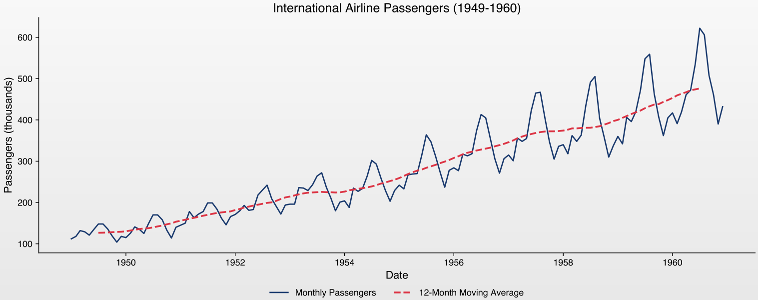
Other Domains

- Weather/temperature
- Website traffic
- Hospital admissions
- Transportation usage
- Electricity demand

Why It Matters

Ignoring seasonality leads to biased forecasts and invalid inference!

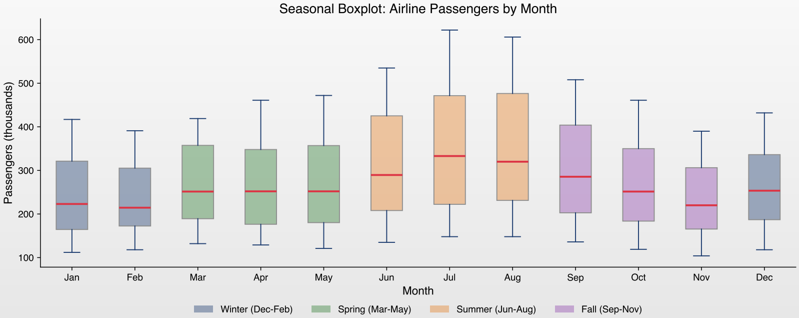
Example: Airline Passengers Data



- Monthly international airline passengers (1949–1960)
- Clear **upward trend** and **growing seasonal amplitude**
- Summer peaks reflect vacation travel patterns

TSA_ch4_airline_data

Visualizing Seasonal Patterns



- Box plot reveals consistent seasonal pattern across years
- July–August show highest passenger counts (summer travel)
- November–February show lowest counts (winter months)

TSA_ch4_seasonal_boxplot

Deterministic vs Stochastic Seasonality

Deterministic Seasonality

- **Fixed pattern**: $Y_t = \sum_{j=1}^{s} \gamma_j D_{jt} + \varepsilon_t$
  - D_{jt} are seasonal dummies
- Pattern constant over time
- Same amplitude every year
- Removed by regression on dummies
- ACF: sharp cutoff at seasonal lags
- **Example:** University enrollment peaks every September by the same amount

Stochastic Seasonality

- **Evolving pattern**: $\Delta_s Y_t = Y_t - Y_{t-s}$
  - Exhibits dependence structure
- Pattern evolves over time
- Amplitude may grow or shrink
- Requires seasonal differencing
- ACF: slow decay at seasonal lags
- **Example:** Retail sales peaks grow larger each December

How to decide?

- Slow ACF decay at lags $s, 2s, 3s, \ldots \Rightarrow$ stochastic (use Δ_s)
- Sharp cutoff $\Rightarrow$ deterministic (use dummies)
- Use HEGY or Canova-Hansen tests to confirm

Detecting Seasonality

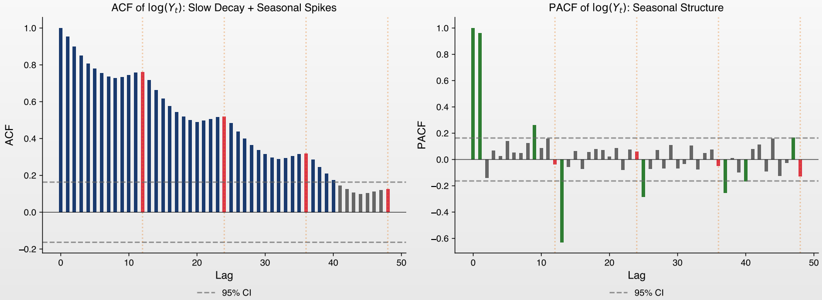
Visual Methods (Primary Approach)

- **Time series plot** – look for repeating patterns
- **Seasonal boxplot** – compare distributions across seasons
- **ACF plot** – spikes at seasonal lags $(s, 2s, 3s, \ldots)$

ACF Signature of Seasonality

- Strong spikes at lags $s, 2s, 3s, \ldots$ indicate seasonal pattern
- Slow decay at seasonal lags $\Rightarrow$ stochastic seasonality (needs differencing)
- Quick cutoff at seasonal lags $\Rightarrow$ deterministic seasonality (use dummies)

ACF Reveals Seasonal Structure



- **Slow decay** at all lags indicates non-stationarity (trend)
- **Spikes at lags 12, 24, 36** confirm seasonal pattern ($s = 12$)
- Slow decay at seasonal lags $\Rightarrow$ needs seasonal differencing $(1 - L^{12})$

TSA_ch4_acf_seasonality

F-Test for Seasonal Dummy Variables: Intuition

What does this test do?

