Blood Flow in the Human Circulatory System

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Hemodynamics refers to study of bloods motion, herein, we report mathematical models that govern such kinematics within the human vessel.

Keywords: *computational hemodynamics, PINN's, Deep-Reisz, discontinuous galerkin, Lax-Wendroff, fluid-structure ineraction (FSI)*

summarize final contributions here and update keywords

Contents

Work in Progress (WIP). Tips to keep in mind:

- Start w/ literature suvery, summarizing each articles contribution once. (i.e., limit repeat citation refs.), then prioritize sections by technicality
- Tag material by relevance, e.g., foundational. vs. tangential vs. speculative.
- Comparative analsyis contrasting results, models, or point out contradictions across papers
- Always consist in notation: Actively managing acroynms, abrevs., notation, and preliminaries.
- Look into [Zotero] to manage refs.

0.1 Acroynms and Abbreviations

a.e.	almost everywhere
e.g.	"exempli gratia" (for example)
i.e.	"id est" (that means)
s.s.	sufficiently smooth
s.t.	such that
r.t.	refers to
w.r.t.	with respect to
m.b.s.	must be shown
i.m.b.s.	it must be shown
i.r.t.s.	it remains to show
w.a.t.s.	we aim to show
bpm	beats per minute
wlog	without loss of generality
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PDES	System of Partial Differential Equations
IC	Initial Condition
BC	Boundary Condition
0D	Zero dimensional
1D	One dimensional
2D	Two dimensional
3D	Three dimensional
FSI	Fluid-Structure Interaction
SB	Stenotic Blockage
RBC	Red Blood Cell
CVD	Cardiovascular disease & CVDs r.t. such diseases

0.2 Mathematical Notation

$\therefore$	consequently
$\because$	because
$\implies$	implies
$\iff$	if and only if
$:=$	defines
$\equiv$	equivalence
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}^-$	set of negative real numbers
$\mathbb{R}^n$	n-dimensional real vector space
$\mathbb{N}$	set of natural numbers
$\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$	general basis of $\mathbb{R}^n$
$\{\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}\}$	standard basis of $\mathbb{R}^n$
$[n] \subset \mathbb{N}$	set $\{1, 2, \ldots, n\}$
$\Omega \subset \mathbb{R}^n$	a connected open subset of $\mathbb{R}^n$
$\overline{\Omega}$	the closure of Ω
$\partial\Omega$	the boundary of Ω
$C^k(\Omega)$	space of k times continuously differentiable functions on Ω
$C_0^k(\Omega)$	space of k times continuously differentiable functions with compact support in Ω
$C_0^k(\overline{\Omega})$	space of k times continuously differentiable functions which have bounded and uni continuous derivatives up to order k with compact support in Ω
$C_0^\infty(\Omega)$	space of smooth functions with compact support in Ω
$L^p(\Omega)$	Lebesgue space of p-integrable functions on Ω

Locally integrable, lipshitz, Holder continuous, Sobolev spaces, weak derivatives, distributions, test functions, multi-index notation

$d\boldsymbol{x}$	Lebesgue measure on $\mathbb{R}^n$
$dS_{\boldsymbol{x}}$	surface measure on $\partial\Omega \subset \mathbb{R}^n$
dV	volume measure on domain $\Omega \subset \mathbb{R}^3$
dS	surface measure on boundary $\partial\Omega \subset \mathbb{R}^3$
∇	gradient operator
$\Delta = \nabla^2 = \nabla \cdot \nabla(\cdot)$	Laplace operator
div	divergence of a vector field
div	divergence of a tensor
$\boldsymbol{v}_i$	i-th component of vector $\boldsymbol{v}$
$\langle \cdot, \cdot \rangle_X$	inner product on vector space X
$\langle \boldsymbol{u},\ \boldsymbol{v} \rangle \equiv \langle \boldsymbol{u},\ \boldsymbol{v} \rangle_{\mathbb{R}^n}$	inner product of vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$
$\dfrac{\partial}{\partial \hat{\boldsymbol{n}}} = \langle \nabla, \hat{\boldsymbol{n}} \rangle$	normal derivative on $\partial\Omega$
$\| \cdot \|$	L^2-norm

0.3 Domain Specific Notation

R	radius of vessel with diameter $2R$
η	dynamic viscosity $\quad \left[Pa \cdot s\right]$
μ	kinematic viscosity $\quad \left[\frac{m^2}{s}\right]$
τ	shear stress
$\dot{\gamma}$	shear rate
ρ	density field $\quad \left[\frac{kg}{m^3}\right]$
p	pressure field
$\boldsymbol{u}$	velocity field
W_0	Womersley number $\left[-\right]$
Re	Reynolds number $\left[-\right]$
Pe	Péclet number $\left[-\right]$
c	concentration of a material element
D	diffusion coefficient $\quad \left[\frac{m^2}{s}\right]$
t	time $\quad \left[s\right]$
T	terminal time $\quad \left[s\right], t > 0$
ω	angular frequency $\quad \left[\frac{rad}{s}\right]$
$\boldsymbol{f_b}$	body force per unit volume $\quad \left[\frac{N}{m^3}\right]$

0.4 Mathematical Foundations

We assume Zermelo-Fraenkel set theory with the axiom of choice (ZFC), and according to Cohen, it's consistent to assume the the continuum hypothesis (CH); a necessary posulate in continuum mechanics. A *domain* r.t. an open and bounded subset of $\mathbb{R}^{N+1}$ with nonempty interior Ω°, for $N \in \{1, 2, 3\}$. By the Heine-Borel theorem, every domain Ω has a well-defined boundary $\partial\Omega$ with compact closure $\overline{\Omega} = \Omega \cup \partial\Omega$. The compact domains of $\mathbb{R}^N$ are precisely the closed and bounded subsets of $\mathbb{R}^N$. When every path between two points in Ω may be continously contrated to a point without leaving Ω, we say that Ω is *simply connected*. A *Lipschitz domain* is a domain Ω s.t for every point $x \in \partial\Omega$, there exists a neighborhood U_x of x s.t $\Omega \cap U_x$ is the region above the graph of a Lipschitz continuous function (after a suitable rotation and translation of coordinates).

