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Abstract. An interface between biology and mathematics has initiated and fostered new mathematical areas, where the ideas from biology and mathematics are synergistically applied. Study of fluid dynamics plays a significant role in fluid flow inside the human body, and modeling of blood flow is an important field in circulatory physics. However, models have been developed are very complex with three-dimensional analysis. This project presents a novel and simple mathematical model of blood flow. Assuming blood is a Newtonian fluid which is governed by the Navier-Stokes equations and continuity equation and with making use of the Navier-Stokes equation, a simple differential equation called as the circulatory system equation is derived. Then by applying the logical assumptions on this model, the general mathematical model of the normal blood flow rate is developed. Using Poisuelli's equation, the sirculatory system equation is also used to develop a model for blood pressure. These two models are then analyzed against surface, pressure gradient and the vessel's length using MATLAB.

Keywords: Fluid Dynamics; Circulatory System; Navier Stokes; Pressure Gradient.

1. Introduction

Mathematical model is a description of a system using mathematical concept and language [1]. The process of developing a mathematical model is termed mathematical modelling. Mathematical models is used not only in the natural science and engineering disciplines, but also in the social sciences [2]. A model may help to explain a system and to study the effects of different components and to make predictions about behaviour. Blood flow is the continuous circulation of blood in the cardiovascular system. The cardiovascular system in the body consists of three component which is blood, heart, and blood vessel. This process blood flow in the cardiovasuclar system ensures the transportation of nutrients, hormones, metabolic wastes, oxygen, and carbon dioxide through the body to maintain cell, level metabolism, the regulation of the pH, osmotic pressure and temperature of the whole body, and the protection from microbial and mechanical harms [3]. The science dedicated to describe the physics of blood flow is called hemodynamics. For the basic understanding it is important to be familiar with anatomy of the cardiovascular system and hydrodynamics. However it is crucial to mention that blood is the non-Newtonian fluid and blood vessels are not rigid tubes [4]. For recent researchers about mass or heat transference under Fourier-Fick in a stratified non-Newtonian fluid with variable fluid characteristics also are stated in reference [5]. In many previous analytical studies in concept with blood flow have been carried out in the recent past, the non linear velocity profile of pulsatile flow of blood through the descending aorta have been satisfactorily investigated by [6] and [7] who treated the artery as an uniformly tapered thick cylindrical tube of isotropic and incompressible material. Moreover, some of studies have been performed on the measurement of electrical activity [8], deformation [9], flow [10], modeling of the heart [11], and computational design of cardiac activity in the body [12]. This study will focus on mathematical modeling of blood blow problems. Although blood is the non-Newtonian fluid [13,14], in this study will considered the blood as Newtonian fluid (properties of blood become linear) which is governed by the Navier-Stokes equation and the continuity equation for solving blood flow problems [15,16]. In this study, we will focus on the derivation of cardiovascular system equation or called as master equation with the help of continuity equation and Navier-Stokes equation in order to developed general equation of normal blood flow and extended normal blood pressure equation. In whole literature, some assumptions have been considered, which include that although the blood vessels are different in size, they are all considered being cylindrical shaped and deformable components with circular cross-sections. They expand as the blood enters into and contract as the blood leaves it. Although the blood needs the help of the lungs for the supply of oxygen, its properties remain unchanged by the addition of that oxygen. Another assumption is required, and that is, the blood has both the radial and axial flow in only one direction z-direction in a three dimensional system. So, the other two components x and y-direction are vanished. The problems of this study will be solved using the fluid dynamic assumption assist by differential equation and by using MATLAB and MATLAB to show the solution and to verify the validity of the problems. The results of this study will give benefits to the fields of mathematics, physics, engineering, biology and medical. This research will lead to further investigation in the mathematical modeling of blood flow theory and method which is frequently used in mathematics, physics, and mostly in medical field. Besides that, the result of the research can be a guide line to determine the cardiovascular system equation, blood flow rate equation and normal blood pressure equation in getting the strong solution for blood flow problems. This investigation also gives a chance for readers to know more about benefits of this equations model in solving blood flow problems.

