Modeling Blood Flow in Stenotic Arteries

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October 24, 2025

Collection of notes pertaining to modeling blood flow in stenotic arteries. This document is a work in progress, and will be updated as new information is learned.

Contents

1 Introduction

Current methods for predicting coronary artery stenosis are rudimentary; and often prediction does mean coronary artery stenosis obstruction. (ref)

Are we merely intrested in predicting an obstruction, or, are we interested in predicting the risk associated with the obstruction? TODO: continue to define research problem(s) of interest.

We perform a literature survey of arterial blood flow using known methods from the literature, with the hope of understanding the computational challenges and tradeoffs of various *mathematical models*.

The human cardiovascular system supplys the human organs with blood, also known as the circulatory system. We study various mathematical models describing the dynamics of the circularty system. The principal quantities of our models are the bloods velocity $\boldsymbol{u}$ and pressure $\boldsymbol{p}$, and throughout it will be assumed that we are modeling a segment of the circulatory sytem restrict corresponding to an arterial region. Knowing the pressure and velocity for a portion of time allows us to compute stresses to which an arterial wall is subjected to due to the blood movement.

Note, our arterial boundary $\partial\Omega$ is a surface in $\mathbb{R}^3$ that evolves in time which we refer to as the interface. Note, an explicit representation of $\partial\Omega$ maps $(t, \theta, r, \theta, z) \mapsto (x, y, z)$. At times, it may be advantagoues to consider $\partial\Omega$ as an implicit interface, where the isocontors of our boundary represent the interface... .

The region enclosed our interface is Ω, and we aim to model the velocity and pressure fields on Ω; which also is evolving in time.

This is a bit weird, look into the cylindrical coordinates implicit dynamical surface representations of our arterory

In *continuum mechanics*, matter is modeled by *continuous fields*. For instance, to model the behavior of an *incompressible* fluid in time, one could use a **pressure field**:

$$p : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \quad (t, x) \mapsto p(t, x)$$

and a **velocity field**:

$$\mathbf{u} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (t, x) \mapsto \mathbf{u}(t, x)$$

$$\mathbf{u}(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$$

At *atomic scales*, this model breaks down—fluids are discrete collections of molecules, not continuous fields. However, one can hope that such models remain accurate at macroscopic scales.

Make some comment about ”correct terminology for describing the types of coronary arterial stenosis is ”coronary artery steno-

2 Preliminaries

2.1 The Navier-Stokes Equations

The fundamental mathematical model for incompressible fluids (e.g., water) is the **viscous incompressible Navier-Stokes equations** (NS):

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p, & \text{(Momentum equation: Newton's law } F = ma\text{)} \\ \nabla\cdot\mathbf{u} = 0, & \text{(Incompressibility condition)} \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & \text{(Initial conditions)} \end{cases}$$

where $\nu > 0$ is the *viscosity* of the fluid (the limiting case $\nu = 0$ corresponds to the *Euler equations*).
The initial velocity field is given by:

$$\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3, \quad \nabla\cdot\mathbf{u}_0 = 0.$$

Global Regularity Problem

Given smooth, spatially localized initial data

$$\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3,$$

do there exist smooth solutions

$$\mathbf{u} : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}^3, \quad p : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$$

to the Navier-Stokes equations?

Local Existence Theorem

For any smooth, spatially localized initial data $\mathbf{u}_0$, there exists a **maximal time of existence** $0 < T_* \leq +\infty$ and a solution.
If $T_* \neq +\infty$, then **blowup** occurs in the sense that

$$\sup_{x\in\mathbb{R}^3} |\mathbf{u}(t,x)| \to +\infty \quad \text{as} \quad t \to T_*.$$

If $T_* = +\infty$, no blowup occurs, and $|\mathbf{u}| \to 0$ as $t \to +\infty$.

Turbulence and Regularity

Numerical simulations suggest that global regularity is generally **true**—for *most* choices of initial data $\mathbf{u}_0$, one has a global smooth solution to (NS).

However, for large initial data, turbulence can occur. Energy moves from coarse to fine scales in a complex way before dissipating due to viscous effects.

Dimensional Analysis

The Navier-Stokes equation consists of a balance between:

- The **transport term**: $\mathbf{u} \cdot \nabla \mathbf{u}$
- The **dissipation term**: $\nu \Delta \mathbf{u}$

Heuristically, if $\nu \Delta \mathbf{u} \gg (\mathbf{u} \cdot \nabla)\mathbf{u}$, one expects linear, non-turbulent behavior (*global regularity*).

If $(\mathbf{u} \cdot \nabla)\mathbf{u} \gg \nu \Delta \mathbf{u}$, one expects **nonlinear turbulence** (and possibly **blowup**).

Kinetic Energy Considerations

The kinetic energy is given by:

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}(t, x)|^2 \, dx.$$

This quantity is *decreasing in time* due to dissipation. Heuristically:

$$E(t) \gtrsim V^2 L^3.$$

This leads to a bound:

$$V = O(L^{-3/2}).$$

Blowup Scenarios

This suggests a **possible blowup scenario**. If the velocity field concentrates in a ball of radius L_1, we estimate:

$$|\mathbf{u}| \approx L_1^{-3/2}.$$

At a later time t_2, the energy has concentrated further into a ball of radius $L_2 = L_1/2$, so:

$$|\mathbf{u}| \approx L_2^{-3/2}.$$

This leads to **potential infinite velocity** at a single spatial point:

$$\sum_n L_n^{r_2} < \infty.$$

At each stage, the dissipative forces are negligible compared to the transport effects. This results in an approximately self-similar solution.

Perfectly self-similar solutions are known to not exist for Navier-Stokes.

Final Question

But can one actually <u>construct</u> such an approximately self-similar solution?

Thanks to Tarrance Tao for the wonderful lecture.

3 Literature Review

3.1 Review of Modeling Blood Flow in Human Circulatory System

We start by simulating a cut region, i.e., an arterial segment. We perform numerical simulations of various models to understand the following:

> Suppose there are n stenotic regions within our arterial segment and j regions are determined to be high risk. Standard protocol suggests to remove the j regions regions; but in what order? Also, how are the blood flow dynamics effected at each step of cardiologists intervention. Additionally, consider the first step of intervention, which region should be focused on first? How does the removal of the first stenotic region effect the remaining $n-1$ regions (it may release debry to only get blocked in a later region). What is the best by-pass configuration?

3.2 Extended 1d model for blood flow within a stenotic artery

4 Apendix

References

Code Listings

Optional Space for sumplementary code listings of computations done while investigating

Code 1: Algorithm 16.5

```
function foo()
    println("Hello World)
end
```