Blood Flow in the Human Circulatory System

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Hemodynamics studies blood's motion, herein, we report mathematical models describing the kinematics of blood in the human macrocirculatory system. We review physiological, rheological, and mathematical foundations necessary for modeling blood flow in large vessels. We discuss numerical methods for solving the governing equations of blood flow, the incompressible Navier-Stokes equations. Finally, we present recent advances in computational hemodynamics, including physics-informed neural networks (PINN's) and fluid-structure interaction (FSI) models.

Keywords: *computational hemodynamics, physics-informed neural networks, Deep-Riesz, discontinuous Galerkin, Lax–Wendroff, fluid–structure interaction (FSI)*

summarize final contributions here and update keywords

Contents

Taging conventions are as follows:

- Red Text means check accuracy or clarity
- Blue Text means misplaced or not necessary
- Green Text means to possibly expand or add more references

Work in Progress (WIP). Tips to keep in mind:
- Start w/ literature suvery, summarizing each articles contribution once. (i.e., limit repeat citation refs.), then prioritize sections by technicality
- Tag material by relevance, e.g., foundational. vs. tangential vs. speculative.
- Comparative analsyis contrasting results, models, or point out contradictions across papers
- Always consist in notation: Actively managing acroynms, abrevs., notation, and preliminaries.
- Look into [Zotero] to manage refs.

0.1 Acroynms and Abbreviations

a.e.	almost everywhere
e.g.	”exempli gratia” (for example)
i.e.	”id est” (that means)
s.s.	sufficiently smooth
s.t.	such that
r.t.	refers to
w.r.t.	with respect to
m.b.s.	must be shown
i.m.b.s.	it must be shown
i.r.t.s.	it remains to show
w.a.t.s.	we aim to show
bpm	beats per minute
wlog	without loss of generality
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PDES	System of Partial Differential Equations
IC	Initial Condition
BC	Boundary Condition
0D	Zero dimensional
1D	One dimensional
2D	Two dimensional
3D	Three dimensional
FSI	Fluid-Structure Interaction
SB	Stenotic Blockage
RBC	Red Blood Cell
CVD	Cardiovascular disease & CVDs r.t. such diseases

0.2 Mathematical Notation

$\therefore$	consequently
$\because$	because
$\implies$	implies
$\iff$	if and only if
$:=$	defines
$\equiv$	equivalence
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	positive real numbers
$\mathbb{R}^-$	negative real numbers
$\mathbb{R}^n$	n-dimensional real vector space
$\mathbb{N}$	natural numbers
$\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$	general basis of $\mathbb{R}^n$
$\{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$	standard basis of $\mathbb{R}^n$
$[n] \subset \mathbb{N}$	set $\{1, 2, \dots, n\}$
$\Omega \subset \mathbb{R}^n$	bounded domain
$\overline{\Omega}$	the closure of Ω
$\partial\Omega$	the boundary of Ω
$C^k(\Omega)$	k times continuously differentiable functions in Ω
$C_0^k(\Omega)$	k times continuously differentiable functions with compact support in Ω
$C_0^k(\overline{\Omega})$	k times continuously differentiable functions which have bounded and uniformly conti derivatives up to order k with compact support in Ω
$C_0^\infty(\Omega)$	smooth functions with compact support in Ω
$L^p(\Omega)$	Lebesgue space of p-integrable functions on Ω

Locally integrable, lipshitz, Holder continuous, Sobolev spaces, weak derivatives, distributions, test functions, multi-index notation

$d\boldsymbol{x}$	Lebesgue measure on $\mathbb{R}^n$
$dS_{\boldsymbol{x}}$	surface measure on $\partial\Omega \subset \mathbb{R}^n$
dV	volume measure on domain $\Omega \subset \mathbb{R}^3$
dS	surface measure on boundary $\partial\Omega \subset \mathbb{R}^3$
∇	gradient operator
$\Delta = \nabla^2 = \nabla \cdot \nabla(\cdot)$	Laplace operator
div	divergence of a vector field
div	divergence of a tensor field
$\boldsymbol{v}_i$	i-th component of vector $\boldsymbol{v}$
$\langle \cdot,\ \cdot \rangle_X$	inner product on vector space X
$\langle \boldsymbol{u},\ \boldsymbol{v} \rangle \equiv \langle \boldsymbol{u},\ \boldsymbol{v} \rangle_{\mathbb{R}^n}$	inner product of vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$
$\dfrac{\partial}{\partial \hat{\boldsymbol{n}}} = \langle \nabla,\ \hat{\boldsymbol{n}} \rangle$	normal derivative on $\partial\Omega$
$\\| \cdot \\|$	L^2-norm

0.3 Domain Specific Notation

R	radius of vessel with diameter $2R$
η	dynamic viscosity $\left[\, Pa \cdot s \,\right]$
μ	kinematic viscosity $\left[\, \frac{cm^2}{s} \,\right]$
τ	shear stress
$\dot{\gamma}$	shear rate
ρ	density field $\left[\, \frac{kg}{cm^3} \,\right]$
p	pressure field
$\boldsymbol{u}$	velocity field $\left[\, \frac{cm}{s} \,\right]$
W_0	Womersley number $\left[\, - \,\right]$
Re	Reynolds number $\left[\, - \,\right]$
Pe	Péclet number $\left[\, - \,\right]$
c	concentration of a material element
D	diffusion coefficient $\left[\, \frac{cm^2}{s} \,\right]$
t	time $\left[\, s \,\right]$
T	terminal time $\left[\, s \,\right], t > 0$
ω	angular frequency $\left[\, \frac{rad}{s} \,\right]$
$\boldsymbol{f_b}$	body force per unit volume $\left[\, \frac{N}{cm^3} \,\right]$

0.4 Mathematical Foundations

Assume Zermelo-Fraenkel set theory with the axiom of choice (ZFC), and according to Cohen, it's consistent if we take the continuum hypothesis (CH); a necessary posulate in continuum mechanics. A *domain* r.t. an open and bounded subset of $\mathbb{R}^n$ with nonempty interior Ω°, for $N \in [3]$. By the Heine-Borel theorem, every domain Ω has a well-defined boundary $\partial\Omega$ with compact closure $\overline{\Omega} = \Omega \cup \partial\Omega$. The compact domains of $\mathbb{R}^N$ are precisely the closed and bounded subsets of $\mathbb{R}^N$. When every path between two points in Ω may be continously contrated to a point without leaving Ω, we say that Ω is *simply connected*.

Given domain Ω, the function sp. $\mathcal{F}(\Omega)$ is the vector sp. $(F(\Omega),\, K)$ of scalar valued functions $f : \Omega \to K$. E.g. $C^0(\Omega)$ r.t. $(C(\Omega),\, \mathbb{R})$ all continuous real valued functions on Ω. Let sp. $C^k(\Omega)$ be the continuous and k-times continuously differentiable functions on Ω, then $C^k(\overline{\Omega})$ r.t. $f \in C^k(\Omega)$ s.t f and it's derivatives up to order k may be continuously extended to the boundary $\partial\Omega$. The sp. $C^\infty(\Omega)$ is the intersection of all $C^k(\Omega)$ for $k \geq 0$, i.e. the infinitely differentiable functions on Ω. And $C^k(\Omega;\, \mathbb{R}^m)$ denotes the functions $\boldsymbol{f} : \Omega \to \mathbb{R}^m$ with components in $C^k(\Omega)$, i.e., $\boldsymbol{f} = (f_i)_{i\in[m]}$ with $f_i \in C^k(\Omega)$ for all $i \in [m]$.

Let $n = \dim(\Omega)$, then we say

$$\begin{aligned}
&\phi : \Omega \to \mathbb{R} \;\text{ r.t. a } \textit{scalar field},\\
&\boldsymbol{f} : \Omega \to \mathbb{R}^n \;\text{ r.t. a } \textit{vector field},\\
&\boldsymbol{T} : \Omega \to \mathbb{R}^{n\times n} \;\text{ r.t. a } \textit{(second-order) tensor field},
\end{aligned}$$

and we write $\phi(\boldsymbol{x})$, $\boldsymbol{f}(\boldsymbol{x})$, and $\boldsymbol{T}(\boldsymbol{x})$[1] for the values at a point $\boldsymbol{x} \in \Omega$. The k-th partial derivative of ϕ w.r.t. coordinate x_i is

$$\partial^k_{x_i}\phi \;\equiv\; \frac{\partial^k \phi}{\partial x_i^k}.$$

The generalized derivative of order $|\alpha|$ of scalar field ϕ is

$$D^\alpha \phi \;\equiv\; \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\,\ldots\,\partial x_n^{\alpha_n}},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. For $|\alpha| = 1$, $D^\alpha\phi$ is a first-order partial derivative of ϕ, e.g. $\partial_{x_i}\phi$ where $\alpha_i = 1$ and $\alpha_j = 0 \;\forall\; j \neq i$.