2. Mathematical formulation and modelling

Mathematical model uses mathematical language to describe a system in the real world. The process of developing a mathematical model is termed as mathematical modelling or modelling. Cardiovascular system is the blood distribution network in the body. The cardiovascular system in the body consists of three components: blood, heart and blood vessels. When blood flows through the vessels, pressure is detected on the wall which is termed as the blood pressure. Blood pressure depends mainly on flow rate and size of the vessels and on the pressure gradient. There are three major types of blood vessels: arteries, capillaries, and veins. Arteries are large blood vessels that carry blood from the heart to all regions of the body [17]. The arterioles further divide into smaller vessels called capillaries. Capillaries are the anatomic units that connect the arterial and venous circulatory system. The veins form a low-pressure collecting system to return the oxygen-poor blood to the heart [18]. In this analysis, all the vessels are assumed to be same in nature excluding their size, length and cross-sectional area.

To develop the model of the blood flow and blood pressure, some assumptions has been considered. These include that the blood vessels are the cylindrical, deformable components with circular cross-sections. They change their size as the blood flows through it. The blood is considered to be the Newtonian fluid which is governed by the Navier-Stokes equation and by the continuity equation. Although the blood needs the help of the lungs for the supply of oxygen, its properties remain unchanged by the addition of that oxygen. Another assumption is required, and that is the blood has both the radial and axial flow in only one direction, which is z-direction in a three-dimensional system. So, the other two components (x-direction and y-direction) are vanished.

2.1. Developing the cardiovascular system

The velocity components in the x, y and z directions are typically named u, v and w respectively. Let ρ be the density of blood, P be the blood pressure and μ is the kinematic viscosity of blood. Taking the incompressible flow assumption into account and assuming constant viscosity, the Navier Stokes equations will read, in Cartesian coordinates form by the following equation [11]:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_u, \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_v, \tag{2}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_w. \tag{3}$$

Then neglecting the orientation of gravity inside the body of above equation, the Navier-Stokes equation in the Cartesian coordinates is given by the following equations:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{4}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{5}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right). \tag{6}$$

Hence, equation (4), (5), (6) are equations of Navier-Stokes equation in the Cartesian coordinates. The equation of Navier-Stokes equation in Cylindrical is given by the following equations:

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_c}{r}\frac{\partial v_z}{\partial c}-\frac{v_c^2}{r}+v_z\frac{\partial v_r}{\partial z}=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\upsilon\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial c^2}-\frac{2}{r^2}\frac{\partial v_c}{\partial c}+\frac{\partial^2 v_r}{\partial z^2}\right). \tag{7}$$

Now, equation (7) becomes:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$+v\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}+\frac{\partial^2 u}{\partial z^2}\right), \tag{8}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial \theta}$$
$$+v\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right)+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u}{\partial \theta}+\frac{\partial^2 v}{\partial z^2}\right), \tag{9}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right). \tag{10}$$

The equation (8), (9), (10) are cylindrical representation of the incompressible Navier Stokes equation which is the second most commonly seen. Cylindrical coordinates are chosen to take advantage of symmetry, so that the velocity components can disappear. A very common case is axisymmetric flow with the assumption of no tangential velocity $(v = 0)$ and the remaining quantities are independent of z. A change of variables on the above equations will yields the following system of equation in the cylindrical coordinate system:

$$\frac{\partial w}{\partial t}+f\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\right), \tag{11}$$

$$\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial r}+w\frac{\partial f}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{\partial^2 f}{\partial z^2}-\frac{f}{r^2}\right), \tag{12}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rf)+\frac{\partial w}{\partial z}=0. \tag{13}$$

The equation (11) and (12) can be simplify as (14) and (15)

$$\frac{\partial w}{\partial t}+f\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right), \tag{14}$$

$$\frac{\partial w}{\partial t}+f\frac{\partial f}{\partial r}+w\frac{\partial f}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+v\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial z^2}-\frac{f}{r^2}\right), \tag{15}$$

where ρ=density, $v = \mu/\rho$ viscosity of fluid, P= pressure, f be the radial flow component, and w be the axial flow component in z-direction.

equation of continuity can be written as:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho w)}{\partial z}=0. \tag{16}$$

Usually, any problem that involves couple of the fluid mechanics and the vessel wall mechanics, $R(z,t)$ could be derived as a part of the solution instead of having its specific form