TODO:
- Metric sp, normed sp., inner-product sp. defs and notation; include stmt. that Norm's induce metrics, metrics induce norms.
- Completeness, Banach sp., Hilbert sp. defs and notation. Finite dim sp.s are complete, hence banach, and hilbert under a particular choice of inner-product; state such inner-product for our sp.s.

Given domain Ω, the function space $\mathcal{F}(\Omega)$ is the vector space (vec. sp.) $(F(\Omega), K)$ of scalar valued functions $f : \Omega \to K$. E.g. $C^0(\Omega)$ r.t. $(C(\Omega), \mathbb{R})$, the vec. sp. of continuous real valued functions on Ω. Let $C^k(\Omega)$ be the sp. of continuous and k-times continuously differentiable functions on Ω. Then $C^k(\overline{\Omega})$ is the sp. of functions $f \in C^k(\Omega)$ s.t f and it's derivatives up to order k may be continuously extended to the boundary $\partial\Omega$. The space $L^p(\Omega)$ denotes the Lebesgue sp. of p-integrable functions on Ω. .

TODO: - Stmt that we forgoe defining the def. of derivative, and what it means for func f to be integrable on domain Ω.
For now assume integrable means $\int_\Omega |f|\, d\boldsymbol{x}$ is defined and finite.
Assume deriv. is std. unless otherwise sated. Reiz represenation theorem.

compact support, locally integrable, lipschitz sp., etc.

Let $n = \dim(\Omega)$, then we say

$\phi : \Omega \to \mathbb{R}$ r.t. a *scalar field*,

$\boldsymbol{f} : \Omega \to \mathbb{R}^n$ r.t. a *vector field*,

$\boldsymbol{T} : \Omega \to \mathbb{R}^{n\times n}$ r.t. a *(second-order) tensor field*,

and we say that $\phi(\boldsymbol{x})$, $\boldsymbol{f}(\boldsymbol{x})$, and $\boldsymbol{T}(\boldsymbol{x})$ are the values of the fields at point $\boldsymbol{x} \in \Omega$.[1]

If $f \in C^1(\Omega)$, the partial derivative w.r.t. coordinate x_i is denoted

$$\partial_{x_i} f = \frac{\partial f}{\partial x_i},$$

[1] The $\boldsymbol{T}\boldsymbol{x} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, and once a basis is fixed, $\exists\, A \in \mathbb{R}^{n\times n}$ s.t. $\boldsymbol{T}(\boldsymbol{x}) = A\,\boldsymbol{x}$ for all $\boldsymbol{x} \in \Omega$.

the gradient of f is $\nabla f = (\partial_{x_i} f)_{i\in[n]}$ and the Laplacian of f is $\Delta f = \sum_{i\in[n]} \partial^2_{x_i} f$. [2]

For Lebesgue measure $d\boldsymbol{x}$ on $\mathbb{R}^n$ the integral of f in Ω is

$$\int_\Omega f(\boldsymbol{x})\, d\boldsymbol{x} \;\equiv\; \int_\Omega f.$$

If $\partial\Omega \in C^1$, then $f \in C^1(\overline{\Omega}) \supset L^1(\partial\Omega)$ and the integral at the boundary is

$$\int_{\partial\Omega} f(\boldsymbol{x})\, dS_{\boldsymbol{x}} \;\equiv\; \int_{\partial\Omega} f.$$

[2] In coordinate free-terms the gradient $\nabla f(\boldsymbol{x})$ is defined s.t. $df = \langle \nabla f,\ d\boldsymbol{x} \rangle$ for all infinitesimal displacements $d\boldsymbol{x}$.

For vector valued $\boldsymbol{f} \in C^1(\Omega; \mathbb{R}^N)$, the divergence operator is

$$\operatorname{div} \boldsymbol{f} \;:=\; \nabla \cdot \boldsymbol{f} = \langle \boldsymbol{f},\, \nabla \rangle_{\mathbb{R}^n} = \sum_{i=1}^{N} \partial_{x_i} f_i,$$

and the Laplacian and integral are defined component-wise as

$$\Delta \boldsymbol{f} \;:=\; (\Delta f_i)_{i\in[N]} \quad \text{and} \quad \int_\Omega \boldsymbol{f} \;:=\; \left(\int_\Omega f_i \, d\boldsymbol{x} \right)_{i\in[N]}.$$

For tensor valued $\boldsymbol{T} \in C^1(\Omega; \mathbb{R}^{N\times N})$, the divergence operator is

$$\mathbf{div}\, \boldsymbol{T} \;:=\; (\operatorname{div} \boldsymbol{T}_{i\cdot})_{i\in[N]}\,,$$

where $\boldsymbol{T}_{i\cdot}$ is the i-th row of $\boldsymbol{T}$. The Laplacian and integral are defined component-wise as

$$\Delta \boldsymbol{T} \;:=\; (\Delta T_{ij})_{i,j\in[N]} \quad \text{and} \quad \int_\Omega \boldsymbol{T} \;:=\; \left(\int_\Omega T_{ij} \, d\boldsymbol{x} \right)_{i,j\in[N]}.$$

Work In Progress (WIP)

1 Introduction

Hemodynamics studies the kinematics of blood. Our interest is the kinematic motion of blood within the human macrocirculatory system, i.e. the flow of blood in large vessels such as arteries and veins. Blood is observed as a complex fluid of formed elements suspended in plasma, thus, the rheological behavior of blood is non-trivial. We report techniques and methodologies for modeling blood's motion in large vessels.

Our report is organized as follows.

After stating our report's motivation, Sec. 1.1 provides a brief physiological review of the human circulatory system. Then Sec. 1.2 reviews the continuum hypothesis, a necessary postulate in fluid mechanics which treats blood as a continuous medium. This framing reduces the hemodynamic problem to describing blood's motion in a continuum. Then Sec. 1.3 discusses rheological assumptions that lead to constitutive relations between the material properties of blood. By conservation, the kinematic-viscosity Navier–Stokes (NS) equations are obtained (Sec. 1.5). The NS equations are a system of coupled system of nonlinear partial differential equations (PDEs), the foundation of our hemodynamic models.