For real-valued $\phi \in C^1(\Omega)$, the partial derivative w.r.t. coordinate x_i is

$$\partial_{x_i}\phi = \frac{\partial \phi}{\partial x_i}.$$

In coordinate-free notation, for $\boldsymbol{x} \in \Omega$ and $\hat{\boldsymbol{n}} \in \mathbb{R}^n$, the normal derivative is

define as declared in notation sec

[1]Note, for tensor $\boldsymbol{T}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$, $\exists$ a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\boldsymbol{T}(\boldsymbol{x})(\boldsymbol{u}) \mapsto A\,\boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

$$\partial_{\hat{\boldsymbol{n}}}\phi(\boldsymbol{x}) \;:=\; \langle\nabla\phi(\boldsymbol{x}),\; \hat{\boldsymbol{n}}\rangle_{\mathbb{R}^n}.$$

For Lebesgue measure $d\boldsymbol{x}$ on $\mathbb{R}^n$ the integral of ϕ in Ω is

$$\int_\Omega \phi(\boldsymbol{x})\,d\boldsymbol{x} \;\equiv\; \int_\Omega \phi.$$

If $\partial\Omega \in C^1$, then $\phi \in C^1(\overline{\Omega}) \supset L^1(\partial\Omega)$ and the integral at the boundary is

$$\int_{\partial\Omega} \phi(\boldsymbol{x})\,dS_{\boldsymbol{x}} \;\equiv\; \int_{\partial\Omega} \phi.$$

The gradient and Laplacian of ϕ are (respectively):

$$\begin{aligned}
&\phi \;\mapsto\; \boldsymbol{\nabla}(\phi) \;\equiv\; \nabla\phi \;=\; \left(\partial_{x_1},\,\ldots,\,\partial_{x_N}\right)^\top \;=\; (\partial_{x_i}\phi)_{\forall i[n]}\,.\\
&\phi \;\mapsto\; \Delta\phi \;\equiv\; \nabla\cdot\nabla\phi \;=\; \sum_{\forall i[n]} \partial^2_{x_i}\phi.
\end{aligned}$$

For vector-valued $\boldsymbol{f} \in C^1(\Omega;\mathbb{R}^n)$ with components $\boldsymbol{f} = (f_i)_{\forall i[n]}$, the divergence is

$$\operatorname{div}\boldsymbol{f} \;:=\; \nabla\cdot\boldsymbol{f} \;=\; \sum_{\forall i[n]} \partial_{x_i} f_i \;=\; \langle\boldsymbol{f},\; \nabla\rangle_{\mathbb{R}^n}$$

The Laplacian and domain integral of $\boldsymbol{f}$ are defined component-wise:

$$\begin{aligned}
&\boldsymbol{f} \;\mapsto\; \Delta\,\boldsymbol{f} \;:=\; (\Delta f_i)_{\forall i[n]}\\
&\boldsymbol{f} \;\mapsto\; \int_\Omega \boldsymbol{f} \;:=\; \left(\int_\Omega f_i\,d\boldsymbol{x}\right)_{\forall i[n]}.
\end{aligned}$$

For tensor-valued $\boldsymbol{T} \in C^1(\Omega;\mathbb{R}^{n\times n})$ with entries $\boldsymbol{T} = (T_{ij})^n_{i,j=1}$, the divergence operator is

$$\mathbf{div}\,\boldsymbol{T} \;:=\; (\operatorname{div}\boldsymbol{t}_i)_{\forall i[n]} \;:\; \boldsymbol{t}_i \text{ is the ith row of } \boldsymbol{T}$$

The Laplacian and domain integral of $\boldsymbol{T}$ are again defined component-wise:

$$\Delta\,\boldsymbol{T} \;:=\; (\Delta T_{ij})_{\forall i,j[n]} \qquad \text{and} \qquad \int_\Omega \boldsymbol{T} \;:=\; \left(\int_\Omega T_{ij}\,d\boldsymbol{x}\right)_{\forall i,j[n]}.$$

Let K be compact (e.g. $\overline{\Omega}$), then the sp. $(C^k(K),\,\mathbb{R})$ is complete with norm

$$\|\phi\|_{C^k(K)} \;:=\; \sum_{\forall|\alpha|\in[k]} \sup_{\boldsymbol{x}\in K} |D^\alpha\phi(\boldsymbol{x})|\,.$$

meaning that every Cauchy sequence $\{\phi_j\}_{\forall j\in\mathbb{N}} \subset C^k(K)$ converges to a limit in $C^k(K)$. So the complete normed sp. $(C^k(K),\, \|\cdot\|_{C^k(K)})$ is banach, by the definition of banach sp. In particular, for $k = 0$, we have

$$\|\phi\|_{C(\overline{\Omega})} \;:=\; \sup_{\boldsymbol{x}\in\overline{\Omega}} |\phi(\boldsymbol{x})| \;\equiv\; \max_{\boldsymbol{x}\in\overline{\Omega}} |\phi(\boldsymbol{x})| \;=\; \|\phi\|_{\infty} \quad \text{(since } \overline{\Omega} \text{ is compact).}$$

Let $\phi \in L^p(\Omega)$ be a real-valued measurable function on Ω, then the Lebesgue sp. $L^p(\Omega)$ is defined as

$$L^p(\Omega) \;:=\; \left\{\phi : \Omega \to \mathbb{R} \;\middle|\; \|\phi\|_{L^p(\Omega)} < \infty\right\} \quad \text{s.t.} \quad \|\cdot\|_{L^p(\Omega)} : \phi \mapsto \left(\int_{\Omega} |\phi(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p} \quad 1 \le p < \infty.$$

For $p = \infty$, the sp. $L^\infty(\Omega)$ is defined as

$$L^\infty(\Omega) \;:=\; \left\{\phi : \Omega \to \mathbb{R} \;\middle|\; \|\phi\|_{L^\infty(\Omega)} < \infty\right\} \quad \text{s.t.} \quad \|\cdot\|_{L^\infty(\Omega)} : \phi \mapsto \sup_{\boldsymbol{x}\in\Omega} |\phi(\boldsymbol{x})|.$$

where sup here r.t. the essential supremum. It follows that $\boldsymbol{f} \in L^p(\Omega;\, \mathbb{R}^m)$ denotes the functions whose components are in $L^p(\Omega)$, i.e., $\boldsymbol{f} = (f_i)_{i\in[m]}$ with $f_i \in L^p(\Omega)$ for all $i \in [m]$. The sp $(L^p(\Omega),\, \|\cdot\|_{L^p(\Omega)})$ is complete, hence a banach sp. for all $1 \le p \le \infty$.

When refering to a physical quantity, it's measure units are indicated as $\big[\cdot\big]$; whereas dimensionless quantities are indicated as $\big[-\big]$.

TODO:

- forgoe until we work with a Lipschitz boundary: Hölder spaces, then Hilbert and Sobolev spaces. defs and notation, then trace thm., Poincare inequality, Rellich-Kondrachov comp. thm., etc. Distributional derivatives, test functions, multi-index notation, weak derivatives, weak formulation, etc.
- Add def. and notation for Lipschitz continuous functions/sp., $Lip(\Omega)$ and it's norm.
- Add relation of $Lip(\Omega)$ to $C^0(\Omega)$, brenner's book. Don't state why, just cite.
- Proceed into .

Work In Progress (WIP): Misc items.