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as input. $R(z,t)$ is known explicity and hence our attention would be centered of the haemodynamic factor only. For convenience, we define a new variable which is the radial coordinate transformation, given by:

$$\gamma = \frac{r}{R(z,t)}, \tag{17}$$

where $R(z,t)$ denote the inner radius of the vessels, r is radial direction and γ be the arterial wall viscosity.Which has the effect of immobilizing the vessel wall in the transformed coordinate x. By using radial coordinates transformation (17), the governing equation for the z component of momentum in (14) will be transform to the following form:

$$\frac{\partial w}{\partial t}+\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial R}\cdot\frac{\partial R}{\partial t}\right)+f\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial r}\right)+w\left(\frac{\partial w}{\partial z}+\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial R}\cdot\frac{\partial R}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial P}{\partial z}+$$
$$\frac{\mu}{\rho}\left[\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial r}\right)\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial r}\right)+\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial z}\right)\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial z}\right)\right], \tag{18}$$

where $v = \mu/\rho$.

Then substituting $\frac{\partial\gamma}{\partial R} = \frac{\partial}{\partial R}\left(\frac{r}{R}\right)$, $\frac{\partial\gamma}{\partial r} = \frac{\partial}{\partial r}\left(\frac{r}{R}\right)$ and $\frac{\partial\gamma}{\partial z} = \frac{\partial}{\partial z}\left(\frac{r}{R}\right)$ in (18), we will obtain

$$\begin{aligned}&\frac{\partial w}{\partial t}+\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial R}\left(\frac{r}{R}\right)\cdot\frac{\partial R}{\partial t}\right)+f\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial r}\left(\frac{r}{R}\right)\right)\\&+w\left(\frac{\partial w}{\partial z}+\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial R}\left(\frac{r}{R}\right)\cdot\frac{\partial R}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial P}{\partial z}\\&+\frac{\mu}{\rho}\left[\begin{array}{l}\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial r}\left(\frac{r}{R}\right)\right)\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial r}\left(\frac{r}{R}\right)\right)+\frac{1}{r}\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial r}\left(\frac{r}{R}\right)\right)\\+\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial z}\left(\frac{r}{R}\right)\right)\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial}{\partial z}\left(\frac{r}{R}\right)\right)\end{array}\right],\end{aligned} \tag{19}$$

Solving equation (19), we will get

$$\begin{aligned}&\frac{\partial w}{\partial t}+\left(\frac{\partial w}{\partial \gamma}\cdot\left(-\frac{r}{R^2}\right)\cdot\frac{\partial R}{\partial t}\right)+f\left(\frac{\partial w}{\partial \gamma}\cdot\left(\frac{1}{R}\right)\right)\\&+w\left(\frac{\partial w}{\partial z}+\frac{\partial w}{\partial \gamma}\cdot\left(-\frac{r}{R^2}\right)\cdot\frac{\partial R}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial P}{\partial z}\\&+\frac{\mu}{\rho}\left[\left(\frac{\partial w}{\partial \gamma}\cdot\left(\frac{1}{R}\right)\right)\left(\frac{\partial w}{\partial \gamma}\cdot\left(\frac{1}{R}\right)\right)+\frac{1}{r}\left(\frac{\partial w}{\partial \gamma}\cdot\left(\frac{1}{R}\right)\right)\right],\end{aligned} \tag{20}$$

from (20), we will simplify, factorizing and simplifying the left hand side, then we will obtain

$$\frac{\partial w}{\partial t}-\frac{1}{R}\left[\gamma\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial z}\right)-f\right]\frac{\partial w}{\partial \gamma}-w\frac{\partial w}{\partial z}= -\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{\mu}{\rho R^2}\left[\frac{\partial^2 w}{\partial \gamma^2}+\frac{1}{\gamma}\frac{\partial w}{\partial \gamma}\right]. \tag{21}$$

Equation (13) cannot be solve directly to get the axial velocity component, $w(\gamma, z, t)$. In this case, the continuity equation (13) will be transform by using radial coordinate transformation and we write this equation as

$$\frac{\partial f}{\partial r}+\frac{f}{r}+\frac{\partial w}{\partial z}=0. \tag{22}$$