Motivation The World Health Organization (WHO) claims cardiovascular diseases (CVDs) resulted in $\approx$ 32% of all deaths in 2022, the leading cause of death globally, where $\approx$ 85% of the deaths where from heart attack or stroke. In such cases the underlying CVD is often coronary artery stonosis (CAS), the narrowing of a coranary vessel due to the buildup of plaque. Such plaque is r.t. as a stenotic blockage. There's a need for evidience-based tools to predict and assess the severity of SBs, as often predicition of CAS doesn't mean an obstruction [[1]] and severity assesments often use a simple 0D-lummped-parameter model. Cariological interventionists need better tools for predicting and treating CAS.

Current clinical methods for assessing the severity of a SB rely on imaging techniques such as angiography, intravascular ultrasound (IVUS), and optical coherence tomography (OCT) to visualize the arteries and identify areas of narrowing. Such methods provide valuable information about the anatomy of the arteries, but they do not provide direct information about the functional significance of the CAS. Functional assessment of CAS typically involves measuring the fractional flow reserve (FFR), which is the ratio of the blood pressure downstream of the stenosis to the blood pressure upstream of the stenosis during maximum blood flow. However, measuring FFR requires the use of a pressure wire, which can be invasive and carries some risks. Therefore, there is a need for non-invasive methods to assess the functional significance of CAS.

cleanup, add refs

1.1 Physiology

The circulatory system is the human heart, vascular network, lungs, and organs. The systems source is the heart, transporting oxygen-rich blood to the organs and deoxygenated (and carbon dioxide-enriched) blood back to the lungs. Lungs discharge $CO2$ and enrich the blood with Oxygen. These processes are r.t. the

pulmonary circulation and the *systemic circulation* (resp.). The *macrocirculatory system* consists of the heart and the large vessels in the systemic circulation. Particularly, the arteries of the macrocirculatory system transport oxygenated blood from the heart, driving the return of deoxygenated blood in large vessels back to the heart.

A single beat of the heart propels blood through the macrocirculatory system, the "lub-dub" sound. The beat and the following sequence of events until the successive beat as the *cardiac cycle*. The cardiac cycle consists of two main phases: systole and diastole, during which the heart chamber is accumulating blood and releasing blood (resp.). The beat can be recognized as a pulse wave in large vessels, characterized by the Wormersley number

$$W_0 := 2R \cdot \sqrt{\frac{\omega\rho}{\eta}}, \quad : \ \rho \text{ is fixed blood density}$$

a dimensionless parameter comparing the frequency ω of the pulse wave to the blood's dynamic viscosity η and the vessel diameter $2R$. The Reynolds number Re characterizes flow in blood vessels

$$Re := 2R \cdot \frac{\rho U}{\eta}, \quad : \ \rho \text{ is fixed blood density}$$

Low Re indicates laminar flow, while high Re suggests turbulent flow.

Double check W_0 and Re definitions with book1 and hemodynamics refs

Observed Cardiac Cycle Characteristics

Normal resting heart rate is considered to be $\omega = 70$ bpm, so the cardiac cycle is approximately $0.86s$., consisting of:

1. Systole (ventricular contraction) ≈ 0.3 seconds.
2. Diastole (ventricular relaxation) ≈ 0.7 seconds.

The blood volume of a human is approximately 5.7-6.0 liters of blood, flowing a full cycle roughly every minute. The energy driving the flow comes from oxygen and nutrients absorbed from food, creating waste products that must be removed; the *coronary artery*'s responsibility. The buildup of waste products results in Arteriosclerosis, a narrowing of the coronary artery, leading to reduced and turbulent blood flow. (Add citations here of turbulence in the presence of stenotic arteries).

Note [3, Table 1.1, p. 10, §1.1] shows $W_0 \propto 2R$ and $Re \propto (2R)^{-1}$; we observe large pulses and turbulent flow in large vessels and small pulses and laminar flow in small vessels.

Constituents and hematocrit. Blood consists of plasma and formed elements which we call cells. Red blood cells (RBCs) comprise $\approx 97\%$ of the cellular volume, and cellular volume is approximately $\approx 45\%$ of the blood volume. The remaining $\approx 55\%$ of blood volume is plasma, which is $\approx 90\%$ water. The ratio of

RBC volume to total blood volume is the *hematocrit value* H, a key metric governing apparent viscosity η: as H increases, η typically increases (cf. S6.5.1 [2]). The formed elements suspended in plasma include white blood cells (WBCs) and platelets.

1.2 Continuum

W.a.t. simulate blood flow in a time-dependent fluid domain $\Omega_B(t) \subset \mathbb{R}^N$, where typically $N = 3$. For each $t \in I_T := [0, T] \subset \mathbb{R}$, with $T > 0$, we define

$$\Omega_B(t) := \{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{x} \text{ lies inside the vessel at time } t\}.$$

Formally, we model the fluid region as a time-dependent family of open sets $\{\Omega_B(t)\}_{t \in I_T} \subset \mathbb{R}^N$ occupied by blood at time t. Let the fluids spatial-temporal domain be

$$\Omega \ := \ \{(x, t) \in \mathbb{R}^{N+1} \ : \ x \in \Omega(t), \ t \in I_T\}.$$

For each $t \in I_T$, herein assume $\Omega_B(t)$ is simply connected and has Lipschitz continuous boundary. Thus, any closed curve in $\Omega_B(t)$ can be continuously contracted to a point within $\Omega_B(t)$, and standard results from potential theory (e.g. Poincaré-type lemmas) apply when introducing scalar potentials for irrotational vector fields. Under these regularity assumptions on $\partial\Omega_B$, standard trace theorems hold for Sobolev spaces (e.g. $H^1(\Omega_B(t))$). In particular, the outward unit normal is defined a.e. on $\partial\Omega_B$, so we can meaningfully speak of normal and tangential components of vector fields on the boundary.

Remark. Often we may impose stronger conditions on $\partial\Omega_B$, e.g., require that the boundary be a C^1 closed surface. This is reasonable since the vessel wall is typically smooth in healthy vessels. The additional regularity allows us to globally define geometric quantities such as curvature and surface differential operators on the fluids boundary $\partial\Omega_B$.