- **Goal**: test whether mean values differ significantly across seasons
- **Logic**: if the mean in January $\neq$ February $\neq \ldots \neq$ December $\Rightarrow$ seasonality
- **Method**: compare a model WITH seasonal dummy variables vs. a model WITHOUT

Models compared

- **Restricted**: $Y_t = \alpha + \varepsilon_t$ **Unrestricted**: $Y_t = \alpha + \sum_{j=1}^{s-1} \gamma_j D_{jt} + \varepsilon_t$
- where $D_{jt} = 1$ if observation t is in season j, 0 otherwise

Key idea

- If adding seasonal dummy variables **significantly reduces** prediction errors, then seasonality is present

F-Test for Seasonal Dummy Variables: Formula and Example

F-statistic formula

- **Formula**: $F = \frac{(SSR_R - SSR_U)/(s-1)}{SSR_U/(n-s)} \sim F_{s-1,n-s}$
  - SSR_R: sum of squared residuals from the restricted model (no dummies)
  - SSR_U: sum of squared residuals from the unrestricted model (with dummies)
  - $s-1$: number of restrictions (monthly: 11, quarterly: 3)

Numerical example (Monthly data, n=120)

- $SSR_R = 15000$, $SSR_U = 8500$, $s = 12$
- $F = \frac{(15000-8500)/11}{8500/108} = \frac{590.9}{78.7} = 7.51$
- Critical value $F_{0.05,11,108} \approx 1.87$. Since $7.51 > 1.87$: **Reject** $H_0 \Rightarrow$ Seasonality present!

Kruskal-Wallis Test: Intuition

What does this test do?

- **Non-parametric test**: checks whether observations from different seasons come from the same distribution
- **Mechanism**: ranks all observations from smallest to largest
- **Check**: whether ranks are uniformly distributed across seasons
- **Conclusion**: if one season consistently has higher/lower ranks $\Rightarrow$ seasonality

Why use it instead of the F-test?

- **No normality assumption** – works with any distribution
- **Robust to outliers** – extreme values do not distort results

Limitation

- Less powerful than the F-test when data ARE normally distributed

Kruskal-Wallis Test: Formula and Example

Test statistic

- $H = \frac{12}{N(N+1)} \sum_{j=1}^{s} \frac{R_j^2}{n_j} - 3(N+1)$ where N = total obs., n_j = obs. in season j, R_j = rank sum

Example: Quarterly sales (n=20, s=4)

- Data ranked 1–20. Rank sums: Q1: $R_1 = 15$, Q2: $R_2 = 35$, Q3: $R_3 = 70$, Q4: $R_4 = 90$
- $H = \frac{12}{20 \times 21} \left(\frac{15^2}{5} + \frac{35^2}{5} + \frac{70^2}{5} + \frac{90^2}{5} \right) - 3(21) = 19.6$
- Critical value $\chi^2_{0.05,3} = 7.81$. Since $19.6 > 7.81$: **Reject** $H_0 \Rightarrow$ Seasonality!

In Python

- **Implementation**: `scipy.stats.kruskal(q1, q2, q3, q4)`

HEGY Test: What Problem Does It Solve?

Key question

- **Problem**: given a seasonal series, we need to determine the type of differencing
- **Regular differencing** $(1-L)$? $\Rightarrow$ set $d=1$; **Seasonal differencing** $(1-L^s)$? $\Rightarrow$ set $D=1$
- **HEGY**: tests for both types of unit roots simultaneously!

Why not just use ADF?

- **ADF**: tests only for a regular unit root at frequency zero
- **Limitation**: seasonal data may have unit roots at seasonal frequencies that ADF misses!

HEGY tests multiple frequencies

- **Quarterly**: tests at 0, π, $\pm\pi/2$
- **Monthly**: tests at 0, π, $\pm\pi/6$, $\pm\pi/3$, $\pm\pi/2$, $\pm 2\pi/3$, $\pm 5\pi/6$

HEGY Test: Auxiliary Regression (Quarterly)

HEGY auxiliary regression

- **Quarterly data** $(s = 4)$: $\Delta_4 y_t = \pi_1 z_{1,t-1} + \pi_2 z_{2,t-1} + \pi_3 z_{3,t-2} + \pi_4 z_{4,t-2} + \sum_{j=1}^{k} \phi_j \Delta_4 y_{t-j} + \varepsilon_t$

Transformed variables

- z_{1t}: $(1 + L + L^2 + L^3)y_t = y_t + y_{t-1} + y_{t-2} + y_{t-3}$
- z_{2t}: $-(1 - L + L^2 - L^3)y_t = -y_t + y_{t-1} - y_{t-2} + y_{t-3}$
- z_{3t}: $-(1 - L^2)y_t = -y_t + y_{t-2}$
- z_{4t}: $-(L - L^3)y_t = -y_{t-1} + y_{t-3}$

Hypotheses

- $H_0 : \pi_1 = 0$: unit root at frequency 0
- $H_0 : \pi_2 = 0$: unit root at frequency π
- $H_0 : \pi_3 = \pi_4 = 0$: unit root at frequency $\pm\pi/2$

HEGY Test: Decision Rules with Examples

HEGY critical values (5%, n=100, with constant)

Test	Statistic	Critical value	If NOT rejected...
t_1 $(\pi_1 = 0)$	t-stat	-2.88	Requires $d = 1$
t_2 $(\pi_2 = 0)$	t-stat	-2.88	Requires $D = 1$
F_{34} $(\pi_3 = \pi_4 = 0)$	F-stat	6.57	Requires $D = 1$

Example: Quarterly GDP

- **HEGY results**: $t_1 = -1.52$, $t_2 = -4.21$, $F_{34} = 2.15$
- $t_1 = -1.52 > -2.88$: Cannot reject $\Rightarrow$ **requires** $d = 1$
- $t_2 = -4.21 < -2.88$: Reject $\Rightarrow$ no unit root at π
- $F_{34} = 2.15 < 6.57$: Cannot reject $\Rightarrow$ **requires** $D = 1$
- **Conclusion**: Use SARIMA with $d = 1, D = 1$

Canova-Hansen Test: The Opposite of HEGY

HEGY vs Canova-Hansen: Different null hypotheses!