Now, upon applying the radial coordinate transformation (17), equation (22) becomes

$$\frac{\partial f}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial r}+\frac{f}{r}+\left(\frac{\partial w}{\partial \gamma}\cdot\frac{\partial \gamma}{\partial R}\cdot\frac{\partial R}{\partial z}\right)+\frac{\partial w}{\partial z}=0, \tag{23}$$

Finally, by simplying and substituting $r = \gamma R$, we will obtain

$$\frac{\partial f}{\partial \gamma}\frac{1}{R}+\frac{f}{\gamma R}-\frac{\partial w}{\partial \gamma}\frac{\gamma}{R}\frac{\partial R}{\partial z}+\frac{\partial w}{\partial z}=0. \tag{24}$$

For the axial and radial velocity components, first, we need to simplifying equation (24) in order to get the radial velocity component, $u(\gamma, z, t)$. Multiplying equation (24) by γR, then t become,

$$\gamma\frac{\partial f}{\partial \gamma}+f-\gamma^2\frac{\partial w}{\partial \gamma}\cdot\frac{\partial R}{\partial z}+\gamma R\frac{\partial w}{\partial z}=0, \tag{25}$$

integrating the equation with respect to γ from the limits 0 to γ to find,

$$\int_0^\gamma \gamma\frac{\partial f}{\partial \gamma}dS+\int_0^\gamma f dS+\int_0^\gamma \gamma R\frac{\partial w}{\partial z}dS-\int_0^\gamma \gamma^2\frac{\partial w}{\partial \gamma}\cdot\frac{\partial R}{\partial z}dS=0, \tag{26}$$

As we know $w(0, z, t) = f(0, z, t) = 0$ the above equation becomes

$$\gamma f(\gamma, z, t)+R\int_0^\gamma \gamma\frac{\partial w}{\partial z}dS-\frac{\partial R}{\partial z}\gamma^2 w+\frac{\partial R}{\partial z}\int_0^\gamma 2w\gamma dS=0,$$

rearranging and dividing the above equation, we find

$$\gamma f(\gamma, z, t)=-R\int_0^\gamma \gamma\frac{\partial w}{\partial z}dS+\frac{\partial R}{\partial z}\gamma^2 w-\frac{\partial R}{\partial z}\int_0^\gamma 2w\gamma dS,$$

dividing the above equation by γ we find

$$f(\gamma,z,t)=\gamma\frac{\partial R}{\partial z}w-\frac{R}{\gamma}\int_0^\gamma \gamma\frac{\partial w}{\partial z}dS-\frac{2}{\gamma}\frac{\partial R}{\partial z}\int_0^\gamma w\gamma dS, \tag{27}$$

Now, we need to prove above equation before we applying to the equation problem.

Let, $$f(\gamma)=-\frac{2}{N}\sum_{k=1}^{N}\frac{k+1}{k}\left(\gamma^{2k}-1\right) \tag{28}$$

Therefore, we solving equation above and get

$$-\int_0^1 \gamma\frac{\partial w}{\partial z}d\gamma=\int_0^1 \gamma f(\gamma)\frac{1}{R}\frac{\partial R}{\partial t}d\gamma+\frac{2}{R}\frac{\partial R}{\partial z}\int_0^1 \gamma w d\gamma,$$

$$-\int_0^1 \gamma\frac{\partial w}{\partial z}d\gamma=\int_0^1\left[\gamma f(\gamma)\frac{1}{R}\frac{\partial R}{\partial t}+\frac{2}{R}\frac{\partial R}{\partial z}\gamma w d\gamma\right],$$

$$-\int_0^1 \gamma\frac{\partial w}{\partial z}d\gamma=\int_0^1 \gamma\left[\frac{f(\gamma)}{R}\frac{\partial R}{\partial t}+\frac{2w}{R}\frac{\partial R}{\partial z}\right]d\gamma. \tag{29}$$