For derivations that follow, we let $N \equiv 3$ so that $\Omega_B(t) \in \mathbb{R}^3$. Bases of $\mathbb{R}^3$ are denoted $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and we let $\{\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}\}$ r.t. the standard orthonormal cartesian basis of $\mathbb{R}^3$.
Assume blood fluid is a continuum that deforms continously, i.e., at every point $\boldsymbol{x} \in \Omega_B(t)$ and time $t \in I_T$, the blood's kinematic quantities are described by sufficiently smooth fields. At microscopic scales the continuum hypothesis breaks down, since matter is a discrete collection of molecules, but at macroscopic scales empirical evidence suggests such models remain accurate. Let blood's velocity, thermodynamic pressure,

and density fields be

$$\boldsymbol{u} : \Omega_B(t) \to \mathbb{R}^3,\ (x,y,z) \mapsto (u_1(x,y,z),\, u_2(x,y,z),\, u_3(x,y,z))^\top, \quad \left[\ \frac{m}{s}\ \right]$$
$$p : \Omega_B(t) \to \mathbb{R},\ (x,y,z) \mapsto p(x,y,z), \quad \left[\ \text{Pa} \equiv \frac{N}{m^2}\ \right]$$
$$\rho : \Omega_B(t) \to \mathbb{R}^+,\ (x,y,z) \mapsto \rho(x,y,z), \quad \left[\ \frac{kg}{m^3}\ \right].$$

A fluid that deforms independent of time is simple, such fluids deformation and rate of deformation aren't subject to material memory effects (also r.t. as viscoelastic effects). The contrary are complex fluids, their deformation is subject to both viscous and elastic characteristics.

Definition 1. Let

$$D_t\phi \ :=\ \partial_t\phi + \boldsymbol{u}\cdot\text{div}(\phi),$$

be the material derivative $D_t\phi$ measuring the instantanous rate of change of quantity ϕ moving with velocity field $\boldsymbol{u}$. The literature also refers to D_t as the *substantial derivative*, *advective derivative*, *lagrangian derivative*, or *convective derivative*.

For ϕ subject to the flow $\boldsymbol{u}$, note that

$$D_t\phi(\boldsymbol{x},t) \ =\ \lim_{\Delta t\to 0} \frac{\phi(\boldsymbol{x}+\boldsymbol{u}(\boldsymbol{x},t)\,\Delta t, t+\Delta t) - \phi(\boldsymbol{x},t)}{\Delta t},$$

i.e., $D_t\phi$ is the instantanous rate of change of quantity ϕ if a material fluid element $V(t)_{\boldsymbol{x}}$ positioned at $\boldsymbol{x} \in \Omega(t)$ and moving with velocity $\boldsymbol{u}(x,t)$. By the incompressibile assumption of blood, the material density of $V(t)_{\boldsymbol{x}}$ is $\rho(\boldsymbol{x},t) \equiv \rho_0$ for all $\boldsymbol{x} \in \Omega(t)$ and $t \in I_T$. Implying $D_t\rho = 0$, so mass conservation in $\Omega(t)$ reads

$$\partial_t\rho + \text{div}(\rho\boldsymbol{u}) = 0 \quad \Longleftarrow \text{ incompressibility assumption}$$

By assumption blood is Newtonian, linear momentum balance follows from Newton's 2nd Law ($F = ma$) as

$$\rho(\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) \ =\ \mathbf{div}(\boldsymbol{T}) + \rho\boldsymbol{f_b}, \qquad \text{in } \Omega(t)$$

where $\boldsymbol{T}$ is the Cauchy stress tensor describing the fluids deformation, $\boldsymbol{f_b}$ may be some body force per unit mass $\left[\frac{m}{s^2}\right]$. Note that **div** acts rowise on $\boldsymbol{T}$, implying $\exists$ matrix $A\ :\ \boldsymbol{T}\boldsymbol{u} = \boldsymbol{T}\ \because$ blood is a simple fluid. Let

the rate-of-deformation tensor be defined as the symmetric part of the velocity gradient, i.e.,

$$\boldsymbol{D}(\boldsymbol{u}) := \tfrac{1}{2}\big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\big) \quad \text{s.t.} \quad \nabla \boldsymbol{u} := \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{bmatrix}.$$

Note that $\boldsymbol{u} \mapsto \boldsymbol{D}(\boldsymbol{u})$ captures spatial deformations of element $V_{\boldsymbol{x}}$ in Ω under flow $\boldsymbol{u}$.

Definition 2. A fluid is *Newtonian* if its Cauchy stress tensor $\boldsymbol{T}$ depends linearly on the rate-of-deformation tensor $\boldsymbol{D}(\boldsymbol{u})$.

Definition 3. A fluid is *isotropic* if its constitutive response is independent of the coordinate system. Writing the Cauchy stress as $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{D})$, isotropy means that for every orthogonal rotator $\boldsymbol{Q} \in \mathrm{SO}(3)$,

$$\boldsymbol{Q}\,\boldsymbol{T}(\boldsymbol{D})\,\boldsymbol{Q}^\top = \boldsymbol{T}\big(\boldsymbol{Q}\,\boldsymbol{D}\,\boldsymbol{Q}^\top\big).$$

Definition 4 (Newtonian, isotropic constitutive law)**.** For a Newtonian, isotropic fluid the Cauchy stress is

$$\boldsymbol{T} = -\,p\,\mathrm{I} + 2\,\eta\,\boldsymbol{D}(\boldsymbol{u}) + \lambda\,\mathrm{div}(\boldsymbol{u})\,\mathrm{I},$$

where $\eta > 0$ is the dynamic (shear) viscosity and λ the bulk viscosity.

So an isotropic fluid at rest (quiescent state $\boldsymbol{u} \equiv \boldsymbol{0}$) sustains only hydrostatic stress:

$$\implies \boldsymbol{T} = -\,p\,\mathrm{I} \quad \text{when } \boldsymbol{u} \equiv \boldsymbol{0}.$$

One may make a distinction between incompressible fluids and incompressible flows. feild $\boldsymbol{u}$.

Definition 5 (Incompressibile fluid)**.** An element $V \in \Omega$ with constant density $\rho(\boldsymbol{x}, t)$ for all $\boldsymbol{x} \in V(t), t \in I_T$ is an *incompressible fluid*.

Definition 6 (Incompressible flow)**.** An element $V \in \Omega$ subject to $\boldsymbol{u}$ with constant rate of material density change (in both space and time so that $D_t\rho = 0$ for all $t \in I_T$) undergoes an *incompressible flow* in Ω.