	HEGY	**Canova-Hansen**
H_0	Seasonal unit root	**No** seasonal unit root
H_1	No seasonal unit root	Seasonal unit root
Reject H_0	Use seasonal dummies	Use differencing $(1-L^s)$
Do not reject	Use differencing $(1-L^s)$	Use seasonal dummies

Why does it matter?

- HEGY: "Prove there is NO unit root" (conservative towards differencing)
- CH: "Prove there IS a unit root" (conservative towards dummies)
- Use **both** tests for robust conclusions!

Canova-Hansen Test: Formula

Testing procedure

- **Step 1**: Regress y_t on seasonal dummies: $y_t = \sum_{j=1}^{s} \gamma_j D_{jt} + u_t$
- **Step 2**: Compute partial sums at seasonal frequency λ_i:
  - $S_{it}^{(c)} = \sum_{j=1}^{t} \hat{u}_j \cos(\lambda_i j), \quad S_{it}^{(s)} = \sum_{j=1}^{t} \hat{u}_j \sin(\lambda_i j)$

LM test statistic

- $LM_i = \frac{1}{T^2 \hat{\omega}_i} \left[\sum_{t=1}^{T} (S_{it}^{(c)})^2 + \sum_{t=1}^{T} (S_{it}^{(s)})^2 \right]$
- where $\hat{\omega}_i$ = consistent estimator of the spectral density at frequency λ_i

Decision

- **Rule**: reject H_0 (stationarity) if $LM >$ critical value $\Rightarrow$ seasonal differencing is needed

Summary: Choosing the Right Seasonality Test

Test	H_0	If rejected	Best for
F-test	No seasonality	Seasonality exists	Normal data
Kruskal-Wallis	No difference across seasons	Seasonality exists	Non-normal, outliers
HEGY	Unit root exists	Use dummies	Determining d, D
Canova-Hansen	No unit root	Use $(1-L^s)$	Confirming stability

Key idea

- **F-test / Kruskal-Wallis**: "*Does seasonality exist?*"
- **HEGY / Canova-Hansen**: "*What type?*" (deterministic vs stochastic)

The Seasonal Difference Operator

Definition 2 (Seasonal Difference)

The **seasonal difference operator** Δ_s is defined as:

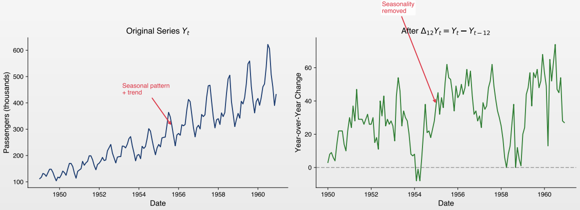
$$\Delta_s Y_t = (1 - L^s)Y_t = Y_t - Y_{t-s}$$

where $L^s Y_t = Y_{t-s}$ is the seasonal lag operator.

Examples

- Monthly data ($s = 12$): $\Delta_{12} Y_t = Y_t - Y_{t-12}$
  Compares each month to the same month last year
- Quarterly data ($s = 4$): $\Delta_4 Y_t = Y_t - Y_{t-4}$
  Compares each quarter to the same quarter last year

Seasonal Difference: Visual Illustration



Effect of Seasonal Differencing

Left: Original series with clear seasonal pattern. Right: After $\Delta_{12} = (1 - L^{12})$, seasonal pattern is removed. Year-over-year comparison eliminates seasonal effects.

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Proof: Seasonal Differencing Removes Deterministic Seasonality

Claim: If $Y_t = \mu_t + \varepsilon_t$ where $\mu_t = \mu_{t-s}$ (periodic mean), then $\Delta_s Y_t$ removes the seasonal mean.

Proof: Let $Y_t = \mu_t + \varepsilon_t$ where μ_t has period s. Apply seasonal difference:

$$\begin{aligned}\Delta_s Y_t &= Y_t - Y_{t-s} = (\mu_t + \varepsilon_t) - (\mu_{t-s} + \varepsilon_{t-s}) \\ &= \mu_t - \mu_{t-s} + \varepsilon_t - \varepsilon_{t-s} \\ &= 0 + \varepsilon_t - \varepsilon_{t-s} \quad (\text{since } \mu_t = \mu_{t-s})\end{aligned}$$

Properties of $\Delta_s Y_t = \varepsilon_t - \varepsilon_{t-s}$:

- $\mathbb{E}[\Delta_s Y_t] = 0$ (constant mean)
- $\text{Var}(\Delta_s Y_t) = 2\sigma^2$ (constant variance)
- Autocovariance: $\gamma(s) = -\sigma^2$, $\gamma(k) = 0$ for $k \neq 0, s$

Result

Seasonal differencing transforms periodic seasonal pattern into MA(1) at seasonal lag.