Taking approximation of considering the equality between the integral to integrands for equation (29)

$$-\frac{\partial w}{\partial z}=\frac{f(\gamma)}{R}\frac{\partial R}{\partial t}+\frac{2w}{R}\frac{\partial R}{\partial z}, \tag{30}$$

rearranging and substituting the value of from (28) into (30)

$$-\frac{\partial w}{\partial z}=\frac{2w}{R}\frac{\partial R}{\partial z}+\frac{1}{R}\frac{\partial R}{\partial t}\left[-\frac{2}{N}\sum_{k=1}^{N}\frac{k+1}{k}\left(\gamma^{2k}-1\right)\right],$$

$$\frac{\partial w}{\partial z}=-\frac{2w}{R}\frac{\partial R}{\partial z}+\frac{1}{R}\frac{\partial R}{\partial t}\left[\frac{2}{N}\sum_{k=1}^{N}\frac{k+1}{k}\left(\gamma^{2k}-1\right)\right]. \tag{31}$$

Substituting (31) into (27)

$$f(\gamma,z,t)=\gamma\frac{\partial R}{\partial z}w-\frac{R}{\gamma}\int_0^x \gamma\left[-\frac{2w}{R}\frac{\partial R}{\partial z}+\frac{2}{R}\frac{\partial R}{\partial t}\left(\frac{1}{N}\sum_{k=1}^{N}\frac{k+1}{k}\left(\gamma^{2k}-1\right)\right)\right]d\gamma$$
$$-\frac{2}{\gamma}\frac{\partial R}{\partial z}\int_0^x w\gamma d\gamma, \tag{32}$$

simplifying and solving

$$f(\gamma,z,t)=\gamma\frac{\partial R}{\partial z}w+\frac{R}{\gamma}\int_0^x\gamma\frac{2w}{R}\frac{\partial R}{\partial z}d\gamma-\frac{R}{\gamma}\int_0^x\frac{2\gamma}{R}\frac{\partial R}{\partial t}\frac{1}{N}\sum_{k=1}^{N}\frac{k+1}{k}\left(\gamma^{2k}-1\right)d\gamma$$
$$-\frac{2}{\gamma}\frac{\partial R}{\partial z}\int_0^x w\gamma d\gamma,$$
$$f(\gamma,z,t)=\gamma\frac{\partial R}{\partial z}w+\frac{2}{\gamma}\frac{\partial R}{\partial z}\int_0^x w\gamma d\gamma-\frac{2}{\gamma N}\frac{\partial R}{\partial t}\sum_{k=1}^{N}\frac{k+1}{k}\int_0^x\left(\gamma^{2k+1}-\gamma\right)d\gamma$$
$$-\frac{2}{\gamma}\frac{\partial R}{\partial z}\int_0^x w\gamma d\gamma,$$

$$f(\gamma,z,t)=\gamma\frac{\partial R}{\partial z}w-\frac{2}{\gamma N}\frac{\partial R}{\partial t}\sum_{k=1}^{N}\frac{k+1}{k}\int_0^x\left(\gamma^{2k+1}-\gamma\right)d\gamma. \tag{33}$$

Integrating and expanding (33), the velocity profile in the radial direction is expressed as:

$$f(\gamma,z,t)=\gamma\frac{\partial R}{\partial z}w+\gamma\frac{\partial R}{\partial t}-\frac{\gamma}{N}\frac{\partial R}{\partial t}\sum_{k=1}^{N}\frac{1}{k}\left(\gamma^{2k}-1\right). \tag{34}$$

The basic idea of this hemodynamic modeling, described by E. Belardinelli and S. Cavalcanti, is to assume that the velocity profile in the axial direction can be expressed as the following polynomial form:

$$w(\gamma,z,t)=\sum_{k=1}^{N}q_k\left(\gamma^{2k}-1\right). \tag{35}$$

For simplicity, we choose N=1, the equation (34) and (35) becomes:

$$f(\gamma,z,t)=\frac{\partial R}{\partial z}\gamma w+\frac{\partial R}{\partial t}\gamma-\frac{\partial R}{\partial t}\gamma\left(\gamma^{2k}-1\right), \quad \text{and} \tag{36}$$

$$w(\gamma,z,t)=q(z,t)\left(\gamma^2-1\right). \tag{37}$$

Using the help of equations of axial and radial velocity profile, radial coordinate and the continuity equation, we plugging equations (36) and (37) into equations and (21), the Navier-Stokes equations get the forms as below to determine the variable q(z,t) and R(z,t):

$$\frac{\partial q}{\partial t}-\frac{4q}{R}\frac{\partial R}{\partial t}-\frac{2q^2}{R}\frac{\partial R}{\partial z}+\frac{4\mu}{R^2}q+\frac{1}{\rho}\frac{\partial P}{\partial z}=0, \quad \text{and} \tag{38}$$

$$2\frac{\partial R}{\partial t}+\frac{R}{2}\frac{\partial q}{\partial z}+q\frac{\partial R}{\partial z}=0. \tag{39}$$