- Write cylindrical operators
- Write material on directional derivatives and gradients
- Write material on Jacobian and Hessian matrices
- More detail on integration? Fubini's thm., change of variables, etc.
- Scaling argument for completeness of $C^k(K)$ sp. norm and nondimensionalization
- Define metric sp, normed sp., inner-product sp. defs and notation; include stmt. that norm's induce metrics, metrics induce norms.
- Finite dim sp.s are complete, hence banach, and hilbert under a particular choice of inner-product; state such inner-product for their sp.s.

1 Introduction

Hemodynamics studies the kinematics of blood. Our interest is the kinematic motion of blood within the human macrocirculatory system, i.e. the flow of blood in large vessels such as arteries and veins. Blood is observed as a complex fluid of formed elements suspended in plasma, thus, the rheological behavior of blood is non-trivial. We report techniques and methodologies for modeling blood's motion in large vessels.

Our report is organized as follows.

After stating our report's motivation, Sec. 1.1 provides a brief physiological review of the human circulatory system. Then Sec. 1.2 reviews the continuum hypothesis, a necessary postulate in fluid mechanics which treats blood as a continuous medium. This framing reduces the hemodynamic problem to describing blood's motion in a continuum. Then Sec. 1.5 discusses rheological assumptions that lead to constitutive relations between the material properties of blood. By conservation, the kinematic-viscosity Navier–Stokes (NS) equations are obtained (Sec. 1.4). The NS equations are a system of coupled nonlinear partial differential equations (PDEs), the foundation of our hemodynamic models.

Motivation The World Health Organization (WHO) claims cardiovascular diseases (CVDs) resulted in $\approx 32\%$ of all deaths in 2022, the leading cause of death globally, where $\approx 85\%$ of the deaths where from heart attack or stroke. In such cases the underlying CVD is often coronary artery stenosis (CAS), the narrowing of a coranary vessel due to the buildup of plaque. Such plaque is r.t. as a stenotic blockage. There is a need for evidience-based tools to predict and assess the severity of SBs, as often predicition of CAS doesn't mean an obstruction [[1]] and severity assesments often use a simple 0D lummped-parameter model. Cardiological interventionists need better tools for predicting and treating CAS.

Current clinical methods for assessing the severity of a SB rely on imaging techniques such as angiography, intravascular ultrasound (IVUS), and optical coherence tomography (OCT) to visualize the arteries and identify areas of narrowing. Such methods provide valuable information about the anatomy of the arteries, but they do not provide direct information about the functional significance of the CAS. Functional assessment of CAS typically involves measuring the fractional flow reserve (FFR), which is the ratio of the blood pressure downstream of the stenosis to the blood pressure upstream of the stenosis during maximum blood flow. However, measuring FFR requires the use of a pressure wire, which can be invasive and carries some risks. Therefore, there is a need for non-invasive methods to assess the functional significance of CAS. cleanup, add refs

1.1 Physiology

The circulatory system is the human heart, vascular network, lungs, and organs. The systems source is the heart, transporting oxygen-rich blood to the organs and deoxygenated (and carbon dioxide-enriched) blood back to the lungs. Lungs discharge CO_2 and enrich the blood with Oxygen, referred to as the *pulmonary* *circulation* and the *systemic circulation* (resp.). The *macrocirculatory system* consists of the heart and the large vessels in the systemic circulation. The arteries of the macrocirculatory system transport oxygenated blood from the heart, driving the return of deoxygenated blood in large vessels back to the heart.

A single beat of the heart propels blood through the macrocirculatory system, the "lub-dub" sound. The beat and the following sequence of events following until the successive beat is known as the *cardiac cycle*. The cardiac cycle consists of two main phases: systole and diastole, during which the heart chamber is accumulating blood and releasing blood (resp.). Normal resting heart rate is considered to be $\omega = 70$ bpm, so the cardiac cycle is approximately $0.86s$. and consists of:

1. Systole (ventricular contraction) ≈ 0.3 seconds.
2. Diastole (ventricular relaxation) ≈ 0.7 seconds.

During ventricular contraction, blood is ejected from the left ventricle into the aorta, creating a pressure wave that propagates through the arterial network. The Womersley number W_0 characterizes the pulse waves in large vessels by comparing the pulse frequency ω to the viscous effects determined by the kinematic viscosity $\mu := \eta/\rho \ \left[\ \frac{cm^2}{s}\ \right]$:

$$W_0 := \sqrt{\frac{\rho\omega U}{\eta U L^{-}2}} = \sqrt{L^2 \cdot \frac{\omega}{\mu}} \left[\ -\ \right] \quad \text{s.t.} \quad \begin{cases} L: \text{ characteristic length scale } \left[\ cm\ \right] \\ U: \text{ characteristic velocity scale } \left[\ \frac{cm}{s}\ \right] \\ \eta: \text{ dynamic viscosity } \left[\ Pa \cdot s\ \right] \\ \rho: \text{ blood density } \left[\ \frac{kg}{cm^3}\ \right] \end{cases}$$

High W_0 indicates large, rapid pulses where inertial effects dominate viscous effects while low W_0 indicates small, slow pulses where viscous effects dominate inertial effects. The Reynolds number Re characterizes flow in blood vessel

$$Re := L \cdot \frac{\rho U}{\eta} \left[\ -\ \right].$$

Low Re indicates laminar flow while high Re suggests turbulent flow. Note [4, Table 1.1, p. 10, §1.1] shows $W_0 \propto L$ and $Re \propto (L)^{-1}$; we observe large pulses and turbulent flow in large vessels and small pulses and laminar flow in small vessels. Note, most often we set $L = 2R$ where R is the radius of the vessel.

Observed Cardiac Cycle Characteristics

The blood volume of a human is approximately 5.7-6.0 liters of blood, flowing a full cycle roughly every minute. The energy driving the flow comes from oxygen and nutrients absorbed from food, creating waste products that must be removed; the *coronary artery*'s responsibility. The buildup of waste products results in Arteriosclerosis, a narrowing of the coronary artery, leading to reduced and turbulent blood flow. (Add citations here of turbulence in the presence of stenotic arteries).

Is there a relationship between W_0 and Re we can exploit to simplify our models? Appears so...

Constituents and hematocrit. Blood consists of plasma and formed elements which we call cells. Red blood cells (RBCs) comprise $\approx 97\%$ of the cellular volume, and cellular volume is approximately $\approx 45\%$ of the blood volume. The remaining $\approx 55\%$ of blood volume is plasma, which is $\approx 90\%$ water. The ratio of RBC volume to total blood volume is the *hematocrit value* H, a key metric governing apparent viscosity η: as H increases, η typically increases (cf. S6.5.1 [3]). The formed elements suspended in plasma include white blood cells (WBCs) and platelets.

1.2 Continuum of Blood

W.a.t. simulate blood flow in a time-dependent fluid domain $\Omega_B \subset \mathbb{R}^{N+1}$; we'll take $N \equiv 3$, the natural setting for the derivations that follow. For each $t \in I_T := [0,T] \subset \mathbb{R}$, with $T > 0$, we define

$$\Omega_B(t) := \{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{x} \text{ lies inside the vessel at time } t\}.$$

Formally, the fluid region is a time-dependent family of open sets $\{\Omega_B(t)\}_{\forall t \in I_T} \subset \mathbb{R}^N$ occupied by blood at time t. Let $\Omega_B(0)$ r.t. the *reference* configuration and $\Omega_B(t)$ r.t. the *current* configuration. The fluids computational domain r.t. all possible configurations of $\Omega_B(t)$, i.e., the spatial-temporal lumen

$$\Omega_B \;:=\; I_T \times \Omega_B(t) \;\equiv\; \{(t, \boldsymbol{x}) \in \mathbb{R}^{N+1} \;:\; x \in \Omega(t),\; t \in I_T\}.$$

For each material particle $\boldsymbol{\xi} \in \Omega_B(0)$, let's assume $\boldsymbol{u}(t, \boldsymbol{\xi}(t))$ is the flow feild determined from a system of $PDEs$ governing $\boldsymbol{\xi}$'s motion in Ω_B. Our aim here is to derive such PDEs – the Navier-Stokes system.