I.m.b.s. for all material elements $V \in \Omega$, 5 implies 6. Note, the converse is not generally true: an incompressible flow ($D_t\rho = 0$) only preserves density along particle paths and allows $\rho = \rho(\boldsymbol{x})$ to vary spatially; but, particularly, in our case the initial density is distributed uniformly in space, and here incompressible flow implies incompressible fluid.

Remark (Divergence-free condition). By assuming blood is incompressible fluid and flow, we have

$$
\begin{aligned}
& D_t\rho = 0 \\
& \iff \partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0 \\
& \iff \operatorname{div}(\rho_0\boldsymbol{u}) = 0 \quad (\because\ \rho = \rho_0) \\
& \iff \nabla\cdot(\rho_0\boldsymbol{u}) = 0 \\
& \iff \langle\rho_0\boldsymbol{u}, \nabla\rangle = 0 \\
& \iff \rho_0\langle\boldsymbol{u}, \nabla\rangle = 0 \\
& \iff \langle\boldsymbol{u}, \nabla\rangle = 0 \\
& \iff \nabla\cdot\boldsymbol{u} = 0,
\end{aligned}
$$

and we r.t. $\nabla\cdot\boldsymbol{u} = \operatorname{div}(\boldsymbol{u}) = 0$ as the *divergence-free condition* of $\boldsymbol{u}$ in $\Omega(t)$.

Consequently, the constitutive law simplifies as follows.

Definition 7 (Incompressible stress tensor)**.** The Cauchy stress 4 simplifies to

$$\boldsymbol{T} \;=\; -p\,\mathrm{I} + 2\,\eta\,\boldsymbol{D}(\boldsymbol{u}).$$

Remark (Dynamic vs. Kinematic Viscosity). In our model assumptions, *dynamic viscosity* $\eta \in \mathbb{R}^+$ and *kinematic viscosity* $\mu \in \mathbb{R}^+$ relate as

$$\mu \;:=\; \frac{\eta}{\rho} \;=\; \frac{\eta}{\rho_0} \;\in\; \mathbb{R}^+.$$

Here η quantifies the internal resistance of blood to shear deformation, i.e., $\eta := \dfrac{\tau}{\dot\gamma}$, with units $[Pa\cdot s]$. Moreover μ adjusts η by the density ρ, capturing the viscous diffusion of momentum per unit mass, with units $\left[\frac{m^2}{s}\right]$. Intuitively, η measures how "thick" or "sticky" the fluid is, while μ measures how quickly momentum diffuses through the fluid due to viscosity.

Remark (Newtonian Blood Justification). When diameter d and hematocrit effects are needed, one may use a Non-Newtonian model with relative viscosity $\eta_r(H,d)$ that scales an absolute baseline η:

$$\eta_{\text{eff}} \;=\; \eta_r(H,d)\,\eta \quad \text{(effective viscosity)}$$

An empirical fit from [4]

$$\eta_r \;=\; 1 + (\eta_{0.45} - 1)\,\frac{(1-H)^C - 1}{(1-0.45)^C - 1} \text{ s.t. } \begin{cases} \eta_{0.45} = 6\,e^{-0.085\,d} + 3.2 - 2.44\,e^{-0.06\,d^{0.645}}, \\ C = \left(0.8 + e^{-0.075\,d}\right)\left(\dfrac{1}{1+10^{-11}\,d^{12}} - 1\right) + \dfrac{1}{1+10^{-11}\,d^{12}}, \end{cases}$$

where $d := 2R/(1.0\mu m)$ is the (scaled) vessel diameter. In large vessels, η_r is often constant, justifying the Newtonian assumption. [[3], sec. 3.1]

1.3 Blood Model

One chooses a model based upon the specific application, computational resources, and desired accuracy. We construct models of blood flow in various geometries, starting from a single vessel, then extending our approach to bifurcations and arterial networks. Our strategy involves a *domain decomposition approach.*

We seek solutions to inital and boundary value problems of Eq. 15.

Definition 8. Let $\boldsymbol{u_0} : \Omega(0) \to R^3,\ \boldsymbol{x} \mapsto \boldsymbol{u_0}(x)\ :\ \boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u_0}$. We refer to $\boldsymbol{u_0}$ as the initial condition of velocity feild $\boldsymbol{u}$

Definition 9. Let $\boldsymbol{u^0}(t)$ and $\boldsymbol{u^1}(t)$ be the velocity feilds at S_0 and S_1.

In practice, $\boldsymbol{u_0}(z)$ may be prescribed or determined from sensor data.

1.4 Dimension-Reduced Models of Blood Flow

We start our disscusion with 1D and 0D models, reducing the d.o.f. in the NS system 15 by imposing further simplifying assumptions. Namely, w.a.t. compute average pressures and velocities after a relatively short simulation time by solving the 1D NS system in a compliant vessel with suitable side conditions. Because our model averages pressure and velocity over a surface-area, we obtain a uniform distribution of WSS on the vessel.

One may start by introducing a rigid-vessel assumption, which leads to a no slip condition that $\boldsymbol{u}\big|_{\partial\Omega(t)} = \boldsymbol{0}$. Instead we seek to model the link between blood flow and the deformation of the vessel wall. We begin our derivation of Dimension-Reduced models by assuming the following transformation exists of our fluid domain boundary $\Omega(t)$ to a simplified geometry.