For equation (39), firstly, we need to multiplying equation (39) with π and its become as:

$$2\pi\frac{\partial R}{\partial t}+\frac{\pi R}{2}\frac{\partial q}{\partial z}+\pi q\frac{\partial R}{\partial z}=0. \tag{40}$$

By using the chain rule, $\frac{\partial Q}{\partial z}$ and $\frac{\partial S}{\partial t}$ are obtained as:

$$\frac{\partial Q}{\partial z}=\pi qR\frac{\partial R}{\partial z}+\frac{\pi R^2}{2}\frac{\partial q}{\partial z},$$

and

$$\frac{\partial S}{\partial t}=2\pi R\frac{\partial R}{\partial t}.$$

After substituting above equation into (40), the differential equation is obtained as follows:

$$\frac{\partial S}{\partial t}+\frac{\partial Q}{\partial z}=0. \tag{41}$$

After inserting the value of $\frac{\partial q}{\partial t},\frac{\partial q}{\partial z},\frac{\partial R}{\partial t},\frac{\partial R}{\partial z}$ in (39), another differential equation are obtained as:

$$\frac{\partial Q}{\partial t}+\frac{3Q}{S}\frac{\partial Q}{\partial z}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}. \tag{42}$$

After combining (41) and (42) a simple differential equation is obtained as follows:

$$\frac{\partial Q}{\partial t}-\frac{3Q}{S}\frac{\partial S}{\partial t}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0, \tag{43}$$

where $\frac{\partial Q}{\partial z}=-\frac{\partial S}{\partial t}$

Equation (43) is now called as the 'master equation' or cardiovascular system equation. The model of the blood flow rate and blood pressure can be now got by applying some assumptions on this master equation.

3. Finding and discussion

3.1. Modeling of the blood flow rate

To develop the model of the blood flow, it is assumed that the cross-section area of the blood vessel remains unchanged with time and it is also assumed to be constant over distance and the pressure gradient is assumed to be constant over the distance. By applying $\frac{\partial S}{\partial t}=0$ and $\frac{\partial S}{\partial z}=0$ on (43), the master equation reduces to:

$$\frac{\partial Q}{\partial t}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0. \tag{44}$$

This is the one dimensional mathematical model of the blood flow rate. The required boundary condition and the values of the other parameters to solve this equation are obtained from the past works in this field [19,20]. Such as:

Pressure gradient, $\frac{\partial P}{\partial z}$ = 100 to 40 mmHg
Kinematic viscosity of blood, $\mu = 0.0035 \text{cm}^2/\text{s}$
Density of blood, $\rho = 1.043$ to 1.057 g/cm^3

3.2. Analysis of blood flow rate

In this section of this research, analysis of blood flow rate has been performed to verify the validity of the proposed model. The equation of the mathematical model of blood flow rate can be solved using the MATLAB. Figure 1 represents this solution for different cross sectional area. The significance of this plot is that the blood flow rate increases with the cross-sectional area. This result supports figure 2 which is derived from the Poisuelli's equation.

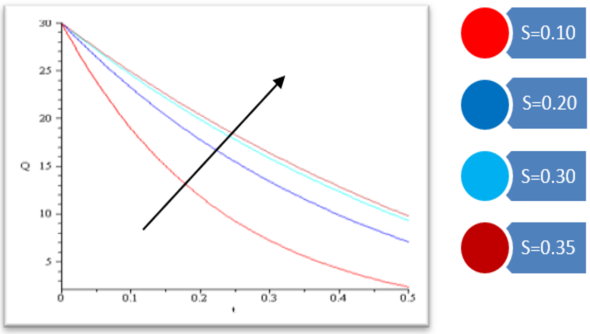

Figure 1: Blood flow rate versus time for different increasing cross-sectional area, S (from 0.10 to 0.35 cm^2)