Blood Dynamics

The spatial configuration $\Omega_B(t)$ is determined by a Lagrangian mapping

$$\mathcal{L}_t : \Omega_B(0) \mapsto \Omega_B(t), \quad \boldsymbol{\xi} \mapsto \boldsymbol{x}(t, \boldsymbol{\xi}) \;=\; \mathcal{L}_t(\boldsymbol{\xi})$$

Let's impose that $\mathcal{L}_t$ is a continuous bijection $\forall t \in I_T$ in $\overline{\Omega_B}$, then $\exists\ \mathcal{L}_t^{-1} : \overline{\Omega_B(t)} \mapsto \overline{\Omega_B(0)}$ s.t. $\boldsymbol{\xi} = \mathcal{L}_t^{-1}(\boldsymbol{x})$. I.e., there is a one-to-one correspondence between material fluid particles in the reference configuration and spatial points in the current configuration at each time t. The variables $(t, \boldsymbol{x})$ and $(t, \boldsymbol{\xi})$ are refered to as the Eulerian and Lagrangian coordinates, resp..

Remark 1.2.1 (Eulerian vs. Lagrangian). Informally, the Eulerian approach focuses our attention to $\boldsymbol{x} \in \Omega_B(t)$, namely, some fluid particle located at $\boldsymbol{x}$ at a particular time t. Whereas, the Lagrangian approach tracks an individual fluid particle $\boldsymbol{\xi} \in \Omega_B(0)$ along it's trajectory $\boldsymbol{T_\xi} := \{(t, \boldsymbol{\xi}) : t \in I_T\}$ as it moves through space and time along a characteristic curve defined by the velocity field $\boldsymbol{u}$.

For $\phi \in C^k(\Omega_B)$ s.t. $(t, \boldsymbol{x}) \mapsto \phi(t, \boldsymbol{x})$ in the Eulerian variables, we'll let $\hat{\phi}(t, \xi) := \phi(t, \boldsymbol{x})$ s.t. $\boldsymbol{x} = \mathcal{L}_t(\boldsymbol{\xi})$. Or equivalently, we'll let $\hat{\phi} = \phi \circ \mathcal{L}_t$ and it follows that $\phi = \hat{\phi} \circ \mathcal{L}_t^{-1}$.

Definition 1.2.2. Let the Lagrangian velocity feild be defined as as the time derivative along the trajectory $T_{\boldsymbol{\xi}}$ of a flulid particle $\boldsymbol{\xi} \in \Omega_B(0)$

$$\hat{\boldsymbol{u}} := \partial_t \boldsymbol{x}(t, \boldsymbol{\xi}) \mapsto \hat{\boldsymbol{u}}(t, \boldsymbol{\xi}) \quad \text{in} \;\; \Omega_B(t)$$

A change of corrdinates in 1.2.2 yeilds the velocity feild $\boldsymbol{u}$ in the Eulerian frame, i.e.,

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \circ \mathcal{L}_t^{-1} \quad \Longleftrightarrow \quad \boldsymbol{u}(t, \boldsymbol{x}) = \hat{\boldsymbol{u}}(t, \mathcal{L}_t(\boldsymbol{x})^{-1})$$

Then if the velocity feild $\hat{\boldsymbol{u}}$ and the reference configuration $\Omega(0)$ are known, we arrive at the Cauchy problem

$$\begin{cases} \partial_t \boldsymbol{x}(t, \boldsymbol{\xi}) = \hat{\boldsymbol{u}}(t, \boldsymbol{\xi}), \forall t \in I_T \\ \boldsymbol{x}(0, \xi) = \boldsymbol{\xi}. \end{cases}$$

Material Derivatives

Definition 1.2.3. Let $\phi \in C^k(\Omega_B)$ and $\hat{\phi} = \phi \circ \mathcal{L}_t$, then the *material derivative* $D_t\phi$ is the derivative of ϕ w.r.t. t in the Lagrangian frame expressed as a function in the Eulerian frame, i.e.,

$$\begin{aligned} D_t\big(\phi(t, \boldsymbol{x})\big) &:= \partial_t \hat{\phi}(t, \boldsymbol{\xi}), \quad \text{s.t. } \boldsymbol{\xi} = \mathcal{L}_t^{-1}(\boldsymbol{x}) \\ &= \frac{d}{dt}\phi(t, \boldsymbol{x}(t, \boldsymbol{\xi})), \quad \forall \boldsymbol{\xi} \in \Omega_B(0). \end{aligned}$$

By the multivariable chain rule, $D_t\phi$ in 1.2.3 may be written as

$$D_t\big(\phi(t, \boldsymbol{x})\big) = \partial_t \phi + \langle \nabla\phi, \, \boldsymbol{u} \rangle_{\mathbb{R}^N} \quad \text{in } \Omega_B$$

where we may see that the material derivative $D_t\phi$ measures the rate of variation of ϕ along trajectory $T_{\boldsymbol{\xi}}$. The literature also r.t. D_t as the *substantial derivative*, *advective derivative*, *lagrangian derivative*, or *convective derivative*.

Now differentiating $\hat{\boldsymbol{u}}$ w.r.t. t yeilds the acceleration vector feild $\hat{\boldsymbol{a}}$ in the Lagrangian frame

$$\begin{aligned}
\hat{\boldsymbol{a}} \;&:=\; \partial_t \hat{\boldsymbol{u}}(t,\boldsymbol{\xi}) = \partial_t^2 \boldsymbol{x}(t,\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Omega_B(0) \\
\iff \boldsymbol{a} \;&=\; D_t \boldsymbol{u} \;=\; \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \\
\iff \boldsymbol{a}(t,\boldsymbol{x}) \;&=\; \partial_t \boldsymbol{u}(t,\boldsymbol{x}) + \sum_{i=1}^{N} u_i(t,\boldsymbol{x}) \partial_{x_i} \boldsymbol{u}(t,\boldsymbol{x}) \quad \forall (t,\boldsymbol{x}) \in \Omega_B.
\end{aligned}$$

The acceleration $\boldsymbol{a}$ measures the rate of change of velocity $\boldsymbol{u}$ along the trajectory $T_{\boldsymbol{\xi}}$, i.e., we applied the material derivative to the velocity feild $\boldsymbol{u}$ component-wise. In other words, $\boldsymbol{a}$ is the material derivative 1.2.3 applied component-wise to the velocity field $\boldsymbol{u}$.
(see pg. 19 [2] for more details and examples)

The defermation gradient $\hat{\boldsymbol{F}}(t)$ in the Lagrangian frame is defined as

$$\begin{aligned}
\hat{\boldsymbol{F}}(t) \;&:=\; \nabla_{\boldsymbol{\xi}} \mathcal{L}_t(\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \Omega_B(0) \\
\iff \hat{\boldsymbol{F}}(t)(\boldsymbol{x}) \;&=\; \nabla_{\boldsymbol{\xi}} \mathcal{L}_t(\mathcal{L}_t^{-1}(\boldsymbol{x})) \;=\; \partial_{\boldsymbol{\xi}} \boldsymbol{x} \quad \forall (t,\boldsymbol{x}) \in \Omega_B
\end{aligned}$$

The deformation gradient $\hat{\boldsymbol{F}} : \Omega_B(0) \to \mathbb{R}^{N \times N}$ measures the local spatial deformation of fluid particles in the reference configuration as they move to the current configuration under $\mathcal{L}_t$ along trajectory $T_{\boldsymbol{\xi}}$. To ensure $\hat{\boldsymbol{F}}(t)$ is invertible, it is sufficient to note that $\mathcal{L}_t$ is a continuous bijection in $\overline{\Omega_B}$ with continuous inverse $\mathcal{L}_t^{-1}$. Thus by the inverse function theorem $\hat{\boldsymbol{F}}(t)$ is invertible $\forall \boldsymbol{\xi} \in \Omega_B(0)$. So we may define the Jacobian determinant $\hat{J}(t)$ in the Lagrangian frame as

$$\begin{aligned}
\hat{J}(t)(\boldsymbol{\xi}) \;&:=\; \det\left(\hat{\boldsymbol{F}}(t)(\boldsymbol{\xi})\right) \quad \forall \boldsymbol{\xi} \in \Omega_B(0) \\
\iff J(t)(\boldsymbol{x}) \;&=\; \det\left(\hat{\boldsymbol{F}}(t)(\mathcal{L}_t^{-1}(\boldsymbol{x}))\right) \quad \forall (t,\boldsymbol{x}) \in \Omega_B
\end{aligned}$$

The Jacobian determinant $J(t)(\boldsymbol{x})$ measures the local change in volume of an infinitesimal fluid element as it moves from the reference configuration to the current configuration at time t along trajectory $T_{\boldsymbol{\xi}}$. We'll later see that $\partial_t J$ relates to the fluids divergence $\operatorname{div}(\boldsymbol{u})$.