Combining Regular and Seasonal Differencing

Full Differencing

For series with both trend and seasonality:

$$\Delta\Delta_s Y_t = (1 - L)(1 - L^s)Y_t$$

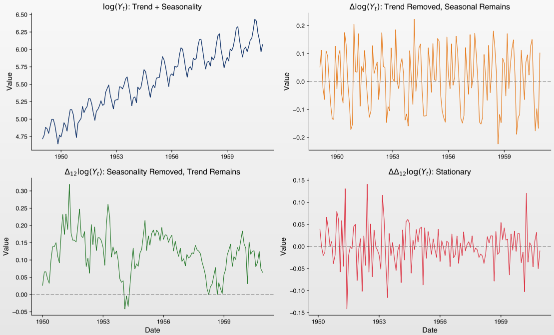
Expansion

$(1 - L)(1 - L^s)Y_t = Y_t - Y_{t-1} - Y_{t-s} + Y_{t-s-1}$. For monthly: $\Delta\Delta_{12} Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$

Order of Differencing

d: regular differences (trend removal); D: seasonal differences (seasonal trend removal)

Effect of Differencing Operations



- Regular differencing removes trend but seasonal pattern remains
- Seasonal differencing removes seasonality but trend pattern remains
- **Both differences** needed to achieve stationarity

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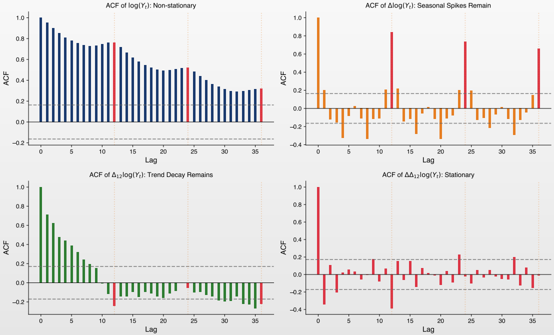
ACF Before and After Differencing



- Original ACF: slow decay indicates non-stationarity
- After Δ: seasonal spikes remain at lags 12, 24, 36
- After Δ_{12}: trend decay remains at early lags
- After $\Delta\Delta_{12}$: ACF cuts off $\Rightarrow$ **stationary**

TSA_ch4_acf_differencing

Seasonal Integration

Definition 3 (Seasonally Integrated Process)

A series Y_t is **seasonally integrated** of order $(d, D)_s$, written $Y_t \sim I(d, D)_s$, if:

$$(1 - L)^d (1 - L^s)^D Y_t$$

is stationary.

Common Cases

- $I(1, 0)_{12}$: Regular unit root only (monthly)
- $I(0, 1)_{12}$: Seasonal unit root only
- $I(1, 1)_{12}$:
  - Both regular and seasonal unit roots

SARIMA Model Definition

Definition 4 (SARIMA$(p, d, q) \times (P, D, Q)_s$)

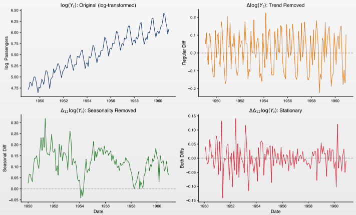
The **Seasonal ARIMA** model is:

$$\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D Y_t = c + \theta(L)\Theta(L^s)\varepsilon_t$$

Components

- $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$: Non-seasonal AR
- $\Phi(L^s) = 1 - \Phi_1 L^s - \cdots - \Phi_P L^{Ps}$: Seasonal AR
- $\theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$: Non-seasonal MA
- $\Theta(L^s) = 1 + \Theta_1 L^s + \cdots + \Theta_Q L^{Qs}$: Seasonal MA
- $(1-L)^d$:
  - Regular differencing; $(1-L^s)^D$: Seasonal differencing

SARIMA: Visual Illustration



Differencing Strategy

Progressive transformation: Original → regular difference (removes trend) → seasonal difference (removes seasonality) → both. Apply minimum differencing needed to achieve stationarity.

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Proof: Multiplicative Seasonal Structure

Why multiplicative? Consider SARIMA$(1,0,0) \times (1,0,0)_s$:

$$(1-\phi L)(1-\Phi L^s)Y_t = \varepsilon_t$$

Expand: $(1-\phi L)(1-\Phi L^s)Y_t = Y_t - \phi Y_{t-1} - \Phi Y_{t-s} + \phi\Phi Y_{t-s-1}$

Interpretation (Monthly, $s = 12$)

Y_t depends on: Y_{t-1} (last month), Y_{t-12} (same month last year), Y_{t-13} (interaction).
Parsimony: Multiplicative form uses 2 parameters (ϕ, Φ); additive would need 3+.

SARIMA Notation

Full Specification

SARIMA$(p, d, q) \times (P, D, Q)_s$ has 7 parameters to specify:

Parameter	**Meaning**
p	Non-seasonal AR order
d	Non-seasonal differencing order
q	Non-seasonal MA order
P	Seasonal AR order
D	Seasonal differencing order
Q	Seasonal MA order
s	Seasonal period

Example

SARIMA$(1, 1, 1) \times (1, 1, 1)_{12}$: Monthly data with AR(1), MA(1), seasonal AR(1), seasonal MA(1), and both regular and seasonal differencing.

Common SARIMA Models

Airline Model: SARIMA$(0, 1, 1) \times (0, 1, 1)_s$

$(1 - L)(1 - L^s)Y_t = (1 + \theta L)(1 + \Theta L^s)\varepsilon_t$ – Classic model (Box & Jenkins, 1970)

SARIMA$(1, 0, 0) \times (1, 0, 0)_s$

$(1 - \phi L)(1 - \Phi L^s)Y_t = \varepsilon_t$ – Pure seasonal and non-seasonal AR

SARIMA$(0, 1, 1) \times (0, 1, 0)_s$

$(1 - L)(1 - L^s)Y_t = (1 + \theta L)\varepsilon_t$ – Random walk + seasonal diff + MA(1)

Multiplicative Structure

Why multiplicative?

