Solving Quadratic Equations

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It is well known how to solve polynomial equations of the first degree. For a first degree equation $ax + b = 0$ with $a \neq 0$ the solution is $x = -\frac{b}{a}$. We now look at solving $ax^2 + bx + c = 0$.

To Prove: The equation $ax^2 + bx + c = 0$ with $a \neq 0$ has the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

Proof: We use the method of completing the square to rewrite $ax^2 + bx + c$.

$$\begin{align}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \tag{2}\\
&= a\left(x^2 + \frac{b}{a}x + \frac{b}{2a}\right)^2 - \frac{b}{2a}^2 + c \tag{3}\\
&= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \tag{4}\\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{\left(b^2 - 4ac\right)}{4a}. \tag{5}
\end{align}$$

Therefore $ax^2 + bx + c = 0$ can be rewritten as

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{\left(b^2 - 4ac\right)}{4a} = 0, \tag{6}$$

which can in turn be rearranged as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{\left(b^2 - 4ac\right)}{4a^2}. \tag{7}$$

Taking square roots gives

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \tag{8}$$

which implies

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{9}$$

as required.