1.3 NS-derivation

In the following derivations we let $N \equiv 3$ so that $\Omega_B(t) \subset \mathbb{R}^3$. We assume blood fluid is a continuum that deforms continously, i.e., at every point $\boldsymbol{x} \in \Omega_B(t)$ and time $t \in I_T$, the blood's kinematic quantities are described by sufficiently smooth fields. At microscopic scales this continuum hypothesis breaks down, since matter is a discrete collection of molecules, but at macroscopic scales empirical evidence suggests such models remain accurate.

For each $t \in I_T$, herein assume $\Omega_B(t)$ is simply connected and boundary $\partial\Omega_B(t) \in C^1$, so, any closed curve in $\Omega_B(t)$ can be continuously contracted to a point within $\Omega_B(t)$; such an assumption is reasonable in a health vessel with a smooth wall. This regularity allows us to globally define geometric quantities such as surface differential operators on the fluids boundary $\partial\Omega_B$, then we may apply stokes thoerem and Green's identities on the C^1 boundaries, useful for obtaining weak integro-differential and variational formulations.

Remark 1.3.1. A weaker regularity condition is that $\partial\Omega_B$ is Lipschitz. Here we must rely on standard trace theorems for Soblev spaces (e.g. $H^1(\Omega_B(t))$). In particular, the outward unit normal is then defined a.e. on $\partial\Omega_B(t)$, so we can meaningfully speak of normal and tangential components of vector feilds on the boundary.

Let $V(t)$ denote a measurable material volume at time t that moves with the fluid flow, i.e.,

$$V(t) \subset \Omega_B(t), \quad \text{for each } t \in I_T$$

Equivalently, $V(t)$ is the image of some reference volume $V(0) \subset \Omega_B(0)$ under the Lagrangian mapping $\mathcal{L}_t$, i.e.,

$$V(t) = \mathcal{L}_t\big(V(0)\big).$$

Theorem 1.3.2 (Reynolds Transport Theorem)**.** *Let $V(0) \subset \Omega_B(0)$ be a material volume in the reference configuration, and $V(t) \subset \Omega_B(t)$ its image in the current configuration under $\mathcal{L}_t$. For any sufficiently smooth scalar field $\phi : \Omega_B \to \mathbb{R}$,*

$$\frac{d}{dt}\int_{V(t)} \phi(t,\boldsymbol{x})\, dV \;=\; \int_{V(t)} \partial_t\phi(t,\boldsymbol{x})\, dV + \int_{\rho V(t)} \phi(t,\boldsymbol{x})\, \langle \boldsymbol{u},\ \hat{\boldsymbol{n}}\rangle\, dS,$$

where $\boldsymbol{u}$ is the fluid velocity field and $\hat{\boldsymbol{n}}$ the outward unit normal on $\rho V(t)$.

Proof. Ref [2], theorem 2.2. pg. 21. □

Let blood's velocity, thermodynamic pressure, and density fields be

$$\begin{aligned}
&\boldsymbol{u} : \Omega_B \to \mathbb{R}^3,\ (t,x,y,z) \mapsto (u_1(t,x,y,z),\, u_2(t,x,y,z),\, u_3(t,x,y,z))^\top, \quad \left[\frac{m}{s}\right]\\
&p : \Omega_B \to \mathbb{R}^+,\ (t,x,y,z) \mapsto p(t,x,y,z), \quad \left[\mathrm{Pa} \equiv \frac{N}{m^2}\right]\\
&\rho : \Omega_B \to \mathbb{R}^+,\ (t,x,y,z) \mapsto \rho(t,x,y,z), \quad \left[\frac{kg}{m^3}\right].
\end{aligned}$$

One may make a distinction between incompressible fluids and incompressible flows.

Definition 1.3.3 (Incompressibile fluid)**.** An element $V \in \Omega_B$ with constant density $\rho(\boldsymbol{x},t)\ \forall t,\ \boldsymbol{x} \in I_T \times V(t)$ is an *incompressible fluid.*

Definition 1.3.4 (Incompressible flow)**.** An element $V \in \Omega_B$ subject to $\boldsymbol{u}$ with constant rate of material density change (in both space and time so that $D_t\rho = 0$ for all $t \in I_T$) undergoes an *incompressible flow* in Ω_B.

I.m.b.s. for all material elements $V \in \Omega$, 1.3.3 implies 1.3.4. Note, the converse is not generally true: an incompressible flow ($D_t\rho = 0$) only preserves density along particle paths and allows $\rho = \rho(\boldsymbol{x})$ to vary spatially; but, particularly, in our case the initial density is distributed uniformly in space, and here incompressible flow also implies incompressible fluid. By the incompressibile assumption of blood fluid, the material density $\rho(\boldsymbol{x}, t) \equiv \rho_0$ for all $\boldsymbol{x} \in \Omega(t)$ and $t \in I_T$. Constant blood density implies $D_t\rho = 0$, i.e., incompressibile flow in $\Omega(t)$

$$\partial_t\rho + \nabla \cdot (\rho\boldsymbol{u}) = 0 \qquad \Longleftarrow \qquad \rho \text{ constant in } \Omega_B(t)$$

By assumption blood is Newtonian, linear momentum balance follows from Newton's 2nd Law ($F = ma$) as

$$\rho(\partial_t\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \;=\; \mathbf{div}(\boldsymbol{T}) + \rho\boldsymbol{f_b}, \qquad \text{in } \Omega_B(t)$$

where $\boldsymbol{T}$ is the Cauchy stress tensor describing the fluids deformation, $\boldsymbol{f_b}$ may be some body force per unit mass $[\frac{m}{s^2}]$. Let the rate-of-deformation tensor be defined as the symmetric part of the velocity gradient, i.e.,

$$\boldsymbol{D}(\boldsymbol{u}) \;:=\; \tfrac{1}{2}\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\big) \quad \text{s.t.} \quad \nabla\boldsymbol{u} \;:=\; \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{bmatrix}.$$

Note that $\boldsymbol{u} \mapsto \boldsymbol{D}(\boldsymbol{u})$ captures spatial deformations of element V in Ω_B under flow $\boldsymbol{u}$.

Definition 1.3.5. A fluid is *Newtonian* if its Cauchy stress tensor $\boldsymbol{T}$ depends linearly on the rate-of-deformation tensor $\boldsymbol{D}(\boldsymbol{u})$.

Definition 1.3.6. A fluid is *isotropic* if its constitutive response is independent of the coordinate system. Writing the Cauchy stress as $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{D})$, isotropy means that for every orthogonal rotator $\boldsymbol{Q} \in \mathrm{SO}(3)$,

$$\boldsymbol{Q}\,\boldsymbol{T}(\boldsymbol{D})\,\boldsymbol{Q}^\top \;=\; \boldsymbol{T}\big(\boldsymbol{Q}\,\boldsymbol{D}\,\boldsymbol{Q}^\top\big).$$

Definition 1.3.7 (Newtonian, isotropic constitutive law)**.** For a Newtonian, isotropic fluid the Cauchy stress is

$$\boldsymbol{T} = -\,p\,\mathrm{I} + 2\,\eta\,\boldsymbol{D}(\boldsymbol{u}) + \lambda\,\mathrm{div}(\boldsymbol{u})\,\mathrm{I},$$

where $\eta > 0$ is the dynamic (shear) viscosity and λ is the bulk viscosity.