- **Principle**: the seasonal and non-seasonal parts multiply
- **AR**: $\phi(L)\Phi(L^s)$ **MA**: $\theta(L)\Theta(L^s)$

Example: SARIMA$(1,0,0) \times (1,0,0)_{12}$

- **Model**: $(1-\phi L)(1-\Phi L^{12})Y_t = \varepsilon_t$
- **Expansion**: $Y_t - \phi Y_{t-1} - \Phi Y_{t-12} + \phi\Phi Y_{t-13} = \varepsilon_t$
- **Cross-term** $\phi\Phi Y_{t-13}$ captures the interaction!

Interpretation

- **Advantage**: the multiplicative structure allows parsimonious modeling of complex seasonal patterns with few parameters

ACF/PACF for Seasonal Models

Key Insight

Seasonal models show patterns at both:

- Non-seasonal lags: $1, 2, 3, \ldots$
- Seasonal lags: $s, 2s, 3s, \ldots$

Model	ACF	PACF
SAR(P)	Decays at $s, 2s, \ldots$	Cuts off after Ps
SMA(Q)	Cuts off after Qs	Decays at $s, 2s, \ldots$
SARMA	Decays at seasonal lags	Decays at seasonal lags

Example: Airline Model ACF/PACF

SARIMA$(0,1,1) \times (0,1,1)_{12}$

After differencing $W_t = (1-L)(1-L^{12})Y_t$: $W_t = (1+\theta L)(1+\Theta L^{12})\varepsilon_t$

Expected ACF Pattern

Spikes at lag 1 (θ), lag 12 (Θ), lag 13 ($\theta \cdot \Theta$ interaction); all other lags near zero.

Expected PACF Pattern

Exponential decay at lags $1, 2, 3, \ldots$ and at lags $12, 24, 36, \ldots$

Model Identification Guidelines

Step-by-Step Process

1. Examine ACF for slow decay at seasonal lags $\Rightarrow$ seasonal differencing
2. After differencing, check ACF/PACF patterns
3. Non-seasonal behavior at lags $1, 2, \ldots, s-1$
4. Seasonal behavior at lags $s, 2s, 3s, \ldots$

Practical Tips

- Start with $d \leq 1$ and $D \leq 1$
- Usually $P, Q \leq 2$ is sufficient
- Use information criteria (AIC, BIC) for final selection
- Auto-SARIMA algorithms can help

Estimation Methods

Maximum Likelihood Estimation

Standard approach for SARIMA:

- Conditional MLE (conditional on initial values)
- Exact MLE (via Kalman filter)

Computational Considerations

- More parameters than ARIMA $\Rightarrow$ more data needed
- Seasonal parameters estimated from lags $s, 2s, \ldots$
- Need sufficient seasonal cycles (at least 3-4 years of monthly data)

Exact Likelihood: Prediction Error Decomposition

Why the Kalman Filter?

- **SARIMA**: has the structure of a state-space model
- **Kalman filter**: recursively computes prediction errors v_t and their variances f_t, without conditioning on initial values

Exact Log-Likelihood (Prediction Error Decomposition)

- **Formula**: $\ell(\theta) = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln(f_t) + \frac{v_t^2}{f_t}\right]$
- v_t: $Y_t - \hat{Y}_{t|t-1}$ (innovation); f_t: $\text{Var}(v_t)$ (innovation variance)

Advantages over Conditional MLE

- Does not require choosing initial values
- Each term $\ln(f_t)$ weights observations differently (variable variance at start)
- Essential for short series where initial values matter
- Implemented by default in `statsmodels.tsa.SARIMAX()` with `method='mle'`

Stationarity and Invertibility

Stationarity Conditions

Both non-seasonal and seasonal AR polynomials must have roots outside the unit circle:

- $\phi(z) = 0 \Rightarrow |z| > 1$
- $\Phi(z^s) = 0 \Rightarrow |z| > 1$

Invertibility Conditions

Both non-seasonal and seasonal MA polynomials must have roots outside the unit circle:

- $\theta(z) = 0 \Rightarrow |z| > 1$
- $\Theta(z^s) = 0 \Rightarrow |z| > 1$

Diagnostic Checking

Residual Analysis

After fitting SARIMA, check that residuals are white noise:

1. Plot residuals over time (no patterns)
2. ACF of residuals (no significant spikes)
3. Ljung-Box test at multiple lags including seasonal
4. Normality tests (Q-Q plot, Jarque-Bera)

Important

Check ACF at **both** non-seasonal and seasonal lags!
Significant ACF at lag 12 suggests inadequate seasonal modeling.

Model Selection Criteria

Information Criteria

Compare competing SARIMA models using:

- AIC $= -2\ln(L) + 2k$
- BIC $= -2\ln(L) + k\ln(n)$
- AICc $=$ AIC $+ \frac{2k(k+1)}{n-k-1}$ (corrected for small samples)

where $k = p + q + P + Q + 1$ (plus 1 for variance).

Auto-SARIMA

Python's `pmdarima.auto_arima()` with `seasonal=True` automatically searches for optimal $(p, d, q) \times (P, D, Q)_s$.

Point Forecasts

Forecast Computation

SARIMA forecasts are computed recursively:

- Replace future ε_{T+h} with 0
- Replace future Y_{T+h} with forecasts $\hat{Y}_{T+h|T}$
- Use known past values $Y_T, Y_{T-1}, \ldots$

Seasonal Pattern in Forecasts

SARIMA forecasts naturally capture seasonality:

- Short-term: influenced by recent values
- Long-term: revert to seasonal pattern

Forecast Intervals

Uncertainty Quantification

$(1-\alpha)\%$ prediction interval:

$$\hat{Y}_{T+h|T} \pm z_{\alpha/2}\sqrt{\text{Var}(e_{T+h})}$$

Variance computed from MA(∞) representation.