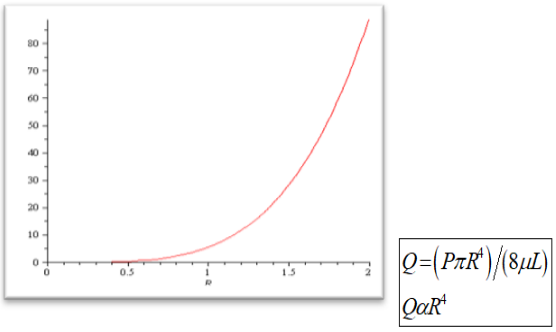

Figure 2: Variation of blood flow rate with vessel radius using Poisuelli's equation

In figure 3, the solution is plotted for various pressure gradients. Accordingly, pressure is higher at the beginning than at the end of vessel, establishing a pressure gradient. The greater the pressure gradient forcing bloods through a vessel, the greater the rate of flow through vessel [19]. Figure 3 also shows that for a given pressure gradient, the blood flow rate decreases with time and as the pressure gradient increases the blood flow rate also increases. The figure 3 is performed with difference pressure gradients from 50 to 200 mmHg.

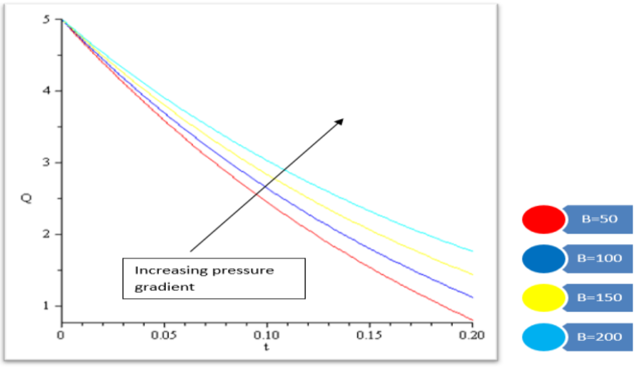

Figure 3: Blood flow rate versus time for different increasing pressure gradient (from 50 to 200mmHg).

3.3. Modeling of the blood pressure

To develop the mathematical model of the blood pressure, Poisuelli's equation is considered. Poisuelli's equation determines the relation between blood flow rate and the pressure which is given as:

$$Q = \frac{\pi R^4}{8L\mu} P, \tag{45}$$

where, L is the length and R is the radius of vessel.

After inserting (45) into (44), the new equation are obtained as follows:

$$\frac{\pi R^4}{8L\mu}\frac{dP}{dt} + \frac{4\pi\mu}{S}\frac{\pi R^4}{8L\mu}P + \frac{S}{2\rho}\frac{\partial P}{\partial z} = 0,$$

$$\frac{dP}{dt} + \frac{4\mu}{R^2}P + \frac{4L\mu}{\rho R^2}\frac{\partial P}{\partial z} = 0. \tag{46}$$

Equation (46) is the mathematical model of the blood pressure in the body. The required boundary condition and the values of the other parameters to solve this equation can be obtained from the past works in this field. Such as:

Pressure gradient, $\frac{\partial P}{\partial z} = 100$ to 40 mmHg
Kinematic viscosity of blood, μ=0.0035cm2/s
Density of blood, ρ=1.043 to 1.057 g/cm3

3.4. Analysis of blood pressure

The equation of the mathematical model of blood pressure can be solved using MATLAB. Figure 4 represents this solution for different increasing cross-sectional area from 0.1 to 0.22 cm^2. The significance of this plot is that the blood pressure decreases with the increasing cross-sectional area. This result supports figure 5 which is derived from the Poisuelli's equation.

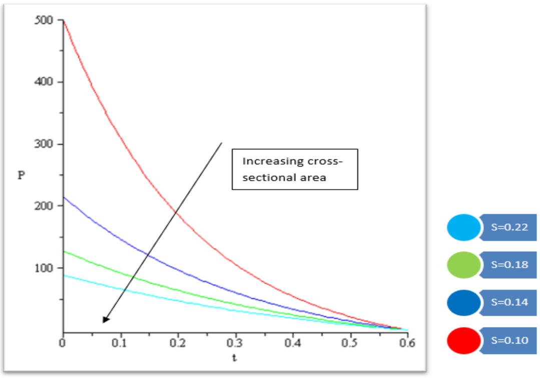

Figure 4: Blood pressure for different with increasing cross-sectional area of vessels (from 0.10 to 0.22 cm^2).