So an isotropic fluid at rest (quiescent state $\boldsymbol{u} \equiv \mathbf{0}$) sustains only hydrostatic stress:

$$\implies \boldsymbol{T} \;=\; -\,p\,\mathrm{I} \quad \text{when } \boldsymbol{u} \equiv \mathbf{0}.$$

Remark 1.3.8 (Divergence-free condition). By assuming blood is incompressible fluid 1.3.3 and flow 1.3.4 we have

$$\begin{aligned}
&D_t\rho = 0\\
&\iff \partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0\\
&\iff \operatorname{div}(\rho_0\boldsymbol{u}) = 0 \quad (\because\ \rho = \rho_0)\\
&\iff \nabla\cdot(\rho_0\boldsymbol{u}) = 0\\
&\iff \langle\rho_0\boldsymbol{u},\ \nabla\rangle = 0\\
&\iff \rho_0\langle\boldsymbol{u},\ \nabla\rangle = 0\\
&\iff \langle\boldsymbol{u},\ \nabla\rangle = 0\\
&\iff \nabla\cdot\boldsymbol{u} = 0,
\end{aligned}$$

and we r.t. $\nabla\cdot\boldsymbol{u} = \operatorname{div}(\boldsymbol{u}) = 0$ as the *divergence-free condition* of $\boldsymbol{u}$ in $\Omega(t)$.

Consequently, the constitutive law1.3.7 simplifies as follows.

Definition 1.3.9 (Incompressible stress tensor)**.** The Cauchy stress 1.3.7 simplifies to

$$\boldsymbol{T} \;=\; -\,p\,\mathrm{I} + 2\,\eta\,\boldsymbol{D}(\boldsymbol{u}).$$

Remark 1.3.10 (Dynamic vs. Kinematic Viscosity). In our model assumptions, *dynamic viscosity* $\eta \in \mathbb{R}^+$ and *kinematic viscosity* $\mu \in \mathbb{R}^+$ relate as

$$\mu \;:=\; \frac{\eta}{\rho} \;=\; \frac{\eta}{\rho_0} \;\in\; \mathbb{R}^+.$$

Here η quantifies the internal resistance of blood to shear deformation, i.e., $\eta \;:=\; \dfrac{\tau}{\dot{\gamma}}$, with units $[Pa\cdot s]$. Moreover μ adjusts η by the density ρ, capturing the viscous diffusion of momentum per unit mass, with units $\left[\frac{m^2}{s}\right]$. Intuitively, η measures how "thick" or "sticky" the fluid is, while μ measures how quickly momentum diffuses through the fluid due to viscosity.

Remark 1.3.11 (Newtonian Blood Justification). When diameter d and hematocrit effects are needed, one may use a Non-Newtonian model with relative viscosity $\eta_r(H,d)$ that scales an absolute baseline η:

$$\eta_{\text{eff}} \;=\; \eta_r(H,d)\,\eta \quad \text{(effective viscosity)}$$

An empirical fit from [5]

$$\eta_r \;=\; 1 + (\eta_{0.45} - 1)\,\frac{(1-H)^C - 1}{(1-0.45)^C - 1} \;\; \text{s.t.} \;\; \begin{cases} \eta_{0.45} = 6\,e^{-0.085\,d} + 3.2 - 2.44\,e^{-0.06\,d^{0.645}}, \\ C = \left(0.8 + e^{-0.075\,d}\right)\left(\dfrac{1}{1+10^{-11}\,d^{12}} - 1\right) + \dfrac{1}{1+10^{-11}\,d^{12}}, \end{cases}$$

where $d \;:=\; 2R/(1.0\mu m)$ is the (scaled) vessel diameter. In large vessels, η_r is often constant, justifying the Newtonian assumption. [[4], sec. 3.1]

A fluid that deforms independent of time is simple, such fluids deformation and rate of deformation aren't subject to material memory effects (also r.t. as viscoelastic effects). The contrary are complex fluids, their deformation is subject to both viscous and elastic characteristics. (Out of place)

1.4 Navier-Stokes

Let $\boldsymbol{f}$ be an external force acting on a continuum of blood fluid. When modeling non-Newtonian effects (when $\eta \neq$ constant), the kinematic viscosity $\mu(\cdot)$ is often chosen by Careau model [3]

$$2\,\mu(|\boldsymbol{D}|^2) = \eta_\infty + (\eta_0 - \eta_\infty)\cdot\left(1 + \kappa|\boldsymbol{D}|^2\right).$$

Where η_0 and η_∞ are chosen to be the viscosity for very small and very large shear rates, resp., and $\kappa \in \mathbb{R}^+$ and $n \in (-0.5, 0)$ are model parameters. According to [[4], pg. 38], we often set

$$\eta_0 = 65.7\cdot 10^{-3}\;Pa\cdot s,\;\; \eta_\infty = 4.45\cdot 10^{-3}\;Pa\cdot s, \kappa = 212.2\;s^2, \;\;\text{and } n = -0.325$$

In the Newtonian case, we choose $\eta = \eta_\infty$ which allows us to determine μ as $\mu(|\boldsymbol{D}|^2) = \eta$. When coupling our momentum balance equation with the divergence-free condition of $\boldsymbol{u}$, we obtain the Navier-Stokes (NS) equations.

Definition 1.4.1 (Conservative-Momentum Balance Form)**.**

$$\begin{cases} \partial_t(\rho\boldsymbol{u}) \;+\; (\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \;=\; -\nabla p \;\; \mathbf{div}\big(2\,\eta\,\boldsymbol{D}(\boldsymbol{u})\big) \;\; \rho\,\boldsymbol{f}, \\ \operatorname{div}(\boldsymbol{u}) \;=\; 0, \qquad \rho \;\equiv\; \rho_0 > 0 \text{ (constant)}. \end{cases}$$

Since $\rho \;\equiv\; \rho_0$ and $\eta \;\equiv\; \mu|\boldsymbol{D}|^2$, the advective form follows from 1.4.1.

Definition 1.4.2 (Generalized-Newtonian Navier-Stokes (NS))**.**

$$\begin{cases} \rho\big(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) = -\nabla p \;+\; \mathbf{div}\Big(2\,\mu(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big) \;+\; \rho\,\boldsymbol{f} \\ \mathrm{div}(\boldsymbol{u}) = 0, \end{cases}$$

We divide ρ and obtain the kinematic-viscosity form from 1.4.2.

Definition 1.4.3 (Kinematic-Viscosity Navier-Stokes (NS))**.**

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\,\frac{\nabla p}{\rho} \;+\; \mathbf{div}\Big(\frac{2}{\rho}\,\mu(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big) \;+\; \boldsymbol{f}, \\ \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\,\frac{\nabla p}{\rho} \;+\; \mathbf{div}\Big(\frac{2}{\rho}\,\mu(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big) \;+\; \boldsymbol{f}, \\ \mathrm{div}(\boldsymbol{u}) = 0, \end{cases}$$

Finally, we write the NS system in operator form.

Definition 1.4.4. We write Eq. 1.4.2 in standard form

$$\begin{cases} F\big(\partial_t \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla p,\, \boldsymbol{u},\, p;\, \rho, \mu\big) \;=\; \boldsymbol{f}, \\ \mathrm{div}(\boldsymbol{u}) \;=\; 0, \end{cases}$$

where $F\big(\partial_t \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla p,\, \boldsymbol{u},\, p;\, \rho, \mu\big) \;:=\; \partial_t \boldsymbol{u} \;+\; (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \;+\; \dfrac{\nabla p}{\rho} \;-\; \mathbf{div}\Big(\dfrac{2}{\rho}\,\mu\,(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big)$

Remark 1.4.5. The NS equations are a non-linear coupled system of PDEs. The first equation follows from the balance of linear momentum and it's terms are characterized as:

- The convective term $\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \rho\begin{bmatrix}\langle \boldsymbol{u},\, \nabla\boldsymbol{u}_1\rangle \\ \langle \boldsymbol{u},\, \nabla\boldsymbol{u}_2\rangle \\ \langle \boldsymbol{u},\, \nabla\boldsymbol{u}_3\rangle\end{bmatrix}$ governs acceleration of fluid (non-linear).
- The diffusive term $\mathbf{div}\Big(2\,\mu(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big)$ describes the viscouselastic behavior (linear since μ is constant).

The second equation is the continuity equation, a consequence of the assumed fluid properties of blood that lead to the divergence-free condition on $\boldsymbol{u}$. The total system comprises of four equations in four unknowns: the three components of the velocity field $\boldsymbol{u}$ and the pressure field p.