Key Properties

- Intervals widen with forecast horizon
- For $I(1,1)_s$ series: intervals grow without bound
- Seasonal pattern visible in point forecasts
- Uncertainty captures both trend and seasonal variation

Long-Horizon Forecasts

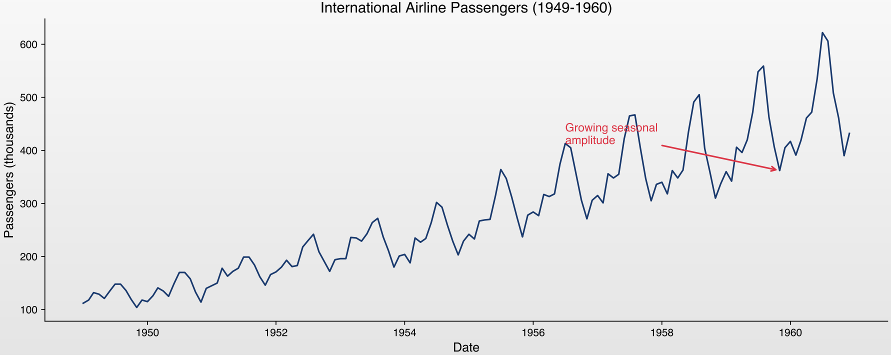
Behavior as $h \to \infty$

- Point forecasts converge to deterministic seasonal pattern
- If drift present: linear trend + seasonal pattern
- Forecast intervals continue to widen

Practical Implication

- Short-term: SARIMA captures both level and season
- Medium-term: Good seasonal forecasts, growing uncertainty
- Long-term:
  - Mainly reflects seasonal pattern, wide intervals

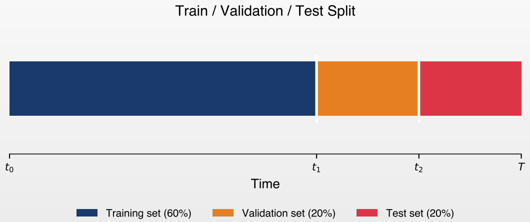
Case Study: Airline Passengers Data



- Classic Box-Jenkins dataset: monthly airline passengers (1949-1960)
- Clear upward trend and increasing seasonal amplitude
- Multiplicative seasonality suggests log transformation

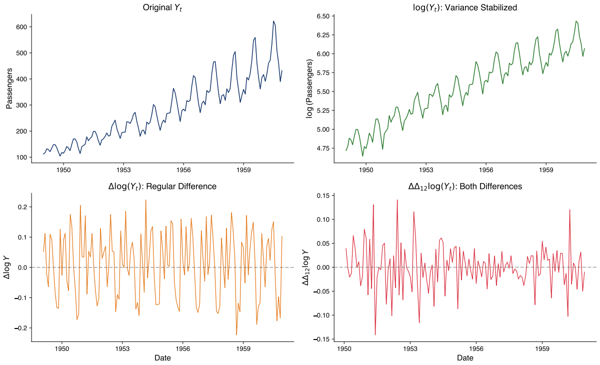
TSA_ch4_case_raw_data

Data Splitting Strategy



- **Training set (70%)** — Fit model parameters
  - Estimate SARIMA coefficients (ϕ, θ, Φ, Θ)
  - Largest portion ensures reliable parameter estimates
- **Validation set (15%)** — Select best model
  - Compare candidate models (different orders)
  - Choose model with lowest validation error
- **Test set (15%)** — Final evaluation
  - Unbiased out-of-sample performance; never used during development

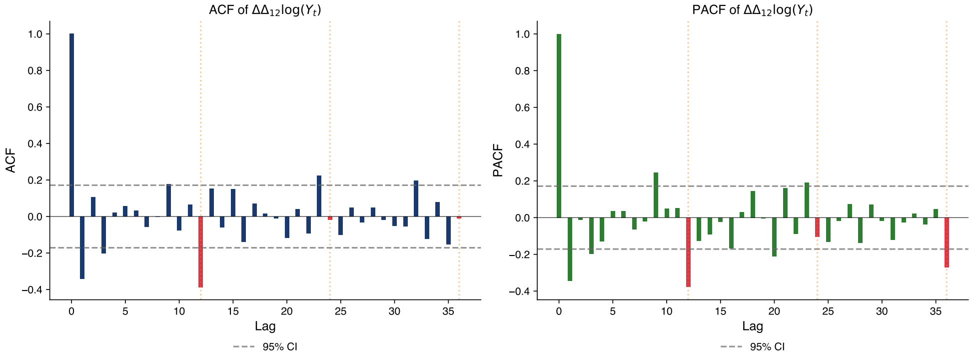
Step 1: Transformations



- Log transform stabilizes variance (multiplicative $\rightarrow$ additive)
- First difference removes trend; seasonal difference removes seasonality
- Double-differenced series appears stationary