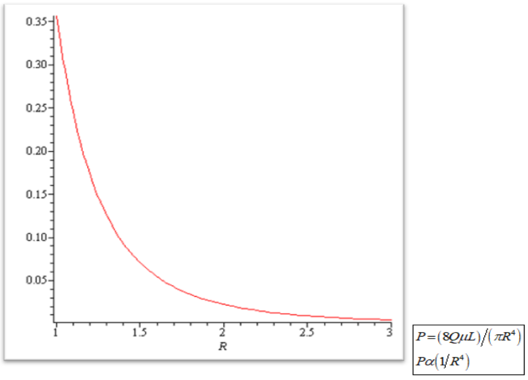

Figure 5: Variation of blood pressure vessel radius using Poisuelli's Equation.

In figure 6, the solution is plotted for various length of the blood vessel. This analysis indicates the increment of blood pressure with the increment of length of the blood vessels. Higher pressure at the beginning and lower pressure at the end and the difference between this two highly varies with length of the blood vessels. This result supports figure 7 which is derived from the Poisuelli's equation.

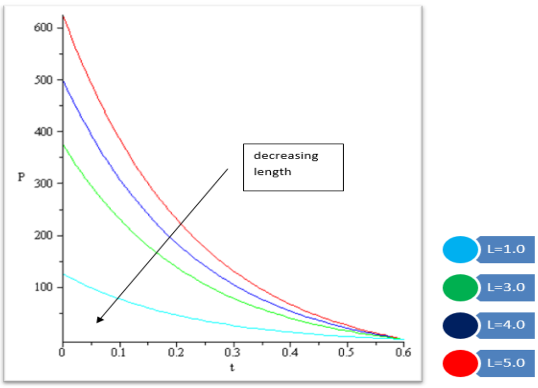

Figure 6: Blood pressure for different with decreasing length of vessels (from 5.0 to 1.0 cm^2).

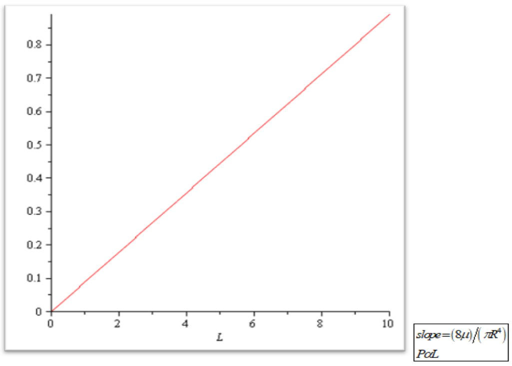

Figure 7: Variation of blood pressure vessel length using Poisuelli's Equation.

Systolic (maximum) blood pressure in the normal adult is in the range of 95 to 140 mmHg, with 120 mmHg being average. These figures are subject to much variation with age, climate, eating habits, and other factors. Normal diastolic blood pressure (lowest pressure between beats) ranges from 60 to 90 mmHg, 80 mmHg being about average. This pressure is usually measured in the brachial artery in the arm [21].

4. Conclusion

The design of computational models of human organs is a new research field which opens new possibilities for medical image analysis and therapy simulation [22]. The goal of this paper was to represent a mathematical model of the blood flow. A limited number of internal parameters were considered in developing the model. So, possible improvements of the study would include the integration of more anatomical structure (valve, size of the heart chambers), more realistic model and a more complex constitutive law. However, the objective of this research was not to build to more complex and faithful heart model ever. Instead, we want to adapt the complexity of the model but can show the sensitivity of the change in cross-sectional area, pressure gradient and length of blood vessel on the blood flow rate and on the blood pressure. Although a large numbers of assumptions have been considered, the model can be treated as a valid one, because, this model is able to show that the blood flow rate is influenced by the change of cross-sectional area and the pressure gradient. The model can also show that the blood pressure is influenced by both of the cross-sectional area and the length of the blood vessel. Furthermore, the problems of this study were solved using the fluid dynamic assumption assist by differential equation and software solution which is by using MATLAB programming. Lastly, the graphical representations in this study help to verify the validity of the proposed model for this study.

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