If pressure p and the velocity $\boldsymbol{u}$ are given, the Cauchy stress $\boldsymbol{T}$ is computed from Eq. 1.3.9. It follows that the wall shear stress (WSS) at the vessel wall is:

$$\mathrm{WSS} \;:=\; \langle \boldsymbol{t_{blood}},\, \boldsymbol{T}\,\hat{n}\rangle \quad : \quad \begin{cases} t_{blood} \text{ is tangent of a flow line through a cross-sectional area} \\ \hat{n} \text{ is outer normal of the cross-sectional area} \end{cases}$$

Forgoing the rigid-wall assumption allows us to model the relationship between the vessel wall and blood flow. Applicable models are referred to as fluid-structure interaction (FSI) models.

discuss in a later section

1.4.1 NS in Cylindrical Coordinates

Let our vessel wall $\partial\Omega = [0, T] \times \partial\Omega(t)$ be a surface in $\mathbb{R}^4$ that evolves in time which we refer to as the interface. Let $\overline{\Omega} = \partial\Omega \cup \Omega$ be the closed and compact region enclosed by our interface. So the region enclosed by our interface is Ω, and we aim to model the velocity and pressure fields on Ω.

The relationship between cartesian and cylidrical coordinates is

$$(x,\ y,\ z)\ \mapsto (r\sin(\theta),\ r\cos(\theta),\ z), \quad r = \sqrt{x^2 + y^2}.$$

Assume a vessel of length L is aligned with the z-axis whose cross-section is circular with radius $R(z, t)$ at axial position z and time t. Our fluid domain becomes

$$\Omega(t)\ =\ \{(r,\ ,\theta\ , z) \in \mathbb{R}^3\ :\ r \in [0, R(z, t)],\ \theta \in [0, 2\pi),\ z \in [0, l)\}$$

where $R(z, t)$ is the vessel radius at axial position z and time t.

Ref. notes for further details on deriving the transformation rules for vector calculus operators, or, notes where I do the derivation directly. Then discuss simplifications for axisymmetric flow.

1.5 Dimension Reduced Models

One chooses a model based upon the specific application, computational resources, and desired accuracy. We construct models of blood flow in various geometries, starting from a single vessel, then extending our approach to bifurcations and arterial networks. Our strategy involves a *domain decomposition approach.*

We seek solutions to inital and boundary value problems of Eq. 1.4.2.

Definition 1.5.1. Let $\boldsymbol{u_0} : \Omega(0) \to R^3,\ \boldsymbol{x} \mapsto \boldsymbol{u_0}(x)\ :\ \boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u_0}$. We refer to $\boldsymbol{u_0}$ as the initial condition of velocity feild $\boldsymbol{u}$

Definition 1.5.2. Let $\boldsymbol{u^0}(t)$ and $\boldsymbol{u^1}(t)$ be the velocity feilds at S_0 and S_1.

In practice, $\boldsymbol{u_0}(z)$ may be prescribed or determined from sensor data.

1.6 Dimension-Reduced Models of Blood Flow

We start our disscusion with 1D and 0D models, reducing the d.o.f. in the NS system 1.4.2 by imposing further simplifying assumptions. Namely, w.a.t. compute average pressures and velocities after a relatively short simulation time by solving the 1D NS system in a compliant vessel with suitable side conditions. Because our model averages pressure and velocity over a surface-area, we obtain a uniform distribution of WSS on the vessel.

One may start by introducing a rigid-vessel assumption, which leads to a no slip condition that $\boldsymbol{u}\big|_{\partial\Omega(t)} = \mathbf{0}$. Instead we seek to model the link between blood flow and the deformation of the vessel wall. We begin our derivation of Dimension-Reduced models by assuming the following transformation exists of our fluid domain boundary $\Omega(t)$ to a simplified geometry.

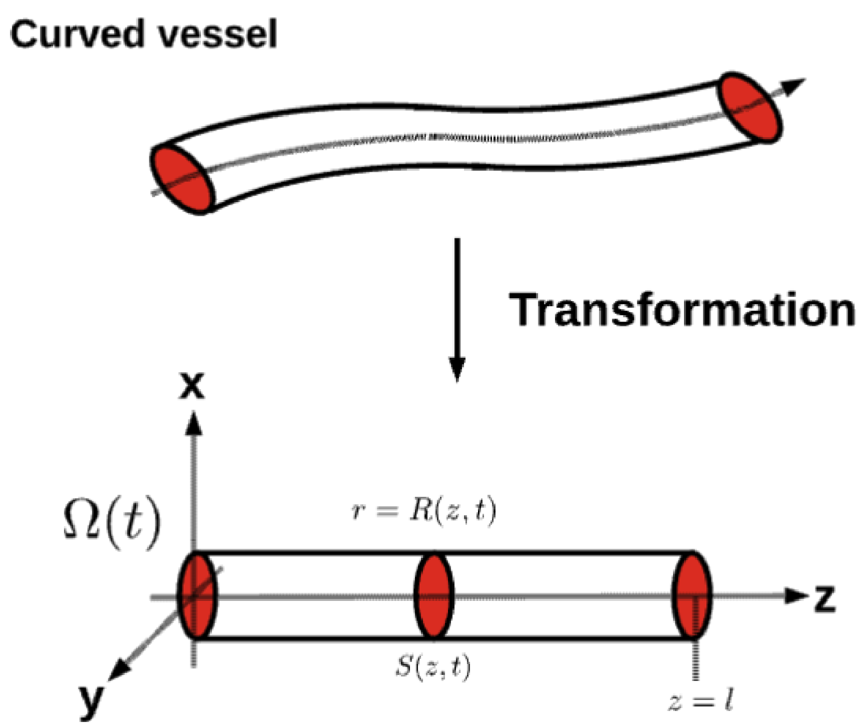

Figure 1: From [4] [Fig. 3.2, pg. 37]

We consider the fluid dynamics of the following fluid element contained in a portion of the lumen $\Omega(t)$ Let

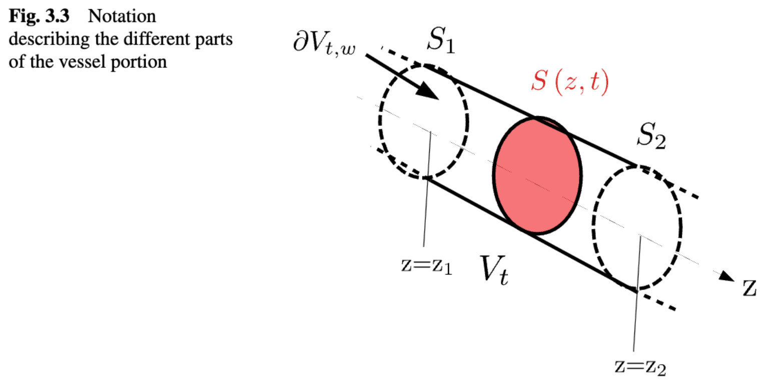

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Figure 2: From [4] [Fig. 3.3, pg. XX]

$S_1(t),\ S_2(t)$ be the time-dependent shaded boundaries at $z = z_1$ and $z = z_2$ s.t. $0 < z_1 < z_2 < \ell$. Let V_t be the fluid element of blood. The boundary of the fluid element is $\partial V_t \ = \ S_1(t) \cup S_2(t) \cup \partial V_{t,w}$ such that $\partial V_{t,w}$ is the vessel wall in contact with the fluid element. According to the Reynold's transport theorem for scalar feild $\phi \in L^1(\Omega(t))$.

$$\frac{d}{dt}\int_{V_t} \phi dV \ = \ \int_{V_t} \frac{\partial \phi}{\partial t} dV \ + \ \int_{\partial V_t} (\boldsymbol{u_b} \cdot \hat{\boldsymbol{n}})\phi dS$$

where $\boldsymbol{u_b}$ is the velocity feild deforming the boundary ∂V_t (Pf. see wiki). If we assume the normal component of $\boldsymbol{u_b} = \boldsymbol{0}$ near the inlet and outlet boundaries S_1 and S_2 (resp.) of Ω, then the motion of the vessel wall is coupled to the blood flow through the fluid element V_t. The velocity $\boldsymbol{u_b}$ is equivalent to the velocity of the vessel wall $\partial\Omega(t)$ in contact with the boundary element ∂V_t. I.e., the vessel wall velocity $\boldsymbol{u_w} = \boldsymbol{u_b}$. Now let $\boldsymbol{w} = \boldsymbol{u_w} - \boldsymbol{u}$ be the relative velocity of the vessel wall w.r.t. the velocity $\boldsymbol{u} = (u_1, u_2, u_3)^\top$ of the blood element V_t. Then it follows that

$$\begin{aligned}\int_{\partial V_t} (\boldsymbol{u_b} \cdot \hat{\boldsymbol{n}})\phi \, dS \ &= \ \int_{\partial V_t} (\boldsymbol{u_w} \cdot \hat{\boldsymbol{n}})\phi \, dS \\ &= \int_{\partial V_t} (\boldsymbol{w} \cdot \hat{\boldsymbol{n}})\phi \, dS \ + \ \int_{\partial V_t} (\boldsymbol{u} \cdot \hat{\boldsymbol{n}})\phi \, dS\end{aligned}$$