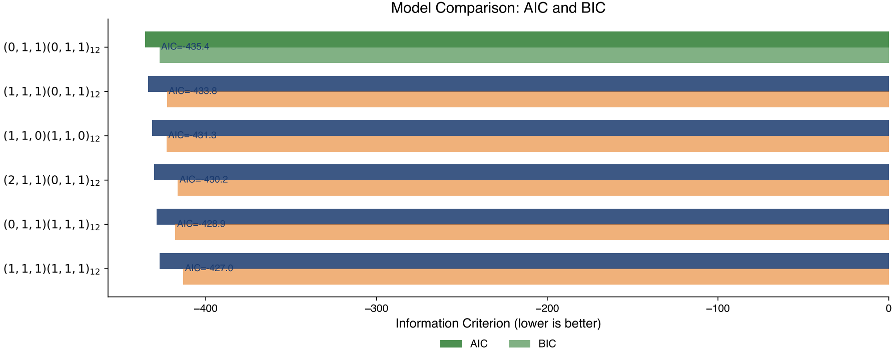
TSA_ch4_case_transformations

Step 2: ACF/PACF Analysis



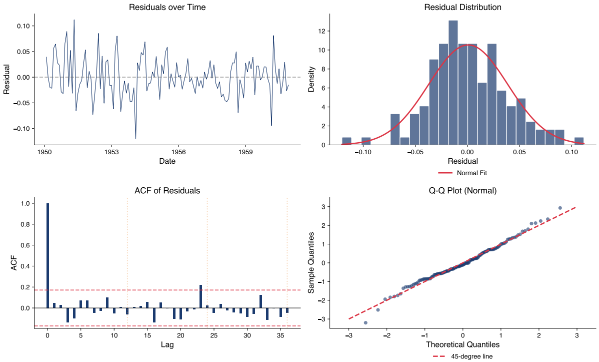
- ACF: Significant spike at lag 1 and lag 12 $\Rightarrow$ MA(1), SMA(1)
- PACF: Exponential decay pattern confirms MA structure
- Suggests SARIMA$(0, 1, 1) \times (0, 1, 1)_{12}$ (airline model)

Step 3: Model Comparison



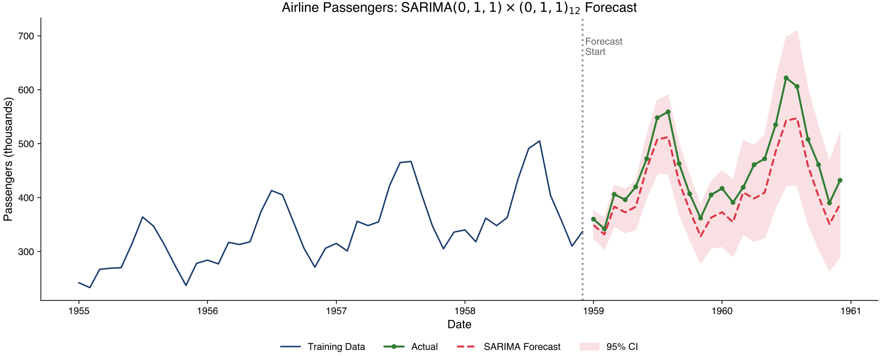
- Compare candidate SARIMA models using AIC criterion
- SARIMA$(0,1,1) \times (0,1,1)_{12}$ provides best fit (lowest AIC)
- This is the famous "airline model" identified by Box & Jenkins

Step 4: Residual Diagnostics



- Residuals appear random with no remaining autocorrelation
- Q-Q plot shows approximate normality
- Model adequately captures both trend and seasonal structure

Step 5: Forecasting



- 24-month forecast with 95% confidence interval
- Model captures seasonal pattern and upward trend
- Prediction intervals widen appropriately with forecast horizon

Key Takeaways

Main Points

1. **Seasonality** is common in economic and business data
2. **Seasonal differencing** $(1 - L^s)$ removes stochastic seasonality
3. **SARIMA**$(p, d, q) \times (P, D, Q)_s$ extends ARIMA for seasonal data
4. **Multiplicative structure** captures seasonal-trend interactions
5. **ACF/PACF** show patterns at both regular and seasonal lags
6. **Model selection**: Use AIC/BIC or auto-SARIMA algorithms

Next Steps

Chapter 5 will cover multivariate time series: VAR models, Granger causality, and cointegration.

Quiz Question 1

Question

For monthly data with annual seasonality, what is the seasonal period s?

(A) $s = 4$

(B) $s = 7$

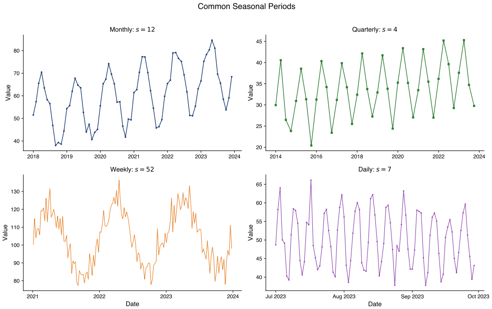
(C) $s = 12$

(D) $s = 52$

Quiz Question 1: Answer

Correct Answer: (C) $s = 12$ (12 months per year)

Common periods: Quarterly=4, Monthly=12, Weekly=52, Daily=7, Hourly=24



Quiz Question 2

Question

What does the seasonal difference operator $(1 - L^{12})$ do to a monthly series?