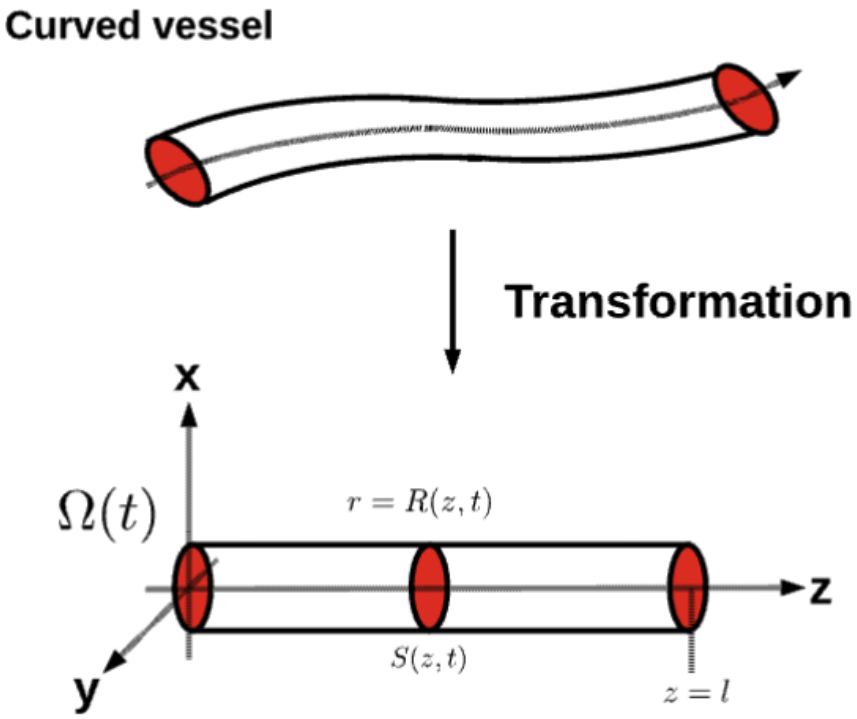
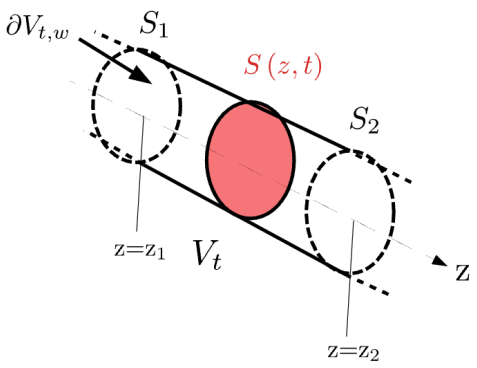


Figure 1: From [3] [Fig. 3.2, pg. 37]

We consider the fluid dynamics of the following fluid element contained in a portion of the lumen $\Omega(t)$ Let

Fig. 3.3 Notation describing the different parts of the vessel portion



‘’

Figure 2: From [3] [Fig. 3.3, pg. XX]

$S_1(t)$, $S_2(t)$ be the time-dependent shaded boundaries at $z = z_1$ and $z = z_2$ s.t. $0 < z_1 < z_2 < \ell$. Let V_t be the fluid element of blood. The boundary of the fluid element is $\partial V_t = S_1(t) \cup S_2(t) \cup \partial V_{t,w}$ such that $\partial V_{t,w}$ is the vessel wall in contact with the fluid element. According to the Reynold's transport theorem for

scalar feild $\phi \in L^1(\Omega(t))$.

$$\frac{d}{dt}\int_{V_t}\phi dV \;=\; \int_{V_t}\frac{\partial\phi}{\partial t}dV \;+\; \int_{\partial V_t}(\boldsymbol{u_b}\cdot\hat{\boldsymbol{n}})\phi dS$$

where $\boldsymbol{u_b}$ is the velocity feild deforming the boundary ∂V_t (Pf. see wiki). If we assume the normal component of $\boldsymbol{u_b} = \boldsymbol{0}$ near the inlet and outlet boundaries S_1 and S_2 (resp.) of Ω, then the motion of the vessel wall is coupled to the blood flow through the fluid element V_t. The velocity $\boldsymbol{u_b}$ is equivalent to the velocity of the vessel wall $\partial\Omega(t)$ in contact with the boundary element ∂V_t. I.e., the vessel wall velocity $\boldsymbol{u_w} = \boldsymbol{u_b}$. Now let $\boldsymbol{w} = \boldsymbol{u_w} - \boldsymbol{u}$ be the relative velocity of the vessel wall w.r.t. the velocity $\boldsymbol{u} = (u_1, u_2, u_3)^\top$ of the blood element V_t. Then it follows that

$$\begin{aligned}\int_{\partial V_t}(\boldsymbol{u_b}\cdot\hat{\boldsymbol{n}})\phi\, dS \;&=\; \int_{\partial V_t}(\boldsymbol{u_w}\cdot\hat{\boldsymbol{n}})\phi\, dS \\ &=\; \int_{\partial V_t}(\boldsymbol{w}\cdot\hat{\boldsymbol{n}})\phi\, dS \;+\; \int_{\partial V_t}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}})\phi\, dS\end{aligned}$$

Let $\bar{\phi}$ denote the average value of ϕ defined over a surface S

$$\bar{\phi} \;:=\; \frac{1}{A}\int_{S(z,t)}\phi\, dS \quad : \quad A(z,t) \;:=\; \int_{S(z,t)} dS$$

Now we may rewrite the volume integral in the LHS of RT theorem

$$\int_{V_t}\phi dV \;=\; \int_{z_1}^{z_2}\int_{S(z,t)}\phi\, dS\, dz \;=\; \int_{z_1}^{z_2} A\cdot\bar{\phi}\, dz$$

where $z_1 < z_2$ are fixed z-coordinates for S_1 and S_2. Then we differentiate the integrands in the above equation w.r.t. t

$$\int_{V_t}\frac{\partial\phi}{\partial t}dV \;=\; \int_{z_1}^{z_2}\frac{\partial}{\partial t}\left[A\cdot\bar{\phi}\right] dz,$$

and we've rewritten the first term in the RHS of the reynolds system. The surface integral in the RHS may be written as

$$\int_{\partial V_t}(\boldsymbol{u_b}\cdot\hat{\boldsymbol{n}})\phi dS = \int_{\partial V_t}(\boldsymbol{u_b}\cdot\hat{\boldsymbol{n}})\phi dS....$$

With a little more work, one may obtain:

cleanup, ref. pg. 43

Definition 10. The 1D Reynolds Transport theorem for both compressible and incompressible fluids:

$$\frac{\partial}{\partial t}\left(A\bar{\phi}\right) \;+\; \frac{\partial}{\partial z}\left(A(\overline{\phi\cdot u_3})\right) \;=\; \int_S\left(\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\boldsymbol{u})\right)dS \;+\; \int_{\partial S}\phi\boldsymbol{w}\cdot\hat{\boldsymbol{n}}\, d\gamma$$