Let $\bar{\phi}$ denote the average value of ϕ defined over a surface S

$$\bar{\phi} \ := \ \frac{1}{A}\int_{S(z,t)} \phi \, dS \quad : \quad A(z,t) \ := \ \int_{S(z,t)} dS$$

Now we may rewrite the volume integral in the LHS of RT theorem

$$\int_{V_t} \phi dV \ = \ \int_{z_1}^{z_2}\int_{S(z,t)} \phi \, dS \, dz \ = \ \int_{z_1}^{z_2} A \cdot \bar{\phi} \, dz$$

where $z_1 < z_2$ are fixed z-coordinates for S_1 and S_2. Then we differentiate the integrands in the above equation w.r.t. t

$$\int_{V_t} \frac{\partial \phi}{\partial t} dV \;=\; \int_{z_1}^{z_2} \frac{\partial}{\partial t}\left[A \cdot \bar{\phi}\right] dz,$$

and we've rewritten the first term in the RHS of the reynolds system. The surface integral in the RHS may be written as

$$\int_{\partial V_t} (\boldsymbol{u_b} \cdot \hat{\boldsymbol{n}})\phi dS = \int_{\partial V_t} (\boldsymbol{u_b} \cdot \hat{\boldsymbol{n}})\phi dS....$$

With a little more work, one may obtain:

cleanup, ref. pg. 43

Definition 1.6.1. The 1D Reynolds Transport theorem for both compressible and incompressible fluids:

$$\frac{\partial}{\partial t}\Big(A\bar{\phi}\Big) \;+\; \frac{\partial}{\partial z}\big(A(\overline{\phi \cdot u_3})\big) \;=\; \int_S \left(\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \boldsymbol{u})\right) dS \;+\; \int_{\partial S} \phi \boldsymbol{w} \cdot \hat{\boldsymbol{n}}\, d\gamma$$

Remark 1.6.2. By taking $f = \rho$ in 1.6.1, mass conservation follows directly. Also, by our assumption that blood is incompressible, we have $\begin{cases} \operatorname{div}(\boldsymbol{u}) = 0 \\ \rho = \text{const.} \end{cases}$ and we simplify 1.6.1 as

$$\frac{\partial A}{\partial t} \;+\; \frac{\partial}{\partial z}\big(A(\overline{u_3})\big) \;=\; \int_{\partial S} \boldsymbol{w} \cdot \hat{\boldsymbol{n}}\, d\gamma$$

The RHS term above describing the transport process across the vessel wall.

complete, pg. 45

Remark 1.6.3. By taking $f = u_3$ in 1.6.1, momentum conservation follows directly. Also, by our assumption that blood is incompressible, we simplify 1.6.1 as

$$\frac{\partial}{\partial t}\Big(Au_3\Big) \;+\; \frac{\partial}{\partial z}\big(A(\overline{u_3^2})\big) \;=\; \int_S \left(\frac{\partial u_3}{\partial t} + \nabla u_3 \cdot \boldsymbol{u}\right) dS \;+\; \int_{\partial S} u_3 \boldsymbol{w} \cdot \hat{\boldsymbol{n}}\, d\gamma$$

The RHS term above describing the transport process across the vessel wall.

Remark 1.6.4 (Tube law from a thin elastic cylindrical wall). We briefly justify the pressure–area relation used in `aq_1d_compliant.jl`

Definition 1.6.5.

$$p(A) - p_{ext} \;=\; \beta\big(\sqrt{A} - \sqrt{A_0}\big)$$

used in the 1D (A, Q) model. Consider a straight cylindrical vessel with (local) lumen radius $R(x,t)$, reference radius R_0, wall thickness $h \ll R$, and internal pressure $p(x,t)$ relative to an external pressure p_{ext} (assumed constant in space and time for simplicity). The corresponding lumen area is

$$A(x,t) = \pi R(x,t)^2, \qquad A_0 = \pi R_0^2.$$

Under the thin–wall assumption, balance of forces in the circumferential direction (Young–Laplace law) yields

Definition 1.6.6.

$$(p - p_{ext})\, 2\pi R \;=\; \sigma_\theta\, 2h\pi$$

where σ_θ is the circumferential (hoop) Cauchy stress in the vessel wall. We model the wall as linearly elastic in the hoop direction, so that

Definition 1.6.7.

$$\sigma_\theta \;=\; E_{eff}\, \varepsilon_\theta$$

with an effective circumferential modulus $E_{eff} > 0$ and circumferential strain

$$\varepsilon_\theta \;=\; \frac{\text{change in circumference} - \text{reference circumference}}{\text{reference circumference}} \;=\; \frac{2\pi R - 2\pi R_0}{2\pi R_0} \;=\; \frac{R - R_0}{R_0}.$$

Equating (1.6.6) and (1.6.7) gives

$$\begin{aligned} (p - p_{ext})\, 2\pi R \;&=\; E_{eff}\, \frac{R - R_0}{R_0}\, 2h\pi \\ \iff (p - p_{ext})\, R \;&=\; E_{eff}\, \frac{R - R_0}{R_0}\, h \end{aligned}$$

Express R and R_0 in terms of the areas A and A_0:

$$\begin{aligned} R &= \sqrt{\frac{A}{\pi}} \\ &= \frac{\sqrt{A}}{\sqrt{\pi}}, \\ R_0 = \sqrt{\frac{A_0}{\pi}} \quad &= \frac{\sqrt{A_0}}{\sqrt{\pi}}, \end{aligned}$$

so that

$$\begin{aligned} R - R_0 \;&=\; \frac{\sqrt{A}}{\sqrt{\pi}} - \frac{\sqrt{A_0}}{\sqrt{\pi}} \\ &=\; \frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big). \end{aligned}$$

Substituting into the expression for $p - p_{ext}$, we obtain

$$
\begin{aligned}
(p - p_{ext})\,\frac{\sqrt{A}}{\sqrt{\pi}} \;&=\; E_{eff}h\,\frac{\frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big)}{R_0} \\
\iff (p - p_{ext})\,\frac{\sqrt{A}}{\sqrt{\pi}} \;&=\; E_{eff}h\,\frac{\frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big)}{R_0}.
\end{aligned}
$$

For moderate deformations where A remains close to A_0, we approximate the factor $1/\sqrt{A}$ by its reference value $1/\sqrt{A_0}$, which yields

$$
\begin{aligned}
(p - p_{ext})\,\frac{\sqrt{A_0}}{\sqrt{\pi}} \;&=\; E_{eff}h\,\frac{\frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big)}{R_0} \\
\iff (p - p_{ext}) \;&=\; \frac{E_{eff}h}{R_0\sqrt{A_0}}\,\big(\sqrt{A} - \sqrt{A_0}\big).
\end{aligned}
$$

Defining the lumped stiffness parameter

$$
\beta \;:=\; \frac{E_{eff}h}{R_0\sqrt{A_0}}.
$$

we arrive at the tube law (**??**) used in the 1D model:

$$
p(A) - p_{ext} \;=\; \beta\big(\sqrt{A} - \sqrt{A_0}\big).
$$

In the numerical experiments below, we take β and A_0 to be constant along the vessel, so that p can be written as a function of A alone.

1.6.1 0D Models

The 0D model, on the other hand, treats the vessel as a lumped parameter system, focusing on overall pressure and flow relationships without spatial resolution.

2 Appendix

Bibliography

References

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Code Listings

Code listings

Code 1: Algorithm 16.5

```
function foo()
    println("Hello World)
end
```