(A) Computes $Y_t - Y_{t-1}$ (month-to-month change)

(B) Computes $Y_t - Y_{t-12}$ (year-over-year change)

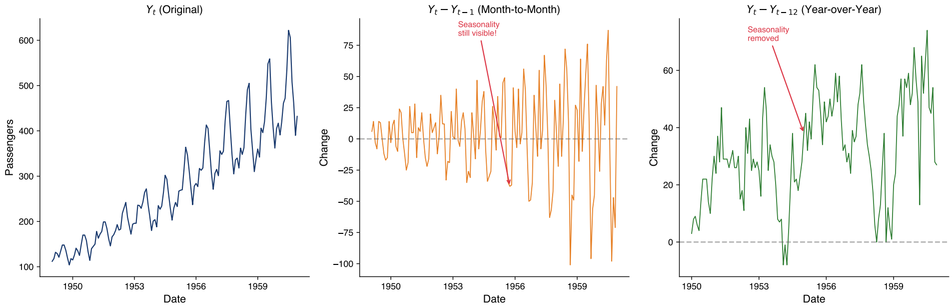
(C) Computes the 12-month moving average

(D) Removes the trend component only

Quiz Question 2: Answer

Correct Answer: (B) Year-over-year change

$(1 - L^{12})Y_t = Y_t - Y_{t-12}$ removes the seasonal pattern by comparing same months.



Quiz Question 3

Question

In SARIMA$(1, 1, 1) \times (1, 1, 1)_{12}$ notation, what does the $(1, 1, 1)_{12}$ part represent?

(A) AR(1), differencing once, MA(1) at the regular level

(B) Seasonal AR(1), seasonal differencing once, seasonal MA(1)

(C) 12 AR terms, 12 differences, 12 MA terms

(D) The model has 12 parameters in total

Quiz Question 3: Answer

Correct Answer: (B)

Seasonal AR(1), seasonal differencing once, seasonal MA(1)

SARIMA Notation Breakdown

SARIMA$(p, d, q) \times (P, D, Q)_s$:

(p, d, q)	Non-seasonal: AR(p), d differences, MA(q)
$(P, D, Q)_s$	Seasonal: SAR(P), D seasonal diffs, SMA(Q)

For $(1, 1, 1) \times (1, 1, 1)_{12}$:

- Non-seasonal: AR(1), one regular difference, MA(1)
- Seasonal: SAR(1) at lag 12, one Δ_{12}, SMA(1) at lag 12

Quiz Question 4

Question

The "Airline Model" is SARIMA$(0,1,1) \times (0,1,1)_{12}$. How many parameters need to be estimated (excluding variance)?

(A) 1
(B) 2
(C) 4
(D) 12

Quiz Question 4: Answer

Correct Answer: (B) — 2 parameters

SARIMA$(0,1,1) \times (0,1,1)_{12}$: $(1-L)(1-L^{12})Y_t = (1+\theta_1 L)(1+\Theta_1 L^{12})\varepsilon_t$
Parameters: θ_1 (non-seasonal MA) and Θ_1 (seasonal MA), plus σ^2.

Why "Airline Model"?

Box & Jenkins (1970) used this model to forecast international airline passengers. Remarkably effective for many seasonal economic series!

Quiz Question 5

Question

You observe significant ACF spikes at lags 12, 24, and 36 in a monthly series. What does this suggest?

(A) The series has a unit root

(B) The series has annual seasonality that needs seasonal differencing

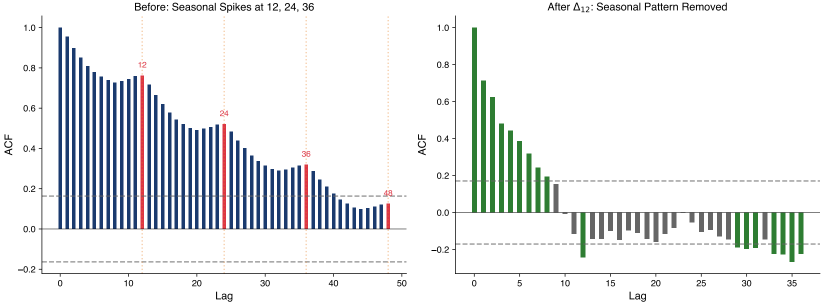
(C) The series follows an AR(36) process

(D) The series is already stationary

Quiz Question 5: Answer

Correct Answer: (B) Needs seasonal differencing

ACF spikes at 12, 24, 36 = stochastic seasonality. Apply $(1 - L^{12})$ to remove it.



Quiz Question 6

Question

After applying $(1-L)(1-L^{12})$ to a monthly series, the ACF shows a significant spike only at lag 1 and lag 12. What SARIMA model is suggested?

(A) SARIMA$(1,1,0) \times (1,1,0)_{12}$

(B) SARIMA$(0,1,1) \times (0,1,1)_{12}$

(C) SARIMA$(1,1,1) \times (1,1,1)_{12}$

(D) SARIMA$(0,1,0) \times (0,1,0)_{12}$

Quiz Question 6: Answer

Correct Answer: (B)

SARIMA$(0,1,1) \times (0,1,1)_{12}$ (The Airline Model)

ACF/PACF Identification Rules

For MA processes, ACF **cuts off** after lag q:

Pattern	Suggests
ACF spike at lag 1 only	MA(1) for non-seasonal part
ACF spike at lag 12 only	SMA(1) for seasonal part

Combined: MA(1) $\times$ SMA(1) $= (0,d,1) \times (0,D,1)_{12}$

With $d = 1$ and $D = 1$ (already differenced): $(0,1,1) \times (0,1,1)_{12}$

References

- Box, G.E.P., Jenkins, G.M., Reinsel, G.C., & Ljung, G.M. (2015). *Time Series Analysis: Forecasting and Control*. 5th ed. Wiley.
- Hyndman, R.J. & Athanasopoulos, G. (2021). *Forecasting: Principles and Practice*. 3rd ed. OTexts.
- Hamilton, J.D. (1994). *Time Series Analysis*. Princeton University Press.
- Brockwell, P.J. & Davis, R.A. (2016). *Introduction to Time Series and Forecasting*. 3rd ed. Springer.