Remark. By taking $f = \rho$ in 10, mass conservation follows directly. Also, by our assumption that blood is

incompressible, we have $\begin{cases} \operatorname{div}(\boldsymbol{u}) = 0 \\ \rho = \text{const.} \end{cases}$ and we simplify 10 as

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}\big(A(\overline{u_3})\big) = \int_{\partial S} \boldsymbol{w} \cdot \hat{\boldsymbol{n}}\, d\gamma$$

The RHS term above describing the transport process across the vessel wall.

complete, pg. 45

Remark. By taking $f = u_3$ in 10, momentum conservation follows directly. Also, by our assumption that blood is incompressible, we simplify 10 as

$$\frac{\partial}{\partial t}\Big(Au_3\Big) + \frac{\partial}{\partial z}\big(A(\overline{u_3^2})\big) = \int_S \left(\frac{\partial u_3}{\partial t} + \nabla u_3 \cdot \boldsymbol{u}\right) dS + \int_{\partial S} u_3 \boldsymbol{w} \cdot \hat{\boldsymbol{n}}\, d\gamma$$

The RHS term above describing the transport process across the vessel wall.

Remark (Tube law from a thin elastic cylindrical wall). We briefly justify the pressure–area relation used in aq_1d_compliant.jl

Definition 11.

$$p(A) - p_{ext} = \beta\big(\sqrt{A} - \sqrt{A_0}\big)$$

used in the 1D (A, Q) model. Consider a straight cylindrical vessel with (local) lumen radius $R(x,t)$, reference radius R_0, wall thickness $h \ll R$, and internal pressure $p(x,t)$ relative to an external pressure p_{ext} (assumed constant in space and time for simplicity). The corresponding lumen area is

$$A(x,t) = \pi R(x,t)^2, \qquad A_0 = \pi R_0^2.$$

Under the thin–wall assumption, balance of forces in the circumferential direction (Young–Laplace law) yields

Definition 12.

$$(p - p_{ext})\, 2\pi R = \sigma_\theta\, 2h\pi$$

where σ_θ is the circumferential (hoop) Cauchy stress in the vessel wall. We model the wall as linearly elastic in the hoop direction, so that

Definition 13.

$$\sigma_\theta = E_{eff}\, \varepsilon_\theta$$

with an effective circumferential modulus $E_{eff} > 0$ and circumferential strain

$$\varepsilon_\theta \;=\; \frac{\text{change in circumference} - \text{reference circumference}}{\text{reference circumference}} \qquad = \; \frac{2\pi R - 2\pi R_0}{2\pi R_0} \qquad = \; \frac{R - R_0}{R_0}.$$

Equating (12) and (13) gives

$$\begin{aligned}
(p - p_{ext})\, 2\pi R \;&=\; E_{eff}\, \frac{R - R_0}{R_0}\, 2h\pi \\
\iff (p - p_{ext})\, R \;&=\; E_{eff}\, \frac{R - R_0}{R_0}\, h
\end{aligned}$$

Express R and R_0 in terms of the areas A and A_0:

$$\begin{aligned}
R = \sqrt{\frac{A}{\pi}} \\
&= \frac{\sqrt{A}}{\sqrt{\pi}}, \\
R_0 = \sqrt{\frac{A_0}{\pi}} \quad &= \frac{\sqrt{A_0}}{\sqrt{\pi}},
\end{aligned}$$

so that

$$\begin{aligned}
R - R_0 \;&=\; \frac{\sqrt{A}}{\sqrt{\pi}} - \frac{\sqrt{A_0}}{\sqrt{\pi}} \\
&=\; \frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big).
\end{aligned}$$

Substituting into the expression for $p - p_{ext}$, we obtain

$$\begin{aligned}
(p - p_{ext})\, \frac{\sqrt{A}}{\sqrt{\pi}} \;&=\; E_{eff} h\, \frac{\frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big)}{R_0} \\
\iff (p - p_{ext})\, \frac{\sqrt{A}}{\sqrt{\pi}} \;&=\; E_{eff} h\, \frac{\frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big)}{R_0}.
\end{aligned}$$

For moderate deformations where A remains close to A_0, we approximate the factor $1/\sqrt{A}$ by its reference value $1/\sqrt{A_0}$, which yields

$$\begin{aligned}
(p - p_{ext})\, \frac{\sqrt{A_0}}{\sqrt{\pi}} \;&=\; E_{eff} h\, \frac{\frac{1}{\sqrt{\pi}}\big(\sqrt{A} - \sqrt{A_0}\big)}{R_0} \\
\iff (p - p_{ext}) \;&=\; \frac{E_{eff} h}{R_0 \sqrt{A_0}}\, \big(\sqrt{A} - \sqrt{A_0}\big).
\end{aligned}$$

Defining the lumped stiffness parameter

$$\beta \;:=\; \frac{E_{eff}h}{R_0\sqrt{A_0}}.$$

we arrive at the tube law (**??**) used in the 1D model:

$$p(A) - p_{ext} \;=\; \beta\big(\sqrt{A} - \sqrt{A_0}\big).$$

In the numerical experiments below, we take β and A_0 to be constant along the vessel, so that p can be written as a function of A alone.

1.4.1 0D Models

The 0D model, on the other hand, treats the vessel as a lumped parameter system, focusing on overall pressure and flow relationships without spatial resolution.

1.5 Navier-Stokes

Let $\boldsymbol{f}$ an external force acting on a continumm of blood fluid. When modeling non-Newtonian effects (when $\eta \neq$ constant), the kinematic viscosity $\mu(\cdot)$ is often chosen by Careau model [2]

$$2\,\mu(|\boldsymbol{D}|^2) = \eta_\infty + (\eta_0 - \eta_\infty)\cdot\big(1 + \kappa|\boldsymbol{D}|^2\big).$$

Where η_0 and η_∞ are chosen to be the viscosity for very small and very large shear rates, resp., and $\kappa \in \mathbb{R}^+$ and $n \in (-0.5, 0)$ are model parameters. According to [[3], pg. 38], we often set

$$\eta_0 = 65.7\cdot 10^{-3}\ Pa\cdot s,\ \eta_\infty = 4.45\cdot 10^{-3}\ Pa\cdot s, \kappa = 212.2\ s^2,\ \text{ and } n = -0.325$$

In the Newtonian case, we choose $\eta = \eta_\infty$ which allows us to determine μ as $\mu(|\boldsymbol{D}|^2) = \eta$. When coupling our momentum balance equation with the divergence-free condition of $\boldsymbol{u}$, we obtain the Navier-Stokes (NS) equations.

Definition 14 (Conservative-Momentum Balance Form)**.**

$$\begin{cases} \partial_t(\rho\boldsymbol{u}) \;+\; (\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \;=\; -\nabla p \;+\; \mathbf{div}\big(2\,\eta\,\boldsymbol{D}(\boldsymbol{u})\big) \;+\; \rho\,\boldsymbol{f}, \\ \mathrm{div}(\boldsymbol{u}) \;=\; 0, \qquad \rho \equiv \rho_0 > 0 \text{ (constant)}. \end{cases}$$

Since $\rho \equiv \rho_0$ and $\eta = \mu|\boldsymbol{D}|^2$, the advective form follows from 14.

Definition 15 (Generalized-Newtonian Navier-Stokes (NS))**.**

$$\begin{cases} \rho\big(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) = -\nabla p \;+\; \mathbf{div}\Big(2\,\mu(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big) \;+\; \rho\,\boldsymbol{f} \\ \operatorname{div}(\boldsymbol{u}) = 0, \end{cases}$$

We divide ρ and obtain the kinematic-viscosity form from 15.

Definition 16 (Kinematic-Viscosity Navier-Stokes (NS))**.**

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{\nabla p}{\rho} \;+\; \mathbf{div}\Big(\frac{2}{\rho}\,\mu(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big) \;+\; \boldsymbol{f}, \\ \operatorname{div}(\boldsymbol{u}) = 0, \end{cases}$$

Finally, we write the NS system in operator form.

Definition 17. We write Eq. 15 in standard form

$$\begin{cases} F\big(\partial_t \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla p,\, \boldsymbol{u},\, p;\, \rho, \mu\big) \;=\; \boldsymbol{f}, \\ \operatorname{div}(\boldsymbol{u}) \;=\; 0, \end{cases}$$

where $F\big(\partial_t \boldsymbol{u},\, \nabla\boldsymbol{u},\, \nabla p,\, \boldsymbol{u},\, p;\, \rho, \mu\big) \;:=\; \partial_t \boldsymbol{u} \;+\; (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \;+\; \dfrac{\nabla p}{\rho} \;-\; \mathbf{div}\Big(\dfrac{2}{\rho}\,\mu\,(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big)$

Remark. The NS equations are a non-linear coupled system of PDEs. The first equation follows from the balance of linear momentum, where the terms:

- $\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \rho \begin{bmatrix} \langle \boldsymbol{u}, \nabla \boldsymbol{u}_1 \rangle \\ \langle \boldsymbol{u}, \nabla \boldsymbol{u}_2 \rangle \\ \langle \boldsymbol{u}, \nabla \boldsymbol{u}_3 \rangle \end{bmatrix}$ is the convective term governing acceleration of fluid (non-linear).
- $\mathbf{div}\Big(2\,\mu(|\boldsymbol{D}|^2)\,\boldsymbol{D}\Big)$ is the diffusive term describing the viscouselastic behavior (linear since μ is constant).

The second equation is the continuity equation, a consequence of the assumed fluid properties of blood that lead to the divergence-free condition on $\boldsymbol{u}$. The total system comprises of four equations in four unknowns: the three components of the velocity field $\boldsymbol{u}$ and the pressure field p.

If pressure p and the velocity $\boldsymbol{u}$ are given, the Cauchy stress $\boldsymbol{T}$ is computed from Eq. 7. It follows that the wall shear stress (WSS) at the vessel wall is:

$$\text{WSS} \;:=\; \langle \boldsymbol{t_{blood}},\, \boldsymbol{T}\,\hat{n}\rangle \quad : \quad \begin{cases} t_{blood} \text{ is tangent of a flow line through a cross-sectional area} \\ \hat{n} \text{ is outer normal of the cross-sectional area} \end{cases}$$

Forgoing the rigid-wall assumption allows us to model the relationship between the vessel wall and blood flow. Applicable models are referred to as fluid-structure interaction (FSI) models.

discuss in a later section

1.5.1 NS in Cylindrical Coordinates

Let our vessel wall $\partial\Omega = [0,T]\times\Omega(t)$ be a surface in $\mathbb{R}^4$ that evolves in time which we refer to as the interface. Let $\overline{\Omega} = \partial\Omega \cup \Omega$ be the closed and compact region enclosed by our interface. So the region enclosed by our interface is Ω, and we aim to model the velocity and pressure fields on Ω.

The relationship between cartesian and cylidrical coordinates is

$$(x,\ y,\ z)\ \mapsto (r\sin(\theta),\ r\cos(\theta),\ z), \quad r = \sqrt{x^2+y^2}.$$

Assume a vessel of length L is aligned with the z-axis whose cross-section is circular with radius $R(z,t)$ at axial position z and time t. Our fluid domain becomes

$$\Omega(t)\ =\ \{(r,\ ,\theta\ ,z)\in\mathbb{R}^3\ :\ r\in[0,R(z,t)],\ \theta\in[0,2\pi),\ z\in[0,l)\}$$

where $R(z,t)$ is the vessel radius at axial position z and time t.

Ref. notes for further details on deriving the transformation rules for vector calculus operators, or, notes where I do the derivation directly. Then discuss simplifications for axisymmetric flow.

2 Appendix

Bibliography

References

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[2] Giovanni P. Galdi et al. *Hemodynamical Flows: Modeling, Analysis and Simulation*. Birkhäuser Basel, 2008. DOI: https://doi.org/10.1007/978-3-7643-7806-6.

[3] Tobias Köppl and Rainer Helmig. *Dimension Reduced Modeling of Blood Flow in Large Arteries. An Introduction for Master Students and First Year Doctoral Students*. Springer Nature Switzerland, 2023.

[4] Gaehtgens P Pries AR Neuhaus D. "Blood viscosity in tube flow: dependence on diameter and hematocrit". In: *Am J Physiol.* (6 Pt 2).263 (1992). DOI: 10.1152/ajpheart.1992.263.6.H1770.

Code Listings

Code listings

Code 1: Algorithm 16.5

```
        function foo()
            println("Hello World)
        end
```